Circuit Theory and Networks

Analysis and Synthesis

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Dedicated to

My Father

Late Shri Ramsagar Singh

and

My Mother

Late Shrimati Premsheela Singh

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Preface

Overview

Circuit Theory and Networks (Analysis and Synthesis) is an important subject for third-semester students of of Electronics & Telecommunication Engineering and Electronics Engineering. With lucid and brief theory, this textbook provides thorough understanding of the topics of this subject. Following a problem-solving approach and discussing both analysis and synthesis of networks, it offers good coverage of dc circuits, network theorems, two-port networks, and network synthesis.

Generally, numerical problems are expected in university examinations in this subject. The weightage given to problems in examinations is more than 70–80%. Questions from important topics of this subject are part of competitive examinations such as IAS, IES, etc. Hence, numerous solved examples and exercise problems are included in each chapter of this book to help students develop and master problem-solving skills required to ace any examination with confidence. Objective-type questions from various competitive examinations are also included at the end of each chapter for easy revision of core concepts.

Salient Features

- Up-to-date and full coverage of the latest syllabus of University of Mumbai
- Covers both analysis and synthesis of networks
- Uses problem-solving approach to explain topics
- Lucid coverage of network theorems, transient analysis, two-port networks, network synthesis
- Extensively supported by illustrations
- Examination-oriented excellent pedagogy:
  - ✔ *Illustrations: 1500+*
  - ✔ *Solved Examples within chapters 539*
  - ✔ *Unsolved Problems: 195*
  - ✔ *Objective Type Questions: 130*

Chapter Organisation

This text is organised into 11 chapters. Chapter 1 covers basic circuit elements and laws comprising networks. Chapter 2 elucidates DC network theorems while AC network theorems are covered in Chapter 3. Chapter 4 discusses about magnetic circuits. Further, Chapter 5 discusses the concepts of graph theory. Chapters 6 and 7 elaborate upon transient analysis in time domain and frequency domain, respectively. Chapters 8 and 9 cover network functions and two-port networks. Chapter 10 deals with network synthesis. Lastly, Chapter 11 describes filters.

Acknowledgements

My acknowledgements would be incomplete without a mention of the contribution of my family members. I feel indebted to my father and mother for their lifelong inspiration. I also send a heartfelt thanks to my wife Nitu; son Aman; and daughter Aditri, for always motivating and supporting me during the preparation of the project. I appreciate the support extended by the team at McGraw Hill Education (India), especially Hemant K Jha, Navneet Kumar, Satinder Singh Baveja, Anuj Shrivastava and Jagriti Kundu during the editorial, copyediting and production stages of this book.

Suggestions for improvements will always be welcome.

Ravish R Singh

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Roadmap to the Syllabus

(As per latest revised syllabus of University of Mumbai)
This text is useful for
Circuit Theory and Networks—ECC304

Module 1: Electrical Circuit Analysis

1.1 Analysis of DC and AC Circuits: Analysis of circuits with and without controlled sources using generalized loop and node matrix methods, circuit theorems: Superposition, Thevenin's, Norton's, Maximum Power Transfer and Reciprocity theorems
1.2 Magnetic Circuits: Concept of self and mutual inductances, coefficient of coupling, dot convention, equivalent circuit, coupled circuit-solution using mesh analysis

GO TO:
CHAPTER 1. BASIC CIRCUIT CONCEPTS
CHAPTER 2. ANALYSIS OF DC CIRCUITS
CHAPTER 3. ANALYSIS OF AC CIRCUITS
CHAPTER 4. MAGNETIC CIRCUITS

Module 2: Graph Theory

2.1 Objectives of graph theory, linear oriented graphs, graph terminologies, matrix representation of a graph: incidence matrix, circuit matrix, cut-set matrix, reduced incident matrix, tieset matrix, f-cutset matrix.
2.2 Relationship between sub matrices A, B and Q.
2.3 KVL and KCL using matrix

GO TO:
CHAPTER 5. GRAPH THEORY

Module 3: Time and Frequency Domain Analysis

3.1 Time Domain Analysis of R-L and R-C Circuits: Forced and natural response, initial and final values solution using first order differential equation for impulse, step, ramp, exponential and sinusoidal signals
3.2 Time Domain Analysis of R-L-C Circuits: Forced and natural response, effect of damping factor, solution using second order equation for step, ramp, exponential and sinusoidal signals.

3.3 Frequency Domain Analysis: Frequency-domain representation of R, L, C, initial value theorem and final value theorem, applications of Laplace Transform in analyzing electrical circuits

GO TO:
CHAPTER 6. TIME DOMAIN ANALYSIS OF RLC CIRCUITS
CHAPTER 7. FREQUENCY DOMAIN ANALYSIS OF RLC CIRCUITS

Module 4: Network Functions

4.1 Network functions for the one port and two port networks, driving point and transfer functions, poles and zeros of network functions, necessary condition for driving point functions, necessary condition for transfer functions, calculation of residues by analytical and graphical methods, time domain behavior as related to the pole-zero plot, stability and causality, testing for Hurwitz polynomial

4.2 Analysis of ladder and symmetrical lattice network

GO TO:
CHAPTER 8. NETWORK FUNCTIONS

Module 5: Two Port Networks

5.1 Parameters: Open circuits, short circuit, transmission and hybrid parameters, relationship among parameters, conditions for reciprocity and symmetry

5.2 Interconnections of two port network T and π representation

5.3 Terminated two port network

GO TO:
CHAPTER 9. TWO-PORT NETWORKS

Module 6: Synthesis of RLC circuits

6.1 Positive Real Functions: Concept of positive real function, testing for necessary and sufficient conditions for positive real functions

6.2 Synthesis of LC, RC and RL Circuits: Properties of LC, RC and RL driving point functions, LC, RC and RL network synthesis in Cauer-I and Cauer-II , Foster-I and Foster-II forms

GO TO:
CHAPTER 10. SYNTHESIS OF RLC CIRCUITS

This text is useful for
Electrical Network Analysis and Synthesis—ELX304

Module 1: Analysis of DC Circuits

1.1 DC Circuit Analysis: Analysis of DC circuits with dependent sources using generalized loop, node matrix analysis.
1.2 Application of Network Theorems to DC Circuits: Superposition, Thevenin, Norton, Maximum Power Transfer and Millman theorems.

GO TO:
CHAPTER 1. BASIC CIRCUIT CONCEPTS
CHAPTER 2. ANALYSIS OF DC CIRCUITS

Module 2: Analysis of AC Circuits

2.1 Analysis of Steady State AC circuits: Analysis of AC circuits with independent sources using generalized loop, node matrix analysis.
2.2 Application of Network Theorems to AC Circuits: Superposition, Thevenin, Norton, Maximum Power Transfer and Millman theorems.
2.3 Analysis of Coupled Circuits: Self and mutual inductances, coefficient of coupling, dot convention, equivalent circuit, solution using loop analysis.

GO TO:
CHAPTER 3. ANALYSIS OF AC CIRCUITS
CHAPTER 4. MAGNETIC CIRCUITS

Module 3: Time and Frequency Domain Analysis of Electrical Networks

3.1 Time Domain Analysis of R-L and R-C Circuits: Forced and natural responses, time constant, initial and final values.
3.2 Solution using First Order Equation for Standard Input Signals: Transient and steady state time response, solution using universal formula.
3.3 Frequency Domain Analysis of RLC Circuits: S-domain representation, concept of complex frequency, applications of Laplace Transform in solving electrical networks, driving point and transfer function, poles and zeros, calculation of residues by analytical and graphical method.

GO TO:
CHAPTER 6. TIME DOMAIN ANALYSIS OF RLC CIRCUITS
CHAPTER 7. FREQUENCY DOMAIN ANALYSIS OF RLC CIRCUITS
CHAPTER 8. NETWORK FUNCTIONS

Module 4: Two Port Networks

4.1 Parameters: Open circuit, short circuit, transmission and hybrid parameters, relationships among parameters, reciprocity and symmetry conditions

4.2 Series/Parallel Connection: T and Pi representations, interconnection of two port networks.

GO TO:
CHAPTER 9. TWO-PORT NETWORKS

Module 5: Synthesis of RLC Circuits

5.1 Positive Real Functions: Concept of positive real function, testing for Hurwitz polynomials, testing for necessary and sufficient conditions for positive real functions.

5.2 Synthesis of RC, RL, LC Circuits: Concepts of synthesis of RC, RL, LC driving point functions.

GO TO:
CHAPTER 10. SYNTHESIS OF RLC CIRCUITS

Module 6: Filters

6.1 Basic Filter Circuits: Low pass, high pass, band pass and band stop filters, transfer function, frequency response, cut-off frequency, bandwidth, quality factor, attenuation constant, phase shift, characteristic impedance.

6.2 Design and Analysis of Filters: Constant K filters

GO TO:
CHAPTER 11. FILTERS

1

Basic Circuit Concepts

1.1 INTRODUCTION

We know that like charges repel each other whereas unlike charges attract each other. To overcome this force of attraction, a certain amount of work or energy is required. When the charges are separated, it is said that a potential difference exists and the work or energy per unit charge utilised in this process is known as voltage or potential difference.

The phenomenon of transfer of charge from one point to another is termed current. Current (I) is defined as the rate of flow of electrons in a conductor. It is measured by the number of electrons that flow in unit time.

Energy is the total work done in the electric circuit. The rate at which the work is done in an electric circuit is called electric power. Energy is measured in joules (J) and power in watts (W).

1.2 RESISTANCE

Resistance is the property of a material due to which it opposes the flow of electric current through it.

Certain materials offer very little opposition to the flow of electric current and are called conductors, e.g., metals, acids and salt solutions. Certain materials offer very high resistance to the flow of electric current and are called insulators, e.g., mica, glass, rubber, Bakelite, etc.

The practical unit of resistance is **ohm** and is represented by the symbol Ω. A conductor is said to have resistance of one ohm if a potential difference of one volt across its terminals causes a current of one ampere to flow through it.

The resistance of a conductor depends on the following factors.

(i) It is directly proportional to its length.
(ii) It is inversely proportional to the area of cross section of the conductor.
(iii) It depends on the nature of the material.
(iv) It also depends on the temperature of the conductor.

Hence,

$$R \propto \frac{l}{A}$$

$$R = \rho \frac{l}{A}$$

where l is length of the conductor, A is the cross-sectional area and ρ is a constant known as specific resistance or resistivity of the material.

1. Power Dissipated in a Resistor We know that $v = R\,i$

When current flows through any resistor, power is absorbed by the resistor which is given by

$$p = v\,i$$

The power dissipated in the resistor is converted to heat which is given by

$$E = \int_0^t v\,i\,dt = \int_0^t R\,i\,i\,dt = i^2\,Rt$$

1.3 INDUCTANCE

Inductance is the property of a coil that opposes any change in the amount of current flowing through it. If the current in the coil is increasing, the self-induced emf is set up in such a direction so as to oppose the rise of current. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage.

Inductance is defined as the ratio of flux linkage to the current flowing through the coil. The practical unit of inductance is **henry** and is represented by the symbol H. A coil is said to have an inductance of one henry if a current of one ampere when flowing through it produces flux linkages of one weber-turn in it.

The inductance of an inductor depends on the following factors.

(i) It is directly proportional to the square of the number of turns.
(ii) It is directly proportional to the area of cross section.
(iii) It is inversely proportional to the length.
(iv) It depends on the absolute permeability of the magnetic material.

Hence,

$$L \propto \frac{N^2 A}{l}$$

$$L = \mu \frac{N^2 A}{l}$$

where l is the mean length, A is the cross-sectional area and μ is the absolute permeability of the magnetic material.

1. Current–Voltage Relationships in an Inductor We know that

$$v = L\frac{di}{dt}$$

Expressing inductor current as a function of voltage,

$$di = \frac{1}{L} v\,dt$$

Integrating both the sides,

$$\int_{i(0)}^{i(t)} di = \frac{l}{L}\int_0^t v\,dt$$

$$i(t) = \frac{1}{L}\int_0^t v\,dt + i(0)$$

The quantity $i(0)$ denotes the initial current through the inductor. When there is no initial current through the inductor,

$$i(t) = \frac{1}{L}\int_0^t v\, dt$$

2. *Energy Stored in an Inductor* Consider a coil of inductance L carrying a changing current I. When the current is changed from zero to a maximum value I, every change is opposed by the self-induced emf produced. To overcome this opposition, some energy is needed and this energy is stored in the magnetic field. The voltage v is given by

$$v = L\frac{di}{dt}$$

Energy supplied to the inductor during interval dt is given by

$$dE = v\, i\, dt = L\frac{di}{dt}i\, dt = L\, i\, dt$$

Hence, total energy supplied to the inductor when current is increased from 0 to I amperes is

$$E = \int_0^I dE = \int_0^I L\, i\, di = \frac{1}{2}L\, I^2$$

1.4 CAPACITANCE

Capacitance is the property of a capacitor to store an electric charge when its plates are at different potentials. If Q coulombs of charge is given to one of the plates of a capacitor and if a potential difference of V volts is applied between the two plates then its capacitance is given by

$$C = \frac{Q}{V}$$

The practical unit of capacitance is **farad** and is represented by the symbol F. A capacitor is said to have capacitance of one farad if a charge of one coulomb is required to establish a potential difference of one volt between its plates.

The capacitance of a capacitor depends on the following factors.

(i) It is directly proportional to the area of the plates.
(ii) It is inversely proportional to the distance between two plates.
(iii) It depends on the absolute permittivity of the medium between the plates.

Hence,

$$C \propto \frac{A}{d}$$

$$C = \varepsilon\frac{A}{d}$$

where d is the distance between two plates, A is the cross-sectional area of the plates and ε is absolute permittivity of the medium between the plates.

1. Current–Voltage Relationships in a Capacitor The charge on a capacitor is given by

$$q = Cv$$

where q denotes the charge and v is the potential difference across the plates at any instant.
We know that

$$i = \frac{dq}{dt} = \frac{d}{dt}Cv = C\frac{dv}{dt}$$

Expressing capacitor voltage as a function of current,

$$dv = \frac{1}{C} i\, dt$$

Integrating both the sides,

$$\int_{v(0)}^{v(t)} dv = \frac{1}{C}\int_0^t i\, dt$$

$$v(t) = \frac{1}{C}\int_0^t i\, dt + v(0)$$

The quantity v (0) denotes the initial voltage across the capacitor. When there is no initial voltage on the capacitor,

$$v(t) = \frac{1}{C}\int_0^t i\, dt$$

2. Energy Stored in a Capacitor Let a capacitor of capacitance C farads be charged from a source of V volts. Then current i is given by

$$i = C\frac{dv}{dt}$$

Energy supplied to the capacitor during interval dt is given by

$$dE = v\, i\, dt = vC\frac{dv}{dt}dt$$

Hence, total energy supplied to the capacitor when potential difference is increased from 0 to V volts is

$$E = \int_0^V dE = \int_0^V Cv\, dv = \frac{1}{2}CV^2$$

1.5 SOURCES

Source is a basic network element which supplies energy to the networks. There are two classes of sources, namely,

1. Independent sources
2. Dependent sources

1.5.1 Independent Sources

Output characteristics of an independent source are not dependent on any network variable such as a current or voltage. Its characteristics, however, may be time-varying. There are two types of independent sources:

1. Independent voltage source
2. Independent current source

1. *Independent Voltage Source* An independent voltage source is a two-terminal network element that establishes a specified voltage across its terminals. The value of this voltage at any instant is independent of the value or direction of the current that flows through it. The symbols for such voltage sources are shown in Fig. 1.1.

The terminal voltage may be a constant, or it may be some specified function of time.

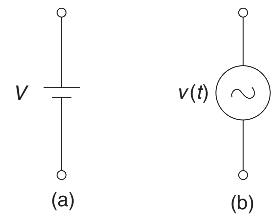


Fig. 1.1 *Symbols for independent voltage source*

2. *Independent Current Source* An independent current source is a two-terminal network element which produces a specified current. The value and direction of this current at any instant is independent of the value or direction of the voltage that appears across the terminals of the source. The symbols for such current sources are shown in Fig. 1.2.

The output current may be a constant or it may be a function of time.

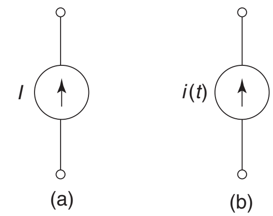


Fig. 1.2 *Symbols for independent current source*

1.5.2 Dependent Sources

If the voltage or current of a source depends in turn upon some other voltage or current, it is called as dependent or controlled source. The dependent sources are of four kinds, depending on whether the control variable is voltage or current and the controlled source is a voltage source or current source.

1. *Voltage-Controlled Voltage Source (VCVS)* A voltage-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to a voltage v_{ab} between two points a and b.

The symbol for such a source is shown in Fig. 1.3.

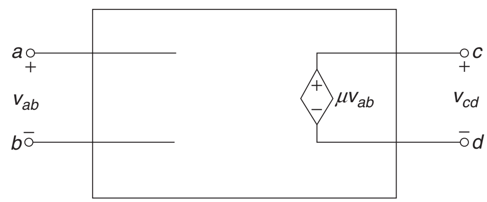


Fig. 1.3 *Symbol for VCVS*

The (+) and (−) sign inside the diamond of the component symbol identifies the component as a voltage source.

$$v_{cd} = \mu\, v_{ab}$$

The voltage v_{cd} depends upon the control voltage v_{ab} and the constant μ, a dimensionless constant called voltage gain.

2. *Voltage-Controlled Current Source (VCCS)* A voltage-controlled current source is a four-terminal network component that establishes a current i_{cd} in a branch of the circuit that is proportional to the voltage v_{ab} between two points a and b.

The symbol for such a source is shown in Fig. 1.4.

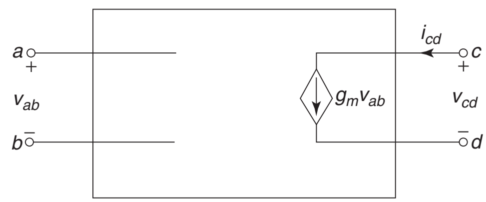


Fig. 1.4 *Symbol for VCCS*

The arrow inside the diamond of the component symbol identifies the component as a current source.

$$i_{cd} = g_m\, v_{ab}$$

The current i_{cd} depends only on the control voltage v_{ab} and the constant g_m, called the transconductance or mutual conductance. The constant g_m has dimension of ampere per volt or siemens (S).

3. ***Current-Controlled Voltage Source (CCVS)*** A current-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to the current i_{ab} in some branch of the circuit.

   The symbol for such a source is shown in Fig. 1.5.

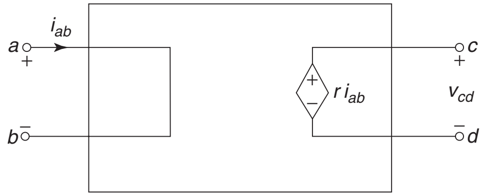


Fig. 1.5 *Symbol for CCVS*

$$v_{cd} = r\, i_{ab}$$

The voltage v_{cd} depends only on the control current i_{ab} and the constant r called the transresistance or mutual resistance. The constant r has dimension of volt per ampere or ohm (Ω).

4. ***Current-Controlled Current Source (CCCS)*** A current-controlled current source is a four-terminal network component that establishes a current i_{cd} in one branch of a circuit that is proportional to the current i_{ab} in some branch of the network.

   The symbol for such a source is shown in Fig. 1.6.

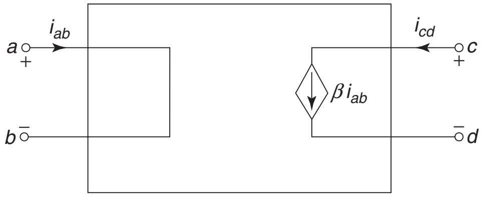


Fig. 1.6 *Symbol for CCCS*

$$i_{cd} = \beta\, i_{ab}$$

The current i_{cd} depends only on the control current i_{ab} and the dimensionless constant β, called the current gain.

1.6 SOME DEFINITIONS

1. ***Network and Circuit*** The interconnection of two or more circuit elements (viz., voltage sources, resistors, inductors and capacitors) is called an *electric network*. If the network contains at least one closed path, it is called an electric circuit. Every circuit is a network, but all networks are not circuits. Figure 1.7(a) shows a network which is not a circuit and Fig. 1.7(b) shows a network which is a circuit.

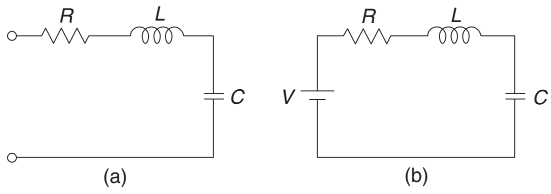


Fig. 1.7 *(a) Network which is not a circuit*
(b) Network which is a circuit

2. ***Linear and Non-linear Elements*** If the resistance, inductance or capacitance offered by an element does not change linearly with the change in applied voltage or circuit current, the element is termed as *linear element*. Such an element shows a linear relation between voltage and current as shown in Fig. 1.8. Ordinary resistors, capacitors and inductors are examples of linear elements.

A non-linear circuit element is one in which the current does not change linearly with the change in applied voltage. A semiconductor diode operating in the curved region of characteristics as shown in Fig. 1.8 is common example of non-linear element.

Other examples of non-linear elements are voltage-dependent resistor (VDR), voltage-dependent capacitor (varactor), temperature-dependent resistor (thermistor), light-dependent resistor (LDR), etc. Linear elements obey Ohm's law whereas non-linear elements do not obey Ohm's law.

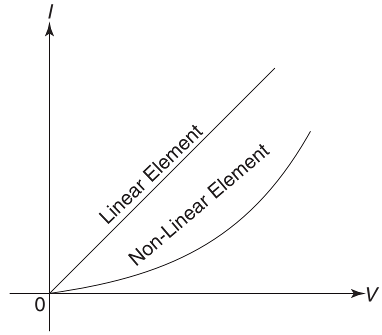


Fig. 1.8 *V-I characteristics of linear and non-linear elements*

3. ***Active and Passive Elements*** An element which is a source of electrical signal or which is capable of increasing the level of signal energy is termed as *active element*. Batteries, BJTs, FETs or OP-AMPs are treated as active elements because these can be used for the amplification or generation of signals. All other circuit elements, such as resistors, capacitors, inductors, VDR, LDR, thermistors, etc., are termed *passive elements.* The behaviour of active elements cannot be described by Ohm's law.
4. ***Unilateral and Bilateral Elements*** If the magnitude of current flowing through a circuit element is affected when the polarity of the applied voltage is changed, the element is termed *unilateral element*. Consider the example of a semiconductor diode. Current flows through the diode only in one direction. Hence, it is called an unilateral element. Next, consider the example of a resistor. When the voltage is applied, current starts to flow. If we change the polarity of the applied voltage, the direction of the current is changed but its magnitude is not affected. Such an element is called a *bilateral element*.
5. ***Lumped and Distributed Elements*** A lumped element is the element which is separated physically, like resistors, inductors and capacitors. Distributed elements are those which are not separable for analysis purposes. Examples of distributed elements are transmission lines in which the resistance, inductance and capacitance are distributed along its length.
6. ***Active and Passive Networks*** A network which contains at least one active element such as an independent voltage or current source is an active network. A network which does not contain any active element is a passive network.
7. ***Time-invariant and Time-variant Networks*** A network is said to be time-invariant or fixed if its input–output relationship does not change with time. In other words, a network is said to time-invariant, if for any time shift in input, an identical time-shift occurs for output. In time-variant networks, the input–output relationship changes with time.

1.7 SERIES AND PARALLEL COMBINATIONS OF RESISTORS

Let R_1, R_2 and R_3 be the resistances of three resistors connected in series across a dc voltage source V as shown in Fig. 1.9. Let V_1, V_2 and V_3 be the voltages across resistances R_1, R_2 and R_3 respectively.

In series combination, the same current flows through each resistor but voltage across each resistor is different.

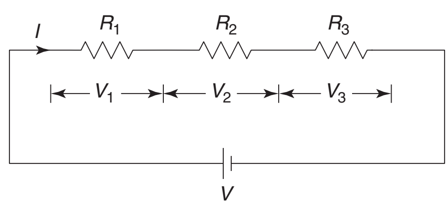


Fig. 1.9 *Series combination of resistors*

$$V = V_1 + V_2 + V_3$$

$$R_T I = R_1 I + R_2 I + R_3 I$$

$$R_T = R_1 + R_2 + R_3$$

Hence, when a number of resistors are connected in series, the equivalent resistance is the sum of all the individual resistance.

1. Voltage Division and Power in a Series Circuit

$$I = \frac{V}{R_1 + R_2 + R_3}$$

$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2 + R_3} V$$

$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2 + R_3} V$$

$$V_3 = R_3 I = \frac{R_3}{R_1 + R_2 + R_3} V$$

Total power $$P_T = P_1 + P_2 + P_3$$

$$= I^2 R_1 + I^2 R_2 + I^2 R_3$$

$$= \frac{V_1^2}{R_1} + \frac{V_2^2}{R_2} + \frac{V_3^2}{R_3}$$

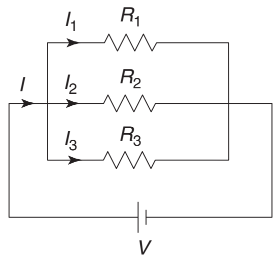


Fig. 1.10 *Parallel combination of resistors*

Figure 1.10 shows three resistors connected in parallel across a dc voltage source V. Let I_1, I_2 and I_3 be the current flowing through resistors R_1, R_2 and R_3 respectively.

In parallel combination, the voltage across each resistor is same but current through each resistor is different.

$$I = I_1 + I_2 + I_3$$

$$\frac{V}{R_T} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$R_T = \frac{R_1 \; R_2 \; R_3}{R_2 \; R_3 + R_3 \; R_1 + R_1 \; R_2}$$

Hence, when a number of resistors are connected in parallel, the reciprocal of the equivalent resistance is equal to the sum of reciprocals of individual resistances.

2. Current Division and Power in a Parallel Circuit

$$V = R_T I = R_1 \, I_1 = R_2 \, I_2 = R_3 \, I_3$$

$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$

$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$

$$I_3 = \frac{V}{R_3} = \frac{R_T I}{R_3} = \frac{R_1 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$

Total power

$$P_T = P_1 + P_2 + P_3$$
$$= I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$$
$$= \frac{V^2}{R_1} + \frac{V^2}{R_2} + \frac{V^3}{R_3}$$

Note: For two branch circuits,

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$
$$V = R_T I = R_1 I_1 = R_2 I_2$$
$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2}{R_1 + R_2} I$$
$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

1.8 SERIES AND PARALLEL COMBINATION OF INDUCTORS

Let L_1, L_2 and L_3 be the inductances of three inductors connected in series across an ac voltage source v as shown in Fig. 1.11. Let v_1, v_2 and v_3 be the voltages across inductances L_1, L_2 and L_3 respectively.

In series combination, the same current flows through each inductor but the voltage across each inductor is different.

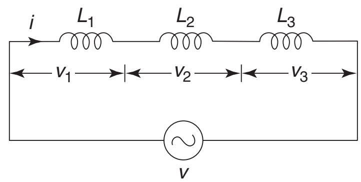


Fig. 1.11 *Series connection of inductors*

$$v = v_1 + v_1 + v_3$$
$$L_T \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$
$$L_T = L_1 + L_2 + L_3$$

Hence, when a number of inductors are connected in series, the equivalent inductance is the sum of all the individual inductances.

Figure 1.12 shows three inductors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each inductance L_1, L_2 and L_3 respectively.

In parallel combination, the voltage across each inductor is same but the current through each inductor is different.

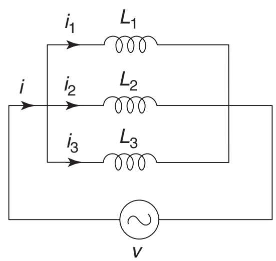


Fig. 1.12 *Parallel connection of inductors*

$$i = i_1 + i_2 + i_3$$

$$\frac{1}{L_T} \int v\,dt = \frac{1}{L_1} \int v\,dt + \frac{1}{L_2} \int v\,dt + \frac{1}{L_3} \int v\,dt$$
$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

$$L_T = \frac{L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}$$

Hence, when a number of inductors are connected in parallel, the reciprocal of the equivalent inductance is equal to the sum of reciprocals of individual inductances.

1.9 SERIES AND PARALLEL COMBINATION OF CAPACITORS

Let C_1, C_2 and C_3 be the capacitances of three capacitors connected in series across an ac voltage source v as shown in Fig 1.13. Let v_1, v_2 and v_3 be the voltages across capacitances C_1, C_2 and C_3 respectively.

In series combination, the charge on each capacitor is same but voltage across each capacitor is different.

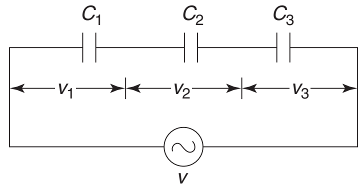


Fig. 1.13 *Series combination of capacitors*

$$v = v_1 + v_2 + v_3$$

$$\frac{1}{C_T}\int i\,dt = \frac{1}{C_1}\int i\,dt + \frac{1}{C_2}\int i\,dt + \frac{1}{C_3}\int i\,dt$$

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_T = \frac{C_1 C_2 C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1}$$

Hence, when a number of capacitors are connected in series, the reciprocal of the equivalent capacitance is equal to the sum of reciprocals of individual capacitances.

1. Voltage Division in a Series Circuit

$$Q = C_T V = C_1 V_1 = C_2 V_2 = C_3 V_3$$

$$V_1 = \frac{Q}{C_1} = \frac{C_T V}{C_1} = \frac{C_2\, C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

$$V_2 = \frac{Q}{C_2} = \frac{C_T V}{C_2} = \frac{C_1\, C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

$$V_3 = \frac{Q}{C_3} = \frac{C_T V}{C_3} = \frac{C_1\, C_2}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

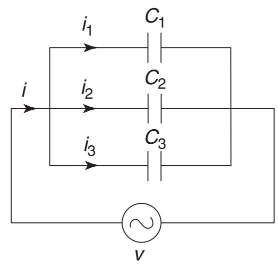


Fig. 1.14 *Parallel combination of capacitors*

Figure 1.14 shows three capacitors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each capacitance C_1, C_2 and C_3 respectively.

In parallel combination, the voltage across each capacitor is same but current through each capacitor is different.

$$i = i_1 + i_2 + i_3$$

$$C_T \frac{dv}{dt} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

$$C_T = C_1 + C_2 + C_3$$

Hence, when a number of capacitors are connected in parallel, the equivalent capacitance is the sum of all the individual capacitance.

1.10 STAR-DELTA TRANSFORMATION

When a circuit cannot be simplified by normal series–parallel reduction technique, the star-delta transformation can be used.

Figure 1.15 (a) shows three resistors R_A, R_B and R_C connected in delta.

Figure 1.15 (b) shows three resistors R_1, R_2 and R_3 connected in star.

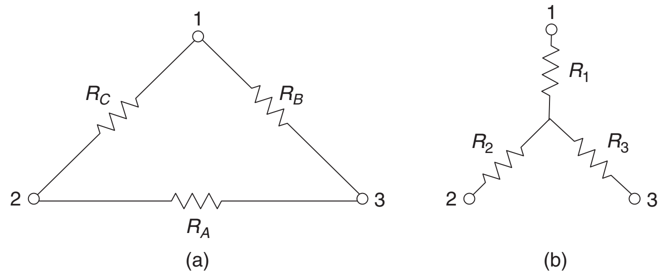


Fig. 1.15 *(a) Delta network (b) Star network*

These two networks will be electrically equivalent if the resistance as measured between any pair of terminals is the same in both the arrangements.

1.10.1 Delta to Star Transformation

Referring to delta network shown in Fig. 1.15 (a),

the resistance between terminals 1 and 2 = $R_C \parallel (R_A + R_B) = \dfrac{R_C(R_A + R_B)}{R_A + R_B + R_C}$

Referring to the star network shown in Fig. 1.15 (b), the resistance between terminals 1 and 2 = $R_1 + R_2$. Since the two networks are electrically equivalent,

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \qquad \text{...(1.1)}$$

Similarly,

$$R_2 + R_3 = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} \qquad \text{...(1.2)}$$

and

$$R_3 + R_1 = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} \qquad \text{...(1.3)}$$

Subtracting Eq. (1.2) from Eq. (1.1),

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \qquad \text{...(1.4)}$$

Adding Eq. (1.4) and Eq. (1.3),

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

Similarly,

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

Thus, star resistor connected to a terminal is equal to the product of the two delta resistors connected to the same terminal divided by the sum of the delta resistors.

1.10.2 Star to Delta Transformation

Multiplying the above equations,

$$R_1 R_2 = \frac{R_A R_B R_C^2}{(R_A + R_B + R_C)^2} \qquad ...(1.5)$$

$$R_2 R_3 = \frac{R_A^2 R_B R_C}{(R_A + R_B + R_C)^2} \qquad ...(1.6)$$

$$R_3 R_1 = \frac{R_A R_B^2 R_C}{(R_A + R_B + R_C)^2} \qquad ...(1.7)$$

Adding Eqs (1.5), (1.6) and (1.7),

$$R_1 R_2 + R_2 R_3 + R_3 R_1 = \frac{R_A R_B R_C (R_A + R_B + R_C)}{(R_A + R_B + R_C)^2} = \frac{R_A R_B R_C}{R_A + R_B + R_C}$$

$$= R_A R_1 = R_B R_2 = R_C R_3$$

Hence,

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} = R_2 + R_3 + \frac{R_2 R_3}{R_1}$$

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} = R_1 + R_3 + \frac{R_3 R_1}{R_2}$$

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} = R_1 + R_2 + \frac{R_1 R_2}{R_3}$$

Thus, delta resistor connected between the two terminals is the sum of two star resistors connected to the same terminals plus the product of the two resistors divided by the remaining third star resistor.

Note: (1) When three equal resistors are connected in delta (Fig. 1.16), the equivalent star resistance is given by

$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta} = \frac{R_\Delta}{3}$$

$$R_\Delta = 3R_Y$$

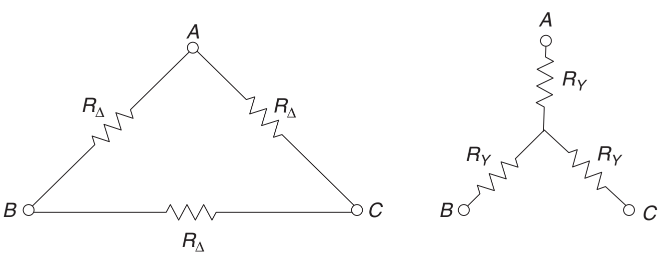


Fig. 1.16 *Equivalent star resistance for three equal delta resistors*

(2) Star-delta transformation can also be applied to network containing inductors and capacitors.

1.11 SOURCE TRANSFORMATION

A voltage source with a series resistor can be converted into a equivalent current source with a parallel resistor. Conversely, a current source with a parallel resistor can be converted into a voltage source with a series resistor as shown in Fig. 1.17.

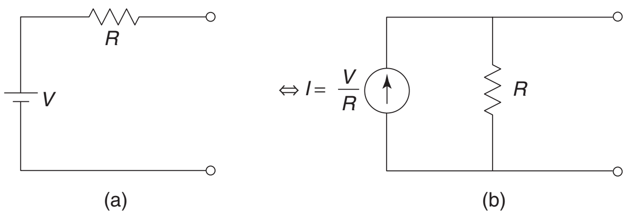


Fig. 1.17 *Source transformation*

Source transformation can be applied to dependent sources as well. The controlling variable, however must not be tampered with any way since the operation of the controlled sources depends on it.

Example 1.1 *Replace the given network of Fig. 1.18 with a single current source and a resistor.*

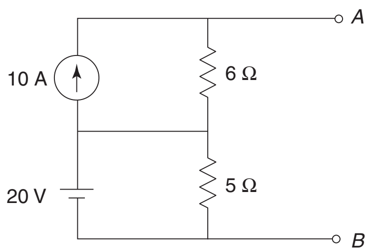


Fig. 1.18

Solution Since the resistor of 5 Ω is connected in parallel with the voltage source of 20 V it becomes redundant. Converting parallel combination of current source and resistor into equivalent voltage source and resistor (Fig. 1.19),

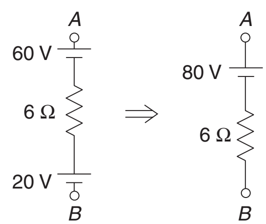


Fig. 1.19

By source transformation (Fig. 1.20),

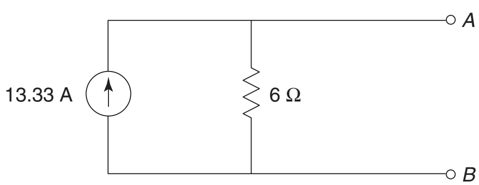


Fig. 1.20

Example 1.2 *Reduce the network shown in Fig. 1.21 into a single source and a single resistor between terminals A and B.*

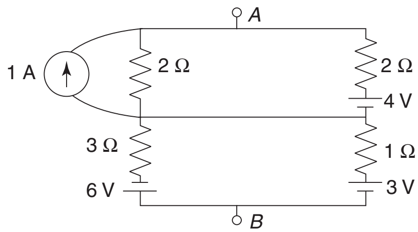


Fig. 1.21

Solution Converting all voltage sources into equivalent current sources (Fig. 1.22),

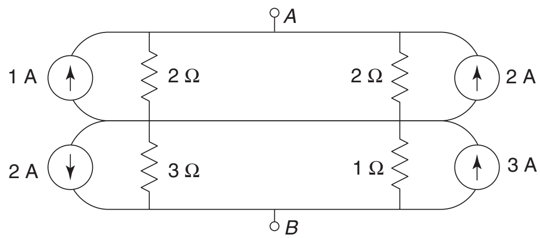


Fig. 1.22

Adding the current sources and simplifying the network (Fig. 1.23),

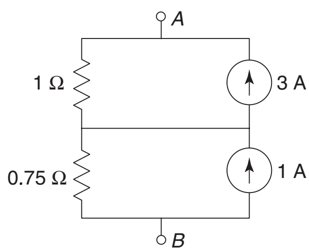


Fig. 1.23

Converting the current sources into equivalent voltage sources (Fig. 1.24),

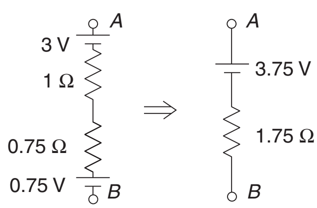


Fig. 1.24

Example 1.3 *Replace the circuit between A and B in Fig. 1.25 with a voltage source in series with a single resistor.*

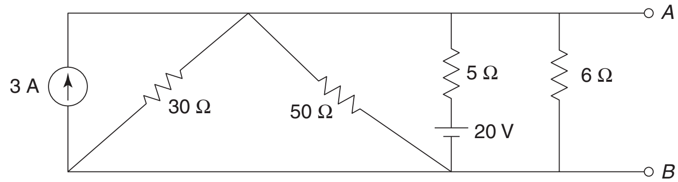


Fig. 1.25

Solution Converting the series combination of voltage source of 20 V and a resistor of 5 Ω into equivalent parallel combination of current source and resistor (Fig. 1.26),

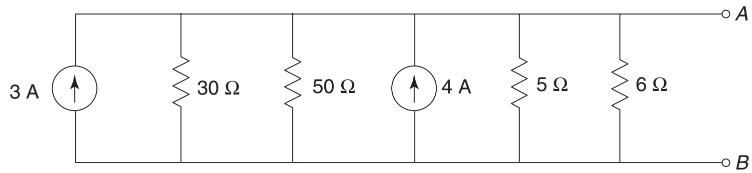


Fig. 1.26

Adding the two current sources and simplifying the circuit (Fig. 1.27),

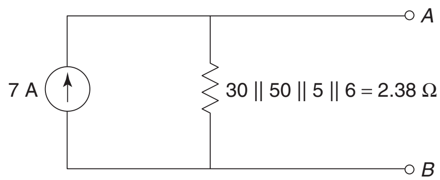


Fig. 1.27

By source transformation (Fig. 1.28),

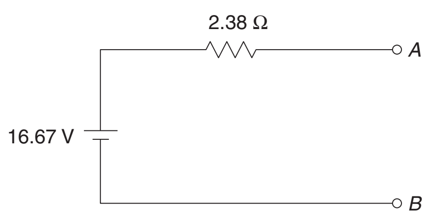


Fig. 1.28

Example 1.4 *Find the power delivered by the 50 V source in the network of Fig. 1.29.*

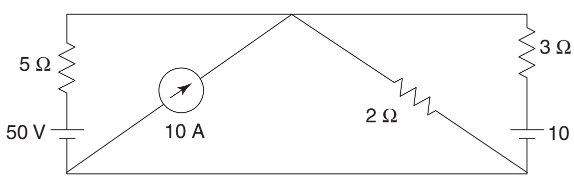


Fig. 1.29

Solution Converting the series combination of voltage source of 10 V and resistor of 3 Ω into equivalent current source and resistor (Fig. 1.30),

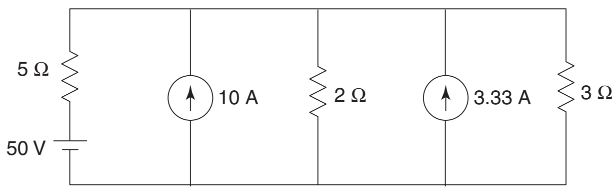


Fig. 1.30

Adding the two current sources and simplifying the network (Fig. 1.31),

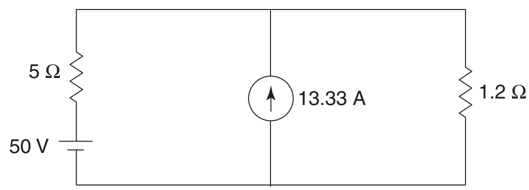


Fig. 1.31

By source transformation (Fig. 1.32),

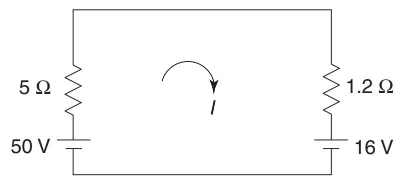


Fig. 1.32

$$I = \frac{50-16}{5+1.2} = 5.48 \text{ A}$$

Power delivered by the 50 V source = 50 × 5.48 = 274 W

Example 1.5 *Find the current in the 4 Ω resistor shown in network of Fig. 1.33.*

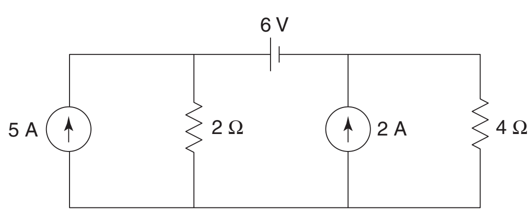


Fig. 1.33

Solution Converting the parallel combination of the current source of 5 A and the resistor of 2 Ω into an equivalent series combination of voltage source and resistor (Fig. 1.34),

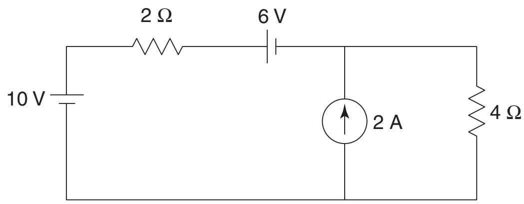


Fig. 1.34

Adding two voltage sources (Fig. 1.35),

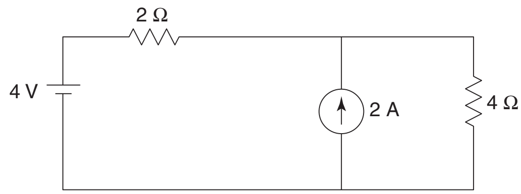


Fig. 1.35

Again by source transformation (Fig. 1.36),

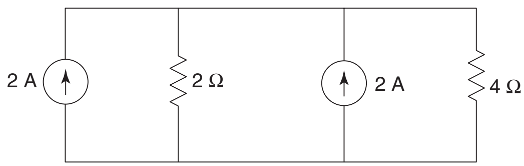


Fig. 1.36

Adding two current sources (Fig. 1.37),

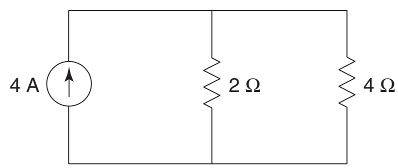


Fig. 1.37

By current-division rule,

$$I_{4\,\Omega} = 4 \times \frac{2}{2+4} = 1.33 \text{ A}$$

Example 1.6 *Find the voltage across the 4 Ω resistor shown in network of Fig. 1.38.*

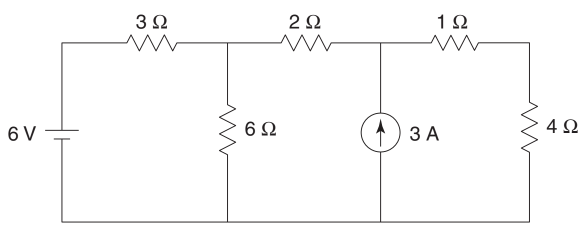


Fig. 1.38

Solution Converting the series combination of the voltage source of 6 V and the resistor of 3 Ω into equivalent current source and resistor (Fig. 1.39),

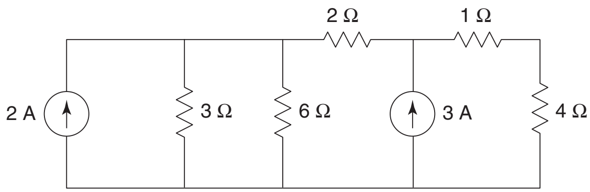


Fig. 1.39

By series–parallel reduction technique (Fig. 1.40),

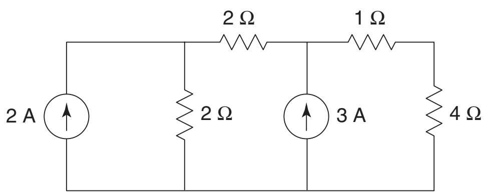


Fig. 1.40

By source transformation (Fig. 1.41),

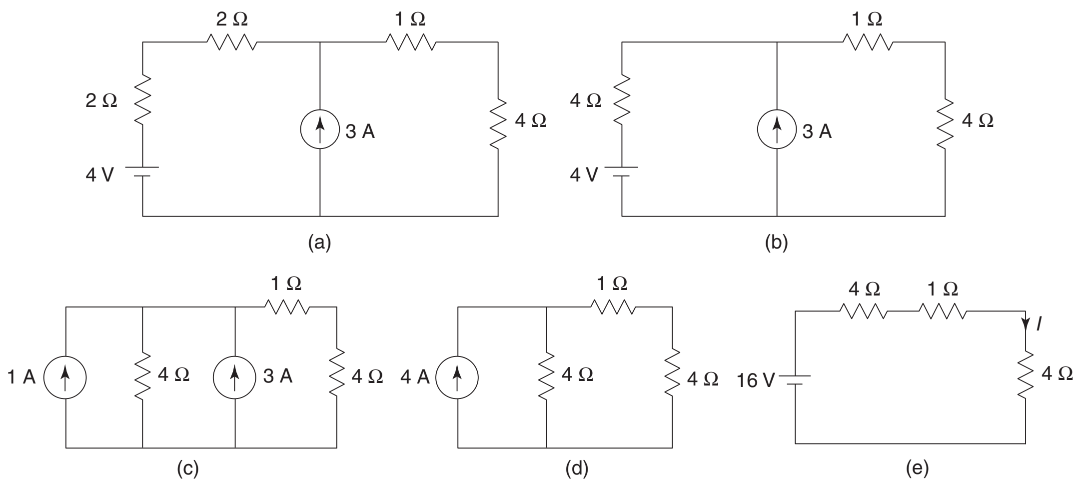


Fig. 1.41

$$I = \frac{16}{4+1+4} = 1.78 \text{ A}$$

Voltage across the 4 Ω resistor $= 4I = 4 \times 1.78 = 7.12$ V

Example 1.7 *Find the voltage at Node 2 of the network shown in Fig. 1.42.*

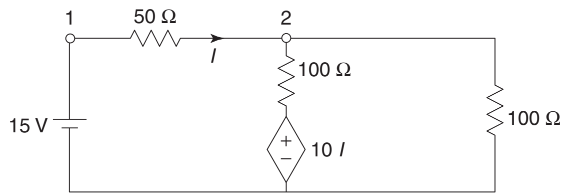


Fig. 1.42

Solution We cannot change the network between nodes 1 and 2 since the controlling current I, for the controlled source, is in the resistor between these nodes. Applying source transformation to series combination of controlled source and the 100 Ω resistor (Fig. 1.43),

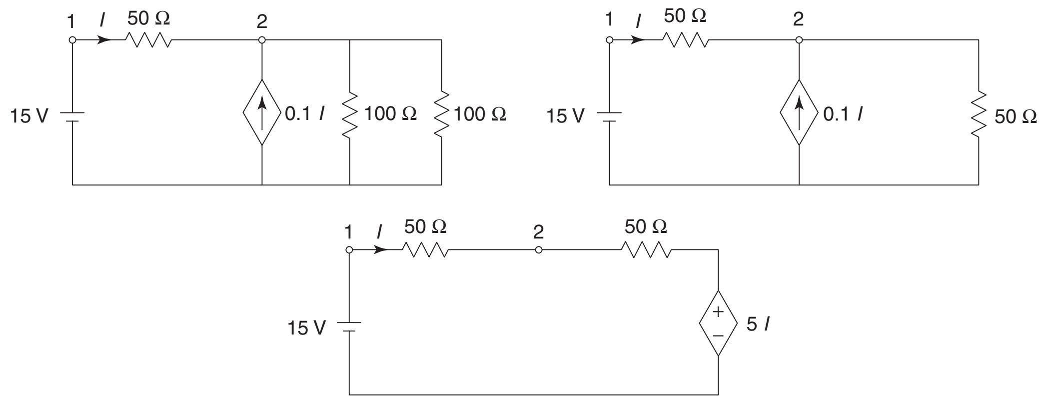


Fig. 1.43

Applying KVL to the mesh,

$$15 - 50I - 50I - 5I = 0$$

$$I = \frac{15}{105} = 0.143 \text{ A}$$

Voltage at Node 2 $= 15 - 50I = 15 - 50 \times 0.143 = 7.86$ V

1.12 SOURCE SHIFTING

Source shifting is the simplification technique used when there is no resistor in series with a voltage source or a resistor in parallel with a current source.

Example 1.8 *Calculate the voltage across the 6 Ω resistor in the network of Fig. 1.44 using source-shifting technique.*

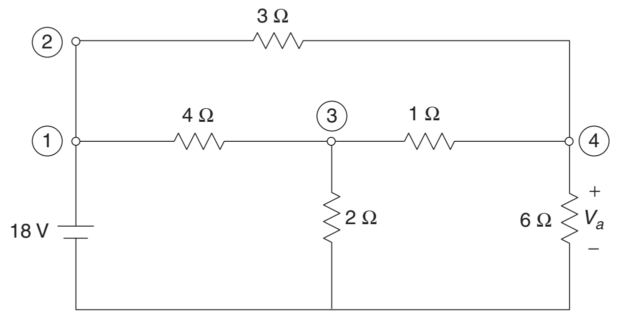


Fig. 1.44

Solution Adding a voltage source of 18 V to the network and connecting to Node 2 (Fig. 1.45), we have

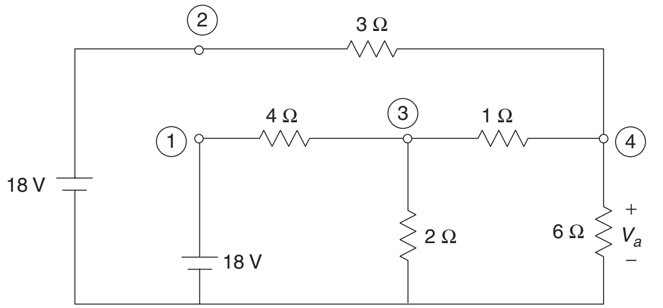


Fig. 1.45

Since nodes 1 and 2 are maintained at the same voltage by the sources, the connection between nodes 1 and 2 is removed. Now the two voltage sources have resistors in series and source transformation can be applied (Fig. 1.46).

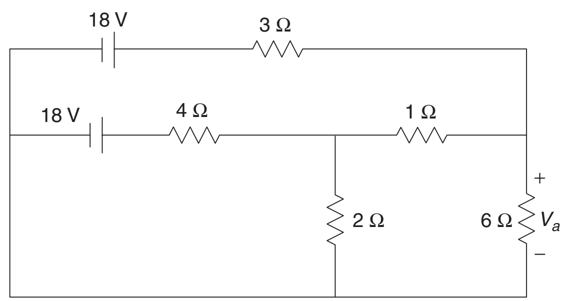


Fig. 1.46

Simplifying the network (Fig. 1.47),

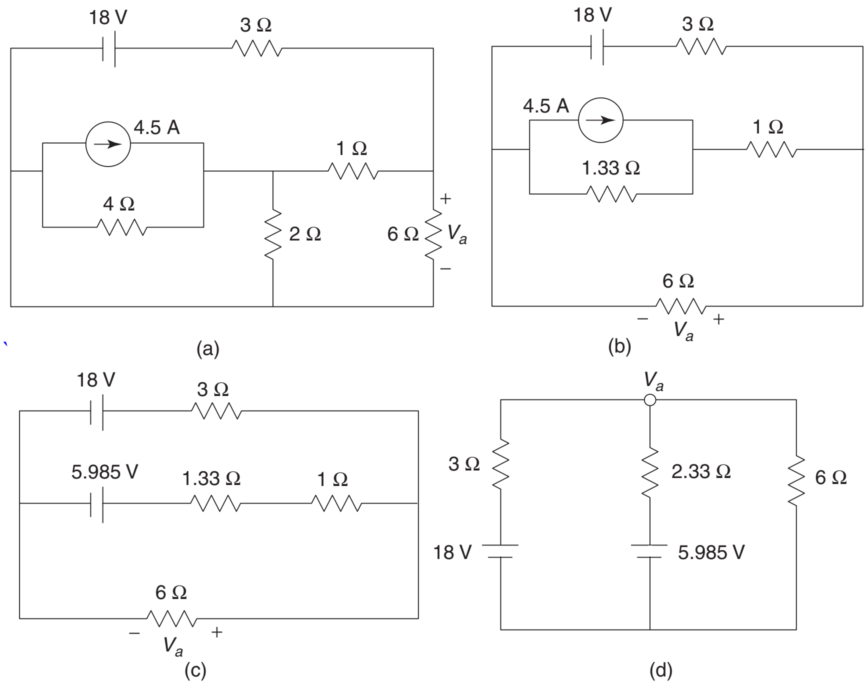


Fig. 1.47

Applying KCL at the node,

$$\frac{V_a - 18}{3} + \frac{V_a - 5.985}{2.33} + \frac{V_a}{6} = 0$$

$$V_a = 9.23 \text{ V}$$

Exercises

1.1 Use source transformation to simplify the network until two elements remain to the left of terminals *A* and *B*.

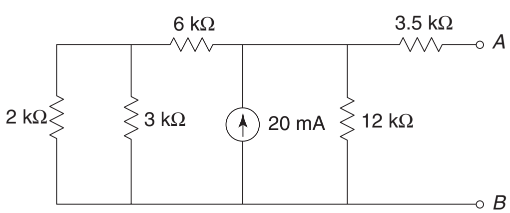


Fig. 1.48

[88.42 V, 7.92 k Ω]

1.2 Determine the voltage V_x in the network of Fig. 1.49 by source-shifting technique.

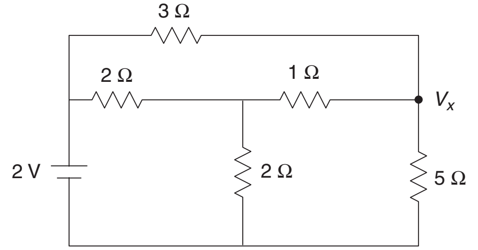


Fig. 1.49

[1.129 V]

Objective-Type Questions

1.1 A network contains linear resistors and ideal voltage sources. If values of all the resistors are doubled then the voltage across each resistor is

(a) halved
(b) doubled
(c) increased by four times
(d) not changed

1.2 Four resistances 80 Ω, 50 Ω, 25 Ω, and R are connected in parallel. Current through 25 Ω resistor is 4 A. Total current of the supply is 10 A. The value of R will be

(a) 66.66 Ω (b) 40.25 Ω
(c) 36.36 Ω (d) 76.56 Ω

1.3 Viewed from the terminal AB, the network of Fig. 1.50 can be reduced to an equivalent network of a single voltage source in series with a single resistor with the following parameters

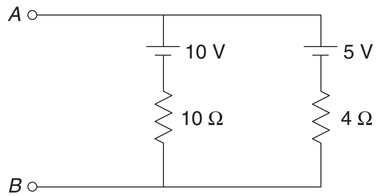


Fig. 1.50

(a) 5 V source in series with a 10 Ω resistor
(b) 1 V source in series with a 2.4 Ω resistor
(c) 15 V source in series with a 2.4 Ω resistor
(d) 1 V source in series with a 10 Ω resistor

1.4 A 10 V battery with an internal resistance of 1 Ω is connected across a nonlinear load whose *V-I* characteristic is given by $7I = V^2 + 2V$. The current delivered by the battery is

(a) 0 (b) 10 A
(c) 5 A (d) 8 A

1.5 If the length of a wire of resistance R is uniformly stretched to n times its original value, its new resistance is

(a) nR (b) $\frac{R}{n}$
(c) n^2R (d) $\frac{R}{n^2}$

1.6 All the resistances in Fig. 1.51 are 1 Ω each. The value of I will be

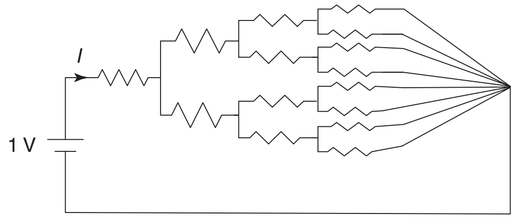


Fig. 1.51

(a) $\frac{1}{15}$ A (b) $\frac{2}{15}$ A
(c) $\frac{4}{15}$ A (d) $\frac{8}{15}$ A

1.7 The current waveform in a pure resistor at 10 Ω is shown in Fig. 1.52. Power dissipated in the resistor is

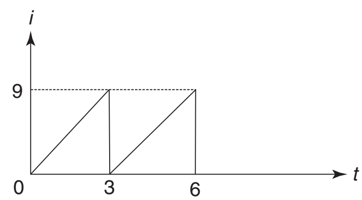


Fig. 1.52

(a) 7.29 W (b) 52.4 W
(c) 135 W (d) 270 W

1.8 Two wires A and B of the same material and length L and $2L$ have radius r and $2r$ respectively. The ratio of their specific resistance will be

(a) 1 : 1 (b) 1 : 2
(c) 1 : 4 (d) 1 : 8

Answers to Objective-Type Questions

1.1. (d)	1.2. (c)	1.3. (b)	1.4. (c)	1.5. (c)	1.6. (d)	1.7. (d)
1.8. (b)						

2

Analysis of DC Circuits

2.1 INTRODUCTION

In Chapter 1, we have studied basic circuit concepts. In network analysis, we have to find currents and voltages in various parts of networks. In this chapter, we will study elementary network theorems like Kirchhoff's laws, mesh analysis and node analysis. These methods are applicable to all types of networks. The first step in analyzing networks is to apply Ohm's law and Kirchhoff's laws. The second step is the solving of these equations by mathematical tools. There are some other methods also to analyse circuits. We will also study superposition theorem, Thevenin's theorem, Norton's theorem, maximum power transfer theorem, Reciprocity theorem and Millman's theorem. We can find currents and voltages in various parts of the circuits with these methods.

2.2 KIRCHHOFF'S LAWS

The entire study of electric network analysis is based mainly on Kirchhoff's laws. But before discussing this, it is essential to familiarise ourselves with the following terms:

Node A node is a junction where two or more network elements are connected together.

Branch An element or number of elements connected between two nodes constitute a branch.

Loop A loop is any closed part of the circuit.

Mesh A mesh is the most elementary form of a loop and cannot be further divided into other loops. All meshes are loops but all loops are not meshes.

1. Kirchhoff's Current Law (KCL) The algebraic sum of currents meeting at a junction or node in an electric circuit is zero.

Consider five conductors, carrying currents I_1, I_2, I_3, I_4 and I_5 meeting at a point O as shown in Fig. 2.1. Assuming the incoming currents to be positive and outgoing currents negative, we have

$$I_1 + (-I_2) + I_3 + (-I_4) + I_5 = 0$$

$$I_1 - I_2 + I_3 - I_4 + I_5 = 0$$

$$I_1 + I_3 + I_5 = I_2 + I_4$$

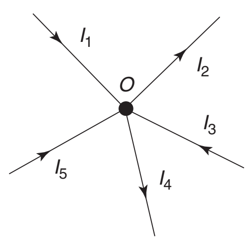


Fig. 2.1 *Kirchhoff's current law*

Thus, the above law can also be stated as the sum of currents flowing towards any junction in an electric circuit is equal to the sum of the currents flowing away from that junction.

2. *Kirchhoff's Voltage Law (KVL)* The algebraic sum of all the voltages in any closed circuit or mesh or loop is zero.

If we start from any point in a closed circuit and go back to that point, after going round the circuit, there is no increase or decrease in potential at that point. This means that the sum of emfs and the sum of voltage drops or rises meeting on the way is zero.

3. *Determination of Sign* A rise in potential can be assumed to be positive while a fall in potential can be considered negative. The reverse is also possible and both conventions will give the same result.

(i) If we go from the positive terminal of the battery or source to the negative terminal, there is a fall in potential and so the emf should be assigned a negative sign (Fig. 2.2a). If we go from the negative terminal of the battery or source to the positive terminal, there is a rise in potential and so the emf should be given a positive sign (Fig. 2.2b).

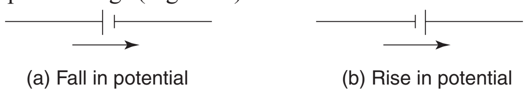


Fig. 2.2 *Sign convention*

(ii) When current flows through a resistor, there is a voltage drop across it. If we go through the resistor in the same direction as the current, there is a fall in the potential and so the sign of this voltage drop is negative (Fig. 2.3a). If we go opposite to the direction of the current flow, there is a rise in potential and hence, this voltage drop should be given a positive sign (Fig. 2.3b).

Fig. 2.3 *Sign convention*

2.3 MESH ANALYSIS

A mesh is defined as a loop which does not contain any other loops within it. Mesh analysis is applicable only for planar networks. A network is said to be planar if it can be drawn on a plane surface without crossovers. In this method, the currents in different meshes are assigned continuous paths so that they do not split at a junction into branch currents. If a network has a large number of voltage sources, it is useful to use mesh analysis. Basically, this analysis consists of writing mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents.

Steps to be Followed in Mesh Analysis

1. Identify the mesh, assign a direction to it and assign an unknown current in each mesh.
2. Assign the polarities for voltage across the branches.
3. Apply KVL around the mesh and use Ohm's law to express the branch voltages in terms of unknown mesh currents and the resistance.
4. Solve the simultaneous equations for unknown mesh currents.

Consider the network shown in Fig. 2.4 which has three meshes. Let the mesh currents for the three meshes be I_1, I_2, and I_3 and all the three mesh currents may be assumed to flow in the clockwise direction. The choice of direction for any mesh current is arbitrary.

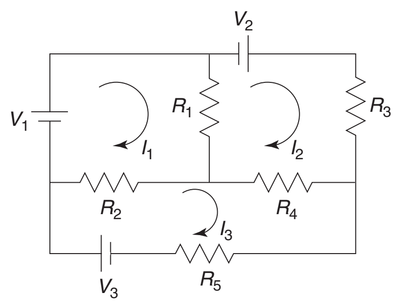


Fig. 2.4 *Circuit for mesh analysis*

Applying KVL to Mesh 1,

$$V_1 - R_1(I_1 - I_2) - R_2(I_1 - I_3) = 0$$

$$(R_1 + R_2)I_1 - R_1 I_2 - R_2 I_3 = V_1 \qquad \text{...(2.1)}$$

Applying KVL to Mesh 2,

$$V_2 - R_3 I_2 - R_4(I_2 - I_3) - R_1(I_2 - I_1) = 0$$

$$-R_1 I_1 + (R_1 + R_3 + R_4)I_2 - R_4 I_3 = V_2 \qquad \text{...(2.2)}$$

Applying KVL to Mesh 3,

$$-R_2(I_3 - I_1) - R_4(I_3 - I_2) - R_5 I_3 + V_3 = 0$$

$$-R_2 I_1 - R_4 I_2 + (R_2 + R_4 + R_5)I_3 = V_3 \qquad \text{...(2.3)}$$

Writing Eqs (2.1), (2.2), and (2.3) in matrix form,

$$\begin{bmatrix} R_1 + R_2 & -R_1 & -R_2 \\ -R_1 & R_1 + R_3 + R_4 & -R_4 \\ -R_2 & -R_4 & R_2 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

In general,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where, R_{11} = Self-resistance or sum of all the resistance of mesh 1
$R_{12} = R_{21}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 2
$R_{13} = R_{31}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 3
R_{22} = Self-resistance or sum of all the resistance of mesh 2
$R_{23} = R_{32}$ = Mutual resistance or sum of all the resistances common to meshes 2 and 3
R_{33} = Self-resistance or sum of all the resistance of mesh 3

If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign, and if the direction of the currents passing through common resistance are opposite then the mutual resistance will have a negative sign. If each mesh current is assumed to flow in the clockwise direction then all self-resistances will always be positive and all mutual resistances will always be negative.

The voltages V_1, V_2 and V_3 represent the algebraic sum of all the voltages in meshes 1, 2 and 3 respectively. While going along the current, if we go from negative terminal of the battery to the positive terminal then its emf is taken as positive. Otherwise, it is taken as negative.

Example 2.1 *Find the current through the 5 Ω resistor is shown in Fig. 2.5.*

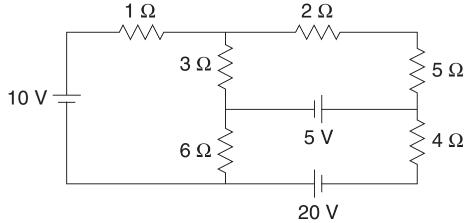


Fig. 2.5

Solution Assign clockwise currents in three meshes as shown in Fig. 2.6.
Applying KVL to Mesh 1,

$$10 - 1I_1 - 3(I_1 - I_2) - 6(I_1 - I_3) = 0$$
$$10I_1 - 3I_2 - 6I_3 = 10 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 5I_2 - 5 = 0$$
$$-3I_1 + 10I_2 = -5 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) + 5 - 4I_3 + 20 = 0$$
$$-6I_1 + 10I_3 = 25 \qquad \ldots\text{(iii)}$$

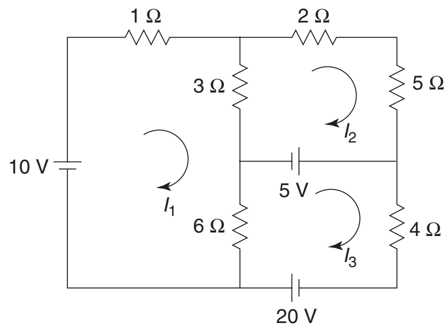


Fig. 2.6

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 10 & -3 & -6 \\ -3 & 10 & 0 \\ -6 & 0 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 25 \end{bmatrix}$$

We can write matrix equation directly from Fig. 2.6,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where R_{11} = Self-resistance of Mesh 1 = 1 + 3 + 6 = 10 Ω
R_{12} = Mutual resistance common to meshes 1 and 2 = –3 Ω

Here, negative sign indicates that the current through common resistance are in opposite direction.

R_{13} = Mutual resistance common to meshes 1 and 3 = –6 Ω

Similarly,

$$R_{21} = -3\ \Omega$$
$$R_{22} = 3 + 2 + 5 = 10\ \Omega$$
$$R_{23} = 0$$
$$R_{31} = -6\ \Omega$$
$$R_{32} = 0$$
$$R_{33} = 6 + 4 = 10\ \Omega$$

For voltage matrix,

$$V_1 = 10\ \text{V}$$
$$V_2 = -5\ \text{V}$$

V_3 = algebraic sum of all the voltages in mesh 3 = 5 + 20 = 25 V

Solving Eqs (i), (ii) and (iii),

$$I_1 = 4.27\ \text{A}$$
$$I_2 = 0.78\ \text{A}$$
$$I_3 = 5.06\ \text{A}$$
$$I_{5\,\Omega} = I_2 = 0.78\ \text{A}$$

Example 2.2 *Determine the current through the 5 Ω resistor of the network shown in Fig. 2.7.*

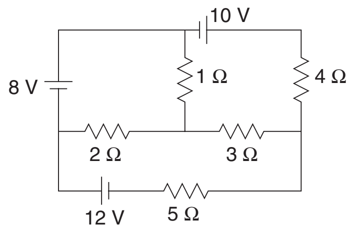


Fig. 2.7

Solution Assign clockwise currents in three meshes as shown in Fig. 2.8.

Applying KVL to Mesh 1,

$$8-1(I_1-I_2)-2(I_1-I_3)=0$$

$$3I_1-I_2-2I_3=8 \qquad \ldots\text{(i)}$$

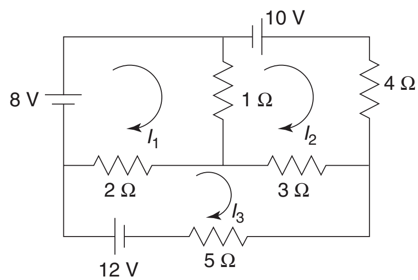


Fig. 2.8

Applying KVL to Mesh 2,

$$10-4I_2-3(I_2-I_3)-1(I_2-I_1)=0$$

$$-I_1+8I_2-3I_3=10 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3-I_1)-3(I_3-I_2)-5I_3+12=0$$

$$-2I_1-3I_2+10I_3=12 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$I_1=6.01\text{ A}$$

$$I_2=3.27\text{ A}$$

$$I_3=3.38\text{ A}$$

$$I_{5\Omega}=I_3=3.38\text{ A}$$

Example 2.3 *Find the current through the 2 Ω resistor in the network of Fig. 2.9.*

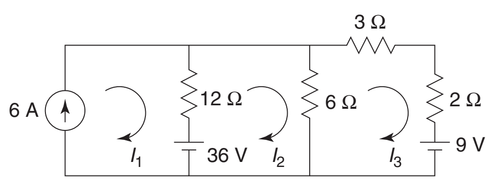


Fig. 2.9

Solution Mesh 1 contains a current source of 6 A. Hence, we can write current equation for Mesh 1. Since direction of current source and mesh current I_1 are same,

$$I_1=6 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$36-12(I_2-I_1)-6(I_2-I_3)=0$$
$$36-12(I_2-6)-6I_2+6I_3=0$$
$$18I_2-6I_3=108 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6(I_3-I_2)-3I_3-2I_3-9=0$$
$$6I_2-11I_3=9 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_3=3\text{ A}$$
$$I_{2\Omega}=I_3=3\text{ A}$$

Example 2.4 *Find the current through the 5 Ω resistor in the network of Fig. 2.10.*

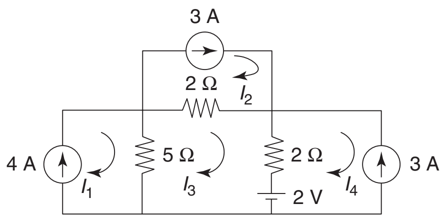


Fig. 2.10

Solution Writing current equations for Meshes 1, 2 and 4,

$$I_1=4 \qquad \ldots\text{(i)}$$
$$I_2=3 \qquad \ldots\text{(ii)}$$
$$I_4=-3 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-5(I_3-I_1)-2(I_3-I_2)-2(I_3-I_4)-2=0 \qquad \ldots\text{(iv)}$$

Substituting Eqs (i), (ii) and (iii) in Eq. (iv),

$$-5(I_3-4)-2(I_3-3)-2(I_3+3)-2=0$$
$$I_3=2\text{ A}$$
$$I_{5\Omega}=I_1-I_3=4-2=2\text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.5 *Obtain the branch currents in the network shown in Fig. 2.11.*

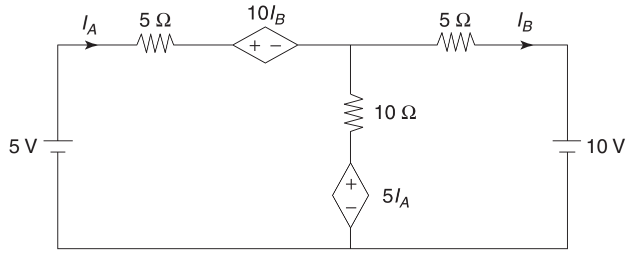


Fig. 2.11

Solution Assign clockwise currents in two meshes as shown in Fig. 2.12.
From Fig. 2.12,

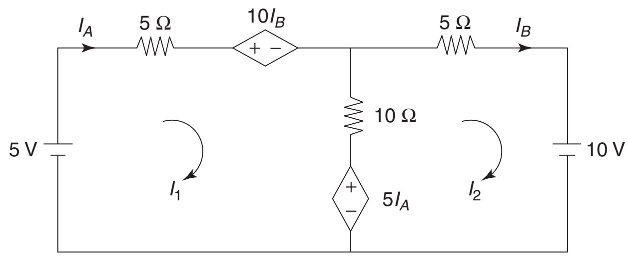


Fig. 2.12

$$I_A = I_1 \quad \text{...(i)}$$
$$I_B = I_2 \quad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$5 - 5I_1 - 10I_B - 10(I_1 - I_2) - 5I_A = 0$$
$$5 - 5I_1 - 10I_2 - 10I_1 + 10I_2 - 5I_1 = 0$$
$$-20I_1 = -5$$
$$I_1 = \frac{1}{4} = 0.25\,\text{A} \quad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$5I_A - 10(I_2 - I_1) - 5I_2 - 10 = 0$$
$$5I_1 - 10I_2 + 10I_1 - 5I_2 = 10$$
$$15I_1 - 15I_2 = 10 \quad \text{...(iv)}$$

Putting $I_1 = 0.25$ A in Eq. (iv),

$$15(0.25) - 15I_2 = 10$$
$$I_2 = -0.416\text{ A}$$

Example 2.6 *Find the mesh currents in the network shown in Fig. 2.13.*

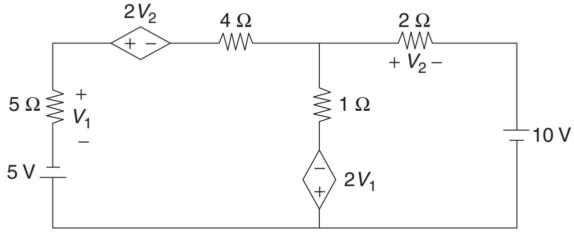


Fig. 2.13

Solution Assign clockwise currents in the two meshes as shown in Fig. 2.14.

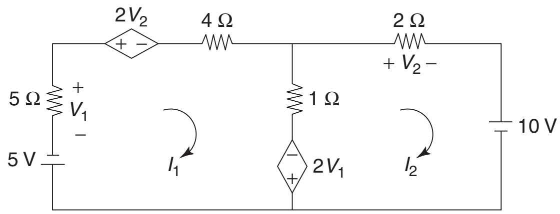


Fig. 2.14

From Fig. 2.14,

$$V_1 = -5I_1 \qquad \ldots\text{(i)}$$

$$V_2 = 2I_2 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$-5 - 5I_1 - 2V_2 - 4I_1 - 1(I_1 - I_2) + 2V_1 = 0$$

$$-5 - 5I_1 - 2(2I_2) - 4I_1 - I_1 + I_2 + 2(-5I_1) = 0$$

$$20I_1 + 3I_2 = -5 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 2,

$$-2V_1 - 1(I_2 - I_1) - 2I_2 - 10 = 0$$

$$-2(-5I_1) - I_2 + I_1 - 2I_2 = 10$$

$$11I_1 - 3I_2 = 10 \qquad \ldots\text{(iv)}$$

Solving Eqs (iii) and (iv),

$$I_1 = 0.161 \text{ A}$$

$$I_2 = -2.742 \text{ A}$$

Example 2.7 *Find currents I_x and I_y of the network shown in Fig. 2.15.*

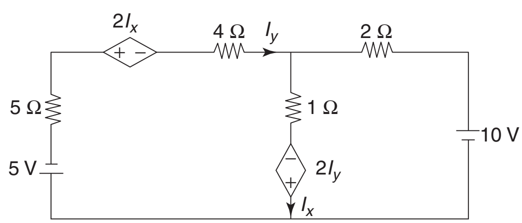


Fig. 2.15

Solution Assign clockwise currents in the two meshes as shown in Fig. 2.16.

From Fig. 2.16,

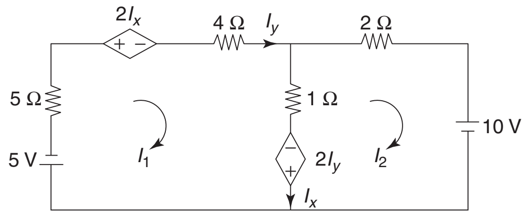


Fig. 2.16

$$I_y = I_1 \qquad \text{...(i)}$$

$$I_x = I_1 - I_2 \qquad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$-5 - 5I_1 - 2I_x - 4I_1 - 1(I_1 - I_2) + 2I_y = 0$$

$$-5 - 5I_1 - 2(I_1 - I_2) - 4I_1 - I_1 + I_2 + 2I_1 = 0$$

$$-5 - 5I_1 - 2I_1 + 2I_2 - 4I_1 - I_1 + I_2 + 2I_1 = 0$$

$$-10I_1 + 3I_2 = 5 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-2I_y - 1(I_2 - I_1) - 2I_2 - 10 = 0$$

$$-2I_1 - I_2 + I_1 - 2I_2 = 10$$

$$-I_1 - 3I_2 = 10 \qquad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$I_1 = -\frac{15}{11} = -1.364 \text{ A}$$

$$I_2 = -2.878 \text{ A}$$

$$I_y = -1.364 \text{ A}$$

$$I_x = I_1 - I_2 = -1.364 + 2.878 = 1.514 \text{ A}$$

Example 2.8 *Find the currents in the three meshes of the network shown in Fig. 2.17.*

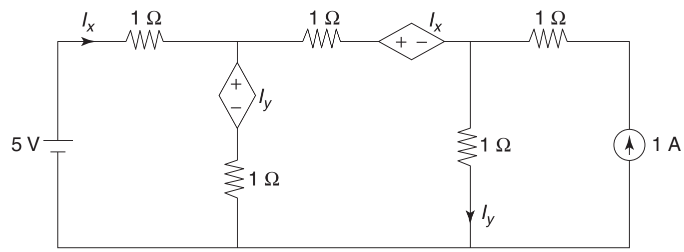


Fig. 2.17

Solution Assign clockwise currents in the three meshes as shown in Fig. 2.18.

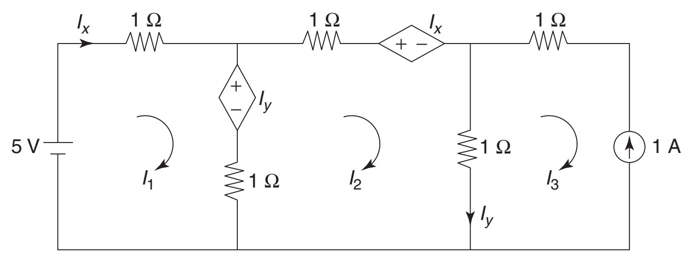


Fig. 2.18

From Fig. 2.18,

$$I_x = I_1 \qquad \text{...(i)}$$

$$I_x = I_1$$

$$I_y = I_2 - I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$5 - 1I_1 - I_y - 1(I_1 - I_2) = 0$$

$$5 - I_1 - (I_2 - I_3) - (I_1 - I_2) = 0$$

$$-2I_1 + I_3 = -5 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + I_y - 1I_2 - I_x - 1(I_2 - I_3) = 0$$

$$-(I_2 - I_1) + (I_2 - I_3) - I_2 - I_1 - (I_2 - I_3) = 0$$

$$-2I_2 = 0 \qquad \text{...(iv)}$$

For Mesh 3,

$$I_3 = -1 \qquad \text{...(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 2\text{ A}$$

$$I_2 = 0$$

$$I_3 = -1\text{ A}$$

Example 2.9 *For the network shown in Fig. 2.19, find the power supplied by the dependent voltage source.*

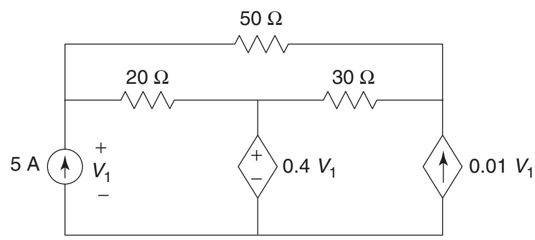


Fig. 2.19

Solution Assign clockwise currents in three meshes as shown in Fig. 2.20.

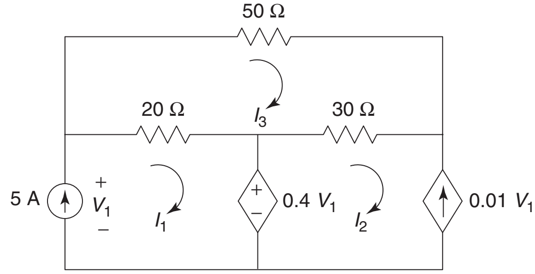


Fig. 2.20

From Fig. 2.20,

$$V_1 - 20(I_1 - I_3) - 0.4\,V_1 = 0$$

$$0.6\,V_1 = 20\,I_1 - 20\,I_3$$

$$V_1 = 33.33\,I_1 - 33.33\,I_3 \qquad \text{...(i)}$$

For Mesh 1,

$$I_1 = 5 \qquad \text{...(ii)}$$

For Mesh 2,

$$I_2 = -0.01\,V_1 = -0.01(33.33\,I_1 - 33.33\,I_3)$$

$$0.33\,I_1 + I_2 - 0.33\,I_3 = 0 \qquad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-50\,I_3 - 30(I_3 - I_2) - 20(I_3 - I_1) = 0$$

$$-20\,I_1 - 30\,I_2 + 100\,I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5 \text{ A}$$

$$I_2 = -1.47 \text{ A}$$

$$I_3 = 0.56 \text{ A}$$

$$V_1 = 33.33\,I_1 - 33.33\,I_3 = 33.33(5) - 33.33(0.56) = 148 \text{ V}$$

Power supplied by the dependent voltage source = $0.4\,V_1(I_1 - I_2) = 0.4\,(148)(5 + 1.47) = 383.02$ W

Example 2.10 *Find the voltage V_x in the network shown in Fig. 2.21.*

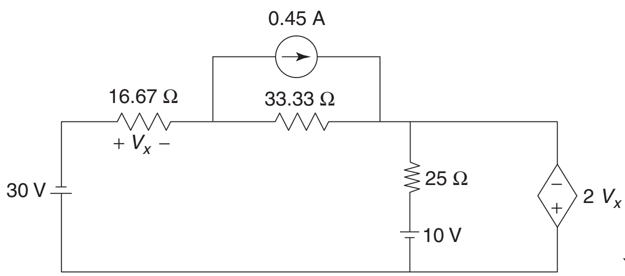


Fig. 2.21

Solution Assign clockwise currents in the three meshes as shown in Fig. 2.22.

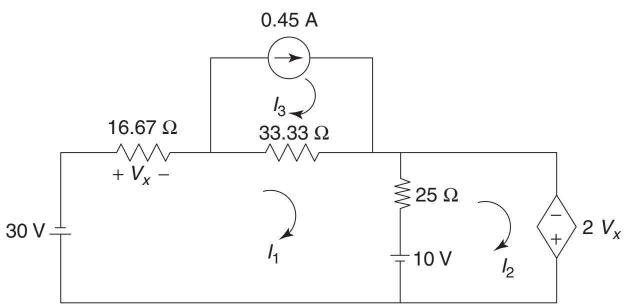


Fig. 2.22

From Fig. 2.22,

$$V_x = 16.67\, I_1 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$-30 - 16.67\, I_1 - 33.33(I_1 - I_3) - 25(I_1 - I_2) - 10 = 0$$
$$-30 - 16.67\, I_1 - 33.33\, I_1 + 33.33\, I_3 - 25\, I_1 + 25\, I_2 - 10 = 0$$
$$-75\, I_1 + 25\, I_2 + 33.33\, I_3 = 40 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$10 - 25(I_2 - I_1) + 2\, V_x = 0$$
$$10 - 25(I_2 - I_1) + 2(16.67\, I_1) = 0$$
$$10 - 25\, I_2 + 25\, I_1 + 33.34\, I_1 = 0$$
$$58.34\, I_1 - 25\, I_2 = -10 \quad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = 0.45 \quad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.9 \text{ A}$$
$$I_2 = -1.7 \text{ A}$$
$$I_3 = 0.45 \text{ A}$$
$$V_x = 16.67\, I_1 = 16.67\,(-0.9) = -15 \text{ V} \quad \ldots\text{(v)}$$

Example 2.11 *For the network shown in Fig. 2.23, find the mesh currents I_1, I_2 and I_3.*

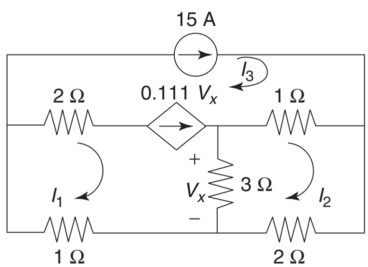


Fig. 2.23

Solution From Fig. 2.23,

$$V_x = 3(I_1 - I_2) \quad \ldots\text{(i)}$$

Writing current equation for the two current sources,

$$I_3 = 15 \quad \ldots\text{(ii)}$$

and

$$0.111\, V_x = I_1 - I_3$$
$$0.111\,[3\,(I_1 - I_2)] = I_1 - I_3$$

$$0.333\,I_1 - 0.333\,I_2 - I_1 + I_3 = 0$$
$$-0.667\,I_1 - 0.333\,I_2 + I_3 = 0 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-3\,(I_2 - I_1) - 1(I_2 - I_3) - 2\,I_2 = 0$$
$$-3\,I_1 + 6\,I_2 - I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 17\text{ A}$$
$$I_2 = 11\text{ A}$$
$$I_3 = 15\text{ A}$$

Example 2.12 *For the network shown in Fig. 2.24, find the magnitude of V_0 and the current supplied by it, given that power loss in $R_L = 2\ \Omega$ resistor is 18 W.*

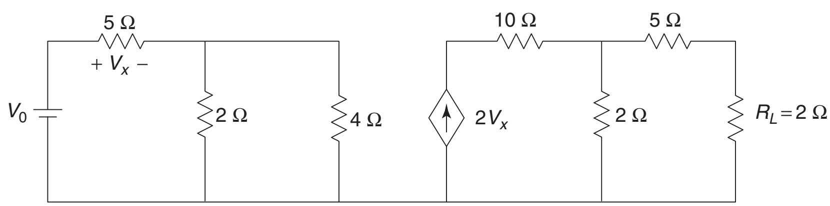


Fig. 2.24

Solution Assign clockwise currents in meshes as shown in Fig. 2.25.

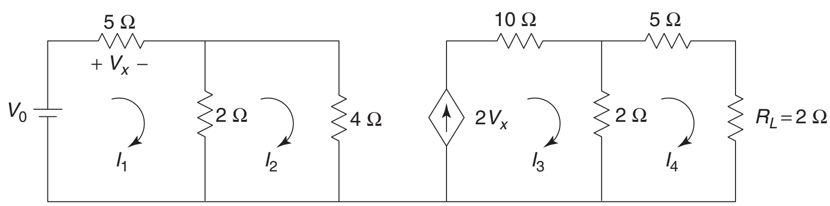


Fig. 2.25

From Fig. 2.25,

$$V_x = 5\,I_1 \qquad \text{...(i)}$$

Also,

$$I_4^2\,R_L = 18$$
$$I_4^2(2) = 18$$
$$I_4 = 3\text{ A} \qquad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$V_0 - 5\,I_1 - 2(I_1 - I_2) = 0$$
$$7\,I_1 - 2\,I_2 = V_0 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 4I_2 = 0$$

$$-2I_1 + 6I_2 = 0 \qquad \ldots\text{(iv)}$$

For Mesh 3,

$$I_3 = 2V_x = 2(5I_1) = 10I_1$$

$$10I_1 - I_3 = 0 \qquad \ldots\text{(v)}$$

Applying KVL to Mesh 4,

$$-2(I_4 - I_3) - 5I_4 - 2I_4 = 0$$

$$-2I_3 + 9I_4 = 0$$

$$-2I_3 + 9(3) = 0$$

$$I_3 = 13.5 \text{ A}$$

From Eq. (v),

$$I_1 = \frac{I_3}{10} = \frac{13.5}{10} = 1.35 \text{ A}$$

From Eq. (iv),

$$-2(1.35) + 6I_2 = 0$$

$$I_2 = 0.45 \text{ A}$$

From Eq. (iii),

$$7(1.35) - 2(0.45) = V_0$$

$$V_0 = 8.55 \text{ V}$$

Current supplied by voltage source $V_0 = I_1 = 1.35$ A

Example 2.13 *In the network shown in Fig. 2.26, find voltage V_2 such that $V_x = 0$.*

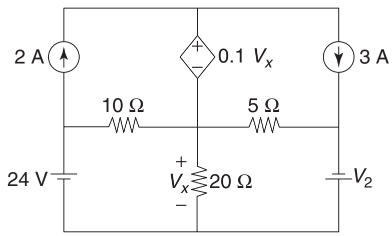


Fig. 2.26

Solution Assign clockwise currents in four meshes as shown in Fig. 2.27.

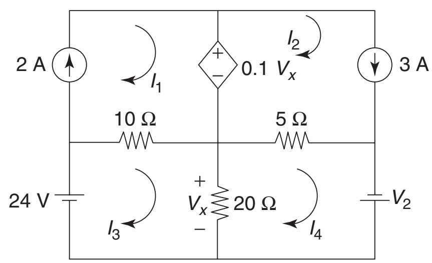


Fig. 2.27

From Fig. 2.27,

$$V_x = 20(I_3 - I_4) \qquad \ldots\text{(i)}$$

Writing current equations for Meshes 1 and 2,

$$I_1 = 2 \qquad \ldots\text{(ii)}$$

$$I_2 = 3 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$24-10(I_3-I_1)-20(I_3-I_4)=0$$

$$24-10(I_3-2)-20(I_3-I_4)=0$$

$$-30I_3+20I_4=-44 \quad \text{...(iv)}$$

Applying KVL to Mesh 4,

$$-20(I_4-I_3)-5(I_4-I_2)+V_2=0$$

$$-20(I_4-I_3)-5(I_4-3)+V_2=0$$

$$20I_3-25I_4=-V_2-15 \quad \text{...(v)}$$

But

$$V_x=0$$

$$20(I_3-I_4)=0$$

$$I_3=I_4$$

From Eq. (iv),

$$-30I_3+20I_3=-44$$

$$I_3=4.4 \text{ A}$$

$$I_4=4.4 \text{ A}$$

From Eq. (v),

$$20\,(4.4)-25\,(4.4)=-V_2-15$$

$$V_2=7 \text{ V}$$

2.4 SUPERMESH ANALYSIS

Meshes that share a current source with other meshes, none of which contains a current source in the outer loop, form a supermesh. A path around a supermesh doesn't pass through a current source. A path around each mesh contained within a supermesh passes through a current source. The total number of equations required for a supermesh is equal to the number of meshes contained in the supermesh. A supermesh requires one mesh current equation, that is, a KVL equation. The remaining mesh current equations are KCL equations.

Example 2.14 *Find the current in the 3 Ω resistor of the network shown in Fig. 2.28.*

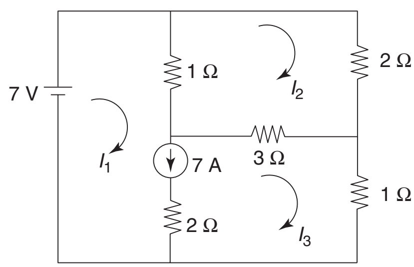


Fig. 2.28

Solution Meshes 1 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_1-I_3=7 \quad \text{...(i)}$$

Applying KVL to the outer path of the supermesh,

$$7-1(I_1-I_2)-3(I_3-I_2)-1I_3=0$$

$$-I_1+4I_2-4I_3=-7 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-1(I_2-I_1)-2I_2-3(I_2-I_3)=0$$

$$I_1-6I_2+3I_3=0 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=9\text{ A}$$

$$I_2=2.5\text{ A}$$

$$I_3=2\text{ A}$$

Current through the 3 Ω resistor = $I_2 - I_3$ = 2.5 – 2 = 0.5 A

Example 2.15 *Find the current in the 5 Ω resistor of the network shown in Fig. 2.29.*

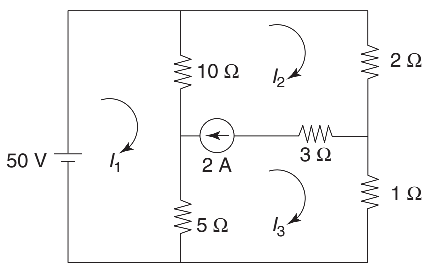


Fig. 2.29

Solution Applying KVL to Mesh 1,

$$50-10(I_1-I_2)-5(I_1-I_3)=0$$

$$15I_1-10I_2-5I_3=50 \quad \text{...(i)}$$

Meshes 2 and 3 will form a supermesh as these two meshes share a common current source of 2 A. Writing current equation for the supermesh,

$$I_2-I_3=2 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-10(I_2-I_1)-2I_2-1I_3-5(I_3-I_1)=0$$

$$-15I_1+12I_2+6I_3=0 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=20\text{ A}$$

$$I_2=17.33\text{ A}$$

$$I_3=15.33\text{ A}$$

Current through the 5 Ω resistor = $I_1 - I_3 = 20 - 15.33 = 4.67$ A

Example 2.16 *Determine the power delivered by the voltage source and the current in the 10 Ω resistor of the network shown in Fig. 2.30.*

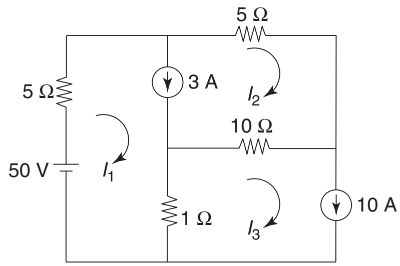


Fig. 2.30

Solution Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 3 \quad \text{...(i)}$$

Applying KVL to the outer path of the supermesh,

$$50 - 5I_1 - 5I_2 - 10(I_2 - I_3) - 1(I_1 - I_3) = 0$$

$$-6I_1 - 15I_2 + 11I_3 = -50 \quad \text{...(ii)}$$

For Mesh 3,

$$I_3 = 10 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 9.76 \text{ A}$$

$$I_2 = 6.76 \text{ A}$$

$$I_3 = 10 \text{ A}$$

Power delivered by the voltage source = $50\,I_1 = 50 \times 9.76 = 488$ W

$$I_{10\,\Omega} = I_3 - I_2 = 10 - 6.76 = 3.24 \text{ A}$$

Example 2.17 *For the network shown in Fig. 2.31, find current through the 8 Ω resistor.*

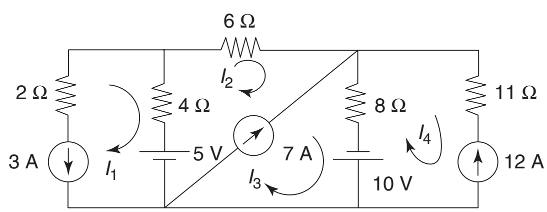


Fig. 2.31

Writing current equations for meshes 1 and 4,

$$I_1 = -3 \quad \text{...(i)}$$

$$I_4 = -12 \quad \text{...(ii)}$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 7 \quad \text{...(iii)}$$

Applying KVL to the outer path of the supermesh,

$$5 - 4(I_2 - I_1) - 6I_2 - 8(I_3 - I_4) + 10 = 0$$

$$5 - 4(I_2 + 3) - 6I_2 - 8(I_3 + 12) + 10 = 0$$

$$-10I_2 - 8I_3 = 93 \quad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$I_2 = -8.28 \text{ A}$$

$$I_3 = -1.28 \text{ A}$$

$$I_{8\,\Omega} = I_3 - I_4 = -1.28 + 12 = 10.72 \text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.18 *In the network of Fig. 2.32, find currents I_1 and I_2.*

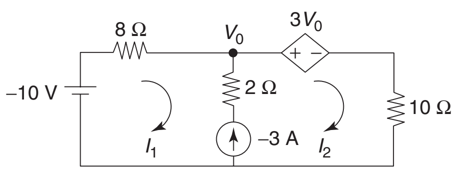


Fig. 2.32

Solution From Fig. 2.32,

$$-10 - 8I_1 - V_0 = 0$$

$$V_0 = -10 - 8I_1 \quad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equations for the supermesh,

$$I_2 - I_1 = -3 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-10 - 8I_1 - 3V_0 - 10I_2 = 0$$

$$-10 - 8I_1 - 3(-10 - 8I_1) - 10I_2 = 0$$

$$16I_1 - 10I_2 = -20 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -8.33 \text{ A}$$

$$I_2 = -11.33 \text{ A}$$

Example 2.19 *In the network of Fig. 2.33, find the current through the 3 Ω resistor.*

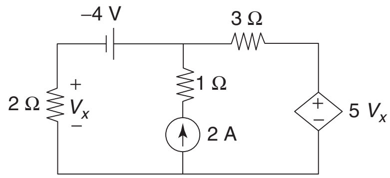


Fig. 2.33

Solution Assign clockwise currents in two meshes as shown in Fig. 2.34.
From Fig. 2.34,

$$V_x = -2\,I_1 \qquad \ldots\text{(i)}$$

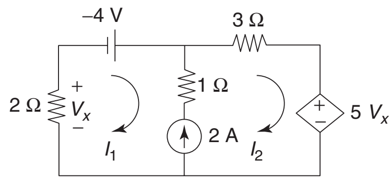


Fig. 2.34

Meshes 1 and 2 will form a supermesh.
Writing current equations for the supermesh,

$$I_2 - I_1 = 2 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-2I_1 - 4 - 3I_2 - 5V_x = 0$$

$$-2I_1 - 4 - 3I_2 - 5(-2I_1) = 0$$

$$8\,I_1 - 3I_2 = 4 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 2\text{ A}$$

$$I_2 = 4\text{ A}$$

$$I_{3\Omega} = I_2 = 4\text{ A}$$

Example 2.20 *Find the currents I_1 and I_2 at the network shown in Fig. 2.35.*

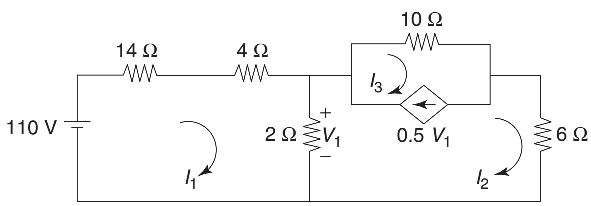


Fig. 2.35

Solution From Fig. 2.35,

$$V_1 = 2(I_1 - I_2)$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 0.5\text{ V}_1 = 0.5 \times 2(I_1 - I_2) = I_1 - I_2$$

$$I_3 = I_1$$

Applying KVL to outer path of the supermesh,

$$-2(I_2 - I_1) - 10I_3 - 6I_2 = 0$$
$$-2I_2 + 2I_1 - 10I_1 - 6I_2 = 0$$
$$I_1 = -I_2$$

Applying KVL to Mesh 1,

$$110 - 14I_1 - 4I_1 - 2(I_1 - I_2) = 0$$
$$110 - 20I_1 + 2I_2 = 0$$
$$110 + 20I_1 + 2I_2 = 0$$
$$I_2 = -5 \text{ A}$$
$$I_1 = -I_2 = 5 \text{ A}$$

Example 2.21 *For the network of Fig. 2.36, find current through the 8 Ω resistor.*

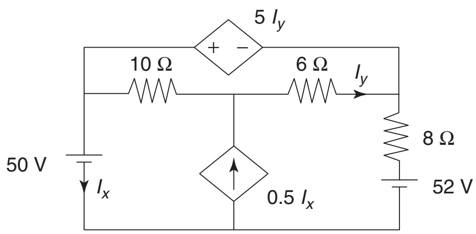


Fig. 2.36

Solution Assign clockwise currents to the three meshes as shown in Fig. 2.37.
From Fig. 2.37,

$$I_x = -I_1 \qquad \ldots\text{(i)}$$
$$I_y = I_2 - I_3 \qquad \ldots\text{(ii)}$$

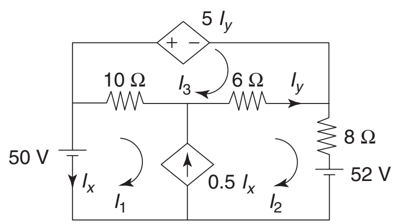


Fig. 2.37

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 0.5\,I_x = 0.5\,(-I_1)$$
$$-0.5\,I_1 + I_2 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to the outer path of the supermesh,

$$50 - 10(I_1 - I_3) - 6(I_2 - I_3) - 8I_2 - 52 = 0$$
$$-10I_1 - 14I_2 + 16I_3 = 2 \qquad \ldots\text{(iv)}$$

Applying KVL to Mesh 3,

$$-5\,I_y - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$
$$-5\,(I_2 - I_3) - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$
$$10\,I_1 + I_2 - 11\,I_3 = 0 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = -1.56\text{ A}$$
$$I_2 = -0.58\text{ A}$$
$$I_3 = -1.11\text{ A}$$
$$I_{8\,\Omega} = I_2 = -0.58\text{ A}$$

Example 2.22 *For the network shown in Fig. 2.38, find the current through the 10 Ω resistor.*

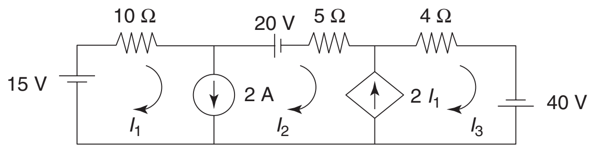


Fig. 2.38

Solution Meshes 1, 2 and 3 will form a supermesh.
Writing current equations for the supermesh,

$$I_1 - I_2 = 2 \quad \text{...(i)}$$
$$I_3 - I_2 = 2\,I_1$$

and
$$2\,I_1 + I_2 - I_3 = 0 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$15 - 10\,I_1 - 20 - 5\,I_2 - 4I_3 + 40 = 0$$
$$10\,I_1 + 5\,I_2 + 4\,I_3 = 35 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.96\text{ A}$$
$$I_2 = -0.04\text{ A}$$
$$I_3 = 3.89\text{ A}$$
$$I_{10\,\Omega} = I_1 = 1.96\text{ A}$$

Example 2.23 *In the network shown in Fig. 2.39, find the power delivered by the 4 V source and voltage across the 2 Ω resistor.*

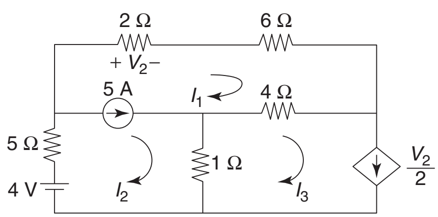


Fig. 2.39

Solution From Fig. 2.39,

$$V_2 = 2\,I_1 \quad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$4 - 5I_2 - 2I_1 - 6I_1 - 4(I_1 - I_3) - 1(I_2 - I_3) = 0$$

$$-12I_1 - 6I_2 + 5I_3 = -4 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = \frac{V_2}{2} = \frac{2I_1}{2} = I_1$$

$$I_1 - I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -2 \text{ A}$$

$$I_2 = 3 \text{ A}$$

$$I_3 = -2 \text{ A}$$

Power delivered by the 4 V source $= 4I_2 = 4(3) = 12$ W

$$V_{2\Omega} = 2I_1 = 2(-2) = -4 \text{ V}$$

Example 2.24 *Find currents I_1, I_2, I_3, I_4 of the network shown in Fig. 2.40.*

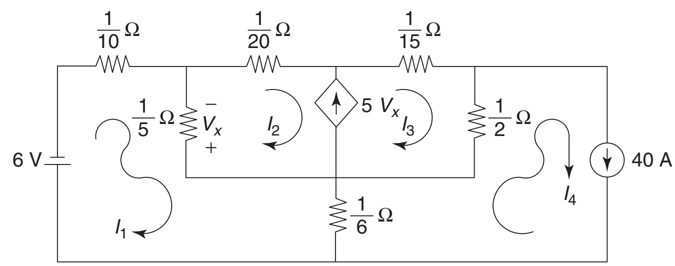


Fig. 2.40

From Fig. 2.40,

$$V_x = \frac{1}{5}(I_2 - I_1) \qquad \ldots\text{(i)}$$

For Mesh 4,

$$I_4 = 40 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}(I_1 - I_2) - \frac{1}{6}(I_1 - I_4) = 0$$

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}I_1 + \frac{1}{5}I_2 - \frac{1}{6}I_1 + \frac{1}{6}(40) = 0$$

$$-\frac{7}{15}I_1 + \frac{1}{5}I_2 = -\frac{2}{3} \qquad \ldots\text{(iii)}$$

Mesh 2 and 3 will form a supermesh.

Writing current equation for the supermesh,

$$I_3 - I_2 = 5\,V_x = 5\left[\frac{1}{5}(I_2 - I_1)\right] = I_2 - I_1$$

$$I_1 - 2I_2 + I_3 = 0 \qquad \ldots\text{(iv)}$$

Applying KVL to the outer path of the supermesh,

$$-\frac{1}{5}(I_2 - I_1) - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}(I_3 - I_4) = 0$$

$$-\frac{1}{5}I_2 + \frac{1}{5}I_1 - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}I_3 + \frac{1}{2}(40) = 0$$

$$\frac{1}{5}I_1 - \frac{1}{4}I_2 - \frac{17}{30}I_3 = -20 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 10\text{ A}$$
$$I_2 = 20\text{ A}$$
$$I_3 = 30\text{ A}$$
$$I_4 = 40\text{ A}$$

2.5 NODE ANALYSIS

Node analysis is based on Kirchhoff's current law which states that the algebraic sum of currents meeting at a point is zero. Every junction where two or more branches meet is regarded as a node. One of the nodes in the network is taken as *reference node* or *datum node*. If there are *n* nodes in any network, the number of simultaneous equations to be solved will be $(n - 1)$.

Steps to be followed in Node Analysis

1. Assuming that a network has *n* nodes, assign a reference node and the reference directions, and assign a current and a voltage name for each branch and node respectively.
2. Apply KCL at each node except for the reference node and apply Ohm's law to the branch currents.
3. Solve the simultaneous equations for the unknown node voltages.
4. Using these voltages, find any branch currents required.

Example 2.25 *Determine the current through the 5 Ω resistor for the network shown in Fig. 2.41.*

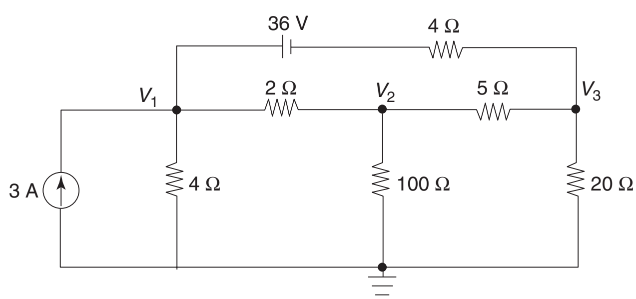


Fig. 2.41

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1}{4}+\frac{V_1-V_2}{2}+\frac{V_1-36-V_3}{4}=3$$

$$\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right)V_1-\frac{1}{2}V_2-\frac{1}{4}V_3=3+\frac{36}{4}$$

$$V_1-0.5\,V_2-0.25\,V_3=12 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{2}+\frac{V_2}{100}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{2}V_1+\left(\frac{1}{2}+\frac{1}{100}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=0$$

$$-0.5\,V_1+0.71\,V_2-0.2\,V_3=0 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{20}+\frac{V_3-(-36)-V_1}{4}=0$$

$$-\frac{1}{4}V_1-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{20}+\frac{1}{4}\right)V_3=-9$$

$$-0.25\,V_1-0.2\,V_2+0.5\,V_3=-9 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=13.41\text{ V}$$

$$V_2=7.06\text{ V}$$

$$V_3=-8.47\text{ V}$$

$$\text{Current through the } 5\ \Omega \text{ resistor} = \frac{V_2-V_3}{5}=\frac{7.06-(-8.47)}{5}=3.11\text{ A}$$

Example 2.26 *Find the power dissipated in the 6 Ω resistor for the network shown in Fig. 2.42.*

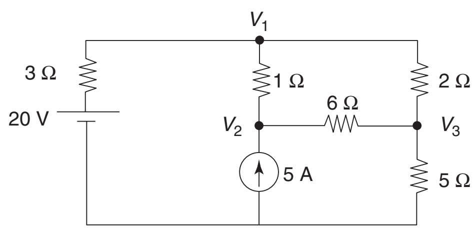


Fig. 2.42

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1-20}{3}+\frac{V_1-V_2}{1}+\frac{V_1-V_3}{2}=0$$

$$\left(\frac{1}{3}+1+\frac{1}{2}\right)V_1-V_2-\frac{1}{2}V_3=\frac{20}{3}$$

$$1.83\,V_1-V_2-0.5\,V_3=6.67 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{1}+\frac{V_2-V_3}{6}=5$$

$$-V_1+\left(1+\frac{1}{6}\right)V_2-\frac{1}{6}V_3=5$$

$$-V_1+1.17\,V_2-0.17\,V_3=5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_1}{2}+\frac{V_3}{5}+\frac{V_3-V_2}{6}=0$$

$$-\frac{1}{2}V_1-\frac{1}{6}V_2+\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{6}\right)V_3=0$$

$$-0.5\,V_1-0.17\,V_2+0.87\,V_3=0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=23.82\text{ V}$$

$$V_2=27.4\text{ V}$$

$$V_3=19.04\text{ V}$$

$$I_{6\,\Omega}=\frac{V_2-V_3}{6}=\frac{27.4-19.04}{6}=1.39\text{ A}$$

Power dissipated in the 6 Ω resistor $=(1.39)^2\times 6=11.59$ W

Example 2.27 *Find V_1 and V_2 for the network shown in Fig. 2.43.*

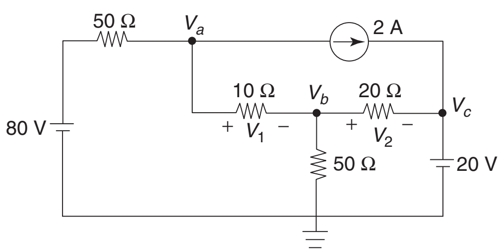


Fig. 2.43

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node a,

$$\frac{V_a-80}{50}+\frac{V_a-V_b}{10}+2=0$$

$$\left(\frac{1}{50}+\frac{1}{10}\right)V_a-\frac{1}{10}V_b=\frac{80}{50}-2$$

$$0.12\,V_a-0.1\,V_b=-0.4 \qquad \ldots\text{(i)}$$

Applying KCL at Node b,

$$\frac{V_b-V_a}{10}+\frac{V_b}{50}+\frac{V_b-V_c}{20}=0$$

$$-\frac{1}{10}V_a+\left(\frac{1}{10}+\frac{1}{50}+\frac{1}{20}\right)V_b-\frac{1}{20}V_c=0$$

$$-0.1\,V_a+0.17\,V_b-0.05\,V_c=0 \qquad \ldots\text{(ii)}$$

Node c is directly connected to a voltage source of 20 V. Hence, we can write voltage equation at Node c.

$$V_c=20 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_a=3.08\text{ V}$$

$$V_b=7.69\text{ V}$$

$$V_1=V_a-V_b=3.08-7.69=-4.61\text{ V}$$

$$V_2=V_b-V_c=7.69-20=-12.31\text{V}$$

Example 2.28 *Find the voltage across the 100 Ω resistor for the network shown in Fig. 2.44.*

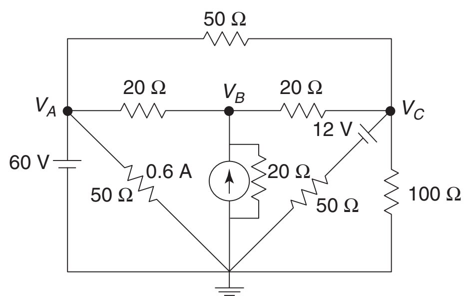


Fig. 2.44

Solution

Node A is directly connected to a voltage source of 60 V. Hence, we can write voltage equation at Node A.

$$V_A=60 \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.

Applying KCL at Node B,

$$\frac{V_B - V_A}{20} + \frac{V_B - V_C}{20} + \frac{V_B}{20} = 0.6$$

$$-\frac{1}{20}V_A + \left(\frac{1}{20} + \frac{1}{20} + \frac{1}{20}\right)V_B - \frac{1}{20}V_C = 0.6$$

$$-0.05\,V_A + 0.15\,V_B - 0.05\,V_C = 0.6 \qquad \ldots\text{(ii)}$$

Applying KCL at Node C,

$$\frac{V_C - V_A}{50} + \frac{V_C - V_B}{20} + \frac{V_C - 12}{50} + \frac{V_C}{100} = 0$$

$$-\frac{1}{50}V_A - \frac{1}{20}V_B + \left(\frac{1}{50} + \frac{1}{20} + \frac{1}{50} + \frac{1}{100}\right)V_C = \frac{12}{50}$$

$$-0.02\,V_A - 0.05\,V_B + 0.1\,V_C = 0.24 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_C = 31.68 \text{ V}$$

$$\text{Voltages across the } 100\ \Omega \text{ resistor} = 31.68 \text{ V}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.29 *Find the voltage across the 5 Ω resistor in the network shown in Fig. 2.45.*

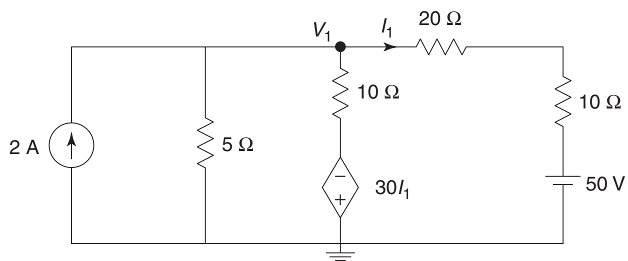


Fig. 2.45

Solution From Fig. 2.45,

$$I_1 = \frac{V_1 - 50}{20 + 10} = \frac{V_1 - 50}{30} \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the node.

Applying KCL at Node 1,

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\,I_1}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\left(\frac{V_1 - 50}{30}\right)}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{2V_1 - 50}{10} + \frac{V_1 - 50}{30} \qquad \ldots\text{(ii)}$$

Solving Eq. (ii),

$$V_1 = 20 \text{ V}$$

Voltage across the 5 Ω resistor = 20 V

Example 2.30 *For the network shown in Fig. 2.46, find the voltage V_x.*

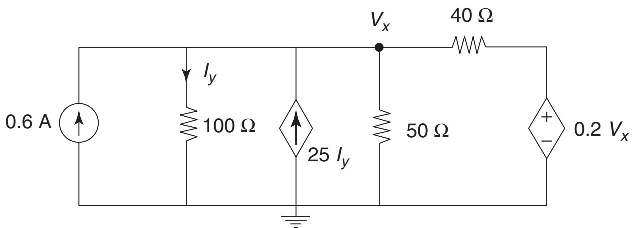


Fig. 2.46

Solution From Fig. 2.46,

$$I_y = \frac{V_x}{100} \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the node.
Applying KCL at Node x,

$$25\,I_y + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{V_x - 0.2\,V_x}{40}$$

$$25\left(\frac{V_x}{100}\right) + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{0.8\,V_x}{40}$$

$$\left(\frac{1}{4} - \frac{1}{100} - \frac{1}{50} - \frac{0.8}{40}\right)V_x = -0.6$$

$$0.2\,V_x = -0.6$$

$$V_x = -3 \text{ V}$$

Example 2.31 *For the network shown in Fig. 2.47, find voltages V_1 and V_2.*

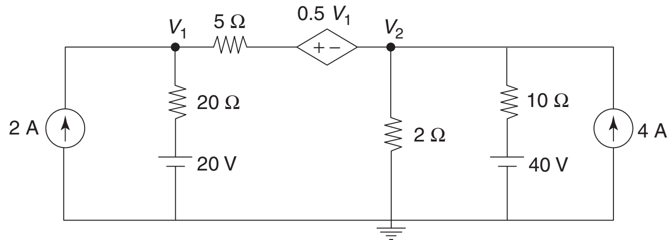


Fig. 2.47

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1 - 20}{20} + \frac{V_1 - 0.5\,V_1 - V_2}{5}$$

$$\left(\frac{1}{20} + \frac{1}{5} - \frac{0.5}{5}\right) V_1 - \frac{1}{5} V_2 = 3$$

$$0.15\,V_1 - 0.2\,V_2 = 3 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 + 0.5\,V_1 - V_1}{5} + \frac{V_2}{2} + \frac{V_2 - 40}{10} = 4$$

$$\left(\frac{0.5}{5} - \frac{1}{5}\right) V_1 + \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{10}\right) V_2 = 4 + \frac{40}{10}$$

$$-0.1\,V_1 + 0.8\,V_2 = 8 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 40 \text{ V}$$

$$V_2 = 15 \text{ V}$$

Example 2.32 *Determine the voltages V_1 and V_2 in the network of Fig. 2.48.*

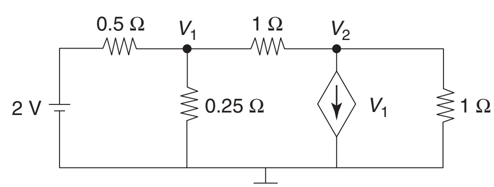


Fig. 2.48

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 2}{0.5} + \frac{V_1}{0.25} + \frac{V_1 - V_2}{1} = 0$$

$$\left(\frac{1}{0.5} + \frac{1}{0.25} + 1\right) V_1 - V_2 = \frac{2}{0.5}$$

$$7V_7 - V_2 = 4 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} + V_1 = 0$$

$$2V_2 = 0$$

$$V_2 = 0 \qquad \text{...(ii)}$$

From Eq. (i),

$$V_1 = \frac{4}{7}\text{ V}$$

Example 2.33 *In the network of Fig. 2.49, find the node voltages V_1, V_2 and V_3.*

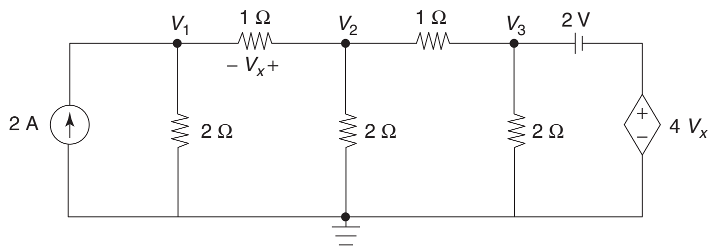


Fig. 2.49

Solution From Fig. 2.49,

$$V_x = V_2 - V_1 \qquad \text{...(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1}{2} + \frac{V_1 - V_2}{1}$$

$$\left(\frac{1}{2} + 1\right) V_1 - V_2 = 2$$

$$1.5V_1 - V_2 = 2 \qquad \text{...(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{2} + \frac{V_2 - V_3}{1} = 0$$

$$-V_1 + \left(1 + \frac{1}{2} + 1\right) V_2 - V_3 = 0$$

$$-V_1 + 2.5\, V_2 - V_3 = 0 \qquad \text{...(iii)}$$

At Node 3,

$$V_3 - 4\, V_x = 2$$

$$V_3 - 4(V_2 - V_1) = 2$$

$$4\, V_1 - 4\, V_2 + V_3 = 2 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$V_1 = -1.33 \text{ V}$$

$$V_2 = -4 \text{ V}$$

$$V_3 = -8.67 \text{ V}$$

Example 2.34 *For the network shown in Fig. 2.50, find the node voltages V_1 and V_2.*

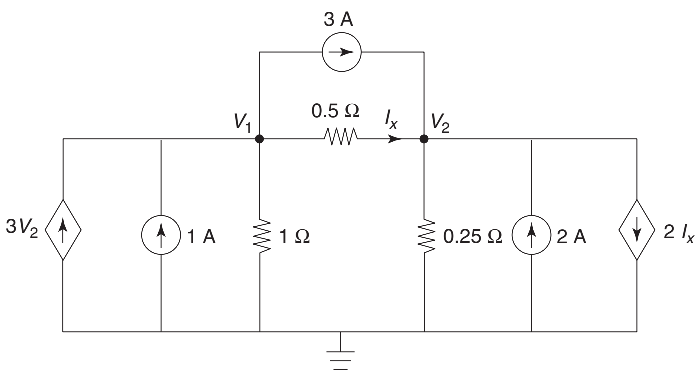


Fig. 2.50

Solution From Fig. 2.50,

$$I_x = \frac{V_1 - V_2}{0.5} \qquad \text{...(i)}$$

Assume that the currents are moving away from the nodes.

Applying KCL at Node 1,

$$3V_2 + 1 = \frac{V_1}{1} + \frac{V_1 - V_2}{0.5} + 3$$

$$\left(1 + \frac{1}{0.5}\right)V_1 - \left(3 + \frac{1}{0.5}\right)V_2 = -2$$

$$3V_1 - 5V_2 = -2 \qquad \text{...(ii)}$$

Applying KCL at Node 2,

$$3 + 2 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2I_x$$

$$5 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2\left(\frac{V_1 - V_2}{0.5}\right)$$

$$\left(-\frac{1}{0.5} + \frac{2}{0.5}\right)V_1 + \left(\frac{1}{0.5} + \frac{1}{0.25} - \frac{2}{0.5}\right)V_2 = 5$$

$$2V_1 + 2V_2 = 5 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = 1.31 \text{ V}$$
$$V_2 = 1.19 \text{ V}$$

Example 2.35 *Find voltages V_1 and V_2 in the network shown in Fig. 2.51.*

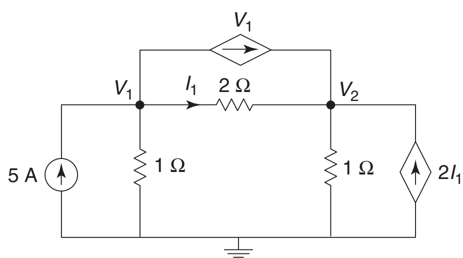


Fig. 2.51

Solution From Fig. 2.51,

$$I_1 = \frac{V_1 - V_2}{2} \qquad \text{...(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$5 = \frac{V_1}{1} + \frac{V_1 - V_2}{2} + V_1$$

$$\left(1 + \frac{1}{2} + 1\right)V_1 - \frac{1}{2}V_2 = 5$$

$$2.5V_1 - 0.5V_2 = 5 \qquad \text{...(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} = 2I_1 + V_1$$

$$V_2 - V_1 + V_2 = 2\left(\frac{V_1 - V_2}{2}\right) + V_1$$

$$3\,V_1 = 3\,V_2$$

$$V_1 = V_2 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = 2.5 \text{ V}$$

$$V_2 = 2.5 \text{ V}$$

Example 2.36 *Find the power supplied by the 10 V source in the network shown in Fig. 2.52.*

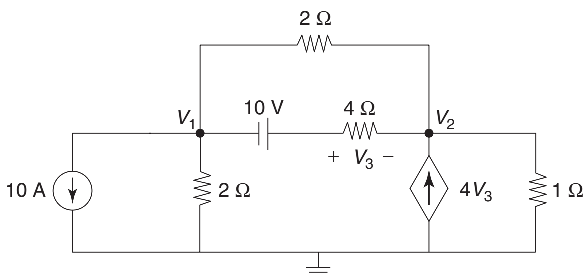


Fig. 2.52

Solution

From Fig. 2.52,

$$V_3 = V_1 + 10 - V_2 \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$10 + \frac{V_1}{2} + \frac{V_1 + 10 - V_2}{4} + \frac{V_1 - V_2}{2} = 0$$

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right)V_1 - \left(\frac{1}{4} + \frac{1}{2}\right)V_2 = -10 - \frac{10}{4}$$

$$1.25\,V_1 - 0.75\,V_2 = -12.5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - 10 - V_1}{4} + \frac{V_2 - V_1}{2} + \frac{V_2}{1} = 4\,V_3$$

$$\frac{V_2 - 10 - V_1}{4} + \frac{V_2 - V_1}{2} + \frac{V_2}{1} = 4(V_1 + 10 - V_2)$$

$$\left(-\frac{1}{4}-\frac{1}{2}-4\right)V_1+\left(\frac{1}{4}+\frac{1}{2}+1+4\right)V_2=\frac{10}{4}+40$$

$$-4.75\,V_1+5.75\,V_2=42.5 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1=-11.03\text{ V}$$

$$V_2=-1.72\text{ V}$$

$$I_{10\text{ V}}=\frac{V_1+10-V_2}{4}=\frac{-11.03+10-(-1.72)}{4}=0.173\text{ A}$$

Power supplied by the 10 V source = 10 × 0.173 = 1.73 W

Example 2.37 *For the network shown in Fig. 2.53, find voltages V_1 and V_2.*

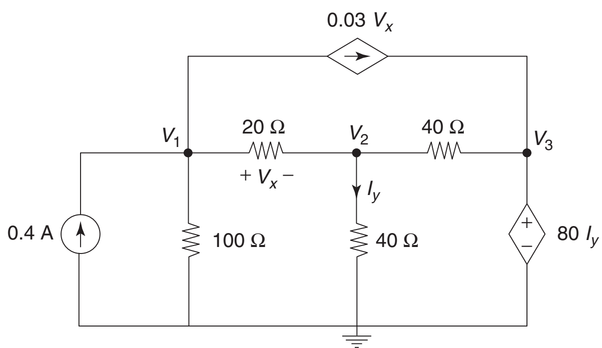


Fig. 2.53

Solution From Fig. 2.53,

$$V_x=V_1-V_2 \qquad \ldots\text{(i)}$$

$$I_y=\frac{V_2}{40} \qquad \ldots\text{(ii)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$0.4=\frac{V_1}{100}+\frac{V_1-V_2}{20}+0.03\,V_x$$

$$0.4=\frac{V_1}{100}+\frac{V_1-V_2}{20}+0.03(V_1-V_2)$$

$$\left(\frac{1}{100}+\frac{1}{20}+0.03\right)V_1-\left(\frac{1}{20}+0.03\right)V_2=0.4$$

$$0.09\,V_1-0.08\,V_2=0.4 \qquad \ldots\text{(iii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{20} + \frac{V_2}{40} + \frac{V_2 - V_3}{40} = 0$$

$$-\frac{1}{20}V_1 + \left(\frac{1}{20} + \frac{1}{40} + \frac{1}{40}\right)V_2 - \frac{1}{40}V_3 = 0$$

$$-0.05\,V_1 + 0.1\,V_2 - 0.025\,V_3 = 0 \qquad \text{...(iv)}$$

For Node 3,

$$V_3 = 80\,I_y = 80\left(\frac{V_2}{40}\right) = 2\,V_2$$

$$2\,V_2 - V_3 = 0 \qquad \text{...(v)}$$

Solving Eqs (iii), (iv) and (v),

$$V_1 = 40 \text{ V}$$
$$V_2 = 40 \text{ V}$$
$$V_3 = 80 \text{ V}$$

Example 2.38 *Find voltages V_a, V_b and V_c in the network shown in Fig. 2.54.*

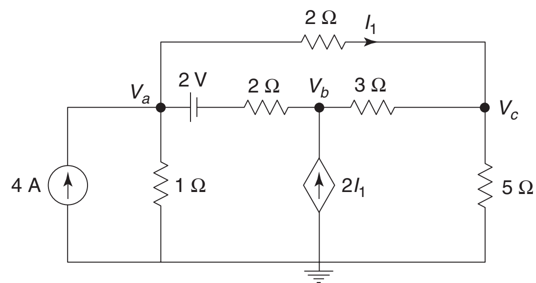


Fig. 2.54

Solution From Fig. 2.54,

$$I_1 = \frac{V_a - V_c}{2}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node *a*,

$$4 = \frac{V_a}{1} + \frac{V_a - V_c}{2} + \frac{V_a - 2 - V_b}{2}$$

$$\left(1 + \frac{1}{2} + \frac{1}{2}\right)V_a - \frac{1}{2}V_b - \frac{1}{2}V_c = 5$$

$$2\,V_a - 0.5\,V_b - 0.5\,V_c = 5 \qquad \text{...(i)}$$

Applying KCL at Node b,

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2I_1$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2\left(\frac{V_a - V_c}{2}\right)$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = V_a - V_c$$

$$\left(-\frac{1}{2} - 1\right)V_a + \left(\frac{1}{2} + \frac{1}{3}\right)V_b + \left(1 - \frac{1}{3}\right)V_c = -1$$

$$-1.5\,V_a + 0.83\,V_b + 0.67\,V_c = -1 \qquad \ldots\text{(ii)}$$

Applying KCL at Node c,

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = I_1$$

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = \frac{V_a - V_c}{2}$$

$$-\frac{1}{2}V_a - \frac{1}{3}V_b + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{2}\right)V_c = 0$$

$$-0.5\,V_a - 0.33\,V_b + 1.033\,V_c = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_a = 4.303 \text{ V}$$

$$V_b = 3.88 \text{ V}$$

$$V_c = 3.33 \text{ V}$$

2.6 SUPERNODE ANALYSIS

Nodes that are connected to each other by voltage sources, but not to the reference node by a path of voltage sources, form a *supernode*. A supernode requires one node voltage equation, that is, a KCL equation. The remaining node voltage equations are KVL equations.

Example 2.39 *Determine the current in the 5 Ω resistor for the network shown in Fig. 2.55.*

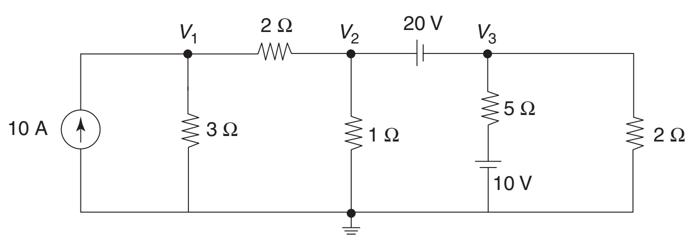


Fig. 2.55

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

$$\left(\frac{1}{3} + \frac{1}{2}\right) V_1 - \frac{1}{2} V_2 = 10$$

$$0.83\, V_1 - 0.5\, V_2 = 10 \qquad \text{...(i)}$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_2 - V_3 = 20 \qquad \text{...(ii)}$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

$$-\frac{1}{2} V_1 + \left(\frac{1}{2} + 1\right) V_2 + \left(\frac{1}{5} + \frac{1}{2}\right) V_3 = 2$$

$$-0.5\, V_1 + 1.5\, V_2 + 0.7\, V_3 = 2 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 19.04 \text{ V}$$
$$V_2 = 11.6 \text{ V}$$
$$V_3 = -8.4 \text{ V}$$

$$I_{5\,\Omega} = \frac{V_3 - 10}{5} = \frac{-8.4 - 10}{5} = -3.68 \text{ A}$$

Example 2.40 *Find the power delivered by the 5 A current source in the network shown in Fig. 2.56.*

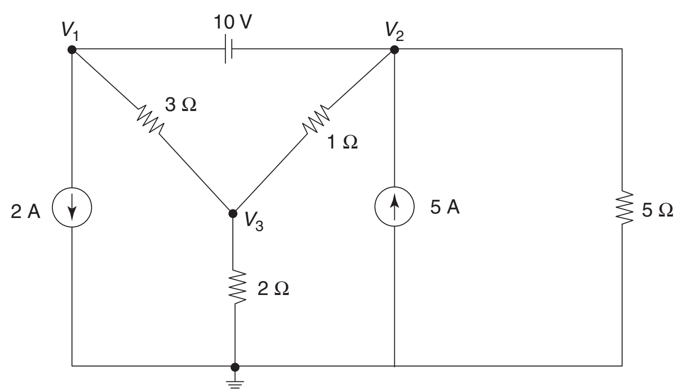


Fig. 2.56

Solution Assume that the currents are moving away from the nodes.
Nodes 1 and 2 will form a supernode.
Writing voltage equation for the supernode,

$$V_1 - V_2 = 10 \qquad \ldots\text{(i)}$$

Applying KCL at the supernode,

$$2 + \frac{V_1 - V_3}{3} + \frac{V_2}{5} + \frac{V_2 - V_3}{1} = 5$$

$$\frac{1}{3}V_1 + \left(\frac{1}{5} + 1\right)V_2 - \left(\frac{1}{3} + 1\right)V_3 = 3$$

$$0.33\,V_1 + 1.2\,V_2 - 1.33\,V_3 = 3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{3} + \frac{V_3 - V_2}{1} + \frac{V_3}{2} = 0$$

$$-\frac{1}{3}V_1 - V_2 + \left(\frac{1}{3} + 1 + \frac{1}{2}\right)V_3 = 0$$

$$-0.33\,V_1 - V_2 + 1.83\,V_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 13.72 \text{ V}$$

$$V_2 = 3.72 \text{ V}$$

$$V_3 = 4.51 \text{ V}$$

Power delivered by the 5 A source = 5 V_2 = 5 × 3.72 = 18.6 W

Example 2.41 *In the network of Fig. 2.57, find the node voltages V_1, V_2 and V_3.*

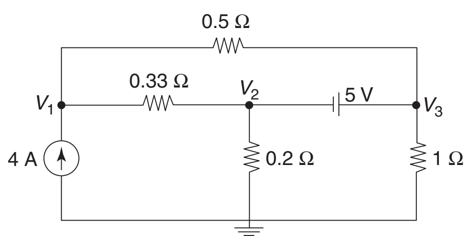


Fig. 2.57

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$4 = \frac{V_1 - V_2}{0.33} + \frac{V_1 - V_3}{0.5}$$

$$\left(\frac{1}{0.33} + \frac{1}{0.5}\right)V_1 - \frac{1}{0.33}V_2 - \frac{1}{0.5}V_3 = 4$$

$$5.03\,V_1 - 3.03\,V_2 - 2\,V_3 = 4 \qquad \ldots\text{(i)}$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_3 - V_2 = 5 \qquad \ldots\text{(ii)}$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{0.33} + \frac{V_2}{0.2} + \frac{V_3}{1} + \frac{V_3 - V_1}{0.5} = 0$$

$$\left(-\frac{1}{0.33} - \frac{1}{0.5}\right)V_1 + \left(\frac{1}{0.33} + \frac{1}{0.2}\right)V_2 + \left(1 + \frac{1}{0.5}\right)V_3 = 0$$

$$-5.03\,V_1 + 8.03\,V_2 + 3\,V_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 2.62 \text{ V}$$
$$V_2 = -0.17 \text{ V}$$
$$V_3 = 4.83 \text{ V}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.42 *For the network shown in Fig. 2.58, determine the voltage V_x.*

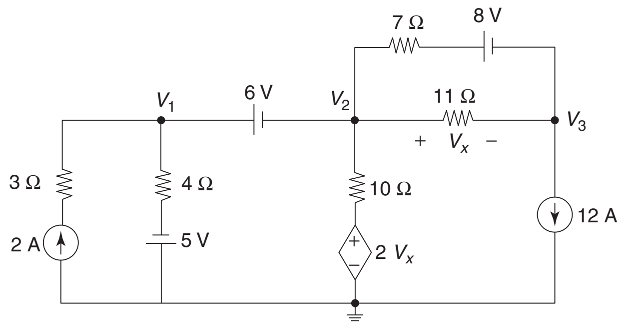


Fig. 2.58

Solution From Fig. 2.58,

$$V_x = V_2 - V_3$$

Assume that the currents are moving away from the nodes.
Node 1 and 2 will form a supernode.
Writing voltage equations for the supernode,

$$V_1 - V_2 = 6 \qquad \ldots\text{(i)}$$

Applying KCL at the supernode,

$$2 = \frac{V_1 + 5}{4} + \frac{V_2 - 2V_x}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$2 = \frac{V_1 + 5}{4} + \frac{V_2 - 2(V_2 - V_3)}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$\frac{1}{4}V_1+\left(\frac{1}{10}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}\right)V_2+\left(\frac{1}{5}-\frac{1}{7}-\frac{1}{11}\right)V_3=2-\frac{5}{4}+\frac{8}{7}$$

$$0.25\,V_1+0.133\,V_2-0.033\,V_3=1.89 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{11}+\frac{V_3+8-V_2}{7}+12=0$$

$$\left(-\frac{1}{11}-\frac{1}{7}\right)V_2+\left(\frac{1}{11}+\frac{1}{7}\right)V_3=-12-\frac{8}{7}$$

$$-0.233\,V_2+0.233\,V_3=-13.14 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=1.8\text{ V}$$
$$V_2=-4.2\text{ V}$$
$$V_3=-60.6\text{ V}$$

Example 2.43 *Find the node voltages in the network shown in Fig. 2.59.*

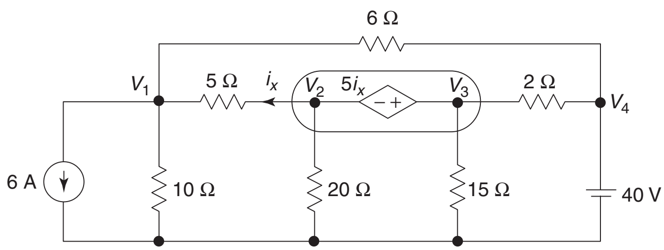


Fig. 2.59

Solution From Fig. 2.59,

$$I_x=\frac{V_2-V_1}{5} \qquad \ldots\text{(i)}$$

For Node 4,

$$V_4=40 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 1,

$$6+\frac{V_1}{10}+\frac{V_1-V_2}{5}+\frac{V_1-V_4}{6}=0$$

$$6+\frac{V_1}{10}+\frac{V_1-V_2}{5}+\frac{V_1-40}{6}=0$$

$$\left(\frac{1}{10}+\frac{1}{5}+\frac{1}{6}\right)V_1-\frac{1}{5}V_2=\frac{40}{6}-6$$

$$\frac{7}{15}V_1-\frac{1}{5}V_2=\frac{2}{3} \qquad \ldots\text{(iii)}$$

Nodes 2 and 3 will form a supernode,
Writing voltage equation for the supernode,

$$V_3 - V_2 = 5\,I_x = 5\left(\frac{V_2 - V_1}{5}\right) = V_2 - V_1$$

$$V_1 - 2V_2 + V_3 = 0 \qquad \text{...(iv)}$$

Applying KCL to the supernode,

$$\frac{V_2 - V_1}{5} + \frac{V_2}{20} + \frac{V_3}{15} + \frac{V_3 - V_4}{2} = 0$$

$$\frac{V_2 - V_1}{5} + \frac{V_2}{20} + \frac{V_3}{15} + \frac{V_3 - 40}{2} = 0$$

$$-\frac{1}{5}V_1 + \left(\frac{1}{5} + \frac{1}{20}\right)V_2 + \left(\frac{1}{15} + \frac{1}{2}\right)V_3 = 20$$

$$-\frac{1}{5}V_1 + \frac{1}{4}V_2 + \frac{17}{30}V_3 = 20 \qquad \text{...(v)}$$

Solving Eqs (iii), (iv) and (v),

$$V_1 = 10\text{ V}$$

$$V_2 = 20\text{ V}$$

$$V_3 = 30\text{ V}$$

$$V_4 = 40\text{ V}$$

Example 2.44 *Find the node voltages in the network shown in Fig. 2.60.*

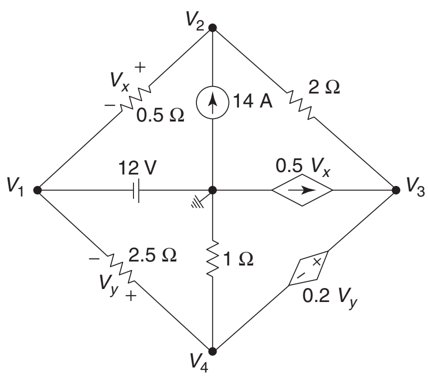


Fig. 2.60

Solution Selecting the central node as reference node,

$$V_1 = -12\text{ V} \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{0.5} + \frac{V_2 - V_3}{2} = 14$$

$$-\frac{1}{0.5}V_1 + \left(\frac{1}{0.5} + \frac{1}{2}\right)V_2 - \frac{1}{2}V_3 = 14$$

$$-2\,V_1 + 2.5\,V_2 - 0.5\,V_3 = 14 \qquad \text{...(ii)}$$

Nodes 3 and 4 will form a supernode,

Writing voltage equation for the supernode,

$$V_3 - V_4 = 0.2V_y = 0.2(V_4 - V_1)$$

$$0.2V_1 + V_3 - 1.2V_4 = 0 \qquad \text{...(iii)}$$

Applying KCL to the supernode,

$$\frac{V_3 - V_2}{2} - 0.5V_x + \frac{V_4}{1} + \frac{V_4 - V_1}{2.5} = 0$$

$$\frac{V_3 - V_2}{2} - 0.5(V_2 - V_1) + V_4 + \frac{V_4 - V_1}{2.5} = 0$$

$$\left(0.5 - \frac{1}{2.5}\right)V_1 - \left(\frac{1}{2} + 0.5\right)V_2 + \frac{1}{2}V_3 + \left(1 + \frac{1}{2.5}\right)V_4 = 0$$

$$0.1\,V_1 - V_2 + 0.5\,V_3 + 1.4\,V_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (i), (ii), (iii) and (iv),

$$V_1 = -12 \text{ V}$$

$$V_2 = -4 \text{ V}$$

$$V_3 = 0$$

$$V_4 = -2 \text{ V}$$

2.7 SUPERPOSITION THEOREM

It states that '*in a linear network containing more than one independent source and dependent source, the resultant current in any element is the algebraic sum of the currents that would be produced by each independent source acting alone, all the other independent sources being represented meanwhile by their respective internal resistances.*'

The independent voltage sources are represented by their internal resistances if given or simply with zero resistances, i.e., short circuits if internal resistances are not mentioned. The independent current sources are represented by infinite resistances, i.e., open circuits.

The dependent sources are not sources but dissipative components—hence they are active at all times. A dependent source has zero value only when its control voltage or current is zero.

A linear network is one whose parameters are constant, i.e., they do not change with voltage and current.

Explanation Consider the network shown in Fig. 2.61. Suppose we have to find current I_4 through resistor R_4.

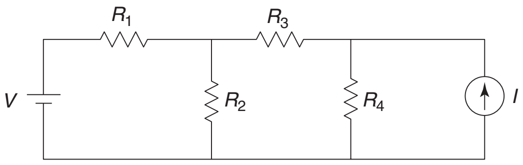


Fig. 2.61 *Network to illustrate superposition theorem*

The current flowing through resistor R_4 due to constant voltage source V is found to be say I_4' (with proper direction), representing constant current source with infinite resistance, i.e., open circuit.

The current flowing through resistor R_4 due to constant current source I is found to be say I_4'' (with proper direction), representing the constant voltage source with zero resistance or short circuit.

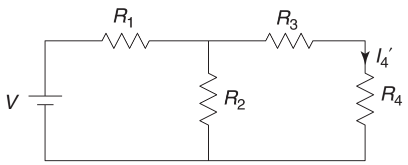


Fig. 2.62 *When voltage source V is acting alone*

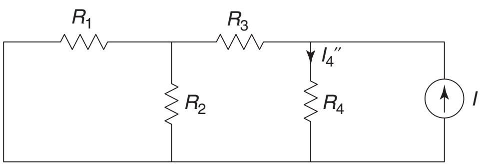


Fig. 2.63 *When current source I is acting alone*

The resultant current I_4 through resistor R_4 is found by superposition theorem.

$$I_4 = I_4' + I_4''$$

Steps to be followed in Superposition Theorem

1. Find the current through the resistance when only one independent source is acting, replacing all other independent sources by respective internal resistances.
2. Find the current through the resistance for each of the independent sources.
3. Find the resultant current through the resistance by the superposition theorem considering magnitude and direction of each current.

Example 2.45 *Find the current through the 4 Ω resistor in Fig. 2.64.*

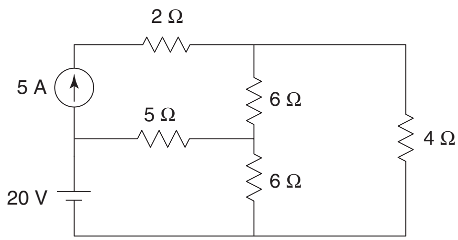


Fig. 2.64

Solution

Step I When the 5 A source is acting alone (Fig. 2.65)

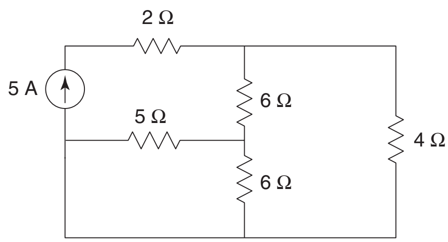


Fig. 2.65

By series-parallel reduction technique (Fig. 2.66),

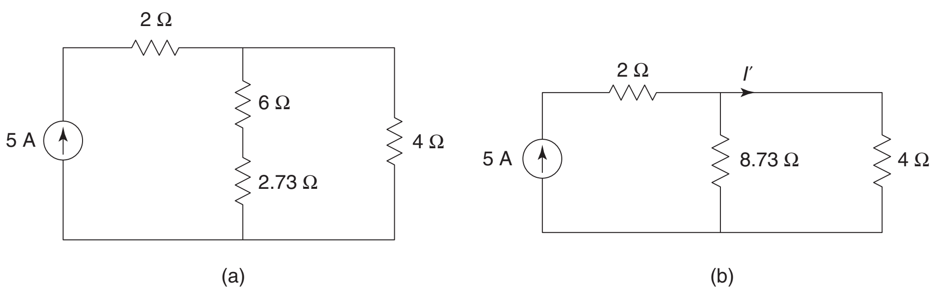


Fig. 2.66

$$I' = 5 \times \frac{8.73}{8.73+4} = 3.43 \text{ A}(\downarrow)$$

Step II When the 20 V source is acting alone (Fig. 2.67)

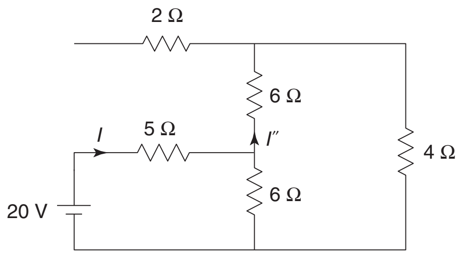


Fig. 2.67

By series-parallel reduction technique (Fig. 2.68),

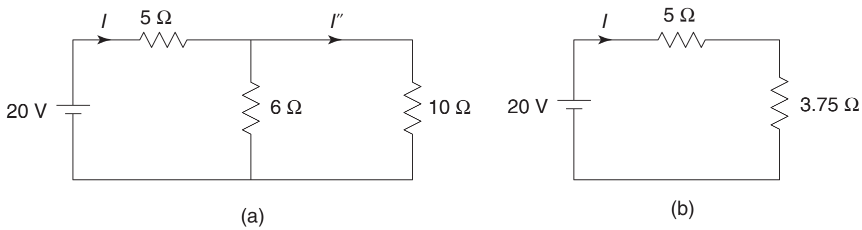


Fig. 2.68

$$I = \frac{20}{5+3.75} = 2.29 \text{ A}$$

From Fig. 2.68(a), by current-division rule,

$$I'' = 2.29 \times \frac{6}{6+10} = 0.86 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.43 + 0.86 = 4.29 \text{ A } (\downarrow)$$

Example 2.46 *Find the current through the 3 Ω resistor in Fig. 2.69.*

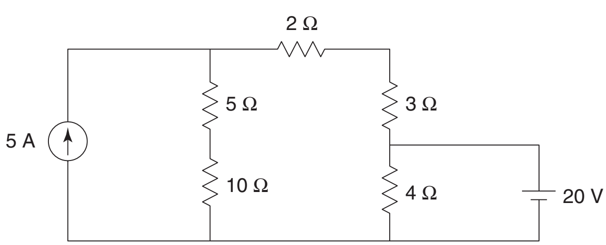


Fig. 2.69

Solution

Step I When the 5 A source is acting alone (Fig. 2.70)

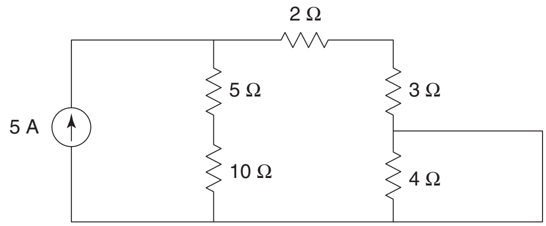


Fig. 2.70

By series-parallel reduction technique (Fig. 2.71),

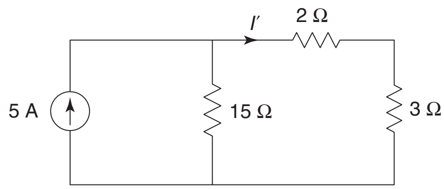


Fig. 2.71

$$I' = 5 \times \frac{15}{15+2+3} = 3.75 \text{ A}(\downarrow)$$

Step II When the 20 V source is acting alone (Fig. 2.72)

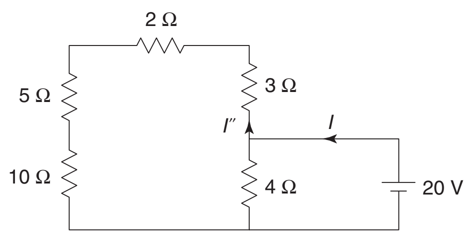


Fig. 2.72

By series-parallel reduction technique (Fig. 2.73),

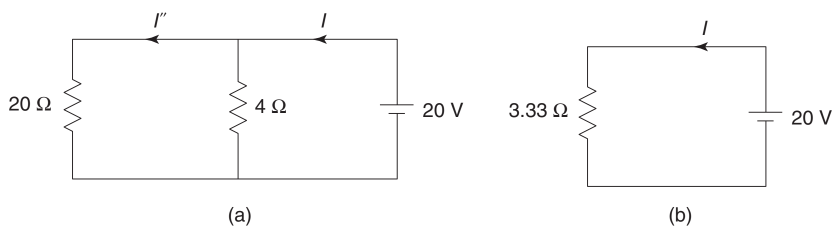


Fig. 2.73

$$I = \frac{20}{3.33} = 6 \text{ A}$$

From Fig. 2.73(a), by current-division rule,

$$I'' = 6 \times \frac{4}{20+4} = 1 \text{ A}(\uparrow) = -1 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.75 - 1 = 2.75 \text{ A } (\downarrow)$$

Example 2.47 *Find the current in the 1 Ω resistors in Fig. 2.74.*

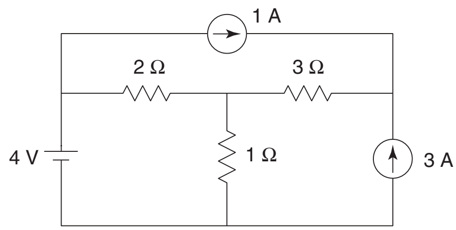


Fig. 2.74

Solution

Step I When the 4 V source is acting alone (Fig. 2.75)

$$I' = \frac{4}{2+1} = 1.33\ \text{A}(\downarrow)$$

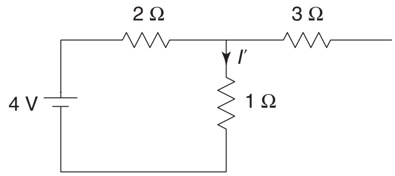


Fig. 2.75

Step II When the 3 A source is acting alone (Fig. 2.76)
By current-division rule,

$$I'' = 3 \times \frac{2}{1+2} = 2\ \text{A}\ (\downarrow)$$

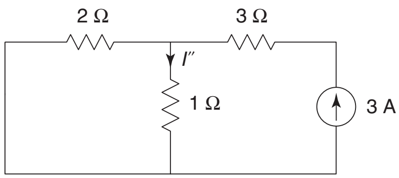


Fig. 2.76

Step III When the 1 A source is acting alone (Fig. 2.77)

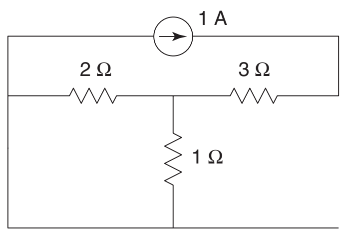


Fig. 2.77

Redrawing the network (Fig. 2.78),
By current-division rule,

$$I''' = 1 \times \frac{2}{2+1} = 0.66\ \text{A}(\downarrow)$$

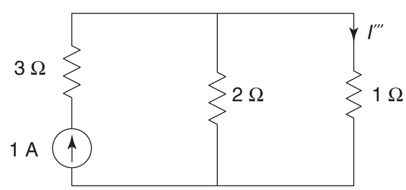


Fig. 2.78

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 1.33 + 2 + 0.66 = 4\ \text{A}\ (\downarrow)$$

Example 2.48 *Find the voltage V_{AB} in Fig. 2.79.*

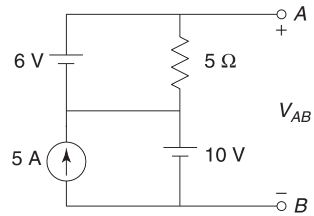


Fig. 2.79

Solution

Step I When the 6 V source is acting alone (Fig. 2.80)

$$V'_{AB} = 6 \text{ V}$$

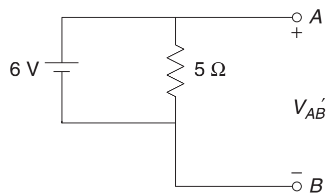


Fig. 2.80

Step II When the 10 V source is acting alone (Fig. 2.81)
Since the resistor of 5 Ω is shorted, the voltage across it is zero.

$$V''_{AB} = 10 \text{ V}$$

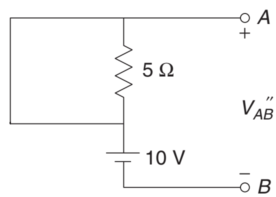


Fig. 2.81

Step III When the 5 A source is acting alone (Fig. 2.82)
Due to short circuit in both the parts,

$$V'''_{AB} = 0$$

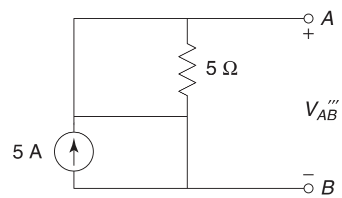


Fig. 2.82

Step IV By superposition theorem,

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB} = 6 + 10 + 0 = 16 \text{ V}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.49 *Find the current through the 6 Ω resistor in Fig. 2.83.*

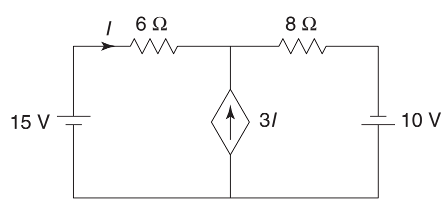


Fig. 2.83

Solution

Step I When the 15 V source is acting alone (Fig. 2.84)

From Fig. 2.84,

$$I' = I_1 \quad \text{...(i)}$$

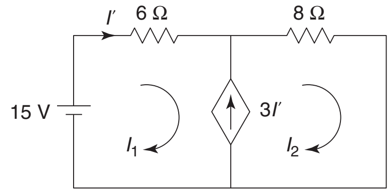


Fig. 2.84

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 3I' = 3I_1$$

$$4I_1 - I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$15 - 6I_1 - 8I_2 = 0$$

$$6I_1 + 8I_2 = 15 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.39\text{A}$$
$$I_2 = 1.57\text{A}$$
$$I' = I_1 = 0.39 \text{ A } (\rightarrow)$$

Step II When the 10 V source is acting alone (Fig. 2.85)

From Fig. 2.85,

$$I'' = I_1 \quad \text{...(i)}$$

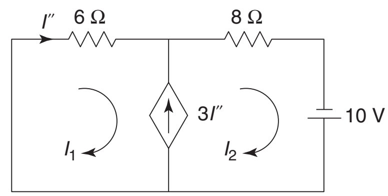


Fig. 2.85

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 3I'' = 3I_1$$
$$4I_1 - I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-6I_1 - 8I_2 + 10 = 0$$
$$6I_1 + 8I_2 = 10 \quad \text{.....(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.26 \text{ A}$$
$$I_2 = 1.05 \text{ A}$$
$$I'' = I_1 = 0.26 \text{ A} \quad (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.39 + 0.26 = 0.65 \text{ A } (\rightarrow)$$

Example 2.50 *Find the current I_x in Fig. 2.86.*

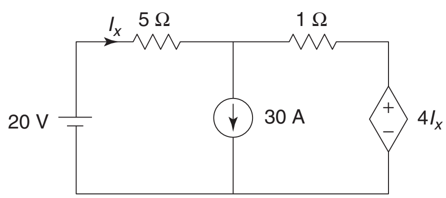


Fig. 2.86

Solution

Step I When the 30 A source is acting alone (Fig. 2.87)
From Fig. 2.87,

$$I'_x = I_1 \qquad \ldots\text{(i)}$$

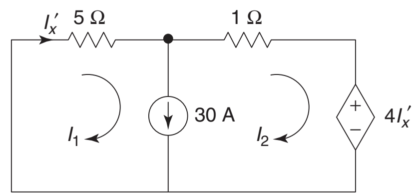


Fig. 2.87

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 30 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 1I_2 - 4I'_x = 0$$
$$-5I_1 - I_2 - 4I_1 = 0$$
$$9I_1 + I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 3\text{ A}$$
$$I_2 = -27\text{ A}$$
$$I'_x = I_1 = 3\text{ A } (\rightarrow)$$

Step II When the 20 V source is acting alone (Fig. 2.88)
Applying KVL to the mesh,

$$20 - 5I''_x - 1I''_x - 4I''_x = 0$$
$$I''_x = 2\text{ A}(\rightarrow)$$

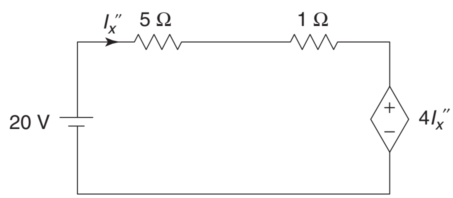


Fig. 2.88

Step III By superposition theorem,

$$I_x = I'_x + I''_x = 3 + 2 = 5\text{ A}(\rightarrow)$$

Example 2.51 *Find the current I_1 in Fig. 2.89.*

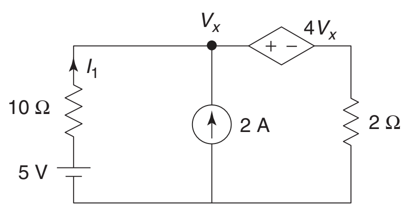


Fig. 2.89

Solution

Step I When the 5 V source is acting alone (Fig. 2.90)
From Fig. 2.90,

$$V_x = 5 - 10I_1'$$

Applying KVL to the mesh,

$$5 - 10I'_1 - 4V_x - 2I'_1 = 0$$

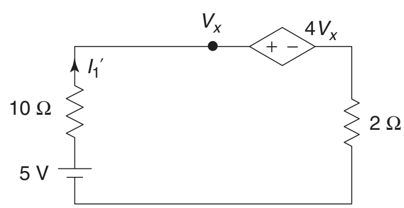


Fig. 2.90

$$5 - 10I_1' - 4\,(5 - 10I_1') - 2I_1' = 0$$
$$5 - 10I_1' - 20 + 40I_1' - 2I_1' = 0$$
$$I_1' = \frac{15}{28} = 0.54\text{ A}(\uparrow)$$

Step II When the 2 A source is acting alone (Fig. 2.91)

From Fig. 2.91,

$$V_x = -10I_1' \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1' = 2 \qquad \ldots\text{(ii)}$$

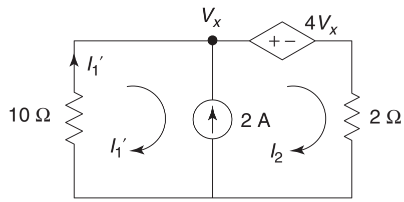


Fig. 2.91

Applying KVL to the outer path of the supermesh,

$$-10I_1' - 4V_x - 2I_2 = 0$$
$$-10I_1' - 4(-10I_1') - 2I_2 = 0$$
$$30I_1' - 2I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.14\text{ A }(\uparrow)$$
$$I_2 = 2.14\text{ A}$$

Step III By superposition theorem,

$$I_1 = I_1' + I_1'' = 0.54 + 0.14 = 0.68\text{ A }(\uparrow)$$

Example 2.52 *Determine the current through the 10 Ω resistor in Fig. 2.92.*

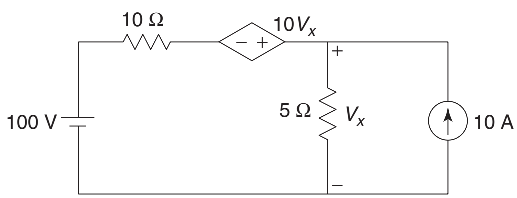


Fig. 2.92

Solution

Step I When the 100 V source is acting alone (Fig. 2.93)

From Fig. 2.93,

$$V_x = 5I'$$

Applying KVL to the mesh,

$$100 - 10I' + 10V_x - 5I' = 0$$
$$100 - 10I' + 10(5I') - 5I' = 0$$

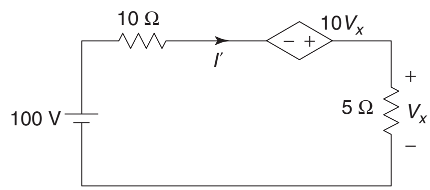


Fig. 2.93

$$I' = -2.86 \text{ A} (\rightarrow)$$

Step II When the 10 A source is acting alone (Fig. 2.94)

From Fig. 2.94,

$$V_x = 5(I_1 - I_2) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$-10I_1 + 10V_x - 5(I_1 - I_2) = 0$$

$$-10I_1 + 10\{5(I_1 - I_2)\} - 5(I_1 - I_2) = 0$$

$$35I_1 - 45I_2 = 0 \qquad \ldots\text{(ii)}$$

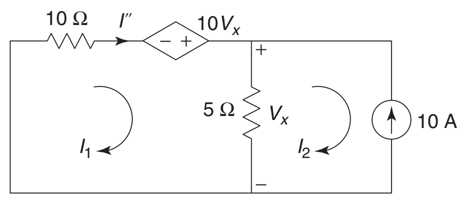


Fig. 2.94

For Mesh 2,

$$I_2 = -10 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -12.86 \text{ A}$$

$$I_2 = -10 \text{ A}$$

$$I'' = I_1 = -12.86 \text{ A} (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = -2.86 - 12.86 = -15.72 \text{ A} (\rightarrow)$$

Example 2.53 *Find the current I in the network of Fig. 2.95.*

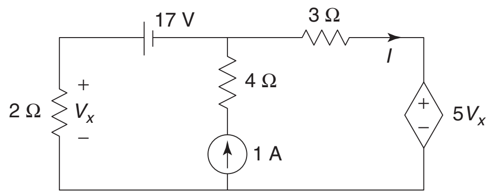


Fig. 2.95

Solution

Step I When the 17 V source is acting alone (Fig. 2.96)

From Fig. 2.96,

$$V_x = -2I'$$

Applying KVL to the mesh,

$$-2I' - 17 - 3I' - 5V_x = 0$$

$$-2I' - 17 - 3I' - 5(-2I') = 0$$

$$I' = 3.4 \text{ A} (\rightarrow)$$

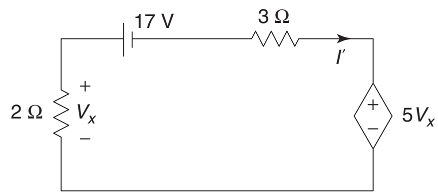


Fig. 2.96

Step II When the 1 A source is acting alone (Fig. 2.97)

From Fig. 2.97,

$$V_x = -2I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(ii)}$$

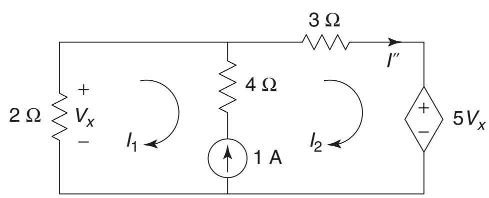


Fig. 2.97

Applying KVL to the outer path of the supermesh,

$$-2I_1 - 3I_2 - 5V_x = 0$$
$$-2I_1 - 3I_2 - 5(-2I_1) = 0$$
$$8I_1 - 3I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.6 \text{ A}$$
$$I_2 = 1.6 \text{ A}$$
$$I'' = I_2 = 1.6 \text{ A} (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.4 + 1.6 = 5 \text{ A} (\rightarrow)$$

Example 2.54 *Find the voltage V_1 in Fig. 2.98.*

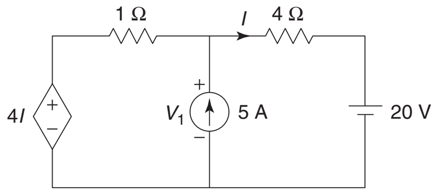


Fig. 2.98

Solution

Step I When the 5 A source is acting alone (Fig. 2.99)

From Fig. 2.99,

$$I = \frac{V_1'}{4}$$

Applying KCL at Node 1,

$$\frac{V_1' - 4I}{1} + \frac{V_1'}{4} = 5$$

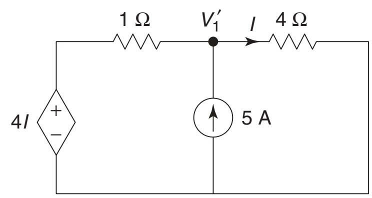


Fig. 2.99

$$V_1' - 4\left(\frac{V_1'}{4}\right) + \frac{V_1'}{4} = 5$$

$$V_1' = 20 \text{ V}$$

Step II When the 20 V source is acting alone (Fig. 2.100)

Applying KVL to the mesh,

$$4I - I - 4I - 20 = 0$$
$$I = -20 \text{ A}$$
$$V_1'' = 4I - 1(I) = 3I = 3(-20) = -60 \text{ V}$$

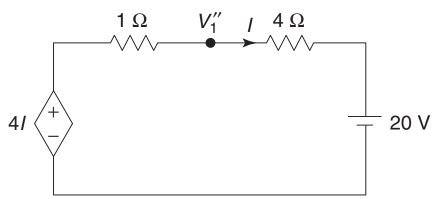


Fig. 2.100

Step III By superposition theorem,

$$V_1 = V_1' + V_1'' = 20 - 60 = -40 \text{ V}$$

Example 2.55 *Find the current in the 6 Ω resistor in Fig. 2.101.*

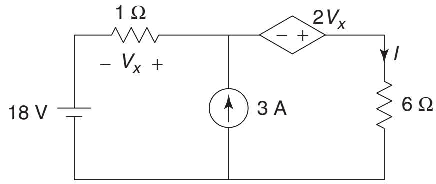


Fig. 2.101

Solution

Step I When the 18 V source is acting alone (Fig. 2.102)
From Fig. 2.102,

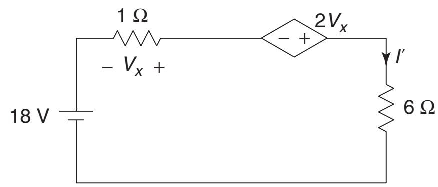


Fig. 2.102

$$V_x = -I'$$

Applying KVL to the mesh,

$$18 - I' + 2V_x - 6I' = 0$$
$$18 - I' - 2I' - 6I' = 0$$
$$I' = 2 \text{ A} (\downarrow)$$

Step II When the 3 A source is acting alone (Fig. 2.103)
From Fig. 2.103,

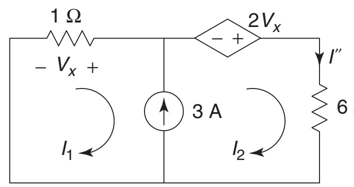


Fig. 2.103

$$V_x = -1\,I_1 = -I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3 \qquad \ldots\text{(ii)}$$

Applying KVL to the outerpath of the supermesh,

$$-1I_1 + 2V_x - 6I_2 = 0$$
$$-I_1 + 2(-I_1) - 6I_2 = 0$$
$$3I_1 + 6I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -2 \text{ A}$$
$$I_2 = 1 \text{ A}$$
$$I'' = I_2 = 1 \text{ A} (\downarrow)$$

Step III By superposition theorem,

$$I_{6\,\Omega} = I' + I'' = 2 + 1 = 3 \text{ A} (\downarrow)$$

Example 2.56 *Find the current I_y in Fig. 2.104.*

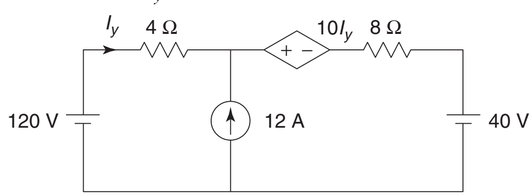


Fig. 2.104

Solution

Step I When the 120 V source is acting alone (Fig. 2.105)

Applying KVL to the mesh,

$$120 - 4I_y' - 10I_y' - 8I_y' = 0$$

$$I_y' = 5.45 \text{ A } (\rightarrow)$$

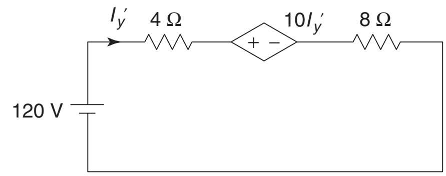


Fig. 2.105

Step II When the 12 A source is acting alone (Fig. 2.106)

From Fig. 2.106,

$$I_y'' = I_1 \quad \text{...(i)}$$

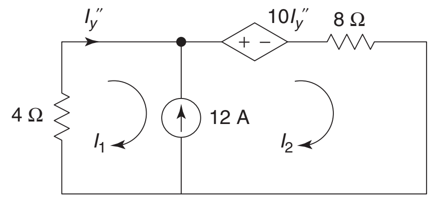


Fig. 2.106

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 12 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-4I_1 - 10I_y'' - 8I_2 = 0$$

$$-4I_1 - 10I_1 - 8I_2 = 0$$

$$14I_1 + 8I_2 = 0 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -4.36 \text{ A}$$

$$I_2 = 7.64 \text{ A}$$

$$Iy'' = I_1 = -4.36 \text{ A } (\rightarrow)$$

Step III When the 40 V source is acting alone (Fig. 2.107)

Applying KVL to the mesh,

$$-4\,I_y''' - 10I_y''' - 8I_y''' - 40 = 0$$

$$I_y''' = -\frac{40}{22} = -1.82 \text{ A } (\rightarrow)$$

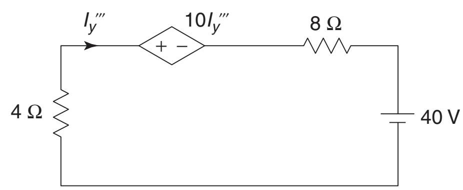


Fig. 2.107

Step IV By superposition theorem,

$$I_y = I_y' + I_y'' + I_y''' = 5.45 - 4.36 - 1.82 = -0.73 \text{ A } (\rightarrow)$$

Example 2.57 *Find the voltage V_x in Fig. 2.108.*

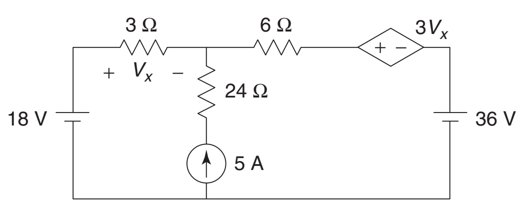


Fig. 2.108

Solution

Step I When the 18 V source is acting alone (Fig. 2.109)

From Fig. 2.109,

$$V_x' = 3I$$

Applying KVL to the mesh,

$$18 - 3I - 6I - 3V_x' = 0$$
$$18 - 3I - 6I - 3\,(3I) = 0$$
$$I = 1\text{ A}$$
$$V_x' = 3\text{ V}$$

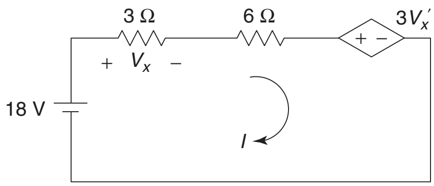


Fig. 2.109

Step II When the 5 A source is acting alone (Fig. 2.110)

From Fig. 2.110,

$$V_x'' = -3I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \qquad \ldots\text{(ii)}$$

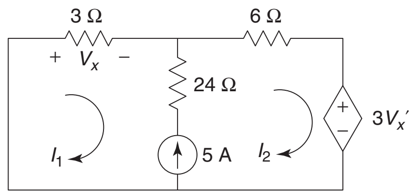


Fig. 2.110

Applying KVL to the outer path of the supermesh,

$$-3I_1 - 6I_2 - 3V_x'' = 0$$
$$-3I_1 - 6I_2 - 3(3I_1) = 0$$
$$12I_1 + 6I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -1.67\text{A}$$
$$I_2 = 3.33\text{ A}$$
$$V_x'' = 3I_1 = 3(-1.67) = -5\text{ V}$$

Step III When the 36 V source is acting alone (Fig. 2.111)

From Fig. 2.111,

$$V_x''' = -3I$$

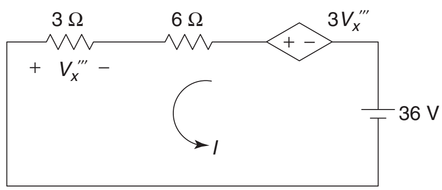


Fig. 2.111

Applying KVL to the mesh,

$$36 + 3V_x''' - 6I - 3I = 0$$

$$36 + 3V_x''' - 6\left(\frac{-V_x'''}{3}\right) - 3\left(\frac{-V_x'''}{3}\right) = 0$$

$$36 + 3V_x''' + 2V_x''' + V_x''' = 0$$

$$V_x''' = -6\text{ V}$$

Step IV By superposition theorem,

$$V_x = V_x' + V_x'' + V_x''' = 3 - 5 - 6 = -8\text{ V}$$

Example 2.58 *Find the voltage V in the network of Fig. 2.112.*

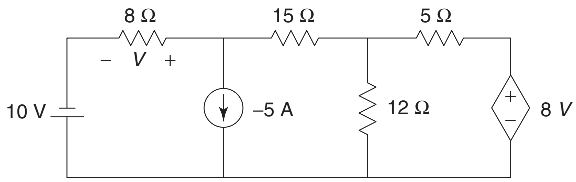


Fig. 2.112

Solution

Step I When the 10 V source is acting alone (Fig. 2.113)

From Fig. 2.113,

$$V' = -8I_1 \quad \text{...(i)}$$

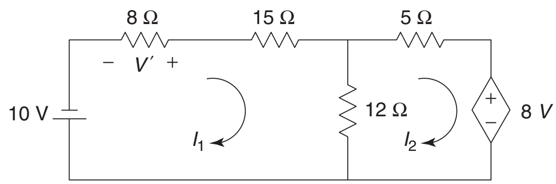


Fig. 2.113

Applying KVL to Mesh 1,

$$-10 - 8I_1 - 15\,I_1 - 12\,(I_1 - I_2) = 0$$

$$35I_1 - 12I_2 = -10 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-12(I_2 - I_1) - 5I_2 - 8V' = 0$$

$$-12I_2 + 12I_1 - 5I_2 - 8(-8I_1) = 0$$

$$76I_1 - 17I_2 = 0 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.54 \text{ A}$$

$$I_2 = 2.4 \text{ A}$$

$$V' = -8I_1 = -8(0.54) = -4.32 \text{ V}$$

Step II When the –5 A source is acting alone (Fig. 2.114)

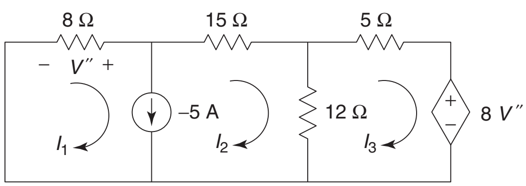


Fig. 2.114

From Fig. 2.114,

$$V'' = -8I_1 \quad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = -5 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-8I_1 - 15I_2 - 12(I_2 - I_3) = 0$$

$$-8I_1 - 27I_2 + 12I_3 = 0 \qquad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-12(I_3 - I_2) - 5I_3 - 8V'' = 0$$

$$-12I_3 + 12I_2 - 5I_3 - 8(-8I_1) = 0$$

$$64I_1 + 12I_2 - 17I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.97 \text{ A}$$

$$I_2 = 9.97 \text{ A}$$

$$I_3 = 25.74 \text{ A}$$

$$V'' = -8I_1 = -8(-4.97) = -39.76 \text{ V}$$

Step III By superposition theorem,

$$V = V' + V'' = -4.32 - 39.76 = -44.08 \text{ V}$$

Example 2.59 *For the network shown in Fig. 2.115, find the voltage V_0.*

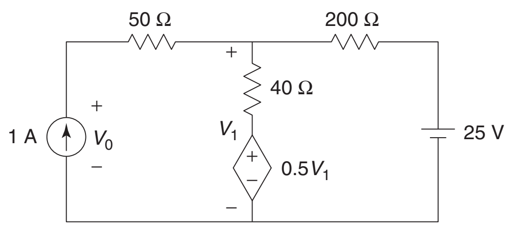


Fig. 2.115

Solution

Step I When the 1 A source is acting alone (Fig. 2.116)

From Fig. 2.116,

$$V_1 = 200\,I_2 \qquad \text{...(i)}$$

For Mesh 1,

$$I_1 = 1 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$0.5V_1 - 40(I_2 - I_1) - 200\,I_2 = 0$$

$$0.5(200I_2) - 40I_2 + 40I_1 - 200I_2 = 0$$

$$40I_1 - 140\,I_2 = 0 \qquad \text{...(iii)}$$

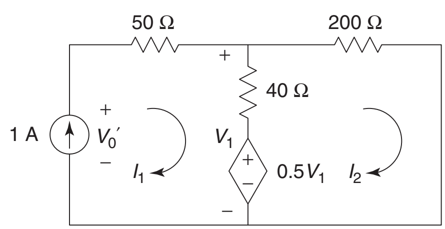


Fig. 2.116

Solving Eqs (ii) and (iii),

$$I_1 = 1 \text{ A}$$

$$I_2 = 0.29 \text{ A}$$

$$V_0' - 50\;I_1 - 200I_2 = 0$$

$$V_0' - 50(1) - 200(0.29) = 0$$

$$V_0' = 108 \text{ V}$$

Step II When the 25 V source is acting alone (Fig. 2.117)
From Fig. 2.117,

$$V_1 - 200I - 25 = 0$$

$$V_1 = 200I + 25 \qquad \text{...(i)}$$

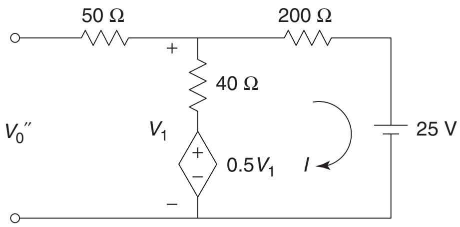


Fig. 2.117

Applying KVL to Mesh 1,

$$0.5V_1 - 40I - 200I - 25 = 0$$

$$0.5(200I + 25) - 40I - 200I - 25 = 0$$

$$I = -0.09 \text{ A}$$

$$V_0'' = V_1 = 200I + 25 = 200(-0.09) + 25 = 7 \text{ V}$$

Step III By superposition theorem,

$$V_0 = V_0' + V_0'' = 108 + 7 = 115 \text{ V}$$

Example 2.60 *For the network shown in Fig. 2.118, find the voltage V_x.*

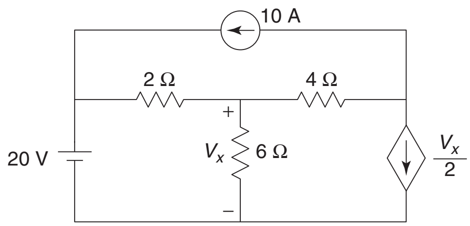


Fig. 2.118

Solution

Step I When the 20 V source is acting alone (Fig. 2.119)
From Fig. 2.119,

$$V_x' = 6(I_1 - I_2) \qquad \text{...(i)}$$

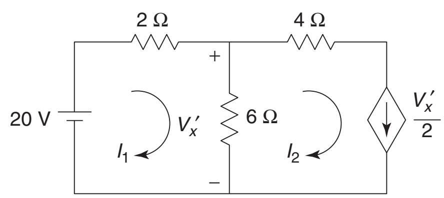


Fig. 2.119

Applying KVL to Mesh 1,

$$20 - 2I_1 - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 = 20 \qquad \text{...(ii)}$$

For Mesh 2,

$$I_2 = \frac{V_x'}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 5.71 \text{ A}$$

$$I_2 = 4.29 \text{ A}$$

$$V_x' = 6(I_1 - I_2) = 6(5.71 - 4.29) = 8.52 \text{ V}$$

Step II When the 10 A source is acting alone (Fig. 2.120)
From Fig. 2.120,

$$V_x'' = 6(I_1 - I_2) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2(I_1 - I_3) - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 - 2I_3 = 0 \qquad \text{...(ii)}$$

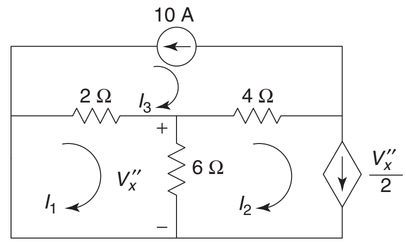


Fig. 2.120

For Mesh 2,

$$I_2 = \frac{V_x''}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \qquad \text{...(iii)}$$

For Mesh 3,

$$I_3 = -10 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -5.71 \text{ A}$$

$$I_2 = -4.29 \text{ A}$$

$$I_3 = -10 \text{ A}$$

$$V_x'' = 6(I_1 - I_2) = 6(-5.71 + 4.29) = -8.52 \text{ V}$$

Step III By superposition theorem,

$$V_x = V_x' + V_x'' = 8.52 - 8.52 = 0$$

Example 2.61 *Calculate the current I in the network shown in Fig. 2.121.*

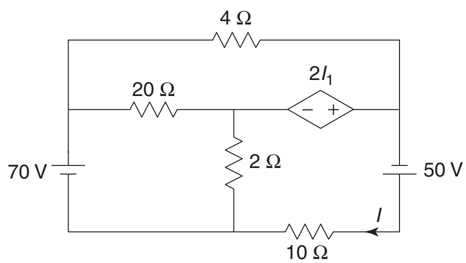


Fig. 2.121

Solution

Step I When the 70 V source is acting alone (Fig. 2.122)
From Fig. 2.122,

$$I' = I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$70 - 20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 70 \qquad \text{...(iii)}$$

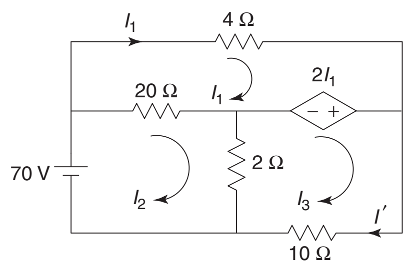


Fig. 2.122

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = 0 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 8.94\text{ A}$$
$$I_2 = 11.62\text{ A}$$
$$I_3 = 3.43\text{ A}$$
$$I' = I_3 = 3.43\text{ A} (\leftarrow)$$

Step II When the 50 V source is acting alone (Fig. 2.123)
From Fig. 2.123,

$$I'' = I_3 \quad \text{...(i)}$$

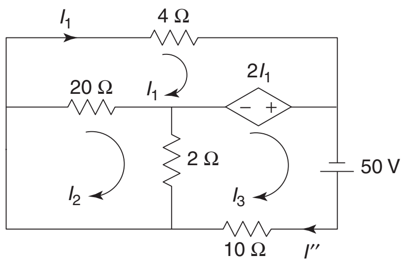


Fig. 2.123

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 0 \quad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 + 50 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = -50 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii), and (iv),

$$I_1 = 1.06\text{ A}$$
$$I_2 = 1.38\text{ A}$$
$$I_3 = 4.57\text{ A}$$
$$I'' = I_3 = 4.57\text{ A} (\leftarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.43 + 4.57 = 8\text{ A} (\leftarrow)$$

Example 2.62 *Find the voltage V_0 in the network of Fig. 2.124.*

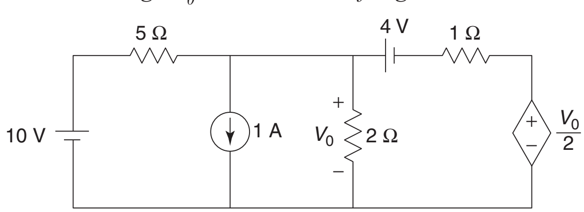


Fig. 2.124

Solution

Step I When the 10 V source is acting alone (Fig. 2.125)

Applying KCL at the node,

$$\frac{V_0'-10}{5}+\frac{V_0'}{2}+\frac{V_0'-\dfrac{V_0'}{2}}{1}=0$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0'=2$$

$$V_0'=1.67\text{ V}$$

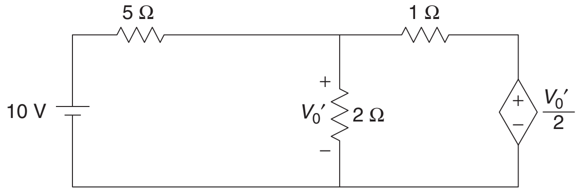


Fig. 2.125

Step II When the 1A current source is acting alone (Fig. 2.126)

Applying KCL at the node,

$$\frac{V_0''}{5}+1+\frac{V_0''}{2}+\frac{V_0''-\dfrac{V_0''}{2}}{1}$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0''=-1$$

$$V_0''=-0.83\text{ V}$$

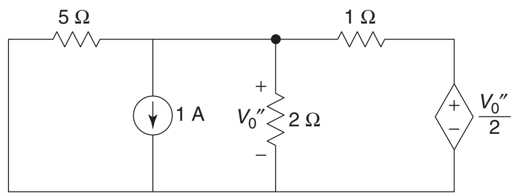


Fig. 2.126

Step III When the 4 V source is acting alone (Fig. 2.127)

Applying KCL at the node,

$$\frac{V_0'''}{5}+\frac{V_0'''}{2}+\frac{V_0'''-4-\dfrac{V_0'''}{2}}{1}=0$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0'''=4$$

$$V_0'''=3.33\text{ V}$$

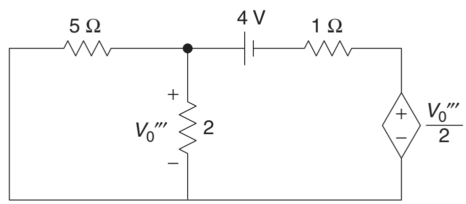


Fig. 2.127

Step IV By superposition theorem,

$$V_0=V_0'+V_0''+V_0'''=1.67-0.83+3.33=4.17\text{ V}$$

2.8 THEVENIN'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent voltage source and an equivalent series resistance. The voltage source is the voltage across the two terminals with load, if any, removed. The series resistance is the resistance of the network measured between two terminals with load removed and constant voltage source being replaced by its internal resistance (or if it is not given with zero resistance, i.e., short circuit) and constant current source replaced by infinite resistance, i.e., open circuit.*'

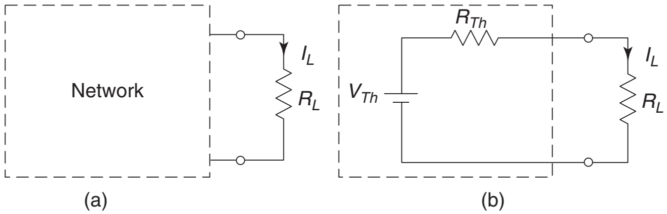


Fig. 2.128 *Network illustrating Thevenin's theorem*

Explanation Consider a simple network as shown in Fig. 2.129.

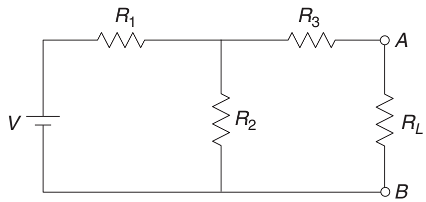


Fig. 2.129 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate open circuit voltage V_{Th} across points A and B as shown in Fig. 2.130.

$$V_{\text{Th}} = \frac{R_2}{R_1 + R_2} \text{ V}$$

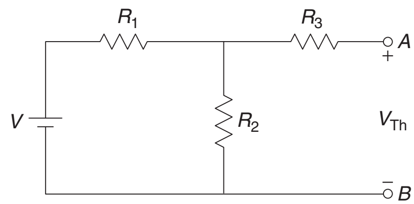


Fig. 2.130 *Calculation of V_{Th}*

For finding series resistance R_{Th}, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 2.131.

$$R_{\text{Th}} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

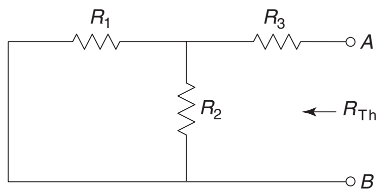


Fig. 2.131 *Calculation of R_{Th}*

Thevenin's equivalent network is shown in Fig. 2.132.

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L}$$

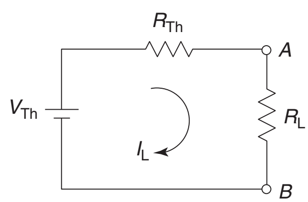


Fig. 2.132 *Thevenin's equivalent network*

If the network contains both independent and dependent sources, Thevenin's resistance R_{Th} is calculated as,

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N}$$

where I_N is the short-circuit current which would flow in a short circuit placed across the terminals A and B. Dependent sources are active at all times. They have zero values only when the control voltage or current is zero. R_{Th} may be negative in some cases which indicates negative resistance region of the device, i.e., as voltage increases, current decreases in the region and vice-versa.

If the network contains only dependent sources then

$$V_{\mathrm{Th}} = 0$$
$$I_N = 0$$

For finding R_{Th} in such a network, a known voltage V is applied across the terminals A and B and current is calculated through the path AB.

$$R_{\mathrm{Th}} = \frac{V}{I}$$

or a known current source I is connected across the terminals A and B and voltage is calculated across the terminals A and B.

$$R_{\mathrm{Th}} = \frac{V}{I}$$

Thevenin's equivalent network for such a network is shown in Fig. 2.133.

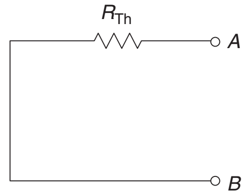


Fig. 2.133 *Thevenin's equivalent network*

Steps to be Followed in Thevenin's Theorem

1. Remove the load resistance R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Replace the network by a voltage source V_{Th} in series with resistance R_{Th}.
5. Find the current through R_L using Ohm's law.

$$I_L = \frac{V_{\mathrm{Th}}}{R_{\mathrm{Th}} + R_L}$$

Example 2.63 *Determine the current through the 24 Ω resistor in Fig. 2.134.*

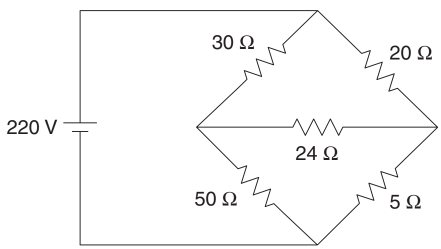


Fig. 2.134

Solution

Step I Calculation of V_{Th} (Fig. 2.135)

$$I_1 = \frac{220}{30+50} = 2.75 \text{ A}$$

$$I_2 = \frac{220}{20+5} = 8.8 \text{ A}$$

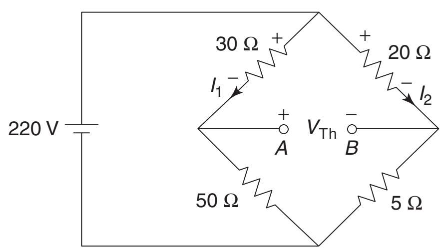


Fig. 2.135

Writing the V_{Th} equation,

$$V_{Th} + 30I_1 - 20I_2 = 0$$

$$V_{Th} = 20I_2 - 30I_1 = 20(8.8) - 30(2.75) = 93.5 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 2.136)

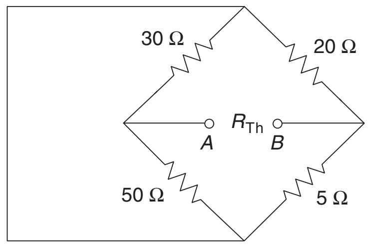


Fig. 2.136

Redrawing the circuit (Fig. 2.137),

$$R_{Th} = (30 \,||\, 50) + (20 \,||\, 5) = 22.75\ \Omega$$

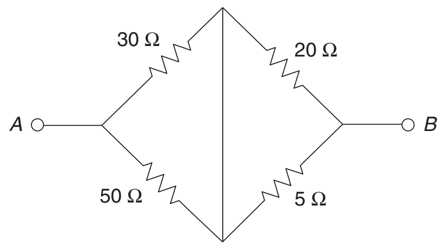


Fig. 2.137

Step III Calculation of I_L (Fig. 2.138)

$$I_L = \frac{93.5}{22.75 + 24} = 2 \text{ A}$$

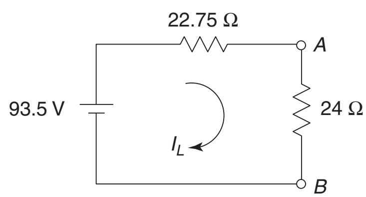


Fig. 2.138

Example 2.64 *Find the current through the 20 Ω resistor in Fig. 2.139.*

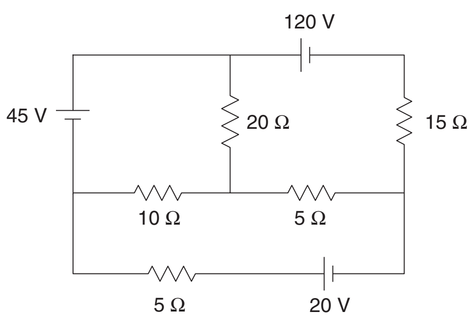


Fig. 2.139

Solution

Step I Calculation of V_{Th} (Fig. 2.140)

Applying KVL to Mesh 1,

$$45 - 120 - 15I_1 - 5(I_1 - I_2) - 10(I_1 - I_2) = 0 \qquad \text{...(i)}$$
$$30I_1 - 15I_2 = -75$$

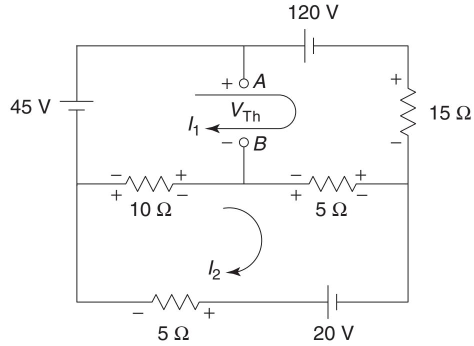


Fig. 2.140

Applying KVL to Mesh 2,

$$20 - 5I_2 - 10(I_2 - I_1) - 5(I_2 - I_1) = 0 \qquad \text{...(ii)}$$
$$-15I_1 + 20I_2 = 20$$

Solving Eqs (i) and (ii),

$$I_1 = -3.2 \text{ A}$$
$$I_2 = -1.4 \text{ A}$$

Writing the V_{Th} equation,

$$45 - V_{\mathrm{Th}} - 10(I_1 - I_2) = 0$$
$$V_{\mathrm{Th}} = 45 - 10(I_1 - I_2) = 45 - 10[-3.2 - (-1.4)] = 63 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 2.141)

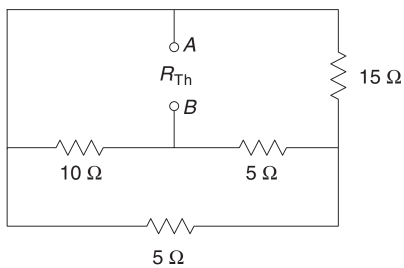


Fig. 2.141

Converting the delta formed by resistors of 10 Ω, 5 Ω and 5 Ω into an equivalent star network (Fig. 2.142),

$$R_1 = \frac{10 \times 5}{20} = 2.5\ \Omega$$
$$R_2 = \frac{10 \times 5}{20} = 2.5\ \Omega$$
$$R_3 = \frac{5 \times 5}{20} = 1.25\ \Omega$$

Simplifying the network (Fig. 2.143 and Fig. 2.144),

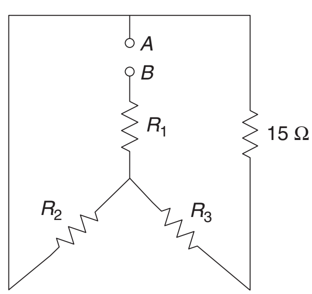


Fig. 2.142

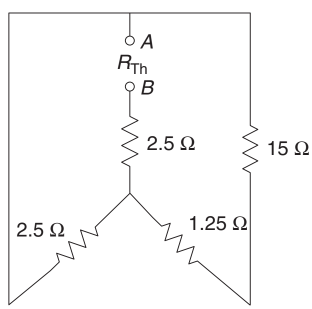


Fig. 2.143

$$R_{\text{Th}} = (16.25\,||\,2.5) + 2.5 = 4.67\ \Omega$$

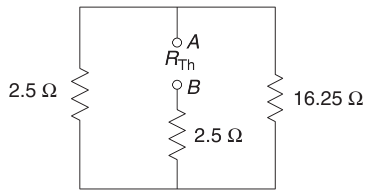


Fig. 2.144

Step III Calculation of I_L (Fig. 2.145)

$$I_L = \frac{63}{4.67 + 20} = 2.55\ \text{A}$$

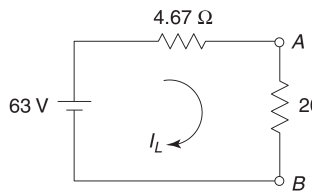


Fig. 2.145

Example 2.65 *Find the current through the 10 Ω resister in Fig. 2.146.*

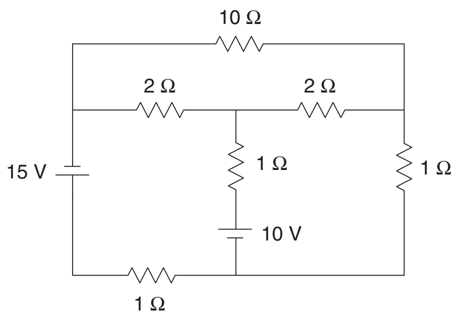


Fig. 2.146

Solution

Step I Calculation of V_{Th} (Fig. 2.147)

Applying KVL to Mesh 1,

$$-15 - 2I_1 - 1(I_1 - I_2) - 10 - 1I_1 = 0$$

$$4I_1 - I_2 = -25 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$10 - 1(I_2 - I_1) - 2I_2 - 1I_2 = 0$$

$$-I_1 + 4I_2 = 10 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = -6\ \text{A}$$

$$I_2 = 1\ \text{A}$$

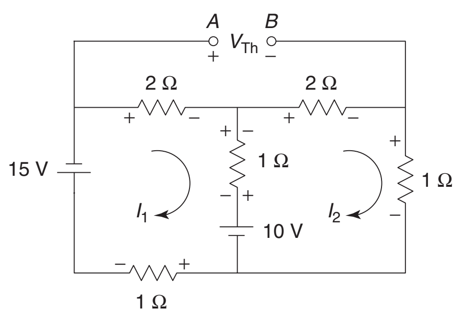


Fig. 2.147

Writing the V_{Th} equation,

$$-V_{\text{Th}} + 2I_2 + 2I_1 = 0$$

$$V_{\text{Th}} = 2I_1 + 2I_2 = 2(-6) + 2(1) = -10\text{ V}$$
$$= 10\text{ V (the terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 2.148)

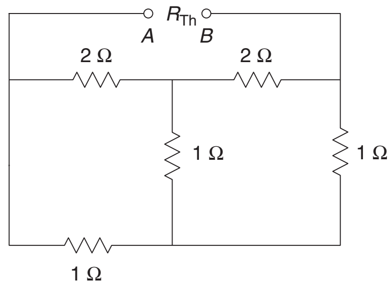


Fig. 2.148

Converting the star network formed by resistors of $2\,\Omega, 2\,\Omega$ and $1\,\Omega$ into an equivalent delta network (Fig. 2.149),

$$R_1 = 2 + 2 + \frac{2\times 2}{1} = 8\,\Omega$$
$$R_2 = 2 + 1 + \frac{2\times 1}{2} = 4\,\Omega$$
$$R_3 = 2 + 1 + \frac{2\times 1}{2} = 4\,\Omega$$

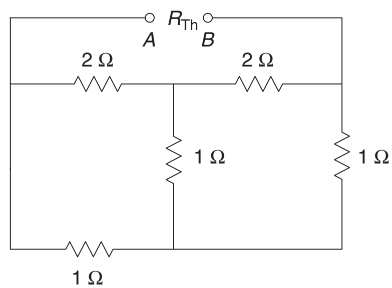


Fig. 2.149

Simplifying the network (Fig. 2.150),

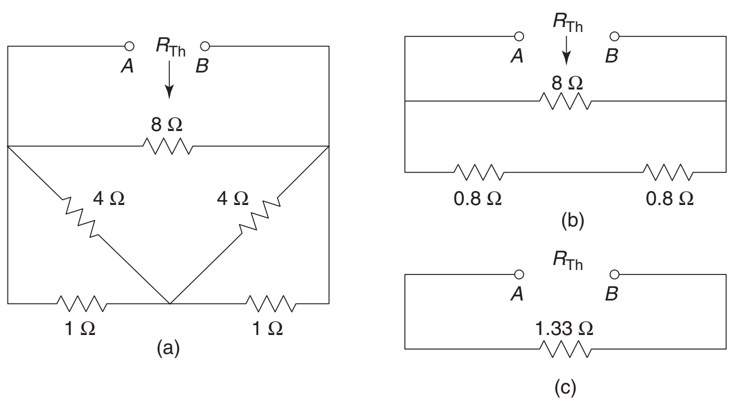


Fig. 2.150

$$R_{\text{Th}} = 1.33\ \Omega$$

Step III Calculation of I_L (Fig. 2.151)

$$I_L = \frac{10}{1.33+10} = 0.88\text{ A }(\uparrow)$$

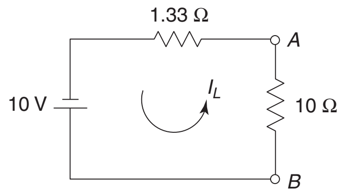


Fig. 2.151

Example 2.66 *Find the current through the 1 Ω resistor in Fig. 2.152.*

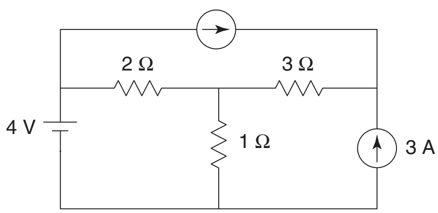


Fig. 2.152

Solution

Step I Calculation of V_{Th} (Fig. 2.153)

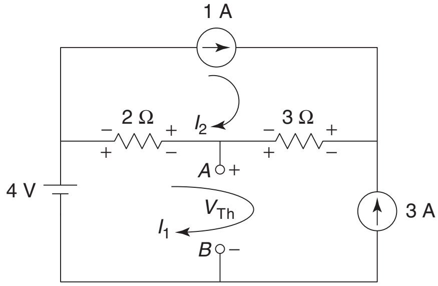


Fig. 2.153

Writing the current equations for Meshes 1 and 2,

$$I_1 = -3$$

$$I_2 = 1$$

Writing the V_{Th} equation,

$$4 - 2(I_1 - I_2) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 4 - 2(I_1 - I_2) = 4 - 2(-4) = 12\text{ V}$$

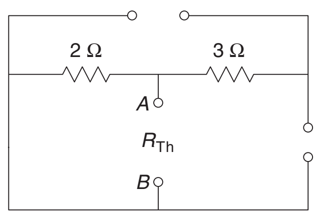


Fig. 2.154

Step II Calculation of R_{Th} (Fig. 2.154)

$$R_{\text{Th}} = 2\ \Omega$$

Step III Calculation of I_L (Fig. 2.155)

$$I_L = \frac{12}{2+1} = 4\text{ A}$$

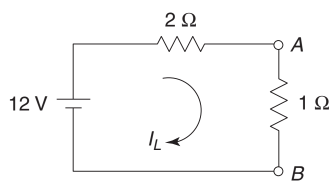


Fig. 2.155

EXAMPLES WITH DEPENDENT SOURCES

Example 2.67 *Obtain the Thevenin equivalent network for the given network of Fig. 2.156 at terminals A and B.*

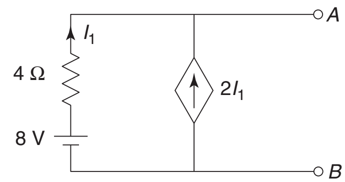


Fig. 2.156

Solution

Step I Calculation of V_{Th} (Fig. 2.157)

From Fig. 2.157,

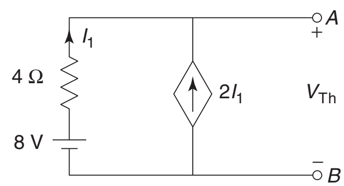


Fig. 2.157

$$I_1 = -2I_1$$
$$3I_1 = 0$$
$$I_1 = 0$$

Writing the V_{Th} equation,

$$8 - 0 - V_{Th} = 0$$
$$V_{Th} = 8\text{ V}$$

Step II Calculation of I_N (Fig. 2.158),

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

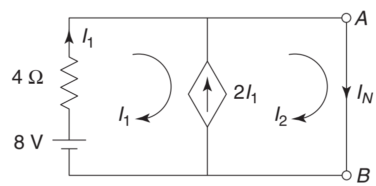


Fig. 2.158

$$I_2 - I_1 = 2I_1$$
$$3I_1 - I_2 = 0 \qquad \text{...(i)}$$

Applying KVL to the outer path of the supermesh,

$$8 - 4I_1 = 0$$
$$I_1 = 2 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = 6\text{ A}$$
$$I_N = I_2 = 6\text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{8}{6} = 1.33\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.159)

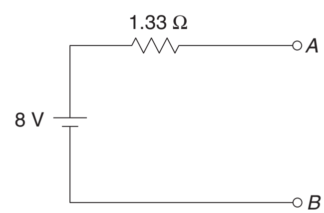


Fig. 2.159

Example 2.68 *Find Thevenin's equivalent network of Fig. 2.160.*

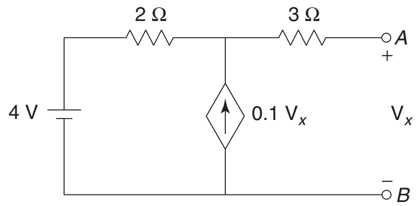


Fig. 2.160

Solution

Step I Calculation of V_{Th} (Fig. 2.161)

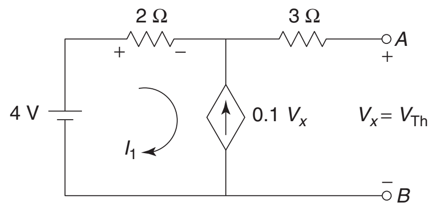


Fig. 2.161

$$V_x = V_{\text{Th}}$$

$$I_1 = -0.1V_x$$

Writing the V_{Th} equation,

$$4 - 2I_1 - V_x = 0$$

$$4 - 2(-0.1V_x) - V_x = 0$$

$$4 - 0.8V_x = 0$$

$$V_x = 5 \text{ V}$$

$$V_x = V_{\text{Th}} = 5 \text{ V}$$

Step II Calculation of I_N (Fig. 2.162)

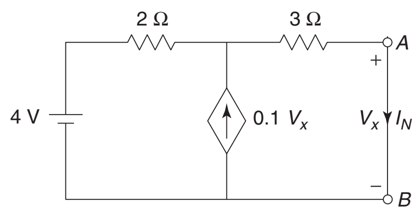


Fig. 2.162

From Fig. 2.162,

$$V_x = 0$$

The dependent source 0.1 V_x depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e., 0.1 $V_x = 0$ as shown in Fig. 2.163.

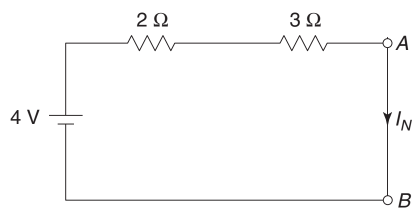


Fig. 2.163

$$I_N = \frac{4}{2+3} = 0.8 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{5}{0.8} = 6.25\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.164)

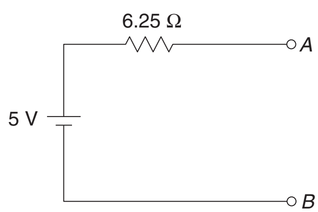


Fig. 2.164

Example 2.69 *Obtain the Thevenin equivalent network of Fig. 2.165 for the terminals A and B.*

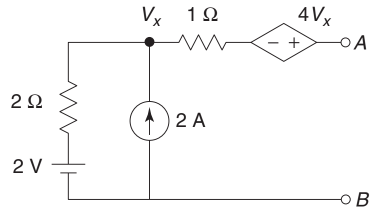


Fig. 2.165

Solution

Step I Calculation of V_{Th} (Fig. 2.166)

From Fig. 2.166,

$$2 - 2I_1 - V_x = 0$$
$$V_x = 2 - 2I_1$$

For Mesh 1,

$$I_1 = -2 \text{ A}$$
$$V_x = 2 - 2(-2) = 6 \text{ V}$$

Writing the V_{Th} equation,

$$2 - 2I_1 - 0 + 4V_x - V_{Th} = 0$$
$$2 - 2(-2) - 0 + 4(6) - V_{Th} = 0$$
$$V_{Th} = 30 \text{ V}$$

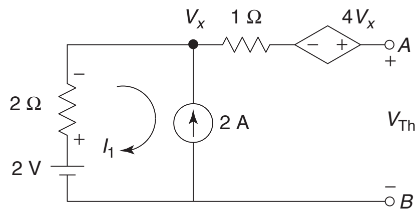


Fig. 2.166

Step II Calculation of I_N (Fig. 2.167)

From Fig. 2.167,

$$V_x = 2 - 2I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for the supermesh

$$I_2 - I_1 = 2 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$2 - 2I_1 - 1I_2 + 4V_x = 0$$
$$2 - 2I_1 - I_2 + 4(2 - 2I_1) = 0$$
$$10I_1 + I_2 = 10 \qquad \ldots\text{(iii)}$$

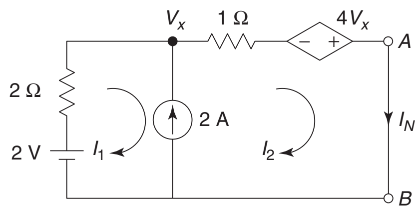


Fig. 2.167

Solving Eqs (ii) and (iii),

$$I_1 = 0.73 \text{ A}$$
$$I_2 = 2.73 \text{ A}$$
$$I_N = I_2 = 2.73 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{2.73} = 10.98\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.168)

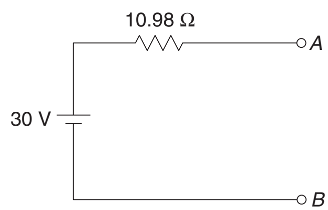


Fig. 2.168

Example 2.70 *Find the Thevenin equivalent network of Fig. 2.169 for the terminals A and B.*

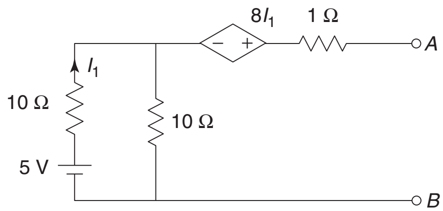


Fig. 2.169

Solution

Step I Calculation of V_{Th} (Fig. 2.170)
Applying KVL to the mesh,

$$5 - 10I_1 - 10I_1 = 0$$

$$I_1 = \frac{5}{20} = 0.25 \text{ A}$$

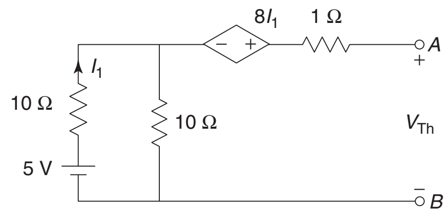


Fig. 2.170

Writing the V_{Th} equation,

$$5 - 10I_1 + 8I_1 - 0 - V_{Th} = 0$$

$$V_{Th} = 5 - 2I_1 = 5 - 2(0.25) = 4.5 \text{ V}$$

Step II Calculation of I_N (Fig. 2.171)
Applying KVL to Mesh 1,

$$5 - 10I_1 - 10(I_1 - I_2) = 0$$

$$20I_1 - 10I_2 = 5 \qquad \text{...(i)}$$

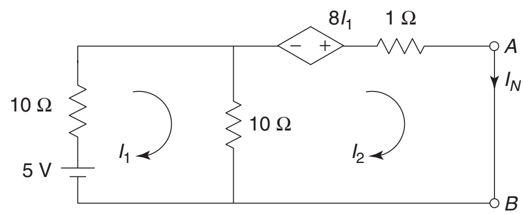


Fig. 2.171

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) + 8I_1 - 1I_2 = 0$$

$$18I_1 - 11I_2 = 0 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 1.375 \text{ A}$$

$$I_2 = 2.25 \text{ A}$$

$$I_N = I_2 = 2.25 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{4.5}{2.25} = 2\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.172)

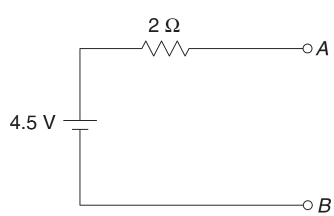


Fig. 2.172

Example 2.71 *Find V_{Th} and R_{Th} between terminals A and B of the network shown in Fig. 2.173.*

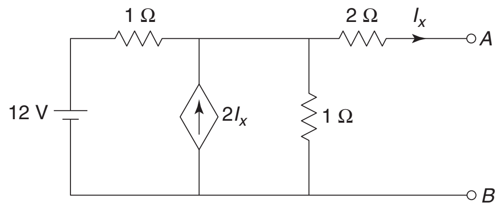


Fig. 2.173

Solution

Step I Calculation of V_{Th} (Fig. 2.174)

$$I_x = 0$$

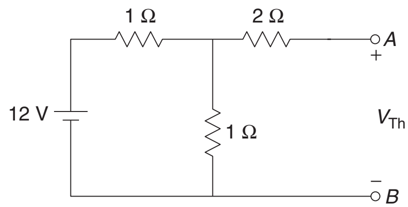


Fig. 2.174

The dependent source $2I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $2I_x = 0$ as shown in Fig. 2.174.

Writing the V_{Th} equation,

$$V_{Th} = 12 \times \frac{1}{1+1} = 6 \text{ V}$$

Step II Calculation of I_N (Fig. 2.175)
From Fig. 2.175,

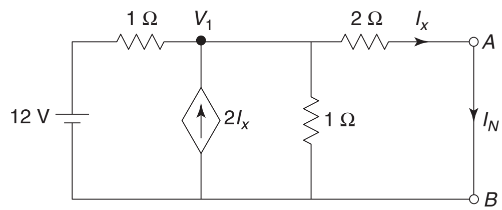


Fig. 2.175

$$I_x = \frac{V_1}{2}$$

Applying KCL at Node 1,

$$\frac{V_1 - 12}{1} + \frac{V_1}{1} + \frac{V_1}{2} = 2I_x$$

$$V_1 + V_1 + \frac{V_1}{2} - 12 = 2\left(\frac{V_1}{2}\right)$$

$$V_1 = 8 \text{ V}$$

$$I_N = \frac{V_1}{2} = \frac{8}{2} = 4 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{6}{4} = 1.5\ \Omega$$

Example 2.72 *Obtain the Thevenin equivalent network of Fig. 2.176 for the given network at terminals a and b .*

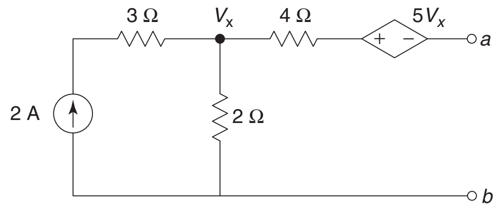


Fig. 2.176

Solution

Step I Calculation of V_{Th} (Fig. 2.177)
Applying KCL at Node x,

$$2 = \frac{V_x}{2}$$

$$V_x = 4 \text{ V}$$

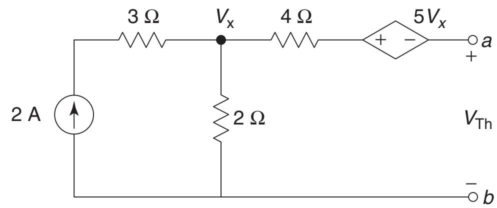


Fig. 2.177

Writing the V_{Th} equation,

$$V_{\text{Th}} = V_x - 5V_x = -4V_x$$

$$= -16 \text{ V} \quad (\text{the terminal } a \text{ is negative w.r.t. } b)$$

Step II Calculation of I_N (Fig. 2.178)
Applying KCL at Node x,

$$2 = \frac{V_x}{2} + \frac{V_x - 5V_x}{4}$$

$$2 = \frac{V_x}{2} - V_x = -\frac{V_x}{2}$$

$$V_x = -4 \text{ V}$$

$$I_N = \frac{V_x - 5V_x}{4} = -V_x = 4 \text{ A}$$

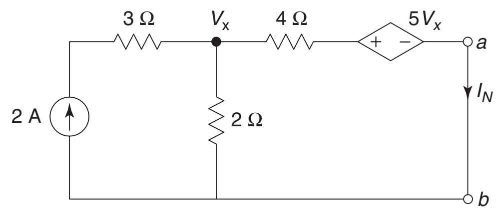


Fig. 2.178

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{-16}{4} = -4\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.179)

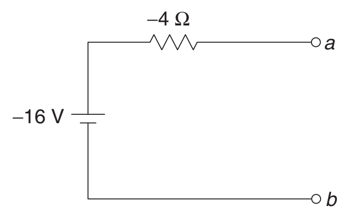


Fig. 2.179

Example 2.73 *Obtain the Thevenin equivalent network of Fig. 2.180 for the given network.*

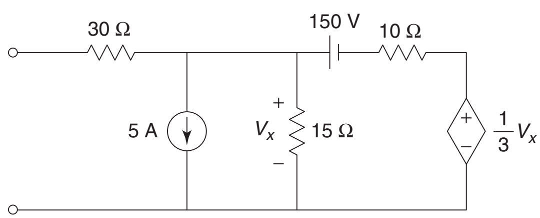


Fig. 2.180

Solution

Step I Calculation of V_{Th} (Fig. 2.181)
From Fig. 2.181,

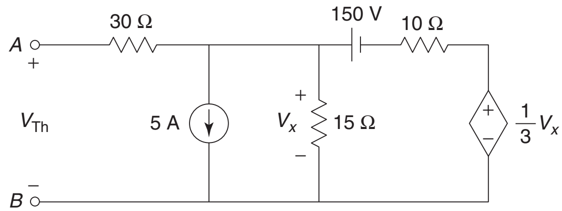


Fig. 2.181

$$V_x = V_{\text{Th}}$$

Applying KCL at the node,

$$\frac{V_x - 150 - \frac{1}{3}V_x}{10} + \frac{V_x}{15} + 5 = 0$$

$$V_x = 75 \text{ V}$$

$$V_{\text{Th}} = 75 \text{ V}$$

Step II Calculation of I_N (Fig. 2.182)
Applying KCL at Node x,

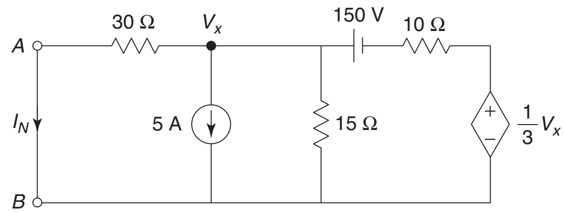


Fig. 2.182

$$\frac{V_x}{30} + 5 + \frac{V_x}{15} + \frac{V_x - 150 - \frac{1}{3}V_x}{10} = 0$$

$$\frac{V_x}{30} + \frac{V_x}{15} + \frac{V_x}{10} - \frac{V_x}{30} = 15 - 5$$

$$V_x = 60 \text{ V}$$

$$I_N = \frac{V_x}{30} = \frac{60}{30} = 2 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{75}{2} = 37.5\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.183)

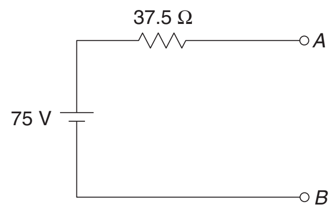


Fig. 2.183

Example 2.74 *Find the Thevenin's equivalent network of the network to the left of A-B in the Fig. 2.184.*

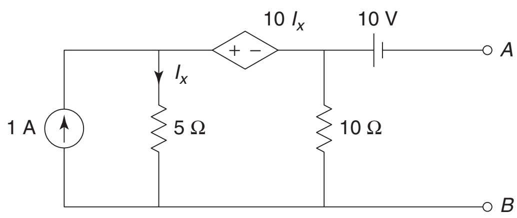


Fig. 2.184

Solution

Step I Calculation of V_{Th} (Fig. 2.185)
From Fig. 2.185,

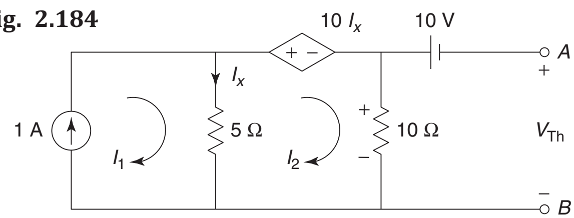


Fig. 2.185

$$I_x = I_1 - I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 1 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10I_2 = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10I_2 = 0$$

$$5I_1 + 5I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 1\text{ A}$$

$$I_2 = -1\text{ A}$$

$$I_x = I_1 - I_2 = 1 - (-1) = 2\text{ A}$$

Writing the V_{Th} equation,

$$10I_2 - 10 - V_{\text{Th}} = 0$$

$$10(-1) - 10 - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = -20\text{ V}$$

Step II Calculation of I_N (Fig. 2.186)

From Fig. 2.186,

$$I_x = I_1 - I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 1 \qquad \ldots\text{(ii)}$$

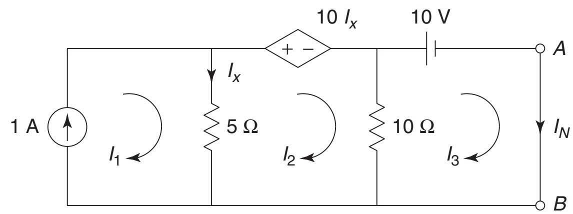


Fig. 2.186

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10(I_2 - I_3) = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10(I_2 - I_3) = 0$$

$$-5I_1 - 5I_2 + 10I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-10(I_3 - I_2) - 10 = 0$$

$$10I_2 - 10I_3 = 10 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 1\text{ A}$$

$$I_2 = 3\text{ A}$$

$$I_3 = 2\text{ A}$$

$$I_N = I_3 = 2\text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{-20}{2} = -10\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 2.187)

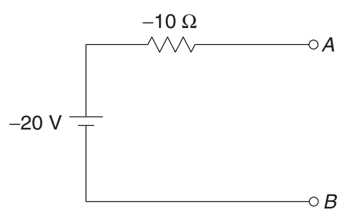


Fig. 2.187

Example 2.75 *Find Thevenin's equivalent network at terminals A and B in the network of Fig. 2.188.*

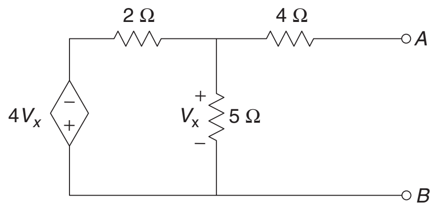


Fig. 2.188

Solution

Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$

$$I_N = 0$$

But the R_{Th} can be calculated by applying a known voltage source of 1 V at the terminals A and B as shown in Fig. 2.189.

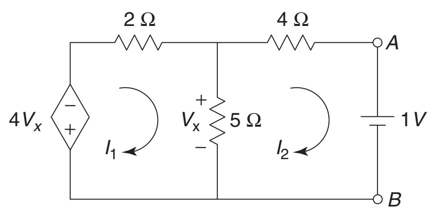


Fig. 2.189

$$R_{\text{Th}} = \frac{V}{I} = \frac{1}{I}$$

From Fig. 2.189,

$$V_x = 5(I_1 - I_2) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4V_x - 2I_1 - 5(I_1 - I_2) = 0$$

$$-4[5(I_1 - I_2] - 2I_1 - 5I_1 + 5I_2 = 0$$

$$-27I_1 + 25I_2 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 4I_2 - 1 = 0$$

$$5I_1 - 9I_2 = 1 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -0.21 \text{ A}$$

$$I_2 = -0.23 \text{ A}$$

Hence, current supplied by voltage source of 1 V is 0.23 A.

$$R_{\text{Th}} = \frac{1}{0.23} = 4.35\ \Omega$$

Hence, Thevenin's equivalent network is shown in Fig. 2.190.

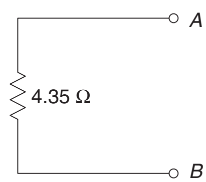


Fig. 2.190

Example 2.76 *Find the current in the 9 Ω resistor in Fig. 2.191.*

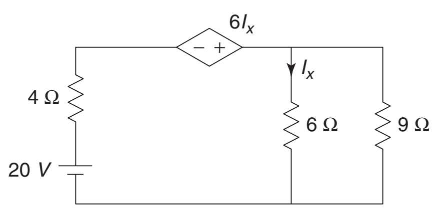


Fig. 2.191

Solution

Step I Calculation of V_{Th} (Fig. 2.192)

Applying KVL to the mesh,

$$20 - 4I_x + 6I_x - 6I_x = 0$$

$$I_x = 5 \text{ A}$$

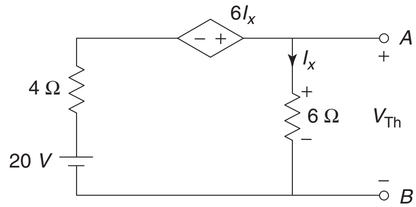


Fig. 2.192

Writing the V_{Th} equation,

$$6I_x - V_{Th} = 0$$

$$6(5) - V_{Th} = 0$$

$$V_{Th} = 30 \text{ V}$$

Step II Calculation of I_N (Fig. 2.193).

From Fig. 2.193,

$$I_x = 0$$

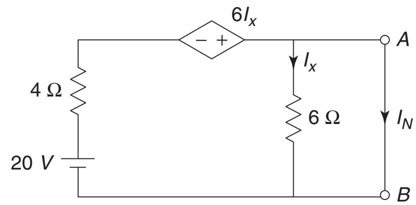


Fig. 2.193

The dependent source $6I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $6I_x = 0$ as shown in Fig. 2.194.

$$I_N = \frac{20}{4} = 5 \text{ A}$$

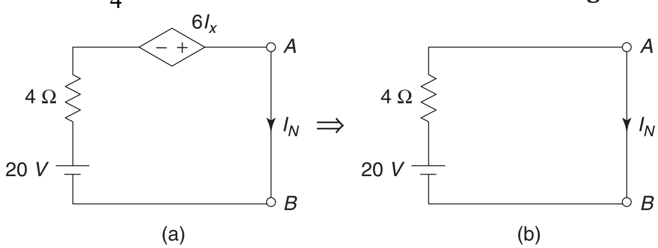


Fig. 2.194

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6\ \Omega$$

Step IV Calculation of I_L (Fig. 2.195)

$$I_L = \frac{30}{6+9} = 2 \text{ A}$$

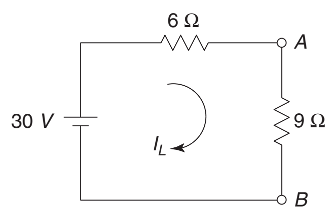


Fig. 2.195

Example 2.77 *Determine the current in the 16 Ω resistor in Fig. 2.196.*

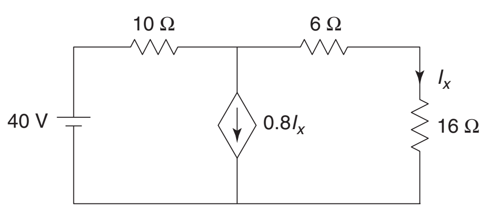


Fig. 2.196

Solution

Step I Calculation of V_{Th} (Fig. 2.197)

From Fig. 2.197,

$$I_x = 0$$

The dependent source $0.8I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, as shown in Fig. 2.198.

i.e.,
$$0.8I_x = 0$$
$$V_{Th} = 40 \text{ V}$$

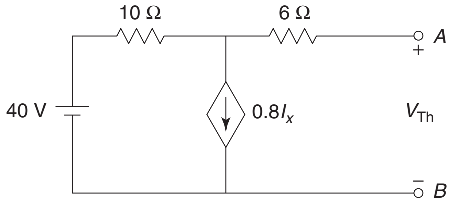


Fig. 2.197

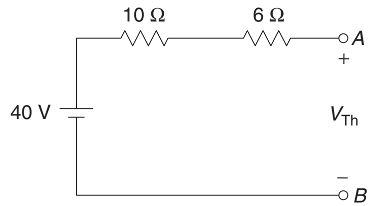


Fig. 2.198

Step II Calculation of I_N (Fig. 2.199)

From Fig. 2.199,

$$I_x = I_2 \quad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for the supermesh,

$$I_1 - I_2 = 0.8\, I_x = 0.8\, I_2$$
$$I_1 - 1.8\, I_2 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$40 - 10\, I_1 - 6\, I_2 = 0$$
$$10\, I_1 + 6\, I_2 = 40 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 3 \text{ A}$$
$$I_2 = \frac{5}{3} \text{ A}$$
$$I_N = I_2 = \frac{5}{3} \text{ A}$$

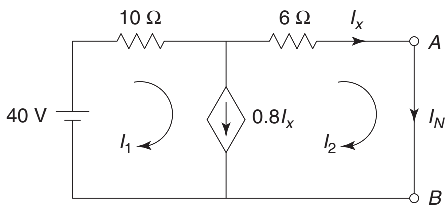


Fig. 2.199

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{40}{\frac{5}{3}} = 24\ \Omega$$

Step IV Calculation of I_L (Fig. 2.200)

$$I_L = \frac{40}{24+16} = 1 \text{ A}$$

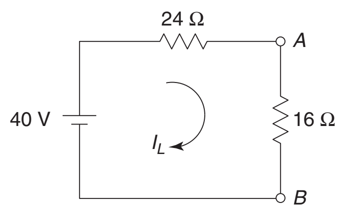


Fig. 2.200

Example 2.78 *Find the current in the 6 Ω resistor in Fig. 2.201.*

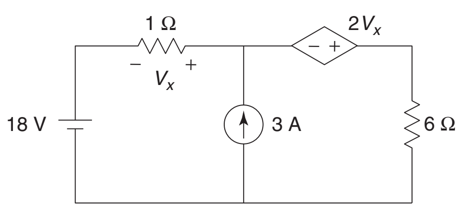


Fig. 2.201

Solution

Step I Calculation of V_{Th} (Fig. 2.202)

From Fig. 2.202,

$$V_x = -1I_1 = -I_1 \qquad \text{...(i)}$$

For Mesh 1,

$$I_1 = -3\text{ A} \qquad \text{...(ii)}$$

$$V_x = 3\text{ V}$$

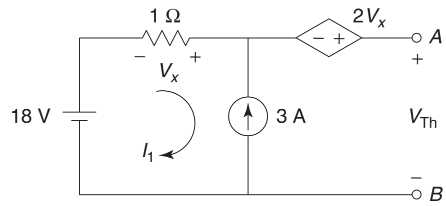


Fig. 2.202

Writing the V_{Th} equation,

$$18 - 1\,I_1 + 2\,V_x - V_{\text{Th}} = 0$$

$$18 + 3 + 2(3) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 27\text{ V}$$

Step II Calculation of I_N (Fig. 2.203)

From Fig. 2.203,

$$V_x = -I_1 \qquad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for supermesh,

$$I_2 - I_1 = 3 \qquad \text{...(ii)}$$

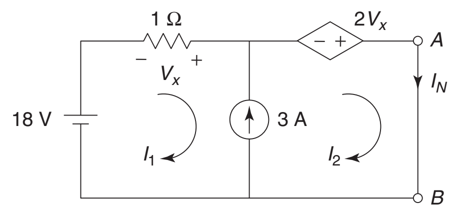


Fig. 2.203

Applying KVL to the outer path of the supermesh,

$$18 - 1I_1 + 2\,V_x = 0$$

$$18 - I_1 + 2(-I_1) = 0 \qquad \text{...(iii)}$$

$$I_1 = 6\text{ A}$$

Solving Eqs (ii) and (iii),

$$I_2 = 9\text{ A}$$

$$I_N = I_2 = 9\text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{27}{9} = 3\,\Omega$$

Step IV Calculation of I_L (Fig. 2.204)

$$I_L = \frac{27}{3+6} = 3\text{ A}$$

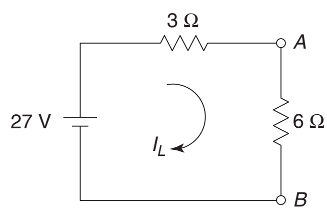


Fig. 2.204

Example 2.79 *Find the current in the 10 Ω resistor.*

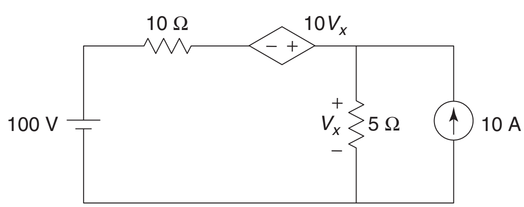


Fig. 2.205

Solution

Step I Calculation of V_{Th} (Fig. 2.206)

From Fig. 2.206,

$$V_x = 10 \times 5 = 50\text{ V}$$

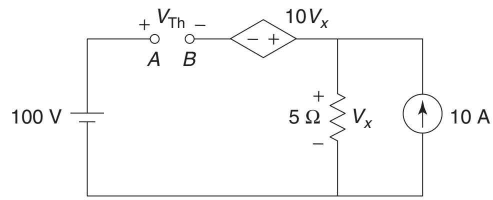


Fig. 2.206

Writing the V_{Th} equation,

$$100 - V_{Th} + 10V_x - V_x = 0$$
$$100 - V_{Th} + 9V_x = 0$$
$$100 - V_{Th} + 9(50) = 0$$
$$V_{Th} = 550\text{ V}$$

Step II Calculation of I_N (Fig. 2.207)

From Fig. 2.207,

$$V_x = 5(I_N + 10)$$

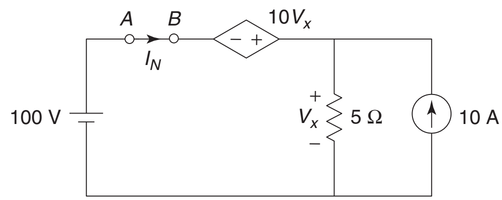


Fig. 2.207

Applying KVL to Mesh 1,

$$100 + 10V_x - V_x = 0$$
$$V_x = -\frac{100}{9}$$
$$-\frac{100}{9} = 5I_N + 50$$
$$I_N = -\frac{550}{45}\text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{550}{-\frac{550}{45}} = -45\ \Omega$$

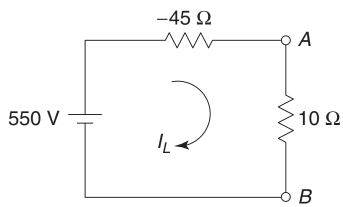


Fig. 2.208

Step IV Calculation of I_L (Fig. 2.208)

$$I_L = \frac{550}{-45+10} = -\frac{110}{7}\text{ A}$$

2.9 NORTON'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent current source and an equivalent parallel resistance.*' The constant current is equal to the current which would flow in a short circuit placed across the terminals. The parallel resistance is the resistance of the network when viewed from these open-circuited terminals after all voltage and current sources have been removed and replaced by internal resistances.

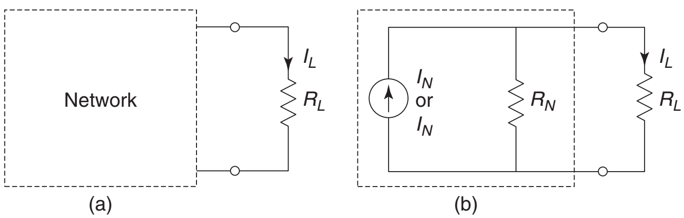


Fig. 2.209 *Network illustrating Norton's theorem*

Explanation Consider a simple network as shown in Fig. 2.210.

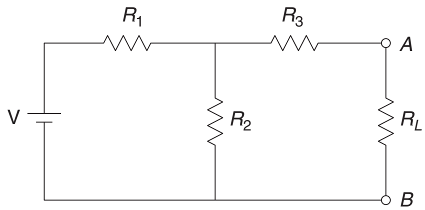


Fig. 2.210 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate short circuit current I_{SC} or I_N which would flow in a short circuit placed across terminals A and B as shown in Fig. 2.211.

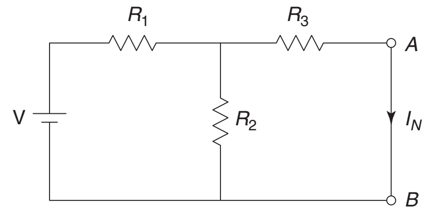


Fig. 2.211 *Calculation of I_N*

For finding parallel resistance R_N, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 2.212.

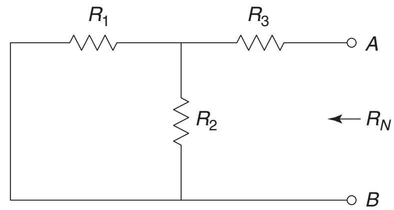


Fig. 2.212 *Calculation of R_N*

$$R_N = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

Norton's equivalent network is shown in Fig. 2.213.

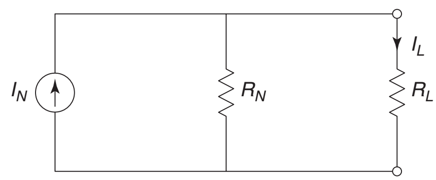


Fig. 2.213 *Norton's equivalent network*

$$I_L = I_N \frac{R_N}{R_N + R_L}$$

If the network contains both independent and dependent sources, Norton's resistances R_N is calculated as

$$R_N = \frac{V_{\text{Th}}}{I_N}$$

where V_{Th} is the open-circuit voltage across terminals A and B. If the network contains only dependent sources, then

$$V_{\text{Th}} = 0$$

$$I_N = 0$$

To find R_{Th} in such network, a known voltage V or current I is applied across the terminals A and B, and the current I or the voltage V is calculated respectively.

$$R_N = \frac{V}{I}$$

Norton's equivalent network for such a network is shown in Fig. 2.214.

Fig. 2.214 *Norton's equivalent network*

Steps to be followed in Norton's Theorem

1. Remove the load resistance R_L and put a short circuit across the terminals.
2. Find the short-circuit current I_{SC} or I_N.
3. Find the resistance R_N as seen from points A and B.
4. Replace the network by a current source I_N in parallel with resistance R_N.
5. Find current through R_L by current–division rule.

$$I_L = \frac{I_N R_N}{R_N + R_L}$$

Example 2.80 *Find the current through the 10 Ω resistor in Fig. 2.215.*

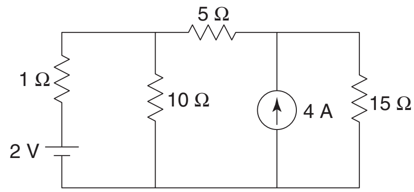


Fig. 2.215

Solution

Step I Calculation of I_N (Fig. 2.216)

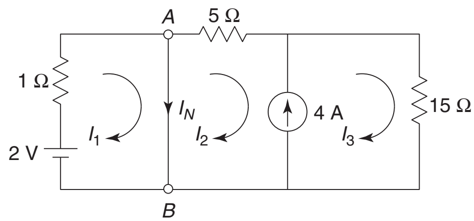


Fig. 2.216

Applying KVL to Mesh 1,

$$2 - 1I_1 = 0$$

$$I_1 = 2 \qquad \text{...(i)}$$

Meshes 2 and 3 will form a supermesh.

Writing the current equation for the supermesh,

$$I_3 - I_2 = 4 \qquad \text{...(ii)}$$

Applying KVL to the supermesh,

$$-5I_2 - 15I_3 = 0 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 2\text{ A}$$

$$I_2 = -3\text{ A}$$

$$I_3 = 1\text{ A}$$

$$I_N = I_1 - I_2 = 2 - (-3) = 5\text{ A}$$

Step II Calculation of R_N (Fig. 2.217)

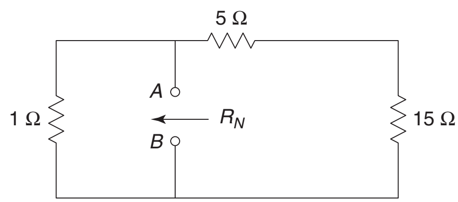


Fig. 2.217

$$R_N = 1 \,||\, (5 + 15) = 0.95\ \Omega$$

Step III Calculation of I_L (Fig. 2.218)

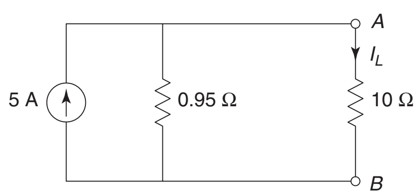


Fig. 2.218

$$I_L = 5 \times \frac{0.95}{0.95 + 10} = 0.43\text{ A}$$

Example 2.81 *Find the current through the 10 Ω resistor in Fig. 2.219.*

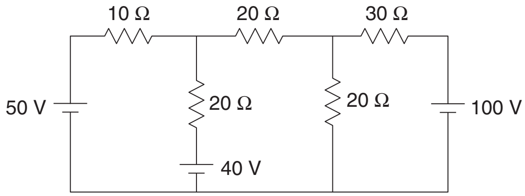


Fig. 2.219

Solution

Step I Calculation of I_N (Fig. 2.220)

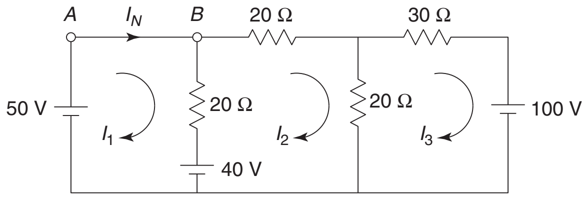


Fig. 2.220

Applying KVL to Mesh 1,

$$50 - 20(I_1 - I_2) - 40 = 0$$

$$20I_1 - 20I_2 = 10 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$40 - 20(I_2 - I_1) - 20I_2 - 20(I_2 - I_3) = 0$$

$$-20I_1 + 60I_2 - 20I_3 = 40 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-20(I_3 - I_2) - 30I_3 - 100 = 0$$

$$-20I_2 + 50I_3 = -100 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.81\,\text{A}$$

$$I_N = I_1 = 0.81\,\text{A}$$

Step II Calculation of R_N (Fig. 2.221)

$$R_N = [(20 \,||\, 30) + 20] \,||\, 20 = 12.3\,\Omega$$

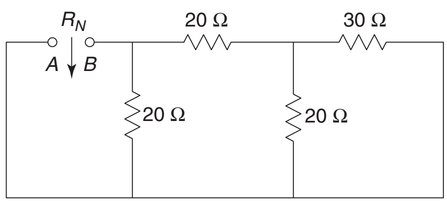


Fig. 2.221

Step III Calculation of I_L (Fig. 2.222)

$$I_L = 0.81 \times \frac{12.3}{12.3 + 10} = 0.45\,\text{A}$$

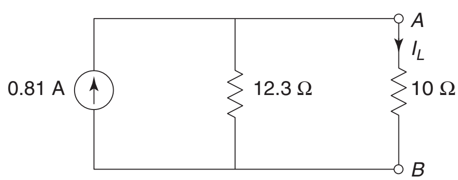


Fig. 2.222

Example 2.82 *Find the current through the 8 Ω resistor in Fig. 2.223.*

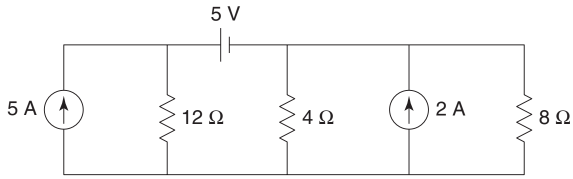


Fig. 2.223

Solution

Step I Calculation of I_N (Fig. 2.224)

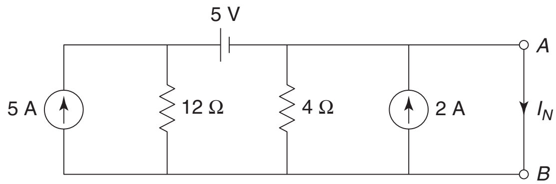


Fig. 2.224

The resistor of the 4 Ω gets shorted as it is in parallel with the short circuit. Simplifying the network by source transformation (Fig. 2.225),

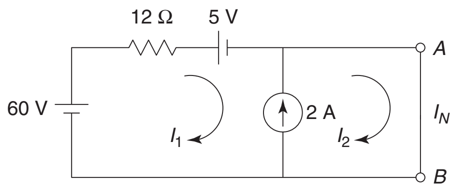


Fig. 2.225

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \text{...(i)}$$

Applying KVL to the supermesh,

$$60 - 12I_1 - 5 = 0$$
$$12I_1 = 55 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 4.58\,\text{A}$$
$$I_2 = 6.58\,\text{A}$$
$$I_N = I_2 = 6.58\,\text{A}$$

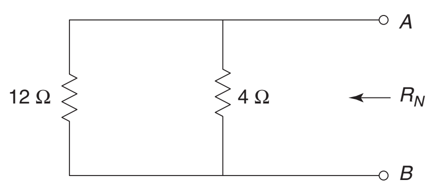


Fig. 2.226

Step II Calculation of R_N (Fig. 2.226)

$$R_N = 12 \,||\, 4 = 3\,\Omega$$

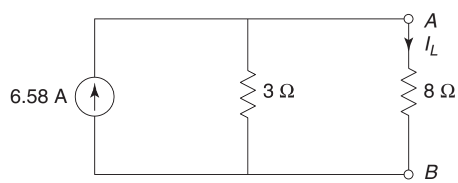


Fig. 2.227

Step III Calculation of I_L (Fig. 2.227)

$$I_L = 6.58 \times \frac{3}{3+8} = 1.79\,\text{A}$$

Example 2.83 *Find the current through the 1 Ω resistor in Fig. 2.228.*

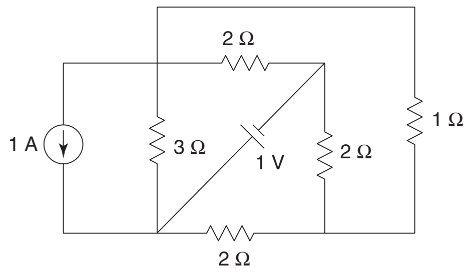


Fig. 2.228

Solution

Step I Calculation of I_N (Fig. 2.229)

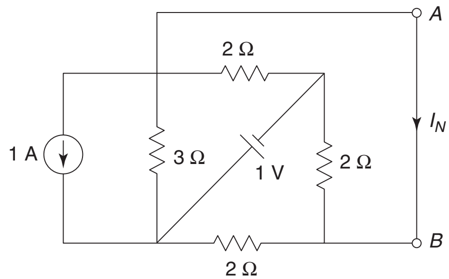


Fig. 2.229

By source transformation (Fig. 2.230),

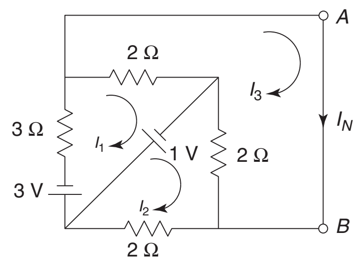


Fig. 2.230

Applying KVL to Mesh 1,

$$-3-3I_1-2(I_1-I_3)+1=0$$

$$5I_1-2I_3=-2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-1-2(I_2-I_3)-2I_2=0$$
$$4I_2-2I_3=-1 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3-I_1)-2(I_3-I_2)=0$$
$$-2I_1-2I_2+4I_3=0 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=-0.64\,\text{A}$$
$$I_2=-0.55\,\text{A}$$
$$I_3=-0.59\,\text{A}$$
$$I_N=I_3=-0.59\,\text{A}$$

Step II Calculation of R_N (Fig. 2.231)

$$R_N=2.2\,\Omega$$

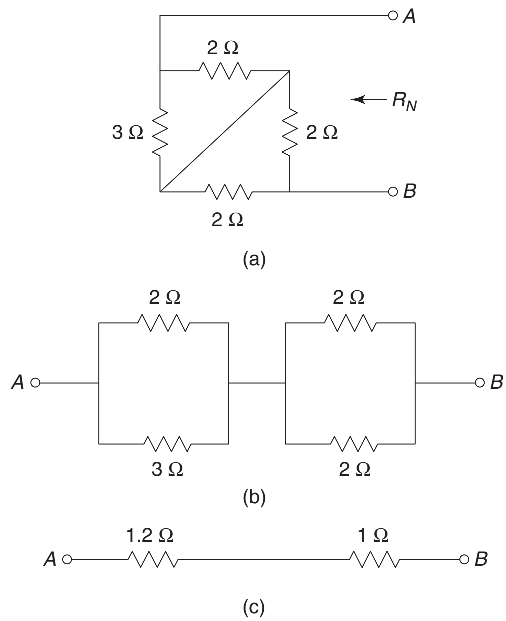


Fig. 2.231

Step III Calculation of I_L (Fig. 2.232)

$$I_L=0.59\times\frac{2.2}{2.2+1}=0.41\,\text{A}$$

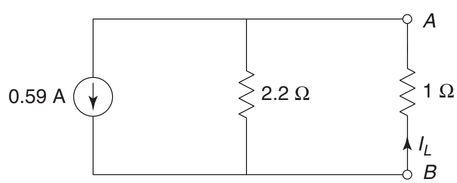


Fig. 2.232

EXAMPLES WITH DEPENDENT SOURCES

Example 2.84 *Find Norton's equivalent network across terminals A and B of Fig. 2.233.*

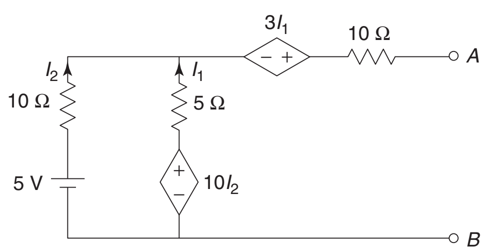


Fig. 2.233

Solution

Step I Calculation of V_{Th} (Fig. 2.234)

From Fig. 2.234,

$$I_2 = I_x$$
$$I_1 = -I_x$$

Applying KVL to the mesh,

$$5 - 10I_x - 5I_x - 10I_2 = 0$$
$$5 - 10I_x - 5I_x - 10I_x = 0$$
$$I_x = 0.2 \text{ A}$$
$$I_1 = -0.2 \text{ A}$$

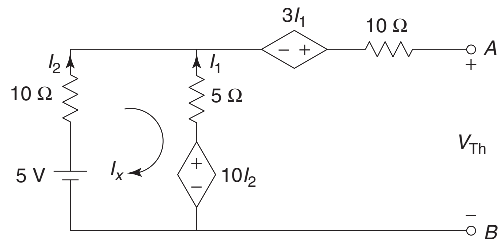


Fig. 2.234

Writing the V_{Th} equation,

$$5 - 10I_x + 3I_1 - V_{\text{Th}} = 0$$
$$5 - 10(0.2) + 3(-0.2) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 2.4 \text{ V}$$

Step II Calculation of I_N (Fig. 2.235)

From Fig. 2.235,

$$I_2 = I_x \qquad \ldots\text{(i)}$$
$$I_1 = I_y - I_x \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$5 - 10I_x - 5(I_x - I_y) - 10I_2 = 0$$
$$5 - 10I_x - 5I_x + 5I_y - 10I_x = 0$$
$$25I_x - 5I_y = 5 \qquad \ldots\text{(iii)}$$

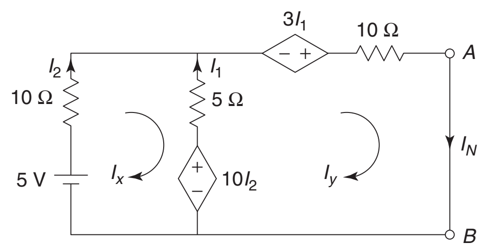


Fig. 2.235

Applying KVL to Mesh 2,

$$10I_2 - 5(I_y - I_x) + 3I_1 - 10I_y = 0$$
$$10I_x - 5I_y + 5I_x + 3(I_y - I_x) - 10I_y = 0$$
$$12I_x - 12I_y = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (iii) and (iv),

$$I_x = 0.25 \text{ A}$$
$$I_y = 0.25 \text{ A}$$
$$I_N = I_y = 0.25 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{2.4}{0.25} = 9.6\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 2.236)

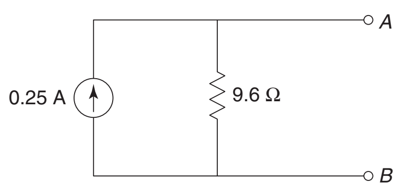


Fig. 2.236

Example 2.85 *For the network shown in Fig. 2.237, find Norton's equivalent network.*

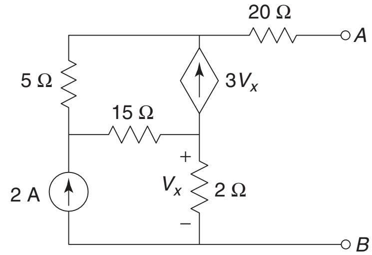


Fig. 2.237

Solution

Step I Calculation of V_{Th} (Fig. 2.238)

From Fig. 2.238,

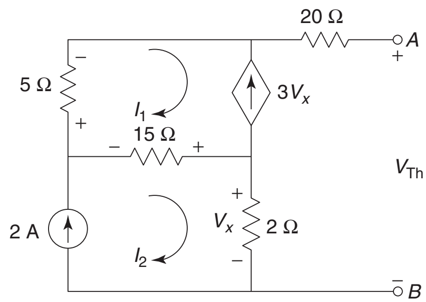


Fig. 2.238

$$V_x = 2I_2 \qquad \text{...(i)}$$

For Mesh 1,

$$I_1 = -3V_x = -3(2I_2) = -6I_2 \qquad \text{...(ii)}$$

For Mesh 2,

$$I_2 = 2 \qquad \text{...(iii)}$$

$$I_1 = -6I_2 = -6(2) = -12 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 0 + 5I_1 + 15(I_1 - I_2) - 2I_2 = 0$$

$$V_{Th} + 5(-12) + 15(-12 - 2) - 2(2) = 0$$

$$V_{Th} = 274 \text{ V}$$

Step II Calculation of I_N (Fig. 2.239)

From Fig. 2.239,

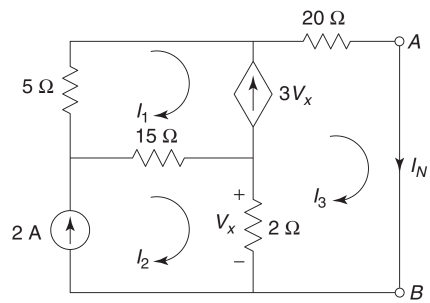


Fig. 2.239

$$V_x = 2(I_2 - I_3) \qquad \text{...(i)}$$

For Mesh 2,

$$I_2 = 2 \qquad \text{...(ii)}$$

Meshes 1 and 3 will form a supermesh.

Writing the current equation for the supermesh,

$$I_3 - I_1 = 3V_x = 3[2(I_2 - I_3)] = 6I_2 - 6I_3$$

$$I_1 + 6I_2 - 7I_3 = 0 \qquad \text{...(iii)}$$

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 20I_3 - 2(I_3 - I_2) - 15(I_1 - I_2) = 0$$

$$-20I_1 + 17I_2 - 22I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.16 \text{ A}$$
$$I_2 = 2 \text{ A}$$
$$I_3 = 1.69 \text{ A}$$
$$I_N = I_3 = 1.69 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{274}{1.69} = 162.13\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 2.240)

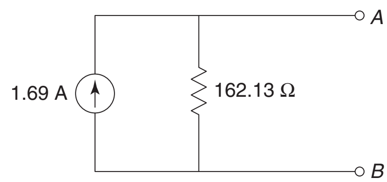


Fig. 2.240

Example 2.86 *Obtain Norton's equivalent network across A-B in the network of Fig. 2.241.*

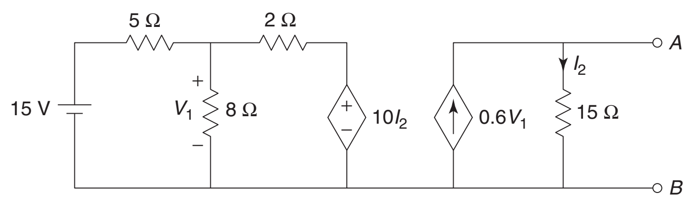


Fig. 2.241

Solution

Step I Calculation of V_{Th} (Fig. 2.242)

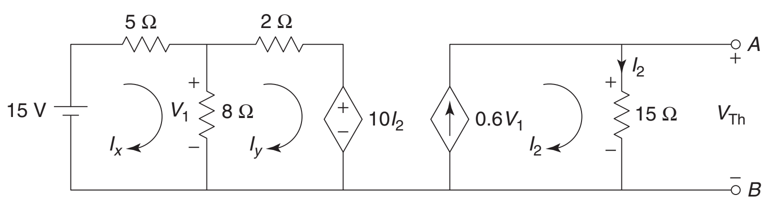


Fig. 2.242

From Fig. 2.242,

$$V_1 = 8(I_x - I_y) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$
$$13I_x - 8I_y = 15 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y - 10I_2 = 0$$
$$8I_x - 10I_y - 10I_2 = 0 \qquad \text{...(iii)}$$

For Mesh 3,

$$I_2 = 0.6V_1 = 0.6\left[8(I_x - I_y)\right]$$

$$4.8I_x - 4.8I_y - I_2 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_x = 3.28 \text{ A}$$
$$I_y = 3.45 \text{ A}$$
$$I_2 = -0.83 \text{ A}$$

Writing the V_{Th} equation,

$$15I_2 - V_{\text{Th}} = 0$$
$$15(-0.83) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -12.45 \text{ V}$$

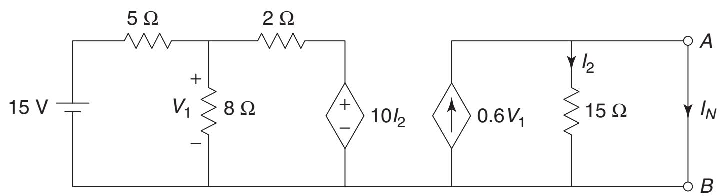


Fig. 2.243

Step II Calculation of I_N (Fig. 2.243)
From Fig. 2.243,

$$I_2 = 0$$

The dependent source of 10 I_2 depends on the controlling variable I_2. When $I_2 = 0$, the dependent source vanishes, i.e. $10I_2 = 0$ as shown in Fig. 2.244.

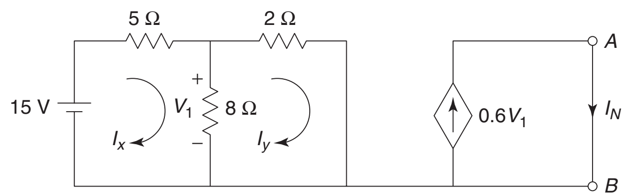


Fig. 2.244

From Fig. 2.244,

$$V_1 = 8(I_x - I_y) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$
$$13I_x - 8I_y = 15 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y = 0$$
$$-8I_x + 10I_y = 0 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_x = 2.27 \text{ A}$$
$$I_y = 1.82 \text{ A}$$
$$V_1 = 8(I_x - I_y) = 8(2.27 - 1.82) = 3.6 \text{ V}$$

For Mesh 3,

$$I_N = 0.6V_1 = 0.6(3.6) = 2.16 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-12.45}{2.16} = -5.76\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 2.245)

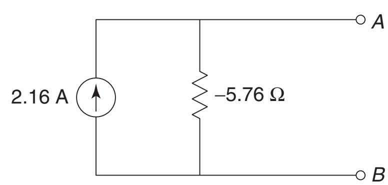


Fig. 2.245

Example 2.87 *Find Norton's equivalent network of Fig. 2.246.*

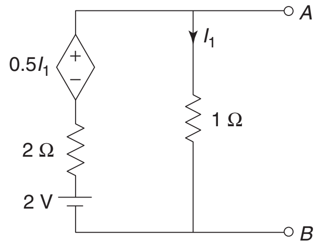


Fig. 2.246

Solution

Step I Calculation of V_{Th} (Fig. 2.247)

Applying KVL to the mesh,

$$2 - 2I_1 + 0.5I_1 - 1I_1 = 0$$

$$2 - 2.5I_1 = 0$$

$$I_1 = 0.8 \text{ A}$$

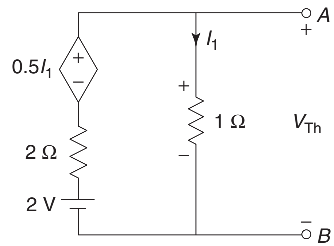


Fig. 2.247

Writing the V_{Th} equation,

$$1I_1 - V_{\text{Th}} = 0$$

$$1(0.8) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 0.8 \text{ V}$$

Step II Calculation of I_N (Fig. 2.248)

When a short circuit is placed across the 1 Ω resistor, it gets shorted.

$$I_1 = 0$$

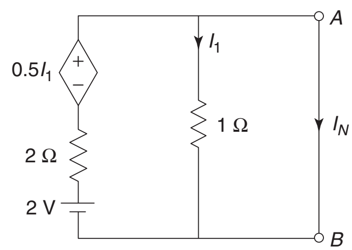


Fig. 2.248

The dependent source of $0.5I_1$ depends on the controlling variable I_1. When $I_1 = 0$, the dependent source vanishes, i.e. $0.5\ I_1 = 0$ as shown in Fig. 2.249.

$$I_N = \frac{2}{2} = 1 \text{ A}$$

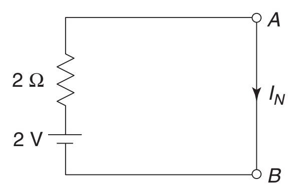


Fig. 2.249

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{0.8}{1} = 0.8\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 2.250)

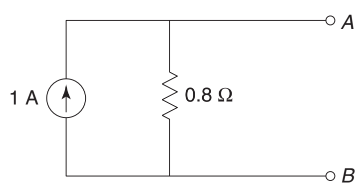


Fig. 2.250

Example 2.88 *Find Norton's equivalent network at the terminals A and B of Fig. 2.251.*

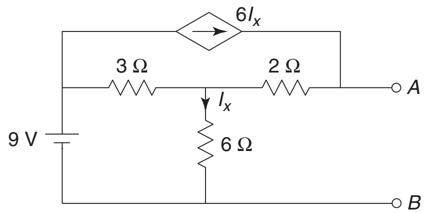


Fig. 2.251

Solution

Step I Calculation of V_{Th} (Fig. 2.252)

From Fig. 2.252,

$$I_x = I_1 \quad \text{...(i)}$$

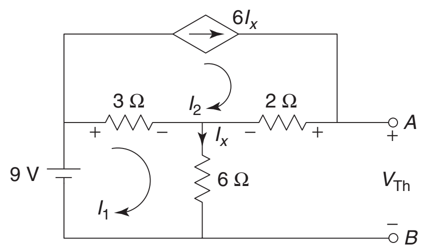


Fig. 2.252

Applying KVL to Mesh 1,

$$9-3(I_1-I_2)-6I_1=0$$
$$9I_1-3I_2=9 \quad \text{...(ii)}$$

For Mesh 2,

$$I_2 = 6I_x = 6I_1$$
$$6I_1 - I_2 = 0 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -1 \text{ A}$$
$$I_2 = -6 \text{ A}$$

Writing the V_{Th} equation,

$$9-3(I_1-I_2)+2I_2-V_{Th}=0$$
$$9-3(-1+6)+2(-6)-V_{Th}=0$$
$$V_{Th}=-18 \text{ V}$$

Step II Calculation of I_N (Fig. 2.253)

From Fig. 2.253,

$$I_x = I_1 - I_3 \quad \text{...(i)}$$

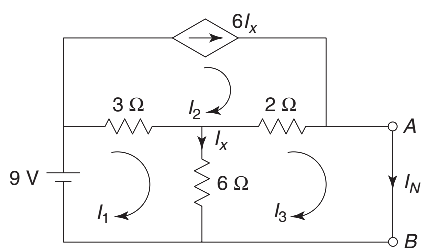


Fig. 2.253

Applying KVL to Mesh 1,

$$9-3(I_1-I_2)-6(I_1-I_3)=0$$
$$9I_1-3I_2-6I_3=9 \quad \text{...(ii)}$$

For Mesh 2,

$$I_2 = 6I_x = 6(I_1-I_3)$$
$$6I_1 - I_2 - 6I_3 = 0 \quad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-6(I_3-I_1)-2(I_3-I_2)=0$$
$$-6I_1-2I_2+8I_3=0 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5\text{ A}$$
$$I_2 = 3\text{ A}$$
$$I_3 = 4.5\text{ A}$$
$$I_N = I_3 = 4.5\text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 2.254)

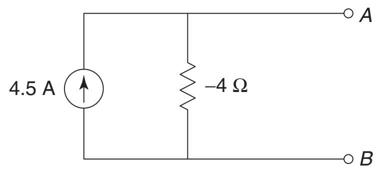


Fig. 2.254

Example 2.89 *Find Norton's equivalent network to the left of terminal A-B in Fig. 2.255.*

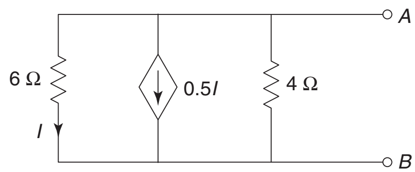


Fig. 2.255

Solution Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$
$$I_N = 0$$

But R_N can be calculated by applying a known current source of 1 A at the terminals A and B as shown in Fig. 2.256.

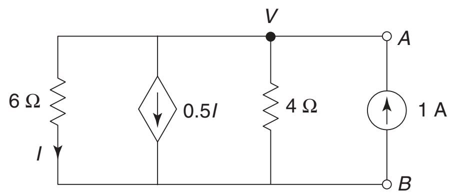


Fig. 2.256

From Fig. 2.256,

$$I = \frac{V}{6}$$

Applying KCL at the node,

$$\frac{V}{6} + 0.5I + \frac{V}{4} = 1$$

$$\frac{V}{6} + 0.5\left(\frac{V}{6}\right) + \frac{V}{4} = 1$$

$$\left(\frac{1}{6} + \frac{0.5}{6} + \frac{1}{4}\right)V = 1$$

$$V = 2$$

$$R_N = \frac{V}{1} = \frac{2}{1} = 2\ \Omega$$

Hence, Norton's equivalent network is shown in Fig. 2.257.

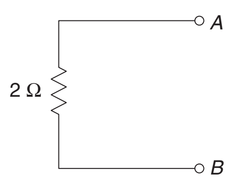


Fig. 2.257

Example 2.90 *Find the current through the 2 Ω resistor in the network shown in Fig. 2.258.*

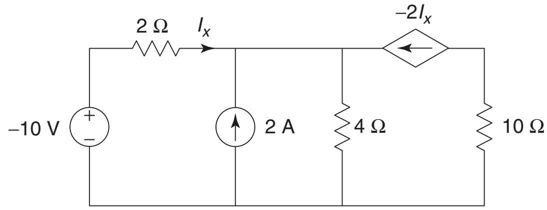


Fig. 2.258

Solution

Step I Calculation of V_{Th} (Fig. 2.259)
From Fig. 2.259,

$$I_x = 0$$

The dependent source of $-2\,I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e. $-2I_x = 0$ as shown in Fig. 2.260.

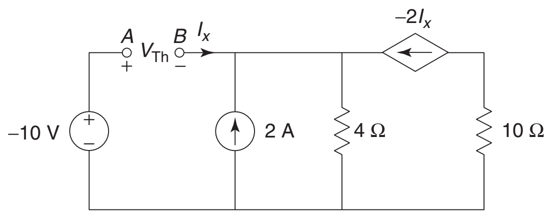


Fig. 2.259

$$I_1 = 2$$

Writing the V_{Th} equation,

$$-10 - V_{\text{Th}} - 4I_1 = 0$$

$$-10 - V_{\text{Th}} - 4(2) = 0$$

$$V_{\text{Th}} = -18 \text{ V}$$

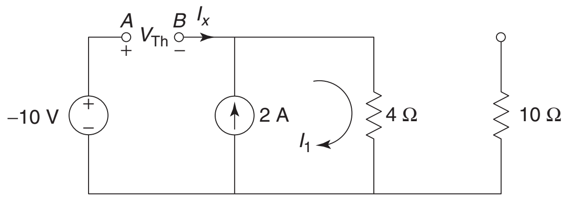


Fig. 2.260

Step II Calculation of I_N (Fig. 2.261)
From Fig. 2.261,

$$I_x = I_1 \qquad \text{...(i)}$$

Mesh 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \text{...(ii)}$$

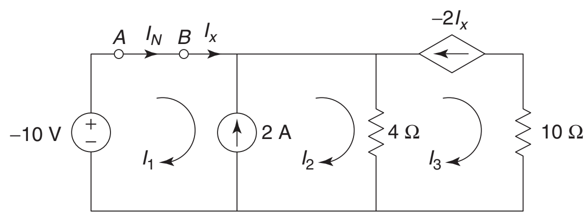


Fig. 2.261

Applying KVL to the outer path of the supermesh,

$$-10 - 4(I_2 - I_3) = 0$$

$$-4I_2 + 4I_3 = 10 \qquad \text{...(iii)}$$

For Mesh 3,

$$I_3 = -(-2I_x) = 2I_x = 2I_1$$

$$2I_1 - I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.5 \text{ A}$$
$$I_2 = 6.5 \text{ A}$$
$$I_3 = 9 \text{ A}$$
$$I_N = I_1 = 4.5 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4\ \Omega$$

Step IV Calculation of I_L (Fig. 2.262)

$$I_L = 4.5 \times \frac{-4}{-4+2} = 9 \text{ A}$$

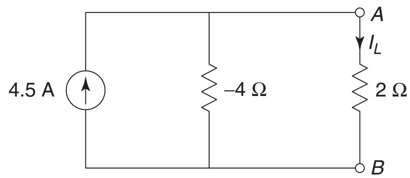


Fig. 2.262

Example 2.91 *Find the current through the 2Ω resistor in the network of Fig. 2.263.*

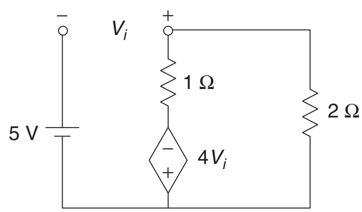


Fig. 2.263

Solution

Step I Calculation of V_{Th} (Fig. 2.264)

From Fig. 2.264,

$$5 + V_i + 4V_i = 0$$
$$V_i = -1 \text{ V}$$

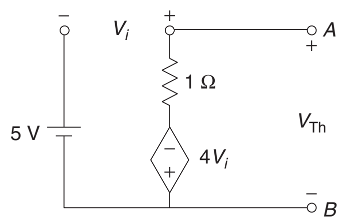


Fig. 2.264

Writing the V_{Th} equation,

$$-4V_i - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -4V_i = -4(-1) = 4 \text{ V}$$

Step II Calculation of I_N (Fig. 2.265)

From Fig. 2.265,

$$5 + V_i = 0$$
$$V_i = -5 \text{ V}$$

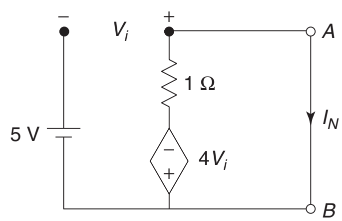


Fig. 2.265

Applying KVL to the mesh,

$$-4V_i - 1I_N = 0$$
$$I_N = -4V_i = -4(-5) = 20 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{4}{20} = 0.2\ \Omega$$

Step IV Calculation of I_L (Fig. 2.266)

$$I_L = 20 \times \frac{0.2}{0.2+2} = 1.82\ \text{A}$$

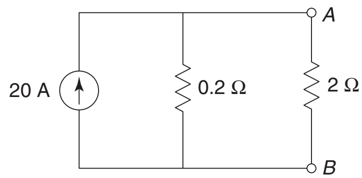


Fig. 2.266

Example 2.92 *Find the current in the 2 Ω resistor in the network of Fig. 2.267.*

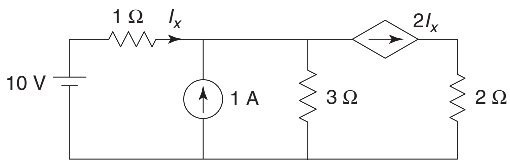


Fig. 2.267

Solution

Step I Calculation of V_{Th} (Fig. 2.268)

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \text{...(i)}$$

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3I_2 = 0$$

$$I_1 + 3I_2 = 10 \qquad \text{...(ii)}$$

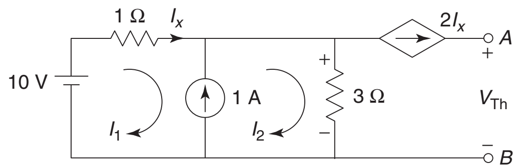


Fig. 2.268

Solving Eqs (i) and (ii),

$$I_1 = 1.75\ \text{A}$$

$$I_2 = 2.75\ \text{A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{\text{Th}} = 0$$

$$3(2.75) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 8.25\ \text{V}$$

Step II Calculation of I_N (Fig. 2.269)

From Fig. 2.269,

$$I_x = I_1 \qquad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \text{...(ii)}$$

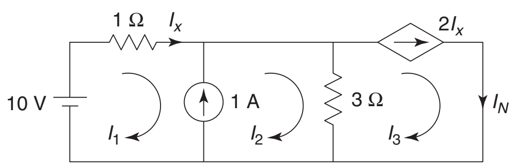


Fig. 2.269

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3(I_2 - I_3) = 0$$

$$I_1 + 3I_2 - 3I_3 = 10 \quad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = 2I_x = 2I_1$$

$$2I_1 - I_3 = 0 \quad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -3.5 \text{ A}$$

$$I_2 = -2.5 \text{ A}$$

$$I_3 = -7 \text{ A}$$

$$I_N = I_3 = -7 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8.25}{-7} = -1.18\ \Omega$$

Step IV Calculation of I_L (Fig. 2.270)

$$I_L = -7 \times \frac{-1.18}{-1.18 + 2} = 10.07 \text{ A}$$

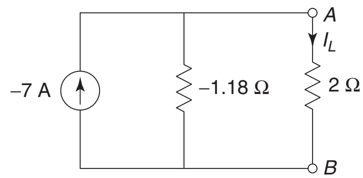


Fig. 2.270

Example 2.93 *Find the current through the 10 Ω resistor for the network of Fig. 2.271.*

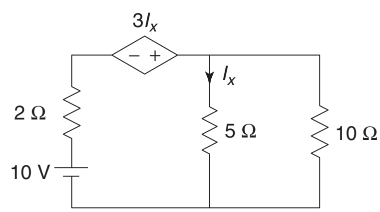


Fig. 2.271

Solution

Step I Calculation of V_{Th} (Fig. 2.272)
Applying KVL to the mesh,

$$10 - 2I_x + 3I_x - 5I_x = 0$$

$$I_x = 2.5 \text{ A}$$

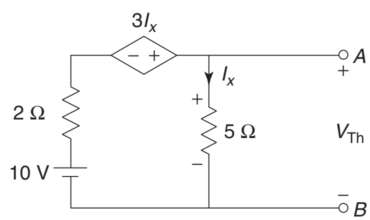


Fig. 2.272

Writing the V_{Th} equation,

$$5I_x - V_{\text{Th}} = 0$$

$$5(2.5) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 12.5 \text{ V}$$

Step II Calculation of I_N (Fig. 2.273)
From Fig. 2.273,

$$I_x = 0$$

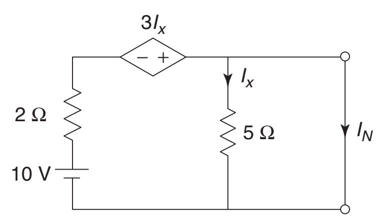


Fig. 2.273

The dependent source of 3 I_x depends on the controlling variable I_x. When $I_x = 0$, the dependent source 3 I_x vanishes, i.e. 3 $I_x = 0$ as shown in Fig. 2.274.

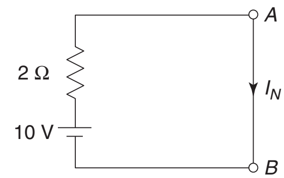


Fig. 2.274

$$I_N = \frac{10}{2} = 5 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{12.5}{5} = 2.5\ \Omega$$

Step IV Calculation of I_L (Fig. 2.275)

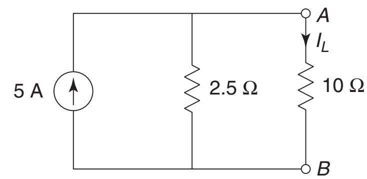


Fig. 2.275

$$I_L = 5 \times \frac{2.5}{2.5+10} = 1 \text{ A}$$

Example 2.94 *Find the current through the 5 Ω resistor in the network of Fig. 2.276.*

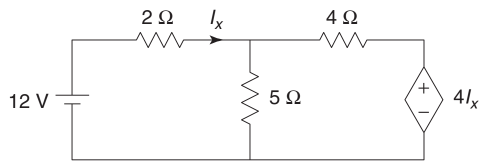


Fig. 2.276

Solution

Step I Calculation of V_{Th} (Fig. 2.277)

Applying KVL to the mesh,

$$12 - 2I_x - 4I_x - 4I_x = 0$$
$$12 - 10I_x = 0$$
$$I_x = 1.2 \text{ A}$$

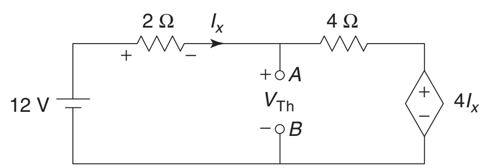


Fig. 2.277

Writing the V_{Th} equation,

$$12 - 2I_x - V_{\text{Th}} = 0$$
$$12 - 2(1.2) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 9.6 \text{ V}$$

Step II Calculation of I_N (Fig. 2.278)

From Fig. 2.278,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$12 - 2I_1 = 0$$
$$I_1 = 6 \text{ A} \qquad \ldots\text{(ii)}$$

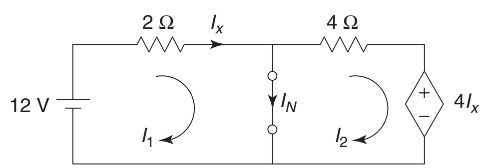


Fig. 2.278

Applying KVL to Mesh 2,

$$-4I_2 - 4I_x = 0$$
$$-4I_2 - 4I_1 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_2 = -6 \text{ A}$$
$$I_N = I_1 - I_2 = 6 - (-6) = 12 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{9.6}{12} = 0.8\ \Omega$$

Step IV Calculation of I_L (Fig. 2.279)

$$I_L = 12 \times \frac{0.8}{0.8+5} = 1.66\ \text{A}$$

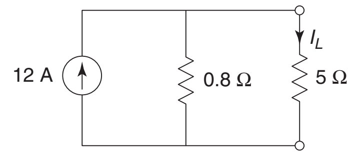


Fig. 2.279

Example 2.95 *Find the current through the 10 Ω resistor for the network of Fig. 2.280.*

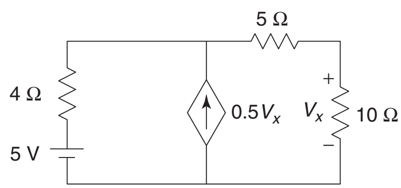


Fig. 2.280

Solution

Step I Calculation of V_{Th} (Fig. 2.281)
For the mesh,

$$I = -0.5V_x = -0.5V_{\text{Th}}$$

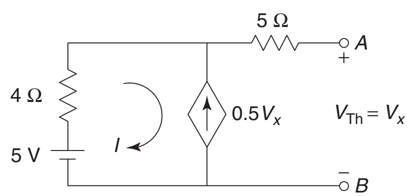


Fig. 2.281

Writing the V_{Th} equation,

$$5 - 4I - 0 - V_{\text{Th}} = 0$$

$$5 - 4(-0.5V_{\text{Th}}) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = -5\ \text{V}$$

Step II Calculation of I_N (Fig. 2.282)
From Fig. 2.282,

$$V_x = 0$$

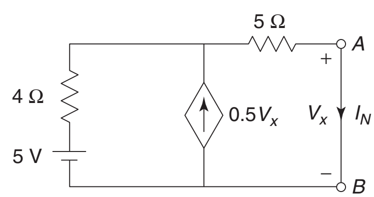


Fig. 2.282

The dependent source of 0.5 V_x depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. 0.5 $V_x = 0$ as shown in Fig. 2.283.

$$I_N = \frac{5}{4+5} = \frac{5}{9}\ \text{A}$$

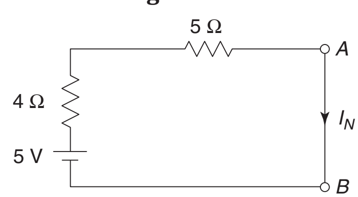


Fig. 2.283

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-5}{\frac{5}{9}} = -9\ \Omega$$

Step IV Calculation of I_L (Fig. 2.284)

$$I_L = \frac{5}{9} \times \frac{-9}{-9+10} = -5\ \text{A}$$

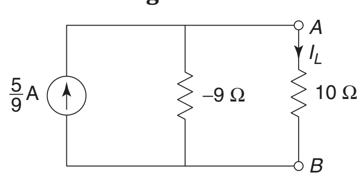


Fig. 2.284

Example 2.96 *Find the current through the 10 Ω resistor in the network shown in Fig. 2.285.*

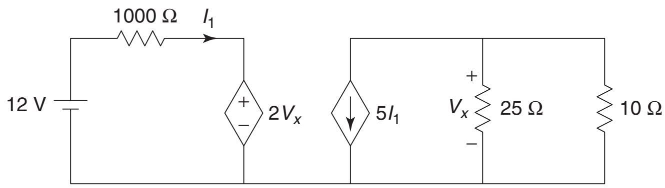


Fig. 2.285

Solution

Step I Calculation of V_{Th} (Fig. 2.286)

From Fig. 2.286,

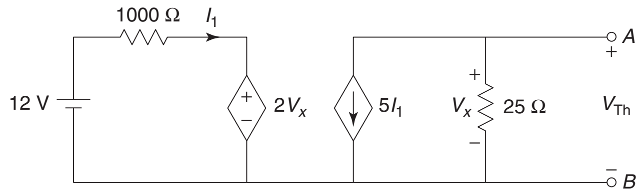


Fig. 2.286

$$V_x = -25(5I_1) = -125\,I_1 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$12 - 1000I_1 - 2V_x = 0$$

$$12 - 1000I_1 - 2(-125I_1) = 0 \qquad \ldots\text{(ii)}$$

$$I_1 = 0.016 \text{ A}$$

$$V_x = -125I_1 = -125(0.016) = -2 \text{ V}$$

Writing the V_{Th} equation,

$$V_{Th} = V_x = -2 \text{ V}$$

Step II Calculation of I_N (Fig. 2.287)

From Fig. 2.287,

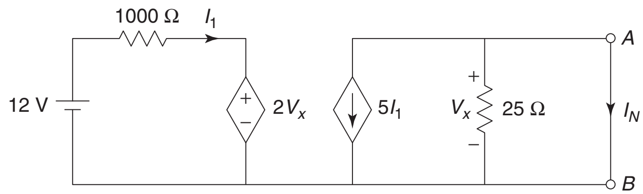


Fig. 2.287

$$V_x = 0$$

The dependent source of $2V_x$ depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. $2\,V_x = 0$, as shown in Fig. 2.288.

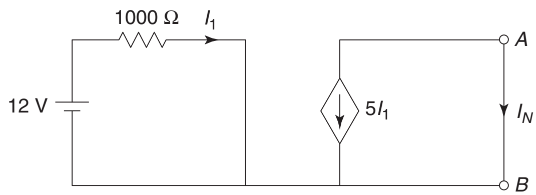


Fig. 2.288

$$I_1 = \frac{12}{1000} = 0.012 \text{ A}$$

$$I_N = -5I_1 = -5(0.012) = -0.06 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{-2}{-0.06} = 33.33\ \Omega$$

Step IV Calculation of I_L (Fig. 2.289)

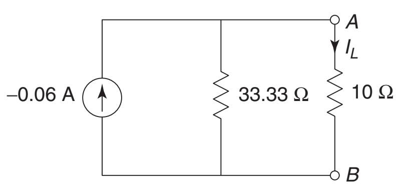


Fig. 2.289

$$I_L = -0.06 \times \frac{33.33}{33.33 + 10} = -0.046 \text{ A}$$

Example 2.97 *Find the current through the 5 Ω resistor for the network of Fig. 2.290.*

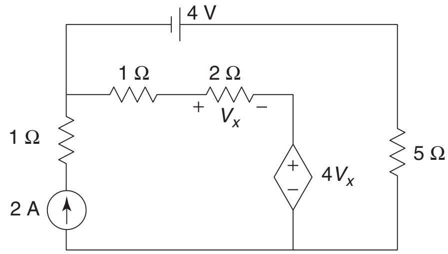


Fig. 2.290

Solution

Step I Calculation of V_{Th} (Fig. 2.291)

From Fig. 2.291,

$$V_x = 2I \quad \ldots\text{(i)}$$

For the mesh,

$$I = 2 \quad \ldots\text{(ii)}$$

$$V_x = 2(2) = 4 \text{ V}$$

Writing the V_{Th} equation,

$$4V_x + 2I + 1I + 4 - V_{Th} = 0$$

$$4(4) + 2(2) + 2 + 4 - V_{Th} = 0$$

$$V_{Th} = 26 \text{ V}$$

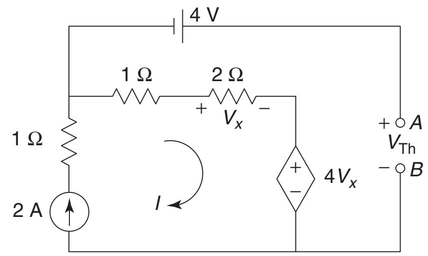


Fig. 2.291

Step II Calculation of I_N (Fig. 2.292)

From Fig. 2.292,

$$V_x = 2(I_1 - I_2) \quad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 2 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$4V_x - 2(I_2 - I_1) - 1(I_2 - I_1) + 4 = 0$$

$$4[2(I_1 - I_2)] - 2I_2 + 2I_1 - I_2 + I_1 + 4 = 0$$

$$11I_1 - 11I_2 = -4 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$

$$I_2 = 2.36 \text{ A}$$

$$I_N = I_2 = 2.36 \text{ A}$$

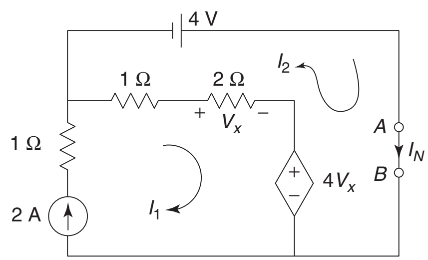


Fig. 2.292

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{26}{2.36} = 11.02\ \Omega$$

Step IV Calculation of I_L (Fig. 2.293)

$$I_L = 2.36 \times \frac{11.02}{11.02 + 5} = 1.62 \text{ A}$$

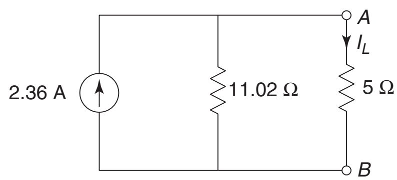


Fig. 2.293

Example 2.98 *Find the current through the 1 Ω resistor in the network of Fig. 2.294.*

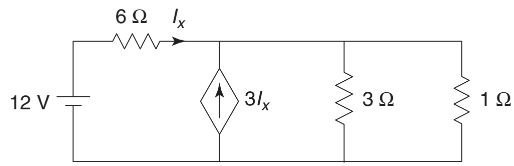


Fig. 2.294

Solution

Step I Calculation of V_{Th} (Fig. 2.295)

From Fig. 2.295,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

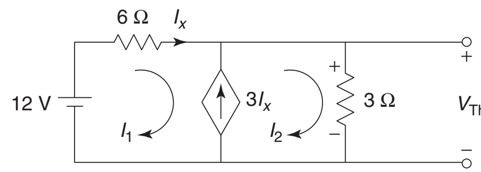


Fig. 2.295

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 - 3I_2 = 0$$

$$6I_1 + 3I_2 = 12 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.67\text{ A}$$

$$I_2 = 2.67\text{ A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{Th} = 0$$

$$3(2.67) - V_{Th} = 0$$

$$V_{Th} = 8\text{ V}$$

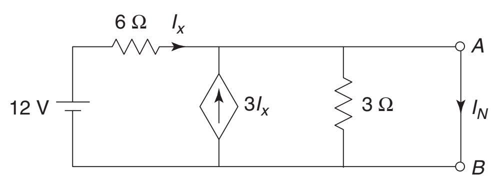


Fig. 2.296

Step II Calculation of I_N (Fig. 2.296)

When a short circuit is placed across a 3 Ω resistor, it gets shorted as shown in Fig. 2.297.

From Fig. 2.297,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

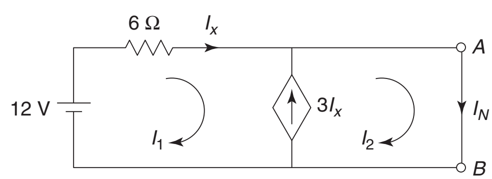


Fig. 2.297

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 = 0$$

$$I_1 = 2 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 8 \text{ A}$$
$$I_N = I_2 = 8 \text{ A}$$

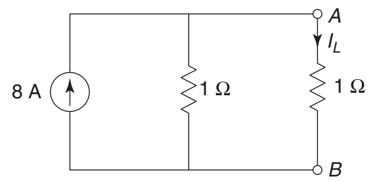


Fig. 2.298

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8}{8} = 1\,\Omega$$

Step IV Calculation of I_L (Fig. 2.298)

$$I_L = 8 \times \frac{1}{1+1} = 4 \text{ A}$$

Example 2.99 *Find the current through the 1.6 Ω resistor in the network of Fig. 2.299.*

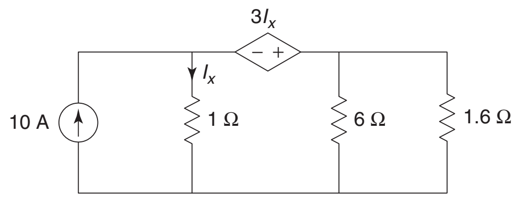


Fig. 2.299

Solution

Step I Calculation of V_{Th} (Fig. 2.300)

From Fig. 2.300,

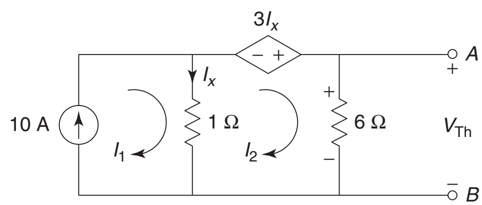


Fig. 2.300

$$I_x = I_1 - I_2 \quad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x - 6I_2 = 0$$
$$-I_2 + I_1 + 3(I_1 - I_2) - 6I_2 = 0$$
$$4I_1 - 10I_2 = 0 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 4 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - V_{\text{Th}} = 0$$
$$6(4) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 24 \text{ V}$$

Step II Calculation of I_N (Fig. 2.301)

When a short circuit is placed across the 3 Ω resistor, it gets shorted as shown in Fig. 2.302.

From Fig. 2.302,

$$I_x = I_1 - I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \qquad \ldots\text{(ii)}$$

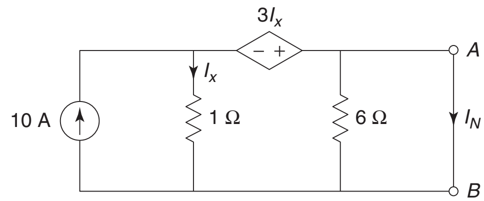


Fig. 2.301

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x = 0$$
$$-I_2 + I_1 + 3(I_1 - I_2) = 0$$
$$4I_1 - 4I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 10 \text{ A}$$
$$I_N = I_2 = 10 \text{ A}$$

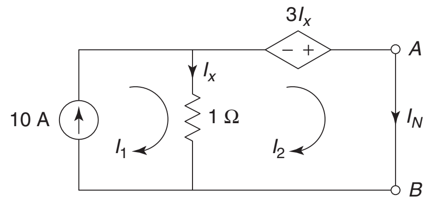


Fig. 2.302

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{24}{10} = 2.4\ \Omega$$

Step IV Calculation of I_L (Fig. 2.303)

$$I_L = 10 \times \frac{2.4}{2.4 + 1.6} = 6 \text{ A}$$

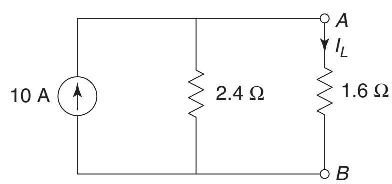


Fig. 2.303

2.10 MAXIMUM POWER TRANSFER THEOREM

It states that '*the maximum power is delivered from a source to a load when the load resistance is equal to the source resistance.*'

Proof From Fig. 2.304,

$$I = \frac{V}{R_s + R_L}$$

Power delivered to the load $R_L = P = I^2\ R_L = \dfrac{V^2 R_L}{(R_s + R_L)^2}$

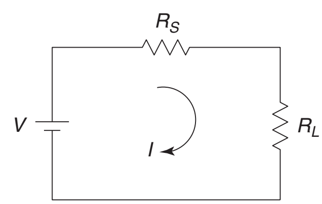


Fig. 2.304 *Network illustrating maximum power transfer theorem*

To determine the value of R_L for maximum power to be transferred to the load,

$$\frac{dP}{dR_L} = 0$$

$$\frac{dP}{dR_L} = \frac{d}{dR_L} \frac{V^2}{(R_s + R_L)^2} R_L$$

$$= \frac{V^2[(R_s + R_L)^2 - (2R_L)(R_s + R_L)]}{(R_s + R_L)^4}$$

$$(R_s + R_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s^{\,2} + R_L^{\,2} + 2R_sR_L - 2R_LR_s - 2R_L^2 = 0$$

$$R_s = R_L$$

Hence, the maximum power will be transferred to the load when load resistance is equal to the source resistance.

Steps to be followed in Maximum Power Transfer Theorem

1. Remove the variable load resistor R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Find the resistance R_L for maximum power transfer.

$$R_L = R_{\text{Th}}$$

5. Find the maximum power (Fig. 2.305).

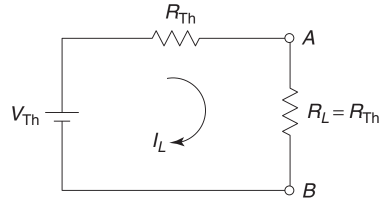


Fig. 2.305 *Thevenin's equivalent network*

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L} = \frac{V_{\text{Th}}}{2R_{\text{Th}}}$$

$$P_{\max} = I_L^2\,R_L = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}^2} \times R_{\text{Th}} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}}$$

Example 2.100 *For the value of resistance R_L in Fig. 2.306 for maximum power transfer and calculate the maximum power.*

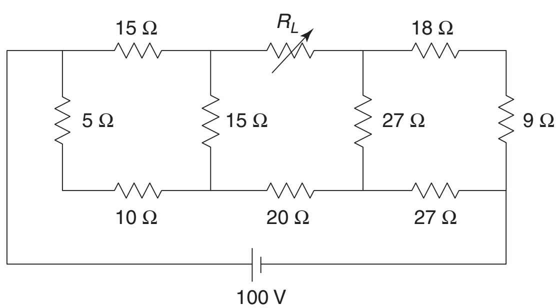


Fig. 2.306

Solution

Step I Calculation of V_{Th} (Fig. 2.307)

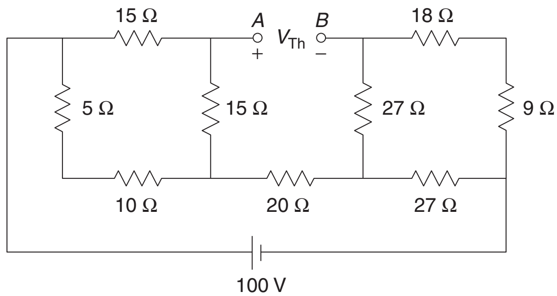


Fig. 2.307

By star-delta transformation (Fig. 2.308),

$$I = \frac{100}{5+5+20+9+9} = 2.08 \text{ A}$$

Writing the V_{Th} equation,

$$100 - 5I - V_{Th} - 9I = 0$$

$$V_{Th} = 100 - 14I$$

$$= 100 - 14(2.08)$$

$$= 70.88 \text{ V}$$

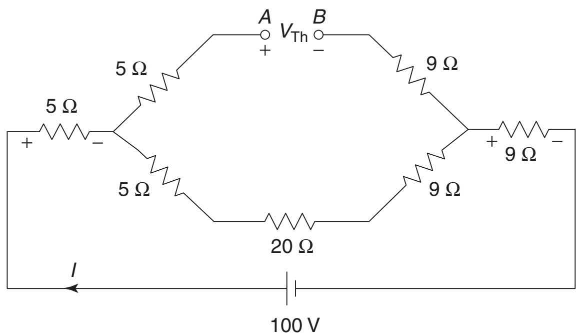


Fig. 2.308

Step II Calculation of R_{Th} (Fig. 2.309)

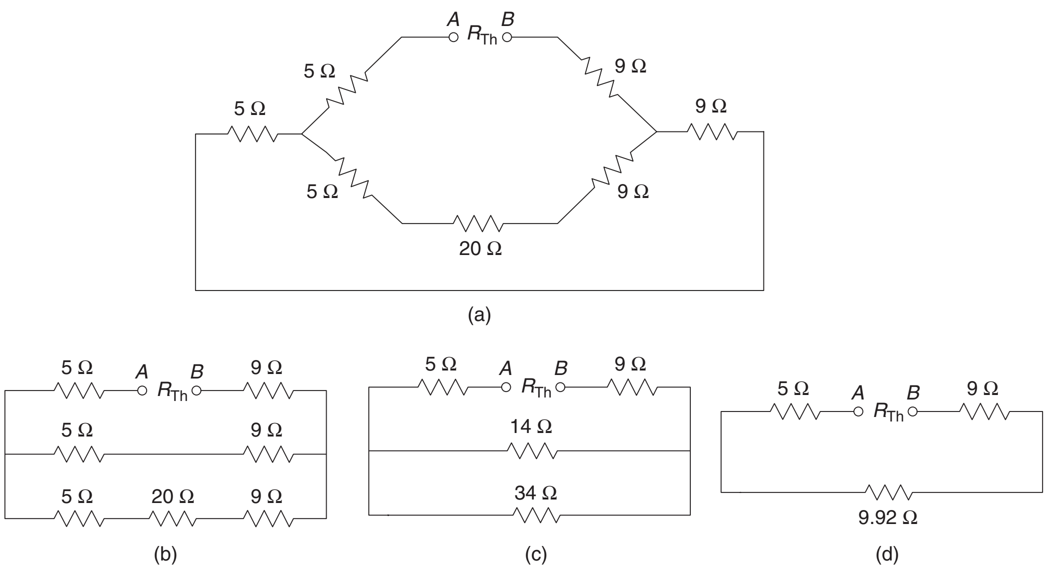


Fig. 2.309

$$R_{Th} = 23.92\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 23.92\ \Omega$$

Step IV Calculation of P_{max} (Fig. 2.310)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(70.88)^2}{4 \times 23.92} = 52.51\ \text{W}$$

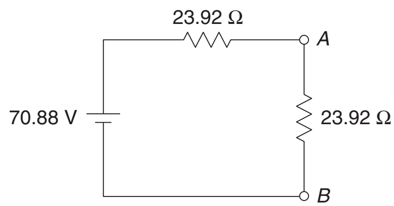


Fig. 2.310

Example 2.101 *For the value of resistance R_L in Fig. 2.311 for maximum power transfer and calculate the maximum power.*

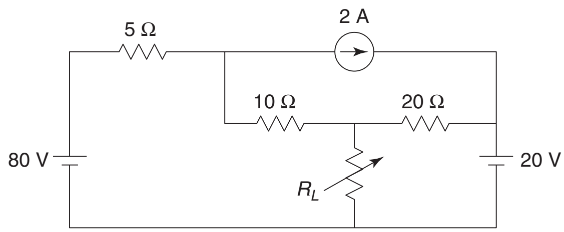


Fig. 2.311

Solution

Step I Calculation of V_{Th} (Fig. 2.312)

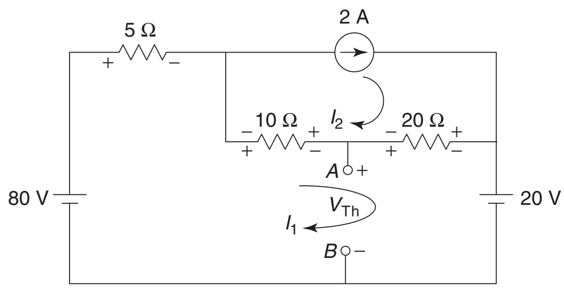


Fig. 2.312

Applying KVL to Mesh 1,

$$80 - 5I_1 - 10(I_1 - I_2) - 20(I_1 - I_2) - 20 = 0$$

$$35I_1 - 30I_2 = 60 \qquad \text{...(i)}$$

Writing the current equation for Mesh 2,

$$I_2 = 2 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 3.43 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 20\,(I_1 - I_2) - 20 = 0$$

$$V_{Th} = 20(3.43 - 2) + 20 = 48.6 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 2.313)

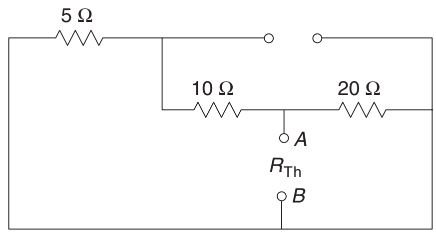


Fig. 2.313

$$R_{Th} = 15 \,||\, 20 = 8.57\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 8.57\ \Omega$$

Step IV Calculation of P_{max} (Fig. 2.314)

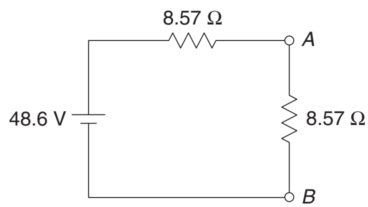


Fig. 2.314

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(48.6)^2}{4 \times 8.57} = 68.9 \text{ W}$$

Example 2.102 *For the value of resistance R_L in Fig. 2.315 for maximum power transfer and calculate the maximum power.*

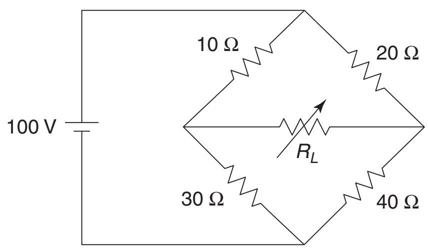


Fig. 2.315

Solution

Step I Calculation of V_{Th} (Fig. 2.316)

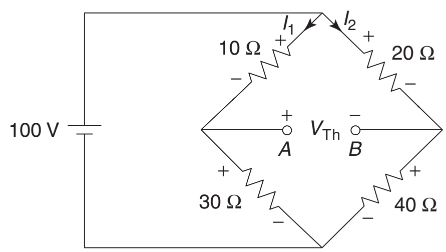


Fig. 2.316

$$I_1 = \frac{100}{10 + 30} = 2.5 \text{ A}$$

$$I_2 = \frac{100}{20 + 40} = 1.66 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} + 10\,I_1 - 20I_2 = 0$$

$$V_{Th} = 20I_2 - 10I_1 = 20(1.66) - 10(2.5) = 8.2 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 2.317)

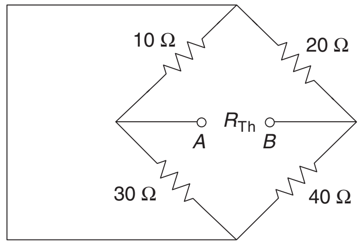


Fig. 2.317

Redrawing the network (Fig. 2.318),

$$R_{Th} = (10 \,||\, 30) + (20 \,||\, 40) = 20.83\ \Omega$$

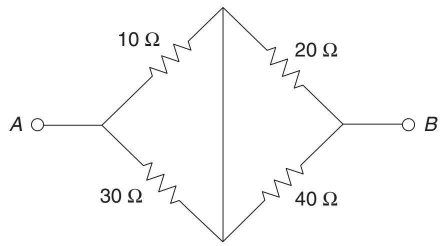


Fig. 2.318

Step III Value of R_L

For maximum power transfer,

$$R_L = R_{Th} = 20.83\ \Omega$$

Step IV Calculation of P_{max} (Fig. 2.319)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(8.2)^2}{4 \times 20.83} = 0.81\ \text{W}$$

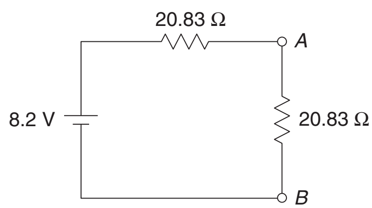


Fig. 2.319

Example 2.103 *For the value of resistance R_L in Fig. 2.320 for maximum power transfer and calculate the maximum power.*

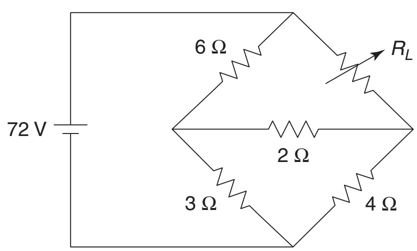


Fig. 2.320

Solution

Step I Calculation of V_{Th} (Fig. 2.321)

Applying KVL to Mesh 1,

$$72 - 6I_1 - 3(I_1 - I_2) = 0$$

$$9I_1 - 3I_2 = 72 \quad \ldots\text{(i)}$$

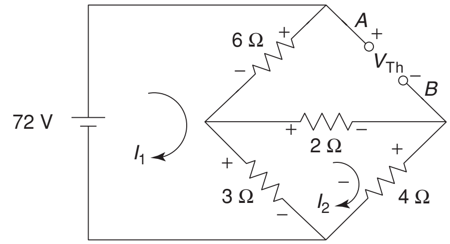


Fig. 2.321

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 4I_2 = 0$$

$$-3I_1 + 9I_2 = 0 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 9\text{ A}$$

$$I_2 = 3\text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 6I_1 - 2I_2 = 0$$

$$V_{Th} = 6I_1 + 2I_2 = 6(9) + 2(3) = 60\text{ V}$$

Step II Calculation of R_{Th} (Fig. 2.322)

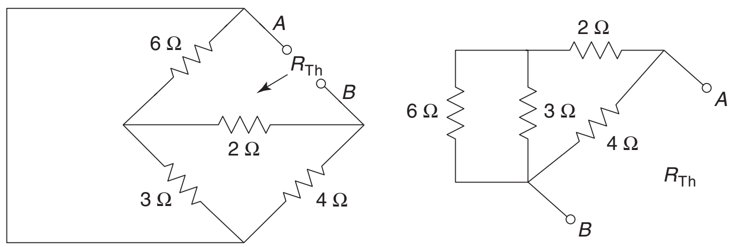


Fig. 2.322

$$R_{Th} = [(6 \,\|\, 3) + 2] \,\|\, 4 = 2\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 2\ \Omega$$

Step IV Calculation of P_{max} (Fig. 2.323)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(60)^2}{4 \times 2} = 450\text{ W}$$

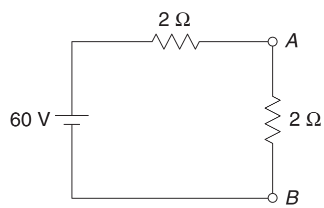


Fig. 2.323

EXAMPLES WITH DEPENDENT SOURCES

Example 2.104 *For the network shown in Fig. 2.324, find the value of R_L for maximum power transfer. Also, calculate maximum power.*

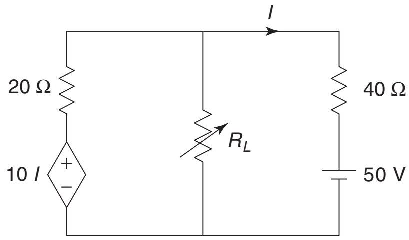


Fig. 2.324

Solution

Step I Calculation of V_{Th} (Fig. 2.325)

Applying KVL to the mesh,

$$10I - 20I - 40I - 50 = 0$$
$$I = -1 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 40I - 50 = 0$$
$$V_{Th} - 40(-1) - 50 = 0$$
$$V_{Th} = 10 \text{ V}$$

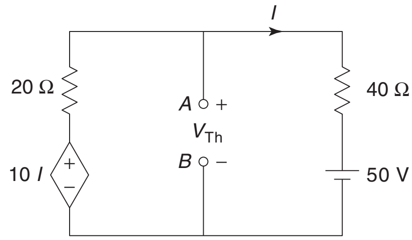


Fig. 2.325

Step II Calculation of I_N (Fig. 2.326)

From Fig. 2.326,

$$I = I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$10I - 20I_1 = 0$$

$$10I_2 - 20I_1 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-40I_2 - 50 = 0$$
$$I_2 = -1.25 \text{ A} \qquad \ldots\text{(iii)}$$

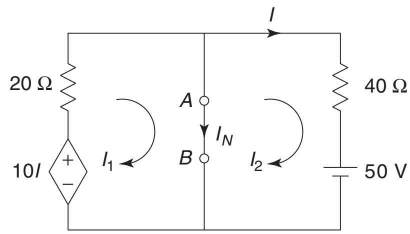


Fig. 2.326

Solving Eqs (i), (ii) and (iii),

$$I_1 = -0.625 \text{ A}$$
$$I_N = I_1 - I_2 = -0.625 + 1.25 = 0.625 \text{ A}$$

Step III Calculation of R_N

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{10}{0.625} = 16\ \Omega$$

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 16\ \Omega$$

Step V Calculation of P_{max} (Fig. 2.327)

$$P_{max} = \frac{V_{Th}}{4R_{Th}} = \frac{(10)^2}{4 \times 16} = 1.56 \text{ W}$$

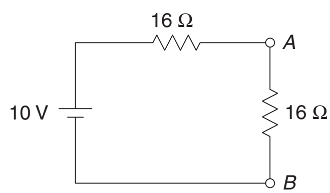


Fig. 2.327

Example 2.105 *For the network shown in Fig. 2.328, calculate the maximum power that may be dissipated in the load resistor R_L.*

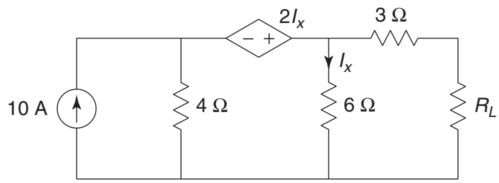


Fig. 2.328

Solution

Step I Calculation of V_{Th} (Fig. 2.329)

From Fig. 2.329,

$$I_x = I_2 \quad \ldots(i)$$

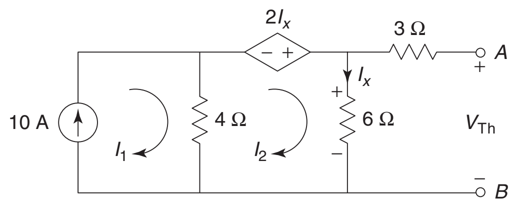


Fig. 2.329

For Mesh 1,

$$I_1 = 10 \quad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6I_2 = 0$$
$$-4I_2 + 4I_1 + 2I_2 - 6I_2 = 0$$
$$4I_1 - 8I_2 = 0 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 5 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - 0 - V_{Th} = 0$$
$$V_{Th} = 6I_2 = 6(5) = 30 \text{ V}$$

Step II Calculation of I_N (Fig. 2.330)

From Fig. 2.330,

$$I_x = I_2 - I_3 \quad \ldots(i)$$

For Mesh 1,

$$I_1 = 10 \quad \ldots(ii)$$

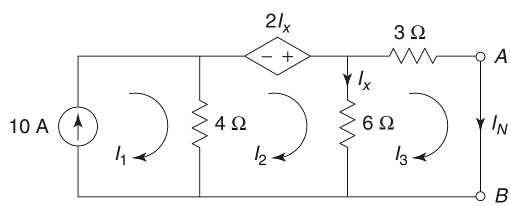


Fig. 2.330

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6(I_2 - I_3) = 0$$
$$-4I_2 + 4I_1 + 2(I_2 - I_3) - 6I_2 + 6I_3 = 0$$
$$4I_1 - 8I_2 + 4I_3 = 0 \quad \ldots(iii)$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 3I_3 = 0$$

$$6I_2 - 9I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 10 \text{ A}$$

$$I_2 = 7.5 \text{ A}$$

$$I_3 = 5 \text{ A}$$

$$I_N = I_3 = 5 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{30}{5} = 6\ \Omega$$

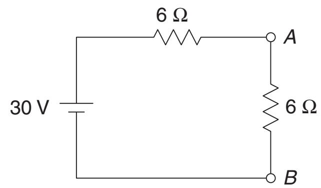


Fig. 2.331

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{\text{Th}} = 6\ \Omega$$

Step V Calculation of $P_{\max}$ (Fig. 2.331)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(30)^2}{4 \times 16} = 37.5 \text{ W}$$

Example 2.106 *For the network shown in Fig. 2.332, find the value of R_L for maximum power transfer. Also, find maximum power.*

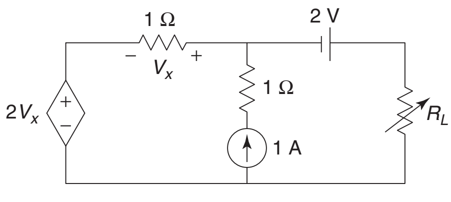


Fig. 2.332

Solution

Step I Calculation of V_{Th} (Fig. 2.333)

From Fig. 2.333,

$$V_x = -1I = -I \qquad \text{...(i)}$$

For Mesh 1,

$$I = -1 \qquad \text{...(ii)}$$

$$V_x = 1 \text{ V}$$

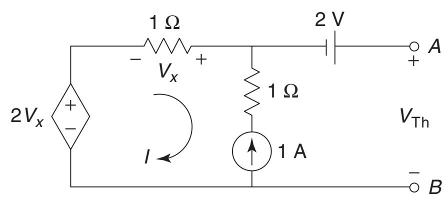


Fig. 2.333

Writing the V_{Th} equation,

$$2V_x - 1I + 2 - V_{Th} = 0$$
$$2(1) - (-1) + 2 - V_{Th} = 0$$
$$V_{Th} = 5 \text{ V}$$

Step II Calculation of I_N (Fig. 2.334)

From Fig. 2.334,

$$V_x = -1I_1 = -I_1 \qquad \ldots\text{(i)}$$

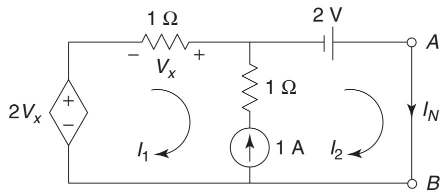


Fig. 2.334

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$2V_x - 1I_1 + 2 = 0$$
$$2(-I_1) - I_1 + 2 = 0$$
$$3I_1 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.67 \text{ A}$$
$$I_2 = 1.67 \text{ A}$$
$$I_N = I_2 = 1.67 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{5}{1.67} = 3\ \Omega$$

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3\ \Omega$$

Step V Calculation of P_{max} (Fig. 2.335)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(5)^2}{4 \times 3} = 2.08 \text{ W}$$

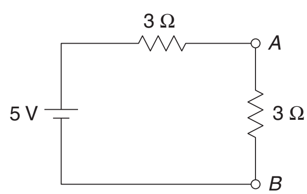


Fig. 2.335

Example 2.107 *What will be the value of R_L in Fig. 2.336 to get maximum power delivered to it? What is the value of this power?*

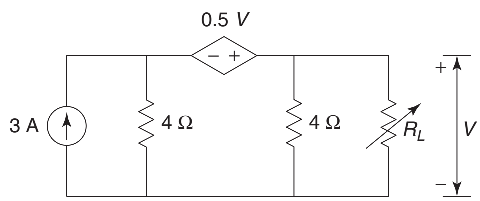


Fig. 2.336

Solution

Step I Calculation of V_{Th} (Fig. 2.337)
By source transformation,
From Fig. 2.337,

$$V_{Th} = 4I$$

Applying KVL to the mesh,

$$12 - 4I + 0.5\,V_{Th} - 4I = 0$$

$$12 - V_{Th} + 0.5\,V_{Th} - V_{Th} = 0$$

$$V_{Th} = 8\text{ V}$$

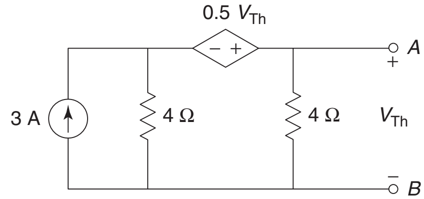


Fig. 2.337

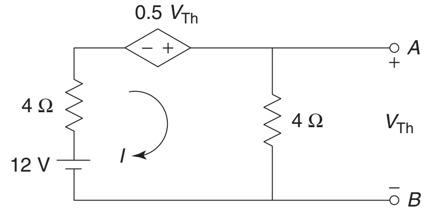


Fig. 2.338

Step II Calculation of I_N (Fig. 2.338)

If two terminals A and B are shorted, the 4 Ω resistor gets shorted.

$$V = 0$$

Dependent source 0.5 V depends on the controlling variable V. When $V = 0$, the dependent source vanishes, i.e. 0.5 $V = 0$ as shown in Fig. 2.339 and Fig. 2.340.

$$I_N = \frac{12}{4} = 3\text{ A}$$

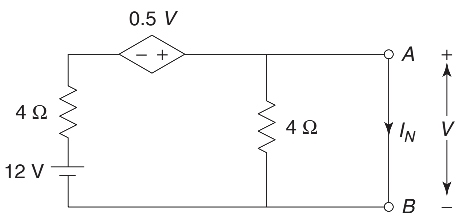


Fig. 2.339

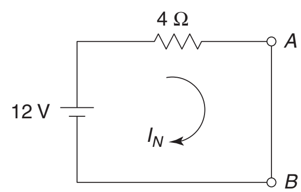


Fig. 2.340

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{8}{3} = 2.67\ \Omega$$

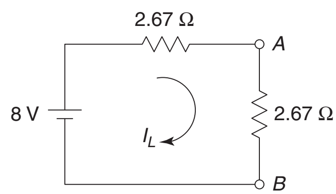


Fig. 2.341

Step IV *Calculation of* R_L
For maximum power transfer,

$$R_L = R_{Th} = 2.67\ \Omega$$

Step V Calculation of P_{max} (Fig. 2.341)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(8)^2}{4 \times 2.67} = 6\text{ W}$$

2.11 RECIPROCITY THEOREM

It states that '*in a linear, bilateral, active, single source network, the ratio of excitation to response remains same when the positions of excitation and response are interchanged*.'

In other words, it may be stated as 'if a single voltage source V_a in the branch '*a*' produces a current I_b in the branch '*b*' then if the voltage source V_a is removed and inserted in the branch '*b*', it will produce a current I_b in branch '*a*''.

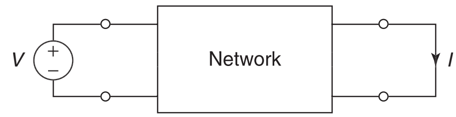


Fig. 2.342 *Network*

Fig. 2.343 *Network when excitation and response are interchanged*

Explanation Consider a network shown in Fig. 2.342.

When the voltage source V is applied at the port 1, it produces a current I at the port 2. If the positions of the excitation (source) and response are interchanged, i.e., if the voltage source is applied at the port 2 then it produces a current I at the port 1 (Fig. 2.343).

The limitation of this theorem is that it is applicable only to a single-source network. This theorem is not applicable in the network which has a dependent source. This is applicable only in linear and bilateral networks. In the reciprocity theorem, position of any passive element (R, L, C) do not change. Only the excitation and response are interchanged.

Steps to be followed in Reciprocity Theorem

1. Identify the branches between which reciprocity is to be established.
2. Find the current in the branch when excitation and response are not interchanged.
3. Find the current in the branch when excitation and response are interchanged.

Example 2.108 *Calculate current I and verify the reciprocity theorem for the network shown in Fig. 2.344.*

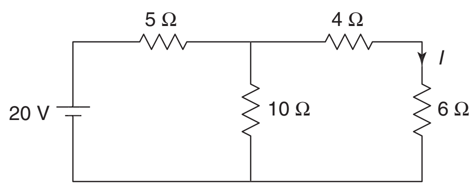


Fig. 2.344

Solution

Case I Calculation of current I when excitation and response are not interchanged (Fig. 2.345)

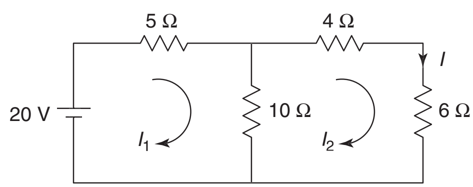


Fig. 2.345

Applying KVL to Mesh 1,

$$20 - 5I_1 - 10(I_1 - I_2) = 0$$

$$15I_1 - 10I_2 = 20 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) - 4I_2 - 6I_2 = 0$$

$$-10I_1 + 20I_2 = 0 \qquad \text{...(ii)}$$

Solving. Eqs (i) and (ii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 1 \text{ A}$$
$$I = I_2 = 1 \text{ A}$$

Case II Calculation of current I when excitation and response are interchanged (Fig. 2.346).

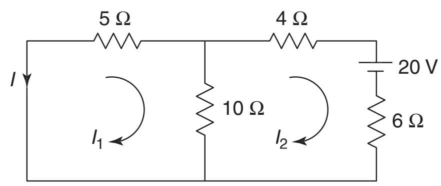


Fig. 2.346

Applying KVL to Mesh 1,

$$-5I_1 - 10(I_1 - I_2) = 0$$
$$15I_1 - 10I_2 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) - 4I_2 - 20 - 6I_2 = 0$$
$$-10I_1 + 20I_2 = -20 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = -1 \text{ A}$$
$$I_2 = -1.5 \text{ A}$$
$$I = -I_1 = 1 \text{ A}$$

Since the current I remains the same in both the cases, reciprocity theorem is verified.

Example 2.109 *Find the current I and verify reciprocity theorem for the network shown in Fig. 2.347.*

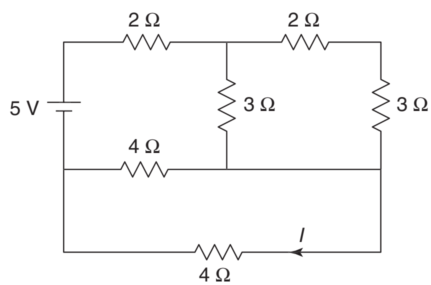


Fig. 2.347

Solution

Case I Calculation of the current I when excitation and response are not interchanged (Fig. 2.348)

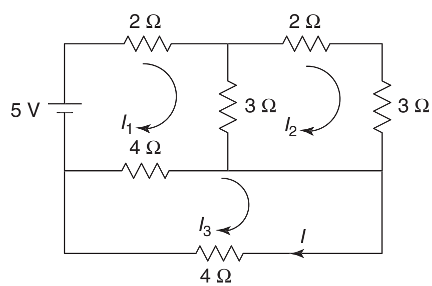


Fig. 2.348

Applying KVL to Mesh 1,

$$5 - 2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$
$$9I_1 - 3I_2 - 4I_3 = 5 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$
$$-3I_1 + 8I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4(I_3 - I_1) - 4I_3 = 0$$

$$-4I_1 + 8I_3 = 0 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.85 \text{ A}$$
$$I_2 = 0.32 \text{ A}$$
$$I_3 = 0.43 \text{ A}$$
$$I = I_3 = 0.43 \text{ A}$$

Case II Calculation of current I when excitation and response are interchanged (Fig. 2.349).

Applying KVL to Mesh 1,

$$-2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 4I_3 = 0 \qquad \text{...(i)}$$

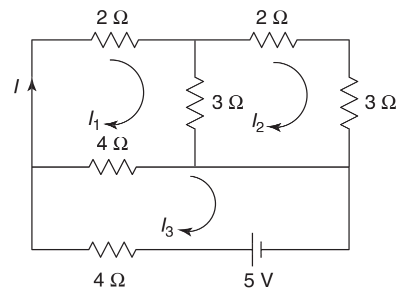


Fig. 2.349

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$

$$-3I_1 + 8I_2 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-4(I_3 - I_1) + 5 - 4I_3 = 0$$

$$-4I_1 + 8I_3 = 5 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.43 \text{ A}$$
$$I_2 = 0.16 \text{ A}$$
$$I_3 = 0.84 \text{ A}$$
$$I = I_1 = 0.43 \text{ A}$$

Since the current I remains the same in both the cases, reciprocity theorem is verified.

Example 2.110 *Find the voltage V and verify reciprocity theorem for the network shown in Fig. 2.350.*

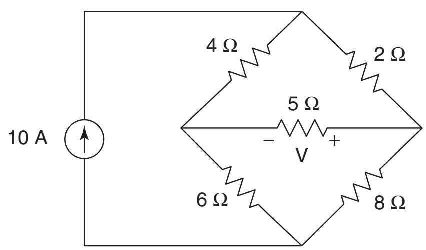


Fig. 2.350

Solution

Case I Calculation of the voltage V when excitation and response are not interchanged (Fig. 2.351)

For Mesh 1,

$$I_1 = 10 \qquad \ldots(\text{i})$$

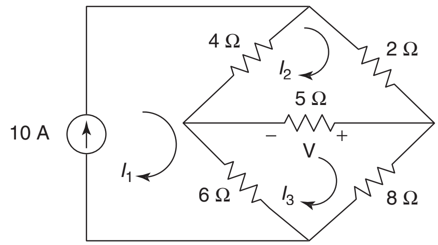


Fig. 2.351

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) - 2I_2 - 5(I_2 - I_3) = 0$$
$$-4I_1 + 11I_2 - 5I_3 = 0 \qquad \ldots(\text{ii})$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) - 5(I_3 - I_2) - 8I_3 = 0$$
$$-6I_1 - 5I_2 + 19I_3 = 0 \qquad \ldots(\text{iii})$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 5.76 \text{ A}$$
$$I_3 = 4.67 \text{ A}$$
$$V = 5(I_2 - I_3) = 5(5.76 - 4.67) = 5.45 \text{ V}$$

Case II Calculation of voltage V when excitation and response are interchanged (Fig. 2.352).

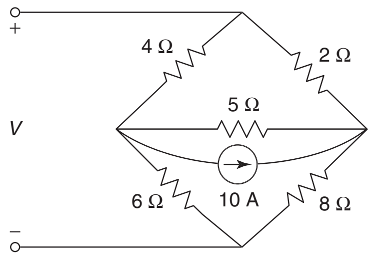


Fig. 2.352

By source transformation (Fig. 2.353),

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 50 - 5(I_1 - I_2) = 0$$
$$11I_1 - 5I_2 = -50 \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-6I_2 - 5(I_2 - I_1) + 50 - 8I_2 = 0$$
$$-5I_1 + 19I_2 = 50 \qquad \ldots(\text{ii})$$

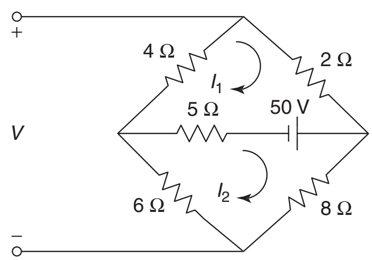


Fig. 2.353

Solving Eqs (i) and (ii),

$$I_1 = -3.8 \text{ A}$$
$$I_2 = 1.63 \text{ A}$$

From Fig. 2.353,

$$V + 4I_1 + 6I_2 = 0$$

$$V + 4(-3.8) + 6(1.63) = 0$$

$$V = 5.42 \text{ V}$$

Since the voltage V is same in both the cases, the reciprocity theorem is verified.

2.12 MILLMAN'S THEOREM

It states that '*if there are n voltage sources* $V_1, V_2, \ldots, V_n$ *with internal resistances* $R_1, R_2, \ldots, R_n$ *respectively connected in parallel then these voltage sources can be replaced by a single voltage source* V_m *and a single series resistance* R_m, '(Fig. 2.354).

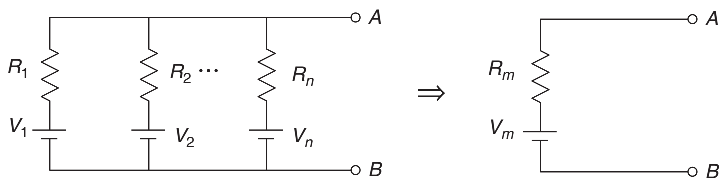


Fig. 2.354 *Millman's network*

where
$$V_m = \frac{V_1G_1 + V_2G_2 + \ldots + V_nG_n}{G_1 + G_2 + \ldots + G_n}$$

and
$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

Explanation By source transformation, each voltage source in series with a resistance can be converted to a current source in parallel with a resistance as shown in Fig. 2.355.

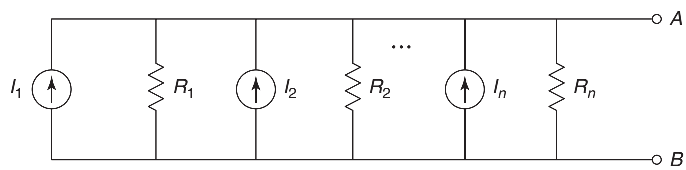


Fig. 2.355 *Equivalent network*

Let I_m be the resultant current of the parallel current sources and R_m be the equivalent resistance as shown in Fig. 2.356.

$$I_m = I_1 + I_2 + \ldots + I_n = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \ldots + \frac{V_n}{R_n} = V_1\, G_1 + V_2\, G_2 + \ldots V_n\, G_n$$

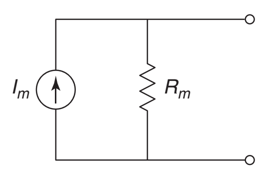


Fig. 2.356 *Equivalent network*

$$\frac{1}{R_m} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$
$$G_m = G_1 + G_2 + \ldots + G_n$$

By source transformation, the parallel circuit can be converted into a series circuit as shown in Fig. 2.357.

$$V_m = I_m R_m = \frac{I_m}{G_m} = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

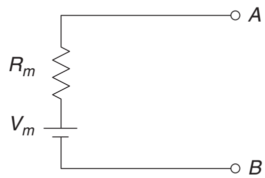


Fig. 2.357 *Millman's equivalent network*

Dual of Millman's Theorem

It states that '*if there are n current sources $I_1, I_2, \ldots, I_n$ with internal resistances $R_1, R_2, \ldots, R_n$ respectively, connected in series then these current sources can be replaced by a single current source I_m and a single parallel resistance R_m*' (Fig. 2.358).

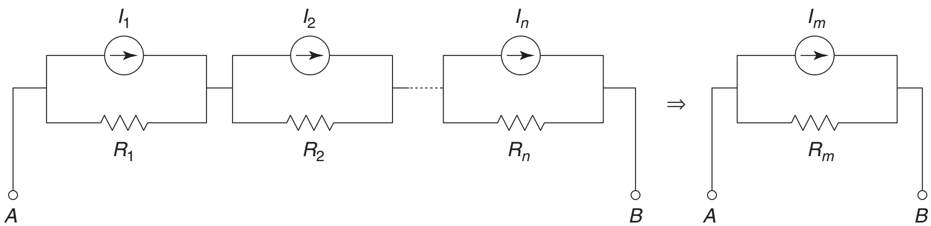


Fig. 2.358 *Millman's network*

where
$$I_m = \frac{I_1 R_1 + I_2 R_2 + \ldots + I_n R_n}{R_1 + R_2 + \ldots + R_n}$$
$$R_m = R_1 + R_2 + \ldots + R_n$$

Steps to be followed in Millman's Theorem

1. Remove the load resistance R_L.
2. Find Millman's voltage across points *A* and *B*.

$$V_m = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

3. Find the resistance R_m between points *A* and *B*.

$$R_m = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

4. Replace the network by a voltage source V_m in series with the resistance R_m.
5. Find the current through R_L using ohm's law.

$$I_L = \frac{V_m}{R_m + R_L}$$

Example 2.111 *Find Millman's equivalent for the left of the terminals A-B in Fig. 2.359.*

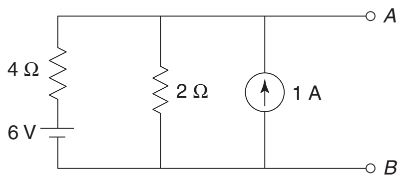


Fig. 2.359

Solution By source transformation, the network is redrawn as shown in Fig. 2.360.

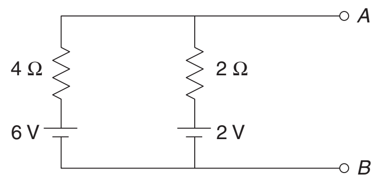


Fig. 2.360

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{6\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right)}{\frac{1}{4} + \frac{1}{2}} = 3.33 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{4} + \frac{1}{2}} = 1.33\ \Omega$$

Step III Millman's Equivalent Network (Fig. 2.361)

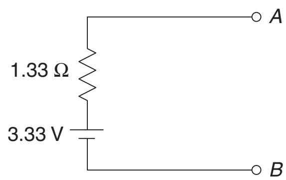


Fig. 2.361

Example 2.112 *Find the current through the 10 Ω resistor in the network of Fig. 2.362.*

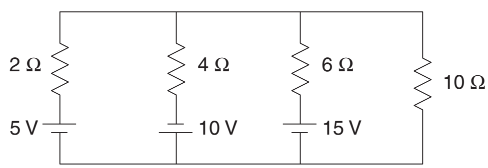


Fig. 2.362

Solution

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{5\left(\frac{1}{2}\right) - 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{6}\right)}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 2.73 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}} = 1.09\ \Omega$$

Step III Calculation of I_L (Fig. 2.363)

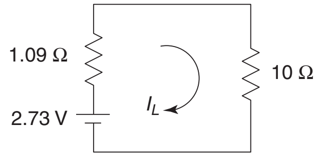


Fig. 2.363

$$I_L = \frac{2.73}{1.09+10} = 0.25\ \text{A}$$

Example 2.113 *Find the current through the 10 Ω resistor in the network of Fig. 2.364.*

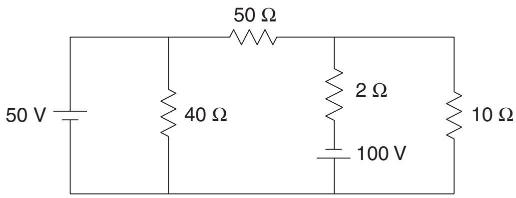


Fig. 2.364

Solution

Since the 40 Ω resistor is connected in parallel with the 50 V source, it becomes redundant. The network can be redrawn as shown in Fig. 2.365.

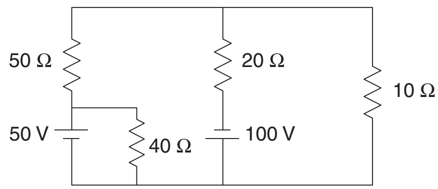


Fig. 2.365

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{50\left(\frac{1}{50}\right) - 100\left(\frac{1}{20}\right)}{\frac{1}{50}+\frac{1}{20}} = -57.15\ \text{V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{50}+\frac{1}{20}} = 14.29\ \Omega$$

Step III Calculation of I_L (Fig. 2.366)

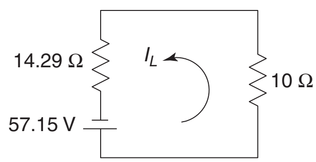


Fig. 2.366

$$I_L = \frac{57.15}{14.29+10} = 2.35\ \text{A}$$

Example 2.114 *Draw Millman's equivalent network across terminals AB in the network of Fig. 2.367.*

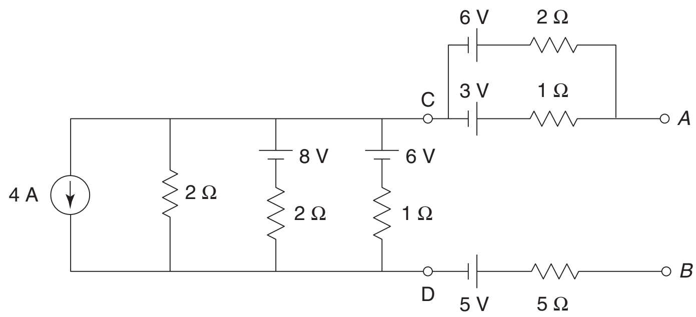


Fig. 2.367

Step I By source transformation, the network is redrawn as shown in Fig. 2.368.

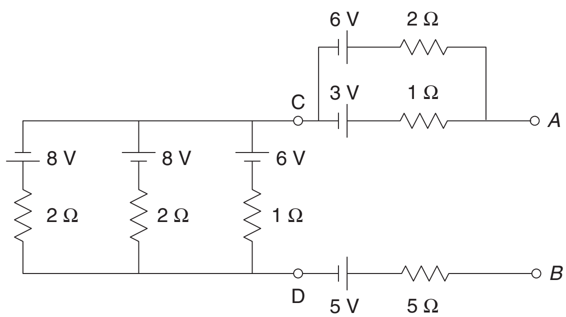


Fig. 2.368

Step II Applying Millman's theorem at terminals *CD,*

$$V_{m_1} = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{-8\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right) + 6(1)}{\frac{1}{2} + \frac{1}{2} + 1} = 3 \text{ V}$$

$$R_{m_1} = \frac{1}{G_{m_1}} = \frac{1}{G_1 + G_2 + G_3} = \frac{1}{\frac{1}{2} + \frac{1}{2} + 1} = 0.5 \ \Omega$$

Step III Applying Millman's theorem at terminals *CA*,

$$V_{m_2} = \frac{V_4 G_4 + V_5 G_5}{G_4 + G_5} = \frac{6\left(\frac{1}{2}\right) + 3(1)}{\frac{1}{2} + 1} = 4 \text{ V}$$

$$R_{m_2} = \frac{1}{G_{m_2}} = \frac{1}{G_4 + G_4} = \frac{1}{\frac{1}{2} + 1} = 0.67 \ \Omega$$

Step IV Millman's Equivalent Network (Fig. 2.369)
Simplifying Fig. 2.369 further, the Millman's equivalent network is shown in Fig. 2.370.

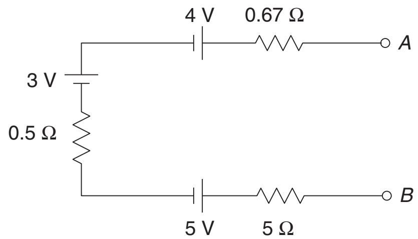


Fig. 2.369

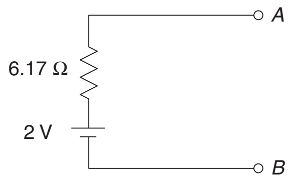


Fig. 2.370

Exercises

Mesh Analysis

2.1 Find currents I_x and I_y in the network shown in Fig. 2.371.

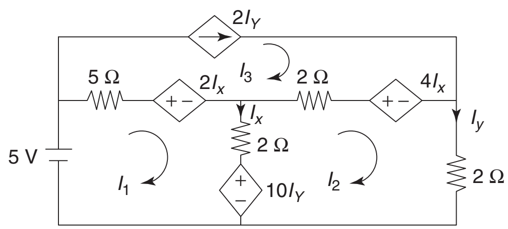


Fig. 2.371

[0.5 A, 0.1 A]

2.2 In the network shown in Fig. 2.372, find V_3 if element A is a

(i) short circuit
(ii) 5 Ω resistor
(iii) 20 V independent voltage source, positive reference on the right
(iv) dependent voltage source of 1.5 i_1, with positive reference on the right
(v) dependent current source 5 i_1, arrow directed to the right

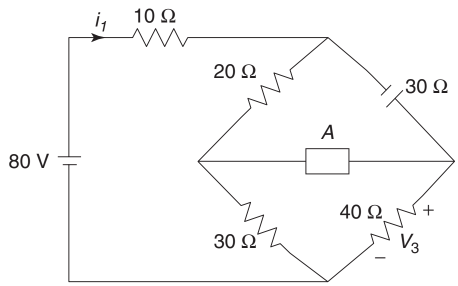


Fig. 2.372

[69.4 V, 72.38 V, 73.68 V, 70.71 V, 97.39 V]

2.3 Find currents I_1, I_2, and I_3 in the network shown in Fig. 2.373.

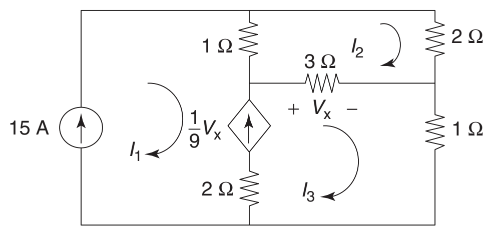


Fig. 2.373

[15 A, 11 A, 17 A]

2.4 Find currents I_x in the network shown in Fig. 2.374.

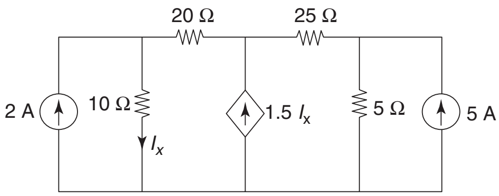


Fig. 2.374

[8.33 A]

2.5 Find currents I_1 in the network shown in Fig. 2.375.

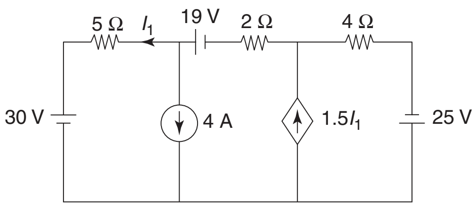


Fig. 2.375

[–12 A]

Node Analysis

2.6 Find the voltage V_x in the network shown in Fig. 2.376.

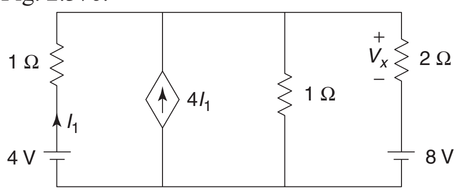


Fig. 2.376

[–4.31 V]

2.7 Find the currents V_x in the network shown in Fig. 2.377.

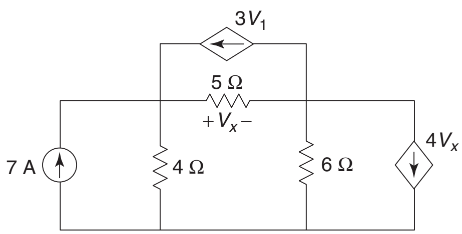


Fig. 2.377

[2.09 V]

2.8 Find the voltage V_x in the network shown in Fig. 2.378.

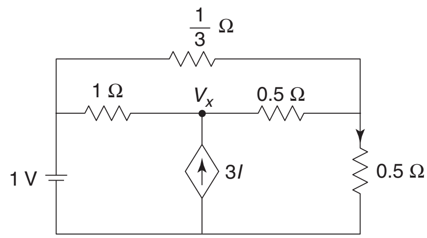


Fig. 2.378

[6.2 V]

2.9 Determine V_1 in the network shown in Fig. 2.379.

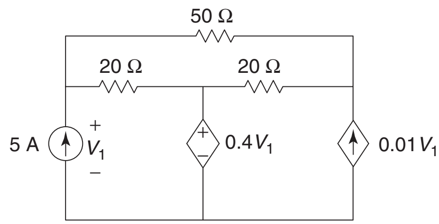


Fig. 2.379

[140 V]

2.10 Find the voltage V_y in the network shown in Fig. 2.380.

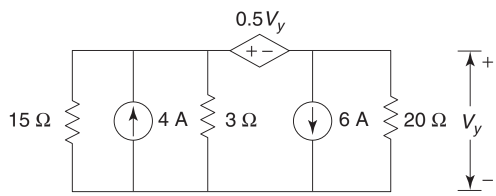


Fig. 2.380

[–10 V]

2.11 Find the voltage V_2 in the network shown in Fig. 2.381.

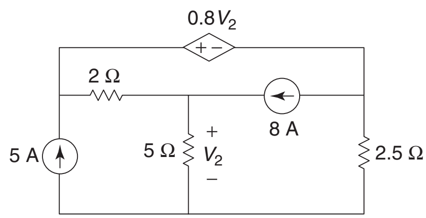


Fig. 2.381

[25.9 V]

Superposition Theorem

2.12 Find the voltage V_x in Fig. 2.382.

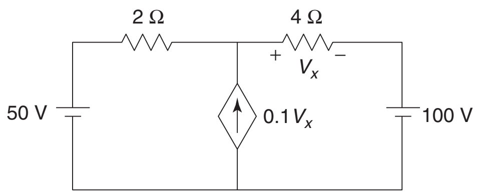


Fig. 2.382

$[-38.5 \text{ V}]$

2.13 Determine the voltages V_1 and V_2 in Fig. 2.383.

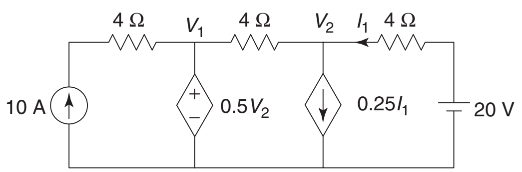


Fig. 2.383

$[6 \text{ V}, 12 \text{ V}]$

2.14 Find the voltage V_x in Fig. 2.384.

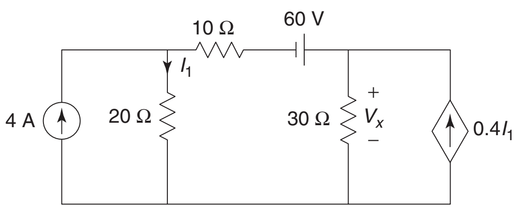


Fig. 2.384

$[7.5 \text{ V}]$

Thevenin's Theorem

2.15 Determine Thevenin's equivalent network for figures 2.385 to 2.388 shown below.

(i)

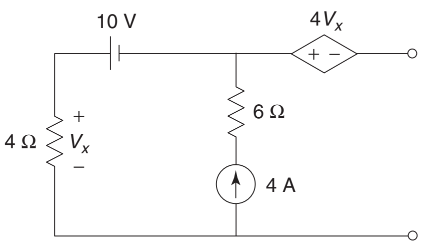


Fig. 2.385

$[-58 \text{ V}, 12\,\Omega]$

(ii)

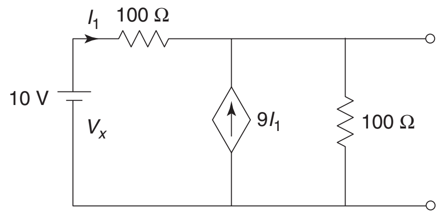


Fig. 2.386

$[9.09 \text{ V}, 9.09\,\Omega]$

(iii)

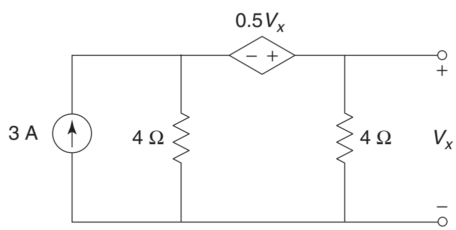


Fig. 2.387

$[8 \text{ V}, 2.66\,\Omega]$

(iv)

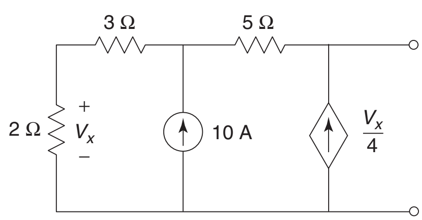


Fig. 2.388

$[150 \text{ V}, 20\,\Omega]$

2.16 Find the current I_x in Fig. 2.389.

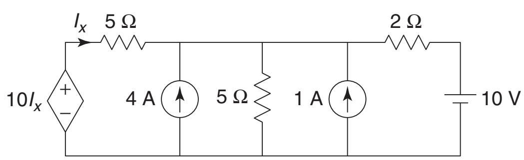


Fig. 2.389

$[4 \text{ A}]$

2.17 Find the current in the 24 Ω resistor in Fig. 3.390.

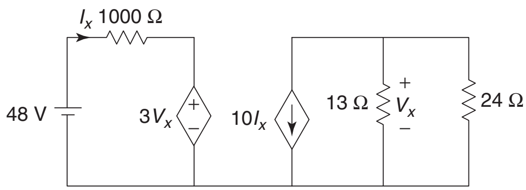


Fig. 3.390

[0.225 A]

Norton's Theorem

2.18 Find Norton's equivalent network and hence find the current in the 10 Ω resistor in Fig. 2.391.

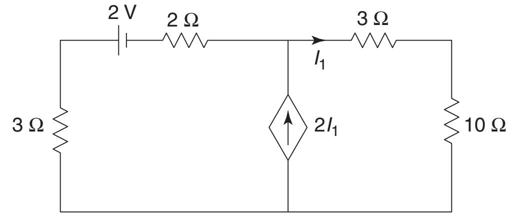


Fig. 2.391

[0.25 A]

2.19 Find Norton's equivalent network in Fig. 2.392.

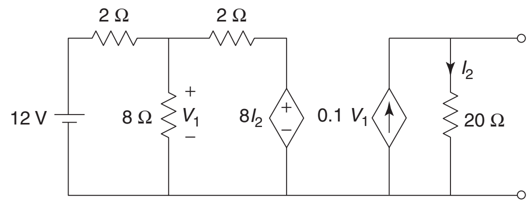


Fig. 2.392

[0.533 A, 31 Ω]

Objective-Type Questions

2.1 Two electrical sub-networks N_1 and N_2 are connected through three resistors as shown in Fig. 2.393. The voltages across the 5 Ω resistor and 1 Ω resistor are given to be 10 V and 5 V respectively. Then the voltage across the 15 Ω resistor is

(a) −105 V (b) 105 V
(c) −15 V (d) 15 V

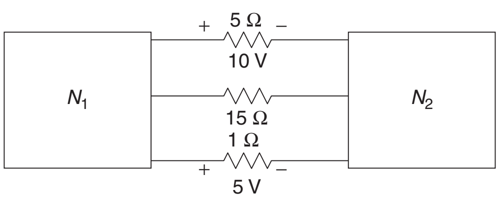


Fig. 2.393

2.2 The nodal method of circuit analysis is based on

(a) KVL and Ohm's law
(b) KCL and Ohm's law
(c) KCL and KVL
(d) KCL, KVL and Ohm's law

2.3 The voltage across terminals *a* and *b* in Fig. 2.394 is

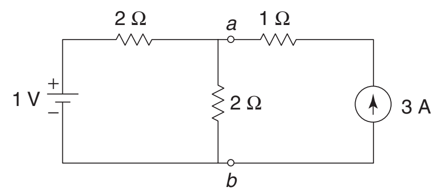


Fig. 2.394

(a) (a) 0.5 V (b) 3 V
(c) 3.5 V (d) 4 V

2.4 The voltage V_0 in Fig. 2.395 is

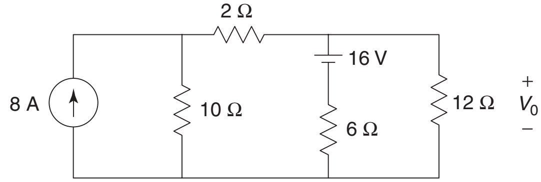


Fig. 2.395

(a) 48 V (b) 24 V
(c) 36 V (d) 28 V

2.5 The dependent current source shown in Fig. 2.396.

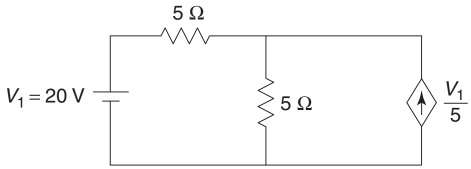


Fig. 2.396

(a) delivers 80 W (b) absorbs 80 W
(c) delivers 40 W (d) absorbs 40 W

2.6 If $V = 4$ in Fig. 2.397, the value of I_s is given by

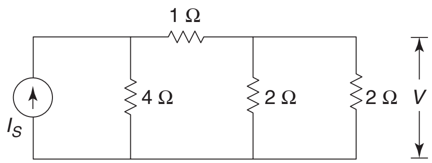


Fig. 2.397

(a) 6 A (b) 2.5 A
(c) 12 A (d) none of these

2.7 The value of V_x, V_y and V_z in Fig. 2.398 shown are

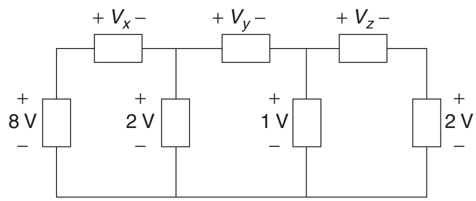


Fig. 2.398

(a) −6, 3, −3 (b) −6, −3, 1
(c) 6, 3, 3 (d) 6, 1, 3

2.8 The circuit shown in Fig. 2.399 is equivalent to a load of

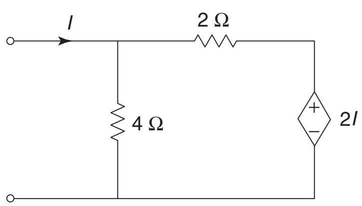


Fig. 2.399

(a) $\frac{4}{3}\,\Omega$ (b) $\frac{8}{3}\,\Omega$
(c) 4 Ω (d) 2 Ω

2.9 In the network shown in Fig. 2.400, the effective resistance faced by the voltage source is

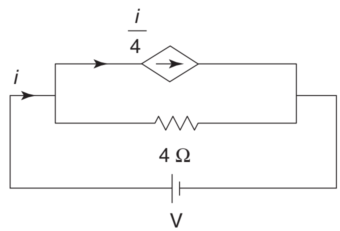


Fig. 2.400

(a) 4 Ω (b) 3 Ω
(c) 2 Ω (d) 1 Ω

2.10 A network contains only an independent current source and resistors. If the values of all resistors are doubled, the value of the node voltages will
(a) become half
(b) remain unchanged
(c) become double
(d) none of these

2.11 The value of the resistance *R* connected across the terminals *A* and *B* in Fig. 2.401, which will absorb the maximum power is

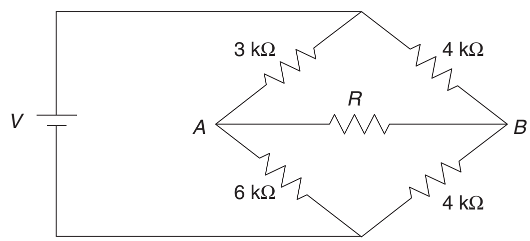


Fig. 2.401

(a) 4 kΩ (b) 4.11 kΩ
(c) 8 kΩ (d) 9 k Ω

2.12 Superposition theorem is not applicable to networks containing
(a) nonlinear elements
(b) dependent voltage source

(c) dependent current source
(d) transformers

2.13 The value of R required for maximum power transfer in the network shown in Fig. 2.402 is

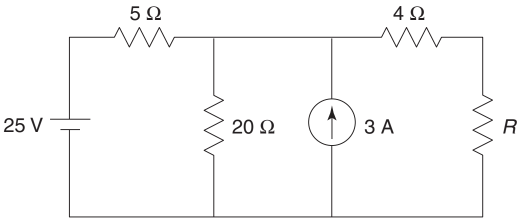


Fig. 2.402

(a) 2 Ω (b) 4 Ω
(c) 8 Ω (d) 16 Ω

2.14 In the network of Fig. 2.403, the maximum power is delivered to R_L if its value is

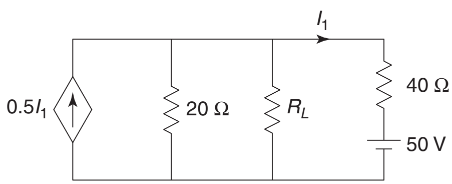


Fig. 2.403

(a) 16 Ω (b) $\frac{40}{3}$ Ω
(c) 60 Ω (d) 20 Ω

2.15 The maximum power that can be transferred to the load R_L from the voltage source in Fig. 2.404 is

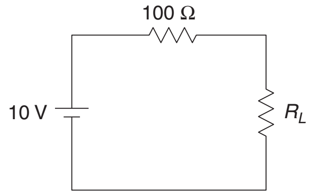


Fig. 2.404

(a) 1 W (b) 10 W
(c) 0.25 W (d) 0.5 W

2.16 For the circuit shown in Fig. 2.405, Thevenin's voltage and Thevenin's equivalent resistance at terminals *a-b* is

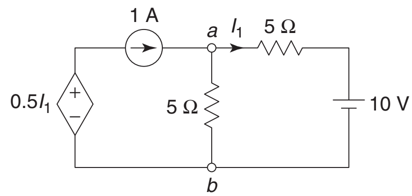


Fig. 2.405

(a) 5 V and 2 Ω (b) 7.5 V and 2.5 Ω
(c) 4 V and 2 Ω (d) 3 V and 2.5 Ω

2.17 The value of R_L in Fig. 2.406 for maximum power transfer is

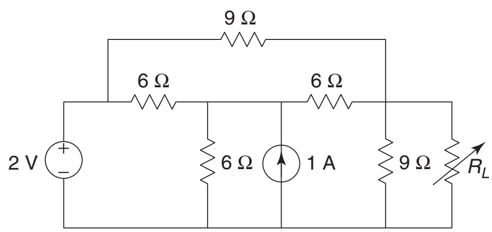


Fig. 2.406

(a) 3 Ω (b) 1.125 Ω
(c) 4.1785 Ω (d) none of these

Answers to Objective-Type Questions

2.1. (a)	2.2. (b)	2.3. (c)	2.4. (d)	2.5. (a)	2.6. (d)	2.7. (a)
2.8. (a)	2.9. (d)	2.10. (b)	2.11. (a)	2.12. (a)	2.13. (c)	2.14. (a)
2.15. (c)	2.16. (b)	2.17. (a)				

3

Analysis of AC Circuits

3.1 INTRODUCTION

We have discussed the network theorems with reference to resistive load and dc sources. Now, all the theorems will be discussed when a network consists of ac sources, resistors, inductors and capacitors. All the theorems are also valid for ac sources.

3.2 MESH ANALYSIS

Mesh analysis is useful if a network has a large number of voltage sources. In this method, currents are assigned in each mesh. We can write mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents,

Example 3.1 *Find mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the network of Fig. 3.1.*

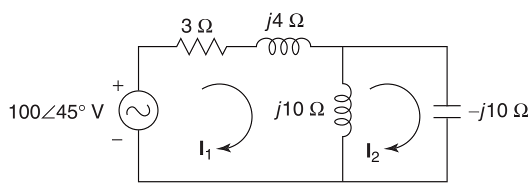


Fig. 3.1

Solution Applying KVL to Mesh 1,

$$100\angle 45^\circ - (3 + j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j14)\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 45^\circ \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) + j10(\mathbf{I}_2) = 0$$

$$j10\mathbf{I}_1 = 0 \quad \text{...(ii)}$$

$$\mathbf{I}_1 = 0$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$-j10\mathbf{I}_2 = 100\angle 45^\circ$$

$$\mathbf{I}_2 = \frac{100\angle 45^\circ}{-j10} = 10\angle 135^\circ \text{ A}$$

Example 3.2 *Find mesh current I_1, I_2 and I_3 in the network of Fig. 3.2.*

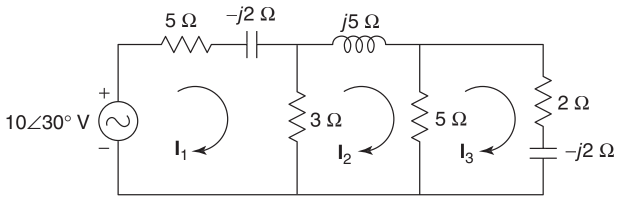


Fig. 3.2

Solution Applying KVL to Mesh 1,

$$10\angle 30° - (5 - j2)\,\mathbf{I}_1 - 3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 - j2)\,\mathbf{I}_1 - 3\mathbf{I}_2 = 10\angle 30° \qquad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-3(\mathbf{I}_2 - \mathbf{I}_1) - j5\mathbf{I}_2 - 5(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-3\mathbf{I}_1 + (8 + j5)\,\mathbf{I}_2 - 5\mathbf{I}_3 = 0 \qquad \ldots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-5(\mathbf{I}_3 - \mathbf{I}_2) - (2 - j2)\,\mathbf{I}_3 = 0$$

$$-5\mathbf{I}_2 + (7 - j2)\,\mathbf{I}_3 = 0 \qquad \ldots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii),

$$\begin{bmatrix} 8-j2 & -3 & 0 \\ -3 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 30° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 30° & -3 & 0 \\ 0 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{vmatrix}}{\begin{vmatrix} 8-j2 & -3 & 0 \\ -3 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{vmatrix}} = 1.43\angle 38.7° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8-j2 & 10\angle 30° & 0 \\ -3 & 0 & -5 \\ 0 & 0 & 7-j2 \end{vmatrix}}{\begin{vmatrix} 8-j2 & -3 & 0 \\ -3 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{vmatrix}} = 0.693\angle -2.2° \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 8-j2 & -3 & 10\angle 30° \\ -3 & 8+j5 & 0 \\ 0 & -5 & 0 \end{vmatrix}}{\begin{vmatrix} 8-j2 & -3 & 0 \\ -3 & 8+j5 & -5 \\ 0 & -5 & 7-j2 \end{vmatrix}} = 0.476\angle 13.8° \text{ A}$$

Example 3.3 *In the network of Fig. 3.3, find the value of V_2 so that the current through (2 + j3) ohm impedance is zero.*

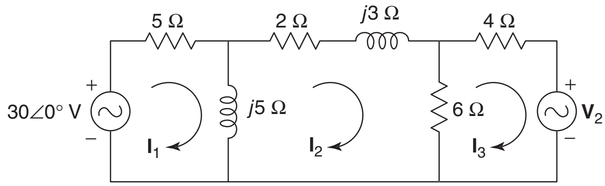


Fig. 3.3

Solution Applying KVL to Mesh 1,

$$30\angle 0° - 5\mathbf{I}_1 - j5(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(5 + j5)\mathbf{I}_1 - j5\mathbf{I}_2 = 30\angle 0° \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j5(\mathbf{I}_2 - \mathbf{I}_1) - (2 + j3)\mathbf{I}_2 - 6(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j5\mathbf{I}_1 + (8 + j8)\mathbf{I}_2 - 6\mathbf{I}_3 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-6(\mathbf{I}_3 - \mathbf{I}_2) - 4\mathbf{I}_3 - \mathbf{V}_2 = 0$$

$$-6\mathbf{I}_2 + 10\mathbf{I}_3 = -\mathbf{V}_2 \qquad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 30\angle 0° \\ 0 \\ -\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5+j5 & 30\angle 0° & 0 \\ -j5 & 0 & -6 \\ 0 & -\mathbf{V}_2 & 10 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{vmatrix}} = 0$$

$$(5 + j5)(-6\mathbf{V}_2) - (30)(-j50) = 0$$

$$\mathbf{V}_2 = \frac{j1500}{30 + j30} = 35.36\angle 45° \text{ V}$$

Example 3.4 *Find the value of the current I_3 in the network shown in Fig. 3.4.*

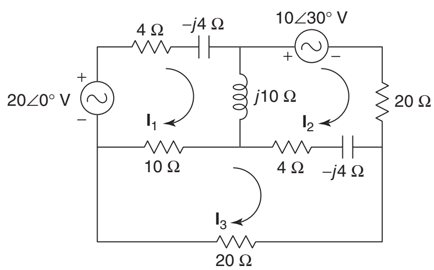


Fig. 3.4

Solution Applying KVL to Mesh 1,

$$20\angle 0° - (4 - j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) - 10(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(14 + j6)\mathbf{I}_1 - j10\mathbf{I}_2 - 10\mathbf{I}_3 = 20\angle 0° \quad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) - 10\angle 30° - 20\mathbf{I}_2 - (4 - j4)(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j10\mathbf{I}_1 + (24 + j6)\mathbf{I}_2 - (4 - j4)\mathbf{I}_3 = -10\angle 30° \quad \ldots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-10(\mathbf{I}_3 - \mathbf{I}_1) - (4 - j4)(\mathbf{I}_3 - \mathbf{I}_2) - 20\mathbf{I}_3 = 0$$

$$-10\mathbf{I}_1 - (4 - j4)\mathbf{I}_2 + (34 - j4)\mathbf{I}_3 = 0 \quad \ldots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 14 + j6 & -j10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 20\angle 0° \\ -10\angle 30° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 14 + j6 & -j10 & 20\angle 0° \\ -j10 & 24 + j6 & -10\angle 30° \\ -10 & -(4 - j4) & 0 \end{vmatrix}}{\begin{vmatrix} 14 + j6 & -10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{vmatrix}} = 0.44\angle -14° \text{ A}$$

Example 3.5 *Find the voltage V_{AB} in the network of Fig. 3.5.*

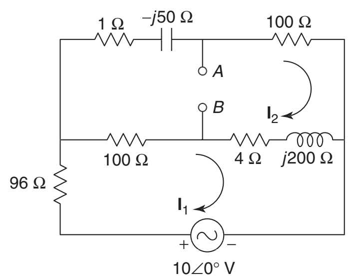


Fig. 3.5

Solution Applying KVL to Mesh 1,

$$-96\,\mathbf{I}_1 - (100 + 4 + j200)(\mathbf{I}_1 - \mathbf{I}_2) + 10\angle 0° = 0$$

$$(200 + j200)\,\mathbf{I}_1 - (104 + j200)\,\mathbf{I}_2 = 10\angle 0° \quad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-(1 - j50 - 100)\,\mathbf{I}_2 - (100 + 4 + j200)(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$-(104 + j200)\,\mathbf{I}_1 + (205 + j150)\,\mathbf{I}_2 = 0 \quad \ldots \text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 200 + j200 & -(104 + j200) \\ -(104 + j200) & 205 + j150 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -(104+j200) \\ 0 & 205+j150 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.051\angle 2.72\times 10^{-3}\,{}^\circ\text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 200+j200 & 10\angle 0^\circ \\ -(104+j200) & 0 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.045\angle 26.34^\circ \text{ A}$$

$$\begin{aligned}\mathbf{V}_{AB} &= 100\mathbf{I}_2 - (4+j200)(\mathbf{I}_1 - \mathbf{I}_2) \\ &= 100(0.045\angle 26.34^\circ) - (4+j200)(0.051\angle 2.72\times 10^{-3}\,{}^\circ - 0.045\angle 26.34^\circ) \\ &= 0.058\angle -92.65^\circ \text{ V}\end{aligned}$$

Example 3.6 *For the network shown in Fig. 3.6, find the voltage across the capacitor.*

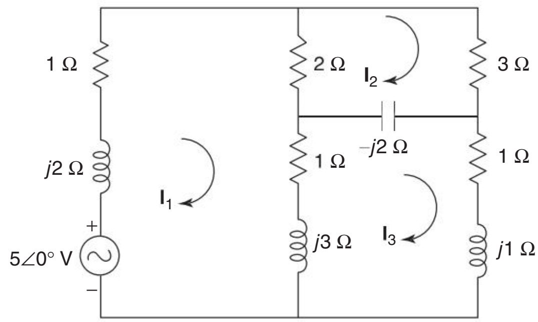


Fig. 3.6

Solution Applying KVL to Mesh 1,

$$5\angle 0^\circ - (1+j2)\mathbf{I}_1 - 2(\mathbf{I}_1 - \mathbf{I}_2) - (1+j3)(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(4+j5)\mathbf{I}_1 - 2\mathbf{I}_2 - (1+j3)\mathbf{I}_3 = 5\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-2(\mathbf{I}_2 - \mathbf{I}_1) - 3\mathbf{I}_2 + j2(\mathbf{I}_2 - \text{I}_3) = 0$$

$$-2\mathbf{I}_1 + (5-j2)\mathbf{I}_2 + j2\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-(1+j3)(\mathbf{I}_3 - \mathbf{I}_1) + j2(\mathbf{I}_3 - \text{I}_2) - (1+j1)\mathbf{I}_3 = 0$$

$$-(1+j3)\mathbf{I}_1 + j2\mathbf{I}_2 + (2+j2)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 5\angle 0^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4+j5 & 5\angle 0^\circ & -(1+j3) \\ -2 & 0 & j2 \\ -(1+j3) & 0 & 2+j2 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.65\angle 130.51^\circ \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4+j5 & -2 & 5\angle 0^\circ \\ -2 & 5-j2 & 0 \\ -(1+j3) & j2 & 0 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.91\angle -21.51^\circ \text{ A}$$

$$\mathbf{V}_c = (-j2)(\mathbf{I}_3 - \mathbf{I}_2) = (-j2)(0.91\angle -21.51^\circ - 0.65\angle 130.51^\circ) = 3.03\angle -123.12^\circ \text{ V}$$

Example 3.7 *Find the voltage across the 2 Ω resistor in the network of Fig. 3.7.*

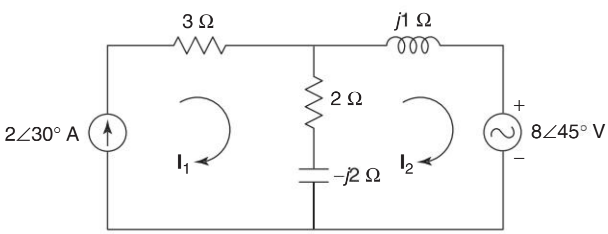


Fig. 3.7

Solution For Mesh 1,

$$\mathbf{I}_1 = 2\angle 30^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-(2-j2)(\mathbf{I}_2 - \mathbf{I}_1) - j1\mathbf{I}_2 - 8\angle 45^\circ = 0$$

$$(2-j2)\mathbf{I}_1 - (2-j1)\mathbf{I}_2 = 8\angle 45^\circ \qquad \ldots\text{(ii)}$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$(2-j2)(2\angle 30^\circ) - (2-j1)\mathbf{I}_2 = 8\angle 45^\circ$$

$$\mathbf{I}_2 = \frac{-(8\angle 45^\circ) + (2-j2)(2\angle 30^\circ)}{2-j1} = 3.19\angle -65^\circ \text{ A}$$

$$\mathbf{V}_{2\,\Omega} = 2(\mathbf{I}_1 - \mathbf{I}_2) = 2(2\angle 30^\circ - 3.19\angle -65^\circ) = 7.82\angle 84.37^\circ \text{ V}$$

Example 3.8 *Find the current through 3 Ω resistor in the network of Fig. 3.8.*

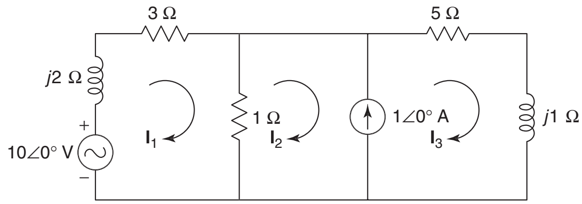


Fig. 3.8

Solution Applying KVL to Mesh 1,

$$10\angle 0° - j2\mathbf{I}_1 - 3\mathbf{I}_1 - 1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(4 + j2)\mathbf{I}_1 - \mathbf{I}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$\mathbf{I}_3 - \mathbf{I}_2 = 1\angle 0° \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-1(\mathbf{I}_2 - \mathbf{I}_1) - 5\mathbf{I}_3 - j1\mathbf{I}_3 = 0$$

$$\mathbf{I}_1 - \mathbf{I}_2 - (5 + j1)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 1\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0° & -1 & 0 \\ 1\angle 0° & -1 & 1 \\ 0 & -1 & -(5+j1) \end{vmatrix}}{\begin{vmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{vmatrix}} = 2.11\angle -28.01° \text{ A}$$

$$\mathbf{I}_{3\,\Omega} = \mathbf{I}_1 = 2.11\angle -28.01° \text{ A}$$

Example 3.9 *Find the currents $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ in the network of Fig. 3.9.*

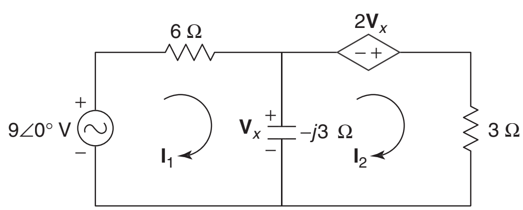


Fig. 3.9

Solution From Fig. 3.9,

$$\mathbf{V}_x = -j3(\mathbf{I}_1 - \mathbf{I}_2) \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$9\angle 0° - 6\mathbf{I}_1 + j3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$(6 - j3)\mathbf{I}_1 + j3\mathbf{I}_2 = 9\angle 0° \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$j3(\mathbf{I}_2 - \mathbf{I}_1) + 2\mathbf{V}_x - 3\mathbf{I}_2 = 0$$
$$j3\mathbf{I}_2 - j3\mathbf{I}_1 + 2[-j3(\mathbf{I}_1 - \mathbf{I}_2)] - 3\mathbf{I}_2 = 0$$
$$j9\mathbf{I}_1 + (3 - j9)\mathbf{I}_2 = 0 \quad \text{...(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 9\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 9\angle 0° & j3 \\ 0 & 3 - j9 \end{vmatrix}}{\begin{vmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{vmatrix}} = 1.3\angle 2.49° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 6 - j3 & 9\angle 0° \\ j9 & 0 \end{vmatrix}}{\begin{vmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{vmatrix}} = 1.24\angle -15.95° \text{ A}$$

Example 3.10 *Find the voltage across the 4 Ω resistor in the network of Fig. 3.10.*

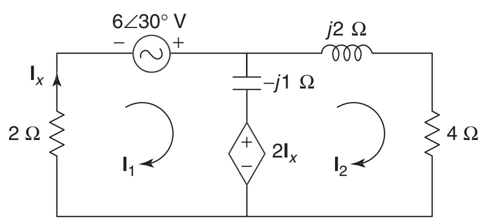


Fig. 3.10

Solution From Fig. 3.10,

$$\mathbf{I}_x = \mathbf{I}_1 \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2\mathbf{I}_1 + 6\angle 30° + j1(\mathbf{I}_1 - \mathbf{I}_2) - 2\mathbf{I}_x = 0$$
$$-2\mathbf{I}_1 + 6\angle 30° + j1\mathbf{I}_1 - j1\mathbf{I}_2 - 2\mathbf{I}_1 = 0$$
$$(4 - j1)\mathbf{I}_1 + j1\mathbf{I}_2 = 6\angle 30° \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$2\mathbf{I}_x + j1(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$2\mathbf{I}_1 + j1\mathbf{I}_2 - j1\,\mathbf{I}_1 - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$(2 - j1)\mathbf{I}_1 - (4 + j1)\mathbf{I}_2 = 0 \qquad \text{...(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 6\angle 30^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4 - j1 & 6\angle 30^\circ \\ 2 - j1 & 0 \end{vmatrix}}{\begin{vmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{vmatrix}} = 0.74\angle -2.91^\circ \text{ A}$$

$$V_{4\Omega} = 4\mathbf{I}_2 = 4(0.74\angle -2.91^\circ) = 2.96\angle -2.91^\circ \text{ V}$$

3.3 NODE ANALYSIS

Node analysis uses Kirchhoff's current law for finding currents and voltages in a network. For ac networks, Kirchhoff's current law states that the phasor sum of currents meeting at a point is equal to zero.

Example 3.11 *In the network shown in Fig. 3.11, determine $\mathbf{V}_a$ and $\mathbf{V}_b$.*

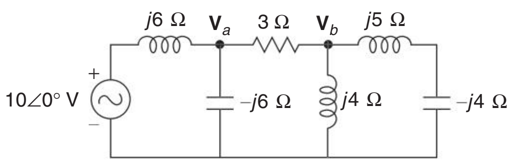


Fig. 3.11

Solution Applying KCL at Node a,

$$\frac{\mathbf{V}_a - 10\angle 0^\circ}{j6} + \frac{\mathbf{V}_a}{-j6} + \frac{\mathbf{V}_a - \mathbf{V}_b}{3} = 0$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)\mathbf{V}_a - \frac{1}{3}\mathbf{V}_b = \frac{10\angle 0^\circ}{j6}$$

$$0.33\mathbf{V}_a - 0.33\mathbf{V}_b = 1.67\angle -90^\circ \qquad \text{...(i)}$$

Applying KCL at Node b,

$$\frac{\mathbf{V}_b - \mathbf{V}_a}{3} + \frac{\mathbf{V}_b}{j4} + \frac{\mathbf{V}_b}{j1} = 0$$

$$-\frac{1}{3}\mathbf{V}_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)\mathbf{V}_b = 0$$

$$-0.33\mathbf{V}_a + (0.33 - j1.25)\mathbf{V}_b = 0 \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii),

$$-j1.25\mathbf{V}_b = 1.67\angle-90^\circ$$

$$\mathbf{V}_b = \frac{1.67\angle-90^\circ}{-j1.25} = 1.34\angle0^\circ \text{ V}$$

Substituting $\mathbf{V}_b$ in Eq. (i),

$$0.33\mathbf{V}_a - 0.33(1.34\angle0^\circ) = 1.67\angle-90^\circ$$

$$\mathbf{V}_a = \frac{1.73\angle75.17^\circ}{0.33} = 5.24\angle-75.17^\circ \text{ V}$$

Example 3.12 *For the network shown in Fig. 3.12, find the voltages V_1 and V_2.*

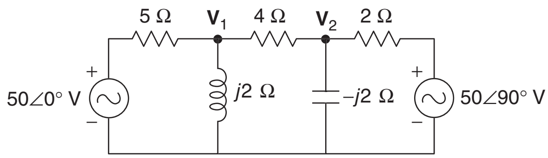


Fig. 3.12

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle0^\circ}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_2 = 10\angle0^\circ$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_2 = 10\angle0^\circ \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_2}{-j2} + \frac{\mathbf{V}_2 - 50\angle90^\circ}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_2 = 25\angle90^\circ$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\mathbf{V}_2 = 25\angle90^\circ \qquad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle0^\circ \\ 25\angle90^\circ \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle0^\circ & -0.25 \\ j25 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 24.7\angle72.25^\circ \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.45 - j0.5 & 10\angle0^\circ \\ -0.25 & 25\angle90^\circ \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 34.34\angle52.82^\circ \text{ V}$$

Example 3.13 *Find the voltage V_{AB} in the network of Fig. 3.13.*

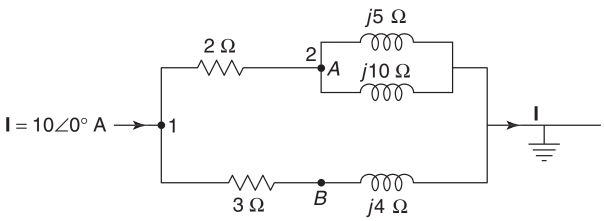


Fig. 3.13

Solution Applying KCL at Node 1,

$$10\angle 0^\circ = \frac{\mathbf{V}_1 - \mathbf{V}_2}{2} + \frac{\mathbf{V}_1}{3 + j4}$$

$$\left(\frac{1}{2} + \frac{1}{3 + j4}\right)\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 = 10\angle 0^\circ$$

$$(0.62 - j0.16)\mathbf{V}_1 - 0.5\mathbf{V}_2 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{2} + \frac{\mathbf{V}_2}{j5} + \frac{\mathbf{V}_2}{j10} = 0$$

$$-\frac{1}{2}\mathbf{V}_1 + \left(\frac{1}{2} + \frac{1}{j5} + \frac{1}{j10}\right)\mathbf{V}_2 = 0$$

$$-0.5\mathbf{V}_1 + (0.5 - j0.3)\mathbf{V}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -0.5 \\ 0 & 0.5 - j0.3 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 21.8\angle 56.42^\circ \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.62 - j0.16 & 10\angle 0^\circ \\ -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 18.7\angle 87.42^\circ \text{ V}$$

$$\mathbf{V}_A = \mathbf{V}_2 = 18.7\angle 87.42^\circ \text{ V}$$

$$\mathbf{V}_B = \frac{\mathbf{V}_1}{3 + j4}(j4) = \frac{21.8\angle 56.42^\circ}{3 + j4}(j4) = 17.45\angle 93.32^\circ \text{ V}$$

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B = (18.7\angle 87.42^\circ) - (17.45\angle 93.32^\circ) = 2.23\angle 34.1^\circ \text{ V}$$

Example 3.14 *Find the node voltages V_1 and V_2 in the network of Fig. 3.14.*

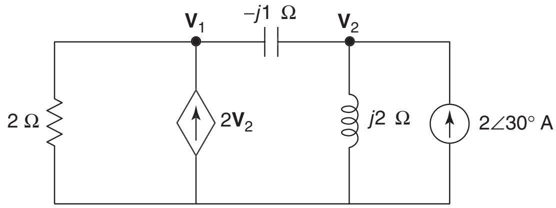


Fig. 3.14

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1}{2}+\frac{\mathbf{V}_1-\mathbf{V}_2}{-j1}=2\mathbf{V}_2$$

$$\left(\frac{1}{2}+\frac{1}{-j1}\right)\mathbf{V}_1-\left(2-\frac{1}{j1}\right)\mathbf{V}_2=0$$

$$(0.5+j1)\,\mathbf{V}_1-(2+j1)\mathbf{V}_2=0 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2-\mathbf{V}_1}{-j1}+\frac{\mathbf{V}_2}{j2}=2\angle 30^\circ$$

$$\frac{1}{j1}\mathbf{V}_1+\left(\frac{1}{-j1}+\frac{1}{j2}\right)\mathbf{V}_2=2\angle 30^\circ$$

$$-j1\,\mathbf{V}_1+j\,0.5\,\mathbf{V}_2=2\angle 30^\circ \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix}0.5+j1 & -(2+j1)\\ -j1 & j0.5\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}0\\ 2\angle 30^\circ\end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1=\frac{\begin{vmatrix}0 & -(2+j1)\\ 2\angle 30^\circ & j0.5\end{vmatrix}}{\begin{vmatrix}0.5+j1 & -(2+j1)\\ -j1 & j0.5\end{vmatrix}}=2.46\angle 130.62^\circ\text{ V}$$

$$\mathbf{V}_2=\frac{\begin{vmatrix}0.5+j1 & 0\\ -j1 & 2\angle 30^\circ\end{vmatrix}}{\begin{vmatrix}0.5+j1 & -(2+j1)\\ -j1 & j0.5\end{vmatrix}}=1.23\angle 167.49^\circ\text{ V}$$

Example 3.15 *In the network of Fig. 3.15, find the voltage V_2 which results in zero current through 4 Ω resistor.*

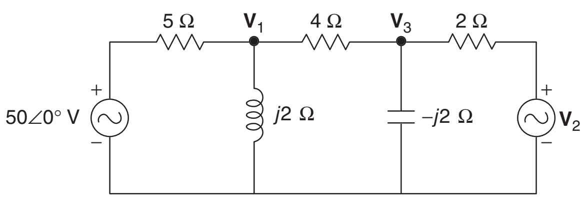


Fig. 3.15

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_3 = 10\angle 0^\circ$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_3 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 3,

$$\frac{\mathbf{V}_3 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_3}{-j2} + \frac{\mathbf{V}_3 - \mathbf{V}_2}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_3 = 0.5\mathbf{V}_2$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\,\mathbf{V}_3 = 0.5\mathbf{V}_2 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0.5\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -0.25 \\ 0.5\mathbf{V}_2 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{10(0.75 + j0.5) + 0.125\,\mathbf{V}_2}{0.55\angle -15.95^\circ}$$

$$\mathbf{V}_3 = \frac{\begin{vmatrix} 0.45 - j\,0.5 & 10\angle 0^\circ \\ -0.25 & 0.5\mathbf{V}_2 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{0.5\,\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$\mathbf{I}_{4\,\Omega} = \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\mathbf{V}_1 = \mathbf{V}_3$$

$$\frac{10(0.75 + j0.5) + 0.125\mathbf{V}_2}{0.55\angle -15.95^\circ} = \frac{0.5\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$7.5 + 0.125\mathbf{V}_2 - j\,5 = 2.5 + 0.225\mathbf{V}_2 - j0.25\mathbf{V}_2$$

$$5 + j\,5 = \mathbf{V}_2(0.1 - j0.25)$$

$$\mathbf{V}_2 = \frac{5 + j5}{0.1 - j0.25} = 26.26\angle 113.2^\circ \text{ V}$$

Example 3.16 *Find the voltage across the capacitor in the network of Fig. 3.16.*

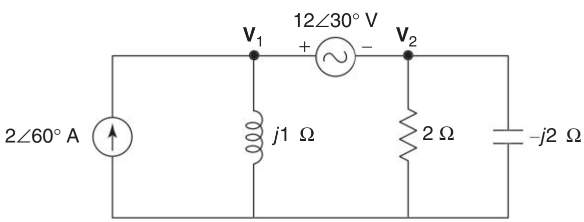


Fig. 3.16

Solution Nodes 1 and 2 will form a supernode.
Writing the voltage equation for the supernode,

$$\mathbf{V}_1 - \mathbf{V}_2 = 12\angle 30° \qquad \ldots\text{(i)}$$

Applying KCL to the supernode,

$$\frac{\mathbf{V}_1}{j1} + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2}{-j2} = 2\angle 60°$$

$$(-j1)\,\mathbf{V}_1 + (0.5 + j0.5)\,\mathbf{V}_2 = 2\angle 60° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 12\angle 30° \\ 2\angle 60° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 1 & 12\angle 30° \\ -j1 & 2\angle 60° \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{vmatrix}} = 18.55\angle 157.42° \text{ V}$$

$$\mathbf{V}_c = \mathbf{V}_2 = 18.55\angle 157.42° \text{ V}$$

3.4 SUPERPOSITION THEOREM

The superposition theorem can be used to analyse an ac network containing more than one source. The superposition theorem states that *in a network containing more than one voltage source or current source, the total current or voltage in any branch of the network is the phasor sum of currents or voltages produced in that branch by each source acting separately*. As each source is considered, all of the other sources are replaced by their internal impedances. This theorem is valid only for linear systems.

Example 3.17 *Find the current through the 3 + j4 ohm impedance.*

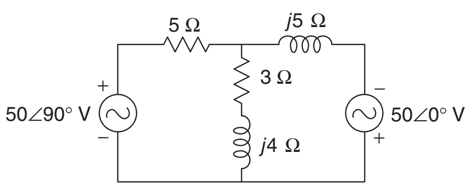


Fig. 3.17

Solution

Step I When the 50 ∠90° V source is acting alone (Fig. 3.18)

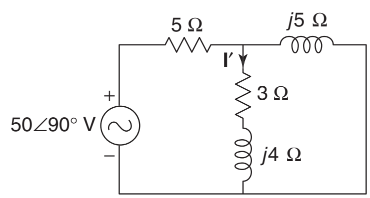


Fig. 3.18

$$\mathbf{Z}_T = 5 + \frac{(3 + j4)(j5)}{3 + j9} = 6.35\angle 23.2° \ \Omega$$

$$\mathbf{I}_T = \frac{50\angle 90°}{6.35\angle 23.2°} = 7.87\angle 66.8° \text{ A}$$

By current division rule,

$$\mathbf{I}' = (7.87\angle 66.8^\circ)\left(\frac{j5}{3+j9}\right) = 4.15\angle 85.3^\circ \text{ A}(\downarrow)$$

Step II When the $50\angle 0^\circ$ V source is acting alone (Fig. 3.19)

$$\mathbf{Z}_T = j5 + \frac{5(3+j4)}{8+j4} = 6.74\angle 68.2^\circ\ \Omega$$

$$\mathbf{I}_T = \frac{50\angle 0^\circ}{6.74\angle 68.2^\circ} = 7.42\angle -68.2^\circ \text{ A}$$

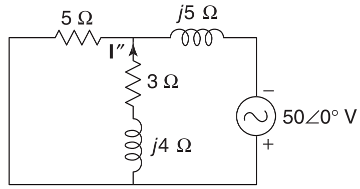


Fig. 3.19

By current division rule,

$$\mathbf{I}'' = (7.42\angle -68.2^\circ)\left(\frac{5}{8+j4}\right) = 4.15\angle -94.77^\circ \text{ A}(\uparrow) = 4.15\angle 85.3^\circ \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 4.15\ \angle 85.3^\circ + 4.15\ \angle 85.3^\circ = 8.31\ \angle 85.3^\circ\text{A}\ (\downarrow)$$

Example 3.18 *Determine the voltage across the (2 + j5) ohm impedance for the network shown in Fig. 3.20.*

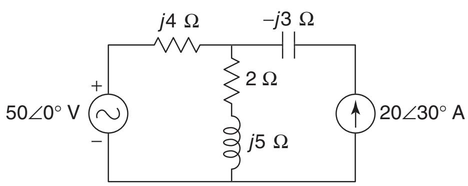


Fig. 3.20

Solution

Step I When the $50\angle 0^\circ$ V source is acting alone (Fig. 3.21)

$$\mathbf{I} = \frac{50\angle 0^\circ}{2+j4+j5} = 5.42\angle -77.47^\circ \text{ A}$$

Voltage cross $(2+j5)\ \Omega$ impedance

$$\mathbf{V}' = (2+j5)\ (5.42\ \angle -77.47^\circ) = 29.16\ \angle -9.28^\circ \text{ V}$$

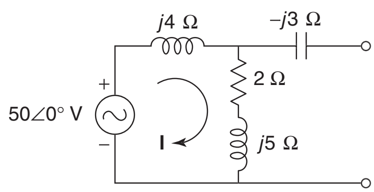


Fig. 3.21

Step II When the $20\angle 30^\circ$ A source is acting alone (Fig. 3.22)
By current division rule,

$$\mathbf{I} = (20\angle 30^\circ)\left(\frac{j4}{2+j9}\right) = 8.68\angle 42.53^\circ \text{ A}$$

Voltage across $(2+j5)\ \Omega$ impedance

$$\mathbf{V}'' = (2+j5)\ (8.68\ \angle 42.53^\circ) = 46.69\ \angle 110.72^\circ \text{ V}$$

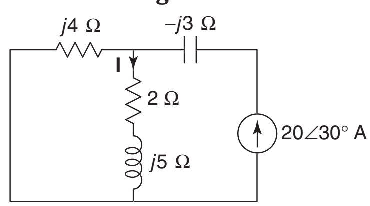


Fig. 3.22

Step III By superposition theorem,

$$\mathbf{V} = \mathbf{V}' + \mathbf{V}'' = 29.16 \angle{-9.28°} + 46.69 \angle 110.72° = 40.85 \angle 72.53° \text{ V}$$

Example 3.19 *Determine the voltage $\mathbf{V}_{AB}$ for the network shown in Fig. 3.23.*

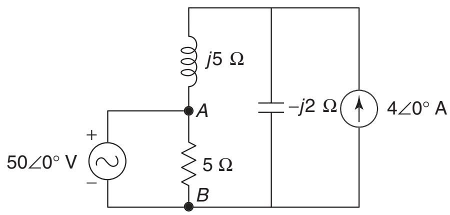


Fig. 3.23

Solution

Step I When the 50∠0° V source is acting alone (Fig. 3.24)

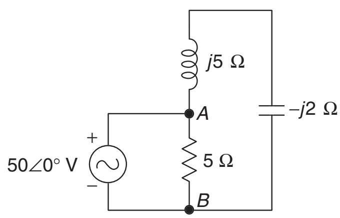


Fig. 3.24

$$\mathbf{V}'_{AB} = 50\angle 0° \text{ V}$$

Step II When the 4∠0° A source is acting alone (Fig. 3.25)

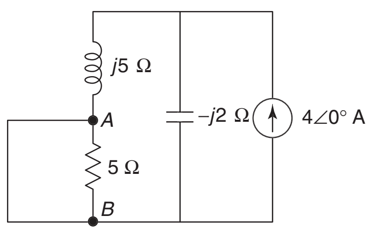


Fig. 3.25

$$\mathbf{V}''_{AB} = 0$$

Step III By superposition theorem,

$$\mathbf{V}_{AB} = \mathbf{V}'_{AB} + \mathbf{V}''_{AB} = 50\angle 0° + 0 = 50\angle 0° \text{ V}$$

Example 3.20 *Find the current **I** in the network shown in Fig. 3.26.*

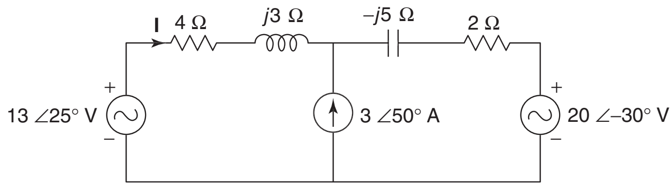


Fig. 3.26

Solution

Step I When the 13∠25° V source is acting alone (Fig. 3.27)

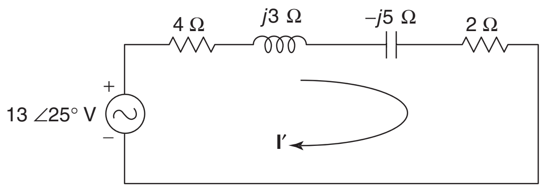


Fig. 3.27

$$\mathbf{I}' = \frac{13\angle 25^\circ}{6 - j2} = 2.057\angle 43.43^\circ \text{ A} \quad (\rightarrow)$$

Step II When the 20∠–30° V source is acting alone (Fig. 3.28)

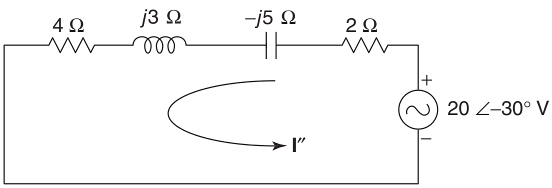


Fig. 3.28

$$\mathbf{I}'' = \frac{20\angle -30^\circ \text{ V}}{6 - j2} = 3.16\angle -11.57^\circ \text{ A}(\leftarrow) = 3.16\angle 168.43^\circ \text{ A}(\rightarrow)$$

Step III When the 3∠50° A source is acting alone (Fig. 3.29)

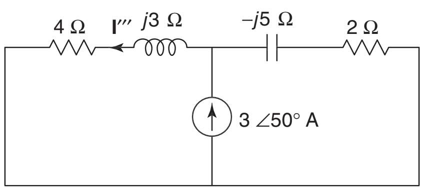


Fig. 3.29

By current division rule,

$$\mathbf{I}''' = 3\angle 50° \times \frac{2 - j5}{6 - j2} = 2.56\angle 0.23° \text{ A}(\leftarrow) = 2.56\angle -179.77° \text{ A}(\rightarrow)$$

Step IV By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 2.057\ \angle 43.13° + 3.16\ \angle 168.43° + 2.56\ \angle -179.77° \text{ A} = 4.62\ \angle 153.99° \text{ A } (\rightarrow)$$

Example 3.21 *Find the current through the j3 Ω reactance in the network of Fig. 3.30.*

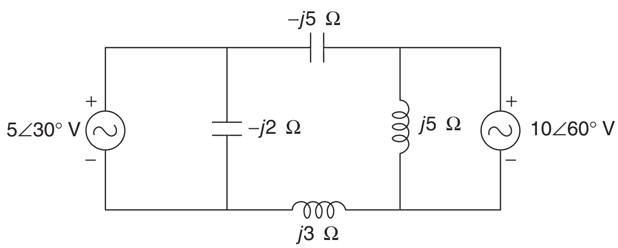


Fig. 3.30

Solution

Step I When the $5\angle 30°$ V source is acting alone (Fig. 3.31)

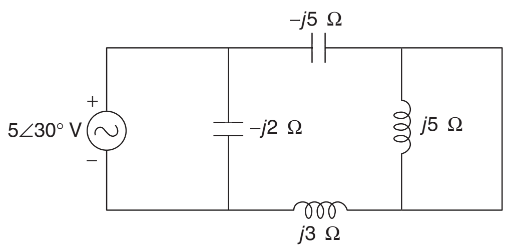


Fig. 3.31

When a short circuit is placed across $j15\ \Omega$ reactance, it gets shorted as shown in Fig. 3.32.

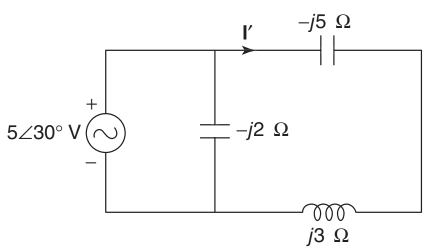


Fig. 3.32

$$\mathbf{I}' = \frac{5\angle 30°}{-j5 + j3} = 2.5\angle 120° \text{ A}(\leftarrow)$$

Step II When the $10\angle 60°$ V source is acting alone (Fig. 3.33)

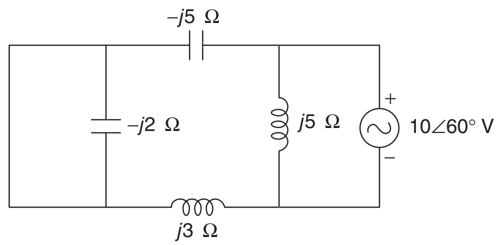


Fig. 3.33

When a short circuit is placed across the $-j2\ \Omega$ reactance, it gets shorted as shown in Fig. 3.34

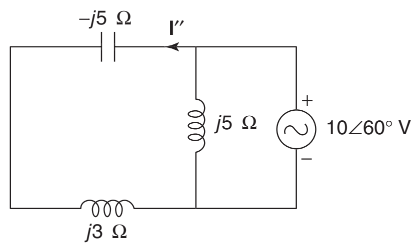


Fig. 3.34

$$\mathbf{I}'' = \frac{10\angle 60°}{-j5 + j3} = 5\angle 150°\ \text{A}\ (\rightarrow) = 5\angle -30°\ \text{A}\ (\leftarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 2.5\angle 120° + 5\angle -30° = 3.1\angle -6.21°\ \text{A} \quad (\leftarrow)$$

Example 3.22 *Find the current $\mathbf{I}_0$ in the network of Fig. 3.35.*

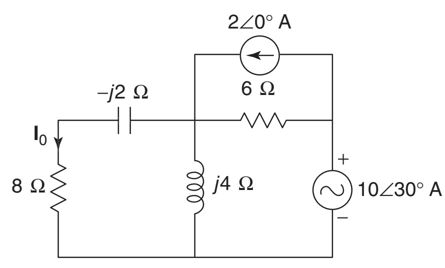


Fig. 3.35

Solution

Step I When the $10\angle 30°$ V source is acting alone (Fig. 3.36)

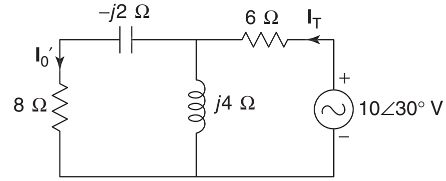


Fig. 3.36

$$\mathbf{Z}_T = 6 + \frac{j4(8 - j2)}{j4 + 8 - j2} = 8.64\angle 24.12°\ \Omega$$

$$\mathbf{I}_T = \frac{10\angle 30°}{8.64\angle 24.12°} = 1.16\angle 5.88°\ \text{A}$$

By current division rule,

$$\mathbf{I}_0' = 1.16\angle 5.88^\circ \times \frac{j4}{8 - j2 + j4} = 0.56\angle 81.84^\circ \text{ A} \quad (\downarrow)$$

Step II When the $2\angle 0^\circ$ A source is acting alone (Fig. 3.37)

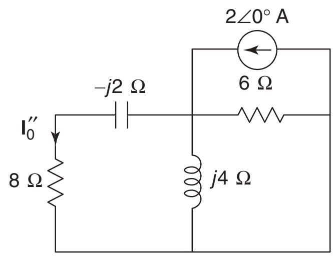


Fig. 3.37

The network can be redrawn as shown in Fig. 3.38.

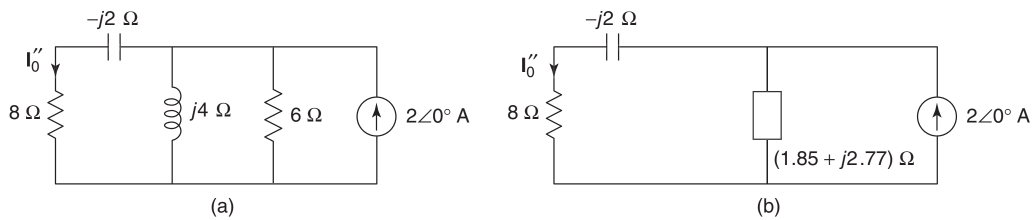


Fig. 3.38

By current division rule,

$$\mathbf{I}_0'' = 2\angle 0^\circ \times \frac{1.85 + j2.77}{1.85 + j2.77 + 8 - j2} = 0.67\angle 51.83^\circ \text{ A } (\downarrow)$$

Step III By superposition theorem,

$$\mathbf{I}_0 = \mathbf{I}_0' + \mathbf{I}_0'' = 0.56\angle 81.84^\circ + 0.67\angle 51.83^\circ = 1.19\angle 65.46^\circ \text{ A } (\downarrow)$$

Example 3.23 *Find the current through the j5 Ω branch for the network shown in Fig. 3.39.*

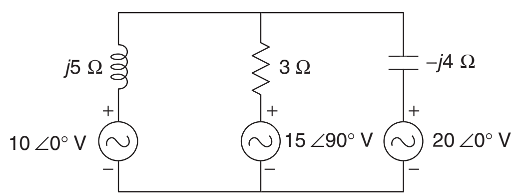


Fig. 3.39

Solution

Step I When the 10∠0° V source is acting alone (Fig. 3.40)

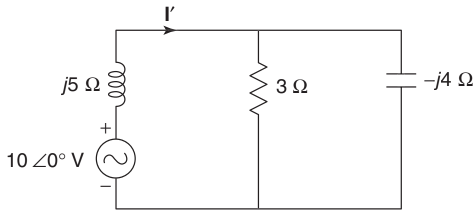


Fig. 3.40

$$\mathbf{Z}_T = j5 + \frac{3(-j4)}{3-j4} = 4.04\angle 61.66°\ \Omega$$

$$\mathbf{I}' = \frac{10\angle 0°}{4.04\angle 61.66°} = 2.48\angle -61.66° \text{A} \quad (\rightarrow)$$

Step II When the 15∠90° V source is acting alone (Fig. 3.41)

$$\mathbf{Z}_T = 3 + \frac{(j5)(-j4)}{j5-j4} = 20.22\angle -81.47°\ \Omega$$

$$\mathbf{I}_T = \frac{15\angle 90°}{20.22\angle -81.47°} = 0.74\angle 171.47° \text{ A}$$

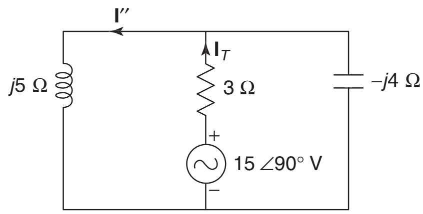


Fig. 3.41

By current division rule,

$$\mathbf{I}'' = 0.74\angle 171.47° \times \frac{-j4}{-j4+j5} = 2.96\angle -8.53° \text{ A } (\leftarrow) = 2.96\angle 171.47° \text{ A} \quad (\rightarrow)$$

Step III When the 20 ∠0° V source is acting along (Fig. 3.42)

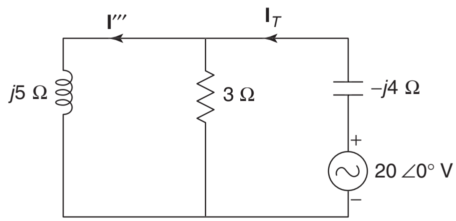


Fig. 3.42

$$\mathbf{Z}_T = -j4 + \frac{3(j5)}{3+j5} = 3.47\angle -50.51°\ \Omega$$

$$\mathbf{I}_T = \frac{20\angle 0°}{3.47\angle -50.51°} = 5.76\angle 50.51° \text{ A}$$

By current division rule,

$$\mathbf{I}''' = 5.76\angle 50.51° \times \frac{3}{3+j5} = 2.96\angle -8.53° \text{ A } (\leftarrow) = 2.96\angle 171.47° \text{ A } (\rightarrow)$$

Step IV By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 2.48\angle-61.66° + 2.96\angle 171.47° + 2.96\angle 171.47° = 4.86\angle -164.41° \text{ A}$$

Example 3.24 *Find the voltage drop across the capacitor for the network shown in Fig. 3.43.*

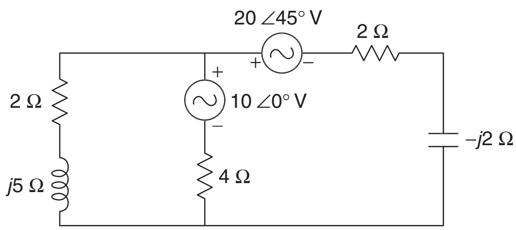


Fig. 3.43

Solution

Step I When the $10\angle 0°$ V source is acting alone (Fig. 3.44)

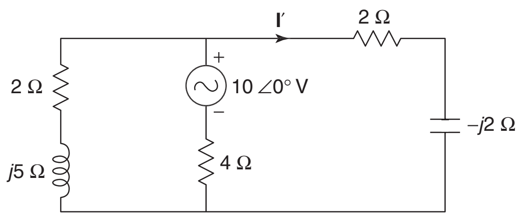


Fig. 3.44

$$\mathbf{Z}_T = 4 + \frac{(2+j5)(2-j2)}{2+j5+2-j2}$$
$$= 7\angle -5.91° \ \Omega$$
$$\mathbf{I}_T = \frac{10\angle 0°}{7\angle -5.91°} = 1.43\angle 5.91° \text{ A}$$

By current division rule,

$$\mathbf{I}' = (1.43\angle 5.91°)\left(\frac{2+j5}{2+j5+2-j2}\right) = 1.54\angle 37.24° \text{ A} \quad (\rightarrow)$$

Step II When the $20\angle 45°$ V source is acting alone (Fig. 3.45)

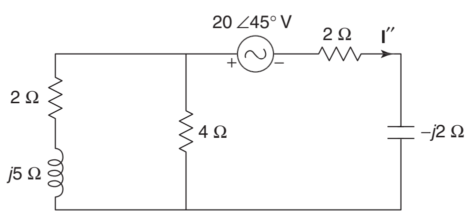


Fig. 3.45

$$\mathbf{Z}_T = (2-j2) + \frac{4(2+j5)}{4+2+j5} = 4.48\angle -8.84° \ \Omega$$
$$\mathbf{I}'' = \frac{20\angle 45°}{4.48\angle -8.84°} = 4.46\angle 53.84° \text{ A} \quad (\leftarrow) = -4.46\angle 53.84° \text{ A} \quad (\rightarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 1.54\angle 37.24 - 4.46\angle 53.84° = 3.01\angle -117.78° \text{ A}$$

$$\mathbf{V}_c = (-j2)\mathbf{I} = (-j2)\ (3.01\angle -117.78°) = 6.02\angle 152.22° \text{ V}$$

Example 3.25 *Find the node voltage V_2 in the network of Fig. 3.46.*

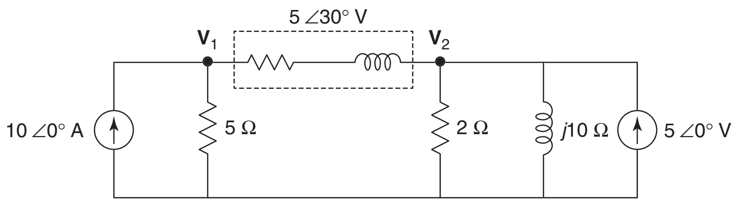


Fig. 3.46

Solution

Step I When the $10\angle 0°$ A source is acting alone (Fig. 3.47)

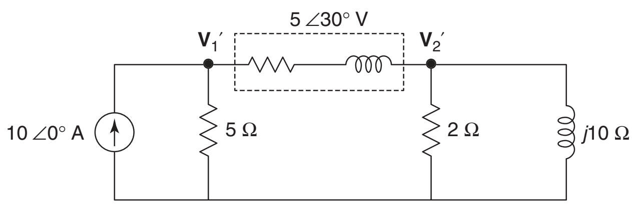


Fig. 3.47

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1'}{5} + \frac{\mathbf{V}_1' - \mathbf{V}_2'}{5\angle 30°} = 10\angle 0°$$

$$\left(\frac{1}{5} + \frac{1}{5\angle 30°}\right)\mathbf{V}_1' - \frac{1}{5\angle 30°}\mathbf{V}_2' = 10\angle 0°$$

$$(0.37 - j0.1)\mathbf{V}_1' - (0.17 - j0.1)\mathbf{V}_2' = 10\angle 0° \quad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2' - \mathbf{V}_1'}{5\angle 30°} + \frac{\mathbf{V}_2'}{2} + \frac{\mathbf{V}_2'}{j10} = 0$$

$$-\frac{1}{5\angle 30°}\mathbf{V}_1' + \left(\frac{1}{5\angle 30°} + \frac{1}{2} + \frac{1}{j10}\right)\mathbf{V}_2' = 0$$

$$-(0.17 - j0.1)\mathbf{V}_1' + (0.67 - j0.2)\mathbf{V}_2' = 0 \quad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.37 - j0.1 & -(0.17 - j0.1) \\ -(0.17 - j0.1) & 0.67 - j0.2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_2' = \frac{\begin{vmatrix} 0.37 - j0.1 & 10\angle 0^\circ \\ -(0.17 - j0.1) & 0 \end{vmatrix}}{\begin{vmatrix} 0.37 - j0.1 & -(0.17 - j0.1) \\ -(0.17 - j0.1) & 0.67 - j0.2 \end{vmatrix}} = 8.57\angle -3.36^\circ \text{ V}$$

Step II When the $5\angle 0^\circ$ A source is acting alone (Fig. 3.48)

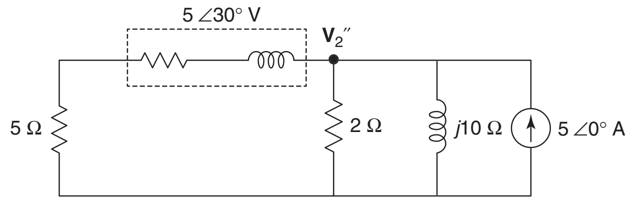


Fig. 3.48

$$\frac{\mathbf{V}_2''}{5\angle 30^\circ + 5} + \frac{\mathbf{V}_2''}{2} + \frac{\mathbf{V}_2''}{j10} = 5\angle 0^\circ$$

$$(0.61\angle -11.93^\circ)\mathbf{V}_2'' = 5\angle 0^\circ$$

$$\mathbf{V}_2'' = 8.2\angle 11.93^\circ \text{ V}$$

Step III By superposition theorem,

$$\mathbf{V}_2 = \mathbf{V}_2' + \mathbf{V}_2'' = 8.57\angle -3.36^\circ + 8.2\angle 11.93^\circ = 16.62\angle 4.12^\circ \text{ V}$$

Example 3.26 *Find current through inductor in the network of Fig. 3.49.*

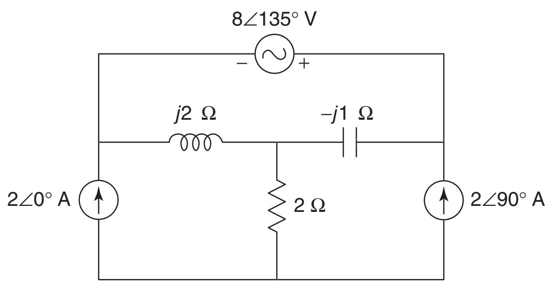


Fig. 3.49

Solution

Step I When the $8\angle 135^\circ$ V source is acting alone (Fig. 3.50)

Applying KVL to the mesh,

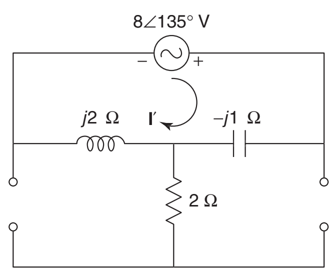


Fig. 3.50

$$8\angle 135^\circ - (-j1)\mathbf{I}' - j2\mathbf{I}' = 0$$

$$\mathbf{I}' = \frac{8\angle 135^\circ}{j1} = 8\angle 45^\circ \text{ A } (\leftarrow) = 8\angle -135^\circ \text{ A } (\rightarrow)$$

Step II When the $2\angle 0°$ A source is acting alone (Fig. 3.51)

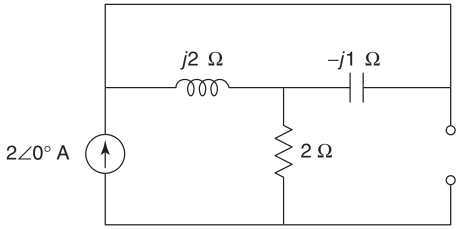


Fig. 3.51

The network can be redrawn as shown in Fig. 3.52.

By current division rule,

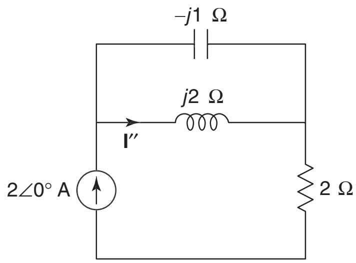


Fig. 3.52

$$\mathbf{I}'' = 2\angle 0°\left(\frac{-j1}{-j1+j2}\right) = 2\angle 0°\left(\frac{-j1}{j1}\right) = 2\angle 180° \text{ A}(\rightarrow)$$

Step III When the $2\angle 90°$ A source is acting alone (Fig. 3.53)

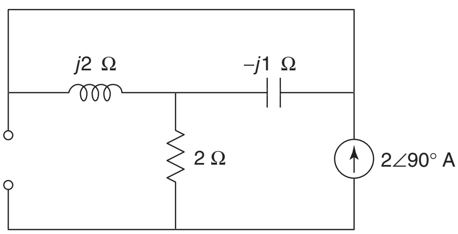


Fig. 3.53

The network can be redrawn as shown in Fig. 3.54.

By current division rule,

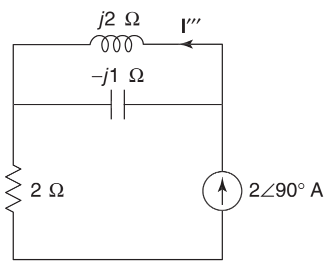


Fig. 3.54

$$\mathbf{I}''' = 2\angle 90°\left(\frac{-j1}{-j1+j2}\right) = 2\angle -90° \text{ A } (\leftarrow) = 2\angle 90° \text{ A } (\rightarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 8\angle{-135°} + 2\angle 180° + 2\angle 90° = 8.49\angle{-154.47°}\text{A}$$

Example 3.27 *Determine the source voltage V_S so that the current through 2 Ω resistor is zero in the network of Fig. 3.55.*

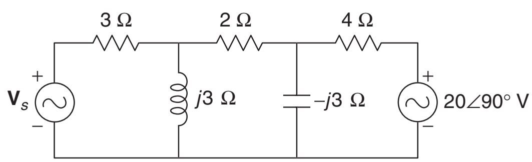


Fig. 3.55

Solution

Step I When the voltage source $\mathbf{V}_s$ is acting alone (Fig. 3.56)

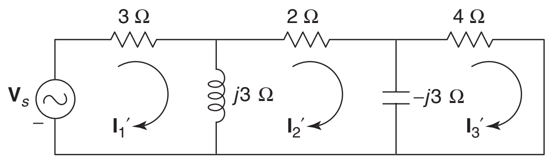


Fig. 3.56

Appling KVL to Mesh 1,

$$\mathbf{V}_s - 3\mathbf{I}_1' - j3(\mathbf{I}_1' - \mathbf{I}_2') = 0$$

$$(3 + j3)\mathbf{I}_1' - j3\mathbf{I}_2' = \mathbf{V}_s \qquad \text{...(i)}$$

Appling KVL to Mesh 2,

$$-j3(\mathbf{I}_2' - \mathbf{I}_1') - 2\mathbf{I}_2' + j3(\mathbf{I}_2' - \mathbf{I}_3') = 0$$

$$-j3\mathbf{I}_1' + 2\mathbf{I}_2' + j3\mathbf{I}_3' = 0 \qquad \text{...(ii)}$$

Appling KVL to Mesh 3,

$$-j3(\mathbf{I}_3' - \mathbf{I}_2') - 4\mathbf{I}_3' = 0$$

$$j3\mathbf{I}_2' + (4 - j3)\mathbf{I}_3' = 0 \qquad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1' \\ \mathbf{I}_2' \\ \mathbf{I}_3' \end{bmatrix} = \begin{bmatrix} \mathbf{V}_s \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2' = \frac{\begin{vmatrix} 3+j3 & \mathbf{V}_s & 0 \\ -j3 & 0 & j3 \\ 0 & 0 & 4-j3 \end{vmatrix}}{\begin{vmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{vmatrix}} = \frac{(9+j12)\,\mathbf{V}_s}{\Delta}$$

Step II When the $20\angle 90°$ V source is acting alone (Fig. 3.57)

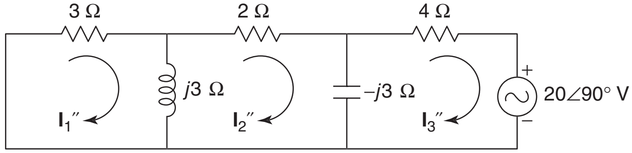


Fig. 3.57

Applying KVL to Mesh 1,

$$-3\mathbf{I}_1'' - j3(\mathbf{I}_1'' - \mathbf{I}_2'') = 0$$

$$(3 + j3)\,\mathbf{I}_1'' - j3\mathbf{I}_2'' = 0 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j3(\mathbf{I}_2'' - \mathbf{I}_1'') - 2\mathbf{I}_2'' + j3(\mathbf{I}_2'' - \mathbf{I}_3'') = 0$$

$$-j3\mathbf{I}_1'' + 2\mathbf{I}_2'' + j3\mathbf{I}_3'' = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$j3(\mathbf{I}_3'' - \mathbf{I}_2'') - 4\mathbf{I}_3'' - 20\angle 90° = 0$$

$$j3\mathbf{I}_2'' + (4 - j3)\mathbf{I}_3'' = -20\angle 90° \qquad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1'' \\ \mathbf{I}_2'' \\ \mathbf{I}_3'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -20\angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2'' = \frac{\begin{vmatrix} 3+j3 & 0 & 0 \\ -j3 & 0 & j3 \\ 0 & -20\angle 90° & 4-j3 \end{vmatrix}}{\begin{vmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{vmatrix}} = \frac{-180 - j180}{\Delta}$$

Step III By superposition theorem,

$$\mathbf{I}_2 = \mathbf{I}_2' + \mathbf{I}_2'' = \frac{(9 + j12)\mathbf{V}_s + (-180 - j180)}{\Delta} = 0$$

$$(9 + j12)\mathbf{V}_s + (-180 - j180) = 0$$

$$(9 + j12)\mathbf{V}_s = 180 + j180$$

$$\mathbf{V}_s = 16.97\angle -8.13° \text{ V}$$

3.5 THEVENIN'S THEOREM

Thevenin's theorem gives us a method for simplifying a network. In Thevenin's theorem, *any linear network can be replaced by a voltage source* $\mathbf{V}_{\text{Th}}$ *in series with an impedance* $\mathbf{Z}_{\text{Th}}$.

Example 3.28 *Obtain Thevenin's equivalent network for the terminals A and B in Fig. 3.58.*

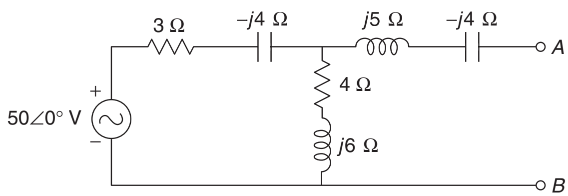


Fig. 3.58

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.59)

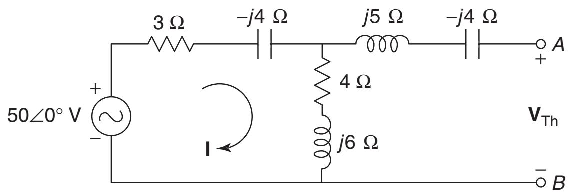


Fig. 3.59

Applying KVL to the mesh,

$50\ \angle 0° - (3 - j4)\,\mathbf{I} - (4 + j6)\,\mathbf{I} = 0$

$$\mathbf{I} = \frac{50\angle 0°}{(3 - j4) + (4 + j6)} = 6.87\angle -15.95°\ \text{A}$$

$$\mathbf{V}_{\text{Th}} = (4 + j6)\,\mathbf{I} = (4 + j6)\ (6.87\ \angle -15.95°) = 49.5\ \angle 40.35°\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.60)

$$\mathbf{Z}_{\text{Th}} = (j5 - j4) + \frac{(3 - j4)(4 + j6)}{(3 - j4) + (4 + j6)} = 4.83\angle -1.13°\ \Omega$$

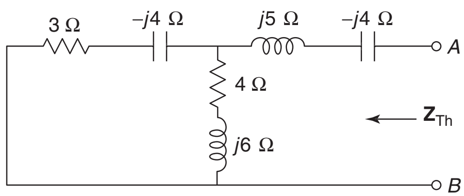


Fig. 3.60

Step III Thevenin's Equivalent Network (Fig. 3.61)

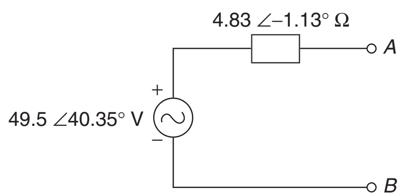


Fig. 3.61

Example 3.29 *Find Thevenin's equivalent network for Fig. 3.62.*

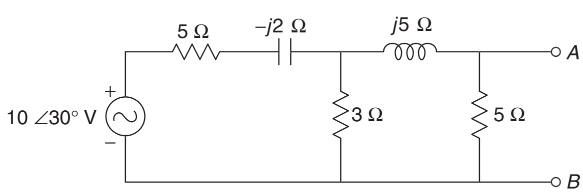


Fig. 3.62

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 3.63)

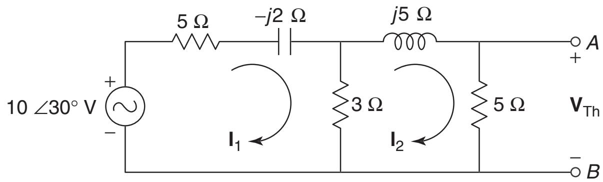


Fig. 3.63

Applying KVL to Mesh 1,

$$10\angle 30° - (5 - j2)\,\mathbf{I}_1 - 3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 - j2)\,\mathbf{I}_1 - 3\mathbf{I}_2 = 10\angle 30° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(\mathbf{I}_2 - \mathbf{I}_1) - j5\,\mathbf{I}_2 - 5\,\mathbf{I}_2 = 0$$

$$-3\mathbf{I}_1 + (8 + j5)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form;

$$\begin{bmatrix} 8-j2 & -3 \\ -3 & 8+j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 30° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8-j2 & 10\angle 30° \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} 8-j2 & -3 \\ -3 & 8+j5 \end{vmatrix}} = 0.433\angle 9.7° \text{ A}$$

$$\mathbf{V}_{Th} = 5\mathbf{I}_2 = 5(0.433\angle 9.7°) = 2.16\angle 9.7° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 3.64)

$$\mathbf{Z}_{Th} = \left[\left\{\frac{(5-j2)3}{5-j2+3}\right\} + j5\right] \| 5$$

$$= [1.94 - j0.265 + j5] \| 5 = (1.94 + j4.735) \| 5$$

$$= \frac{(1.94 + j4.735)5}{6.94 + j4.735} = 3.04\angle 33.4° \ \Omega$$

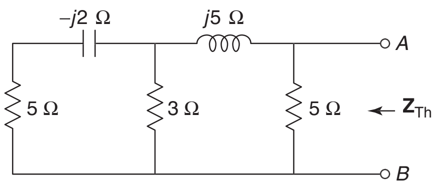


Fig. 3.64

Step III Thevenin's equivalent Network (Fig. 3.65)

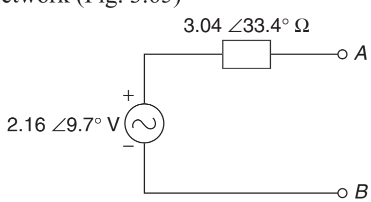


Fig. 3.65

Example 3.30 *Obtain Thevenin's equivalent network for Fig. 3.66.*

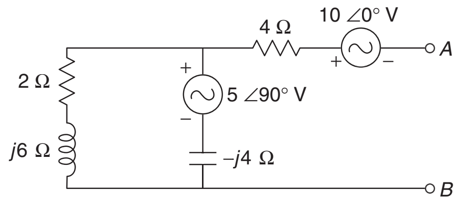


Fig. 3.66

Solution

Step I Calculation of $\mathbf{V}_{Th}$

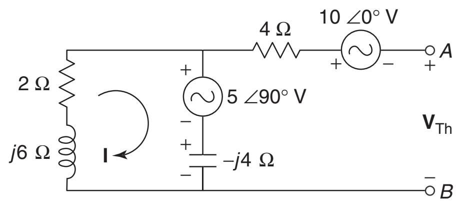


Fig. 3.67

Applying KVL to the mesh,

$$(2+j6-j4)\mathbf{I}-5\angle 90° = 0$$

$$\mathbf{I} = \frac{5\angle 90°}{2+j2} = 1.77\angle 45° \text{ A}$$

$$\mathbf{V}_{Th} = (-j4)\,\mathbf{I} + 5\angle 90° - 10\angle 0° = (4\angle -90°)\,(1.77\angle 45°) + 5\angle 90° - 10\angle 0° = 18\angle 146.31° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 3.67)

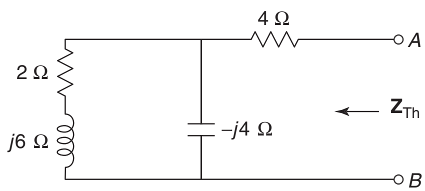


Fig. 3.68

$$\mathbf{Z}_{Th} = 4 + \frac{(2+j6)(-j4)}{2+j2} = 11.3\angle -44.93° \ \Omega$$

Step III Thevenin's Equivalent Network

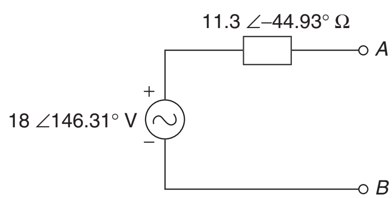


Fig. 3.69

Example 3.31 *Obtain Thevenin's equivalent network for Fig. 3.70.*

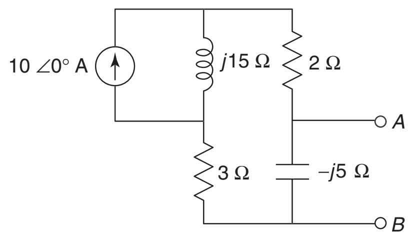


Fig. 3.70

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.71)
By current division rule,

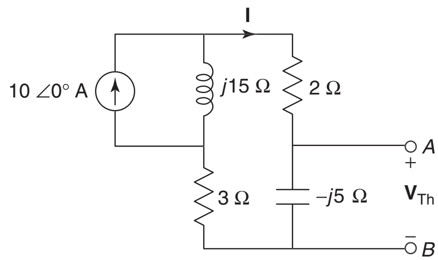


Fig. 3.71

$$\mathbf{I} = \frac{(10\angle 0°)(j15)}{5 - j5 + j15} = 13.42\angle 26.57° \text{ A}$$

$$\begin{aligned}\mathbf{V}_{\text{Th}} &= (-j5)\,\mathbf{I} \\ &= (5\ \angle -90°)\ (13.42\ \angle 26.57°) = 67.08\ \angle -63.43° \text{ V}\end{aligned}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.72)

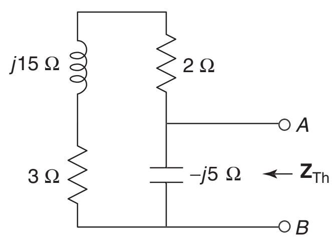


Fig. 3.72

$$\mathbf{Z}_{\text{Th}} = \frac{(-j5)(5 + j15)}{-j5 + 5 + j15} = 7.07\angle -81.86° \ \Omega$$

Step III Thevenin's Equivalent Network

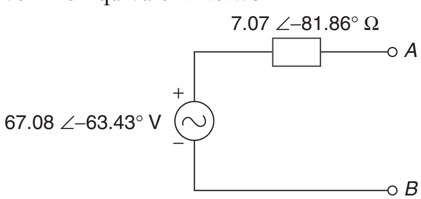


Fig. 3.73

Example 3.32 *Obtain Thevenin's equivalent network for Fig. 3.74.*

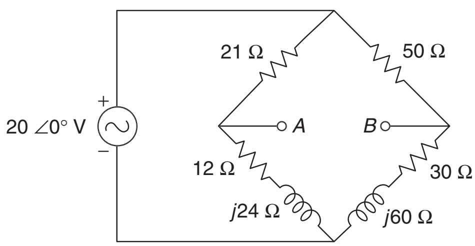


Fig. 3.74

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.75)

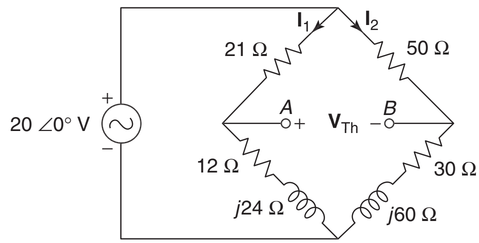


Fig. 3.75

$$\mathbf{I}_1 = \frac{20\angle 0^\circ}{21+12+j24} = 0.49\angle -36.02^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{20\angle 0^\circ}{80+j60} = 0.2\angle -36.86^\circ \text{ A}$$

$$\begin{aligned}\mathbf{V}_{\text{Th}} &= (12+j24)\,\mathbf{I}_1 - (30+j60)\,\mathbf{I}_2 \\ &= (26.83\angle 63.43^\circ)(0.49\angle -36.02^\circ) - (67.08\angle 63.43^\circ)(0.2\angle -36.86^\circ) \\ &= 0.33\angle 171.12^\circ \text{ V}\end{aligned}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.76)

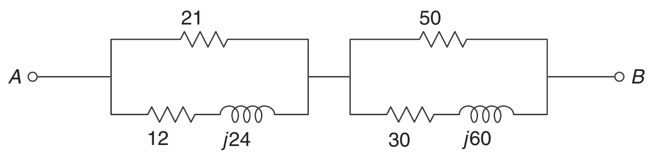


Fig. 3.76

$$\mathbf{Z}_{\text{Th}} = \frac{21(12+j24)}{33+j24} + \frac{50(30+j60)}{80+j60} = 47.4\angle 26.8^\circ\ \Omega$$

Step III Thevenin's Equivalent Network

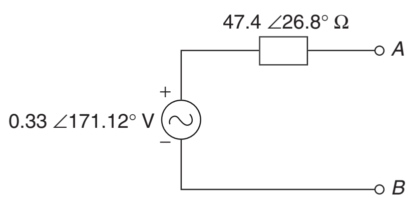


Fig. 3.77

Example 3.33 *Find Thevenin's equivalent network across terminals A and B for Fig. 3.78.*

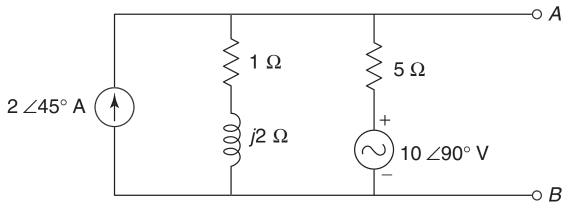


Fig. 3.78

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.79)

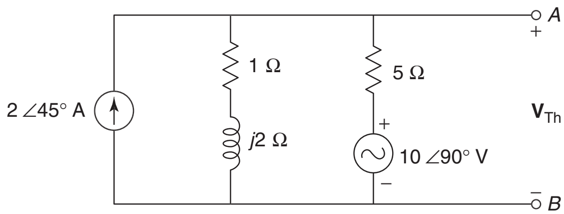


Fig. 3.79

Applying KCL at the node,

$$\frac{\mathbf{V}_{\text{Th}}}{1+j2}+\frac{\mathbf{V}_{\text{Th}}-10\angle 90^\circ}{5}=2\angle 45^\circ$$

$$\left(\frac{1}{1+j2}+\frac{1}{5}\right)\mathbf{V}_{\text{Th}}=2\angle 45^\circ+2\angle 90^\circ$$

$$(0.57\angle -45^\circ)\mathbf{V}_{\text{Th}}=3.7\angle 67.5^\circ$$

$$\mathbf{V}_{\text{Th}}=6.49\angle 112.5^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.80)

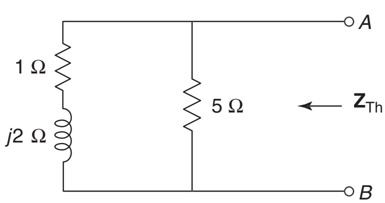


Fig. 3.80

$$\mathbf{Z}_{\text{Th}}=\frac{5(1+j2)}{5+1+j2}=1.77\angle 45^\circ\ \Omega$$

Step III Thevenin's Equivalent Network (Fig. 3.81)

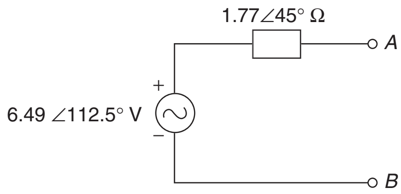


Fig. 3.81

Example 3.34 *Find the current through the* $(5 + j2)\ \Omega$ *impedance in the network of Fig. 3.82.*

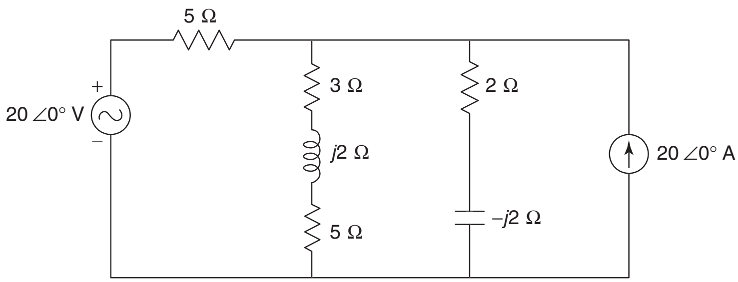


Fig. 3.82

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.83)

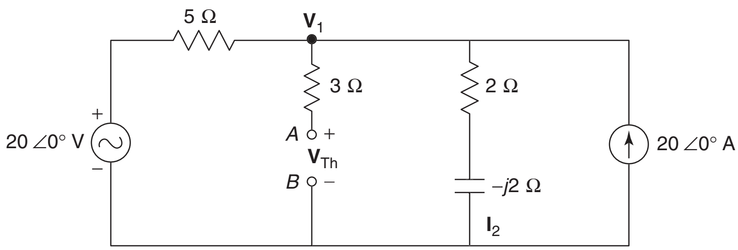


Fig. 3.83

Applying KCL at the node,

$$\frac{\mathbf{V}_1 - 20\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{2 - j2} = 20\angle 0^\circ$$

$$\left(\frac{1}{5} + \frac{1}{2 - j2}\right)\mathbf{V}_1 = 20\angle 0^\circ + 4\angle 0^\circ$$

$$0.51\angle 29.05^\circ\ \mathbf{V}_1 = 24\angle 0^\circ$$

$$\mathbf{V}_1 = 47.06\angle -29.05^\circ\ \text{V}$$

$$\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = 47.06\angle -29.05^\circ\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.84)

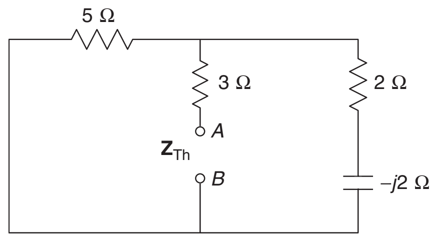


Fig. 3.84

$$\mathbf{Z}_{\text{Th}} = 3 + \frac{5(2-j2)}{5+2-j2} = 4.79\angle -11.35°\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.85)

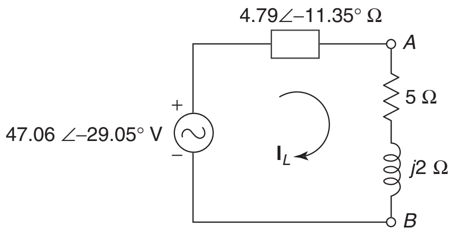


Fig. 3.85

$$\mathbf{I}_L = \frac{47.06\angle -29.05°}{4.79\angle -11.35° + 5 + j2} = 4.73\angle -39.96°\ \text{A}$$

Example 3.35 *Find the current through the 5 Ω resistor in the network of Fig. 3.86.*

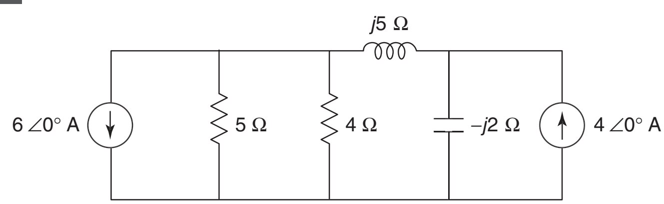


Fig. 3.86

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.87)

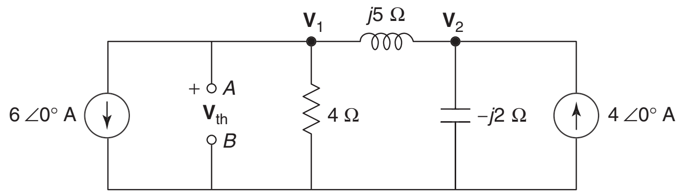


Fig. 3.87

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1}{4}+\frac{\mathbf{V}_1-\mathbf{V}_2}{j5}+6\angle 0°=0$$

$$\left(\frac{1}{4}+\frac{1}{j5}\right)\mathbf{V}_1-\frac{1}{j5}\mathbf{V}_2=-6\angle 0°$$

$$(0.25-j0.2)\mathbf{V}_1+j0.2\mathbf{V}_2=-6\angle 0° \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2-\mathbf{V}_1}{j5}+\frac{\mathbf{V}_2}{-j2}=4\angle 0°$$

$$\left(-\frac{1}{j5}\right)\mathbf{V}_1+\left(\frac{1}{j5}-\frac{1}{j2}\right)\mathbf{V}_2=4\angle 0°$$

$$j0.2\,\mathbf{V}_1+j0.3\,\mathbf{V}_2=4\angle 0° \qquad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix}0.25-j0.2 & j0.2\\ j0.2 & j0.3\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}-6\angle 0°\\ 4\angle 0°\end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1=\frac{\begin{vmatrix}-6\angle 0° & j0.2\\ 4\angle 0° & j0.3\end{vmatrix}}{\begin{vmatrix}0.25-j0.2 & j0.2\\ j0.2 & j0.3\end{vmatrix}}=20.8\angle -126.87°\text{ V}$$

$$\mathbf{V}_{\text{Th}}=\mathbf{V}_1=20.8\angle -126.87°\text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.88)

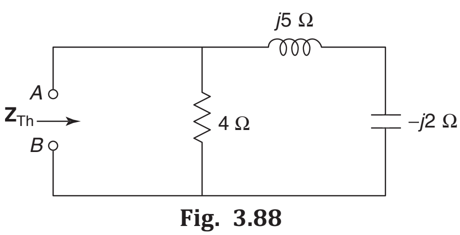


Fig. 3.88

$$\mathbf{Z}_{\text{Th}}=\frac{4(-j2+j5)}{4-j2+j5)}=2.4\angle 53.13°\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.89)

$$\mathbf{I}_L=\frac{20.8\angle -126.87°}{2.4\angle 53.13°+5}=3.1\angle -143.47°\text{ A}$$

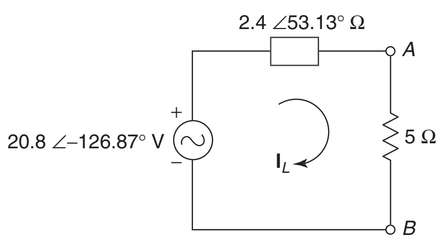


Fig. 3.89

Example 3.36 *In the network of Fig. 3.90, find the current through the 10 Ω resistor.*

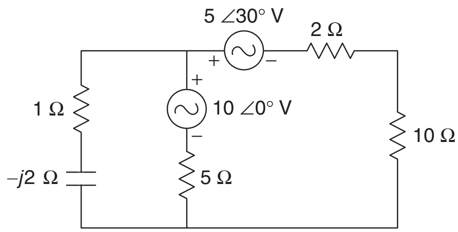


Fig. 3.90

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.91)

Applying KVL to the mesh,

$$j2\mathbf{I} - 1\mathbf{I} - 10\angle 0^\circ - 5\mathbf{I} = 0$$

$$(j2 - 6)\mathbf{I} = 10\angle 0^\circ$$

$$\mathbf{I} = 1.58\angle -161.57^\circ \text{ A}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$5\mathbf{I} + 10\angle 0^\circ - 5\angle 30^\circ - 0 - \mathbf{V}_{\text{Th}} = 0$$

$$5(1.58\angle -161.57^\circ) - 10\angle 0^\circ - 5\angle 30^\circ - \mathbf{V}_{\text{Th}} = 0$$

$$\mathbf{V}_{\text{Th}} = 5.32\angle -110.06^\circ \text{ V}$$

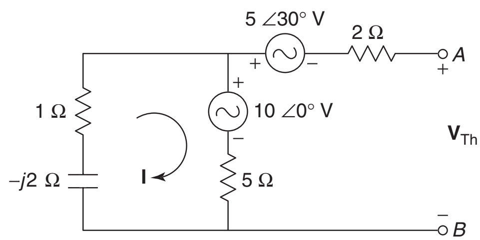


Fig. 3.91

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.92)

$$\mathbf{Z}_{\text{Th}} = 2 + \frac{5(1 - j2)}{5 + 1 - j2} = 3.48\angle -21.04^\circ \ \Omega$$

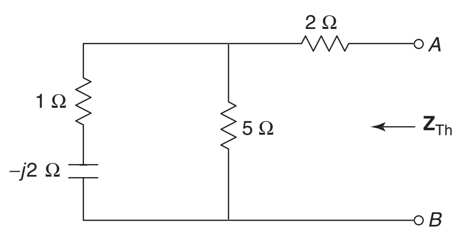


Fig. 3.92

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.93)

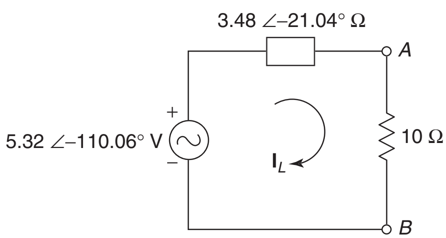


Fig. 3.93

$$\mathbf{I}_L = \frac{5.32\angle -110.06^\circ}{3.48\angle -21.04^\circ + 10} = 0.4\angle -104.67^\circ \text{ A}$$

Example 3.37 *Find the current through $(4+j6)\ \Omega$ impedance in the network of Fig. 3.94.*

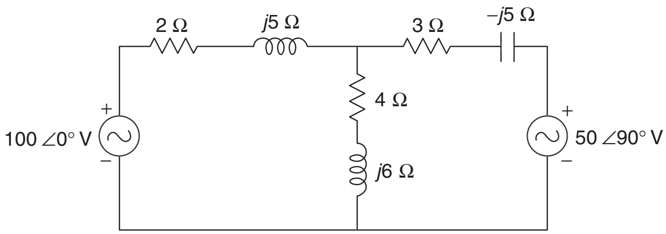


Fig. 3.94

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.95)

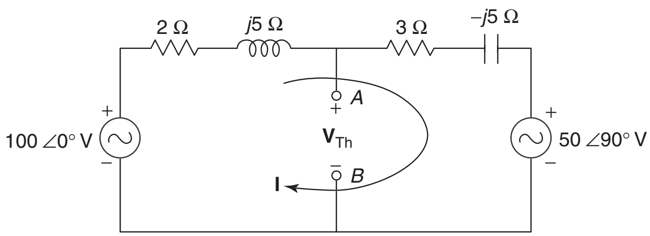


Fig. 3.95

Applying KVL to the mesh,

$$100\angle 0^\circ - 2\mathbf{I} - j5\mathbf{I} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90^\circ = 0$$

$$\mathbf{I} = 22.36\angle -26.57^\circ\ \text{A}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$\mathbf{V}_{\text{Th}} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90^\circ = 0$$

$$\mathbf{V}_{\text{Th}} - (3 - j5)(22.36\angle -26.57^\circ) - 50\angle 90^\circ = 0$$

$$\mathbf{V}_{\text{Th}} = 80.61\angle -82.88^\circ\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.96)

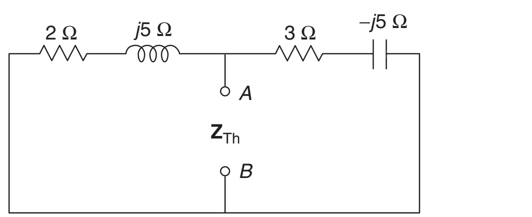


Fig. 3.96

$$\mathbf{Z}_{\text{Th}} = \frac{(2+j5)(3-j5)}{2+j5+3-j5} = 6.28\angle 9.16^\circ\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.97)

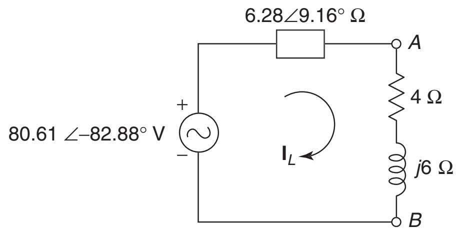


Fig. 3.97

$$\mathbf{I}_L = \frac{80.61\angle -82.88^\circ}{6.28\angle 9.16^\circ + 4 + j6} = 6.52\angle -117.34^\circ \text{ A}$$

Example 3.38 *Obtain Thevenin's equivalent network across terminals A and B in Fig. 3.98.*

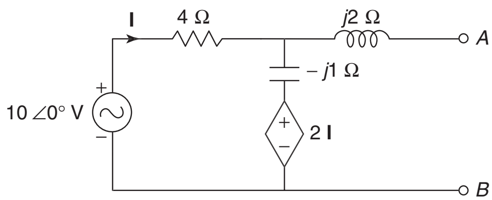


Fig. 3.98

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 3.99)

Applying KVL to the mesh,

$$10\angle 0^\circ - 4\mathbf{I} + j1\mathbf{I} - 2\mathbf{I} = 0$$

$$\mathbf{I} = 1.64\angle 9.46^\circ \text{ A}$$

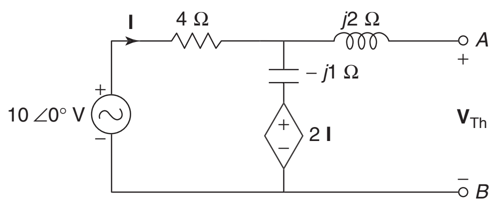


Fig. 3.99

Writing $\mathbf{V}_{Th}$ equation,

$$10\angle 0^\circ - 4\mathbf{I} - 0 - \mathbf{V}_{Th} = 0$$

$$10\angle 0^\circ - 4(1.64\angle 9.46^\circ) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 3.69\angle -17^\circ \text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 3.100)

From Fig. 3.100,

$$\mathbf{I} = \mathbf{I}_1$$

Applying KVL to Mesh 1,

$$10\angle 0^\circ - 4\mathbf{I}_1 + j1(\mathbf{I}_1 - \mathbf{I}_2) - 2\mathbf{I} = 0$$

$$10\angle 0^\circ - 4\mathbf{I}_1 + j1\mathbf{I}_1 - j1\mathbf{I}_2 - 2\mathbf{I}_1 = 0$$

$$(6 - j1)\,\mathbf{I}_1 + j1\mathbf{I}_2 = 10\angle 0^\circ \quad \ldots\text{(i)}$$

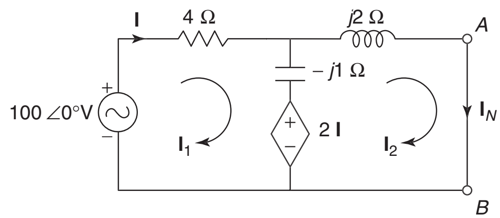


Fig. 3.100

Applying KVL to Mesh 2,

$$2\mathbf{I} + j1(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 = 0$$

$$2\mathbf{I}_1 + j1\mathbf{I}_2 - j1\mathbf{I}_1 - j_2\mathbf{I}_2 = 0$$

$$(2-j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 0 \qquad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 6-j1 & j1 \\ 2-j1 & -j1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 6-j1 & 10\angle 0^\circ \\ 2-j1 & 0 \end{vmatrix}}{\begin{vmatrix} 6-j1 & j1 \\ 2-j1 & -j1 \end{vmatrix}} = 2.71\angle -102.53^\circ \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 2.71\angle -102.53^\circ \text{ A}$$

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{3.69\angle -17^\circ}{2.71\angle -102.53^\circ} = 1.36\angle 85.53^\circ\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.101)

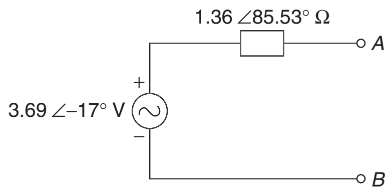


Fig. 3.101

Example 3.39 *Find Thevenin's equivalent network across terminals A and B for Fig. 3.102.*

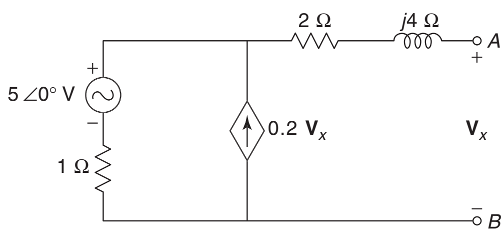


Fig. 3.102

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.103)

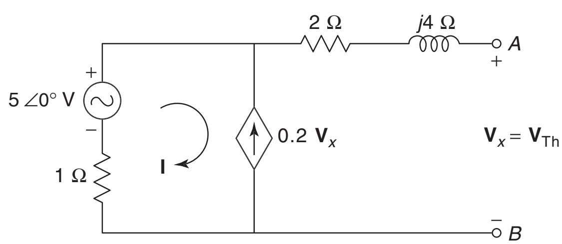


Fig. 3.103

From Fig. 3.103,

$$\mathbf{I} = -0.2\,\mathbf{V}_x \qquad \text{... (i)}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$-\mathbf{I} + 5\angle 0^\circ - 0 - \mathbf{V}_x = 0$$

$$0.2\mathbf{V}_x + 5\angle 0° - \mathbf{V}_x = 0$$
$$\mathbf{V}_x = 6.25\angle 0° \text{ V}$$
$$\mathbf{V}_{\text{Th}} = \mathbf{V}_x = 6.25\angle 0° \text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 3.104)

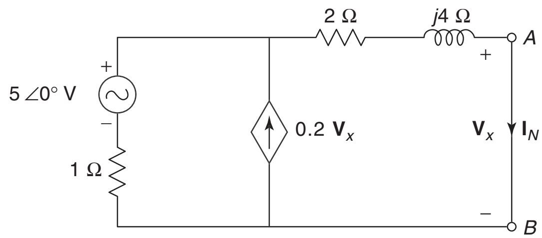


Fig. 3.104

From Fig. 3.104,

$$\mathbf{V}_x = 0$$

The dependent source depends on the controlling variable $\mathbf{V}_x$. When $\mathbf{V}_x = 0$, the dependent source vanishes, i.e. $0.2\,\mathbf{V}_x = 0$ as shown in Fig. 3.105.

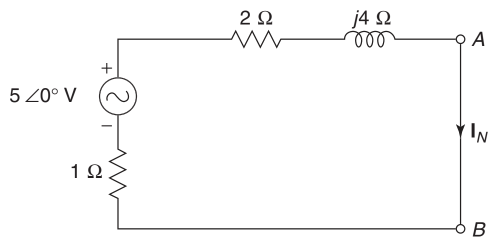


Fig. 3.105

$$\mathbf{I}_N = \frac{5\angle 0°}{1+2+j4} = 1\angle -53.13° \text{ A}$$

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{6.25\angle 0°}{1\angle -53.13°} = 6.25\angle 53.13° \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.106)

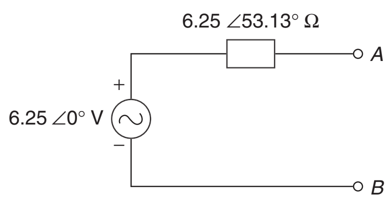


Fig. 3.106

3.6 NORTON'S THEOREM

Norton's theorem states that *any linear network can be replaced by a current source* $\mathbf{I}_N$ *parallel with an impedance* $\mathbf{Z}_N$ *where* $\mathbf{I}_N$ *is the current flowing through the short-circuited path placed across the terminals*.

Example 3.40 *Obtain Norton's equivalent network between terminals A and B as shown in Fig. 3.107.*

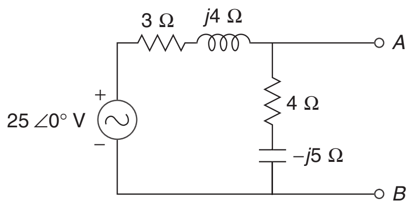


Fig. 3.107

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.108)

When a short circuit is placed across $(4 - j4)\ \Omega$ impedance, it gets shorted as shown in Fig. 3.109.

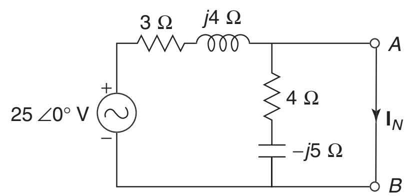


Fig. 3.108

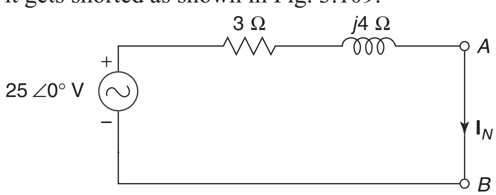


Fig. 3.109

$$\mathbf{I}_N = \frac{25\angle 0^\circ}{3 + j4} = 5\angle -53.13^\circ \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.110)

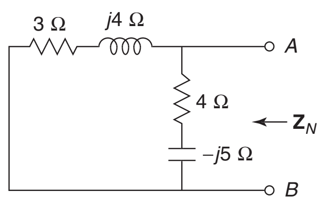


Fig. 3.110

$$\mathbf{Z}_N = \frac{(3 + j4)(4 - j5)}{3 + j4 + 4 - j5} = 4.53\angle 9.92^\circ\ \Omega$$

Step III Norton's Equivalent Network

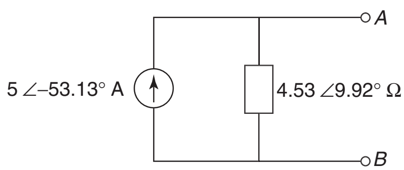


Fig. 3.111

Example 3.41 *Obtain Norton's equivalent network at the terminals A and B in Fig. 3.112.*

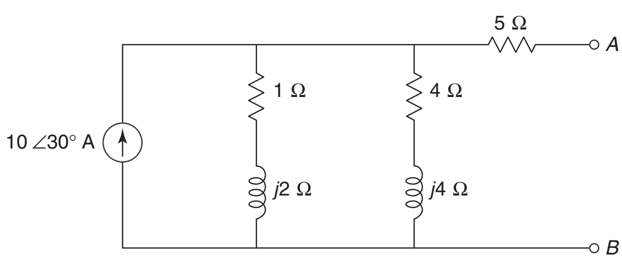


Fig. 3.112

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.113)

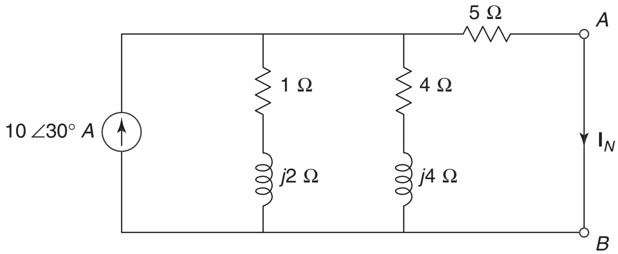


Fig. 3.113

By series-parallel reduction technique (Fig. 3.114)

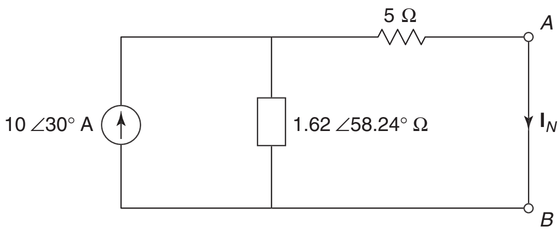


Fig. 3.114

$$\mathbf{I}_N = (10\angle 30°)\left(\frac{1.62\angle 58.24°}{1.62\angle 58.24° + 5}\right) = 2.69\angle 75° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.115)

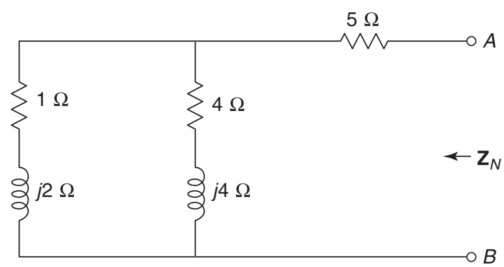


Fig. 3.115

$$\mathbf{Z}_N = 5 + \frac{(1+j2)(4+j4)}{1+j2+4+j4} = 6.01\angle 13.24° \ \Omega$$

Step III Norton's Equivalent Network (Fig. 3.116)

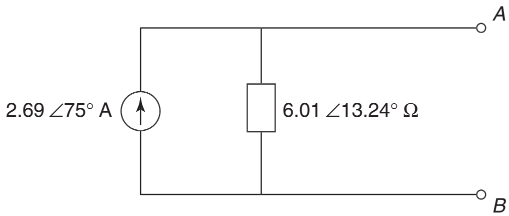


Fig. 3.116

Example 3.42 *Find Norton's equivalent network across terminals A and B in Fig. 3.117.*

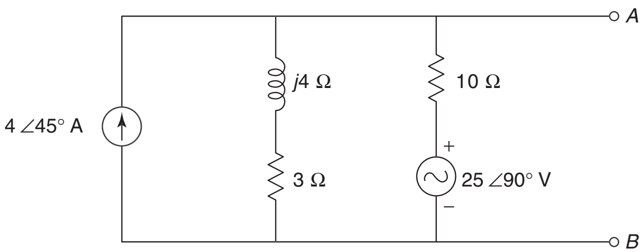


Fig. 3.117

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.118)

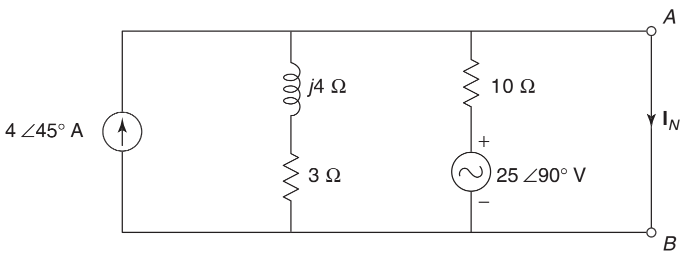


Fig. 3.118

When a short circuit is placed across the $(3+j4)\ \Omega$ impedance, it gets shorted as shown in Fig. 3.119.

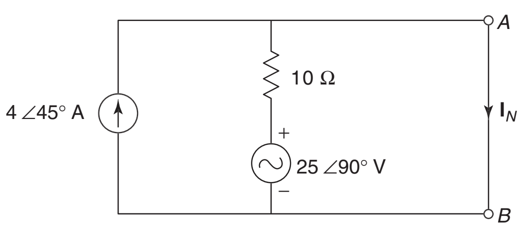


Fig. 3.119

By source transformation, the network is redrawn as shown in Fig. 3.120.

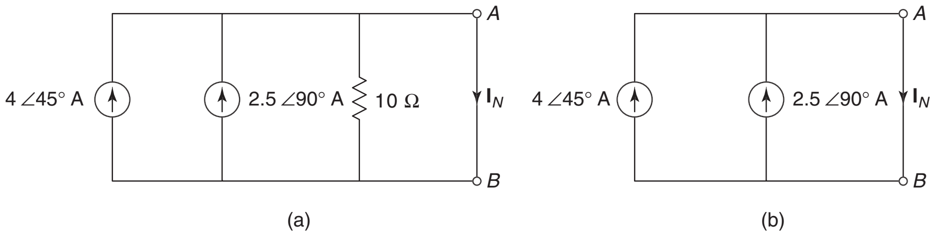


Fig. 3.120

$$I_N = 4\angle 45° + 2.5\angle 90° = 6.03\angle 62.04° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.121)

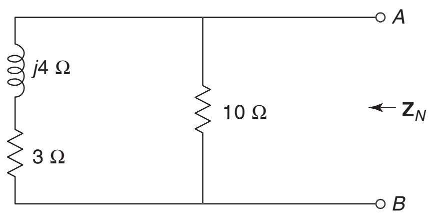


Fig. 3.121

$$\mathbf{Z}_N = \frac{10(3+j4)}{10+3+j4} = 3.68\angle 36.03° \ \Omega$$

Step III Norton's Equivalent Network (Fig. 3.122)

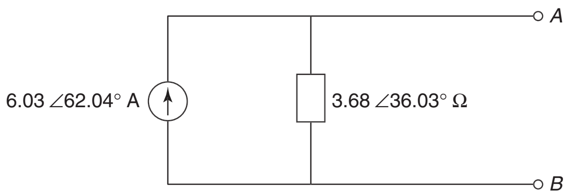


Fig. 3.122

Example 3.43 *Obtain the Norton's equivalent network for Fig. 3.123.*

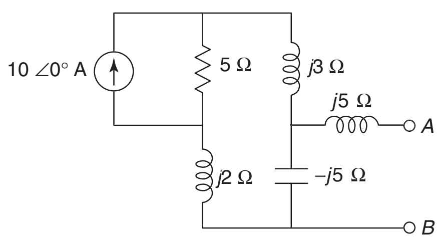


Fig. 3.123

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.124)

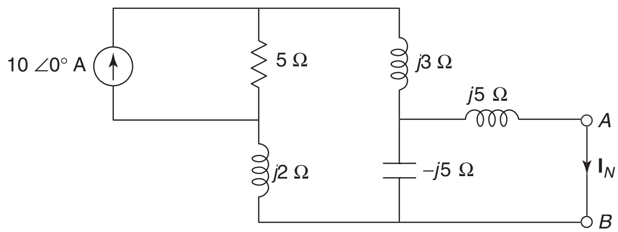


Fig. 3.124

By source transformation, the network can be redrawn as shown in Fig. 3.125.

Writing KVL equations in matrix form,

$$\begin{bmatrix} 5 & j5 \\ j5 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

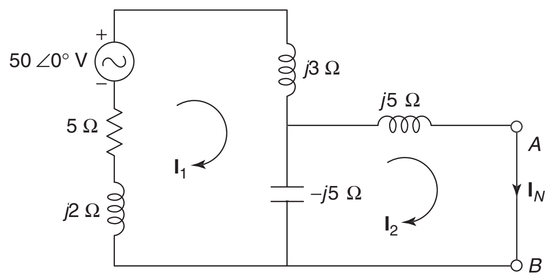


Fig. 3.125

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5 & 50\angle 0^\circ \\ j5 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & j5 \\ j5 & 0 \end{vmatrix}} = 10\angle -90^\circ \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 10\angle -90^\circ \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.126)

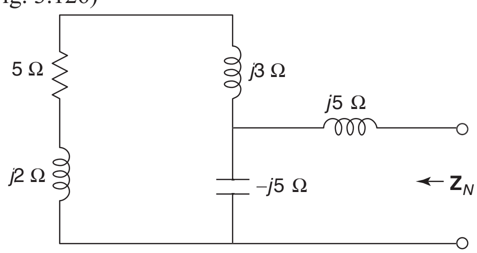


Fig. 3.126

$$\mathbf{Z}_N = j5 + \frac{(5+j5)(-j5)}{5+j5-j5} = 5\ \Omega$$

Step III Norton's Equivalent Network (Fig. 3.127)

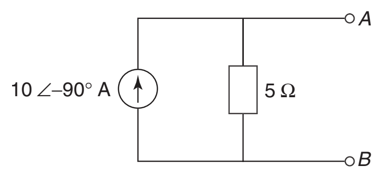


Fig. 3.127

Example 3.44 *Obtain the Norton's equivalent network for Fig. 3.128.*

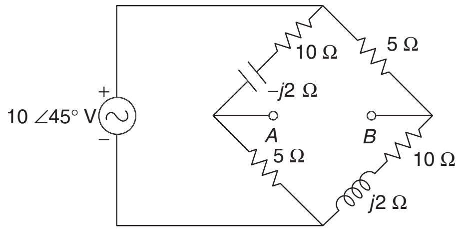


Fig. 3.128

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.129)

Writing KVL equations in matrix form,

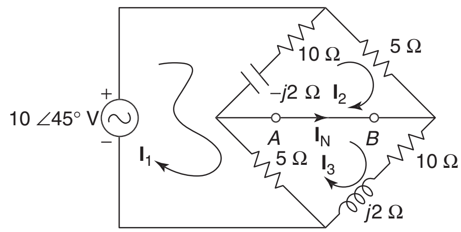


Fig. 3.129

$$\begin{bmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 15-j2 & 10\angle 45^\circ & -5 \\ -10+j2 & 0 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}}{\begin{vmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}} = 1\angle 41.28^\circ \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 15-j2 & -10+j2 & 10\angle 45^\circ \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}} = 0.49\angle 37.41^\circ \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_3 - \mathbf{I}_2 = 0.49\angle 37.41 - 1\angle 41.28^\circ = 0.51\angle -135^\circ \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.130)

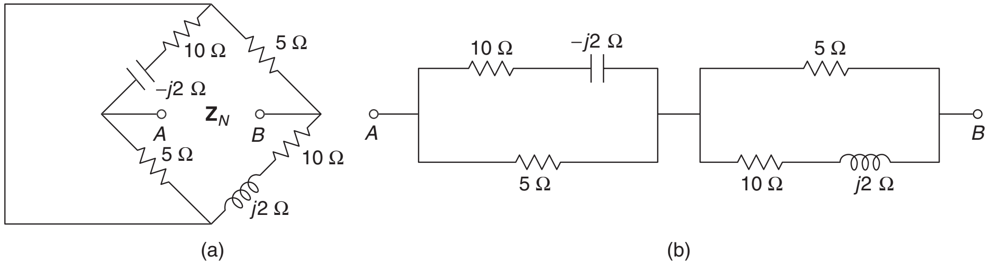


Fig. 3.130

$$\mathbf{Z}_N = \frac{5(10-j2)}{5+10-j2} + \frac{5(10+j2)}{5+10+j2} = 6.72\ \Omega$$

Step III Norton's Equivalent Network (Fig. 3.131)

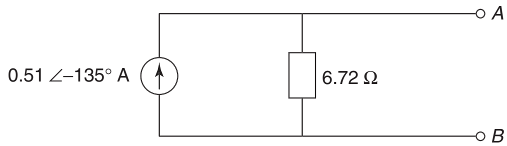


Fig. 3.131

Example 3.45 *Find the current through the 8 Ω resistor in the Network of Fig. 3.132.*

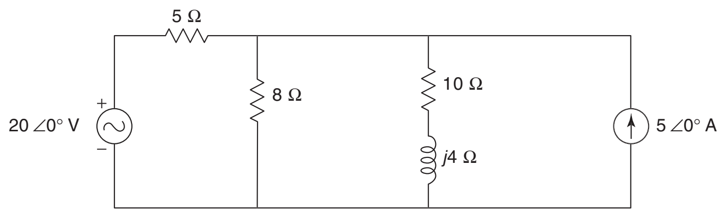


Fig. 3.132

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 3.133)

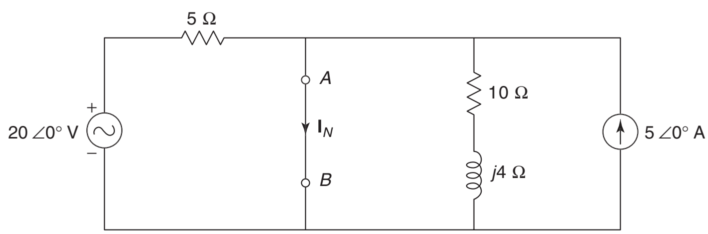


Fig. 3.133

When a short circuit is placed across the $(10 + j4)\ \Omega$ impedance, it gets shorted as shown in Fig. 3.134.

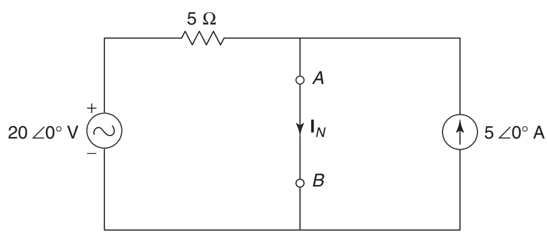


Fig. 3.134

By source transformation, the network is redrawn as shown in Fig. 3.135.

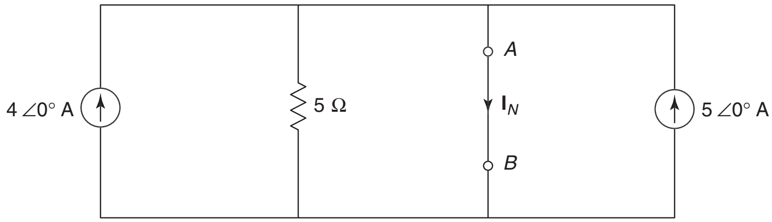


Fig. 3.135

$$\mathbf{I}_N = 4\angle 0° + 5\angle 0° = 9\angle 0° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 3.136)

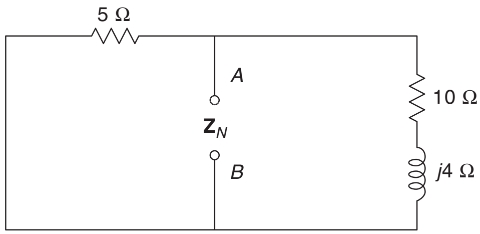


Fig. 3.136

$$\mathbf{Z}_N = \frac{5(10+j4)}{5+10+j4} = 3.47\angle 6.87° \ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.137)

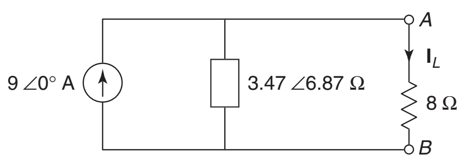


Fig. 3.137

$$\mathbf{I}_L = \frac{9\angle 0°}{3.47\angle 6.87° + 8} = 0.79\angle -2.08° \text{ A}$$

Example 3.46 *Obtain Norton's equivalent network across the terminals A and B in Fig. 3.138.*

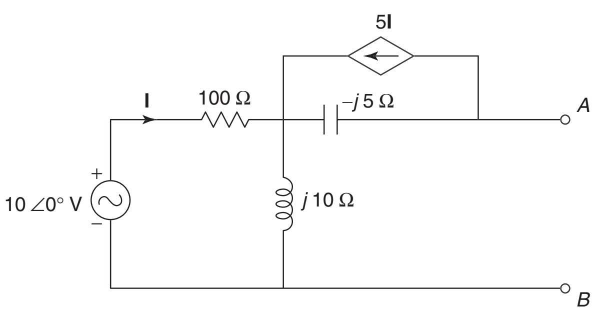


Fig. 3.138

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.139)

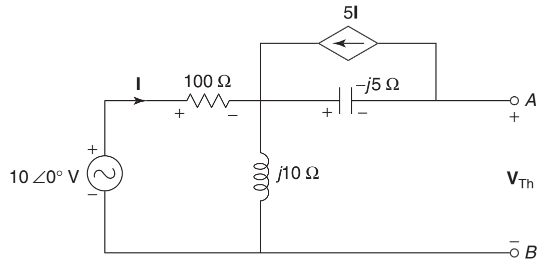


Fig. 3.139

$$\mathbf{I} = \frac{10\angle 0°}{100 + j10} = 0.1\angle -5.71°\ \text{A}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$10\angle 0° - 100\,\mathbf{I} - (-j5)(5\mathbf{I}) - \mathbf{V}_{\text{Th}} = 0$$

$$10\angle 0° - 100\,(0.1\angle -5.71°) + (j5)(5)\,(0.1\angle -5.71°) - \mathbf{V}_{\text{Th}} = 0$$

$$\mathbf{V}_{\text{Th}} = 3.5\angle 85.1°\ \text{V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 3.140)

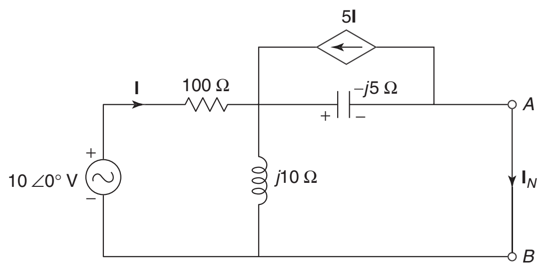


Fig. 3.140

By source transformation, the network is redrawn as shown in Fig. 3.141.

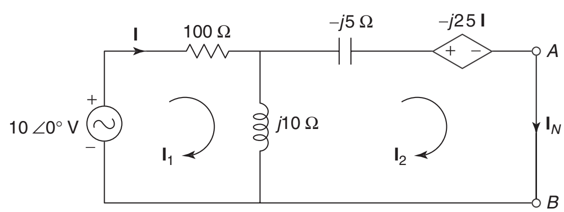


Fig. 3.141

From Fig. 3.141,

$$\mathbf{I} = \mathbf{I}_1 \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$10\angle 0° - 100\,\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(100 + j10)\,\mathbf{I}_1 - j10\mathbf{I}_2 = 10\angle 0° \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) + j5\mathbf{I}_2 + j25\mathbf{I} = 0$$

$$-j10\mathbf{I}_2 + j10\mathbf{I}_1 + j5\mathbf{I}_2 + j25\mathbf{I}_1 = 0$$

$$j35\mathbf{I}_1 - j5\mathbf{I}_2 = 0 \quad \ldots\text{(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 100 + j10 & -j10 \\ j35 & -j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 100 + j10 & 10\angle 0° \\ j35 & 0 \end{vmatrix}}{\begin{vmatrix} 100 + j10 & -j10 \\ j35 & -j5 \end{vmatrix}} = 0.6\angle 30.96° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 0.6\angle 30.96° \text{ A}$$

Step III Calculation of $\mathbf{Z}_N$

$$\mathbf{Z}_N = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{3.5\angle 85.1°}{0.6\angle 30.96°} = 5.83\angle 54.14°\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.142)

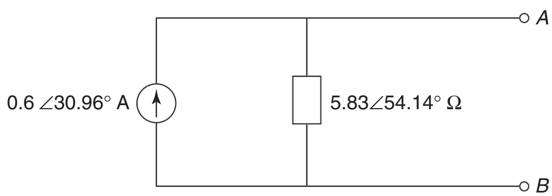


Fig. 3.142

3.7 MAXIMUM POWER TRANSFER THEOREM

This theorem is used to determine the value of load impedance for which the source will transfer maximum power.

Consider a simple network as shown in Fig. 3.143. There are three possible cases for load impedance $\mathbf{Z}_L$.

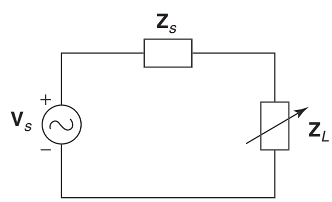


Fig. 3.143

Case (i) When the load impedance is variable resistance (Fig. 3.144)

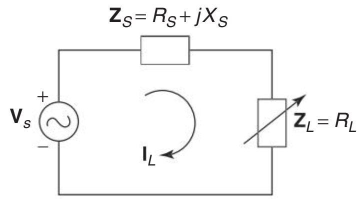


Fig. 3.144 *Purely resistive load*

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L} = \frac{\mathbf{V}_s}{R_s + jX_s + R_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + X_s^2}}$$

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2 R_L = \frac{|\mathbf{V}_s|^2 R_L}{\sqrt{(R_s + R_L)^2 + X_s^2}}$$

For power to be maximum,

$$\frac{dP_L}{dR_L} = 0$$

$$|\mathbf{V}_s|^2 \left[\frac{\{(R_s + R_L)^2 + X_s^2\} - 2R_L(R_s + R_L)}{[(R_s + R_L)^2 + X_s^2]^2} \right] = 0$$

$$(R_s + R_L)^2 + X_s^2 - 2R_L(R_s + R_L) = 0$$

$$R_s^2 + 2R_sR_L + R_L^2 + X_s^2 - 2R_LR_s - 2R_L^2 = 0$$

$$R_s^2 + X_s^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2 + X_s^2$$

$$R_L = \sqrt{R_s^2 + X_s^2} = |Z_s|$$

Hence, load resistance R_L should be equal to the magnitude of the source impedance for maximum power transfer.

Case (ii) When the load impedance is a complex impedance with variable resistance and variable reactance (Fig. 3.145)

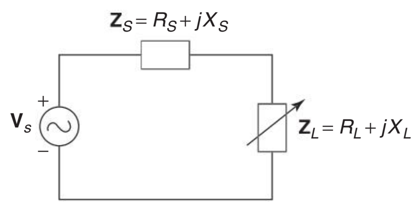


Fig. 3.145 *Complex impedance load*

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2 R_L = \frac{|\mathbf{V}_S|^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

For maximum value of P_L, denominator of the equation should be small, ie. $X_L = -X_s$.

$$P_L = \frac{|\mathbf{V}_S|^2 R_L}{(R_s + R_L)^2}$$

Differentiating the above equation w.r.t. R_L and equating to zero,

$$\frac{dP_L}{dR_L} = |\mathbf{V}_s|^2 \left[\frac{(R_s + R_L)^2 - 2\,R_L\,(R_s + R_L)}{(R_s + R_L)^2} \right] = 0$$

$$(R_s + R_L)^2 - 2\,R_L\,(R_s + R_L) = 0$$

$$R_s^2 + 2\,R_s R_L + R_L^2 - 2\,R_L R_s - 2R_L^2 = 0$$

$$R_s^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2$$

$$R_L = R_s$$

Hence, load resistance R_L should be equal to source resistance R_L and load reactance X_L should be equal to negative value of source reactance for maximum power transfer.

$$\mathbf{Z}_L = \mathbf{Z}_s^* = R_s - jX_s$$

i.e. load impedance should be a complex conjugate of the source impedance.

Case (iii) When the load impedance is a complex impedance with variable resistance and fixed reactance (Fig. 3.146)

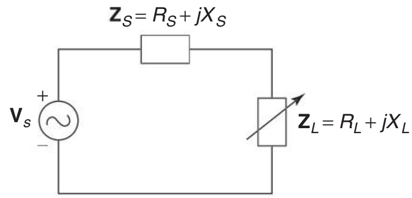


Fig. 3.146 *Complex impedance load*

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2\,R_L = \frac{|\mathbf{V}_s|^2\,R_L}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

For maximum power,

$$\frac{dP_L}{dR_L} = 0$$

$$|\mathbf{V}_s|^2 \left[\frac{(R_s + R_L)^2 + (X_s + X_L)^2 - 2\,R_L(R_s + R_L)}{\{(R_s + R_L)^2 + (X_s + X_L)^2\}^2} \right] = 0$$

$$(R_s + R_L)^2 + (X_s + X_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s^2 + 2\,R_s\;R_L + R_L^2 + (X_s + X_L)^2 - 2\,R_L R_s - 2\,R_L^2 = 0$$

$$R_s^2 + (X_s + X_L)^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2 + (X_s + X_L)^2$$

$$R_L = \sqrt{R_s^2 + (X_s + X_L)^2}$$

$$= |R_s + j(X_s + X_L)|$$

$$= |R_s + jX_s + jX_L|$$

$$= |Z_s + jX_L|$$

Hence, load resistance R_L should be equal to the magnitude of the impedance $\mathbf{Z}_s + jX_L$, i.e. $|\mathbf{Z}_s + jX_L|$ for maximum power transfer.

Example 3.47 *For maximum power transfer, find the value of $\mathbf{Z}_L$ in the network of Fig. 3.147 if (i) $\mathbf{Z}_L$ is an impedance, and (ii) $\mathbf{Z}_L$ is pure resistance.*

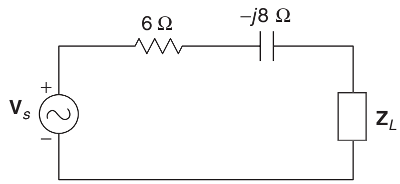


Fig. 3.147

Solution

$$\mathbf{Z}_s = (6 - j8)\ \Omega$$

(i) If $\mathbf{Z}_L$ is an impedance

For maximum power transfer, $\mathbf{Z}_L = \mathbf{Z}_s^* = (6 + j8)\ \Omega$

(ii) If $\mathbf{Z}_L$ is a resistance

For maximum power transfer, $\mathbf{Z}_L = |\mathbf{Z}_S| = |6 + j8| = 10\ \Omega$

Example 3.48 *For the maximum power transfer, find the value of $\mathbf{Z}_L$ in the network of Fig. 3.148 for the following cases:*

(i) $\mathbf{Z}_L$ is variable resistance, (ii) $\mathbf{Z}_L$ is complex impedance, with variable resistance and variable reactance, and (iii) $\mathbf{Z}_L$ is complex impedance with variable resistance and fixed reactance of j5 Ω.

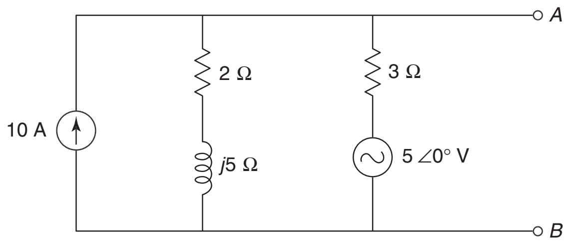


Fig. 3.148

Solution Thevenin's impedance can be calculated by replacing voltage source by a short circuit and current source by an open circuit.

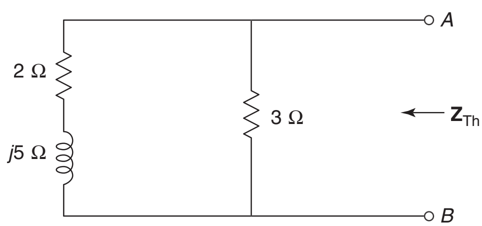


Fig. 3.149

$$\mathbf{Z}_{\text{Th}} = \frac{3(2+j5)}{3+2+j5} = (2.1+j0.9)\ \Omega$$

For maximum power transfer, value of $\mathbf{Z}_L$ will be,

(i) $\mathbf{Z}_L$ is variable resistance

$$\mathbf{Z}_L = |\mathbf{Z}_{\text{Th}}| = |2.1+j0.9| = 2.28\ \Omega$$

(ii) $\mathbf{Z}_L$ is complex impedance with variable resistance and variable reactance

$$\mathbf{Z}_L = \mathbf{Z}^*_{\text{Th}} = (2.1-j0.9)\ \Omega$$

(iii) $\mathbf{Z}_L$ is complex impedance with variable resistance and fixed reactance of $j5\ \Omega$

$$\mathbf{Z}_L = |\mathbf{Z}_{\text{Th}} + j5| = |2.1+j0.9+j5| = 6.26\ \Omega$$

Example 3.49 *Find the impedance $\mathbf{Z}_L$ so that maximum power can be transferred to it in the network of Fig. 3.150. Find maximum power.*

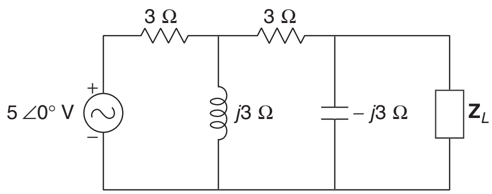


Fig. 3.150

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.151)

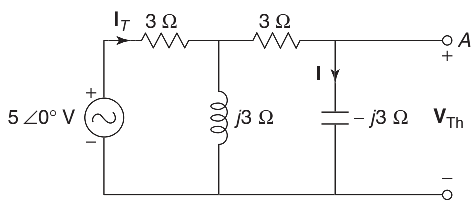


Fig. 3.151

$$\mathbf{Z}_T = 3 + \frac{j3(3-j3)}{3+j3-j3} = 6.71\angle 26.57°\ \Omega$$

$$\mathbf{I}_T = \frac{5\angle 0°}{6.71\angle 26.57°} = 0.75\angle -26.57°\ \text{A}$$

By current division rule,

$$\mathbf{I} = 0.75\angle -26.57° \times \frac{j3}{3+j3-j3} = 0.75\angle 63.43°\ \text{A}$$

$$\mathbf{V}_{\text{Th}} = (-j3)(0.75\angle 63.43°) = 2.24\angle -26.57°\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 3.152)

$$\mathbf{Z}_{Th} = [(3 \parallel j3) + 3] \parallel (-j3)$$
$$= 3 \angle{-53.12°}\ \Omega$$
$$= (1.8 - j2.4)\ \Omega$$

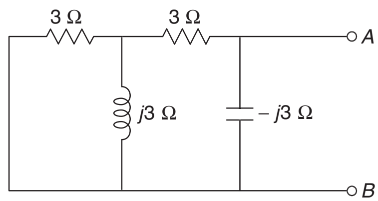


Fig. 3.152

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be a complex conjugate of the source impedance.

$$\mathbf{Z}_L = (1.8 + j2.4)\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.153)

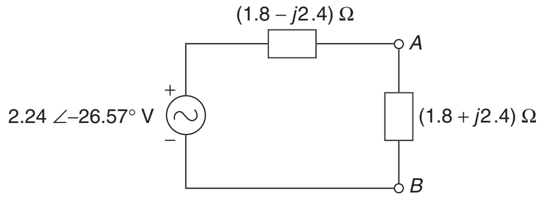


Fig. 3.153

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{|2.24|^2}{4 \times 1.8} = 0.7\ \text{W}$$

Example 3.50 *Find the value of $\mathbf{Z}_L$ for maximum power transfer in the network shown and find maximum power.*

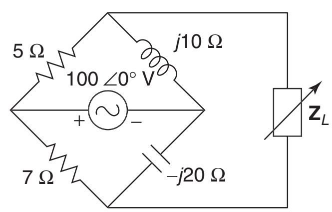


Fig. 3.154

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 3.155)

$$\mathbf{I}_1 = \frac{100\angle 0°}{5 + j10} = 8.94\angle{-63.43°}\ \text{A}$$

$$\mathbf{I}_2 = \frac{100\angle 0°}{7 - j20} = 4.72\angle 70.7°\ \text{A}$$

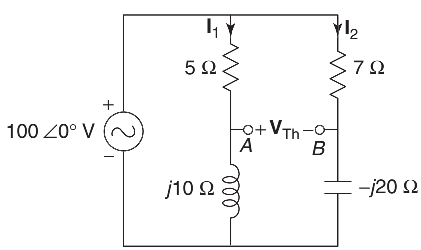


Fig. 3.155

$$\mathbf{V}_{\text{Th}} = \mathbf{V}_A - \mathbf{V}_B = (8.94\angle -63.43°)(j10) - (4.72\angle 70.7°)(-j20) = 71.76\angle 97.3° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.156)

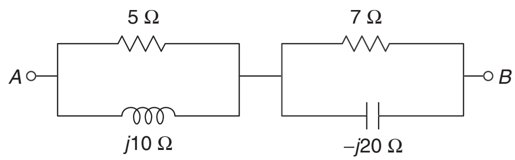


Fig. 3.156

$$\mathbf{Z}_{\text{Th}} = \frac{5(j10)}{5+j10} + \frac{7(-j20)}{7-j20} = \frac{50\angle 90°}{11.18\angle 63.43°} + \frac{140\angle -90°}{21.19\angle -70.7°} = (10.23 - j0.18)\ \Omega$$

Step III For maximum power transfer, the load impedance should be complex conjugate of the source impedance.

$$\mathbf{Z}_L = (10.23 + j0.18)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.157)

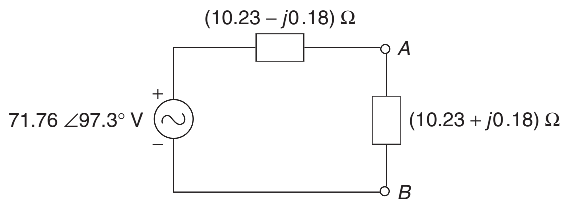


Fig. 3.157

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{|71.76|^2}{4 \times 10.23} = 125.84 \text{ W}$$

Example 3.51 *Find the value of load impedance $\mathbf{Z}_L$ so that maximum power can be transferred to it in the network of Fig. 3.158. Find maximum power.*

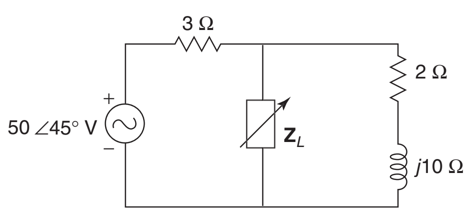


Fig. 3.158

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.159)

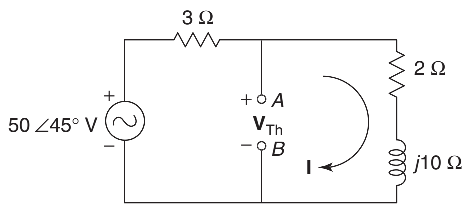


Fig. 3.159

$$\mathbf{I} = \frac{50\angle 45^\circ}{3+2+j10} = 4.47\angle -18.43^\circ \text{ A}$$

$$\mathbf{V}_{\text{Th}} = (2+j10)\,\mathbf{I} = (2+j10)(4.47\angle -18.43^\circ) = 45.6\angle 60.26^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.160)

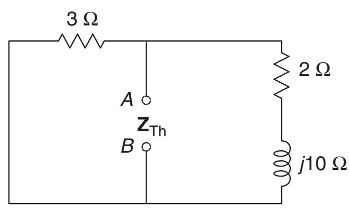


Fig. 3.160

$$\mathbf{Z}_{\text{Th}} = \frac{3(2+j10)}{3+2+j10} = (2.64 + j0.72)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be complex conjugate of the source impedance.

$$\mathbf{Z}_L = (2.64 - j0.72)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.161)

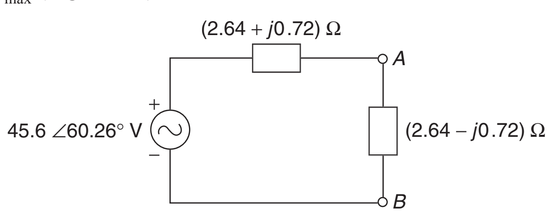


Fig. 3.161

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{|45.6|^2}{4\times 2.64} = 196.91 \text{ W}$$

Example 3.52 *Determine the load $\mathbf{Z}_L$ required to be connected in the network of Fig. 3.162 for maximum power transfer. Determine the maximum power drawn.*

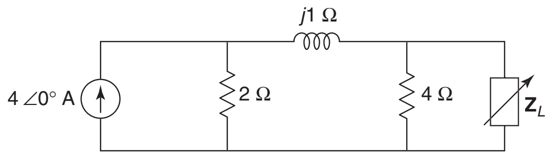


Fig. 3.162

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 3.163)

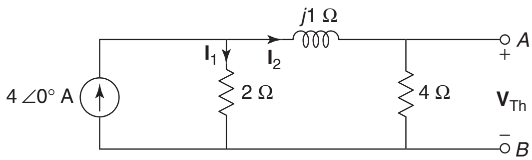


Fig. 3.163

$$\mathbf{I}_2 = 4\angle 0° \times \frac{2}{6 + j1} = 1.315\angle -9.46° \text{ A}$$

$$\mathbf{V}_{Th} = 4\mathbf{I}_2 = 4(1.315\angle -9.46°) = 5.26\angle -9.46° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 3.164)

$$\mathbf{Z}_{Th} = \frac{4(2 + j1)}{4 + 2 + j1} = 1.47\angle 17.1° = (1.41 + j0.43)\ \Omega$$

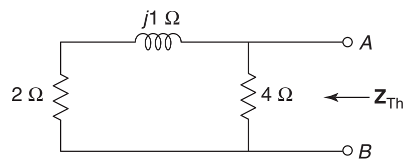


Fig. 3.164

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$\mathbf{Z}_L = (1.41 - j0.43)\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.165)

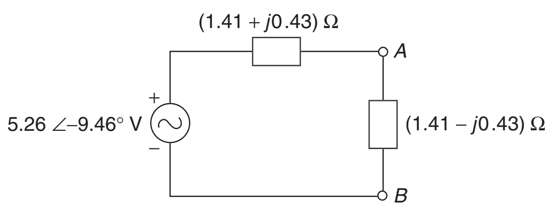


Fig. 3.165

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{|5.26|^2}{4 \times 1.41} = 4.91 \text{ W}$$

Example 3.53 *In the network shown in Fig. 3.166, find the value of* $\mathbf{Z}_L$ *for which the power transferred will be maximum. Also find maximum power.*

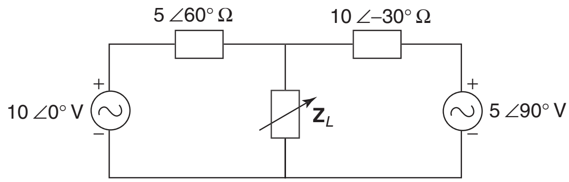


Fig. 3.166

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.167)

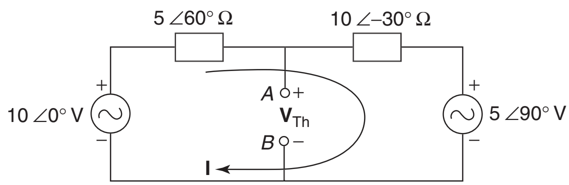


Fig. 3.167

Applying KVL to the mesh,

$$10\angle 0° - (5\angle 60°)\,\mathbf{I} - (10\angle -30°)\,\mathbf{I} - 5\angle 90° = 0$$
$$11.18\angle -26.57° - (11.18\angle -3.43°)\,\mathbf{I} = 0$$
$$\mathbf{I} = 1\angle -23.14° \text{ A}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$10\angle 0° - (5\angle 60°)\mathbf{I} - \mathbf{V}_{\text{Th}} = 0$$
$$10\angle 0° - (5\angle 60°)(1\angle -23.14°) - \mathbf{V}_{\text{Th}} = 0$$
$$\mathbf{V}_{\text{Th}} = 6.71\angle -26.56° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 3.168)

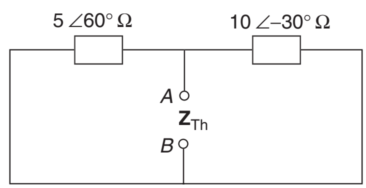


Fig. 3.168

$$\mathbf{Z}_{\text{Th}} = \frac{(5\angle 60°)(10\angle -30°)}{5\angle 60° + 10\angle -30°} = 4.47\angle 33.43°\ \Omega = (3.73 + j2.46)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$Z_L = Z_{\text{Th}}^* = (3.73 - j2.46)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.169)

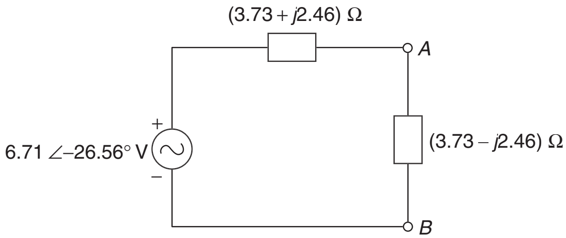


Fig. 3.169

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{(6.71)^2}{4\times 3.73} = 3.02\ \text{W}$$

Example 3.54 *In the network shown in Fig. 3.170, find the value of $\mathbf{Z}_L$ so that power transfer from the source is maximum. Also find maximum power.*

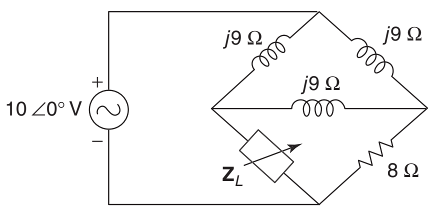


Fig. 3.170

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 3.171)

Applying Star-delta transformation (Fig. 3.172)

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \frac{(j9)(j9)}{j9 + j9 + j9} = j3\ \Omega$$

$\mathbf{V}_{\text{Th}}$ = Voltage drop across $(8 + j3)\Omega$ impedence

$$= (8 + j3)\frac{10\angle 0^\circ}{8 + j3 + j3} = 8.54\angle -16.31^\circ\ \text{V}$$

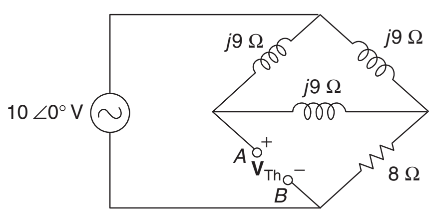


Fig. 3.171

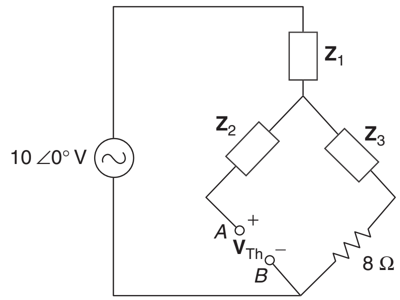


Fig. 3.172

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 3.173)

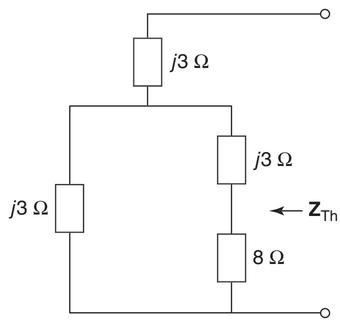


Fig. 3.173

$$\mathbf{Z}_{Th} = j3 + \frac{j3(8+j3)}{j3+8+j3)} = 5.51\angle 82.49°\ \Omega = (0.72 + j5.46)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$\mathbf{Z}_L = \mathbf{Z}^*_{Th} = (0.72 - j5.46)\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.174)

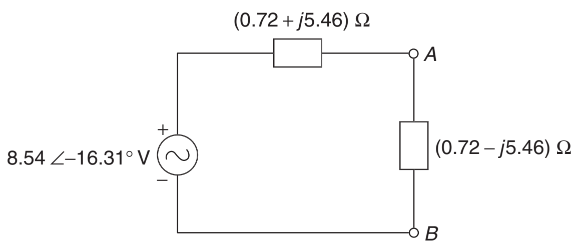


Fig. 3.174

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{(8.54)^2}{4 \times 0.72} = 25.32\ \text{W}$$

Example 3.55 *For the network shown in Fig. 3.175, find the value of $\mathbf{Z}_L$ that will transfer maximum power from the source. Also find maximum power.*

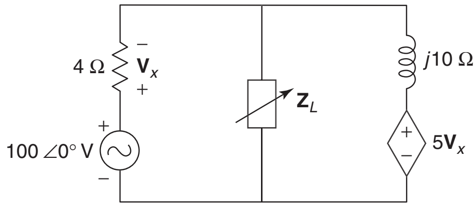


Fig. 3.175

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 3.176)

From Fig. 3.176,

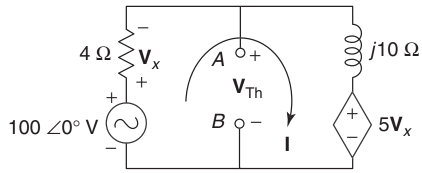


Fig. 3.176

$$\mathbf{V}_x = 4\mathbf{I}$$

Applying KVL to the mesh,

$$100\angle 0° - 4\mathbf{I} - j\,10\,\mathbf{I} - 5\mathbf{V}_x = 0$$

$$100\angle 0° - (4 + j10)\,\mathbf{I} - 5(4\mathbf{I}) = 0$$

$$\mathbf{I} = \frac{100\angle 0°}{24 + j10} = 3.85\angle -22.62°\text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$100\angle 0° - 4\mathbf{I} - \mathbf{V}_{Th} = 0$$

$$100\angle 0° - 4(3.85\angle -22.62°) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 86\angle 3.95°\text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 3.177)

From Fig. 3.177,

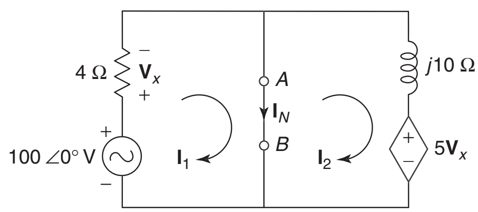


Fig. 3.177

$$\mathbf{V}_x = 4\mathbf{I}_1$$

Applying KVL to Mesh 1,

$$100\angle 0° - 4\mathbf{I}_1 = 0$$

$$\mathbf{I}_1 = 25\text{ A}$$

Applying KVL to Mesh 2,

$$-j10\mathbf{I}_2 - 5\mathbf{V}_x = 0$$

$$-j10\mathbf{I}_2 - 5(4\mathbf{I}_1) = 0$$

$$-j10\mathbf{I}_2 - 5(100) = 0$$

$$\mathbf{I}_2 = 50\angle 90°\text{A}$$

$$\mathbf{I}_N = \mathbf{I}_1 - \mathbf{I}_2 = 25 - 50\angle 90° = 55.9\angle -63.43°\text{A}$$

Step III Calculation of $\mathbf{Z}_{Th}$

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_{Th}}{\mathbf{I}_N} = \frac{86\angle 3.95°}{55.9\angle -63.43°} = 1.54\angle 67.38°\ \Omega = (0.59 + j1.42)\ \Omega$$

Step IV Calculation of $\mathbf{Z}_L$

For maximum power transfer, $\mathbf{Z}_L = \mathbf{Z}^*_{\text{Th}} = (0.59 - j1.42)\ \Omega$

Step V Calculation of $P_{\max}$ (Fig. 3.178)

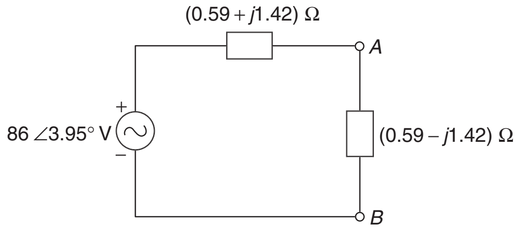


Fig. 3.178

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{(86)^2}{4 \times 0.59} = 3133.9 \text{ W}$$

3.8 RECIPROCITY THEOREM

The Reciprocity theorem states that '*In a linear, bilateral, active, single-source network, the ratio of excitation to response remains same when the positions of excitation and response are interchanged.*'

Example 3.56 *Find the current through the 6 Ω resistor and verify the reciprocity theorem.*

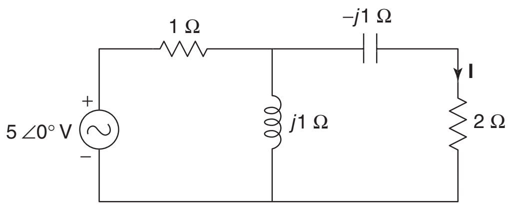


Fig. 3.179

Solution

Case I Calculation of current **I** when excitation and response are not interchanged (Fig. 3.180)

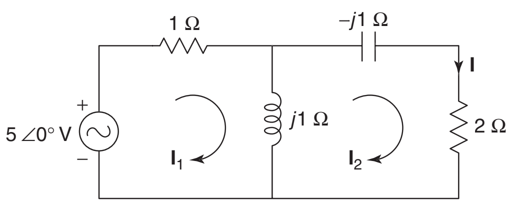


Fig. 3.180

Applying KVL to Mesh 1,

$$5\angle 0° - 1\mathbf{I}_1 - j1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(1 + j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 5\angle 0° \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j1(\mathbf{I}_2-\mathbf{I}_1)+j1\mathbf{I}_2-2\mathbf{I}_2=0$$
$$-j1\mathbf{I}_1+2\mathbf{I}_2=0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix}1+j1 & -j1\\ -j1 & 2\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}=\begin{bmatrix}5\angle 0^\circ\\ 0\end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2=\frac{\begin{vmatrix}1+j1 & 5\angle 0^\circ\\ -j1 & 0\end{vmatrix}}{\begin{vmatrix}1+j1 & -j1\\ -j1 & 2\end{vmatrix}}=1.39\angle 56.31^\circ\text{ A}$$

$$\mathbf{I}=\mathbf{I}_2=1.39\angle 56.31^\circ\text{ A}$$

Case II Calculation of current **I** when excitation and response are interchanged (Fig. 3.181)

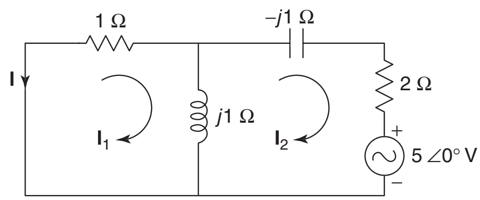


Fig. 3.181

Applying KVL to Mesh 1,

$$-1\mathbf{I}_1-j1(\mathbf{I}_1-\mathbf{I}_2)=0$$
$$(1+j1)\mathbf{I}_1-j1\mathbf{I}_2=0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j1(\mathbf{I}_2-\mathbf{I}_1)+j1\mathbf{I}_2-2\mathbf{I}_2-5\angle 0^\circ=0$$
$$-j1\mathbf{I}_1+2\mathbf{I}_2=-5\angle 0^\circ \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix}1+j1 & -j1\\ -j1 & 2\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}=\begin{bmatrix}0\\ -5\angle 0^\circ\end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1=\frac{\begin{vmatrix}0 & -j1\\ -5\angle 0^\circ & 2\end{vmatrix}}{\begin{vmatrix}1+j1 & -j1\\ -j1 & 2\end{vmatrix}}=1.38\angle -123.69^\circ\text{ A}$$

$$\mathbf{I}=-\mathbf{I}_1=1.39\angle 56.31^\circ\text{ A}$$

Since the current **I** is same in both the cases, the reciprocity theorem is verified.

Example 3.57 *In the network of Fig. 3.182, find the voltage $\mathbf{V}_x$ and verify the reciprocity theorem.*

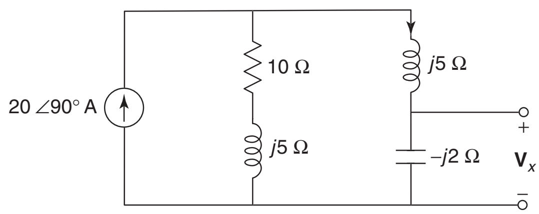


Fig. 3.182

Solution

Case I Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged. (Fig. 3.183)

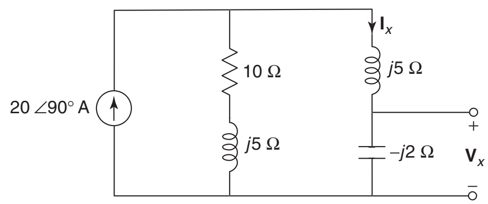


Fig. 3.183

By current division rule,

$$\mathbf{I}_x = (20\angle 90°)\frac{(10+j5)}{(10+j5)+(j5-j2)} = 17.46\angle 77.91° \text{ A}$$

$$\mathbf{V}_x = (-j2)\mathbf{I}_x = (-j2)(17.46\angle 77.91°) = 34.92\ \angle -12.09° \text{ V}$$

Case II Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged (Fig. 3.184)

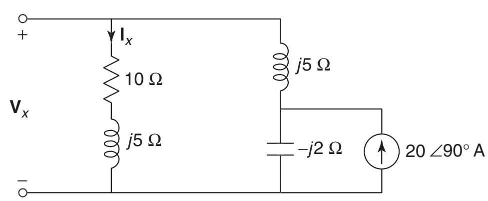


Fig. 3.184

$$\mathbf{I}_x = (20\angle 90°)\frac{(-j2)}{(-j2)+(10+j5+j5)} = 3.12\angle -38.66° \text{A}$$

$$\mathbf{V}_x = (10+j5)\mathbf{I}_x = (10+j5)(3.12\angle -38.66°) = 34.88\angle -12.09° \text{V}$$

Since the voltage $\mathbf{V}_x$ is same in both the cases, the reciprocity theorem is verified.

Example 3.58 *Find **I** and verify the reciprocity theorem for the network shown in Fig. 3.185.*

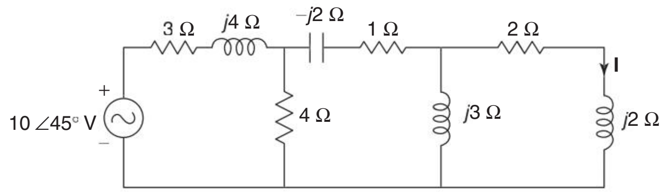


Fig. 3.185

Solution

Case I Calculation of **I** when excitation and response are not interchanged (Fig. 3.186)

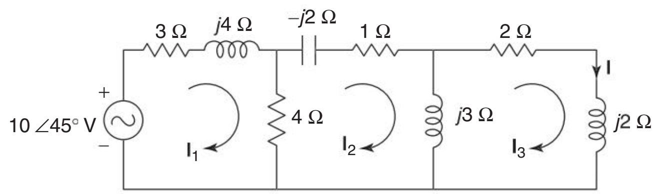


Fig. 3.186

Applying KVL to Mesh 1,

$$10\angle 45° - (3+j4)\mathbf{I}_1 - 4(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(7+j4)\mathbf{I}_1 - 4\mathbf{I}_2 = 10\angle 45° \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4(\mathbf{I}_2 - \mathbf{I}_1) - (1-j2)\,\mathbf{I}_2 - j3(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-4\mathbf{I}_1 + (5+j1)\,\mathbf{I}_2 - j3\mathbf{I}_3 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-j3(\mathbf{I}_3 - \mathbf{I}_2) - 2\mathbf{I}_3 - j2\mathbf{I}_3 = 0$$

$$-j3\mathbf{I}_2 + (2+j5)\mathbf{I}_3 = 0 \quad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 45° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 7+j4 & -4 & 10\angle 45° \\ -4 & 5+j1 & 0 \\ 0 & -j3 & 0 \end{vmatrix}}{\begin{vmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{vmatrix}} = 0.704\angle 30.72° \text{ A}$$

$$\mathbf{I} = \mathbf{I}_3 = 0.704\angle 30.72° \text{A}$$

Case II Calculation of **I** when excitation and response are interchanged (Fig. 3.187)

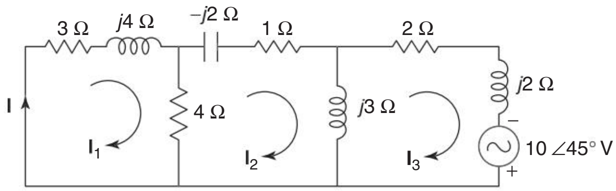


Fig. 3.187

Applying KVL to Mesh 1,

$$-(3+j4)\mathbf{I}_1 - 4(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(7+j4)\mathbf{I}_1 - 4\mathbf{I}_2 = 0 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-4(\mathbf{I}_2 - \mathbf{I}_1) - (1-j2)\,\mathbf{I}_2 - j3(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-4\mathbf{I}_1 + (5+j1)\mathbf{I}_2 - j3\mathbf{I}_3 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-j3(\mathbf{I}_3 - \mathbf{I}_2) - 2\mathbf{I}_3 - j2\mathbf{I}_3 + 10\angle 45^\circ = 0$$

$$-j3\mathbf{I}_2 + (2+j5)\mathbf{I}_3 = 10\angle 45^\circ \qquad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10\angle 45^\circ \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 0 & -4 & 0 \\ 0 & 5+j1 & -j3 \\ 10\angle 45^\circ & -j3 & 2+j5 \end{vmatrix}}{\begin{vmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{vmatrix}} = 0.704\angle 30.72^\circ\text{A}$$

$$\mathbf{I} = \mathbf{I}_1 = 0.704\angle 30.72^\circ\text{A}$$

Since the current **I** is same in both the cases, the reciprocity theorem, is verified.

3.9 MILLMAN'S THEOREM

Millman's theorem states that '*If there are n voltage sources* $\boldsymbol{V_1}$, $\boldsymbol{V_2}$. … $\boldsymbol{V_n}$ *with internal impedances* $\boldsymbol{Z_1}$, $\boldsymbol{Z_2}$, … $\boldsymbol{Z_n}$ *respectively connected in parallel then these voltage sources can be replaced by a single voltage source* $\boldsymbol{V_m}$ *and a single series impedance* $\boldsymbol{Z_m}$.

$$\mathbf{V}_m = \frac{\mathbf{V}_1\,\mathbf{Y}_1 + \mathbf{V}_2\,\mathbf{Y}_2 + \ldots + \mathbf{V}_n\,\mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

Example 3.59 *Find the current through the 40 Ω resistor for the network shown in Fig. 3.188.*

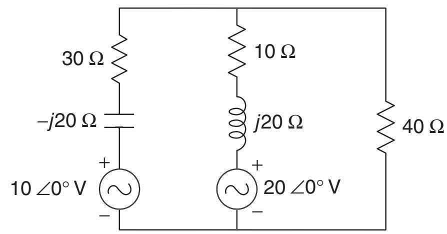


Fig. 3.188

Solution

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{(10\angle 0°)\left(\dfrac{1}{30 - j20}\right) + (20\angle 0°)\left(\dfrac{1}{10 + j20}\right)}{\dfrac{1}{30 - j20} + \dfrac{1}{10 + j20}} = 18.2\angle{-15.95}°\text{V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{30 - j20} + \dfrac{1}{10 + j20}} = 20.15\angle 29.74°\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 3.189)

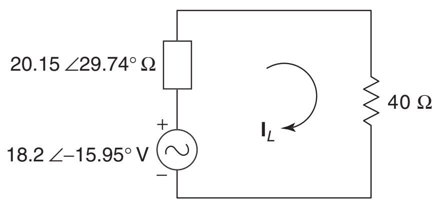


Fig. 3.189

$$\text{I}_L = \frac{18.2\angle{-15.95}°}{20.15\angle 29.74° + 40} = 0.31\angle{-25.81}°\ \text{A}$$

Example 3.60 *Find the current **I** in the network shown in Fig. 3.190.*

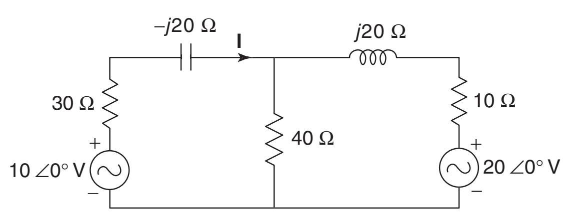


Fig. 3.190

Solution The network is redrawn as shown in Fig. 3.191.

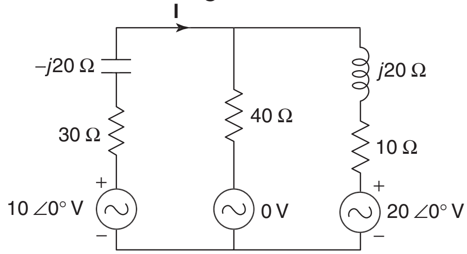


Fig. 3.191

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{0\left(\frac{1}{40}\right) + (20\angle 0°)\left(\frac{1}{10+j20}\right)}{\frac{1}{40} + \frac{1}{10+j20}} = 14.86\angle -21.8° \text{ V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\frac{1}{40} + \frac{1}{10+j20}} = 16.61\angle 41.63° \ \Omega$$

Step III Calculation of **I** (Fig. 3.192)

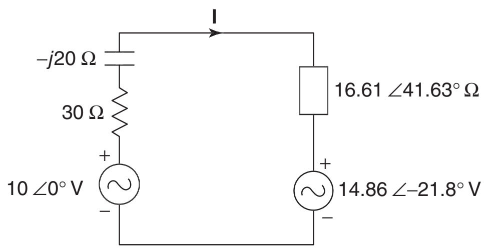


Fig. 3.192

$$\mathbf{I} = \frac{10 - 14.86\angle -21.8°}{30 - j20 + 16.61\angle 41.63°} = 0.15\angle 136.47° \text{ A}$$

Example 3.61 *Apply the dual Millman theorem and find the power loss in the (2 + j2) Ω impedance.*

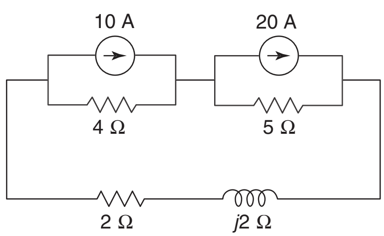


Fig. 3.193

Solution

Step I Calculation of $\mathbf{I}_m$

$$\mathbf{I}_m = \frac{\mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_2\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(10)(4)+(20)(5)}{4+5} = 15.56\text{ A}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \mathbf{Z}_1 + \mathbf{Z}_2 = 4+5 = 9\ \Omega$$

Step III Calculation of P (Fig. 3.194)

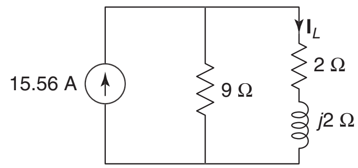


Fig. 3.194

By current division rule,

$$\mathbf{I}_L = 15.56 \times \frac{9}{9+2+j2} = 12.53\angle -10.3^\circ\text{ A}$$

$$P = \mathbf{I}_L^2 R_L = (12.53)^2 \times 2 = 314\text{ W}$$

Example 3.62 *In the network shown in Fig. 3.195, what load Z_L will receive the maximum power. Also find maximum power.*

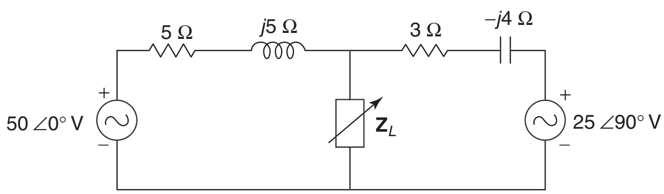


Fig. 3.195

Solution

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{(50\angle 0^\circ)\left(\dfrac{1}{5+j5}\right) + (25\angle 90^\circ)\left(\dfrac{1}{3-j4}\right)}{\dfrac{1}{5+j5} + \dfrac{1}{3-j4}} = 9.81\angle -78.69^\circ\text{V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{5+j5} + \dfrac{1}{3-j4}} = 4.39\angle -15.26^\circ\ \Omega = (4.23 - j1.15)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer $\quad \mathbf{Z}_L = \mathbf{Z}^*_{\text{Th}} = (4.23 + j1.15)\ \Omega$

Step IV Calculation of $P_{\max}$ (Fig. 3.196)

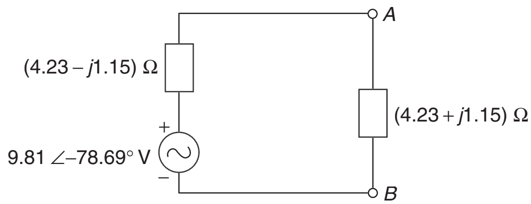


Fig. 3.196

$$P_{\max} = \frac{|V_m|^2}{4R_L} = \frac{(9.81)^2}{4\times 4.23} = 5.69\text{ W}$$

Exercises

MESH ANALYSIS

3.1 Find the current through the 3 + $j4\ \Omega$ impedance in Fig. 3.197.

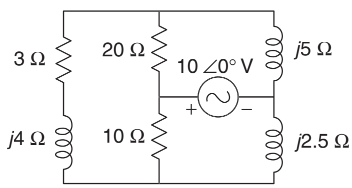


Fig. 3.197

[0]

3.2 In the network of Fig. 3.198, find $\mathbf{V}_0$.

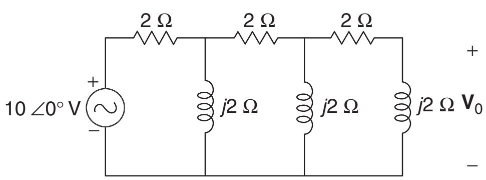


Fig. 3.198

[1.56 ∠128.7° V]

3.3 Find the current $\mathbf{I}_3$ in the network of Fig. 3.199.

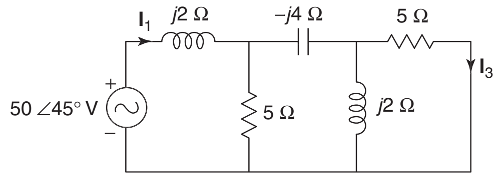


Fig. 3.199

[11.6 ∠113.2° A]

3.4 In the network of Fig. 3.200, find $\mathbf{V}_2$ which results in zero current through the 4 Ω resistor.

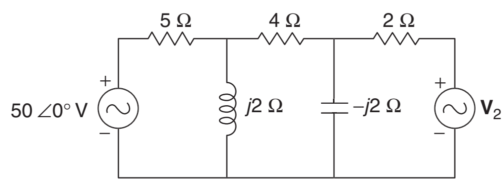


Fig. 3.200

[26.3 ∠113.2° V]

NODE ANALYSIS

3.5 For the network shown in Fig. 3.201, find the voltage $\mathbf{V}_{AB}$.

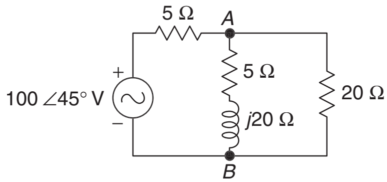


Fig. 3.201

[75.4 ∠55.2° V]

3.6 Find the voltages at nodes 1 and 2 in the network of Fig. 3.202.

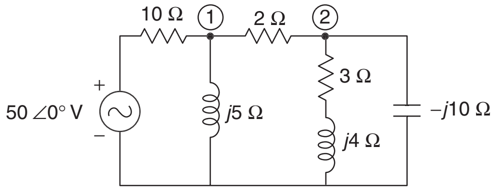


Fig. 3.202

[15.95 ∠49.94° V, 12.9 ∠55.5° V]

3.7 In the network of Fig. 3.203, find the current in the 10 ∠30° V source.

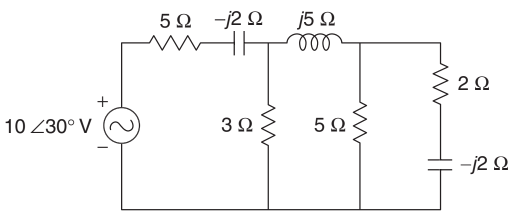


Fig. 3.203

[1.44 ∠38.8° A]

SUPERPOSITION THEOREM

3.8 For the network shown in Fig. 3.204, find the current in the 10 Ω resistor.

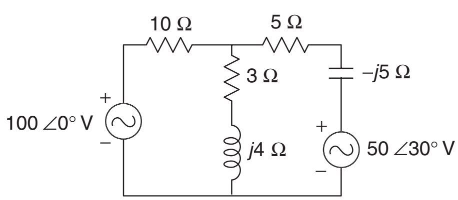


Fig. 3.204

[73.4 ∠−21.84° A]

3.9 In the network of Fig. 3.205, find the current through capacitance.

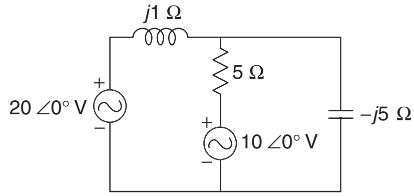


Fig. 3.205

[4.86 ∠80.8° A]

THEVENIN'S THEOREM

3.10 Obtain Thevenin's equivalent network for the network shown in Fig. 3.206.

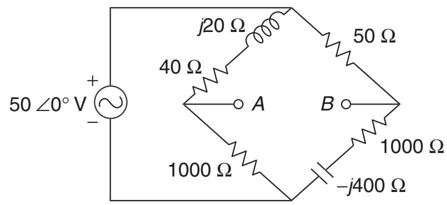


Fig. 3.206

[0.192 ∠−43.4° V, 88.7 ∠11.55° Ω]

3.11 Obtain Thevenin's equivalent network for Fig. 3.207.

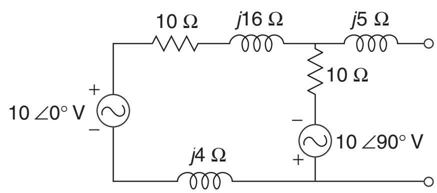


Fig. 3.207

[11.17 ∠−63.4° V, 10.6 ∠45° Ω]

NORTON'S THEOREM

3.12 Find Norton's equivalent network for Fig. 3.208.

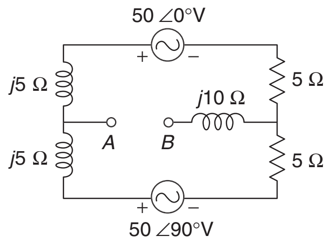


Fig. 3.208

[2.77 ∠–33.7° A, 2.5 + *j*12.5 Ω]

3.13 Find the current through the (3 + *j*4) Ω impedance in the network of Fig. 3.209.

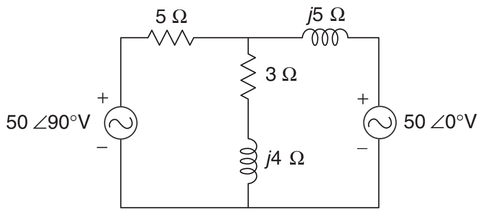


Fig. 3.209

[8.3 ∠85.2° A]

MAXIMUM POWER TRANSFER THEOREM

3.14 Determine the maximum power delivered to the load in the network shown in Fig. 3.210.

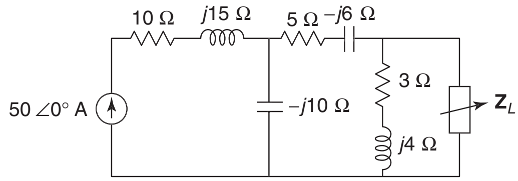


Fig. 3.210

[1032.35 W]

3.15 For the network shown in Fig. 3.211, find the value of $\mathbf{Z}_L$ that will receive the maximum power. Determine also this power.

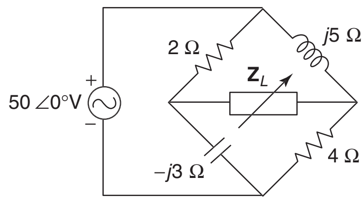


Fig. 3.211

[3.82 – *j*1.03 Ω, 54.5 W]

Objective-Type Questions

3.1 In Fig. 3.212, the equivalent impedance seen across terminals *a*, *b*, is

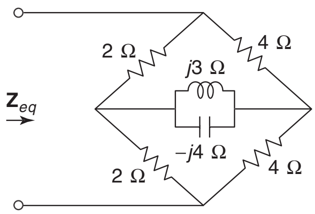


Fig. 3.212

(a) $\frac{16}{3}$ Ω (b) $\frac{8}{3}$ Ω

(C) $\left(\frac{8}{3}+j12\right)$ Ω (d) none of the above

3.2 The Thevenin equivalent voltage $\mathbf{V}_{\text{Th}}$ appearing between the terminals *A* and *B* of the network shown in Fig. 3.213 is given by

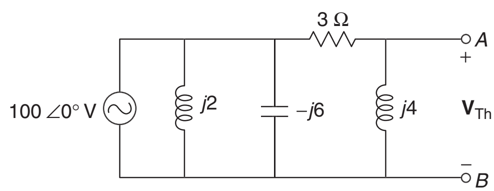


Fig. 3.213

(a) $j16(3 - j4)$ (b) $j16(3 + j4)$

(c) $16(3 + j4)$ (d) $16\,(3 - j4)$

3.3 A source of angular frequency of 1 rad/s has a source impedance consisting of a 1 Ω resistance in series with a 1 H inductance. The load that will obtain the maximum power transfer is

(a) 1 Ω resistance
(b) 1 Ω resistance in parallel with 1 H inductance
(c) 1 Ω resistance in series with 1 F capacitance
(d) 1 Ω resistance in parallel with 1 F capacitance

3.4 For the network shown in Fig. 3.214, the instantaneous current $\mathbf{I}_1$ is

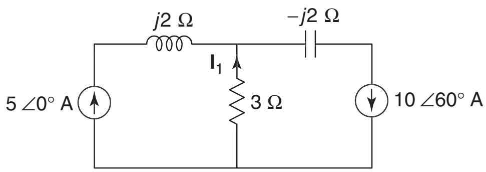


Fig. 3.214

(a) $\frac{10\sqrt{3}}{2}\angle 90°$ A (b) $\frac{10\sqrt{3}}{2}\angle -90°$ A
(c) 5 ∠60° A (d) 5∠−60° A

3.5 In the network shown in Fig. 3.215, the current supplied by the sinusoidal current source **I** is

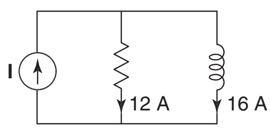


Fig. 3.215

(a) 28 A
(b) 4 A
(c) 20 A
(d) cannot be determined

3.6 In the network of Fig. 3.216, the magnitudes of $\mathbf{V}_L$ and $\mathbf{V}_C$ are twice that of $\mathbf{V}_R$. The inductance of the coil is

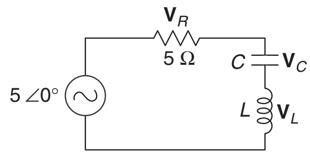


Fig. 3.216

(a) 2.14 mH (b) 5.3 H
(c) 31.8 mH (d) 1.32 H

3.7 Phase angle of the current **I** with respect to the voltage $\mathbf{V}_1$ in the circuit shown in Fig. 3.217.

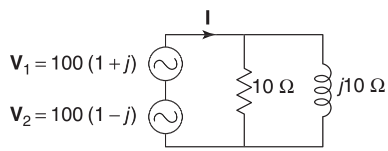


Fig. 3.217

(a) 0° (b) 45°
(c) −45° (d) −90°

Answers to Objective-Type Questions

3.1 (b) 3.2 (d) 3.3 (c) 3.4 (a) 3.5 (c) 3.6 (c) 3.7 (d)

4 Magnetic Circuits

4.1 INTRODUCTION

Two circuits are said to be coupled circuits when energy transfer takes place from one circuit to the other without having any electrical connection between them. Such coupled circuits are frequently used in network analysis and synthesis. Common examples of coupled circuits are transformer, gyrator, etc. In this chapter, we will discuss self and mutual inductance, magnetically coupled circuits, dot conventions and tuned circuits.

4.2 SELF-INDUCTANCE

Consider a coil of N turns carrying a current i as shown in Fig. 4.1.

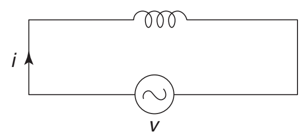


Fig. 4.1 *Coil carrying current*

When current flows through the coil, a flux ϕ is produced in the coil. The flux produced by the coil links with the coil itself. If the current flowing through the coil changes, the flux linking the coil also changes. Hence, an emf is induced in the coil. This is known as self-induced emf. The direction of this emf is given by *Lenz's law*.

We know that

$$\phi \propto i$$

$$\frac{\phi}{i} = k, \text{ a constant}$$

$$\phi = k\, i$$

Hence, rate of change of flux = $k \times$ rate of change of current

$$\frac{d\phi}{dt} = k\frac{di}{dt}$$

According to Faraday's laws of electromagnetic induction, a self-induced emf can be expressed as

$$v = -N\frac{d\phi}{dt} = -Nk\frac{di}{dt} = -N\frac{\phi}{i}\frac{di}{dt} = -L\frac{di}{dt}$$

where $L = \dfrac{N\phi}{i}$ and is called coefficient of self-inductance.

The property of a coil that opposes any change in the current flowing through it is called self-inductance or inductance of the coil. If the current in the coil is increasing, the self-induced emf is set up in such a direction so as to oppose the rise in current, i.e., the direction of self-induced emf is opposite to that of the applied voltage. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage. Self-inductance does not prevent the current from changing, it serves only to delay the change.

4.3 MUTUAL INDUCTANCE

If the flux produced by one coil links with the other coil, placed closed to the first coil, an emf is induced in the second coil due to change in the flux produced by the first coil. This is known as mutually induced emf.

Consider two coils 1 and 2 placed adjacent to each other as shown in Fig. 4.2. Let Coil 1 has N_1 turns while Coil 2 has N_2 turns.

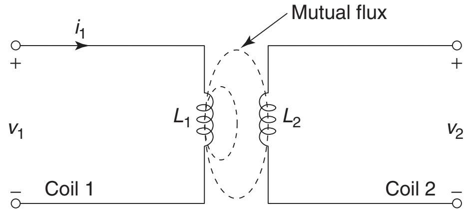


Fig. 4.2 *Two adjacent coils*

If a current i_1 flows in Coil 1, flux is produced and a part of this flux links Coil 2. The emf induced in Coil 2 is called mutually induced emf.

We know that

$$\phi_2 \propto i_1$$

$$\frac{\phi_2}{i_1} = k, \text{ a constant}$$

$$\phi_2 = k\, i_1$$

Hence, rate of change of flux = $k \times$ rate of change of current i_1

$$\frac{d\phi_2}{dt} = k\frac{di_1}{dt}$$

According to Faraday's law of electromagnetic induction, the induced emf is expressed as

$$v_2 = -N_2\frac{d\phi_2}{dt} = -N_2\, k\frac{di_1}{dt} = -N_2\frac{\phi_2}{i_1}\frac{di_1}{dt} = -M\frac{di_1}{dt}$$

where $M = \dfrac{N_2\,\phi_2}{i_1}$ and is called *coefficient of mutual inductance*.

4.4 COEFFICIENT OF COUPLING (k)

The coefficient of coupling (k) between coils is defined as fraction of magnetic flux produced by the current in one coil that links the other.

Consider two coils having number of turns N_1 and N_2 respectively. When a current i_1 is flowing in Coil 1 and is changing, an emf is induced in Coil 2.

$$M = \frac{N_2\,\phi_2}{i_1}$$

Let
$$k_1 = \frac{\phi_2}{\phi_1}$$
$$\phi_2 = k_1\ \phi_1$$
$$M = \frac{N_2\ k_1\ \phi_1}{i_1} \qquad \ldots(4.1)$$

If the current i_2 is flowing in Coil 2 and is changing, an emf is induced in Coil 1,
$$M = \frac{N_1\ \phi_1}{i_2}$$

Let
$$k_2 = \frac{\phi_1}{\phi_2}$$
$$\phi_1 = k_2\ \phi_2$$
$$M = \frac{N_1\ k_2\ \phi_2}{i_2} \qquad \ldots(4.2)$$

Multiplying Eqs (4.1) and (4.2),
$$M^2 = k_1 k_2 \times \frac{N_1\phi_1}{i_1} \times \frac{N_2\phi_2}{i_2} = k^2 L_1 L_2$$
$$M = k\sqrt{L_1 L_2}$$

where
$$k = \sqrt{k_1 k_2}$$

4.5 INDUCTANCES IN SERIES

1. Cumulative Coupling Figure 4.3 shows two coils 1 and 2 connected in series, so that currents through the two coils are in the same direction in order to produce flux in the same direction. Such a connection of two coils is known as *cumulative coupling*.

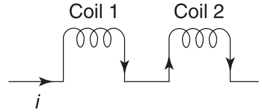


Fig. 4.3 *Cumulative coupling*

Let L_1 = coefficient of self-inductance of Coil 1

L_2 = coefficient of self-inductance of Coil 2

M = coefficient of mutual inductance

If the current in the coil increases by di amperes in dt seconds then

Self-induced emf in Coil 1 $= -L_1 \dfrac{di}{dt}$

Self-induced emf in Coil 2 $= -L_2 \dfrac{di}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M \dfrac{di}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M \dfrac{di}{dt}$

Total induced emf
$$v = -(L_1 + L_2 + 2M)\frac{di}{dt} \qquad \ldots(4.3)$$

If L is the equivalent inductance then total induced emf in that single coil would have been

$$v = -L\frac{di}{dt} \qquad \ldots(4.4)$$

Equating Eqs (4.3) and (4.4),

$$L = L_1 + L_2 + 2M$$

2. *Differential Coupling* Figure 4.4 shows the coils connected in series but the direction of current in Coil 2 is now opposite to that in 1. Such a connection of two coils is known as *differential coupling*.

Hence, total induced emf in coils 1 and 2.

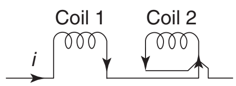


Fig. 4.4 *Differential coupling*

$$v = -L_1\frac{di}{dt} - L_2\frac{di}{dt} + 2M\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

Coils 1 and 2 connected in series can be considered as a single coil with equivalent inductance L. The induced emf in the equivalent single coil with same rate of change of current is given by,

$$v = -L\frac{di}{dt}$$

$$-L\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

$$L = L_1 + L_2 - 2M$$

4.6 INDUCTANCES IN PARALLEL

1. *Cumulative Coupling* Figure 4.5 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the same direction. Such a connection of two coils is known as cumulative coupling.

Let L_1 = coefficient of self-inductance of Coil 1

L_2 = coefficient of self-inductance of Coil 2

M = coefficient of mutual inductance

If the current in the coils changes by di amperes in dt seconds then

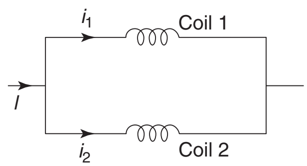


Fig. 4.5 *Cumulative coupling*

Self-induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2\dfrac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M\dfrac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M\dfrac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt} - M\dfrac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \dfrac{di_2}{dt} - M \dfrac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in both the coils must be equal.

$$-L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = -L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} - M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} - M \frac{di_2}{dt}$$

$$(L_1 - M)\frac{di_1}{dt} = (L_2 - M)\frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} \qquad \text{...(4.5)}$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M} + 1\right)\frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 - 2M}{L_1 - M}\right)\frac{di_2}{dt} \qquad \text{...(4.6)}$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L\frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L\frac{di}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}\left(L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}\right)$$

$$= \frac{1}{L}\left[L_1 \left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} + M \frac{di_2}{dt}\right]$$

$$= \frac{1}{L}\left[L_1 \left(\frac{L_2 - M}{L_1 - M}\right) + M\right]\frac{di_2}{dt} \qquad \text{...(4.7)}$$

Substituting Eq. (4.6) in Eq. (4.7),

$$\left(\frac{L_1 + L_2 - 2M}{L_1 - M}\right)\frac{di_2}{dt} = \frac{1}{L}\left[L_1 \left(\frac{L_2 - M}{L_1 - M}\right) + M\right]\frac{di_2}{dt}$$

$$L = \frac{L_1\left(\dfrac{L_2 - M}{L_1 - M}\right) + M}{\dfrac{L_1 + L_2 - 2M}{L_1 - M}}$$

$$= \frac{L_1 L_2 - L_1 M + L_1 M - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

2. *Differential Coupling* Figure 4.6 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the opposite direction. Such a connection of two coils is known as differential coupling.

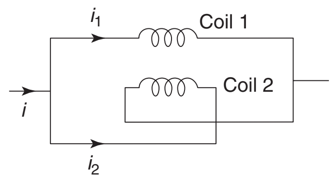


Fig. 4.6 *Differential coupling*

Self-induced emf in Coil 1 $= -L_1 \frac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2 \frac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= M \frac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= M \frac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in the coils must be equal.

$$-L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} + M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_2}{dt}$$

$$(L_1 + M) \frac{di_1}{dt} = (L_2 + M) \frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 + M}{L_1 + M} \right) \frac{di_2}{dt} \qquad \text{...(4.8)}$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M} \right) \frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M} + 1 \right) \frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 + 2M}{L_1 + M} \right) \frac{di_2}{dt} \qquad \text{...(4.9)}$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L\frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L\frac{di}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}\left(L_1\frac{di_1}{dt} - M\frac{di_2}{dt}\right)$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right)\frac{di_2}{dt} - M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt} \qquad \text{...(4.10)}$$

Substituting Eq. (4.9) in Eq. (4.10),

$$\left(\frac{L_1+L_2+2M}{L_1+M}\right)\frac{di_2}{dt} = \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt}$$

$$L = \frac{L_1\left(\dfrac{L_2+M}{L_1+M}\right) - M}{\dfrac{L_1+L_2+2M}{L_1+M}}$$

$$= \frac{L_1L_2 + L_1M - L_1M - M^2}{L_1+L_2+2M}$$

$$= \frac{L_1L_2 - M^2}{L_1+L_2+2M}$$

Example 4.1 *The combined inductance of two coils connected in series is 0.6 H or 0.1 H depending on relative directions of currents in the two coils. If one of the coils has a self-inductance of 0.2 H, find (a) mutual inductance, and (b) coefficient of coupling.*

Solution

$$L_1 = 0.2\text{ H}, \quad L_{\text{diff}} = 0.1\text{ H}, \quad L_{\text{cum}} = 0.6\text{ H}$$

(a) Mutual inductance

$$L_{\text{cum}} = L_1 + L_2 + 2M = 0.6 \qquad \text{...(i)}$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.1 \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii),

$$2(L_1+L_2) = 0.7$$

$$L_1 + L_2 = 0.35$$

$$L_2 = 0.35 - 0.2 = 0.15\text{ H}$$

Subtracting Eqs (ii) from Eqs (i),

$$4M = 0.5$$
$$M = 0.125 \text{ H}$$

(b) Coefficient of coupling

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.125}{\sqrt{0.2 \times 0.15}} = 0.72$$

Example 4.2 *Two coils with a coefficient of coupling of 0.6 between them are connected in series so as to magnetise in (a) same direction, and (b) opposite direction. The total inductance in the same direction is 1.5 H and in the opposite direction is 0.5 H. Find the self-inductance of the coils.*

Solution

$$k = 0.6, \quad L_{\text{diff}} = 0.5 \text{ H}, \quad L_{\text{cum}} = 1.5 \text{ H}$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.5 \qquad \text{...(i)}$$
$$L_{\text{cum}} = L_1 + L_2 + 2M = 1.5 \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$4M = 1$$
$$M = 0.25 \text{ H}$$

Adding Eq. (i) and (ii),

$$2(L_1 + L_2) = 2$$
$$L_1 + L_2 = 1 \qquad \text{...(iii)}$$
$$k = \frac{M}{\sqrt{L_1 L_2}}$$
$$0.6 = \frac{0.25}{\sqrt{L_1 L_2}}$$
$$L_1 L_2 = 0.1736 \qquad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$L_1 = 0.22 \text{ H}$$
$$L_2 = 0.78 \text{ H}$$

Example 4.3 *Two coils having self-inductances of 4 mH and 7 mH respectively are connected in parallel. If the mutual inductance between them is 5 mH, find the equivalent inductance.*

Solution

$$L_1 = 4 \text{ mH}, \quad L_2 = 7 \text{ mH}, \quad M = 5 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 - 2(5)} = 3 \text{ mH}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 + 2(5)} = 0.143 \text{ mH}$$

Example 4.4 *Two inductors are connected in parallel. Their equivalent inductance when the mutual inductance aids the self-inductance is 6 mH and it is 2 mH when the mutual inductance opposes the self-inductance. If the ratio of the self- inductances is 1:3 and the mutual inductance between the coils is 4 mH, find the self-inductances.*

Solution

$$L_{\text{cum}} = 6 \text{ mH}, \quad L_{\text{diff}} = 2 \text{ mH}, \quad \frac{L_1}{L_2} = 1.3, \quad M = 4 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$6 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 - 2(4)}$$

$$6 = \frac{L_1 L_2 - 16}{L_1 + L_2 - 8} \qquad \text{...(i)}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$2 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 + 8}$$

$$2 = \frac{L_1 L_2 - 16}{L_1 + L_2 + 8} \qquad \text{...(ii)}$$

From Eqs (i) and (ii),

$$2(L_1 + L_2 + 8) = 6(L_1 + L_2 - 8)$$

$$L_1 + L_2 + 8 = 3L_1 + 3L_2 - 24$$

$$L_1 + L_2 = 16$$

But

$$\frac{L_1}{L_2} = 1.3$$

$$1.3\, L_2 + L_2 = 16$$

$$2.3\, L_2 = 16$$

$$L_2 = 6.95 \text{ mH}$$

$$L_1 = 1.3\, L_2 = 9.035 \text{ mH}$$

4.7 DOT CONVENTION

Consider two coils of inductances L_1 and L_2 respectively connected in series as shown in Fig. 4.7. Each coil will contribute the same mutual flux (since it is in a series connection, the same current flows through L_1 and L_2) and hence, same mutual inductance (M). If the mutual fluxes of the two coils aid each other as shown in Fig 4.7 (a), the inductances of each coil will be increased by M, i.e., the inductance of coils will become $(L_1 + M)$ and $(L_2 + M)$. If the mutual fluxes oppose each other as shown in Fig. 4.7 (b), inductance of the coils will become $(L_1 - M)$ and $(L_2 - M)$. Whether the two mutual fluxes aid to each other or oppose will depend upon the manner in which coils are wound. The method described above is very inconvenient because we have to include the pictures of the coils in the circuit. There is another simple method of defining the directions of currents in the coils. This is known as dot convention.

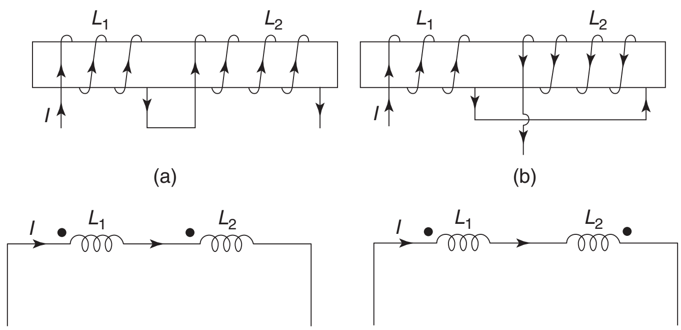


Fig. 4.7 *Dot convention*

Figure 4.7 shows the schematic connection of the two coils. It is not possible to state from Fig. 4.7(a) and Fig. 4.7(b) whether the mutual fluxes are additive or in opposition. However dot convention removes this confusion.

If the current enters from both the dotted ends of Coil 1 and Coil 2, the mutual fluxes of the two coils aid each other as shown in Fig. 4.7(c). If the current enters from the dotted end of Coil 1 and leaves from the dotted end of Coil 2, the mutual fluxes of the two coils oppose each other as shown in Fig. 4.7(d).

When two mutual fluxes aid each other, the mutual inductance is positive and polarity of the mutually induced emf is same as that of the self-induced emf. When two mutual fluxes oppose each other, the mutual inductance is negative and polarity of the mutually induced emf is opposite to that of the self-induced emf.

Example 4.5 *Obtain the dotted equivalent circuit for Fig. 4.8 shown below.*

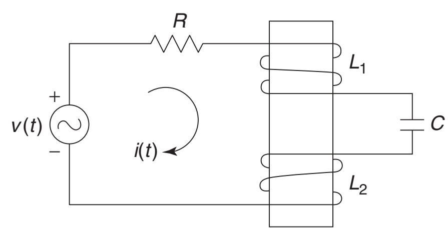


Fig. 4.8

Solution The current in the two coils is shown in Fig. 4.9. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

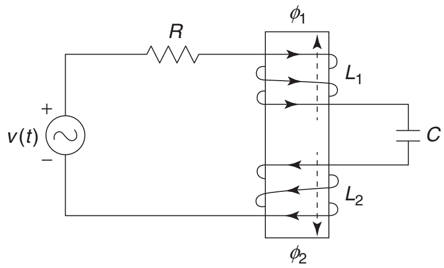


Fig. 4.9

From Fig. 4.9, it is seen that, the flux ϕ_1 is in upward direction in Coil 1, and flux ϕ_2 is in downward direction in Coil 2. Hence, fluxes are opposing each other. The mutual inductances are negative and mutually induced emfs have opposite polarities as that of self-induced emf. The dots are placed in two coils to illustrate these conditions. Hence, current $i(t)$ enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2.

The dotted equivalent circuit is shown in Fig. 4.10.

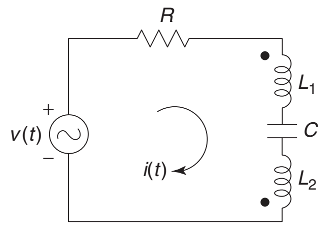


Fig. 4.10

Example 4.6 *Obtain the dotted equivalent circuit for the circuit of Fig. 4.11.*

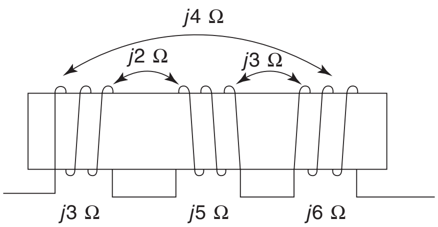


Fig. 4.11

Solution The current in the three coils is shown in Fig. 4.12. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

From Fig. 4.12, it is seen that the flux is towards the left in Coil 1, towards the right in Coil 2 and towards the left in Coil 3. Hence, fluxes ϕ_1 and ϕ_2 oppose each other in coils 1 and 2, fluxes ϕ_2 and ϕ_3 oppose each other in coils 2 and 3, and fluxes ϕ_1 and ϕ_3 aid each other in coils 1 and 3. The dots are placed in three coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1, leaves from the dotted end in Coil 2 and enters from the dotted end in Coil 3.

The dotted equivalent circuit is shown in Fig. 4.13.

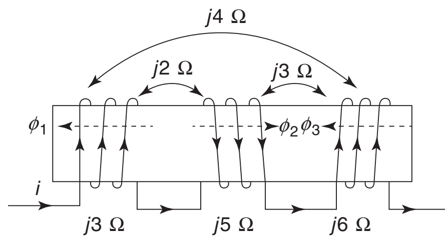


Fig. 4.12

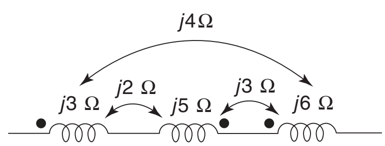


Fig. 4.13

Example 4.7 *Obtain the dotted equivalent circuit for the circuit shown in Fig. 4.14.*

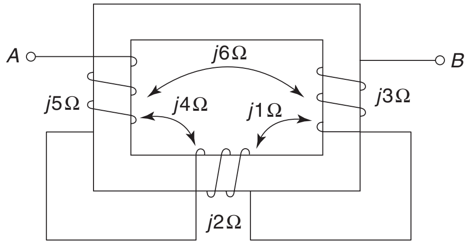


Fig. 4.14

Solution The current in the three coils is shown in Fig. 4.15. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

From Fig. 4.15, it is seen that all the three fluxes ϕ_1, ϕ_2, ϕ_3 aid each other. Hence, all the mutual reactances are positive and mutually induced emfs have same polarities as that of self-induced emfs. The dots are placed in three coils to illustrate these conditions. Hence, currents enter from the dotted end in each of the three coils. The dotted equivalent circuit is shown in Fig. 4.16.

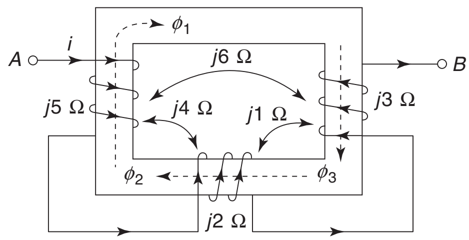


Fig. 4.15

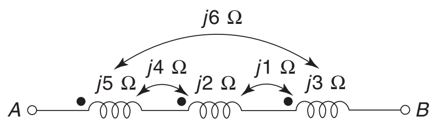


Fig. 4.16

Example 4.8 *Obtain the dotted equivalent circuit for the coupled circuit of Fig. 4.17.*

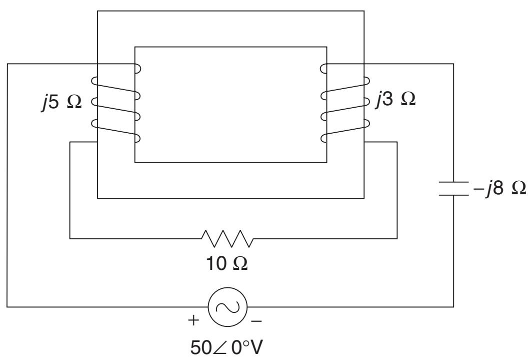


Fig. 4.17

Solution The current in the two coils is shown in Fig. 4.18 . The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

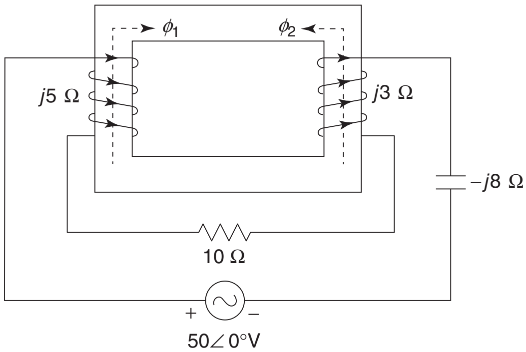


Fig. 4.18

From Fig. 4.18, it is seen that the flux ϕ_1 is in clockwise direction in Coil 1 and in anti-clockwise direction in Coil 2. Hence, fluxes are opposing each other. The dots are placed in two coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2. The dotted equivalent circuit is shown in Fig. 4.19.

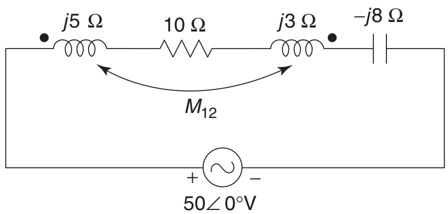


Fig. 4.19

Example 4.9 *Find the equivalent inductance of the network shown in Fig. 4.20.*

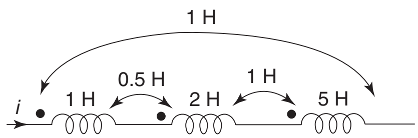


Fig. 4.20

Solution

$$
\begin{aligned}
L &= (L_1 + M_{12} + M_{13}) + (L_2 + M_{23} + M_{21}) + (L_3 + M_{31} + M_{32}) \\
&= (1 + 0.5 + 1) + (2 + 1 + 0.5) + (5 + 1 + 1) \\
&= 13\ \text{H}
\end{aligned}
$$

Example 4.10 *Find the equivalent inductance of the network shown in Fig. 4.21.*

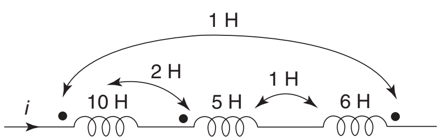


Fig. 4.21

Solution

$$
\begin{aligned}
L &= (L_1 + M_{12} - M_{13}) + (L_2 - M_{23} + M_{21}) + (L_3 - M_{31} - M_{23}) \\
&= (10 + 2 - 1) + (5 - 1 + 2) + (6 - 1 - 1) = 21\ \text{H}
\end{aligned}
$$

Example 4.11 *Find the equivalent inductance of the network shown in Fig. 4.22.*

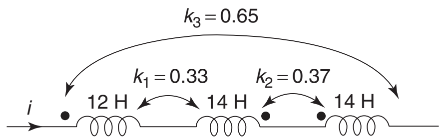


Fig. 4.22

Solution

$$M_{12} = M_{21} = k_1\sqrt{L_1 L_2} = 0.33\sqrt{(12)(14)} = 4.28 \text{ H}$$
$$M_{23} = M_{32} = k_2\sqrt{L_2 L_3} = 0.37\sqrt{(14)(14)} = 5.18 \text{ H}$$
$$M_{31} = M_{13} = k_3\sqrt{L_3 L_1} = 0.65\sqrt{(12)(14)} = 8.42 \text{ H}$$
$$\begin{aligned} L &= (L_1 - M_{12} + M_{13}) + (L_2 - M_{23} - M_{21}) + (L_3 + M_{31} - M_{32}) \\ &= (12 - 4.28 + 8.42) + (14 - 5.18 - 4.28) + (14 + 8.42 - 5.18) \\ &= 37.92 \text{ H} \end{aligned}$$

Example 4.12 *Find the equivalent inductance of the network shown in Fig. 4.23.*

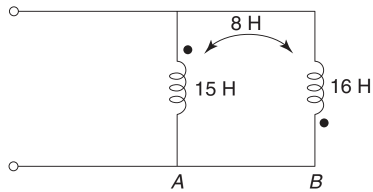


Fig. 4.23

Solution For Coil A,

$$L_A = L_1 - M_{12} = 15 - 8 = 7 \text{ H}$$

For Coil B,

$$L_B = L_2 - M_{12} = 16 - 8 = 8 \text{ H}$$
$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} = \frac{1}{7} + \frac{1}{8} = \frac{15}{56}$$
$$L = \frac{56}{15} = 3.73 \text{ H}$$

Example 4.13 *Find the equivalent inductance of the network shown in Fig. 4.24.*

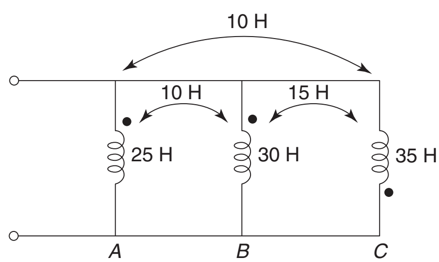


Fig. 4.24

Solution For Coil A,

$$L_A = L_1 + M_{12} - M_{13} = 25 + 10 - 10 = 25 \text{ H}$$

For Coil B,

$$L_B = L_2 - M_{23} + M_{21} = 35 - 15 + 10 = 25 \text{ H}$$

For Coil C,

$$L_C = L_3 - M_{32} - M_{31} = 35 - 15 - 10 = 10 \text{ H}$$

$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} + \frac{1}{L_C} = \frac{1}{25} + \frac{1}{25} + \frac{1}{10} = \frac{9}{50}$$

$$L = \frac{50}{9} = 5.55 \text{ H}$$

Example 4.14 *Find the equivalent impedance across the terminals A and B in Fig. 4.25.*

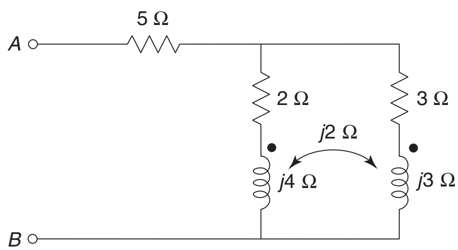


Fig. 4.25

Solution $\mathbf{Z}_1 = 5\,\Omega, \quad \mathbf{Z}_2 = (2 + j4)\,\Omega, \quad \mathbf{Z}_3 = (3 + j3)\,\Omega, \quad \mathbf{Z}_M = j2\,\Omega$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3 - \mathbf{Z}_M^2}{\mathbf{Z}_2 + \mathbf{Z}_3 - 2\mathbf{Z}_M} = 5 + \frac{(2 + j4)(3 + j3) - (j2)^2}{2 + j4 + 3 + j3 - 2(j2)} = 6.9 \angle 24.16^\circ\,\Omega$$

4.8 COUPLED CIRCUITS

Consider two coils located physically close to one another as shown in Fig. 4.26.

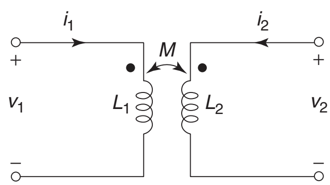


Fig. 4.26 *Coupled circuit*

When current i_1 flows in the first coil and $i_2 = 0$ in the second coil, flux ϕ_1 is produced in the coil. A fraction of this flux also links the second coil and induces a voltage in this coil. The voltage v_1 induced in the first coil is

$$v_1 = L_1 \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

The voltage v_2 induced in the second coil is

$$v_2 = M \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

The polarity of the voltage induced in the second coil depends on the way the coils are wound and it is usually indicated by dots. The dots signify that the induced voltages in the two coils (due to single current) have the same polarities at the dotted ends of the coils. Thus, due to i_1, the induced voltage v_1 must be positive at the dotted end of Coil 1. The voltage v_2 is also positive at the dotted end in Coil 2.

The same reasoning applies if a current i_2 flows in Coil 2 and $i_1 = 0$ in Coil 1. The induced voltages v_2 and v_1 are

$$v_2 = L_2 \left.\frac{di_2}{dt}\right|_{i_1 = 0}$$

and
$$v_1 = M \left.\frac{di_2}{dt}\right|_{i_1 = 0}$$

The polarities of v_1 and v_2 follow the dot convention. The voltage polarity is positive at the doted end of inductor L_2 when the current direction for i_2 is as shown in Fig. 4.26. Therefore, the voltage induced in Coil 1 must be positive at the dotted end also.

Now if both currents i_1 and i_2 are present, by using superposition principle, we can write

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This can be represented in terms of dependent sources, as shown in Fig. 4.27.

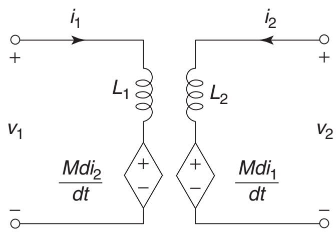


Fig. 4.27 *Equivalent circuit*

Now consider the case when the dots are placed at the opposite ends in the two coils, as shown in Fig. 4.28.

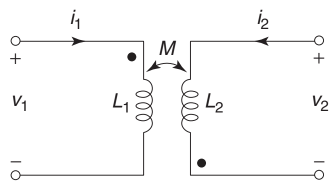


Fig. 4.28 *Coupled circuit*

Due to i_1, with $i_2 = 0$, the dotted end in Coil 1 is positive, so the induced voltage in Coil 2 is positive at the dot, which is the reverse of the designated polarity for v_2. Similarly, due to i_2, with $i_1 = 0$, the dotted ends have negative polarities for the induced voltages. The mutually induced voltages in both cases have polarities that are the reverse of terminal voltages and the equations are

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

This can be repressed in terms of dependent sources as shown in Fig. 4.29.

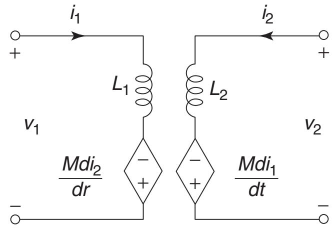


Fig. 4.29 *Equivalent circuit*

The various cases are summarised in the table shown in Fig. 4.30.

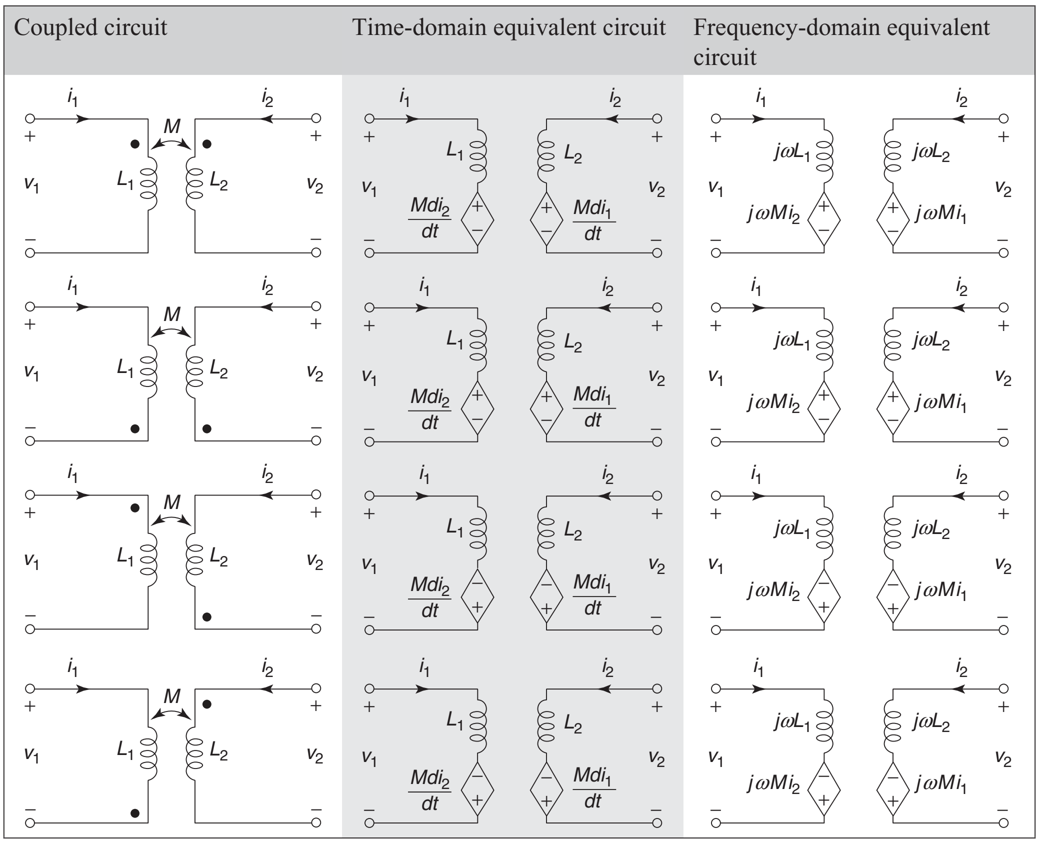


Fig. 4.30 *Coupled circuits for various cases*

Example 4.15 *Write mesh equations for the network shown in Fig. 4.31.*

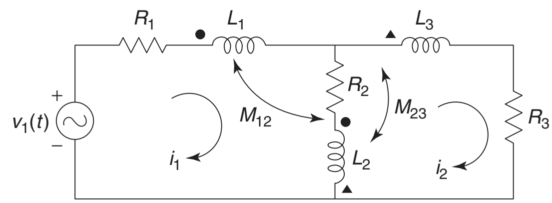


Fig. 4.31

Solution Coil 1 is magnetically coupled to Coil 2. Similarly, Coil 2 is magnetically coupled with Coil 1 and Coil 3. By applying dot convention, the equivalent circuit is drawn with the dependent sources.

The equivalent circuit in terms of dependent sources is shown in Fig. 4.32.

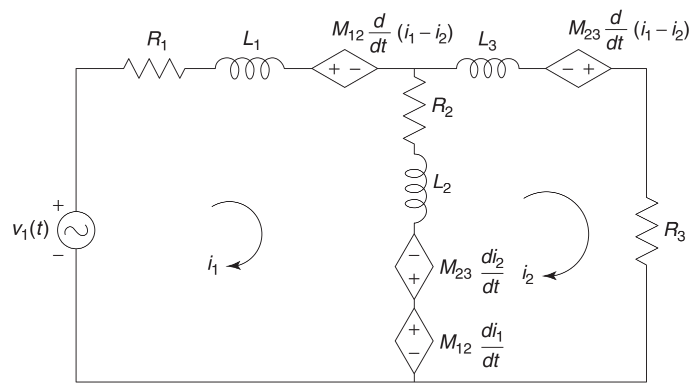


Fig. 4.32

(a) In Coil 1, there is a mutually induced emf due to current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is same as that of self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from the dotted ends.

(b) In Coil 2, there are two mutually induced emfs, one due to current i_1 in Coil 1 and the other due to current i_2 in Coil 3. The polarity of the mutually induced emf in Coil 2 due to the current i_1 is same as that of the self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from dotted ends. The polarity of the mutually induced emf in Coil 2 due to the current i_2 is opposite to that of the self-induced emf because current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

(c) In Coil 3, there is a mutually induced emf due to the current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is opposite to that of self-induced emf because the current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

Applying KVL to Mesh 1,

$$v_1(t) - R_1\, i_1 - L_1 \frac{di_1}{dt} - M_{12}\frac{d}{dt}(i_1 - i_2) - R_2(i_1 - i_2) - L_2\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{di_2}{dt} - M_{12}\frac{di_1}{dt} = 0$$

$$(R_1+R_2)\,i_1+(L_1+L_2+2M_{12})\frac{di_1}{dt}-R_2 i_2-(L_2+M_{12}+M_{23})\frac{di_2}{dt}=v_1(t) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$M_{12}\frac{di_1}{dt}-M_{23}\frac{di_2}{dt}-L_2\frac{d}{dt}(i_2-i_1)-R_2(i_2-i_1)-L_3\frac{di_2}{dt}+M_{23}\frac{d}{dt}(i_1-i_2)-R_3 i_2=0$$

$$-R_2\,i_1-(L_2+M_{12}+M_{23})\frac{di_1}{dt}+(R_2+R_3)\,i_2+(L_2+L_3+2M_{23})\frac{di_2}{dt}=0 \qquad \ldots\text{(ii)}$$

Example 4.16 *Write KVL equations for the circuit shown in Fig. 4.33.*

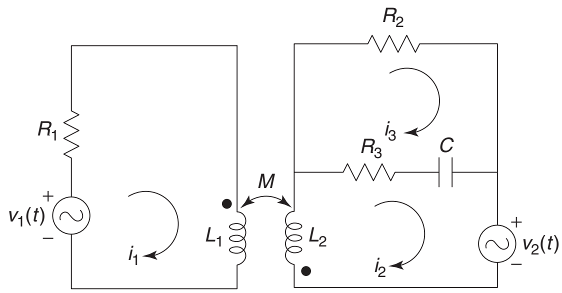


Fig. 4.33

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.34.

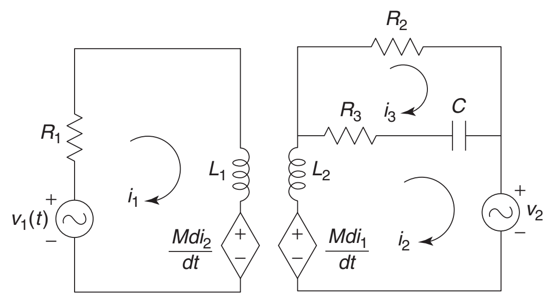


Fig. 4.34

Applying KVL to Mesh 1,

$$v_1(t)-R_1\,i_1-L_1\frac{di_1}{dt}-M\frac{di_2}{dt}=0$$

$$R_1\,i_1+L_1\frac{di_1}{dt}+M\frac{di_2}{dt}=v_1(t) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$M\frac{di_1}{dt}-L_2\frac{di_2}{dt}-R_3(i_2-i_3)-\frac{1}{C}\int_0^t(i_2-i_3)\,dt-v_2(t)=0$$

$$M\frac{di_1}{dt} - L_2\frac{di_2}{dt} - R_3(i_2 - i_3) - \frac{1}{C}\int_0^t (i_2 - i_3)\, dt = v_2(t) \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-R_2 i_3 - \frac{1}{C}\int_0^t (i_3 - i_2) dt - R_3(i_3 - i_2) = 0 \qquad \text{...(iii)}$$

Example 4.17 *Write down the mesh equations for the network shown in Fig. 4.35.*

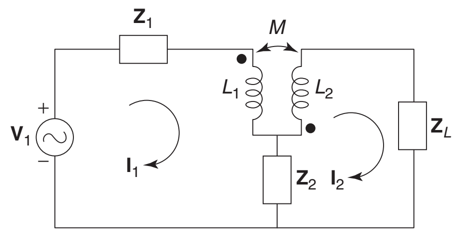


Fig. 4.35

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.36.

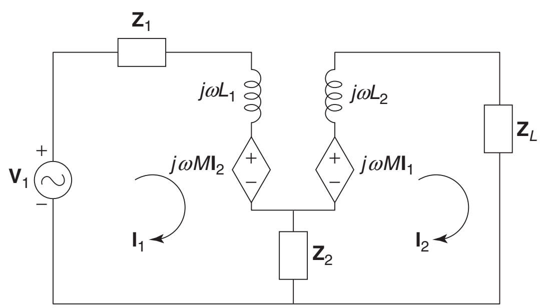


Fig. 4.36

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{Z}_1\mathbf{I}_1 - j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(\mathbf{Z}_1 + j\omega L_1 + \mathbf{Z}_2)\,\mathbf{I}_1 - (\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) + j\omega M\mathbf{I}_1 - j\omega L_2\mathbf{I}_2 - \mathbf{Z}_L\mathbf{I}_2 = 0$$

$$-(\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_1 + (\mathbf{Z}_2 + j\omega L_2 + \mathbf{Z}_L)\,\mathbf{I}_2 = 0 \qquad \text{...(ii)}$$

Example 4.18 *Write mesh equations for the network shown in Fig. 4.37.*

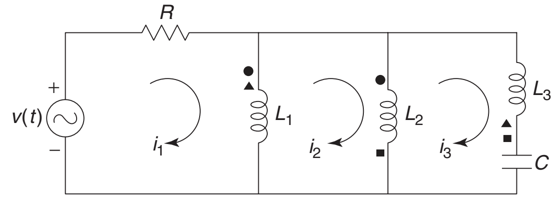


Fig. 4.37

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.38.

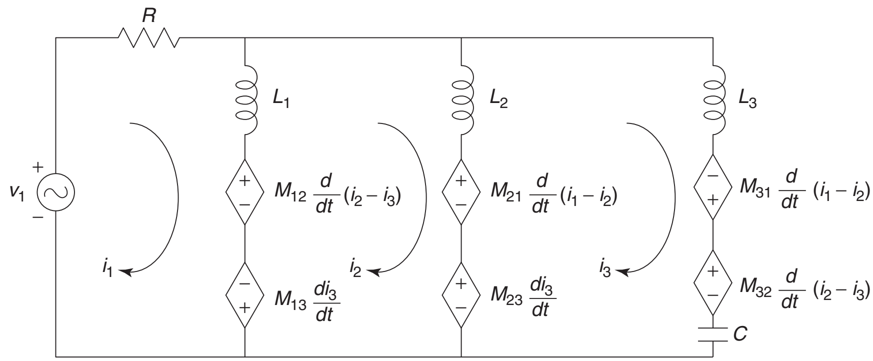


Fig. 4.38

Applying KVL to Mesh 1,

$$v(t) - Ri_1 - L_1\frac{d}{dt}(i_1 - i_2) - M_{12}\frac{d}{dt}(i_2 - i_3) + M_{13}\frac{di_3}{dt} = 0$$

$$Ri_1 + L_1\frac{d}{dt}(i_1 - i_2) + M_{12}\frac{d}{dt}(i_2 - i_3) - M_{13}\frac{di_3}{dt} = v(t) \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-M_{13}\frac{di_3}{dt} + M_{12}\frac{d}{dt}(i_2 - i_3) - L_1\frac{d}{dt}(i_2 - i_1) - L_2\frac{d}{dt}(i_2 - i_3) - M_{21}\frac{d}{dt}(i_1 - i_2) - M_{23}\frac{di_3}{dt} = 0$$

$$M_{13}\frac{di_3}{dt} - M_{12}\frac{d}{dt}(i_2 - i_3) + L_1\frac{d}{dt}(i_2 - i_1) + L_2\frac{d}{dt}(i_2 - i_3) + M_{21}\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{di_3}{dt} = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$M_{23}\frac{di_3}{dt} + M_{21}\frac{d}{dt}(i_1 - i_2) - L_2\frac{d}{dt}(i_3 - i_2) - L_3\frac{di_3}{dt} + M_{31}\frac{d}{dt}(i_1 - i_2) - M_{32}\frac{d}{dt}(i_2 - i_3) - \frac{1}{C}\int i_3\,dt = 0$$

$$-M_{23}\frac{di_3}{dt} - M_{21}\frac{d}{dt}(i_1 - i_2) + L_2\frac{d}{dt}(i_3 - i_2) + L_3\frac{di_3}{dt} - M_{31}\frac{d}{dt}(i_1 - i_2) + M_{32}\frac{d}{dt}(i_2 - i_3) + \frac{1}{C}\int i_3\,dt = 0 \qquad \text{...(iii)}$$

Example 4.19 *Write KVL equations for the network shown in Fig. 4.39.*

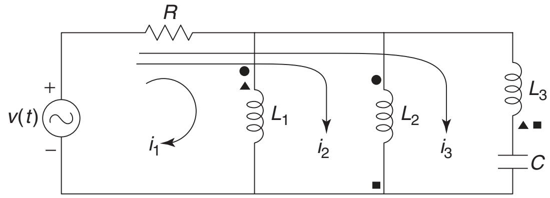


Fig. 4.39

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.40.

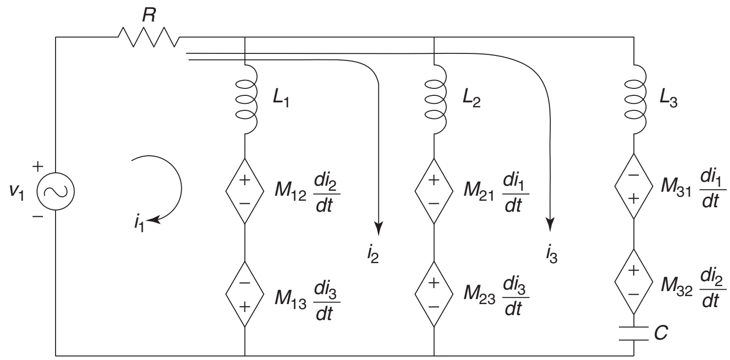


Fig. 4.40

Applying KVL to Loop 1,

$$v(t) - R(i_1 + i_2 + i_3) - L_1 \frac{di_1}{dt} - M_{12} \frac{di_2}{dt} + M_{13} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} - M_{13} \frac{di_3}{dt} = v(t) \qquad \ldots\text{(i)}$$

Applying KVL to Loop 2,

$$v(t) - R(i_1 + i_2 + i_3) - L_2 \frac{di_2}{dt} - M_{21} \frac{di_1}{dt} - M_{23} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt} + M_{23} \frac{di_3}{dt} = v(t) \qquad \ldots\text{(ii)}$$

Applying KVL to Loop 3,

$$v(t) - R(i_1 + i_2 + i_3) - L_3 \frac{di_3}{dt} + M_{31} \frac{di_1}{dt} - M_{32} \frac{di_2}{dt} - \frac{1}{C}\int i_3 \, dt = 0$$

$$R(i_1 + i_2 + i_3) + L_3 \frac{di_3}{dt} - M_{31} \frac{di_1}{dt} + M_{32} \frac{di_2}{dt} + \frac{1}{C}\int i_3 \, dt = v(t) \qquad \ldots\text{(iii)}$$

Example 4.20 *In the network shown in Fig. 4.41, find the voltages V_1 and V_2.*

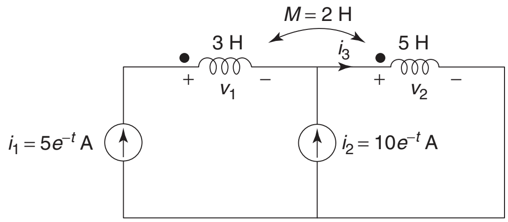


Fig. 4.41

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.42.

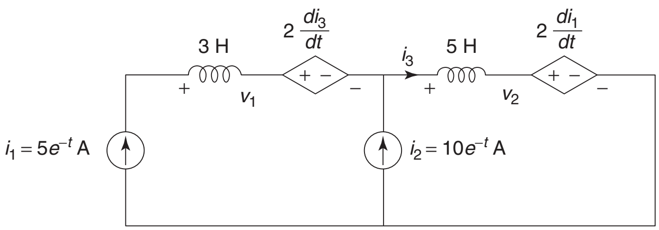


Fig. 4.42

From Fig. 4.42,

$$i_3 = i_1 + i_2 = 5\,e^{-t} + 10\,e^{-t} = 15\,e^{-t}\text{ A}$$

$$v_1 = 3\frac{di_1}{dt} + 2\frac{di_3}{dt} = 3\frac{d}{dt}(5\,e^{-t}) + 2\frac{d}{dt}(15\,e^{-t}) = -15\,e^{-t} - 30\,e^{-t} = -45\,e^{-t}\text{V}$$

$$v_2 = 5\frac{di_3}{dt} + 2\frac{di_1}{dt} = 5\frac{d}{dt}(15\,e^{-t}) + 2\frac{d}{dt}(5\,e^{-t}) = -75\,e^{-t} - 10\,e^{-t} = -85\,e^{-t}\text{V}$$

Example 4.21 *In the network shown in Fig. 4.43, find the voltages V_1 and V_2.*

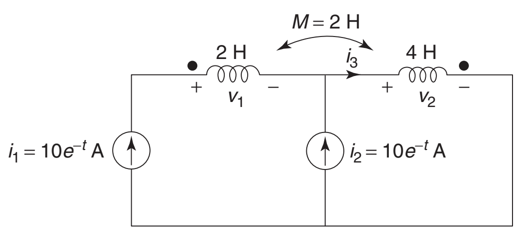


Fig. 4.43

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.44.

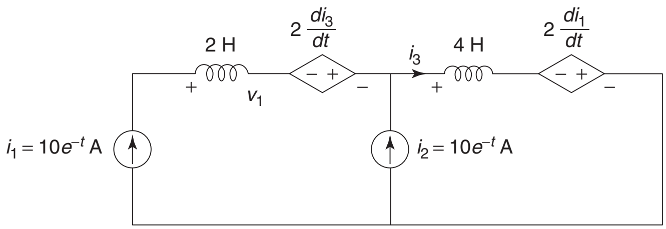


Fig. 4.44

From Fig. 4.44,

$$i_3 = i_1 + i_2 = 10\,e^{-t} + 10\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_1 = 2\frac{di_1}{dt} - 2\frac{di_3}{dt} = 2\frac{d}{dt}(10\,e^{-t}) - 2\frac{d}{dt}(20\,e^{-t}) = -20\,e^{-t} + 40\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_2 = 4\frac{di_3}{dt} - 2\frac{di_1}{dt} = 4\frac{d}{dt}(20\,e^{-t}) - 2\frac{d}{dt}(10\,e^{-t}) = -80\,e^{-t} + 20\,e^{-t} = -60\,e^{-t}\text{A}$$

Example 4.22 *Calculate the current $i_2(t)$ in the coupled circuit of Fig. 4.45.*

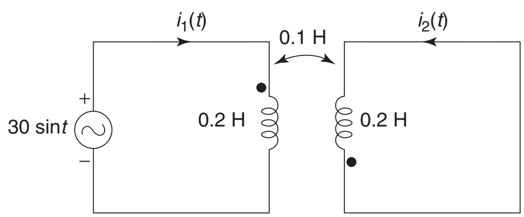


Fig. 4.45

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.46.

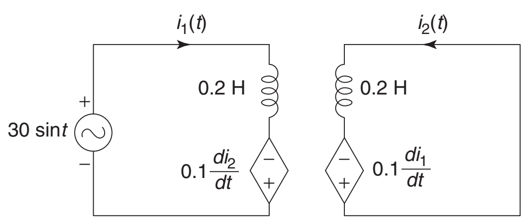


Fig. 4.46

Applying KVL to Mesh 1,

$$30\sin t - 0.2\frac{di_1}{dt} + 0.1\frac{di_2}{dt} = 0 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-0.2\frac{di_2}{dt}+0.1\frac{di_1}{dt}=0$$

$$\frac{di_1}{dt}=2\frac{di_2}{dt} \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$30\sin t-0.2\left(2\frac{di_2}{dt}\right)+0.1\frac{di_2}{dt}=0$$

$$0.3\frac{di_2}{dt}=30\sin t$$

$$\frac{di_2}{dt}=100\sin t$$

$$di_2=100\sin t\,dt$$

Integrating both the sides,

$$i_2\,(t)=100\int_0^t \sin t\,dt$$

$$=100\left[-\cos t\right]_0^t$$

$$=100\,(1-\cos t)$$

Example 4.23 *Find the voltage V_2 in the circuit shown in Fig. 4.47 such that the current in the left-hand loop (Loop 1) is zero.*

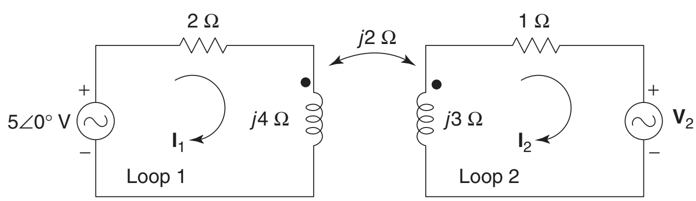


Fig. 4.47

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.48.

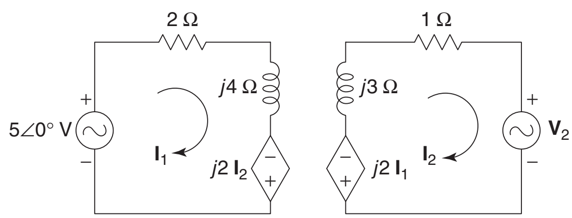


Fig. 4.48

Applying KVL to Loop 1,

$$5\angle 0° - 2\,\mathbf{I}_1 - j4\mathbf{I}_1 + j2\mathbf{I}_2 = 0$$

$$(2+j4)\,\mathbf{I}_1 - j2\mathbf{I}_2 = 5\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Loop 2,

$$-j2\mathbf{I}_1 - j3\mathbf{I}_2 - 1\mathbf{I}_2 - \mathbf{V}_2 = 0$$

$$-j2\mathbf{I}_1 - (1+j3)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 2+j4 & -j2 \\ -j2 & -(1+j3) \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 5\angle 0° \\ \mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 5\angle 0° & -j2 \\ \mathbf{V}_2 & -(1+j3) \end{vmatrix}}{\begin{vmatrix} 2+j4 & -j2 \\ -j2 & -(1+j3) \end{vmatrix}}$$

But $\mathbf{I}_1 = 0$.

$$\therefore -(5\angle 0°)(1+j3) + j2\,\mathbf{V}_2 = 0$$

$$\mathbf{V}_2 = \frac{(5\angle 0°)(1+j3)}{j2} = 7.91\angle -18.43° \text{ V}$$

Example 4.24 *Determine the ratio* $\frac{V_2}{V_1}$ *in the circuit of Fig. 4.49, if* $I_1 = 0$.

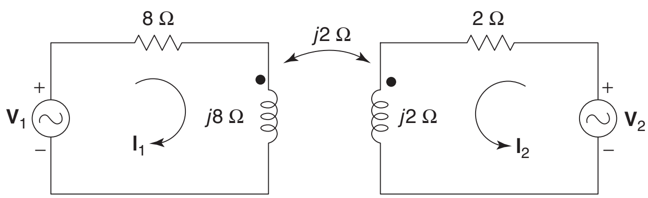


Fig. 4.49

Solution The equivalent circuit in terms of dependent sources is as shown in Fig. 4.50.

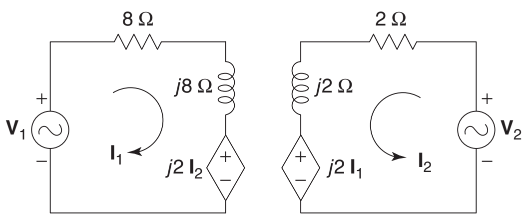


Fig. 4.50

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 8\mathbf{I}_1 - j8\mathbf{I}_1 - j2\mathbf{I}_2 = 0$$

$$(8+j8)\,\mathbf{I}_1 + j2\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Putting $\mathbf{I}_1 = 0$ in Eq (i),

$$j2\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - 2\mathbf{I}_2 - j2\mathbf{I}_2 - j2\mathbf{I}_1 = 0$$

$$j2\mathbf{I}_1 + (2+j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \text{...(iii)}$$

Putting $\mathbf{I}_1 = 0$ in Eq (iii),

$$(2+j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \text{...(iv)}$$

From Eqs (ii) and (iv),

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{(2+j2)\mathbf{I}_2}{j2\mathbf{I}_2} = \frac{2+j2}{j2} = 1.41\angle -45^\circ\text{ V}$$

Example 4.25 *For the coupled circuit shown in Fig. 4.51, find input impedance at terminals A and B.*

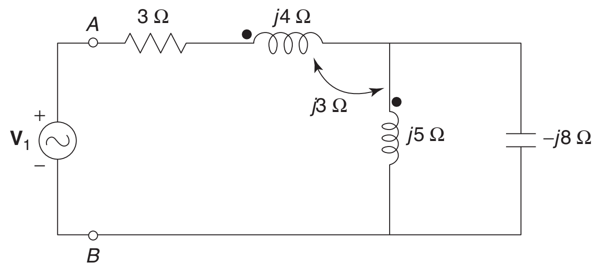


Fig. 4.51

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.52.

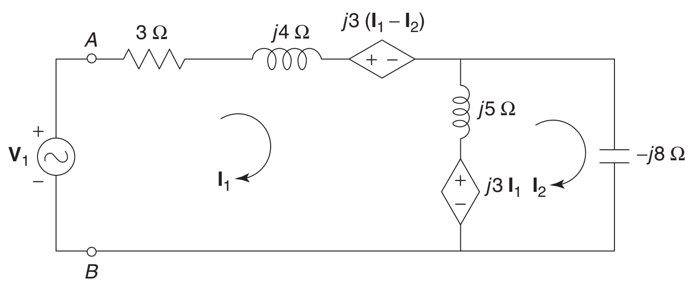


Fig. 4.52

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 3\mathbf{I}_1 - j4\mathbf{I}_1 - j3\,(\mathbf{I}_1 - \mathbf{I}_2) - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - j3\mathbf{I}_1 = 0$$

$$(3+j15)\,\mathbf{I}_1 - j8\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$

$$j8\,\mathbf{I}_1 + j3\,\mathbf{I}_2 = 0$$

$$\mathbf{I}_2 = -\frac{j8}{j3}\mathbf{I}_1 = -2.67\,\mathbf{I}_1 \qquad \text{...(ii)}$$

Substituting Eq (ii) in Eq (i),

$$(3+j15)\mathbf{I}_1 - j8(-2.67\,\mathbf{I}_1) = \mathbf{V}_1$$

$$(3+j\,36.36)\,\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = (3+j\,36.36)\,\Omega = 36.48\angle 85.28°\,\Omega$$

Example 4.26 *Find equivalent impedance of the network shown in Fig. 4.53.*

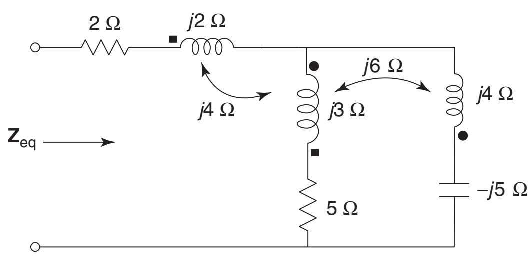


Fig. 4.53

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.54.

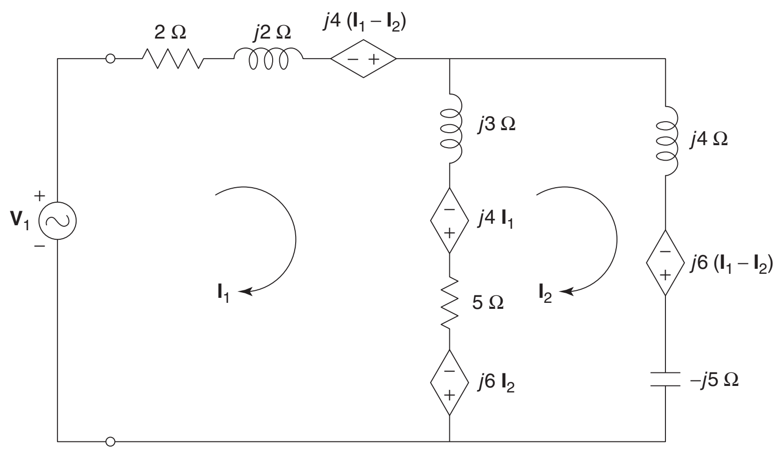


Fig. 4.54

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 2\mathbf{I}_1 - j2\mathbf{I}_1 + j4(\mathbf{I}_1 - \mathbf{I}_2) - j3(\mathbf{I}_1 - \mathbf{I}_2) + j4\,\mathbf{I}_1 - 5(\mathbf{I}_1 - \mathbf{I}_2) + j6\mathbf{I}_2 = 0$$

$$(7 - j3)\,\mathbf{I}_1 - (5 + j5)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j6\mathbf{I}_2 - 5(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_1 - j3(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_2 + j6(\mathbf{I}_1 - \mathbf{I}_2) + j5\mathbf{I}_2 = 0$$

$$(5 + j5)\mathbf{I}_1 = (5 + j4)\,\mathbf{I}_2$$

$$\mathbf{I}_2 = \left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$(7 - j3)\mathbf{I}_1 - (5 + j5)\left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 7 - j3 - \frac{(5 + j5)\,(5 + j5)}{5 + j4} = 5.63\angle -47.15^\circ\ \Omega$$

Example 4.27 *Find the voltage across the 5 Ω resistor in Fig. 4.55 using mesh analysis.*

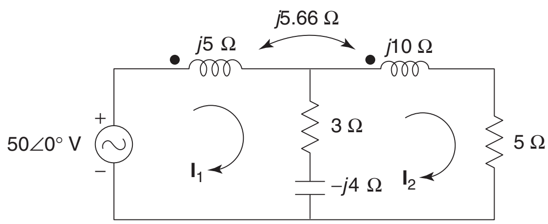


Fig. 4.55

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.56.

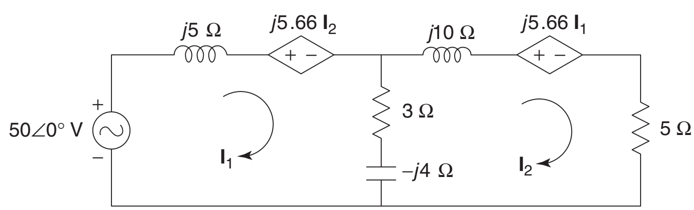


Fig. 4.56

Applying KVL to Mesh 1,

$$50\angle 0^\circ - j5\,\mathbf{I}_1 - j\,5.66\,\mathbf{I}_2 - (3 - j4)\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j\,1)\,\mathbf{I}_1 - (3 - j\,9.66)\,\mathbf{I}_2 = 50\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-(3 - j4)\,(\mathbf{I}_2 - \mathbf{I}_1) - j\,10\mathbf{I}_2 - j\,5.66\,\mathbf{I}_1 - 5\mathbf{I}_2 = 0$$

$$-(3 - j\,9.66)\,\mathbf{I}_1 + (8 + j6)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3+j1 & 50\angle 0° \\ -(3-j9.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{vmatrix}} = 3.82\angle -112.14° \text{ A}$$

$$\mathbf{V}_{5\,\Omega} = 5\,\mathbf{I}_2 = 5\,(3.82\angle -112.14°) = 19.1\angle -112.14° \text{ V}$$

Example 4.28 *Find the voltage across the 5 Ω resistor in Fig. 4.57 using mesh analysis.*

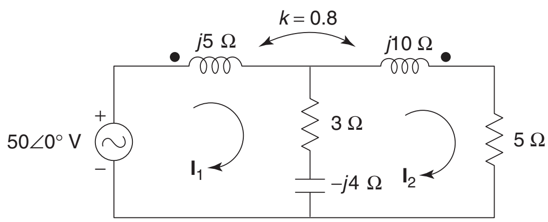


Fig. 4.57

Solution For a magnetically coupled circuit,

$$X_M = k\sqrt{X_{L_1} X_{L_2}}$$
$$= 0.8\sqrt{(5)\,(10)}$$
$$= 5.66\ \Omega$$

The equivalent circuit in terms of dependent sources is shown in Fig. 4.58.

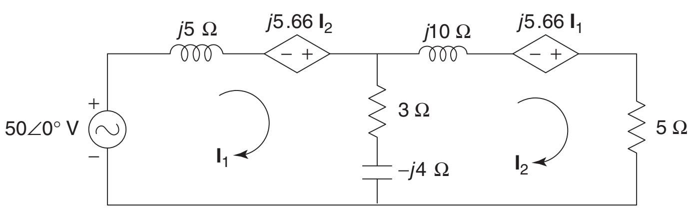


Fig. 4.58

Applying KVL to Mesh 1,

$$50\angle 0° - j5\,\mathbf{I}_1 + j\,5.66\,\mathbf{I}_2 - (3-j4)\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3+j1)\,\mathbf{I}_1 - (3+j1.66)\,\mathbf{I}_2 = 50\angle 0° \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-(3-j4)(\mathbf{I}_2-\mathbf{I}_1)-j\,10\mathbf{I}_2+j\,5.66\,\mathbf{I}_1-5\mathbf{I}_2=0$$

$$-(3+j\,1.66)\,\mathbf{I}_1+(8+j6)\,\mathbf{I}_2=0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j\,1 & -(3+j\,1.66) \\ -(3+j\,1.66) & 8+j\,6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}=\begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2=\frac{\begin{vmatrix} 3+j\,1 & 50\angle 0^\circ \\ -(3+j\,1.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j\,1 & -(3+j\,1.66) \\ -(3+j\,1.66) & 8+j6 \end{vmatrix}}=8.62\angle-24.79^\circ\text{ A}$$

$$\mathbf{V}_{5\,\Omega}=5\,\mathbf{I}_2=5\,(8.62\angle-24.79^\circ)=43.1\angle-24.79^\circ\text{ A}$$

Example 4.29 *Find the current through the capacitor in Fig. 4.59 using mesh analysis.*

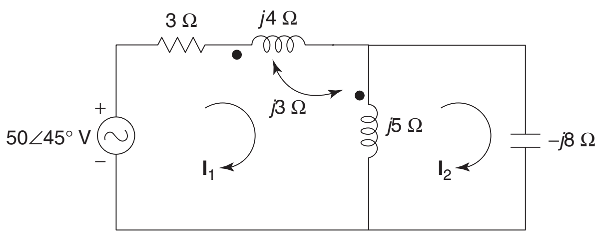


Fig. 4.59

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.60.

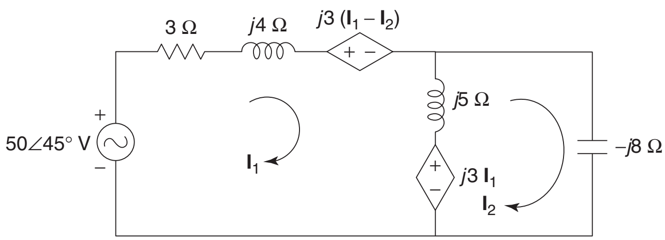


Fig. 4.60

Applying KVL to Mesh 1,

$$50\angle 45^\circ-(3+j4)\,\mathbf{I}_1-j3(\mathbf{I}_1-\mathbf{I}_2)-j5\,(\mathbf{I}_1-\mathbf{I}_2)-j3\,\mathbf{I}_1=0$$

$$(3+j\,15)\,\mathbf{I}_1-j\,8\,\mathbf{I}_2=50\angle 45^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$
$$-j8\,\mathbf{I}_1 - j3\,\mathbf{I}_2 = 0 \qquad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3+j15 & 50\angle 45^\circ \\ -j8 & 0 \end{vmatrix}}{\begin{vmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{vmatrix}} = 3.66\angle -310.33^\circ \text{ A}$$

$$\mathbf{I}_C = \mathbf{I}_2 = 3.66\angle -310.33^\circ\text{A}$$

Example 4.30 *Find the voltage across the 15 Ω resistor in Fig. 4.61 using mesh analysis.*

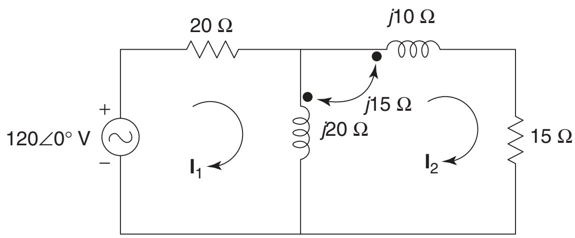


Fig. 4.61

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.62.

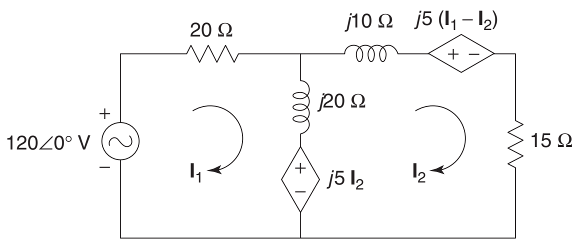


Fig. 4.62

Applying KVL to Mesh 1,

$$120\angle 0^\circ - 20\,\mathbf{I}_1 - j20(\mathbf{I}_1 - \mathbf{I}_2) - j5\,\mathbf{I}_2 = 0$$
$$(20 + j\,20)\,\mathbf{I}_1 - j\,15\,\mathbf{I}_2 = 120\angle 0^\circ \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$j5\,\mathbf{I}_2 - j\,20\,(\mathbf{I}_2 - \mathbf{I}_1) - j10\,\mathbf{I}_2 - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - 15\,\mathbf{I}_2 = 0$$

$$-j15\,\mathbf{I}_1 + (15 + j20)\,\mathbf{I}_2 = 0 \qquad \text{...(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120 \angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 20 + j\,20 & 120 \angle 0° \\ -j15 & 0 \end{vmatrix}}{\begin{vmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{vmatrix}} = 2.53 \angle 10.12°\text{A}$$

$$\mathbf{V}_{15\,\Omega} = 15\,\mathbf{I}_2 = 15\,(2.53 \angle 10.12°) = 37.95 \angle 10.12°\text{V}$$

Example 4.31 *Find the current through the 6 Ω resistor in Fig. 4.63 using mesh analysis.*

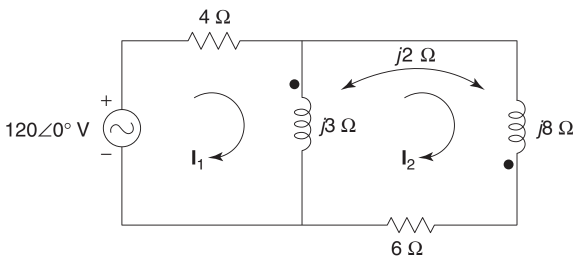


Fig. 4.63

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.64.

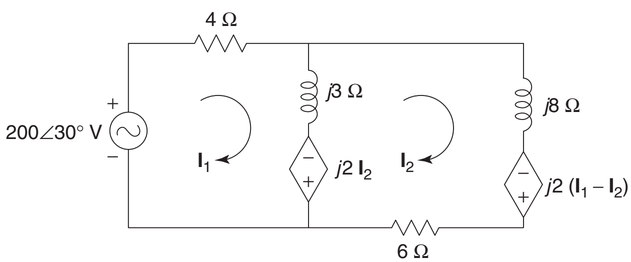


Fig. 4.64

Applying KVL to Mesh 1,

$$120 \angle 0° - 4\,\mathbf{I}_1 - j3(\mathbf{I}_1 - \mathbf{I}_2) + j2\,\mathbf{I}_2 = 0$$

$$(4 + j\,3)\,\mathbf{I}_1 - j\,5\,\mathbf{I}_2 = 120 \angle 0° \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j\,2\,\mathbf{I}_2 - j\,3\,(\mathbf{I}_2 - \mathbf{I}_1) - j8\,\mathbf{I}_2 + j2\,(\mathbf{I}_1 - \mathbf{I}_2) - 6\,\mathbf{I}_2 = 0$$

$$-j5\,\mathbf{I}_1 + (6 + j15)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 4+j\,3 & -j5 \\ -j5 & 6+j\,15 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4+j\,3 & 120\angle 0^\circ \\ -j5 & 0 \end{vmatrix}}{\begin{vmatrix} 4+j\,3 & -j5 \\ -j5 & 6+j\,15 \end{vmatrix}} = 7.68\angle 2.94^\circ\text{A}$$

Example 4.32 *Determine the mesh current $\boldsymbol{I}_3$ in the network of Fig. 4.65.*

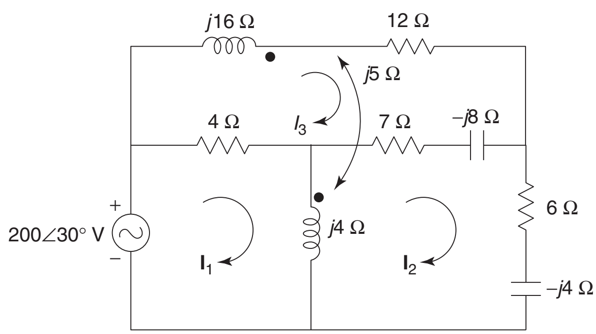


Fig. 4.65

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.66.

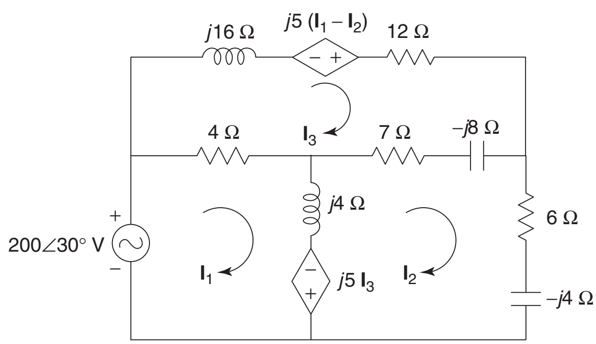


Fig. 4.66

Applying KVL to Mesh 1,

$$200 \angle 30° - 4\,(\mathbf{I}_1 - \mathbf{I}_3) - j4(\mathbf{I}_1 - \mathbf{I}_2) + j5\,\mathbf{I}_3 = 0$$

$$(4 + j\,4)\,\mathbf{I}_1 - j\,4\,\mathbf{I}_2 - (4 + j5)\,\mathbf{I}_3 = 200 \angle 30° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j\,5\,\mathbf{I}_3 - j\,4\,(\mathbf{I}_2 - \mathbf{I}_1) - (7 - j8)\,(\mathbf{I}_2 - \mathbf{I}_3) - (6 - j\,4)\,\mathbf{I}_2 = 0$$

$$-j4\,\mathbf{I}_1 + (13 - j8)\,\mathbf{I}_2 - (7 - j13)\,\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-j\,16\,\mathbf{I}_3 + j\,5\,(\mathbf{I}_1 - \mathbf{I}_2) - 12\,\mathbf{I}_3 - (7 - j\,8)\,(\mathbf{I}_3 - \mathbf{I}_2) - 4(\mathbf{I}_3 - \mathbf{I}_1) = 0$$

$$-(4 + j5)\,\mathbf{I}_1 - (7 - j13)\,\mathbf{I}_2 + (23 + j8)\,\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs. (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 200 \angle 30° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4 + j4 & -j4 & 200 \angle 30° \\ -j4 & 13 - j8 & 0 \\ -(4 + j5) & -(7 - j13) & 0 \end{vmatrix}}{\begin{vmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{vmatrix}} = 16.28 \angle 16.87° \text{A}$$

Example 4.33 *Obtain the dotted equivalent circuit for the coupled circuit shown in Fig. 4.67 and find mesh currents. Also find the voltage across the capacitor.*

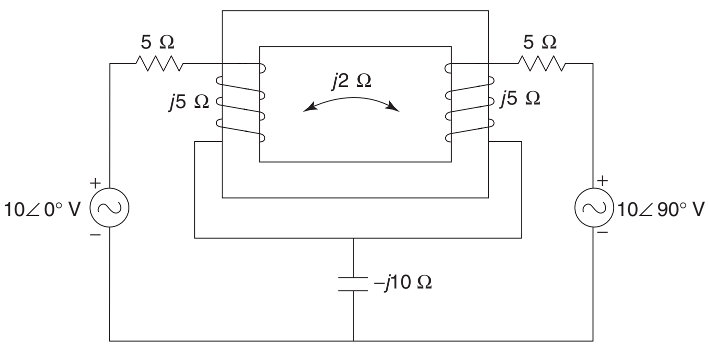


Fig. 4.67

Solution The currents in the coils are as shown in Fig. 4.68. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

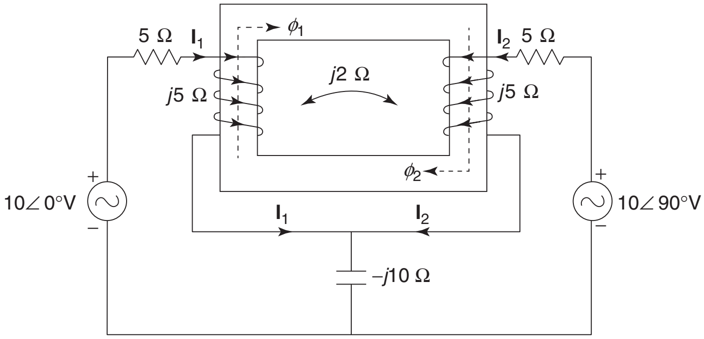


Fig. 4.68

From Fig. 4.68, it is seen that two fluxes ϕ_1 and ϕ_2 aid each other. Hence, dots are placed at the two coils as shown in Fig. 4.69.

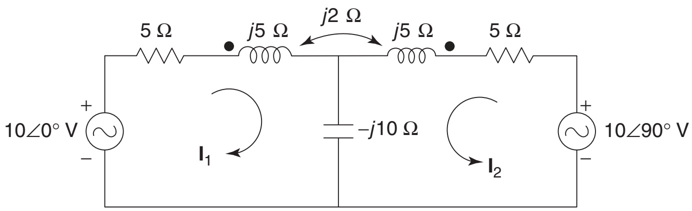


Fig. 4.69

The equivalent circuit in terms of dependent sources is shown in Fig. 4.70.

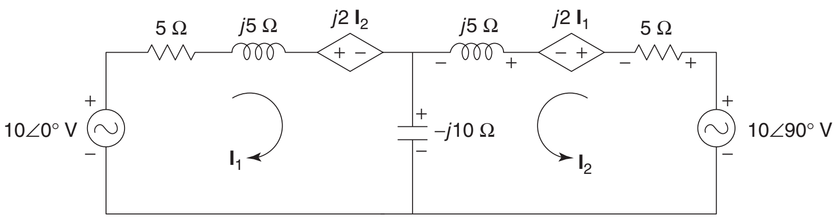


Fig. 4.70

Applying KVL to Mesh 1,

$$10 \angle 0° - (5 + j5)\, \mathbf{I}_1 - j2\, \mathbf{I}_2 + j10\, (\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$(5 - j\,5)\, \mathbf{I}_1 - j\,8\, \mathbf{I}_2 = 10 \angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j10\, (\mathbf{I}_2 + \mathbf{I}_1) + j5\, \mathbf{I}_2 - j2\, \mathbf{I}_1 + 5\, \mathbf{I}_2 - 10 \angle 90° = 0$$

$$-j\,8\, \mathbf{I}_1 + (5 - j5)\, \mathbf{I}_2 = 10 \angle 90° \qquad \ldots\text{(ii)}$$

Writing Eqs. (i) and (ii) in matrix form,

$$\begin{bmatrix} 5 - j\,5 & -j8 \\ -j8 & 5 - j\,5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10 \angle 0° \\ 10 \angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -j8 \\ 10\angle 90^\circ & 5-j5 \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 0.72\angle -82.97^\circ \text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5-j5 & 10\angle 0^\circ \\ -j8 & 10\angle 90^\circ \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 1.71\angle 106.96^\circ \text{A}$$

$$\begin{aligned} \mathbf{V}_C &= -j\,10\,(\mathbf{I}_1 + \mathbf{I}_2) = (-j\,10)\,(0.72\angle -82.97^\circ + 1.71\angle 106.96^\circ \text{ A}) \\ &= 10.08\angle 24.03^\circ \text{V} \end{aligned}$$

4.9 CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS

For simplifying circuit analysis, it is desirable to replace a magnetically coupled circuit with an equivalent circuit called conductively coupled circuit. In this circuit, no magnetic coupling is involved. The dot convention is also not needed in the conductively coupled circuit.

Consider a coupled circuit as shown in Fig. 4.71.

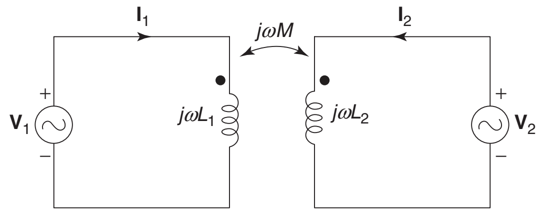


Fig. 4.71 *Coupled circuit*

The equivalent circuit in terms of dependent sources is shown in Fig. 4.72.

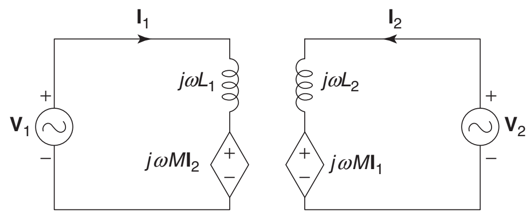


Fig. 4.72 *Equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - j\omega L_1\,\mathbf{I}_1 - j\omega M\,\mathbf{I}_2$$

$$j\omega L_1\,\mathbf{I}_1 + j\omega M\,\mathbf{I}_2 \qquad \ldots(4.11)$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - j\omega L_2\,\mathbf{I}_2 - j\omega M\,\mathbf{I}_1 = 0$$

$$j\omega M\,\mathbf{I}_1 + j\omega L_2\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots(4.12)$$

Writing Eqs (4.11) and (4.12) in matrix form,

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \qquad \ldots(4.13)$$

Consider a T-network as shown in Fig. 4.73.

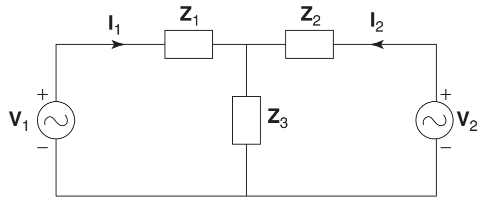


Fig. 4.73 *T-network*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{Z}_1\,\mathbf{I}_1 - \mathbf{Z}_3(\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$(\mathbf{Z}_1 + \mathbf{Z}_3)\,\mathbf{I}_1 + \mathbf{Z}_3\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(4.14)$$

Applying KVL to Mesh 2,

$$R_{\text{Th}} = [(2\,||\,12) + 1]\,||\,3 = 1.43\ \Omega \qquad \ldots(4.15)$$

Writing Eqs (4.14) and (4.15) in matrix form,

$$\begin{bmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & \mathbf{Z}_3 \\ \mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Comparing matrix equations,

$$\mathbf{Z}_1 + \mathbf{Z}_3 = j\,\omega\,L_1$$

$$\mathbf{Z}_3 = j\,\omega\,M$$

$$\mathbf{Z}_2 + \mathbf{Z}_3 = j\,\omega\,L_2$$

Solving these equations,

$$\mathbf{Z}_1 = j\,\omega\,L_1 - j\,\omega\,M = j\,\omega(L_1 - M)$$

$$\mathbf{Z}_2 = j\,\omega\,L_2 - j\,\omega\,M = j\,\omega(L_2 - M)$$

$$\mathbf{Z}_3 = j\,\omega\,M$$

Hence, the conductively coupled circuit of a magnetically coupled circuit is shown in Fig. 4.74.

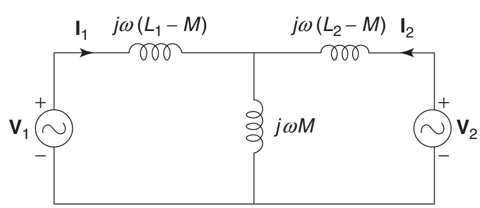


Fig. 4.74 *Conductively coupled equivalent circuit*

Example 4.34 *Find the conductively coupled equivalent circuit for the network shown in Fig. 4.75.*

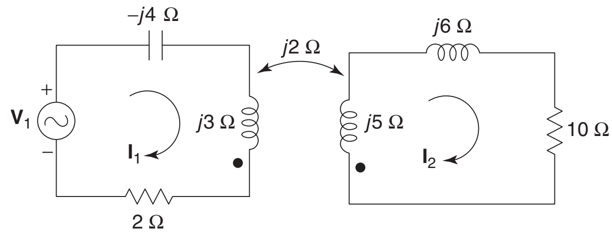


Fig. 4.75

Solution The current $\mathbf{I}_1$ leaves from the dotted end and $\mathbf{I}_2$ enters from the dotted end. Hence, mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j3 - j2 = j1\,\Omega$$

$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j5 - j2 = j3\,\Omega$$

$$\mathbf{Z}_3 = j\omega M = j2\,\Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.76.

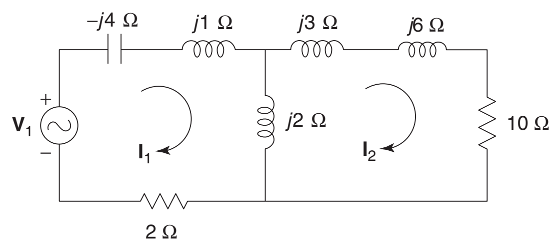


Fig. 4.76

Example 4.35 *Draw the conductively coupled equivalent circuit of Fig. 4.77.*

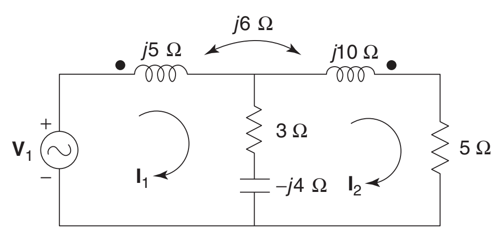


Fig. 4.77

Solution The current $\mathbf{I}_1$ enters from the dotted end and $\mathbf{I}_2$ leaves from the dotted end. Hence, the mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j5 - j6 = -j1\,\Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j10 - j6 = j4\,\Omega$$
$$\mathbf{Z}_3 = j\omega M = j6\,\Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.78.

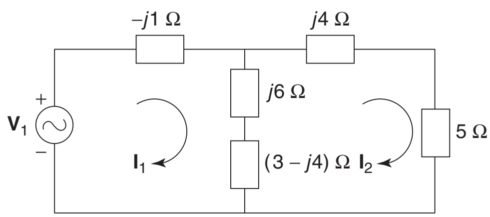


Fig. 4.78

Example 4.36 *Find the conductively coupled equivalent circuit of the network in Fig. 4.79.*

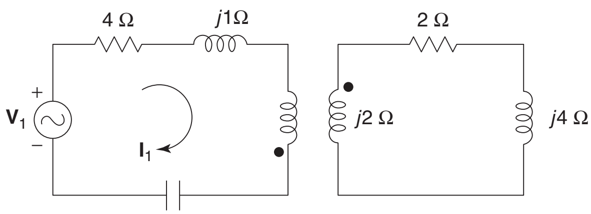


Fig. 4.79

Solution The currents $\mathbf{I}_1$ and $\mathbf{I}_2$ leave from the dotted terminals. Hence, mutual inductance is positive. In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 + M) = j\omega L_1 + j\omega M = j4 + j2 = j6\,\Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 + M) = j\omega L_2 + j\omega M = j2 + j2 = j4\,\Omega$$
$$\mathbf{Z}_3 = -j\omega M = -j2\,\Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.80.

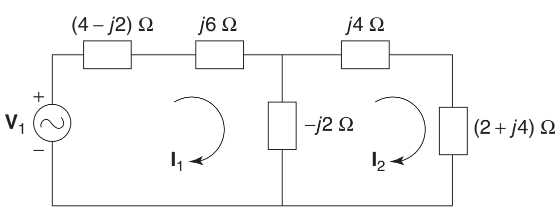


Fig. 4.80

Exercises

4.1 Two coupled coils have inductances of 0.8 H and 0.2 H. The coefficient of coupling is 0.90. Find the mutual inductance and the turns ratio $\dfrac{N_1}{N_2}$.

[0.36 H, 2]

4.2 Two coils with coefficient of coupling 0.5 are connected in such a way that they magnetise (i) in the same direction, and (ii) in opposite directions. The corresponding equivalent inductances are 1.9 H and 0.7 H. Find self-inductances of the two coils and the mutual inductance between them.

[0.4 H, 0.9 H, 0.3 H]

4.3 Two coils having 3000 and 2000 turns are wound on a magnetic ring. 60% of the flux produced in the first coil links with the second coil. A current of 3 A produce a flux of 0.5 mwb in the first coil and 0.3 mwb in the second coil. Determine the mutual inductance and coefficient of coupling.

[0.2 H, 0.63]

4.4 Find the equivalent inductance of the network shown in Fig. 4.81.

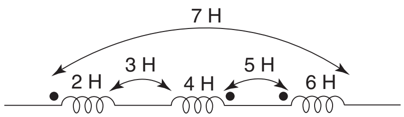


Fig. 4.81

[10 H]

4.5 Find the effective inductance of the network shown in Fig. 4.82.

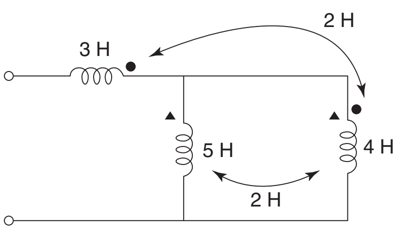


Fig. 4.82

[4.8 H]

4.6 Write mesh equations of the network shown in Fig. 4.83.

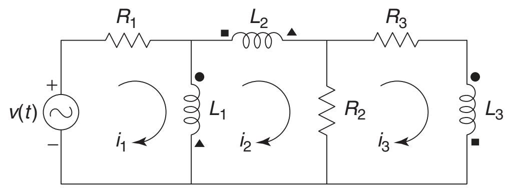


Fig. 4.83

$$\left[\begin{array}{l} v = i_1R_1 + L_1\dfrac{d}{dt}(i_1 - i_2) + M_{12}\dfrac{di_2}{dt} + M_{13}\dfrac{di_3}{dt} \\ R_2(i_3 - i_2) + R_3 i_3 + L_3\dfrac{di_3}{dt} + M_{13}\dfrac{d}{dt}(i_1 - i_2) \\ -M_{23}\dfrac{di_2}{dt} = 0 \end{array}\right]$$

4.7 Find the input impedance at terminals *AB* of the coupled circuits shown in Fig. 4.84 to 4.85.

(i)

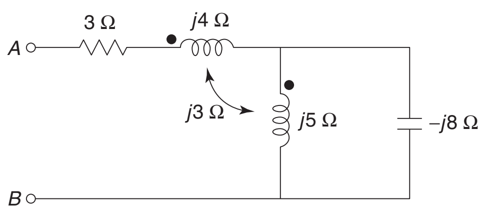


Fig. 4.84

(ii)

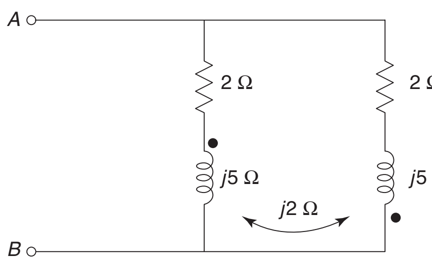


Fig. 4.85

(iii)

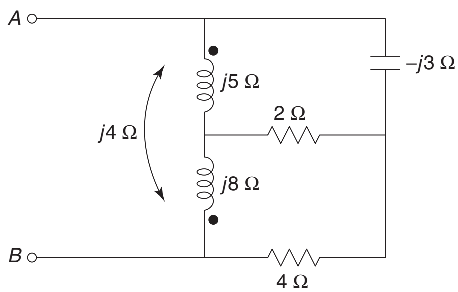


Fig. 4.86

$$\begin{bmatrix} \text{(a)}\,(3+j36.3)\,\Omega \quad \text{(b)}\,(1+j1.5)\,\Omega \\ \text{(c)}\,(6.22+j4.65)\,\Omega \end{bmatrix}$$

4.8 In the coupled circuit shown in Fig. 4.87, find $\mathbf{V}_2$ for which $\mathbf{I}_1 = 0$. What voltage appears at the 8 Ω inductive reactance under this condition?

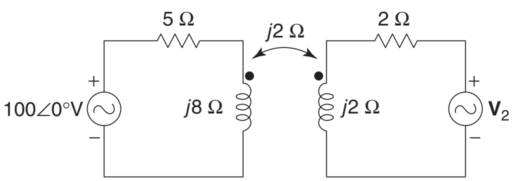


Fig. 4.87

[141.5∠−45°V, 100∠0°V]

4.9 For the coupled circuit shown in Fig. 4.88, find the components of the current $\mathbf{I}_2$ resulting from each source $\mathbf{V}_1$ and $\mathbf{V}_2$.

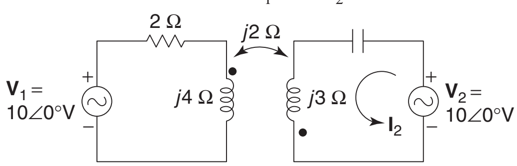


Fig. 4.88

[0.77∠112.6°A, 1.72∠86.05°A]

4.10 Find the voltage across the 5 Ω resistor in the network shown in Fig. 4.89.

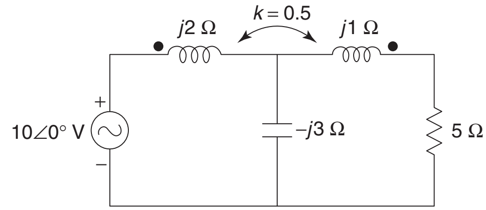


Fig. 4.89

[19.2∠−33.02°V]

4.11 Find the power dissipated in the 5 Ω resistor in the network of Fig. 4.90.

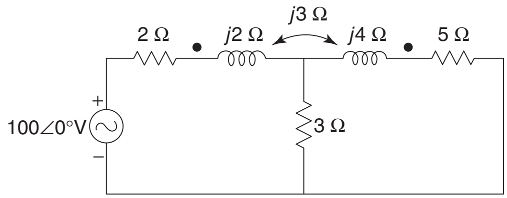


Fig. 4.90

[668.16 W]

4.12 Find the current **I** in the circuit of Fig. 4.91.

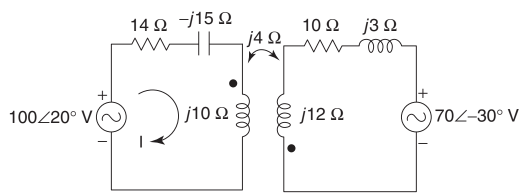


Fig. 4.91

[7.07∠45° V, 1.

4.13 Obtain a conductively coupled circuit for the circuit shown in Fig. 4.92.

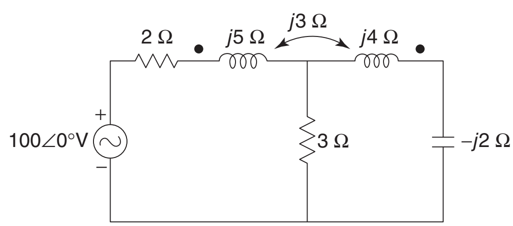


Fig. 4.92

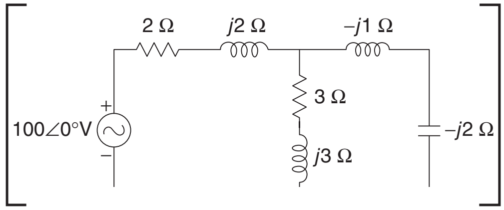


Fig. 4.93

Objective-Type Questions

4.1 Two coils are wound on a common magnetic core. The sign of mutual inductance M for finding out effective inductance of each coil is positive if the

(a) two coils are wound in the same sense.
(b) fluxes produced by the two coils are equal
(c) fluxes produced by the coils act in the same direction
(d) fluxes produced by the two coils act in opposition

4.2 When two coils having self-inductances of L_1 and L_2 are coupled through a mutual inductance M, the coefficient of coupling k is given by

(a) $k = \dfrac{M}{\sqrt{2L_1L_2}}$ (b) $k = \dfrac{M}{\sqrt{L_1L_2}}$

(c) $k = \dfrac{2M}{\sqrt{L_1L_2}}$ (d) $k = \dfrac{L_1L_2}{M}$

4.3 The overall inductance of two coils connected in series, with mutual inductance aiding self-inductance is L_1; with mutual inductance opposing self-inductance, the overall inductance is L_2. The mutual inductance M is given by

(a) $L_1 + L_2$ (b) $L_1 - L_2$

(c) $\dfrac{1}{4}(L_1 - L_2)$ (d) $\dfrac{1}{2}(L_1 + L_2)$

4.4 Consider the following statements:
The coefficient of coupling between two oils depends upon
1. Orientation of the coils
2. Core material
3. Number of turns on the two coils
4. Self-inductance of the two coils
of these statements,
(a) 1, 2 and 3 are correct
(b) 1 and 2 are correct
(c) 3 and 4 are correct
(d) 1, 2 and 4 are correct

4.5 Two coupled coils connected in series have an equivalent inductance of 16 mH or 8 mH depending on the inter connection.
Then the mutual inductance M between the coils is

(a) 12 mH (b) $8\sqrt{2}$ mH
(c) 4 mH (d) 2 mH

4.6 Two coupled coils with $L_1 = L_2 = 0.6$ H have a coupling coefficient of $k = 0.8$. The turns ratio $\dfrac{N_1}{N_2}$ is

(a) 4 (b) 2
(c) 1 (d) 0.5

4.7 The coupling between two magnetically coupled coils is said to be ideal if the coefficient of coupling is

(a) zero (b) 0.5
(c) 1 (d) 2

4.8 The mutual inductance between two coupled coils is 10 mH . If the turns in one coil are doubled and that in the other are halved then the mutual inductance will be

(a) 5 mH (b) 10 mH
(c) 14 mH (d) 20 mH

4.9 Two perfectly coupled coils each of 1 H self-inductance are connected in parallel so as to aid each other. The overall inductance in henrys is

(a) 2 (b) 1

(c) $\frac{1}{2}$ (d) Zero

4.10 The impedance Z as shown in Fig. 4.94 is

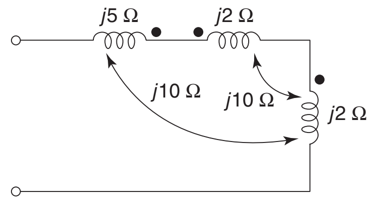


Fig. 4.94

(a) $j\,29\ \Omega$ (b) $j9\ \Omega$

(c) $j19\ \Omega$ (d) $j39\ \Omega$

Answers to Objective-Type Questions

4.1 (c)	4.2 (b)	4.3 (c)	4.4 (d)	4.5 (d)	4.6 (c)
4.7 (c)	4.8 (b)	4.9 (b)	4.10 (b)		

5 Graph Theory

5.1 INTRODUCTION

The purpose of network analysis is to find voltage across and current through all the elements. When the network is complicated and has a large number of nodes and closed paths, network analysis can be done conveniently by using 'Network Topology'. This theory does not make any distinction between different types of physical elements of the network but makes the study based on a geometric pattern of the network. The basic elements of this theory are nodes, branches, loops and meshes.

Node It is defined as a point at which two or more elements have a common connection.

Branch It is a line connecting a pair of nodes, the line representing a single element or series connected elements.

Loop Whenever there is more than one path between two nodes, there is a circuit or loop.

Mesh It is a loop which does not contain any other loops within it.

5.2 GRAPH OF A NETWORK

A linear graph is a collection of nodes and branches. The nodes are joined together by branches.

The graph of a network is drawn by first marking the nodes and then joining these nodes by lines which correspond to the network elements of each branch. All the voltage and current sources are replaced by their internal impedances. The voltage sources are replaced by short circuits as their internal impedances are zero whereas current sources are replaced by open circuits as their internal impedances are infinite. Nodes and branches are numbered. Figure 5.1 shows a network and its associated graphs.

Each branch of a graph may be given an orientation or a direction with the help of an arrow head which represents the assigned reference direction for current. Such a graph is then referred to as a directed or oriented graph.

Branches whose ends fall on a node are said to be incident at that node. Degree of a node is defined as the number of branches incident to it. Branches 2, 3 and 4 are incident at Node 2 in Fig. 5.1(c). Hence, the degree of Node 2 is 3.

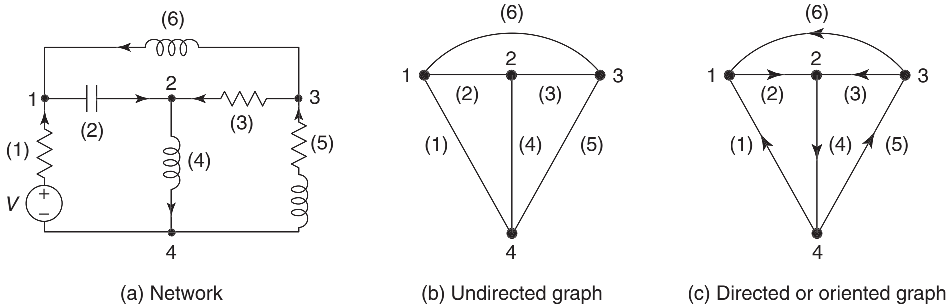


Fig. 5.1 *Network and its graphs*

5.3 GRAPH TERMINOLOGIES

1. Planar Graph A graph drawn on a two-dimensional plane is said to be planar if two branches do not intersect or cross at a point which is other than a node. Figure 5.2 shows such graphs.

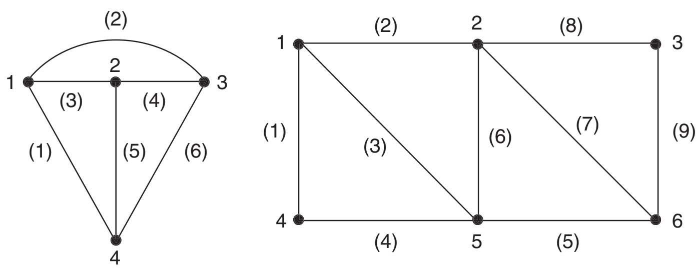


Fig. 5.2 *Planar graphs*

2. Non-planar Graph A graph drawn on a two-dimensional plane is said to be non-planar if there is intersection of two or more branches at another point which is not a node. Figure 5.3 shows non-planar graphs.

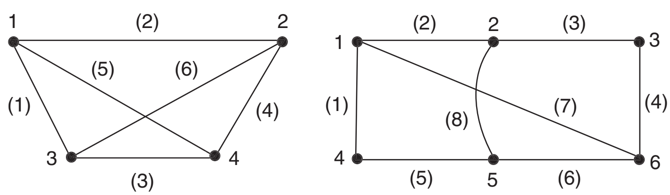


Fig. 5.3 *Non-planar graphs*

3. Sub-graph It is a subset of branches and nodes of a graph. It is a proper sub-graph if it contains branches and nodes less than those on a graph. A sub-graph can be just a node or only one branch of the graph. Figure 5.4 shows a graph and its proper sub-graph.

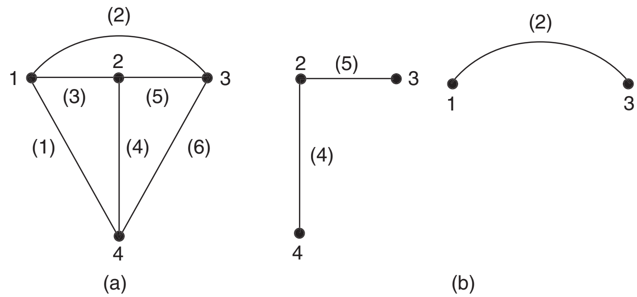


Fig. 5.4 *(a) Graph (b) Proper sub-graph*

4. Path It is an improper sub-graph having the following properties:

1. At two of its nodes called terminal nodes, there is incident only one branch of sub-graph.
2. At all remaining nodes called internal nodes, there are incident two branches of a graph.

In Fig. 5.5, branches 2, 5 and 6 together with all the four nodes, constitute a path.

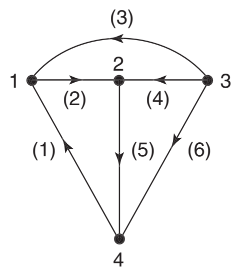


Fig. 5.5 *Path*

5. Connected Graph A graph is said to be connected if there exists a path between any pair of nodes. Otherwise, the graph is disconnected.

6. Rank of a Graph If there are n nodes in a graph, the rank of the graph is $(n - 1)$.

7. Loop or Circuit A loop is a connected sub-graph of a connected graph at each node of which are incident exactly two branches. If two terminals of a path are made to coincide, it will result in a loop or circuit. Figure 5.6 shows two loops.

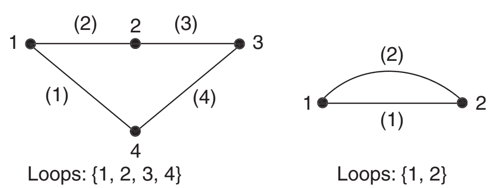


Fig. 5.6 *Loops*

Loops of a graph have the following properties:

1. There are at least two branches in a loop.
2. There are exactly two paths between any pair of nodes in a circuit.
3. The maximum number of possible branches is equal to the number of nodes.

8. Tree A tree is a set of branches with every node connected to every other node in such a way that removal of any branch destroys this property.

Alternately, a tree is defined as a connected sub-graph of a connected graph containing all the nodes of the graph but not containing any loops.

Branches of a tree are called twigs. A tree contains $(n - 1)$ twigs where n is the number of nodes in the graph. Figure 5.7 shows a graph and its trees.

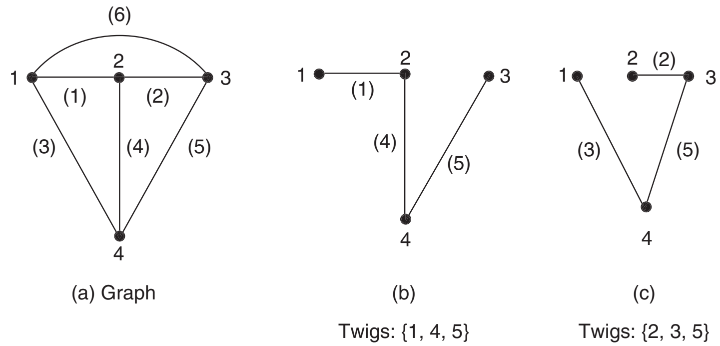


Fig. 5.7 *Graph and its trees*

Trees have the following properties:

1. There exists only one path between any pair of nodes in a tree.
2. A tree contains all nodes of the graph.
3. If n is the number of nodes of the graph, there are $(n - 1)$ branches in the tree.
4. Trees do not contain any loops.
5. Every connected graph has at least one tree.
6. The minimum terminal nodes in a tree are two.

9. Co-tree Branches which are not on a tree are called links or chords. All links of a tree together constitute the compliment of the corresponding tree and is called the co-tree.

A co-tree contains $b - (n - 1)$ links where b is the number of branches of the graph.
In Fig. 5.7 (b) and (c) the links are {2, 3, 6} and {1, 4, 6} respectively.

Example 5.1 *Draw directed graph of the networks shown in Fig. 5.8.*

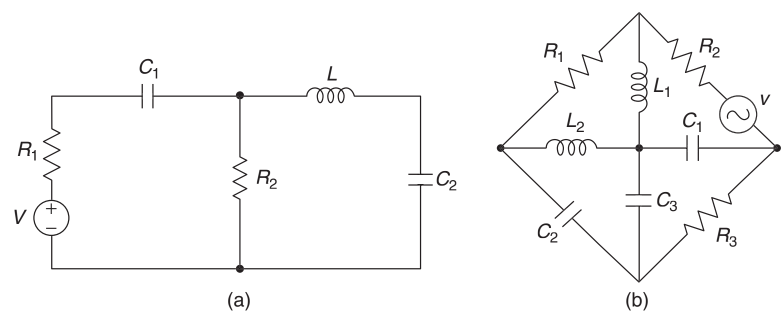


Fig. 5.8

Solution For drawing the directed graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the voltage source by a short-circuit,
3. assume directions of branch currents, and
4. number all the nodes and branches.

The directed graph for the two networks are shown in Fig. 5.9.

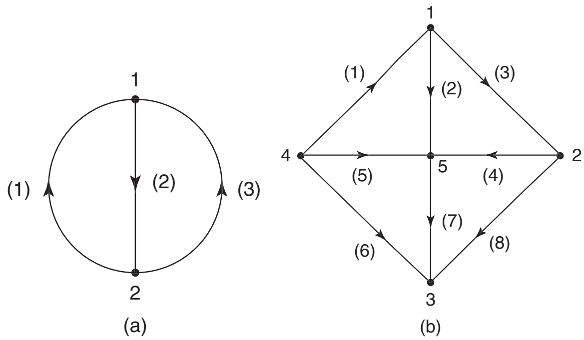


Fig. 5.9

Example 5.2 *Figure 5.10 shows a graph of the network. Show all the trees of this graph.*

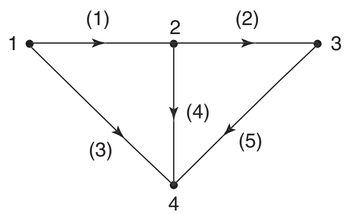


Fig. 5.10

Solution A graph has many trees. A tree is a connected sub-graph of a connected graph containing all the nodes of the graph but not containing any loops. Figure 5.11 shows various trees of the given graph.

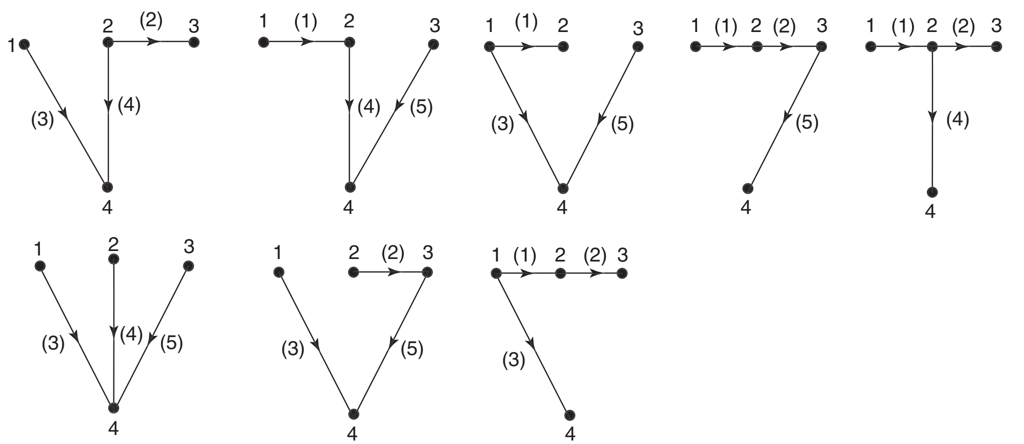


Fig. 5.11

5.4 INCIDENCE MATRIX

A linear graph is made up of nodes and branches. When a graph is given, it is possible to tell which branches are incident at which nodes and what are its orientations relative to the nodes.

5.4.1 Complete Incidence Matrix (A_a)

For a graph with n nodes and b branches, the complete incidence matrix is a rectangular matrix of order $n \times b$. Elements of this matrix have the following values:

$a_{ij} = 1$, if branch j is incident at node i and is oriented away from node i.
$= -1$, if branch j is incident at node i and is oriented towards node i.
$= 0$, if branch j is not incident at node i.

For the graph shown in Fig. 5.12, branch 1 is incident at nodes 1 and 4. It is oriented away from Node 1 and oriented towards Node 4. Hence, $a_{11} = 1$ and $a_{41} = -1$. Since branch 1 is not incident at nodes 2 and 3, $a_{21} = 0$ and $a_{31} = 0$. Similarly, other elements of the complete incidence matrix are written.

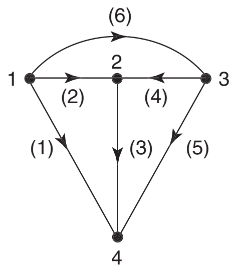


Fig. 5.12 *Graph*

Nodes ↓ / Branches →	1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	−1	1	−1	0	0
3	0	0	0	1	1	−1
4	−1	0	−1	0	−1	0

The complete incidence matrix is

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

It is seen from the matrix A_a that the sum of the elements in any column is zero. Hence, any one row of the complete incidence matrix can be obtained by the algebraic manipulation of other rows.

5.4.2 Reduced Incidence Matrix (A)

The reduced incidence matrix A is obtained from the complete incidence matrix A_a by eliminating one of the rows. It is also called *incidence matrix*. It is of order $(n - 1) \times b$.

Eliminating the third row of matrix A_a,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

When a tree is selected for the graph as shown in Fig. 5.13, the incidence matrix is obtained by arranging a column such that the first $(n - 1)$ column corresponds to twigs of the tree and the last $b - (n - 1)$ branches corresponds to the links of the selected tree.

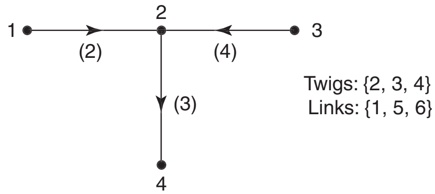


Fig. 5.13 *Tree*

$$A = \begin{array}{c} \begin{array}{cccccc} \text{Twigs} & & & \text{Links} & & \\ 2 & 3 & 4 & 1 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

The matrix A can be subdivided into submatrices A_t and A_l.

$$A = [A_t : A_l]$$

Where A_t the is twig matrix and A_l is the link matrix.

5.4.3 Number of Possible Trees of a Graph

Let the transpose of the reduced incidence matrix A be A^T. It can be shown that the number of possible trees of a graph will be given by

$$\text{Number of possible trees} = |AA^T|$$

For the graphs shown in Fig. 5.12, the reduced incidence matrix is given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Then transpose of this matrix will be

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence, number of all possible trees of the graph

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$|AA^T| = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3(9-1) + (1)(-3-1) - 1(1+3) = 16$$

Thus, 16 different trees can be drawn.

5.5 LOOP MATRIX OR CIRCUIT MATRIX

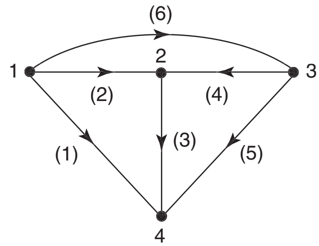
When a graph is given, it is possible to tell which branches constitute which loop or circuit. Alternately, if a loop matrix or circuit matrix is given, we can reconstruct the graph.

For a graph having n nodes and b branches, the loop matrix B_a is a rectangular matrix of order b columns and as many rows as there are loops.

Its elements have the following values:

b_{ij} = 1, if branch j is in loop i and their orientations coincide.
= – 1, if branch j is in loop i and their orientations do not coincide.
= 0, if branch j is not in loop i.

A graph and its loops are shown in Fig. 5.14.



Loop 1: {1, 2, 3}
Loop 2: {3, 4, 5}
Loop 3: {2, 4, 6}
Loop 4: {1, 2, 4, 5}
Loop 5: {1, 5, 6}
Loop 6: {2, 3, 5, 6}
Loop 7: {1, 3, 4, 6}

Fig. 5.14 *Graph*

All the loop currents are assumed to be flowing in a clockwise direction.

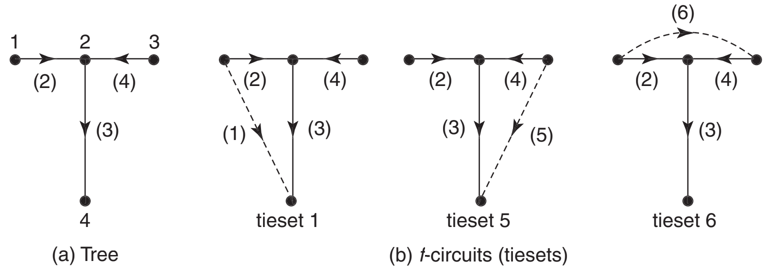
Loops ↓	Branches → 1	2	3	4	5	6
1	–1	1	1	0	0	0
2	0	0	–1	–1	1	0
3	0	–1	0	1	0	1
4	–1	1	0	–1	1	0
5	–1	0	0	0	1	1
6	0	–1	–1	0	1	1
7	–1	0	1	1	0	1

$$B_a = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

5.5.1 Fundamental Circuit (Tieset) and Fundamental Circuit Matrix

When a graph is given, first select a tree and remove all the links. When a link is replaced, a closed loop or circuit is formed. Circuits formed in this way are called fundamental circuits or *f*-circuits or tiesets.

Orientation of an *f*-circuit is given by the orientation of the connecting link. The number of *f*-circuits is same as the number of links for a graph. In a graph having b branches and n nodes, the number of *f*-circuits or tiesets will be $(b - n + 1)$. Figure 5.15 shows a tree and *f*-circuits (tiesets) for the graph shown in Fig. 5.14.



tieset 1: {1, 2, 3}
tieset 5: {5, 3, 4}
tieset 6: {6, 2, 4}

Fig. 5.15 *Tree and f-circuits*

Here, $b = 6$ and $n = 4$.

$$\text{Number of tiesets} = b - n + 1 = 6 - 4 + 1 = 3$$

f-circuits are shown in Fig. 5.15. The orientation of each *f*-circuit is given by the orientation of the corresponding connecting link.

The branches 1, 2 and 3 are in the tieset 1. Orientation of tieset 1 is given by orientation of branch 1. Since the orientation of branch 1 coincides with orientation of tieset 1, $b_{11} = 1$. The orientations of branches 2 and 3 do not coincide with the orientation of tieset 1, Hence, $b_{12} = -1$ and $b_{13} = -1$. The branches 4, 5 and 6 are not in tieset 1. Hence, $b_{14} = 0$, $b_{15} = 0$ and $b_{16} = 0$. Similarly, other elements of the tieset matrix are written.
Then, the tieset schedule will be written as

Tiesets ↓	Branches → 1	2	3	4	5	6
1	1	–1	–1	0	0	0
5	0	0	–1	–1	1	0
6	0	–1	0	1	0	1

Hence, an *f*-circuit matrix or tieset matrix will be given as

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Usually, the *f*-circuit matrix B is rearranged so that the first $(n - 1)$ columns correspond to the twigs and $b - (n - 1)$ columns to the links of the selected tree.

$$\begin{array}{c} \quad\quad \text{Twigs} \quad\quad\quad \text{Links} \\ \quad\; 2 \quad\; 3 \quad\; 4 \quad\; 1 \quad\; 5 \quad\; 6 \\ B = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix B can be partitioned into two matrices B_t and B_l.

$$B = [B_t : B_l] = [B_t : U]$$

where B_t is the twig matrix, B_l is the link matrix and U is the unit matrix.

5.5.2 Orthogonal Relationship between Matrix *A* and Matrix *B*

For a linear graph, if the columns of the two matrices A_a and B_a are arranged in the same order, it can be shown that

$$A_a B_a^T = 0$$

or

$$B_a A_a^T = 0$$

The above equations describe the orthogonal relationship between the matrices A_a and B_a.

If the reduced incidence matrix A and the f-circuit matrix B are written for the same tree, it can be shown that

$$A B^T = 0$$

or

$$B A^T = 0$$

These two equations show the orthogonal relationship between matrices A and B.

5.6 CUTSET MATRIX

Consider a linear graph. By removing a set of branches without affecting the nodes, two connected sub-graphs are obtained and the original graph becomes unconnected. The removal of this set of branches which results in cutting the graph into two parts are known as a *cutset*. The cutset separates the nodes of the graph into two groups, each being in one of the two groups.

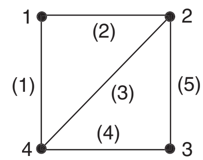


Fig. 5.16 *Graph*

Figure 5.16 shows a graph.

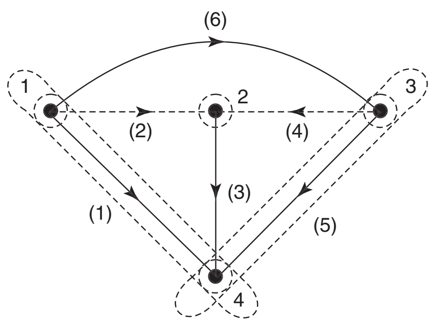
Branches 1, 3 and 4 will form a cutset. This set of branches separates the graph into two parts. One having an isolated node 4 and other part having branches 2 and 5 and nodes 1, 2 and 3.

Similarly, branches 1 and 2 will form a cutset. Each branch of the cutset has one of its terminals incident at a node in one part and its other end incident at other nodes in the other parts. The orientation of a cutset is made to coincide with orientation of defining branch.

For a graph having n nodes and b branches, the cutset matrix Q_a is a rectangular matrix of order b columns and as many rows as there are cutsets. Its elements have the following values:

q_{ij} = 1, if the branch j is in the cutset i and the orientation coincide.
= −1, if the branch j is in the cutset i and the orientations do not coincide.
= 0, if the branch j is not in the cutset i.

Figure 5.17 shows a directed graph and its cutsets.



Cutset 1: {1, 2, 6}
Cutset 2: {2, 3, 4}
Cutset 3: {3, 1, 5}
Cutset 4: {4, 5, 6}
Cutset 5: {5, 2, 3, 6}
Cutset 6: {6, 1, 3, 4}

Cutsets ↓	Branches → 1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	1	−1	1	0	0
3	1	0	1	0	1	0
4	0	0	0	1	1	−1
5	0	−1	1	0	1	−1
6	1	0	1	−1	0	1

Fig. 5.17 *Directed graph*

For the cutset 2, which cuts the branches 2, 3 and 4 and is shown by a dotted circle, the entry in the cutset schedule for the branch 2 is 1, since the orientation of this cutset is given by the orientation of the branch 2 and hence it coincides. The entry for branch 3 is −1 as orientation of branch 3 is opposite to that of cutset 2,

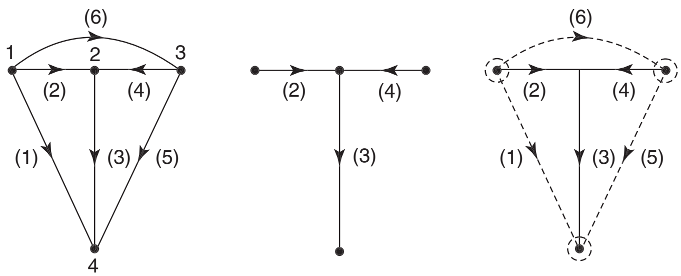
i.e., branch 2 goes into cutset while branch 3 goes out of cutset. The entry for branch 4 is 1 as the branch 2 and the branch 4 go into the cutset. Thus their orientations coincide.
Hence, the cutset matrix Q_a is given as

$$Q_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

5.6.1 Fundamental Cutset and Fundamental Cutset Matrix

When a graph is given, first select a tree and note down its twigs. When a twig is removed from the tree, it separates a tree into two parts (one of the separated part may be an isolated node). Now, all the branches connecting one part of the disconnected tree to the other along with the twig removed, constitutes a cutset. This set of branches is called a fundamental cutset or *f*-cutset. A matrix formed by these *f*-cutsets is called an *f*-cutset matrix. The orientation of the *f*-cutset is made to coincide with the orientation of the defining branch, i.e., twig. The number of *f*-cutsets is the same as the number of twigs for a graph.

Figure 5.18 shows a graph, selected tree and *f*-cutsets corresponding to the selected tree.



(a) Graph (b) Tree (c) *f*-cutsets

f-cutset 2: {2, 1, 6}
f-cutset 3: {3, 1, 5}
f-cutset 4: {4, 5, 6}

Fig. 5.18 *Graph, selected tree and f-cutsets*

The branches 2, 1, and 6 are in the *f*-cutset 2. Orientation of *f*-cutset 2 is given by orientation of the branch 2 which is moving away from the *f*-cutset 2. Since the orientations of branches 2, 1 and 6 coincide with the orientation of the *f*-cutset 2, $q_{11} = 1$, $q_{12} = 1$ and $q_{16} = 0$. The branches 3, 4 and 5 are not in the *f*-cutset 2. Hence, $q_{13} = 0$, $q_{14} = 0$ and $q_{15} = 0$.

Similarly, other elements of the *f*-cutset matrix are written.

The cutset schedule is

f-cutsets ↓	Branches → 1	2	3	4	5	6
2	1	1	0	0	0	1
3	1	0	1	0	1	0
4	0	0	0	1	1	–1

Hence, the *f*-cutset matrix Q is given by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

The *f*-cutset matrix Q is rearranged so that the first $(n - 1)$ columns correspond to twigs and $b - (n - 1)$ columns to links of the selected tree.

$$\begin{array}{c} \\ \\ Q = \end{array} \begin{array}{c} \begin{array}{cccccc} \multicolumn{3}{c}{\text{Twigs}} & \multicolumn{3}{c}{\text{Links}} \\ 2 & 3 & 4 & 1 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

The matrix Q can be subdivided into matrices Q_t and Q_l.

$$Q = [Q_t : Q_l] = [U : Q_l]$$

where Q_t is the twig matrix, Q_l is the link matrix and U is the unit matrix.

5.6.2 Orthogonal Relationship between Matrix *B* and Matrix Q

For a linear graph, if the columns of two matrices B_a and Q_a are arranged in the same order, it can be shown that

$$Q_a B_a^T = 0$$

or

$$B_a Q_a^T = 0$$

If the *f*-circuit matrix B and the *f*-cutset matrix Q are written for the same selected tree, it can be shown that

$$B Q^T = 0$$

or

$$Q B^T = 0$$

These two equations show the orthogonal relationship between matrices A and B.

5.7 RELATIONSHIP AMONG SUBMATRICES OF A, B AND Q

Arranging the columns of matrices A, B and Q with twigs for a given tree first and then the links, we get the partitioned forms as

$$A = [A_t : A_l]$$
$$B = [B_t : B_l] = [B_t : U]$$
$$Q = [Q_t : Q_l] = [U : Q_l]$$

From the orthogonal relation, $AB^T = 0$,

$$AB^T = [A_t : A_l] \begin{bmatrix} B_t^T \\ \cdots \\ B_l^T \end{bmatrix}$$

$$A_t B_t^T + A_l B_l^T = 0$$
$$A_t B_t^T = -A_l B_l^T$$

Since A_t is non-singular, i.e., $|A| \neq 0$, A_t^{-1} exists.
Premultiplying with A_t^{-1},

$$B_t^T = -A_t^{-1} A_l B_l^T$$

$$B_t = -B_l (A_t^{-1} . A_l)^T$$

Since B_l is a unit matrix

$$B_t = -(A_t^{-1} . A_l)^T$$

Hence, matrix B is written as

$$B = [-(A_t^{-1} \cdot A_l)^T : U] \qquad \ldots(5.1)$$

We know that $AB^T = 0$

$$A_l B_l^T = -A_t B_t^T$$

Postmultiplying with $(B_l^T)^{-1}$,

$$A_l = -A_t B_t^T (B_l^T)^{-1} = -A_t B_t^T (B_l^{-1})^T = -A_t (B_l^{-1} \cdot B_t)^T$$

Hence matrix A can be written as

$$\begin{aligned} A &= [A_l : -A_t (B_l^{-1} \cdot B_t)^T] \\ &= A_t [U : -(B_l^{-1} \cdot B_t)^T] \end{aligned} \qquad \ldots(5.2)$$

Similarly we can prove that

$$Q = [U : -(B_l^{-1} \cdot B_t)^T] \qquad \ldots(5.3)$$

From Eqs (5.2) and (5.3), we can write

$$A = A_t Q$$

$$Q = A_t^{-1} A = A_t^{-1} [A_t : A_l] = [U : A_t^{-1} A_l]$$

We have shown that $B_t = -(A_t^{-1} \cdot A_l)^T$

$$B_t^T = -(A_t^{-1} \cdot A_l)$$

Hence, Q can be written as

$$Q = [U : -B_t^T]$$

$$Q_1 = -B_t^T$$

Example 5.3 *For the circuit shown in Fig. 5.19, draw the oriented graph and write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

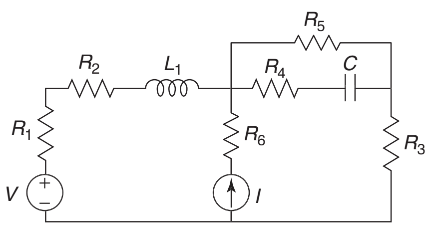


Fig. 5.19

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the voltage source by short circuit and the current source by an open circuit,
3. assume the directions of branch currents arbitrarily, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 5.20.

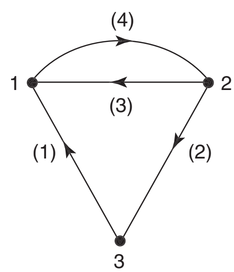


Fig. 5.20

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4
1	−1	0	−1	1
2	0	1	1	−1
3	1	−1	0	0

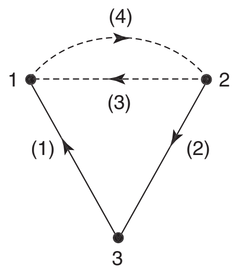
$$A_a = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Eliminating the third row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 5.21.



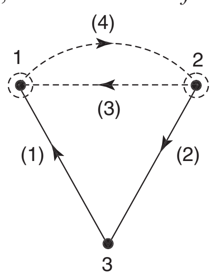
Links: {3, 4}
Tieset 3: {3, 1, 2}
Tieset 4: {4, 1, 2}

$$B = \begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 5.21

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 5.22.



Twigs: {1, 2}
f-cutset 1: {1, 3, 4}
f-cutset 2: {2, 3, 4}

$$Q = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 5.22

Example 5.4 *For the network shown in Fig. 5.23, draw the oriented graph and write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

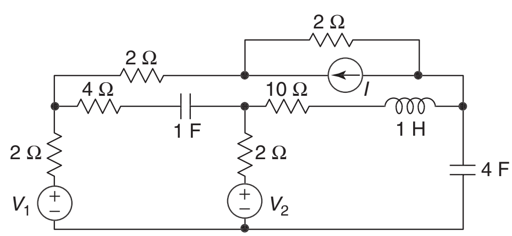


Fig. 5.23

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace all voltage sources by short circuits and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 5.24.

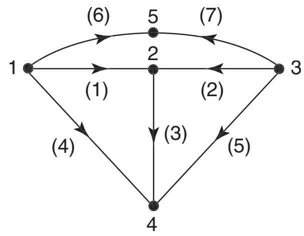


Fig. 5.24

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4	5	6	7
1	1	0	0	1	0	1	0
2	–1	–1	1	0	0	0	0
3	0	1	0	0	1	0	1
4	0	0	–1	–1	–1	0	0
5	0	0	0	0	0	–1	–1

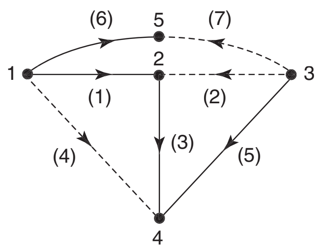
$$A_a = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Eliminating the last row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 5.25.



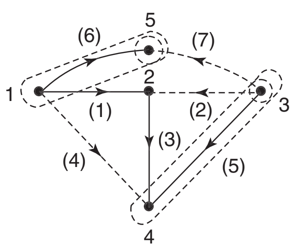
Links: {2, 4, 7}
Tieset 2: {2, 3, 5}
Tieset 4: {4, 1, 3}
Tieset 7: {7, 6, 1, 3, 5}

$$\begin{array}{c} \\ B = \begin{array}{c} 2 \\ 4 \\ 7 \end{array} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.25

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 5.26.



Twigs: {1, 3, 5, 6}
f-cutset 1: {1, 4, 7}
f-cutset 3: {3, 4, 7, 2}
f-cutset 5: {5, 2, 7}
f-cutset 6: {6, 7}

$$\begin{array}{c} \\ Q = \begin{array}{c} 1 \\ 3 \\ 5 \\ 6 \end{array} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.26

Example 5.5 *For the circuit shown in Fig. 5.27, draw the oriented graph and write (a) incidence matrix, (b) tieset matrix, and (c) cutset matrix.*

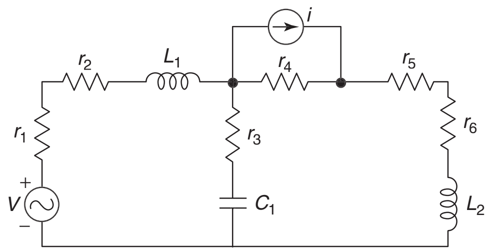


Fig. 5.27

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace voltage source by short circuit and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number the nodes and branches.

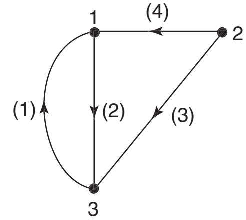


Fig. 5.28

The oriented graph is shown in Fig. 5.28.

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4
1	−1	1	0	−1
2	0	0	1	1
3	1	−1	−1	0

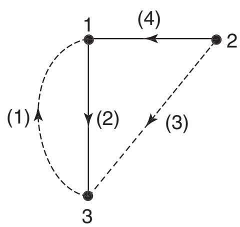
$$A_a = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

Eliminating the third row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 5.29.



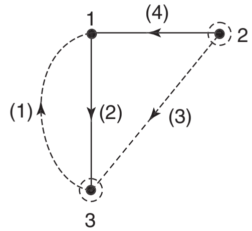
Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2, 4}

$$B = \begin{array}{c} \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 5.29

(c) *f*-cutset Matrix (Q)

The oriented graph, selected tree and *f*-cutsets are shown in Fig. 5.30.



Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3}

$$Q = \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.30

Example 5.6 *For the circuit shown in Fig. 5.31, (a) draw its graph, (b) draw its tree, and (c) write the fundamental cutset matrix.*

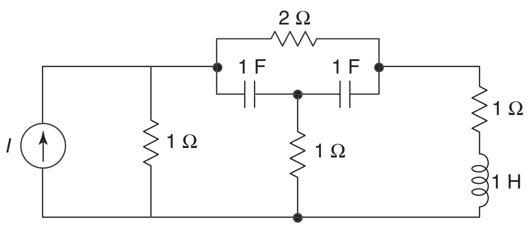


Fig. 5.31

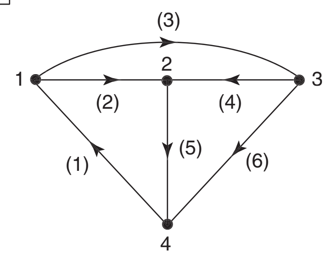


Fig. 5.32

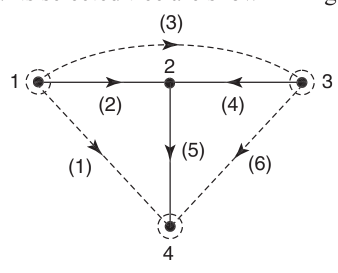
Solution

(a) For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the current source by an open circuit,
3. assume directions of branch currents, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 5.32.

(b) Tree
The oriented graph and its selected tree are shown in Fig. 5.33.



Twigs: {2, 4, 5}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3, 6}
f-cutset 5: {5, 1, 6}

Fig. 5.33

(c) Fundamental Cutset Matrix (Q)

$$Q = \begin{array}{c} \\ 2 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 5.7 *The graph of a network is shown in Fig. 5.34. Write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

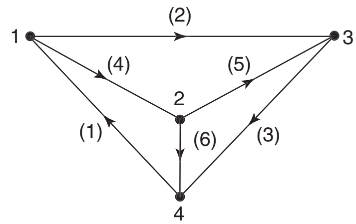


Fig. 5.34

Solution

(a) Incidence Matrix (A)

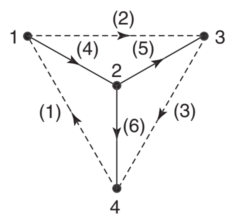
$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The incidence matrix A is obtained by eliminating any row from the matrix A_a.

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

(b) Tieset, Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 5.35.



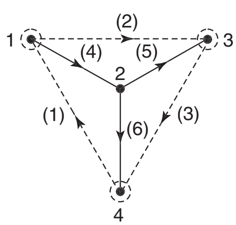
Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 4, 5}
Tieset 3: {3, 5, 6}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 5.35

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 5.36.



Twigs: {4, 5, 6}
f-cutset 4: {4, 1, 2}
f-cutset 5: {5, 2, 3}
f-cutset 6: {6, 1, 3}

$$Q = \begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 5.36

Example 5.8 *For the graph shown in Fig. 5.37, write the incidence matrix, tieset matrix and f-cutset matrix.*

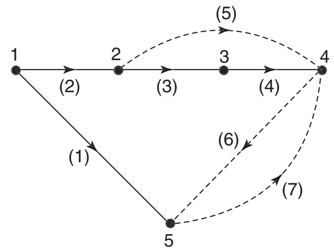


Fig. 5.37

Solution

(a) Incidence Matrix (A)

$$A_a = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

The incidence matrix A is obtained by eliminating any row from the matrix A_a.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 \end{bmatrix}$$

(b) Tieset Matrix (B)

Links: $\{5, 6, 7\}$
Tieset 5: $\{5, 3, 4\}$
Tieset 6: $\{6, 1, 2, 3, 4\}$
Tieset 7: $\{7, 1, 2, 3, 4\}$

$$B = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 5.38.

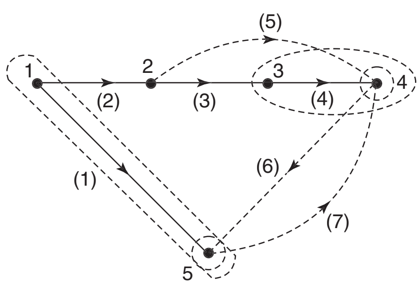


Fig. 5.38

Twigs: {1, 2, 3, 4}
f-cutset 1: {1, 6, 7}
f-cutset 2: {2, 6, 7}
f-cutset 3: {3, 5, 6, 7}
f-cutset 4: {4, 5, 6, 7}

$$Q = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Example 5.9 *For the graph shown in Fig. 5.39, write the incidence matrix, tieset matrix and f-cutset matrix.*

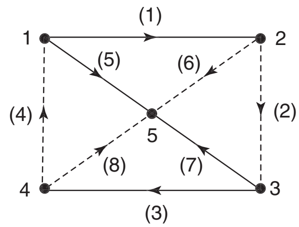


Fig. 5.39

Solution

(a) Incidence Matrix (A)

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

The incidence matrix is obtained by eliminating any one row.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Tieset Matrix (B)

Links: {2, 4, 6, 8}
Tieset 2: {2, 7, 5, 1}
Tieset 4: {4, 5, 7, 3}
Tieset 6: {6, 5, 1}
Tieset 8: {8, 7, 3}

$$B = \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

(c) *f*-cutset Matrix (Q)

The oriented graph, selected tree and *f*-cutsets are shown in Fig. 5.40.

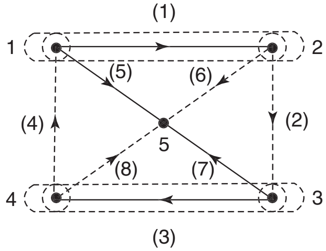


Fig. 5.40

Twigs: {1, 3, 5, 7}
f-cutset 1: {1, 6, 2}
f-cutset 3: {3, 4, 8}
f-cutset 5: {5, 4, 6, 2}
f-cutset 7: {7, 2, 8, 4}

$$Q = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 7 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 5.10 *How many trees are possible for the graph of the network of Fig. 5.41.*

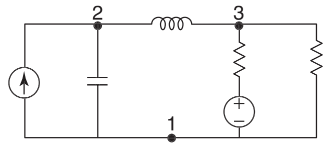


Fig. 5.41

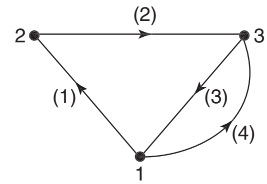


Fig. 5.42

Solution To draw the graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace voltage source by short circuit and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 5.42.
The complete Incidence Matrix (A_a) is written as

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

The reduced incidence matrix A is obtained by eliminating the last row from matrix A_a.

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

The number of possible trees $= \left|AA^T\right| = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 6 - 1 = 5.$

Example 5.11 *Draw the oriented graph from the complete incidence matrix given below;*

Nodes ↓	Branches → 1	2	3	4	5	6	7	8
1	1	0	0	0	1	0	0	1
2	0	1	0	0	–1	1	0	0
3	0	0	1	0	0	–1	1	–1
4	0	0	0	1	0	0	–1	0
5	–1	–1	–1	–1	0	0	0	0

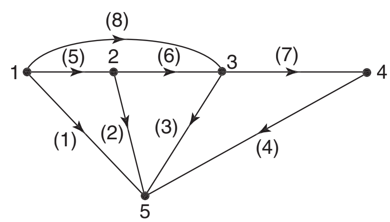


Fig. 5.43

Solution First, note down the nodes 1, 2, 3, 4, 5 as shown in Fig. 5.43. From the complete incidence matrix, it is clear that the branch number 1 is between nodes 1 and 5 and it is going away from node 1 and towards node 5 as the entry against node 1 is 1 and that against 5 is –1. Hence, connect the nodes 1 and 5 by a line, point the arrow towards 5 and call it branch 1 as shown in Fig. 5.43. Similarly, draw the other oriented branches.

Example 5.12 *The reduced incidence matrix of an oriented graph is given below. Draw the graph.*

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution First, writing the complete incidence matrix from the matrix A such that the sum of all entries in each column of A_a will be zero, we have

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

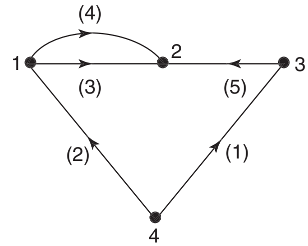


Fig. 5.44

Now, the oriented graph can be drawn with matrix A_a as shown in Fig. 5.44.

Example 5.13 *The reduced incidence matrix of an oriented graph is*

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Draw the graph. (b) How many trees are possible for this graph? (c) Write the tieset and cutset matrices.

Solution

(a) First, writing the complete incidence matrix A_a such that the sum of all the entries in each column of A_a is zero, we have

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

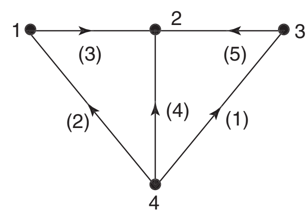


Fig. 5.45

Now, the oriented graph can be drawn with the matrix A_a, as shown in Fig. 5.45.

(b) The number of possible trees = $|AA^T|$

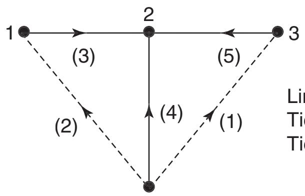
$$AA^T = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$|AA^T| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(6-1)+1(-2) = 8$$

The number of possible trees = 8.

(c) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 5.46.



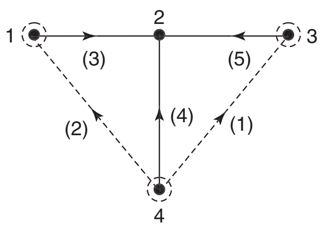
Links: {1, 2}
Tieset 1: {1, 4, 5}
Tieset 2: {2, 3, 4}

$$B = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 5.46

f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 5.47



Twigs: {3, 4, 5}
f-cutset 3: {3, 2}
f-cutset 4: {4, 2, 1}
f-cutset 5: {5, 1}

$$Q = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 5.47

Example 5.14 *The fundamental cutset matrix of a network is given as follows;*

Twigs			*Links*		
a	c	e	b	d	f
1	0	0	1	0	1
0	1	0	0	1	1
0	0	1	1	1	1

Draw the oriented graph.

Solution No. of links $l = b - n + 1$

No. of nodes $n = b - l + 1 = 6 - 3 + 1 = 4$

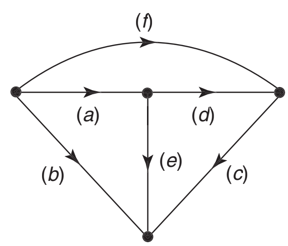
f-cutsets are written as,

f-cutsets a: $\{a, b, f\}$

f-cutsets c: $\{c, d, f\}$

f-cutsets e: $\{e, b, d, f\}$

The oriented graph is drawn as shown in Fig. 5.48.



Twigs: {a, c, e}
Links: {b, d, f}

Fig. 5.48

Example 5.15 *Draw the oriented graph of a network with the f-cutset matrix as shown below:*

$$\begin{array}{cccc|ccc} \multicolumn{4}{c}{\textit{Twigs}} & \multicolumn{3}{c}{\textit{Links}} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Solution No. of links $l = b - n + 1$

No. of nodes $n = b - l + 1 = 7 - 3 + 1 = 5$

f-cutsets are written as

f-cutset 1: {1, 5}

f-cutset 2: {2, 5, 7}

f-cutset 3: {3, 6, 7}

f-cutset 4: {4, 6}

Then oriented graph can be drawn as shown in Fig. 5.49.

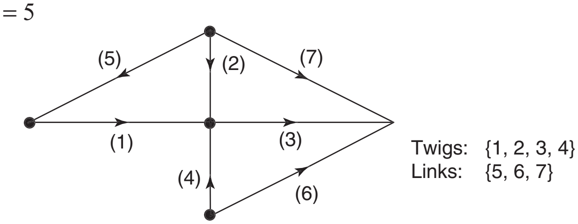


Fig. 5.49

5.8 KIRCHHOFF'S VOLTAGE LAW

KVL states that if v_k is the voltage drop across the k^{th} branch, then

$$\sum v_k = 0$$

the sum being taken over all the branches in a given loop. If l is the number of loops or *f*-circuits, then there will be l number of KVL equations, one for each loop. The KVL equation for the *f*-circuit or loop 'l' can be written as

$$\sum_{k=1}^{b} b_{ik}\, v_k = 0 \qquad (k = 1, 2, \ldots, l)$$

where b_{ik} is the elements of the tieset matrix B, b being the number of branches. The set of l KVL equations can be written in matrix form.

$$BV_b = 0$$

where $V_b = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix}$ is a column vector of branch voltages.

and B is the fundamental circuit matrix.

5.9 KIRCHHOFF'S CURRENT LAW

KCL states that if i_k is the current in the k^{th} branch then at a given node

$$\sum i_k = 0$$

the sum being taken over all the branches incident at a given node. If there are 'n' nodes, there will 'n' such equations, one for each node

$$\sum_{k=1}^{b} a_{ik}\, i_k = 0 \qquad (k = 1, 2,\dots, n)$$

so that set of n equations can be written in matrix form.

$$A_a I_b = 0 \qquad \dots(5.4)$$

where $$I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$$ is a column vector of branch currents.

and A_a is the complete incidence matrix.

If one node is taken as reference node or datum node, we can write the Eq. (5.4) as,

$$AI_b = 0 \qquad \dots(5.5)$$

where A is the incidence matrix of order $(n - 1) \times b$.

We know that $$A = A_t\, Q$$

Equation (5.5) can be written as

$$A_t\, QI_b = 0$$

Premultiplying with A_t^{-1},

$$A_t^{-1} A_t\, Q\, I_b = A_t^{-1} \cdot 0$$

$$I\, Q\, I_b = 0$$

$$Q\, I_b = 0$$

where Q is the f-cutset matrix.

5.10 RELATION BETWEEN BRANCH VOLTAGE MATRIX V_b, TWIG VOLTAGE MATRIX V_t AND NODE VOLTAGE MATRIX V_n

We know that $$B V_b = 0$$

$$[B_t : B_l] \begin{bmatrix} V_t \\ \cdots \\ V_l \end{bmatrix} = 0$$

$$B_t V_t + B_l V_l = 0$$

$$B_l V_l = -B_t V_t$$

Premultiplying with B_l^{-1}.

$$V_l = -B_l^{-1} B_t V_t = -(B_l^{-1} B_t) V_t$$

Now $$V_b = \begin{bmatrix} V_t \\ \cdots \\ V_l \end{bmatrix}$$

$$= \begin{bmatrix} V_t \\ \cdots \\ -\left(B_l^{-1} B_t\right) V_t \end{bmatrix} = \begin{bmatrix} U \\ \cdots \\ -\left(B_l^{-1} B_t\right) \end{bmatrix} \cdot V_t \qquad \dots(5.6)$$

Also,

$$V_b = Q^T V_t$$
$$Q = A_t^{-1} A$$
$$Q^T = A^T (A_t^{-1})^T = A^T (A_t^T)^{-1}$$

Hence Eq. (5.6) can be written as

$$V_b = A^T (A_t^T)^{-1} V_t = A^T \left\{(A_t^T)^{-1} V_t\right\} = A^T V_n$$

where $V_n = (A_t^T)^{-1} V_t$ is node voltage matrix.

5.11 RELATION BETWEEN BRANCH CURRENT MATRIX I_b AND LOOP CURRENT MATRIX I_l

We know that, $A I_b = 0$

$$[A_t : A_l]\begin{bmatrix} I_t \\ \cdots \\ I_l \end{bmatrix} = 0$$

$$A_t I_t + A_l I_l = 0$$
$$A_t I_t = -A_l I_l$$

Premultiplying with A_t^{-1},

$$I_t = -A_t^{-1} A_l I_l = -(A_t^{-1} A_l) I_l$$

Now

$$I_b = \begin{bmatrix} I_t \\ \cdots \\ I_l \end{bmatrix} = \begin{bmatrix} -(A_t^{-1} A_l) I_l \\ \cdots \\ I_l \end{bmatrix} = \begin{bmatrix} -(A_t^{-1} A_l) \\ \cdots \\ U \end{bmatrix} \cdot I_l$$

$$I_b = B^T I_l$$

5.12 NETWORK EQUILIBRIUM EQUATION

5.12.1 KVL Equation

1. If there is a voltage source v_{sk} in the branch k having impedance z_k and carrying current i_k, as shown in Fig. 5.50,

$$v_k = z_k i_k - v_{sk} \qquad (k = 1, 2, \ldots, b)$$

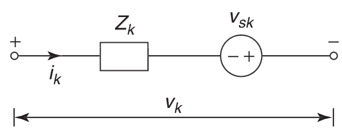


Fig. 5.50 *Circuit diagram*

In matrix form,

$$V_b = Z_b I_b - V_s$$

where Z_b is the branch impedance matrix, I_b is the column vector of branch currents and V_s is the column vector of source voltages. Hence, KVL equation can be written as

$$B V_b = 0$$
$$B (Z_b I_b - V_s) = 0$$
$$B Z_b I_b = B V_s$$

Also, $$I_b = B^T I_l$$

$$B\,Z_b\,B^T\,I_l = BV_s$$

$$Z\,I_l = E$$

where $$E = B\,V_s$$

and $$Z = B\,Z_b\,B^T$$

The matrix Z is called *loop impedance matrix*.

2. If there is a voltage source in series with an impedance and a current source in parallel with the combination as shown in Fig. 5.51,

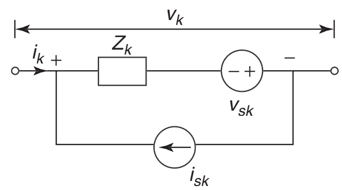


Fig. 5.51 *Circuit diagram*

$$i_k = \frac{(v_k + v_{sk})}{zk} - i_{sk}$$

$$v_k = z_k\,i_k + z_k\,i_{sk} - v_{sk}$$

In matrix form,

$$V_b = Z_b\,I_b + Z_b\,I_s - V_s$$

KVL equation is $B\,V_b = 0$.

$$B\,V_b = B(Z_b\,I_b + Z_b\,I_s - V_s) = 0$$

$$B\,Z_b\,I_b = B\,V_s - B\,Z_b\,I_s$$

Now $$I_b = B^T I_l$$

$$B\,Z_b B^T I_l = B\,V_s - B\,Z_b\,I_s$$

$$Z\,I_l = B\,V_s - B\,Z_b\,I_s$$

where $Z = B\,Z_b\,B^T$ is the loop impedance matrix. This is the generalised KVL equation.

5.12.2 KCL Equation

1. If the branch k contains an input current source i_{sk} and an admittance y_k as shown in Fig. 5.52,

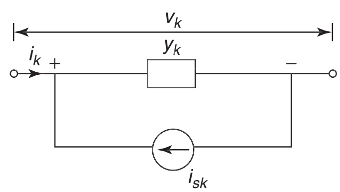


Fig. 5.52 *Circuit diagram*

$$i_k = y_k\,v_k - i_{sk} \qquad (k = 1, 2, \ldots, b)$$

In the matrix form,

$$I_b = Y_b\,V_b - I_s$$

where Y_b is the branch admittance matrix.
Hence KCL equation is given by,

$$A\,I_b = 0$$

$$A(Y_b\,V_b - I_s) = 0$$

$$A\,Y_b\,V_b = A\,I_s$$

Also $$V_b = A^T\,V_n$$

$$A\,Y_b\,A^T\,V_n = A\,I_s$$

$$Y\,V_n = I$$

where $$Y = A\,Y_b\,A^T$$

and $$I = A\,I_s$$

The matrix Y is called admittance matrix. This is the KCL equation in matrix form.

In terms of f-cutset matrix, the KCL equation can be written as

$$Q\,I_b = 0$$
$$Q(Y_b V_b - V_s) = 0$$
$$Q\,Y_b V_b = Q\,I_s$$

Also $$V_b = Q^T V_t$$

$$Q\,Y_b\,Q^T V_t = Q\,I_s$$
$$Y\,V_t = I$$

where $$Y = Q\,Y_b\,Q^T$$

and $$I = Q\,I_s$$

This is the KCL equation in matrix form.

2. If there is a voltage source in series with an impedance and a current source in parallel with the combination as shown in Fig. 5.53,

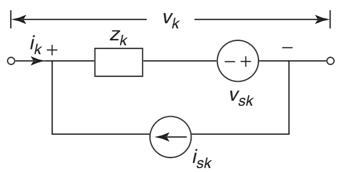


Fig. 5.53 *Circuit diagram*

$$y_k = \frac{1}{z_k}$$
$$i_k = y_k\,v_k + y_k\,v_{sk} - i_{sk}$$

In matrix form,

$$I_b = Y_b V_b + Y_b V_s - I_s$$

KCL equation will be given by,

$$A\,I_b = 0$$
$$A(Y_b V_b + Y_b V_s - I_s) = 0$$
$$A\,Y_b V_b = A\,I_s - A\,Y_b V_s$$

Also $$V_b = A^T V_n$$

$$A\,Y_b\,A^T V_n = A\,I_s - A\,Y_b V_s$$
$$Y\,V_n = A\,I_s - A V_b V_s$$

where $Y = A\,Y_b\,A^T$ is the node admittance matrix. This is a generalised KCL equation.

In terms of f-cutset matrix, the KCL equation can be written as

$$Q\,I_b = 0$$
$$Q(Y_b V_b + Y_b V_s - I_s) = 0$$
$$Q\,Y_b V_b = Q\,I_s - Q\,Y_b V_s$$

Also $$V_b = Q^T V_t$$

$$Q\,Y_b\,Q^T V_t = Q\,I_s - Q\,Y_b V_s$$
$$Y\,V_t = Q\,I_s - Q\,Y_b V_s$$

This is a generalised KCL equation.

Note

(i) For a graph having b branches, the branch impedance matrix Z_b is a square matrix of order b, having branch impedances as diagonal elements and the mutual impedances between the branches as non-diagonal elements. For a network having no mutual impedances, only diagonal elements will be present in the branch impedance matrix.

(ii) For a graph having b branches, the branch admittance matrix Y_b is a square matrix of order b, having branch admittances as diagonal elements and the mutual admittances between the branches as non-diagonal elements. For a network having no mutual admittances, only diagonal elements will be present in the branch admittance matrix.

(iii) For a graph having b branches, the voltage source matrix or vector V_s is a rectangular matrix of order $b \times 1$, having the value of the voltage source in the particular branch. The value will be positive if there is a voltage rise in the direction of current and will be negative if there is a voltage fall in the direction of current.

(iv) For a graph having b branches, the current source matrix or vector I_s is a rectangular matrix of order $b \times 1$, having the value of the current source in the particular parallel branch. The value will be positive if the direction of the current source and the corresponding parallel branch current are not same. The value will be negative if the directions of the current source and corresponding parallel branch current are same.

Example 5.16 *Write the incidence matrix of the graph of Fig. 5.54 and express branch voltages in terms of node voltages. Write the tieset matrix and express branch currents in terms of loop currents.*

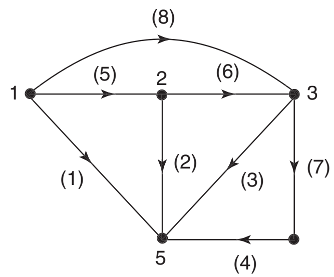


Fig. 5.54

Solution

(a) Incidence Matrix

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ A_a = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The incidence matrix is obtained by eliminating any one row.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

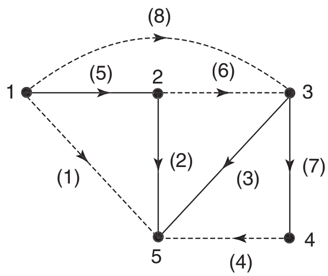
(b) Branch voltages in terms of node voltages

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \\ V_{n_4} \end{bmatrix}$$

(c) Tieset Matrix

Selected tree and tiesets are shown in Fig. 5.55.



Links: {1, 4, 6, 8}
Tieset 1: {1, 2, 5}
Tieset 4: {4, 3, 7}
Tieset 6: {6, 2, 3}
Tieset 8: {8, 2, 3, 5}

$$B = \begin{array}{c} \\ 1 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 5.55

(d) Branch currents in terms of loop currents

$$I_b = B^T I_l$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_4} \\ I_{l_6} \\ I_{l_8} \end{bmatrix}$$

Example 5.17 *Branch current and loop-current relationships are expressed in matrix form as*

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \\ I_{l_4} \end{bmatrix}$$

Draw the oriented graph.

Solution Writing the equation in matrix form,

$$I_b = B^T I_l$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \therefore B = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

From tieset matrix,

No. of links $l = 4$

No of branches $b = 8$

No of nodes $n = b - l + 1 = 8 - 4 + 1 = 1$

The oriented graph is shown in Fig. 5.56.

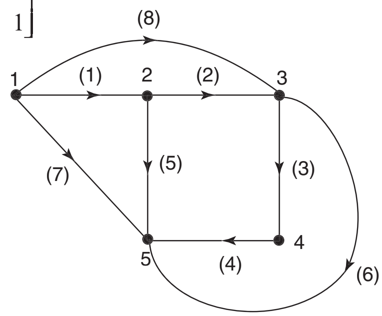


Fig. 5.56

Example 5.18 *For the given graph shown in Fig. 5.57, write down the basic tieset matrix and taking a tree of branches 2, 4, 5, write down KVL equations from the matrix.*

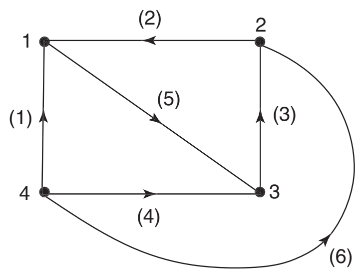


Fig. 5.57

Solution Selecting branches 2, 4, and 5 as the tree as shown in Fig. 5.58,

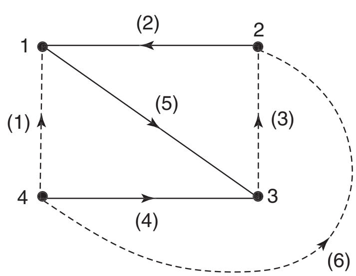


Fig. 5.58

Links: {1, 3, 6}
Tieset 1: {1, 5, 4}
Tieset 3: {3, 2, 5}
Tieset 6: {6, 2, 5, 4}

$$B = \begin{array}{c} \\ 1 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

The KVL equation in matrix form is given by

$$BV_b = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = 0$$

$$V_1 - V_4 + V_5 = 0$$
$$V_2 + V_3 + V_5 = 0$$
$$V_2 - V_4 + V_5 + V_6 = 0$$

Example 5.19 *Obtain the f-cutset matrix for the graph shown in Fig. 5.59 taking 1, 2, 3, 4 as tree branches. Write down the network equations from the f-cutset matrix.*

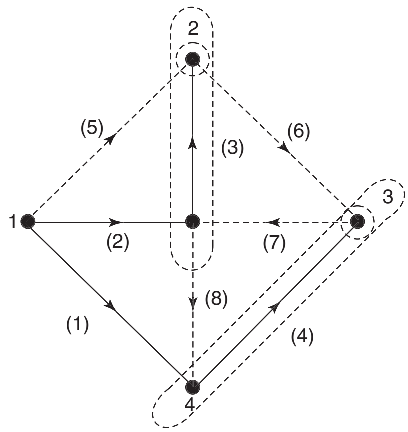


Fig. 5.59

Solution Twigs: $\{1, 2, 3, 4\}$

f-cutset 1 : $\{1, 6, 7, 8\}$

f-cutset 2 : $\{2, 5, 6, 7, 8\}$

f-cutset 3 : $\{3, 5, 6\}$

f-cutset 4 : $\{4, 6, 7\}$

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

The KCL equation in matrix form is given by

$$Q\,I_b = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = 0$$

$$I_1 + I_6 - I_7 + I_8 = 0$$
$$I_2 + I_5 - I_6 + I_7 - I_8 = 0$$
$$I_3 + I_5 - I_6 = 0$$
$$I_4 + I_6 - I_7 = 0$$

Example 5.20 *The reduced incidence matrix of a graph is given as*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Express branch voltages in terms of node voltages.

Solution For the given graph,

No of branches $b = 5$

No of nodes $n = 3$

Branch voltages can be expressed in terms of node voltages by

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \end{bmatrix}$$

$$V_1 = V_{n_1} - V_{n_2}$$
$$V_2 = -V_{n_2}$$
$$V_3 = -V_{n_2} + V_{n_3}$$
$$V_4 = -V_{n_3}$$
$$V_5 = -V_{n_1}$$

Example 5.21 *The fundamental cutset matrix of a graph is given as*

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Express branch voltages in terms of twig voltages.

Solution For the given graph,

No. of branches $b = 5$

No. of twigs = 3

Branch voltages are expressed in terms of twig voltages by

$$V_b = Q^T V_t$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{t_1} \\ V_{t_2} \\ V_{t_3} \end{bmatrix}$$

$$V_1 = -V_{t_1} + V_{t_3}$$
$$V_2 = V_{t_1}$$
$$V_3 = V_{t_2}$$
$$V_4 = V_{t_3}$$
$$V_5 = -V_{t_1} - V_{t_2}$$

Example 5.22 *For this network shown in Fig. 5.60, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL. Calculate the loop currents and branch currents.*

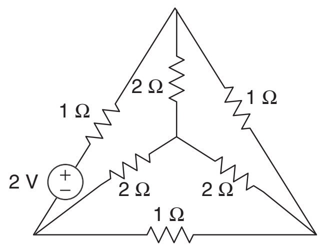
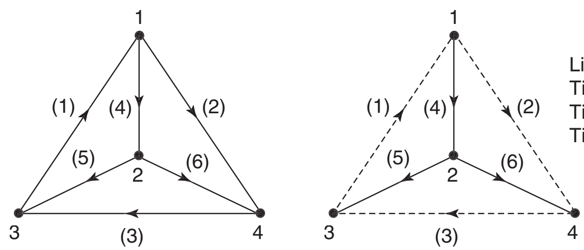


Fig. 5.60

Solution The oriented graph and one of its trees are shown in Fig. 5.61.



Links: {1, 2, 3}
Tieset 1: {1, 4, 5}
Tieset 2: {2, 4, 6}
Tieset 3: {3, 5, 6}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.61

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B\,V_s - B\,Z_b\,I_s$$

Here, $$I_s = 0,$$

$$B\,Z_b\,B^T\,I_l = B\,V_s$$

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$B\,V_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The KVL equation in matrix form is given by

$$\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = \frac{6}{7}\,\text{A}$$

$$I_{l_2} = \frac{4}{7}\,\text{A}$$

$$I_{l_3} = \frac{4}{7}\,\text{A}$$

The branch currents are given by

$$I_b = B^T\,I_l$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{6}{7} \\ \frac{4}{7} \\ \frac{4}{7} \\ \frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ \frac{4}{7} \\ \frac{4}{7} \\ \frac{2}{7} \\ \frac{2}{7} \\ 0 \end{bmatrix}$$

Example 5.23 *For the network shown in Fig. 5.62, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL. Calculate loop currents.*

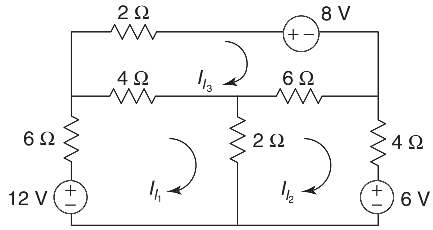
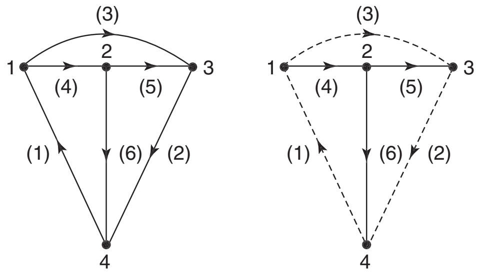


Fig. 5.62

Solution The oriented graph and its selected tree are shown in Fig. 5.63.



Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 5, 6}
Tieset 3: {3, 5, 4}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 5.63

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B V_s - B\,Z_b\,I_s$$

Here, $\quad I_s = 0,$

$$B\,Z_b\,B^T I_l = B V_s$$

$$Z_b = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix};\ V_s = \begin{bmatrix} 12 \\ -6 \\ -8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 0 & 6 & -2 \\ 0 & 0 & 2 & -4 & -6 & 0 \end{bmatrix}$$

$$BZ_b\,B^T = \begin{bmatrix} 6 & 0 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 0 & 6 & -2 \\ 0 & 0 & 2 & -4 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & -2 & -4 \\ -1 & 12 & -6 \\ -4 & -6 & 12 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ -6 \\ -8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ -8 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 12 & -2 & -4 \\ -2 & 12 & -6 \\ -4 & -6 & 12 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ -8 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = 0.55\text{A}$$
$$I_{l_2} = -0.866\text{A}$$
$$I_{l_3} = -0.916\text{A}$$

Example 5.24 *For the network shown in Fig. 5.64, draw the oriented graph. Write the tieset schedule and hence obtain the equilibrium equation on loop basis. Calculate the values of branch currents and branch voltages.*

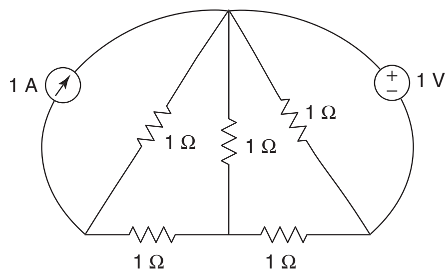


Fig. 5.64

Solution The oriented graph and one of its trees are shown in Fig. 5.65.

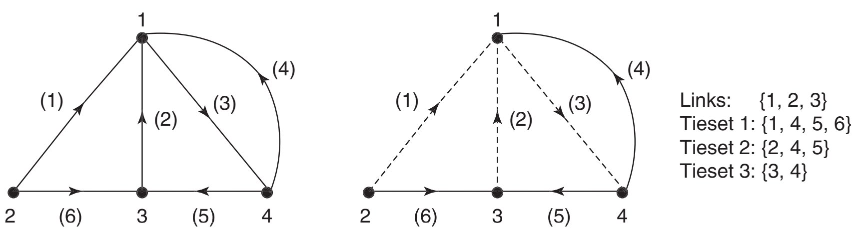


Fig. 5.65

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B V_s - B\,Z_b\,I_s$$

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix};\ V_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix};\ I_s = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$BZ_b\,B^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B\,V_s = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$B\,Z_b\,I_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = \frac{1}{5}\text{A}$$

$$I_{l_2} = -\frac{3}{5}\text{A}$$

$$I_{l_3} = 1\text{A}$$

The branch currents are given by

$$I_b = B^T\, I_l + I_s$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ \frac{7}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

The branch voltages are given by

$$V_b = Z_b\, I_b - V_s$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ \frac{7}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ -1 \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

Example 5.25 *For the network shown in Fig. 5.66, obtain the loop equation in matrix form.*

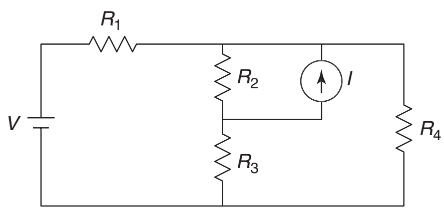
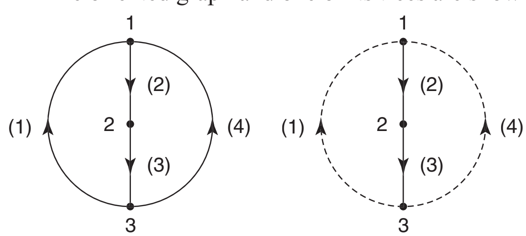


Fig. 5.66

Solution The oriented graph and one of its trees are shown in Fig. 5.67.



Links: {1, 4}
Tieset 1: {1, 2, 3}
Tieset 4: {4, 2, 3}

$$B = \begin{array}{c} \\ 1 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.67

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B V_s - B\,Z_b I_s$$

$$Z_b = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix};\ V_s = \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \end{bmatrix};\ I_s = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 + R_3 & R_2 + R_3 \\ R_2 + R_3 & R_2 + R_3 + R_4 \end{bmatrix}$$

$$B V_s = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

$$B\,Z_b\,I_s = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_2 I \\ R_2 I \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} R_1 + R_2 + R_3 & R_2 + R_3 \\ R_2 + R_3 & R_2 + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_4} \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} - \begin{bmatrix} R_2 & I \\ R_2 & I \end{bmatrix} = \begin{bmatrix} V - R_2 I \\ -R_2 I \end{bmatrix}$$

Example 5.26 *For the network shown in Fig. 5.68, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

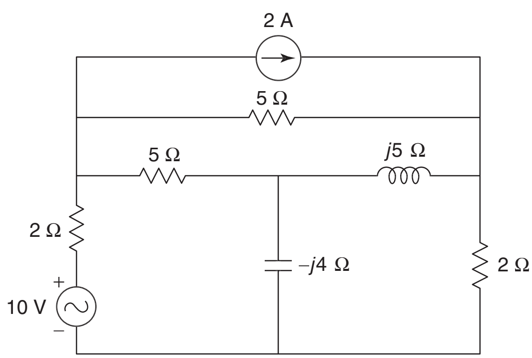
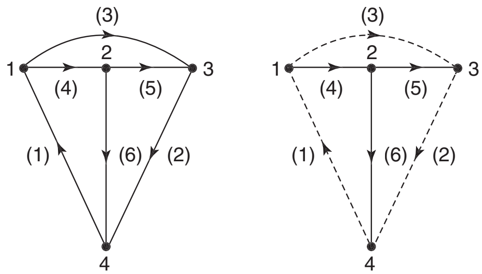


Fig. 5.68

Solution The oriented graph and its selected tree are shown in Fig. 5.69.



Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 5, 6}
Tieset 3: {3, 5, 4}

$$B = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 5.69

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B V_s - B\,Z_b I_s$$

$$Z_b = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & j5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j4 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}; V_s = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; I_s = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & j5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 7-j4 & j4 & -5 \\ j4 & 2+j1 & -j5 \\ -5 & -j5 & 10+j5 \end{bmatrix}$$

$$B V_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b\,I_s = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

Hence, the KCL equation in matrix form is given by

$$\begin{bmatrix} 7-j4 & j4 & -5 \\ j4 & 2+j1 & -j5 \\ -5 & -j5 & 10+j5 \end{bmatrix}\begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}$$

Example 5.27 *For the network shown in Fig. 5.70, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

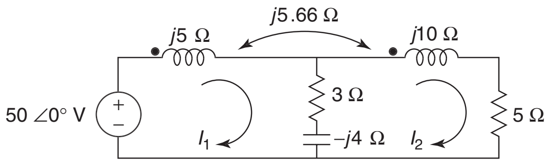
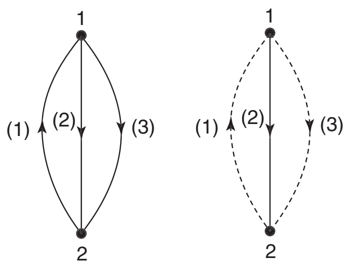


Fig. 5.70

Solution The branch currents are so chosen that they assume the direction out of the dotted terminals. Because of this choice of current direction, the mutual inductance is positive. The oriented graph and its selected tree are shown in Fig. 5.71.



Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2}

$$B = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.71

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = BV_s - B\,Z_b\,I_s$$

Here, $\qquad I_s = 0,$

$$B\,Z_b\,B^T\,I_l = BV_s$$

$$Z_b = \begin{bmatrix} j5 & 0 & j5.66 \\ 0 & 3-j4 & 0 \\ j5.66 & 0 & 5+j10 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix};\ V_s = \begin{bmatrix} 50\angle 0^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} j5 & 0 & j5.66 \\ 0 & 3-j4 & 0 \\ j5.66 & 0 & 5+j10 \end{bmatrix} = \begin{bmatrix} j5 & 3-j4 & j5.66 \\ j5.66 & -3+j4 & 5+j10 \end{bmatrix}$$

$$B\,Z_b B^T = \begin{bmatrix} j5 & 3-j4 & j5.66 \\ j5.66 & -3+j4 & 5+j10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+j1 & -3+j9.66 \\ -3+j9.66 & 8+j6 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 50\angle 0^\circ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 3+j1 & -3+j9.66 \\ -3+j9.66 & 8.j6 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

Example 5.28 *For the network shown in Fig. 5.72, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

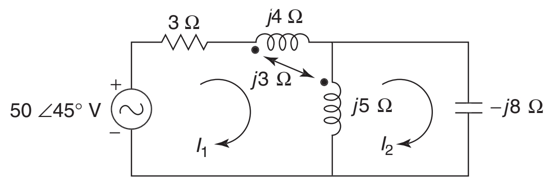
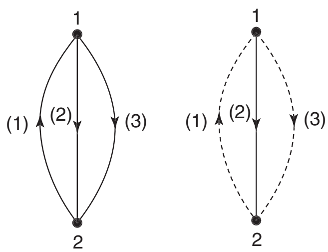


Fig. 5.72

Solution The branch currents are so chosen that they assume the direction out of the dotted terminals. Because of this choice of current direction, the mutual inductance is positive. The oriented graph and its selected tree are shown in Fig. 5.73.



Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2}

$$B = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.73

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B\,V_s - B\,Z_b\,I_s$$

Here, $I_s = 0,$

$$B\,Z_b\,B^T\,I_l = B\,V_s$$

$$Z_b = \begin{bmatrix} 3+j4 & j3 & 0 \\ j3 & j5 & 0 \\ 0 & 0 & -j8 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 50\angle 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3+j4 & j3 & 0 \\ j3 & j5 & 0 \\ 0 & 0 & -j8 \end{bmatrix} = \begin{bmatrix} 3+j7 & j8 & 0 \\ -j3 & -j5 & -j8 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} 3+j7 & j8 & 0 \\ -j3 & -j5 & -j8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}$$

$$B\,V_s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 50\angle 45^\circ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by,

$$\begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}\begin{bmatrix} I_{l_1} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

Example 5.29 *For the network shown in Fig. 5.74, obtain branch voltages using KCL equation on node basis.*

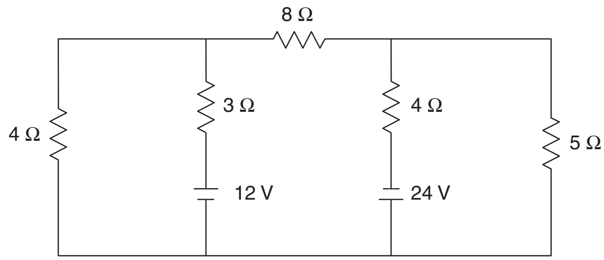


Fig. 5.74

Solution The oriented graph is shown in Fig. 5.75.

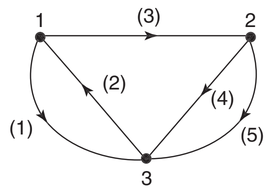


Fig. 5.75

The complete incidence matrix for the graph is

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Eliminating the last row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

The KCL equation in matrix form is given by

$$A Y_b A^T V_n = A I_s - A Y_b V_s$$

$$I_s = 0,$$

Here,

$$A Y_b A^T V_n = -A Y_b V_s$$

$$Y_b = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}; A^T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 0 \\ 12 \\ 0 \\ 24 \\ 0 \end{bmatrix}$$

$$A Y_b = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$A Y_b A^T = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{23}{40} \end{bmatrix}$$

$$A Y_b V_s = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 12 \\ 0 \\ 24 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Hence, the KCL equation in matrix form is given by

$$\begin{bmatrix} \frac{17}{24} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{23}{40} \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \end{bmatrix} = -\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Solving this matrix equation,

$$V_{n_1} = 3.96\,\text{V}$$

$$V_{n_2} = -9.57\,\text{V}$$

Branch voltages are given by,

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3.96 \\ -9.57 \end{bmatrix} = \begin{bmatrix} 3.96 \\ -3.96 \\ 13.53 \\ -9.57 \\ -9.57 \end{bmatrix}$$

Example 5.30 *For the network shown in Fig. 5.76, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL.*

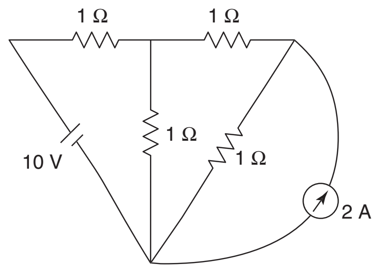
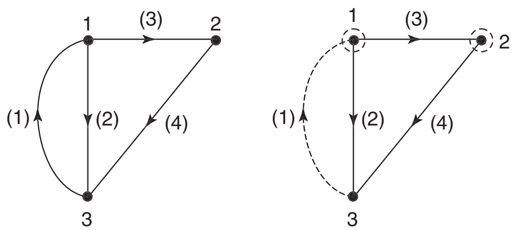


Fig. 5.76

Solution The oriented graph and its selected tree are shown in Fig. 5.77.



Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3}

$$Q = \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.77

The KCL equation in matrix form is given by

$$Q\,Y_b\,Q^T\,V_t = Q\,I_s - Q\,Y_b\,V_s$$

$$Y_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ I_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix};\ V_s = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix};\ Q^T = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$Q\,Y_b = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$Q\,Y_b\,Q^T = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$QI_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$QY_b V_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Hence, the KCL equation is given by

$$\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} V_{t_2} \\ V_{t_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -10 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

Solving this matrix equation,

$$V_{t_2} = 4.4\text{ V}$$
$$V_{t_4} = 3.2\text{ V}$$

Example 5.31 *Calculate the twig voltages using KCL equation for the network shown in Fig. 5.78.*

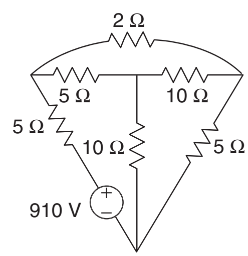
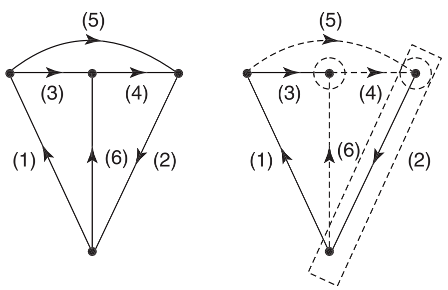


Fig. 5.78

Solution The oriented graph and one of the trees are shown in Fig. 5.79.



Twigs: {1, 2, 3}
f-cutset 1: {1, 4, 5, 6}
f-cutset 2: {2, 4, 5}
f-cutset 3: {3, 4, 6}

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 5.79

The network equilibrium equation on node basis can be written as

$$QY_b Q^T V_t = QI_s - QY_b V_s$$

Here, $$I_s = 0,$$

$$QY_b Q^T V_t = -QY_b V_s$$

$$Y_b = \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 910 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$QY_b = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix}$$

$$QY_bQ^T = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.6 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

$$QY_bV_s = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 910 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 182 \\ 0 \\ 0 \end{bmatrix}$$

Hence, KCL equation can be written as,

$$\begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.6 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} v_{t_1} \\ v_{t_2} \\ v_{t_3} \end{bmatrix} = \begin{bmatrix} -182 \\ 0 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = -460 \text{ V}$$
$$v_{t_2} = 320 \text{ V}$$
$$v_{t_3} = 200 \text{ V}$$

Example 5.32 *For the network shown in Fig. 5.80, obtain equilibrium equation on node basis.*

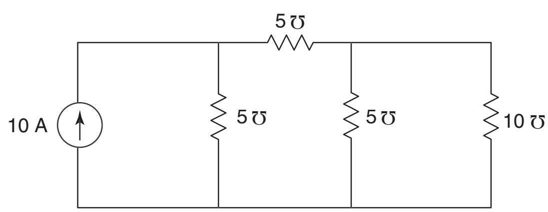
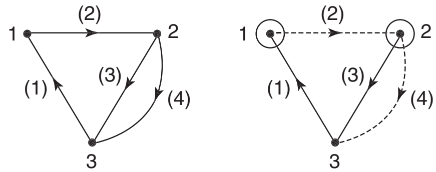


Fig. 5.80

Solution The oriented graph and its selected tree are shown in Fig. 5.81.



Twigs: {1, 3}
f-cutset 1: {1, 2}
f-cutset 3: {3, 2, 4}

$$Q = \begin{array}{c} \\ 1 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.81

The KCL equation in matrix form is given by

$$Q Y_b Q^T V_t = Q I_s - Q Y_b V_s$$

Here, $V_s = 0,$

$$Q Y_b Q^T V_t = Q I_s$$

$$Y_b = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}; I_s = \begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q Y_b = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 0 & 0 \\ 0 & -5 & 5 & 10 \end{bmatrix}$$

$$Q Y_b Q^T = \begin{bmatrix} 5 & -5 & 0 & 0 \\ 0 & -5 & 5 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 20 \end{bmatrix}$$

$$Q I_s = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Hence, KCL equation will be written as

$$\begin{bmatrix} 10 & 5 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} v_{t_1} \\ v_{t_3} \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = -\frac{8}{7}\text{ V}$$

$$v_{t_3} = \frac{2}{7}\text{ V}$$

Example 5.33 *For the network shown in Fig. 5.82, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL and calculate v.*

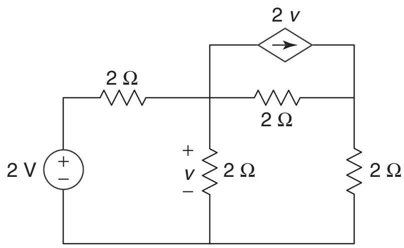
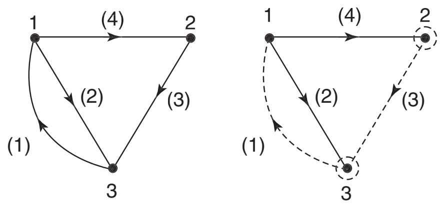


Fig. 5.82

Solution The oriented graph and its selected tree are shown in Fig. 5.83
Since voltage v is to be determined, Branch 2 is chosen as twig,



Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 3: {4, 3}

$$Q = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 5.83

The KCL equation in matrix form is given by

$$Q\,Y_b\,Q^T\,V_t = Q\,I_s - Q\,Y_b\,V_s$$

$$Y_b = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix};\ Q^T = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix};\ I_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2v \end{bmatrix};\ V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q\,Y_b = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix}$$

$$Q\,Y_b\,Q^T = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$Q\,I_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2v \end{bmatrix} = \begin{bmatrix} 0 \\ -2v \end{bmatrix}$$

$$Q\,Y_b\,V_s = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$Q\,I_s - Q\,Y_b\,V_s = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

Hence, the KCL equation can be written as

$$\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} v_{t_2} \\ v_{t_4} \end{bmatrix} = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

From Fig. 5.82, $v_{t_2} = v$

Solving this matrix equation,

$$v_{t_2} = 0.44 \text{ V}$$
$$v_{t_4} = 0.66 \text{ V}$$
$$v = v_{t_2} = 0.44 \text{ V}$$

Example 5.34 *For the network shown in Fig. 5.84, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL and calculate v.*

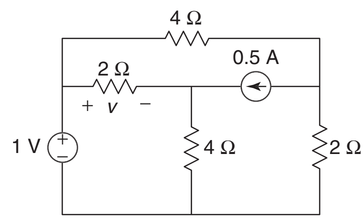


Fig. 5.84

Solution The voltage and current sources are converted into accompanied sources by source–shifting method as shown in Fig. 5.85.

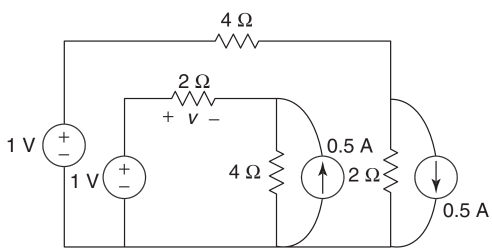
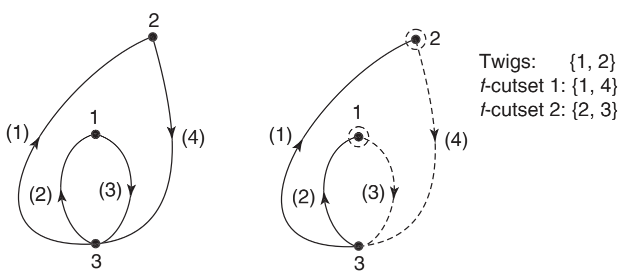


Fig. 5.85

The oriented graph and its selected tree are shown in Fig. 5.86.



$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ Q = & \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 5.86

The KCL equation in the matrix form is given by

$$Q\,Y_b\,Q^T\,V_t = Q\,I_s - Q\,Y_b\,V_s$$

$$Y_b = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}; \; I_s = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}; \; V_s = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q\,Y_b = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix}$$

$$Q\,Y_b\,Q^T = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix}$$

$$Q\,I_s = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$$Q\,Y_b\,V_s = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

$$Q\,I_s - Q\,Y_b\,V_s = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}$$

Hence, the KCL equation can be written as

$$\begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix} \begin{bmatrix} v_{t_1} \\ v_{t_2} \end{bmatrix} = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = 0.33 \text{ V}$$

$$v_{t_2} = -1.33 \text{ V}$$

From Fig. 5.85, $$v = 1 + v_{t_2} = -0.33 \text{ V}$$

Exercises

5.1 For the networks shown in Fig. 5.87–5.90, write the incidence matrix, tieset matrix and *f*-cutset matrix.

(i)

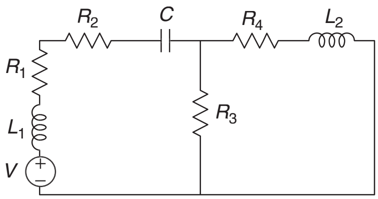


Fig. 5.87

(ii)

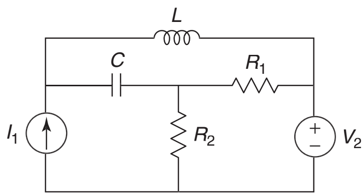


Fig. 5.88

(iii)

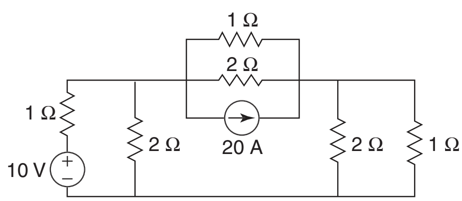


Fig. 5.89

(iv)

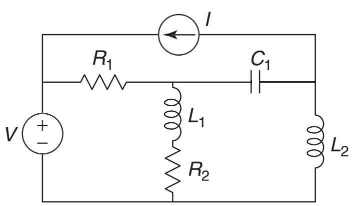


Fig. 5.90

5.2 For the graph shown in Fig. 5.91, write the incidence matrix, tieset matrix and *f*-cutset matrix.

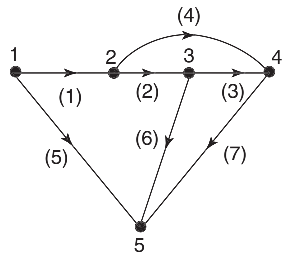


Fig. 5.91

5.3 The incidence matrix is given as follows:

Branches →

1	2	3	4	5	6	7	8
–1	–1	0	0	0	0	1	0
0	1	1	0	1	0	0	0
0	0	–1	–1	0	1	0	0
1	0	0	1	0	0	0	1

Draw oriented graph and write tieset matrix.

5.4 The incidence matrix is given below:

Branches →

1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	0	1	0	0
0	–1	–1	0	0	0	–1	0	0	–1
–1	1	0	0	0	0	0	–1	–1	1
1	0	0	0	–1	–1	1	0	0	0

Draw the oriented graph.

5.5 For the network shown in Fig. 5.92, draw the oriented graph and obtain the tieset matrix. Use this matrix to calculate the current *i*.

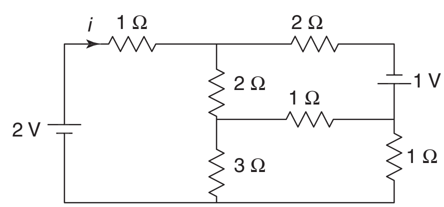


Fig. 5.92

[0.91 A]

5.6 Using the principles of network topology, write the loop/node equation in matrix form for the network shown in Fig. 5.93.

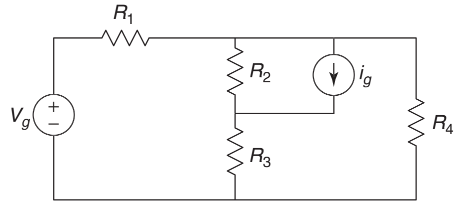


Fig. 5.93

Objective-Type Questions

5.1 The number of independent loops for a network with n nodes and b branches is
(a) n − 1
(b) b − n
(c) b − n + 1
(d) independent of the number of nodes

5.2 A network has 7 nodes and 5 independent loops. The number of branched in the network is
(a) 13 (b) 12
(c) 11 (d) 10

5.3 Identify which of the following is NOT a tree of the graph shown in Fig. 5.94.

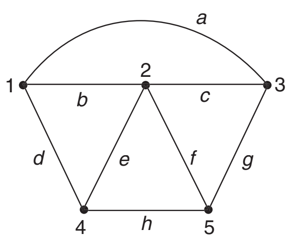


Fig. 5.94

(a) *begh* (b) *defg*
(c) *adfg* (d) *aegh*

5.4 The minimum number of equations required to analyze the circuit shown in Fig. 5.95 is

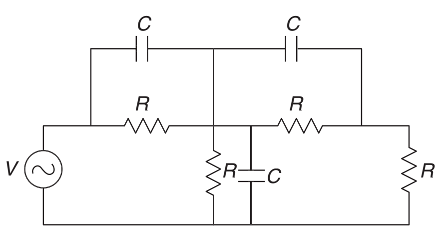


Fig. 5.95

(a) 3 (b) 4
(c) 6 (d) 7

5.5 Consider the network graph shown in Fig. 5.96.

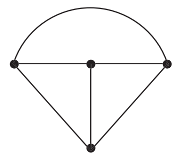
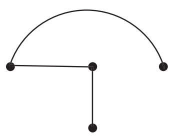
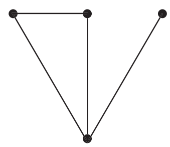
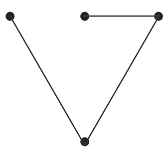
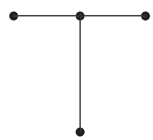


Fig. 5.96

Which one of the following is NOT a tree of this graph?

(a)

(b)

(c)

(d)


5.6 Figure 5.97 below shows a network and its graph is drawn aside. A proper tree chosen for analyzing the network will contain the edges.

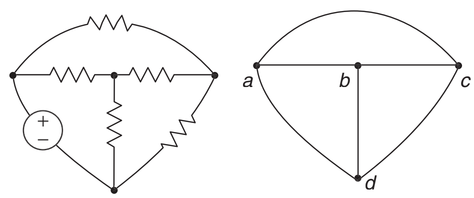


Fig. 5.97

(a) *ab, bc, ad* (b) *ab, bc, ca*
(c) *ab, bd, ca?* (d) *ac, bd, ad*

5.7 The graph of an electrical network has n nodes and b branches. The number of links with respect to the choice of a tree is given by

(a) $b - n + 1$ (b) $b + n$
(c) $n - b + 1$ (d) $n - 2b - 1$.

5.8 In the graph shown in Fig. 5.98, one possible tree is formed by the branches 4, 5, 6, 7. Then one possible fundamental cutset is

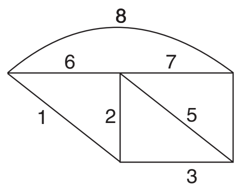


Fig. 5.98

(a) 1, 2, 3, 8 (b) 1, 2, 5, 6
(c) 1, 5, 6, 8 (d) 1, 2, 3, 7, 8

5.9 Which one of the following represents the total number of trees in the graph given in Fig. 5.99?

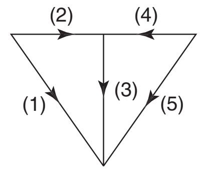


Fig. 5.99

(a) 4 (b) C
(c) 5 (d) 8

5.10 Which one of the following is a cutset of the graph shown in Fig. 5.100?

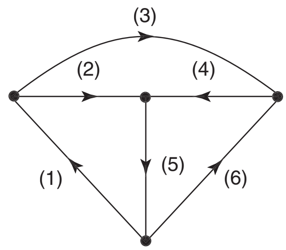


Fig. 5.100

(a) 1, 2, 3 and 4 (b) 2, 3, 4 and 6
(c) 1, 4, 5 and 6 (d) 1, 2, 4 and 5

Answers to Objective-Type Questions

5.1. (c)	5.2. (c)	5.3. (c)	5.4. (b)	5.5. (b)
5.6. (d)	5.7. (a)	5.8. (d)	5.9. (d)	5.10. (d)

6

Time Domain Analysis of RLC Circuits

6.1 INTRODUCTION

Whenever a network containing energy storage elements such as inductor or capacitor is switched from one condition to another, either by change in applied source or change in network elements, the response current and voltage change from one state to the other state. The time taken to change from an initial steady state to the final steady state is known as the *transient period*. This response is known as *transient response* or *transients*. The response of the network after it attains a final steady value is independent of time and is called the steady-state response. The complete response of the network is determined with the help of a differential equation.

6.2 INITIAL CONDITIONS

In solving the differential equations in the network, we get some arbitary constant. Initial conditions are used to determine these arbitrary constants. It helps us to know the behaviour of elements at the instant of switching.

To differentiate between the time immediately before and immediately after the switching, the signs '–' and '+' are used. The conditions existing just before switching are denoted as $i\ (0^-)$, $v\ (0^-)$, etc. Conditions just after switching are denoted as $i\ (0^+)$, $v\ (0^+)$.

Sometimes conditions at $t = \infty$ are used in the evaluation of arbitrary constants. These are known as *final conditions*.

In solving the problems for initial conditions in the network, we divide the time period in the following ways:

1. Just before switching (from $t = -\infty$ to $t = 0^-$)
2. Just after switching (at $t = 0^+$)
3. After switching (for $t > 0$)

If the network remains in one condition for a long time without any switching action, it is said to be under steady-state condition.

1. Initial Conditions for the Resistor For a resistor, current and voltage are related by $v(t) = Ri(t)$. The current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, the voltage will change instantaneously if the current changes instantaneously.

2. Initial Conditions for the Inductor For an inductor, current and voltage are related by,

$$v(t) = L\frac{di}{dt}$$

Voltage across the inductor is proportional to the rate of change of current. It is impossible to change the current through an inductor by a finite amount in zero time. This requires an infinite voltage across the inductor. An inductor does not allow an abrupt change in the current through it.

The current through the inductor is given by,

$$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current through the inductor.

If there is no current flowing through the inductor at $t = 0^-$, the inductor will act as an open circuit at $t = 0^+$. If a current of value I_0 flows through the inductor at $t = 0^-$, the inductor can be regarded as a current source of I_0 ampere at $t = 0^+$.

3. Initial Conditions for the Capacitor For the capacitor, current and voltage are related by,

$$i(t) = C\frac{dv(t)}{dt}$$

Current through a capacitor is proportional to the rate of change of voltage. It is impossible to change the voltage across a capacitor by a finite amount in zero time. This requires an infinite current through the capacitor. A capacitor does not allow an abrupt change in voltage across it.

The voltage across the capacitor is given by,

$$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

where $v(0)$ is the initial voltage across the capacitor.

If there is no voltage across the capacitor at $t = 0^-$, the capacitor will act as a short circuit at $t = 0^+$. If the capacitor is charged to a voltage V_0 at $t = 0^-$, it can be regarded as a voltage source of V_0 volt at $t = 0^+$. These conditions are summarized in Fig. 6.1.

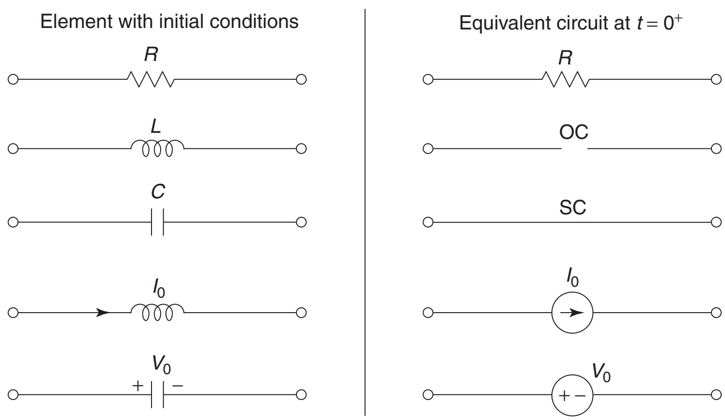


Fig. 6.1 *Initial conditions*

Similarly, we can draw the chart for final conditions as shown in Fig. 6.2

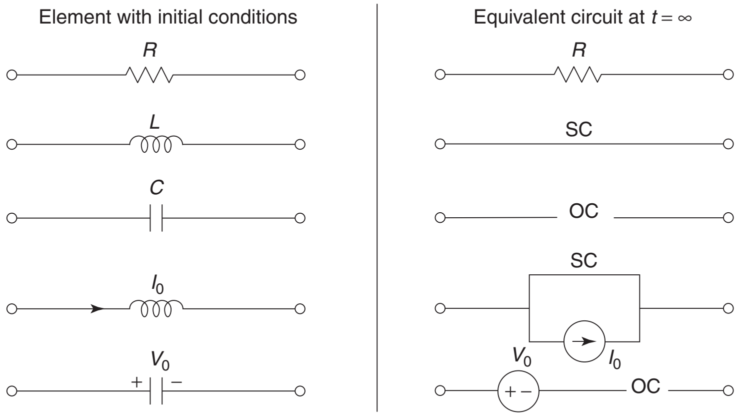


Fig. 6.2 *Final conditions*

4. Procedure for Evaluating Initial Conditions

(a) Draw the equivalent network at $t = 0^-$. Before switching action takes place, i.e., for $t = -\infty$ to $t = 0^-$, the network is under steady-state conditions. Hence, find the current flowing through the inductors $i_L(0^-)$ and voltage across the capacitor $v_C(0^-)$.

(b) Draw the equivalent network at $t = 0^+$, i.e., immediately after switching. Replace all the inductors with open circuits or with current sources $i_L(0^+)$ and replace all capacitors by short circuits or voltage sources $v_C(0^+)$. Resistors are kept as it is in the network.

(c) Initial voltages or currents in the network are determined from the equivalent network at $t = 0^+$.

(d) Initial conditions, i.e., $\frac{di}{dt}(0^+), \frac{dv}{dt}(0^+), \frac{d^2i}{dt^2}(0^+), \frac{d^2v}{dt^2}(0^+)$ are determined by writing integro-differential equations for the network for $t > 0$, i.e., after the switching action by making use of initial condition.

Example 6.1 *In the given network of Fig. 6.3, the switch is closed at t = 0. With zero current in the inductor, find i,* $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at t = 0⁺.*

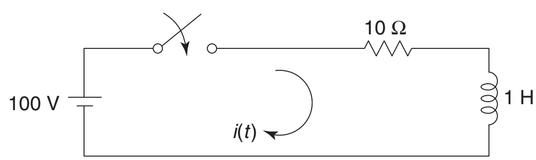


Fig. 6.3

Solution

At $t = 0^-$, no current flows through the inductor.

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.4.

At $t = 0^+$, the inductor acts as an open circuit.

$$i(0^+) = 0$$

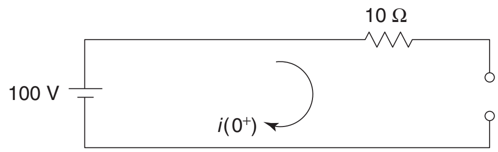


Fig. 6.4

For $t > 0$, the network is shown in Fig. 6.5.
Writing the KVL equation for $t > 0$,

$$100 - 10i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

$$\frac{di}{dt} = 100 - 10i \qquad \ldots\text{(ii)}$$

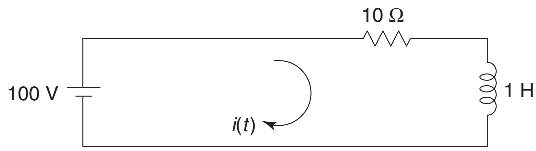


Fig. 6.5

At $t = 0^+$,
$$\frac{di}{dt}(0^+) = 100 - 10i(0^+) = 100 - 10(0) = 100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -10\frac{di}{dt}$$

At $t = 0^+$,
$$\frac{d^2i}{dt^2}(0^+) = -10\frac{di}{dt}(0^+) = -10(100) = -1000 \text{ A/s}^2$$

Example 6.2 *In the network of Fig. 6.6, the switch is closed at t = 0. With the capacitor uncharged, find value for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at t = 0⁺.*

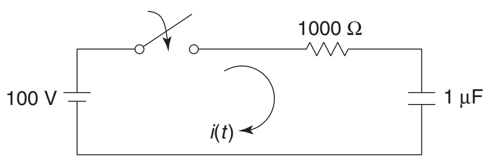


Fig. 6.6

Solution

At $t = 0^-$, the capacitor is uncharged.

$$v_C(0^-) = 0$$
$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.7.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$
$$i(0^+) = \frac{100}{1000} = 0.1 \text{ A}$$

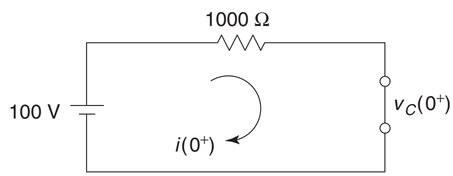


Fig. 6.7

For $t > 0$, the network is shown in Fig. 6.8.
Writing the KVL equation for $t > 0$,

$$100 - 1000i - \frac{1}{1\times 10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

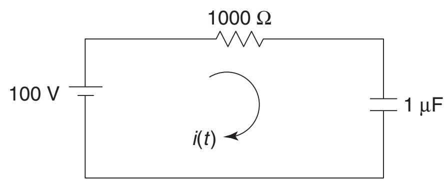


Fig. 6.8

Differentiating Eq. (i),

$$0 - 1000\frac{di}{dt} - 10^6 i = 0$$

$$\frac{di}{dt} = -\frac{10^6}{1000}i \qquad \ldots\text{(ii)}$$

At $t = 0^+$,

$$\frac{di}{dt}(0^+) = -\frac{10^6}{1000}i(0^+) = -\frac{10^6}{1000}(0.1) = -100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -\frac{10^6}{1000}\frac{di}{dt}$$

At $t = 0^+$,

$$\frac{d^2i}{dt^2}(0^+) = -\frac{10^6}{1000}\frac{di}{dt}(0^+) = -\frac{10^6}{1000}(-100) = 10^5 \text{ A/s}^2$$

Example 6.3 *In the network shown in Fig. 6.9, the switch is closed. Assuming all initial conditions as zero, find* i, $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

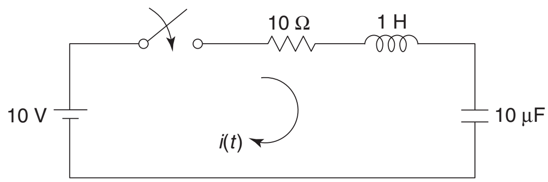


Fig. 6.9

Solution

At $t = 0^-$,

$$i(0^-) = 0$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.10.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i(0^+) = 0$$

$$v_C(0^+) = 0$$

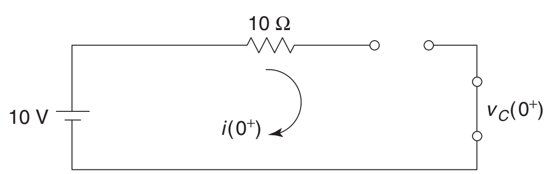


Fig. 6.10

For $t > 0$, the network is shown in Fig. 6.11.

Writing the KVL equation for $t > 0$,

$$10 - 10i - 1\frac{di}{dt} - \frac{1}{10\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

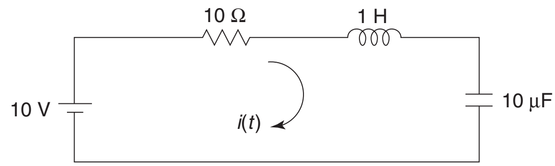


Fig. 6.11

At $t = 0^+$, $\quad 10 - 10i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$

$$\frac{di}{dt}(0^+) = 10\text{A/s}$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - \frac{d^2i}{dt^2} - \frac{1}{10\times10^{-6}}i = 0$$

At $t = 0^+$, $\quad 0 - 10\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) - \frac{1}{10^{-5}}i(0^+) = 0$

$$\frac{d^2i}{dt^2}(0^+) = -10\times10 = -100\ \text{A/s}^2$$

Example 6.4 *In the network shown in Fig. 6.12, at $t = 0$, the switch is opened. Calculate v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at $t = 0^+$.*

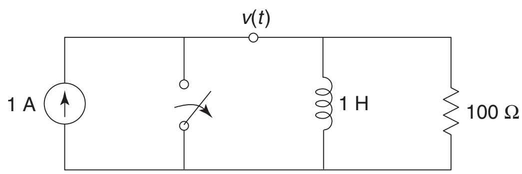


Fig. 6.12

Solution At $t = 0^-$, the switch is closed. Hence, no current flows through the inductor.

$$i_L(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.13.

At $t = 0^+$, the inductor acts as an open circuit.

$$i_L(0^+) = 0$$

$$v(0^+) = 100\times1 = 100\ \text{V}$$

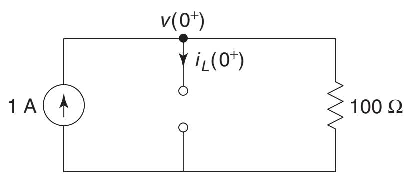


Fig. 6.13

For $t > 0$, the network is shown in Fig. 6.14.
Writing the KCL equation for $t > 0$,

$$\frac{v}{100} + \frac{1}{1}\int_0^t v\, dt = 1 \qquad \ldots\text{(i)}$$

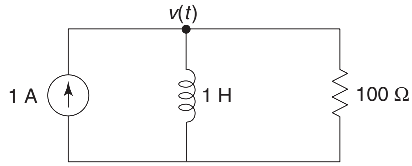


Fig. 6.14

Differentiating Eq. (i),

$$\frac{1}{100}\frac{dv}{dt} + v = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$\frac{dv}{dt}(0^+) = -100v(0^+) = -100 \times 100 = -10000 \text{ V/s}$$

Differentiating Eq. (ii),

$$\frac{1}{100}\frac{d^2v}{dt^2} + \frac{dv}{dt} = 0$$

At $t = 0^+$,
$$\frac{d^2v}{dt^2}(0^+) = -100\frac{dv}{dt}(0^+) = -100 \times (-10^4) = 10^6 \text{ V/s}^2$$

Example 6.5 *In the given network of Fig. 6.15, the switch is opened at t = 0. Solve for* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

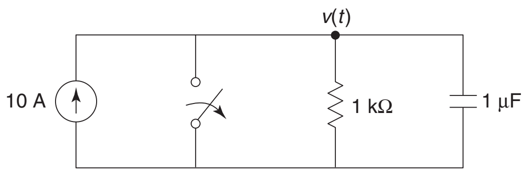


Fig. 6.15

Solution At $t = 0^-$, switch is closed. Hence, the voltage across the capacitor is zero.

$$v(0^-) = v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.16.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v(0^+) = v_C(0^+) = 0$$

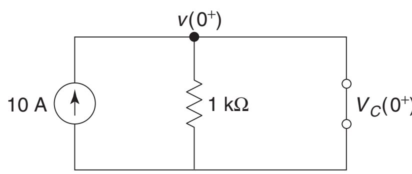


Fig. 6.16

For $t > 0$, the network is shown in Fig. 6.17.
Writing the KCL equation for $t > 0$,

$$\frac{v}{1000} + 10^{-6}\frac{dv}{dt} = 10 \qquad \ldots\text{(i)}$$

At $t = 0^+$,
$$\frac{v(0^+)}{1000} + 10^{-6}\frac{dv}{dt}(0^+) = 10$$

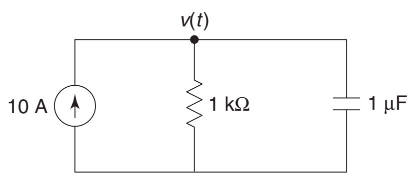


Fig. 6.17

$$\frac{dv}{dt}(0^+) = \frac{10}{10^{-6}} = 10 \times 10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{1000}\frac{dv}{dt} + 10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{1}{1000}\frac{dv}{dt}(0^+) + 10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

$$\frac{d^2v}{dt^2}(0^+) = -\frac{1}{1000 \times 10^{-6}} \times 10 \times 10^6 = -10 \times 10^9 \text{ V/s}^2$$

Example 6.6 *For the network shown in Fig. 6.18, the switch is closed at t = 0, determine* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

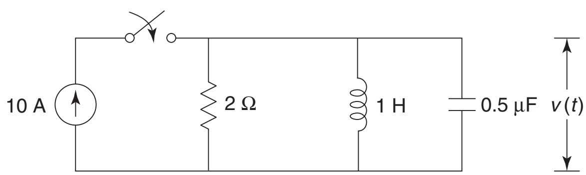


Fig. 6.18

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$

$$v(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.19.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i_L(0^+) = 0$$

$$v(0^+) = 0$$

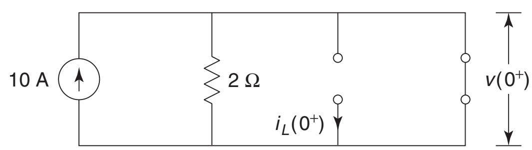


Fig. 6.19

For $t > 0$, the network is shown in Fig. 6.20.

Writing the KCL equation for $t > 0$,

$$\frac{v}{2} + \frac{1}{1}\int_{1}^{t} v\,dt + 0.5 \times 10^{-6}\frac{dv}{dt} = 10 \qquad \ldots\text{(i)}$$

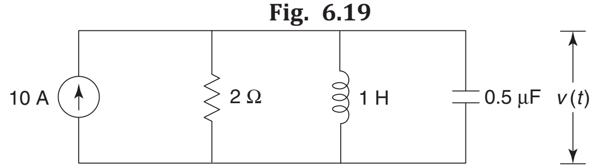


Fig. 6.20

At $t = 0^+$,

$$\frac{v(0^+)}{2} + 0 + 0.5 \times 10^{-6}\frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = 20 \times 10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{2}\frac{dv}{dt}+v+0.5\times10^{-6}\frac{d^2v}{dt^2}=0$$

At $t = 0^+$,

$$\frac{1}{2}\frac{dv}{dt}(0^+)+v(0^+)+0.5\times10^{-6}\frac{d^2v}{dt^2}(0^+)=0$$

$$\frac{d^2v}{dt^2}(0^+)=-20\times10^{12}\ \text{V/s}^2$$

Example 6.7 *In the network shown in Fig. 6.21, the switch is changed from the position 1 to the position 2 at $t = 0$, steady condition having reached before switching. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.*

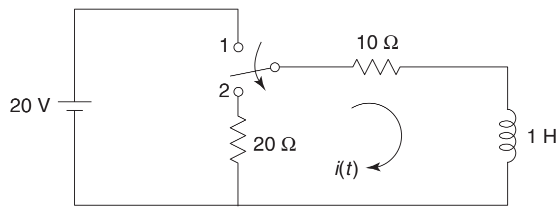


Fig. 6.21

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

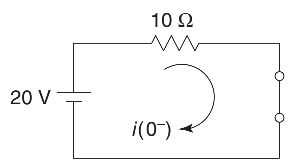


Fig. 6.22

$$i(0^-)=\frac{20}{10}=2\ \text{A}$$

At $t = 0^+$, the network is shown in Fig. 6.23.

At $t = 0^+$, the inductor acts as a current source of 2 A.

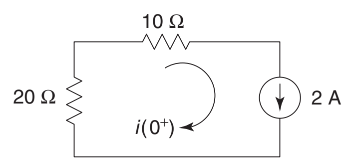


Fig. 6.23

$$i(0^+)=2\ \text{A}$$

For $t > 0$, the network is shown in Fig. 6.24.

Writing the KVL equation for $t > 0$,

$$-20i-10i-1\frac{di}{dt}=0 \qquad \ldots\text{(i)}$$

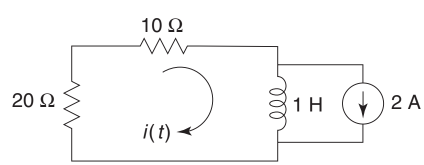


Fig. 6.24

At $t = 0^+$,

$$-30i(0^+)-\frac{di}{dt}(0^+)=0$$

$$\frac{di}{dt}(0^+)=-30\times2=-60\ \text{A/s}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt}-\frac{d^2i}{dt^2}=0$$

At $t = 0^+$,

$$-30\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$

$$\frac{d^2i}{dt^2}(0^+) = 1800 \text{ A/s}^2$$

Example 6.8 *In the network shown in Fig.6.25, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* i, $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

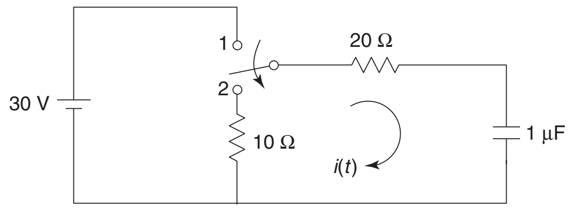


Fig. 6.25

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

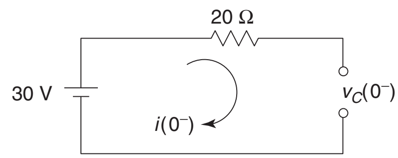


Fig. 6.26

$$v_C(0^-) = 30 \text{ V}$$

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.27.
At $t = 0^+$, the capacitor acts as a voltage source of 30 V.

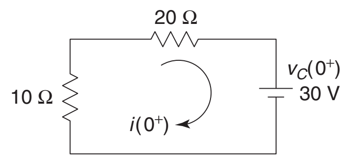


Fig. 6.27

$$v_C(0^+) = 30 \text{ V}$$

$$i(0^+) = -\frac{30}{30} = -1 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.28.
Writing the KVL equation for $t > 0$,

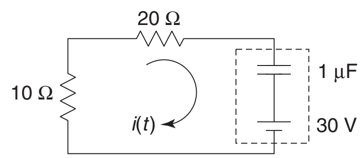


Fig. 6.28

$$-10i - 20i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt - 30 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt} - 10^6 i = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,

$$-30\frac{di}{dt}(0^+) - 10^6 i(0^+) = 0$$

$$\frac{di}{dt}(0^+) = \frac{10^6(-1)}{30} = 0.33\times10^5 \text{ A/s}$$

Differentiating Eq. (ii),

$$-30\frac{d^2i}{dt^2}-10^6\frac{di}{dt}=0$$

At $t = 0^+$,
$$-30\frac{d^2i}{dt^2}(0^+)-10^6\frac{di}{dt}(0^+)=0$$

$$\frac{d^2i}{dt^3}(0^+)=-\frac{10^6\times0.33\times10^5}{30}=-1.1\times10^9\ \text{A/s}^2$$

Example 6.9 *In the network shown in Fig. 6.29, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* i, $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

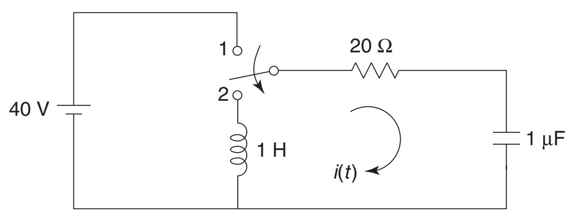


Fig. 6.29

Solution At $t = 0^-$, the network attains steady state. Hence, the capacitor acts as an open circuit.

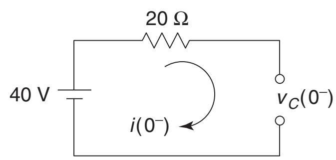


Fig. 6.30

$$v_C(0^-)=40\text{ V}$$

$$i(0^-)=0$$

At $t=0^+$, the network is shown in Fig. 6.31.

At $t = 0^+$, the capacitor acts as a voltage source of 40 V and the inductor acts as an open circuit.

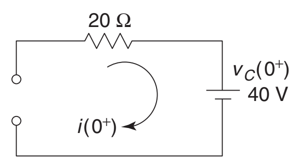


Fig. 6.31

$$v_C(0^+)=40\text{ V}$$

$$i(0^+)=0$$

For $t>0$, the network is shown in Fig. 6.32.

Writing the KVL equation fo $t > 0$,

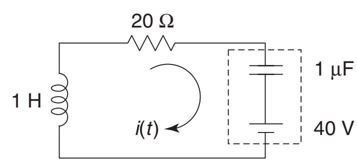


Fig. 6.32

$$-1\frac{di}{dt}-20i-\frac{1}{1\times10^{-6}}\int_0^t i\,dt-40=0 \qquad \ldots\text{(i)}$$

At $t = 0^+$,
$$-\frac{di}{dt}(0^+)-20i(0^+)-0-40=0$$

$$\frac{di}{dt}(0^+)=-40\text{ A/s}$$

Differentiating Eq. (i),

$$-\frac{d^2i}{dt^2} - 20\frac{di}{dt} - 10^6 i - 0 = 0$$

At $t = 0^+$,
$$-\frac{d^2i}{dt^2}(0^+) - 20\frac{di}{dt}(0^+) - 10^6 i(0^+) = 0$$

$$\frac{d^2i}{dt^2}(0^+) = 800 \text{ A/s}^2$$

Example 6.10 *In the network of Fig. 6.33, the switch is changed from the position 'a' to 'b' at $t = 0$. Solve for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.*

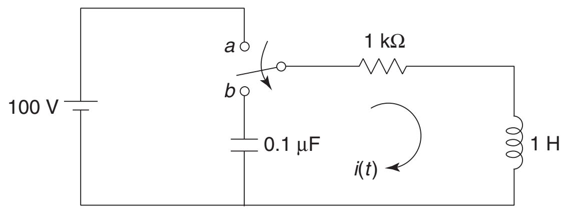


Fig. 6.33

Solution At $t = 0^-$, the network attains steady condition. Hence, the inductor acts as a short circuit.

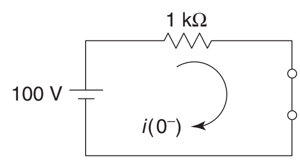


Fig. 6.34

$$i(0^-) = \frac{100}{1000} = 0.1 \text{ A}$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.35.

At $t = 0^+$, the inductor acts as a current source of 0.1 A and the capacitor acts as a short circuit.

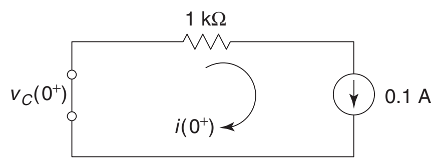


Fig. 6.35

$$i(0^+) = 0.1 \text{ A}$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.36.

Writing the KVL equation for $t > 0$,

$$-\frac{1}{0.1\times 10^{-6}}\int_0^t i\, dt - 1000i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

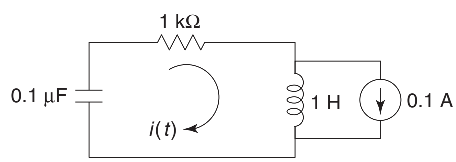


Fig. 6.36

At $t = 0^+$,
$$-0 - 1000i(0^+) - \frac{di}{dt}(0^+) = 0$$
$$\frac{di}{dt}(0^+) = -1000\,i(0^+) = -1000 \times 0.1 = -100 \text{ A/s}$$

Differentiating Eq. (i),
$$-\frac{1}{10^{-7}}i - 1000\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-10^7 i(0^+) - 1000\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$
$$\frac{d^2i}{dt^2}(0^+) = -10^7(0.1) - 1000(-100) = -9 \times 10^5 \text{ A/s}^2$$

Example 6.11 *The network of Fig. 6.37 attains steady-state with the switch closed. At $t = 0$, the switch is opened. Find the voltage across the switch v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.*

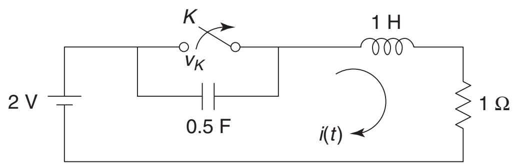


Fig. 6.37

Solution: At $t = 0^-$, the network is shown in Fig. 6.38. At $t = 0^-$, the network attains steady-state condition. The capacitor acts as an open circuit and the inductor acts as a short circuit.

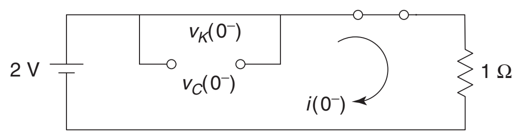


Fig. 6.38

$$i(0^-) = \frac{2}{1} = 2 \text{ A}$$
$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.39.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as a current source of 2 A.

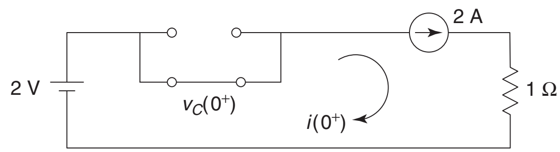


Fig. 6.39

$$i(0^+) = 2 \text{ A}$$
$$v_C(0^+) = 0$$
$$v_K(0^+) = 0$$

Also,
$$v_K = \frac{1}{C}\int i\,dt$$
$$\frac{dv_K}{dt} = \frac{i}{C}$$

At $t = 0^+$,
$$\frac{dv_K}{dt}(0^+) = \frac{i(0^+)}{C} = \frac{2}{0.5} = 4 \text{ A/s}$$

Example 6.12 *In the network shown in Fig. 6.40, assuming all initial conditions as zero, find* $i_1(0^+), i_2(0^+), \frac{di_1}{dt}(0^+), \frac{di_2}{dt}(0^+), \frac{d^2 i_1}{dt^2}(0^+)$ *and* $\frac{d^2 i_2}{dt^2}(0^+)$.

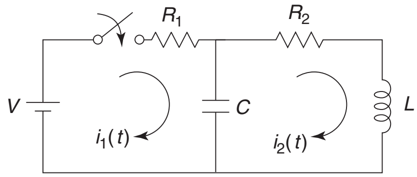


Fig. 6.40

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.41.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

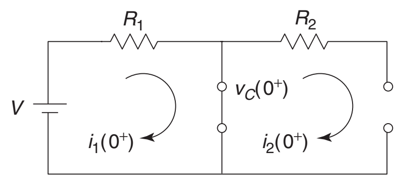


Fig. 6.41

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.42.

Writing the KVL equations for two meshes for $t > 0$,

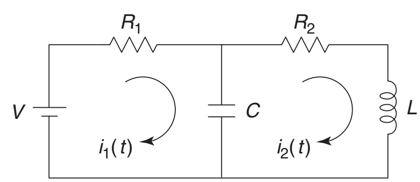


Fig. 6.42

$$V - R_1 i_1 - \frac{1}{C}\int_0^t (i_1 - i_2)\,dt = 0 \qquad \ldots\text{(i)}$$

and
$$-\frac{1}{C}\int (i_2 - i_1)\,dt - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

From Eq. (ii), at $t = 0^+$,

$$-\frac{1}{C}\int_0^{0^+}(i_2 - i_1)dt - R_2 i_2(0^+) - L\frac{di_2}{dt}(0^+) = 0$$

$$\frac{di_2}{dt}(0^+) = 0$$

Differentiating Eq. (i),

$$0 - R_1\frac{di_1}{dt} - \frac{1}{C}(i_1 - i_2) = 0 \qquad \ldots\text{(iii)}$$

At $t = 0^+$,

$$0 - R_1\frac{di_1}{dt}(0^+) - \frac{1}{C}i_1(0^+) + \frac{1}{C}i_2(0^+) = 0$$

$$R_1\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{V}{R_1} = 0$$

$$\frac{di_1}{dt}(0^+) = -\frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$-R_1\frac{d^2 i_1}{dt^2} - \frac{1}{C}\frac{di_1}{dt} + \frac{1}{C}\frac{di_2}{dt} = 0$$

At $t = 0^+$,

$$-R_1\frac{d^2 i_1}{dt^2}(0^+) - \frac{1}{C}\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{di_2}{dt}(0^+) = 0$$

$$\frac{d^2 i_1}{dt^2}(0^+) = \frac{V}{R_1^3 C^2}$$

Differentiating Eq. (ii),

$$-\frac{1}{C}(i_2 - i_1) - R_2\frac{di_2}{dt} - L\frac{d^2 i_2}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{d^2 i_2}{dt^2}(0^+) = -\frac{R_2}{L}\frac{di_2}{dt}(0^+) - \frac{1}{LC}[i_2(0^+) - i_1(0^+)] = \frac{V}{R_1 LC}$$

Example 6.13 *In the network shown in Fig. 6.43, assuming all initial conditions as zero, find* $i_1, i_2\ \frac{di_1}{dt}, \frac{di_2}{dt}, \frac{d^2 i_1}{dt^2}$ *and* $\frac{d^2 i_2}{dt^2}$ *at* $t = 0^+$.

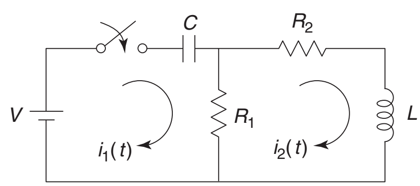


Fig. 6.43

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.44.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as an open circuit.

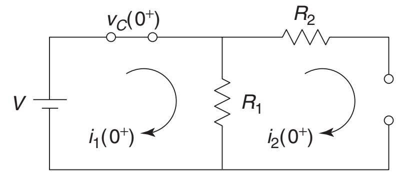


Fig. 6.44

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.45.

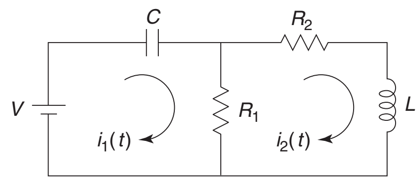


Fig. 6.45

Writing the KVL equation for $t > 0$,

$$V - \frac{1}{C}\int_0^t i_1\, dt - R_1(i_1 - i_2) = 0 \qquad \ldots(\text{i})$$

and $$-R_1(i_2 - i_1) - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots(\text{ii})$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{1}{L}[R_1 i_1 - (R_1 + R_2)i_2] \qquad \ldots(\text{iii})$$

At $t = 0^+$, $$\frac{di_2}{dt}(0^+) = \frac{1}{L}[R_1 i_1(0^+) - (R_1 + R_2)i_2(0^+)] = \frac{1}{L}\left[R_1\frac{V}{R_1} - (R_1 + R_2)0\right] = \frac{V}{L}$$

Differentiating Eq. (i),

$$0 - \frac{i_1}{C} - R_1\frac{di_1}{dt} + R_1\frac{di_2}{dt} = 0$$
$$\frac{di_1}{dt} = \frac{di_2}{dt} - \frac{i_1}{R_1 C} \qquad \ldots(\text{iv})$$

At $t = 0^+$, $$\frac{di_1}{dt}(0^+) = \frac{di_2}{dt}(0^+) - \frac{i_1(0^+)}{R_1 C} = \frac{V}{L} - \frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$\frac{d^2 i_2}{dt^2} = \frac{1}{L}\left[R_1\frac{di_1}{dt} - (R_1 + R_2)\frac{di_2}{dt}\right]$$

At $t = 0^+$, $$\frac{d^2 i_2}{dt^2}(0^+) = -V\left(\frac{1}{R_1 LC} + \frac{R_2}{L^2}\right)$$

Differentiating Eq. (iv),

$$\frac{d^2 i_1}{dt^2} = \frac{d^2 i_2}{dt^2} - \frac{1}{R_1 C}\frac{di_1}{dt}$$

At $t = 0^+$, $\frac{d^2 i_1}{dt^2}(0^+) = \frac{d^2 i_2}{dt^2}(0^+) - \frac{1}{R_1 C}\frac{di_1}{dt}(0^+) = -\frac{V}{R_1 LC} - \frac{VR_2}{L^2} - \frac{1}{R_1 C}\left(\frac{V}{L} - \frac{V}{R_1^2 C}\right) = \frac{V}{R_1^3 C^2} - \frac{2V}{R_1 LC} - \frac{VR_2}{L^2}$

Example 6.14 *In the network shown in Fig. 6.46, a steady state is reached with the switch open. At t = 0, the switch is closed. For the element values given, determine the value of* $v_a(0^-)$, $v_b(0^-)$, $v_a(0^+)$ *and* $v_b(0^+)$.

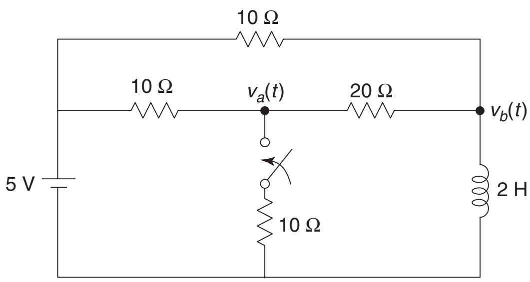


Fig. 6.46

Solution At $t = 0^-$, the network is shown in Fig. 6.47.

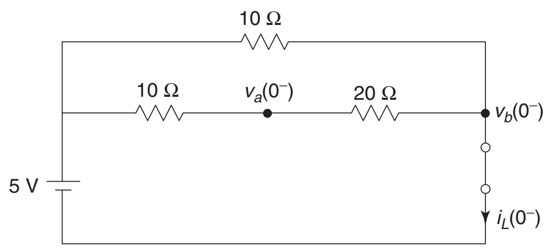


Fig. 6.47

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{5}{(10 \,||\, 30)} = \frac{5}{7.5} = \frac{2}{3}\text{ A}$$

$$v_b(0^-) = 0$$

$$v_a(0^-) = 5 \times \frac{20}{30} = 3.33\text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.48.

At $t = 0^+$, the inductor acts as a current source of $\frac{2}{3}$ A.

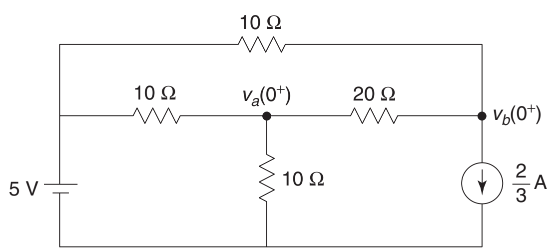


Fig. 6.48

$$i_L(0^+) = \frac{2}{3}\text{ A}$$

Writing the KCL equations at $t = 0^+$,

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

and $$\frac{v_b(0^+)-v_a(0^+)}{20}+\frac{v_b(0^+)-5}{10}+\frac{2}{3}=0$$

Solving these two equations,

$$v_a(0^+)=1.9 \text{ V}$$
$$v_b(0^+)=-0.477 \text{ V}$$

Example 6.15 *In the accompanying Fig. 6.49 is shown a network in which a steady state is reached with switch open. At t = 0, switch is closed. Determine $v_a(0^-)$, $v_a(0^+)$, $v_b(0^-)$ and $v_b(0^+)$.*

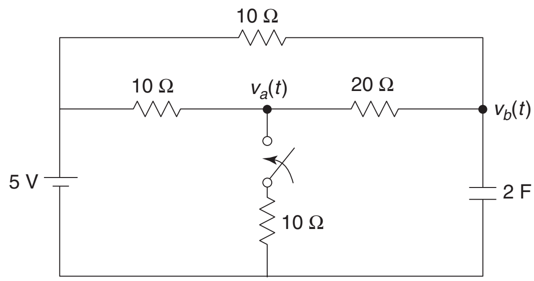


Fig. 6.49

Solution At $t=0^-$, the network is shown in Fig. 6.50.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_a(0^-)=5 \text{ V}$$
$$v_b(0^-)=5 \text{ V}$$

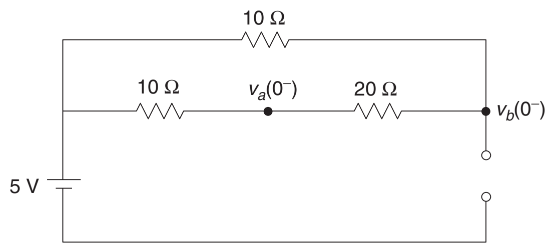


Fig. 6.50

At $t=0^+$, the network is shown in Fig. 6.51.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$v_b(0^+)=5 \text{ V}$$

Writing the KCL equation at $t = 0^+$,

$$\frac{v_a(0^+)-5}{10}+\frac{v_a(0^+)}{10}+\frac{v_a(0^+)-5}{20}=0$$
$$0.25v_a(0^+)=0.75$$
$$v_a(0^+)=3 \text{ V}$$

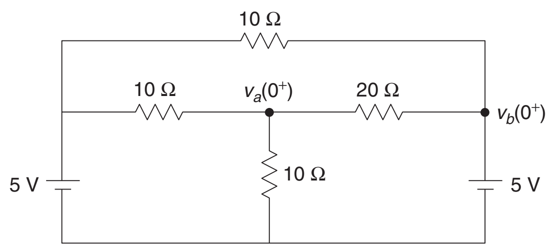


Fig. 6.51

Example 6.16 *The network shown in Fig. 6.52 has two independent node pairs. If the switch is opened at t = 0. Find $v_1, v_2, \frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0^+$.*

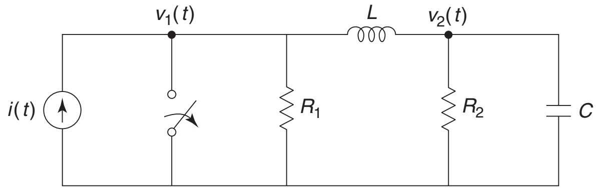


Fig. 6.52

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$

$$v_C(0^-) = v_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.53.
At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i_L(0^+) = 0$$

$$v_1(0^+) = R_1\, i(0^+)$$

$$v_2(0^+) = 0$$

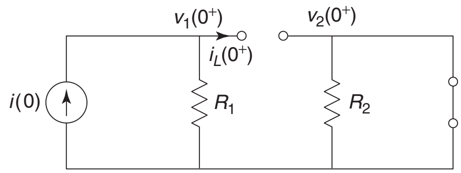


Fig. 6.53

For $t > 0$, the network is shown in Fig. 6.54.
Writing the KCL equation at Node 1 for $t > 0$,

$$\frac{v_1}{R_1} + \frac{1}{L}\int_0^t (v_1 - v_2)dt = i(t) \qquad \ldots\text{(i)}$$

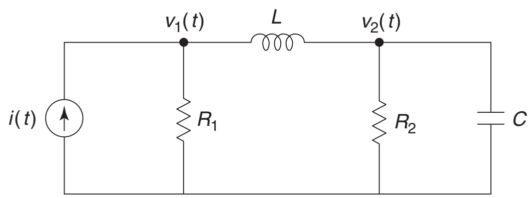


Fig. 6.54

Differentiating Eq. (i),

$$\frac{1}{R_1}\frac{dv_1}{dt} + \frac{1}{L}(v_1 - v_2) = \frac{di}{dt}$$

At $t = 0^+$,

$$\frac{dv_1}{dt}(0^+) = R_1\left[\frac{di}{dt}(0^+) - \frac{1}{L}R_1 i(0^+)\right]$$

Writing the KCL equation at Node 2 for $t > 0$,

$$\frac{1}{L}\int_0^t (v_2 - v_1)dt + \frac{v_2}{R_2} + C\frac{dv_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,

$$0 + \frac{v_2(0^+)}{R_2} + C\frac{dv_2}{dt}(0^+) = 0$$

$$\frac{dv_2}{dt}(0^+) = 0$$

Example 6.17 *In the network shown in Fig. 6.55, the switch is closed at t = 0, with zero capacitor voltage and zero inductor current. Solve for* $v_1, v_2, \frac{dv_1}{dt}, \frac{dv_2}{dt}$ *and* $\frac{d^2v_2}{dt^2}$ *at* $t = 0^+$.

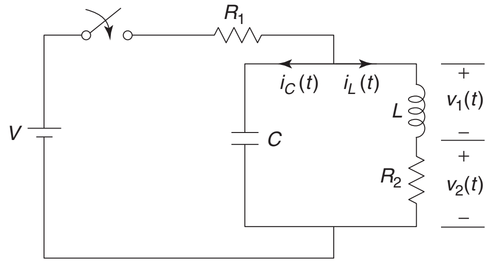


Fig. 6.55

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$v_1(0^-) = 0$$

$$v_2(0^-) = 0$$

$$i_L(0^-) = 0$$

$$i_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.56.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$

$$v_1(0^+) = 0$$

$$v_2(0^+) = 0$$

$$i_L(0^+) = 0$$

$$i_C(0^+) = \frac{V}{R_1}$$

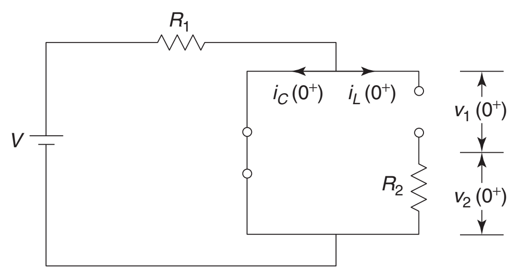


Fig. 6.56

For $t > 0$, the network is shown in Fig. 6.57.

Writing the KVL equation for $t > 0$,

$$v_C(t) = v_1(t) + v_2(t) \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$\frac{dv_C}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt} \qquad \ldots\text{(ii)}$$

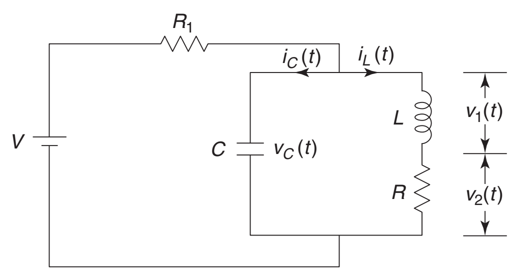


Fig. 6.57

Now,
$$v_C = \frac{1}{C}\int_0^t i_C \, dt \qquad \ldots\text{(iii)}$$

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

At $t = 0^+$,
$$\frac{dv_C}{dt}(0^+) = \frac{i_C(0^+)}{C} = \frac{V}{R_1 C} \text{ V/s}$$

Also
$$v_1 = L\frac{di_L}{dt} \qquad \ldots\text{(iv)}$$

$$\frac{di_L}{dt} = \frac{v_1}{L} \qquad \ldots\text{(v)}$$

At $t = 0^+$,
$$\frac{di_L}{dt}(0^+) = \frac{v_1(0^+)}{L} = 0$$

Also,
$$v_2 = R_2 i_L \qquad \ldots\text{(vi)}$$

$$\frac{dv_2}{dt} = R_2\frac{di_L}{dt} \qquad \ldots\text{(vii)}$$

At $t = 0^+$,
$$\frac{dv_2}{dt}(0^+) = R_2\frac{di_L}{dt}(0^+) = 0$$

$$\frac{dv_C}{dt}(0^+) = \frac{dv_1}{dt}(0^+) + \frac{dv_2}{dt}(0^+)$$

$$\frac{dv_1}{dt}(0^+) = \frac{V}{R_1 C} \text{ V/s}$$

Differentiating Eq. (vii),

$$\frac{d^2 v_2}{dt^2} = R_2\frac{d^2 i_L}{dt^2}$$

At $t = 0^+$,
$$\frac{d^2 v_2}{dt^2}(0^+) = R_2\frac{d^2 i_L}{dt}(0^+)$$

Differentiating Eq. (v),

$$\frac{d^2 i_L}{dt^2} = \frac{1}{L}\frac{dv_1}{dt}$$

At $t = 0^+$,
$$\frac{d^2 i_L}{dt^2}(0^+) = \frac{1}{L}\frac{dv_1}{dt}(0^+) = \frac{1}{L}\frac{V}{R_1 C}$$

$$\frac{d^2 v_2}{dt^2}(0^+) = \frac{R_2 V}{R_1 L C} \text{ V/s}^2$$

Example 6.18 *In the network shown in Fig. 6.58, a steady state is reached with switch open. At $t = 0$, switch is closed. Find the three loop currents at $t = 0^+$.*

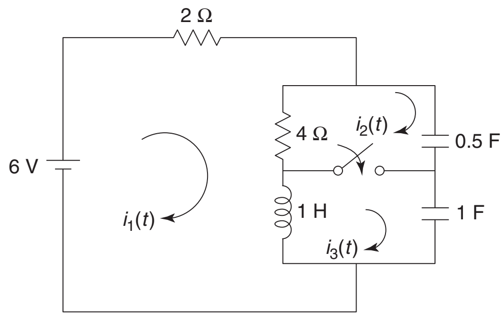


Fig. 6.58

Solution At $t = 0^-$, the network is shown in Fig. 6.59.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor act as a short circuit and the capacitors act as open circuits.

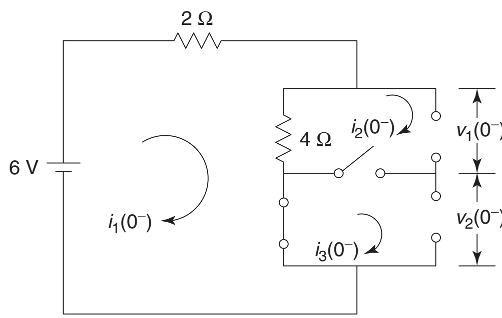


Fig. 6.59

$$i_{4\Omega}(0^-) = i_1(0^-) = \frac{6}{6} = 1 \text{ A}$$

$$i_2(0^-) = 0$$

$$i_3(0^-) = 0$$

$$v_1(0^-) + v_2(0^-) = 6 \times \frac{4}{6} = 4 \text{ V}$$

Since the charges on capacitors are equal when connected in series,

$$Q_1 = Q_2$$

$$C_1 v_1 = C_2 v_2$$

$$\frac{v_1(0^-)}{v_2(0^-)} = \frac{C_2}{C_1} = \frac{1}{0.5} = 2$$

$$v_1(0^-) = \frac{8}{3} \text{ V}$$

and

$$v_2(0^-) = \frac{4}{3} \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.60.

At $t = 0^+$, the inductor is replaced by a current source of 1 A and the capacitors are replaced by a voltage source of $\frac{8}{3}$ V and $\frac{4}{3}$ V respectively.

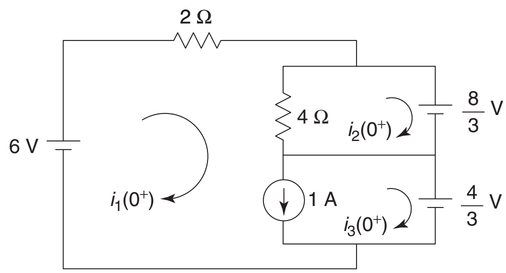


Fig. 6.60

$$v_1(0^+) = \frac{8}{3}\text{ V}$$

$$v_2(0^+) = \frac{4}{3}\text{ V}$$

At $t = 0^+$,

$$6 - 2i_1(0^+) - \frac{8}{3} - \frac{4}{3} = 0$$

$$i_1(0^+) = 1\text{ A}$$

Now,

$$i_1(0^+) - i_3(0^+) = 1$$

$$i_3(0^+) = 0$$

Writing the KVL equation for Mesh 2,

$$-4[i_2(0^+) - i_1(0^+)] - \frac{8}{3} = 0$$

$$-4i_2(0^+) + 4 - \frac{8}{3} = 0$$

$$i_2(0^+) = \frac{1}{3}\text{ A}$$

Example 6.19 *In the network shown in Fig. 6.61, the switch K is closed at $t = 0$ connecting a voltage $V_0 \sin \omega t$ to the parallel RL-RC circuit. Find (a) $i_1(0^+)$ and $i_2(0^+)$ (b) $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.*

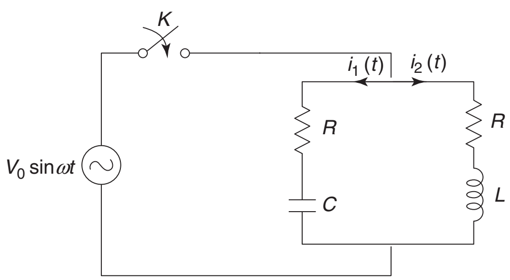


Fig. 6.61

Solution At $t = 0^-$, no current flows in the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.62.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit. The voltage source $V_0 \sin \omega t$ acts as a short circuit.

$$i_1(0^+) = 0$$

$$i_2(0^+) = 0$$

$$v_C(0^+) = 0$$

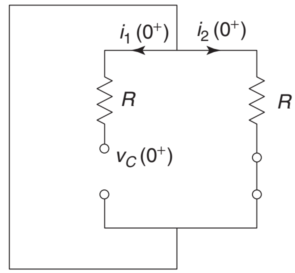


Fig. 6.62

For $t > 0$, the network is shown in Fig. 6.63.

Writing the KVL equation for $t > 0$,

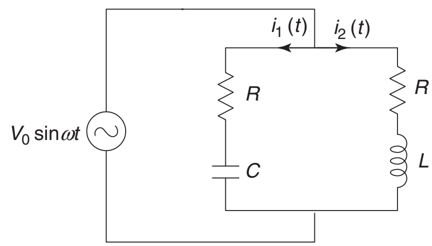


Fig. 6.63

$$V_0 \sin \omega t - R\, i_1 - \frac{1}{C}\int i_1 \, dt = 0 \qquad \ldots\text{(i)}$$

and

$$V_0 \sin \omega t - R\, i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

Differentiating Eq. (i),

$$V_0\, \omega \cos \omega t - R\frac{di_1}{dt} - \frac{i_1}{C} = 0$$

$$\frac{di_1}{dt} = \frac{V_0 \omega}{R}\cos \omega t - \frac{i_1}{RC} \qquad \ldots\text{(iii)}$$

At $t = 0^+$,

$$\frac{di_1}{dt}(0^+) = \frac{V_0 \omega}{R}\cos \omega t\bigg|_{t=0^+} - \frac{i_1(0^+)}{RC} = \frac{V_0 \omega}{R}$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{V_0}{L}\sin \omega t - \frac{R}{L} i_2$$

At $t = 0^+$,

$$\frac{di_2}{dt}(0^+) = \frac{V_0}{L}\sin \omega t\bigg|_{t=0^+} - \frac{R}{L} i_2(0^+) = 0$$

Example 6.20 *In the network of Fig. 6.64, the switch K is changed from 'a' to 'b' at $t = 0$ (a steady state having been established at the position a). Find i_1, i_2 and i_3 at $t = 0^+$.*

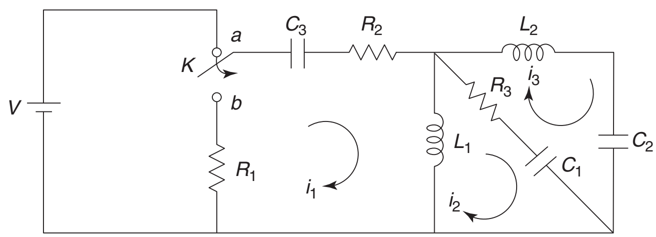


Fig. 6.64

Solution At $t = 0^-$, the network is shown in Fig. 6.65.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitors act as open circuits and inductors act as short circuits.

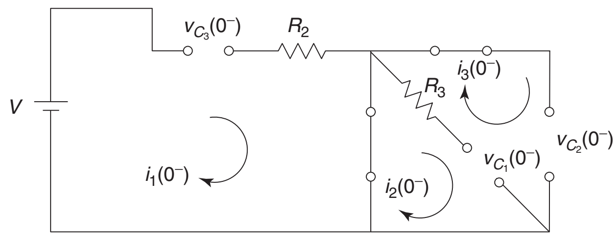


Fig. 6.65

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_3(0^-) = 0$$
$$v_{C_3}(0^-) = V$$
$$v_{C_2}(0^-) = 0$$
$$v_{C_1}(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.66.

At $t = 0^+$, the capacitor C_3 acts as a voltage source of V volts and capacitors C_1 and C_2 act as short circuts. The inductors act as open circuits.

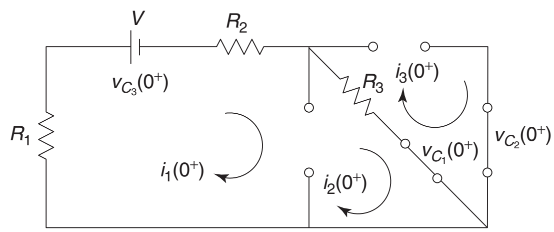


Fig. 6.66

$$i_1(0^+) = i_2(0^+) = -\frac{V}{R_1 + R_2 + R_3}$$

$$i_3(0^+) = 0$$

Example 6.21 *In the network of Fig. 6. 67, the switch K_1 has been closed for a long time prior to $t = 0$. At $t = 0$, the switch K_2 is closed. Find $v_C(0^+)$ and $i_C(0^+)$.*

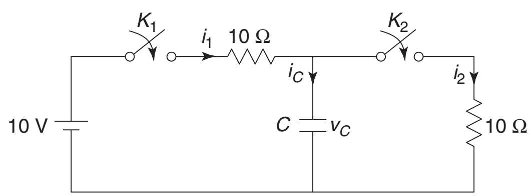


Fig. 6.67

Solution At $t = 0^-$, the network is shown in Fig. 6.68. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

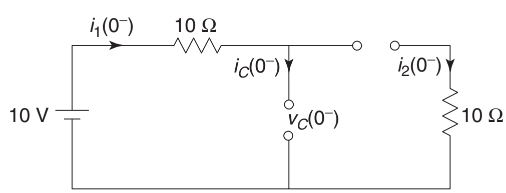


Fig. 6.68

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_C(0^-) = 0$$
$$v_C(0^-) = 10 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.69.

At $t = 0^+$, the capacitor acts as a voltage source of voltage V.

$$v_C(0^+) = 10 \text{ V}$$

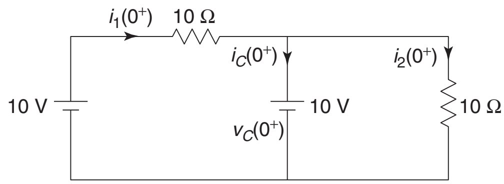


Fig. 6.69

Writing the KVL equation at $t = 0^+$,

$$10 - 10\, i_1(0^+) - 10 = 0$$

and

$$10 - 10\, i_2(0^+) = 0$$

$$i_1(0^+) = 0$$

$$i_2(0^+) = -1 \text{ A}$$

$$i_1(0^+) = i_C(0^+) + i_2(0^+)$$

$$i_C(0^+) = 1 \text{ A}$$

Example 6.22 *In the network shown in Fig. 6.70, a steady state is reached with the switch open. At $t = 0$, the switch is closed. Determine $v_C(0^-)$, $i_1(0^+)$, $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.*

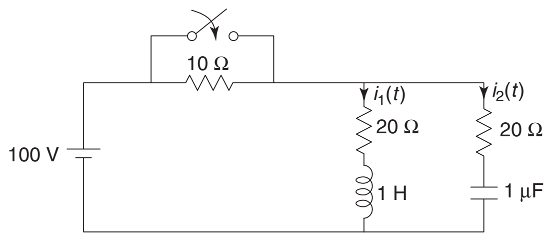


Fig. 6.70

Solution At $t = 0^-$, the network is shown in Fig. 6.71.

At $t = 0^-$, the network is in steady-state. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

$$v_C(0^-) = 100 \times \frac{20}{20 + 10} = 66.67 \text{ V}$$

$$i_1(0^-) = \frac{66.67}{20} = 3.33 \text{ A}$$

$$i_2(0^-) = 0$$

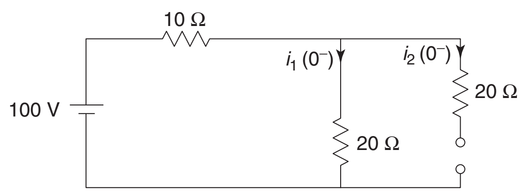


Fig. 6.71

At $t = 0^+$, the network is shown in Fig. 6.72.

At $t = 0^+$, the inductor acts as a current source of 3.33 A and the capacitor acts as a voltage source of 66.67 V.

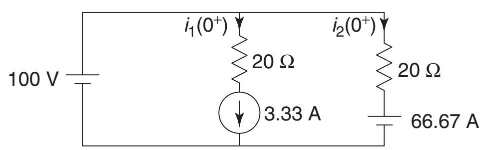


Fig. 6.72

$$v_C(0^+) = 66.67 \text{ V}$$
$$i_1(0^+) = 3.33 \text{ A}$$
$$i_2(0^+) = \frac{100 - 66.67}{20} = 1.67 \text{ A}$$

For $t > 0^-$, the network is shown in Fig. 6.73.

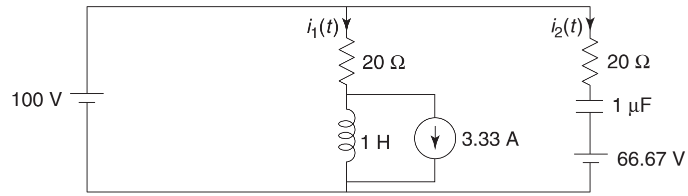


Fig. 6.73

Writing the KVL equations for $t > 0$,

$$100 - 20i_1 - 1\frac{di_1}{dt} = 0 \qquad \text{...(i)}$$

and
$$100 - 20\, i_2 - \frac{1}{10^{-6}}\int i_2\, dt - 66.67 = 0 \qquad \text{...(ii)}$$

At $t = 0^+$,
$$\frac{di_1}{dt}(0^+) = 100 - 20i_1(0^+) = 100 - 20\,(3.33) = 33.3 \text{ A/s}$$

Differentiating Eq. (ii),

$$0 - 20\frac{di_2}{dt} - 10^6\, i_2 = 0$$

At $t = 0^+$,
$$20\frac{di_2}{dt}(0^+) = -10^6\, i_2(0^+)$$
$$\frac{di_2}{dt}(0^+) = -\frac{10^6}{20} \times 1.67 = -83500 \text{ A/s}^2$$

6.3 RESISTOR-INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 6.74. The switch is closed at time $t = 0$. The inductor in the circuit is initially un-energised.

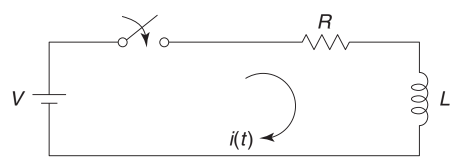


Fig. 6.74 *Series RL circuit*

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} = 0$$

This is a linear differential equation of first order. It can be solved if the variables can be separated.

$$(V - Ri)\, dt = L\, di$$
$$\frac{L\, di}{V - Ri} = dt$$

Integrating both the sides,

$$-\frac{L}{R} l_n (V - Ri) = t + k$$

where l_n denotes that the logarithm is of base e and k is an arbitrary constant. k can be evaluated from the initial condition. In the circuit, the switch is closed at $t = 0$, i.e., just before closing the switch, the current in the inductor is zero. Since the inductor does not allow sudden change in current, at $t = 0^+$, just after the switch is closed, the current remains zero.

Setting $i = 0$ at $t = 0$,

$$-\frac{L}{R} l_n V = k$$

$$-\frac{L}{R} l_n (V - Ri) = t - \frac{L}{R} l_n V$$

$$-\frac{L}{R}\left[l_n (V - Ri) - l_n V\right] = t$$

$$\frac{V - Ri}{V} = e^{-\frac{R}{L}t}$$

$$V - Ri = Ve^{-\frac{R}{L}t}$$

$$Ri = V - Ve^{-\frac{R}{t}t}$$

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

The complete response is composed of two parts, the steady-state response or forced response or zero state response $\frac{V}{R}$ and transient response or natural response or zero input response $\frac{V}{R} e^{-\frac{R}{L}t}$.

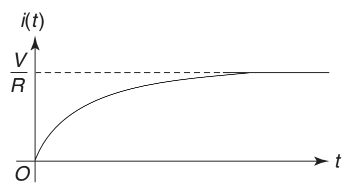


Fig. 6.75 *Current response of series RL circuit*

The natural response is a characteristic of the circuit. Its form may be found by considering the source-free circuit. The forced response has the characteristics of forcing function, i.e., applied voltage. Thus, when the switch is closed, response reaches the steady-state value after some time interval as shown in Fig. 6.75.

Here, the transient period is defined as the time taken for the current to reach its final or steady state value from its initial value.

The term $\frac{L}{R}$ is called time constant and is denoted by T.

$$T = \frac{L}{R}$$

At one time constant, the current reaches 63.2 per cent of its final value $\frac{V}{R}$.

$$i(T) = \frac{V}{R} - \frac{V}{R} e^{-\frac{1}{T}t} = \frac{V}{R} - \frac{V}{R} e^{-1} = \frac{V}{R} - 0.368\frac{V}{R} = 0.632\frac{V}{R}$$

Similarly,

$$i(2T) = \frac{V}{R} - \frac{V}{R}e^{-2} = \frac{V}{R} - 0.135\frac{V}{R} = 0.865\frac{V}{R}$$

$$i(3T) = \frac{V}{R} - \frac{V}{R}e^{-3} = \frac{V}{R} - 0.0498\frac{V}{R} = 0.950\frac{V}{R}$$

$$i(5T) = \frac{V}{R} - \frac{V}{R}e^{-5} = \frac{V}{R} - 0.0067\frac{V}{R} = 0.993\frac{V}{R}$$

After 5 time constants, the current reaches 99.33 per cent of its final value. The voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= V\left(1 - e^{-\frac{R}{L}t}\right) \qquad \text{for } t > 0$$

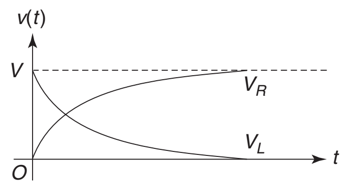


Fig. 6.76 *Voltage response of series RL circuit*

Similarly, the voltage across the inductor is

$$v_L = \frac{di}{dt} = L\frac{V}{R}\frac{d}{dt}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= Ve^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

Note:

1. Consider a homogeneous equation,

$$\frac{di}{dt} + Pi = 0 \qquad \text{where } P \text{ is a constant.}$$

   The solution of this equation is given by,

$$i(t) = ke^{-Pt}$$

   The value of k is obtained by putting $t = 0$ in the equation for $i(t)$.

2. Consider a non-homogeneous equation,

$$\frac{di}{dt} + Pi = Q$$

   where P is a constant and Q may be a function of the independent variable t or a constant.
   The solution of this equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + ke^{-Pt}$$

   The value of k is obtained by putting $t = 0$ in the equation of $i(t)$.

Example 6.23 *In the network of Fig. 6.77, the switch is initially at the position 1. On the steady state having reached, the switch is changed to the position 2. Find current i(t).*

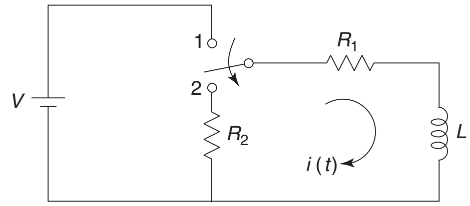


Fig. 6.77

Solution At $t = 0^-$, the network is shown in Fig. 6.78.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

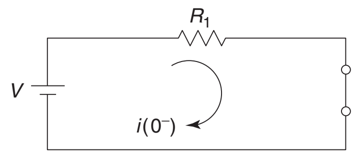


Fig. 6.78

$$i(0^-) = \frac{V}{R_1}$$

Since the inductor does not allow sudden change in current,

$$i(0^+) = \frac{V}{R_1}$$

For $t > 0$, the network is shown in Fig. 6.79.

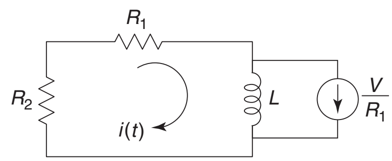


Fig. 6.79

Writing the KVL equation for $t > 0$,

$$-R_2 i - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{(R_1 + R_2)}{L} i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = \frac{R_1 + R_2}{L}$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt}$$

$$i(t) = k\,e^{-\left(\frac{R_1 + R_2}{L}\right)t}$$

At $t = 0$, $i(0) = \frac{V}{R_1}$

$$\frac{V}{R_1} = k\,e^0 = k$$

$$i(t) = \frac{V}{R_1} e^{-\left(\frac{R_1 + R_2}{L}\right)t} \quad \text{for } t > 0$$

Example 6.24 *In the network shown in Fig. 6.80, the switch is closed at t = 0, a steady state having previously been attained. Find the current i (t).*

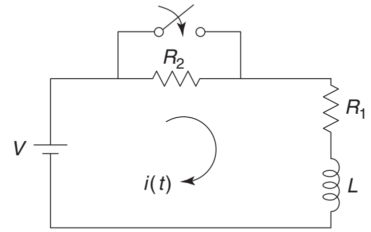


Fig. 6.80

Solution At $t = 0^-$, the network is shown in Fig. 6.81.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

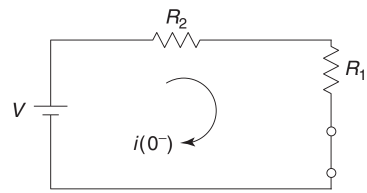


Fig. 6.81

$$i(0^-) = \frac{V}{R_1 + R_2}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = \frac{V}{R_1 + R_2}$$

For $t > 0$, the network is shown in Fig. 6.82.

Writing the KVL equation for $t > 0$,

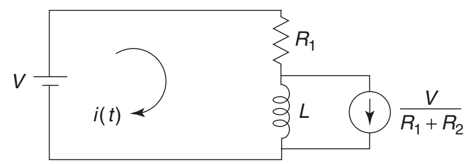


Fig. 6.82

$$V - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R_1}{L} i = \frac{V}{L}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = \frac{R_1}{L}, \quad Q = \frac{V}{L}$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Qe^{Pt}\, dt + k\,e^{-Pt} \\ &= e^{-\frac{R_1}{L}t} \int \frac{V}{L} e^{-\frac{R_1}{L}t}\, dt + k\,e^{-\frac{R_1}{L}t} \\ &= \frac{V}{R_1} + k\,e^{-\frac{R_1}{L}t} \end{aligned}$$

At $t = 0$, $i(0) = \dfrac{V}{R_1 + R_2}$

$$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + k$$

$$k = -\frac{VR_2}{R_1(R_1 + R_2)}$$

$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} e^{-\frac{R_1}{L}t}\right) \quad \text{for } t > 0$$

Example 6.25 *In the network of Fig. 6.83, a steady state is reached with the switch K open. At $t = 0$, the switch K is closed. Find the current $i(t)$ for $t > 0$.*

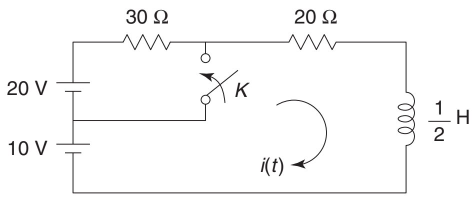


Fig. 6.83

Solution At $t = 0^-$, the network is shown in Fig. 6.84.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

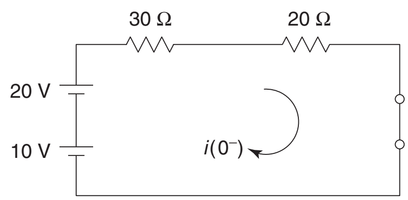


Fig. 6.84

$$i(0^-) = \frac{20 + 10}{30 + 20} = 0.6 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 0.6 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.85.

Writing the KVL equation for $t > 0$.

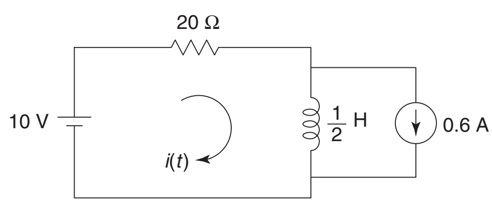


Fig. 6.85

$$10 - 20i - \frac{1}{2}\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 40i = 20$$

Comparing with the differential equation $\frac{di}{dt}+Pi=Q$,

$$P=40, \quad Q=20$$

The solution of this differential equation is given by,

$$i(t)=e^{-Pt}\int Q\,e^{Pt}\,dt+k\,e^{-Pt}$$
$$=e^{-40t}\int 20\,e^{40t}\,dt+k\,e^{-40t}$$
$$=\frac{20}{40}+k\,e^{-40t}$$
$$=0.5+k\,e^{-40t}$$

At $t=0, i(0)=0.6$ A

$$0.6=0.5+k$$
$$k=0.1$$
$$i(t)=0.5+0.1\,e^{-40t} \qquad \text{for } t>0$$

Example 6.26 *The network of Fig. 6.86 is under steady state with switch at the position 1. At t = 0, switch is moved to position 2. Find i (t).*

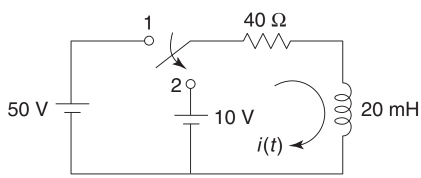


Fig. 6.86

Solution At $t=0^-$, the network is shown in Fig. 6.87.

At t = 0^-, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-)=\frac{50}{40}=1.25 \text{ A}$$

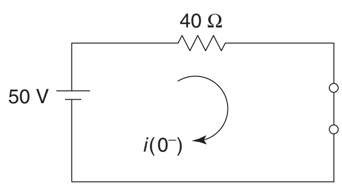


Fig. 6.87

Since current through the inductor cannot change instantaneously,

$$i(0^+)=1.25 \text{ A}$$

For $t>0$, the network is shown in Fig. 6.88.

Writing the KVL equation for $t>0$,

$$10-40i-20\times10^{-3}\frac{di}{dt}=0$$

$$\frac{di}{dt}+2000i=500$$

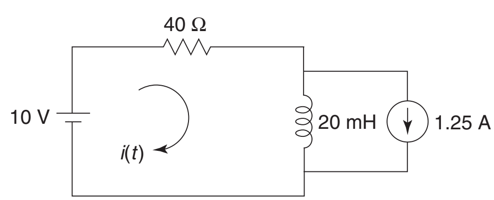


Fig. 6.88

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 2000, \quad Q = 500$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-2000t} \int 500\, e^{2000t} + k\, e^{-2000t} \\ &= \frac{500}{2000} + k\, e^{-2000t} \\ &= 0.25 + k\, e^{-2000t} \end{aligned}$$

At $t = 0,\ i(0) = 1.25$ A

$$1.25 = 0.25 + k$$

$$k = 1$$

$$i(t) = 0.25 + e^{-2000t} \qquad \text{for } t > 0$$

Example 6.27 *In the network of Fig. 6.89, the switch is moved from 1 to 2 at t = 0. Determine i(t).*

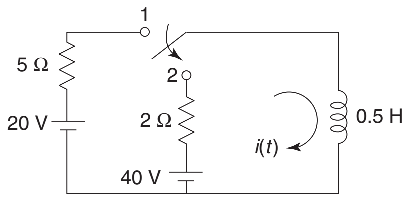


Fig. 6.89

Solution At $t = 0^-$, the network is shown in Fig. 6.90.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

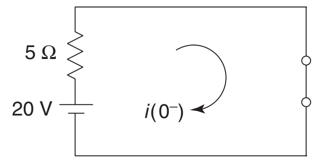


Fig. 6.90

$$i(0^-) = \frac{20}{5} = 4 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 4 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.91.

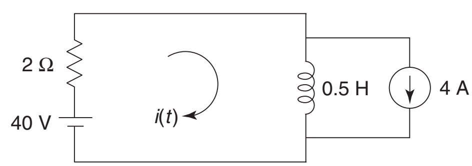


Fig. 6.91

Writing the KVL equation for $t > 0$,

$$40 - 2i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 4i = 80$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 4, \quad Q = 80$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-4t} \int 80\, e^{4t}\, dt + k\, e^{-4t} \\ &= \frac{80}{4} + k\, e^{-4t} \\ &= 20 + k\, e^{-4t} \end{aligned}$$

At $t = 0$, $i(0) = 4$ A

$$4 = 20 + k$$

$$k = -16$$

$$i(t) = 20 - 16\, e^{-4t} \qquad \text{for } t > 0$$

Example 6.28 *For the network shown in Fig. 6.92, steady state is reached with the switch closed. The switch is opened at t = 0. Obtain expressions for i_L (t) and v_L (t).*

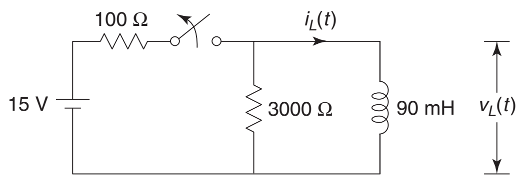


Fig. 6.92

Solution At $t = 0^-$, the network is shown in Fig. 6.93.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

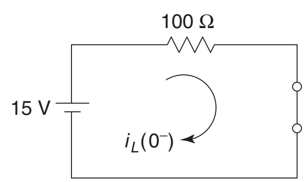


Fig. 6.93

$$i_L(0^-) = \frac{15}{100} = 0.15 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 0.15 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.94.

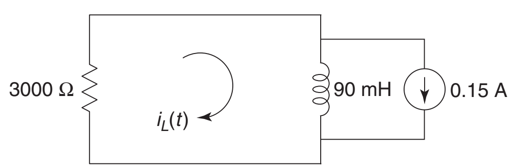


Fig. 6.94

Writing the KVL equation for $t > 0$,

$$-3000 i_L - 90 \times 10^{-3} \frac{di_L}{dt} = 0$$

$$\frac{di_L}{dt} + 33.33 \times 10^3\, i_L = 0$$

Comparing with the differential equation $\frac{di}{dt}+Pi=0$,

$$P = 33.33\times10^3$$

The solution of this differential equation is given by,

$$i_L(t) = k\,e^{-Pt}$$

$$i_L(t) = k\,e^{-33.33\times10^3 t}$$

At $t = 0$, $i_L(0) = 0.15$ A

$$0.15 = k$$

$$i_L(t) = 0.15\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Also,

$$v_L(t) = L\frac{di_L}{dt}$$

$$= 90\times10^{-3}\frac{d}{dt}(0.15\,e^{-33.33\times10^3 t})$$

$$= -90\times10^{-3}\times0.15\times33.33\times10^3\times e^{-33.33\times10^3 t}$$

$$= -450\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Example 6.29 *In the network of Fig. 6.95, the switch is open for a long time and it closes at t = 0. Find i (t).*

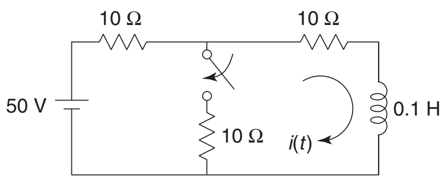


Fig. 6.95

Solution At $t = 0^-$, the network is shown in Fig. 6.96.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

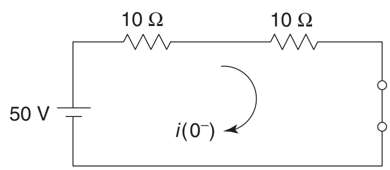


Fig. 6.96

$$i(0^-) = \frac{50}{10+10} = 2.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.97.

For $t > 0$, representing the network to the left of the inductor by Thevenin's equivalent network,

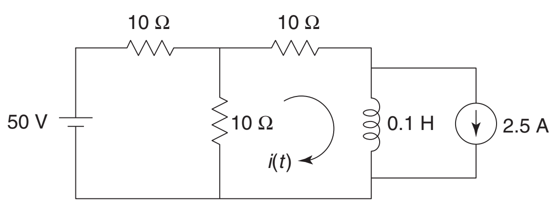


Fig. 6.97

$$V_{eq} = 50\times\frac{10}{10+10} = 25 \text{ V}$$

$$R_{eq} = (10\,\|\,10)+10 = 15\ \Omega$$

For $t > 0$, Thevenin's equivalent network is shown in Fig. 6.98.

Writing the KVL equation for $t > 0$,

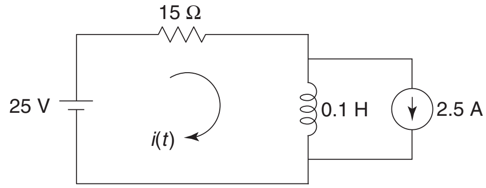


Fig. 6.98

$$25 - 15i - 0.1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 150i = 250$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 150, \quad Q = 250$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-150t}\int 250\, e^{150t}\, dt + k\, e^{-Pt}$$

$$= \frac{250}{150} + k\, e^{-150t}$$

$$= 1.667 + k\, e^{-150t}$$

At $t = 0$, $i(0) = 2.5$ A

$$2.5 = 1.667 + k$$

$$k = 0.833$$

$$i(t) = 1.667 + 0.833\, e^{-150t} \qquad \text{for } t > 0$$

Example 6.30 *In Fig. 6.99, the switch is closed at t = 0. Find i (t) for t > 0.*

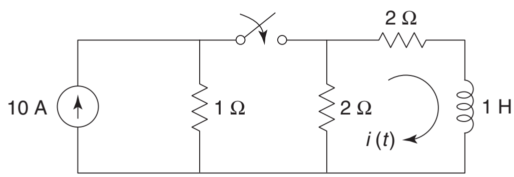


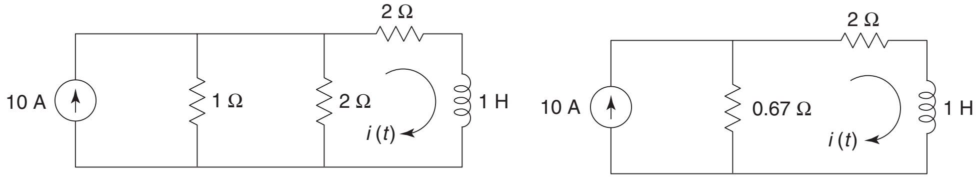
Fig. 6.99

Solution At $t = 0^-$, $\quad i(0^-) = 0$

Since current through inductor cannot change instantaneously,

$$i(0^+) = 0$$

For $t > 0$, simplifying the network by source-transformation technique as shown in Fig. 6.100.

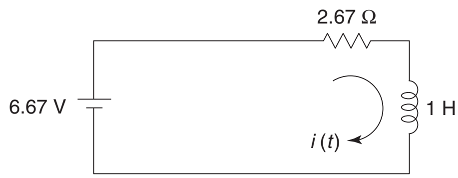


(a) (b)

Writing the KVL equation for $t > 0$,

$$6.67 - 2.67i - 1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2.67i = 6.67$$



(c)

Fig. 6.100

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 2.67,\ Q = 6.67$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt} \\ &= e^{-2.67t}\int 6.67\,e^{2.67t}\,dt + k\,e^{-2.67t} \\ &= \frac{6.67}{2.67} + k\,e^{-2.67t} \\ &= 2.5 + k\,e^{-2.67t} \end{aligned}$$

At $t = 0$, $i\,(0) = 0$

$$\begin{aligned} 0 &= 2.5 + k \\ k &= -2.5 \\ i(t) &= 2.5 - 2.5\,e^{-2.67t} \\ &= 2.5(1 - e^{-2.67t}) \qquad \text{for } t > 0 \end{aligned}$$

Example 6.31 *Find the current i (t) for t > 0.*

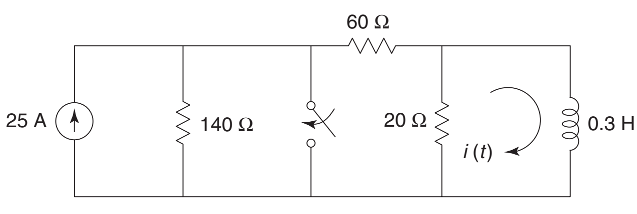


Fig. 6.101

Solution At $t = 0^-$, the inductor acts as a short circuit. Simplifying the network as shown in Fig. 6.102.

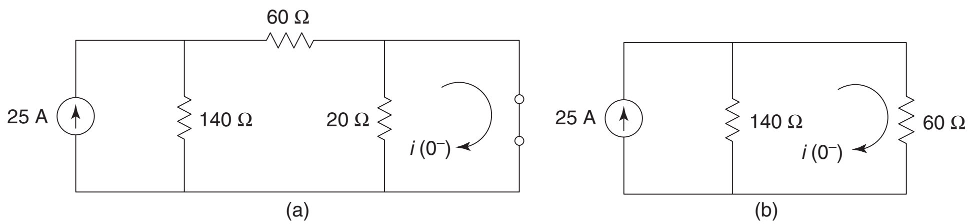


Fig. 6.102

$$i(0^-) = 25 \times \frac{140}{140+60} = 17.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 17.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.103.

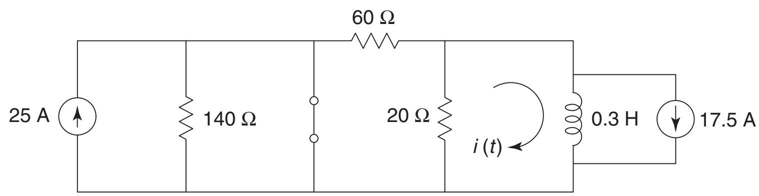


Fig. 6.103

Simplifying the network by source transformation as shown in Fig. 6.104,

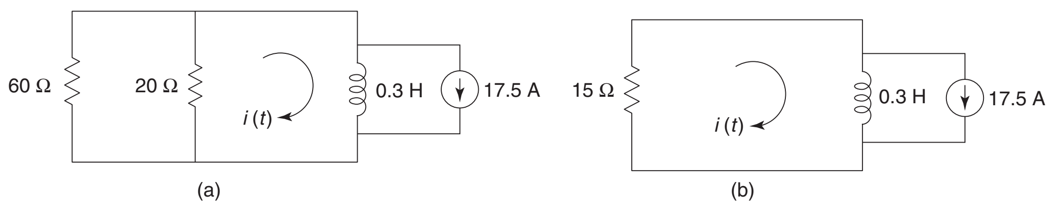


Fig. 6.104

Writing the KVL equation for $t > 0$,

$$-15i - 0.3\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 50i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = 50$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50t}$$

At $t = 0$, $i\,(0) = 17.5$ A

$$k = 17.5$$

$$i(t) = 17.5\,e^{-50t} \qquad \text{for } t > 0$$

Example 6.32 *In the network of Fig. 6.105, the switch is in position 'a' for a long time. At t = 0, the switch is moved from a to b. Find v_2 (t). Assume that the initial current in the 2 H inductor is zero.*

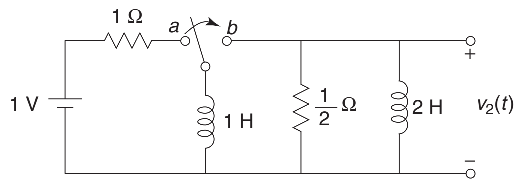


Fig. 6.105

Solution At $t = 0^-$, the switch is in the position *a*. The network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Current through the 1 H inductor is given by

$$i(0^-) = \frac{1}{1} = 1\text{ A}$$

$$v_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1\text{ A}$$

$$v_2(0^+) = -1 \times \frac{1}{2} = -0.5\text{ V}$$

For $t > 0$, the network is shown in Fig. 6.106.

Writing the KCL equation for $t > 0$,

$$\frac{1}{1}\int_0^t v_2 dt + 1 + \frac{v_2}{\frac{1}{2}} + \frac{1}{2}\int_0^t v_2 dt = 0 \qquad \ldots\text{(i)}$$

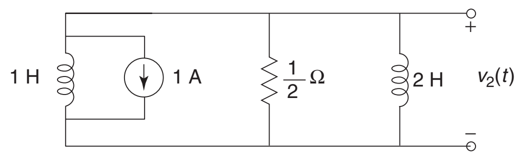


Fig. 6.106

Differentiating Eq. (i),

$$v_2 + 2\frac{dv_2}{dt} + \frac{1}{2}v_2 = 0$$

$$\frac{dv_2}{dt} + \frac{3}{4}v_2 = 0$$

Comparing with the differential equation $\dfrac{dv}{dt} + Pv = 0$,

$$P = \frac{3}{4}$$

The solution of this differential equation is given by,

$$v_2(t) = K e^{-Pt} = k e^{-\frac{3}{4}t}$$

At $t = 0$, $v_2(0) = -0.5$ V

$$-0.5 = k e^{\circ}$$
$$k = -0.5$$
$$v_2(t) = -0.5\, e^{-\frac{3}{4}t} \qquad \text{for } t > 0$$

Example 6.33 *In the network shown in Fig. 6.107, a steady-state condition is achieved with switch open. At $t = 0$ switch is closed. Find $v_a(t)$.*

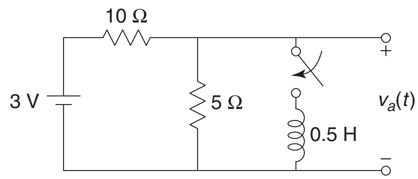


Fig. 6.107

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = 0$$
$$v_a(0^-) = 3 \times \frac{5}{10+5} = 1\text{ V}$$

Since current through inductor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v_a(0^+) = 1\text{ V}$$

For $t > 0$, the network is shown in Fig. 6.108.

Writing the KCL equation for $t > 0$,

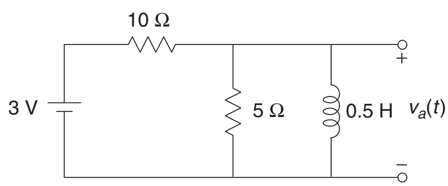


Fig. 6.108

$$\frac{1}{0.5}\int_0^t v_a\, dt + \frac{v_a}{5} + \frac{v_a - 3}{10} = 0$$

Differentiating Eq. (i),

$$2v_a + 0.2\frac{dv_a}{dt} + 0.1\frac{dv_a}{dt} = 0$$
$$\frac{dv_a}{dt} + \frac{20}{3}v_a = 0$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = 0$,

$$P = \frac{20}{3}$$

The solution of this differential equation is given by,

$$v_a(t) = k\,e^{-Pt} = k\,e^{-\frac{20}{3}t}$$

At $t = 0$, $v_a(0) = 1$ V

$$1 = k$$

$$v_a(t) = e^{-\frac{20}{3}t} \qquad \text{for } t > 0$$

Example 6.34 *In the network of Fig. 6.109, determine currents $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$.*

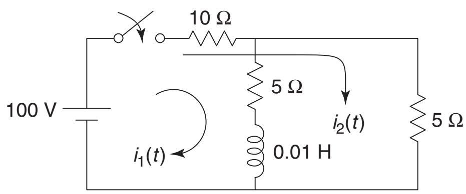


Fig. 6.109

Solution At $t = 0^-$,

$$i_1(0^-) = i_2(0^-) = 0$$

At $t = 0^+$,

$$i_1(0^+) = 0$$

$$i_2(0^+) = \frac{100}{15} = 6.67 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.110.

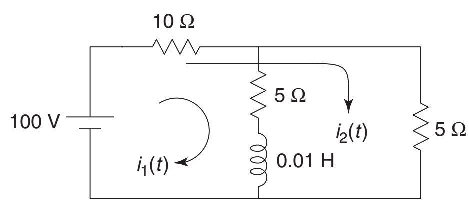


Fig. 6.110

Writing the KVL equations for $t > 0$,

$$100 - 10(i_1 + i_2) - 5i_1 - 0.01\frac{di_1}{dt} = 0 \qquad \ldots\text{(i)}$$

and

$$100 - 10(i_1 + i_2) - 5i_2 = 0 \qquad \ldots\text{(ii)}$$

From Eq. (ii),

$$i_2 = \frac{100 - 10i_1}{15}$$

Substituting in Eq. (i),

$$\frac{di_1}{dt} + 833i_1 = 3333$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 833, Q = 3333$$

The solution of this differential equation is given by,

$$\begin{aligned} i_1(t) &= e^{-Pt}\int Q e^{Pt} dt + k e^{-Pt} \\ &= e^{-833t}\int 3333\, e^{833t} dt + k e^{-833t} \\ &= \frac{3333}{833} + k e^{-833t} \\ &= 4 + k e^{-833t} \end{aligned}$$

At $t = 0$, $i_1(0) = 0$

$$\begin{aligned} 0 &= 4 + k \\ k &= -4 \\ i_1(t) &= 4 - 4e^{-833t} \\ &= 4(1 - e^{-833t}) \qquad \text{for } t > 0 \\ i_2(t) &= \frac{100 - 10 i_1}{15} \\ &= \frac{100 - 10(4 - 4e^{-833t})}{15} \\ &= 4 + 2.67 e^{-833t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.35 *The switch in the network shown in Fig. 6.111 is closed at $t = 0$. Find $v_2(t)$ for all $t > 0$. Assume zero initial current in the inductor.*

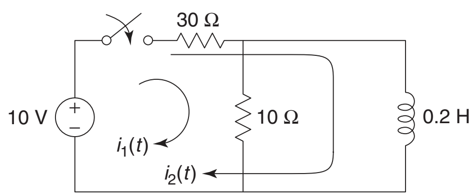


Fig. 6.111

Solution At $t = 0^-$,

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_2(0^+) = 0$$
$$i_1(0^+) = \frac{10}{30 + 10} = 0.25 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.112.

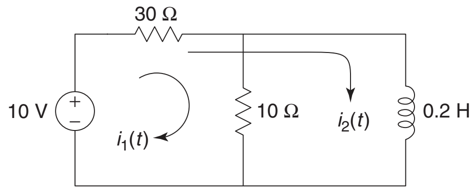


Fig. 6.112

Writing the KVL equations for $t > 0$,

$$10 - 30(i_1 + i_2) - 10\, i_1 = 0 \qquad \ldots\text{(i)}$$

and $$10 - 30(i_1 + i_2) - 0.2\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

From Eq. (i),

$$i_1 = \frac{10 - 30\, i_2}{40} = 0.25 - 0.75\, i_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) into Eq. (ii),

$$\frac{di_2}{dt} + 37.5\, i_2 = 2.5$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 37.5, Q = 2.5$$

The solution of this differential equation is given by,

$$\begin{aligned} i_2(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-37.5t}\int 2.5\, e^{37.5t}\, dt + k\, e^{-37.5t} \\ &= \frac{2.5}{37.5} + k\, e^{-37.5t} \\ &= 0.067 + k\, e^{-37.5t} \end{aligned}$$

At $t = 0, i_2(0) = 0$

$$\begin{aligned} 0 &= 0.067 + k \\ k &= -0.067 \\ i_2(t) &= 0.067 - 0.067\, e^{-37.5t} \\ v_2(t) &= 0.2\frac{di_2}{dt} \\ &= 0.2\frac{d}{dt}(0.067 - 0.067\, e^{-37.5t}) \\ &= 0.5\, e^{-37.5t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.36 *For the network shown in Fig. 6.113, find the current $i(t)$ when the switch is changed from the position 1 to 2 at $t = 0$.*

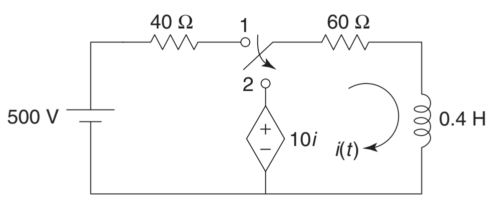


Fig. 6.113

Solution At $t = 0^-$, the network is shown in Fig. 6.114.
At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

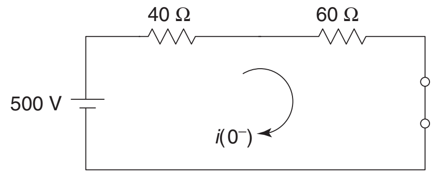


Fig. 6.114

$$i(0^-) = \frac{500}{40+60} = 5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.115.
Writing the KVL equation for $t > 0$,

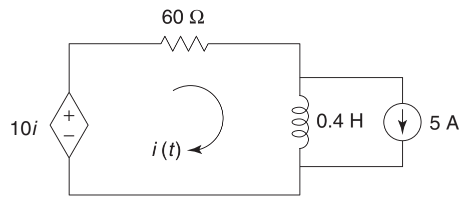


Fig. 6.115

$$10i - 60i - 0.4\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 125i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0,$

$$P = 125$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-125t}$$

At $t = 0, i(0) = 5$ A

$$5 = k$$

$$i(t) = 5e^{-125t} \qquad \text{for } t > 0$$

Example 6.37 *For the network shown in Fig. 6.116, find the current in the 20 Ω resistor when the switch is opened at $t = 0$.*

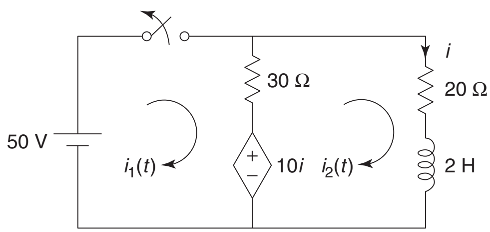


Fig. 6.116

Solution At $t = 0^-$, the network is shown in Fig. 6.117.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

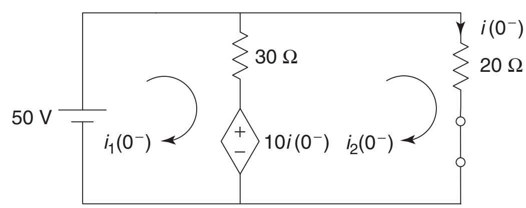


Fig. 6.117

$$i(0^-) = i_2(0^-)$$

Writing the KVL equations at $t = 0^-$,

$$50 - 30(i_1 - i_2) - 10i_2 = 0$$
$$10i_2 - 30(i_2 - i_1) - 20i_2 = 0$$

Solving these equations,

$$i_1(0^-) = 3.33 \text{ A}$$
$$i_2(0^-) = 2.5 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i_2(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.118.

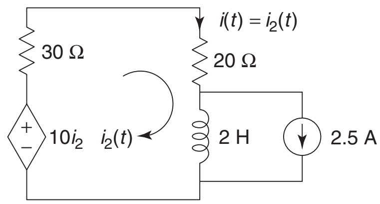


Fig. 6.118

Writing the KVL equation for $t > 0$,

$$10i_2 - 30i_2 - 20i_2 - 2\frac{di_2}{dt} = 0$$
$$\frac{di_2}{dt} + 20i_2 = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = 20$$

The solution of this differential equation is given by,

$$i_2(t) = k\,e^{-Pt} = k\,e^{-20t}$$

At $t = 0, i_2(0) = 2.5$ A

$$2.5 = k$$
$$i_2(t) = 2.5\,e^{-20t} \qquad \text{for } t > 0$$

Example 6.38 *In the network of Fig. 6.119, an exponential voltage $v(t) = 4\,e^{-3t}$ is applied at $t = 0$. Find the expression for current $i(t)$. Assume zero current through inductor at $t = 0$.*

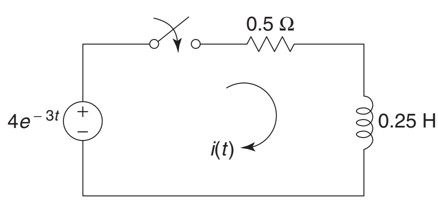


Fig. 6.119

Solution At $t = 0^-$, $\quad i(0^-) = 0$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 0$$

Writing the KVL equation for $t > 0$,

$$4e^{-3t} - 0.5i - 0.25\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2i = 16e^{-3t}$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 2, \quad Q = 16\,e^{-3t}$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q e^{Pt}\, dt + k\,e^{-Pt} \\ &= e^{-2t}\int 16e^{-3t}\, e^{2t}\, dt + +k\,e^{-2t} \\ &= 16\,e^{-2t}\int e^{-t}\, dt + k\,e^{-2t} \\ &= -16\,e^{-3t} + k\,e^{-2t} \end{aligned}$$

At $t = 0, i(0) = 0$

$$0 = -16 + k$$

$$k = 16$$

$$i(t) = -16\,e^{-3t} + 16\,e^{-2t} \qquad \text{for } t > 0$$

Example 6.39 *For the network shown in Fig. 6.120, a sinusoidal voltage source $v = 150\sin(500t + \theta)$ volts is applied at a time when $\theta = 0$. Find the expression for the current $i(t)$.*

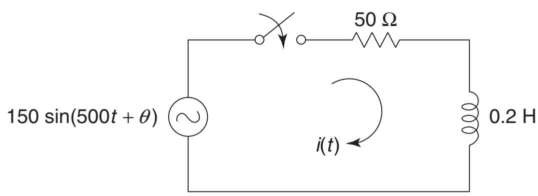


Fig. 6.120

Solution

Writing the KVL equation for $t > 0$,

$$150\sin(500t + \theta) - 50i - 0.2\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 250i = 750\sin(500t + \theta)$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 250, \quad Q = 750\sin(500t + \theta)$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q e^{Pt}\,dt + k e^{-Pt}$$

$$= e^{-250t}\int 750\sin(500t+\theta)e^{250t} + k e^{-250t}$$

$$= 750\, e^{-250t}\left[\frac{e^{250t}}{(250)^2+(500)^2}\{250\sin(500t+\theta) - 500\cos(500t+\theta)\}\right] + k e^{-250t}$$

$$= 0.6\sin(500t+\theta) - 1.2\cos(500t+\theta) + k e^{-250t}$$

Let $$A\cos\phi = 0.6$$

and $$A\sin\phi = 1.2$$

$$A^2\cos^2\phi + A^2\sin^2\phi = (0.6)^2 + (1.2)^2 = 1.8$$

$$A = 1.342$$

and $$\phi = \tan^{-1}\left(\frac{1.2}{0.6}\right) = 63.43°$$

$$i(t) = 1.342\cos(63.43°)\sin(500t+\theta) - 1.342\sin(63.43°)\cos(500t+\theta) + k e^{-250t}$$

$$i(t) = 1.342\sin(500t+\theta-63.43°) + k e^{-250t}$$

At $t = 0, \theta = 0, i(0) = 0$

$$0 = 1.342\sin(-63.43°) + k$$

$$k = 1.2$$

$$i(t) = 1.342\sin(500t+\theta-63.43°) + 1.2\, e^{-250t} \qquad \text{for } t > 0$$

Example 6.40 *For the network shown in Fig. 6.121, find the transient current when the switch is moved from the position 1 to 2 at $t = 0$. The network is in steady state with the switch in the position 1. The voltage applied to the network is $v = 150\cos(200t + 30°)\,V$.*

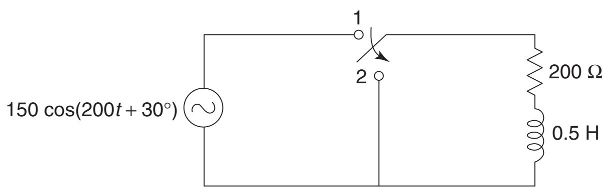


Fig. 6.121

Solution At $t = 0^-$, the network is shown in Fig 6.122.

At $t = 0^-$ the network attains steady-state condition.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{150\angle 30°}{200 + j\times 200\times 0.5} = 0.67\angle 3.43° \text{ A}$$

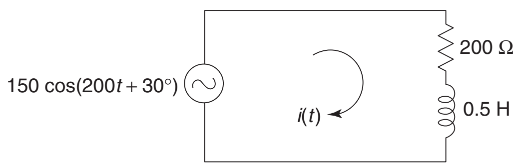


Fig. 6.122

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 0.67\cos(200t + 3.43°) \qquad \ldots\text{(i)}$$

For $t > 0$, the network is shown in Fig. 6.123.

Writing the KVL equation for $t > 0$,

$$-200\,i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 400\,i = 0$$

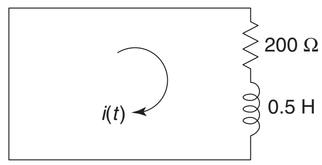


Fig. 6.123

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 400$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-pt} = k\,e^{-400t} \qquad \text{...(ii)}$$

From Eqs (i) and (ii),

$$0.67\cos(200t + 3.43°) = k\,e^{-400t}$$

$$0.67\cos(3.43°) = k$$

At $t = 0$,

$$k = 6.67$$

$$i(t) = 0.67\,e^{-400t} \qquad \text{for } t > 0$$

6.4 RESISTOR–CAPACITOR CIRCUIT

Consider a series RC circuit as shown in Fig. 6.124. The switch is closed at time $t = 0$. The capacitor is initially uncharged. Applying KVL to the circuit for $t > 0$,

$$V - Ri - \frac{1}{C}\int_0^1 i\,dt = 0$$

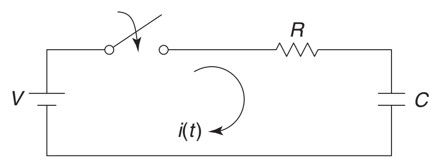


Fig. 6.124 *Series RC circuit*

Differentiating the above equation,

$$0 - R\frac{di}{dt} - \frac{i}{C} = 0$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

This is a linear differential equation of first order. The variables may be separated to solve the equation.

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating both the sides,

$$l_n i = -\frac{1}{RC}t + k$$

The constant k can be evaluated from initial condition. In the circuit shown, the switch is closed at $t = 0$. Since the capacitor never allows sudden change in voltage, it will act as short circuit at $t = 0^+$. Hence, current in the circuit at $t = 0^+$ is $\frac{V}{R}$.

Setting $i = \frac{V}{R}$ at $t = 0$,

$$l_n \frac{V}{R} = k$$

$$l_n i = -\frac{1}{RC}t + l_n \frac{V}{R}$$

$$l_n i - l_n \frac{V}{R} = -\frac{1}{RC}t$$

$$l_n \frac{i}{\left(\frac{V}{R}\right)} = -\frac{1}{RC}t$$

$$\frac{i}{\frac{V}{R}} = e^{-\frac{1}{RC}t}$$

$$i = \frac{V}{R}e^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

When the switch is closed, the response decays with time as shown in Fig. 6.125(a).

The term RC is called time constant and is denoted by T.

$$T = RC$$

After 5 time constants, the current drops to 99 per cent of its initial value.

The voltage across the resistor is

$$v_R = Ri = R\frac{V}{R}e^{-\frac{1}{RC}t}$$

$$= Ve^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

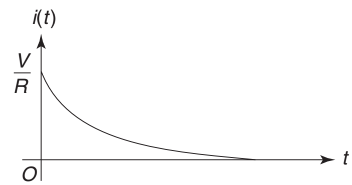


Fig. 6.125(a) *Current response of series RC circuit*

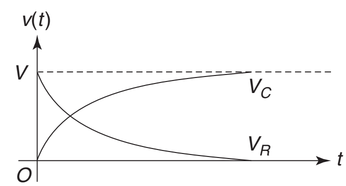


Fig. 6.125(b) *Voltage response of series RC circuit*

Similarly, the voltage across the capacitor is

$$v_C = \frac{1}{C}\int_0^t i\,dt$$

$$= \frac{1}{C}\int_0^t \frac{V}{R}e^{-\frac{1}{RC}t}$$

$$= -Ve^{-\frac{1}{RC}t} + k$$

At $t = 0$, $v_C(0) = 0$

$$k = V$$

Hence,
$$v_C = V\left(1 - e^{-\frac{1}{RC}t}\right) \quad \text{for } t > 0$$

Example 6.41 *The switch in the circuit of Fig. 6.126 is moved from the position 1 to 2 at $t = 0$. Find $v_C(t)$.*

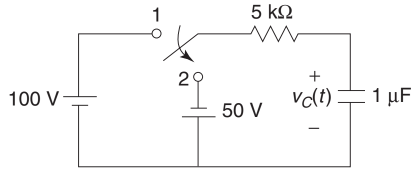


Fig. 6.126

Solution At $t = 0^-$, the network is shown in Fig. 6.127.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 100 \text{ V}$$

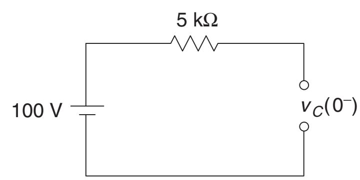


Fig. 6.127

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 100 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.128.

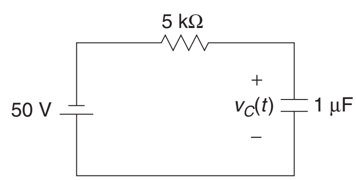


Fig. 6.128

Writing the KCL equation for $t > 0$,

$$1 \times 10^{-6} \frac{dv_C}{dt} + \frac{v_C + 50}{5000} = 0$$

$$\frac{dv_C}{dt} + 200 v_C = 10^4$$

Comparing with the differential equation $\dfrac{dv}{dt} + Pv = Q$,

$$P = 200, \quad Q = 10^4$$

Solution of this differential equation is given by,

$$v_C(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-200t} \int 10^4\, e^{200t}\, dt + k\, e^{-200t}$$

$$= \frac{10^4}{200} + k\, e^{-200t}$$

$$= -50 + k\, e^{-200t}$$

At $t = 0$, $v_C(0) = 100$ V

$$100 = -50 + k$$
$$k = 150$$
$$v_C(t) = -50 + 150\,e^{-200t} \qquad \text{for } t > 0$$

Example 6.42 *In the network shown in Fig. 6.129, the switch closes at $t = 0$. The capacitor is initially uncharged. Find $v_C(t)$ and $i_C(t)$.*

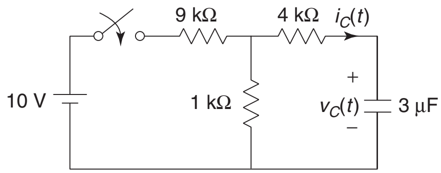


Fig. 6.129

Solution At $t = 0^-$, the capacitor is uncharged. Hence, it acts as a short circuit.

$$v_C(0^-) = 0$$
$$i_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.130.

Since voltage across the capacitor cannot change instantaneously,

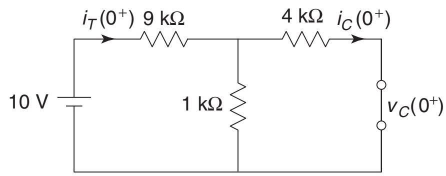


Fig. 6.130

$$v_C(0^+) = 0$$

At $t = 0^+$, $$i_T(0^+) = \left[\frac{10}{9\text{ k} + (4\text{ k} \,\|\, 1\text{ k})}\right] = \frac{10}{9.8\text{ k}} = 1.02 \text{ mA}$$

$$i_C(0^+) = 1.02\text{ m} \times \frac{1\text{ k}}{1\text{ k} + 4\text{ k}} = 0.204 \text{ mA}$$

For $t > 0$, the network is shown in Fig. 6.131.

For $t > 0$, representing the network to the left of the capacitor by Thevenin's equivalent network,

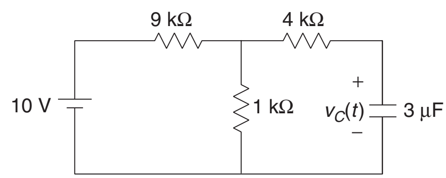


Fig. 6.131

$$V_{eq} = 10 \times \frac{1\text{ k}}{9\text{ k} + 1\text{ k}} = 1 \text{ V}$$
$$R_{eq} = (9\text{ k} \,\|\, 1\text{ k}) + 4\text{ k} = 4.9 \text{ k}\Omega$$

For $t > 0$, Thevenin's equivalent network is shown in Fig. 6.132.

Writing the KCL equation for $t > 0$,

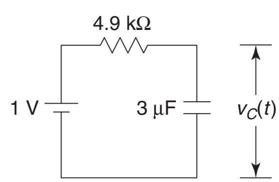


Fig. 6.132

$$3 \times 10^{-6}\frac{dv_C}{dt} + \frac{v_C - 1}{4.9 \times 10^3} = 0$$
$$\frac{dv_C}{dt} + 68.02\, v_C = 68.02$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 68.02, \quad Q = 68.02$$

The solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-68.02t}\int 68.02\, e^{68.02t}\, dt + k\, e^{-68.02t}$$

$$= 1 + k\, e^{-68.02t}$$

At $t = 0, v_C(0) = 0$

$$0 = 1 + k$$

$$k = -1$$

$$v_C(t) = 1 - e^{-68.02t} \qquad \text{for } t > 0$$

$$i_C(t) = C\frac{dv_C}{dt}$$

$$= 3\times 10^{-6}\frac{d}{dt}(1 - e^{-68.02t})$$

$$= 3\times 10^{-6} \times 68.02 e^{-68.02t}$$

$$= 204.06\times 10^{-6} e^{-68.02t} \qquad \text{for } t > 0$$

Example 6.43 *For the network shown in Fig. 6.133, the switch is open for a long time and closes at t = 0. Determine $v_C(t)$.*

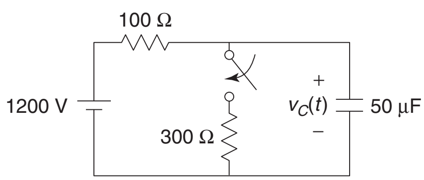


Fig. 6.133

Solution At $t = 0^-$, the network is shown in Fig. 6.134.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 1200 \text{ V}$$

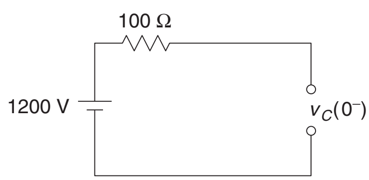


Fig. 6.134

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 1200 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.135.

Writing the KCL equation for $t > 0$,

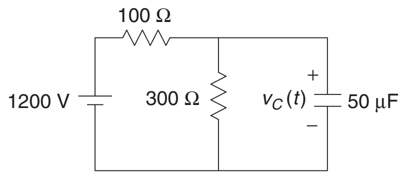


Fig. 6.135

$$50 \times 10^{-6} \frac{dv_C}{dt} + \frac{v_C}{300} + \frac{v_C - 1200}{100} = 0$$

$$\frac{dv_C}{dt} + 266.67\, v_C = 0.24 \times 10^6$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 266.67, \quad Q = 0.24 \times 10^6$$

The solution of this differential equation is given by,

$$v_C(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-266.67t} \int 0.24 \times 10^6\, e^{266.67t}\, dt + k\, e^{-266.67t}$$

$$= \frac{0.24 \times 10^6}{266.67} + k\, e^{-266.67t}$$

$$= 900 + k\, e^{-266.67t}$$

At $t = 0, v_C(0) = 1200$ V

$$1200 = 900 + k$$

$$k = 300$$

$$v_C(t) = 900 + 300\, e^{-266.67t} \qquad \text{for } t > 0$$

Example 6.44 *In Fig. 6.136, the switch is closed at $t = 0$ Find $v_C(t)$ for $t > 0$.*

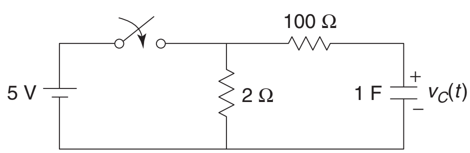


Fig. 6.136

Solution At $t = 0^-$, $\quad v_C(0^-) = 0$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

Since the resistor of 2 Ω is connected in parallel with the voltage source of 5 V, it becomes redundant.

For $t > 0$, the network is as shown in Fig. 6.137.

Writing KCL equation for $t > 0$,

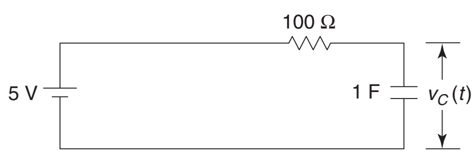


Fig. 6.137

$$\frac{v_C - 5}{100} + 1 \frac{dv_C}{dt} = 0$$

$$100 \frac{dv_C}{dt} + v_C = 5$$

$$\frac{dv_C}{dt} + 0.01 v_C = 0.05$$

Comparing with the differential equation $\dfrac{dv}{dt} + Pv = Q$,

$$P = 0.01, \quad Q = 0.05$$

The solution of this differential equation is given by,

$$\begin{aligned} v_C(t) &= e^{-Pt} \int Q e^{Pt}\, dt + k e^{-Pt} \\ &= e^{-0.01} \int 0.05\, e^{0.01t}\, dt + k e^{-0.01t} \\ &= \frac{0.05}{0.01} + k e^{-0.01t} \\ &= 5 + k e^{-0.01t} \end{aligned}$$

At $t = 0, v_C(0) = 0$

$$\begin{aligned} 0 &= 5 + k \\ k &= -5 \\ v_C(t) &= 5 - 5e^{-0.01t} \\ &= 5(1 - e^{-0.01t}) \qquad \text{for } t > 0 \end{aligned}$$

Example 6.45 *In the network shown, the switch is shifted to position b at t = 0. Find v (t) for t > 0.*

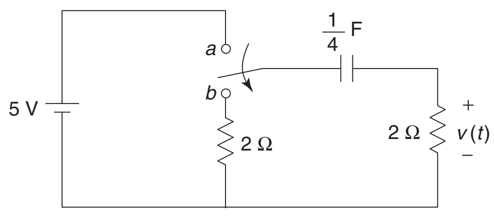


Fig. 6.138

Solution At $t = 0^-$, the network is shown in Fig. 6.139.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 5 \text{ V}$$

$$v(0^-) = 0$$

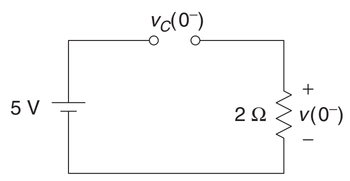


Fig. 6.139

At $t = 0^+$, the network is shown in Fig. 6.140.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$i(0^+) = -\frac{5}{4} = -1.25 \text{ A}$$

$$v(0^+) = -1.25 \times 2 = -2.5 \text{ V}$$

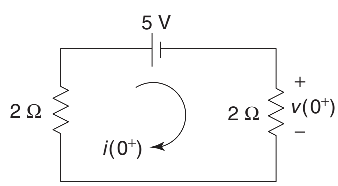


Fig. 6.140

For $t > 0$, the network is shown in Fig. 6.141.
Writing the KVL equation for $t > 0$,

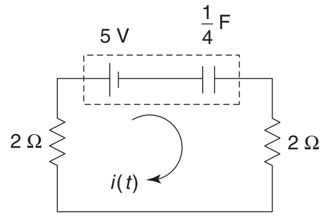


Fig. 6.141

$$-2i - 5 - \frac{1}{\frac{1}{4}}\int_0^t i\,dt - 2i = 0 \qquad \text{...(i)}$$

Differentiating Eq. (i),

$$-4\frac{di}{dt} - 4i = 0$$

$$\frac{di}{dt} + i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-t}$$

At $t = 0$, $i(0) = -1.25$ A

$$k = -1.25$$

$$i(t) = -1.25\,e^{-t} \quad \text{for } t > 0$$

$$v(t) = 2i(t)$$

$$= -2.5e^{-t} \quad \text{for } t > 0$$

Example 6.46 *In the network of Fig. 6.142, the switch is open for a long time and at $t = 0$, it is closed. Determine $v_2(t)$.*

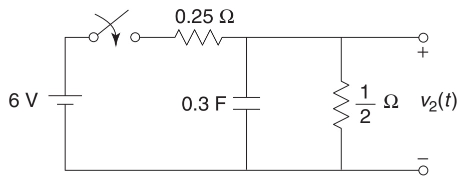


Fig. 6.142

Solution At $t = 0^-$, the switch is open.

$$v_2(0^-) = 0$$

Since voltage across capacitor cannot change instantaneously,

$$v_2(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.143.

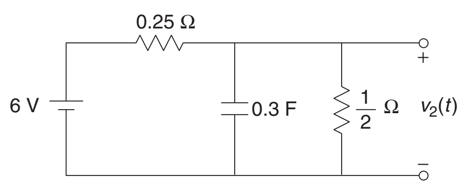


Fig 6.143

Writing KCL equation for $t > 0$,

$$\frac{v_2}{\frac{1}{2}} + 0.3\frac{dv_2}{dt} + \frac{v_2 - 6}{0.25} = 0$$

$$\frac{dv_2}{dt} + 20v_2 = 80$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 20, \quad Q = 80$$

The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-20t} \int 80\, e^{20t}\, dt + k\, e^{-20t} \\ &= \frac{80}{20} + k\, e^{-20t} \\ v_2 t &= 4 + k\, e^{-20t} \end{aligned}$$

At $t = 0, v_2(0) = 0$

$$\begin{aligned} 0 &= 4 + k \\ k &= -4 \\ v_2(t) &= 4 - 4e^{-20t} \\ &= 4(1 - e^{-20t}) \qquad \text{for } t > 0. \end{aligned}$$

Example 6.47 *The switch is moved from the position a to b at t = 0, having been in the position a for a long time before t = 0. The capacitor C_2 is uncharged at t = 0. Find i (t) and v_2 (t) for t > 0.*

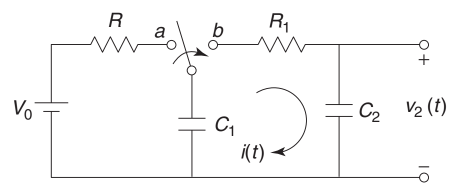


Fig. 6.144

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor C_1 acts as an open circuit and it will charge to V_0 volt.

$$v_{C_1}(0^-) = V_0$$
$$v_{C_2}(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_{C_1}(0^+) = V_0$$
$$v_{C_2}(0^+) = 0$$
$$i(0^+) = \frac{V_0}{R_1}$$

For $t > 0$, the network is shown in Fig. 6.145.

Writing the KVL equation for $t > 0$,

$$V_0 - \frac{1}{C_1}\int_0^t i\, dt - R_1 i - \frac{1}{C_2}\int_0^t i\, dt = 0 \qquad \text{...(i)}$$

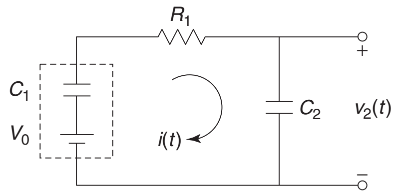


Fig. 6.145

Differentiating Eq. (i),

$$-\frac{i}{C_1} - R_1\frac{di}{dt} - \frac{i}{C_2} = 0$$

$$\frac{di}{dt} + \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right)i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right)$$

The solution of this differential equation is given by,

$$i(t) = k\, e^{-Pt} = k\, e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

At $t = 0$, $i(0) = \dfrac{V_0}{R_1}$

$$k = \frac{V_0}{R_1}$$

$$i(t) = \frac{V_0}{R_1} e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

$$= \frac{V_0}{R_1} e^{-\frac{1}{R_1 C}t} \qquad \text{where, } C = \frac{C_1 C_2}{C_1 + C_2}$$

$$v_2(t) = \frac{1}{C_2}\int_0^t i\, dt$$

$$= \frac{1}{C_2}\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{R_1 C}} dt$$

$$= \frac{V_0}{R_1 C_2} R_1 C\left(1 - e^{-\frac{1}{R_1 C}t}\right)$$

$$= \frac{V_0 C_1}{C_1 + C_2}\left(1 - e^{-\frac{t}{R_1 C}t}\right) \qquad \text{for } t > 0$$

Example 6.48 *For the network shown in Fig.6.146, the switch is opened at $t = 0$. Find v (t) for* $t > 0$.

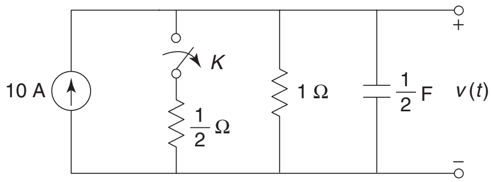


Fig. 6.146

Solution At $t = 0^-$, the network is shown in Fig. 6.147.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 0$$

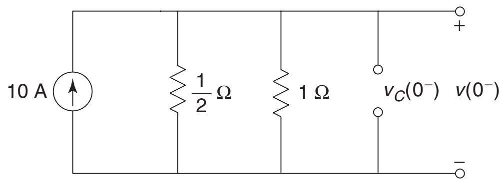


Fig. 6.147

Writing the KCL equation at $t = 0^-$,

$$\frac{v(0^-)}{1} + \frac{v(0^-)}{\frac{1}{2}} = 10$$

$$3v(0^-) = 10$$

$$v(0^-) = 3.33 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = v(0^+) = 3.33 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.148.

Writing the KCL equation for $t > 0$,

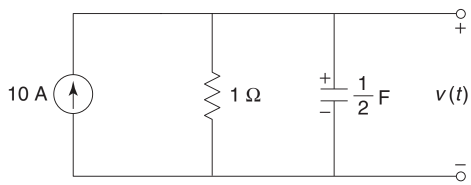


Fig. 6.148

$$\frac{1}{2}\frac{dv}{dt} + \frac{v}{1} = 10$$

$$\frac{dv}{dt} + 2v = 20$$

Comparing with the differential equation $\dfrac{dv}{dt} + Pv = Q$,

$$P = 2, \quad Q = 20$$

The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-2t}\int 20\, e^{-2t}\, dt + k\, e^{-2t} \\ &= \frac{20}{2} + k\, e^{-2t} \\ &= 10 + k\, e^{-2t} \end{aligned}$$

At $t = 0$, $v(0) = 3.33$ V

$$3.33 = 10 + k$$

$$k = 6.67$$

$$v(t) = 10 + 6.67\, e^{-2t}$$

Example 6.49 *For the network shown in Fig. 6.149, find the current i(t) when the switch is opened at t = 0.*

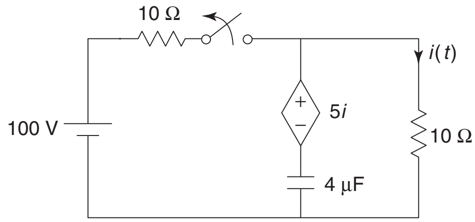


Fig. 6.149

Solution At $t = 0^-$, the network is shown in Fig. 6.150.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as open circuit.

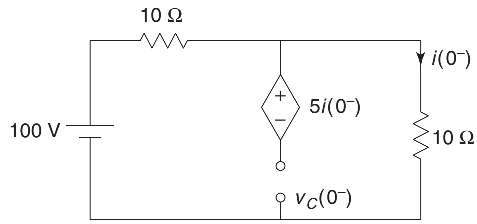


Fig. 6.150

$$i(0^-) = \frac{100}{10+10} = 5 \text{ A}$$

$$v_C(0^-) = 100 - 10i(0^-) - 5i(0^-) = 100 - 10(5) - 5(5) = 25 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.151.

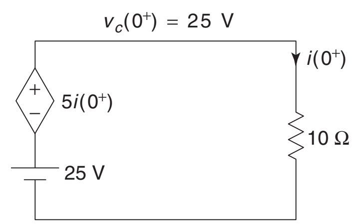


Fig. 6.151

$$25 + 5i(0^-) - 10i(0^+) = 0$$

$$i(0^+) = 5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.152.

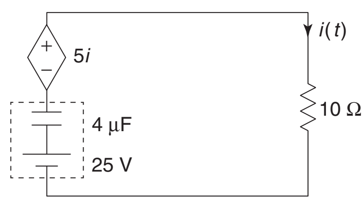


Fig. 6.152

$$25 - \frac{1}{4 \times 10^{-6}} \int_0^t i\, dt + 5i - 10i = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$0 - 0.25 \times 10^6\, i + 5\frac{di}{dt} - 10\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 50000\, i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = 50000$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50000\,t}$$

At $t = 0,\quad i(0) = 5$ A

$$5 = k$$

$$i(t) = 5\,e^{-50000\,t} \qquad \text{for } t > 0$$

Example 6.50 *For the network shown in Fig. 6.153, find the current i(t) when the switch is opened at t = 0.*

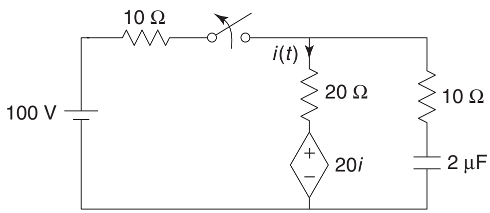


Fig. 6.153

Solution At $t = 0^-$, the network is shown in Fig. 6.154. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

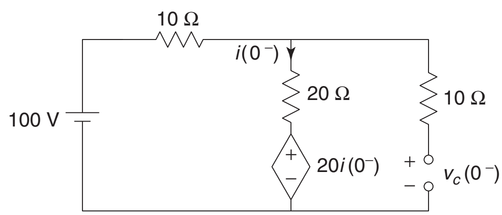


Fig. 6.154

Writing the KVL equation at $t = 0^-$,

$$100 - 10\,i(0^-) - 20\,i(0^-) - 20\,i(0^-) = 0$$

$$i(0^-) = 2 \text{ A}$$

Also, $$20\,i(0^-) + 20\,i(0^-) - 0 - v_C(0^-) = 0$$

$$v_C(0^-) = 40\,i(0^-) = 40(2) = 80 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.155.

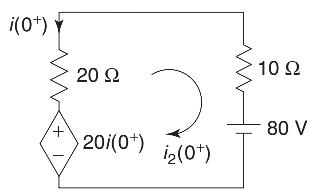


Fig. 6.155

From Fig. 6.155, $i(0^+) = -i_2(0^+)$

$$20\,i(0^+) - 20\,i_2(0^+) - 10\,i_2(0^+) - 80 = 0$$

$$20\,i(0^+) + 20\,i(0^+) + 10\,i(0^+) - 80 = 0$$

$$i(0^+) = 1.6 \text{ A}$$

$$v_C(0^+) = 80 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.156.

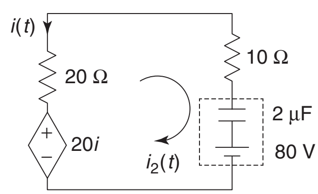


Fig. 6.156

From Fig. 6.156, $i(t) = -i_2(t)$

Writing the KVL equation for $t > 0$,

$$20i - 20i_2 - 10i_2 - \frac{1}{2\times 10^{-6}}\int_0^t i_2\,dt - 80 = 0$$

$$20i + 20i_2 + 10i_2 + \frac{1}{2\times10^{-6}}\int_0^t i\,dt - 80 = 0$$

$$50i + \frac{1}{2\times10^{-6}}\int i\,dt - 80 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$50\frac{di}{dt} + 5\times10^5 i = 0$$

$$\frac{di}{dt} + 1\times10^4 i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0,$

$$P = 1\times10^4$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-1\times10^4 t}$$

At $t = 0,\quad i(0) = 1.6$ A

$$1.6 = k$$

$$i(t) = 1.6\,e^{-1\times10^4 t} \qquad \text{for } t > 0$$

Example 6.51 *In the network of Fig. 6.157, an exponential voltage $4e^{-5t}$ is applied at time $t = 0$. Find the expression for current $i(t)$. Assume zero voltage across the capacitor at $t = 0$.*

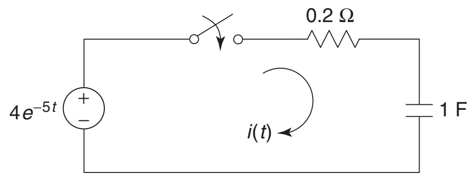


Fig. 6.157

Solution At $t = 0^-$,

$$v_C(0^-) = 0$$

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.158.
Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{4}{0.2} = 20 \text{ A}$$

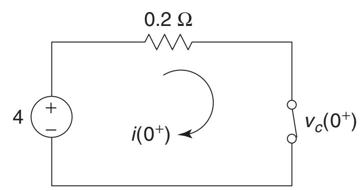


Fig. 6.158

Writing the KVL equation for $t > 0$,

$$4e^{-5t} - 0.2i - \frac{1}{1}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-20e^{-5t} - 0.2\frac{di}{dt} - i = 0$$

$$\frac{di}{dt} + 5i = -100e^{-5t}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 5, \quad Q = -100e^{-5t}$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-5t}\int -100e^{-5t} e^{5t}\, dt + k\, e^{-5t}$$

$$= -100t\, e^{-5t} + k\, e^{-5t}$$

At $t = 0$, $i(0) = 20$ A

$$20 = k$$

$$i(t) = -100\, t\, e^{-5t} + 20\, e^{-5t} \qquad \text{for } t > 0$$

Example 6.52 *In the network shown in Fig. 6.159, the switch is closed at $t = 0$ connecting a source e^{-t} to the network. At $t = 0$, $v_C(0) = 0.5$ V. Determine $v(t)$.*

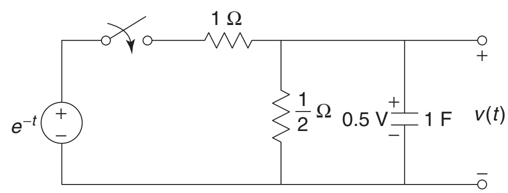


Fig. 6.159

Solution At $t = 0^-$, $$v(0) = v_C(0^-) = 0.5 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = v_C(0^+) = 0.5 \text{ V}$$

Writing the KCL equation for $t > 0$,

$$\frac{v - e^{-t}}{1} + \frac{v}{\frac{1}{2}} + 1\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + 3v = e^{-t}$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 3, \quad Q = e^{-t}$$

The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-3t}\int e^{-t}\, e^{3t}\, dt + k\, e^{-3t} \\ &= e^{-3t}\int e^{2t}\, dt + k\, e^{-3t} \\ &= \frac{1}{2}e^{-t} + k\, e^{-3t} \end{aligned}$$

At $t = 0$, $\quad v(0) = 0.5$ V

$$\begin{aligned} 0.5 &= \frac{1}{2} + k \\ k &= 0 \\ v(t) &= 0.5\, e^{-t} \end{aligned}$$

Example 6.53 *In the network shown in Fig. 6.160, a sinusoidal voltage $v = 100 \sin(500t + \theta)$ volts is applied to the circuit at a time corresponding to $\theta = 45°$. Obtain the expression for the current $i(t)$.*

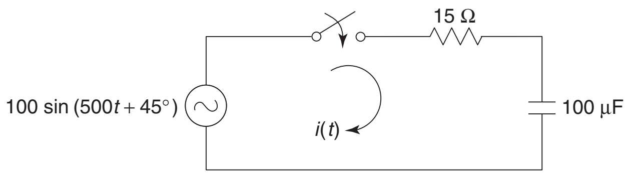


Fig. 6.160

Solution

Writing the KVL equation for $t > 0$,

$$100\sin(500t + 45°) - 15i - \frac{1}{100\times 10^{-6}}\int_0^t i\, dt = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$\begin{aligned} (100)(500)\cos(500t + 45°) - 15\frac{di}{dt} - 10^4 i &= 0 \\ \frac{di}{dt} + 666.67i &= 3333.33\cos(500t + 45°) \end{aligned}$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 666.67,\ Q = 3333.33\cos(500t + 45°)$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-666.67t}\int 3333.33\cos(500t + 45°)\, e^{666.67t} + k\, e^{-666.67t} \end{aligned}$$

$$= 3333.33\, e^{-666.67t}\left[\frac{e^{666.67t}}{(666.67)^2+(500)^2}\{666.67\cos(500t+45°)+500\sin(500t+45°)\}\right]+k\,e^{-666.67t}$$

$$= 3.2\cos(500t+45°)+2.4\sin(500t+45°)+k\,e^{-666.67t}$$

Let $$A\sin\phi = 3.2$$

and $$A\cos\phi = 2.4$$

$$A^2\sin^2\phi + A^2\cos^2\phi = (3.2)^2+(2.4)^2 = 16$$

$$A = 4$$

and $$\phi = \tan^{-1}\left(\frac{3.2}{2.4}\right) = 53.13°$$

$$i(t) = 4\sin(53.13°)\cos(500t+45°)+4\cos(53.13°)\sin(500t+45°)+k\,e^{-666.67t}$$

$$= 4\sin(500t+98.13°)+k\,e^{-666.67t}$$

Putting $t = 0$, in Eq. (i),

$$100\sin(45°) - 15\,i(0) - 0 = 0$$

$$i(0) = 4.71$$

$$4.71 = 4\sin(98.13°)+k$$

$$k = 0.75$$

$$i(t) = 4\sin(500t+98.13°)+0.75\,e^{-666.67t} \qquad \text{for } t>0$$

Example 6.54 *In the network of Fig. 6.161, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression $i(t)$, when $\omega = 1000$ rad/s.*

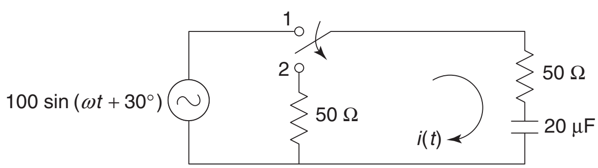


Fig. 6.161

Solution At $t = 0^-$, the network is shown in Fig. 6.162. At $t = 0^-$, the network attains steady-state condition.

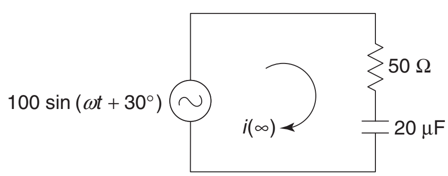


Fig. 6.162

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100\angle 30°}{50 - j\dfrac{1}{1000\times20\times10^{-6}}} = 1.41\angle 75°\text{ A}$$

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 1.41\sin(1000t+75°) \qquad \text{...(i)}$$

For $t > 0$, the network is as shown in Fig. 6.163.
Writing the KVL equation for $t > 0$,

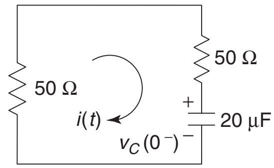


Fig. 6.163

$$-50i - 50i - \frac{1}{20\times 10^{-6}}\int_0^t i\,dt - v_C(0) = 0 \qquad \ldots\text{(ii)}$$

Differentiating Eq. (ii),

$$-50\frac{di}{dt} - 50\frac{di}{dt} - \frac{1}{20\times 10^{-6}} i = 0$$

$$\frac{di}{dt} + 500i = 0$$

Comparing with differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = 500$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-500t} \qquad \ldots\text{(iii)}$$

From Eqs (i) and (ii),

$$1.41\sin(1000t + 75°) = k\,e^{-500t}$$

At $t = 0$,

$$1.41\sin(75°) = k$$

$$k = 1.36$$

$$i(t) = 1.36\,e^{-500t} \quad \text{for } t > 0$$

6.5 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series RLC circuit as shown in Fig. 6.164. The switch is closed at time $t = 0$. The capacitor and inductor are initially uncharged.

Applying KVL to the circuit for $t > 0$,

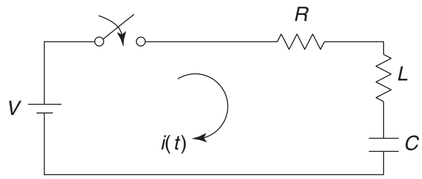


Fig. 6.164 *Series RLC circuit*

$$V - Ri - L\frac{di}{dt} - \frac{1}{C}\int_0^t i\,dt = 0$$

Differentiating the above equation,

$$0 - R\frac{di}{dt} - L\frac{d^2i}{dt^2} - \frac{1}{C}i = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a second-order differential equation. The auxiliary equation or characteristic equation will be given by,

$$s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$$

Let s_1 and s_2 be the roots of the equation.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where
$$\alpha = \frac{R}{2L}$$
$$\omega_0 = \frac{1}{\sqrt{LC}}$$
and
$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

The solution of the above second-order differential equation will be given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation.
Now depending upon the values of α and ω_0, we have 3 cases of the response.

Case I *When $\alpha > \omega_0$*

i.e.,
$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

The roots are real and unequal and it gives an overdamped response.
In this case, the solution is given by,

$$i = e^{-\alpha t}(k_1 e^{\beta t} + k_2 e^{-\beta t})$$

or
$$i = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad \text{for } t > 0$$

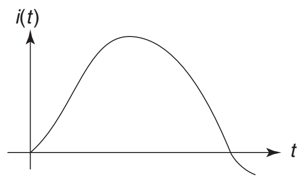


Fig. 6.165 *Overdamped response*

The current curve for an overdamped case is shown in Fig. 6.165.

Case II *When $\alpha = \omega_0$*

i.e.,
$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

The roots are real and equal and it gives a critically damped response.
In this case the solution is given by,

$$i = e^{-\alpha t}(k_1 + k_2 t) \quad \text{for } t > 0$$

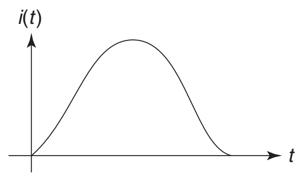


Fig. 6.166 *Critically damped response*

The current curve for critically damped case is shown in Fig. 6.166.

Case III *When $\alpha < \omega_0$*

i.e.,
$$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

The roots are complex conjugate and it gives an underdamped response.

In this case, the solution is given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $\quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 + \omega_0^2}$

Let $\quad \sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$

where $\quad j = \sqrt{-1}$

and $\quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Hence,

$$i(t) = e^{-\alpha t}(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t})$$

$$= e^{-\alpha t}\left\{(k_1 + k_2)\left[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right] + j(k_1 - k_2)\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{j2}\right]\right\}$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \quad \text{for } t > 0$$

The current curve for an underdamped case is shown in Fig. 6.167.

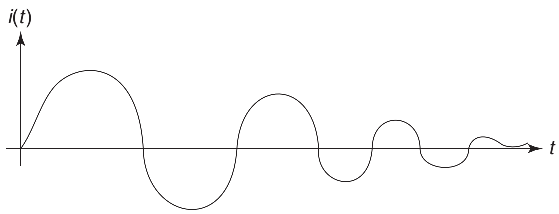


Fig. 6.167 *Underdamped response*

Example 6.55 *In the network of Fig. 6.168, the switch is closed at $t = 0$. Obtain the expression for current $i(t)$ for $t > 0$.*

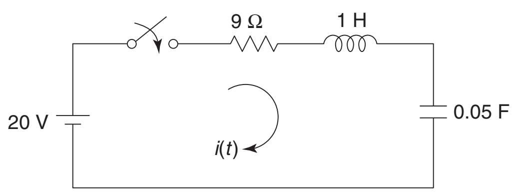


Fig. 6.168

Solution At $t = 0^-$, the switch is open.

$$i(0^-) = 0$$

$$v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i(0^+) = 0$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.169.

Writing the KVL equation for $t > 0$,

$$20 - 9i - 1\frac{di}{dt} - \frac{1}{0.05}\int_0^t i\,dt = 0 \qquad \ldots(\text{i})$$

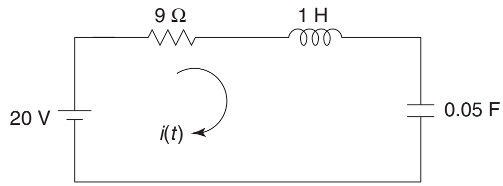


Fig. 6.169

Differentiating Eq. (i),

$$0 - 9\frac{di}{dt} - \frac{d^2i}{dt^2} - 20i = 0$$

$$\frac{d^2i}{dt^2} + 9\frac{di}{dt} + 20i = 0$$

$$(D^2 + 9D + 20)i = 0$$

$$D_1 = -4,\ D_2 = -5$$

The solution of this differential equation is given by,

$$i(t) = k_1 e^{-4t} + k_2 e^{-5t} \qquad \ldots(\text{ii})$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -4k_1 e^{-4t} - 5k_2 e^{-5t} \qquad \ldots(\text{iii})$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \ldots(\text{iv})$$

$$\frac{di}{dt}(0) = -4k_1 - 5k_2 \qquad \ldots(\text{v})$$

Putting $t = 0$ in Eq. (i),

$$20 - 9i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 20 - 9i(0^+) = 20 \text{ A/s}$$

From Eq. (v),

$$20 = -4k_1 - 5k_2 \qquad \ldots(\text{vi})$$

Solving Eqs (iv) and (vi),

$$k_1 = 20$$

$$k_2 = 20$$

$$i(t) = 20e^{-4t} - 20e^{-5t} \qquad \text{for } t > 0$$

Example 6.56 *In the network shown in Fig. 6.170, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Determine the expression for the current $i(t)$.*

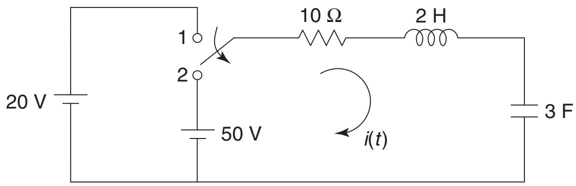


Fig. 6.170

Solution At $t = 0^-$, the network is shown in Fig. 6.171.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

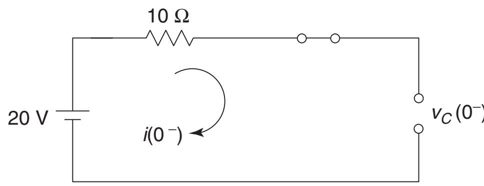


Fig. 6.171

$$v_C(0^-) = 20 \text{ V}$$
$$i(0^-) = 0$$

Since the current through the inductor and the voltage across the capacitor cannot change instaneously,

$$v_C(0^+) = 20 \text{ V}$$
$$i(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.172.

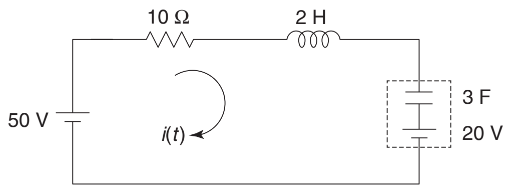


Fig. 6.172

Writing the KVL equation for $t > 0$,

$$50 - 10i - 2\frac{di}{dt} - \frac{1}{3}\int_0^t i\, dt - 20 = 0 \qquad \text{...(i)}$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - 2\frac{d^2i}{dt^2} - \frac{1}{3}i - 0 = 0$$
$$\frac{d^2i}{dt^2} + 5\frac{di}{dt} + \frac{1}{6}i = 0$$
$$\left(D^2 + 5D + \frac{1}{6}\right)i = 0$$
$$D_1 = -0.03,\ D_2 = -4.97$$

The solution of this differential equation is given by,

$$i(t) = k_1\, e^{-0.03t} + k_2\, e^{-4.97t} \qquad \text{...(ii)}$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -0.03k_1\, e^{-0.03t} - 4.97k_2\, e^{-4.97t} \qquad \text{...(iii)}$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \text{...(iv)}$$

$$\frac{di}{dt}(0) = -0.03\, k_1 - 4.97\, k_2 \qquad \text{...(v)}$$

Putting $t = 0$ in Eq. (i),

$$20 - 10\, i(0^+) - 2\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = \frac{20 - 10\, i(0^+)}{2} = 10 \text{ A/s}$$

From Eq. (v),

$$10 = -0.03\, k_1 - 4.97\, k_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (iv) and (vi),

$$k_1 = 2.02$$

$$k_2 = 2.02$$

$$i(t) = 2.02\, e^{-0.03t} - 2.02\, e^{-4.97t} \quad \text{for } t > 0$$

Example 6.57 *In the network of Fig. 6.173, the switch is closed and a steady state is reached in the network. At $t = 0$, the switch is opened. Find the expression for the current $i_2(t)$ in the inductor.*

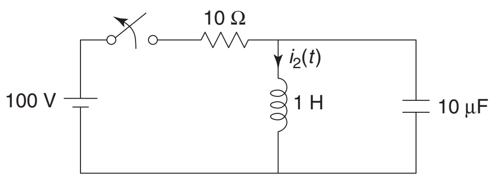


Fig. 6.173

Solution At $t = 0^-$, the network is shown in Fig. 6.174.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

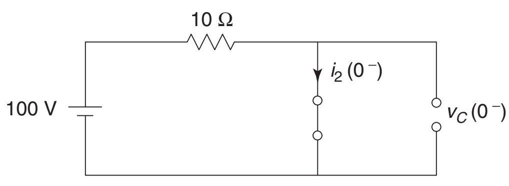


Fig. 6.174

$$i_2(0^-) = \frac{100}{10} = 10 \text{ A}$$

$$v_C(0^-) = 0$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_2(0^+) = 10 \text{ A}$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.175.

Writing the KVL equation for $t > 0$,

$$-1\frac{di_2}{dt} - \frac{1}{10 \times 10^{-6}}\int_0^t i\, dt = 0 \qquad \ldots\text{(i)}$$

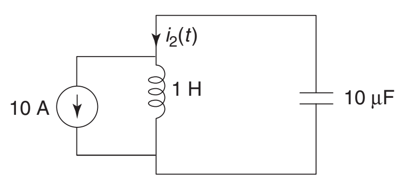


Fig. 6.175

Differentiating Eq. (i),

$$-\frac{d^2 i_2}{dt^2} - 10^5 i = 0$$

$$\frac{d^2 i_2}{dt^2} + 10^5 i = 0$$

$$(D^2 + 10^5)i = 0$$

$$D_1 = j\,316,\ D_2 = -j\,316$$

The solution of this differential equation is given by,

$$i_2(t) = k_1 \cos 316t + k_2 \sin 316t \qquad \text{...(ii)}$$

Differentiating Eq. (ii),

$$\frac{di_2}{dt} = -316\, k_1 \sin 316t + 316\, k_2 \cos 316t \qquad \text{...(iii)}$$

At $t = 0,\ i_2(0) = 10\ \text{A}$

$$10 = k_1 \qquad \text{...(iv)}$$

$$\frac{di_2}{dt}(0) = 316\, k_2 \qquad \text{...(v)}$$

Putting $t = 0$ in Eq. (i),

$$-\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 0$$

From Eq. (v),

$$0 = 316\, k_2$$

$$k_2 = 0$$

$$i_2(t) = 10 \cos 316t \quad \text{for } t > 0$$

Example 6.58 *In the network of Fig. 6.176, capacitor C has an initial voltage $v_c(0^-)$ of 10 V and at the same instant, current in the inductor L is zero. The switch is closed at time $t = 0$. Obtain the expression for the voltage $v(t)$ across the inductor L.*

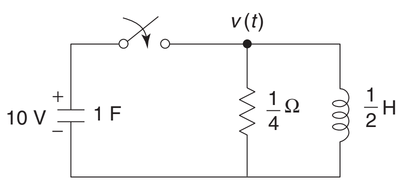


Fig. 6.176

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = v_C(0^-) = 10\ \text{V}$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$

$$v(0^+) = v_C(0^+) = 10 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.177.

Writing the KCL equation for $t > 0$,

$$1\frac{dv}{dt} + \frac{v}{\frac{1}{4}} + \frac{1}{\frac{1}{2}}\int_0^t v\,dt = 0 \qquad \ldots\text{(i)}$$

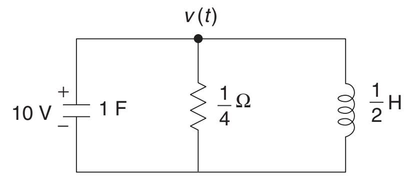


Fig. 6.177

Differentiating Eq. (i),

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 2v = 0$$

$$(D^2 + 4D + 2)v = 0$$

$$D_1 = -1, D_2 = -3$$

The solution of this differential equation is given by,

$$v(t) = k_1 e^{-t} + k_2 e^{-3t} \qquad \ldots\text{(ii)}$$

Differentiating Eq. (iii),

$$\frac{dv}{dt} = -k_1 e^{-t} - 3k_2 e^{-3t} \qquad \ldots\text{(iii)}$$

At $t = 0, v(0) = 10$ V

$$10 = k_1 + k_2 \qquad \ldots\text{(iv)}$$

$$\frac{dv}{dt}(0) = -k_1 - 3k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$\frac{dv}{dt}(0^+) + 4v(0) + 0 = 0$$

$$\frac{dv}{dt}(0^+) = -40 \text{ V/s}$$

From Eq. (v),

$$-40 = -k_1 - 3k_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (iv) and (vi),

$$k_1 = -5$$

$$k_2 = 15$$

$$v(t) = -5e^{-t} + 15e^{-3t} \qquad \text{for } t > 0$$

Example 6.59 *In the network of Fig. 6.178, the switch is opened at $t = 0$ obtain the expression for $v(t)$. Assume zero initial conditions.*

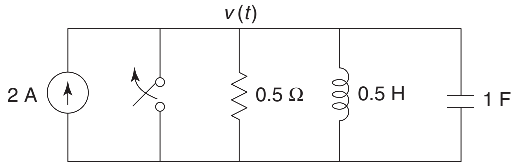


Fig. 6.178

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$
$$v(0^-) = v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor can not change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.179.

Writing the KCL equation for $t > 0$,

$$\frac{v}{0.5} + \frac{1}{0.5}\int_0^t v\,dt + 1\frac{dv}{dt} = 2 \qquad \ldots\text{(i)}$$

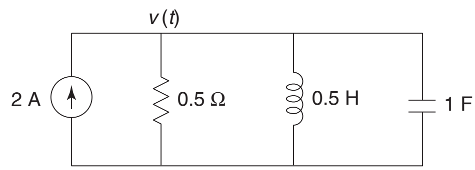


Fig. 6.179

Differentiating Eq. (i),

$$2\frac{dv}{dt} + 2v + \frac{d^2v}{dt^2} = 0$$
$$\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + 2v = 0$$
$$(D^2 + 2D + 2)v = 0$$
$$D_1 = -1 + j1, \quad D_2 = -1 - j1$$

The solution of this differential equation is given by,

$$v(t) = e^{-t}(k_1 \cos t + k_2 \sin t) \qquad \ldots\text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{dv}{dt} = -e^{-t}(k_1 \cos t + k_2 \sin t) + e^{-t}(-k_1 \sin t + k_2 \cos t)$$
$$= e^{-t}[-k_1(\cos t + \sin t) + k_2(\cos t - \sin t)] \qquad \ldots\text{(iii)}$$

At $t = 0$, $v(0) = 0$

$$0 = k_1 \qquad \ldots\text{(iv)}$$
$$\frac{dv}{dt}(0) = -k_1 + k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$2v(0) + 0 + \frac{dv}{dt}(0) = 2$$

$$\frac{dv}{dt}(0) = 2\ V/s$$

From Eq. (v),

$$2 = -k_1 + k_2 \qquad \ldots\text{(vi)}$$

Solving Eq. (iv) and (vi),

$$k_1 = 0$$
$$k_2 = 2$$
$$v(t) = 2\,e^{-t}\sin t \qquad \text{for } t > 0$$

Example 6.60 *The network shown in Fig. 6.180, a sinusoidal voltage $v = 150\sin(200t + \phi)$ is applied at $\phi = 30°$. Determine the current i(t).*

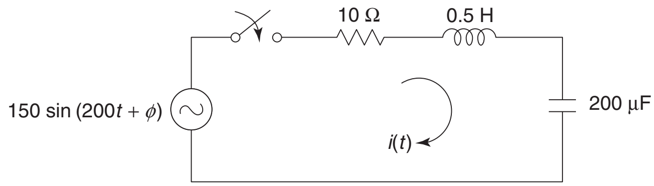


Fig. 6.180

Solution Writing the KVL equation for $t > 0$,

$$150\sin(200t + 30°) - 10i - 0.5\frac{di}{dt} - \frac{1}{200\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$30000\cos(200t + 30°) - 10\frac{di}{dt} - 0.5\frac{d^2i}{dt^2} - 5000i = 0$$
$$\frac{d^2i}{dt^2} + 20\frac{di}{dt} + 10000i = 60000\cos(200t + 30°)$$
$$(D^2 + 20D + 10000)i = 60000\cos(200t + 30°) \qquad \ldots\text{(ii)}$$

The roots of the characteristic equation are

$$D_1 = -10 + j\,99.5, \quad D_2 = -10 - j\,99.5$$

The complimentary function is

$$i_C = e^{-10t}(K_1\cos 99.5t + K_2\sin 99.5t)$$

Let the particular function be

$$i_P = A\cos(200t + 30°) + B\sin(200t + 30°)$$
$$i_P' = -200A\sin(200t + 30°) + 200B\cos(200t + 30°)$$
$$i_P'' = -40000A\cos(200t + 30°) - 40000B\sin(200t + 30°)$$

Substituting these values in Eq. (ii),

$$-40000\ A\cos(200t+30°)-40000\ B\sin(200t+30°)+20[-200\ A\sin(200t+30°)+200\ B\cos(200t+30°)]$$
$$+10000[A\cos(200t+30°)+B\sin(200t+30°)]=60000\cos(200t+30°)$$

$$(-40000\ B-4000\ A+10000\ B)\sin(200\text{t}+30°)+(-40000\ A+4000\ B+10000\ A)\cos(200t+30°)$$
$$=60000\cos(200t+30°)$$

Equating the coefficients,

$$-40000\ B-4000\ A+10000\ B=0$$
$$-40000\ A+4000\ B+10000\ A=60000$$

Solving these equations,

$$A=-1.97$$
$$B=0.26$$
$$i_P=-1.97\cos(200t+30°)+0.26\sin(200t+30°)$$

Let $\qquad A\sin\phi=-1.97$

and $\qquad A\cos\phi=0.26$

$$A^2\sin^2\phi+A^2\cos^2\phi=(-1.97)^2+(0.26)^2=3.95$$
$$A=1.987$$

and $\qquad \phi=\tan^{-1}\left(\dfrac{-1.97}{0.26}\right)=-82.48°$

$$\begin{aligned}i_P&=1.987\sin(-82.48°)\cos(200t+30°)+1.987\cos(-82.48°)\sin(200t+30°)\\&=1.987\sin(200t+30°-82.48°)\\&=1.987\sin(200t-52.48°)\end{aligned}$$

The solution of the differential equation is given by,

$$i(t)=e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+1.987\sin(200t-52.48°) \qquad \text{...(iii)}$$

Differentiating Eq. (iii),

$$\begin{aligned}\frac{di}{dt}&=e^{-10t}(-99.5\ k_1\sin 99.5t+99.5\ k_2\cos 99.5t)\\&\quad-10\ e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+(1.987)(200)\cos(200t-52.48°)\end{aligned}$$

At $t=0$, $\quad i(0)=0$

$$0=k_1+1.987\sin(-52.48°)$$
$$k_1=1.58 \qquad \text{...(iv)}$$

$$\begin{aligned}\frac{di}{dt}(0)&=99.5\ k_2-10\ k_1+(1.987)(200)\cos(-52.48°)\\&=99.5\ k_2-10(1.58)+242.03\\&=99.5\ k_2+226.23\end{aligned} \qquad \text{...(v)}$$

Putting $t = 0$ in Eq. (i),

$$150\sin(30°) - 10(0) - 0.5\frac{di}{dt}(0) - 0 = 0$$

$$\frac{di}{dt}(0) = 150 \text{ A/s}$$

From Eq. (v),

$$150 = 99.5\,k_2 + 226.23$$

$$k_2 = -0.77$$

$$i(t) = e^{-10t}(1.58\cos 99.5t - 0.77\sin 99.5t) + 1.987\sin(200t - 52.48°) \quad \text{for } t > 0$$

Example 6.61 *The switch in the network of Fig. 6.181 is opened at t = 0. Find i (t) for t > 0 if,*

(a) $L = \frac{1}{2}$ H and C = 1 F *(b) L = 1 H and C = 1 F* *(c) L = 5 H and C = 1 F*

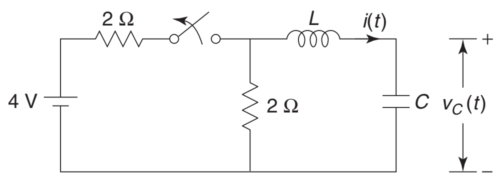


Fig. 6.181

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an pen circuit.

$$v_C(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

$$i(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 2 \text{ V}$$

$$i(0^+) = 0$$

Case I *When $R = 2\ \Omega$, $L = \frac{1}{2}$ H, C = 1 F*

$$\alpha = \frac{R}{2L} = \frac{2}{2 \times \frac{1}{2}} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times 1}} = \frac{1}{\sqrt{0.5}} = 1.414$$

$$\alpha > \omega_0$$

This indicates an overdamped case.

$$i(t) = A_1 e^{s_1 t} - A_2 e^{s_2 t}$$

where, $$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2 - \sqrt{4-2} = -2 - \sqrt{2} = -3.414$$

and $$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2 + \sqrt{2} = -0.586$$

$$i(t) = k_1 e^{-3.414t} + k_2 e^{-0.586t}$$

At $t = 0$, $i(0) = 0$

$$k_1 + k_2 = 0 \quad \ldots\text{(i)}$$

Also $v_L(0^+) + v_C(0^+) + v_R(0^+) = 0$

$$v_L(0^+) = -v_R(0^+) - v_C(0^+) = -2i(0^+) - v_C(0^+) = -2 \text{ V} \quad \ldots\text{(ii)}$$

$$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{0.5} = -4 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$-3.414\, k_1 - 0.586\, k_2 = -4 \quad \ldots\text{(iii)}$$

Solving Eqs (i) and (iii), we get

$$k_1 = 1.414 \quad \text{and} \quad k_2 = -1.414$$

$$i(t) = 1.414(e^{-3.414t} - e^{-0.586t}) \qquad \text{for } t > 0$$

Case II *When $R = 2\ \Omega$, $L = 1\ H$, $C = 1\ F$*

$$\alpha = \frac{R}{2L} = \frac{2}{2\times 1} = \frac{2}{2} = 1$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1}} = 1$$

$$\alpha = \omega_0$$

This indicates a critically damped case.

$$i(t) = e^{-\alpha t}(k_1 + k_2 t) = e^{-t}(k_1 + k_2 t)$$

At $t = 0$, $i(0) = 0$

$$k_1 = 0$$

Also, $$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{1} = -2 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$\left.\frac{di}{dt}\right|_{t=0} = -k_1 + k_2 = -2$$

$$k_2 = -2$$

$$i(t) = -2t\,e^{-t} \qquad \text{for } t > 0$$

Case III *When $R = 2\ \Omega$, $L = 5\ H$, $C = 1\ F$*

$$\alpha = \frac{R}{2L} = \frac{2}{10} = 0.2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5}} = 0.447$$

$$\alpha < \omega_0$$

This indicates an underdamped case.

$$i\,(t) = e^{-\alpha t}\,(B_1 \cos \omega_d\, t + B_2 \sin \omega_d\, t)$$

where, $$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{(0.447)^2 - (0.2)^2} = 0.4$$

$$s_{1,2} = -\alpha \pm j\omega_d = -0.2 \pm j0.4$$

$$i(t) = e^{-0.2t}\,(B_1 \cos 0.4t + B_2 \sin 0.4t)$$

Applying the initial condition,

$$i(0^+) = 0$$

and $$\frac{di}{dt}(0^+) = -\frac{v_L(0^+)}{L} = -\frac{2}{5}$$

$$B_1 = i(0) = 0$$

$$B_2 = -1$$

$$i(t) = -e^{-0.2t} \sin 0.4t \qquad \text{for } t > 0$$

Exercises

6.1 The switch in Fig. 6.182 is moved from the position a to b at $t = 0$, the network having been in steady state in the position a. Determine $i_1(0^+), i_2(0^+), i_3(0^+), \frac{di_2}{dt}(0^+)$ and $\frac{di_3}{dt}(0^+)$.

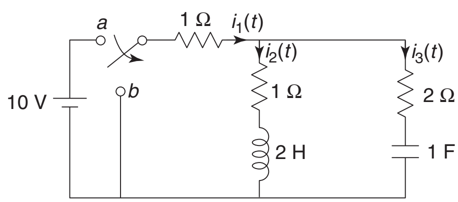


Fig. 6.182

[1.66 A, 5 A, –3.33 A, –3.33 A/s, 2.22 A/s]

6.2 The switch K is closed at $t = 0$ in the network shown in Fig. 6.183. Determine $i(0^+), \frac{di}{dt}(0^+)$ and $\frac{d^2i}{dt^2}(0^+)$.

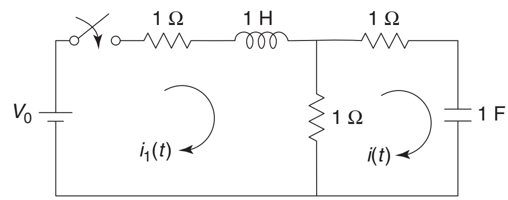


Fig. 6.183

$$\left[0, \frac{1}{2}V_0, -V_0\right]$$

6.3 In the network of Fig. 6.184, the switch K is closed at $t = 0$. At $t = 0^-$, all capacitor voltages and inductor currents are zero. Find $v_1, \dfrac{dv_1}{dt}, v_2, \dfrac{dv_2}{dt}, v_3$ and $\dfrac{dv_3}{dt}$ at $t = 0^+$.

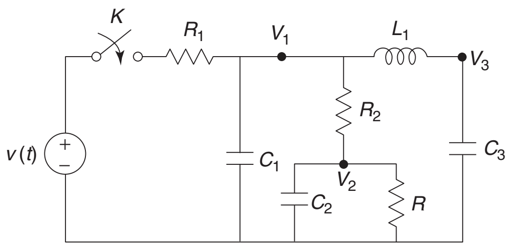


Fig. 6.184

$$\left[0, \frac{1}{C}\frac{v(0^+)}{R_1}, 0, 0, 0, 0\right]$$

6.4 In the network at Fig. 6.185, the capacitor C_1 is charged to voltage 1000 V and the switch K is closed at $t = 0$. Find $\dfrac{d^2 i_2}{dt^2}$ at $t = 0^+$.

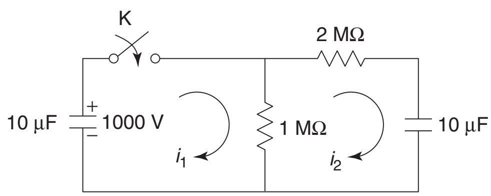


Fig. 6.185

$$\left[\frac{17}{400000}\text{A/s}^2\right]$$

6.5 In the network shown in Fig. 6.186, switch is closed at $t = 0$. Obtain the current $i_2(t)$.

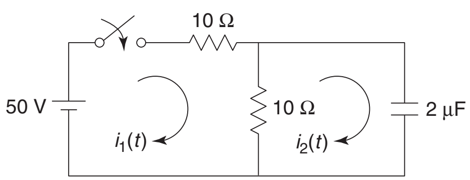


Fig. 6.186

$[i_2(t) = 5\,e^{-100000t}]$

6.6 The network shown in Fig. 6.187 is under steady-state when the switch is closed. At $t = 0$, it is opened. Obtain an expression for $i(t)$.

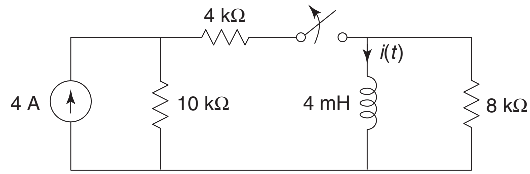


Fig. 6.187

$[i(t) = 2.857\,e^{-2\times10^6\,t}]$

6.7 The switch in Fig. 6.188 is open for a long time and closes at $t = 0$. Determine $i(t)$ for $t > 0$.

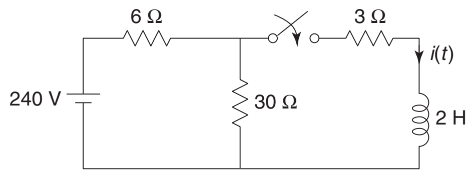


Fig. 6.188

$[i(t) = 25(1 - e^{-4t})]$

6.8 In the network shown in Fig. 6.189, the steady state is reached with the switch open. At $t = 0$, the switch is closed Find $v_C(t)$ for $t > 0$.

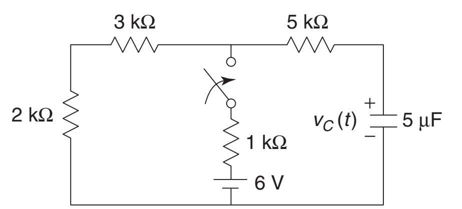


Fig. 6.189

$[v_C(t) = 5e^{-20t}]$

6.9 The circuit shown in Fig. 6.190 has acquired steady state before switching at $t = 0$.

(i) Obtain $v_C(0^+)$, $v_C(0^-)$, $i(0^+)$ and $i(0^-)$.
(ii) Obtain time constant for $t > 0$.
(iii) Find current $i(t)$ for $t > 0$.

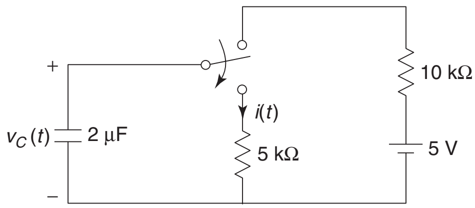


Fig. 6.190

[(i) 5 V, 5 V, 1 mA, 0, (ii) 0.01 s, (iii) $e^{-100\,t}$ mA]

6.10 In the network shown in Fig. 6.191, the switch is initially at the position 1 for a long time. At $t = 0$, the switch is changed to the position 2. Find current $i(t)$ for $t > 0$.

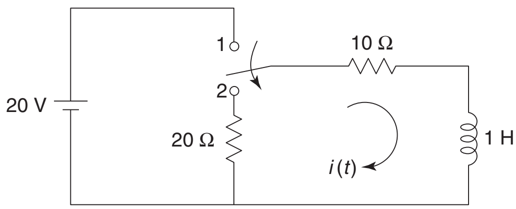


Fig. 6.191

$[i(t) = 2\,e^{-30t}]$

6.11 In the network shown in Fig. 6.192, the switch is closed at $t = 0$. Find $v(t)$ for $t > 0$.

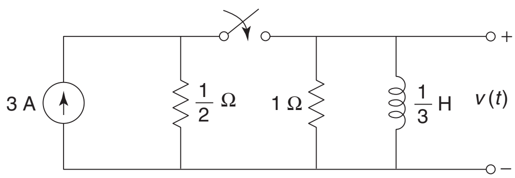


Fig. 6.192

$[v(t) = e^{-t}]$

6.12 In the network shown in Fig. 6.193, the switch is in the position 1 for a long time and at $t = 0$, the switch is moved to the position . Find $v(t)$ for $t > 0$.

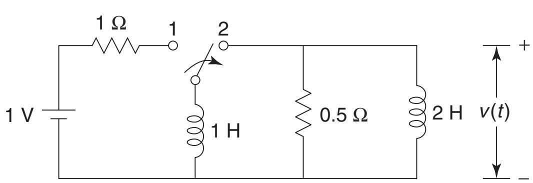


Fig. 6.193

$[v(t) = -0.5\,e^{-\frac{3}{4}t}]$

6.13 In Fig. 6.194, the switch is open until time $t = 100$ seconds and is closed for all times thereafter. Find $v(t)$ for all times greater than 100 if $v(100) = -3$ V.

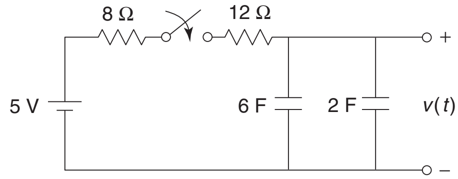


Fig. 6.194

$$\left[v(t) = 5 - 8e^{-\frac{(t-100)}{160}}\right]$$

6.14 A series RL circuit shown in Fig. 6.195 has a constant voltage V applied at $t = 0$. At what time does $v_R = v_L$.

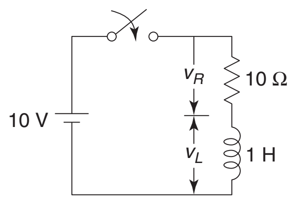


Fig. 6.195

[0.0693 s]

6.15 In the circuit shown in Fig. 6.196, at time $t = 0$, the voltage across the capacitor is zero and the switch is moved to the position y. The switch is kept at position y for 20 seconds and then moved to position z and kept in that position thereafter. Find the voltage across the capacitor at $t = 30$ seconds.

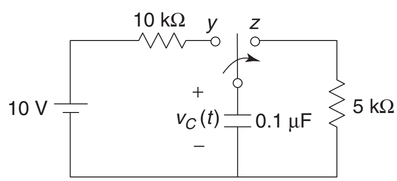


Fig. 6.196

[0]

6.16 Determine whether RLC series circuit shown in Fig 6.197 is underdamped, overdamped or critically damped. Also, find $v_L(0^+)$, $\dfrac{di}{dt}(0^+)$ and $i\,(\infty)$.

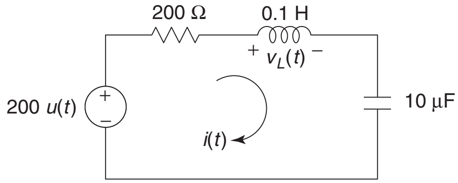


Fig. 6.197

[critically damped, 200 V, 2000 A/s, 0]

6.17 Determine whether *RLC* circuit of Fig 6.198 is underdamped, overdamped or critically damped. Also find $v_L(0^+)$, $\dfrac{di}{dt}(0^+)$, $\dfrac{d^2v}{dt^2}(0^+)$ if $v(t) = u(t)$.

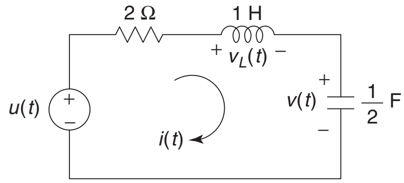


Fig. 6.198

[underdamped 1 V, 1 A/s, 2 V/s^2]

Objective-Type Questions

6.1 The voltages v_{C_1}, v_{C_2} and v_{C_3} across the capacitors in the circuit in Fig. 6.199 under steady state are respectively

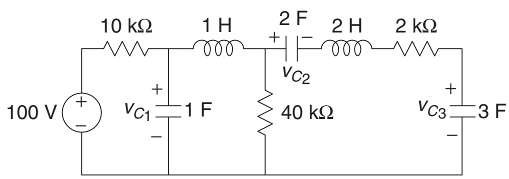


Fig. 6.199

(a) 80 V, 32 V, 48 V
(b) 80 V, 48 V, 32 V
(c) 20 V, 8 V, 12 V
(d) 20 V, 12 V, 8 V

6.2 In the circuit of Fig. 6.200, the voltage $v(t)$ is

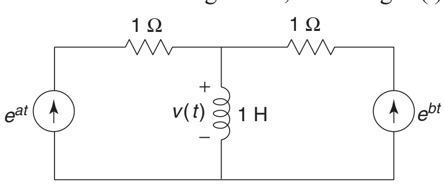


Fig. 6.200

(a) $e^{at} - e^{bt}$ (b) $e^{at} + e^{bt}$
(c) $ae^{dt} - be^{bt}$ (d) $ae^{at} + be^{bt}$

6.3 The differential equation for the current $i(t)$ in the circuit of Fig. 6.201 is

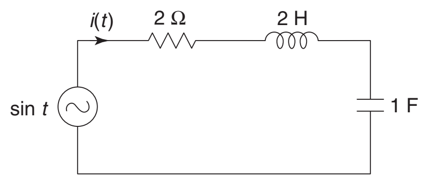


Fig. 6.201

(a) $2\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + i(t) = \sin t$

(b) $\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + 2i(t) = \cos t$

(c) $2\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + i(t) = \cos t$

(d) $\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + 2i(t) = \sin t$

6.4 At $t = 0^+$, the current i_1 in Fig. 6.202 is

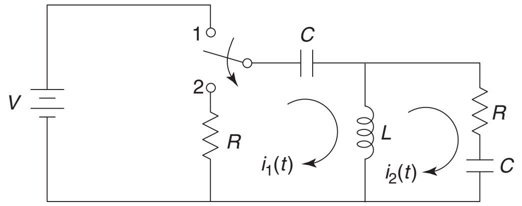


Fig. 6.202

(a) $-\dfrac{V}{2R}$ (b) $-\dfrac{V}{R}$

(c) $-\dfrac{V}{4R}$ (d) zero

6.5 For the circuit shown in Fig. 6.203, the time constant $RC = 1$ ms. The input voltage is $v_i(t) = \sqrt{2}\sin 10^3 t$. The output voltage $v_0(t)$ is equal to

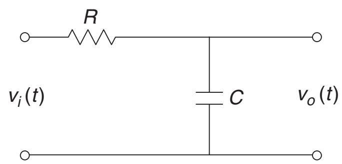


Fig. 6.203

(a) $\sin(10^3 - 45°)$ (b) $\sin(10^3 t + 45°)$

(c) $\sin(10^3 t - 53)$ (d) $\sin(10^3 t + 53°)$

6.6 For the *RL* circuit shown in Fig. 6.204, the input voltage $v_i(t) = u(t)$. The current $i(t)$ is

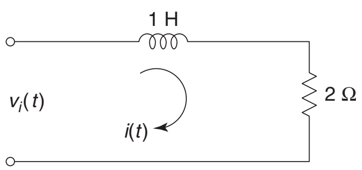
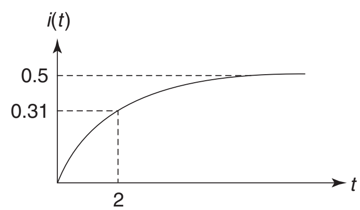
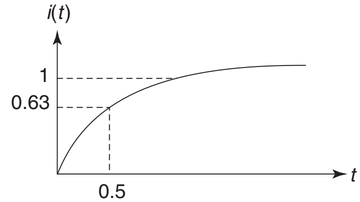


Fig. 6.204

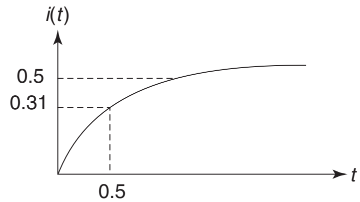
(a)



(b)



(c)



(d)

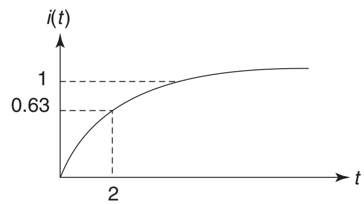


Fig. 6.205

6.7 The condition on *R*, *L* and *C* such that the step response $v(t)$ in Fig. 6.206 has no oscillations is

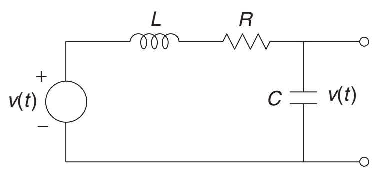


Fig. 6.206

(a) $R \geq \dfrac{1}{2}\sqrt{\dfrac{L}{C}}$ (b) $R \geq \sqrt{\dfrac{L}{C}}$

(c) $R \geq 2\sqrt{\dfrac{L}{C}}$ (d) $R = \dfrac{1}{\sqrt{LC}}$

6.8 The switch *S* in Fig. 6.207 closed at $t = 0$. If $v_2(0) = 10$ V and $v_g(0) = 0$ respectively, the voltages across capacitors in steady state will be

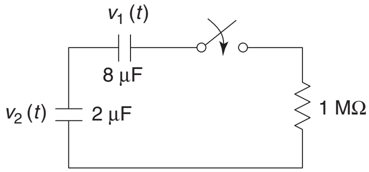


Fig. 6.207

(a) $v_2(\infty) = v_1(\infty) = 0$

(b) $v_2(\infty) = 2$ V, $v_1(\infty) = 8$ V

(c) $v_2(\infty) = v_1(\infty) = 8$ V

(d) $v_2(\infty) = 8$ V, $v_1(\infty) = 2$ V

6.9 The time constant of the network shown in Fig. 6.208 is

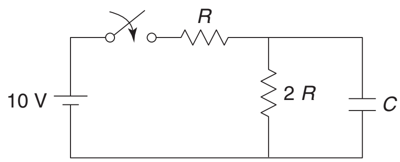


Fig. 6.208

(a) $2\,RC$ (b) $3\,RC$

(c) $\frac{1}{2}RC$ (d) $\frac{2}{3}RC$

6.10 In the series *RC* circuit shown in Fig. 6.209, the voltage across *C* starts increasing when the dc source is switched on. The rate of increase of voltage across *C* at the instant just after the switch is closed i.e., at $t = 0^+$ will be

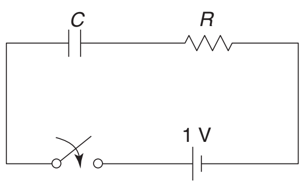


Fig. 6.209

(a) zero (b) infinity

(c) RC (d) $\frac{1}{RC}$

6.11 The $v - i$ characteristic as seen from the terminal pair ($A - B$) of the network of Fig. 6.210(a) is shown in Fig. 6.210(b). If an inductance of value 6 mH is connected across the terminal pair, the time constant of the system will be

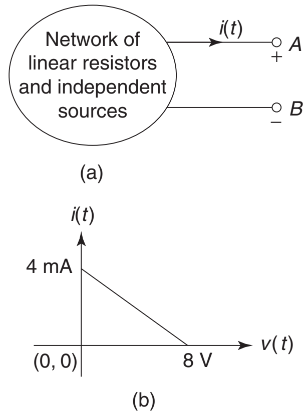


Fig. 6.210

(a) 3 μs

(b) 12 s

(c) 32 s

(d) unknown, unless actual network is specified

6.12 In the network shown in Fig. 6.211, the circuit was initially in the steady-state condition with the switch *K* closed. At the instant when the switch is opened, the rate of decay of current through inductance will be

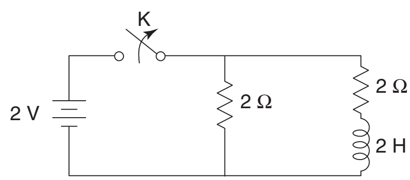


Fig. 6.211

(a) zero (b) 0.5 A/s

(c) 1 A/s (d) 2 A/s

6.13 A step function voltage is applied to an *RLC* series circuit having $R = 2\ \Omega$, $L = 1$ H and $C = 1$ F. The transient current response of the circuit would be

(a) over damped (b) critically damped

(c) under damped (d) none of these

Answers to Objective-Type Questions

6.1 (b)	6.2 (d)	6.3 (c)	6.4 (d)	6.5 (a)	6.6 (b)	6.7 (c)
6.8 (d)	6.9 (d)	6.10 (d)	6.11 (a)	6.12 (d)	6.13 (b)	

7

Frequency Domain Analysis of RLC Circuits

7.1 INTRODUCTION

Time-domain analysis is the conventional method of analysing a network. For a simple network with first-order differential equation of network variable, this method is very useful. But as the order of network variable equation increases, this method of analysis becomes very tedious. For such applications, frequency domain analysis using Laplace transform is very convenient. Time-domain analysis, also known as *classical method,* is difficult to apply to a differential equation with excitation functions which contain derivatives. Laplace transform methods prove to be superior. The Laplace transform method has the following advantages:

(1) Solution of differential equations is a systematic procedure.
(2) Initial conditions are automatically incorporated.
(3) It gives the complete solution, i.e., both complementary and particular solution in one step.

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

7.2 LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

where s is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^\infty |f(t)|\,e^{-\sigma t}\,\mathrm{d}t < \infty$$

where σ is real and positive.
The inverse Laplace transform $L^{-1}\ \{F(s)\}$ is

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds$$

7.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

1. *Constant Function k*

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty k e^{-st} dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}$$

2. *Function t^n*

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

Putting $st = x,\ dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx = \frac{\sqrt{n+1}}{s^{n+1}},\ s > 0,\ n+1 > 0$$

If n is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. *Unit-Step Function*

The unit-step function (Fig 7.1) is defined by the equation,

$$\begin{aligned} u(t) &= 1 \qquad & t > 0 \\ &= 0 \qquad & t < 0 \end{aligned}$$

The Laplace transform of unit step function is

$$L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st}\ dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Fig. 7.1 *Unit-step function*

4. *Delayed or Shifted Unit-Step Function*

The delayed or shifted unit-step function (Fig 7.2) is defined by the equation

$$\begin{aligned} u\ (t - a) &= 1 \qquad & t > a \\ &= 0 \qquad & t < a \end{aligned}$$

The Laplace transform of $u\ (t - a)$ is

$$L\{u(t-a)\} = \int_a^\infty 1 \cdot e^{-st}\ dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

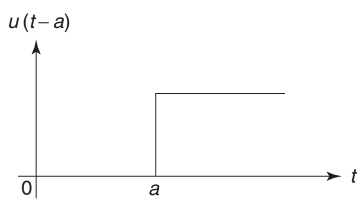


Fig. 7.2 *Shifted unit-step function*

5. *Unit-Ramp Function*

The unit-ramp function (Fig 7.3) is defined by the equation

$$\begin{aligned} r(t) &= t & t > 0 \\ &= 0 & t < 0 \end{aligned}$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^\infty t\, e^{-st} = \frac{1}{s^2}$$

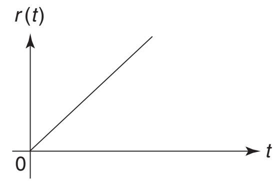


Fig. 7.3 *Unit-ramp function*

6. *Delayed Unit-Ramp Function*

The delayed unit-ramp function (Fig 7.4) is defined by the equation

$$\begin{aligned} r(t-a) &= t & t > a \\ &= 0 & t < a \end{aligned}$$

The Laplace transform of $r(t-a)$ is

$$L\{r(t-a)\} = \int_a^\infty t\, e^{-st}\, dt = \frac{e^{-as}}{s^2}$$

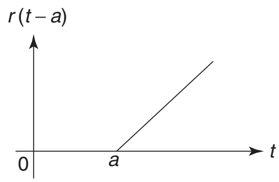


Fig. 7.4 *Delayed unit-ramp function*

7. *Unit-Impulse Function*

The unit-impulse function (Fig 7.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad t = 0$$

The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = 1$$

Fig. 7.5 *Unit-impulse function*

8. *Exponential Function (e^{at})*

The Laplace transform of the exponential function (Fig 7.6) is

$$L\{e^{at}\} = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

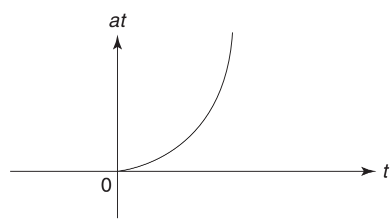


Fig. 7.6 *Exponential function*

9. *Sine Function*

We know that $\sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin \omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

10. *Cosine Function*

We know that $\cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos \omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

11. Hyperbolic sine function

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2 - \omega^2}$$

12. Hyperbolic cosine function

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2 - \omega^2}$$

13. Exponentially Damped Function

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t)\, dt\right\} = \int_{0^-}^{\infty} f(t) e^{-at}\, e^{-st}\, dt = \int_{0^-}^{\infty} f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s + a)$ in place of s in the transform of $f(t)$.

$$L\left\{e^{-at} \sin \omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \sinh \omega t\right\} = \frac{\omega}{(s+a)^2 - \omega^2}$$

$$L\left\{e^{-at} \cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \cosh \omega t\right\} = \frac{s+a}{(s+a)^2 - \omega^2}$$

7.4 PROPERTIES OF LAPLACE TRANSFORM

7.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^{\infty} \{af_1(t) + bf_2(t)\} e^{-st} dt = a\int_0^{\infty} f_1(t) e^{-st} dt + b\int_0^{\infty} f_2(t) e^{-st}\, dt = aF_1(s) + bF_2(s)$$

7.4.2 Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{f(at)\} = \int_0^{\infty} f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \dfrac{dx}{a}$

$$L\{(f(at)\} = \int_0^\infty f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^\infty f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

7.4.3 Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\ d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\ dt = \int_0^\infty f(t) e^{-(s+a)t}\ dt = F(s+a)$$

7.4.4 Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as}F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st}\ dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st}\ dt$$

Putting $\quad t - a = x, \quad dt = dx$

When $\quad t = a, \quad x = 0$

$\quad t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x) e^{-s(a+x)}\ dx = e^{-as}\int_0^\infty f(x) e^{-sx}\ dx = e^{-as}\int_0^\infty f(t)\, e^{-st}\ dt = e^{-as}F(s)$$

7.4.5 Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\, f(t)\} = -\dfrac{d}{ds}F(s)$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\ dt$$

Differentiating both the sides w.r.t s using DUIS,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t) e^{-st} dt = \int_0^\infty \frac{\partial}{\partial s} f(t) e^{-st}\ dt$$

$$= \int_0^\infty (-t)\ f(t) e^{-st}\ dt = \int_0^\infty \{-t\, f(t)\} e^{-st} dt = -L\{t\ f(t)\}$$

$$L\{t\ f(t)\} = (-1)\frac{d}{ds}F(s)$$

7.4.6 Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

Proof

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st}\; dt$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^{\infty} F(s)\, ds = \int_s^{\infty}\int_0^{\infty} f(t) e^{-st} dt\; ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^{\infty} F(s) ds = \int_0^{\infty}\left[\int_s^{\infty} f(t) e^{-st} ds\right] dt = \int_0^{\infty}\left[\frac{1}{-t} f(t) e^{-st}\right]_s^{\infty} dt = \int_0^{\infty} \frac{f(t)}{t} e^{-st} dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s) ds$$

7.4.7 Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$ then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

In general,

$$L\{f''(t)\} = s''F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{(n-1)}(0)$$

Proof

$$L\{f'(t)\} = \int_0^{\infty} f'(t)\, e^{-st}\; dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t) e^{-st}\right]_0^{\infty} - \int_0^{\infty} (-s) f(t) e^{-st} dt = -f(0) + s\int_0^{\infty} f(t) e^{-st} dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\, t\} = -f' + sL\{f'\, t\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right] = -f'(0) - sf(0) + s^2 L\{f(t)\}$$

In general, $$L\left\{f^n(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$$

7.4.8 Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t) dt\right\} = \dfrac{F(s)}{s}$

Proof $$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \left[\int_0^t f(t)\, dt\right] e^{-st}\, dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right]dt = \int_0^\infty \frac{1}{s} f(t)e^{-st}dt = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

7.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\, e^{-st}\, dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to\infty}[f'(t)e^{-st}]dt + f(0) = 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

7.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\, e^{-st}\, dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to 0}\int_0^\infty f'(t)\, e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to 0}\left[f'(t)e^{-st}\right]dt + f(0) = \int_0^\infty f'(t)dt + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) = \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

7.5 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion

7.5.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 7.1 *Find the inverse Laplace transform of* $\dfrac{s^2-3s+4}{s^3}$.

Solution

$$F(s)=\frac{s^2-3s+4}{s^3}=\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}$$

$$L^{-1}\{F(s)\}=1-3t+2t^2$$

Example 7.2 *Find the inverse Laplace transform of* $\dfrac{3s+4}{s^2+9}$.

Solution

$$F(s)=\frac{3s+4}{s^2+9}=\frac{3s}{s^2+9}+\frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\}=3\cos 3t+\frac{4}{3}\sin 3t$$

Example 7.3 *Find the inverse Laplace transform of* $\dfrac{4s+15}{16s^2-25}$.

Solution

$$F(s)=\frac{4s+15}{16s^2-25}=\frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}=\frac{1}{4}\frac{s}{s^2-\frac{25}{16}}+\frac{15}{16}\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\}=\frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

Example 7.4 *Find the inverse Laplace transform of* $\dfrac{2s+2}{s^2+2s+10}$.

Solution

$$F(s)=\frac{2s+2}{s^2+2s+10}=\frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\}=2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}=2e^{-t}\cos 3t$$

Example 7.5 *Find the inverse Laplace transform of* $\dfrac{3s+7}{s^2-2s-3}$.

Solution

$$F(s)=\frac{3s+7}{s^2-2s-3}=\frac{3(s-1)+10}{(s-1)^2-4}=3\frac{(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}$$

$$L^{-1}\{F(s)\}=3e^{t}L^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^{t}L^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^{t}\cosh 2t+5e^{t}\sinh 2t$$

7.5.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I Factors are linear and distinct,

$$F(s)=\frac{P(s)}{(s+a)(s+b)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B}{s+b}$$

Case II Factors are linear and repeated,

$$F(s)=\frac{P(s)}{(s+a)(s+b)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B_1}{s+b}+\frac{B_2}{(s+b)^2}+\ldots+\frac{B_n}{(s+b)^n}$$

Case III Factors are quadratic and distinct,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{Cs+D}{s^2+cs+d}$$

Case IV Factors are quadratic are repeated,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{C_1s+D_1}{s^2+cs+d}+\frac{C_2s+D_2}{(s^2+cs+d)^2}+\ldots+\frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 7.6 *Find the inverse Laplace transform of* $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

$$F(s)=\frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$A=sF(s)\big|_{s=0}=\left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0}=\frac{2}{3}$$

$$B = (s+1)F(s)\big|_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$C = (s+3)F(s)\big|_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Example 7.7 *Find the inverse Laplace transform of* $\frac{s+2}{s^2(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$\begin{aligned} s+2 &= As(s+3) + B(s+3) + Cs^2 \\ &= As^2 + 3As + Bs + 3B + Cs^2 \\ &= (A+C)s^2 + (3A+B)s + 3B \end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned} A + C &= 0 \\ 3A + B &= 1 \\ 3B &= 2 \end{aligned}$$

Solving these equations,

$$A = \frac{1}{9},\ B = \frac{2}{3},\ C = -\frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7.8 *Find the inverse Laplace transform of* $\frac{s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

$$F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$\begin{aligned}5s^2-15s-11&=A(s-2)^2+B(s+1)(s-2)+C(s+1)\\&=A(s^2-4s+4)+B(s^2-s-2)+C(s+1)\\&=As^2-4As+4A+Bs^2-Bs-2B+Cs+c\\&=(A+B)s^2-(4A+B-C)s+(4A-2B+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+B&=5\\4A+B-C&=15\\4A-2B+C&=-11\end{aligned}$$

Solving these equations,

$$\begin{aligned}A&=1\\B&=4\\C&=-7\end{aligned}$$

$$F(s)=\frac{1}{s+1}+\frac{4}{s-2}-\frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}-7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}=e^{-t}+4e^{2t}-7te^{2t}$$

Example 7.9 *Find the inverse Laplace transform of* $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

$$F(s)=\frac{3s+1}{(s+1)(s^2+2)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+2}$$

$$\begin{aligned}3s+1&=A(s^2+2)+(Bs+C)(s+1)\\&=As^2+2A+Bs^2+Bs+Cs+C\\&=(A+B)s^2+(B+C)s+(2A+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+B&=0\\B+C&=3\\2A+C&=1\end{aligned}$$

Solving these equations,

$$A=-\frac{2}{3},\ B=\frac{2}{3},\ C=\frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\} = -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

Example 7.10 *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

$$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

7.6 FREQUENCY DOMAIN REPRESENTAION OF RLC CIRCUITS

Voltage–current relationships of network elements can also be represented in the frequency domain.

1. Resistor For the resistor, the v–i relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 7.7.

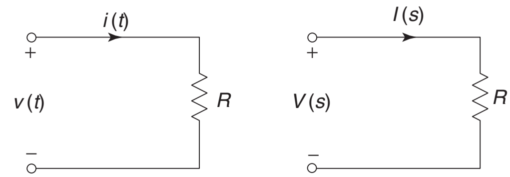


Fig. 7.7 *Resistor*

2. Inductor For the inductor, the v–i relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 7.8.

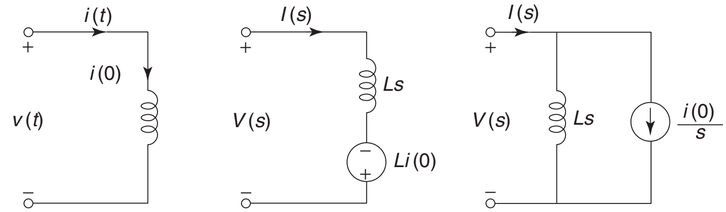


Fig. 7.8 *Inductor*

3. Capacitor For capacitor, the v–i relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 7.9.

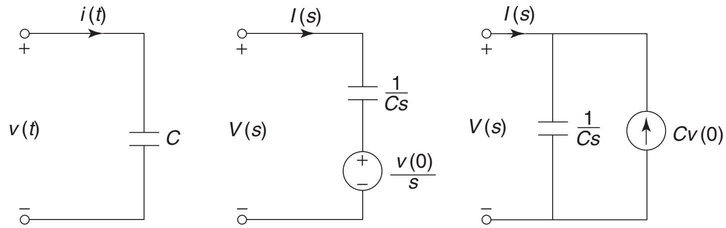


Fig. 7.9 *Capacitor*

7.7 RESISTOR–INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 7.10. The switch is closed at time $t = 0$.

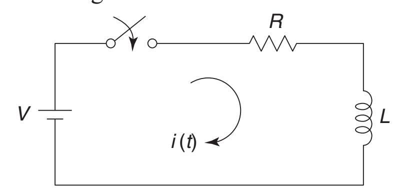


Fig. 7.10 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.11.

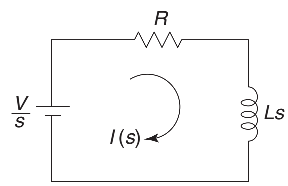


Fig. 7.11 *Transformed network*

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{R}{L}}$$

$$A = sI(s)\big|_{s=0} = s \times \left.\frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}\right|_{s=0} = \frac{V}{R}$$

$$B = \left(s+\frac{R}{L}\right)I(s)\bigg|_{s=-\frac{R}{L}} = \left(s+\frac{R}{L}\right)\times\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\Bigg|_{s=-\frac{R}{L}} = -\frac{V}{R}$$

$$I(s) = \frac{\frac{V}{R}}{s} + \frac{\left(-\frac{V}{R}\right)}{s+\frac{R}{L}}$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}$$

$$= \frac{V}{R}\left[1-e^{-\frac{R}{L}t}\right] \qquad \text{for } t > 0$$

Example 7.11 *In the network of Fig. 7.12, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.*

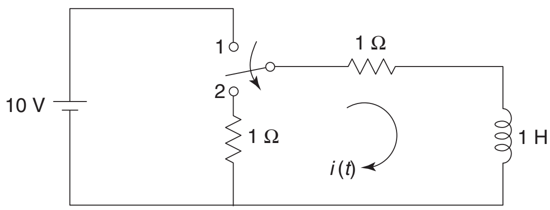


Fig. 7.12

Solution At $t = 0^-$, the network is shown in Fig 7.13. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

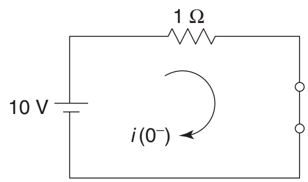


Fig. 7.13

$$i(0^-) = \frac{10}{1} = 10 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i\,(0^+) = 10 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.14.

Applying KVL to the mesh for $t > 0$,

$$-I(s) - I(s) - sI(s) + 10 = 0$$

$$I(s)(s+2) = 10$$

$$I(s) = \frac{10}{s+2}$$

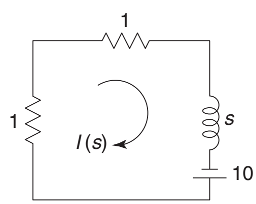


Fig. 7.14

Taking inverse Laplace transform,

$$i\,(t) = 10e^{-2t} \quad \text{for } t > 0$$

Example 7.12 *The network of Fig. 7.15 was initially in the steady state with the switch in the position a. At t = 0, the switch goes from a to b. Find an expression for voltage v (t) for t > 0.*

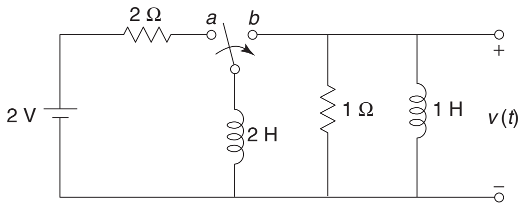


Fig. 7.15

Solution At $t = 0^-$, the network is shown in Fig 7.16. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

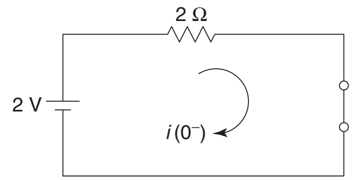


Fig. 7.16

$$i(0^-) = \frac{2}{2} = 1\text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 1\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.17.

Applying KCL at the node for $t > 0$,

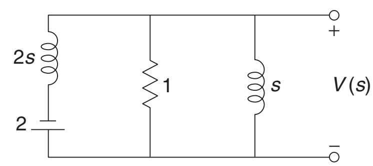


Fig. 7.17

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

$$V(s)\left(1+\frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v\,(t) = -\,e^{-1.5\,t} \qquad \text{for } t > 0$$

Example 7.13 *In the network of Fig. 7.18, the switch is opened at t = 0. Find i(t).*

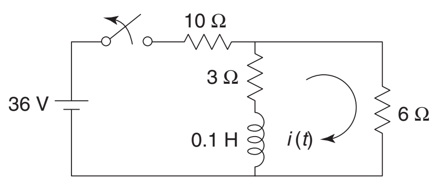


Fig. 7.18

Solution At $t = 0^-$, the network is shown in Fig. 7.19. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the inductor acts as a short circuit.

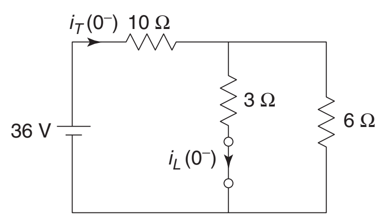


Fig. 7.19

$$i_T(0^-) = \frac{36}{10+(3\,||\,6)} = \frac{36}{10+2} = 3\text{ A}$$

$$i_L(0^-) = 3\times\frac{6}{6+3} = 2\text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.20

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\, I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = -0.2$$

$$I(s) = \frac{-0.2}{0.1s+9} = \frac{-2}{s+90}$$

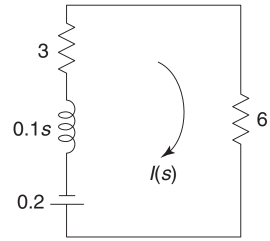


Fig. 7.20

Taking inverse Laplace transform,

$$i(t) = -2e^{-90\,t} \qquad \text{for } t > 0$$

Example 7.14 *The network shown in Fig. 7.21 has acquired steady-state with the switch closed for $t < 0$. At $t = 0$, the switch is opened. Obtain $i(t)$ for $t > 0$.*

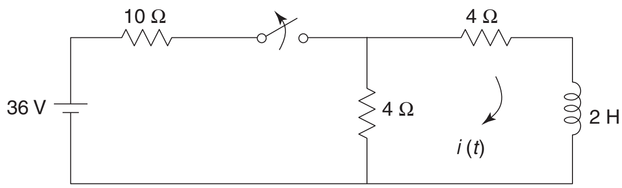


Fig. 7.21

Solution At $t = 0^-$, the network is shown in Fig 7.22. At $t = 0^-$, the switch is closed and the network has acquired steady-state. Hence, the inductor acts as a short circuit.

$$i_T(0^-) = \frac{36}{10 + (4 \,\|\, 4)} = \frac{36}{10+2} = 3\text{ A}$$

$$i(0^-) = 3 \times \frac{4}{4+4} = 1.5\text{ A}$$

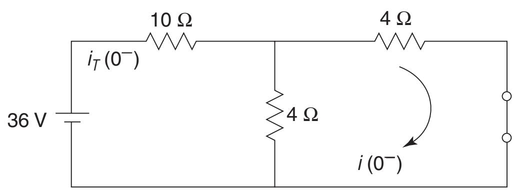


Fig. 7.22

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.5\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.23.

Applying KVL to the mesh for $t > 0$,

$$-4I(s) - 4I(s) - 2sI(s) + 3 = 0$$

$$8I(s) + 2sI(s) = 3$$

$$I(s) = \frac{3}{2s+8} = \frac{1.5}{s+4}$$

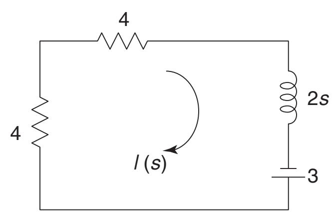


Fig. 7.23

Taking the inverse Laplace transform,

$$i(t) = 1.5e^{-4\,t} \qquad \text{for } t > 0$$

Example 7.15 *In the network shown in Fig. 7.24, the switch is closed at $t = 0$, the steady-state being reached before $t = 0$. Determine current through inductor of 3 H.*

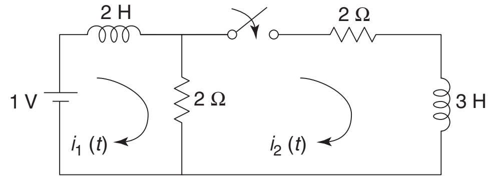


Fig. 7.24

Solution At $t = 0^-$, the network is shown in Fig. 7.25. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

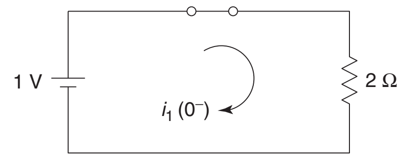


Fig. 7.25

$$i_1(0^-) = \frac{1}{2}\text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2}\text{ A}$$

$$i_2(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 17.26.

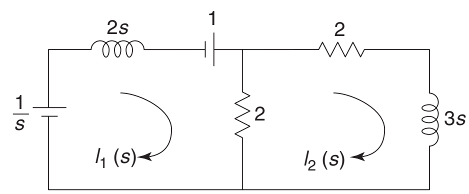


Fig. 7.26

Applying KVL to Mesh 1,

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2+2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2\,(s) = 0$$

$$-2I_1(s) + (4+3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3s^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

$$C = \left(s+\frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

Example 7.16 *In the network of Fig. 7.27, the switch is closed at t = 0 with the network previously unenergised. Determine currents $i_1(t)$.*

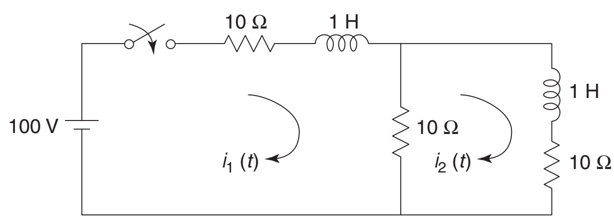


Fig. 7.27

Solution For $t > 0$, the transformed network is shown in Fig. 7.28.

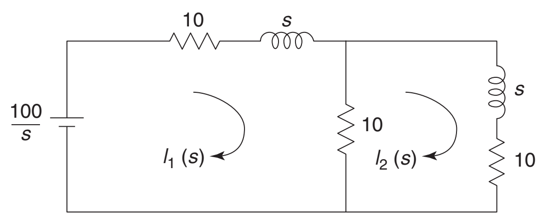


Fig. 7.28

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\, I_2(s) - 10\, I_2(s) = 0$$

$$-10\, I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = s\, I_1(s)\,|_{s=0} = \left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+10)I_1(s)\,|_{s=-10} = \left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_1(s)\,|_{s=-30} = \left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30} = -\frac{5}{3}$$

$$I_1(s) = \frac{20}{3}\frac{1}{s} - \frac{5}{s+10} - \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+20 & \dfrac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\dfrac{1000}{s}}{(s+20)^2 - 100} = \frac{1000}{s(s^2+40s+300)} = \frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = sI_2(s)\,|_{s=0} = \left.\frac{1000}{(s+10)(s+30)}\right|_{s=0} = \frac{10}{3}$$

$$B = (s+10)I_2(s)\,|_{s=-10} = \left.\frac{1000}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_2(s)\,|_{s=-30} = \left.\frac{1000}{s(s+10)}\right|_{s=-30} = \frac{5}{3}$$

$$I_2(s) = \frac{10}{3}\frac{1}{s} - \frac{5}{s+10} + \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t} \qquad \text{for } t > 0$$

7.8 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 7.29. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 7.30.

Applying KVL to the mesh,

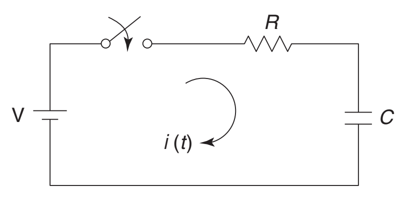


Fig. 7.29 *RC circuit*

$$\frac{V}{s} - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{R + \frac{1}{Cs}} = \frac{\frac{V}{s}}{\frac{RCs+1}{Cs}} = \frac{\frac{V}{R}}{s + \frac{1}{RC}}$$

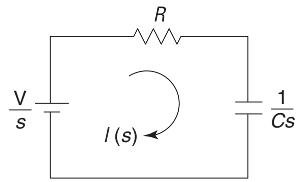


Fig. 7.30 *Transformed network*

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

Example 7.17 *In the network of Fig. 7.31, the switch is moved from a to b at t = 0. Determine i (t) and v_c (t).*

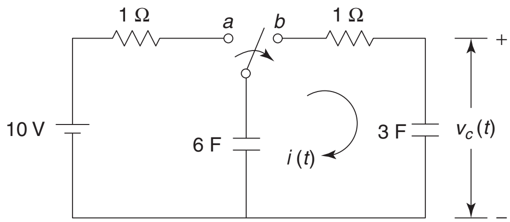


Fig. 7.31

Solution At $t = 0^-$, the network is shown in Fig. 7.32. At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,\text{F}}(0^-) = 10 \text{ V}$$
$$i(0^-) = 0$$
$$v_{3\,\text{F}}(0^-) = 0$$

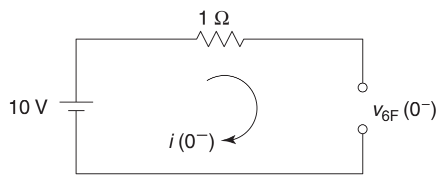


Fig. 7.32

Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,\text{F}}(0^+) = 10 \text{ V}$$
$$v_{3\,\text{F}}(0^+) = 0$$

For $t > 0$, the transformed network is shown in 7.33.

Applying KVL to the mesh for $t > 0$,

$$\frac{10}{s} - \frac{1}{6s}I(s) - I(s) - \frac{1}{3s}I(s) = 0$$

$$\frac{1}{6s}I(s) + I(s) + \frac{1}{3s}I(s) = \frac{10}{s}$$

$$I(s) = \frac{10}{s\left(1 + \frac{1}{6s} + \frac{1}{3s}\right)} = \frac{60}{6s+3} = \frac{10}{s+0.5}$$

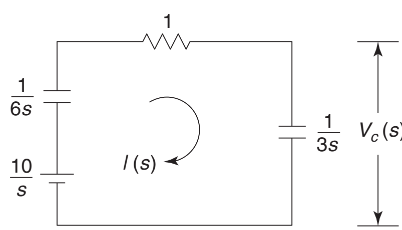


Fig. 7.33

Taking the inverse Laplace transform,

$$i(t) = 10e^{-0.5t} \qquad \text{for } t > 0$$

Voltage across the 3 F capacitor is given by

$$V_c(s) = \frac{1}{3s} I(s) = \frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\Big|_{s=0} = \frac{10}{3(s+0.5)}\Bigg|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\Big|_{s=-0.5} = \frac{10}{3s}\Bigg|_{s=-0.5} = -\frac{20}{3}$$

$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1 - e^{-0.5t}) \qquad \text{for } t > 0$$

Example 7.18 *The switch in the network shown in Fig. 7.34 is closed at t = 0. Determine the voltage cross the capacitor.*

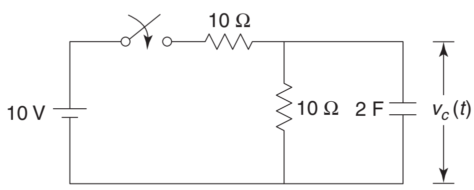


Fig. 7.34

Solution At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.35.

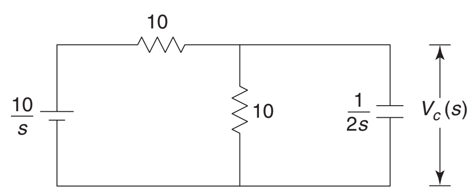


Fig. 7.35

Applying KCL at the node for $t > 0$,

$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.1}$$

$$A = sV_c(s)\Big|_{s=0} = \frac{0.5}{s+0.1}\Bigg|_{s=0} = \frac{0.5}{0.1} = 5$$

$$B = (s+0.1)V_c(s)\Big|_{s=-0.1} = \frac{0.5}{s}\Bigg|_{s=-0.1} = -\frac{0.5}{0.1} = -5$$

$$V_c(s) = \frac{5}{s} - \frac{5}{s+0.1}$$

Taking inverse Laplace transform,

$$v_c(t) = 5 - 5e^{-0.1t} \qquad \text{for } t > 0$$

Example 7.19 *In the network of Fig. 7.36, the switch is closed for a long time and at t = 0, the switch is opened. Determine the current through the capacitor.*

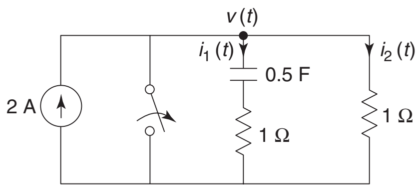


Fig. 7.36

Solution At $t = 0^-$, the network is shown in Fig. 7.37. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v_c(0^-) = 0$$

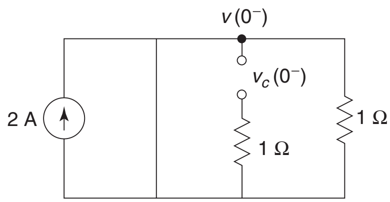


Fig. 7.37

Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.38.

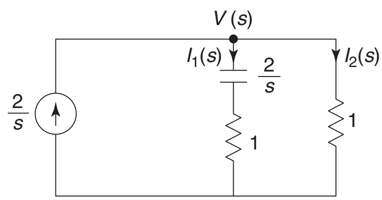


Fig. 7.38

Applying KVL to two parallel branches,

$$\frac{2}{s}I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

$$\frac{2}{s}I_1(s)+I_1(s)=\frac{2}{s}-I_1(s)$$
$$\frac{2}{s}I_1(s)+2I_1(s)=\frac{2}{s}$$
$$I_1(s)=\frac{\frac{2}{s}}{\frac{2}{s}+2}=\frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t)=e^{-t} \qquad \text{for } t>0$$

Example 7.20 *In the network of Fig. 7.39, the switch is moved from a to b, at t = 0. Find v(t).*

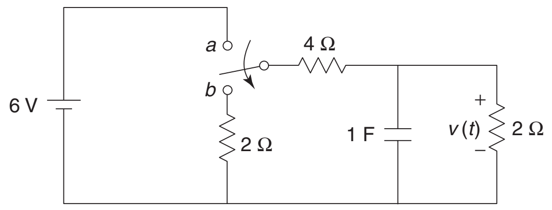


Fig. 7.39

Solution At $t = 0^-$, the network is shown in Fig 7.40. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v(0^-)=6\times\frac{2}{4+2}=2\text{ V}$$

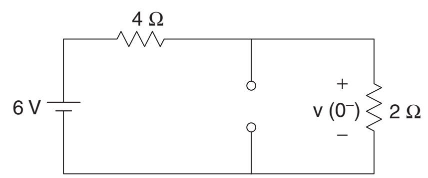


Fig. 7.40

Since voltage across the capacitor cannot change instantaneously,

$$v\,(0^+)=2\text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.41.

Applying KCL at the node for $t > 0$,

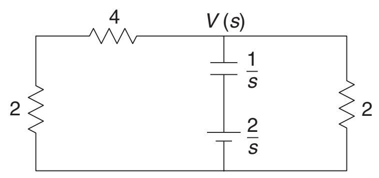


Fig. 7.41

$$\frac{V(s)}{6}+\frac{V(s)-\frac{2}{s}}{\frac{1}{s}}+\frac{V(s)}{2}=0$$
$$V(s)\left(\frac{2}{3}+s\right)=2$$
$$V(s)=\frac{2}{s+\frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t)=2e^{-\frac{2}{3}t} \qquad \text{for } t>0$$

Example 7.21 *The network shown in Fig. 7.42 has acquired steady-state at $t < 0$ with the switch open. The switch is closed at $t = 0$. Determine $v(t)$.*

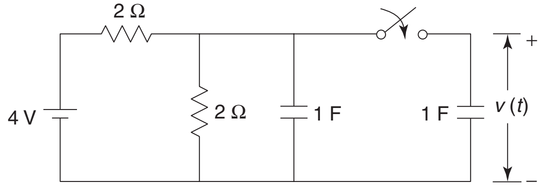


Fig. 7.42

Solution At $t = 0^-$, the network is shown in Fig 7.43. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

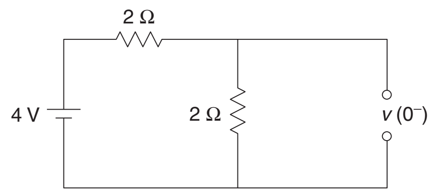


Fig. 7.43

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.44.

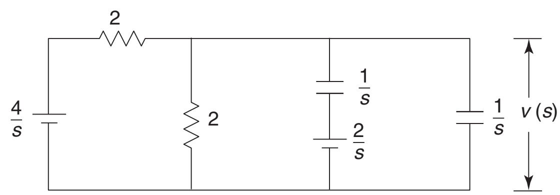


Fig. 7.44

Applying KCL at the node for $t > 0$,

$$\frac{V(s) - \frac{4}{s}}{2} + \frac{V(s)}{2} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{\frac{1}{s}} = 0$$

$$2sV(s) + V(s) = \frac{2}{s} + 2$$

$$V(s) = \frac{\frac{2}{s} + 2}{2s+1} = \frac{2s+2}{s(2s+1)} = \frac{2}{s} - \frac{2}{2s+1} = \frac{2}{s} - \frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t) = 2 - e^{-0.5t} \qquad \text{for } t > 0$$

7.9 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit shown in Fig. 7.45. The switch is closed at time $t = 0$.

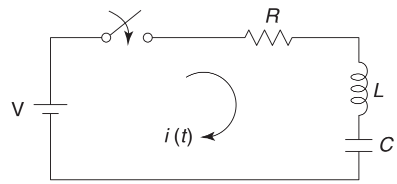


Fig. 7.45 *RLC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.46.

Applying KVL to the mesh,

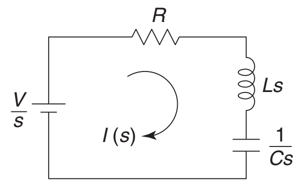


Fig. 7.46 *Transformed network*

$$\frac{V}{s} - RI(s) - Ls\,I(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + Ls + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$\left(\frac{LCs^2 + RCs + 1}{Cs}\right)I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{\frac{LCs^2 + RCs + 1}{Cs}} = \frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)}$$

where s_1 and s_2 are the roots of the equation $s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

By partial-fraction expansion, of $I(s)$,

$$I(s) = \frac{A}{s-s_1} + \frac{B}{s-s_2}$$

$$A = (s-s_1)\,I(s)\Big|_{s=s_1} = \frac{\frac{V}{L}}{s_1 - s_2}$$

$$B = (s-s_2)\,I(s)\Big|_{s=s_2} = \frac{\frac{V}{L}}{s_2 - s_1} = -\frac{\frac{V}{L}}{s_1 - s_2}$$

$$I(s) = \frac{V}{L(s_1 - s_2)}\left[\frac{1}{s-s_1} - \frac{1}{s-s_2}\right]$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{L(s_1 - s_2)}\left[e^{s_1 t} - e^{s_2 t}\right] = k_1\, e^{s_1 t} + k_2\, e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation. Now depending upon the values of s_1 and s_2, we have 3 cases of the response.

Case I When the roots are real and unequal, it gives an overdamped response.

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$\alpha > \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\left(k_1 e^{\beta t} + k_2 e^{-\beta t}\right)$$

or $$i(t) = k_1\, e^{s_1 t} + k_2\, e^{s_2 t} \qquad \text{for } t > 0$$

Case II When the roots are real and equal, it gives a critically damped response.

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\alpha = \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\,(k_1 + k_2\, t) \qquad \text{for } t > 0$$

Case III When the roots are complex conjugate, it gives an underdamped response.

$$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

$$\alpha < \omega_0$$

In this case, the solution is given by

$$i(t) = k_1\, e^{s_1 t} + k_2\, e^{s_2 t}$$

where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Let $$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$$

where $$j = \sqrt{-1}$$

and $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hence

$$i(t) = e^{-\alpha t}\left(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(k_1+k_2)\left\{\frac{e^{j\omega_d t}+e^{-j\omega_d t}}{2}\right\} + j(k_1-k_2)\left\{\frac{e^{j\omega_d t}-e^{-j\omega_d t}}{2j}\right\}\right]$$

$$= e^{-\alpha t}\left[(k_1+k_2)\cos\omega_d t + j(k_1-k_2)\sin\omega_d t\right] \qquad \text{for } t>0$$

Example 7.22 *The switch in Fig. 7.47 is opened at time t = 0. Determine the voltage v(t) for t > 0.*

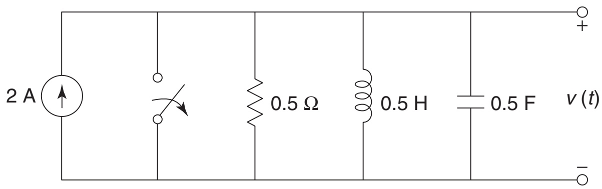


Fig. 7.47

Solution At $t = 0^-$, the network is shown in Fig. 7.48. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

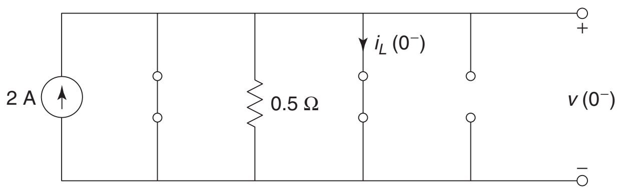


Fig. 7.48

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.49.

Applying KCL at the node for $t > 0$,

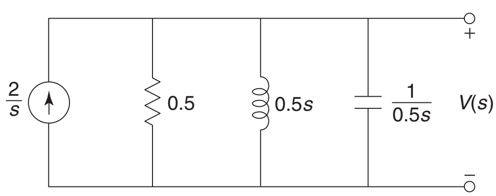


Fig. 7.49

$$\frac{V(s)}{0.5} + \frac{V(s)}{0.5s} + \frac{V(s)}{\dfrac{1}{0.5s}} = \frac{2}{s}$$

$$2V(s) + \frac{2}{s}V(s) + 0.5sV(s) = \frac{2}{s}$$

$$V(s) = \frac{\dfrac{2}{s}}{\dfrac{2}{s}+0.5s+2} = \frac{4}{s^2+4s+4} = \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform,

$$v(t) = 4t\, e^{-2t} \qquad \text{for } t > 0$$

Example 7.23 *In the network of Fig. 7.50, the switch is closed and steady-state is attained. At $t = 0$, switch is opened. Determine the current through the inductor.*

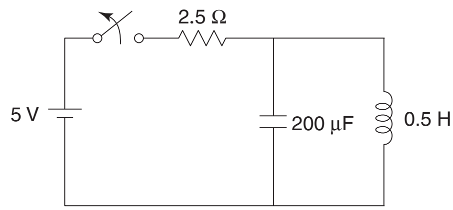


Fig. 7.50

Solution At $t = 0^-$, the network is shown in Fig. 7.51. At $t = 0^-$, the switch is closed and steady-state condition is attained. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit. Current through inductor is same as the current through the resistor.

$$i_L(0^-) = \frac{5}{2.5} = 2 \text{ A}$$

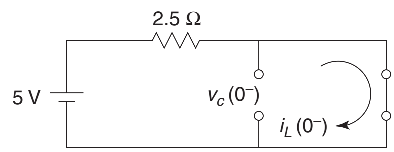


Fig. 7.51

Voltage across the capacitor is zero as it is connected in parallel with a short.

$$v_c(0^-) = 0$$

Since voltage across the capacitor and current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.52.

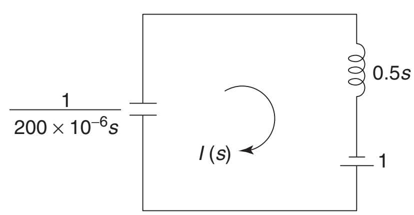


Fig. 7.52

Applying KVL to the mesh for $t > 0$,

$$-\frac{1}{200 \times 10^{-6} s} I(s) - 0.5s\, I(s) + 1 = 0$$

$$0.5s\, I(s) - 1 + 5000 \frac{I(s)}{s} = 0$$

$$I(s) = \frac{1}{0.5s + \dfrac{5000}{s}} = \frac{2s}{s^2 + 10000}$$

Taking inverse Laplace transform,

$$i(t) = 2 \cos 100t \qquad \text{for } t > 0$$

Example 7.24 *In the network shown in Fig. 7.53, the switch is opened at t =0. Steady-state condition is achieved before t = 0. Find i(t).*

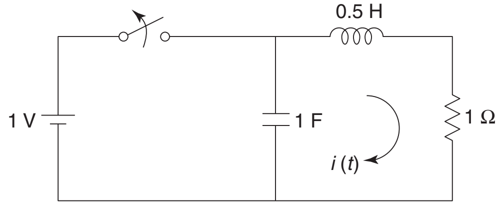


Fig. 7.53

Solution At $t = 0^-$, the network is shown in Fig 7.54. At $t = 0^-$, the switch is closed and steady-state condition is achieved. Hence, the capacitor acts as an open circuit and the inductor acts as a short circuit.

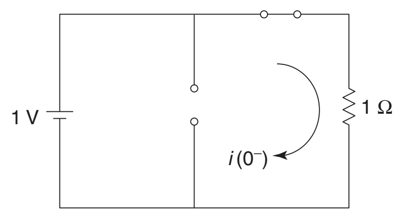


Fig. 7.54

$$v_c(0^-) = 1 \text{ V}$$
$$i(0^-) = 1 \text{ A}$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 1 \text{ V}$$
$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.55.

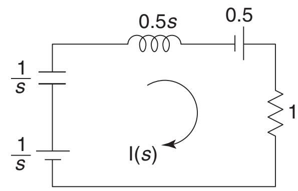


Fig. 7.55

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s} - \frac{1}{s} I(s) - 0.5s\, I(s) + 0.5 - I(s) = 0$$

$$0.5 + \frac{1}{s} = \frac{1}{s} I(s) + 0.5\, s\, I(s) + I(s)$$

$$I(s)\left[1 + \frac{1}{s} + 0.5s\right] = 0.5 + \frac{1}{s}$$

$$I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$i(t) = e^{-t}\cos t + e^{-t}\sin t \qquad \text{for } t > 0$$

Example 7.25 *In the network shown in Fig. 7.56, the switch is closed at t = 0. Find the currents $i_1(t)$ and $i_2(t)$ when initial current through the inductor is zero and initial voltage on the capacitor is 4 V.*

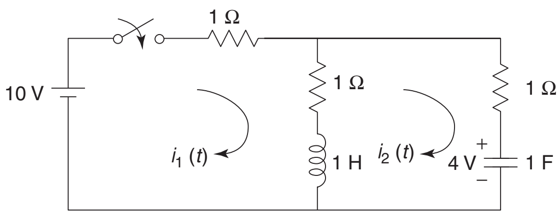


Fig. 7.56

Solution For $t > 0$, the transformed network is shown in Fig. 7.57.

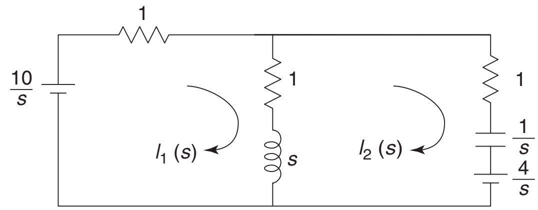


Fig. 7.57

Applying KVL to Mesh 1,

$$\frac{10}{s} - I_1(s) - (1+s)[I_1(s) - I_2(s)] = 0$$

$$(s+2)I_1(s) - (s+1)I_2(s) = \frac{10}{s}$$

Applying KVL to Mesh 2,

$$-(s+1)[I_2(s) - I_1(s)] - I_2(s) - \frac{1}{s}I_2(s) - \frac{4}{s} = 0$$

$$-(s+1)I_1(s) + \left(s + 2 + \frac{1}{s}\right)I_2(s) = -\frac{4}{s}$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{10}{s} & -(s+1) \\ -\frac{4}{s} & s+2+\frac{1}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{\left(\frac{10}{s}\right)\left(\frac{s^2+2s+1}{s}\right) - (s+1)\left(\frac{4}{s}\right)}{(s+2)\left(\frac{s^2+2s+1}{s}\right) - (s+1)^2} = \frac{\frac{10}{s^2}(s+1)^2 - (s+1)\frac{4}{s}}{(s+2)\frac{(s+1)^2}{s} - (s+1)^2}$$

$$= \frac{\frac{10}{s^2}(s+1) - \frac{4}{s}}{(s+2)\frac{(s+1)}{s} - (s+1)} = \frac{3s+5}{s(s+1)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+1}$$

$$A = sI_1(s)\big|_{s=0} = \left.\frac{3s+5}{s+1}\right|_{s=0} = 5$$

$$B = (s+1)I_1(s)\big|_{s=-1} = \left.\frac{3s+5}{s}\right|_{s=-1} = -2$$

$$I_1(s) = \frac{5}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform,

$$i_1(t) = 5 - 2e^{-t} \qquad \text{for } t > 0$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+2 & \frac{10}{s} \\ -(s+1) & -\frac{4}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{3s+1}{(s+1)^2} = \frac{3s+3-2}{(s+1)^2} = \frac{3(s+1)-2}{(s+1)^2} = \frac{3}{s+1} - \frac{2}{(s+1)^2}$$

Taking inverse Laplace transform,

$$i_2(t) = 3e^{-t} - 2te^{-t} \qquad \text{for } t > 0$$

7.10 RESPONSE OF RL CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RL* circuit shown in Fig. 7.58. When the switch is closed at $t = 0$, $i(0^-) = i(0^+) = 0$.

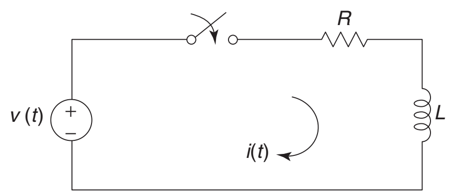


Fig. 7.58 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.59.
Applying KVL to the mesh,

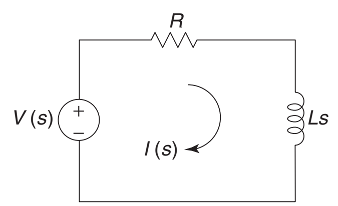


Fig. 7.59 *Transformed network*

$$V(s) - RI(s) - LsI(s) = 0$$

$$I(s) = \frac{V(s)}{R+Ls} = \frac{1}{L}\frac{V(s)}{s+\frac{R}{L}}$$

(a) When the unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{1}{L}\frac{\frac{1}{s}}{s+\frac{R}{L}}$$

$$= \frac{1}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{1}{L}\left(\frac{A}{s} + \frac{B}{s+\frac{R}{L}}\right)$$

$$A = s\,I(s)\big|_{s=0} = \left.\frac{1}{s+\frac{R}{L}}\right|_{s=0} = \frac{L}{R}$$

$$B = \left(s+\frac{R}{L}\right)I(s)\big|_{s=-\frac{R}{L}} = \left.\frac{1}{s}\right|_{s=-\frac{R}{L}} = -\frac{L}{R}$$

$$I(s) = \frac{1}{L}\left(\frac{L}{R}\frac{1}{s} - \frac{L}{R}\frac{1}{s+\frac{R}{L}}\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t \qquad \text{for } t > 0$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)}$$

By partial-faction expansion,

$$\frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+\frac{R}{L}}$$

$$\frac{1}{L} = As\left(s+\frac{R}{L}\right) + B\left(s+\frac{R}{L}\right) + Cs^2$$

Putting $s = 0$,

$$B = \frac{1}{R}$$

Putting $s = -\frac{R}{L}$,

$$C = \frac{L}{R^2}$$

Comparing coefficients of s^2,

$$A + C = 0$$

$$A = -C = -\frac{L}{R^2}$$

$$I(s) = -\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L}{R^2}\frac{1}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = -\frac{L}{R^2} + \frac{1}{R}t + \frac{L}{R^2}e^{-\left(\frac{R}{L}\right)t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(c) When unit impulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{1}{L}\frac{1}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{L}e^{-\left(\frac{R}{L}\right)t} \qquad \text{for } t > 0$$

Example 7.26 *At $t = 0$, unit pulse voltage of unit width is applied to a series RL circuit as shown in Fig. 7.60. Obtain an expression for $i(t)$.*

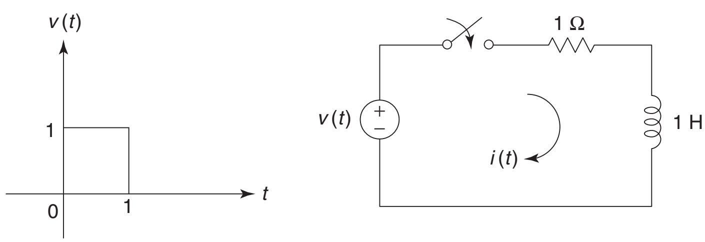


Fig. 7.60

Solution

$$v(t) = u(t) - u(t-1)$$

$$V(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

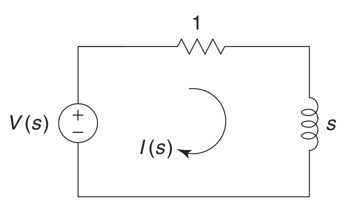


Fig. 7.61

For $t > 0$, the transformed network is shown in Fig. 7.61.
Applying KVL to the mesh,

$$V(s) - I(s) - sI(s) = 0$$

$$I(s) = \frac{V(s)}{s+1}$$

$$= \frac{1-e^{-s}}{s(s+1)}$$

$$= \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$

Taking inverse Laplace transform,

$$i(t) = u(t) - e^{-t}u(t) - u(t-1) + e^{-(t-1)}u(t-1)$$

$$= (1-e^{-t})u(t) - [1-e^{-(t-1)}]u(t-1) \qquad \text{for } t > 0$$

Example 7.27 *For the network shown in Fig. 7.62, determine the current i(t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

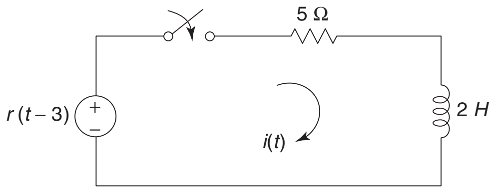


Fig. 7.62

Solution For $t > 0$, the transformed network is shown in Fig. 7.63.

Applying KVL to the mesh for $t > 0$,

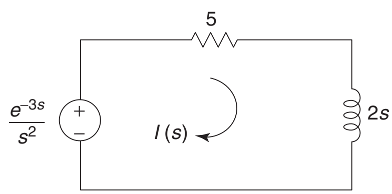


Fig. 7.63

$$\frac{e^{-3s}}{s^2} - 5\,I(s) - 2s\,I(s) = 0$$

$$5\,I(s) + 2s\,I(s) = \frac{e^{-3s}}{s^2}$$

$$I(s) = \frac{e^{-3s}}{s^2(2s+5)} = \frac{0.5\,e^{-3s}}{s^2(s+2.5)}$$

By partial-fraction expansion,

$$\frac{0.5}{s^2(s+2.5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2.5}$$

$$0.5 = As(s+2.5) + B(s+2.5) + Cs^2$$

$$= As^2 + 2.5As + Bs + 2.5B + Cs^2$$

$$= (A+C)s^2 + (2.5A+B)s + 2.5\,B$$

Comparing coefficients of s^2, s and s^0,

$$A + C = 0$$

$$2.5A + B = 0$$

$$2.5B = 0.5$$

Solving these equations,

$$A = -0.08$$
$$B = 0.2$$
$$C = 0.08$$

$$I(s) = e^{-3s}\left(-\frac{0.08}{s} + \frac{0.2}{s^2} + \frac{0.08}{s+2.5}\right)$$
$$= -0.08\frac{e^{-3s}}{s} + 0.2\frac{e^{-3s}}{s^2} + 0.08\frac{e^{-3s}}{s+2.5}$$

Taking inverse Laplace transform,

$$i(t) = -0.08u(t-3) + 0.2r(t-3) + 0.08e^{-2.5(t-3)}u(t-3)$$

Example 7.28 *Determine the expression for v_L (t) in the network shown in Fig. 7.64. Find $v_L(t)$ when (i) $v_s(t) = \delta(t)$, and (ii) $v_s(t) = e^{-t}u(t)$.*

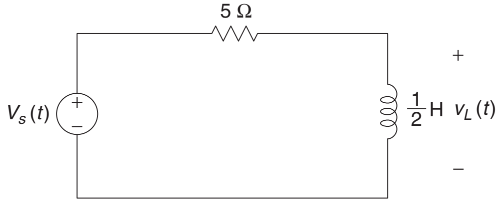


Fig. 7.64

Solution For $t > 0$, the transformed network is shown in Fig. 7.65.
By voltage-division rule,

$$V_L(s) = V_s(s) \times \frac{\frac{s}{2}}{\frac{s}{2}+5} = \frac{s}{s+10}V_s(s)$$

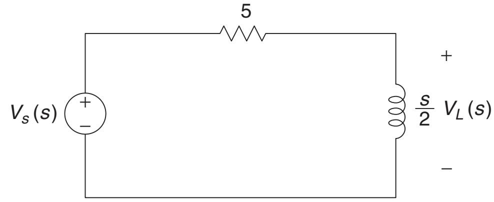


Fig. 7.65

(a) For impulse input,

$$V_s(s) = 1$$
$$V_L(s) = \frac{s}{s+10} = \frac{s+10-10}{s+10} = 1 - \frac{10}{s+10}$$

Taking inverse Laplace transform,

$$V_L(t) = \delta(t) - 10e^{-10t}u(t) \qquad \text{for } t > 0$$

(b) For $v_s(t) = e^{-t}u(t)$,

$$V_s(s) = \frac{1}{s+1}$$
$$V_L(s) = \frac{s}{(s+10)(s+1)}$$

By partial-fraction expansion,

$$V_L(s) = \frac{A}{s+10} + \frac{B}{s+1}$$

$$A = (s+10)V_L(s)\big|_{s=-10} = \frac{s}{s+1}\bigg|_{s=-10} = \frac{10}{9}$$

$$B = (s+1)V_L(s)\big|_{s=-1} = \frac{s}{s+10}\bigg|_{s=-1} = -\frac{1}{9}$$

$$V_L(s) = \frac{10}{9}\frac{1}{s+10} - \frac{1}{9}\frac{1}{s+1}$$

Taking inverse Laplace transform,

$$v_L(t) = \frac{10}{9}e^{-10t}u(t) - \frac{1}{9}e^{-t}u(t)$$

$$= \left(\frac{10}{9}e^{-10t} - \frac{1}{9}e^{-t}\right)u(t) \qquad \text{for } t > 0$$

Example 7.29 *For the network shown in Fig. 7.66, determine the current i (t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

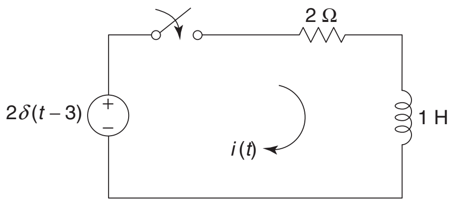


Fig. 7.66

Solution For $t > 0$, the transformed network is shown in Fig. 7.67.

Applying KVL to the mesh for $t > 0$,

$$2e^{-3s} - 2I(s) - sI(s) = 0$$

$$2I(s) + sI(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s+2}$$

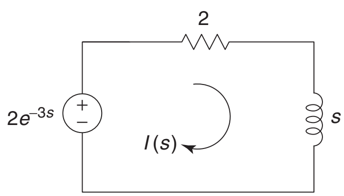


Fig. 7.67

Taking inverse Laplace transform,

$$i(t) = 2e^{-2(t-3)}u(t-3) \qquad \text{for } t > 0$$

Example 7.30 *Determine the current i(t) in the network shown in Fig. 7.68, when the switch is closed at t = 0.*

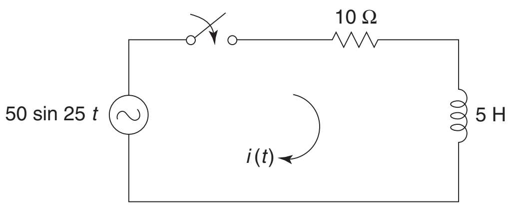


Fig. 7.68

Solution For $t > 0$, the transformed network is shown in Fig. 7.69.

Applying KVL to the mesh for $t > 0$,

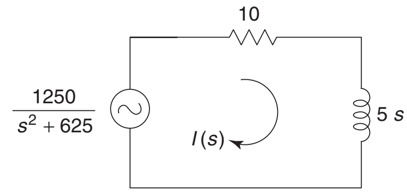


Fig. 7.69

$$\frac{1250}{s^2+625} - 10I(s) - 5sI(s) = 0$$

$$I(s) = \frac{250}{(s^2+625)(s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+625} + \frac{C}{s+2}$$

$$250 = (As+B)(s+2) + C(s^2+625)$$

$$= (A+C)s^2 + (2A+B)s + (2B+625C)$$

Comparing coefficients,

$$A + C = 0$$

$$2A + B = 0$$

$$2B + 625C = 250$$

Solving the equations,

$$A = -0.397$$

$$B = 0.795$$

$$C = 0.397$$

$$I(s) = \frac{-0.397s+0.795}{s^2+625} + \frac{0.397}{s+2} = -\frac{0.397s}{s^2+625} + \frac{0.795}{s^2+625} + \frac{0.397}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = -0.397\cos 25t + 0.032\sin 25t + 0.397e^{-2t} \qquad \text{for } t > 0$$

Example 7.31 *Find impulse response of the current i(t) in the network shown in Fig. 7.70.*

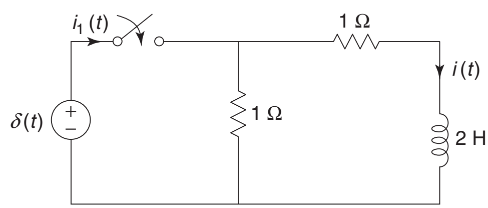


Fig. 7.70

Solution The transformed network is shown in Fig. 7.71.

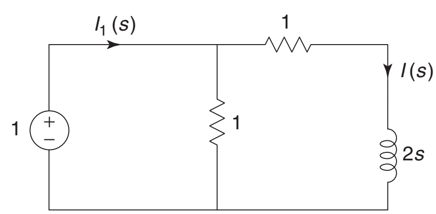


Fig. 7.71

$$Z(s) = \frac{1(2s+1)}{2s+1+1} = \frac{2s+1}{2s+2}$$

$$I_1(s) = \frac{V(s)}{Z(s)} = \frac{1}{\frac{2s+1}{2s+2}} = \frac{2s+2}{2s+1}$$

By current-division rule,

$$I(s) = I_1(s) \times \frac{1}{2s+2} = \frac{1}{2s+2} \times \frac{2s+2}{2s+1} = \frac{1}{2s+1} = \frac{1}{2}\frac{1}{s+0.5}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}e^{-0.5t}u(t) \qquad \text{for } t > 0$$

Example 7.32 *The network shown in Fig. 7.72 is at rest for $t < 0$. If the voltage $v(t) = u(t)\cos t + A\delta(t)$ is applied to the network, determine the value of A so that there is no transient term in the current response i(t).*

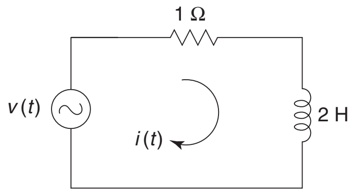


Fig. 7.72

$$v(t) = u(t)\cos t + A\delta(t)$$

$$V(s) = \frac{s}{s^2+1} + A$$

Solution For $t > 0$, the transformed network is shown in Fig. 7.73. Applying KVL to the mesh for $t > 0$,

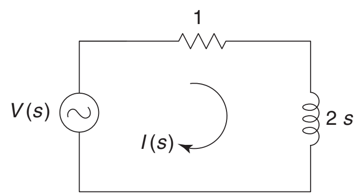


Fig. 7.73

$$V(s) = 2sI(s) + I(s) = \frac{s}{s^2+1} + A$$

$$I(s) = \frac{s + A(s^2+1)}{2\left(s+\frac{1}{2}\right)(s^2+1)} = \frac{K_1}{s+\frac{1}{2}} + \frac{K_2 s + K_3}{s^2+1}$$

The transient part of the response is given by the first term. Hence, for the transient term to vanish, $K_1 = 0$.

$$K_1 = \left(s+\frac{1}{2}\right)I(s)\Big|_{s=-\frac{1}{2}} = \frac{\frac{-1}{2} + A\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)}$$

When $K_1 = 0$

$$\frac{5}{4}A = \frac{1}{2}$$

$$A = \frac{2}{5} = 0.4$$

7.11 RESPONSE OF RC CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RC* circuit as shown in Fig. 7.74.

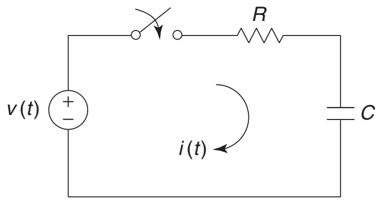


Fig. 7.74 *RC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.75.

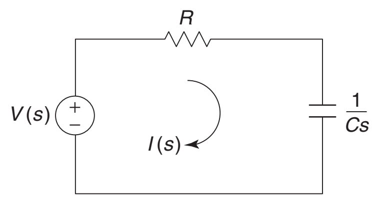


Fig. 7.75 *Transformed network*

Applying KVL to the mesh,

$$V(s) - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$I(s) = \frac{V(s)}{\frac{1}{Cs} + R} = \frac{sV(s)}{R\left(s + \frac{1}{RC}\right)}$$

(a) When unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{s \times \frac{1}{s}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{s \times \frac{1}{s^2}}{R\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{R}}{s\left(s + \frac{1}{RC}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}}$$

$$A = s\, I(s)\big|_{s=0} = \left.\frac{\frac{1}{R}}{s + \frac{1}{RC}}\right|_{s=0} = C$$

$$B = \left(s + \frac{1}{RC}\right) I(s)\Big|_{s=-\frac{1}{RC}} = \left.\frac{\frac{1}{R}}{s}\right|_{s=-\frac{1}{RC}} = -C$$

$$I(s) = \frac{C}{s} - \frac{C}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = C - Ce^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(c) When unit inpulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{s}{R\left(s + \frac{1}{RC}\right)} = \frac{s + \frac{1}{RC} - \frac{1}{RC}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left(1 - \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC} e^{-\frac{1}{RC}t}\right] \qquad \text{for } t > 0$$

Example 7.33 *A rectangular voltage pulse of unit height and T-seconds duration is applied to a series RC network at t = 0. Obtain the expression for the current i(t). Assume the capacitor to be initially uncharged.*

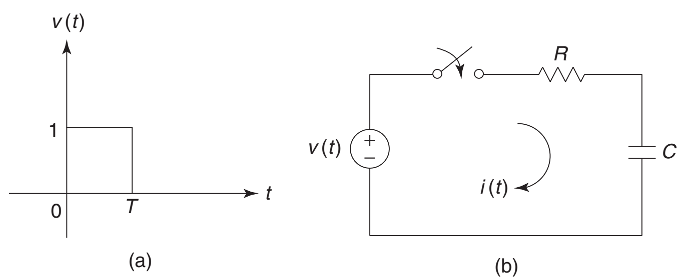


Fig. 7.76

Solution

$$v(t) = u(t) - u(t-T)$$

$$V(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}$$

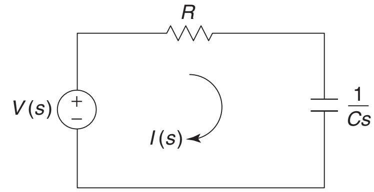


Fig. 7.77

For $t > 0$, the transformed network is shown in Fig. 7.77.
Applying KVL to the mesh for $t > 0$,

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}} = \frac{\frac{1}{R}s}{s + \frac{1}{RC}} V(s) = \frac{1-e^{-sT}}{R\left(s+\frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{1}{s+\frac{1}{RC}} - \frac{e^{-sT}}{s+\frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[e^{-\left(\frac{1}{RC}\right)t} u(t) - e^{-\left(\frac{1}{RC}\right)(t-T)} u(t-T)\right] \qquad \text{for } t > 0$$

Example 7.34 *For the network shown in Fig. 7.78, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

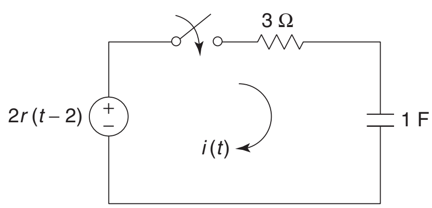


Fig. 7.78

Solution For $t > 0$, the transformed network is shown in Fig. 7.79.
Applying KVL to the mesh for $t > 0$,

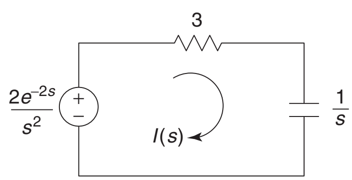


Fig. 7.79

$$\frac{2e^{-2s}}{s^2} - 3\,I(s) - \frac{1}{s} I(s) = 0$$

$$\left(3 + \frac{1}{s}\right) I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s^2\left(3+\frac{1}{s}\right)} = \frac{0.67e^{-2s}}{s(s+0.33)}$$

By partial-fraction expansion,

$$\frac{0.67}{s(s+0.33)} = \frac{A}{s} + \frac{B}{s+0.33}$$

$$A = \left.\frac{0.67}{s+0.33}\right|_{s=0} = 2$$

$$B = \left.\frac{0.67}{s}\right|_{s=-0.33} = -2$$

$$I(s) = e^{-2s}\left(\frac{2}{s} - \frac{2}{s+0.33}\right) = 2\frac{e^{-2s}}{s} - 2\frac{e^{-2s}}{s+0.33}$$

Taking inverse Laplace transform,

$$i(t) = 2u(t-2) - 2e^{-0.33(t-2)}u(t-2) \qquad \text{for } t > 0$$

Example 7.35 *For the network shown in Fig. 7.80, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

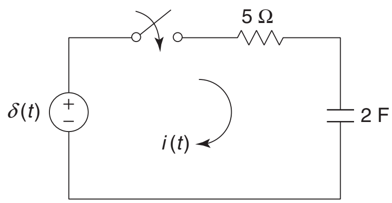


Fig. 7.80

Solution For $t > 0$, the transformed network is shown in Fig. 7.81.
Applying KVL to the mesh for $t > 0$,

$$1 - 5\,I(s) - \frac{1}{2s}I(s) = 0$$

$$\left(5 + \frac{1}{2s}\right)I(s) = 1$$

$$I(s) = \frac{1}{5 + \frac{1}{2s}}$$

$$= \frac{2s}{10s+1}$$

$$= \frac{0.2s}{s+0.1}$$

$$= \frac{0.2(s+0.1-0.1)}{s+0.1}$$

$$= 0.2\left(1 - \frac{0.1}{s+0.1}\right)$$

$$= 0.2 - \frac{0.02}{s+0.1}$$

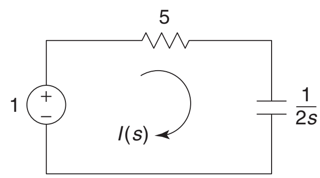


Fig. 7.81

Taking inverse Laplace transform,

$$i(t) = 0.2\delta(t) - 0.02\,e^{-0.1t}u(t)$$

Example 7.36 *For the network shown in Fig. 7.82, find the response $v_0(t)$.*

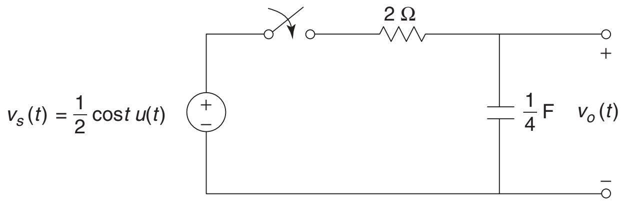


Fig. 7.82

Solution For $t > 0$, the transformed network is shown in Fig. 7.83.

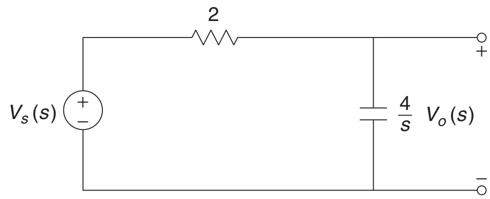


Fig. 7.83

$$V_s(s) = \frac{1}{2}\frac{s}{s^2+1}$$

By voltage-division rule,

$$V_o(s) = V_s(s) \times \frac{\frac{4}{s}}{2+\frac{4}{s}} = \frac{2V_s(s)}{s+2} = \frac{s}{(s^2+1)(s+2)}$$

By partial-fraction expansion,

$$V_o(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

$$s = (As+B)(s+2) + c(s^2+1)$$

$$s = (A+C)s^2 + (2A+B)s + (2B+C)$$

Comparing coefficient of s^2, s and s^0,

$$A + C = 0$$

$$2A + B = 1$$

$$2B + C = 0$$

Solving the equations,

$$A = 0.4, \quad B = 0.2, \quad C = -0.4$$

$$V_o(s) = \frac{0.4s+0.2}{s^2+1} - \frac{0.4}{s+2} = \frac{0.4s}{s^2+1} + \frac{0.2}{s^2+1} - \frac{0.4}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \qquad \text{for } t > 0$$

Example 7.37 *Find the impulse response of the voltage across the capacitor in the network shown in Fig. 7.84. Also determine response $v_c(t)$ for step input.*

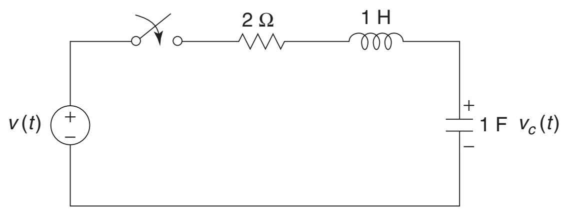


Fig. 7.84

Solution For $t > 0$, the transformed network is shown in Fig. 7.85.

By voltage-division rule,

$$V_c(s) = V(s) \times \frac{\frac{1}{s}}{2 + s + \frac{1}{s}}$$

$$= \frac{V(s)}{s^2 + 2s + 1} = \frac{V(s)}{(s+1)^2}$$

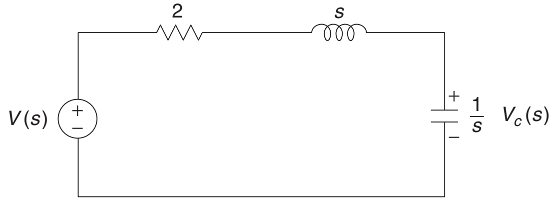


Fig. 7.85

(a) For impulse input,

$$V(s) = 1$$

$$V_c(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = te^{-t}u(t) \qquad \text{for } t > 0$$

(b) For step input,

$$V(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(s+1)^2}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$

$$= A(s^2 + 2s + 1) + B(s^2 + s) + Cs$$

$$= (A+B)s^2 + (2A + B + C)s + A$$

Comparing coefficient of s^2, s^1 and s^0,

$$A = 1$$

$$A + B = 0$$

$$B = -A = -1$$

$$2A + B + C = 0$$

$$C = -2A - B = -2 + 1 = -1$$

$$V_c(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$

$$= (1 - e^{-t} - te^{-t})ut \qquad \text{for } t > 0$$

Example 7.38 *For the network shown in Fig. 7.86, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

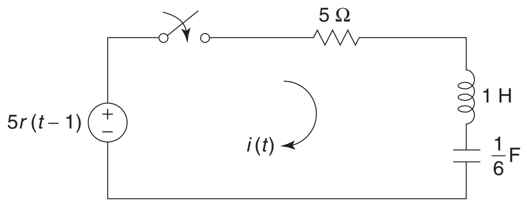


Fig. 7.86

Solution For $t > 0$, the transformed network is shown in Fig. 7.87.

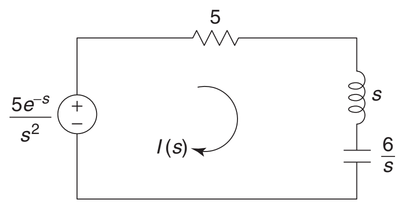


Fig. 7.87

Applying KVL to the mesh for $t > 0$,

$$\frac{5e^{-s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2+5s+6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right] = \frac{5}{6}\frac{e^{-s}}{s} + \frac{5}{3}\frac{e^{-s}}{s+3} - \frac{5}{2}\frac{e^{-s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1) \qquad \text{for } t > 0$$

Example 7.39 *For the network shown in Fig. 7.88, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions.*

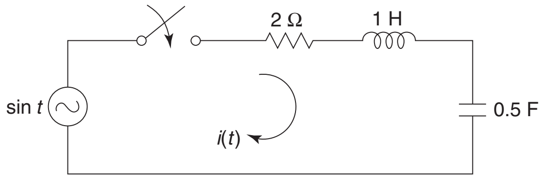


Fig. 7.88

Solution For $t > 0$, the transformed network is shown in Fig. 7.89.

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s^2+1} - 2I(s) - s\,I(s) - \frac{2}{s}I(s) = 0$$

$$\left(2+s+\frac{2}{s}\right)I(s) = \frac{1}{s^2+1}$$

$$I(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$$

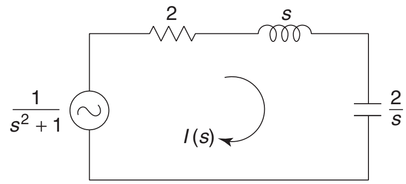


Fig. 7.89

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing coefficients of s^3, s^2, s^1 and s^0,

$$A + C = 0$$

$$2A + B + D = 0$$

$$2A + 2B + C = 1$$

$$2B + D = 0$$

Solving these equations,

$$A = 0.2,\ B = 0.4,\ C = -0.2,\ D = -0.8$$

$$I(s) = \frac{0.2s+0.4}{s^2+1} - \frac{0.2s+0.8}{s^2+2s+2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2s+0.2+0.6}{(s+1)^2+(1)^2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2(s+1)}{(s+1)^2+1} - \frac{0.6}{(s+1)^2+1}$$

Taking inverse Laplace transform,

$$i(t) = 0.2\cos t + 0.4\sin t - 0.2\,e^{-t}\cos t - 0.6\,e^{-t}\sin t$$

$$= 0.2\cos t + 0.4\sin t - e^{-t}(0.2\cos t + 0.6\sin t) \qquad \text{for } t > 0$$

Example 7.40 *For the network shown in Fig. 7.90, the switch is closed at $t = 0$. Determine the current $i(t)$ assuming zero initial conditions in the network elements.*

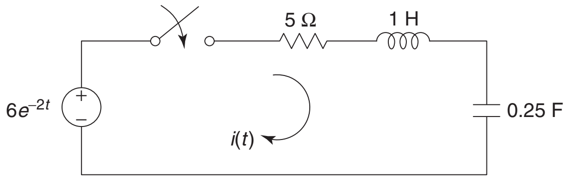


Fig. 7.90

Solution For $t > 0$, the transformed network is shown in Fig. 7.91.

Applying KVL to the mesh for $t > 0$,

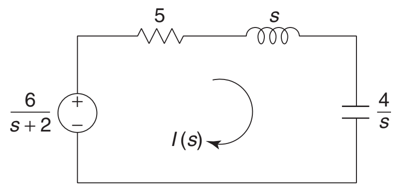


Fig. 7.91

$$\frac{6}{s+2} - 5I(s) - s\,I(s) - \frac{4}{s}I(s) = 0$$

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6s}{(s+2)(s^2+5s+4)}$$

$$= \frac{6s}{(s+2)(s+1)(s+4)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = (s+2)I(s)\,|_{s=-2} = \left.\frac{6s}{(s+1)(s+4)}\right|_{s=-2} = 6$$

$$B = (s+1)I(s)\,|_{s=-1} = \left.\frac{6s}{(s+2)(s+4)}\right|_{s=-1} = -2$$

$$C = (s+4)I(s)\,|_{s=-4} = \left.\frac{6s}{(s+2)(s+1)}\right|_{s=-4} = -4$$

$$I(s) = \frac{6}{s+2} - \frac{2}{s+1} - \frac{4}{s+4}$$

Taking inverse Laplace transform,

$$i(t) = 6e^{-2t}u(t) - 2e^{-t}u(t) - 4\,e^{-4t}u(t) \qquad \text{for } t > 0$$

Example 7.41 *The network shown has zero initial conditions. A voltage $v_i(t) = \delta(t)$ applied to two terminal network produces voltage $v_o(t) = [e^{-2t} + e^{-3t}]\,u(t)$. What should be $v_i(t)$ to give $v_o(t) = t\,e^{-2t}\,u(t)$?*

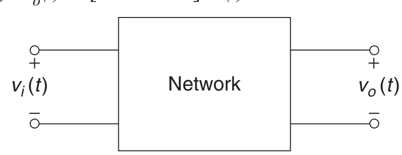


Fig. 7.92

Solution For $v_i(t) = \delta(t)$,

$$V_i(s) = 1$$
$$v_o(t) = [e^{-2t} + e^{-3t}]u(t)$$
$$V_o(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

System function $H(s) = \dfrac{V_o(s)}{V_i(s)}$

$$= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)} \qquad \ldots\text{(i)}$$

For $v_o(t) = te^{-2t}u(t)$,

$$V_o(s) = \frac{1}{(s+2)^2}$$

From Eq. (i),

$$V_i(s) = \frac{V_o(s)}{H(s)} = \frac{1}{(s+2)^2} \times \frac{(s+2)(s+3)}{2s+5} = \frac{(s+3)}{2(s+2.5)(s+2)}$$

By partial-fraction expansion,

$$V_i(s) = \frac{A}{s+2} + \frac{B}{s+2.5}$$
$$A = 1$$
$$B = -0.5$$
$$V_i(s) = \frac{1}{s+2} - \frac{0.5}{s+2.5}$$

Taking inverse Laplace transform,

$$v_i(t) = e^{-2t} - 0.5e^{-2.5t} \qquad \text{for } t > 0$$

Example 7.42 *A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal.*

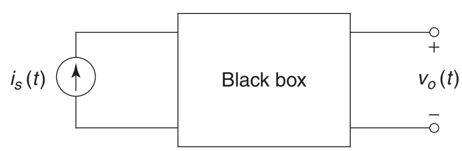


Fig. 7.93

Solution

$$v_o(t) = 2e^{-t} - e^{-3t}$$
$$V_o(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

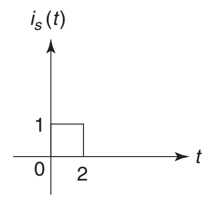


Fig. 7.94

When $i_s(t) = \delta(t)$,

$$I_s(s) = 1$$

$$V_o(s) = Z(s)\, I_s(s) \qquad \ldots\text{(i)}$$

$$Z(s) = \frac{V_o(s)}{I_s(s)} = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t)$ is a pulse of 1 A height and a duration of 2 seconds then,

$$i_s(t) = u(t) - u(t-2)$$

$$I_s(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

From Eq. (i),

$$V_o(s) = \left[\frac{2}{s+1} - \frac{1}{s+3}\right]\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{2}{s(s+1)} - \frac{1}{s(s+3)} - \frac{2e^{-2s}}{s(s+1)} + \frac{e^{-2s}}{s(s+3)}$$

$$= 2\left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right] - 2e^{-2s}\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{e^{-2s}}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace transform,

$$v(t) = 2[u(t) - e^{-t}u(t)] - \frac{1}{3}[u(t) - e^{-3t}u(t)] - 2[u(t-2) - e^{-(t-2)}u(t-2)] + \frac{1}{3}[u(t-2) - e^{-3(t-2)}u(t-2)]$$

for $t > 0$

Exercises

7.1 For the network shown in Fig 7.95, the switch is closed at $t = 0$. Find the current i1(t) for $t > 0$.

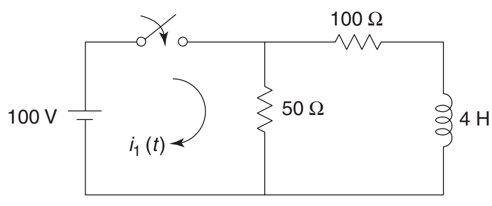


Fig. 7.95

$[i_1(t) = 3 - e^{-25\,t}]$

7.2 Determine the current $i(t)$ in the network of Fig. 7.96, when the switch is closed at $t = 0$. The inductor is initially unenergized.

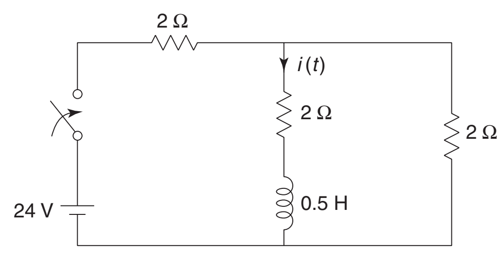


Fig. 7.96

$[i(t) = 4(1 - e^{-6t})]$

7.3 In the network of Fig. 7.97, after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.

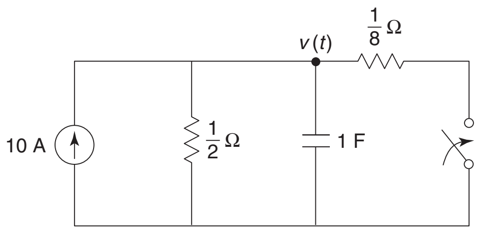


Fig. 7.97

$[v(t) = 1 + 4\,e^{-10t}]$

7.4 The circuit of Fig. 7.98, has been in the condition shown for a long time. At $t = 0$, switch is closed. Find $v(t)$ for $t > 0$.

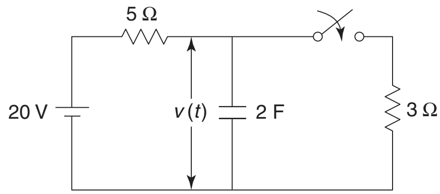


Fig. 7.98

$[v(t) = 7.5 + 12.5\,e^{-(4/15)t}]$

7.5 Figure 7.99 shows a circuit which is in the steady-state with the switch open. At $t = 0$, the switch is closed. Determine the current $i(t)$. Find its value at $t = 0.114$ μ seconds.

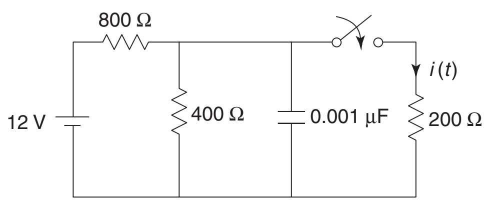


Fig. 7.99

$[i(t) = 0.00857 + 0.01143\,e^{-8.75 \times 10^6 t}, 0.013\text{ A}]$

7.6 Find $i(t)$ for the network shown in Fig. 7.100.

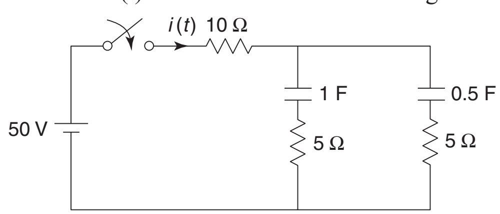


Fig. 7.100

$[i(t) = 0.125\,e^{-0.308t} + 3.875\,e^{-0.052t}]$

7.7 Determine $v(t)$ in the network of Fig. 7.101 where $i_L(0^-) = 15$ A and $v_c(0^-) = 5$ V.

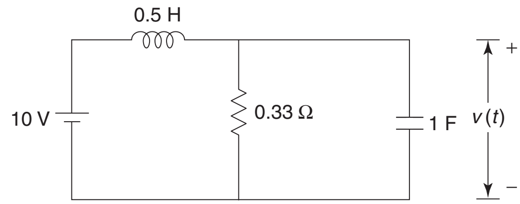


Fig. 7.101

$[v(t) = 10 - 10e^{-t} + 5e^{-2t}]$

7.8 The network shown in Fig. 7.102 has acquired steady state with the switch at position 1 for $t < 0$. At $t = 0$, the switch is thrown to the position 2. Find $v(t)$ for $t > 0$.

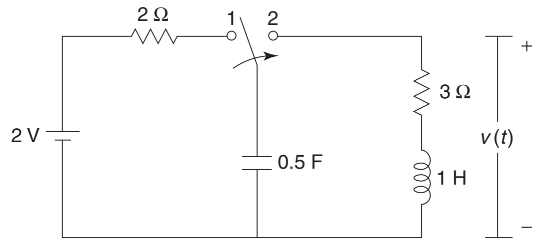


Fig. 7.102

$[v(t) = 4e^{-t} - 2\,e^{-2t}]$

7.9 In the network shown in Fig. 7.103, the switch is closed at $t = 0$. Find current $i_1(t)$ for $t > 0$.

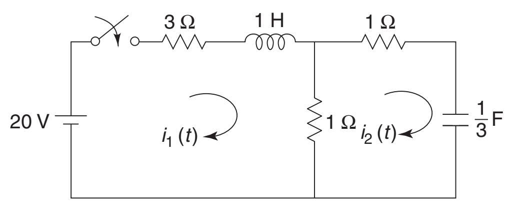


Fig. 7.103

$[i_1(t) = 5 + 5e^{-2t} - 10e^{-3t}]$

7.10 In the network shown in Fig. 7.104, the switch is closed at $t = 0$. Find the current through the 30 Ω resistor.

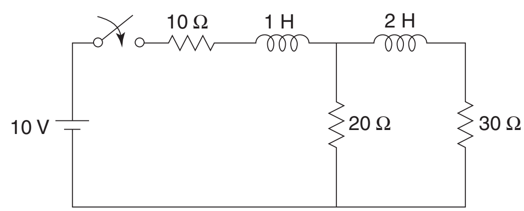


Fig. 7.104

$[i(t) = 0.1818 - 0.265\,e^{-13.14t} + 0.083\,e^{-41.86t}]$

7.11 The network shown in Fig. 7.105 is in steady state with s_1 closed and s_2 open. At $t = 0$, s_1 is opened and s_2 is closed. Find the current through the capacitor.

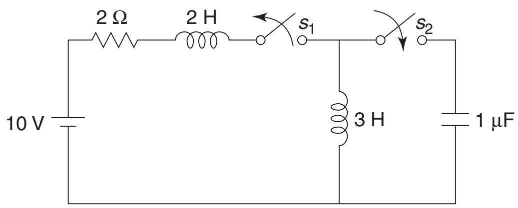


Fig. 7.105

$[i(t) = 5 \cos (0.577 \times 10^3\, t)]$

7.12 In the network shown in Fig. 7.106, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

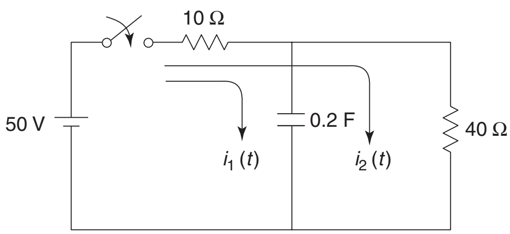


Fig. 7.106

$[i_1(t) = 5\, e^{-0.625t},\ i_2(t) = 1 - e^{-0.625t}]$

7.13 For the network shown in Fig. 7.107, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

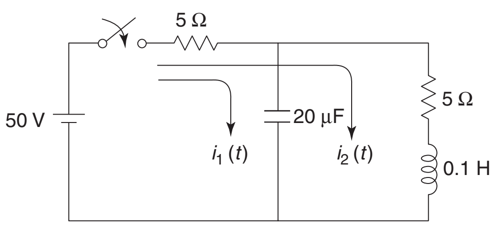


Fig. 7.107

$$\begin{bmatrix} i_1(t) = 0.101e^{-100.5t} + 10.05e^{-9949.5t} \\ i_2(t) = 5 - 5.05e^{-100.5t} + 0.05e^{-9949.5t} \end{bmatrix}$$

7.14 In the network shown in Fig. 7.108, the switch is opened at $t = 0$, the steady state having been established previously. Find $i(t)$ for $t > 0$.

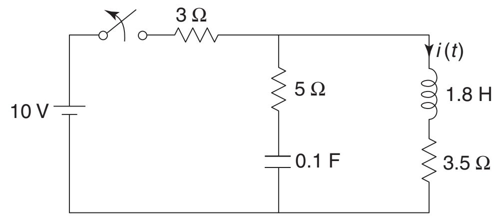


Fig. 7.108

$[i(t) = 1.5124e^{-2.22t} + 3.049e^{-2.5t}]$

7.15 Find the current $i(t)$ in the network of Fig. 7.109, if the switch is closed at $t = 0$. Assume initial conditions to be zero.

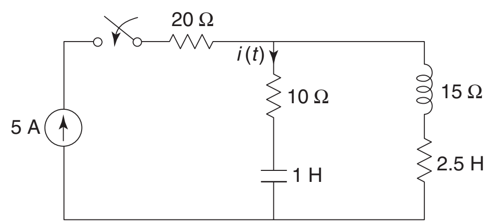


Fig. 7.109

$[i(t) = 3 + 0.57e^{-7.14\,t}]$

7.16 In the network shown in Fig. 7.110, find the voltage $v(t)$ for $t > 0$.

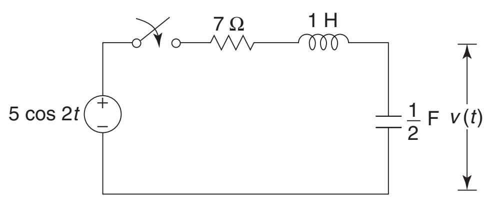


Fig. 7.110

$$\left[v(t) = -\frac{6}{5}e^{-t} + \frac{9}{10}e^{-6t} + \frac{3}{10}\cos 2t + \frac{21}{10}\sin 2t \right]$$

7.17 For the network shown in Fig. 7.111, determine $v(t)$ when the input is

(i) an impulse function $[e^{-t}\, u(t)]$

(ii) $i(t) = 4e^{-t}\, u(t)$ $[4t\, e^{-t}\, u(t)]$

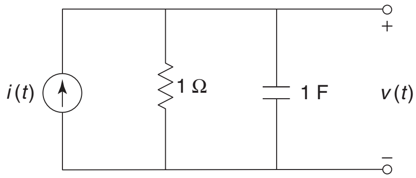


Fig. 7.111

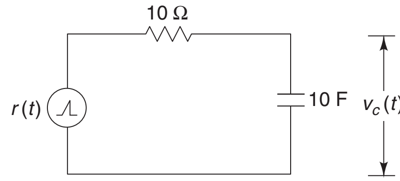


Fig. 7.112

7.18 For a unit-ramp input shown in Fig. 7.112, find the response $v_c(t)$ for $t > 0$.

$[v_c(t) = -100\, u(t) + 100e^{-0.01t}\, u(t) + tu(t)]$

Objective-Type Questions

7.1 If the Laplace transform of the voltage across a capacitor of value $\frac{1}{2}$ F is

$$V_c(s) = \frac{1}{s^2+1}$$

the value of the current through the capacitor at $t = 0^+$ is

(a) 0 (b) 2 A

(c) $\frac{1}{2}$ A (d) 1 A

7.2 The response of an initially relaxed linear constant parameter network to a unit impulse applied at $t = 0$ is $4\, e^{-2t}\, u(t)$. The response of this network to a unit-step function will be

(a) $2[1 - e^{-2t}]\, u(t)$
(b) $4[e^{-t} - e^{-2t}]\, u(t)$
(c) $\sin 2t$
(d) $(1 - 4\, e^{-4t})\, u(t)$

7.3 The Laplace transform of a unit-ramp function starting at $t = a$ is

(a) $\frac{1}{(s+a)^2}$ (b) $\frac{e^{-as}}{(s+a)^2}$

(c) $\frac{e^{-as}}{s^2}$ (d) $\frac{a}{s^2}$

7.4 The Laplace transform of $e^{at} \cos \alpha t$ is equal to

(a) $\frac{s-\alpha}{(s-\alpha)^2+\alpha^2}$

(b) $\frac{s+\alpha}{(s-\alpha)^2+\alpha^2}$

(c) $\frac{1}{(s-\alpha)^2}$

(d) none of the above

7.5 The circuit shown in Fig. 7.113 has initial current $i(0^-) = 1$ A through the inductor and an initial voltage $v_c(0^-) = -1$ V across the capacitor. For input $v(t) = u(t)$, the Laplace transform of the current $i(t)$ for $t \geq 0$ is

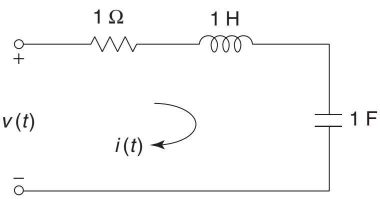


Fig. 7.113

(a) $\frac{s}{s^2+s+1}$ (b) $\frac{s+2}{s^2+s+1}$

(c) $\frac{s-2}{s^2+s+1}$ (d) $\frac{s-2}{s^2+s+1}$

7.6 A square pulse of 3 volts amplitude is applied to an *RC* circuit shown in Fig. 7.114. The capacitor is initially uncharged. The output voltage v_0 at time $t = 2$ seconds is

(a) 3 V (b) −3 V
(c) 4 V (d) −4 V

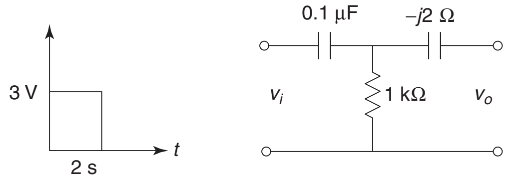


Fig. 7.114

7.7 A 2 mH inductor with some initial current can be represented as shown in Fig. 7.115. The value of the initial current is

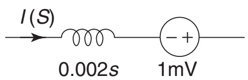


Fig. 7.115

(a) 0.5 A (b) 2 A
(c) 1 A (d) 0

7.8 A current impulse 5 $\delta(t)$ is forced through a capacitor C. The voltage $v_c(t)$ across the capacitor is given by

(a) $5\,t$ (b) $5\,u(t) - C$

(c) $\frac{5}{C}t$ (d) $\frac{5u(t)}{C}$

7.9 In the circuit shown in Fig. 7.116, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be

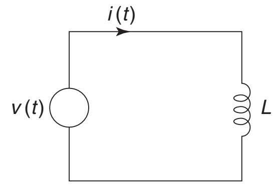


Fig. 7.116

(a) a constant voltage
(b) a linearly increasing voltage
(c) an ideal impulse
(d) as exponential increasing voltage

7.10 When a unit-impulse voltage is applied to an inductor of 1 H, the energy supplied by the source is

(a) ∞ (b) 1 J

(c) $\frac{1}{2}$ J (d) 0

Answers to Objective-Type Questions

7.1 (c)	7.2 (a)	7.3 (c)	7.4 (a)	7.5 (b)	7.6 (b)
7.7 (a)	7.8 (d)	7.9 (c)	7.10 (c)		

8 Network Functions

8.1 INTRODUCTION

A network function gives the relation between currents or voltages at different parts of the network. It is broadly classified as *driving point* and *transfer function*. It is associated with terminals and ports.

Any network may be represented schematically by a rectangular box. Terminals are needed to connect any network to any other network or for taking some measurements. Two such associated terminals are called *terminal pair* or *port*. If there is only one pair of terminals in the network, it is called a one-port network. If there are two pairs of terminals, it is called a two-port network. The port to which energy source is connected is called the *input port*. The port to which load is connected is known as the *output port*. One such network having only one pair of terminals (1 – 1′) is shown in Fig. 8.1 (a) and is called *one-port network*. Figure 8.1 (b) shows a two-port network with two pairs of terminals. The terminals 1 – 1′ together constitute a port. Similarly, the terminals 2 – 2′ constitute another port.

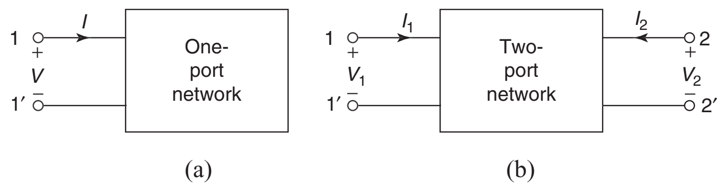


Fig. 8.1 *(a) One-port network (b) Two-port network*

A voltage and current are assigned to each of the two ports. V_1 and I_1 are assigned to the input port, whereas V_2 and I_2 are assigned to the output port. It is also assumed that currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2 respectively.

8.2 DRIVING-POINT FUNCTIONS

If excitation and response are measured at the same ports, the network function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function (and its reciprocal) can be defined.

1. ***Driving-point Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at the same port. It is denoted by $Z(s)$.

$$Z(s) = \frac{V(s)}{I(s)}$$

2. ***Driving-point Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at the same port. It is denoted by $Y(s)$.

$$Y(s) = \frac{I(s)}{V(s)}$$

For a two-port network, the driving-point impedance function and driving-point admittance function at port 1 are

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

Similarly, at port 2,

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

8.3 TRANSFER FUNCTIONS

The transfer function is used to describe networks which have at least two ports. It relates a voltage or current at one port to the voltage or current at another port. These functions are also defined as the ratio of a response transform to an excitation transform. Thus, there are four possible forms of transfer functions.

1. ***Voltage Transfer Function*** It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$.

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_{21}(s) = \frac{V_1(s)}{V_2(s)}$$

2. ***Current Transfer Function*** It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$.

$$\alpha_{12}(s) = \frac{I_2(s)}{I_1(s)}$$

$$\alpha_{21}(s) = \frac{I_1(s)}{I_2(s)}$$

3. ***Transfer Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted by $Z(s)$.

$$Z_{12}(s)=\frac{V_2(s)}{I_1(s)}$$

$$Z_{21}(s)=\frac{V_1(s)}{I_2(s)}$$

4. ***Transfer Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at another port. It denoted by Y (s).

$$Y_{12}(s)=\frac{I_2(s)}{V_1(s)}$$

$$Y_{21}(s)=\frac{I_1(s)}{V_2(s)}$$

Example 8. 1 *Determine the driving-point impedance function of a one-port network shown in Fig. 8.2.*

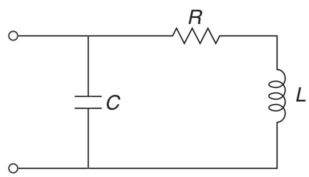


Fig. 8.2

Solution The transformed network is shown in Fig. 8.3.

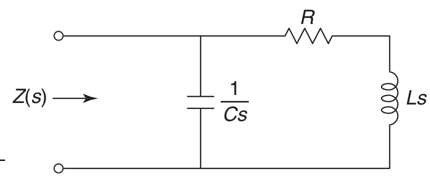


Fig. 8.3

$$Z(s)=\frac{\frac{1}{Cs}(R+Ls)}{\frac{1}{Cs}+(R+Ls)}=\frac{R+Ls}{LCs^2+RCs+1}=\frac{1}{C}\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example 8.2 *Determine the driving-point impedance of the network shown in Fig. 8.4.*

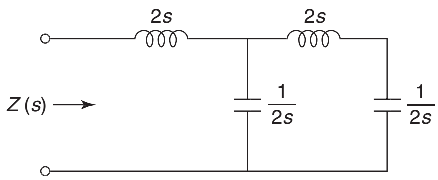


Fig. 8.4

Solution

$$Z(s)=2s+\frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{1}{2s}+2s+\frac{1}{2s}}=2s+\frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{2+4s^2}{2s}}=2s+\frac{2s+\frac{1}{2s}}{2+4s^2}=\frac{4s+8s^3+2s+\frac{1}{2s}}{2+4s^2}=\frac{16s^4+12s^2+1}{8s^3+4s}$$

Example 8.3 *Determine the driving-point impedance of the network shown in Fig. 8.5.*

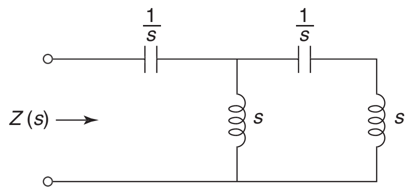


Fig. 8.5

Solution

$$Z(s)=\frac{1}{s}+\frac{s\left(\frac{1}{s}+s\right)}{s+\frac{1}{s}+s}=\frac{1}{s}+\frac{(1+s^2)\,s}{2s^2+1}=\frac{1}{s}+\frac{s+s^3}{2s^2+1}=\frac{s^4+3s^2+1}{2s^3+s}$$

Example 8.4 *Find the driving-point admittance function of the network shown in Fig. 8.6.*

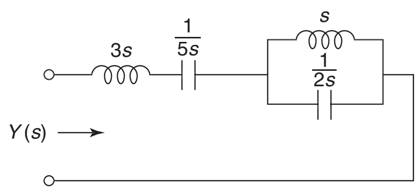


Fig. 8.6

Solution

$$Z(s)=3s+\frac{1}{5s}+\frac{s\left(\frac{1}{2s}\right)}{s+\frac{1}{2s}}=3s+\frac{1}{5s}+\frac{s}{2s^2+1}=\frac{30s^4+15s^2+2s^2+1+5s^2}{5s\,(2s^2+1)}=\frac{30s^4+22s^2+1}{5s\,(2s^2+1)}$$

$$Y(s)=\frac{1}{Z(s)}=\frac{5s\,(2s^2+1)}{30s^4+22s^2+1}$$

Example 8.5 *Find the transfer impedance function $Z_{12}(s)$ for the network shown in Fig. 8.7.*

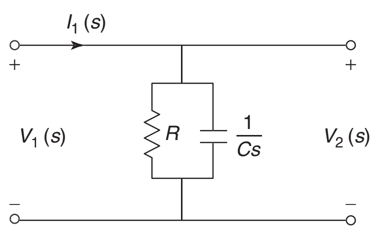


Fig. 8.7

Solution

$$V_2(s) = I_1(s)\frac{R\left(\frac{1}{Cs}\right)}{R+\frac{1}{Cs}}$$

$$\frac{V_2(s)}{I_1(s)} = \frac{R}{RCs+1}$$

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{C\left(s+\frac{1}{RC}\right)}$$

Example 8.6 *Find voltage transfer function of the two-port network shown in Fig. 8.8.*

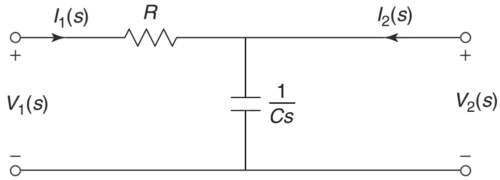


Fig. 8.8

Solution By voltage division rule,

$$V_2(s) = V_1(s)\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = V_1(s)\frac{1}{RCs+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Voltage transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

8.4 ANALYSIS OF LADDER NETWORKS

The network functions of a ladder network can be obtained by a simple method. This method depends upon the relationships that exist between the branch currents and node voltages of the ladder network. Consider the network shown in Fig. 8.9 where all the impedances are connected in series branches and all the admittances are connected in parallel branches.

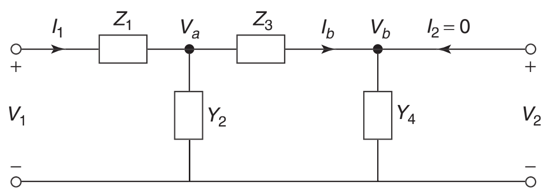


Fig. 8.9 *Ladder network*

Analysis is done by writing the set of equations. In writing these equations, we begin at the port 2 of the ladder and work towards the port 1.

$$V_b = V_2$$
$$I_b = Y_4 V_2$$
$$V_a = Z_3 I_b + V_2 = (Z_3 Y_4 + 1) V_2$$
$$I_1 = Y_2 V_a + I_b = [Y_2 (Z_3 Y_4 + 1) + Y_4] V_2$$
$$V_1 = Z_1 I_1 + V_a = [Z_1 \{Y_2 (Z_3 Y_4 + 1) + Y_4\} + (Z_3 Y_4 + 1)] V_2$$

Thus, each succeeding equation takes into account one new impedance or admittance. Except the first two equations, each subsequent equation is obtained by multiplying the equation just preceding it by imittance (either impedance or admittance) that is next down the line and then adding to this product the equation twice preceding it. After writing these equations, we can obtain any network function.

Example 8.7 *For the network shown in Fig. 8.10, determine transfer function* $\frac{V_2}{V_1}$.

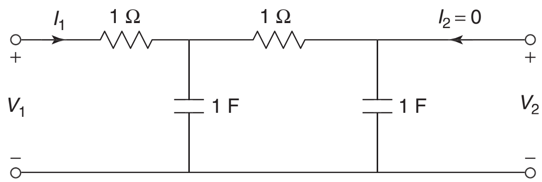


Fig. 8.10

Solution The transformed network is shown in Fig. 8.11.

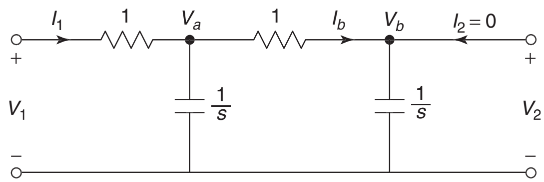


Fig. 8.11

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_a = 1I_b + V_2$$

$$= sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s+1)V_2 + sV_2 = (s^2+2s)V_2$$

$$V_1 = 1I_1 + V_a = (s^2+2s)V_2 + (s+1)V_2 = (s^2+3s+1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$$

Example 8.8 *For the network shown in Fig. 8.12, determine the voltage transfer function* $\frac{V_2}{V_1}$.

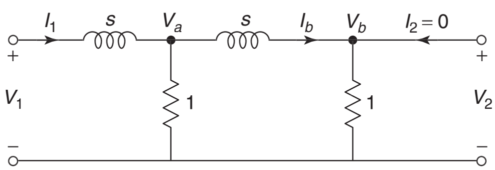


Fig. 8.12

Solution

$$V_b = V_2$$

$$I_b = \frac{V_2}{1} = V_2$$

$$V_a = sI_b + V_2 = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{1} + I_b = (s+1)V_2 + V_2 = (s+2)V_2$$

$$V_1 = sI_1 + V_a = s(s+2)V_2 + (s+1)V_2 = (s^2+3s+1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$$

Example 8.9 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$ *and* $\frac{V_2}{I_1}$ *for the network shown in Fig. 8.13.*

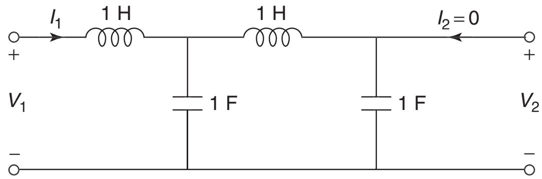


Fig. 8.13

Solution The transformed network is shown in Fig. 8.14.

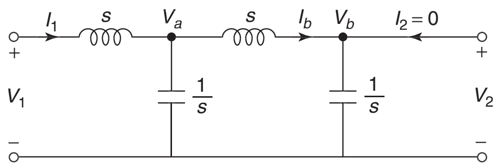


Fig. 8.14

$$V_b = V_2$$
$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$
$$V_a = sI_b + V_2 = s(sV_2) + V_2 = (s^2+1)V_2$$
$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s^2+1)V_2 + sV_2 = (s^3+2s)V_2$$
$$V_1 = sI_1 + V_a = s(s^3+2s)V_2 + (s^2+1)V_2 = (s^4+2s^2+s^2+1)V_2 = (s^4+3s^2+1)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{s^4+3s^2+1}{s^3+2s}$$
$$\frac{V_2}{V_1} = \frac{1}{s^4+3s^2+1}$$
$$\frac{V_2}{I_1} = \frac{1}{s^3+2s}$$

Example 8.10 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$, *and* $\frac{V_2}{I_1}$ *for the network in Fig. 8.15.*

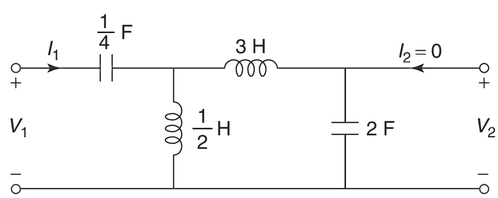


Fig. 8.15

Solution The transformed network is shown in Fig. 8.16.

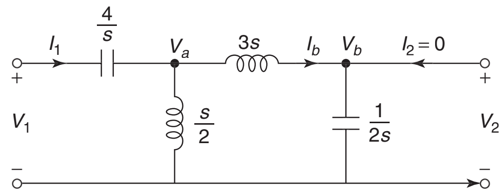


Fig. 8.16

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{2s}} = 2sV_2$$

$$V_a = 3sI_b + V_2 = 3s(2sV_2) + V_2 = (6s^2+1)V_2$$

$$I_1 = \frac{V_a}{\frac{s}{2}} + I_b = \frac{2}{s}(6s^2+1)V_2 + 2sV_2 = \left(\frac{14s^2+2}{s}\right)V_2$$

$$V_1 = \frac{4}{s}I_1 + V_a = \frac{4}{s}\left(\frac{14s^2+2}{s}\right)V_2 + (6s^2+1)V_2 = \left(\frac{6s^4+57s^2+8}{s^2}\right)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{6s^4+57s^2+8}{14s^3+2s}$$

$$\frac{V_2}{V_1} = \frac{s^2}{6s^4+57s^2+8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2+2}$$

Example 8.11 *For the ladder network of Fig. 8.17, find the driving point-impedance at the 1 – 1′ terminal with 2 – 2′ open.*

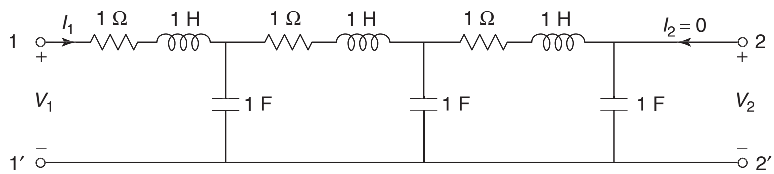


Fig. 8.17

Solution The transformed network is shown in Fig. 8.18.

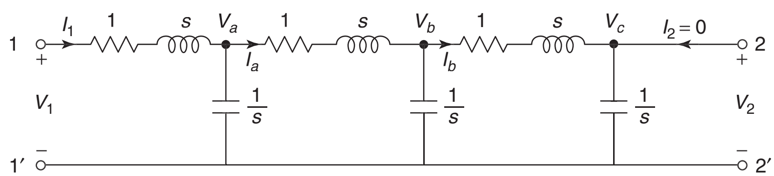


Fig. 8.18

$$V_c = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = s\,V_2$$

$$V_b = (s+1)\,I_b + V_2 = (s+1)sV_2 + V_2 = (s^2 + s + 1)V_2$$

$$I_a = \frac{V_b}{\frac{1}{s}} + I_b = sV_b + I_b = s(s^2 + s + 1)V_2 + sV_2 = (s^3 + s^2 + 2s)V_2$$

$$V_a = (s+1)\,I_a + V_b = (s+1)(s^3 + s^2 + 2s)V_2 + (s^2 + s + 1)V_2 = (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(s^4 + 2s^3 + 4s^2 + 3s + 1)V_2 + (s^3 + s^2 + 2s)V_2$$

$$= (s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2$$

$$V_1 = (s+1)\,I_1 + V_a = (s+1)(s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2 + (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$= (s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1)V_2$$

Hence, $$Z_{11} = \frac{V_1}{I_1} = \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

Example 8.12 *Determine the voltage transfer function* $\frac{V_2}{V_1}$ *for the network shown in Fig. 8.19.*

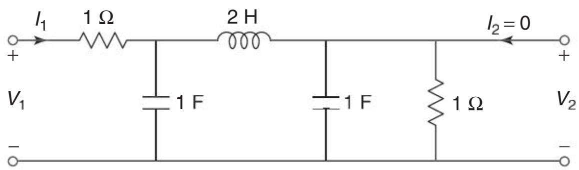


Fig. 8.19

Solution The transformed network is shown in Fig. 8.20.

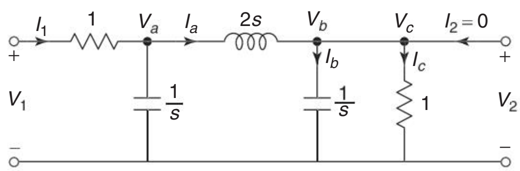


Fig. 8.20

$$V_c = V_b = V_2$$

$$I_a = I_b + I_c = \frac{V_2}{\frac{1}{s}} + \frac{V_2}{1} = s\,V_2 + V_2 = (s+1)V_2$$

$$V_a = 2s\,I_a + V_2 = 2s(s+1)V_2 + V_2$$

$$= (2s^2 + 2s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(2s^2 + 2s + 1)V_2 + (s+1)V_2 = (2s^3 + 2s^2 + 2s + 1)V_2$$

$$V_1 = 1I_1 + V_a = (2s^3 + 2s^2 + 2s + 1)V_2 + (2s^2 + 2s + 1)V_2 = (2s^3 + 4s^2 + 4s + 2)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{2s^3 + 4s^2 + 4s + 2}$$

Example 8.13 *For the network shown in Fig. 8.21, determine the transfer function* $\frac{I_2}{V_1}$.

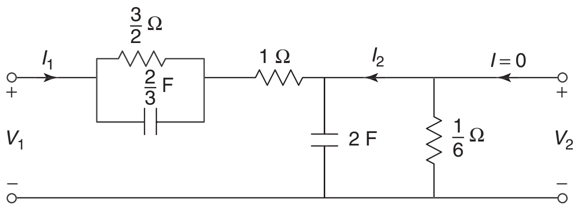


Fig. 8.21

Solution The transformed network is shown in Fig. 8.22.

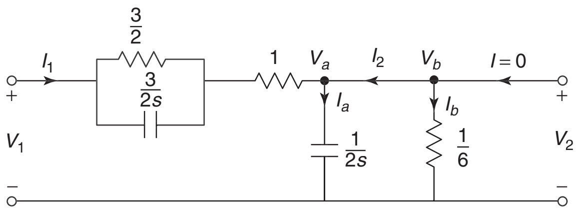


Fig. 8.22

$$V_b = V_a = V_2$$

$$I_1 = I_a + I_b = \frac{V_2}{\frac{1}{2s}} + \frac{V_2}{\frac{1}{6}} = 2sV_2 + 6V_2 = (2s+6)V_2$$

$$V_1 = \left(\frac{\frac{3}{2}\times\frac{3}{2s}}{\frac{3}{2}+\frac{3}{2s}} + 1\right)I_1 + V_2 = \left(\frac{9}{6s+6}+1\right)I_1 + V_2 = \frac{(6s+15)}{6s+6}(2s+6)V_2 + V_2 = \left[\frac{(2s+5)(s+3)}{(s+1)}+1\right]V_2$$

$$= \left[\frac{(2s+5)(s+3)+(s+1)}{s+1}\right]V_2 = \left(\frac{2s^2+6s+5s+15+s+1}{s+1}\right)V_2 = \left(\frac{2s^2+12s+16}{s+1}\right)V_2$$

$$= \frac{2(s^2+6s+8)}{s+1}V_2 = \frac{2(s+4)(s+2)}{s+1}V_2$$

Also, $\quad I_2 = -I_b = -6\,V_2$

$$\frac{I_2}{V_1} = -\frac{3(s+1)}{(s+4)(s+2)}$$

Example 8.14 *For the network shown in Fig. 8.23, compute* $\alpha_{12}(s) = \frac{I_2}{I_1}$ *and* $Z_{12}(s) = \frac{V_2}{I_1}$.

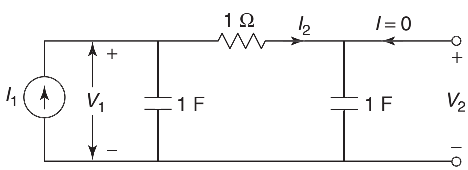


Fig. 8.23

Solution The transformed network is shown in Fig. 8.24.

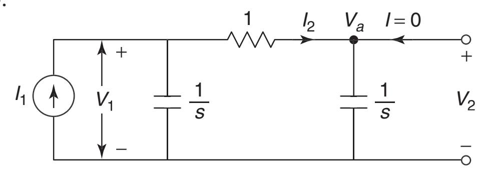


Fig. 8.24

$$V_a = V_2$$

$$I_2 = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_1 = 1I_2 + V_a = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_1}{\frac{1}{s}} + I_2 = sV_1 + I_2 = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

Hence, $$\alpha_{12}(s) = \frac{I_2}{I_1} = \frac{1}{s+2}$$

and $$Z_{12}(s) = \frac{V_2}{I_1} = \frac{1}{s^2 + 2s}$$

Example 8.15 *Determine the voltage ratio* $\frac{V_2}{V_1}$, *current ratio* $\frac{I_2}{I_1}$, *transfer impedance* $\frac{V_2}{I_1}$ *and driving- point impedance* $\frac{V_1}{I_1}$ *for the network shown in Fig. 8.25.*

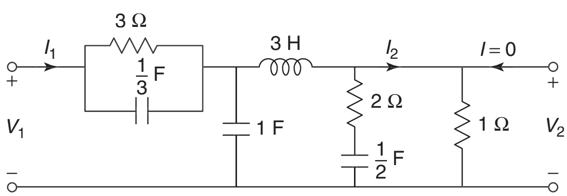


Fig. 8.25

Solution The transformed network is shown in Fig. 8.26.

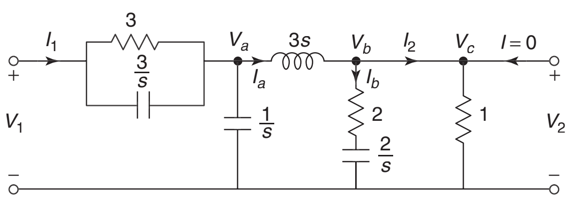


Fig. 8.26

$$V_c = V_b = V_2$$

$$I_2 = \frac{V_2}{1} = V_2$$

$$I_a = I_b + I_2 = \frac{V_2}{2+\frac{2}{s}} + \frac{V_2}{1} = \frac{s}{2s+2}V_2 + V_2 = \left(\frac{3s+2}{2s+2}\right)V_2$$

$$V_a = 3s\,I_a + V_2 = \frac{3s(3s+2)}{2s+2}V_2 + V_2 = \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = \frac{s(9s^2+8s+2)}{2s+2}V_2 + \left(\frac{3s+2}{2s+2}\right)V_2 = \left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2$$

$$V_1 = \left(\frac{3\times\frac{3}{s}}{3+\frac{3}{s}}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)\left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2 + \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$= \left[\frac{27s^3+24s^2+15s+6+9s^3+8s^2+2s+9s^2+8s+2}{(s+1)(2s+2)}\right]V_2 = \frac{(36s^3+41s^2+25s+8)}{(s+1)(2s+2)}V_2$$

$$= \left(\frac{36s^3+41s^2+25s+8}{2s^2+4s+2}\right)V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{2s^2+4s+2}{36s^3+41s^2+25s+8}$$

$$\frac{I_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_1}{I_1} = \frac{36s^3+41s^2+25s+8}{(s+1)(9s^3+8s^2+5s+2)} = \frac{36s^3+41s^2+25s+8}{9s^4+17s^3+13s^2+7s+2}$$

Example 8.16 *For the resistive two-port network of Fig. 8.27, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

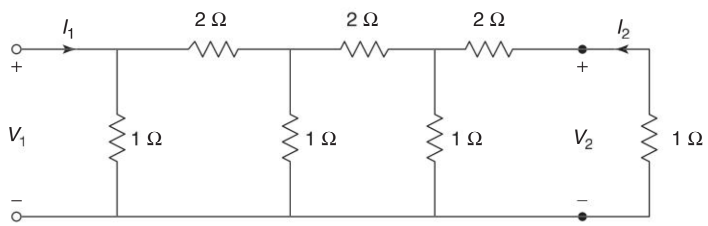


Fig. 8.27

Solution The network is redrawn as shown in Fig. 8.28.

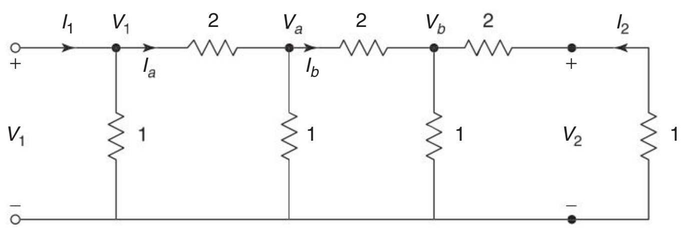


Fig. 8.28

$$I_2 = -\frac{V_2}{1} = -V_2$$

$$V_b = -3I_2 = 3V_2$$

$$I_b = \frac{V_b}{1} + \frac{V_b}{3} = \frac{4}{3}V_b = 4V_2$$

$$V_a = 2I_b + V_b = 8V_2 + 3V_2 = 11V_2$$

$$I_a = \frac{V_a}{1} + I_b = 11V_2 + 4V_2 = 15V_2$$

$$V_1 = 2I_a + V_a = 30V_2 + 11V_2 = 41V_2$$

$$I_1 = \frac{V_1}{1} + I_a = 41V_2 + 15V_2 = 56V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{1}{41}$$

$$\frac{V_2}{I_1} = \frac{1}{56}\,\Omega$$

$$\frac{I_2}{V_1} = -\frac{1}{41}\ \mho$$

$$\frac{I_2}{I_1} = -\frac{1}{56}$$

Example 8.17 *Find the network function $\frac{V_2}{V_1}$ for the network shown in Fig. 8.29.*

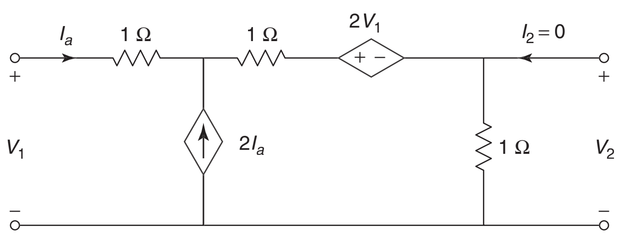


Fig. 8.29

Solution The network is redrawn as shown in Fig. 8.30.

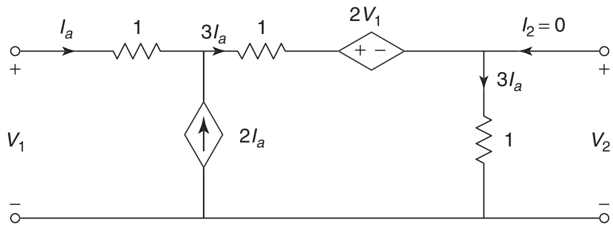


Fig. 8.30

From Fig. 8.30,

$$V_2 = 1\,(3\,I_a) = 3\,I_a$$

Applying KVL to the outermost loop,

$$V_1 - 1\,(I_a) - 1\,(3I_a) - 2\,V_1 - 1\,(3I_a) = 0$$

$$V_1 = -7I_a$$

Hence,

$$\frac{V_2}{V_1} = -\frac{3}{7}$$

Example 8.18 *Find the network function $\dfrac{I_2}{I_1}$ for the network shown in Fig. 8.31.*

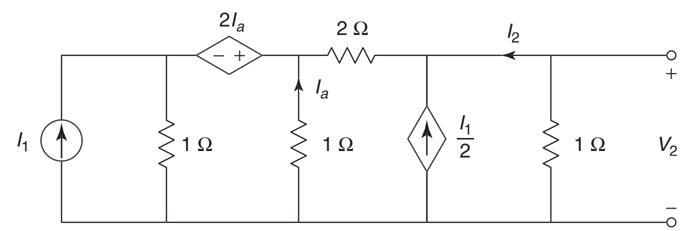


Fig. 8.31

Solution The network is redrawn as shown in Fig. 8.32.

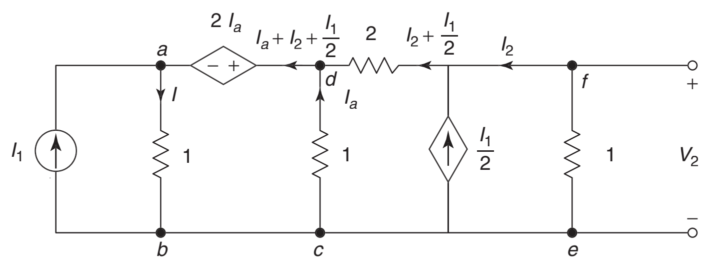


Fig. 8.32

From Fig. 8.32,

$$I = I_1 + I_a + I_2 + \frac{I_1}{2}$$

$$= \frac{3}{2}I_1 + I_a + I_2 \qquad \text{...(i)}$$

Applying KVL to the loop *abcda*,

$$-1I - 1I_a - 2I_a = 0$$

$$-I - 3I_a = 0$$

$$I + 3I_a = 0 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$\frac{3}{2}I_1 + I_a + I_2 + 3I_a = 0$$

$$\frac{3}{2}I_1 + I_2 + 4I_a = 0 \qquad \text{...(iii)}$$

Applying KVL to the loop *dcefd,*

$$1I_a - 1I_2 - 2\left(I_2 + \frac{I_1}{2}\right) = 0$$

$$I_a - 3I_2 - I_1 = 0$$

$$I_a = 3I_2 + I_1 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (iii),

$$\frac{3}{2}I_1 + I_2 + 4(3I_2 + I_1) = 0$$

$$\frac{3}{2}I_1 + I_2 + 12I_2 + 4I_1 = 0$$

$$\frac{11}{2}I_1 + 13I_2 = 0$$

$$13I_2 = -\frac{11}{2}I_1$$

Hence,

$$\frac{I_2}{I_1} = -\frac{11}{26}$$

8.5 ANALYSIS OF NON-LADDER NETWORKS

The above method is applicable for ladder networks. There are other network configurations to which the technique described is not applicable. Figure 8.33 shows one such network.

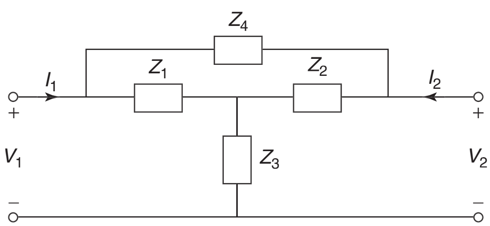


Fig. 8.33 *Non-ladder network*

For such a type of network, it is necessary to express the network functions as a quotient of determinants, formulated on KVL and KCL basis.

Example 8.19 *For the resistive bridged T network shown in Fig. 8.34, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

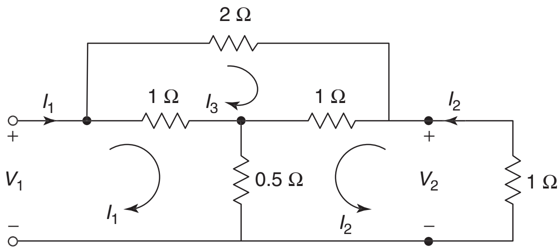


Fig. 8.34

Solution Applying KVL to Mesh 1,

$$V_1 = 1.5 I_1 + 0.5 I_2 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = 0.5 I_1 + 2.5 I_2 + I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -I_1 + I_2 + 4 I_3 \qquad \text{...(iii)}$$

Writing these equations in matrix form,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad \text{...(iv)}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & 0.5 & -1 \\ 0 & 2.5 & 1 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{V_1(10-1)}{9} = V_1 \qquad \text{...(iv)}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 1.5 & V_1 & -1 \\ 0.5 & 0 & 1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{-V_1(2+1)}{9} = -\frac{1}{3} V_1 \qquad \text{...(v)}$$

From Fig. 8.34, $V_2 = -1\,(I_2) = -I_2$

From Eq. (v), $V_1 = -3 I_2$

From Eqs. (iv) and (v), $I_2 = -\frac{1}{3} V_1 = -\frac{1}{3} I_1$

$$I_1 = -3 I_2$$

Hence, $\frac{I_2}{V_1} = -\frac{1}{3}\,℧$

$$\frac{I_2}{I_1} = \frac{-\frac{1}{3}V_1}{V_1} = -\frac{1}{3}$$

$$\frac{V_2}{V_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}$$

$$\frac{V_2}{I_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}\,\Omega$$

Example 8.20 *For the network of Fig. 8.35, find Z_{11}, Z_{12} and G_{12}.*

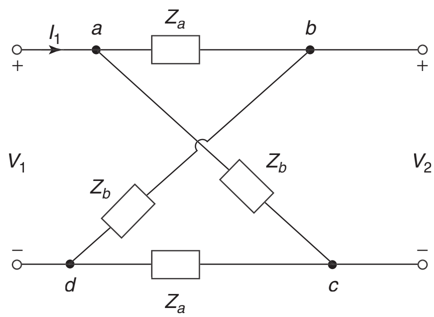


Fig. 8.35

Solution The network can be redrawn as shown in Fig. 8.36. Since the network consists of two identical impedances connected in parallel, the current in I_1 divides equally in each branch.

$$V_1 = (Z_a + Z_b)\frac{I_1}{2}$$

$$Z_{11} = \frac{V_1}{I_1} = \frac{Z_a + Z_b}{2}$$

$$V_2 = Z_b\frac{I_1}{2} - Z_a\left(\frac{I_1}{2}\right) = (Z_b - Z_a)\frac{I_1}{2}$$

$$Z_{12} = \frac{V_2}{I_1} = \frac{Z_b - Z_a}{2}$$

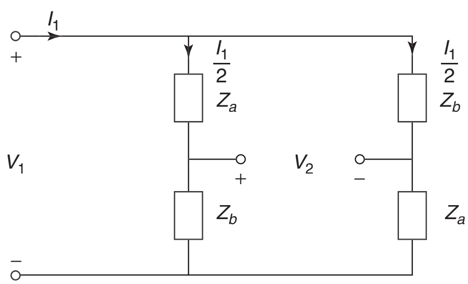


Fig. 8.36

By voltage-division rule,

$$V_2 = \frac{Z_b}{Z_a + Z_b}V_1 - \frac{Z_a}{Z_a + Z_b}V_1 = \frac{Z_b - Z_a}{Z_a + Z_b}V_1$$

$$G_{12} = \frac{V_2}{V_1} = \frac{Z_b - Z_a}{Z_a + Z_b}$$

Example 8.21 *For the network shown in Fig. 8.37, determine Z_{11} (s), G_{12} (s) and Z_{12} (s).*

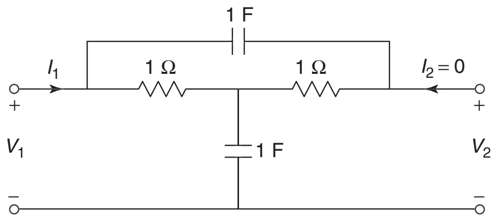


Fig. 8.37

Solution The transformed network is shown in Fig. 8.38.

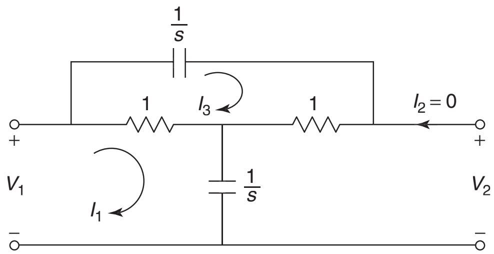


Fig. 8.38

Applying KVL to Mesh 1,

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{s} I_1 + I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + \left(2+\frac{1}{s}\right) I_3 = 0$$

$$I_3 = \left(\frac{s}{2s+1}\right) I_1 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eqs. (i) and (ii),

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - \left(\frac{s}{2s+1}\right) I_1 = \left(\frac{s+1}{s} - \frac{s}{2s+1}\right) I_1 = \left[\frac{s^2+3s+1}{s(2s+1)}\right] I_1$$

$$V_2 = \frac{1}{s} I_1 + \frac{s}{2s+1} I_1 = \left[\frac{s^2+2s+1}{s(2s+1)}\right] I_1$$

Hence,

$$Z_{11}(s) = \frac{V_1}{I_1} = \frac{s^2+3s+1}{s(2s+1)}$$

$$Z_{12}(s) = \frac{V_2}{I_1} = \frac{s^2+2s+1}{s(2s+1)}$$

$$G_{12}(s) = \frac{V_2}{V_1} = \frac{s^2+2s+1}{s^2+3s+1}$$

Example 8.22 *For the network shown in Fig.8.39, find the driving-point admittance Y_{11} and transfer admittance Y_{12}.*

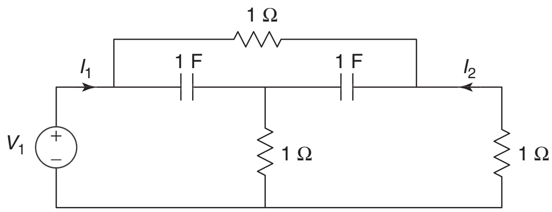


Fig. 8.39

Solution The transformed network is shown in Fig. 8.40.

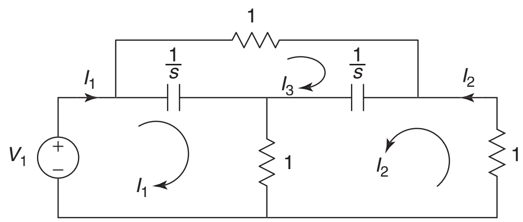


Fig. 8.40

Applying KVL to Mesh 1,

$$V_1 = \left(\frac{1}{s}+1\right)I_1 + I_2 - \frac{1}{s}I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = I_1 + \left(2+\frac{1}{s}\right)I_2 + \frac{1}{s}I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -\frac{1}{s}I_1 + \frac{1}{s}I_2 + \left(\frac{2}{s}+1\right)I_3 \qquad \text{...(iii)}$$

Writing these equations in matrix from,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = \left(\frac{1}{s}+1\right)\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right]-1\left[(1)\left(\frac{2}{s}+1\right)+\frac{1}{s^2}\right]-\frac{1}{s}\left[(1)\left(\frac{1}{s}\right)+\left(\frac{1}{s}\right)\left(2+\frac{1}{s}\right)\right]$$

$$= \frac{s^2+5s+2}{s^2}$$

$$\Delta_1 = \begin{vmatrix} V_1 & 1 & -\frac{1}{s} \\ 0 & 2+\frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = V_1\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right] = V_1\left(\frac{2s^2+5s+1}{s^2}\right)$$

$$I_1 = V_1\left(\frac{2s^2+5s+1}{s^2+5s+2}\right)$$

Hence,
$$Y_{11} = \frac{I_1}{V_1} = \frac{2s^2+5s+1}{s^2+5s+2}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{s}+1 & V_1 & -\frac{1}{s} \\ 1 & 0 & \frac{1}{s} \\ -\frac{1}{s} & 0 & \frac{2}{s}+1 \end{vmatrix} = -V_1\left(\frac{2}{s}+1+\frac{1}{s^2}\right) = -V_1\left(\frac{s^2+2s+1}{s^2}\right)$$

$$I_2 = -V_1\left(\frac{s^2+2s+1}{s^2+5s+2}\right)$$

Hence,
$$Y_{12} = \frac{I_2}{V_1} = -\frac{s^2+2s+1}{s^2+5s+2}$$

8.6 POLES AND ZEROS OF NETWORK FUNCTIONS

The network function $F(s)$ can be written as ratio of two polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}{b_m s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}$$

where a_0, a_1, …, a_n and b_0, b_1, …, b_m are the coefficients of the polynomials $N(s)$ and $D(s)$. These are real and positive for networks of passive elements. Let $N(s) = 0$ have n roots as z_1, z_2, ……, z_n and $D(s) = 0$ have m roots as p_1, p_2, ……, p_m. Then $F(s)$ can be written as

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

where $\frac{a_n}{b_m}$ is a constant called *scale factor* and $z_1, z_2, \ldots, z_n, p_1, p_2, \ldots, p_m$ are complex frequencies. When the variable s has the values $z_1, z_2, \ldots, z_n$, the network function becomes zero; such complex frequencies are known as the zeros of the network function. When the variable s has values $p_1, p_2, \ldots, p_m$, the network function becomes infinite; such complex frequencies are known as the poles of the network function. A network function is completely specified by its poles, zeros and the scale factor.

If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles or multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

The total number of zeros is equal to the total number of poles. For any network function, poles and zeros at zero and infinity are taken into account in addition to finite poles and zeros.

Poles and zeros are critical frequencies. The network function becomes infinite at poles, while the network function becomes zero at zeros. The network function has a finite, non-zero value at other frequencies.

Poles and zeros provide a representation of a network function as shown in Fig. 8.41. The zeros are shown by circles and the poles by crosses. This diagram is referred to as pole-zero plot.

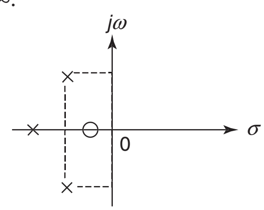


Fig. 8.41 *Pole-zero plot*

8.7 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR DRIVING-POINT FUNCTIONS [COMMON FACTORS IN *N*(*s*) AND *D*(*s*) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real and positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of all poles and zeros, must be negative or zero, i.e., the poles and zeros must lie in left half of s plane.

(4) If the real part of pole or zero is zero, then that pole or zero must be simple.

(5) The polynomials $N(s)$ and $D(s)$ may not have missing terms between those of highest and lowest degree, unless all even or all odd terms are missing.

(6) The degree of $N(s)$ and $D(s)$ may differ by either zero or one only. This condition prevents multiple poles and zeros at $s = \infty$.

(7) The terms of lowest degree in $N(s)$ and $D(s)$ may differ in degree by one at most. This condition prevents multiple poles and zeros at $s = 0$.

8.8 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR TRANSFER FUNCTIONS [COMMON FACTORS IN *N*(S) AND *D*(S) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real, and those for $D(s)$ must be positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of poles must be negative or zero. If the real part is zero, then that pole must be simple.

(4) The polynomial $D(s)$ may not have any missing terms between that of highest and lowest degree, unless all even or all odd terms are missing.

(5) The polynomial $N(s)$ may have terms missing between the terms of lowest and highest degree, and some of the coefficients may be negative.

(6) The degree of $N(s)$ may be as small as zero, independent of the degree of $D(s)$.

(7) For voltage and current transfer functions, the maximum degree of $N(s)$ is the degree of $D(s)$.

(8) For transfer impedance and admittance functions, the maximum degree of $N(s)$ is the degree of $D(s)$ plus one.

Example 8.23 *Test whether the following represent driving-point immittance functions.*

(a) $\dfrac{5s^4+3s^2-2s+1}{s^3+6s+20}$ (b) $\dfrac{s^3+s^2+5s+2}{s^4+6s^3+9s^2}$ (c) $\dfrac{s^2+3s+2}{s^2+6s+2}$

Solution

(a) The numerator and denominator polynomials have a missing term between those of highest and lowest degree and one of the coefficient is negative in numerator polynomial. Hence, the function does not represent a driving-point immittance function.

(b) The term of lowest degree in numerator and denominator polynomials differ in degree by two. Hence, the function does not represent a driving-point immittance function.

(c) The function satisfies all the necessary conditions. Hence, it represents a driving-point immittance function.

Example 8.24 *Test whether the following represent transfer functions.*

(a) $G_{21}=\dfrac{3s+2}{5s^3+4s^2+1}$ (b) $\alpha_{12}=\dfrac{2s^2+5s+1}{s+7}$ (c) $Z_{21}=\dfrac{1}{s^3+2s}$

Solution

(a) The polynomial $D(s)$ has a missing term between that of highest and lowest degree. Hence, the function does not represent a transfer function.

(b) The degree of $N(s)$ is greater than $D(s)$. Hence the function does not represent a transfer function.

(c) The function satisfies all the necessary conditions. Hence, it represents a transfer function.

Example 8.25 *Obtain the pole-zero plot of the following functions.*

(a) $F(s)=\dfrac{s(s+2)}{(s+1)(s+3)}$ (b) $F(s)=\dfrac{s(s+1)}{(s+2)^2(s+3)}$

(c) $F(s)=\dfrac{s(s+2)}{(s+1+j1)(s+1-j1)}$ (d) $F(s)=\dfrac{(s+1)^2(s+5)}{(s+2)(s+3+j2)(s+3-j2)}$

(e) $F(s)=\dfrac{s^2+4}{(s+2)(s^2+9)}$

Solution

(a) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1$ and $s = -3$. The pole-zero plot is shown in Fig. 8.42.

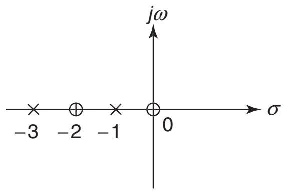


Fig. 8.42

(b) The function $F(s)$ has zeros at $s = 0$ and $s = -1$ and poles at $s = -2, -2$ and $s = -3$. The pole-zero plot is shown in Fig. 8.43.

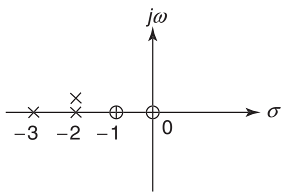


Fig. 8.43

(c) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1 - j1$ and $s = -1 + j1$. The pole-zero plot is shown in Fig. 8.44.

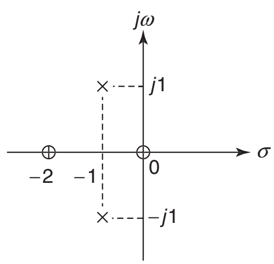


Fig. 8.44

(d) The function $F(s)$ has zeros at $s = -1, -1$ and $s = -5$ and poles at $s = -2$, $s = -3 + j2$ and $s = -3 - j2$. The pole-zero plot is shown in Fig. 8.45.

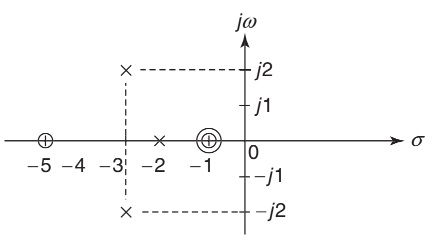


Fig. 8.45

(e) The function $F(s)$ has zeros at $s = j2$ and $s = -j2$ and poles at $s = -2$, $s = j3$ and $s = -j3$. The pole-zero plot is shown in Fig. 8.46.

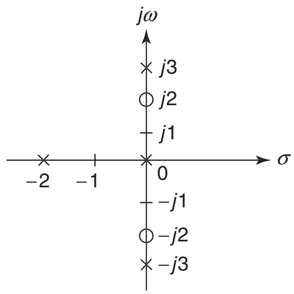


Fig. 8.46

Example 8.26 *Find poles and zeros of the impedance of the network shown in Fig. 8.47 and plot them on the s-plane.*

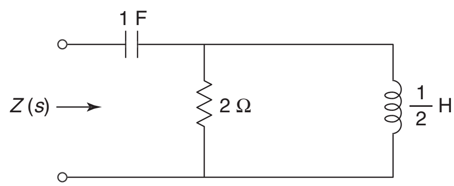


Fig. 8.47

Solution The transformed network is shown in Fig. 8.48.

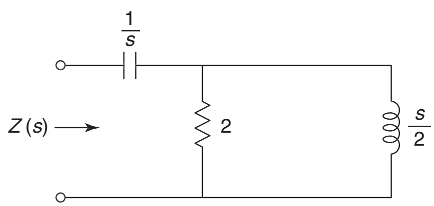


Fig. 8.48

$$Z(s) = \frac{1}{s} + \frac{\frac{s}{2}\times 2}{\frac{s}{2}+2} = \frac{1}{s} + \frac{2s}{s+4} = \frac{2s^2+s+4}{s(s+4)} = \frac{2(s^2+0.5s+2)}{s(s+4)}$$

$$= \frac{2(s+0.25+j1.4)(s+0.25-j1.4)}{s(s+4)}$$

The function $Z(s)$ has zeros at $s = -0.25 + j1.4$ and $s = -0.25 - j1.4$ and poles at $s = 0$ and $s = -4$ as shown in Fig. 8.49.

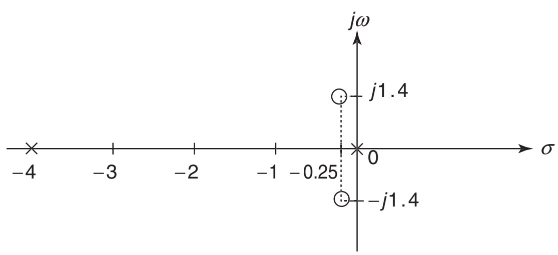


Fig. 8.49

Example 8.27 *Determine the poles and zeros of the impedance function Z (s) in the network shown in Fig. 8.50.*

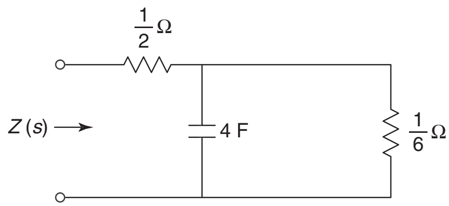


Fig. 8.50

Solution The transformed network is shown in Fig. 8.51.

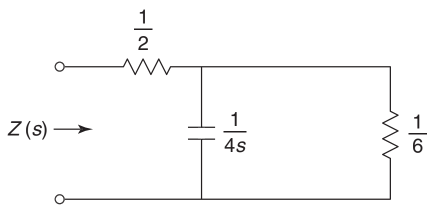


Fig. 8.51

$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4s} \times \frac{1}{6}}{\frac{1}{4s} + \frac{1}{6}} = \frac{1}{2} + \frac{1}{4s+6} = \frac{4s+8}{2(4s+6)} = \frac{s+2}{2s+3} = \frac{0.5(s+2)}{s+1.5}$$

The function $Z(s)$ has zero at $s = -2$ and pole at $s = -1.5$.

Example 8.28 *Determine Z(s) in the network shown in Fig. 8.52. Find poles and zeros of Z(s) and plot them on s-plane.*

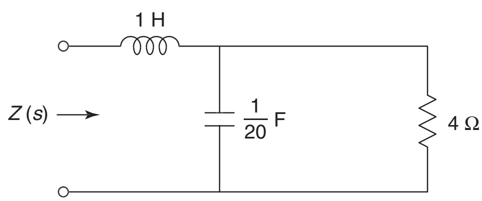


Fig. 8.52

Solution The transformed network is shown in Fig. 8.53.

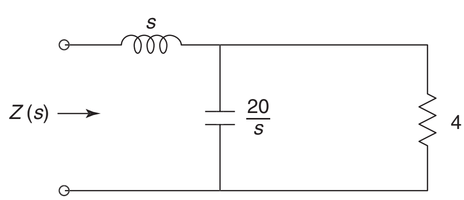


Fig. 8.53

$$Z(s) = s + \frac{\frac{20}{s} \times 4}{\frac{20}{s} + 4} = s + \frac{80}{4s + 20} = s + \frac{20}{s+5} = \frac{s(s+5)+20}{s+5} = \frac{s^2 + 5s + 20}{s+5}$$

$$= \frac{(s+2.5+j3.71)(s+2.5-j3.71)}{s+5}$$

The function $Z(s)$ has zeros at $s = -2.5 + j3.71$ and $s = -2.5 - j3.71$ and pole at $s = -5$. The pole-zero diagram is shown in Fig. 8.54.

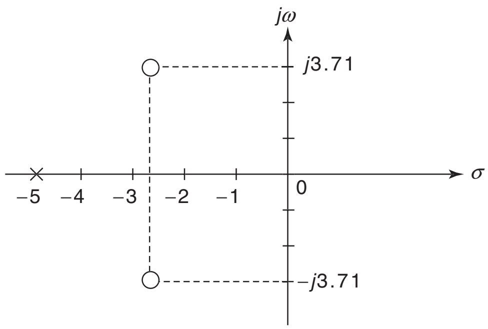


Fig. 8.54

Example 8.29 *For the network shown in Fig. 8.55, plot poles and zeros of function* $\frac{I_0}{I_i}$.

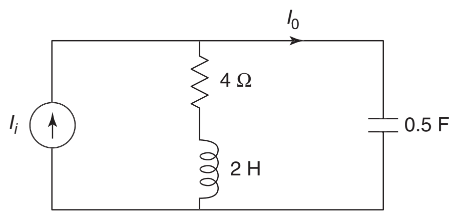


Fig. 8.55

Solution The transformed network is shown in Fig. 8.56.

By current-division rule,

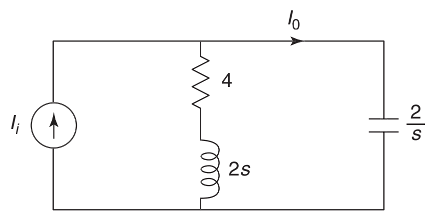


Fig. 8.56

$$I_0 = I_i \left(\frac{4 + 2s}{4 + 2s + \frac{2}{s}} \right)$$

$$\frac{I_0}{I_i} = \frac{s(4+2s)}{4s + 2s^2 + 2} = \frac{s(s+2)}{s^2 + 2s + 1} = \frac{s(s+2)}{(s+1)(s+1)}$$

The function has zeros at $s = 0$ and $s = -2$ and double poles at $s = -1$.

The pole-zero diagram is shown in Fig. 8.57.

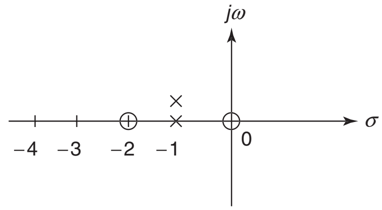


Fig. 8.57

Example 8.30 *Draw the pole-zero diagram of $\frac{I_2}{I_1}$ for the network shown in Fig. 8.58.*

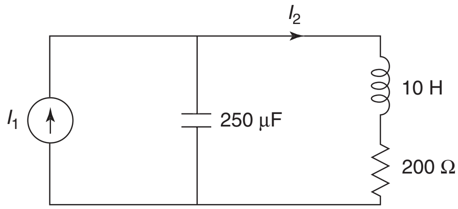


Fig. 8.58

Solution The transformed network is shown in Fig. 8.59.

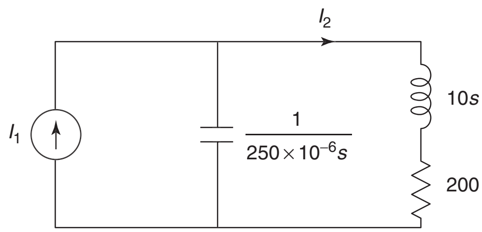


Fig. 8.59

By current-division rule,

$$I_2 = I_1 \frac{\frac{1}{250 \times 10^{-6} s}}{\frac{1}{250 \times 10^{-6} s} + 10s + 200}$$

$$\frac{I_2}{I_1} = \frac{400}{s^2 + 20s + 400} = \frac{400}{(s+10-j17.32)(s+10+j17.32)}$$

The function has no zero and poles at $s = -10 + j17.32$ and $s = -10 - j17.32$. The pole-zero diagram is shown in Fig. 8.60.

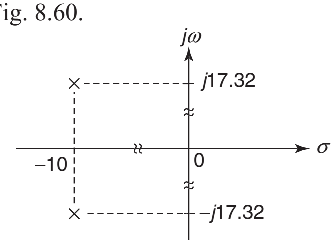


Fig. 8.60

Example 8.31 *For the network shown in Fig. 8.61, draw pole-zero plot of* $\frac{V_c}{V_1}$.

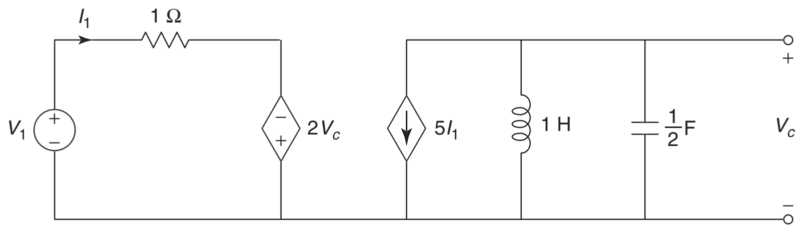


Fig. 8.61

Solution The transformed network is shown in Fig. 8.62.

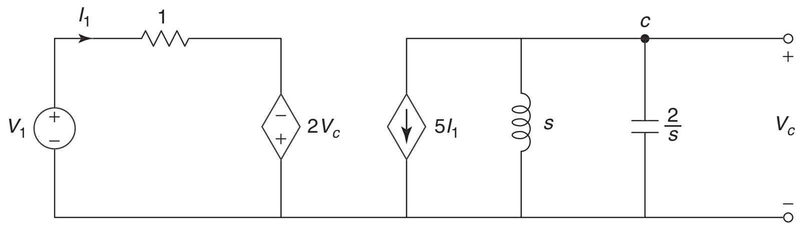


Fig. 8.62

Applying KVL to the left loop,

$$V_1 - 1I_1 + 2V_c = 0$$

$$I_1 = V_1 + 2V_c$$

Applying KCL at Node C,

$$5\,I_1 + \frac{V_c}{s} + \frac{V_c}{\frac{2}{s}} = 0$$

$$5(V_1 + 2V_c) + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$5V_1 + 10V_c + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$V_c\left(\frac{20s + 2 + s^2}{2s}\right) = -5V_1$$

$$\frac{V_c}{V_1} = -\frac{10s}{s^2 + 20s + 2} = -\frac{10s}{(s+0.1)(s+19.9)}$$

The function has zero at $s = 0$ and poles at $s = -0.1$ and $s = -19.9$.
The pole-zero diagram is shown in Fig. 8.63.

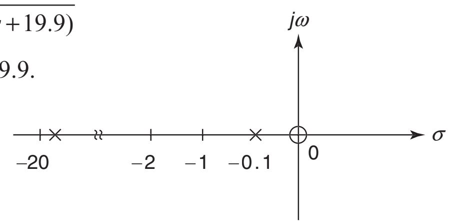


Fig. 8.63

Example 8.32 *Find the driving point admittance function and draw pole-zero plot for the network shown in Fig. 8.64.*

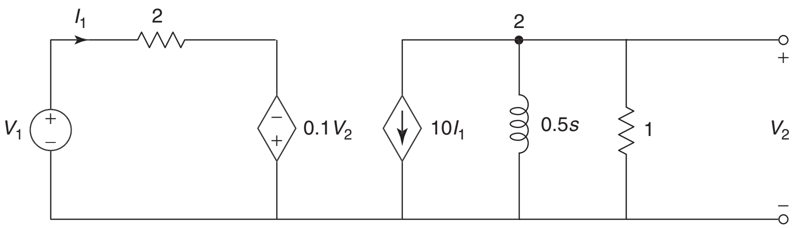


Fig. 8.64

Solution Applying KVL to the left loop,

$$V_1 - 2\,I_1 + 0.1\,V_2 = 0$$

$$I_1 = \frac{V_1 + 0.1\,V_2}{2} \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$10\,I_1 + \frac{V_2}{0.5s} + \frac{V_2}{1} = 0$$

$$10\,I_1 + \frac{2}{s}V_2 + V_2 = 0$$

$$10\,I_1 + \left(\frac{2}{s} + 1\right)V_2 = 0$$

$$10\,I_1 + \left(\frac{2+s}{s}\right)V_2 = 0$$

$$\left(\frac{s+2}{s}\right)V_2 = -10\,I_1$$

$$V_2 = -\frac{10s}{s+2}\,I_1 \qquad \text{...(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{V_1 + 0.1\left(-\frac{10s}{s+2}\right)I_1}{2} = 0.5V_1 + 0.05\left(-\frac{10s}{s+2}\right)I_1$$

$$I_1\left(1 + \frac{0.5s}{s+2}\right) = 0.5V_1$$

Hence,

$$Y_{11} = \frac{I_1}{V_1} = \frac{0.5}{1 + \frac{0.5s}{s+2}} = \frac{0.5\,(s+2)}{s + 0.5s + 2} = \frac{0.5s + 1}{1.5s + 2}$$

The function has zero at $s = -2$ and pole at $s = -1.33$. The pole-zero diagram is shown in Fig 8.65.

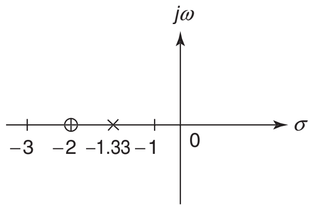


Fig. 8.65

Example 8.33 *For the network shown in Fig. 8.66, determine* $\dfrac{V_2}{I_g}$. *Plot the pole-zero diagram of* $\dfrac{V_2}{I_g}$.

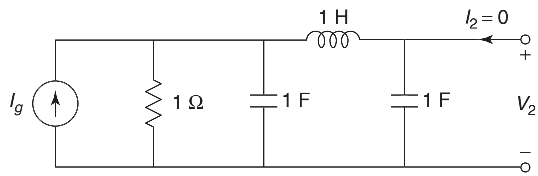


Fig. 8.66

Solution The transformed network is shown in Fig. 8.67.

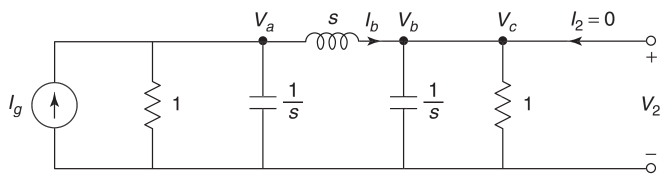


Fig. 8.67

$$V_c = V_b = V_2$$

$$I_b = \frac{V_b}{\frac{1}{s}} + \frac{V_c}{1} = sV_2 + V_2 = (s+1)V_2$$

$$V_a = sI_b + V_b = s(s+1)V_2 + V_2 = (s^2+s+1)V_2$$

$$I_g = \frac{V_a}{1} + \frac{V_a}{\frac{1}{s}} + I_b = (s^2+s+1)V_2 + s(s^2+s+1)V_2 + (s+1)V_2 = (s^3+2s^2+3s+2)V_2$$

Hence,
$$\frac{V_2}{I_g} = \frac{1}{s^3+2s^2+3s+2}$$

The function has no zeros. It has poles at $s = -1$, $s = -0.5 + j1.32$ and $s = -0.5 - j1.32$,
The pole-zero diagram is shown in Fig. 8.68.

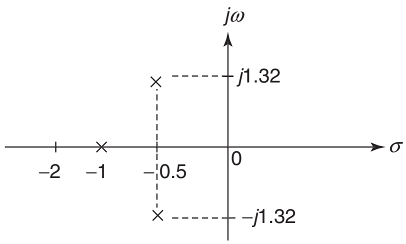


Fig. 8.68

Example 8.34 *For the transfer function* $H(s) = \dfrac{V_0}{V_i} = \dfrac{10}{s^2 + 3s + 10}$, *realise the function using the network shown in Fig. 8.69. Find L and C when* $R = 5\ \Omega$.

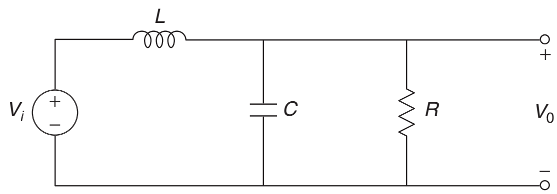


Fig. 8.69

Solution The transformed network is shown in Fig. 8.70.

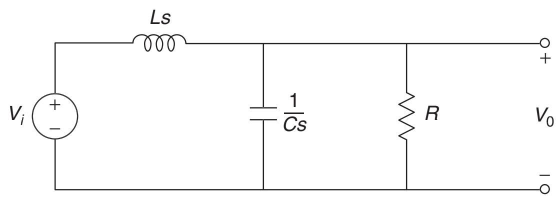


Fig. 8.70

Simplifying the network as shown in Fig. 8.71,

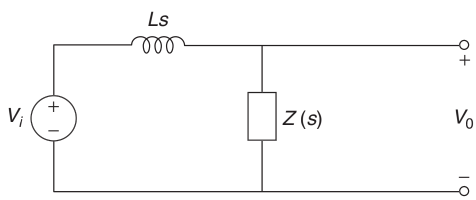


Fig. 8.71

$$Z(s) = \frac{R \times \dfrac{1}{Cs}}{R + \dfrac{1}{Cs}} = \frac{R}{RCs + 1}$$

$$V_0 = V_i \frac{\dfrac{R}{RCs+1}}{Ls + \dfrac{R}{RCs+1}}$$

$$\frac{V_0}{V_i} = \frac{R}{RLC\,s^2 + Ls + R} = \frac{\dfrac{1}{LC}}{s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}} \qquad \text{...(i)}$$

But
$$\frac{V_0}{V_i} = \frac{10}{s^2 + 3s + 10} \qquad \text{...(ii)}$$

and
$$R = 5\ \Omega$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{RC} = 3$$

$$\frac{1}{LC} = 10$$

Solving the above equations,

$$L = 1.5\text{ H}$$

$$C = \frac{1}{15}\text{F}$$

Example 8.35 *Obtain the impedance function Z(s) for which pole-zero diagram is shown in Fig. 8.72.*

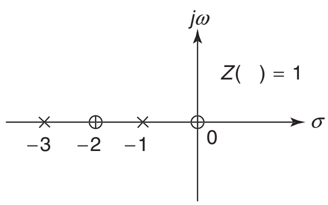


Fig. 8.72

Solution The function $Z(s)$ has poles at $s = -1$ and $s = -3$ and zeros at $s = 0$ and $s = -2$.

$$Z(s) = H\frac{s(s+2)}{(s+1)(s+3)} = H\frac{s^2\left(1+\dfrac{2}{s}\right)}{s^2\left(1+\dfrac{1}{s}\right)\left(1+\dfrac{3}{s}\right)}$$

For $s = \infty$,

$$Z(\infty) = H\frac{1}{(1)(1)} = H$$

When
$$Z(\infty) = 1,$$

$$H = 1$$

$$Z(s) = \frac{s\,(s+2)}{(s+1)(s+3)}$$

Example 8.36 *Obtain the admittance function Y(s) for which the pole-zero diagram is shown in Fig. 8.73.*

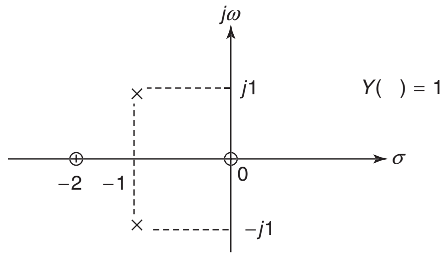


Fig. 8.73

Solution The function $Y(s)$ has poles at $s = -1 + j1$ and $s = -1 - j1$ and zeros at $s = 0$ and $s = -2$.

$$Y(s) = H\frac{s(s+2)}{(s+1+j1)(s+1-j1)} = H\frac{s(s+2)}{(s+1)^2-(j1)^2} = H\frac{s(s+2)}{s^2+2s+2} = H\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

For $s = \infty$,

$$Y(\infty) = H\frac{(1)}{(1)} = H$$

When $Y(\infty) = 1$,

$$H = 1$$

$$Y(s) = \frac{s(s+2)}{s^2+2s+2}$$

Example 8.37 *A network and its pole-zero configuration are shown in Fig. 8.74. Determine the values of R, L and C if Z (j0) = 1.*

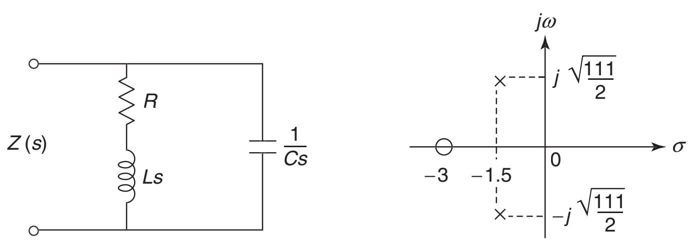


Fig. 8.74

Solution

$$Z(s) = \frac{(Ls+R)\frac{1}{Cs}}{(Ls+R)+\frac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}} \qquad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -3$ and poles are at $s = -1.5 + j\dfrac{\sqrt{111}}{2}$ and $s = -1.5 - j\dfrac{\sqrt{111}}{2}$

$$Z(s) = H\frac{s+3}{\left(s+1.5+j\dfrac{\sqrt{111}}{2}\right)\left(s+1.5-j\dfrac{\sqrt{111}}{2}\right)}$$

$$Z(s) = H\frac{s+3}{(s+1.5)^2 - \left(j\dfrac{\sqrt{111}}{2}\right)^2} = H\frac{s+3}{s^2+3s+30}$$

When $\quad Z(j0) = 1,$

$$1 = H\left(\frac{3}{30}\right)$$

$$H = 10$$

$$Z(s) = \frac{10(s+3)}{s^2+3s+30} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 3$$

$$\frac{1}{C} = 10$$

$$\frac{1}{LC} = 30$$

Solving the above equations,

$$C = \frac{1}{10}\,\text{F}$$

$$L = \frac{1}{3}\,\text{H}$$

$$R = 1\,\Omega$$

Example 8.38 *A network is shown in Fig. 8.75. The poles and zeros of the driving-point function Z(s) of this network are at the following places:*

Poles at $-\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$

Zero at –1

If Z (j0) = 1, determine the values of R, L and C.

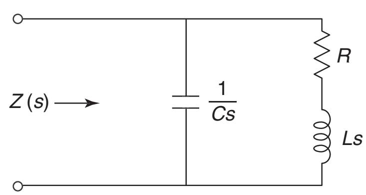


Fig. 8.75

Solution

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \qquad \text{...(i)}$$

The poles are at $-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$ and zero is at –1.

$$Z(s)=H\frac{s+1}{\left(s+\dfrac{1}{2}+j\dfrac{\sqrt{3}}{2}\right)\left(s+\dfrac{1}{2}-j\dfrac{\sqrt{3}}{2}\right)}=H\frac{s+1}{\left(s+\dfrac{1}{2}\right)^2-\left(j\dfrac{\sqrt{3}}{2}\right)^2}=H\frac{s+1}{s^2+s+1}$$

When $Z(j0)=1$,

$$1=H\frac{(1)}{(1)}$$

$$H=1$$

$$Z(s)=\frac{s+1}{s^2+s+1} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$C=1$$

$$\frac{R}{L}=1$$

$$\frac{1}{LC}=1$$

Solving the above equations,

$$C=1\text{ F}$$
$$L=1\text{ H}$$
$$R=1\ \Omega$$

Example 8.39 *The pole-zero diagram of the driving-point impedance function of the network of Fig. 8.76 is shown below. At dc, the input impedance is resistive and equal to 2 W. Determine the values of R, L and C.*

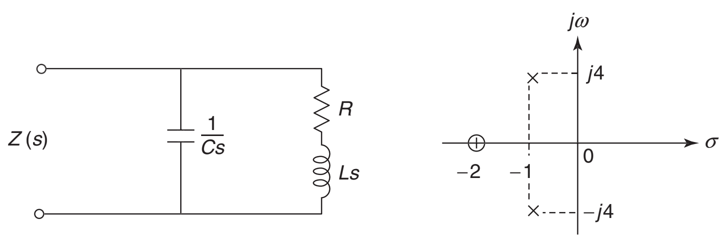


Fig. 8.76

Solution

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \qquad \text{...(i)}$$

From the pole-zero diagram, zero is at $s=-2$ and poles are at $s=-1+j4$ and $s=-1-j4$.

$$Z(s) = H\frac{s+2}{(s+1+j4)(s+1-j4)} = H\frac{s+2}{(s+1)^2-(j4)^2} = H\frac{s+2}{s^2+2s+17}$$

At dc, i.e.,

$$\omega = 0, Z(j0) = 2$$

$$2 = H\frac{2}{17}$$

$$H = 17$$

$$Z(s) = 17\frac{s+2}{s^2+2s+17} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 17$$

$$\frac{R}{L} = 2$$

$$\frac{1}{LC} = 17$$

Solving the above equations,

$$C = \frac{1}{17}\text{ F}$$

$$L = 1\text{ H}$$

$$R = 2\ \Omega$$

Example 8.40 *The network shown in Fig. 8.77 has the driving-point admittance Y (s) of the form*

$$Y(s) = H\frac{(s-s_1)(s-s_2)}{(s-s_3)}$$

(*a*) *Express s_1, s_2, s_3 in terms of R, L and C.*

(*b*) *When $s_1 = -10 + j10^4$, $s_2 = -10 - j10^4$ and $Y(j0) = 10^{-1}$ mho, find the values of R, L and C and determine the value of s_3.*

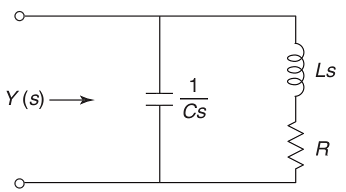


Fig. 8.77

Solution

(a)

$$Y(s) = Cs + \frac{1}{Ls+R} = \frac{(Ls+R)\,Cs+1}{Ls+R} = \frac{LCs^2+RCs+1}{Ls+R} = \frac{C\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s+\frac{R}{L}} \qquad \text{...(i)}$$

But

$$Y(s) = \frac{H(s-s_1)(s-s_2)}{(s-s_3)}$$

where $$s_1, s_2 = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_3 = -\frac{R}{L}$$

(b) When $$s_1 = -10 + j10^4$$
$$s_2 = -10 - j10^4$$

$$Y(s) = H\frac{(s+10-j10^4)(s+10+j10^4)}{s-s_3} = H\frac{s^2+20s+10^8}{s-s_3} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 20$$
$$s_3 = -20$$
$$Y(s) = H\frac{(s^2+20s+10^8)}{(s+20)}$$

At $s = j0$,

$$Y(j0) = H\frac{(10^8)}{20} = 10^{-1}$$
$$H = 0.02\times10^{-6}$$
$$Y(s) = 0.02\times10^{-6}\frac{(s^2+20s+10^8)}{(s+20)} \qquad \text{...(iii)}$$

Comparing Eq. (iii) with Eq. (i),

$$C = 0.02\times10^{-6}\,\text{F} = 0.02\,\mu\text{F}$$
$$\frac{1}{LC} = 10^8$$
$$L = \frac{1}{2}\,\text{H}$$
$$\frac{R}{L} = 20$$
$$R = 10\,\Omega$$

Example 8.41 *A network and pole-zero diagram for driving-point impedance Z(s) are shown in Fig. 8.78. Calculate the values of the parameters R, L, G and C if Z(j0) = 1.*

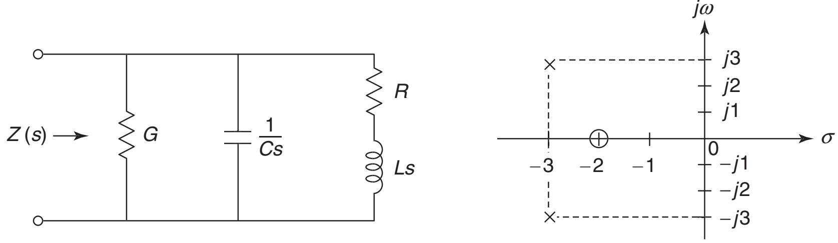


Fig. 8.78

Solution It is easier to calculate $Y(s)$ and then invert it to obtain $Z(s)$.

$$Y(s) = G + Cs + \frac{1}{Ls+R} = \frac{(G+Cs)(Ls+R)+1}{Ls+R} = \frac{LCs^2 + (GL+RC)\,s + 1 + GR}{Ls+R}$$

$$Z(s) = \frac{1}{Y(s)} = \frac{Ls+R}{LCs^2 + (GL+RC)\,s + 1 + GR} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2 + \left(\frac{G}{C}+\frac{R}{L}\right)s + \left(\frac{1+GR}{LC}\right)} \qquad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -2$ and poles are at $s = -3 \pm j3$.

$$Z(s) = H\frac{(s+2)}{(s+3-j3)(s+3+j3)} = H\frac{(s+2)}{(s+3)^2 - (j3)^2} = H\frac{s+2}{s^2+6s+18}$$

When $Z(j0) = 1,$

$$1 = H\frac{2}{18}$$

$$H = 9$$

$$Z(s) = \frac{9\,(s+2)}{(s^2+6s+18)} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 9$$

$$\frac{R}{L} = 2$$

$$\frac{G}{C} + \frac{R}{L} = 6$$

$$\frac{1+GR}{LC} = 18$$

Solving the above equation,

$$C = \frac{1}{9}\ \text{F}$$

$$L = \frac{9}{10}\ \text{H}$$

$$G = \frac{4}{9}\ \mho$$

$$R = \frac{9}{5}\ \Omega$$

Example 8.42 *A series R-L-C circuit has its driving-point admittance and pole-zero diagram is shown in Fig. 8.79. Find the values of R, L and C.*

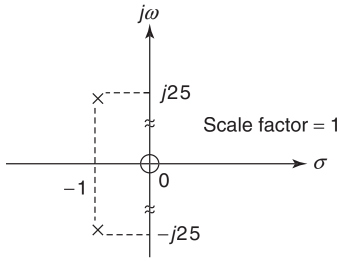


Fig. 8.79

Solution The function $Y(s)$ has poles at $s = -1 + j25$ and $s = -1 - j25$ and zero at $s = 0$.

$$Y(s) = H\frac{s}{(s+1+j25)(s+1-j25)} = H\frac{s}{(s+1)^2-(j25)^2} = H\frac{s}{s^2+2s+626}$$

Scale factor $\quad H = 1$

$$Y(s) = \frac{s}{s^2+2s+626} \qquad \text{...(i)}$$

For a series *RLC* circuit,

$$Z(s) = R + Ls + \frac{1}{Cs} = \frac{LCs^2+RCs+1}{Cs} = \frac{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{s}{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \qquad \text{...(ii)}$$

Comparing Eq. (i) with Eq. (ii),

$$L = 1\text{ H}$$

$$\frac{1}{LC} = 626$$

$$C = \frac{1}{626}\text{ F}$$

$$\frac{R}{L} = 2$$

$$R = 2\ \Omega$$

8.9 TIME-DOMAIN BEHAVIOUR FROM THE POLE-ZERO PLOT

The time-domain behaviour of a system can be determined from the pole-zero plot. Consider a network function

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

The poles of this function determine the time-domain behaviour of $f(t)$. The function $f(t)$ can be determined from the knowledge of the poles, the zeros and the scale factor H. Figure 8.80 shows some pole locations and the corresponding time-domain response.

(i) When pole is at origin, i.e., at $s = 0$, the function $f(t)$ represents steady-state response of the circuit i.e., dc value. (Fig. 8.80)

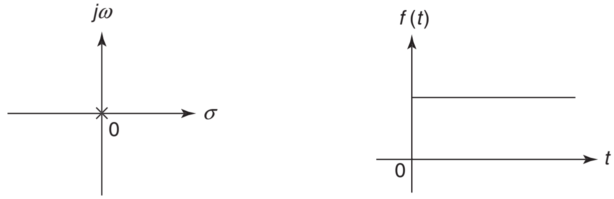


Fig. 8.80 *Pole at origin*

(ii) When pole lies in the left half of the s-plane, the response decreases exponentially. (Fig. 8.81)

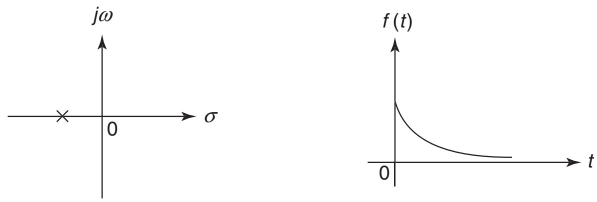


Fig. 8.81 *Pole in left half of the s-plane*

(iii) When pole lies in the right half of the s-plane, the response increases exponentially. A pole in the right-half plane gives rise to unbounded response and unstable system. (Fig. 8.82)

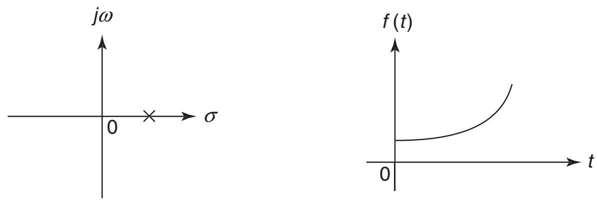


Fig. 8.82 *Pole in right half of the s-plane*

(iv) For $s = 0 + j\omega_n$, the response becomes $f(t) = Ae^{\pm j\omega_n t} = A(\cos \omega_n t \pm j \sin \omega_n t)$.The exponential response $e^{\pm j\omega_n t}$ may be interpreted as a rotating phasor of unit length. A positive sign of exponential $e^{j\omega_n t}$ indicates counterclockwise rotation, while a negative sign of exponential $e^{-j\omega_n t}$ indicates clockwise rotation. The variation of exponential function $e^{j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state. (Fig. 8.83)

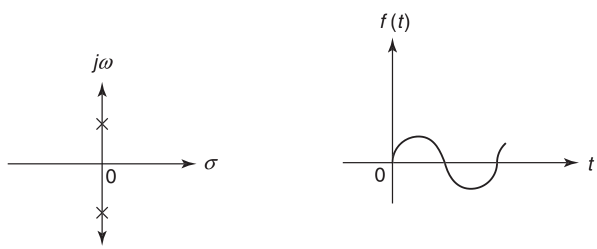
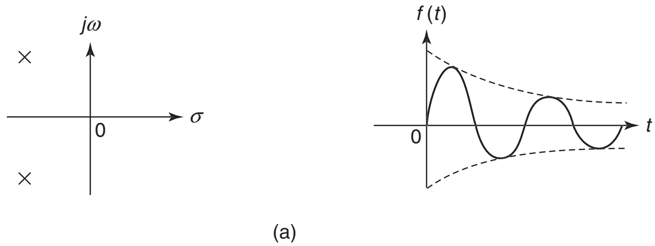


Fig. 8.83 *Poles on jω-axis*

(v) For $s = \sigma_n + j\omega_n$, the response becomes $f(t) = Ae^{st} = Ae^{(\sigma_n + j\omega_n t)} = Ae^{\sigma_n t}\, e^{j\omega_n t}$. The response $e^{\sigma_n t}$ is an exponentially increasing or decreasing function. The response $e^{j\omega_n t}$ is a sinusoidal function. Hence, the response of the product of these responses will be over damped sinusoids or under damped sinusoids (Fig. 8.84).



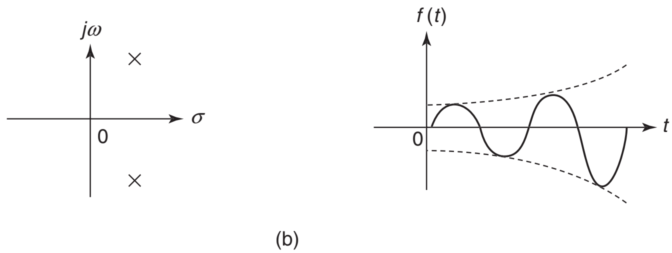


Fig. 8.84 *(a) Complex conjugate poles in left half of the S-plane*
(b) Complex conjugate poles in right half of the S-plane

(vi) The real part s of the pole is the displacement of the pole from the imaginary axis. Since σ is also the damping factor, a greater value of σ (i.e., a greater displacement of the pole from the imaginary axis) means that response decays more rapidly with time. The poles with greater displacement from the real axis correspond to higher frequency of oscillation (Fig. 8.85).

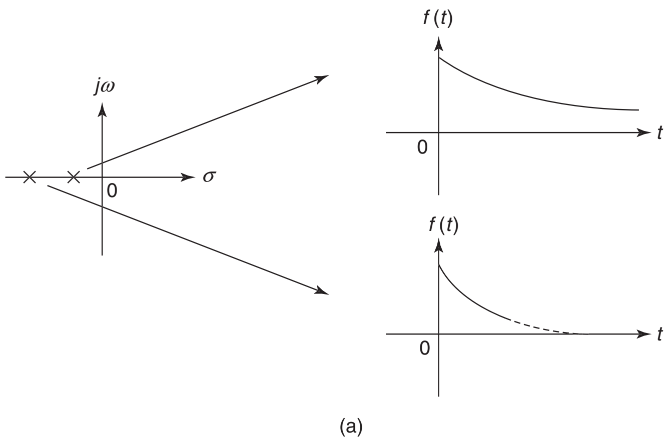
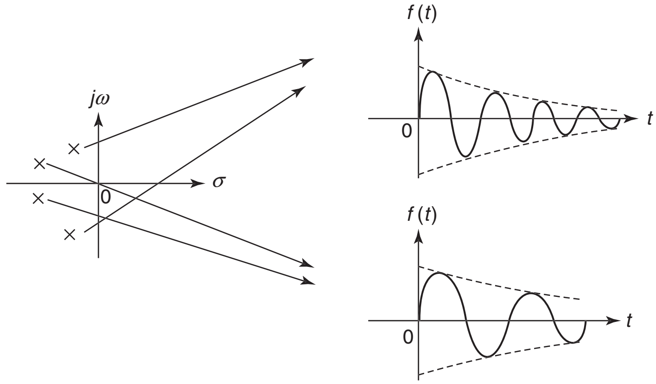


Fig. 8.85 *Nature of response with different positions of poles*



(b)

Fig.8.85 *(Continued)*

8.9.1 Stability of the Network

Stability of the network is directly related to the location of poles in the s-plane.

(i) When all the poles lie in the left half of the s-plane, the network is said to be stable.
(ii) When the poles lie in the right half of the s-plane, the network is said to be unstable.
(iii) When the poles lie on the $j\omega$ axis, the network is said to be marginally stable.
(iv) When there are multiple poles on the $j\omega$ axis, the network is said to be unstable.
(v) When the poles move away from $j\omega$ axis towards the left half of the s-plane, the relative stability of the network improves.

8.10 GRAPHICAL METHOD FOR DETERMINATION OF RESIDUE

Consider a network function,

$$F(s) = H\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

By partial fraction expansion,

$$F(s) = \frac{K_1}{(s-p_1)} + \frac{K_2}{(s-p_2)} + \cdots + \frac{K_m}{(s-p_m)}$$

The residue K_i is given by

$$K_i = (s-p_i)F(s)\big|_{s\to p_i} = H\frac{(p_i-z_1)(p_i-z_2)\cdots(p_i-z_n)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_m)}$$

Each term $(p_i - z_i)$ represents a phasor drawn from zero z_i to pole p_i.
Each term $(p_i - p_k)$, $i \neq k$, represents a phasor drawn from other poles to the pole p_i.

$$K_i = H\frac{\text{Product of phasors (polar form) from each zero to } p_i}{\text{Product of phasors (polar form) from other poles to } p_i}$$

The residues can be obtained by graphical method in the following way:

(1) Draw the pole-zero diagram for the given network function.
(2) Measure the distance from each of the other poles to a given pole.
(3) Measure the distance from each of the other zeros to a given pole.
(4) Measure the angle from each of the other poles to a given pole.
(5) Measure the angle from each of the other zeros to a given pole.
(6) Substitute these values in the required residue equation.

The graphical method can be used if poles are simple and complex. But it cannot be used when there are multiple poles.

Example 8.43 *The current I(s) in a network is given by* $I(s) = \dfrac{2s}{(s+1)(s+2)}$. *Plot the pole-zero pattern in the s-plane and hence obtain i(t).*

Solution Poles are at $s = -1$ and $s = -2$ and zero is at $s = 0$. The pole-zero plot is shown in Fig. 8.86.

By partial-fraction expansion,

$$I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

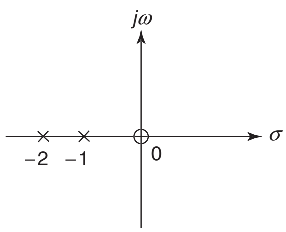


Fig. 8.86

The coefficients K_1 and K_2, often referred as residues, can be evaluated from he pole-zero diagram. From Fig. 8.87,

$$K_1 = H\frac{\text{Phasor from zero at origin to pole at } A}{\text{Phasor from pole at } B \text{ to pole at } A} = 2\left(\frac{1\angle 180^\circ}{1\angle 0^\circ}\right) = 2\angle 180^\circ = -2$$

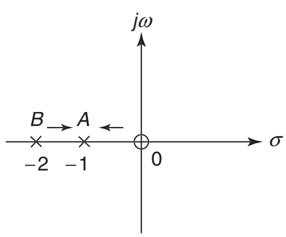


Fig. 8.87

From Fig. 8.88,

$$K_2 = H\frac{\text{Phasor from zero at origin to pole at } B}{\text{Phasor from pole at } A \text{ to pole at } B} = 2\left(\frac{2\angle 180^\circ}{1\angle 180^\circ}\right) = 4$$

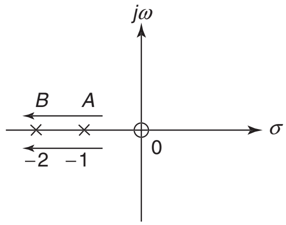


Fig. 8.88

$$I(s) = -\frac{2}{s+1} + \frac{4}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = -2e^{-t} + 4e^{-2t}$$

Example 8.44 *The voltage V(s) of a network is given by*

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)}$$

Plot its pole-zero diagram and hence obtain v (t).

Solution

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)} = \frac{3s}{(s+2)(s+1+j1)(s+1-j1)}$$

Poles are at $s = -2$ and $s = -1 \pm j1$ and zero is at $s = 0$ as shown in Fig. 8.89.

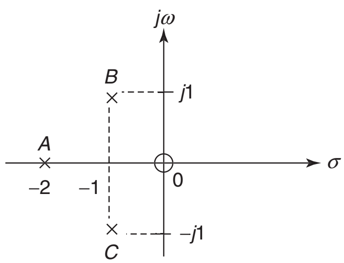


Fig. 8.89

By partial-fraction expansion,

$$V(s) = \frac{K_1}{s+2} + \frac{K_2}{s+1-j1} + \frac{K_2^*}{s+1+j1}$$

The coefficients K_1, K_2 and K_2^* can be evaluated from the pole-zero diagram.
From Fig. 8.90,

$$K_1 = \frac{3\,(\overline{OA})}{(\overline{BA})\,(\overline{CA})} = 3\left[\frac{2\angle 180^\circ}{(\sqrt{2}\angle -135^\circ)\,(\sqrt{2}\angle 135^\circ)}\right] = 3\sqrt{180^\circ} = -3$$

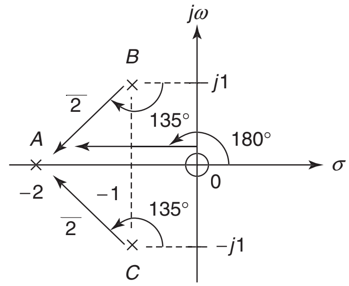


Fig. 8.90

From Fig. 8.91,

$$K_2 = \frac{3\,(\overline{OB})}{(\overline{AB})\,(\overline{CB})} = 3\left[\frac{(\sqrt{2}\angle 135°)}{(\sqrt{2}\angle 45°)\,(2\angle 90°)}\right] = \frac{3}{2}$$

$$K_2^* = \frac{3}{2}$$

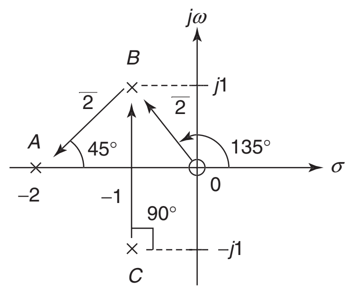


Fig. 8.91

$$V(s) = -\frac{3}{(s+2)} + \frac{\frac{3}{2}}{(s+1-j1)} + \frac{\frac{3}{2}}{(s+1+j1)}$$

Taking inverse Laplace transform,

$$v(t) = -3e^{-2t} + \frac{3}{2}\left[e^{(-1+j1)t} + e^{(-1-j1)t}\right] = -3e^{-2t} + 2\times\frac{3}{2}e^{-t}\left(\frac{e^{j1}+e^{-j1}}{2}\right) = -3e^{-2t} + 3e^{-t}\cos t$$

Example 8.45 *Find the function v(t) using the pole-zero plot of following function:*

$$V(s) = \frac{(s+2)(s+6)}{(s+1)(s+5)}$$

Solution If the degree of the numerator is greater or equal to the degree of the denominator, we have to divide the numerator by the denominator such that the remainder can be expanded into partial fractions.

$$V(s) = \frac{s^2 + 8s + 12}{s^2 + 6s + 5} = 1 + \frac{2s+7}{s^2 + 6s + 5} = 1 + \frac{2(s+3.5)}{(s+1)(s+5)}$$

By partial fraction expansion,

$$V(s) = 1 + \frac{K_1}{s+1} + \frac{K_2}{s+5}$$

K_1 and K_2 can be evaluated from the pole-zero diagram shown in Fig. 8.92 and Fig. 8.93.

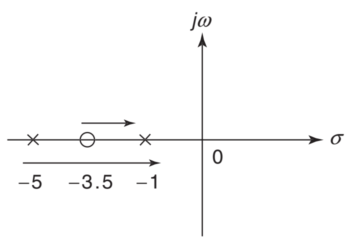


Fig. 8.92

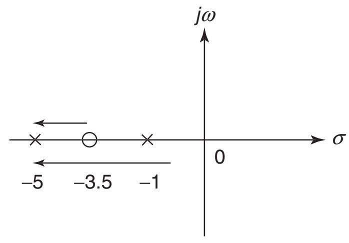


Fig. 8.93

From Fig. 8.92

$$K_1 = 2\left(\frac{2.5\angle 0°}{4\angle 0°}\right) = \frac{5}{4}$$

From Fig. 8.93

$$K_2 = 2\left(\frac{1.5\angle 180°}{4\angle 180°}\right) = \frac{3}{4}$$

$$V(s) = 1 + \frac{\frac{5}{4}}{s+1} + \frac{\frac{3}{4}}{s+5}$$

Taking inverse Laplace transform,

$$v(t) = \delta(t) + \frac{5}{4}e^{-t} + \frac{3}{4}e^{-5t}$$

Example 8.46 *The pole-zero plot of the driving-point impedance of a network is shown in Fig. 8.94. Find the time-domain response.*

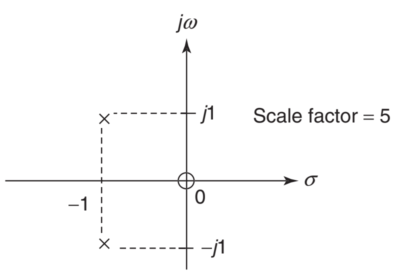


Fig. 8.94

Solution The function $Z(s)$ has poles at $s = -1 + j1$ and $s = -1 - j1$ and zero at $s = 0$.

$$Z(s) = H\frac{s}{(s+1+j1)(s+1-j1)}$$

Scale factor $\qquad H = 5$

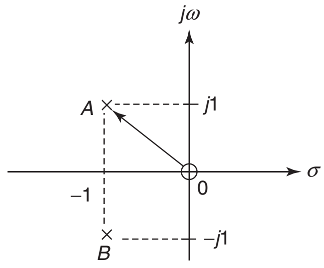


Fig. 8.95

$$Z(s) = \frac{5s}{(s+1+j1)(s+1-j1)}$$

By partial fraction expansion,

$$Z(s) = \frac{K_1}{s+1+j1} + \frac{K_1^*}{s+1-j1}$$

The coefficients K_1 and K_1^* can be evaluated from the pole-zero diagram. From Fig. 8.95,

$$K_1 = \frac{5\,(\overline{OA})}{(\overline{BA})} = \frac{5\,(\sqrt{2}\angle 135^\circ)}{2\angle 90^\circ} = 3.54\angle 45^\circ$$

$$K_1^* = 3.54\angle -45^\circ$$

$$Z(s) = \frac{3.54\angle 45^\circ}{s+1+j1} + \frac{3.54\angle -45^\circ}{s+1-j1}$$

Taking inverse Laplace transform,

$$z(t) = 3.54\angle 45^\circ\, e^{(-1-j1)t} + 3.54\angle -45^\circ e^{(-1+j1)t}$$

Example 8.47 *Evaluate amplitude and phase of the network function* $F(s) = \dfrac{4s}{s^2+2s+2}$ *from the pole-zero plot at* $s = j2$.

Solution

$$F(s) = \frac{4s}{s^2+2s+2} = \frac{4s}{(s+1+j1)(s+1-j1)}$$

The pole-zero plot is shown in Fig. 8.96.

At $s = j2$,

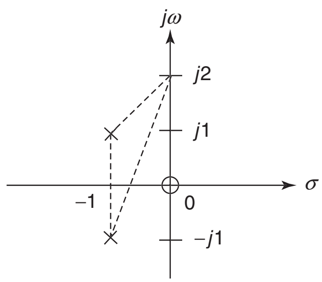


Fig. 8.96

$$|F(j2)| = \frac{\text{Product of phasor magnitudes from all zero to } j2}{\text{Product of phasor magnitudes from all poles to } j2} = \frac{2}{(\sqrt{2})(\sqrt{10})} = 0.447$$

$$\phi(\omega) = \tan^{-1}\left(\frac{2}{0}\right) - \tan^{-1}\left(\frac{3}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = 90° - 71.56° - 45° = -26.56°$$

Example 8.48 *Using the pole-zero plot, find magnitude and phase of the function*

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)} \text{ at } s = j4.$$

Solution

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

The pole-zero plot is shown in Fig. 8.97.

At $s = j4$,

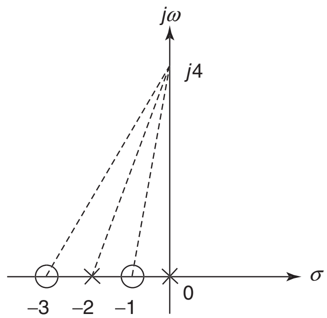


Fig. 8.97

$$|F(j4)| = \frac{\text{Product of phasor magnitudes from all zeros to } j4}{\text{product of phasor magnitudes from all poles to } j4} = \frac{(5)(\sqrt{17})}{(\sqrt{20})(4)} = 1.15$$

$$\phi(\omega) = \tan^{-1}\left(\frac{4}{1}\right) + \tan^{-1}\left(\frac{4}{3}\right) - \tan^{1}\left(\frac{4}{0}\right) - \tan^{-1}\left(\frac{4}{2}\right) = 75.96° + 53.13° - 90° - 63.43° = -24.34°$$

Example 8.49 *Plot amplitude and phase response for*

$$F(s) = \frac{s}{s+10}$$

Solution

$$F(j\omega) = \frac{j\omega}{j\omega + 10}$$

$$|F(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 100}}$$

ω	$\lvert F(j\omega)\rvert$
0	0
10	0.707
100	0.995
1000	1

The amplitude response is shown in Fig. 8.98.

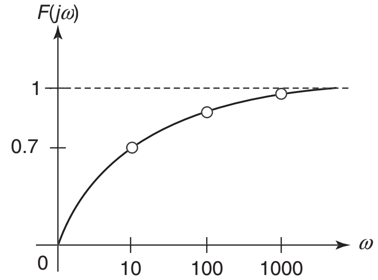


Fig. 8.98

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) = 90° - \tan^{-1}\left(\frac{\omega}{10}\right)$$

ω	$\phi(\omega)$
0	90°
10	45°
100	5.7°
1000	0°

The phase response is shown in Fig. 8.99.

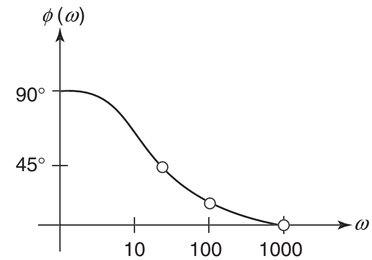


Fig. 8.99

Example 8.50 *Sketch amplitude and phase response for* $F(s) = \dfrac{s+10}{s-10}$

Solution

$$F(j\omega) = \frac{j\omega + 10}{j\omega - 10}$$

$$\lvert F(f\omega)\rvert = \frac{\sqrt{\omega^2 + 100}}{\sqrt{\omega^2 + 100}}$$

For all ω, magnitude is unity.
The amplitude response is shown in Fig. 8.100.

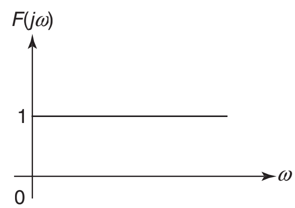


Fig. 8.100

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(-\frac{\omega}{10}\right) = 2\tan^{-1}\left(\frac{\omega}{10}\right)$$

The phase response is shown in Fig. 8.101.

ω	$\phi(\omega)$
0	0°
10	90°
100	168.6°
1000	178.9°

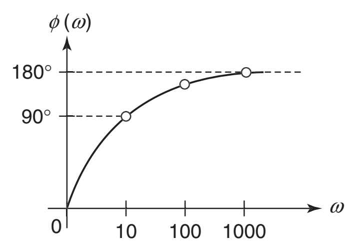


Fig. 8.101

Exercises

8.1 Determine the driving-point impedance $\dfrac{V_1}{I_1}$, transfer impedance $\dfrac{V_2}{I_1}$ and voltage transfer ratio $\dfrac{V_2}{V_1}$ for the network shown in Fig. 8.102.

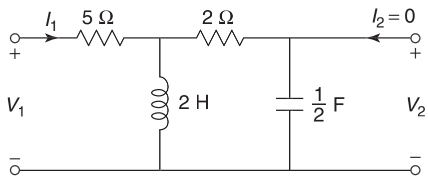


Fig. 8.102

$$\left[\frac{V_1}{I_1} = \frac{7s^2+7s+5}{s2+s+1}; \frac{V_2}{I_1} = \frac{2s}{s^2+s+1}; \frac{V_2}{V_1} = \frac{2s}{7s^2+7s+5}\right]$$

8.2 For the network shown in Fig. 8.103, determine $\dfrac{V_2}{V_1}$ and $\dfrac{V_2}{I_1}$.

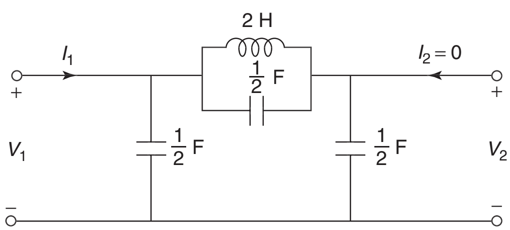


Fig. 8.103

$$\left[\frac{V_2}{V_1} = \frac{s^2+1}{2s^2+1}; \frac{V_2}{I_1} = \frac{2s^2+2}{s(3s^2+2)}\right]$$

8.3 Find the open-circuit transfer impedance Z_{21} and open-circuit voltage ratio G_{21} for the ladder network shown in Fig. 8.104.

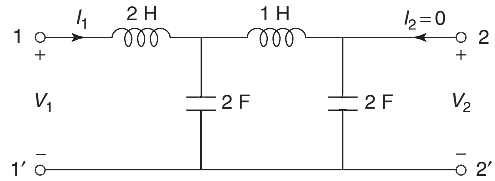


Fig. 8.104

$$\left[Z_{21} = \frac{1}{2s^3+3s}, G_{21} = \frac{1}{4s^4+7s^2+1}\right]$$

8.4 For the two-port network shown in Fig. 8.105, determine Z_{11}, Z_{21} and voltage transfer ratio $G_{21}(s)$.

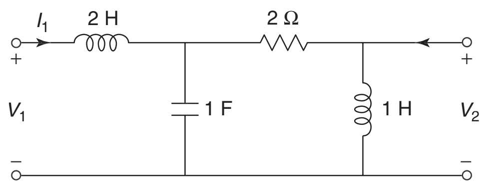


Fig. 8.105

$$\left[Z_{11} = \frac{2s^3+4s^2+3s+2}{s^2+2s+1}, Z_{21} = \frac{s}{s^2+2s+1}, G_{21} = \frac{s}{2s^3+4s^2+3s+2}\right]$$

8.5 Draw the pole-zero diagram of the following network functions:

(i) $F(s) = \dfrac{s^2+4}{s^2+6s+4}$

(ii) $F(s) = \dfrac{5s-12}{s^2+4s+13}$

(iii) $F(s) = \dfrac{s+1}{(s^2+2s+2)^2}$

(iv) $F(s) = \dfrac{s(s^2+5)}{s^4+2s^2+1}$

(v) $F(s) = \dfrac{s^2 + s + 2}{s^4 + 5s^3 + 6s^2}$

(vi) $F(s) = \dfrac{s^2 - s}{s^3 + 2s^2 - s - 2}$

(vii) $F(s) = \dfrac{s^2 + 3s + 2}{s^2 + 3s}$

(viii) $F(s) = \dfrac{(s^2 + 4)(s + 1)}{(s^2 + 1)(s^2 + 2s + 5)}$

8.6 For the network shown in Fig. 8.106, draw the pole-zero plot of the impedance function $Z(s)$.

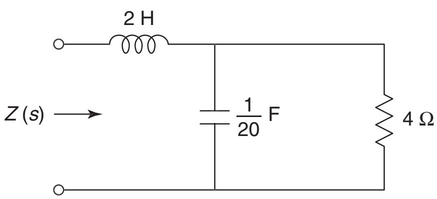


Fig. 8.106

$$Z(s) = \left[\frac{(s + 2.5 - j1.94)(s + 2.5 + j1.94)}{s + 5}\right]$$

8.7 For the network shown in Fig. 8.107, draw the pole-zero plot of driving-point impedance function $Z(s)$.

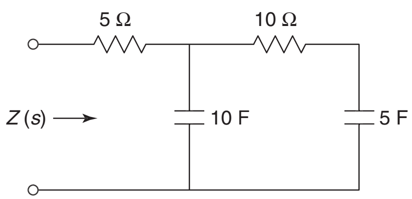


Fig. 8.107

$$\left[Z(s) = \frac{5(s + 0.01)(s + 0.04)}{s(s + 0.03)}\right]$$

8.8 Find the driving-point impedance of the network shown in Fig. 8.108. Also, find poles and zeros.

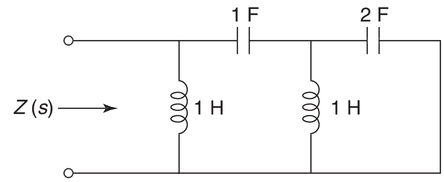


Fig. 8.108

$$\left[Z(s) = \frac{1.5s(s^2 + 0.33)}{(s^2 + 1.707)(s^2 + 0.293)}\right]$$

8.9 Find network functions $\dfrac{V_2}{V_1}$ and $\dfrac{V_1}{I_1}$ for the network shown in Fig. 8.109 and plot poles and zeros of $\dfrac{V_2(s)}{V_1(s)}$.

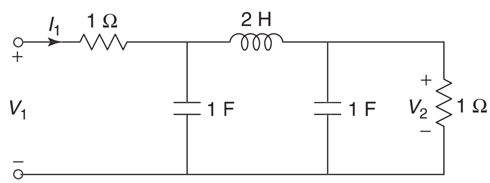


Fig. 8.109

$$\left[\frac{V_2}{V_1} = \frac{1}{2(s^3 + 2s^2 + 2s + 1)}, \frac{V_1}{I_1} = \frac{2(s^3 + 2s^2 + 2s + 1)}{2s^3 + 2s^2 + 2s + 1}\right]$$

8.10 For the network shown in Fig. 8.110, determine $\dfrac{V_1}{I_1}$ *and* $\dfrac{V_2}{I_1}$. Plot the poles and zeros of $\dfrac{V_2}{I_1}$.

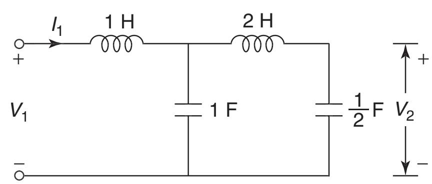


Fig. 8.110

$$\left[\frac{V_1}{I_1} = \frac{2s^4 + 5s^2 + 2}{2s^3 + 3s}, \frac{V_2}{I_1} = \frac{2}{2s^3 + 3s}\right]$$

8.11 For the network shown in Fig. 8.111, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the pole and zeros for $\frac{V_2}{V_1}$.

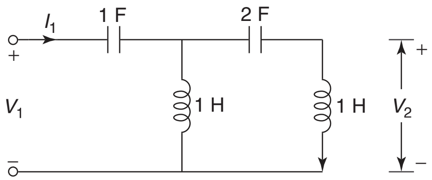


Fig. 8.111

$$\left[\frac{V_1}{I_1}=\frac{s^4+3s^2+1}{2s^3+s}\right]$$

8.12 For the network shown in Fig. 8.112, plot the poles and zeros of transfer impedance function.

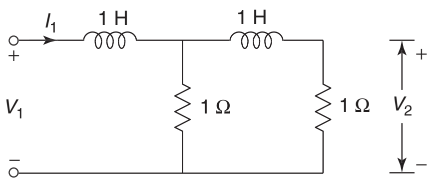


Fig. 8.112

$$\left[\frac{V_2}{I_1}=\frac{1}{s+2}\right]$$

8.13 For the network shown in Fig. 8.113, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the poles and zeros of transfer impedance function.

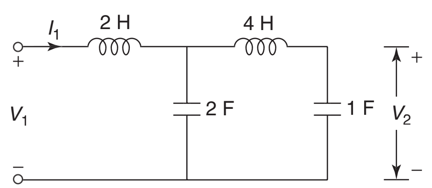


Fig. 8.113

$$\left[\frac{V_1}{I_1}=\frac{16s^4+10s^2+1}{8s^3+3s}, \frac{V_2}{I_1}=\frac{1}{8s^3+3s}, \frac{V_2}{V_1}=\frac{1}{16s^4+10s^2+1}\right]$$

8.14 Obtain the impedance function for which the pole-zero diagram is shown in Fig. 8.114.

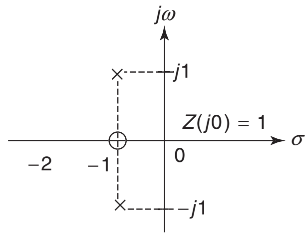


Fig. 8.114

$$\left[Z(s)=\frac{2\,(s+1)}{s^2+2s+2}\right]$$

8.15 For the network shown in Fig. 8.115, poles and zeros of driving point function $Z(s)$ are,

Poles: $(-1 \pm j4)$; zero: -2

If $Z\,(j0) = 1$, find the values of R, L and C.

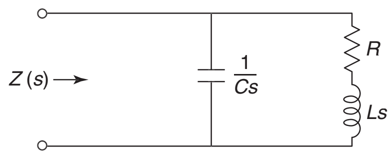


Fig. 8.115

$$\left[1\,\Omega,\ 0.5\ \text{H}, \frac{2}{17}\ \text{F}\right]$$

8.16 For the two-port network shown in Fig. 8.116, find R_1, R_2 and C. $\frac{V_2}{V_1}=\frac{0.2}{s^2+3s+2}$

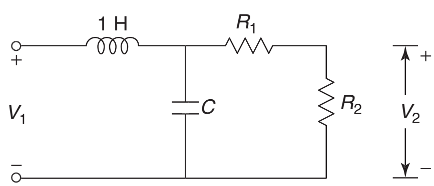


Fig. 8.116

$$\left[\frac{3}{5}\,\Omega, \frac{1}{15}\,\Omega,\ 0.5\ \text{F}\right]$$

8.17 For the given network function, draw the pole-zero diagram and hence obtain the time domain voltage.

$$V(s) = \frac{5(s+5)}{(s+2)(s+7)}$$

$[v(t) = 3e^{-2t} + 2e^{-7t}]$

8.18 A transfer function is given by $Y(s) = \dfrac{10s}{(s+5+j15)(s+5-j15)}$. Find time-domain response using graphical method.

$\left[5.26\angle 18.4^\circ\, e^{-(5+j15)t} + 5.26\angle -18.4^\circ\, e^{-(5-j15)t}\right]$

Objective-Type Questions

8.1 Of the four networks N_1, N_2, N_3 and N_4 of Fig. 8.117, the networks having identical driving-point functions are

(a) N_1 and N_2 (b) N_2 and N_4
(c) N_1 and N_3 (d) N_1 and N_4

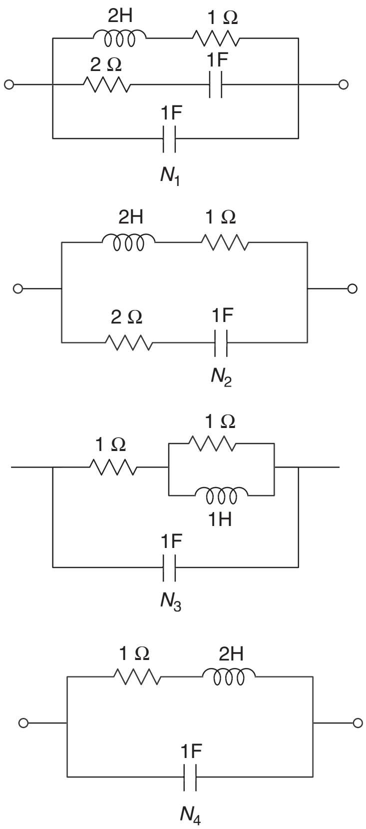


Fig. 8.117

8.2 The driving-point impedance $Z(s)$ of a network has the pole-zero locations as shown in Fig. 8.118. If $Z(0) = 3$, then Z(s) is

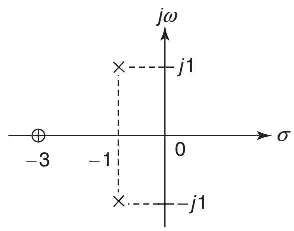


Fig. 8.118

(a) $\dfrac{3(s+3)}{s^2+2s+3}$ (b) $\dfrac{2(s+3)}{s^2+2s+2}$

(c) $\dfrac{3(s-3)}{s^2-2s-2}$ (d) $\dfrac{2(s-3)}{s^2-2s-3}$

8.3 For the circuit shown in Fig. 8.119, the initial conditions are zero. Its transfer function $H(s) = \dfrac{V_0(s)}{V_1(s)}$ is

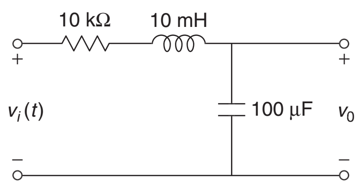


Fig. 8.119

(a) $\dfrac{1}{s^2+10^6 s+10^6}$ (b) $\dfrac{10^6}{s^2+10^3 s+10^6}$

(c) $\dfrac{10^3}{s^2+10^3 s+10^6}$ (d) $\dfrac{10^6}{s^2+10^6 s+10^6}$

8.4 In Fig. 8.120, assume that all the capacitors are initially uncharged. If $v_i(t) = 10\,u(t)$, then $v_0(t)$ is given by

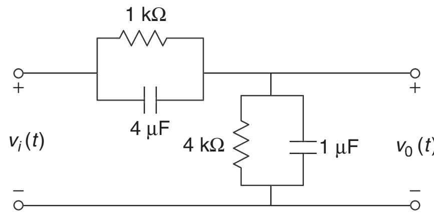


Fig. 8.120

(a) $8\,e^{-0.004\,t}$ (b) $8\,(1-e^{-0.004\,t})$

(c) $8\,u(t)$ (d) 8

8.5 A system is represented by the transfer function $\dfrac{10}{(s+1)(s+2)}$. The dc gain of this system is

(a) 1 (b) 2

(c) 5 (d) 10

8.6 Which one of the following is the ratio $\dfrac{V_{24}}{V_{13}}$ of the network shown in Fig. 8.121.

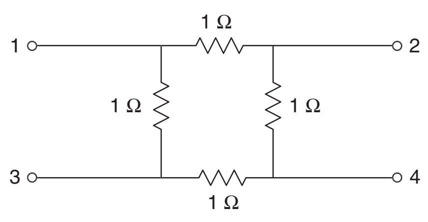


Fig. 8.121

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{3}{4}$ (d) $\dfrac{4}{3}$

8.7 A network has response with time as shown in Fig. 8.122. Which one of the following diagrams represents the location of the poles of this network?

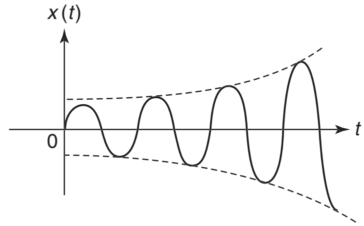


Fig. 8.122

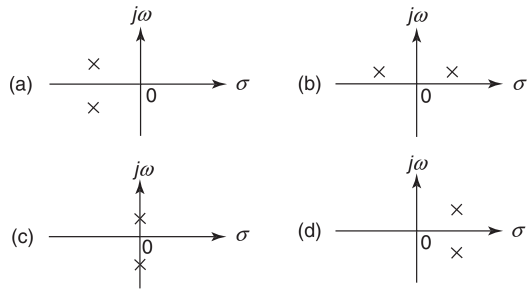


Fig. 8.123

8.8 The transfer function of a low-pass *RC* network is

(a) $(RCs)\,(1 + RCs)$ (b) $\dfrac{1}{1+RCs}$

(c) $\dfrac{RCs}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

8.9 The driving-point admittance function of the network shown in Fig. 8.124 has a

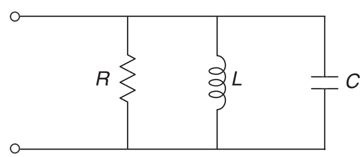


Fig. 8.124

(a) pole at $s = 0$ and zero at $s = \infty$
(b) pole at $s = 0$ and pole at $s = \infty$
(c) pole at $s = \infty$ and zero at $s = 0$
(d) pole at $s = \infty$ and zero at $s = \infty$

8.10 The transfer function $Y_{12}(s) = \dfrac{I_2(s)}{V_1(s)}$ for the network shown in Fig. 8.125 is

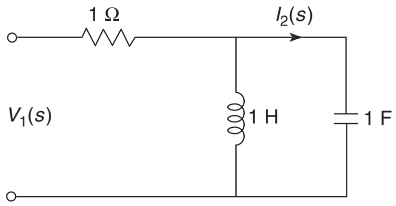


Fig. 8.125

(a) $\dfrac{s^2}{s^2+s+1}$ (b) $\dfrac{s}{s+1}$

(c) $\dfrac{1}{s+1}$ (d) $\dfrac{s+1}{s^2+1}$

8.11 As the poles of a network shift away from the *x* axis, the response

(a) remains constant
(b) becomes less oscillating
(c) becomes more oscillating
(d) none of these

Answers to Objective-Type Questions

8.1. (c)	8.2. (b)	8.3. (d)	8.4. (c)	8.5. (c)	8.6. (a)	8.7. (d)
8.8. (b)	8.9. (a)	8.10. (a)	8.11. (c)			

9

Two-Port Networks

9.1 INTRODUCTION

A two-port network has two pairs of terminals, one pair at the input known as *input port* and one pair at the output known as *output port* as shown in Fig. 9.1. There are four variables V_1, V_2, I_1 and I_2 associated with a two-port network. Two of these variables can be expressed in terms of the other two variables. Thus, there will be two dependent variables and two independent variables. The number of possible combinations generated by four variables taken two at a time is ${}^4C_2$, i.e., six. There are six possible sets of equations describing a two-port network.

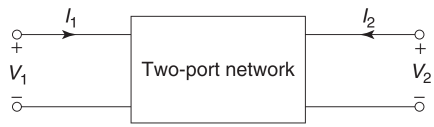


Fig. 9.1 *Two-port network*

Table 9.1 *Two–port parameters*

Parameter	*Variables*		*Equation*
	Express	*In terms of*	
Open-Circuit Impedance	V_1, V_2	I_1, I_2	$V_1 = Z_{11} I_1 + Z_{12} I_2$ $V_2 = Z_{21} I_1 + Z_{22} I_2$
Short-Circuit Admittance	I_1, I_2	V_1, V_2	$I_1 = Y_{11} V_1 + Y_{12} V_2$ $I_2 = Y_{21} V_1 + Y_{22} V_2$
Transmission	V_1, I_1	V_2, I_2	$V_1 = AV_2 - BI_2$ $I_1 = CV_2 - DI_2$
Inverse Transmission	V_2, I_2	V_1, I_1	$V_2 = A' V_1 - B' I_1$ $I_2 = C' V_1 - D' I_1$
Hybrid	V_1, I_2	I_1, V_2	$V_1 = h_{11} I_1 + h_{12} V_2$ $I_2 = h_{21} I_1 + h_{22} V_2$
Inverse Hybrid	I_1, V_2	V_1, I_2	$I_1 = g_{11} V_1 + g_{12} I_2$ $V_2 = g_{21} V_1 + g_{22} I_2$

9.2 OPEN-CIRCUIT IMPEDANCE PARAMETERS (Z PARAMETERS)

The Z parameters of a two-port network may be defined by expressing two-port voltages V_1 and V_2 in terms of two-port currents I_1 and I_2.

$$(V_1, V_2) = f(I_1, I_2)$$
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$[V] = [Z][I]$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero.

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

where Z_{11} is the driving-point impedance with the output port open-circuited. It is also called *open-circuit input impedance*.

Similarly,

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

where Z_{21} is the transfer impedance with the output port open-circuited. It is also called *open-circuit forward transfer impedance*.

Case 2 When input port is open-circuited, i.e., $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where Z_{12} is the transfer impedance with the input port open-circuited. It is also called *open-circuit reverse transfer impedance*.

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

where Z_{22} is the open-circuit driving-point impedance with the input port open-circuited. It is also called *open circuit output impedance*.

As these impedance parameters are measured with either the input or output port open-circuited, these are called *open-circuit impedance parameters*.

The equivalent circuit of the two-port network in terms of Z parameters is shown in Fig. 9.2.

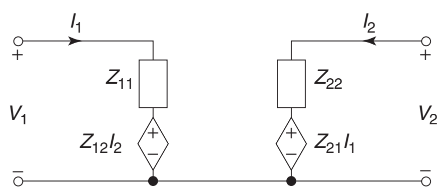


Fig. 9.2 *Equivalent circuit of the two-port network in terms of Z parameter*

9.2.1 Condition for Reciprocity

A network is said to be reciprocal if the ratio of excitation at one port to response at the other port is same if excitation and response are interchanged.

(a) As shown in Fig. 9.3, voltage V_s is applied at the input port with the output port short-circuited.

Fig. 9.3 *Network for deriving condition for reciprocity*

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

From the Z-parameter equations,

$$V_s = Z_{11}\, I_1 - Z_{12}\, I_2'$$
$$0 = Z_{21}\, I_1 - Z_{22}\, I_2'$$
$$I_1 = \frac{Z_{22}}{Z_{21}}\, I_2'$$
$$V_s = Z_{11} \frac{Z_{22}}{Z_{21}} I_2' - Z_{12}\, I_2'$$
$$\frac{V_s}{I_2'} = \frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{21}}$$

(b) As shown in Fig. 9.4, voltage V_s is applied at the output port with input port short-circuited.

i.e.,

$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1 = -I_1'$$

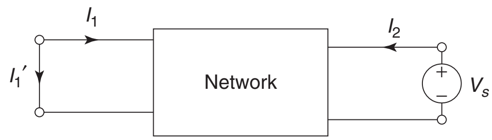


Fig. 9.4 *Network for deriving condition for reciprocity*

From the Z-parameter equations,

$$0 = -Z_{11}\, I_1' + Z_{12}\, I_2$$
$$V_s = -Z_{21}\, I_1' + Z_{22}\, I_2$$
$$I_2 = \frac{Z_{11}}{Z_{12}}\, I_1'$$
$$V_s = -Z_{21} I_1' + Z_{22} \frac{Z_{11}}{Z_{12}}\, I_1'$$
$$\frac{V_s}{I_1'} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{12}}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_1'} = \frac{V_s}{I_2'}$$

i.e.,

$$Z_{12} = Z_{21}$$

9.2.2 Condition for Symmetry

For a network to be symmetrical, the voltage-to-current ratio at one port should be the same as the voltage-to-current ratio at the other port with one of the ports open-circuited.

(a) When the output port is open-circuited, i.e., $I_2 = 0$

From the Z-parameter equation,

$$V_s = Z_{11}\, I_1$$
$$\frac{V_s}{I_1} = Z_{11}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$

From the Z-parameter equation,

$$V_s = Z_{22}\, I_2$$
$$\frac{V_s}{I_2} = Z_{22}$$

Hence, for the network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $$Z_{11} = Z_{22}$$

Example 9.1 *Test results for a two-port network are (a) $I_1 = 0.1 \angle 0°$ A, $V_1 = 5.2 \angle 50°$ V, $V_2 = 4.1 \angle{-25}°$ V with Port 2 open-circuited (b) $I_2 = 0.1 \angle 0°$A, $V_1 = 3.1 \angle{-80}°$V, $V_2 = 4.2 \angle 60°$V, with Port 1 open-circuited. Find Z parameters.*

Solution

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{5.2\angle 50°}{0.1\angle 0°} = 52 \angle 50°\ \Omega, \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{3.1\angle -80°}{0.1\angle 0°} = 31 \angle -80°\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{4.1\angle -25°}{0.1\angle 0°} = 41 \angle -25°\ \Omega, \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{4.2\angle 60°}{0.1\angle 0°} = 42 \angle 60°\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 52\angle 50° & 31\angle -80° \\ 41\angle -25° & 42\angle 60° \end{bmatrix}$$

Example 9.2 *Find the Z parameters for the network shown in Fig. 9.5.*

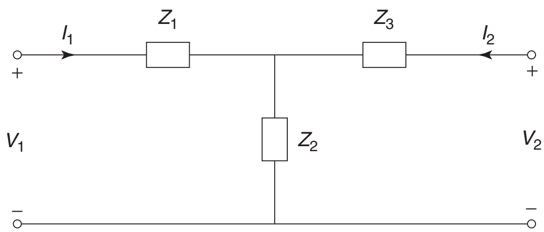


Fig. 9.5

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.
Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)\, I_1$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2$$

Also $$V_2 = Z_2\, I_1$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2$$

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$.
Applying KVL to Mesh 2,

$$V_2 = (Z_2 + Z_3)\, I_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3$$

Also

$$V_1 = Z_2 I_2$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_2 \\ Z_2 & Z_2 + Z_3 \end{bmatrix}$$

Second Method

The network is redrawn as shown in Fig. 9.6.

Applying KVL to Mesh 1,

$$V_1 = Z_1 I_1 + Z_2(I_1 + I_2)$$
$$= (Z_1 + Z_2)I_1 + Z_2 I_2 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = Z_3 I_2 + Z_2(I_1 + I_2)$$
$$= Z_2 I_1 + (Z_2 + Z_3)I_2 \quad \ldots\text{(ii)}$$

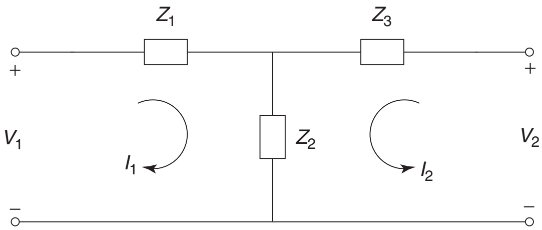


Fig. 9.6

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_2 \\ Z_2 & Z_2 + Z_3 \end{bmatrix}$$

Example 9.3 *Find Z-parameter for the network shown in Fig. 9.7.*

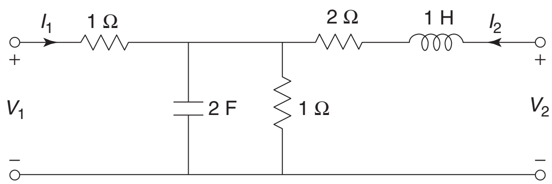


Fig. 9.7

Solution The transformed network is shown in Fig. 9.8.

$$Z_1 = 1$$

$$Z_2 = \frac{\left(\frac{1}{2s}\right)(1)}{\frac{1}{2s} + 1} = \frac{1}{2s+1}$$

$$Z_3 = s + 2$$

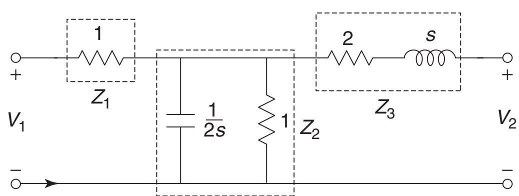


Fig. 9.8

From definition of Z-parameters,

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2 = 1 + \frac{1}{2s+1} = \frac{2s+2}{2s+1}, \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2 = \frac{1}{2s+1}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2 = \frac{1}{2s+1}, \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3 = \frac{1}{2s+1} + s + 2 = \frac{2s^2 + 5s + 3}{2s+1}$$

Example 9.4 *Find Z-parameters for the network shown in Fig. 9.9.*

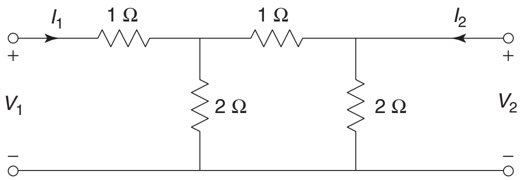


Fig. 9.9

Solution The network is redrawn as shown in Fig. 9.10.

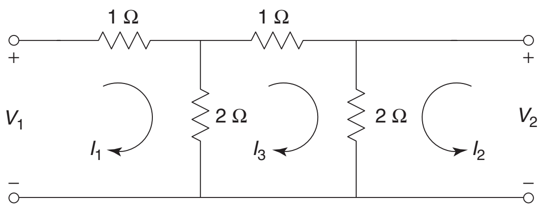


Fig. 9.10

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \frac{4}{5}I_1 + \frac{4}{5}I_2$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + \frac{4}{5}I_1 - \frac{4}{5}I_2$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{6}{5} \end{bmatrix}$$

Example 9.5 *Find the Z-parameters for the network shown in Fig. 9.11.*

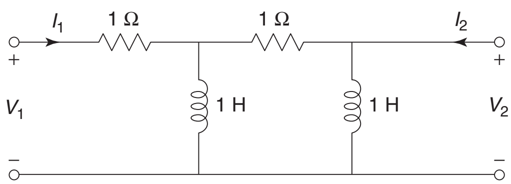


Fig. 9.11

Solution The transformed network is shown in Fig. 9.12.

Applying KVL to Mesh 1,

$$V_1 = (s+1)I_1 - sI_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = sI_2 + sI_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-sI_1 + sI_2 + (2s+1)I_3 = 0$$

$$I_3 = \frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2 \qquad \ldots\text{(iii)}$$

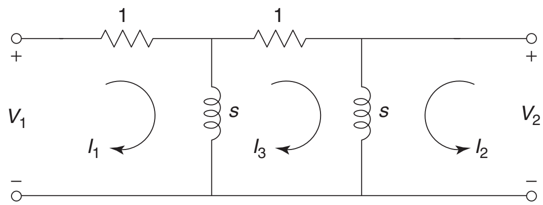


Fig. 9.12

Substituting Eq. (iii) in Eq. (i),

$$V_1 = (s+1)I_1 - s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2+3s+1}{2s+1}\right)I_1 + \left(\frac{s^2}{2s+1}\right)I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = sI_2 + s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2}{2s+1}\right)I_1 + \left(\frac{s^2+s}{2s+1}\right)I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{2s+1} & \dfrac{s^2}{2s+1} \\ \dfrac{s^2}{2s+1} & \dfrac{s^2+s}{2s+1} \end{bmatrix}$$

Example 9.6 *Find the open-circuit impedance parameters for the network shown in Fig. 9.13. Determine whether the network is symmetrical and reciprocal.*

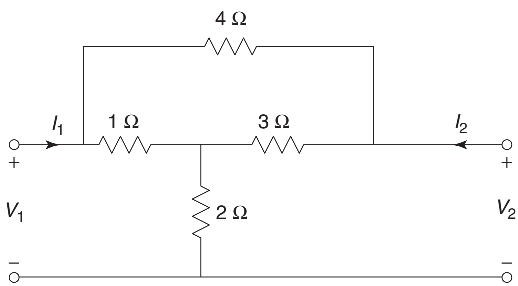


Fig. 9.13

Solution The network is redrawn as shown in Fig. 9.14.

Applying KVL to Mesh 1,

$$V_1 - 1(I_1 - I_3) - 2(I_1 + I_2) = 0$$

$$V_1 = 3I_1 + 2I_2 - I_3 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 3(I_2 + I_3) - 2(I_1 + I_2) = 0$$

$$V_2 = 2I_1 + 5I_2 + 3I_3 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4I_3 - 3(I_2 + I_3) - 1(I_3 - I_1) = 0$$

$$I_1 - 3I_2 + 8I_3 = 0$$

$$I_3 = \frac{1}{8}I_1 - \frac{3}{8}I_2 \quad \ldots\text{(iii)}$$

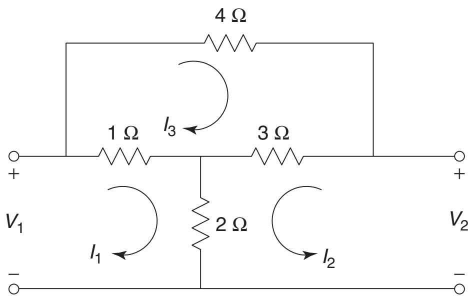


Fig. 9.14

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{23}{8}I_1 + \frac{19}{8}I_2 \quad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 5I_2 + 3\left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{19}{8}I_1 + \frac{31}{8}I_2 \quad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{23}{8} & \frac{19}{8} \\ \frac{19}{8} & \frac{31}{8} \end{bmatrix}$$

Since $Z_{11} \neq Z_{22}$, the network is not symmetrical.
Since $Z_{12} = Z_{21}$, the network is reciprocal.

9.3 SHORT-CIRCUIT ADMITTANCE PARAMETERS (Y PARAMETERS)

The Y parameters of a two-port network may be defined by expressing the two-port currents I_1 and I_2 in terms of the two-port voltages V_1 and V_2.

$$(I_1, I_2) = f(V_1, V_2)$$

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

In matrix form, we can write

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$[I] = [Y][V]$$

The individual Y parameters for a given network can be defined by setting each of the port voltages equal to zero.

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

where Y_{11} is the driving-point admittance with the output port short-circuited. It is also called *short-circuit input admittance*.

Similarly,

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

where Y_{21} is the transfer admittance with the output port short-circuited. It is also called *short-circuit forward transfer admittance*.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

where Y_{12} is the transfer admittance with the input port short-circuited. It is also called *short-circuit reverse transfer admittance*.

Similarly,

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

where Y_{22} is the short-circuit driving-point admittance with the input port short-circuited. It is also called the *short circuit output admittance*.

As these admittance parameters are measured with either input or output port short-circuited, these are called *short-circuit admittance parameters*.

The equivalent circuit of the two-port network in terms of Y parameters is shown in Fig. 9.15.

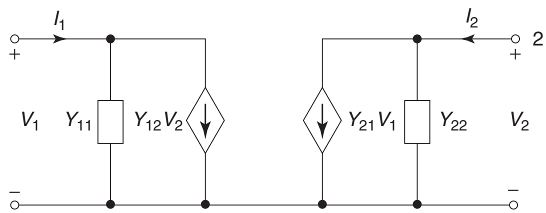


Fig. 9.15 *Equivalent circuit of the two-port network in terms of Y-parameters*

9.3.1 Condition for Reciprocity

(a) As shown in Fig. 9.16, voltage V_s is applied at input port with the output port short-circuited.

i.e,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

From the Y-parameter equation,

$$-I_2' = Y_{21} V_s$$
$$\frac{I_2}{V_s} = -Y_{21}$$

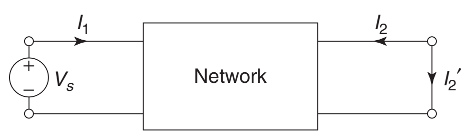


Fig. 9.16 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 9.17, voltage V_s is applied at output port with the input port short-circuited.

i.e,
$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1 = -I_1'$$

From the Y-parameter equation,
$$-I_1' = Y_{12} V_s$$
$$\frac{I_1'}{V_s} = -Y_{12}$$

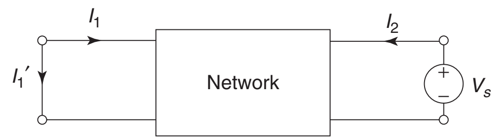


Fig. 9.17 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,
$$\frac{I_2'}{V_s} = \frac{I_1'}{V_s}$$

i.e,
$$Y_{12} = Y_{21}$$

9.3.2 Condition for Symmetry

(a) When the output port is short-circuited, i.e., $V_2 = 0$.
From the Y-parameter equation,
$$I_1 = Y_{11} V_s$$
$$\frac{V_s}{I_1} = \frac{1}{Y_{11}}$$

(b) When the input port is short-circuited, i.e., $V_1 = 0$.
From the Y-parameter equation,
$$I_2 = Y_{22} V_s$$
$$\frac{V_s}{I_2} = \frac{1}{Y_{22}}$$

Hence, for the network to be symmetrical,
$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e.,
$$Y_{11} = Y_{22}$$

Example 9.7 *Test results for a two-port network are*

(a) Port 2 short-circuited: $V_1 = 50\angle 0°\ V, I_1 = 2.1\angle -30°\ A, I_2 = -1.1\angle -20°\ A$

(b) Port 1 short-circuited: $V_2 = 50\angle 0°\ V, I_2 = 3\angle -15°\ A, I_1 = -1.1\angle -20°\ A.$
Find Y-parameters.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.1\angle -30°}{50\angle 0°} = 0.042\angle -30°\ \mho, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-1.1\angle -20°}{50\angle 0°} = -0.022\angle -20°\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-1.1\angle -20°}{50\angle 0°} = -0.022\angle -20°\ \mho, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3\angle -15°}{50\angle 0°} = 0.06\angle -15°\ \mho$$

Hence, the Y-parameters are
$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.042\angle -30° & -0.022\angle -20° \\ -0.022\angle -20° & 0.06\angle -15° \end{bmatrix}$$

Example 9.8 *Find Y-parameters for the network shown in Fig. 9.18.*

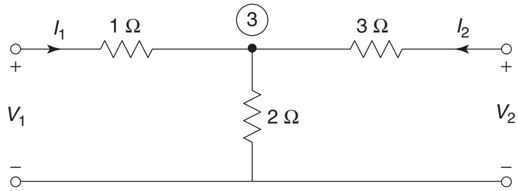


Fig. 9.18

Solution

First Method

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 9.19,

$$R_{eq} = 1 + \frac{2\times 3}{2+3} = 1 + \frac{6}{5} = \frac{11}{5}\ \Omega$$

Now,

$$V_1 = \frac{11}{5} I_1$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{5}{11}\ \mho$$

Also,

$$I_2 = \frac{2}{5}(-I_1) = -\frac{2}{5}\times\frac{5}{11}V_1 = -\frac{2}{11}V_1$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{2}{11}\ \mho$$

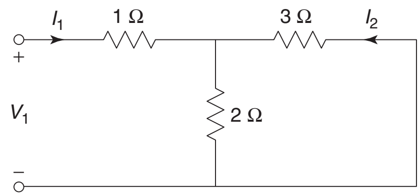


Fig. 9.19

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 9.20,

$$R_{eq} = 3 + \frac{1\times 2}{1+2} = 3 + \frac{2}{3} = \frac{11}{3}\ \Omega$$

Now

$$V_2 = \frac{11}{3} I_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3}{11}\ \mho$$

Also

$$I_1 = \frac{2}{3}(-I_2) = -\frac{2}{3}\times\frac{3}{11}V_2 = -\frac{2}{11}V_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{2}{11}\ \mho$$

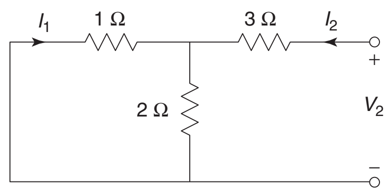


Fig. 9.20

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{5}{11} & -\dfrac{2}{11} \\ -\dfrac{2}{11} & \dfrac{3}{11} \end{bmatrix}$$

Second Method (Refer Fig. 9.18)

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \text{...(i)}$$

$$I_2 = \frac{V_2 - V_3}{3}$$

$$= \frac{V_2}{3} - \frac{V_3}{3} \qquad \text{...(ii)}$$

Applying KCL at Node 3,

$$I_1 + I_2 = \frac{V_3}{2} \qquad \text{...(iii)}$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$V_1 - V_3 + \frac{V_2}{3} - \frac{V_3}{3} = \frac{V_3}{2}$$

$$V_1 + \frac{V_2}{3} = \frac{11}{6} V_3$$

$$V_3 = \frac{6}{11} V_1 + \frac{2}{11} V_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = V_1 - \frac{6}{11} V_1 - \frac{2}{11} V_2$$

$$= \frac{5}{11} V_1 - \frac{2}{11} V_2 \qquad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{V_2}{3} - \frac{1}{3}\left(\frac{6}{11} V_1 + \frac{2}{11} V_2\right)$$

$$= -\frac{2}{11} V_1 + \frac{3}{11} V_2 \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Example 9.9 *Find Y-parameters of the network shown in Fig. 9.21.*

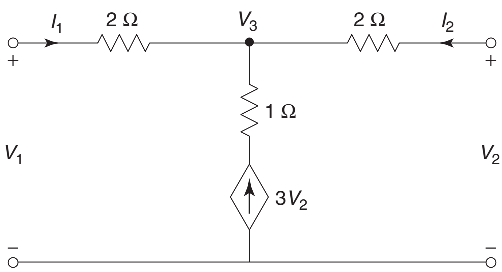


Fig. 9.21

Solution From Fig. 9.21,

$$I_1 = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2} V_1 - \frac{1}{2} V_3 \qquad \text{...(i)}$$

$$I_2 = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$I_1 + I_2 + 3V_2 = 0 \qquad \ldots\text{(iii)}$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} + 3V_2 = 0$$

$$2V_3 = V_1 + 7V_2$$

$$V_3 = \frac{1}{2}V_1 + \frac{7}{2}V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = \frac{1}{2}V_1 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= \frac{1}{4}V_1 - \frac{7}{4}V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{1}{2}V_2 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= -\frac{1}{4}V_1 - \frac{5}{4}V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}$$

Example 9.10 *Determine Y-parameters for the network shown in Fig. 9.22. Determine whether the network is symmetrical and reciprocal.*

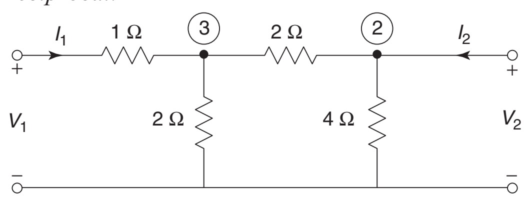


Fig. 9.22

Solution From Fig. 9.22,

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \ldots\text{(i)}$$

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{2} + \frac{V_3 - V_2}{2}$$

$$= V_3 - \frac{V_2}{2} \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{4} + \frac{V_2 - V_3}{2}$$

$$= \frac{3}{4}V_2 - \frac{V_3}{2} \qquad \text{...(iii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = V_3 - \frac{V_2}{2}$$

$$V_3 = \frac{V_1}{2} + \frac{V_2}{4} \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = \frac{V_1}{2} + \frac{V_2}{4} - \frac{V_2}{2}$$

$$= \frac{V_1}{2} - \frac{V_2}{4} \qquad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = \frac{3}{4}V_2 - \frac{1}{2}\left(\frac{V_1}{2} + \frac{V_2}{4}\right)$$

$$= -\frac{V_1}{4} + \frac{5V_2}{8} \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{8} \end{bmatrix}$$

Since $Y_{11} \neq Y_{22}$, the network is not symmetrical.
Since $Y_{12} = Y_{21}$, the network is reciprocal.

Example 9.11 *Determine the short-circuit admittance parameters for the network shown in Fig. 9.23.*

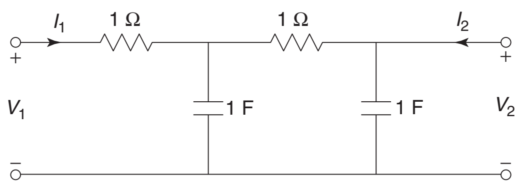


Fig. 9.23

Solution The transformed network is shown in Fig. 9.24.
From Fig. 9.24,

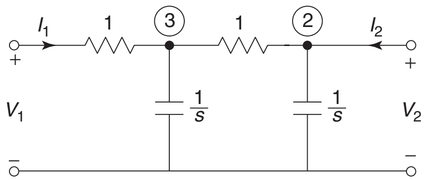


Fig. 9.24

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \text{...(i)}$$

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{\frac{1}{s}} + \frac{(V_3 - V_2)}{1}$$

$$= (s+1)V_3 - V_2 \qquad \text{...(ii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + \frac{(V_2 - V_3)}{1}$$

$$= (s+1)V_2 - V_3 \qquad \text{...(iii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = (s+1)V_3 - V_2$$

$$(s+2)V_3 = V_1 + V_2$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = (s+1)\left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right) - V_2$$

$$= \frac{s+1}{s+2}V_1 - \frac{1}{s+2}V_2 \qquad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = (s+1)V_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\frac{1}{s+2}V_1 + \frac{s^2+3s+1}{s+2}V_2 \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{1}{s+2} \\ -\frac{1}{s+2} & \frac{s^2+3s+1}{s+2} \end{bmatrix}$$

Example 9.12 *Determine Y-parameters for the network shown in Fig. 9.25.*

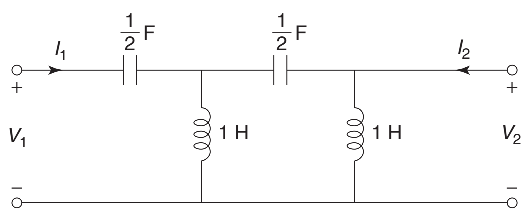


Fig. 9.25

Solution The transformed network is shown in Fig. 9.26.

From Fig. 9.26,

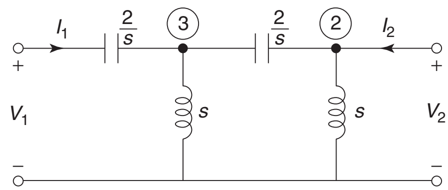


Fig. 9.26

$$I_1 = \frac{V_1 - V_3}{\frac{2}{s}}$$

$$= \frac{s}{2}V_1 - \frac{s}{2}V_3 \qquad \text{...(i)}$$

Applying KCL at Node 3,

$$\frac{s}{2}(V_1 - V_3) = \frac{V_3}{s} + \frac{s}{2}(V_3 - V_2)$$

$$\frac{s}{2}V_3 + \frac{1}{s}V_3 + \frac{s}{2}V_3 = \frac{s}{2}V_1 + \frac{s}{2}V_2$$

$$V_3 = \frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2 \qquad \text{...(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s}{2}V_1 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= \left[\frac{s}{2} - \frac{s^3}{4(s^2+1)}\right]V_1 - \frac{s^3}{4(s^2+1)}V_2$$

$$= \frac{s^3+2s}{4(s^2+1)}V_1 - \frac{s^3}{4(s^2+1)}V_2 \qquad \text{...(iii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{s} + \frac{s}{2}(V_2 - V_3)$$

$$= \frac{s^2+2}{2s}V_2 - \frac{s}{2}V_3 \qquad \text{...(iv)}$$

Substituting Eq. (ii) in Eq. (iv),

$$I_2 = \frac{s^2+2}{2s}V_2 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \left[\frac{s^2+2}{2s} - \frac{s^3}{4(s^2+1)}\right]V_2$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2 \qquad \text{...(v)}$$

Comparing Eqs (iii) and (v) with Y-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^3+2s}{4(s^2+1)} & -\dfrac{s^3}{4(s^2+1)} \\ -\dfrac{s^3}{4(s^2+1)} & \dfrac{s^4+6s^2+4}{4s(s^2+1)} \end{bmatrix}$$

Example 9.13 *Obtain Y-parameters of the network shown in Fig. 9.27.*

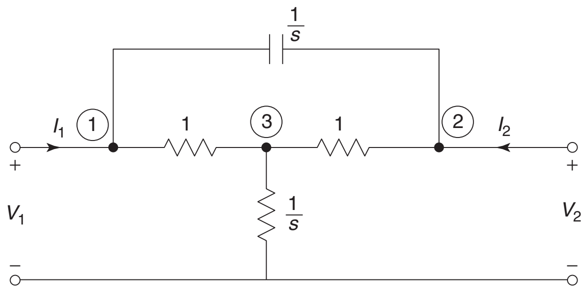


Fig. 9.27

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1 - V_3}{1} + \frac{V_1 - V_2}{\frac{1}{s}}$$

$$= (s+1)\,V_1 - s\,V_2 - V_3 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2 - V_3}{1} + \frac{V_2 - V_1}{\frac{1}{s}}$$

$$= (s+1)\,V_2 - s\,V_1 - V_3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3}{\frac{1}{s}} + \frac{V_3 - V_1}{1} + \frac{V_3 - V_2}{1} = 0$$

$$(s+2)\,V_3 - V_1 - V_2 = 0$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$I_1 = (s+1)\,V_1 - sV_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_1 - \left[\frac{s(s+2)+1}{(s+2)}\right]V_2$$

$$= \left(\frac{s^2+3s+1}{s+2}\right)V_1 - \left(\frac{s^2+2s+1}{s+2}\right)V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$I_2 = (s+1)\,V_2 - s\,V_1 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\left[\frac{s(s+2)+1}{(s+2)}\right]V_1 + \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_2$$

$$= -\left(\frac{s^2+2s+1}{s+2}\right)V_1 + \left(\frac{s^2+3s+1}{s+2}\right)V_2 \qquad \ldots(\text{v})$$

Comparing Eqs (iv) and (v) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{s+2} & -\dfrac{(s^2+2s+1)}{s+2} \\ -\dfrac{(s^2+2s+1)}{s+2} & \dfrac{s^2+3s+1}{s+2} \end{bmatrix}$$

9.4 TRANSMISSION PARAMETERS (ABCD PARAMETERS)

The transmission parameters or chain parameters or *ABCD* parameters serve to relate the voltage and current at the input port to voltage and current at the output port. In equation form,

$$(V_1, I_1) = f(V_2, -I_2)$$

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Here, the negative sign is used with I_2 and not for parameters B and D. The reason the current I_2 carries a negative sign is that in transmission field, the output current is assumed to be coming out of the output port instead of going into the port.

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called transmission matrix.

For a given network, these parameters are determined as follows:

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

where A is the reverse voltage gain with the output port open-circuited.

Similarly,
$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

where C is the transfer admittance with the output port open-circuited.

Case 2 When output port is short-circuited, i.e., $V_2 = 0$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

where B is the transfer impedance with the output port short-circuited.

Similarly,
$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

where D is the reverse current gain with the output port short-circuited.

9.4.1 Condition for Reciprocity

(a) As shown in Fig. 9.28, voltage V_s is applied at the input port with the output port short-circuited.

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

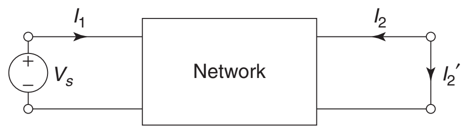


Fig. 9.28 *Network for deriving condition for reciprocity*

From the transmission parameter equations,

$$V_s = B\, I_2'$$
$$\frac{V_s}{I_2'} = B$$

(b) As shown in Fig. 9.29, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,

$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1' = -I_1$$

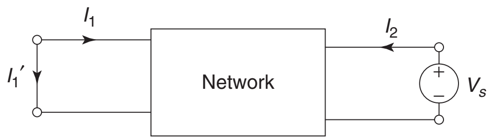


Fig. 9.29 *Network for deriving condition for reciprocity*

From the transmission parameter equations,

$$0 = AV_s - BI_2$$
$$-I_1' = CV_s - DI_2$$
$$I_2 = \frac{A}{B} V_s$$
$$-I_1' = CV_s - \frac{AD}{B} V_s$$
$$\frac{V_s}{I_1'} = \frac{B}{AD - BC}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e.,

$$B = \frac{B}{AD - BC}$$

i.e.,

$$AD - BC = 1$$

9.4.2 Condition for Symmetry

(a) When the output port is open-circuited, i.e., $I_2 = 0$.

From the transmission-parameter equations,

$$V_s = AV_2$$
$$I_1 = CV_2$$
$$\frac{V_s}{I_1} = \frac{A}{C}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$.

From the transmission parameter equation,

$$CV_s = DI_2$$
$$\frac{V_s}{I_2} = \frac{D}{C}$$

Hence, for network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e.,
$$A = D$$

Example 9.14 *Find the transmission parameters for the network shown in Fig. 9.30.*

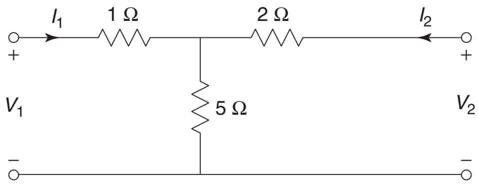


Fig. 9.30

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.

$$V_1 = 6I_1$$

and
$$V_2 = 5I_1$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{6I_1}{5I_1} = \frac{6}{5}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{5}\ \mho$$

Case 2 When the output port is short-circuited, i.e., $V_2 = 0$, as shown in Fig. 9.31,

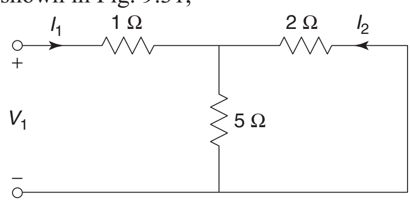


Fig. 9.31

$$R_{eq} = 1 + \frac{5\times 2}{5+2} = 1 + \frac{10}{7} = \frac{17}{7}\ \Omega$$

Now
$$V_1 = \frac{17}{7} I_1$$

and
$$I_2 = \frac{5}{7}(-I_1) = -\frac{5}{7} I_1$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{\frac{17}{7} I_1}{-\frac{5}{7} I_1} = \frac{17}{5}\ \Omega$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{7}{5}$$

Hence, the transmission parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Second Method (Refer Fig. 9.37)

Applying KVL to Mesh 1,

$$V_1 = 6I_1 + 5I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 5I_1 + 7I_2 \qquad \text{...(ii)}$$

Hence,

$$5I_1 = V_2 - 7I_2$$

$$I_1 = \frac{1}{5}V_2 - \frac{7}{5}I_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 6\left(\frac{1}{5}V_2 - \frac{7}{5}I_2\right) + 5I_2$$

$$= \frac{6}{5}V_2 - \frac{17}{5}I_2 \qquad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with transmission parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Example 9.15 *Obtain ABCD parameters for the network shown in Fig. 9.32.*

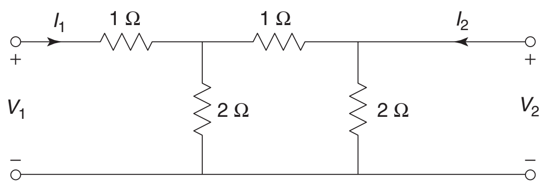


Fig. 9.32

Solution The network is redrawn as shown in Fig. 9.33.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - I_3 - 2(I_3 + I_2) = 0$$

$$5I_3 = 2I_1 - 2I_2$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \qquad \text{...(iii)}$$

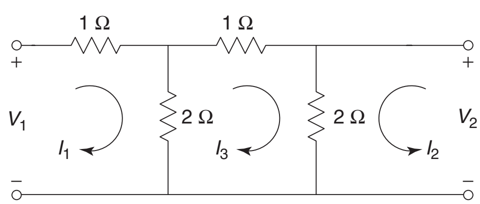


Fig. 9.33

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2$$

$$\frac{4}{5}I_1 = V_2 - \frac{6}{5}I_2$$

$$I_1 = \frac{5}{4}V_2 - \frac{3}{2}I_2 \qquad \ldots(\text{v})$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{11}{5}\left(\frac{5}{4}V_2 - \frac{3}{2}I_2\right) + \frac{4}{5}I_2$$

$$= \frac{11}{4}V_2 - \frac{5}{2}I_2 \qquad \ldots(\text{vi})$$

Comparing Eqs (v) and (vi) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{11}{4} & \frac{5}{2} \\ \frac{5}{4} & \frac{3}{2} \end{bmatrix}$$

Example 9.16 *Determine the transmission parameters for the network shown in Fig. 9.34.*

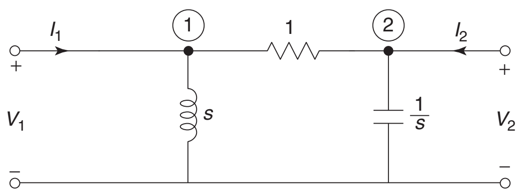


Fig. 9.34

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1}{s} + (V_1 - V_2)$$

$$= \frac{s+1}{s}V_1 - V_2 \qquad \ldots(\text{i})$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + (V_2 - V_1)$$

$$= (s+1)\,V_2 - V_1$$

$$V_1 = (s+1)\,V_2 - I_2 \qquad \ldots(\text{ii})$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s+1}{s}[(s+1)V_2 - I_2] - V_2$$

$$= \left[\frac{(s+1)^2}{s} - 1\right]V_2 - \frac{s+1}{s}I_2$$

$$= \left(\frac{s^2+s+1}{s}\right)V_2 - \left(\frac{s+1}{s}\right)I_2 \qquad \ldots\text{(iii)}$$

Comparing Eqs (ii) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ \dfrac{s^2+s+1}{s} & \dfrac{s+1}{s} \end{bmatrix}$$

Example 9.17 *Find transmission parameters for the two-port network shown in Fig. 9.35.*

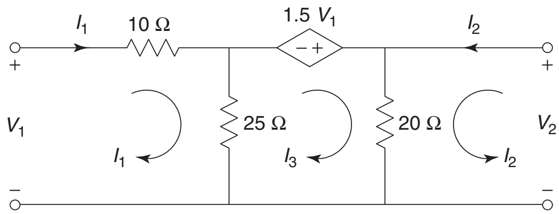


Fig. 9.35

Solution Applying KVL to Mesh 1,

$$V_1 = 10I_1 + 25(I_1 - I_3)$$

$$= 35I_1 - 25I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 20(I_2 + I_3)$$

$$= 20I_2 + 20I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-25(I_3 - I_1) + 1.5V_1 - 20(I_2 + I_3) = 0$$

$$-25I_3 + 25I_1 + 1.5(35I_1 - 25I_3) - 20I_2 - 20I_3 = 0$$

$$-25I_3 + 25I_1 + 52.5I_1 - 37.5I_3 - 20I_2 - 20I_3 = 0$$

$$82.5I_3 = 77.5I_1 - 20I_2$$

$$I_3 = 0.94I_1 - 0.24I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 35I_1 - 25(0.94I_1 - 0.24I_2)$$

$$= 11.5I_1 + 6I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 20I_2 + 20(0.94I_1 - 0.24I_2)$$

$$= 18.8I_1 + 15.2I_2 \qquad \ldots\text{(v)}$$

From Eq. (v),

$$I_1 = 0.053\,V_2 - 0.81 I_2 \qquad \text{...(vi)}$$

Substituting Eq. (vi) in Eq. (iv),

$$V_1 = 11.5(0.053\,V_2 - 0.81 I_2) + 6 I_2$$
$$= 0.61\,V_2 - 3.32 I_2 \qquad \text{...(vii)}$$

Comparing Eqs (vi) and (vii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.61 & -3.32 \\ 0.053 & -0.81 \end{bmatrix}$$

9.5 HYBRID PARAMETERS (*h* PARAMETERS)

The hybrid parameters of a two-port network may be defined by expressing the voltage of input port V_1 and current of output port I_2 in terms of current of input port I_1 and voltage of output port V_2.

$$(V_1, I_2) = f(I_1, V_2)$$

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$

$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The individual *h* parameters can be defined by setting $I_1 = 0$ and $V_2 = 0$.

Case 1 When the output port is short-circuited i.e., $V_2 = 0$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

where h_{11} is the short-circuit input impedance.

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

where h_{21} is the short-circuit forward current gain.

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

where h_{12} is the open-circuit reverse voltage gain.

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

where h_{22} is the open-circuit output admittance.

Since *h* parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called *hybrid parameters*.

The equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 9.36.

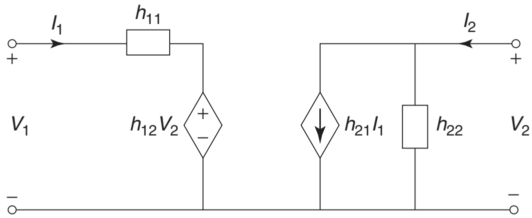


Fig. 9.36 *Equivalent circuit of the two-port network in terms of h-parameters*

9.5.1 Condition for Reciprocity

(a) As shown in Fig. 9.37, voltage V_s is applied at the input port and the output port is short-circuited.

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

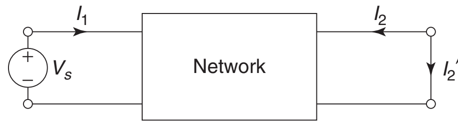


Fig. 9.37 *Network for deriving condition for reciprocity*

From the h-parameter equations,

$$V_s = h_{11}\, I_1$$
$$-I_2' = h_{21}\, I_1$$
$$\frac{V_s}{I_2'} = -\frac{h_{11}}{h_{21}}$$

(b) As shown in Fig. 9.38, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,

$$V_1 = 0$$
$$V_2 = V_s$$
$$I_1 = -I_1'$$

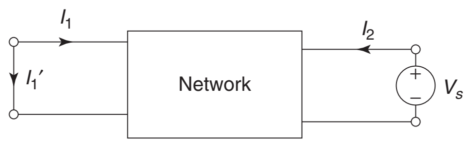


Fig. 9.38 *Network for deriving condition for reciprocity*

From the h-parameter equations,

$$0 = h_{11}I_1 + h_{12}\, V_s$$
$$h_{12}\, V_s = -h_{11}\, I_1 = h_{11}\, I_1'$$
$$\frac{V_s}{I_1'} = \frac{h_{11}}{h_{12}}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e.,

$$h_{21} = -h_{12}$$

9.5.2 Condition for Symmetry

The condition for symmetry is obtained from the Z-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \left.\frac{h_{11}I_1 + h_{12}V_2}{I_1}\right|_{I_2=0} = h_{11} + h_{12}\frac{V_2}{I_1}$$

But with $I_2 = 0$,

$$0 = h_{21}\, I_1 + h_{22}\, V_2$$

$$\frac{V_2}{I_1} = -\frac{h_{21}}{h_{22}}$$

$$Z_{11} = h_{11} - \frac{h_{12}h_{21}}{h_{22}} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

where $\Delta h = h_{11}h_{22} - h_{12}h_{21}$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

With $I_1 = 0$,

$$I_2 = h_{22}\, V_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{1}{h_{22}}$$

For a symmetrical network, $Z_{11} = Z_{22}$

i.e., $$\frac{\Delta h}{h_{22}} = \frac{1}{h_{22}}$$

i.e., $\Delta h = 1$

i.e., $h_{11}\, h_{22} - h_{12}\, h_{21} = 1$

Table 9.2 *Conditions for reciprocity and symmetry*

Parameter	Condition for Reciprocity	Condition for Symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{22}$
T	$AD - BC = 1$	$A = D$
h	$h_{12} = -h_{21}$	$h_{11}\, h_{22} - h_{12}\, h_{21} = 1$

Example 9.18 *In the two-port network shown in Fig. 9.39, compute h-parameters from the following data:*

(a) With the output port short-circuited: $V_1 = 25\ V$, $I_1 = 1\ A$, $I_2 = 2\ A$

(b) With the input port open-circuited: $V_1 = 10\ V$, $V_2 = 50\ V$, $I_2 = 2\ A$

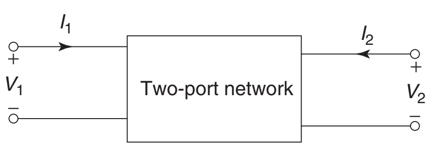


Fig. 9.39

Solution

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{25}{1} = 25\ \Omega, \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{10}{50} = 0.2$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{2}{1} = 2, \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2}{50} = 0.04\ \mho$$

Hence, the h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 25 & 0.2 \\ 2 & 0.04 \end{bmatrix}$$

Example 9.19 *Determine hybrid parameters for the network of Fig. 9.40. Determine whether the network is reciprocal.*

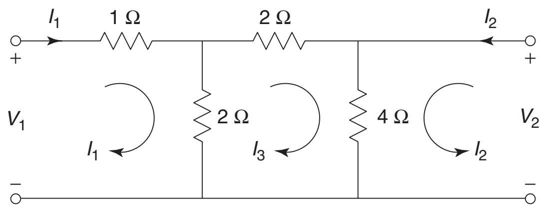


Fig. 9.40

Solution

First Method

Case 1 When Port 2 is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 9.41,

$$R_{eq} = 1 + \frac{2 \times 2}{2+2} = 2\ \Omega$$

Now,

$$V_1 = 2I_1$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 2\ \Omega$$

Also,

$$I_2 = -I_1 \times \frac{2}{2+2} = -\frac{I_1}{2}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\frac{1}{2}$$

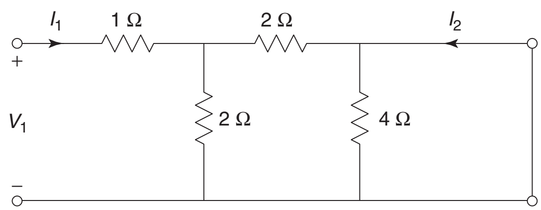


Fig. 9.41

Case 2 When Port 1 is open-circuited, i.e., $I_1 = 0$ as shown in Fig. 9.42,

$$R_{eq} = \frac{(2+2) \times 4}{2+2+4} = 2\ \Omega$$

$$V_1 = 2I_y$$

$$I_y = \frac{I_2}{2}$$

$$V_2 = 4I_x$$

$$I_x = \frac{I_2}{2}$$

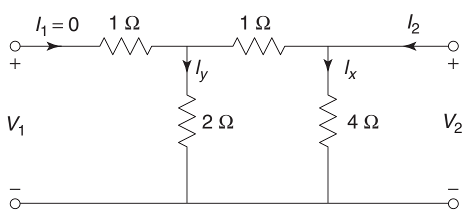


Fig. 9.42

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{2I_y}{4I_x} = \frac{2\times\frac{I_2}{2}}{4\times\frac{I_2}{2}} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2I_x}{4I_x} = \frac{1}{2}\,\mho$$

Hence, the h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Second Method (Refer Fig. 9.40)

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 4I_2 + 4I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2\,(I_3 - I_1) - 2I_3 - 4(I_3 + I_2) = 0$$

$$8I_3 = 2I_1 - 4I_2$$

$$I_3 = \frac{I_1}{4} - \frac{I_2}{2} \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= \frac{5}{2}I_1 + I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 4I_2 + 4\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= 4I_2 + I_1 - 2I_2$$

$$= I_1 + 2I_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{2}V_2 \qquad \text{...(v)}$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{5}{2}I_1 - \frac{1}{2}I_1 + \frac{1}{2}V_2$$

$$= 2I_1 + \frac{1}{2}V_2 \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with h-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since $h_{12} = -\,h_{21}$, the network is reciprocal.

Example 9.20 *Find h-parameters for the network shown in Fig. 9.43.*

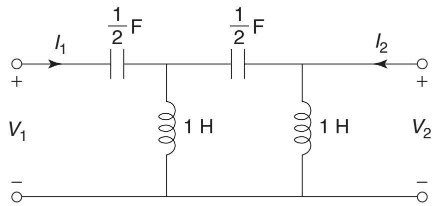


Fig. 9.43

Solution As solved in Example 9.12, derive the equations for I_1 and I_2 in terms of V_1 and V_2.

$$I_1 = \frac{s^3+2s}{4(s^2+1)}V_1 - \frac{s^3}{4(s^2+1)}V_2 \quad \ldots\text{(i)}$$

$$I_2 = -\frac{s^3}{4(s^2+1)}V_1 + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2 \quad \ldots\text{(ii)}$$

From Eq. (i),

$$V_1 = \frac{4(s^2+1)}{s(s^2+2)}I_1 + \frac{s^2}{s^2+2}V_2 \quad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq (ii),

$$I_2 = -\frac{s^3}{4(s^2+1)}\left[\frac{4(s^2+1)}{s(s^2+2)}I_1 + \frac{s^2}{s^2+2}V_2\right] + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2$$

$$= -\frac{s^2}{s^2+2}I_1 + \frac{2(s^2+1)}{s(s^2+2)}V_2 \quad \ldots\text{(iv)}$$

Comparing Eqs (iii) and (iv) with h-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{4(s^2+1)}{s(s^2+2)} & \dfrac{s^2}{s^2+2} \\ -\dfrac{s^2}{s^2+2} & \dfrac{2(s^2+1)}{s(s^2+2)} \end{bmatrix}$$

9.6 INTER-RELATIONSHIPS BETWEEN THE PARAMETERS

When it is required to find out two or more parameters of a particular network then finding each parameter will be tedious. But if we find a particular parameter then the other parameters can be found if the inter-relationship between them is known.

9.6.1 Z-parameters in Terms of Other Parameters

1. Z-parameters in Terms of Y-parameters We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix}}{\begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}} = \frac{Y_{22}I_1 - Y_{12}I_2}{Y_{11}Y_{22} - Y_{12}Y_{21}} = \frac{Y_{22}}{\Delta Y} I_1 - \frac{Y_{12}}{\Delta Y} I_2$$

where $\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21}$

Comparing with $V_1 = Z_{11}I_1 + Z_{12}, I_2,$

$$Z_{11} = \frac{Y_{22}}{\Delta Y}$$

$$Z_{12} = -\frac{Y_{12}}{\Delta Y}$$

Also,

$$V_2 = \frac{\begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix}}{\Delta Y} = \frac{Y_{11}}{\Delta Y} I_2 - \frac{Y_{21}}{\Delta Y} I_1$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{22} = \frac{Y_{11}}{\Delta Y}$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y}$$

2. Z-parameter in Terms of ABCD Parameters We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the second equation,

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{21} = \frac{1}{C}$$

$$Z_{22} = \frac{D}{C}$$

Also,

$$V_1 = A\left[\frac{1}{C} I_1 + \frac{D}{C} I_2\right] - BI_2 = \frac{A}{C} I_1 + \left[\frac{AD}{C} - B\right] I_2 = \frac{A}{C} I_1 + \left[\frac{AD - BC}{C}\right] I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2,$

$$Z_{11} = \frac{A}{C}$$

$$Z_{12} = \frac{AD - BC}{C}$$

3. Z-parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

Rewriting the second equation,

$$V_2 = -\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2$$

Comparing with

$$V_2 = Z_{21} I_1 + Z_{22} I_2,$$

$$Z_{21} = -\frac{h_{21}}{h_{22}}$$

$$Z_{22} = \frac{1}{h_{22}}$$

Also, $V_1 = h_{11} I_1 + h_{12}\left[-\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2\right] = h_{11} I_1 + \frac{h_{12}}{h_{22}} I_2 - \frac{h_{12}h_{21}}{h_{22}} I_1 = \left[\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right] I_1 + \frac{h_{12}}{h_{22}} I_2$

Comparing with $V_1 = Z_{11} I_1 + Z_{12} I_2,$

$$Z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

$$Z_{12} = \frac{h_{12}}{h_{22}}$$

9.6.2 *Y*-parameters in Terms of Other Parameters

1. Y-parameters in terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

By Cramer's rule,

$$I_1 = \frac{\begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}} = \frac{Z_{22} V_1 - Z_{12} V_2}{Z_{11} Z_{22} - Z_{12} Z_{21}} = \frac{Z_{22}}{\Delta Z} V_1 - \frac{Z_{12}}{\Delta Z} V_2$$

where $\Delta Z = Z_{11} Z_{22} - Z_{12} Z_{21}$

Comparing with $I_1 = Y_{11} V_1 + Y_{12} V_2,$

$$Y_{11} = \frac{Z_{22}}{\Delta Z}$$

$$Y_{12} = -\frac{Z_{12}}{\Delta Z}$$

Also,

$$I_2 = \frac{\begin{vmatrix} Z_{11} & V_1 \\ Z_{21} & V_2 \end{vmatrix}}{\Delta Z} = \frac{Z_{11}V_2 - Z_{12}V_1}{\Delta Z} = -\frac{Z_{21}}{\Delta Z} V_1 + \frac{Z_{11}}{\Delta Z} V_2$$

Comparing with $I_2 = Y_{21} V_1 + Y_{22} V_2,$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z}$$

$$Y_{22} = \frac{Z_{11}}{\Delta Z}$$

2. *Y-parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Rewriting the first equation,

$$I_2 = -\frac{1}{B}V_1 + \frac{A}{B}V_2$$

Comparing with $I_2 = Y_{21}V_1 + Y_{22}V_2$,

$$Y_{21} = -\frac{1}{B}$$
$$Y_{22} = \frac{A}{B}$$

Also, $$I_1 = CV_2 - D\left[-\frac{1}{B}V_1 + \frac{A}{B}V_2\right] = \frac{D}{B}V_1 + \left[\frac{BC - AD}{B}\right]V_2$$

Comparing with $I_1 = Y_{11}V_1 + Y_{12}V_2$,

$$Y_{11} = \frac{D}{B}$$
$$Y_{12} = \frac{BC - AD}{B} = -\frac{AD - BC}{B} = -\frac{\Delta T}{B}$$

3. *Y-parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

Rewriting the first equation,

$$I_1 = \frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2$$

Comparing with $I_1 = Y_{11}V_1 + Y_{12}V_2$,

$$Y_{11} = \frac{1}{h_{11}}$$
$$Y_{12} = -\frac{h_{12}}{h_{11}}$$

Also $$I_2 = h_{21}\left[\frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2\right] + h_{22}V_2 = \frac{h_{21}}{h_{11}}V_1 + \left[\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}}\right]V_2$$

Comparing with $I_2 = Y_{21}V_1 + Y_{22}V_2$,

$$Y_{21} = \frac{h_{21}}{h_{11}}$$
$$Y_{22} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}} = \frac{\Delta h}{h_{11}}$$

9.6.3 *ABCD* Parameters in Terms of Other Parameters

1. ABCD Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the second equation,

$$I_1 = \frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{1}{Z_{21}}$$

$$D = \frac{Z_{22}}{Z_{21}}$$

Also, $V_1 = Z_{11}\left[\frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2\right] + Z_{12} I_2 = \frac{Z_{11}}{Z_{21}} V_2 - \frac{Z_{22} Z_{11}}{Z_{21}} I_2 + Z_{12} I_2$

$$= \frac{Z_{11}}{Z_{21}} V_2 - \left[\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}}\right] I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{Z_{11}}{Z_{21}}$$

$$B = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}} = \frac{\Delta Z}{Z_{21}}$$

2. ABCD Parameters in terms of Y-parameters We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the second equation,

$$V_1 = -\frac{Y_{22}}{Y_{21}} V_2 + \frac{1}{Y_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = -\frac{Y_{22}}{Y_{21}}$$

$$B = -\frac{1}{Y_{21}}$$

Also, $I_1 = Y_{11}\left[-\frac{Y_{22}}{Y_{21}} V_2 + \frac{1}{Y_{21}} I_2\right] + Y_{12} V_2 = \left[\frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{21}}\right] V_2 + \frac{Y_{11}}{Y_{21}} I_2$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{21}} = -\frac{\Delta Y}{Y_{21}}$$

$$D = -\frac{Y_{11}}{Y_{21}}$$

3. ABCD Parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$
$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

Rewriting the second equation,

$$I_1 = -\frac{h_{22}}{h_{21}} V_2 + \frac{1}{h_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = -\frac{h_{22}}{h_{21}}$$

$$D = -\frac{1}{h_{21}}$$

Also,
$$V_1 = h_{11}\left[\frac{1}{h_{21}} I_2 - \frac{h_{22}}{h_{21}} V_2\right] + h_{12}\, V_2 = \left[\frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}}\right] V_2 + \frac{h_{11}}{h_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}} = -\frac{\Delta h}{h_{21}}$$

$$B = -\frac{h_{11}}{h_{21}}$$

9.6.4 Hybrid Parameters in Terms of Other Parameters

1. Hybrid Parameters in terms of Z-parameters We know that

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Rewriting the second equation,

$$I_2 = -\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2$$

Comparing with $I_2 = h_{21}\, I_1 + h_{22}\, V_2$,

$$h_{21} = -\frac{Z_{21}}{Z_{22}}$$

$$h_{22} = \frac{1}{Z_{22}}$$

Also,
$$V_1 = Z_{11}\, I_1 + Z_{12}\left[-\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2\right] = \left[\frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}}\right] I_1 + \frac{Z_{12}}{Z_{22}} V_2$$

Comparing with $V_1 = h_{11}\, I_1 + h_{12}\, V_2$,

$$h_{11} = \frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}} = \frac{\Delta Z}{Z_{22}}$$

$$h_{12} = \frac{Z_{12}}{Z_{22}}$$

2. *Hybrid Parameters in terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the first equation,

$$V_1 = \frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{1}{Y_{11}}$$

$$h_{12} = -\frac{Y_{12}}{Y_{11}}$$

Also, $$I_2 = Y_{21}\left[\frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2\right] + Y_{22} V_2 = \left[\frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}}\right] V_2 + \frac{Y_{21}}{Y_{11}} I_1$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2$,

$$h_{21} = \frac{Y_{22}}{Y_{11}}$$

$$h_{22} = \frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}} = \frac{\Delta Y}{Y_{11}}$$

3. *Hybrid Parameters in Terms of ABCD Parameters* We know that

$$V_1 = A V_2 - B I_2$$
$$I_1 = C V_2 - D I_2$$

Rewriting the second equation,

$$I_2 = -\frac{1}{D} I_1 + \frac{C}{D} V_2$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2$,

$$h_{21} = -\frac{1}{D}$$

$$h_{22} = \frac{C}{D}$$

Also, $$V_1 = A V_2 - B\left[-\frac{1}{D} I_1 + \frac{C}{D} V_2\right] = \frac{B}{D} I_1 + \left[\frac{AD - BC}{D}\right] V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{B}{D}$$

$$h_{12} = \frac{AD - BC}{D} = \frac{\Delta T}{D}$$

Table 9.3 *Inter-relationship between parameters*

$$\Delta X = X_{11}X_{22} - X_{12}X_{21}$$

	In terms of			
	[*Z*]	**[*Y*]**	**[*T*]**	**[*h*]**
[Z]	$\begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix}$	$\begin{matrix} \frac{Y_{22}}{\Delta Y} & -\frac{Y_{12}}{\Delta Y} \\ -\frac{Y_{21}}{\Delta Y} & \frac{Y_{11}}{\Delta Y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$
[Y]	$\begin{matrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{matrix}$	$\begin{matrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & -\frac{\Delta T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{matrix}$
[T]	$\begin{matrix} \frac{Z_{11}}{Z_{21}} & \frac{\Delta Z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{matrix}$	$\begin{matrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{\Delta Y}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} -\frac{\Delta h}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{matrix}$
[h]	$\begin{matrix} \frac{\Delta Z}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ -\frac{Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{Y_{11}} & -\frac{Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & \frac{\Delta Y}{Y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$

Example 9.21 *The Z parameters of a two-port network are $Z_{11} = 20\ \Omega$, $Z_{22} = 30\ \Omega$, $Z_{12} = Z_{21} = 10\ \Omega$. Find Y and ABCD parameters.*

Solution

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (20)(30) - (10)(10) = 500$$

***Y*-parameters**

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{30}{500} = \frac{3}{50}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{20}{500} = \frac{2}{50}\ \mho$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{50} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{2}{50} \end{bmatrix}$$

***ABCD* parameters**

$$A = \frac{Z_{11}}{Z_{21}} = \frac{20}{10} = 2, \qquad B = \frac{\Delta Z}{Z_{21}} = \frac{500}{10} = 50$$

$$C = \frac{1}{Z_{21}} = \frac{1}{10} = 0.1, \qquad D = \frac{Z_{22}}{Z_{21}} = \frac{30}{10} = 3$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2 & 50 \\ 0.1 & 3 \end{bmatrix}$$

Example 9.22 *Currents I_1 and I_2 entering at Port 1 and Port 2 respectively of a two-port network are given by the following equations:*

$$I_1 = 0.5V_1 - 0.2V_2$$
$$I_2 = -0.2V_1 + V_2$$

Find Y, Z and ABCD parameters for the network.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.5\ \mho, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -0.2\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -0.2\ \mho, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 1\ \mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 1 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (0.5)(1) - (-0.2)(-0.2) = 0.46$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{1}{0.46} = 2.174\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{0.5}{0.46} = 1.087\ \Omega$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2.174 & 0.434 \\ 0.434 & 1.087 \end{bmatrix}$$

***ABCD* parameters**

$$A = -\frac{Y_{22}}{Y_{21}} = -\frac{1}{-0.2} = 5, \qquad B = -\frac{1}{Y_{21}} = -\frac{1}{-0.2} = 5$$

$$C = -\frac{\Delta Y}{Y_{21}} = -\frac{0.46}{-0.2} = 2.3, \qquad D = -\frac{Y_{11}}{Y_{21}} = -\frac{0.5}{-0.2} = 2.5$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2.3 & 2.5 \end{bmatrix}$$

Example 9.23 *Using the relation $Y = Z^{-1}$, show that $|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$.*

Solution We know that

$$Y = Z^{-1}$$

i.e.,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{bmatrix}$$

$$|Z| = Z_{11} Z_{22} - Z_{12} Z_{21}$$

$$\frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right) = \frac{1}{2}\left(\frac{Z_{22}}{\frac{Z_{22}}{\Delta Z}} + \frac{Z_{11}}{\frac{Z_{11}}{\Delta Z}}\right) = \frac{1}{2}(\Delta Z + \Delta Z) = \frac{1}{2}(2\Delta Z) = \Delta Z = Z_{11} Z_{12} - Z_{12} Z_{21}$$

$$|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$$

Example 9.24 *For the network shown in Fig. 9.44, find Z and Y-parameters.*

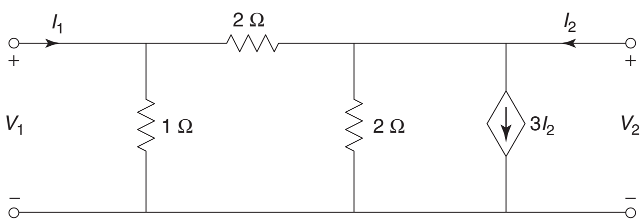


Fig. 9.44

Solution The network is redrawn by source transformation technique as shown in Fig. 9.45.

Applying KVL to Mesh 1,

$$V_1 = I_1 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$\begin{aligned} V_2 &= 2(I_2 + I_3) - 6I_2 \\ &= -4I_2 + 2I_3 \qquad \ldots\text{(ii)} \end{aligned}$$

Applying KVL to Mesh 3,

$$-(I_3 - I_1) - 2I_3 - 2(I_2 + I_3) + 6I_2 = 0$$

$$5I_3 = I_1 + 4I_2$$

$$I_3 = \frac{1}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(iii)}$$

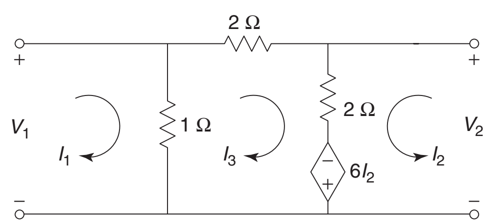


Fig. 9.45

Substituting Eq. (iii) in Eq. (i),

$$V_1 = I_1 - \frac{1}{5}I_1 - \frac{4}{5}I_2$$

$$= \frac{4}{5}I_1 - \frac{4}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = -4I_2 + 2\left(\frac{1}{5}I_1 + \frac{4}{5}I_2\right)$$

$$= \frac{2}{5}I_1 - \frac{12}{5}I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{12}{5} \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}\,Z_{22} - Z_{12}\,Z_{21} = \left(\frac{4}{5}\right)\left(-\frac{12}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{2}{5}\right) = -\frac{40}{25} = -\frac{8}{5}$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{-\frac{12}{5}}{-\frac{8}{5}} = \frac{3}{2}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = \frac{-\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = \frac{-\frac{2}{5}}{-\frac{8}{5}} = \frac{1}{4}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Example 9.25 *Find Z and h-parameters for the network shown in Fig. 9.46.*

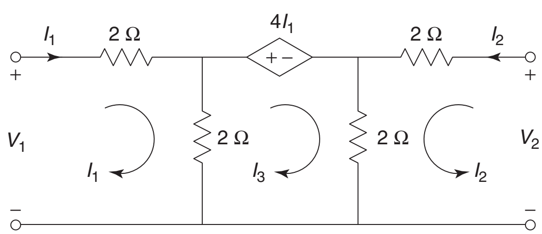


Fig. 9.46

Solution Applying KVL to Mesh 1,

$$V_1 = 2I_1 + 2(I_1 - I_3)$$

$$= 4I_1 - 2I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2(I_2 + I_3)$$

$$= 4I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 4I_1 - 2(I_3 + I_2) = 0$$

$$I_1 + I_2 = -2I_3 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$\begin{aligned} V_1 &= 4I_1 + I_1 + I_2 \\ &= 5I_1 + I_2 \end{aligned} \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$\begin{aligned} V_2 &= 4I_2 - I_1 - I_2 \\ &= -I_1 + 3I_2 \end{aligned} \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

***h*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (5)(3) - (1)(-1) = 15 + 1 = 16$$

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{16}{3}\ \Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{3}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = \frac{1}{3}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{3}\ \mho$$

Hence, the *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{22} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{16}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Example 9.26 *Find Y and Z-parameters for the network shown in Fig. 9.47.*

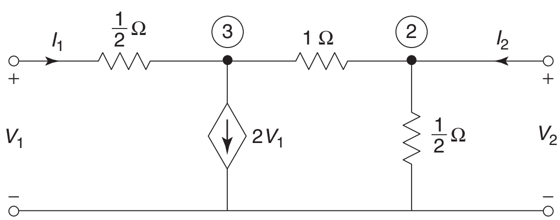


Fig. 9.47

Solution Applying KCL at Node 3,

$$2(V_1 - V_3) = 2V_1 + (V_3 - V_2)$$

$$V_3 = \frac{V_2}{3} \qquad \ldots\text{(i)}$$

Now,

$$\begin{aligned} I_1 &= 2V_1 + (V_3 - V_2) \\ &= 2V_1 + \frac{V_2}{3} - V_2 \\ &= 2V_1 - \frac{2}{3}V_2 \end{aligned} \qquad \ldots\text{(ii)}$$

$$I_2 = 2V_2 + (V_2 - V_3)$$
$$= 3V_2 - \frac{V_2}{3}$$
$$= \frac{8}{3}V_2 \qquad \text{...(iii)}$$

Comparing Eqs (ii) and (iii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (2)\left(\frac{8}{3}\right) - 0 = \frac{16}{3}$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{\frac{16}{3}} = \frac{1}{2}\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{\left(-\frac{2}{3}\right)}{\frac{16}{3}} = \frac{1}{8}\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{0}{\frac{16}{3}} = 0, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{2}{\frac{16}{3}} = \frac{3}{8}\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 9.27 *For the network shown in Fig. 9.48, find Y and Z-parameters.*

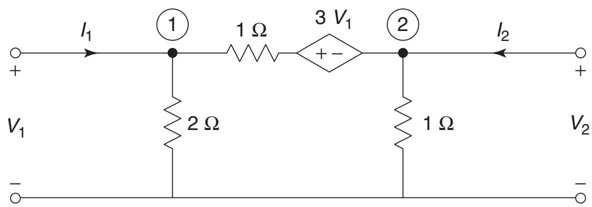


Fig. 9.48

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{2} + \frac{V_1 - 3V_1 - V_2}{1}$$
$$= -\frac{3}{2}V_1 - V_2 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{1} + \frac{V_2 + 3V_1 - V_1}{1} \qquad \text{...(ii)}$$
$$= 2V_1 + 2V_2$$

Comparing Eqs (i) and (ii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -1 \\ 2 & 2 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = \left(-\frac{3}{2}\right)(2) - (-1)(2) = -3 + 2 = -1$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{2}{-1} = -2\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-1)}{(-1)} = -1\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{2}{(-1)} = 2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{-\frac{3}{2}}{-1} = \frac{3}{2}\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & \frac{3}{2} \end{bmatrix}$$

Example 9.28 *Find Z-parameters for the network shown in Fig. 9.49. Hence, find Y and h-parameters.*

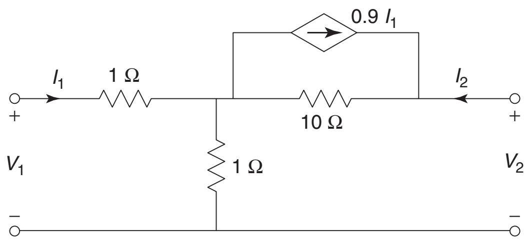


Fig. 9.49

Solution The network is redrawn by source transformation technique as shown in Fig. 9.50.

Applying KVL to Mesh 1,

$$V_1 = 2I_1 + I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 9I_1 + 10I_2 + 1(I_1 + I_2)$$
$$= 10I_1 + 11I_2 \qquad \ldots\text{(ii)}$$

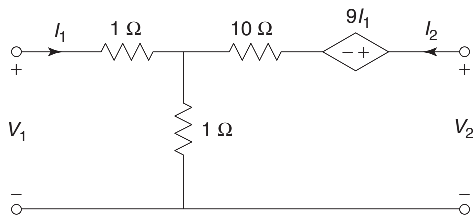


Fig. 9.50

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 10 & 11 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (2)(11) - (1)(10) = 22 - 10 = 12$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{11}{12}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{1}{12}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{12} = -\frac{5}{6}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{2}{12} = \frac{1}{6}\ \mho$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{12} & -\frac{1}{12} \\ -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

***h*-parameters**

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{12}{11}\,\Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{11}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = -\frac{10}{11}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{11}\,\mho$$

Hence, h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{12}{11} & \frac{1}{11} \\ -\frac{10}{11} & \frac{1}{11} \end{bmatrix}$$

Example 9.29 *Find Y and Z-parameters of the network shown in Fig. 9.51.*

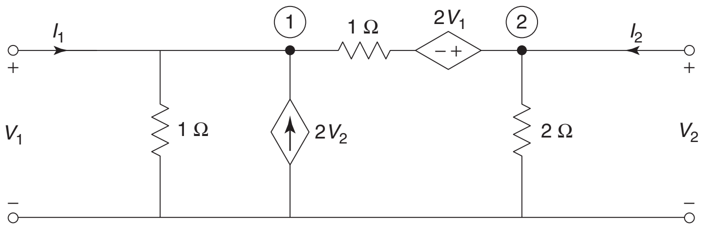


Fig. 9.51

Solution Applying KCL at Node 1,

$$I_1 + 2V_2 = \frac{V_1}{1} + \frac{3V_1 - V_2}{1}$$

$$I_1 = 4V_1 - 3V_2 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + \frac{V_2 - 2V_1 - V_1}{1}$$

$$= -3V_1 + 1.5V_2 \qquad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 1.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (4)(1.5) - (-3)(-3) = -3$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{1.5}{3} = -0.5\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = -\frac{4}{3} = -\frac{4}{3}\,\Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ -1 & -\frac{4}{3} \end{bmatrix}$$

Example 9.30 *Determine Y and Z-parameters for the network shown in Fig. 9.52.*

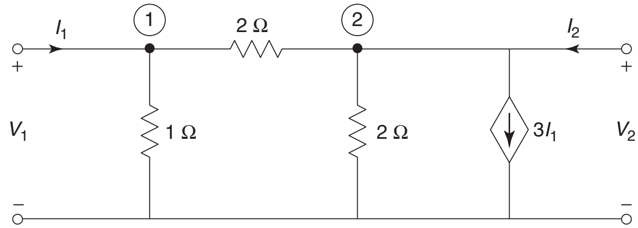


Fig. 9.52

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$

$$= 1.5V_1 - 0.5V_2 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + 3I_1 + \frac{V_2 - V_1}{2}$$

$$= \frac{V_2}{2} + 3(1.5V_1 - 0.5V_2) + \frac{V_2 - V_1}{2}$$

$$= 0.5V_2 + 4.5V_1 - 1.5V_2 + 0.5V_2 - 0.5V_1$$

$$= 4V_1 - 0.5V_2 \qquad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with the Y-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (1.5)(-0.5) - (-0.5)(4) = 1.25$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{0.5}{1.25} = -0.4\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{0.5}{1.25} = 0.4\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{4}{1.25} = -3.2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{1.5}{1.25} = 1.2\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix}$$

Example 9.31 *Determine the Y and Z-parameters for the network shown in Fig. 9.53.*

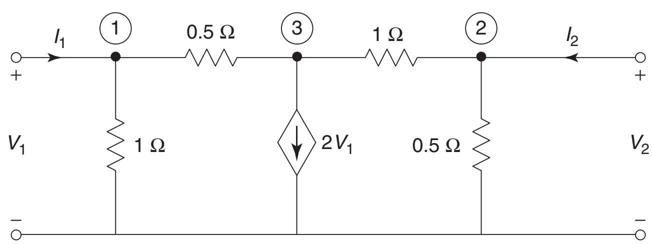


Fig. 9.53

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_3}{0.5}$$
$$= 3V_1 - 2V_3 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{0.5} + \frac{V_2 - V_3}{1}$$
$$= 3V_2 - V_3 \qquad \text{...(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{0.5} + 2V_1 + \frac{V_3 - V_2}{1} = 0$$
$$V_3 = \frac{1}{3}V_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eqs (i) and (ii),

$$I_1 = 3V_1 - \frac{2}{3}V_2 \qquad \text{...(iv)}$$
$$I_2 = 0V_1 + \frac{8}{3}V_2 \qquad \text{...(v)}$$

Comparing Eqs (iv) and (v) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (3)\left(\frac{8}{3}\right) - 0 = 8$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{8} = \frac{1}{3}\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega$$

$$Z_{21} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{Y\Delta} = \frac{3}{8}\,\Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 9.32 *Determine Z and Y-parameters of the network shown in Fig. 9.54.*

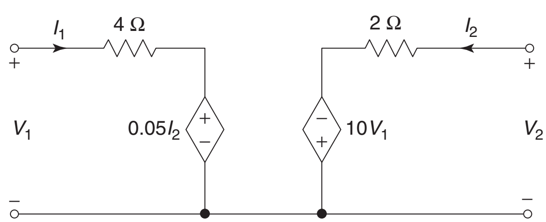


Fig. 9.54

Solution Applying KVL to Mesh 1,

$$V_1 - 4I_1 - 0.05\,I_2 = 0$$

$$V_1 = 4I_1 + 0.05I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 2I_2 + 10V_1 = 0$$

$$V_2 = 2I_2 - 10V_1 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_2 = 2I_2 - 40I_1 - 0.5I_2$$

$$= -40I_1 + 1.5I_2 \qquad \text{...(iii)}$$

Comparing Eqs (i) and (iii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0.05 \\ -40 & 1.5 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (4)(1.5) - (0.05)(-40) = 8$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{1.5}{8}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{0.05}{8}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-40)}{8} = \frac{40}{8}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{4}{8}\,\mho$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1.5}{8} & -\frac{0.05}{8} \\ \frac{40}{8} & \frac{4}{8} \end{bmatrix}$$

Example 9.33 *Determine Z and Y-parameters of the network shown in Fig. 9.55.*

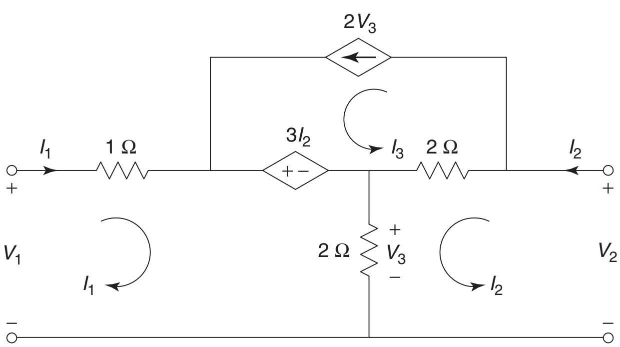


Fig. 9.55

Solution Applying KVL to Mesh 1,

$$V_1 - 1I_1 - 3I_2 - 2(I_1 + I_2) = 0$$

$$V_1 = 3I_1 + 5I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 2(I_2 - I_3) - 2(I_1 + I_2) = 0$$
$$V_2 - 2I_2 + 2I_3 - 2I_1 - 2I_2 = 0$$
$$V_2 = 2I_1 + 4I_2 - 2I_3 \qquad \text{...(ii)}$$

Writing equation for Mesh 3,

$$I_3 = 2V_3 \qquad \text{...(iii)}$$

From Fig. 9.55,

$$V_3 = 2(I_1 + I_2)$$
$$I_3 = 2V_3 = 4I_1 + 4I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$V_2 = -6I_1 - 4I_2 \qquad \text{...(v)}$$

Comparing Eqs (i) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -6 & -4 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (3)(-4) - (5)(-6) = 18$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = -\frac{4}{18} = -\frac{2}{9}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{5}{18}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-6)}{18} = \frac{1}{3}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{3}{18}\,\mho$$

Hence, Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & -\frac{5}{18} \\ \frac{1}{3} & \frac{3}{18} \end{bmatrix}$$

9.7 INTERCONNECTION OF TWO-PORT NETWORKS

We shall now discuss the various types of interconnections of two-port networks, namely, cascade, parallel, series, series-parallel and parallel-series. We shall derive the relation between the input and output quantities of the combined two-port networks.

9.7.1 Cascade Connection

Transmission Parameter Representation Figure 9.56 shows two-port networks connected in cascade. In the cascade connection, the output port of the first network becomes the input port of the second network. Since it is assumed that input and output currents are positive when they enter the network, we have

$$I_1' = -I_2$$

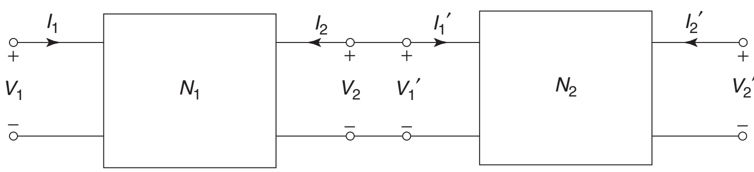


Fig. 9.56 *Cascade Connection*

Let A_1, B_1, C_1, D_1 be the transmission parameters of the network N_1 and A_2, B_2, C_2, D_2 be the transmission parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad ...(9.1)$$

For the network N_2,

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad ...(9.2)$$

Since $V_1' = V_2$ and $I_2' = -I_2$, we can write

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad ...(9.3)$$

Combining Eqs (9.1) and (9.3),

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad ...(9.4)$$

Equation 9.4 shows that the resultant *ABCD* matrix of the cascade connection is the product of the individual *ABCD* matrices.

Example 9.34 *Two identical sections of the network shown in Fig. 9.57 are connected in cascade. Obtain the transmission parameters of the overall connection.*

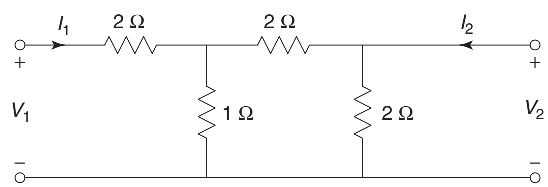


Fig. 9.57

Solution The network is redrawn as shown in Fig. 9.58.

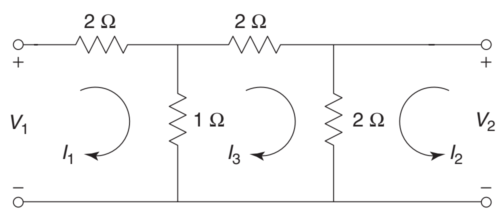


Fig. 9.58

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{1}{5}I_1 - \frac{2}{5}I_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{14}{5}I_1 + \frac{2}{5}I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{2}{5}I_1 + \frac{6}{5}I_2$$

$$I_1 = \frac{5}{2}V_2 - 3I_2 \qquad \text{...(v)}$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{14}{5}\left(\frac{5}{2}V_2 - 3I_2\right) + \frac{2}{5}I_2$$

$$= 7V_2 - 8I_2 \qquad \text{...(vi)}$$

Comparing the Eqs (vi) and (v) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}$$

Hence, transmission parameters of the overall cascaded network are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix} = \begin{bmatrix} 69 & 80 \\ 25 & 29 \end{bmatrix}$$

Example 9.35 *Determine ABCD parameters for the ladder network shown in Fig. 9.59.*

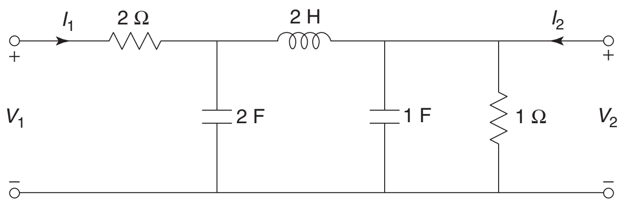


Fig. 9.59

Solution The above network can be considered as a cascade connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 9.60.

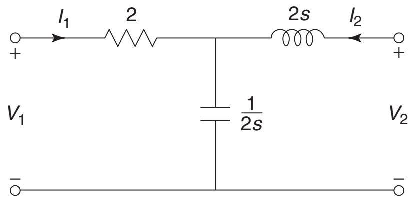


Fig. 9.60

Applying KVL to Mesh 1,

$$V_1 = \left(2 + \frac{1}{2s}\right) I_1 + \frac{1}{2s} I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{2s} I_1 + \left(2s + \frac{1}{2s}\right) I_2 \qquad \text{...(ii)}$$

From Eq. (ii),

$$I_1 = 2s\, V_2 - (4s^2 + 1)\, I_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = \left(2 + \frac{1}{2s}\right)[2s\, V_2 - (4s^2 + 1) I_2] + \frac{1}{2s} I_2$$

$$= (4s + 1)V_2 - (8s^2 + 2s + 2) I_2 \qquad \text{...(iv)}$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 4s+1 & 8s^2 + 2s + 2 \\ 2s & 4s^2 + 1 \end{bmatrix}$$

The network N_2 is shown in Fig. 9.61.

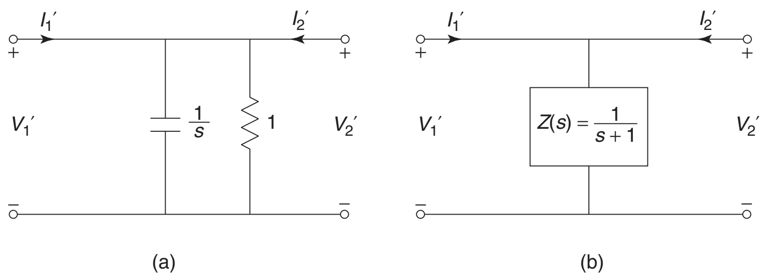


Fig. 9.61

Applying KVL to Mesh 1,

$$V_1' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \text{...(ii)}$$

From Eq. (ii),

$$I_1' = (s+1)V_2' - I_2' \quad \text{...(iii)}$$

Also, $$V_1' = V_2' \quad \text{...(iv)}$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}$$

Hence, overall *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4s+1 & 8s^2+2s+2 \\ 2s & 4s^2+1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} 8s^3+10s^2+8s+3 & 8s^2+2s+2 \\ 4s^3+4s^2+3s+1 & 4s^2+1 \end{bmatrix}$$

9.7.2 Parallel Connection

Figure 9.62 shows two-port networks connected in parallel. In the parallel connection, the two networks have the same input voltages and the same output voltages.

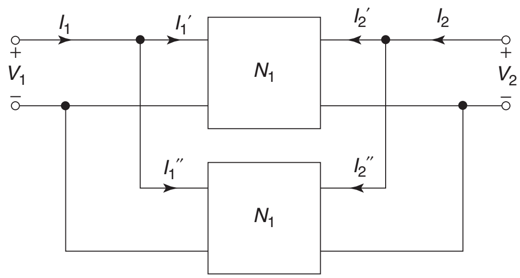


Fig. 9.62 *Parallel connection*

Let $Y_{11}', Y_{12}', Y_{21}', Y_{22}'$ be the *Y*-parameters of the network N_1 and $Y_1'', Y_{12}'', Y_{21}'', Y_{22}''$ be the *Y*-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the combined network, $I_1 = I_1' + I_1''$ and $I_2 = I_2' + I_2''$.

Hence, $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1' + I_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}'+Y_{11}'' & Y_{12}'+Y_{12}'' \\ Y_{21}'+Y_{21}'' & Y_{22}'+Y_{22}'' \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant *Y*-parameter matrix for parallel connected networks is the sum of *Y* matrices of each individual two-port networks.

Example 9.36 *Determine Y-parameters for the network shown in Fig. 9.63.*

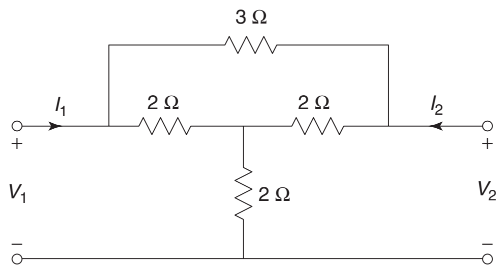


Fig. 9.63

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 9.64.

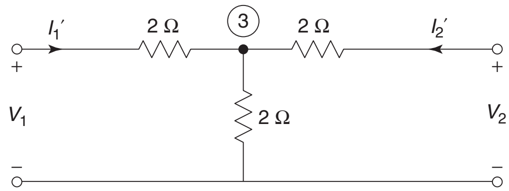


Fig. 9.64

Applying KCL at Node 3,

$$I_1' + I_2' = \frac{V_3}{2} \quad \text{...(i)}$$

From Fig. 9.64,

$$I_1' = \frac{V_1 - V_3}{2} \quad \text{...(ii)}$$

$$I_2' = \frac{V_2 - V_3}{2} \quad \text{...(iii)}$$

Substituting Eqs (ii) and (iii) in Eq (i),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} = \frac{V_3}{2}$$

$$3V_3 = V_1 + V_2$$

$$V_3 = \frac{V_1}{3} + \frac{V_2}{3} \quad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= \frac{1}{3}V_1 - \frac{1}{6}V_2 \quad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= -\frac{1}{6}V_1 + \frac{1}{3}V_2 \quad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

The network N_2 is shown in Fig. 9.65.

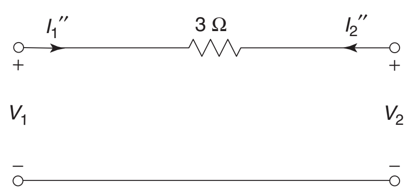


Fig. 9.65

$$I_1'' = -I_2'' = \frac{V_1 - V_2}{3} = \frac{1}{3}V_1 - \frac{1}{3}V_2$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{1}{3} & -\frac{1}{6} - \frac{1}{3} \\ -\frac{1}{6} - \frac{1}{3} & \frac{1}{3} + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

Example 9.37 *Find Y-parameters for the network shown in Fig. 9.66.*

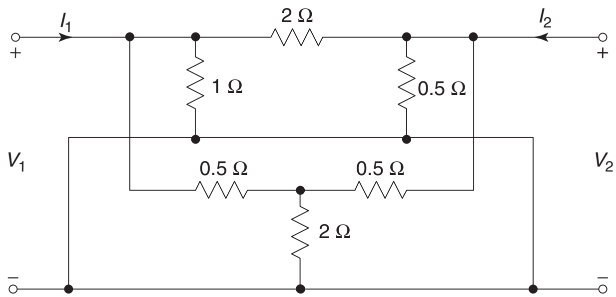


Fig. 9.66

Solution The above network can be considered as a parallel combination of two networks N_1 and N_2. The network N_1 is shown in Fig. 9.67.

Applying KCL at Node 1,

$$I_1' = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$

$$= \frac{3}{2}V_1 - \frac{1}{2}V_2 \qquad \ldots\text{(i)}$$

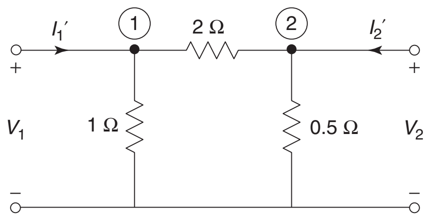


Fig. 9.67

Applying KCL at Node 2,

$$I_2' = \frac{V_2}{0.5} + \frac{V_2 - V_1}{2}$$

$$= -\frac{1}{2}V_1 + \frac{5}{2}V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Y-parameter equation,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

The network N_2 is shown in Fig. 9.68.

Applying KCL at Node 3,

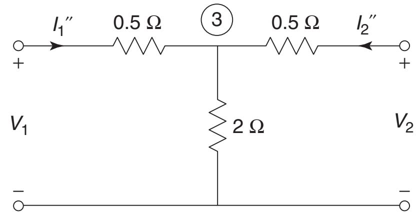


Fig. 9.68

$$I_1'' + I_2'' = \frac{V_3}{2} \qquad \ldots\text{(i)}$$

where
$$I_1'' = \frac{V_1 - V_3}{0.5} = 2V_1 - 2V_3$$

$$I_2'' = \frac{V_2 - V_3}{0.5} = 2V_2 - 2V_3$$

Substituting I_1'' and I_2'' in Eq (i),

$$2V_1 - 2V_3 + 2V_2 - 2V_3 = 0.5V_3$$

$$4.5V_3 = 2V_1 + 2V_2$$

$$V_3 = \frac{4}{9}V_1 + \frac{4}{9}V_2 \qquad \ldots\text{(ii)}$$

$$I_1'' = 2V_1 - 2V_3 = 2V_1 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right) = \frac{10}{9}V_1 - \frac{8}{9}V_2 \qquad \ldots\text{(iii)}$$

and
$$I_2'' = 2V_2 - 2V_3 = 2V_2 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right)$$

$$= -\frac{8}{9}V_1 + \frac{10}{9}V_2 \qquad \ldots\text{(iv)}$$

Comparing Eqs (iii) and (iv) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & -\frac{8}{9} \\ -\frac{8}{9} & \frac{10}{9} \end{bmatrix}$$

Hence, overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{10}{9} & -\frac{1}{2} - \frac{8}{9} \\ -\frac{1}{2} - \frac{8}{9} & \frac{5}{2} + \frac{10}{9} \end{bmatrix} = \begin{bmatrix} \frac{47}{18} & -\frac{25}{18} \\ -\frac{25}{18} & \frac{65}{18} \end{bmatrix}$$

Example 9.38 *Find Y-parameters for the network shown in Fig. 9.69.*

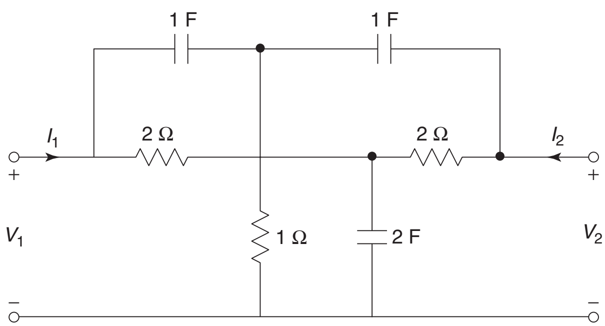


Fig. 9.69

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 9.70.

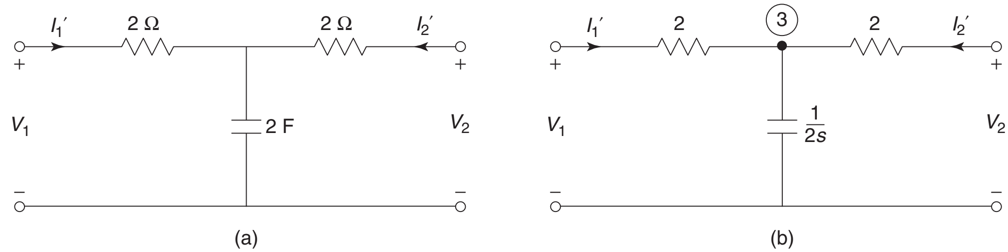


Fig. 9.70

Applying KCL at Node 3,

$$I_1' + I_2' = 2s\,V_3 \qquad \ldots(\text{i})$$

From Fig. 9.70,

$$I_1' = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2}V_1 - \frac{1}{2}V_3 \qquad \ldots(\text{ii})$$

$$I_2' = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \qquad \ldots(\text{iii})$$

Substituting Eq. (ii) and Eq. (iii) in Eq. (i),

$$\frac{V_1}{2} - \frac{V_3}{2} + \frac{V_2}{2} - \frac{V_3}{2} = (2s)\,V_3$$

$$(2s+1)V_3 = \frac{V_1}{2} + \frac{V_2}{2}$$

$$V_3 = \frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2 \qquad \ldots(\text{iv})$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2\right]$$

$$= \left(\frac{4s+1}{8s+4}\right)V_1 - \left(\frac{1}{8s+4}\right)V_2 \qquad \ldots(\text{v})$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 - \frac{1}{2(2s+1)}V_2\right]$$

$$= -\left(\frac{1}{8s+4}\right)V_1 + \left(\frac{4s+1}{8s+4}\right)V_2 \qquad \ldots(\text{vi})$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \dfrac{4s+1}{8s+4} & -\dfrac{1}{8s+4} \\ -\dfrac{1}{8s+4} & \dfrac{4s+1}{8s+4} \end{bmatrix}$$

The network N_2 is shown in Fig. 9.71.

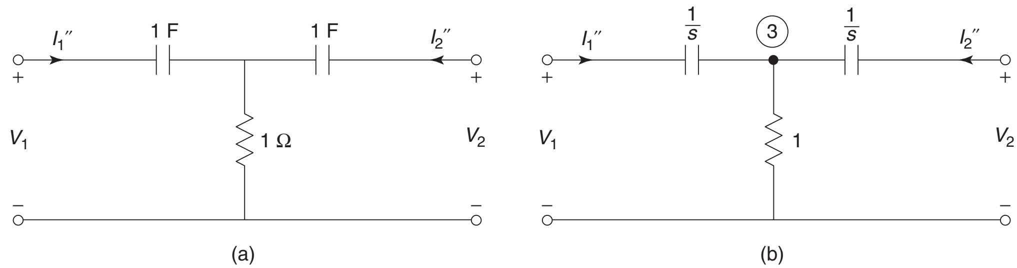


Fig. 9.71

Applying KCL at Node 3,

$$I_1'' + I_2'' = V_3 \qquad \ldots\text{(i)}$$

From Fig. 9.71,

$$I_1'' = \frac{V_1 - V_3}{\frac{1}{s}}$$

$$= sV_1 - sV_3 \qquad \ldots\text{(ii)}$$

$$I_2' = \frac{V_2 - V_3}{\frac{1}{s}}$$

$$= sV_2 - sV_3 \qquad \ldots\text{(iii)}$$

Substituting Eqs (ii) and (iii) in Eq. (i),

$$sV_1 - sV_3 + sV_2 - sV_3 = V_3$$

$$(2s+1)V_3 = sV_1 + sV_2$$

$$V_3 = \left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1'' = sV_1 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \frac{s}{(2s+1)}V_2\right]$$

$$= \left[\frac{s(s+1)}{2s+1}\right]V_1 - \left(\frac{s^2}{2s+1}\right)V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2'' = sV_2 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2\right]$$

$$= -\left(\frac{s^2}{2s+1}\right)V_1 + \left[\frac{s(s+1)}{2s+1}\right]V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{s(s+1)}{2s+1} & -\left(\frac{s^2}{2s+1}\right) \\ -\left(\frac{s^2}{2s+1}\right) & \frac{s(s+1)}{2s+1} \end{bmatrix}$$

Hence, the overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21}' & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}'+Y_{11}'' & Y_{12}'+Y_{12}'' \\ Y_{21}'+Y_{21}'' & Y_{22}'+Y_{22}'' \end{bmatrix} = \begin{bmatrix} \dfrac{4s^2+8s+1}{4(2s+1)} & -\dfrac{(4s^2+1)}{4(2s+1)} \\ -\dfrac{(4s^2+1)}{4(2s+1)} & \dfrac{4s^2+8s+1}{4(2s+1)} \end{bmatrix}$$

9.7.3 Series Connection

Figure 9.72 shows two-port networks connected in series. In a series connection, both the networks carry the same input current. Their output currents are also equal.

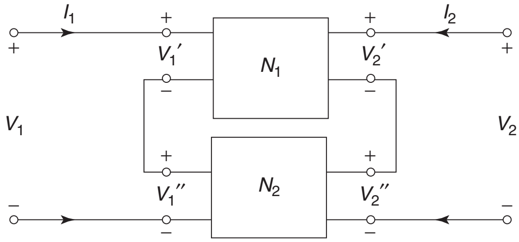


Fig. 9.72 *Series connection*

Let $Z_{11}', Z_{12}', Z_{21}', Z_{22}'$ be the Z-parameters of the network N_1 and $Z_{11}'', Z_{12}'', Z_{21}'', Z_{22}''$ be the Z-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the combined network $\quad V_1 = V_1' + V_1''$ and $V_2 = V_2' + V_2''$.

Hence,
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1'+V_1'' \\ V_2'+V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}'+Z_{11}'' & Z_{12}'+Z_{12}'' \\ Z_{21}'+Z_{21}'' & Z_{22}'+Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus, the resultant Z-parameter matrix for the series-connected networks is the sum of Z matrices of each individual two-port network.

Example 9.39 *Two identical sections of the network shown in Fig. 9.73 are connected in series. Obtain Z-parameters of the overall connection.*

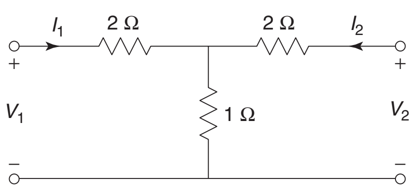


Fig. 9.73

Solution

Applying KVL to Mesh 1,

$$V_1 = 3I_1 + I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = I_1 + 3I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Hence, Z-parameters of the overall connection are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$$

Example 9.40 *Determine Z-parameters for the network shown in Fig. 9.74.*

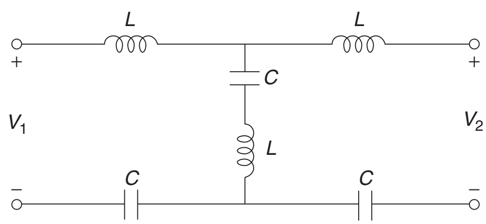


Fig. 9.74

Solution The above network can be considered as a series connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 9.75.

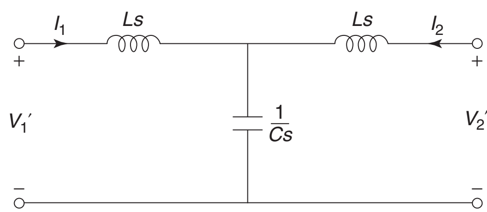


Fig. 9.75

Applying KVL to Mesh 1,

$$V_1' = \left(Ls + \frac{1}{Cs}\right)I_1 + \left(\frac{1}{Cs}\right)I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2' = \left(\frac{1}{Cs}\right)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}' & Z_{12}' \\ Z_{21}' & Z_{22}' \end{bmatrix} = \begin{bmatrix} Ls + \dfrac{1}{Cs} & \dfrac{1}{Cs} \\ \dfrac{1}{Cs} & Ls + \dfrac{1}{Cs} \end{bmatrix}$$

The network N_2 is shown in Fig. 9.76.

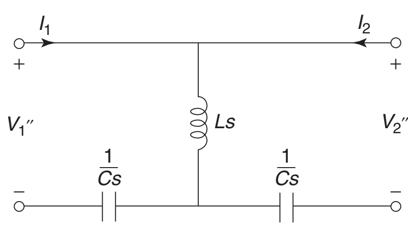


Fig. 9.76

Applying KVL to Mesh 1,

$$V_1'' = \left(Ls + \frac{1}{Cs}\right)I_1 + (Ls)I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2'' = (Ls)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} Ls + \dfrac{1}{Cs} & Ls \\ Ls & Ls + \dfrac{1}{Cs} \end{bmatrix}$$

Hence, the overall Z-parameters of the network are,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix} = \begin{bmatrix} 2Ls + \dfrac{2}{Cs} & Ls + \dfrac{1}{Cs} \\ Ls + \dfrac{1}{Cs} & 2Ls + \dfrac{2}{Cs} \end{bmatrix} = \left(Ls + \frac{1}{Cs} \right) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

9.7.4 Series-Parallel Connection

Figure 9.77 shows two networks connected in series-parallel. Here, the input ports of two networks are connected in series and the output ports are connected in parallel.

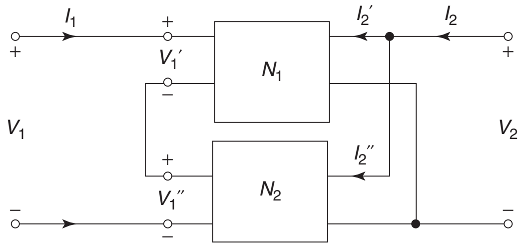


Fig. 9.77 *Series-parallel connection*

Let $h_{11}', h_{12}', h_{21}', h_{22}'$ be the h-parameters of the network N_1 and $h_{11}'', h_{12}'', h_{21}'', h_{22}''$ be the h-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the combined network, $V_1 = V_1' + V_1''$ and $I_2 = I_2' + I_2''$

Hence,
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1' + V_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}' + h_{11}'' & h_{12}' + h_{12}'' \\ h_{21}' + h_{21}'' & h_{22}' + h_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant h-parameter matrix is the sum of h-parameter matrices of each individual two-port networks.

Example 9.41 *Determine h-parameters for the network shown in Fig. 9.78.*

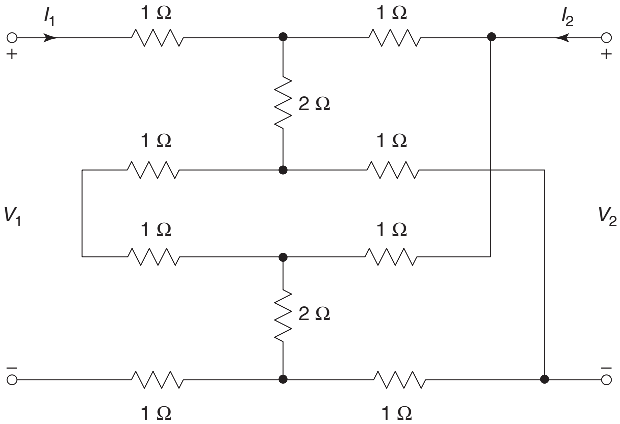


Fig. 9.78

Solution The above network can be considered as a series-parallel connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 9.79.

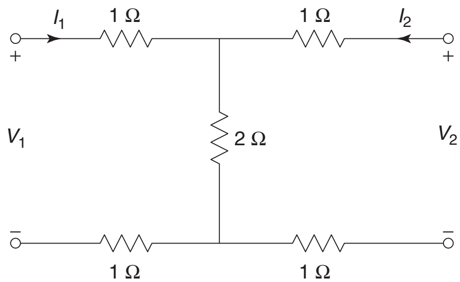


Fig. 9.79

Applying KVL to Mesh 1,

$$V_1 = 4I_1 + 2I_2 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 4I_2 \quad \text{...(ii)}$$

Rewriting Eq. (ii)

$$4I_2 = -2I_1 + V_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{4}V_2 \quad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 4I_1 + 2\left(-\frac{1}{2}I_1 + \frac{1}{4}V_2\right)$$

$$= 3I_1 + \frac{1}{2}V_2 \quad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with h-parameters equations,

$$\begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

For network N_2, h-parameters will be same as the two networks are identical.

$$\begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Hence, the overall h-parameters of the network are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} + \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

9.8 *T*-NETWORK

Any two-port network can be represented by an equivalent *T* network as shown in Fig. 9.80. The elements of the equivalent *T* network may be expressed in terms of *Z*-parameters.

Applying KVL to Mesh 1,

$$V_1 = Z_A I_1 + Z_C(I_1 + I_2)$$
$$= (Z_A + Z_C)I_1 + Z_C I_2 \quad \text{...(9.9)}$$

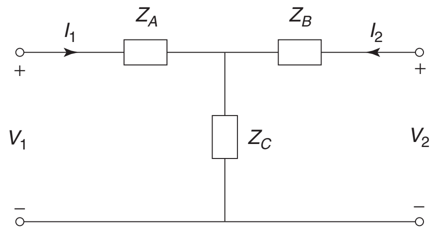


Fig. 9.80 *T-Network*

Applying KVL to Mesh 2,

$$V_2 = Z_B I_2 + Z_C(I_2 + I_1)$$
$$= Z_C I_1 + (Z_B + Z_C)I_2 \quad \text{...(9.10)}$$

Comparing Eqs (9.9) and (9.10) with *Z*-parameter equations,

$$Z_{11} = Z_A + Z_C$$
$$Z_{12} = Z_C$$
$$Z_{21} = Z_C$$
$$Z_{22} = Z_B + Z_C$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12} = Z_{11} - Z_{21}$$
$$Z_B = Z_{22} - ZZ_{21} = Z_{22} - Z_{12}$$
$$Z_C = Z_{12} = Z_{21}$$

9.9 *PI* (π)-NETWORK

Any two-port network can be represented by an equivalent pi (π) network as shown in Fig. 9.81.

Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B(V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \quad \text{...(9.11)}$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B(V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \quad \text{...(9.12)}$$

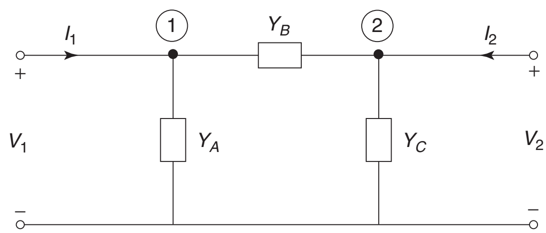


Fig. 9.81 *π-network*

Comparing Eqs (9.11) and (9.12) with *Y*-parameter equations,

$$Y_{11} = Y_A + Y_B$$
$$Y_{12} = -Y_B$$
$$Y_{21} = -Y_B$$
$$Y_{22} = Y_B + Y_C$$

Solving the above equations,

$$Y_A = Y_{11} + Y_{12} = Y_{11} + Y_{21}$$
$$Y_B = -Y_{12} = -Y_{21}$$
$$Y_C = Y_{22} + Y_{12} = Y_{22} + Y_{21}$$

Example 9.42 *The Z-parameters of a two-port network are: $Z_{11} = 10\ \Omega$, $Z_{12} = Z_{21} = 5\ \Omega$, $Z_{22} = 20\ \Omega$. Find the equivalent T-network.*

Solution The *T*-network is shown in Fig. 9.82.

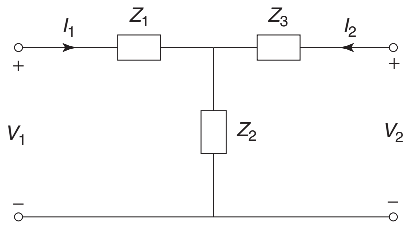


Fig. 9.82

Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)I_1 + Z_2 I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = Z_2 I_1 + (Z_2 + Z_3)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z parameter equations,

$$Z_{11} = Z_1 + Z_2 = 10$$
$$Z_{12} = Z_2 = 5$$
$$Z_{21} = Z_2 = 5$$
$$Z_{22} = Z_2 + Z_3 = 20$$

Solving the above equations,

$$Z_1 = 5\ \Omega$$
$$Z_2 = 5\ \Omega$$
$$Z_3 = 15\ \Omega$$

Example 9.43 *Admittance parameters of a pi network are $Y_{11} = 0.09\ \mho$, $Y_{12} = Y_{21} = -0.05\ \mho$ and $Y_{22} = 0.07\ \mho$. Find the values of R_a, R_b and R_c.*

Solution The pi network is shown in Fig. 9.83.

Applying KCL at Node, 1,

$$I_1 = \frac{V_1}{R_a} + \frac{V_1 - V_2}{R_b}$$
$$= \left(\frac{1}{R_a} + \frac{1}{R_b}\right)V_1 - \frac{1}{R_b}V_2 \qquad \ldots\text{(i)}$$

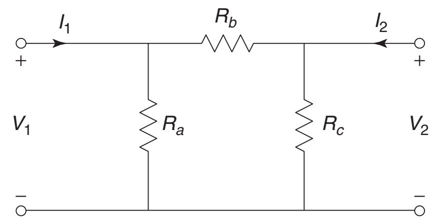


Fig. 9.83

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{R_c} + \frac{V_2 - V_1}{R_b}$$
$$= -\frac{1}{R_b}V_1 + \left(\frac{1}{R_B} + \frac{1}{R_c}\right)V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$Y_{11} = \frac{1}{R_a} + \frac{1}{R_b} = 0.09$$
$$Y_{12} = -\frac{1}{R_b} = -0.05$$
$$Y_{21} = -\frac{1}{R_b} = -0.05$$
$$Y_{22} = \frac{1}{R_b} + \frac{1}{R_c} = 0.07$$

Solving the above equations,

$$R_a = 25\ \Omega$$
$$R_b = 20\ \Omega$$
$$R_c = 50\ \Omega$$

Example 9.44 *Find the parameters YA, YB and YC of the equivalent p network as shown in Fig. 9.84 to represent a two-terminal pair network for which the following measurements were taken:*

(a) With terminal 2 short-circuited, a voltage of $10 \angle 0°$ V applied at terminal pair I resulted in $I_1 = 2.5 \angle 0°$ A and $I_2 = -0.5 \angle 0°$ A.

(b) With terminal 1 short-circuited, the same voltage at terminal pair 2 resulted in $I_2 = 1.5 \angle 0°$ A and $I_1 = -1.1 \angle -20°$ A.

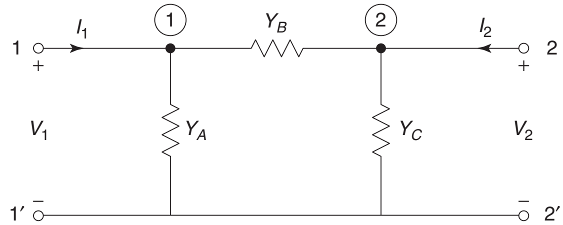


Fig. 9.84

Solution Since measurements were taken with either of the terminal pairs short-circuited, we have to calculate Y-parameters first.

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.5\angle 0°}{10\angle 0°} = 0.25\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-0.5\angle 0°}{10\angle 0°} = -0.05\ \mho$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1.5\angle 0°}{10\angle 0°} = 0.15\ \mho$$

Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B(V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B(V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \qquad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with the Y-parameter equation,

$$Y_{11} = Y_A + Y_B = 0.25$$
$$Y_{12} = Y_{21} = -Y_B = -0.05$$
$$Y_{22} = Y_B + Y_C = 0.15$$

Solving the above equation,

$$Y_A = 0.20\ \mho$$
$$Y_B = 0.05\ \mho$$
$$Y_C = 0.10\ \mho$$

Example 9.45 *A network has two input terminals (a, b) and two output terminals (c, d) as shown in Fig. 9.85. The input impedance with c and d open-circuited is (250 + j100) ohms and with c and d short-circuited is (400 + j 300) ohms. The impedance across c and d with a and b open-circuited is 200 ohms. Determine the equivalent T-network parameters.*

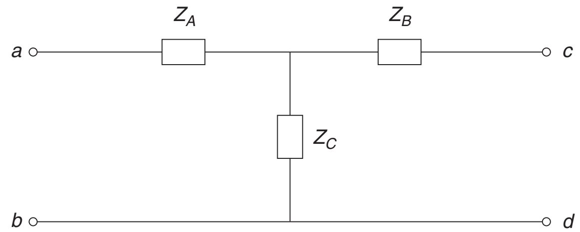


Fig. 9.85

Solution The input impedance with c and d open-circuited is

$$Z_A + Z_C = 250 + j100 \qquad \text{...(i)}$$

The input impedance with c and d short-circuited is,

$$Z_A + \frac{Z_B Z_C}{Z_B + Z_C} = 400 + j300 \qquad \text{...(ii)}$$

The impedance across c and d with a and b open-circuited is

$$Z_B + Z_C = 200 \qquad \text{...(iii)}$$

Subtracting Eq. (i) from (ii),

$$\frac{Z_B Z_C}{Z_B + Z_C} - Z_C = 150 + j200 \qquad \text{...(iv)}$$

From Eq. (iii),

$$Z_B = 200 - Z_C \qquad \text{...(v)}$$

Subtracting the value of Z_B in the equation (iv) and simplifying,

$$Z_C = (100 - j200)\ \Omega \qquad \text{...(vi)}$$

From Eqs (i) and (vi),

$$Z_A = (150 + j300)\ \Omega$$

From Eqs (iii) and (vi),

$$Z_B = (100 + j200)\ \Omega$$

Example 9.46 *Find the equivalent π-network for the T-network shown in Fig. 9.86.*

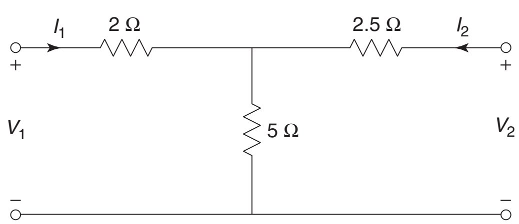


Fig. 9.86

Solution Figure 9.87 shows T-network and π-network.

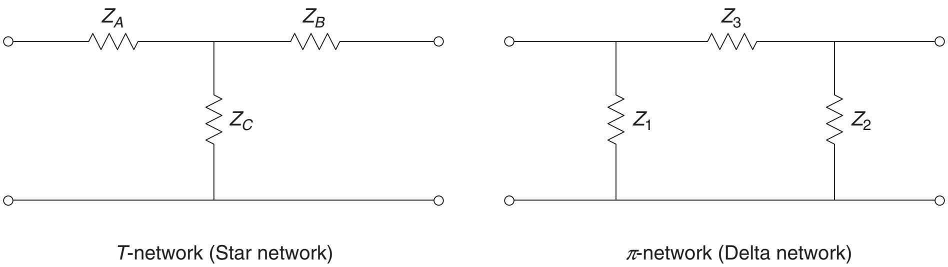


Fig. 9.87

For converting a T-network (star network) into an equivalent π-network (delta network), we can use star-delta transformation technique.

$$Z_1 = Z_A + Z_C + \frac{Z_A Z_C}{Z_B} = 2+5+\frac{2\times 5}{2.5} = 11\,\Omega$$

$$Z_3 = Z_A + Z_B + \frac{Z_A Z_B}{Z_C} = 2+2.5+\frac{2\times 2.5}{5} = 5.5\,\Omega$$

$$Z_2 = Z_B + Z_C + \frac{Z_B Z_C}{Z_A} = 2.5+5+\frac{2.5\times 5}{2} = 13.75\,\Omega$$

The equivalent π-network is shown in Fig. 9.88.

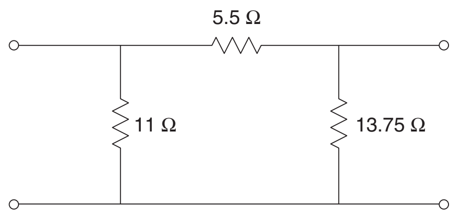


Fig. 9.88

Example 9.47 *For the network shown in Fig. 9.89. Find the equivalent T-network.*

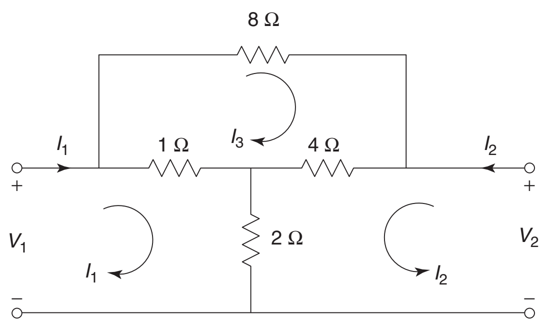


Fig. 9.89

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 6I_2 + 4I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$13I_3 - I_1 + 4I_2 = 0$$

$$I_3 = \frac{1}{13}I_1 - \frac{4}{13}I_2 \qquad \ldots\text{(iii)}$$

Substituting the Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \frac{1}{13}I_1 + \frac{4}{13}I_2$$

$$= \frac{38}{13}I_1 + \frac{30}{13}I_2 \qquad \ldots\text{(iv)}$$

Substituting the Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 6I_2 + 4\left(\frac{1}{13}I_1 - \frac{4}{13}I_2\right)$$

$$= \frac{30}{13}I_1 + \frac{62}{13}I_2 \qquad \ldots\text{(v)}$$

The T-network is shown in Fig. 9.90.

Applying KVL to Mesh 1,

$$V_1 = (Z_A + Z_C)I_1 + Z_C I_2 \qquad \ldots\text{(vi)}$$

Applying KVL to Mesh 2,

$$V_2 = Z_C I_1 + (Z_B + Z_C)I_2 \qquad \ldots\text{(vii)}$$

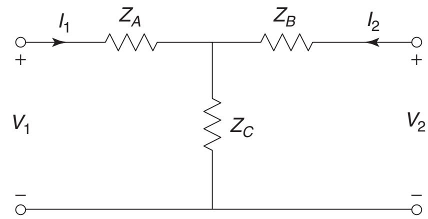


Fig. 9.90

Comparing Eqs (iv) and (v) with Eqs (vi) and (vii),

$$Z_A + Z_C = \frac{38}{13}$$

$$Z_C = \frac{30}{13}$$

$$Z_B + Z_C = \frac{62}{13}$$

Solving the above equations,

$$Z_A = \frac{8}{13}\,\Omega$$

$$Z_B = \frac{32}{13}\,\Omega$$

$$Z_C = \frac{30}{13}\,\Omega$$

9.10 LATTICE NETWORKS

A lattice network is one of the common two-port networks, shown in Fig. 9.91. It is used in filter sections and is also used as attenuator. This network can be represented in terms of z-parameters.

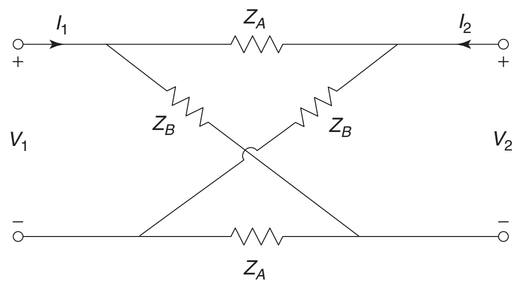


Fig. 9.91 *Lattice network*

The lattice network can be redrawn as a bridge network as shown in Fig. 9.92. This lattice network is symmetric and reciprocal. The current I_1 divides equally between the two arms of the bridge.

When the output port is open-circuited, i.e., $I_2 = 0$

$$V_1 = \frac{I_1}{2}(Z_A + Z_B)$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{Z_A + Z_B}{2}$$

Also

$$V_2 = \frac{I_1}{2}Z_B - \frac{I_1}{2}Z_A = \frac{I_1}{2}(Z_B - Z_A)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{Z_B - Z_A}{2}$$

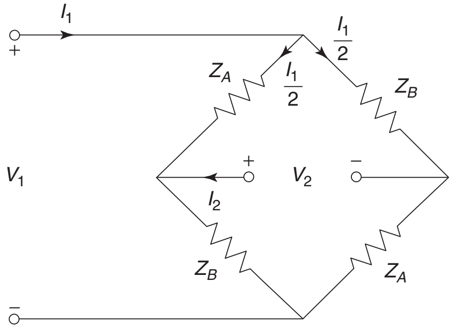


Fig. 9.92 *Bridge network*

Since the network is symmetric,

$$Z_{11} = Z_{22} = \frac{Z_A + Z_B}{2}$$

$$Z_{12} = Z_{21} = \frac{Z_B - Z_A}{2}$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12}$$

$$Z_B = Z_{11} + Z_{12}$$

The lattice network can be represented in terms of other two-port network parameters, with the help of inter-relationship formulae of various parameters.

Example 9.48 *Find the lattice equivalent of a symmetrical T network shown in Fig. 9.93.*

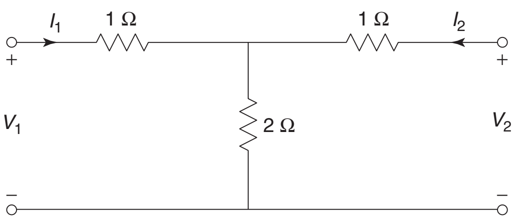


Fig. 9.93

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 3I_2 \quad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$Z_{11} = 3\ \Omega$$

$$Z_{12} = 2\ \Omega$$

$$Z_{21} = 2\ \Omega$$

$$Z_{22} = 3\ \Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 3 - 2 = 1\,\Omega$$
$$Z_B = Z_{11} + Z_{12} = 3 + 2 = 5\,\Omega$$

The lattice network is shown in Fig. 9.94.

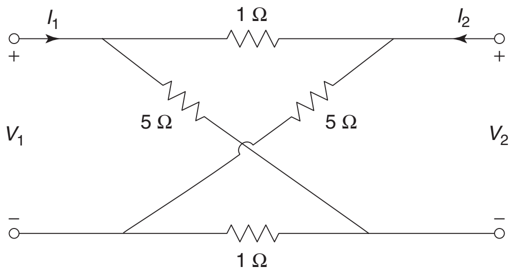


Fig. 9.94

Example 9.49 *Find the lattice equivalent of a symmetric π-network shown in Fig. 9.95.*

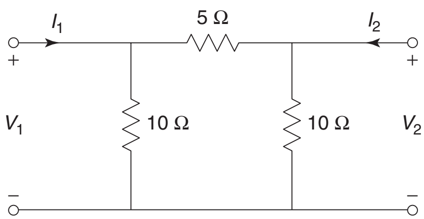


Fig. 9.95

Solution The network is redrawn as shown in Fig. 9.96.

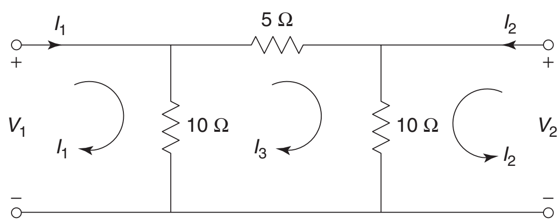


Fig. 9.96

Applying KVL to Mesh 1,

$$V_1 = 10\,I_1 - 10\,I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 10\,I_2 + 10\,I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-10\,I_1 + 10\,I_2 + 25\,I_3 = 0$$

$$I_3 = \frac{2}{5}I - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq (iii) in Eq (i),

$$V_1 = 10\,I_1 - 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 6\,I_1 + 4\,I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq (iii) in Eq (ii),

$$V_2 = 10\,I_2 + 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 4\,I_1 - 6\,I_2 \qquad \ldots\text{(v)}$$

Comparing the Eqs (iv) and (v) with Z-parameter equations,

$$Z_{11} = 6\,\Omega$$
$$Z_{12} = 4\,\Omega$$
$$Z_{21} = 4\,\Omega$$
$$Z_{22} = 6\,\Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 6 - 4 = 2\,\Omega$$
$$Z_B = Z_{11} + Z_{12} = 6 + 4 = 10\,\Omega$$

The lattice network is shown in Fig. 9.97.

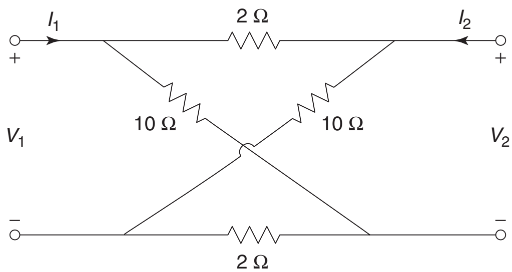


Fig. 9.97

9.11 TERMINATED TWO-PORT NETWORKS

9.11.1 Driving-Point Impedance at Input Port

A two-port network is shown in Fig. 9.98. The output port of the network is terminated in load impedance Z_L. The input impedance of this network can be expressed in terms of parameters of two-port network parameters.

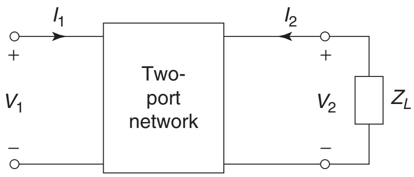


Fig. 9.98 *Terminated two-port network*

1. Input Impedance in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

From Fig. 9.98,

$$V_2 = -Z_L I_2$$
$$-I_2 Z_L = Z_{21} I_1 + Z_{22} I_2$$
$$I_2 = -\frac{Z_{21}}{Z_{22} + Z_L} I_1$$
$$Z_{\text{in}} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\left(-\frac{Z_{21}}{Z_{22} + Z_L}\right) = \frac{Z_{11}Z_{22} + Z_{11}Z_L - Z_{12}Z_{21}}{Z_{22} + Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{\text{in}} = \lim_{Z_L \to \infty} \frac{\dfrac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_L} + Z_{11}}{\dfrac{Z_{22}}{Z_L} + 1} = Z_{11}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{\text{in}} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{22}}$$

2. Input Impedance in Terms of Y-parameters We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

From Fig. 9.98,

$$V_2 = -Z_L I_2$$

$$I_2 = -\frac{V_2}{Z_L} = -Y_L V_2 \qquad \text{where } Y_L = \frac{1}{Z_L}$$

$$-Y_L V_2 = Y_{21}V_1 + Y_{22}V_2$$

$$V_2 = -\frac{Y_{21}}{Y_{22} + Y_L}V_1$$

$$I_1 = Y_{11}V_1 + Y_{12}\left(-\frac{Y_{21}}{Y_{22} + Y_L}\right)V_1 = Y_{11}V_1 - \frac{Y_{21}Y_{12}}{Y_{22} + Y_L}V_1 = \frac{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}{Y_{22} + Y_L}V_1$$

$$Z_{\text{in}} = \frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}$$

When output port is open-circuited, i.e., $Y_L = 0$

$$Z_{\text{in}} = \frac{Y_{22}}{Y_{11}Y_{22} - Y_{12}Y_{21}}$$

When output port is short-circuited, i.e., $Y_L = \infty$,

$$Z_{\text{in}} = \lim_{Y_L \to \infty} \frac{\dfrac{Y_{22}}{Y_L} + 1}{\dfrac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_L} + Y_{11}} = \frac{1}{Y_{11}}$$

3. Input Impedance in Terms of Transmission Parameters We know that

$$V_1 = AV_2 - BV_2$$
$$I_1 = CV_2 - DI_2$$

From Fig. 9.98,

$$V_2 = -Z_L I_2$$

$$I_1 = -CZ_L I_2 - DI_2 = -(CZ_L + D)I_2$$

$$I_2 = -\frac{I_1}{CZ_L + D}$$

$$V_1 = AZ_L I_2 - B = \left(-\frac{I_1}{CZ_L + D}\right) = \left(\frac{AZ_L + B}{CZ_L + D}\right)I_1$$

$$Z_{\text{in}} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{\text{in}} = \frac{A}{C}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{\text{in}} = \frac{B}{D}$$

4. Input Impedance in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$
$$V_2 = -Z_L I_2$$
$$I_2 = h_{21}I_1 - h_{22}Z_L I_2$$
$$I_2 = \frac{h_{21}}{1+h_{22}Z_L} I_L$$
$$V_2 = -\frac{h_{21}Z_L}{1+h_{22}Z_L} I_1$$

Substituting the value of V_2 in V_1,

$$V_1 = h_{11}I_1 + h_{12}\left[\frac{-h_{21}Z_L}{1+h_{22}Z_L} I_L\right] = \left[\frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1+h_{22}Z_L}\right] I_1$$

$$Z_{\text{in}} = \frac{V_1}{I_1} = \frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1+h_{22}Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{\text{in}} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{\text{in}} = h_{11}$$

9.11.2 Driving-Point Impedance at Output Port

A two-port network is shown in Fig. 9.99. The input port is terminated in load impedance Z_L. The output impedance of this network can be expressed in terms of two port network parameters.

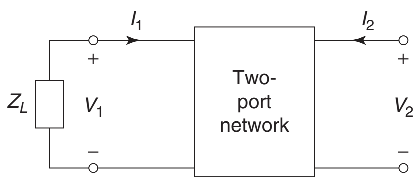


Fig. 9.99 *Terminated two-port network*

1. Output Impedance in terms of Z-parameters We know that

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

From Fig. 9.99,

$$V_1 = -Z_L I_1$$

$$-I_1 Z_1 = Z_{11} I_1 + Z_{12} I_2$$

$$I_1 = \left(-\frac{Z_{12}}{Z_L + Z_{11}}\right) I_2$$

$$V_2 = Z_{21}\left(-\frac{Z_{12}}{Z_L + Z_{11}}\right) I_2 + Z_{22} I_2 = I_2\left(Z_{22} - \frac{Z_{21} Z_{12}}{Z_L + Z_{11}}\right) = \left(\frac{Z_{11} Z_{22} - Z_{12} Z_{21} + Z_{22} Z_L}{Z_{11} + Z_L}\right) I_2$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Z_{11} Z_{22} - Z_{12} Z_{21} + Z_{22} Z_L}{Z_{11} + Z_L}$$

If the input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = Z_{22}$$

If the input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{11}}$$

2. *Output Impedance in Terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

From Fig. 9.99,

$$V_1 = -Z_L I_1$$

$$I_1 = -\frac{V_1}{Z_L} = -Y_L V_1$$

$$-Y_L V_1 = Y_{11} V_1 + Y_{12} V_2$$

$$V_1 = \left(-\frac{Y_{12}}{Y_L + Y_{11}}\right) V_2$$

$$I_2 = Y_{21}\left(-\frac{Y_{12}}{Y_L + Y_{11}}\right) V_2 + Y_{22} V_2 = V_2\left[Y_{22} - \frac{Y_{21} Y_{12}}{Y_L + Y_{11}}\right] = V_2\left[\frac{Y_{11} Y_{22} - Y_{12} Y_{21} + Y_L Y_{22}}{Y_L + Y_{11}}\right]$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Y_L + Y_{11}}{Y_{11} Y_{22} - Y_{12} Y_{21} + Y_L Y_{22}}$$

If input port is open-circuited, i.e., $Y_L = 0$,

$$Z_0 = \frac{Y_{11}}{Y_{11} Y_{22} - Y_{12} Y_{21}}$$

If input port is short-circuited, i.e., $Y_L = \infty$,

$$Z_0 = \frac{1}{Y_{22}}$$

3. *Output Impedance in Terms of ABCD Parameters* We know that

$$V_1 = A V_2 - B I_2$$

$$I_1 = C V_2 - D I_2$$

From Fig. 9.99,

$$V_1 = -Z_L I_1$$

$$\frac{V_1}{I_1} = -Z_L = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

$$V_2(CZ_L + A) = I_2(DZ_L + B)$$

$$Z_0 = \frac{V_2}{I_2} = \frac{DZ_L + B}{CZ_L + A}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{D}{C}$$

If input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{B}{A}$$

4. *Output Impedance in Terms of h-parameters* We know that

$$V_1 = h_{11}I_1 + h_{12}V_{12}$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

From Fig. 9.99,

$$V_1 = -Z_L I_1$$

$$-I_1 Z_L = h_{11}I_1 + h_{22}V_2$$

$$I_1 = \left(-\frac{h_{12}}{h_{11} + Z_L}\right)V_2$$

$$I_2 = h_{21}\left(-\frac{h_{12}}{h_{11} + Z_L}\right)V_2 + h_{22}V_2 = V_2\left[\frac{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}{h_{11} + Z_L}\right]$$

$$Z_0 = \frac{V_2}{I_2} = \frac{h_{11} + Z_L}{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{1}{h_{22}}$$

If input port is short-circuited i.e., $Z_L = 0$,

$$Z_0 = \frac{h_{11}}{h_{11}h_{22} - h_{12}h_{21}}$$

Example 9.50 *Measurements were made on a two-terminal network shown in Fig. 9.100.*

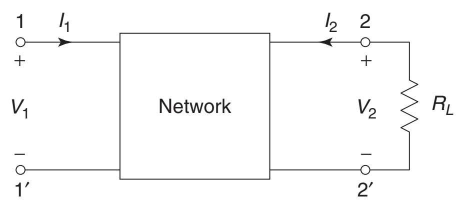


Fig. 9.100

(a) With terminal pair 2 open, a voltage of 100 ∠0° V applied to terminal pair 1 resulted in

$$I_1 = 10\angle 0°\ A,\quad V_2 = 25\angle 0°\ V$$

(b) With terminal pair 1 open, the same voltage applied to terminal pair 2 resulted in

$$I_2 = 20\angle 0°\ A,\quad V_1 = 50\angle 0°\ V$$

Write mesh equations for this network. What will be the voltage across a 10-Ω resistor connected across Terminal pair 2 if a 100 ∠0° V is connected across terminal pair 1?

Solution Since measurements were done with either of the terminal pairs open-circuited, we have to calculate Z-parameters first.

$$Z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0} = \frac{100\angle 0°}{10\angle 0°} = 10\ \Omega$$

$$Z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0} = \frac{25\angle 0°}{10\angle 0°} = 2.5\ \Omega$$

$$Z_{22} = \frac{V_2}{I_2}\bigg|_{I_1=0} = \frac{100\angle 0°}{20\angle 0°} = 5\ \Omega$$

$$Z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0} = \frac{50\angle 0°}{20\angle 0°} = 2.5\ \Omega$$

Putting these values in Z-parameter equations,

$$V_1 = 10I_1 + 2.5I_2 \qquad \text{...(i)}$$

$$V_2 = 2.5I_1 + 5I_2 \qquad \text{...(ii)}$$

When a 10-Ω resistor is connected across terminal pair 1,

$$V_1 = 100\angle 0°\ V$$

$$V_2 = -R_L I_2 = -10I_2$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$100 = 10I_1 + 2.5I_2$$

and

$$-10I_2 = 2.5I_1 + 5I_2$$

$$2.5I_1 = -15I_2$$

$$I_1 = -6I_2$$

$$100 = -60I_2 + 2.5I_2$$

$$I_2 = -\frac{100}{57.5} = -1.74\ \text{A}$$

$$\text{Voltage across the resistor} = -I_2R_L = -10(-1.74) = 17.4\ \text{V}$$

Example 9.51 *The Z-parameters of a two-port network shown in Fig. 9.101 are $Z_{11} = Z_{22} = 10\ \Omega$, $Z_{21} = Z_{22} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, I_2, V_2 and input impedance.*

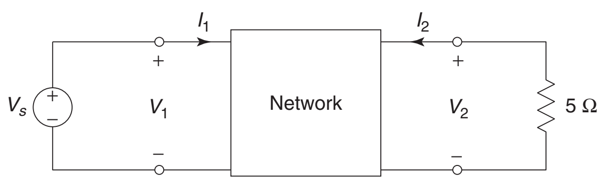


Fig. 9.101

Solution

$$V_1 = V_s = 20 \text{ V}$$
$$V_2 = -20I_2$$

The two-port network can be represented in terms of Z-parameters.

$$V_1 = 10I_1 + 4I_2 \quad \ldots\text{(i)}$$
$$V_2 = 4I_1 + 10I_2 \quad \ldots\text{(ii)}$$
$$-20I_2 = 4I_1 + 10I_2$$
$$4I_1 = -30I_2$$
$$I_1 = -7.5I_2$$

Substituting the value of I_1 in Eq. (i),

$$V_1 = 10(-7.5I_2) + 4I_2 = -71I_2$$
$$20 = -71I_2$$
$$I_2 = -0.28 \text{ A}$$
$$I_1 = -7.5(-0.28) = 2.1 \text{ A}$$
$$V_2 = -20(-0.28) = 56 \text{ V}$$

$$\text{Input impedance } Z_i = \frac{V_1}{I_1} = \frac{20}{2.1} = 9.52\ \Omega$$

Example 9.52 *The Z-parameters of a two-port network shown in Fig. 9.102 are, $Z_{11} = 2\ \Omega$, $Z_{12} = 1\ \Omega$, $Z_{21} = 2\ \Omega$, $Z_{22} = 5\ \Omega$. Calculate the voltage ratio $\frac{V_2}{V_s}$, current ratio $-\frac{I_2}{I_1}$ and input impedance $\frac{V_1}{I_1}$.*

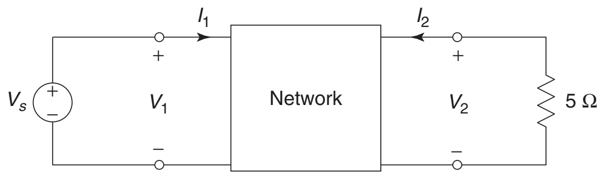


Fig. 9.102

Solution The two-port network can be represented in terms of Z-parameters.

$$V_1 = 2I_1 + I_2 \quad \ldots\text{(i)}$$
$$V_2 = 2I_1 + 5I_2 \quad \ldots\text{(ii)}$$

When the 5 Ω resistor is connected across port-2,

$$V_2 = -5I_2 \quad \ldots\text{(iii)}$$

Applying KVL to the input port,

$$V_s - 1I_1 - V_1 = 0$$
$$V_1 = V_s + I_1 \quad \ldots\text{(iv)}$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$V_s - I_1 = 2I_1 + I_2$$
$$V_s = 3I_1 + I_2 \quad \ldots\text{(v)}$$

and
$$-5I_2 = 2I_1 + 5I_2$$
$$0 = 2I_1 + 10I_2 \qquad \text{...(vi)}$$

Solving Eqs (v) and (vi), we get

$$I_1 = \frac{\begin{vmatrix} V_s & 1 \\ 0 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = \frac{5}{14}V_s$$

$$I_2 = \frac{\begin{vmatrix} 3 & V_s \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = -\frac{1}{14}V_s$$

$$-\frac{I_2}{I_1} = \frac{1}{5}$$

$$V_2 = 2I_1 + 5I_2 = 2\left(\frac{5}{14}V_s\right) + 5\left(-\frac{1}{14}V_s\right) = \frac{5}{14}V_s$$

$$\frac{V_2}{V_s} = \frac{5}{14}$$

$$V_1 = 2I_1 + I_2 = 2\left(\frac{5}{14}V_s\right) - \frac{1}{14}V_s = \frac{9}{14}V_s$$

$$\frac{V_1}{I_1} = \frac{9}{5}\,\Omega$$

Example 9.53 *The following equations give the voltages V_1 and V_2 at the two ports of a two-port network shown in Fig. 9.103.*

$$V_1 = 5I_1 + 2I_2$$
$$V_2 = 2I_1 + I_2$$

A load resistor of 3 Ω is connected across port 2. Calculate the input impedance.

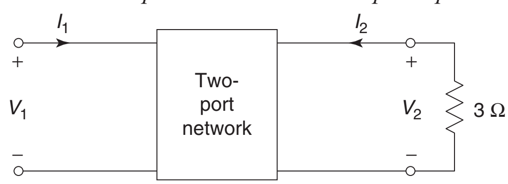


Fig. 9.103

Solution From Fig. 9.103,

$$V_2 = -3I_2 \qquad \text{...(i)}$$

Substituting Eq. (i) in the given equation,

$$-3I_2 = 2I_1 + I_2$$
$$I_2 = -\frac{I_1}{2} \qquad \text{(ii)}$$

Substituting the Eq. (ii) in the given equation.

$$V_1 = 5I_1 - I_1 = 4I_1$$

Input impedance
$$Z_i = \frac{V_1}{I_1} = 4\ \Omega$$

Example 9.54 *The y-parameters for a two-port network shown in Fig. 9.104 are given as, $Y_{11} = 4$ ℧, $Y_{22} = 5$ ℧, $Y_{12} = Y_{21} = 4$ ℧. If a resistor of 1 Ω is connected across port-1 of the network then find the output impedance.*

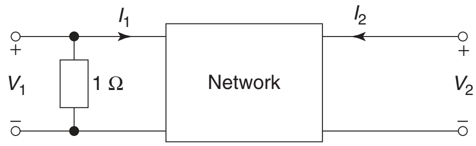


Fig. 9.104

Solution The two-port network can be represented in terms of Y-parameters.

$$I_1 = 4V_1 + 4V_2 \quad \ldots\text{(i)}$$

$$I_2 = 4V_1 + 5V_2 \quad \ldots\text{(ii)}$$

When the 1-Ω resistor is connected across port-1 of the network,

$$V_1 = -1I_1 = -I_1$$

$$I_1 = -V_1$$

Substituting value of I_1 in Eq (i),

$$-V_1 = 4V_1 + 4V_2$$

$$-5V_1 = 4V_2$$

$$V_1 = -\frac{4}{5}V_2$$

Substituting value of V_1 in Eq (ii),

$$I_2 = 4\left(-\frac{4}{5}V_2\right) + 5V_2 = \frac{9}{5}V_2$$

Output impedance $Z_0 = \dfrac{V_2}{I_2} = \dfrac{5}{9}\,\Omega$

Example 9.55 *The following equation gives the voltage and current at the input port of a two-port network shown in Fig. 9.105.*

$$V_1 = 5V_2 - 3I_2$$

$$I_1 = 6V_2 - 2I_2$$

A load resistance of 5 Ω is connected across the output port. Calculate the input impedance.

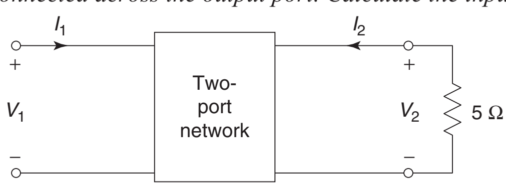


Fig. 9.105

Solution From Fig. 9.105,

$$V_2 = -5I_2$$

Substituting the value of V_2 in the given equations,

$$V_1 = 5(-5I_2) - 3I_2 = -28I_2$$

$$I_1 = 6(-5I_2) - 2I_2 = -32I_2$$

Input impedance $Z_i = \dfrac{V_1}{I_1} = \dfrac{-28I_2}{-32I_2} = \dfrac{7}{8}\,\Omega$

Example 9.56 *The ABCD parameters of a two-port network shown in Fig. 9.106 are $A = 2.5$, $B = 4\ \Omega$, $C = 1\ \mho$, $D = 2$. What must be the input voltage V_1 applied for the output voltage V_2 to be 10 V across the load of 10 Ω connected at Port 2?*

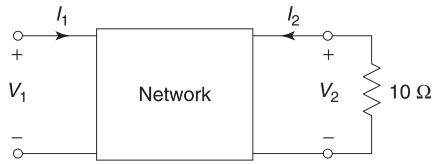


Fig. 9.106

Solution The two-port network can be represented in terms of *ABCD* parameters.

$$V_1 = 2.5\, V_2 - 4I_2 \quad \text{...(i)}$$

$$I_1 = V_2 - 2I_2 \quad \text{...(ii)}$$

When the 10 Ω resistor is connected across Port 2,

$$V_2 = -10 I_2 = 10 \quad \text{...(iii)}$$

$$I_2 = -1\text{A}$$

$$V_1 = 2.5(10) - 4(-1) = 29\text{ V}$$

Example 9.57 *The h-parameters of a two-port network shown in Fig. 9.107 are $h_{11} = 4\ \Omega$, $h_{12} = 1$, $h_{21} = 1$, $h_{22} = 0.5\ \mho$. Calculate the output voltage V_2 when the output port is terminated in a 3 Ω resistance and a 1 V is applied at the input port.*

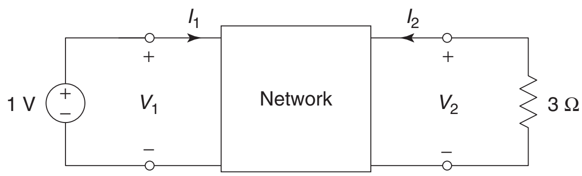


Fig. 9.107

Solution

$$V_1 = 1\text{ V}$$

$$V_2 = -3I_2$$

The two-port network can be represented in terms of *h*-parameters.

$$V_1 = 4I_1 + V_2 \quad \text{...(i)}$$

$$I_2 = I_1 + 0.5\, V_2 \quad \text{...(ii)}$$

$$I_2 = I_1 + 0.5(-3I_2)$$

$$2.5 I_2 = I_1$$

Substituting the value of V_1 and I_1 in Eq (i),

$$1 = 4(2.5 I_2) - 3 I_2$$

$$1 = 7I_2$$

$$I_2 = \frac{1}{7}\text{A}$$

$$V_2 = -3\left(\frac{1}{7}\right) = -\frac{3}{7}\text{V}$$

Example 9.58 *The h-parameters of a two-port network shown in Fig. 9.108 are $h_{11} = 1\ \Omega$, $h_{12} = -h_{21} = 2$, $h_{22} = 1\ \mho$. The power absorbed by a load resistance of $1\ \Omega$ connected across port-2 is 100W. The network is excited by a voltage source of generated voltage V_s and internal resistance $2\ \Omega$. Calculate the value of V_s.*

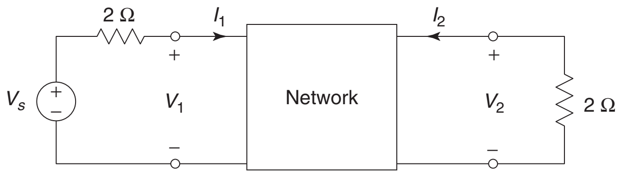


Fig. 9.108

Solution The two-port network can be represented in terms of h-parameters.

$$V_1 = I_1 + 2V_2 \qquad \text{...(i)}$$

$$I_2 = -2I_1 + V_2 \qquad \text{...(ii)}$$

When the 1 Ω resister is connected across port-2,

$$\frac{V_2^2}{1} = 100$$

$$V_2 = 10 \text{ V}$$

$$V_2 = -1 I_2 = 10$$

$$I_2 = -10 \text{ A}$$

Substituting values of I_2 and V_2 in Eq (ii),

$$-10 = -2I_1 + 10$$

$$I_1 = 10 \text{ A}$$

Applying KVL to the input port,

$$V_s - 2I_1 - V_1 = 0$$

$$V_s - 2I_1 - (I_1 + 2V_2) = 0$$

$$V_s - 3I_1 - 2V_2 = 0$$

$$V_s = 3I_1 + 2V_2 = 3(10) + 2(10) = 50 \text{ V}$$

Exercises

9.1 Determine Z-parameters for the network shown in Fig. 9.109.

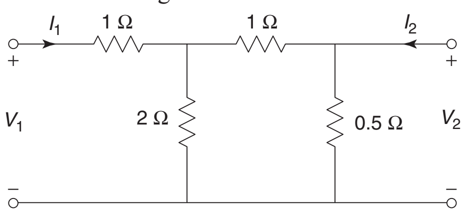


Fig. 9.109

$$Z = \begin{bmatrix} \frac{13}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} \end{bmatrix}$$

9.2 Find Z-parameters for the network shown in Fig. 9.110.

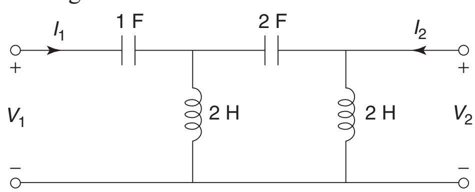


Fig. 9.110

$$Z = \begin{bmatrix} \frac{4s^4 + 6s^2 + 1}{4s^3 + s} & \frac{4s^3}{4s^2 + 1} \\ \frac{4s^3}{4s^2 + 1} & \frac{4s^3 + 2s}{4s^2 + 1} \end{bmatrix}$$

9.3 Find Y-parameters of the network shown in Fig. 9.111.

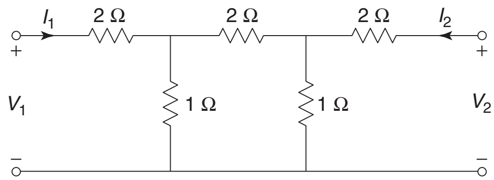


Fig. 9.111

$$Y = \begin{bmatrix} 0.36 & -0.033 \\ -0.033 & -0.36 \end{bmatrix}$$

9.4 Find Y-parameters for the network shown in Fig. 9.112.

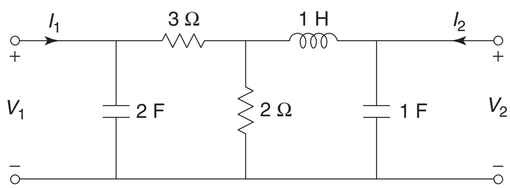


Fig. 9.112

$$Y = \begin{bmatrix} \dfrac{10s^2+13s+2}{5s+6} & -\dfrac{2}{5s+6} \\ -\dfrac{2}{5s+6} & \dfrac{5s^2+6s+5}{5s+6} \end{bmatrix}$$

9.5 Find Y-parameters for the network shown in Fig. 9.113.

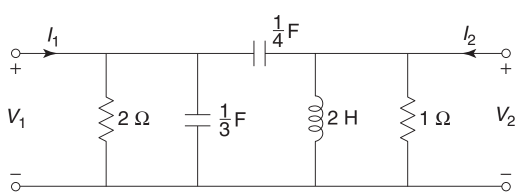


Fig. 9.113

$$Y = \begin{bmatrix} \dfrac{7s+6}{12} & -\dfrac{s}{4} \\ -\dfrac{s}{4} & \dfrac{s^2+4s+2}{4s} \end{bmatrix}$$

9.6 Find Y-parameters for the network shown in Fig. 9.114.

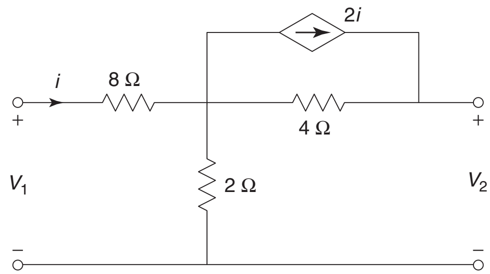


Fig. 9.114

$$Y = \begin{bmatrix} \dfrac{3}{20} & -\dfrac{1}{20} \\ -\dfrac{1}{4} & \dfrac{1}{4} \end{bmatrix}$$

9.7 Show the *ABCD* parameters of the network shown in Fig. 9.115.

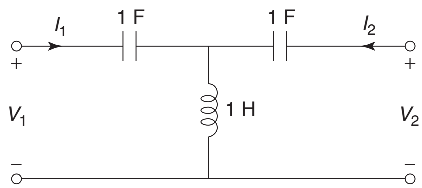


Fig. 9.115

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{1+s^2}{s^2} & \dfrac{1+2s^2}{s^3} \\ \dfrac{1}{s} & \dfrac{1+s^2}{s^2} \end{bmatrix}$$

9.8 Find *ABCD* parameters for the network shown in Fig. 9.116.

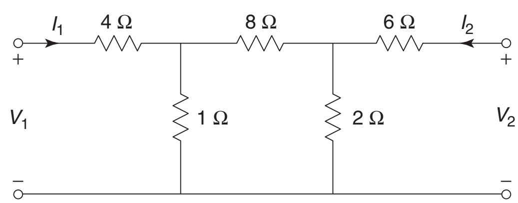


Fig. 9.116

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 27 & 206 \\ \dfrac{11}{2} & 42 \end{bmatrix}$$

9.9 For the network shown in Fig. 9.117, determine parameter h_{21}.

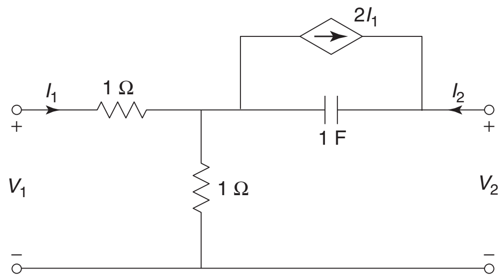


Fig. 9.117

$$\left[h_{21} = \frac{-(2+s)}{1+s}\right]$$

9.10 Determine Y and Z-parameters for the network shown in Fig. 9.118.

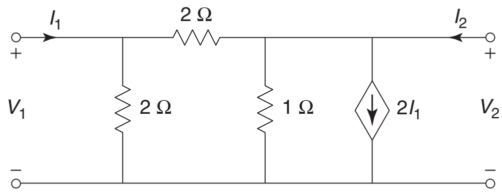


Fig. 9.118

$$\left[Y_{11} = 1\,\mho, Y_{12} = -0.5\,\mho, Y_{21} = 1.5\,\mho, Y_{22} = 0.5\,\mho\right.$$
$$\left.Z_{11} = \frac{2}{5}\Omega, Z_{12} = \frac{2}{5}\Omega, Z_{21} = -\frac{6}{5}\Omega, Z_{22} = \frac{4}{5}\Omega\right]$$

9.11 For the bridged T, R-C network shown in Fig. 9.119 determine Y-parameters using interconnections of two-port networks.

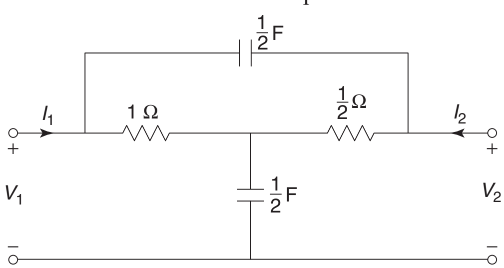


Fig. 9.119

$$Y = \begin{bmatrix} \dfrac{s^2+8s+8}{2(s+6)} & -\dfrac{(s^2+6s+8)}{2(s+6)} \\ -\dfrac{(s^2+6s+8)}{2(s+6)} & \dfrac{s^2+10s+8}{2(s+6)} \end{bmatrix}$$

9.12 For the network of Fig. 9.120, find Y-parameters using interconnection of two-port networks.

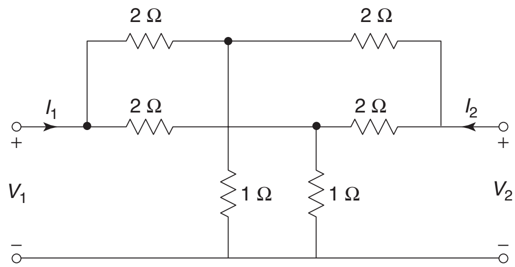


Fig. 9.120

$$Y = \begin{bmatrix} \dfrac{3}{4} & -\dfrac{1}{4} \\ -\dfrac{1}{4} & \dfrac{3}{4} \end{bmatrix}$$

9.13 Two identical sections of the network shown in Fig. 9.121 are connected in parallel. Obtain Y-parameters of the connection.

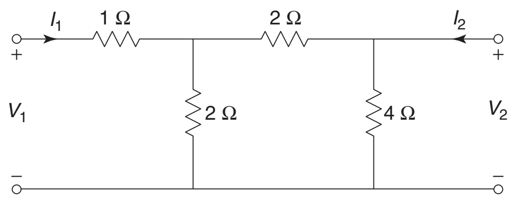


Fig. 9.121

$$Y = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{5}{4} \end{bmatrix}$$

9.14 Determine Y-parameters using interconnection of two-port networks for the network shown in Fig. 9.122.

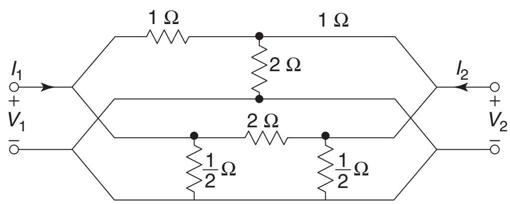


Fig. 9.122

$$Y = \begin{bmatrix} 3.1 & -0.9 \\ -0.9 & 3.1 \end{bmatrix}$$

9.15 Determine the transmission parameters of the network shown in Fig. 9.123 using the concept of interconnection of two two-port networks.

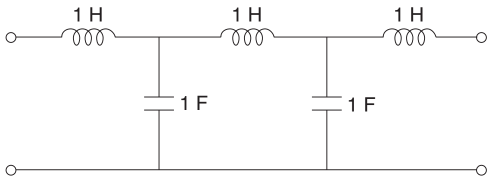


Fig. 9.123

$$\begin{bmatrix} 1+3s^2+s^4 & 3s+4s^3+s^5 \\ 2s+s^3 & 1+3s^2+s^4 \end{bmatrix}$$

9.16 Two networks shown in Fig. 9.124 are connected in series. Obtain the Z-parameters of the resulting network.

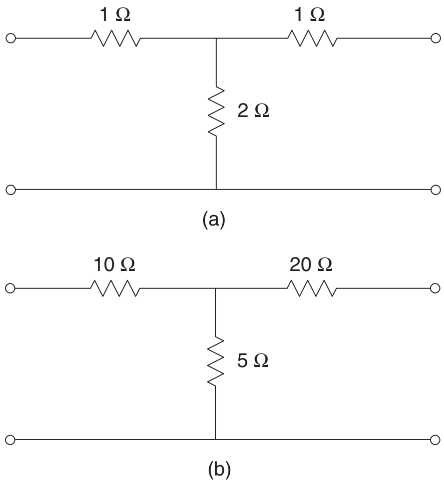


Fig. 9.124

$$\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$$

9.17 Two identical sections of the network shown in Fig. 9.125 are connected in series-parallel. Determine the h-parameters of the overall network.

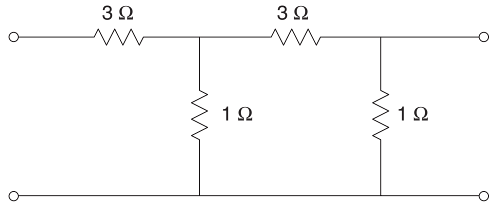


Fig. 9.125

$$\begin{bmatrix} \frac{15}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

9.18 The h-parameters of a two-port network shown in Fig. 9.126 are $h_{11} = 2\ \Omega$, $h_{12} = 4$, $h_{21} = -5$, $h_{22} = 2$ ℧. Determine the supply voltage V_s if the power dissipated in the load resistor of 4 Ω is 25 W and $R_s = 2\ \Omega$.

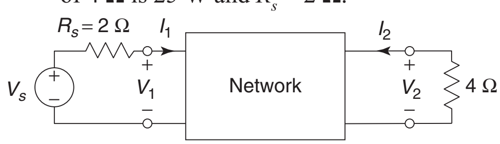


Fig. 9.126

[58 V]

9.19 The Z-parameters of a two-port network are $Z_{11} = 2.1\ \Omega$, $Z_{12} = Z_{21} = 0.6\ \Omega$, $Z_{22} = 1.6\ \Omega$. A resistor of 2 Ω is connected across port 2. What voltage must be applied at port 1 to produce a current of 0.5 A in the 2 Ω resistor.

[6 V]

9.20 If a two-port network has $Z_{11} = 25\ \Omega$, $Z_{12} = Z_{21} = 20\ \Omega$, $Z_{22} = 50\ \Omega$, find the equivalent T-network.

[10 Ω, 30 Ω, 20 Ω]

Objective-Type Questions

9.1 The open-circuit impedance matrix of the two-port network shown in Fig. 9.127 is

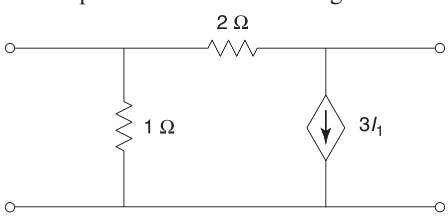


Fig. 9.127

(a) $\begin{bmatrix} -2 & 1 \\ -8 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -8 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

9.2 Two two-port networks are connected in cascade. The combination is to be represented as a single two-port network. The parameters are obtained by multiplying the individual

(a) z-parameter matrix
(b) h-parameter matrix
(c) y-parameter matrix
(d) $ABCD$ parameter matrix

9.3 For a two-port network to be reciprocal

(a) $z_{11} = z_{22}$ (b) $y_{21} = y_{12}$

(c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

9.4 The short-circuit admittance matrix of a two-port network is $\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. The two-port network is

(a) non-reciprocal and passive
(b) non-reciprocal and active
(c) reciprocal and passive
(d) reciprocal and active

9.5 A two-port network is shown in Fig. 9.128. The parameter h_{21} for this network can be given by

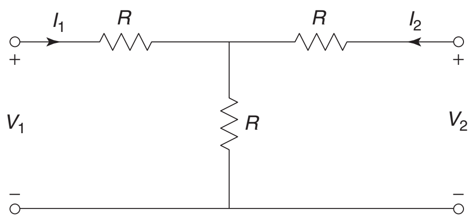


Fig. 9.128

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) $-\frac{3}{2}$ (d) $\frac{3}{2}$

9.6 The admittance parameter Y_{12} in the two-port network in Fig. 9.129.

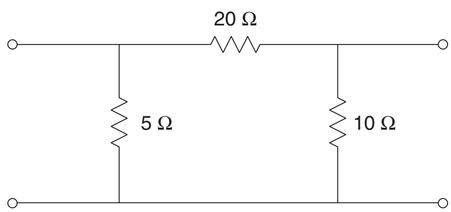


Fig. 9.129

(a) –0.2 mho (b) 0.1 mho

(c) –0.05 mho (d) 0.05 mho

9.7 The Z-parameters Z_{11} and Z_{21} for the two-port network in Fig. 9.130 are,

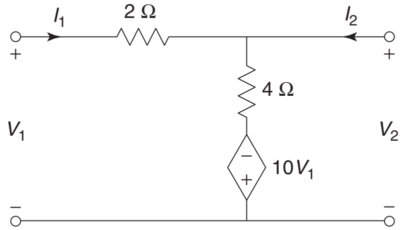


Fig. 9.130

(a) $-\frac{6}{11}\Omega, \frac{16}{11}\Omega$ (b) $\frac{6}{11}\Omega, \frac{4}{11}\Omega$

(c) $\frac{6}{11}\Omega, -\frac{16}{11}\Omega$ (d) $\frac{4}{11}\Omega, \frac{4}{11}\Omega$

9.8 The impedance parameters Z_{11} and Z_{12} of a two-port network in Fig. 9.131.

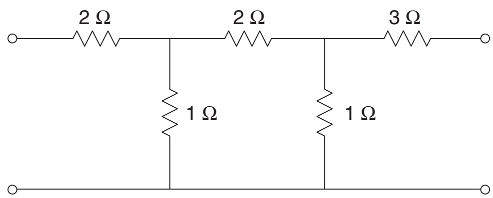


Fig. 9.131

(a) 2.75 Ω, 0.25 Ω (b) 3 Ω, 0.5 Ω

(c) 3 Ω, 0.25 Ω (d) 2.25 Ω, 0.5 Ω

9.9 The h parameters of the circuit shown in Fig. 9.132.

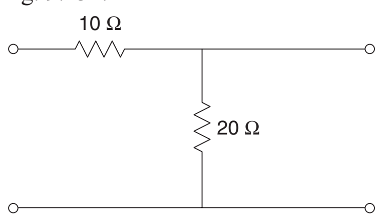


Fig. 9.132

(a) $\begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & -1 \\ 1 & 0.05 \end{bmatrix}$

(c) $\begin{bmatrix} 30 & 20 \\ 20 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 1 \\ -1 & 0.05 \end{bmatrix}$

9.10 A two-port network is represented by *ABCD* parameters given by $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$. If port 2 is terminated by R_L, then the input impedance seen at port 1 is given by

(a) $\dfrac{A+BR_L}{C+DR_L}$ (b) $\dfrac{AR_L+C}{BR_L+D}$

(c) $\dfrac{DR_L+A}{BR_L+C}$ (d) $\dfrac{B+AR_L}{D+CR_L}$

9.11 In the two-port network shown in Fig. 9.133, Z_{12} and Z_{21} are respectively

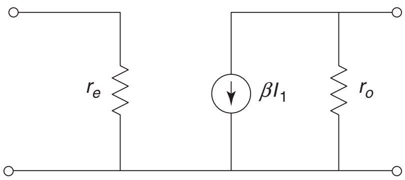


Fig. 9.133

(a) r_e and βr_0 (b) 0 and $-\beta r_0$
(c) 0 and βr_0 (d) r_e and $-\beta r_0$

9.12 If a two-port network is passive, then we have, with the usual notation, the following relationship for symmetrical network

(a) $h_{12} = h_{21}$
(b) $h_{12} = -h_{21}$
(c) $h_{11} = h_{22}$
(d) $h_{11}h_{22} - h_{12}h_{21} = 1$

9.13 A two-port network is defined by the following pair of equations $I_1 = 2V_1 + V_2$ and $I_2 = V_1 + V_2$. Its impedance parameters (Z_{11}, Z_{12}, Z_{21}, Z_{22}) are given by

(a) 2, 1, 1, 1 (b) 1, –1, –1, 2

(c) 1, 1, 1, 2 (d) 2, –1, –1, 1

9.14 A two-port network has transmission parameters $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The input impedance of the network at port 1 will be

(a) $\dfrac{A}{C}$ (b) $\dfrac{AD}{BC}$

(c) $\dfrac{AB}{DC}$ (d) $\dfrac{D}{C}$

9.15 A two-port network is symmetrical if

(a) $Z_{11} Z_{22} - Z_{12} Z_{21} = 1$
(b) $AD - BC = 1$
(c) $h_{11} h_{22} - h_{12} h_{21} = 1$
(d) $Y_{11} Y_{22} - Y_{12} Y_{21} = 1$

9.16 For the network shown in Fig. 9.134 admittance parameters are $Y_{11} = 8$ mho, $Y_{12} = Y_{21} = -6$ mho and $Y_{22} = 6$ mho. The value of Y_A, Y_B and Y_C (in mho) will be respectively

(a) 2, 6, –6 (b) 2,6,0

(c) 2,0,6 (d) 2,6,8

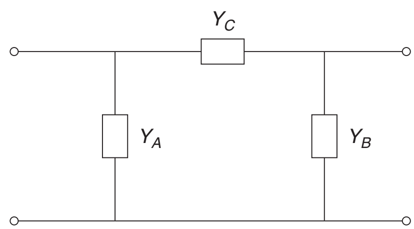


Fig. 9.134

9.17 The impedance matrices of two two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 3 & 25 \end{bmatrix}$. If these two networks are connected in series, the impedance matrix of the resulting two-port network will be

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$ (d) inderminate

9.18 If the π network and T network are equivalent, then the values of R_1, R_2 and R_3 (in ohms) will be respectively

(a) 6, 6, 6 (b) 6, 6, 9

(c) 9, 6, 9 (d) 6, 9, 6

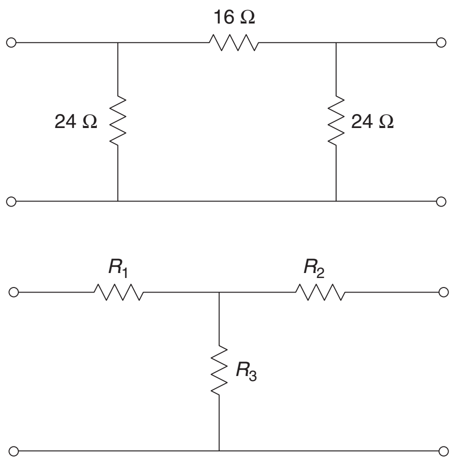


Fig. 9.135

9.19 For a two-port symmetrical bilateral network, if $A = 3$ and $B = 1$, the value of the parameter C will be

(a) 4 (b) 6

(c) 8 (d) 16

9.20 The impedance matrix for the network shown in Fig. 9.136 is

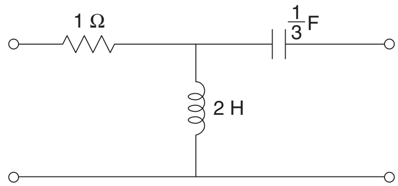


Fig. 9.136

(a) $\begin{bmatrix} 2s+1 & 2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$ (b) $\begin{bmatrix} 2s+1 & -2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$

(c) $\begin{bmatrix} 2s+1 & 2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$ (d) $\begin{bmatrix} 2s+\frac{3}{2} & -2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$

9.21 With the usual notations, a two-port resistive network satisfies the conditions $A = D = \frac{3}{2}B = \frac{4}{3}C$. The Z_{11} of the network is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$

(c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Answers to Objective-Type Questions

9.1. (a)	9.2. (d)	9.3. (b) , (c)	9.4. (b)	9.5. (a)	9.6. (c)	9.7. (c)
9.8. (a)	9.9. (d)	9.10. (d)	9.11. (b)	9.12. (d)	9.13. (b)	9.14. (a)
9.15. (c)	9.16. (c)	9.17. (b)	9.18. (a)	9.19. (c)	9.20. (a)	9.21. (b)

10

Synthesis of RLC Circuits

10.1 INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

10.2 HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.

2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

Example 10.1 *State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Example 10.2 *Test whether the polynomial* $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^3 + 3s\,\big)\,s^4 + 5s^2 + 4\ \big(s \\
\qquad\quad \underline{s^4 + 3s^2} \\
\qquad\qquad 2s^2 + 4\,\big)\,s^3 + 3s\,\big(\tfrac{1}{2}s \\
\qquad\qquad\qquad\quad \underline{s^3 + 2s} \\
\qquad\qquad\qquad\qquad s\,\big)\,2s^2 + 4\ \big(2s \\
\qquad\qquad\qquad\qquad\quad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 4\,\big)\,s\,\big(\tfrac{1}{4}s \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 10.3 *Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\ n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2 \overline{\big)\, s^3 + 5s \,\big(} \; \frac{1}{4}s$$

$$\underline{s^3 + \frac{2}{4}s}$$

$$\frac{9}{2}s \overline{\big)\, 4s^2 + 2 \,\big(} \; \frac{8}{9}s$$

$$\underline{4s^2}$$

$$2 \overline{\big)\, \frac{9}{2}s \,\big(} \; \frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 10.4 *Test whether the polynomial* $P(s) = s^4 + s^3 + 3s^2 + 2s + 12$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 3s^2 + 12$

Odd part of $P(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 2s \overline{\big)\, s^4 + 3s^2 + 12 \,\big(} \; s$$

$$\underline{s^4 + 2s^2}$$

$$s^2 + 12 \overline{\big)\, s^3 + 2s \,\big(} \; s$$

$$\underline{s^3 + 12s}$$

$$-10s \overline{\big)\, s^2 + 12 \,\big(} \; -\frac{1}{10}s$$

$$\underline{s^2}$$

$$12 \overline{\big)\, -10s \,\big(} \; -\frac{10}{12}s$$

$$\underline{-10s}$$

$$0$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 10.5 *Prove that polynomial* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is not Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 2s^2 + 2$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s)s^4 + 2s^2 + 2\ (s$$
$$\underline{s^4 + 3s^2}$$
$$-s^2 + 2)s^3 + 3s(-s$$
$$\underline{s^3 - 2s}$$
$$5s\Big)-s^2 + 2\Big(-\frac{1}{5}s$$
$$\underline{-s^2}$$
$$2\Big)5s\Big(\frac{5}{2}s$$
$$\underline{5s}$$
$$0$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 10.6 *Prove that polynomial* $P(s) = 2s^4 + 5s^3 + 6s^2 + 3s + 1$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^4 + 6s^2 + 1$

Odd part of $P(s) = n(s) = 5s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$5s^3 + 3s\Big)2s^4 + 6s^2 + 1\Big(\frac{2}{5}s$$
$$\underline{2s^4 + \frac{6}{5}s^2}$$
$$\frac{24}{5}s^2 + 1\Big)5s^3 + 3s\Big(\frac{25}{24}s$$
$$\underline{5s^3 + \frac{25}{24}s}$$

$$\frac{47}{24}s\,\Big)\,\frac{24}{5}s^2+1\,\Big(\,\frac{576}{235}s$$

$$\frac{24}{5}s^2$$

$$\overline{}$$

$$1\,\Big)\,\frac{47}{24}s\,\Big(\,\frac{24}{47}s$$

$$\frac{47}{24}s$$

$$\overline{0}$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 10.7 *Test whether the polynomial* $P(s)=s^4+7s^3+6s^2+21s+8$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+6s^2+8$

Odd part of $P(s)=n(s)=7s^3+21s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$7s^3+21s\,\Big)\,s^4+6s^2+8\,\Big(\,\frac{1}{7}s$$

$$s^4+3s^2$$

$$\overline{}$$

$$3s^2+8\,\Big)\,7s^3+21s\,\Big(\,\frac{7}{3}s$$

$$7s^3+\frac{56}{3}s$$

$$\overline{}$$

$$\frac{7}{3}s\,\Big)\,3s^2+8\,\Big(\,\frac{9}{7}s$$

$$3s^2$$

$$\overline{}$$

$$8\,\Big)\,\frac{7}{3}s\,\Big(\,\frac{7}{24}s$$

$$\frac{7}{3}s$$

$$\overline{0}$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 10.8 *Check whether* $P(s)=s^4+5s^3+5s^2+4s+10$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+5s^2+10$

Odd part of $P(s)=n(s)=5s^3+4s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$5s^3 + 4s \overline{\big) s^4 + 5s^2 + 10 \Big(\tfrac{1}{5}s}$$
$$\underline{s^4 + \tfrac{4}{5}s^2}$$
$$\tfrac{21}{5}s^2 + 10 \big) 5s^3 + 4s \Big(\tfrac{25}{21}s$$
$$\underline{5s^3 + \tfrac{250}{21}s}$$
$$-\tfrac{166}{21}s \big) \tfrac{21}{5}s^2 + 10 \Big(-\tfrac{441}{830}s$$
$$\underline{\tfrac{21}{5}s^2}$$
$$10 \big) -\tfrac{166}{21}s \Big(-\tfrac{166}{210}s$$
$$\underline{-\tfrac{166}{21}s}$$
$$0$$

Since the last two quotient terms are negative, the polynomial $P(s)$ is not Hurwitz.

Example 10.9 *Test whether the polynomial* $s^5 + 3s^3 + 2s$ *is Hurwitz.*

Solution Since the given polynomial contains odd functions only, it is not possible to perform a continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 9s^2 + 2$$
$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 9s^2 + 2 \big) s^5 + 3s^3 + 2s \Big(\tfrac{1}{5}s$$
$$\underline{s^5 + \tfrac{9}{5}s^3 + \tfrac{2}{5}s}$$
$$\tfrac{6}{5}s^3 + \tfrac{8}{5}s \big) 5s^4 + 9s^2 + 2 \Big(\tfrac{25}{6}s$$

$$5s^4 + \frac{20}{3}s^2$$

$$\frac{7}{3}s^2 + 2\,\Big)\,\frac{6}{5}s^3 + \frac{8}{5}s\,\Big(\,\frac{18}{35}s$$

$$\frac{6}{5}s^3 + \frac{36}{35}s$$

$$\frac{20}{35}s\,\Big)\,\frac{7}{3}s^2 + 2\,\Big(\,\frac{49}{12}s$$

$$\frac{7}{3}s^2$$

$$2\,\Big)\,\frac{20}{35}s\,\Big(\,\frac{10}{35}s$$

$$\frac{20}{35}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 10.10 *Test whether the polynomial $P(s)$ is Hurwitz.*

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 3s^2 + 1\,\Big)\,s^5 + s^3 + s\,\Big(\,\frac{1}{5}s$$

$$s^5 + \frac{3}{5}s^3 + \frac{1}{5}s$$

$$\frac{2}{5}s^3 + \frac{4}{5}s\,\Big)\,5s^4 + 3s^2 + 1\,\Big(\,\frac{25}{2}s$$

$$5s^4 + 10s^2$$

$$-7s^2 + 1\,\Big)\,\frac{2}{5}s^3 + \frac{4}{5}s\,\Big(\,-\frac{2}{35}s$$

$$\frac{2}{5}s^3 - \frac{2}{35}s$$

$$\frac{26}{35}s\Big)-7s^2+1\Big(-\frac{245}{26}s$$

$$-7s^2$$

$$1\Big)\frac{26}{35}s\Big(\frac{26}{35}s$$

$$\frac{26}{35}s$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

Example 10.11 *Test the polynomial P(s) of Hurwitz property.*

$$P(s)=s^6+3s^5+8s^4+15s^3+17s^2+12s+4$$

Solution Even part of $P(s)=m(s)=s^6+8s^4+17s^2+4$

Odd part of $P(s)=n(s)=3s^5+15s^3+12s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5+15s^3+12s\Big)s^6+8s^4+17s^2+4\Big(\frac{1}{3}s$$

$$s^6+5s^4+4s^2$$

$$3s^4+13s^2+4\Big)3s^5+15s^3+12s\Big(s$$

$$3s^5+13s^3+4s$$

$$2s^3+8s\Big)3s^4+13s^2+4\Big(\frac{3}{2}s$$

$$3s^4+12s^2$$

$$s^2+4\Big)2s^3+8s\Big(2s$$

$$2s^3+8s$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor (s^2+4).

$$P(s)=s^6+3s^5+8s^4+15s^3+17s^2+12s+4=(s^2+4)(s^4+3s^3+4s^2+3s+1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\dfrac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let $$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$

$$m_2(s) = s^4 + 4s^2 + 1$$

$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$3s^3 + 3s \,\big)\, s^4 + 4s^2 + 1 \,\Big(\, \frac{1}{3}s$$
$$\underline{s^4 + s^2}$$
$$3s^2 + 1)\; 3s^3 + 3s \;(s$$
$$\underline{3s^3 + s}$$
$$2s \,\big)\, 3s^2 + 1 \,\Big(\, \frac{3}{2}s$$
$$\underline{3s^2}$$
$$1)\; 2s \;(2s$$
$$\underline{2s}$$
$$0$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.
Hence, $P(s) = (s^2+4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$ is Hurwitz.

Example 10.12 *Test whether the polynomial* $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + s^4 + 8s^2 + 4$

Odd part of $P(s) = n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$$Q(s) = \frac{n(s)}{m(s)}$$

By continued fraction expansion,

$$2s^6+s^4+8s^2+4\overline{\big)\, s^7+2s^5+4s^3+8s\,}\Big(\frac{1}{2}s$$

$$\underline{s^7+\frac{1}{2}s^5+4s^3+2s}$$

$$\frac{3}{2}s^5 \qquad +6s\overline{\big)\,2s^6+s^4+8s^2+4\,}\Big(\frac{4}{3}s$$

$$\underline{2s^6 \qquad +8s^2}$$

$$s^4 \qquad +4\overline{\big)\,\frac{3}{2}s^5+6s\,}\Big(\frac{3}{2}s$$

$$\underline{\frac{3}{2}s^5+6s}$$

$$0$$

Since the division has terminated abruptly it indicates a common factor s^4+4. The polynomial can be written as

$$P(s)=(s^4+4)(s^3+2s^2+2s+1)$$

If both the factor are Hurwitz, $P(s)$ will be Hurwitz.
In the polynomial (s^4+4), the terms s^3, s^2 and s are missing. Hence, it is not Hurwitz.
Therefore, $P(s)$ is not Hurwitz.

Example 10.13 *Test whether the polynomial* $2s^6+s^5+13s^4+6s^3+56s^2+25s+25$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=2s^6+13s^4+56s^2+25$

Odd part of $P(s)=n(s)=s^5+6s^3+25s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^5+6s^3+25s)2s^6+13s^4+56s^2+25(2s$$

$$\underline{2s^6+12s^4+50s^2}$$

$$s^4+6s^2 \quad +25)s^5+6s^3+25s(s$$

$$\underline{s^5+6s^3+25s}$$

$$0$$

The division has terminated abruptly.

$$P(s)=2s^6+s^5+13s^4+6s^3+56s^2+25s+25=(s^4+6s^2+25)\,(2s^2+s+1)$$

Let $\qquad P_1(s)=s^4+6s^2+25$

Since $P_1(s)$ contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 4s^3 + 12s$$

By continued fraction expansion,

$$
\begin{array}{l}
4s^3+12s\,\Big)\,s^4+6s^2+25\,\Big(\frac{1}{4}s \\
\qquad\qquad\quad \underline{s^4+3s^2\qquad\quad} \\
\qquad\qquad\qquad\quad 3s^2+25\,\Big)\,4s^3+\;\;12s\,\Big(\frac{4}{3}s \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{4s^3+\frac{100}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\quad -\frac{64}{3}s\,\Big)\,3s^2+25\,\Big(-\frac{9}{64}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \underline{3s^2\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 25\,\Big)\,-\frac{64}{3}s\,\Big(-\frac{64}{75}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{-\frac{64}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since two of the quotient terms are negative, $P_1(s)$ is not Hurwitz.
We need not test the other factor $(2s^2 + s + 1)$ for being Hurwitz.
Hence, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\qquad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

$$
\begin{array}{c|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}
$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Example 10.14 *Test whether* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Example 10.15 *Determine whether* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 1 & 3 & \\ s^2 & -1 & 2 & \\ s^1 & 5 & 0 & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 10.16 *Test whether* $P(s) = s^5 + 2s^4 + 4s^3 + 6s^2 + 2s + 5$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	4	2
s^4	2	6	5
s^3	1	–0.5	
s^2	7	5	
s^1	–1.21		
s^0	5		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 10.17 *Test whether the polynomial $P(s) = s^5 + s^3 + s$ is Hurwitz.*

Solution The given polynomial contains odd functions only.

$$P'(s) = 5s^4 + 3s^2 + 1$$

The Routh array is given by,

s^5	1	1	1
s^4	5	3	1
s^3	0.4	0.8	
s^2	–7	1	
s^1	0.86		
s^0	1		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 10.18 *Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.*

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

s^8	1	5	2	3	1
s^7	8	30	8	6	0
s^6	1.25	1	2.25	1	
s^5	23.6	–6.4	–0.4	0	
s^4	1.33	2.27	1		
s^3	–46.6	–18.14	0		
s^2	1.75	1			
s^1	8.49				
s^0	1				

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 10.19 *Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^5 & 1 & 45 & 44 \\ s^4 & 12 & 60 & 48 \\ s^3 & 40 & 40 & \\ s^2 & 48 & 48 & \\ s^1 & 0 & 0 & \\ s^0 & & & \end{array}$$

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.

(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.

(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

$$\begin{array}{c|ccc} s^5 & 1 & 45 & 44 \\ s^4 & 12 & 60 & 48 \\ s^3 & 40 & 40 & \\ s^2 & 48 & 48 & \\ s^1 & 96 & 0 & \\ s^0 & 48 & & \end{array}$$

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Example 10.20 *Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 0 & 0 & 0 & \\ s^2 & & & & \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 4 & 12 & & \\ s^2 & 3 & 25 & & \\ s^1 & -21.3 & & & \\ s^0 & 25 & & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 10.21 *Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$

$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$

$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \frac{7}{2}$.

Example 10.22 *Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e., $$\frac{6-k}{3} > 0$$

$$6-k > 0$$

i.e., $k < 6$ and $k > 0$
Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

10.3 POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.

(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,
Re $F(s) \geq 0$ for Re$(s) \geq 0$

10.3.1 Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

10.3.2 Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. Re $F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s - j\omega_0} + \frac{K_1^*}{s + j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} = \frac{m_1 + n_1}{m_2 + n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s) = \frac{m_1 + n_1}{m_2 + n_2} \frac{m_2 - n_2}{m_2 - n_2} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} + \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$\text{Od } F(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\big|_{s=j\omega} = j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.

The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2 - n_2^2\big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1 m_2 - n_1 n_2\big|_{s=j\omega} = A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

Example 10.23 *Test whether* $F(s) = \dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s = -1$ and zero at $s = -3$ as shown in Fig. 10.1.

Thus, pole and zero are in the left half of the s-plane.

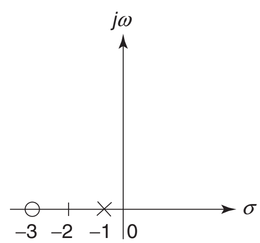


Fig. 10.1

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (3)(1) - (s)(s)\,|_{s=j\omega} = 3 - s^2\,|_{s=j\omega} = 3 + \omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.24

Test whether $F(s) = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14} = \dfrac{(s+5)(s+1)}{(s+7)(s+2)}$

The function $F(s)$ has poles at $s = -7$ and $s = -2$ and zeros at $s = -5$ an $s = -1$ as shown in Fig. 10.2. Thus, all the poles and zeros are in the left half of the s plane.

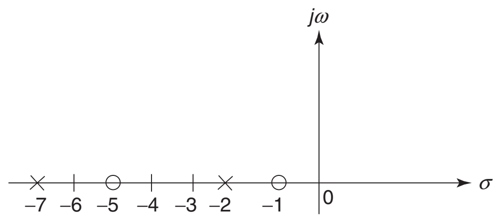


Fig. 10.2

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+5)\,(s^2+14) - (6s)(9s)\,|_{s=j\omega} = s^4 - 35s^2 + 70 \,\,|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.25

Test whether $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)} = \dfrac{s^3 + 8s^2 + 15s}{s^2 + 5s + 4}$

The function $F(s)$ has poles at $s = -1$ and $s = -4$ and zeros at $s = 0$, $s = -3$ and $s = -5$ as shown in Fig. 10.3.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis, hence the residue test is not carried out.

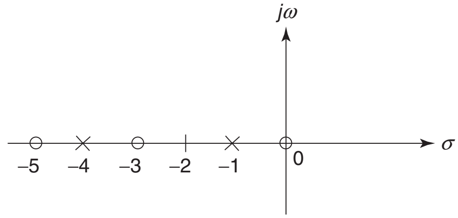


Fig. 10.3

(c) Even part of $N(s) = m_1 = 8s^2$

Odd part of $N(s) = n_1 = s^3 + 15s$

Even part of $D(s) = m_2 = s^2 + 4$

Odd part of $D(s) = n_2 = 5s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (8s^2)(s^2+4) - (s^3+15s)(5s)\,|_{s=j\omega} = 3s^4 - 43s^2\,|_{s=j\omega} = 3\omega^4 + 43\omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.26

Test whether $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 10.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

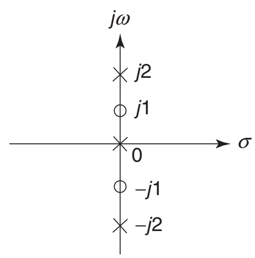


Fig. 10.4

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1 = s\,F(s)\,|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)\,|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (s^2+1)(0) - (0)(s^3+4s)\big|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.27 *Test whether* $F(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \dfrac{2s^3 + 2s^2 + 3s + 2}{(s + j1)(s - j1)}$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 2s^2 + 2$

Odd part of $N(s) = n(s) = 2s^3 + 3s$

By continued fraction expansion,

$$\begin{array}{l}
2s^2+2\,\big)\,2s^3+3s\,\big(\,s \\
\qquad\quad\;\; \underline{2s^3+2s} \\
\qquad\qquad\qquad s\,\big)\,2s^2+2\,\big(\,2s \\
\qquad\qquad\qquad\quad\;\; \underline{2s^2} \\
\qquad\qquad\qquad\qquad\quad 2\,\big)\,s\,\big(\,\frac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s = -j1$ and $s = j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$\begin{array}{r|l}
s^2+1 & 2s^3+2s^2+3s+2 \quad (2s+2 \\
 & \underline{2s^3 \qquad +2s} \\
 & \quad 2s^2+\ s+2 \\
 & \quad \underline{2s^2 \qquad +2} \\
 & \qquad\qquad s
\end{array}$$

$$F(s) = 2s + 2 + \frac{s}{s^2+1}$$

By partial-fraction expansion,

$$F(s) = 2s + 2 + \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1}$$

$$K_1 = (s+j1)F(s)|_{s=-j1} = \frac{-j1}{-j1-j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2 + 2$

Odd part of $N(s) = n_1 = 2s^3 + 3s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 |_{s=j\omega} = (2s^2+2)(s^2+1) - (2s^3+3s)(0)|_{s=j\omega} = 2s^4 + 4s^2 + 2 \ |_{s=j\omega} = 2(\omega^4 - 2\omega^2 + 1)$$
$$= 2(\omega^2 - 1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.28 *Test whether* $F(s) = \dfrac{s^3+6s^2+7s+3}{s^2+2s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+6s^2+7s+3}{s^2+2s+1} = \dfrac{s^3+6s^2+7s+3}{(s+1)(s+1)}$

Since a numerator polynomial cannot be easily factorized, we will test whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 6s^2 + 3$

Odd part of $N(s) = n(s) = s^3 + 7s$

By continued fraction expansion,

$$
\begin{array}{l}
6s^2+3\,\big)\,s^3+7s\,\big(\,\frac{1}{6}s \\
\qquad\qquad \underline{s^3+0.5s} \\
\qquad\qquad\qquad 6.5s\,\big)\,6s^2+3\,\big(\,0.92s \\
\qquad\qquad\qquad\qquad \underline{6s^2\qquad} \\
\qquad\qquad\qquad\qquad\qquad 3\,\big)\,6.5s\,\big(\,2.17s \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{6.5s} \\
\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that the zeros are in the left half of the s plane.

The function $F(s)$ has a double pole at $s = -1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 6s^2 + 3$

Odd part of $N(s) = n_1 = s^3 + 7s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 2s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (6s^2+3)(s^2+1) - (s^3+7s)(2s)\,|_{s=j\omega} = 4s^4 - 5s^2 + 3\,|_{s=j\omega} = 4\omega^4 + 5\omega^2 + 3$$

$A(\omega^2)$ is positive for all $\omega \ge 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 10.29 *Test whether* $F(s) = \dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s) = \frac{N(s)}{D(s)} = \frac{s^2+s+6}{s^2+s+1} = \frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s = -\frac{1}{2} \pm j\frac{\sqrt{23}}{2}$ and poles at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 6$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s)\,|_{s=j\omega} = s^4 + 6s^2 + 6\,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$
This condition is not satisfied.
Hence, the function $F(s)$ is not positve real.

Example 10.30 *Test whether* $F(s) = \dfrac{s^2+4}{s^3+3s^2+3s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+4}{s^3+3s^2+3s+1} = \dfrac{(s+j2)(s-j2)}{(s+1)^3}$

The function $F(s)$ has two zeros at $s = \pm j2$ and three poles at $s = -1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 4$
Odd part of $N(s) = n_1 = 0$
Even part of $D(s) = m_2 = 3s^2 + 1$
Odd part of $D(s) = n_2 = s^3 + 3s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+4)(3s^2+1) - (0)(s^3+3s)\,|_{s=j\omega} = 3s^4 + 13s^2 + 4\,|_{s=j\omega} = 3\omega^4 - 13\omega^2 + 4$$

For $\omega = 1$, $A(\omega)^2 = 3 - 13 + 4 = -6$
This condition is not satisfied.
Hence, the function $F(s)$ is not positive real.

Example 10.31 *Test whether* $F(s) = \dfrac{s^3+5s}{s^4+2s^2+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+5s}{s^4+2s^2+1} = \dfrac{s(s^2+5)}{(s^2+1)^2} = \dfrac{s(s+j\sqrt{5})(s-j\sqrt{5})}{(s\pm j1)(s\pm j1)}$

The function $F(s)$ has zeros at $s = 0$, $s = \pm j\sqrt{5}$ and two poles at $s = j1$ and two poles at $s = -j1$.

Thus, poles on the $j\omega$ axis are not simple.
Hence, the function $F(s)$ not positive real.

Example 10.32 *Test whether* $F(s) = \dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$ *is positive real function.*

Solution

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$$

Here, it is easier to prove that $N(s)$ and $D(s)$ are Hurwitz.

By Routh array,

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 2 \\ s^3 & 3 & 1 & \\ s^2 & \frac{2}{3} & 2 & \\ s^1 & -8 & & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, $N(s)$ is not Hurwitz. Thus, all the zeros are not in the left half of the s plane. The remaining two tests need not be carried out.
Hence, the function $F(s)$ is not positive real.

10.4 ELEMENTARY SYNTHESIS CONCEPTS

We know that impedances and admittances of passive networks are positive real functions. Hence, addition of impedances of the two passive networks gives a function which is also a positive real function. Thus, $Z(s) = Z_1(s) + Z_2(s)$ is a positive real function, if $Z_1(s)$ and $Z_2(s)$ are positive real functions. Similarly, $Y(s) = Y_1(s) + Y_2(s)$ is a positive real function, if $Y_1(s)$ and $Y_2(s)$ are positive real functions. There is a special terminology for synthesis procedure. We have,

$$Z(s) = Z_1(s) + Z_2(s)$$

$$Z_2(s) = Z(s) - Z_1(s)$$

Here, $Z_1(s)$ is said to have been removed from $Z(s)$ in forming the new function $Z_2(s)$ as shown in Fig. 10.5. If the removed network is associated with the pole or zero of the original network impedance then that pole or zero is also said to have been removed.
There are four important removal operations.

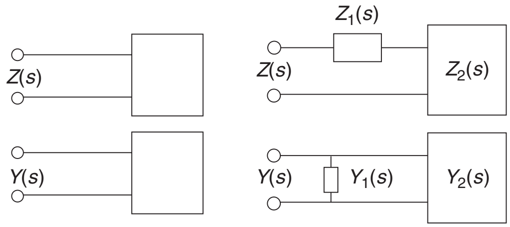


Fig. 10.5 *Network interpretation of the removal of impedance and admittance*

10.4.1 Removal of a Pole at Infinity

Consider an impedance function $Z(s)$ having a pole at infinity which means that the numerator polynomial is one degree greater than the degree of the denominator polynomial.

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Hs + \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

where $\quad H = \dfrac{a_{n+1}}{b_n}$

Let $\quad Z_1(s) = Hs$

and $\quad Z_2(s) = \dfrac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Z(s) - Hs$

$Z_1(s) = Hs$ represents impedance of an inductor of value H. Hence, the removal of a pole at infinity corresponds to the removal of an inductor from the network of Fig. 10.6(a).

If the given function is an admittance function $Y(s)$, then $Y_1(s) = Hs$ represents the admittance of a capacitor $Y_C(s) = Cs$. The network for $Y_1(s)$ is a capacitor of value $C = H$ as shown in Fig. 10.6(b).

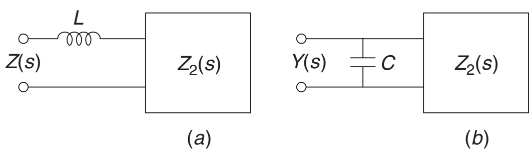


Fig. 10.6 *Network interpretation of the removal of a pole at infinity*

10.4.2 Removal of a Pole at Origin

If $Z(s)$ has a pole at the origin then it may be written as

$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_m s^m} = \frac{K_0}{s} + \frac{d_1 + d_2 s + \ldots + d_n s^{n-1}}{b_1 + b_2 s + \ldots + b_m s^{m-1}} = Z_1(s) + Z_2(s)$$

where $\quad K_0 = \dfrac{a_0}{b_1}$

$Z_1(s) = \dfrac{K_0}{s}$ represents the impedance of a capacitor of value $\dfrac{1}{K_0}$.

If the given function is an admittance function $Y(s)$ then removal of $Y_1(s) = \dfrac{K_0}{s}$ corresponds to an inductor of value $\dfrac{1}{K_0}$.

Thus, removal of a pole from the impedance function $Z(s)$ at the origin corresponds to the removal of a capacitor, and from admittance function $Y(s)$ corresponds to removal of an inductor as shown in Fig. 10.7.

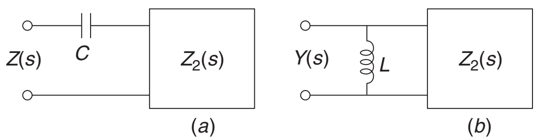


Fig. 10.7 *Network interpretation of the removal of a pole at origin*

10.4.3 Removal of Conjugate Imaginary Poles

If $Z(s)$ contains poles on the imaginary axis, i.e., at $s = \pm j\omega_1$ then $Z(s)$ will have factors $(s + j\omega_1)(s - j\omega_1) = s^2 + \omega_1^2$ in the denominator polynomial

$$Z(s) = \frac{p(s)}{(s^2 + \omega_1^2)\, q_1(s)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s + j\omega_1} + \frac{K_1^*}{s - j\omega_1} + Z_2(s)$$

For a positive real function, $j\omega$ axis poles must themselves be conjugate and must have equal, positive and real residues.

$$K_1 = K_1^*$$

Hence,
$$Z(s) = \frac{2K_1 s}{s^2 + \omega_1^2} + Z_2(s)$$

Thus,
$$Z_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}} = \frac{1}{Y_a + Y_b}$$

where $Y_a = \dfrac{s}{2K_1}$ is the admittance of a capacitor of value $C = \dfrac{1}{2K_1}$

and $Y_b = \dfrac{\omega_1^2}{2K_1 s}$ is the admittance of an inductor of value $L = \dfrac{2K_1}{\omega_1^2}$

If the given function is an admittance function $Y(s)$ then

$$Y_1(s) = \frac{2K_1 s}{s^2+\omega_1^2} = \frac{1}{Z_a+Z_b} = \frac{1}{\dfrac{s}{2K_1}+\dfrac{\omega_1^2}{2K_1 s}}$$

where $Z_a = \dfrac{s}{2K_1}$ is the impedance of an inductor of value $L = \dfrac{1}{2K_1}$ and $Z_b = \dfrac{\omega_1^2}{2K_1 s}$ is the impedance of a capacitor of value $C = \dfrac{2K_1}{\omega_1^2}$.

Thus, removal of conjugate imaginary poles from impedance function $Z(s)$ corresponds to the removal of the parallel combination of $L - C$ and from admittance function $Y(s)$ corresponds to removal of series combination of $L - C$ as shown in Fig. 10.8.

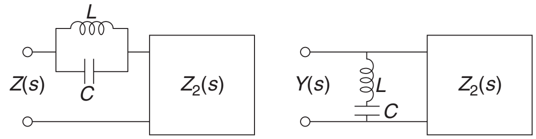


Fig. 10.8 *Network interpretation of the removal of conjugate imaginary poles*

10.4.4 Removal of a Constant

If a real number R_1 is subtracted from $Z(s)$ such that

$$Z_2(s) = Z(s) - R_1$$
$$Z(s) = R_1 + Z_2(s)$$

then R_1 represents a resistor.

If the given function is an admittance function $Y(s)$, then removal of $Y_1(s) = R_1$ represents a conductance of value R_1.

Thus, removal of a constant from impedance function $Z(s)$ corresponds to the removal of a resistance, and from admittance function $Y(s)$ corresponds to removal of a conductance.

Example 10.33 *Synthesize the impedance function* $Z(s) = \dfrac{s^3+4s}{s^2+2}$.

Solution

By long division of $Z(s)$,

$$\begin{array}{r|l} s^2+2 \,\big)\; s^3+4s & \big(\, s \\ s^3+2s & \\ \hline 2s & \end{array}$$

$$Z(s) = s + \frac{2s}{s^2+2} = Z_1(s) + Z_2(s)$$

$Z_1(s) = s$ represents impedance of an inductor of value 1 H.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s^2+2}{2s} = \frac{s^2}{2s} + \frac{2}{2s} = \frac{1}{2}s + \frac{1}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = \dfrac{1}{2}s$ represents the admittance of a capacitor of value $\dfrac{1}{2}$ F.

$Y_4(s) = \dfrac{1}{s}$ represents the admittance of an inductor of value 1 H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.9.

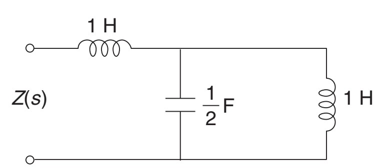


Fig. 10.9

Example 10.34 *Realise the network having impedance function*

$$Z(s)=\frac{s^2+2s+10}{s(s+5)}$$

Solution

By long division of $Z(s)$,

$$\begin{array}{r|l} s^2+5s\,\big) & s^2+2s+10\,\big(\,\frac{2}{s} \\ & \underline{\quad 2s+10 \quad} \\ & s^2 \end{array}$$

$$Z(s)=\frac{2}{s}+\frac{s^2}{s^2+5s}=\frac{2}{s}+\frac{s}{s+5}=Z_1(s)+Z_2(s)$$

$Z_1(s)=\dfrac{2}{s}$ represents the impedance of capacitor of value $\dfrac{1}{2}$F.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s+5}{s}=1+\frac{5}{s}=Y_3(s)+Y_4(s)$$

$Y_3(s)$ = 1 represents the admittance of a resistor of value 1 Ω.

$Y_4(s)=\dfrac{5}{s}$ represents the admittance of an inductor of value $\dfrac{1}{5}$H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.10.

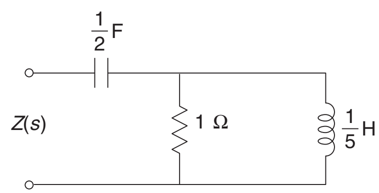


Fig. 10.10

Example 10.35 *Realise the network having impedance function* $Z(s)=\dfrac{6s^3+5s^2+6s+4}{2s^3+2s}$.

Solution By long division of $Z(s)$,

$$\begin{array}{r|l} 2s^3+2s\,\big) & 6s^3+5s^2+6s+4\,\big(\,3 \\ & \underline{6s^3 \qquad\quad +6s \quad} \\ & \qquad 5s^2 \qquad +4 \end{array}$$

$$Z(s)=3+\frac{5s^2+4}{2s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)$ = 3 represents the impedance of a resistor of value 3 Ω.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{2s^3+2s}{5s^2+4}$$

By long division of $Y_2(s)$,

$$5s^2+4\overline{\smash{)}\,2s^3+2s\,}\left(\frac{2}{5}s\right.$$

$$\frac{2s^3+\frac{8}{5}s}{\frac{2}{5}s}$$

$$Y_2(s)=\frac{2}{5}s+\frac{\frac{2}{5}s}{5s^2+4}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{2}{5}s$ represents the admittance of a capacitor of value $\frac{2}{5}$ F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{5s^2+4}{\frac{2}{5}s}=\frac{25s^2+20}{2s}=\frac{25}{2}s+\frac{10}{s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{25}{2}s$ represents the impedance of an inductor of value $\frac{25}{2}$ H.

$Z_6(s)=\frac{10}{s}$ represents the impedance of a capacitor of value $\frac{1}{10}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.11.

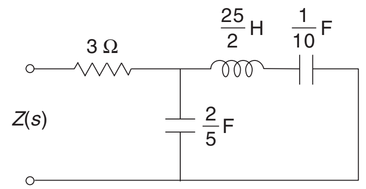


Fig. 10.11

Example 10.36 *Realise the network having impedance function*

$$Z(s)=\frac{s^4+10s^2+7}{s^3+2s}$$

Solution By long division of $Z(s)$,

$$s^3+2s\overline{\smash{)}\,s^4+10s^2+7\,}\left(s\right.$$

$$\frac{s^4+2s^2}{8s^2+7}$$

$$Z(s)=s+\frac{8s^2+7}{s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents the impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^3+2s}{8s^2+7}$$

By long division of $Y_2(s)$,

$$
\begin{array}{r|l}
8s^2+7 & s^3+2s\,\Big(\frac{1}{8}s \\
 & \underline{s^3+\frac{7}{8}s} \\
 & \quad\frac{9}{8}s
\end{array}
$$

$$Y_2(s)=\frac{1}{8}s+\frac{\frac{9}{8}s}{8s^2+7}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{1}{8}s$ represents the admittance of a capacitor of value $\frac{1}{8}$ F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{8s^2+7}{\frac{9}{8}s}=\frac{64}{9}s+\frac{56}{9s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{64}{9}s$ represents the impedance of an inductor of value $\frac{64}{9}$ H.

$Z_6(s)=\frac{56}{9s}$ represents the impedance of a capacitor of value $\frac{9}{56}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.12.

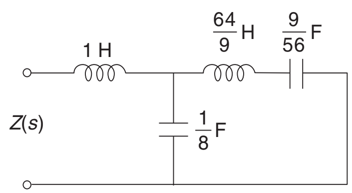


Fig. 10.12

Example 10.37 *Realise the network having admittance function* $Y(s)=\dfrac{4s^2+6s}{s+1}$.

Solution By long division of $Y(s)$,

$$
\begin{array}{r|l}
s+1 & 4s^2+6s\,\Big(4s \\
 & \underline{4s^2+4s} \\
 & \quad 2s
\end{array}
$$

$$Y(s)=4s+\frac{2s}{s+1}=Y_1(s)+Y_2(s)$$

$Y_1(s)=4s$ represents the admittance of a capacitor of value 4 F.

$$Z_2(s)=\frac{1}{Y_2(s)}=\frac{s+1}{2s}=\frac{1}{2}+\frac{1}{2s}=Z_3(s)+Z_4(s)$$

$Z_3(s) = \dfrac{1}{2}$ represents the impedance of a resistor of value $\dfrac{1}{2}\,\Omega$.

$Z_4(s) = \dfrac{1}{2s}$ represents the impedance of a capacitor of value 2 F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.13.

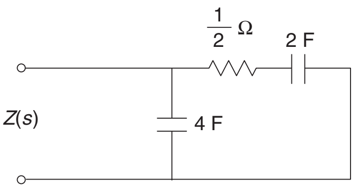


Fig. 10.13

Example 10.38 *Realise the admittance function* $Y(s) = \dfrac{3+5s}{4+2s}$.

Solution By long division,

$$
\begin{array}{r|l}
4+2s & 3+5s \left(\dfrac{3}{4}\right. \\
 & \underline{3+\dfrac{3}{2}s} \\
 & \dfrac{7}{2}s
\end{array}
$$

$$Y(s) = \frac{3}{4} + \frac{\frac{7}{2}s}{4+2s} = Y_1(s) + Y_2(s)$$

$Y_1(s) = \dfrac{3}{4}$ represents the admittance of a resistor of value $\dfrac{4}{3}\,\Omega$.

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{4+2s}{\frac{7}{2}s} = \frac{8+4s}{7s} = \frac{8}{7s} + \frac{4}{7} = Z_3(s) + Z_4(s)$$

$Z_3(s) = \dfrac{8}{7s}$ represents the impedance of a capacitor of value $\dfrac{7}{8}$ F.

$Z_4(s) = \dfrac{4}{7}$ represents the impedance of a resistor of value $\dfrac{4}{7}\,\Omega$.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.14.

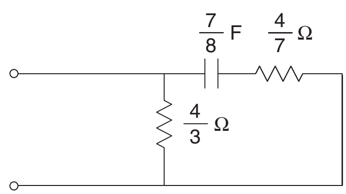


Fig. 10.14

10.5 REALISATION OF *LC* FUNCTIONS

LC driving point immitance functions have the following properties.

1. It is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$-axis.
3. The poles and zeros interlace on the $j\omega$-axis.
4. There must be either a zero or a pole at the origin and infinity.
5. The difference between any two successive powers of numerator and denominator polynomials is at most two. There cannot be any missing terms.

6. The highest powers of numerator and denominator polynomials must differ by unity; the lowest powers also differ by unity.

There are a number of methods of realising an LC function. But we will study only four basic forms—Foster I, Foster II, Cauer I and Cauer II forms. The Foster forms are obtained by partial-fraction expansion of $F(s)$, and the Cauer forms are obtained by continued fraction expansion of $F(s)$.

10.5.1 Foster Realisation

Consider a general LC function $F(s)$ given by

$$F(s)=\frac{H(s^2+\omega_1^2)(s^2+\omega_3^2)...}{s(s^2+\omega_2^2)(s^2+\omega_4^2)...}$$

where $0\le\omega_1^2<\omega_2^2<\omega_3^2\dots$ and H is positive.

By partial-fraction expansion of $F(s)$,

$$F(s)=\frac{K_0}{s}+\frac{K_2}{s+j\omega_2}+\frac{K_2}{s-j\omega_2}+...+K_\infty s$$

Combining terms with conjugate poles,

$$F(s)=\frac{K_0}{s}+\frac{2K_2s}{s+\omega_2^2}...+K_\infty s$$

where K_0, K_i and K_∞ are the residues of $F(s)$ at the origin, at $j\omega_i$ and at infinity respectively. These residues are given by

$$K_0=sF(s)\big|_{s=0}$$

$$K_i=\left.\frac{\left(s^2+\omega_i^2\right)F(s)}{2s}\right|_{s^2=-\omega_i^2}$$

$$K_\infty=\left.\frac{F(s)}{s}\right|_{s\to\infty}$$

Foster I Form If $F(s)$ represents an impedance function, it gives a series connection of impedances.

$$F(s)=Z(s)=\frac{K_0}{s}+\frac{2K_2s}{s^2+\omega_2^2}\dots+K_\infty s=Z_1(s)+Z_2(s)+\dots+Z_n(s)$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_0}$ farad.

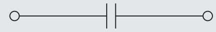
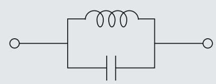
The last term $K_\infty s$ represents the impedance of an inductor of K_∞ henry.

The remaining terms, i.e., $\frac{2K_is}{s^2+\omega_i^2}$ represent the impedance of a parallel combination of capacitor C_i and inductor L_i. For parallel combination of L_i and C_i,

$$Z(s)=\frac{1}{C_is+\frac{1}{L_is}}=\frac{\left(\frac{1}{C_i}\right)s}{s^2+\frac{1}{L_iC_i}}=\frac{2K_is}{s^2+\omega_i^2}$$

$$C_i = \frac{1}{2K_i} \quad \text{and} \quad L_i = \frac{2K_i}{\omega_i^2}$$

Table 10.1 *Realisation of Foster-I form of LC network*

Impedance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{C_0 s}$	 $C_0 = \dfrac{1}{K_0}$
$\dfrac{2K_i s}{s^2+\omega_i^2} = \dfrac{\left(\dfrac{1}{C_i}\right)s}{s^2+\dfrac{1}{L_i C_i}}$	$L_i = \dfrac{2K_i}{\omega_i^2}$  $C_i = \dfrac{1}{2K_i}$
$K_\infty s = Ls$	 $L_\infty = K_\infty$

The network corresponding to Foster I form is shown in Fig. 10.15.

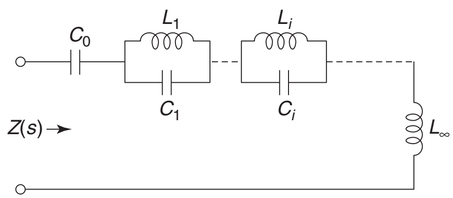


Fig. 10.15 *Foster-I form of LC network*

If $Z(s)$ has no pole at the origin then capacitor C_0 is not present in the network. Similarly, if there is no pole at ∞, inductor L_∞ is not present in the network.

Foster II Form If $F(s)$ represents an admittance function, it gives the parallel combination of admittances.

$$F(s) = Y(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} + \ldots + K_\infty s = Y_1(s) + Y_2(s) + \ldots Y_n(s)$$

The first term $\dfrac{K_0}{s}$ represents the admittance of an inductor of $\dfrac{1}{K_0}$ henry.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farad.

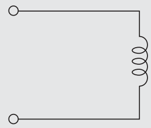
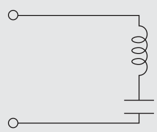
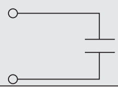
The remaining terms, i.e., $\dfrac{2K_i s}{s^2+\omega_i^2}$ represent the admittance of a series combination of an inductor L_i and a capacitor C_i.

For series combination of L_i and C_i,

$$Y(s) = \frac{1}{L_i s + \dfrac{1}{C_i s}} = \frac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}} = \frac{2K_i s}{s^2 + \omega_i^2}$$

$$L_i = \frac{1}{2K_i} \quad \text{and} \quad C_i = \frac{2K_i}{\omega_i^2}$$

Table 10.2 *Realisation of Foster-II form of LC network*

Admittance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{L_0 s}$	 $L_0 = \dfrac{1}{K_0}$
$\dfrac{2K_i s}{s^2 + \omega_i^2} = \dfrac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}}$	 $L_i = \dfrac{1}{2K_i}$, $C_i = \dfrac{2K_i}{\omega_i^2}$
$K_\infty s = Cs$	 $C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 10.16.

If $Y(s)$ has no pole at the origin then inductor L_0 is not present. Similarly, if there is no pole at infinity, capacitor C_∞ is not present.

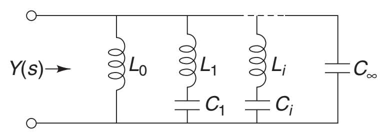


Fig. 10.16 *Foster-II form of LC network*

10.5.2 Cauer Realisation or Ladder Realisation

Cauer I Form Since the numerator and denominator polynomials of an *LC* function always differ in degrees by unity, there is always a zero or a pole at $s = \infty$. The Cauer I Form is obtained by successive removal of a pole or a zero at infinity from the function.

Consider an impedance function $Z(s)$ having a pole at infinity.

By removing the pole at infinity, we get

$$Z_2(s) = Z(s) - L_1 s$$

Now, $Z_2(s)$ has a zero at $s = \infty$. If we invert $Z_2(s)$, $Y_2(s)$ will have a pole at $s = \infty$.

By removing this pole,

$$Y_3(s) = Y_2(s) - C_2 s$$

Now $Y_3(s)$ has a zero at $s = \infty$, which we can invert and remove. This process continues until the remainder is zero. Each time we remove a pole, we remove an inductor or a capacitor depending on whether the function is an impedance or an admittance. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s)=L_1s+\cfrac{1}{C_2s+\cfrac{1}{L_3s+\cfrac{1}{C_4s+\dots}}}$$

Thus, the final structure is a ladder network whose series arms are inductors and shunt arms are capacitors. The Cauer I network is shown in Fig. 10.17.

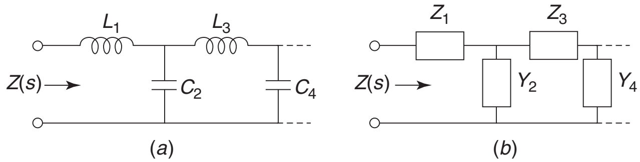


Fig. 10.17 *Cauer I form of LC network*

If the impedance function has zero at infinity, i.e., if degree of numerator is less than that of its denominator by unity, the function is first inverted and continued fraction expansion proceeds as usual. In this case, the first element is a capacitor as shown in Fig. 10.18.

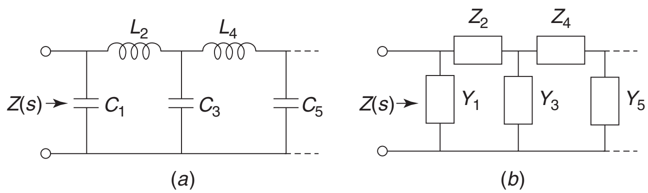


Fig. 10.18 *Cauer-I form of LC network*

Cauer II Form Since the lowest degrees of numerator and denominator polynomials of *LC* function must differ by unity, there is always a zero or a pole at $s = 0$. The Cauer II form is obtained by successive removal of a pole or a zero at $s = 0$ from the function.

In this method, continued fraction expansion of $Z(s)$ is carried out in terms of poles at the origin by removal of the pole at the origin, inverting the resultant function to create a pole at the origin which is removed and this process is continued until the remainder is zero. To do this, we arrange both numerator and denominator polynomials in ascending order and divide the lowest power of the denominator into the lowest power of the numerator. Then we invert the remainder and divide again. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s)=\frac{1}{C_1s}+\cfrac{1}{\cfrac{1}{L_2s}+\cfrac{1}{\cfrac{1}{C_3s}+\cfrac{1}{\cfrac{1}{L_4s}+\dots}}}$$

Thus, the final structure is a ladder network whose first element is a series capacitor and second element is a shunt inductor as shown in Fig. 10.19.

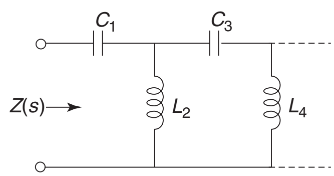


Fig. 10.19 *Cauer II form of LC network*

If the impedance function has a zero at the origin then the first element is a shunt inductor and the second element is a series capacitor as shown in Fig. 10.20.

Thus, the LC function $F(s)$ can be realised in four different forms. All these forms have the same number of elements and the number is equal to the number of poles and zeros of $F(s)$ including any at infinity.

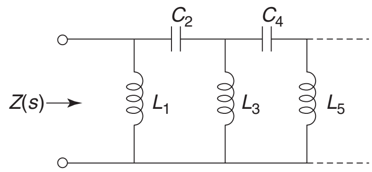


Fig. 10.20 *Cauer–II form of LC network*

Example 10.39 *State whether the following functions are driving point immittance of LC networks are not:*

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$ (b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

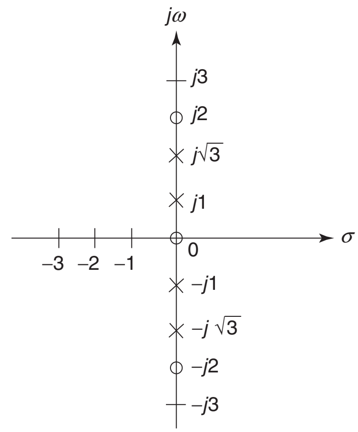


Fig. 10.21

Solution

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$

The function $Z(s)$ has poles at $s = \pm j1$ and $s = \pm j\sqrt{3}$ and zeros at $s = 0$ and $s = \pm j2$ as shown in Fig. 10.21. Since the poles and zeros do not interlace on the $j\omega$ axis, the function $Z(s)$ is not an LC impedance function.

(b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

The function $Z(s)$ has poles at $s = 0$ and $s = \pm j\sqrt{2}$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 10.22. The poles and zeros are simple and lie on the $j\omega$-axis. The poles and zeros interlace on the $j\omega$ axis. Hence, the function $Z(s)$ is an LC impedance function.

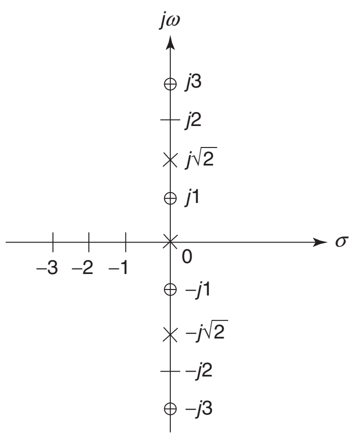


Fig. 10.22

Example 10.40 *Realise the Foster and Cauer forms of the following impedance function*

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = \pm j2$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 10.23.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the $j\omega$ axis. Poles and zeros are interlaced. Hence, the given function is an LC function.

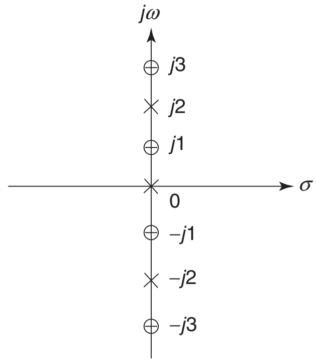


Fig. 10.23

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{4s^4+40s^2+36}{s^3+4s}$$

$$\begin{array}{r|l} s^3+4s) \; 4s^4+40s^2+36 & (4s \\ 4s^4+16s^2 & \\ \hline 24s^2+36 & \end{array}$$

$$Z(s) = 4s + \frac{24s^2+36}{s^3+4s} = 4s + \frac{24s^2+36}{s(s^2+4)}$$

By partial-fraction expansion,

$$Z(s) = 4s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = 4s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+4}$$

where
$$K_0 = sZ(s)\big|_{s=0} = \frac{4(1)(9)}{4} = 9$$

$$K_1 = \frac{(s^2+4)Z(s)}{2s}\bigg|_{s^2=-4} = \frac{4(-4+1)(-4+9)}{2(-4)} = \frac{15}{2}$$

$$Z(s) = 4s + \frac{9}{s} + \frac{15s}{s^2+4}$$

The first term represents the impedance of an inductor of 4 H. The second term represents the impedance of a capacitor of $\frac{1}{9}$ F. The third term represents the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,

$$C = \frac{1}{15}\text{F}$$

$$L = \frac{15}{4}\text{H}$$

The network is shown in Fig. 10.24.

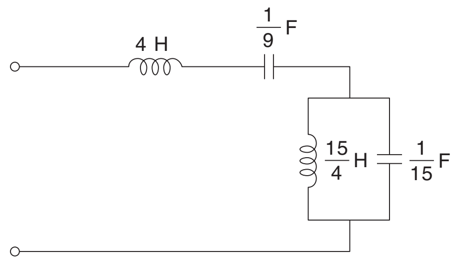


Fig. 10.24

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+4)}{8(-1+9)} = \frac{3}{64}$$

$$K_2 = \frac{(s^2+9)}{2s}Y(s)\bigg|_{s^2=-9} = \frac{(-9+4)}{8(-9+1)} = \frac{5}{64}$$

$$Y(s) = \frac{\left(\frac{3}{32}\right)s}{s^2+1} + \frac{\left(\frac{5}{32}\right)s}{s^2+9}$$

These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{32}{3}\text{H} \qquad C_1 = \frac{3}{32}\text{F}$$

$$L_2 = \frac{32}{5}\text{H} \qquad C_2 = \frac{5}{288}\text{F}$$

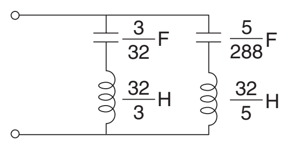


Fig. 10.25

The network is shown in Fig. 10.25.

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{4s^4 + 40s^2 + 36}{s^3 + 4s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$
\begin{array}{l}
s^3+4s\,\big)\,4s^4+40s^2+36\,\big(\,4s \leftarrow Z \\
\qquad\quad \underline{4s^4+16s^2\qquad} \\
\qquad\qquad 24s^2+36\,\big)\,s^3+4s\,\Big(\,\dfrac{1}{24}s \leftarrow Y \\
\qquad\qquad\qquad\qquad \underline{s^3+\dfrac{3}{2}s\qquad} \\
\qquad\qquad\qquad\qquad\quad \dfrac{5}{2}s\,\Big)\,24s^2+36\,\Big(\,\dfrac{48}{5}s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{24s^2\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad 36\,\Big)\,\dfrac{5}{2}s\,\Big(\,\dfrac{5}{72}s \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \underline{\dfrac{5}{2}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; 0
\end{array}
$$

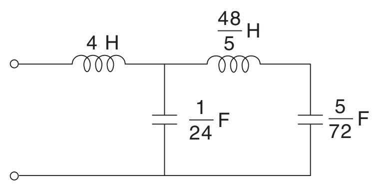


Fig. 10.26

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.26.

Cauer II Form The Cauer II form is obtained from continued fraction expansion about pole at origin.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

The function $Z(s)$ has a pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s)=\frac{36+40s^2+4s^4}{4s+s^3}$$

By continued fraction expansion,

$$
\begin{array}{l}
4s+s^3\,\big)\,36+40s^2+4s^4\,\Big(\,\dfrac{9}{s} \leftarrow Z \\
\qquad\quad \underline{36+\;\;9s^2\qquad\qquad} \\
\qquad\qquad 31s^2+4s^4\,\big)\,4s+s^3\,\Big(\,\dfrac{4}{31s} \leftarrow Y \\
\qquad\qquad\qquad\qquad \underline{4s+\dfrac{16}{31}s^3} \\
\qquad\qquad\qquad\qquad\quad \dfrac{15}{31}s^3\,\Big)\,31s^2+4s^4\,\Big(\,\dfrac{961}{15s} \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{31s^2\qquad\qquad}
\end{array}
$$

$$4s^4 \overline{\big)\, \frac{15}{31}s^3 \,\big(}\, \frac{15}{124s} \leftarrow Y$$
$$\frac{15}{31}s^3$$
$$\overline{\quad 0 \quad}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.27.

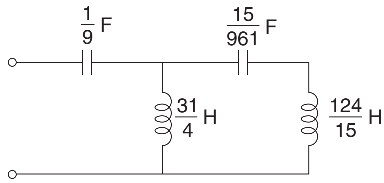


Fig. 10.27

Example 10.41 *Realise Foster forms of the LC impedance function*

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

Solution

Foster I Form The Foster I form is obtained by continued-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} = \frac{s^4+4s^2+3}{s^3+2s}$$

$$s^3+2s \overline{\big)\, s^4+4s^2+3 \,\big(}\, s$$
$$\frac{s^4+2s^2}{\quad 2s^2+3 \quad}$$

$$Z(s) = s + \frac{2s^2+3}{s^3+2s} = s + \frac{2s^2+3}{s(s^2+2)}$$

By partial-fraction expansion,

$$Z(s) = s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+2}$$

where
$$K_0 = sZ(s)|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$
$$K_1 = \frac{(s^2+2)}{2s} Z(s)\Bigg|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)} = \frac{1}{4}$$

$$Z(s) = s + \frac{\left(\frac{3}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)s}{s^2+2}$$

The first term represents the impedance of an inductor of 1 H. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of a parallel *LC* network.

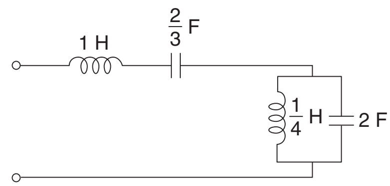
For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = 2\text{F}$$

$$L = \frac{1}{4}\text{H}$$



The network is shown in Fig. 10.28.

Fig. 10.28

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j\sqrt{3}} + \frac{K_2^*}{s-j\sqrt{3}} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+3}$$

where

$$K_1 = \left.\frac{(s^2+1)}{2s}Y(s)\right|_{s^2=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = \left.\frac{(s^2+3)}{2s}Y(s)\right|_{s^2=-3} = \frac{-3+2}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = \frac{\left(\frac{1}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

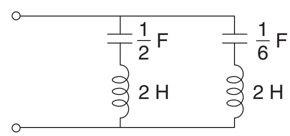
These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = 2\text{H}, \quad C_1 = \frac{1}{2}\text{F}$$

$$L_2 = 2\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$



The network is shown in Fig. 10.29.

Fig. 10.29

Example 10.42 *Realise Foster forms of the following LC impedance function:*

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s)=\frac{K_0}{s}+\frac{K_1}{s+j\sqrt{2}}+\frac{K_1^*}{s-j\sqrt{2}}+\frac{K_2}{s+j2}+\frac{K_2^*}{s-j2}=\frac{K_0}{s}+\frac{2K_1 s}{s^2+2}+\frac{2K_2 s}{s^2+4}$$

where $$K_0=sZ(s)\big|_{s=0}=\frac{(1)(3)}{(2)(4)}=\frac{3}{8}$$

$$K_1=\frac{(s^2+2)}{2s}Z(s)\Bigg|_{s^2=-2}=\frac{(-2+1)(-2+3)}{2(-2)(-2+4)}=\frac{1}{8}$$

$$K_2=\frac{(s^2+4)}{2s}Z(s)\Bigg|_{s^2=-4}=\frac{(-4+1)(-4+3)}{2(-4)(-4+2)}=\frac{3}{16}$$

$$Z(s)=\frac{\frac{3}{8}}{s}+\frac{\left(\frac{1}{4}\right)s}{s^2+2}+\frac{\left(\frac{3}{8}\right)s}{s^2+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The other two terms represent the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s)=\frac{\left(\frac{1}{C}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,

$$C_1=4\text{F},\quad L_1=\frac{1}{8}\text{H}$$

$$C_2=\frac{8}{3}\text{F},\quad L_2=\frac{3}{32}\text{H}$$

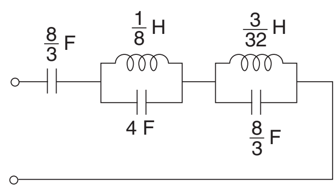


Fig. 10.30

The network is shown in Fig. 10.30.

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s)=\frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}=\frac{s^5+6s^3+8s}{s^4+4s^2+3}$$

Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$\begin{array}{r} s^4+4s^2+3)\,s^5+6s^3+8s\,(s \\ \underline{s^5+4s^3+3s} \\ 2s^3+5s \end{array}$$

$$Y(s)=s+\frac{2s^3+5s}{s^4+4s^2+3}=s+\frac{2s^3+5s}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s)=s+\frac{K_1}{s+j1}+\frac{K_1^*}{s-j1}+\frac{K_2}{s+j\sqrt{3}}+\frac{K_2^*}{s-j\sqrt{3}}=s+\frac{2K_1 s}{s^2+1}+\frac{2K_2 s}{s^2+3}$$

where
$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)(-1+4)}{2(-1+3)} = \frac{3}{4}$$
$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{(-3+2)(-3+4)}{2(-3+1)} = \frac{1}{4}$$
$$Y(s) = s + \frac{\left(\frac{3}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

The first term represents the admittance of capacitor of 1 F. The other two terms represent admittance of a series LC network. For a series LC network,
$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,
$$L_1 = \frac{2}{3}\text{H}, \quad C_1 = \frac{3}{2}\text{F}$$
$$L_2 = 2\,\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$

The network is shown in Fig. 10.31.

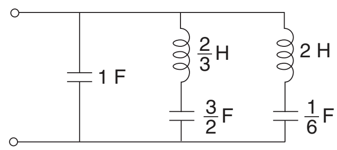


Fig. 10.31

Example 10.43 *Realise Cauer forms of the following LC impedance function:*
$$Z(s) = \frac{10s^4+12s^2+1}{2s^3+2s}$$

Solution

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.
$$Z(s) = \frac{10s^4+12s^2+1}{2s^3+2s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,
$$
\begin{array}{l}
2s^3+2s)\,10s^4+12s^2+1\,(5s \leftarrow Z \\
\qquad\quad \underline{10s^4+10s^2} \\
\qquad\qquad 2s^2+1)\,2s^3+2s\,(s \leftarrow Y \\
\qquad\qquad\qquad \underline{2s^3+s} \\
\qquad\qquad\qquad\quad s)\,2s^2+1\,(2s \leftarrow Z \\
\qquad\qquad\qquad\qquad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\quad 1)\,s\,(s \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.32.

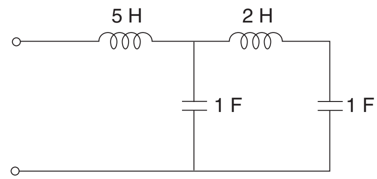


Fig. 10.32

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin.

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

The function $Z(s)$ has a pole at the origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s) = \frac{1 + 12s^2 + 10s^4}{2s + 2s^3}$$

By continued fraction expansion of $Z(s)$,

$$2s + 2s^3 \overline{)\,1 + 12s^2 + 10s^4\,}\left(\frac{1}{2s} \leftarrow Z\right.$$

$$\underline{1 + \quad s^2}$$

$$11s^2 + 10s^4 \overline{)\,2s + 2s^3\,}\left(\frac{2}{11s} \leftarrow Y\right.$$

$$\underline{2s + \frac{20}{11}s^3}$$

$$\frac{2}{11}s^3 \overline{)\,11s^2 + 10s^4\,}\left(\frac{121}{2s} \leftarrow Z\right.$$

$$\underline{11s^2}$$

$$10s^4 \overline{)\,\frac{2}{11}s^3\,}\left(\frac{2}{110s} \leftarrow Y\right.$$

$$\underline{\frac{2}{11}s^3}$$

$$0$$

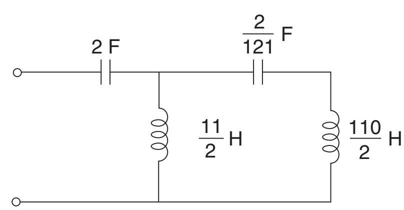


Fig. 10.33

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in Cauer or ladder realisation. The network is shown in Fig. 10.33.

Example 10.44 *Realise the following network function in Cauer I form:*

$$Z(s)=\frac{6s^4+42s^2+48}{s^5+18s^3+48s}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates the presence of a zero at infinity. The admittance function $Y(s)$ has a pole at infinity. Hence, the continued fraction expansion of $Y(s)$ is carried out.

$$Y(s)=\frac{s^5+18s^3+48s}{6s^4+42s^2+48}$$

By continued fraction expansion

$$6s^4+42s^2+s^2\,\Big)\,s^5+18s^3+48s\,\Big(\frac{1}{6}s\leftarrow Y$$

$$\underline{s^5+\;7s^3+\;8s}$$

$$11s^3+40s\,\Big)\,6s^4+42s^2+48\,\Big(\frac{6}{11}s\leftarrow Z$$

$$\underline{6s^4+\frac{240}{11}s^2}$$

$$\frac{222}{11}s^2+30\,\Big)\,11s^3+40s\,\Big(\frac{121}{222}s\leftarrow Y$$

$$\underline{11s^3+\frac{5808}{222}s}$$

$$\frac{3072}{222}s\,\Big)\,\frac{222}{11}s^2+48\,\Big(\frac{49284}{33792}s\leftarrow Z$$

$$\underline{\frac{222}{11}s^2}$$

$$48\,\Big)\,\frac{3072}{222}s\,\Big(\frac{128}{444}s\leftarrow Y$$

$$\underline{\frac{3072}{222}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.34.

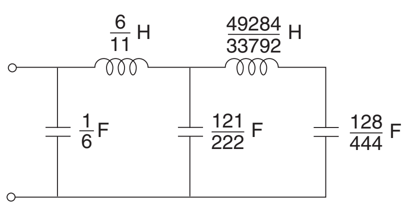


Fig. 10.34

Example 10.45 *Realise Cauer II form of the function:*

$$Z_{LC}(s)=\frac{s(s^4+3s^2+1)}{3s^4+4s^2+1}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin. The given function has a zero at the origin. The admittance function $Y(s)$ has a pole at origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y_{LC}(s)=\frac{3s^4+4s^2+1}{s^5+3s^3+s}=\frac{1+4s^2+3s^4}{s+3s^3+s^5}$$

By continued fraction expansion of $Y(s)$, we have

$$s+3s^3+s^5\overline{\big)1+4s^2+3s^4}\left(\frac{1}{s}\leftarrow Y\right.$$
$$\underline{1+3s^2+\ \ s^4}$$
$$s^2+2s^4\overline{\big)s+3s^3+s^5}\left(\frac{1}{s}\leftarrow Z\right.$$
$$\underline{s+2s^3}$$
$$s^3+s^5\overline{\big)s^2+2s^4}\left(\frac{1}{s}\leftarrow Y\right.$$
$$\underline{s^2+\ \ s^4}$$
$$s^4\overline{\big)s^3+s^5}\left(\frac{1}{s}\leftarrow Z\right.$$
$$\underline{s^3}$$
$$s^5\overline{\big)s^4}\left(\frac{1}{s}\leftarrow Y\right.$$
$$\underline{s^4}$$
$$0$$

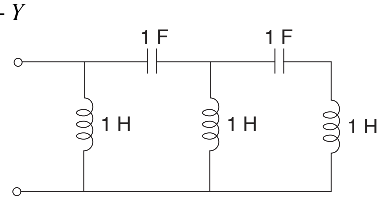


Fig. 10.35

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.35.

Example 10.46 *Obtain the Cauer I form of realisation for the function*

$$Z_{LC}(s)=\frac{s^5+7s^3+10s}{s^4+5s^2+4}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z_{LC}(s)$ about pole at infinity.

$$s^4+5s^2+4\overline{\big)s^5+7s^3+10s\big(}\; s \leftarrow Z$$
$$\underline{s^5+5s^3+\;\;4s}$$
$$2s^3+6s\overline{\big)s^4+5s^2+4\big(}\;\frac{1}{2}s \leftarrow Y$$
$$\underline{s^4+3s^2}$$
$$2s^2+4\overline{\big)2s^3+6s\big(}\; s \leftarrow Z$$
$$\underline{2s^3+4s}$$
$$2s\overline{\big)2s^2+4\big(}\; s \leftarrow Y$$
$$\underline{2s^2}$$
$$4\overline{\big)2s\big(}\;\frac{1}{2}s \leftarrow Z$$
$$\underline{2s}$$
$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.36.

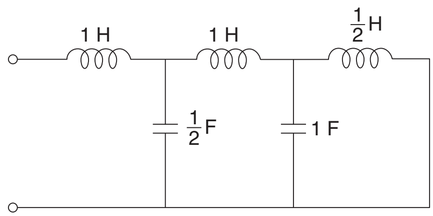


Fig. 10.36

Example 10.47 *Synthesize the following LC impedance function in Cauer II form:*

$$Z(s)=\frac{s^3+2s}{s^4+4s^2+3}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin.

$$Z(s)=\frac{s(s^2+2)}{(s^2+3)(s^2+1)}$$

But $Z(s)$ has a zero at the origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y(s)=\frac{3+4s^2+s^4}{2s+s^3}$$

By continued fraction expansion of $Y(s)$,

$$2s+s^3\overline{\big)\,3+4s^2+s^4\,}\left(\frac{3}{2s}\leftarrow Y\right.$$

$$\underline{3+\frac{3}{2}s^2}$$

$$\frac{5}{2}s^2+s^4\overline{\big)\,2s+s^3\,}\left(\frac{4}{5s}\leftarrow Z\right.$$

$$\underline{2s+\frac{4}{5}s^3}$$

$$\frac{s^3}{5}\overline{\big)\,\frac{5}{2}s^2+s^4\,}\left(\frac{25}{2s}\leftarrow Y\right.$$

$$\underline{\frac{5}{2}s^2}$$

$$s^4\overline{\big)\,\frac{1}{5}s^3\,}\left(\frac{1}{5s}\leftarrow Z\right.$$

$$\underline{\frac{1}{5}s^3}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.37.

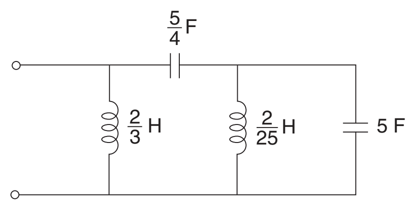


Fig. 10.37

10.6 REALISATION OF *RC* FUNCTIONS

RC driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency nearest to the origin is a pole.
4. The highest critical frequency farthest to the origin is a zero.
5. Residues evaluated at the poles of $Z_{RC}(s)$ are real and positive.
6. The slope $\dfrac{d}{d\sigma}Z_{RC}$ is negative.
7. $Z_{RC}(\infty) < Z_{RC}(0)$.

RC functions can also be realised in four different ways. The impedance function of RC networks is given by,

$$Z(s)=\frac{H(s+\sigma_1)(s+\sigma_3)\ldots}{s(s+\sigma_2)\ldots}$$

10.6.1 Foster Realisation

Foster I Form The Foster I form is obtained by partial-fraction expansion of $Z(s)$.

$$Z(s)=\frac{K_0}{s}+\frac{K_1}{s+\sigma_1}+\frac{K_2}{s+\sigma_2}+\ldots+K_\infty$$

where K_0, K_1, K_2, … K_∞ are residues of $Z(s)$.

$$K_o = sZ(s)\big|_{s=0}$$
$$K_i = (s+\sigma_i)Z(s)\big|\, s = -\sigma_i$$
$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty}$$

The first term $\dfrac{K_0}{s}$ represents the impedance of a capacitor of $\dfrac{1}{K_o}$ farads.

The last term K_∞ represents the impedance of a resistor of K_∞ ohms.

The remaining terms, i.e., $\dfrac{K_i}{s+\sigma_i}$ represent the impedance of the parallel combination of resistor R_i and capacitor C_i. For parallel combinaton of R_i and C_i,

$$Z(s) = \frac{R_i\left(\dfrac{1}{C_i s}\right)}{R_i + \dfrac{1}{C_i s}} = \frac{K_i}{s+\sigma_i}$$

$$R_i = \frac{K_i}{\sigma_i} \quad \text{and} \quad C_i = \frac{1}{K_i}$$

Table 10.3 *Realisation of Foster-I form of RC network*

Impedance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{C_0 s}$	$C_0 = \dfrac{1}{K_0}$
$\dfrac{K_i}{s+\sigma_i} = \dfrac{(R_i)\left(\dfrac{1}{C_i s}\right)}{R_i + \dfrac{1}{C_i s}}$	$Ri = \dfrac{K_i}{\sigma_i}$ $C_i = \dfrac{1}{K_i}$
$K_\infty = R_\infty$	$R_\infty = K_\infty$

The network corresponding to the Foster-I form is shown in Fig. 10.38.

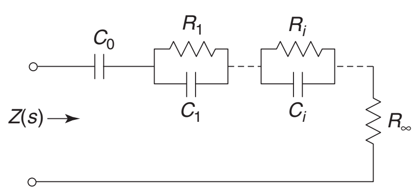


Fig. 10.38 *Foster-I form of RC network*

Foster II Form The Foster-II form is obtained by partial fraction expansion of $Y(s)$. Since $Y(s) = \dfrac{1}{Z(s)}$ has negative residue at its pole, Foster II form is obtained by expanding $\dfrac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{K_o}{s} + \sum_{i=1}^{n} \frac{K_i}{(s+\sigma_i)} + K_\infty$$

Multiplying this equation by s,

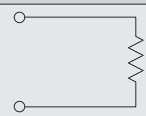
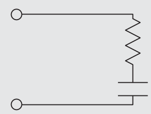
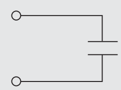
$$Y(s) = K_o + \sum_{i=1}^{n} \frac{K_i s}{(s+\sigma_i)} + K_\infty s$$

The first term K_o represents the conductance of a resistor of $\dfrac{1}{K_o}$ ohms.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farads.

The remaining terms, i.e., $\dfrac{K_i s}{s+\sigma_i}$ represent the admittance of series combination of resistor R_i and capacitor C_i with $R_i = \dfrac{1}{K_i}$ ohms and $C_i = \dfrac{K_i}{\sigma_i}$ farads.

Table 10.4 *Realisation of Foster II form of RC network*

Admittance function	Element
$K_0 = \dfrac{1}{R_0}$	 $R_o = \dfrac{1}{K_o}$
$\dfrac{K_i}{s+\omega_i} = \dfrac{\left(\dfrac{1}{R_i}\right)s}{s+\dfrac{1}{R_i C_i}}$	 $R_i = \dfrac{1}{K_i}$, $C_i = \dfrac{K_i}{\sigma_i}$
$K_\infty s = C_\infty s$	 $C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 10.39.

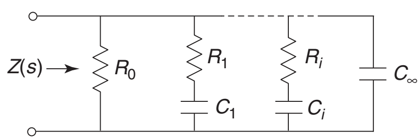


Fig. 10.39 *Foster II form of RC network*

10.6.2 Cauer Realisation

Cauer I Form The Cauer I form is obtained by removal of the pole from the impedance function $Z(s)$ at $s = \infty$. This is the same as a continued fraction expansion of the impedance function about infinity. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \ldots}}}$$

The network is shown in Fig. 10.40.

In the network shown in Fig. 10.40, if $Z(s)$ has a zero at $s = \infty$, the first element is the capacitor C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

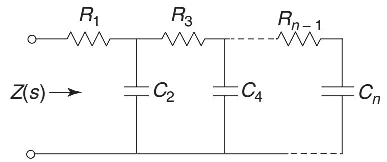


Fig. 10.40 *Cauer-I form of RC network*

Cauer II Form The Cauer II form is obtained by removal of the pole from the impedance function at the origin. This is the same as a continued fraction expansion of an impedance function about the origin.

If the given impedance function has a pole at the origin, it is removed as a capacitor C_1. The reciprocal of the remainder function has a minimum value at $s = 0$ which is removed as a constant of resistor R_2. If the original impedance has no pole at the origin, then the first capacitor is absent and the process is repeated with the removal of the constant corresponding to the resistor R_2.

The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_2} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{R_4} + \dots}}}$$

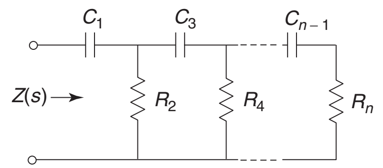


Fig. 10.41 *Caver-II form of RC network*

The network is shown in Fig. 10.41.

In the network shown in Fig. 10.41, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Example 10.48 *Determine whether following functions are RC impedance function or not.*

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$ *(b)* $\dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

Solution

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

The function Z(s) has poles at $s = 0$ and $s = -3$ and zeros at $s = -2$ and $s = -4$ as shown in Fig. 10.42. The poles and zeros are simple and located on the negative real axis of the s plane. The poles and zeros are interlaced. The lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC impedance function.

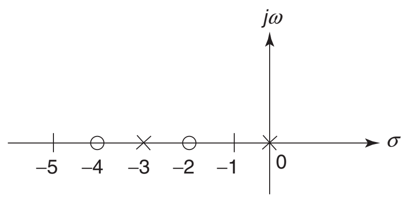


Fig. 10.42

(b) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

The function $Z(s)$ has poles at $s = -2$ and $s = -6$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 10.43. The poles and zeros are simple and located on the negative real axis of the s-plane. The poles and zeros are interlaced. But the lowest critical frequency nearest to the origin is not a pole, but zero. Hence, the function $Z(s)$ is not an RC impedance function.

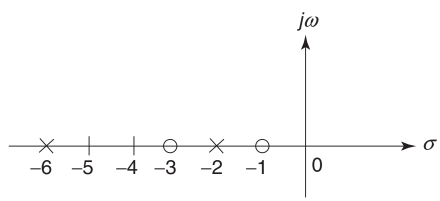


Fig. 10.43

Example 10.49 *Realise the Foster and Cauer forms of the impedance function*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 10.44.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the negative real axis. The poles and zeros are interlaced and the lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC function.

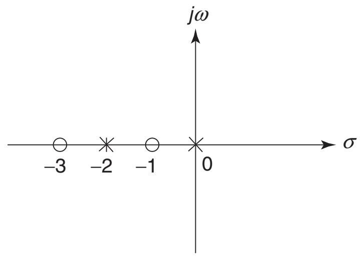


Fig. 10.44

Foster I Form The Foster I form is obtained by partial fraction expansion of impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s}$$

$$s^2+2s\overline{)\,s^2+4s+3\,}(1$$
$$\underline{s^2+2s}$$
$$2s+3$$

$$Z(s) = 1 + \frac{2s+3}{s^2+2s} = 1 + \frac{2s+3}{s(s+2)}$$

By partial-fraction expansion,

$$Z(s) = 1 + \frac{K_1}{s} + \frac{K_2}{s+2}$$

where

$$K_1 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_2 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{-2} = \frac{1}{2}$$

$$Z(s) = 1 + \frac{\frac{3}{2}}{s} + \frac{\frac{1}{2}}{s+2}$$

The first term represents the impedance of a resistor of 1 Ω. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of parallel RC circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R = \frac{1}{4}\,\Omega$$
$$C = 2\text{ F}$$

The network is shown in Fig. 10.45.

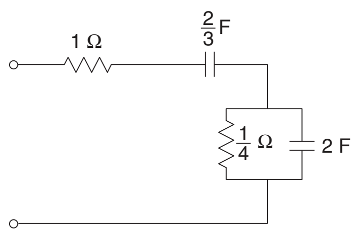


Fig. 10.45

Foster II Form The Foster II form is obtained by the partial-fraction expansion of admittance function $\dfrac{Y(s)}{s}$.

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s+2)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{s+2}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where
$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{1}{2}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

$$Y(s) = \frac{\frac{1}{2}s}{s+1} + \frac{\frac{1}{2}s}{s+3}$$

These two terms represent the admittance of a series RC circuit. For a series RC circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 2\,\Omega, \quad C_1 = \frac{1}{2}\text{ F}$$

$$R_2 = 2\,\Omega, \quad C_2 = \frac{1}{6}\text{ F}$$

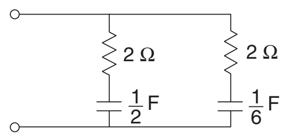


Fig. 10.46

The network is shown in Fig. 10.46.

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s}$$

By continued fraction expansion,

$$\begin{array}{l} s^2+2s\,\big)\,s^2+4s+3\,\big(\,1 \leftarrow Z \\ \qquad\quad\ \ \underline{s^2+2s} \\ \qquad\qquad\quad 2s+3\,\big)\,s^2+2s\,\big(\,\tfrac{1}{2}s \leftarrow Y \end{array}$$

$$\begin{array}{l}
\underline{s^2+\frac{3}{2}s} \\
\qquad \frac{1}{2}s\,\Big)\,2s+3\,\Big(\,4 \leftarrow Z \\
\qquad\qquad \underline{2s\quad} \\
\qquad\qquad\qquad 3\,\Big)\,\frac{1}{2}s\,\Big(\,\frac{1}{6}s \leftarrow Y \\
\qquad\qquad\qquad\quad \underline{\frac{1}{2}s} \\
\qquad\qquad\qquad\quad\ 0
\end{array}$$

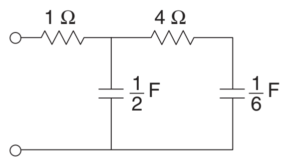


Fig. 10.47

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 10.47.

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s)=\frac{3+4s+s^2}{2s+s^2}$$

By continued fraction expansion,

$$\begin{array}{l}
2s+s^2\,\Big)\,3+4s+s^2\,\Big(\,\frac{3}{2s} \leftarrow Z \\
\quad \underline{3+\frac{3}{2}s\quad} \\
\qquad \frac{5}{2}s+s^2\,\Big)\,2s+s^2\,\Big(\,\frac{4}{5} \leftarrow Y \\
\qquad\qquad\quad \underline{2s+\frac{4}{5}s^2} \\
\qquad\qquad\qquad \frac{1}{5}s^2\,\Big)\,\frac{5}{2}s+s^2\,\Big(\,\frac{25}{2s} \leftarrow Z \\
\qquad\qquad\qquad\qquad \underline{\frac{5}{2}s\quad\ } \\
\qquad\qquad\qquad\qquad\quad s^2\,\Big)\,\frac{1}{5}s^2\,\Big(\,\frac{1}{5} \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad \underline{\frac{1}{5}s^2} \\
\qquad\qquad\qquad\qquad\qquad\ \ 0
\end{array}$$

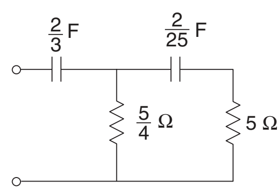


Fig. 10.48

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 10.48.

Example 10.50 *Determine the Foster form of realisation of the RC impedance function.*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where

$$K_0 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{(-2)(-2+4)} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$$

$$K_2 = (s+4)Z(s)\big|_{s=-4} = \frac{(-4+1)(-4+3)}{(-4)(-4+2)} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{3}{8}}{s+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The remaining terms represent the impedance of a parallel RC circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{1}{8}\,\Omega, \qquad C_1 = 4\text{ F}$$

$$R_2 = \frac{3}{32}\,\Omega, \quad C_2 = \frac{8}{3}\text{ F}$$

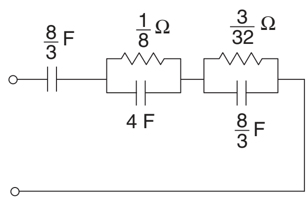


Fig. 10.49

The network is shown in Fig. 10.49.

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2+6s+8}{s^2+4s+3}$$

Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$s^2+4s+3\overline{)\,s^2+6s+8\,}(1$$
$$\underline{s^2+4s+3}$$
$$2s+5$$

$$\frac{Y(s)}{s}=1+\frac{2s+5}{s^2+4s+3}=1+\frac{2s+5}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s}=1+\frac{K_1}{s+1}+\frac{K_2}{s+3}$$

where
$$K_1=(s+1)\frac{Y(s)}{s}\bigg|_{s=-1}=\frac{(-1+2)(-1+4)}{(-1+3)}=\frac{(1)(3)}{2}=\frac{3}{2}$$

$$K_2=(s+3)\frac{Y(s)}{s}\bigg|_{s=-3}=\frac{(-3+2)(-3+4)}{(-3+1)}=\frac{(-1)(1)}{(-2)}=\frac{1}{2}$$

$$\frac{Y(s)}{s}=1+\frac{\frac{3}{2}}{s+1}+\frac{\frac{1}{2}}{s+3}$$

$$Y(s)=s+\frac{\frac{3}{2}s}{s+1}+\frac{\frac{1}{2}s}{s+3}$$

The first term represents the admittance of a capacitor of 1 F. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s)=\frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1=\frac{2}{3}\,\Omega,\quad C_1=\frac{3}{2}\,\text{F}$$

$$R_2=2\,\Omega,\quad C_2=\frac{1}{6}\,\text{F}$$

The network is shown in Fig. 10.50.

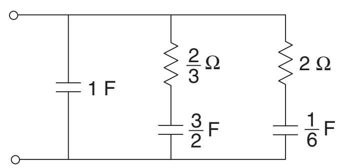


Fig. 10.50

Example 10.51 *Realise Foster forms of the following RC impedance function*

$$Z(s)=\frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$Z(s) = \frac{2s^2 + 12s + 16}{s^2 + 4s + 3}$$

$$s^2 + 4s + 3 \overline{\big)\, 2s^2 + 12s + 16 \,\big(}\, 2$$
$$\underline{2s^2 + \ 8s + \ 6}$$
$$4s + 10$$

$$Z(s) = 2 + \frac{4s+10}{s^2+4s+3} = 2 + \frac{4s+10}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$Z(s) = 2 + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where
$$K_1 = (s+1)Z(s)\Big|_{s=-1} = \frac{2(-1+2)(-1+4)}{(-1+3)} = 3$$
$$K_2 = (s+3)Z(s)\Big|_{s=-3} = \frac{2(-3+2)(-3+4)}{(-3+1)} = 1$$
$$Z(s) = 2 + \frac{3}{s+1} + \frac{1}{s+3}$$

The first term represents the impedance of a resistor of 2 Ω. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 3\ \Omega, \quad C_1 = \frac{1}{3}\ \text{F}$$
$$R_2 = \frac{1}{3}\ \Omega, \quad C_2 = 1\ \text{F}$$

The network is shown in Fig. 10.51.

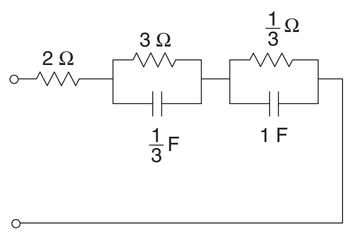


Fig. 10.51

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{(s+1)(s+3)}{2(s+2)(s+4)}$$
$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{2s(s+2)(s+4)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where
$$K_0 = s\frac{Y(s)}{s}\Big|_{s=0} = \frac{(1)(3)}{(2)(2)(4)} = \frac{3}{16}$$
$$K_1 = (s+2)\frac{Y(s)}{s}\Big|_{s=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{(-1)(1)}{2(-2)(2)} = \frac{1}{8}$$

$$K_2 = (s+4)\frac{Y(s)}{s}\bigg|_{s=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{(-3)(-1)}{2(-4)(-2)} = \frac{3}{16}$$

$$\frac{Y(s)}{s} = \frac{\frac{3}{16}}{s} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{16}}{s+4}$$

$$Y(s) = \frac{3}{16} + \frac{\frac{1}{8}s}{s+2} + \frac{\frac{3}{16}s}{s+4}$$

The first term represents the admittance of a resistor of $\frac{16}{3}\ \Omega$. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1 = 8\ \Omega, \qquad C_1 = \frac{1}{16}\ \text{F}$$

$$R_2 = \frac{16}{3}\ \Omega, \quad C_2 = \frac{3}{64}\ \text{F}$$

The network is shown in Fig. 10.52.

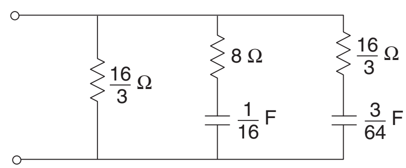


Fig. 10.52

Example 10.52 *Obtain the Cauer forms of the RC impedance function*

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\,\big)\,s^2+8s+12\,\Big(\frac{1}{2} \leftarrow Z$$
$$\underline{s^2+4s+\ 3}$$
$$4s+9\,\big)\,2s^2+8s+6\,\Big(\frac{1}{2}s \leftarrow Y$$
$$\underline{2s^2+\frac{9}{2}s}$$
$$\frac{7}{2}s+6\,\big)\,4s+9\,\Big(\frac{8}{7} \leftarrow Z$$
$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\Big)\,\frac{7}{2}s+6\,\Big(\,\frac{49}{30}s \leftarrow Y$$
$$\frac{7}{2}s$$
$$\overline{\qquad\qquad}$$
$$6\Big)\,\frac{15}{7}\,\Big(\,\frac{5}{14} \leftarrow Z$$
$$\frac{15}{7}$$
$$\overline{\;0\;}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.53.

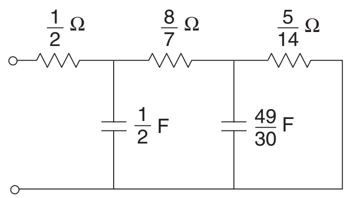


Fig. 10.53

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s)=\frac{12+8s+s^2}{6+8s+2s^2}$$

By continued fraction expansion,

$$6+8s+2s^2\Big)\,12+8s+s^2\,\Big(\,2$$
$$\underline{12+16s+4s^2}$$
$$-8s-3s^2$$

Since negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Big)\,6+8s+2s^2\,\Big(\,\frac{1}{2} \leftarrow Y$$
$$\underline{6+4s+\frac{1}{2}s^2}$$
$$4s+\frac{3}{2}s^2\Big)\,12+8s+s^2\,\Big(\,\frac{3}{s} \leftarrow Z$$
$$\underline{12+\frac{9}{2}s}$$
$$\frac{7}{2}s+s^2\Big)\,4s+\frac{3}{2}+s^2\,\Big(\,\frac{8}{7} \leftarrow Y$$
$$\underline{4s+\frac{8}{7}s^2}$$
$$\frac{5}{14}s^2\Big)\,\frac{7}{2}s+s^2\,\Big(\,\frac{49}{5s} \leftarrow Z$$

$$\frac{7}{2}s$$

$$s^2\,\Big)\,\frac{5}{14}s^2\,\Big(\,\frac{5}{14} \leftarrow Y$$

$$\frac{5}{14}s$$

$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.54.

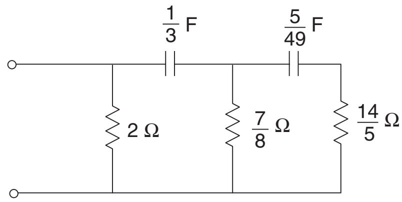


Fig. 10.54

Example 10.53 *Realise the RC impedance in Cauer I and Foster I forms*

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates presence of a zero at infinity. Hence, the admittance function $Y(s)$ has a pole at infinity.

$$Y(s) = \frac{s^2 + 8s + 12}{s+4}$$

By continued fraction expansion,

$$s+4\,\Big)\,s^2+8s+12\,\Big(\,s \leftarrow Y$$

$$\underline{s^2+4s}$$

$$4s+12\,\Big)\,s+4\,\Big(\,\frac{1}{4} \leftarrow Z$$

$$\underline{s+3}$$

$$1\,\Big)\,4s+12\,\Big(\,4s \leftarrow Y$$

$$\underline{4s}$$

$$12\,\Big)\,1\,\Big(\,\frac{1}{12} \leftarrow Z$$

$$\frac{1}{0}$$

The impedances are connected in series branches, whereas the admittances are connected in parallel branches. The network is shown in Fig. 10.55.

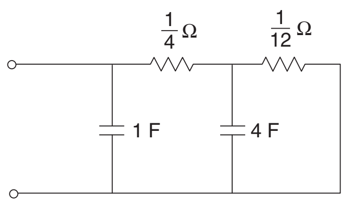


Fig. 10.55

Foster I Form The Foster I form is obtained by partial fraction expansion of $Z(s)$.

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\Big|_{s=-2} = \frac{(-2+4)}{(-2+6)} = \frac{1}{2}$$
$$K_2 = (s+6)Z(s)\Big|_{s=-6} = \frac{(-6+4)}{(-6+2)} = \frac{1}{2}$$
$$Z(s) = \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+6}$$

These two terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad C_1 = 2\text{ F}$$
$$R_2 = \frac{1}{12}\,\Omega, \quad C_2 = 2\text{ F}$$

The network is shown in Fig. 10.56.

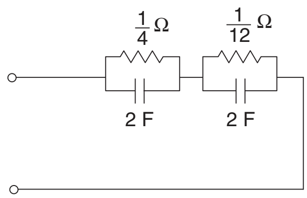


Fig. 10.56

Example 10.54 *The RC driving-point impedance function is given as* $Z(s) = H\dfrac{(s+1)(s+4)}{s(s+3)}$ *Realise the impedance function in the ladder form, given* $Z(-2) = 1$.

Solution Putting $s = -2$,

$$Z(-2) = H\frac{(-2+1)(-2+4)}{(-2)(-2+3)} = H$$
$$H = 1$$
$$Z(s) = \frac{(s+1)(s+4)}{s(s+3)}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,

$$\begin{array}{l}
s^2+3s\,\big)\,s^2+5s+4\,\big(\,1 \leftarrow Z \\
\qquad\quad\;\; \underline{s^2+3s} \\
\qquad\qquad\quad 2s+4\,\big)\,s^2+3s\,\big(\,\frac{1}{2}s \leftarrow Y \\
\qquad\qquad\qquad\qquad \underline{s^2+2s} \\
\qquad\qquad\qquad\qquad\quad s\,\big)\,2s+4\,\big(\,2 \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{2s} \\
\qquad\qquad\qquad\qquad\qquad\quad 4\,\big)\,s\,\big(\,\frac{1}{4}s \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad\qquad\; \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\; 0
\end{array}$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 10.57.

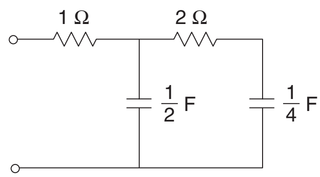


Fig. 10.57

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s) = \frac{4 + 5s + s^2}{3s + s^2}$$

By continued fraction expansion,

$$3s + s^2 \overline{\big)\, 4 + 5s + s^2 \,\big(} \; \frac{4}{3s} \leftarrow Z$$

$$\underline{4 + \frac{4}{3}s}$$

$$\frac{11}{3}s + s^2 \overline{\big)\, 3s + s^2 \,\big(} \; \frac{9}{11} \leftarrow Y$$

$$\underline{3s + \frac{9}{11}s^2}$$

$$\frac{2}{11}s^2 \overline{\big)\, \frac{11}{3}s + s^2 \,\big(} \; \frac{121}{6s} \leftarrow Z$$

$$\underline{\frac{11}{3}s}$$

$$s^2 \overline{\big)\, \frac{2}{11}s^2 \,\big(} \; \frac{2}{11} \leftarrow Y$$

$$\underline{\frac{2}{11}s^2}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.58.

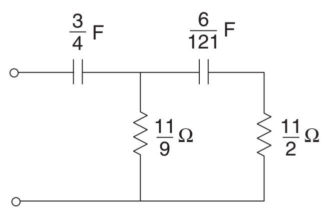


Fig. 10.58

Example 10.55 *An impedance function has the pole-zero diagram as shown in Fig. 10.59. Find the impedance function such that $Z(-4) = \frac{3}{4}$ and realise in Cauer I and Foster II forms.*

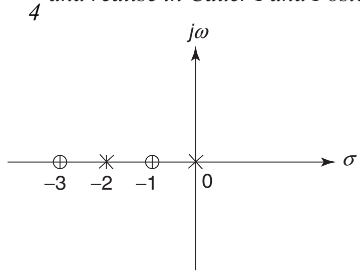


Fig. 10.59

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$.

$$Z(s) = H\frac{(s+1)(s+3)}{s(s+2)}$$

Putting $s = -4$,

$$Z(-4) = H\frac{(-4+1)(-4+3)}{(-4)(-4+2)} = H\frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}\text{H}$$

$$\frac{3}{4} = \frac{3}{8}\text{H}$$

$$H = 2$$

$$Z(s) = \frac{2(s+1)(s+3)}{s(s+2)} = \frac{2s^2+8s+6}{s^2+2s}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.
By continued fraction expansion of $Z(s)$,

$$s^2+2s\,\big)\,2s^2+8s+6\,\big(\,2 \leftarrow Z$$
$$\underline{2s^2+4s}$$
$$4s+6\,\big)\,s^2+2s\,\big(\,\frac{1}{4}s \leftarrow Y$$
$$\underline{s^2+\frac{3}{2}s}$$
$$\frac{1}{2}s\,\big)\,4s+6\,\big(\,8 \leftarrow Z$$
$$\underline{4s}$$
$$6\,\big)\,\frac{1}{2}s\,\big(\,\frac{1}{12}s \leftarrow Y$$
$$\underline{\frac{1}{2}s}$$
$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 10.60.

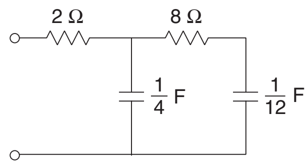


Fig. 10.60

Foster II Form The Foster II form is obtained by partial fraction expansion of $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{s+2}{2(s+1)(s+3)}$$

By partial fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where
$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{2(-3+1)} = \frac{-1}{2(-2)} = \frac{1}{4}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{4}}{s+3}$$

$$Y(s) = \frac{\frac{1}{4}s}{s+1} + \frac{\frac{1}{4}s}{s+3}$$

Two terms represent the admittance of a series RC circuit. For a series RC circuit,

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 4\ \Omega, \quad C_1 = \frac{1}{4}\ \text{F}$$

$$R_2 = 4\ \Omega, \quad C_2 = \frac{1}{12}\ \text{F}$$

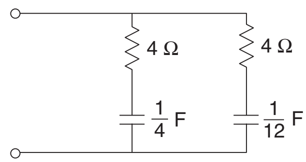


Fig. 10.61

The network is shown in Fig. 10.61.

10.7 REALISATION OF *RL* FUNCTIONS

RL driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.
4. The highest critical frequency is a pole which may be at infinity.
5. Residues evaluated at the poles of $Z_{RL}(s)$ are real and negative while that of $\frac{Z_{RL}(s)}{s}$ are real and positive.
6. The slope $\frac{d}{d\sigma}Z_{RL}$ is positive.
7. $Z_{RL}(0) < Z_{RL}(\infty)$.

The admittance of an inductor is similar to the impedance of a capacitor. Hence, properties of an *RL* admittance are identical to those of an *RC* impedance and vice-versa, i.e.,

$$Z_{RC}(s) = Y_{RL}(s)$$
$$Z_{RL}(s) = Y_{RC}(s)$$

An *RL* admittance can be considered as the dual of an *RC* impedance and vice-versa.

Example 10.56 *Indicate which of the following functions are either RL, RC or LC impedance functions.*

(a) $Z(s) = \frac{4(s+1)(s+3)}{s(s+2)}$ (b) $Z(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$

(c) $Z(s) = \frac{(s+1)(s+4)}{s(s+2)}$ (d) $Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+16)}$

Solution (a) $Z(s) = \frac{4(s+1)(s+3)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(b) $Z(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

(c) $Z(s) = \frac{(s+1)(s+4)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(d) $Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

Example 10.57 *Realise following RL impedance function in Foster-I and Foster-II form.*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(-2+6)} = -\frac{1}{2}$$
$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6+2)} = -\frac{15}{2}$$

Since residues of $Z(s)$ are negative, partial fraction expansion of $\frac{Z(s)}{s}$ is carried out.

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

By partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_0 = s\frac{Z(s)}{s}\bigg|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$
$$K_1 = (s+2)\frac{Z(s)}{s}\bigg|_{s=-2} = \frac{2(-2+1)(-2+3)}{(2)(-2+6)} = \frac{1}{4}$$
$$K_2 = (s+6)\frac{Z(s)}{s}\bigg|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6)(-6+2)} = \frac{5}{4}$$

$$\frac{Z(s)}{s} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{5}{4}}{s+6}$$
$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4}s}{s+2} + \frac{\frac{5}{4}s}{s+6}$$

The first term represents the impedance of the resistor of $\frac{1}{2}\,\Omega$. The other two terms represent the impedance of the parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad L_1 = \frac{1}{8}\,\text{H}$$
$$R_2 = \frac{5}{4}\,\Omega, \quad L_2 = \frac{5}{24}\,\text{H}$$

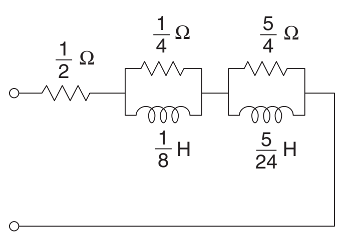


Fig. 10.62

The network is shown in Fig. 10.62.

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$2s^2+8s+6\overline{\big)\,s^2+8s+12\,}\left(\frac{1}{2}\right.$$
$$\frac{s^2+4s+3}{4s+9}$$

$$Y(s)=\frac{1}{2}+\frac{4s+9}{2s^2+8s+6}=\frac{1}{2}+\frac{4s+9}{2(s+1)(s+3)}$$

By partial-fraction expansion,

$$Y_1(s)=\frac{4s+9}{2(s+1)(s+3)}=\frac{K_0}{s+1}+\frac{K_1}{s+3}$$

where
$$K_0=(s+1)Y_1(s)\big|_{s=-1}=\frac{(-4+9)}{2(-1+3)}=\frac{5}{4}$$

$$K_1=(s+3)Y_1(s)\big|_{s=-3}=\frac{(-12+9)}{2(-3+1)}=\frac{3}{4}$$

$$Y(s)=\frac{1}{2}+\frac{\frac{5}{4}}{s+1}+\frac{\frac{3}{4}}{s+3}$$

The first term represents the admittance of a resistor of 2 Ω. The other two terms represent the admittance of a series *RL* circuit. For a series *RL* circuit,

$$Y_{RL}(s)=\frac{\frac{1}{L_i}}{s+\frac{R_i}{L_i}}$$

By direct comparison,

$$R_1=\frac{4}{5}\,\Omega,\quad L_1=\frac{4}{5}\,\text{H}$$

$$R_2=4\,\Omega,\quad L_2=\frac{4}{3}\,\text{H}$$

The network is shown in Fig. 10.63.

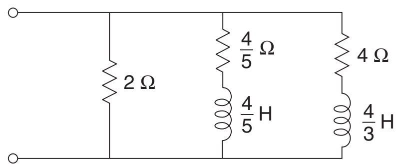


Fig. 10.63

Example 10.58 *Find the Foster forms of the RL impedance function:*

$$Z(s)=\frac{(s+1)(s+4)}{(s+5)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of impedance function $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s}=\frac{(s+1)(s+4)}{s(s+5)(s+3)}$$

By partial-faction expansion,

$$\frac{Z(s)}{s}=\frac{K_0}{s}+\frac{K_1}{s+3}+\frac{K_2}{s+5}$$

where
$$K_0=s\frac{Z(s)}{s}\bigg|_{s=0}=\frac{(1)(4)}{(5)(3)}=\frac{4}{15}$$

$$K_1 = (s+3)\frac{Z(s)}{s}\bigg|_{s=-3} = \frac{(-3+1)(-3+4)}{(-3)(-3+5)} = \frac{(-2)(1)}{(-3)(2)} = \frac{1}{3}$$

$$K_2 = (s+5)\frac{Z(s)}{s}\bigg|_{s=-5} = \frac{(-5+1)(-5+4)}{(-5)(-5+3)} = \frac{(-4)(-1)}{(-5)(-2)} = \frac{2}{5}$$

$$\frac{Z(s)}{s} = \frac{\frac{4}{15}}{s} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{5}}{s+5}$$

$$Z(s) = \frac{4}{15} + \frac{\frac{1}{3}s}{s+3} + \frac{\frac{2}{5}s}{s+5}$$

The first term represents the impedance of the resistor of $\frac{4}{15}\,\Omega$. The other two terms represent the impedance of a parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{3}\,\Omega, \quad L_1 = \frac{1}{9}\,\text{H}$$

$$R_2 = \frac{2}{5}\,\Omega, \quad L_2 = \frac{2}{25}\,\text{H}$$

The network is shown in Fig. 10.64.

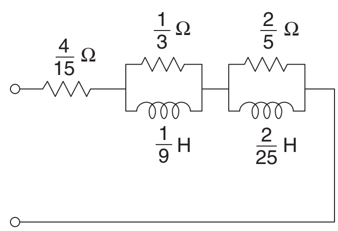


Fig. 10.64

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+5)(s+3)}{(s+1)(s+4)} = \frac{s^2+8s+15}{s^2+5s+4}$$

$$s^2+5s+4)\,s^2+8s+15\,(1$$
$$\underline{s^2+5s+4}$$
$$3s+11$$

$$Y(s) = 1 + \frac{3s+11}{(s+1)(s+4)}$$

By partial-fraction expansion,

$$Y_1(s) = \frac{K_0}{s+1} + \frac{K_1}{s+4}$$

where
$$K_0 = (s+1)Y_1(s)\big|_{s=-1} = \frac{(-3+11)}{(-1+4)} = \frac{8}{3}$$

$$K_1 = (s+4)Y_1(s)\big|_{s=-4} = \frac{(-12+11)}{(-4+1)} = \frac{1}{3}$$

$$Y_1(s) = \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

$$Y(s) = 1 + \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

The first term represents the admittance of a resistor of 1 Ω. The other two terms represent the admittance of a series *RL* circuit.

For a series *RL* circuit,

$$Y_{RL}(s) = \frac{\frac{1}{L_i}}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{3}{8}\,\Omega, \quad L_1 = \frac{3}{8}\,\text{H}$$

$$R_2 = 12\,\Omega, \quad L_2 = 3\,\text{H}$$

The network is shown in Fig. 10.65.

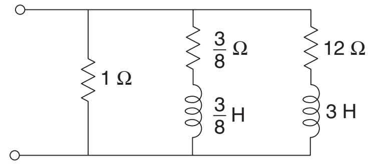


Fig. 10.65

Example 10.59 *Find the Cauer forms of the RL impedance function:*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by a continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{2s^2+8s+6}{s^2+8s+12}$$

By continued fraction expansion,

$$s^2+8s+12)2s^2+8s+6(2 \leftarrow Z$$
$$\underline{2s^2+16s+24}$$
$$-8s-18$$

Since a negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\Big)s^2+8s+12\Big(\frac{1}{2} \leftarrow Y$$
$$\underline{s^2+4s+\ 3}$$
$$4s+9\Big)2s^2+8s+6\Big(\frac{1}{2}s \leftarrow Z$$
$$\underline{2s^2+\frac{9}{2}s}$$
$$\frac{7}{2}s+6\Big)4s+9\Big(\frac{8}{7} \leftarrow Y$$
$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\Bigg)\frac{7}{2}s+6\Bigg(\frac{49}{30}s \leftarrow Z$$

$$\underline{\frac{7}{2}s\qquad\quad}$$

$$6\Bigg)\frac{15}{7}\Bigg(\frac{15}{42} \leftarrow Y$$

$$\underline{\frac{15}{7}}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.66.

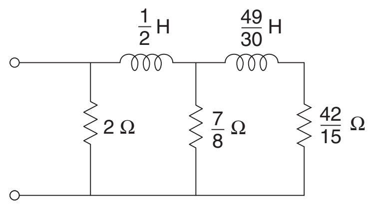


Fig. 10.66

Cauer II Form The Cauer II form is obtained from a continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Bigg)6+8s+2s^2\Bigg(\frac{1}{2} \leftarrow Z$$

$$\underline{6+4s+\frac{1}{2}s^2}$$

$$4s+\frac{3}{2}s^2\Bigg)12+8s+s^2\Bigg(\frac{3}{s} \leftarrow Y$$

$$\underline{12+\frac{9}{2}s\qquad}$$

$$\frac{7}{2}s+s^2\Bigg)4s+\frac{3}{2}s^2\Bigg(\frac{8}{7} \leftarrow Z$$

$$\underline{4s+\frac{8}{7}s^2}$$

$$\frac{5}{14}s^2\Bigg)\frac{7}{2}s+s^2\Bigg(\frac{98}{10s} \leftarrow Y$$

$$\underline{\frac{7}{2}s\qquad\quad}$$

$$s^2\Bigg)\frac{5}{14}s^2\Bigg(\frac{5}{14} \leftarrow Z$$

$$\underline{\frac{5}{14}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.67.

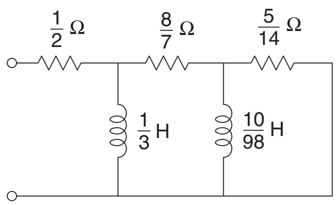


Fig. 10.67

Example 10.60 *Obtain the Foster I and Cauer I forms of the RL impedance function.*

$$Z(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial fraction expansion of $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s} = \frac{(s+4)(s+8)}{(s+1)(s+6)}$$

Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$\begin{array}{r} s^2+7s+6\overline{)\,s^2+12s+32\,}(1 \\ \underline{s^2+7s+6} \\ 5s+26 \end{array}$$

$$\frac{Z(s)}{s} = 1 + \frac{5s+26}{s^2+7s+6} = 1 + \frac{5s+26}{(s+1)(s+6)}$$

By partial-fraction expansion,

$$\frac{Z(s)}{s} = 1 + \frac{K_0}{s+1} + \frac{K_1}{s+6}$$

where

$$K_0 = \left.\frac{5s+26}{s+6}\right|_{s=-1} = \frac{-5+26}{-1+6} = \frac{21}{5}$$

$$K_1 = \left.\frac{5s+26}{s+1}\right|_{s=-6} = \frac{-30+26}{-6+1} = \frac{4}{5}$$

$$\frac{Z(s)}{s} = 1 + \frac{\frac{21}{5}}{s+1} + \frac{\frac{24}{5}}{s+6}$$

$$Z(s) = s + \frac{\frac{21}{5}s}{s+1} + \frac{\frac{4}{5}s}{s+6}$$

The first term represents the impedance of the inductor of 1 H. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \dfrac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{21}{5}\,\Omega, \quad L_1 = \frac{21}{5}\,\text{H}$$

$$R_2 = \frac{4}{5}\,\Omega, \quad L_2 = \frac{4}{30}\,\text{H}$$

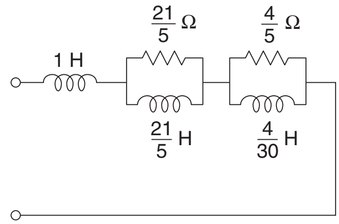


Fig. 10.68

The network is shown in Fig. 10.68.

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{s^3 + 12s^2 + 32s}{s^2 + 7s + 6}$$

By continued fraction expansion,

$$\begin{array}{l}
s^2 + 7s + 6\,\big)\, s^3 + 12s^2 + 32s\,\big(\, s \leftarrow Z \\
\qquad\qquad\quad \underline{s^3 + 7s^2 + 6s} \\
\qquad\qquad\qquad 5s^2 + 26s\,\big)\, s^2 + 7s + 6\,\Big(\, \dfrac{1}{5} \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{s^2 + \dfrac{26}{5}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad \dfrac{9}{5}s + 6\,\Big)\, 5s^2 + 26s\,\Big(\, \dfrac{25}{9}s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{5s^2 + \dfrac{50}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dfrac{28}{3}s\,\Big)\, \dfrac{9}{5}s + 6\,\Big(\, \dfrac{27}{140} \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\dfrac{9}{5}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6\,\Big)\, \dfrac{28}{3}s\,\Big(\, \dfrac{28}{18}s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\dfrac{28}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.69.

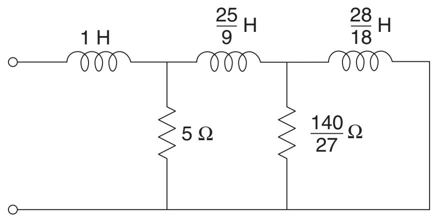


Fig. 10.69

Exercises

10.1 Test the following polynomials for Hurwitz property:

(i) $s^3 + s^2 + 2s + 2$
(ii) $s^4 + s^2 + s + 1$
(iii) $s^3 + 4s^2 + 5s + 2$
(iv) $s^4 + 7s^3 + 6s^2 + 21s + 8$
(v) $s^4 + s^3 + s + 1$
(vi) $s^7 + 3s^6 + 8s^5 + 15s^4 + 17s^3 + 12s^2 + 4s$
(vii) $s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$
(viii) $s^7 + 3s^5 + 2s^3 + s$
(ix) $s^5 + 2s^3 + s$
(x) $s^3 + 2s^2 + 4s + 2$
(xi) $s^4 + s^3 + 4s^2 + 2s + 3$
(xii) $s^5 + 8s^4 + 24s^3 + 28s^2 + 23s + 6$
(xiii) $s^7 - 2s^6 + 2s^5 + 9s^2 + 8s + 4$
(xiv) $s^7 + 3s^5 + 2s^3 + 3$
(xv) $s^5 + s^3 + s$
(xvi) $s^6 + 7s^4 + 5s^3 + s^2 + s$
(xvii) $s^4 + s^3 + 2s^2 + 3s + 2$

10.2 Determine whether the following functions are positive real:

(i) $\dfrac{s^3+5s}{s^4+2s^2+1}$

(ii) $\dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$

(iii) $\dfrac{2s^2+2s+1}{s^3+2s^2+s+2}$

(iv) $\dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$

(v) $\dfrac{2s^3+2s^2+3s+2}{s^2+1}$

(vi) $\dfrac{s^2+2s+1}{s^2+4s+4}$

(vii) $\dfrac{s^3+2s^2+3s+1}{s^3+2s^2+s+2}$

(viii) $\dfrac{s^3+2s^2+s+1}{s^2+s+1}$

(ix) $\dfrac{s+4}{s^2+2s+1}$

(x) $\dfrac{s^2+4s+3}{s^2+6s+8}$

(xi) $\dfrac{s^2+1}{s^3+4s}$

(xii) $\dfrac{s^4+2s^3+3s^2+1}{s^4+s^3+3s^2+2s+1}$

(xiii) $\dfrac{s^2+2s+4}{(s+1)(s+3)}$

(xiv) $\dfrac{2s+4}{s+5}$

(xv) $\dfrac{s^2+2s}{s^2+1}$

(xvi) $\dfrac{s^2+4}{s^3+3s^2+3s+1}$

10.3 Determine whether the following functions are *LC, RC* or *RL* function:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(iii) $Z(s) = \dfrac{(s+1)(s+8)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$

(v) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

(vi) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

(vii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(viii) $F(s) = \dfrac{(s+1)(s+2)}{s(s+3)}$

(ix) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(x) $Z(s) = \dfrac{(s+2)(s+4)}{(s+1)}$

(xi) $Y(s) = \dfrac{4(s+3)}{(s+1)(s+5)}$

(xii) $Y(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(xiii) $Z(s) = \dfrac{s(s^2+4)(s^2+16)}{(s^2+9)(s^2+25)}$

(xiv) $Z(s) = \dfrac{(s^2+1)(s^2+8)}{s(s^2+4)}$

10.4 Realise the following functions in Foster I form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $F(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(v) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

(vi) $Y(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

(vii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

10.5 Realise the following functions in Foster II form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $Z(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Y(s) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)}$

(v) $Z(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

10.6 Realise the following functions in Cauer I form:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(iii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(v) $F(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(vi) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{6(s+2)(s+4)}{s(s+3)}$

(viii) $Z(s) = \dfrac{s^3+2s}{s^4+4s^2+3}$

(ix) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(x) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

10.7 Realise the following function in Cauer II form:

(i) $F(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(ii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iv) $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(v) $F(s) = \dfrac{s^3+12s^2+32s}{s^2+7s+6}$

(vi) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(viii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

10.8 An impedance function has the pole-zero diagram as shown in Fig. 10.70 below. If $Z(-2) = 3$, synthesise the impedance function in Foster and Cauer forms.

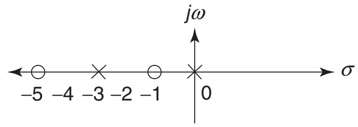


Fig. 10.70

10.9 An impedance function has the pole-zero diagram as shown in Fig. 10.71. Find the impedance function such that $Z(-4) = \dfrac{8}{3}$ and realise in Cauer forms.

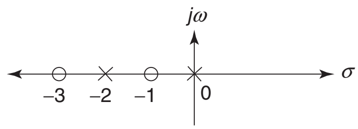


Fig. 10.71

10.10 For the realisation of a given function $F(s)$.

$$F(s) = \frac{K_0}{s} + \sum_{i=1}^{n} \frac{sK_i}{(s^2+\omega_i^2)} + sK_\infty$$

where K_0, K_i ($i = 1, 2, 3, \ldots n$) and K_∞ are constants.

(i) Mention the type of function (*RC*, *RL* or *LC*)

(ii) Given that $K_0 = 6$, $K_1 = 8$, $\omega_1 = 4$, $K_2 = 10$, $\omega_2 = 8$, $K_\infty = 5$, find the component values of realised network for $F(s) = Z(s)$ and $F(s) = Y(s)$. Draw neat diagrams.

Objective-Type Questions

10.1 The necessary and sufficient condition for a rational function $F(s)$ to be the driving-point impedance of an *RC* network is that all poles and zeros should be

(a) simple and lie on the negative real axis in the *s*-plane
(b) complex and lie in the left half of *s*-plane
(c) complex and lie in the right-half of *s*-plane
(d) simple and lie on the positive real axis of the *s*-plane

10.2 The number of roots of $s^3 + 5s^2 + 7s + 3 = 0$ in the left half of *s*-plane is

(a) zero (b) one
(c) two (d) three

10.3 The first and the last critical frequencies of a driving-point impedance function of a passive network having two kinds of elements, are a pole and a zero respectively. The above property will be satisfied by

(a) *RL* network only
(b) *RC* network only
(c) *LC* network only
(d) *RC* as well as *RL* network

10.4 The pole-zero pattern of a particular network is shown in Fig. 10.72. It is that of an

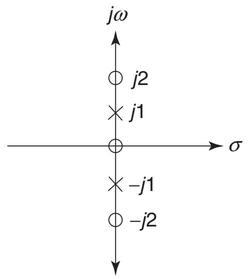


Fig. 10.72

(a) *LC* network (b) *RC* network
(c) *RL* network (d) none of these

10.5 The first critical frequency nearest to the origin of the complex frequency plane for an *RL* driving-point impedance function will be

(a) a zero in the left-half plane
(b) a zero in the right-half plane
(c) a pole in the left-half plane
(d) a pole in the right-half plane

10.6 Consider the following polynomials:
$P_1 = s^8 + 2s^6 + 4s^4$
$P_2 = s^6 - 3s^2 + 2s^2 + 1$
$P_3 = s^4 + 3s^3 + 3s^2 + 2s + 1$
$P_4 = s^7 + 2s^6 + 2s^4 + 4s^3 + 8s^2 + 8s + 4$
which one of these polymials is not Hurwitz?

(a) P_1 (b) P_2
(c) P_3 (d) P_4

10.7 For very high frequencies, the driving-point admittance function, $Y(s) = \dfrac{4(s+1)(s+3)}{s(s+2)(s+4)}$ behaves as

(a) a resistance of $\dfrac{3}{2}\,\Omega$
(b) a capacitance of 4 F
(c) an inductance of $\dfrac{1}{4}$ H
(d) an inductance of 4 H

10.8 The driving-point impedance $Z(s) = \dfrac{s+3}{s+4}$ behaves as

(a) a resistance of 0.75 Ω at low frequencies
(b) a resistance of 1 Ω at high frequencies
(c) both (a) and (b) above
(d) none of the above

10.9 An *RC* driving-point impedance function has zeros at $s = -2$ and $s = -5$. The admissible poles for the functions would be

(a) $s = 0, s = -6$ (b) $s = -1, s = -3$
(c) $s = 0, s = -1$ (d) $s = -3, s = -4$

10.10 Consider the following from the point of view of possible realisation as driving-point impedances using passive elements:

1. $\dfrac{1}{s(s+5)}$ 2. $\dfrac{s+3}{s^2(s+5)}$

3. $\dfrac{s^2+3}{s^2(s^2+5)}$ 4. $\dfrac{s+5}{s(s+5)}$

Of these, the realisable are
(a) 1, 2 and 4 (b) 1, 2 and 3
(c) 3 and 4 (d) none of these

10.11 The poles and zeros of a driving-point function of a network are simple and interlace on the negative real axis with a pole closest to the origin. It can be realised

(a) by an *LC* network
(b) as an *RC* driving point impedance
(c) as an *RC* driving point admittance
(d) only by an *RLC* network

10.12 If $F_1(s)$ and $F_2(s)$ are two positive real functions then the function which is always positive real, is

(a) $F_1(s)\,F_2(s)$ (b) $\dfrac{F_1(s)}{F_2(s)}$

(c) $\dfrac{F_1(s)F_2(s)}{F_1(s)+F_2(s)}$ (d) $F_1(s) - F_2(s)$

10.13 The circuit shown in Fig. 10.73 is

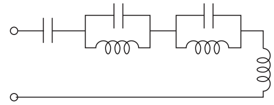


Fig. 10.73

(a) Cauer I form (b) Foster I form
(c) Cauer II form (d) Foster II form

10.14 For an *RC* driving-point impedance function, the poles and zeros

(a) should alternate on the real axis
(b) should alternate only on the negative real axis
(c) should alternate on the imaginary axis
(d) can lie anywhere on the left half-plane

Answers to Objective-Type Questions

10.1 (a)	10.2 (a)	10.3 (b)	10.4 (a)	10.5 (a)	10.6 (b)	10.7 (c)
10.8 (c)	10.9 (b)	10.10 (a)	10.11 (b)	10.12 (c)	10.13 (b)	10.14 (b)

11 Filters

11.1 INTRODUCTION

Filters are frequency-selective networks that attenuate signals at some frequency and allow others to pass with or without attenuation. A filter is constructed from purely reactive elements. Ideally, filters should produce no attenuation in the desired band, called *pass band* and should provide attenuation at all other frequencies called *attenuation band* or *stop band*. The frequency which separates the pass band and the stop band is called *cut-off frequency*. Filter networks are widely used in communication systems to separate various channels in carrier-frequency telephone circuits.

11.2 CLASSIFICATION OF FILTERS

On the basis of frequency characteristics, filters are classified into four categories:

(i) Low-pass filter
(ii) High-pass filter
(iii) Band-pass filter
(iv) Band-stop filter

A *low-pass filter* allows all frequencies up to a certain cut-off frequency to pass through it and attenuates all the other frequencies above the cut-off frequency.

A *high-pass filter* attenuates all the frequency below the cut-off frequency and allows all other frequencies above the cut-off frequency to pass through it.

A *band-pass filter* allows a limited band of frequencies to pass through it and attenuates all other frequencies below or above the frequency band.

A *band-stop filter* attenuates a limited band of frequencies but allows all other frequencies to pass through it.

11.3 T-NETWORK

Figure 11.1 shows a *T* network.

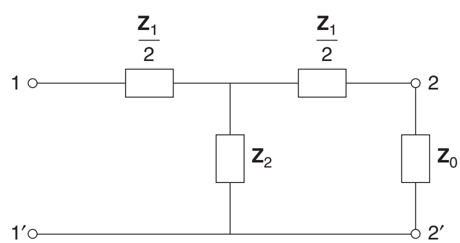


Fig. 11.1 *T-network*

11.3.1 Characteristic Impedance

For a T network, the value of input impedance when it is terminated by characteristic impedance $\mathbf{Z}_0$, is given by

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}$$

But $$\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$$

$$\mathbf{Z}_0 = \frac{\mathbf{Z}_1}{2} + \frac{2\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_2\mathbf{Z}_0}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0}{2(\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0)}$$

$$2\mathbf{Z}_1\mathbf{Z}_0 + 4\mathbf{Z}_2\mathbf{Z}_0 + 4{\mathbf{Z}_0}^2 = \mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0$$

$$4\mathbf{Z}_0^2 = \mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0^2 = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Hence, characteristic impedance for symmetrical T section is given by,

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $$\mathbf{Z}_{oc} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 = \frac{\mathbf{Z}_1 + 2Z_2}{2}$$

Short-circuit impedance $$\mathbf{Z}_{sc} = \frac{\mathbf{Z}_1}{2} + \frac{\frac{\mathbf{Z}_1}{2}\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + 2\mathbf{Z}_2} = \frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}$$

$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \left(\frac{\mathbf{Z}_1 + 2\mathbf{Z}_2}{2}\right)\left(\frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}\right) = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2 = \mathbf{Z}_{0T}^2$$

$$\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

11.3.2 Propagation Constant

The propagation constant γ of the network in Fig. 11.2 is given by,

$$\gamma = \log_e \frac{I_1}{I_2}$$

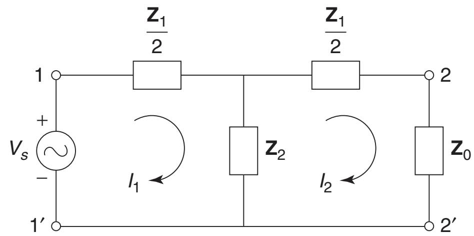


Fig. 11.2 *T-network*

Applying KVL to Mesh 1,

$$V_s - \frac{\mathbf{Z}_1}{2} I_1 - \mathbf{Z}_2 (I_1 - I_2) = 0$$

$$V_s = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 \right) I_1 - \mathbf{Z}_2 I_2$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2 (I_2 - I_1) - \frac{\mathbf{Z}_1}{2} I_2 - \mathbf{Z}_0 I_2 = 0$$

$$-\mathbf{Z}_2 I_1 + \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2 = 0$$

$$\mathbf{Z}_2 I_1 = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2$$

$$\frac{I_1}{I_2} = \frac{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}{\mathbf{Z}_2} = e^{\gamma}$$

$$\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 = \mathbf{Z}_2 e^{\gamma}$$

$$\mathbf{Z}_0 = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

The characteristic impedance of the *T*-network is given by,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

$$\sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2} = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

$$\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2 = \mathbf{Z}_2^2 (e^{\gamma} - 1)^2 + \frac{\mathbf{Z}_1^2}{4} - \mathbf{Z}_1 \mathbf{Z}_2 (e^{\gamma} - 1)$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 (1 + e^{\gamma} - 1) = 0$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 e^{\gamma} = 0$$

$$\mathbf{Z}_2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 e^{\gamma} = 0$$

$$(e^{\gamma} - 1)^2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2} e^{\gamma}$$

$$e^{2\gamma} - 2e^{\gamma} + 1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2 e^{-\gamma}}$$

$$e^{-\gamma} (e^{2\gamma} - 2e^{\gamma} + 1) = \frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$e^{\gamma}+e^{-\gamma}-2=\frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

Dividing both the sides by 2,

$$\frac{e^{\gamma}+e^{-\gamma}}{2}=1+\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\cosh\gamma=1+\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\sinh\gamma=\sqrt{\cosh^2\gamma-1}=\sqrt{\left(1+\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2-1}=\sqrt{1+\frac{\mathbf{Z}_1}{\mathbf{Z}_2}+\frac{\mathbf{Z}_1^2}{4\mathbf{Z}_2^2}-1}=\sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2}+\left(\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2}$$

$$=\frac{1}{\mathbf{Z}_2}\sqrt{\mathbf{Z}_1\mathbf{Z}_2+\frac{\mathbf{Z}_1^2}{4}}=\frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2}$$

$$\tanh\gamma=\frac{\sinh\gamma}{\cosh\gamma}=\frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2\left(1+\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)}=\frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2+\frac{\mathbf{Z}_1}{2}}$$

But $\mathbf{Z}_{0T}=\sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$

and $\mathbf{Z}_{oc}=\frac{\mathbf{Z}_1}{2}+\mathbf{Z}_2$

$$\tanh\gamma=\sqrt{\frac{\mathbf{Z}_{sc}}{\mathbf{Z}_{oc}}}$$

Also, $\sinh\frac{\gamma}{2}=\sqrt{\frac{1}{2}(\cosh\gamma-1)}=\sqrt{\frac{1}{2}\left(1+\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}-1\right)}=\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$

11.4 π NETWORK

Figure 11.3 shows a π-network.

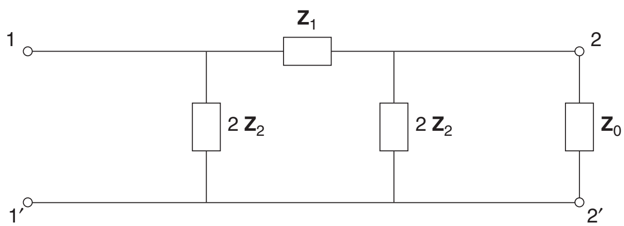


Fig. 11.3 *π-network*

11.4.1 Characteristic Impedance

For a π network, the value of input impedance when it is terminated by impedance $\mathbf{Z}_0$, is given by,

$$\mathbf{Z}_{\text{in}}=\frac{2\mathbf{Z}_2\left(\mathbf{Z}_1+\frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2+\mathbf{Z}_0}\right)}{\mathbf{Z}_1+\frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2+\mathbf{Z}_0}+2\mathbf{Z}_2}$$

But
$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$$

$$\mathbf{Z}_0 = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0}\right)}{\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_2}$$

$$\mathbf{Z}_0\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0^2}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_0\mathbf{Z}_2 = \frac{2\mathbf{Z}_2(2\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_0\mathbf{Z}_1 + 2\mathbf{Z}_0\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_0}$$

$$2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_0^2 + 2\mathbf{Z}_0\mathbf{Z}_2^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 + 2\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2 + 2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_0\mathbf{Z}_2^2$$

$$\mathbf{Z}_1\mathbf{Z}_0^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2$$

$$\mathbf{Z}_0^2(\mathbf{Z}_1 + 4\mathbf{Z}_2) = 4\mathbf{Z}_1\mathbf{Z}_2^2$$

$$\mathbf{Z}_0^2 = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$$

$$\mathbf{Z}_0 = \sqrt{\frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}}$$

Dividing both the sides by $4Z_2$,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}$$

Hence, characteristic impedance of a symmetrical π network is given by

$$\mathbf{Z}_{0\pi} = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}}$$

But
$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $\mathbf{Z}_{oc} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_1 + 2\mathbf{Z}_2} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$

Short-circuit impedance $\mathbf{Z}_{sc} = \dfrac{2\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_2 + \mathbf{Z}_1}$

$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\mathbf{Z}_{0\pi} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

11.4.2 Propagation Constant

The propagation constant of a symmetrical π network is same as that of a symmetrical T network.

11.5 CHARACTERISTIC OF FILTERS

A filter transmits or passes desired range of frequencies without loss and attenuates all undesired frequencies.

The propagation constant

$$\gamma = \alpha + j\beta$$

where α is attenuation constant and β is the phase constant. We know that

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\sinh\left(\frac{\alpha + j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Depending upon the type of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, there are two cases:

Case (i) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are same type of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ and $\sinh\frac{\gamma}{2}$ are real.

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = 0$$

$$\sin\frac{\beta}{2} = 0 \qquad \text{if } \beta = 0 \text{ or } n\pi \text{ where } n = 0, 1, 2, \ldots \quad \left[\because \cosh\frac{\alpha}{2} \text{ cannot be zero}\right]$$

$$\cos\frac{\beta}{2} = 1$$

$$\sinh\frac{\alpha}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\alpha = 2\sinh^{-1}\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Case (ii) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative, i.e. $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < 0$ and $\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$ is purely imaginary.

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0$$

$$j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = j\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

The two conditions of operation of the filter, viz. the pass band and stop band are mathematically obtained from these equations. Both the above equations must be satisfied simultaneously by α and β. Two conditions may arise

(a) $\sinh\dfrac{\alpha}{2} = 0$ i.e. $\alpha = 0$ when $\beta \neq 0$

Since $\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\sin\dfrac{\beta}{2}$ is real,

$$\sin\frac{\beta}{2} = \left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right| \qquad \text{as } \cosh\frac{\alpha}{2} = 1$$

This signifies the region of zero attenuation or pass band which is limited by the upper limit of sine terms.

$$\sin\frac{\beta}{2} = 1$$

$$-1 < \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < 0$$

The phase angle in the pass band is given by,

$$\frac{\beta}{2} = \sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\beta = 2\sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

(b)
$$\cos\frac{\beta}{2} = 0$$

$$\sin\frac{\beta}{2} = \pm 1$$

Since $\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\cosh\dfrac{\alpha}{2}$ is real,

$$\cosh\frac{\alpha}{2} = \left|\sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2}}\right|$$

This signifies a stop band since $\alpha \neq 0$.

$$\frac{\alpha}{2} = \cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\alpha = 2\cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

and
$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < -1$$

11.6 CONSTANT-*k* LOW PASS FILTER

A T or π network is said to be of the constant *k* type if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances satisfying the relation

$$\mathbf{Z}_1\mathbf{Z}_2 = k^2$$

where k is a constant, independent of frequency. k is often referred to as *design impedance* or *nominal impedance* of the constant-k filter. The constant-k, T or π-type filter is also known as the prototype filter because other complex networks can be derived from it. Figure 11.4 shows constant-k, T and π-section filters.

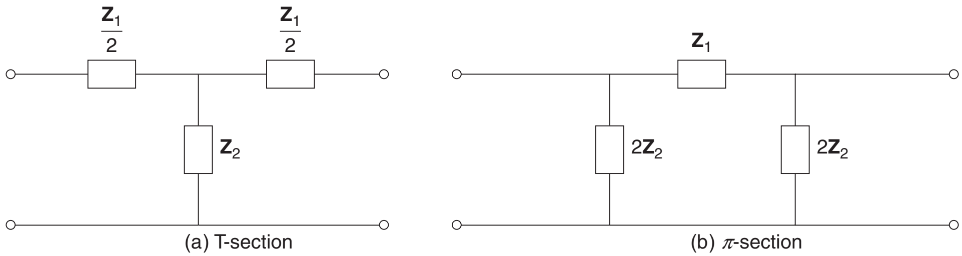


Fig. 11.4 *Constant–k Filter*

In constant-k low pass filter,

$$\mathbf{Z}_1 = j\omega L$$

$$\mathbf{Z}_2 = -j\frac{1}{\omega c} = \frac{1}{j\omega c}$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{(j\omega L)\left(\frac{1}{j\omega c}\right)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$$

$$\mathbf{Z}_1 = 0$$

$$j\omega L = 0$$

$$\omega = 0$$

$$f = 0$$

(ii) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$j\omega L = \frac{4j}{\omega C}$$

$$\omega^2 LC = 4$$

$$\omega^2 = \frac{4}{LC}$$

$$\omega = \omega_c = \frac{2}{\sqrt{LC}}$$

$$f = f_c = \frac{1}{\pi\sqrt{LC}}$$

Hence, the pass band starts at $f = 0$ and continues up to the cut-off frequency f_c. All the frequencies above f_c are in the attenuation or stop band.

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $$\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\sqrt{\frac{\omega^2 LC}{4}} = 2\cosh^{-1}\sqrt{\frac{\omega^2}{\omega_c{}^2}}$$

$$= 2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

The attenuation constant α is zero throughout the pass band but increases gradually from the cut-off frequency. The variation of α is plotted in Fig. 11.5.

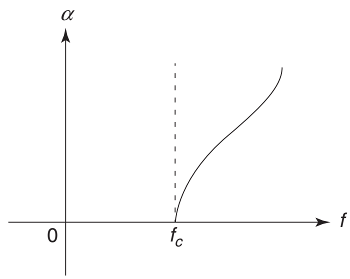


Fig. 11.5 *Variation of α with frequency*

4. Phase Constant

In pass band,

$$\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f}{f_c}\right)$$

In stop band, $\beta = \pi$.

The phase constant β is zero at zero frequency and increases gradually through the pass band, reaches π at cut-off frequency f_c and remains at π for all frequency beyond f_c. The variation of β is plotted in Fig. 11.6.

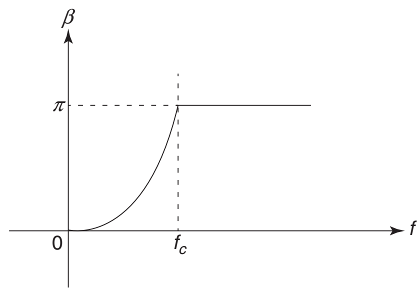


Fig. 11.6 *Variation of β with frequency*

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)} = k\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} = k\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}}$$

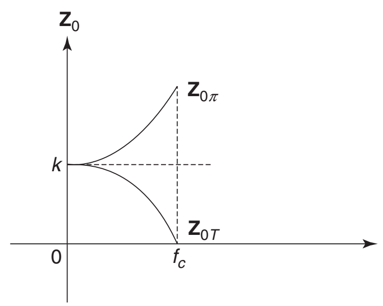


Fig. 11.7 *Variation of characteristic impedance with frequency*

The plot of characteristic impedance versus frequency is shown in Fig. 11.7.

The characteristic impedance $\mathbf{Z}_{0T}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0T} = 0$. For $f > f_c$, $\mathbf{Z}_{0T}$ is imaginary in the stop band, rising to infinite reactance at infinite frequency. $\mathbf{Z}_{0\pi}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0\pi}$ is infinite and for $f > f_c$, $\mathbf{Z}_{0\pi}$ is imaginary.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

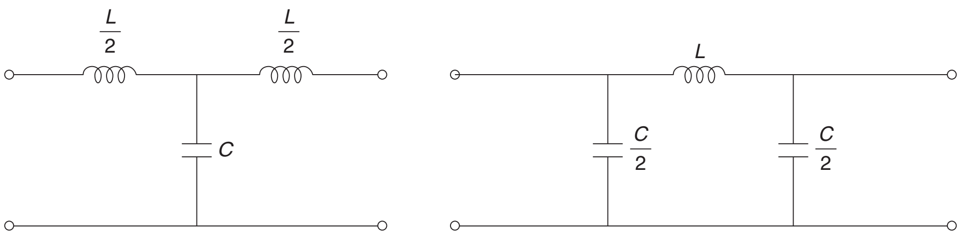
$$f_c = \frac{1}{\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{\pi f_c}$$

$$C = \frac{1}{\pi f_c k}$$

The constant-k, T and π section filters are shown in Fig. 11.8.



(a) T-section (b) π-section

Fig. 11.8 *Constant-k low-pass filter*

Example 11.1 *Find the nominal impedance, cut-off frequency and pass band for the network shown in Fig. 11.9.*

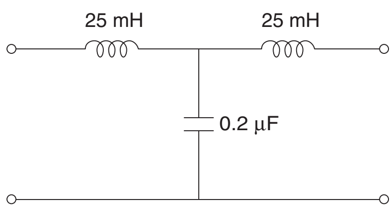


Fig. 11.9

Solution The network is a low-pass filter.

$$\frac{L}{2} = 25 \text{ mH}, \quad C = 0.2 \text{ μF}$$

$$L = 50 \text{ mH}$$

(a) Nominal Impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50 \times 10^{-3}}{0.2 \times 10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{50 \times 10^{-3} \times 0.2 \times 10^{-6}}} = 3.18\ \text{kHz}$$

(c) Pass band

The pass band is from zero to 3.18 kHz.

Example 11.2 *A low-pass filter is composed of a symmetrical π section. Each series branch is a 0.02 H inductor and shunt branch is a 2 μF capacitor. Find (a) cut-off frequency, (b) nominal impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase shift constant at 200 Hz and 2000 Hz.*

Solution

$$L = 0.02\ \text{H}, \quad \frac{C}{2} = 2\ \mu\text{F}$$

$$C = 4\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{0.02 \times 4 \times 10^{-6}}} = 1125.4\ \text{Hz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.02}{4 \times 10^{-6}}} = 70.71\ \Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1 - \left(\frac{200}{1125.4}\right)^2}} = 71.85\ \Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1 - \left(\frac{2000}{1125.4}\right)^2}} = j48.13\ \Omega$$

(d) Attenuation at 200 Hz and 2000 Hz

The 200 Hz frequency lies in the pass band.

$$\alpha = 0$$

At 2000 Hz,

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{2000}{1125.4}\right) = 2.35$$

(e) Phase shift constant at 200 Hz and 2000 Hz

At 200 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right) = 2\sin^{-1}\left(\frac{200}{1125.4}\right) = 20.47°$$

2000 Hz frequency lies in the attenuation band.

$$\beta = 180°$$

Example 11.3 *A π section filter network consists of a series arm inductor of 20 mH and two shunt-arm capacitors of 0.16 μF each. Calculate the cut-off frequency, and attenuation and phase shift at 15 kHz. What is the value of the nominal impedance in the pass band?*

Solution

$$L = 20 \text{ mH}, \quad \frac{C}{2} = 0.16\ \mu\text{F}$$

$$C = 0.32\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{20\times10^{-3}\times0.32\times10^{-6}}} = 3.98 \text{ kHz}$$

(b) Attenuation at 15 kHz

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{15\times10^3}{3.98\times10^3}\right) = 4$$

(c) Phase shift at 15 kHz
15 kHz frequency lies in the attenuation band.

$$\beta = 180°$$

(d) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{20\times10^{-3}}{0.32\times10^{-6}}} = 250\ \Omega$$

Example 11.4 *Each of the two series elements of a T-section low-pass filter consists of an inductor of 60 mH having negligible resistance and a shunt element having a capacitance of 0.2 μF. Calculate (a) the cut-off frequency, (b) nominal impedance, and (c) characteristic impedance at frequencies of 1 kHz and 5 kHz.*

Solution

$$L = 60 \text{ mH}, \quad C = 0.2\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{60\times10^{-3}\times0.2\times10^{-6}}} = 2.91 \text{ kHz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{60\times10^{-3}}{0.2\times10^{-6}}} = 547.72\ \Omega$$

(c) Characteristic impedance at 1 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{1\times10^3}{2.91\times10^3}\right)^2} = 514.36\,\Omega$$

Characteristic impedance at 5 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{5\times10^3}{2.91\times10^3}\right)^2} = j765.29\ \Omega$$

Example 11.5 *Design a constant-k low-pass T and π section filters having cut-off frequency of 4 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_c = 4\text{ kHz},\quad k = 500\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi\times4\times10^3} = 39.79\text{ mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi\times4\times10^3\times500} = 0.16\ \mu\text{F}$$

The T section consists of an inductor of $\frac{L}{2}$, i.e. 19.9 mH in each series branch and a capacitor of 0.16 μF in the shunt branch as shown in Fig. 11.10 (a). The π-section consists of an inductor of 39.79 mH in the series branch and a capacitor of $\frac{C}{2}$, i.e. 0.08 μF in each shunt branch as shown in Fig. 11.10 (b).

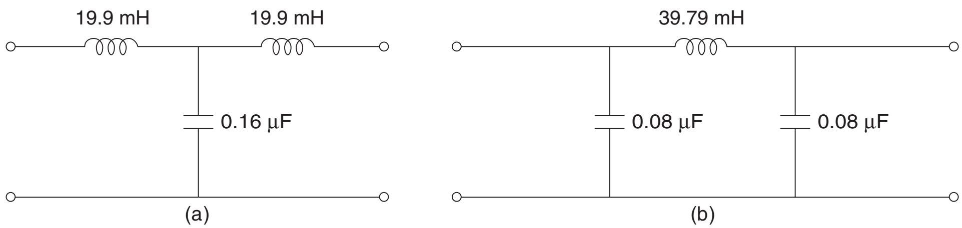


Fig. 11.10

Example 11.6 *Design a constant-k, T section low-pass filter is having 2.5 kHz cut-off frequency and nominal impedance of 700 Ω. Also find the frequency at which this filter produces an attenuation of 19.1 dB. Find its characteristic impedances and phase constant at pass band and stop or attenuation band.*

Solution

$$f_c = 2.5\text{ kHz},\quad k = 700\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{700}{\pi\times2.5\times10^3} = 89.13\text{ mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi\times2.5\times10^3\times700} = 0.18\ \mu\text{F}$$

The T-section filter consists of an inductor of $\frac{L}{2}$, i.e. 44.57 mH in each series arm and a capacitor of 0.18 μF in the shunt arm as shown in Fig. 11.11.

$$\alpha = 19.1 \text{ dB} = 2.197 \text{ nepers}$$

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

$$2.197 = 2\cosh^{-1}\left(\frac{f}{2.5\times10^3}\right)$$

$$f = 4.17 \text{ kHz}$$

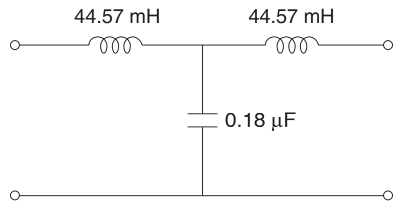


Fig. 11.11

β is imaginary in pass band.

In attenuation band, $\beta = 180°$

11.7 CONSTANT-*k* HIGH-PASS FILTER

A constant-*k* high-pass filter is obtained by changing the positions of series and shunt arm of the constant-*k* low-pass filter, Figure 11.12 shows a constant-*k*, *T* and π section, high-pass filter.

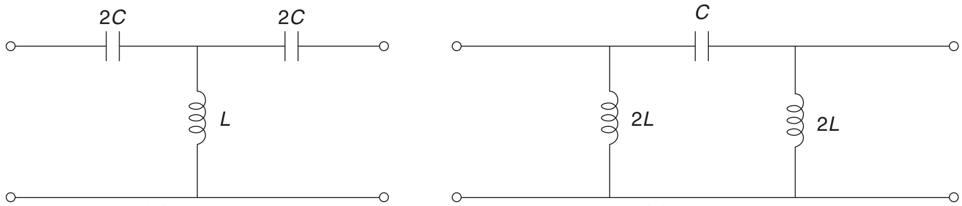


Fig. 11.12 *Constant-k high-pass filter*

In a constant-*k* high-pass filter

$$\mathbf{Z}_1 = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

$$\mathbf{Z}_2 = j\omega L$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\left(\frac{1}{j\omega C}\right)(j\omega L)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When $$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$$

$$\mathbf{Z}_1 = 0$$

$$\frac{1}{j\omega C} = 0$$

$$\omega = \infty$$

$$f = \infty$$

(ii) When $$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$-j\frac{1}{\omega_c} = 4j\omega L$$

$$\omega^2 LC = \frac{1}{4}$$

$$\omega^2 = \frac{1}{4LC}$$

$$\omega = \omega_c = \frac{1}{2\sqrt{LC}}$$

$$f = f_c = \frac{1}{4\pi\sqrt{LC}}$$

Hence, the filter passes all the frequencies beyond f_c. The pass band starts at $f = f_c$ and continues up to infinite frequency. All the frequencies below the cut-off frequency lie in the attenuation or stop band.

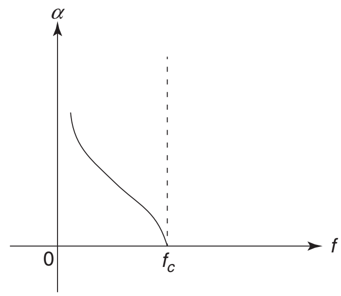


Fig. 11.13 *Variation of α with frequency*

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\left(\frac{f_c}{f}\right)$

The attenuation constant α decreases gradually to zero at the cut-off frequency and remains at zero through the pass band. The variation of α is plotted in Fig. 11.13.

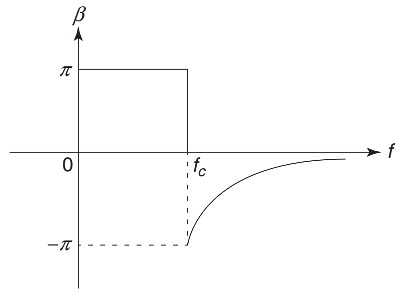


Fig. 11.14 *Variation of β with frequency*

4. Phase Constant

In pass band, $\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f_c}{f}\right)$

In stop band, $\beta = \pi$.

The phase constant β remains at constant value π in the stop band and decreases to $-\pi$ in at f_c and reaches zero value gradually as f increases in the pass band. The variation of β is plotted in Fig. 11.14.

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)} = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

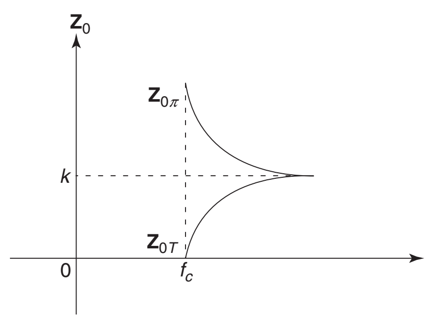


Fig. 11.15 *Variation of characteristic impedance with frequency*

The variation of characteristic impedance is plotted in Fig. 11.15.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{4\pi f_c}$$

$$C = \frac{1}{4\pi f_c k}$$

Example 11.7 *Find the characteristic impedance, cut-off frequency and pass band for the network shown in Fig. 11.16.*

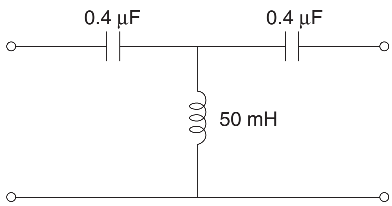


Fig. 11.16

Solution The network is a high-pass filter.

$$2C = 0.4\ \mu\text{F}, \quad L = 50\ \text{mH}$$

$$C = 0.2\ \mu\text{F}$$

(a) Characteristic impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50\times10^{-3}}{0.2\times10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 795.77\ \text{Hz}$$

(c) Pass band
The pass band is from 795.77 Hz to infinite frequency.

Example 11.8 *A high-pass filter section is constructed from two capacitors of 1 μF each and a 15 mH inductance. Find (a) cut-off frequency, (b) infinite frequency characteristic impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase constant at 200 Hz and 2000 Hz.*

Solution

$$L = 15\ \text{mH}, \quad 2C = 1\ \mu\text{F}$$

$$C = 0.5\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{15\times10^{-3}\times0.5\times10^{-6}}} = 918.88 \text{ Hz}$$

(b) Infinite-frequency characteristic impedance

At $f = \infty$,
$$\mathbf{Z}_0 = k = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.5\times10^{-3}}{0.5\times10^{-6}}} = 173.21\ \Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = j776.72\ \Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = 153.85\ \Omega$$

(d) Attenuation at 200 Hz

$$\alpha = 2\cosh^{-1}\left(\frac{f_c}{f}\right) = 2\cosh^{-1}\left(\frac{918.88}{200}\right) = 4.41$$

The frequency of 2000 Hz lies in the pass band.

$$\alpha = 0$$

(e) Phase constant

The frequency 200 Hz lies in the attenuation band.

$$\beta = \pi \text{ radians}$$

At 2000 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{918.88}{2000}\right) = 54.7°$$

Example 11.9 *Design a constant-k high-pass T and π sections filters having a cut-off frequency of 2000 Hz and infinite frequency characteristic impedance of 300 Ω.*

Solution

$$f_c = 2000 \text{ Hz}, \quad k = 300\ \Omega$$

$$L = \frac{k}{4\pi f_c} = \frac{300}{4\pi\times2000} = 11.94 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi\times2000\times300} = 0.13\ \mu\text{F}$$

The T-section filter consists of a capacitor of $2C$, i.e. 0.26 μF in each series arm and an inductor of 11.94 mH in shunt arm as shown in Fig. 11.17 (a).

The π-section filter consists of a capacitor of 0.13 μF in the series arm and an inductor of $2L$, i.e. 23.88 mH in each shunt arm as shown in Fig. 11.17 (b).

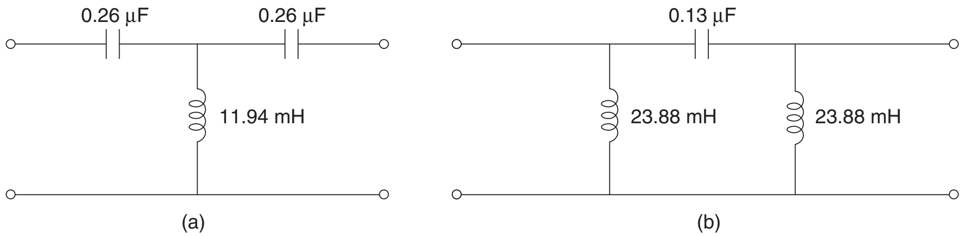


Fig. 11.17

Example 11.10 *Design a T-section constant-k high-pass filter having a cut-off frequency of 10 kHz and a design impedance of 600 Ω. Find its characteristic impedance and phase constant at 25 kHz.*

Solution

$$f_c = 10 \text{ kHz}, \quad k = 600\ \Omega$$

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 10 \times 10^3} = 4.77 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 10 \times 10^3 \times 600} = 0.013\ \mu\text{F}$$

The *T*-section filter consists of a capacitor of $2C$, i.e. 0.026 μF in each series arm and an inductor of 4.77 mH in shunt arm as shown in Fig. 11.18.

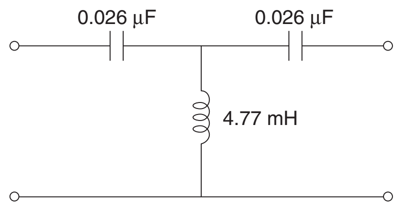


Fig. 11.18

(a) Characteristic impedance at 25 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 600\sqrt{1-\left(\frac{10\times10^3}{25\times10^3}\right)^2} = 549.91\ \Omega$$

(b) Phase constant at 25 kHz

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{10\times10^3}{25\times10^3}\right) = 47.16°$$

The frequency of 25 kHz lies in the attenuation band.

In the attenuation band, $\beta = 180° = \pi$ radians

11.8 BAND-PASS FILTER

A band-pass filter attenuates all the frequencies below a lower cut-off frequency and above an upper cut-off frequency. It passes a band of frequencies without attenuation. A band-pass filter is obtained by using a low pass filter followed by a high-pass filter.

Figure 11.19 shows a band-pass filter. The series arm is a series resonant circuit comprising L_1 and C_1 while its shunt arm is formed by a parallel resonant circuit L_2 and C_2. The resonant frequency of series arm and shunt arm are made equal.

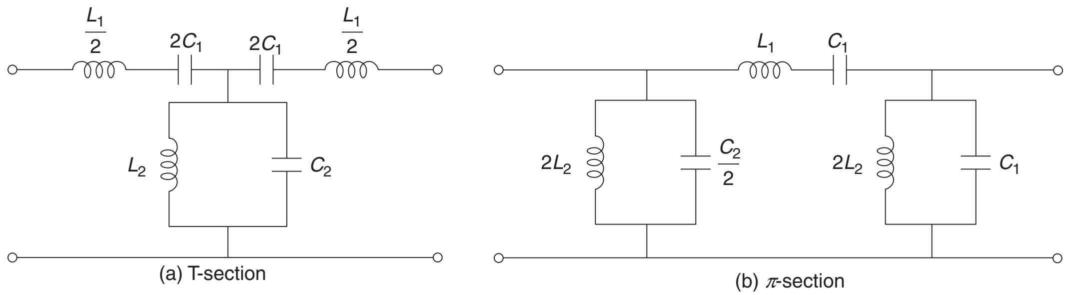


Fig. 11.19 *Band-pass filter*

For series arm,

$$\omega_0 \frac{L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

For shunt arm,

$$\frac{1}{\omega_0 C_2} = \omega_0 L_2$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

For series arm,

$$\mathbf{Z}_1 = j\omega L_1 - \frac{j}{\omega c_1} = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)$$

For shunt arm,

$$\mathbf{Z}_2 = \frac{j\omega L_2 \dfrac{1}{j\omega c_2}}{j\omega L_2 + \dfrac{1}{j\omega c_2}} = \frac{j\omega L_2}{1-\omega^2 L_2 C_2}$$

$$\mathbf{Z}_1\mathbf{Z}_2 = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)\left(\frac{j\omega L_2}{1-\omega^2 L_2 C_2}\right) = -\frac{L_2}{C_1}\left(\frac{\omega^2 L_1 C_1 - 1}{1-\omega^2 L_2 C_2}\right) = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

where k is constant.
For constant k filter, at cut-off frequency,

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1^2 = -4\mathbf{Z}_1\mathbf{Z}_2 = -4k^2$$

$$\mathbf{Z}_1 = \pm j2k$$

i.e. the value of $\mathbf{Z}_1$ at lower cut-off frequency is equal to the negative value of $\mathbf{Z}_1$ at the upper cut-off frequency.

$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -\left(\frac{1}{j\omega_2 C_1} + j\omega_2 L_1\right)$$

$$1 - \omega_1^2 L_1 C_1 = \frac{\omega_1}{\omega_2}\left(\omega_2^2 L_1 C_1 - 1\right)$$

But
$$\omega_0^2 = \frac{1}{L_1 C_1}$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{\omega_1}{\omega_2}\left(\frac{\omega_2^2}{\omega_0^2} - 1\right)$$
$$(\omega_0^2 - \omega_1^2)\omega_2 = \omega_1(\omega_2^2 - \omega_0^2)$$
$$\omega_0^2\omega_2 - \omega_1^2\omega_2 = \omega_1\omega_2^2 - \omega_1\omega_0^2$$
$$\omega_0^2(\omega_1 + \omega_2) = \omega_1\omega_2(\omega_2 + \omega_1)$$
$$\omega_0^2 = \omega_1\omega_2$$
$$\omega_0 = \sqrt{\omega_1\omega_2}$$
$$f_0 = \sqrt{f_1 f_2}$$

Thus, resonant frequency is the geometric mean of the cutoff frequencies. The variation of attenuation, phase constant and characteristic impedance with frequency are shown in Fig. 11.20.

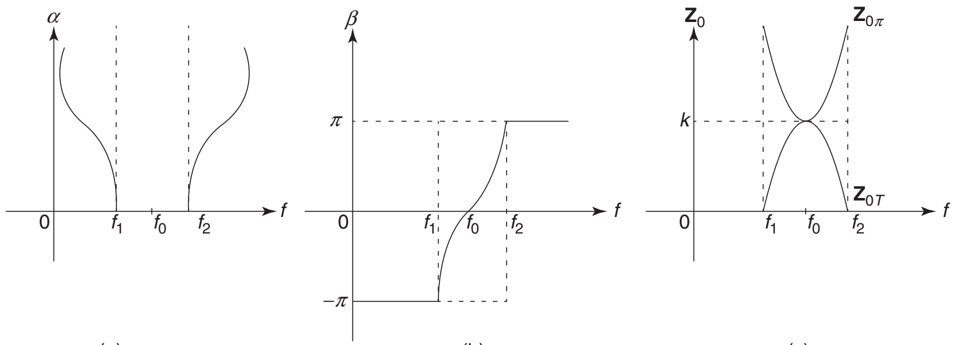


Fig. 11.20 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant–k band-pass filter*

Design of Filter

If the filter is terminated in a load resistance $R = k$ then at lower cut-off frequency,
$$\mathbf{Z}_1 = -2jk$$
$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -2jk$$
$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2k$$
$$1 - \omega_1^2 L_1 C_1 = 2k\omega_1 C_1$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = 2k\omega_1 C_1 \qquad \left(\because\ \omega_0^2 = \frac{1}{L_1 C_1}\right)$$
$$1 - \left(\frac{f_1}{f_0}\right)^2 = 4\pi k f_1 C_1$$

$$1 - \frac{f_1^2}{f_1 f_2} = 4\pi k f_1 C_1 \qquad \left(\because f_0 = \sqrt{f_1 f_2}\right)$$

$$f_2 - f_1 = 4\pi k f_1 f_2 C_1$$

$$C_1 = \frac{f_2 - f_1}{4\pi\, k f_1 f_2}$$

$$L_1 = \frac{1}{\omega_0^2 C_1} = \frac{4\pi\, k f_1 f_2}{\omega_0^2 (f_2 - f_1)} = \frac{4\pi\, k f_1 f_2}{4\pi^2\, f_0^2 \left(f_2 - f_1\right)} = \frac{k}{\pi(f_2 - f_1)}$$

For shunt arm,

$$\mathbf{Z}_1 \mathbf{Z}_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

$$L_2 = C_1 k^2 = \frac{(f_2 - f_1)k}{4\pi\, f_1 f_2}$$

$$C_2 = \frac{L_1}{k^2} = \frac{1}{\pi(f_2 - f_1)k}$$

Example 11.11 *In a constant-k band-pass filter, the ratio of capacitances in the shunt and series arms is 100:1. The resonant frequency of both arms is 1000 Hz. Find the bandwidth,*

Solution

$$\frac{C_2}{C_1} = \frac{100}{1} = 100, \quad f_0 = 1000 \text{ Hz}$$

$$C_2 = \frac{1}{\pi k (f_2 - f_1)}$$

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2}$$

$$\frac{C_2}{C_1} = \frac{1}{\pi k (f_2 - f_1)} \frac{4\pi k f_1 f_2}{f_2 - f_1}$$

$$100 = \frac{4 f_1 f_2}{(f_2 - f_1)^2}$$

$$100 = \frac{4 f_0^2}{(f_2 - f_1)^2}$$

$$f_2 - f_1 = \frac{2 f_0}{\sqrt{100}} = \frac{2 \times 1000}{\sqrt{100}} = 200$$

$$BW = f_2 - f_1 = 200 \text{ Hz}$$

Example 11.12 *Design a band pass constant-k filter with cut-off frequency of 4 kHz and 10 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_1 = 4 \text{ kHz}, \quad f_2 = 10 \text{ kHz}, \quad k = 500\ \Omega$$

$$L_1 = \frac{k}{\pi(f_2 - f_1)} = \frac{500}{\pi(10 \times 10^3 - 4 \times 10^3)} = 26.53 \text{ mH}$$

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2} = \frac{10 \times 10^3 - 4 \times 10^3}{4\pi \times 500 \times 4 \times 10^3 \times 10 \times 10^3} = 0.024\ \mu\text{F}$$

$$L_2 = \frac{(f_2 - f_1)k}{4\pi f_1 f_2} = \frac{(10\times 10^3 - 4\times 10^3)\,500}{4\pi\times 4\times 10^3\times 10\times 10^3} = 5.97\text{ mH}$$

$$C_2 = \frac{1}{\pi(f_2 - f_1)k} = \frac{1}{\pi\times(10\times 10^3 - 4\times 10^3)\times 500} = 0.11\,\mu\text{F}$$

The T-section filter consists of series combination of an inductor of $\frac{L_1}{2}$, i.e. 13.27 mH and a capacitor of 2 C_1, i.e. 0.048 μF in each series arm and a parallel combination of an inductor of 5.97 mH and a capacitor of 0.11 μF in shunt arm as shown in Fig. 11.21.

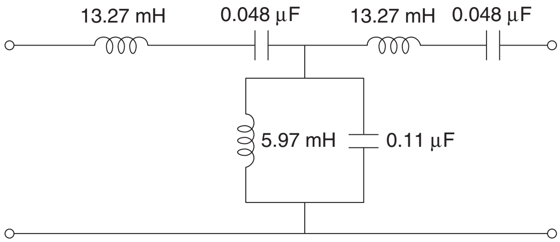


Fig. 11.21

The π-section filter consists of a series combination of an inductor 26.53 mH and a capacitor of 0.024 μF in the series arm and a parallel combination of an inductor of $2L_2$, i.e. 11.94 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.055 μF in each shunt arm as shown in Fig. 11.22.

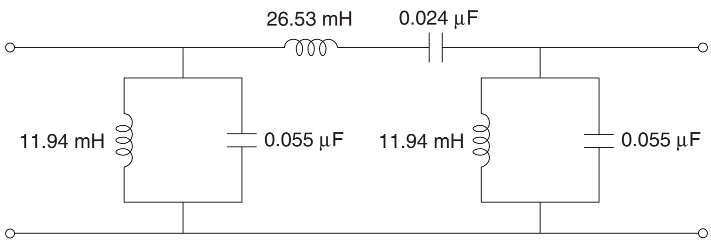


Fig. 11.22

11.9 BAND-STOP FILTER

A band-stop filter attenuates a specified band of frequencies and allows all frequencies below and above this band. A band-stop filter is realised by connecting a low-pass filter in parallel with a high-pass filter. Figure 11.23 shows a band-stop filter.

As in the band-pass filter, the series and shunt arms are chosen to resonate at same frequency ω_0.

For series arm,

$$\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

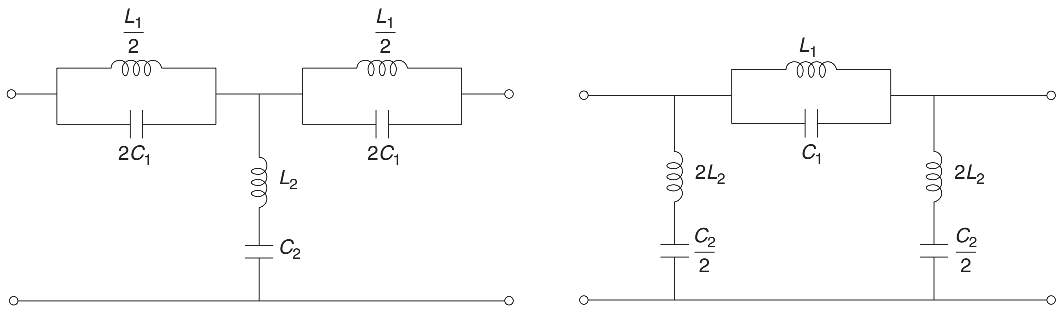


Fig. 11.23 *Band-stop filter*

For shunt arm,

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2}$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

Similarly,
$$\mathbf{Z}_1\mathbf{Z}_2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} = k^2$$

and
$$f_0 = \sqrt{f_1 f_2}$$

At cut-off frequencies,
$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1\mathbf{Z}_2 = -4\mathbf{Z}_2^2 = k^2$$

$$\mathbf{Z}_2 = \pm j\frac{k}{2}$$

The variation of attenuation, phase shift and characteristic impedance with frequency are shown in Fig. 11.24.

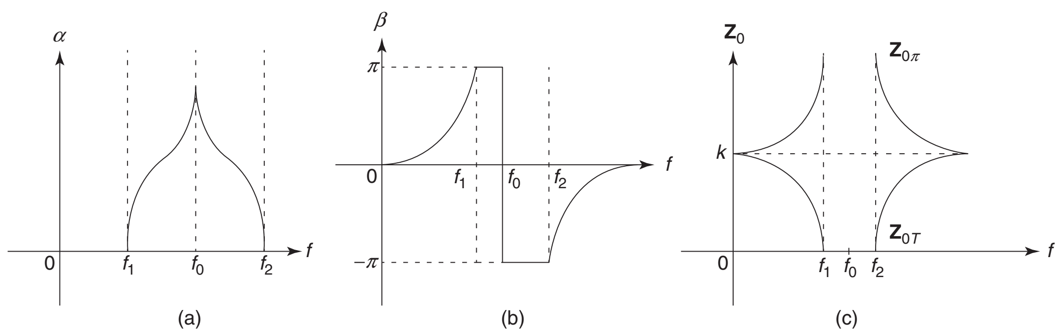


Fig 11.24 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant-k band-stop filter.*

Design of Filter

If the load is terminated in load resistance $R = k$ then at lower cut-off frequency,

$$\mathbf{Z}_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = j\frac{k}{2}$$

$$\frac{1}{\omega_1 C_2} - \omega_1 L_2 = \frac{k}{2}$$

$$1 - \omega_1^2 L_2 C_2 = \omega_1 C_2 \frac{k}{2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{k}{2}\omega_1 C_2 \qquad \left(\because \omega_0^2 = \frac{1}{L_2 C_2}\right)$$

$$1 - \left(\frac{f_1}{f_0}\right)^2 = k\pi f_1 C_2$$

$$C_2 = \frac{1}{k\pi\ f_1}\left[1 - \left(\frac{f_1}{f_0}\right)^2\right] = \frac{1}{k\pi}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = \frac{1}{k\pi}\left(\frac{f_2 - f_1}{f_1 f_2}\right) = \frac{f_2 - f_1}{\pi k\ f_1 f_2}$$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{\pi k f_1 f_2}{\omega_0^2 (f_2 - f_1)} = \frac{\pi k f_1 f_2}{4\pi^2 f_0^2 (f_2 - f_1)} = \frac{k}{4\pi(f_2 - f_1)}$$

$$k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1}$$

$$L_1 = k^2 C_2 = \frac{k(f_2 - f_1)}{\pi f_1 f_2}$$

$$C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi\ k(f_2 - f_1)}$$

Example 11.13 *Design a constant-k band-stop filter having cut-off frequencies at 2000 Hz and 5000 Hz and characteristic resistance of 600 Ω.*

Solution

$$f_1 = 2000 \text{ Hz}, \quad f_2 = 5000 \text{ Hz}, \quad k = 600\ \Omega$$

$$L_1 = \frac{k(f_2 - f_1)}{\pi f_1 f_2} = \frac{600(5000 - 2000)}{\pi \times 2000 \times 5000)} = 57.3 \text{ mH}$$

$$C_1 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{1}{4\pi \times 600 \times (5000 - 2000)} = 0.044\ \mu\text{F}$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)} = \frac{600}{4\pi \times (5000 - 2000)} = 15.92 \text{ mH}$$

$$C_2 = \frac{f_2 - f_1}{\pi k f_1 f_2} = \frac{5000 - 2000}{\pi \times 600 \times 2000 \times 5000} = 0.16\ \mu\text{F}$$

The *T*-section filter consists of a parallel combination of $\frac{L_1}{2}$, i.e. 28.65 mH and a capacitor of 2 C_1, i.e. 0.088 μF in each series arm and a series combination of an inductor of 11.92 mH and a capacitor of 0.16 μF in shunt arm as shown in Fig. 11.25.

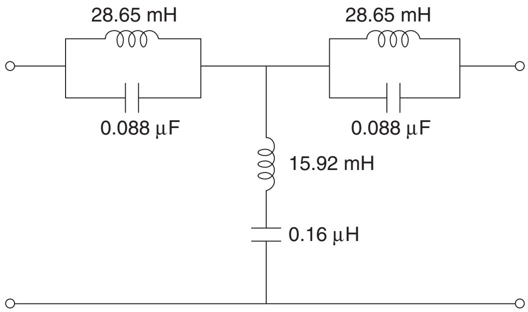


Fig. 11.25

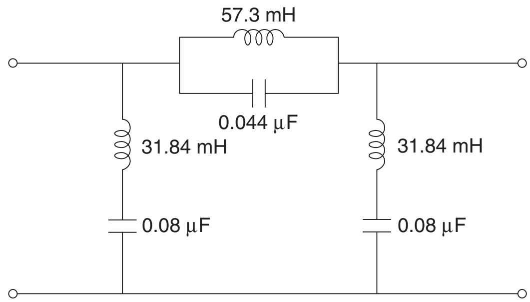


Fig. 11.26

The π-section filter consists of a parallel combination of an inductor of 57.3 mH and a capacitor of 0.044 μF in the series arm and a series combination of an inductor of $2L_2$, i.e. 31.84 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.08 μF in each shunt arm as shown in Fig. 11.26.

11.10 TERMINATING HALF SECTIONS

A filter is composed of a number of sections. As the characteristic impedance of an equivalent T or π section does not match with each other, a half section is used for impedance matching between T and π sections.

The input impedance of the half section is same as the characteristic impedance of a T section, while the output impedance of the half section is same as the characteristic impedance of the π section. Hence, matching can be obtained if a half section is connected in between a T and π section.

Constant-*k* Half Sections

Figure 11.27(a) shows a constant-k T-section filter. If this section is bisected longitudinally, a half section is obtained as shown in Fig. 11.27(b).

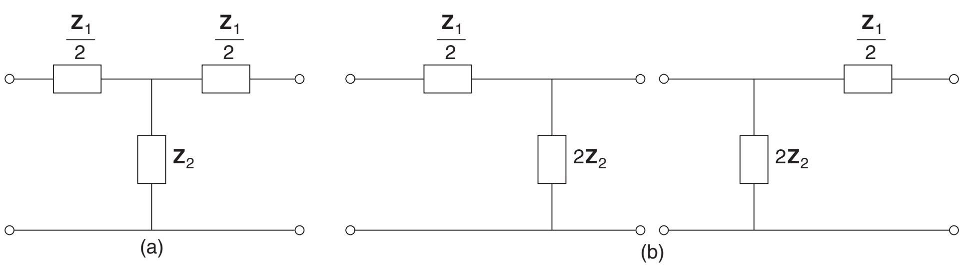


Fig. 11.27 *(a) Constant-k T-section (b) constant-k half T-section*

Similarly, a constant-k π section can be bisected into two half sections as shown in Fig. 11.28.

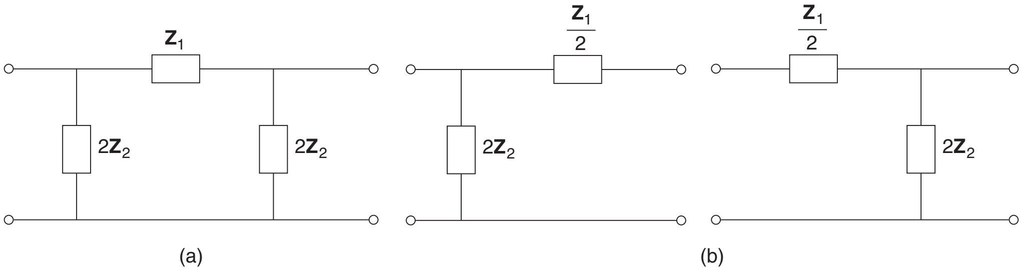


Fig. 11.28 *(a) Constant-k π-section (b) constant-k half π-section*

Figure 11.29 shows a constant-k half section. The image impedance of the section as seen from terminals 1-2 and terminals 3-4 can be found from the open circuit and short-circuit impedances.

$$\mathbf{Z}_{12} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\frac{(2\mathbf{Z}_2)\left(2\mathbf{Z}_2\,\frac{\mathbf{Z}_1}{2}\right)}{2\mathbf{Z}_2 + \frac{\mathbf{Z}_1}{2}}} = \sqrt{\frac{4\mathbf{Z}_1\,\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}} = \mathbf{Z}_{0\pi}$$

$$\mathbf{Z}_{34} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\left(\frac{\mathbf{Z}_1}{2} + 2\mathbf{Z}_2\right)\left(\frac{\mathbf{Z}_1}{2}\right)}$$

$$= \sqrt{\mathbf{Z}_1\,\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)} = \mathbf{Z}_{0T}$$

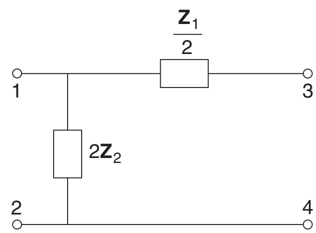


Fig. 11.29 *Constant-k half section*

Thus, the half section has the impedance characteristics of a π section between terminals 1-2 and that of a T section between terminals 3-4. Hence, this half section can be used to match a π section to a T section. It can also be used to match a filter section to a terminating impedance which differs from the characteristic impedance of a π section.

Example 11.14 *Find the values of shunt and series element of each half section of a constant-k T-section high-pass filter. The termination is 600 Ω and the filter is cut off below 20000 Hz.*

Solution $k = 600\ \Omega$, $f_c = 20000$ Hz

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 20000} = 2.39 \text{ mH}$$

$$C = \frac{1}{4\pi f_c\, k} = \frac{1}{4\pi \times 20000 \times 600} = 0.0066\ \mu\text{F}$$

The half section consists of a capacitor of $2C$, i.e. 0.0133 μF in the series branch and an inductor of $2L$, i.e. 4.78 mH in the shunt branch as shown in Fig. 11.30.

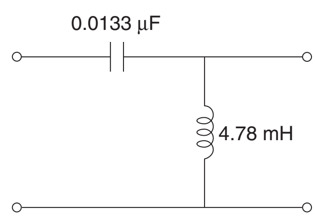


Fig. 11.30

Exercises

11.1 A T-section low-pass filter has a series inductance of 80 mH and a shunt capacitance of 0.022 μF. Find the cut-off frequency and nominal impedance. Also design the equivalent π-section.

$$\left[\begin{array}{r} f_c = 7.587 \text{ kHz}, \quad k = 1.907 \text{ k}\Omega, L = 80 \text{ mH}, \\ \dfrac{C}{2} = 0.011\ \mu\text{F} \end{array}\right]$$

11.2 Calculate the frequency of a constant-k, T-section low-pass filter having a cut-off frequency of 1000 Hz at which it has an attenuation of 10 dB.

$$[f = 1170 \text{ Hz}]$$

11.3 Design a low-pass constant-k (a) T-section, and (b) π-section filter with $f_c = 6$ kHz and $R_O = 500\ \Omega$. Calculate α and β for the filters for $f = 10$ kHz. Also determine the frequency at which the attenuation is 10 dB.

$$\left[\begin{array}{r} \alpha = 2.2 \text{ neper}, \quad \beta = 1.46 \text{ radians}, \\ \beta = \pi \text{ at } 10 \text{ kHz} T \rightarrow \dfrac{L}{2} = 13.25 \text{ mH}, \\ C = 0.106\ \mu\text{F} \pi \rightarrow L = 26.5 \text{ mH}, \\ \dfrac{C}{2} = 0.053\ \mu\text{F} \\ f = 7 \text{ kHz} \end{array}\right]$$

11.4 Design a π-section constant-k high-pass filter having a cut-off frequency $f_c = 8$ kHz and nominal characteristic impedance $k = 600\ \Omega$. Find (a) its characteristic impedance at a frequency of 12 kHz, (b) phase constant at frequency of 12 kHz, (c) attenuation at frequency of 800 Hz.

$$\left[\begin{array}{r} L = 5.971 \text{ mH}, \quad C = 0.016\ \mu\text{F}, \\ Z_{O\pi} = 805\ \Omega, \quad \beta = 83.6°, \quad \alpha = 5 \text{ nepers} \end{array}\right]$$

11.5 Design a band-pass constant-k filter with $f_1 = 2$ kHz and $f_2 = 3$ kHz and $k = 500\ \Omega$.

$$\left[\begin{array}{l} L_1 = 159 \text{ mH}, \quad C_1 = 0.026\ \mu\text{F}, \\ L_2 = 20.8 \text{ mH}, \quad C_2 = 0.637\ \mu\text{F} \end{array}\right]$$

11.6 Design a band-stop constant-k filter with cut-off frequencies of 8 kHz and 12 kHz and nominal characteristic impedance of 500 Ω.

$$\left[\begin{array}{l} L_1 = 6.6 \text{ mH}, \quad C_1 = 0.0398\ \mu\text{F}, \\ L_2 = 9.9 \text{ mH}, \quad C_2 = 0.0265\ \mu\text{F} \end{array}\right]$$

11.7 In a constant-k band-pass filter, the ratio of capacitance in the shunt and series arms is 50:1 and the resonant frequency of both arms is 1000 Hz. Find the bandwidth of the filter.

[282.84 Hz]

Objective–Type Questions

11.1 For the design of low-pass prototype filter Fig. 11.31 of with load resistance R_L = 1 Ω and angular frequency $\omega = 1$ rad/s the values of L and C would be

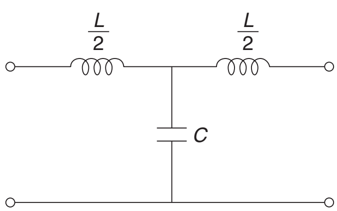


Fig. 11.31

(a) 1 H and I F (b) 1 H and 2 F
(c) 2 H and I F (d) 2 H and 2 F

11.2 The passband of a typical filter network with Z_1 and Z_2 as the series and shunt-arm impedances is characterised by

(a) $-1 < \dfrac{Z_1}{4Z_2} < 0$ (b) $-1 < \dfrac{Z_1}{4Z_2} < 1$

(c) $0 < \dfrac{Z_1}{4Z_2} < 1$ (d) none of the above

11.3 The application of the bisection theorem finds out an equivalent lattice network if the original network is

(a) symmetrical and balanced only
(b) unsymmetrical and balanced only
(c) symmetrical, balanced and unbalanced only
(d) symmetrical and unbalanced only

11.4 Z_1 and Z_2 are the total series and shunt impedances of a T or π-filter. Consider the following zones of operation of filters and the conditions on the impedances

(a) Pass band 1. $\frac{Z_1}{4Z_2} < -1$

(b) Stop band 2. $\frac{Z_1}{4Z_2} = -1$

(c) Transition band 3. $-1 < \frac{Z_1}{4Z_2} < 0$

Tick the correct combination:

	A	B	C
(a)	1	2	3
(b)	3	1	2
(c)	3	2	1
(d)	2	3	1

11.5 Tick out the correct statement in case of a filter.

(a) Characteristic impedance is resistive in stop band
(b) Characteristic impedance is reactive in pass band
(c) Characteristic impedance is resistive in pass band
(d) None of the above

11.6 If L is the total series inductance and C the total shunt capacitacne of a T or π-type low-page filter, the pass band frequency range of the filter is given as

(a) 0 to $\frac{1}{2\pi\sqrt{L_C}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{L_C}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{L_C}}$ Hz

(d) 0 to $\frac{1}{4\pi\sqrt{L_C}}$ Hz

11.7 If C is the total series capatance, L is the total shunt inductance of a T or π-type high-pass filter, the frequency range for the stop band of the filter is

(a) 0 to $\frac{1}{2\pi\sqrt{L_C}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{L_C}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{L_C}}$ Hz

(d) 0 to $\frac{2}{4\pi\sqrt{L_C}}$ Hz

11.8 If f_1 and f_2 are the lower and uppper cut-off frequencies of the band pass filter, the series impedance Z_1 is

(a) capacitive at f_1
(b) inductive at f_1
(c) resistive at f_2
(d) none of the above

11.9 The phase constant β of a filter during stop band is

(a) Zero radian (b) 2
(c) π (d) 2π

Answers to Objective-Type Questions

11.1 (d) 11.2 (a) 11.3 (c) 11.4 (b) 11.5 (c) 11.6 (b) 11.7 (d)
11.8 (a) 11.9 (c)

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