Circuits and Networks

Analysis and Synthesis

Fifth Edition

About the Authors

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Dr Sudhakar has visited several countries like Malaysia, Singapore, Russia, and Sweden to present papers in international conferences. He has published more than 50 papers in various national and international journals and has co-authored several books, all published by McGraw Hill Education (India). He has completed several projects and is working on various new research and sponsored projects funded by AICTE and UGC, Govt of India. His fields of interest include Circuits, Signals and Systems, and Array Antennas. He received the Prathibha Puraskar from T R Mahesh Charitable Trust, Guntur, for his services in the field of Higher Education, Andhra Pradesh on 19th July, 2014. He has received a Certificate of Excellence for his outstanding services, achievements, and contributions to the field of Electronics and Communication Engineering by the IETE Vijayawada centre on the occasion of "IETE Foundation Day" on 2 November, 2014. Dr A Sudhakar, as Head of the Institution, received the Prestigious Indira Gandhi National Service Scheme Award 2013–14 from the President of India for the NSS Unit of RVR & JC College of Engineering.

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Both the authors also have, to their credit, books specific for regional technical universities, such as for Jawaharlal Nehru Technological University, published by McGrawHill Education (India).

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Dedicated to our

Parents

and

Students

Contents

Preface

INTRODUCTION TO THE COURSE

This textbook is exclusively designed for electrical engineering students studying a basic level course in circuit analysis offered to undergraduate students of various universities. This edition is prepared with students and instructors in mind. The principal objectives of the book continue to be to provide an introduction to basic concepts for circuit analysis, and to develop a strong foundation that can be used as the basis for further study. To achieve these objectives, emphasis has been placed on basic laws, theorems, and techniques which are used to develop a working knowledge of the methods of analysis, used in further topics of electrical engineering. The mathematical complexity of the book remains at a level well within the grasp of college first-year undergraduate students.

TARGET AUDIENCE

This book is designed for the third semester of EEE/ECE/EI/CSE students of various universities in the country. This book enables the student have a firm grasp on the basic principles of Circuits and Networks: Analysis and Synthesis. It lays emphasis on the basic laws, theorems, and techniques of analysis which helps students develop the ability to design practical circuits that perform the desired operations.

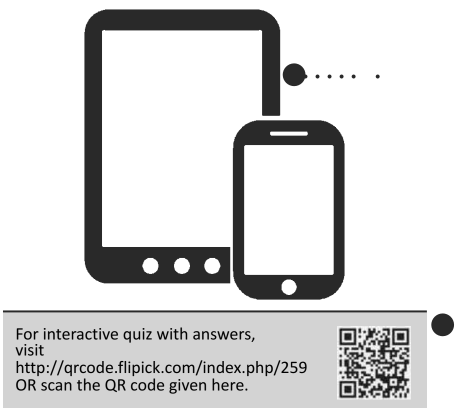
OBJECTIVE OF THE REVISION

The main objective of the revision was to align this extremely popular content with internationally approved learning objectives for the course. Learning Objectives are the heart of every lesson, giving a purpose to learning. They are the foundations for lesson planning so that the students have a sense of purpose to learning and to know what is expected of them.

WHAT'S NEW IN THE FIFTH EDITION

We received a great deal of useful feedback on the fourth edition, and we paid careful attention to it. While attempting to improve the book in all dimensions, our main aim was increasing its usefulness to students and instructors. The revised text and its underlying concepts & principles will be more interesting-to-read, easy-to-understand, and logical-to-follow. Given here is a quick review of the principle newness of the fifth edition over the fourth.

Each of the 19 chapters follows a common structure with a range of learning and assessment tools for instructors and students.



Use of Technology

In bringing out the fifth edition, we have taken advantage of recent technological developments to create a wealth of useful information not present in the physical book. For students using smartphones and tablets, scanning **QR codes** located within the chapters gives them immediate access to more resources.

The QR code appearing at the last page of each chapter gives students access to additional chapter resources which include Interactive Quizzes.

Learning Tools

❖ Learning Objectives

Each chapter begins with a list of key Learning Objectives that are directly tied to the chapter content. These help in focussed planning for instructors and methodical studying for students. The chapters are now more modularised this will help in systematic concept development.

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Draw the tree, co-tree, twigs and links for a given network
LO 2 Describe incidence matrix and its properties; analyse the relationship between KCL and incidence matrix
LO 3 Describe the link currents and tie-set matrix
LO 4 Describe cut-set and tree branch voltages
LO 5 Analyse the network (resistive circuits) using mesh analysis and supermesh analysis and write the mesh equations using inspection method
LO 6 Analyse the network (resistive circuits) using nodal analysis and supernode analysis and write the nodal equations using inspection method
LO 7 Analyse the network (resistive circuits) using source transformation technique

❖ Arrangement of Pedagogy

The pedagogy is arranged as per levels of difficulty to enable the students to evaluate their learning levels. This assessment of levels of difficulty is derived from Bloom's taxonomy.

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

☆☆★ indicates Level 1 and Level 2 i.e., Knowledge and Comprehension based easy-to-solve problems

☆★★ indicates Level 3 and Level 4 i.e., Application and Analysis based medium-difficulty problems

★★★ indicates Level 5 and Level 6 i.e., Synthesis and Evaluation based high-difficulty problems

❖ Definitions and Important Formulae

Features like Definition and Important Formulas are highlighted within the text to draw special attention to important concepts

The average power is expressed in watts. It means the useful power transferred from the source to the load, which is also called true power. If we consider a dc source applied to the network, true power is given by the product of the voltage and the current. In case of sinusoidal voltage applied to the circuit, the product of voltage and current is not the true power or average power. This product is called *apparent power*. The apparent power is expressed in volt amperes, or simply VA.

$\therefore$ ***Apparent power*** $= V_{eff} I_{eff}$

In Eq. (6.10), the average power depends on the value of cos θ; this is called the *power factor* of the circuit.

$$\therefore \qquad \text{Power factor (pf)} = \cos\theta = \frac{P_{av}}{V_{eff}\, I_{eff}}$$

Therefore, power factor is defined as the ratio of average power to the apparent power, whereas apparent power is the product of the effective values of the current and the voltage. ***Power factor*** *is also defined as the factor with which the volt amperes are to be multiplied to get true power in the circuit.*

In the case of sinusoidal sources, the power factor is the cosine of the phase angle between voltage and current

$$pf = \cos\theta$$

As the phase angle between voltage and total current increases, the power factor decreases. The smaller the power factor, the smaller the power dissipation. The power factor varies from 0 to 1. For purely resistive circuits, the phase angle between voltage and current is zero, and hence the power factor is unity. For purely reactive circuits, the phase angle between voltage and current is 90°, and hence the power factor is zero. In an *RC* circuit, the power factor is referred to as *leading* power factor because the current leads the voltage. In an *RL* circuit, the power factor is referred to as lagging power factor because the current lags behind the voltage.

Pedagogy for Student Success

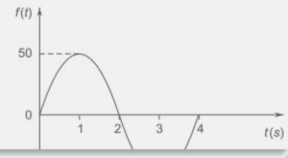
❖ Improved and Expanded In-text Exercises

This is by far the best feature! Exam-friendly pedagogy has been arranged within the text and linked after every Learning Objective. This offers great retention through looping mechanism.

Practice Problems linked to LO 2

☆★★**13-2.1** Use step functions to write the expression for the function shown in Fig. Q.1.

★★★**13-2.2** Step functions can be used to define a window function. Thus, $u(t-1) - u(t-4)$ defines a window 1 unit high and 3 units wide located on the time axis between 1 and 4.
A function $f(t)$ is defined as follows:
$f(t) = 0,\ t \le 0$



EXAMPLE 2.7

Write the mesh current equations in the circuit shown in Fig. 2.27, and determine the currents.

Solution Assume two mesh currents in the direction as indicated in Fig. 2.28.

The mesh current equations are

$$5I_1 + 2(I_1 - I_2) = 10$$

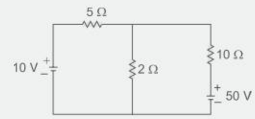


Fig. 2.27

Frequently Asked Questions linked to LO8

☆★★**9-8.1** A three-phase motor can be regarded as a balanced Y-load. A three-phase motor draws 5.6 kW when the line voltage is 220 V and the line current is 18.2 A. Determine the power factor of the motor. **[AU Nov./Dec. 2012]**

☆☆★**9-8.2** In a three-phase balanced delta system the voltage across *R* and *Y* is $400\angle 0^\circ$V. What will be the voltage across *Y* and *B*? Assume *RYB* phase sequence. **[AU April/May 2011]**

☆★★**9-8.3** A balanced Δ-connected load has one phase current $I_{BC} = 2\angle{-90^\circ}$ A. Find the other phase current and the three line currents if the system is an *ABC* system. If the line voltage is 100 V, what is the load impedance? **[AU April/May 2011]**

☆★★**9-8.4** The power consumed in a three-phase, balanced star-connected load is 2 kW at a power factor of 0.8 lagging. The supply voltage is 400 V, 50 Hz. Calculate the resistance and reactance of each phase. **[AU April/May 2011]**

Additional Solved Problems

PROBLEM 9.1

The phase voltage of a star-connected three-phase ac generator is 230 V. Calculate the (a) line voltage, (b) active power output if the line current of the system is 15 A at a power factor of 0.7, and (c) active and reactive components of the phase currents.

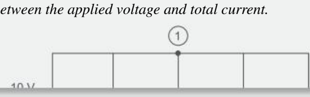
Solution The supply voltage (generator) is always assumed to be balanced

$\therefore\ V_{Ph} = 230$ V; $I_L = I_{Ph} = 15$ A, $\cos\phi = 0.7$, $\sin\phi = 0.71$

PSpice Problems

PROBLEM 5.1

For the parallel circuit shown in Fig. 5.39, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current.



Objective-Type Questions

☆☆★**9.1** The resultant voltage in a closed balanced delta circuit is given by

(a) three times the phase voltage (b) $\sqrt{3}$ times the phase voltage
(c) zero

☆☆★**9.2** Three coils *A*, *B*, *C*, displaced by 120° from each other are mounted on the same axis and rotated in a uniform magnetic field in clockwise direction. If the instantaneous value of emf in coil *A* is $E_{max}\sin\omega t$, the instantaneous value of emf in *B* and *C* coils will be

(a) $E_{max}\sin\left(\omega t - \dfrac{2\pi}{3}\right)$; $E_{max}\sin\left(\omega t - \dfrac{4\pi}{3}\right)$

(b) $E_{max}\sin\left(\omega t + \dfrac{2\pi}{3}\right)$; $E_{max}\sin\left(\omega t + \dfrac{4\pi}{3}\right)$

❖ Chapter-end Exercises

Pedagogy includes Additional Solved Problems, PSpice Problems, and Objective Type Questions, which is also integrated through QR Codes are featured at the end of the chapter.

ORGANIZATION OF THE BOOK

The basic approach of the previous edition has been retained. All the elements with definitions, basic laws, and configurations of the resistive circuits have been introduced in **Chapter 1**. Analysis of dc resistive circuits using graph theory has been discussed in **Chapter 2**. Network theorems on resistive circuits have been presented in **Chapter 3.** The concept of alternating currents and voltages has been introduced in **Chapter 4**. Due emphasis has been laid on finding out the average and rms values of different waveforms. **Chapters 5 and 6** introduce the complex impedance, and the concept of power and power factor respectively.

The steady-state analysis of ac circuits, including network theorems, has been discussed in **Chapter 7.** In all the above chapters, problems, tutorials, and objective questions on dependant sources have been discussed. Resonance phenomenon in series and parallel circuits, and locus diagrams are presented in **Chapter 8**. A comprehensive study of polyphase systems and power measurement in both balanced and unbalanced circuits is presented in **Chapter 9**. A brief study of coupled circuits, tuned circuits, and magnetic circuits is introduced in **Chapter 10.** The transient behaviour of dc and ac circuits and their responses has been discussed in **Chapter 11.** The Fourier methods of waveform analysis and their applications in circuit analysis have been discussed in **Chapter 12**.

Laplace transforms and their applications are presented in **Chapters 13 and 14.** A brief account of *S*-domain analysis is presented in **Chapter 15**. The parameters of two-port networks and their inter-relations have been discussed in **Chapter 16**. Various types of basic filters, attenuators, and equalizers have been discussed in **Chapter 17.** Elements of realizability and synthesis of one-port *RL*, *RC* networks have been briefly discussed in **Chapter 18**. A chapter on introduction to PSpice has been included as **Chapter 19**. The book also includes brief coverage of active filters and the *j*-operator as appendices.

OLC SUPPLEMENTS

The text is supported by an exhaustive website accessible at
http://highered.mheducation.com/sites/9339219600 with the following supplements:

- **For Instructors**
  - Solutions Manual
  - PowerPoint Lecture Slides
- **For Students**
  - Solutions to Frequently Asked Questions

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Despite the best efforts put in by us and our team, it is possible that some unintentional errors might have eluded us. We shall acknowledge with gratitude if any of these is pointed out. Any suggestions or comments from the readers for improving future editions of the book may please be sent to the publisher's email address.

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Piracy-related issues may also be reported.

CHAPTER 1

Circuit Elements and Kirchhoff's Laws

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Explain potential difference and its relationship with current
- LO 2 Understand electrical power and energy
- LO 3 Define circuit and network; classify the network elements
- LO 4 Explain resistance parameter and state Ohm's law
- LO 5 Explain inductance parameter
- LO 6 Explain capacitance parameter
- LO 7 Describe various energy sources
- LO 8 Explain voltage in a series circuit (KVL), voltage division, and power in a series circuit
- LO 9 Analyse current in a parallel circuit (KCL), current division, and power in a parallel circuit

1.1 VOLTAGE

LO 1 Explain potential difference and its relationship with current

According to the structure of an atom, we know that there are two types of charges: positive and negative. A force of attraction exists between these positive and negative charges. A certain amount of energy (work) is required to overcome the force and move the charges through a specific distance. All opposite charges possess a certain amount of potential energy because of the separation between them. The difference in potential energy of the charges is called the *potential difference*.

Potential difference in electrical terminology is known as voltage, and is denoted either by V or v. It is expressed in terms of energy (W) per unit charge (Q), i.e.,

$$V = \frac{W}{Q} \quad \text{or} \quad v = \frac{dw}{dq}$$

dw is the small change in energy, and
dq is the small change in charge.

where energy (W) is expressed in joules (J), charge (Q) in coulombs (C), and voltage (V) in volts (V). One volt is the potential difference between two points when one joule of energy is used to pass one coulomb of charge from one point to the other.

EXAMPLE 1.1

If 70 J of energy is available for every 30 C of charge, what is the voltage?

Solution $V = \frac{W}{Q} = \frac{70}{30} = 2.33 \text{ V}$

1.2 CURRENT

LO 1

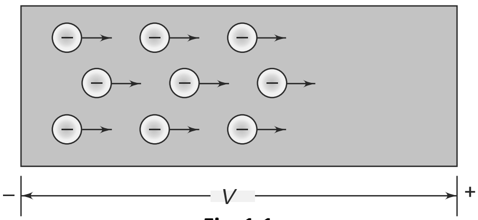


Fig. 1.1

There are free electrons available in all semiconductive and conductive materials. These free electrons move at random in all directions within the structure in the absence of external pressure or voltage. If a certain amount of voltage is applied across the material, all the free electrons move in one direction depending on the polarity of the applied voltage, as shown in Fig. 1.1.

This movement of electrons from one end of the material to the other end constitutes an electric current, denoted by either I or i. The conventional direction of current flow is opposite to the flow of −ve charges, i.e. the electrons.

Current is defined as the rate of flow of electrons in a conductive or semiconductive material. It is measured by the number of electrons that flow past a point in unit time. Expressed mathematically,

$$I = \frac{Q}{t}$$

where I is the current, Q is the charge of electrons, and t is the time, or

$$i = \frac{dq}{dt}$$

where dq is the small change in charge, and dt is the small change in time.

In practice, the unit *ampere* is used to measure current, denoted by A. One ampere is equal to one coulomb per second. One coulomb is the charge carried by 6.25×10^{18} electrons. For example, an ordinary 80 W domestic ceiling fan on 230 V supply takes a current of approximately 0.35 A. This means that electricity is passing through the fan at the rate of 0.35 coulomb every second, i.e. 2.187×10^{18} electrons are passing through the fan in every second; or simply, the current is 0.35 A.

EXAMPLE 1.2

Five coulombs of charge flow past a given point in a wire in 2 s. How many amperes of current is flowing?

Solution $I = \frac{Q}{t} = \frac{5}{2} = 2.5 \text{ A}$

1.3 POWER AND ENERGY

LO 2 Understand electrical power and energy

Energy is the capacity for doing work, i.e. energy is nothing but stored work. Energy may exist in many forms such as mechanical, chemical, electrical and so on. Power is the rate of change of energy, and is denoted by either P or p. If a certain amount of energy is used over a certain length of time, then

$$\text{Power}(P) = \frac{\text{energy}}{\text{time}} = \frac{W}{t} \quad \text{or} \quad p = \frac{dw}{dt}$$

where dw is the change in energy and dt is the change in time.

We can also write $p = \dfrac{dw}{dt} = \dfrac{dw}{dq} \times \dfrac{dq}{dt}$

$= v \times i = vi$ W

Energy is measured in joules (J), time in seconds (s), and power in watts (W).

By definition, one watt is the amount of power generated when one joule of energy is consumed in one second. Thus, the number of joules consumed in one second is always equal to the number of watts. Amounts of power less than one watt are usually expressed in fraction of watts in the field of electronics; viz. milliwatts (mW) and microwatts (μW). In the electrical field, kilowatts (kW) and megawatts (MW) are common units. Radio and television stations also use large amounts of power to transmit signals.

EXAMPLE 1.3

What is the power in watts if energy equal to 50 J is used in 2.5 s?

Solution $P = \dfrac{\text{energy}}{\text{time}} = \dfrac{50}{2.5} = 20$ W

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2*

☆☆★ **1-2.1** A resistor of 30 Ω has a voltage rating of 500 V; what is its power rating?

☆★★ **1-2.2** A 6.8 kΩ resistor has burned out in a circuit. It has to be replaced with another resistor with the same ohmic value. If the resistor carries 10 mA, what should be its power rating?

☆☆★ **1-2.3** A 12 V source is connected to a 10 Ω resistor.
(a) How much energy is used in two minutes?
(b) If the resistor is disconnected after one minute, does the power absorbed in the resistor increase or decrease?

☆☆★ **1-2.4** A capacitor is charged to 50 μC. The voltage across the capacitor is 150 V. It is then connected to another capacitor four times the capacitance of the first capacitor. Find the loss of energy.

☆★★ **1-2.5** The current in the 5 Ω resistance of the circuit shown in Fig. Q.5 is 5 A. Find the current in the 10 Ω resistor. Calculate the power consumed by the 5 Ω resistor.

★★★ **1-2.6** Find the power absorbed by the 5 Ω resistor shown in Fig. Q.6.

☆★★ **1-2.7** Find the power absorbed by each circuit element of Fig. Q.7 it the control for the dependent source is (a) $0.8\,i_x$ and (b) $0.8\,i_y$.

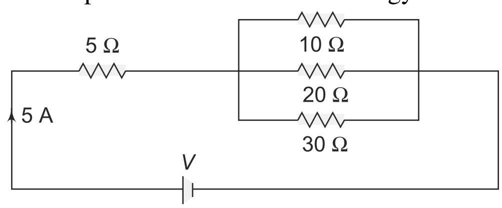


Fig. Q.5

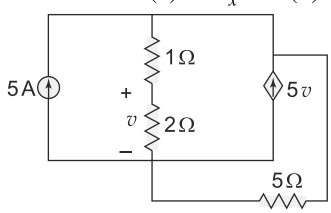


Fig. Q.6

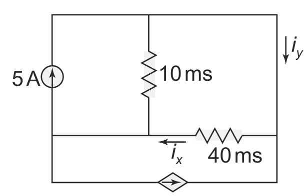


Fig. Q.7

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

☆★★ **1-2.8** Find the power absorbed by each element and show that the algebraic sum of powers is zero in the circuit shown in Fig. Q.8.

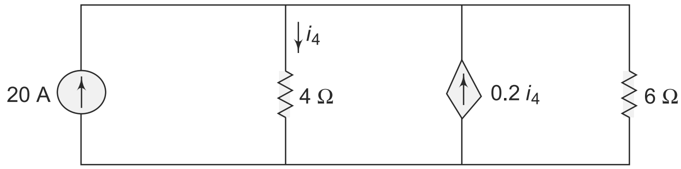


Fig. Q.8

★★★ **1-2.9** Find the power absorbed by the element X in the circuit shown in Fig. Q.9 if it is a (a) 4 kΩ resistor, (b) 20 mA independent current source, reference arrow directed towards right, and (c) dependent current source, reference arrow directed towards right, labeled $2i_x$.

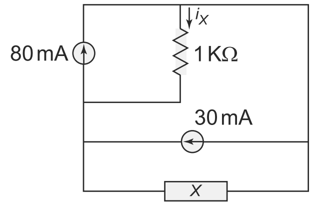


Fig. Q.9

Frequently Asked Questions linked to LO 2*

☆☆★ **1-2.1** A bulb is rated as 230 V, 230 W. Find the rated current and resistance of the filament.

[AU April/May 2011]

1.4 THE CIRCUIT

LO 3 Define circuit and network; classify the network elements

Simply put, an electric circuit consists of three parts: (1) energy source, such as battery or generator, (2) the load or sink, such as lamp or motor, and (3) connecting wires as shown in Fig. 1.2. This arrangement represents a simple circuit. A battery is connected to a lamp with two wires. The purpose of the circuit is to transfer energy from source (battery) to the load (lamp). And this is accomplished by the passage of electrons through wires around the circuit.

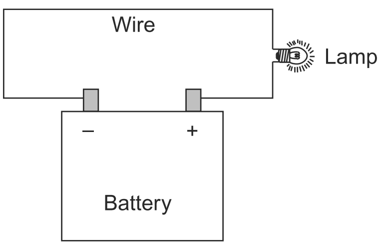


Fig. 1.2

The current flows through the filament of the lamp, causing it to emit visible light. The current flows through the battery by chemical action. A closed circuit is defined as a circuit in which the current has a complete path to flow. When the current path is broken so that current cannot flow, the circuit is called an open circuit.

More specifically, interconnection of two or more simple circuit elements (viz. voltage sources, resistors, inductors and capacitors) is called an electric network. If a network contains at least one closed path, it is called an electric circuit. By definition, a simple circuit element is the mathematical model of two terminal electrical devices, and it can be completely characterised by its voltage and current. Evidently then, a physical circuit must provide means for the transfer of energy.

Broadly, network elements may be classified into four groups, viz.,

1. Active or passive
2. Unilateral or bilateral
3. Linear or nonlinear
4. Lumped or distributed

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

1.4.1 Active and Passive

Energy sources (voltage or current sources) are active elements, capable of delivering power to some external device. Passive elements are those which are capable only of receiving power. Some passive elements like inductors and capacitors are capable of storing a finite amount of energy, and return it later to an external element. More specifically, an active element is capable of delivering an average power greater than zero to some external device over an infinite time interval. For example, ideal sources are active elements. A passive element is defined as one that cannot supply average power that is greater than zero over an infinite time interval. Resistors, capacitors, and inductors fall into this category.

1.4.2 Bilateral and Unilateral

In the bilateral element, the voltage-current relation is the same for current flowing in either direction. In contrast, a unilateral element has different relations between voltage and current for the two possible directions of current. Examples of bilateral elements are elements made of high conductivity materials in general. Vacuum diodes, silicon diodes, and metal rectifiers are examples of unilateral elements.

1.4.3 Linear and Nonlinear Elements

An element is said to be linear, if its voltage-current characteristic is at all times a straight line through the origin. For example, the current passing through a resistor is proportional to the voltage applied through it, and the relation is expressed as $V \propto I$ or $V = IR$. A linear element or network is one which satisfies the principle of superposition, i.e., the principle of homogeneity and additivity. An element which does not satisfy the above principle is called a nonlinear element.

1.4.4 Lumped and Distributed

Lumped elements are those elements which are very small in size and in which simultaneous actions take place for any given cause at the same instant of time. Typical lumped elements are capacitors, resistors, inductors and transformers. Generally, the elements are considered as lumped when their size is very small compared to the wave length of the applied signal. Distributed elements, on the other hand, are those which are not electrically separable for analytical purposes. For example, a transmission line which has distributed resistance, inductance and capacitance along its length may extend for hundreds of miles.

Frequently Asked Questions linked to LO 3

☆☆★**1-3.1** Explain the terms: (a) Linear (b) Bilateral (c) Passive (d) Lumped parameter. **[GTU Dec. 2010]**
☆☆★**1-3.2** Explain the terms: (a) Nonlinear (b) Unilateral (c) Passive (d) Lumped parameter. **[GTU Dec. 2012]**
☆☆★**1-3.3** What are the network elements? Explain them. **[JNTU Nov. 2012]**
☆☆★**1-3.4** What do you mean by a linear bilateral network? **[PTU 2011-2012]**

1.5 RESISTANCE PARAMETER

LO 4 Explain resistance parameter and state Ohm's law

When a current flows in a material, the free electrons move through the material and collide with other atoms. These collisions cause the electrons to lose some of their energy. This loss of energy per unit charge is the drop in potential across the material. The amount of energy lost by the electrons is related to the physical property of the material. These collisions restrict the movement of electrons. The property of a material to restrict the flow of electrons is called resistance, denoted by R. The symbol for the resistor is shown in Fig. 1.3.

Fig. 1.3

The unit of resistance is ohm (Ω). Ohm is defined as the resistance offered by the material when a current of one ampere flows between two terminals with one volt applied across it.

According to Ohm's law, *the current is directly proportional to the voltage and inversely proportional to the total resistance of the circuit,* i.e.,

$$I = \frac{V}{R}$$

or $$i = \frac{v}{R}$$

We can write the above equation in terms of charge as follows.

LO-4

$$V = R\frac{dq}{dt}, \text{ or } i = \frac{v}{R} = Gv$$

where G is the conductance of a conductor. The units of resistance and conductance are ohm (Ω) and mho (℧) respectively.

When current flows through any resistive material, heat is generated by the collision of electrons with other atomic particles. The power absorbed by the resistor is converted to heat. The power absorbed by the resistor is given by

$$P = vi = (iR)i = i^2 R$$

where i is the current in the resistor in amps, and v is the voltage across the resistor in volts. Energy lost in a resistance in time t is given by

$$W = \int_0^t pdt = pt = i^2 Rt = \frac{v^2}{R}t$$

where v is the volts,
R is in ohms,
t is in seconds, and
W is in joules.

EXAMPLE 1.4

A 10 Ω resistor is connected across a 12 V battery. How much current flows through the resistor?

Solution $V = IR$

$$I = \frac{V}{R} = \frac{12}{10} = 1.2 \text{ A}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★ **1-4.1** A resistor with a current of 2 A through it converts 1000 J of electrical energy to heat energy in 15 s. What is the voltage across the resistor?

☆☆★ **1-4.2** The filament of a light bulb in the circuit has a certain amount of resistance. If the bulb operates with 120 V and 0.8 A of current, what is the resistance of its filament?

Frequently Asked Questions linked to LO 4

☆☆★ **1-4.1** State the limitation of Ohm's law. **[AU May/June 2013]**

1.6 INDUCTANCE PARAMETER

LO 5 Explain inductance parameter

A wire of certain length, when twisted into a coil becomes a basic inductor. If current is made to pass through an inductor, an electromagnetic field is formed. A change in the magnitude of the current changes the electromagnetic field. Increase in current expands the fields, and decrease in current reduces it. Therefore, a change in current produces change in the electromagnetic field, which induces a voltage across the coil according to Faraday's law of electromagnetic induction.

The unit of inductance is *henry*, denoted by H. By definition, the inductance is one henry when current through the coil, changing at the rate of one ampere per second, induces one volt across the coil. The symbol for inductance is shown in Fig. 1.4.

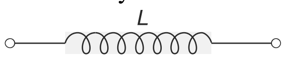


Fig. 1.4

The current-voltage relation is given by

$$v = L\frac{di}{dt}$$

where v is the voltage across inductor in volts, and i is the current through inductor in amps. We can rewrite the above equations as

$$di = \frac{1}{L}\, vdt$$

Integrating both sides, we get

$$\int_0^t di = \frac{1}{L}\int_0^t vdt$$

$$i(t) - i(0) = \frac{1}{L}\int_0^t vdt$$

$$i(t) = \frac{1}{L}\int_0^t vdt + i(0)$$

From the above equation, we note that the current in an inductor is dependent upon the integral of the voltage across its terminals and the initial current in the coil, $i(0)$.

The power absorbed by the inductor is

$$P = vi = Li\frac{di}{dt} \text{ watts}$$

The energy stored by the inductor is

$$W = \int_0^t pdt$$

$$= \int_0^t Li\frac{di}{dt}dt = \frac{Li^2}{2}$$

From the above discussion, we can conclude the following:

1. The induced voltage across an inductor is zero if the current through it is constant. That means an inductor acts as short circuit to dc.
2. A small change in current within zero time through an inductor gives an infinite voltage across the inductor, which is physically impossible. In a fixed inductor, the current cannot change abruptly.
3. The inductor can store finite amount of energy, even if the voltage across the inductor is zero, and
4. A pure inductor never dissipates energy, only stores it. That is why it is also called a non-dissipative passive element. However, physical inductors dissipate power due to internal resistance.

EXAMPLE 1.5

The current in a 2 H inductor varies at a rate of 2 A/s. Find the voltage across the inductor and the energy stored in the magnetic field after 2 s.

Solution

$$v = L\frac{di}{dt}$$
$$= 2\times 4 = 8 \text{ V}$$
$$W = \frac{1}{2}Li^2$$
$$= \frac{1}{2}\times 2\times (4)^2 = 16 \text{ J}$$

1.7 CAPACITANCE PARAMETER

LO 6 Explain capacitance parameter

Any two conducting surfaces separated by an insulating medium exhibit the property of a capacitor. The conducting surfaces are called *electrodes*, and the insulating medium is called *dielectric*. A capacitor stores energy in the form of an electric field that is established by the opposite charges on the two electrodes. The electric field is represented by lines of force between the positive and negative charges, and is concentrated within the dielectric. The amount of charge per unit voltage that is capacitor can store is its capacitance, denoted by *C*. The unit of capacitance is *Farad* denoted by *F*. By definition, one Farad is the amount of capacitance when one coulomb of charge is stored with one volt across the plates. The symbol for capacitance is shown in Fig. 1.5.

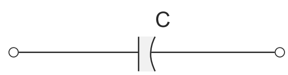


Fig. 1.5

A capacitor is said to have greater capacitance if it can store more charge per unit voltage and the capacitance is given by

$$C = \frac{Q}{V}, \text{ or } C = \frac{q}{v}$$

(lowercase letters stress instantaneous values)

We can write the above equation in terms of current as

$$i = C\frac{dv}{dt} \qquad \left(\because i = \frac{dq}{dt}\right)$$

where v is the voltage across capacitor and i is the current through it.

$$dv = \frac{1}{C} i dt$$

Integrating both sides, we have

$$\int_0^t dv = \frac{1}{C}\int_0^t i\,dt$$

$$v(t) - v(0) = \frac{1}{C}\int_0^t i\,dt$$

$$v(t) = \frac{1}{C}\int_0^t i\,dt + v(0)$$

where $v(0)$ indicates the initial voltage across the capacitor.

From the above equation, the voltage in a capacitor is dependent upon the integral of the current through it, and the initial voltage across it.

The power absorbed by the capacitor is given by

$$p = vi = vC\frac{dv}{dt}$$

The energy stored by the capacitor is

$$W = \int_0^t p\,dt = \int_0^t vC\frac{dv}{dt}dt$$

$$W = \frac{1}{2}Cv^2$$

From the above discussion, we can conclude the following:

1. The current in a capacitor is zero if the voltage across it is constant; that means, the capacitor acts as an open circuit to dc
2. A small change in voltage across a capacitance within zero time gives an infinite current through the capacitor, which is physically impossible. In a fixed capacitance the voltage cannot change abruptly.
3. The capacitor can store a finite amount of energy, even if the current through it is zero, and
4. A pure capacitor never dissipates energy, but only stores it; that is why it is called *non-dissipative passive element*. However, physical capacitors dissipate power due to internal resistance.

EXAMPLE 1.6

A capacitor having a capacitance 2 μF is charged to a voltage of 1000 V. Calculate the stored energy in joules.

Solution $W = \frac{1}{2}Cv^2 = \frac{1}{2}\times 2\times 10^{-6}\times (1000)^2 = 1\text{ J}$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆☆★ **1-6.1** (a) Determine the current in each of the following cases:
(i) 75 C in 1 s (ii) 10 C in 0.5 s (iii) 5 C in 2 s
(b) How long does it take 10 C to flow past a point if the current is 5 A?

☆☆★ **1-6.2** Find the capacitance of a circuit in which an applied voltage of 20 V gives an energy store of 0.3 J.

☆★★ **1-6.3** The voltage across two parallel capacitors of 5 μF and 3 μF changes uniformly from 30 to 75 V in 10 ms. Calculate the rate of change of voltage for (a) each capacitor, and (b) the combination.

Frequently Asked Questions linked to LO 6

★★★**1-6.1** When a dc voltage is applied to a capacitor, voltage across its terminals is found to build up in accordance with $v_c = 50(1 - e^{-100t})$. After 0.01 s, the current flow is equal to 2 mA.
(a) Find the value of capacitance in farad.
(b) How much energy is stored in the electric field? **[AU May/June 2014]**

☆☆★**1-6.2** Give the relation between energy (E) and power (P). Derive the equations for the energy stored in a capacitor (C) and an inductor (L) using $P = VI$ **[GTU May 2011]**

1.8 ENERGY SOURCES

LO 7 Describe various energy sources

According to their terminal voltage–current characteristics, electrical energy sources are categorised into ideal voltage sources and ideal current sources. Further they can be divided into independent and dependent sources.

An ideal voltage source is a two-terminal element in which the voltage v_s is completely independent of the current i_s through its terminals. The representation of ideal constant voltage source is shown in Fig. 1.6 (a).

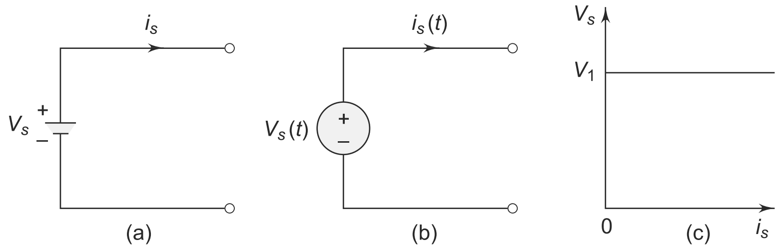


Fig. 1.6

If we observe the v–i characteristics for an ideal voltage source as shown in Fig. 1.6 (c) at any time, the value of the terminal voltage v_s is constant with respect to the value of current i_s. Whenever $v_s = 0$, the voltage source is the same as that of a short circuit. Voltage sources need not have constant magnitude; in many cases the specified voltage may be time-dependent like a sinusoidal waveform. This may be represented as shown in Fig. 1.6 (b). In many practical voltage sources, the internal resistance is represented in series with the source as shown in Fig. 1.7 (a). In this, the voltage across the terminals falls as the current through it increases, as shown in Fig. 1.7 (b).

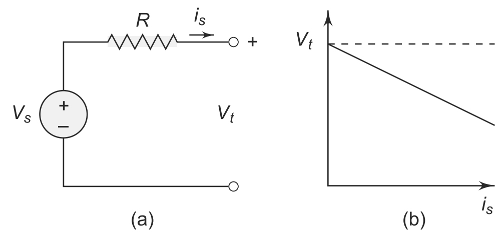


Fig. 1.7

The terminal voltage v_t depends on the source current as shown in Fig. 1.7 (b), where $v_t = v_s - i_s R$.

An ideal constant current source is a two-terminal element in which the current i_s completely independent of the voltage v_s across its terminals. Like voltage sources we can have current sources of constant magnitude i_s or sources whose current varies with time $i_s(t)$. The representation of an ideal current source is shown in Fig. 1.8 (a).

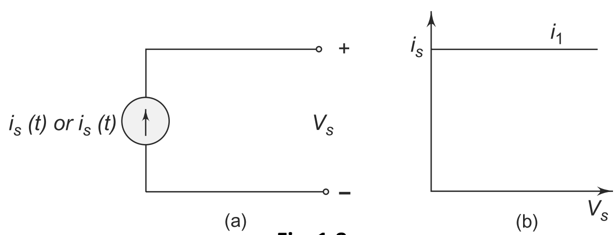


Fig. 1.8

If we observe the $v - i$ characteristics for an ideal current source as shown in Fig. 1.8 (b), at any time the value of the current i_s is constant with respect to the voltage across it. In many practical current sources, the resistance is in parallel with a source as shown in Fig. 1.9 (a). In this the magnitude of the current falls as the voltage across its terminals increases. Its terminal $v - i$ characteristic is shown in Fig. 1.9 (b). The terminal current is given by $i_t = i_s - (v_s/R)$, where R is the internal resistance of the ideal current source.

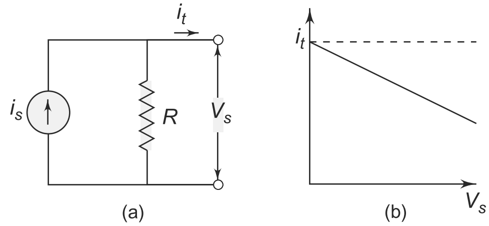


Fig. 1.9

The two types of ideal sources we have discussed are independent sources for which voltage and current are independent and are not affected by other parts of the circuit. In the case of dependent sources, the source voltage or current is not fixed, but is dependent on the voltage or current existing at some other location in the circuit.

Dependent or controlled sources are of the following types:

1. voltage controlled voltage source (VCVS)
2. current controlled voltage source (CCVS)
3. voltage controlled current source (VCCS)
4. current controlled current source (CCCS)

These are represented in a circuit diagram by the symbol shown in Fig. 1.10. These types of sources mainly occur in the analysis of equivalent circuits of transistors.

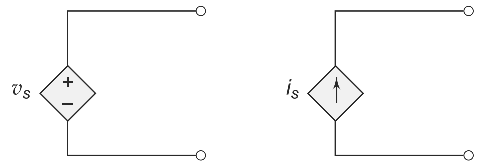


Fig. 1.10

Frequently Asked Questions linked to LO 7

☆☆★**1-7.1** Explain about voltage source and current source. Include ideal, practical, independent and dependent sources in your explanation. **[GTU Dec. 2010]**

☆☆★**1-7.2** Explain following in brief: ideal and practical energy sources. **[GTU Dec. 2010]**

☆☆★**1-7.3** What are the types of sources? Explain them with suitable diagrams and characteristics. **[JNTU Nov. 2012]**

1.9 KIRCHHOFF'S VOLTAGE LAW

LO 8 Explain voltage in a series circuit (KVL), voltage division, and power in a series circuit

Kirchhoff's voltage law states that the algebraic sum of all branch voltages around any closed path in a circuit is always zero at all instants of time. When the current passes through a resistor, there is a loss of energy and, therefore, a voltage drop. In any element, the current always flows from higher potential to lower potential. Consider the circuit in Fig. 1.11. It is customary to take the direction of current I as indicated in the figure, i.e. it leaves the positive terminal of the voltage source and enters into the negative terminal.

As the current passes through the circuit, the sum of the voltage drop around the loop is equal to the total voltage in that loop. Here, the polarities are attributed to the resistors to indicate that the voltages at points a, c, and e are more than the voltages at b, d, and f, respectively, as the current passes from a to f.

$$\therefore \quad V_s = V_1 + V_2 + V_3$$

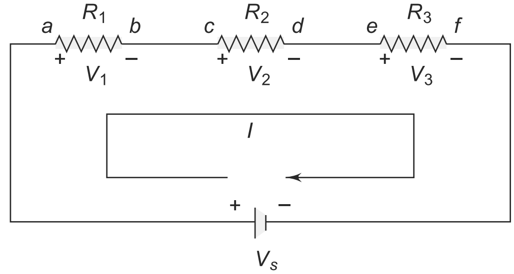


Fig. 1.11

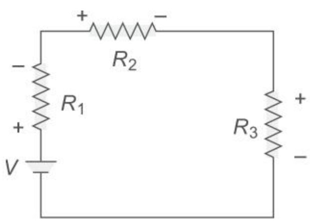


Fig. 1.12

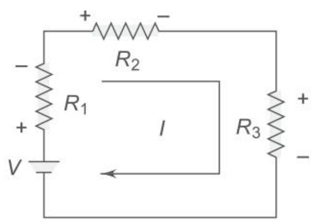


Fig. 1.13

Consider the problem of finding out the current supplied by the source V in the circuit shown in Fig. 1.12.

Our first step is to assume the reference current direction and to indicate the polarities for different elements. (See Fig. 1.13).

By using Ohm's law, we find the voltage across each resistor as follows.

$$V_{R1} = IR_1,\ V_{R2} = IR_2,\ V_{R3} = IR_3$$

where V_{R1}, V_{R2} and V_{R3} are the voltages across R_1, R_2 and R_3, respectively. Finally, by applying Kirchhoff's law, we can form the equations

$$V = V_{R1} + V_{R2} + V_{R3}$$

$$V = IR_1 + IR_2 + IR_3$$

From the above equation, the current delivered by the source is given by

$$I = \frac{V}{R_1 + R_2 + R_3}$$

EXAMPLE 1.7

For the circuit shown in Fig. 1.14, determine the unknown voltage drop V_1.

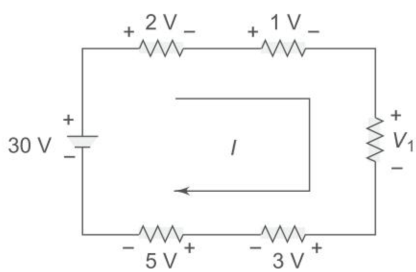


Fig. 1.14

Solution According to Kirchhoff's voltage law, the sum of the potential drops is equal to the sum of the potential rises.

Therefore, $30 = 2 + 1 + V_1 + 3 + 5$

or $V_1 = 30 - 11 = 19\text{ V}$

EXAMPLE 1.8

What is the current in the circuit shown in Fig. 1.15? Determine the voltage across each resistor.

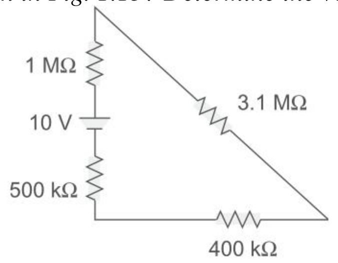


Fig. 1.15

Solution We assume the current I in the clockwise direction and indicate polarities (Fig. 1.16). By using Ohm's law, we find the voltage drops across each resistor.

$$V_{\text{IM}} = I, \quad V_{3.1\text{M}} = 3.1\, I$$
$$V_{500\text{K}} = 0.5\, I, \quad V_{400\text{K}} = 0.4\, I$$

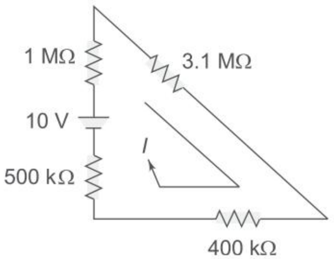


Fig. 1.16

Now, by applying Kirchhoff's voltage law, we form the equation

$$10 = I + 3.1\, I + 0.5\, I + 0.4\, I$$

or $\quad 5\, I = 10$

or $\quad I = 2\ \mu\text{A}$

$\therefore$ voltage across each resistor is as follows:

$$V_{1\text{M}} = 1 \times 2 = 2.0\ \text{V}$$
$$V_{3.1\text{M}} = 3.1 \times 2 = 6.2\ \text{V}$$
$$V_{400\text{K}} = 0.4 \times 2 = 0.8\ \text{V}$$
$$V_{500\text{K}} = 0.5 \times 2 = 1.0\ \text{V}$$

EXAMPLE 1.9

In the circuit given in Fig. 1.17, find (a) the current I, and (b) the voltage across 30 Ω.

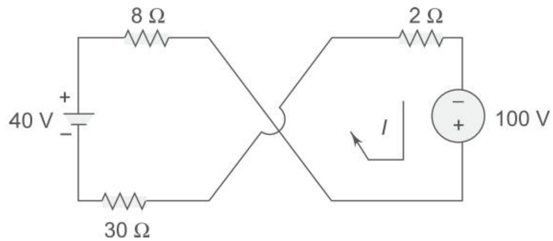


Fig. 1.17

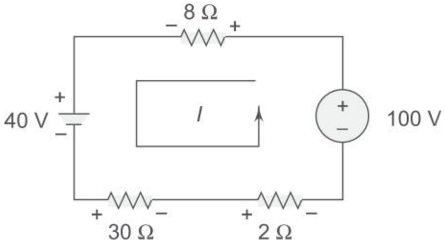


Fig. 1.18

Solution We redraw the circuit as shown in Fig. 1.18 and assume current direction and indicate the assumed polarities of resistors.

By using Ohm's law, we determine the voltage across each resistor as

$$V_8 = 8I, V_{30} = 30I, V_2 = 2I$$

By applying Kirchhoff's law, we get

$$100 = 8I + 40 + 30I + 2I$$

$$40I = 60 \quad \text{or} \quad I = \frac{60}{40} = 1.5\,\text{A}$$

$\therefore$ voltage drop across 30 Ω $= V_{30} = 30 \times 1.5 = 45$ V

1.10 VOLTAGE DIVISION

LO 8

The series circuit acts as a voltage divider. Since the same current flows through each resistor, the voltage drops are proportional to the values of resistors. Using this principle, different voltages can be obtained from a single source, called a voltage divider. For example, the voltage across a 40 Ω resistor is twice that of 20 Ω in a series circuit shown in Fig. 1.19.

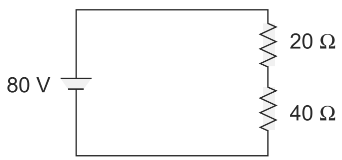


Fig. 1.19

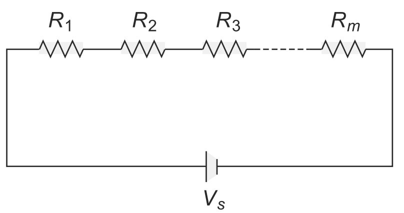


Fig. 1.20

In general, if the circuit consists of a number of series resistors, the total current is given by the total voltage divided by equivalent resistance. This is shown in Fig. 1.20.

The current in the circuit is given by $I = V_s/(R_1 + R_2 + \ldots + R_m)$. The voltage across any resistor is nothing but the current passing through it, multiplied by that particular resistor.

Therefore,
$$V_{R1} = IR_1$$
$$V_{R2} = IR_2$$
$$V_{R3} = IR_3$$
$$\vdots$$
$$V_{R_m} = IR_m$$

or
$$V_{R_m} = \frac{V_s(R_m)}{R_1 + R_2 + \ldots + R_m}$$

From the above equation, we can say that the voltage drop across any resistor, or a combination of resistors in a series circuit is equal to the ratio of that resistance value to the total resistance, multiplied by the source voltage, i.e.,

$$V_m = \frac{R_m}{R_T} V_s$$

where V_m is the voltage across mth resistor, R_m is the resistance across which the voltage is to be determined and R_T is the total series resistance.

EXAMPLE 1.10

What is the voltage across the 10 Ω resistor in Fig. 1.21.

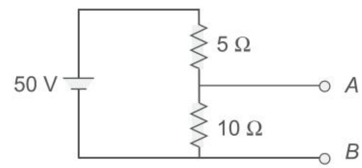


Fig. 1.21

Solution $\text{Voltage across } 10\,\Omega = V_{10} = 50 \times \frac{10}{10+5} = \frac{500}{15} = 33.3\text{ V}$

EXAMPLE 1.11

Find the voltage between A and B in a voltage divider network shown in Fig. 1.22.

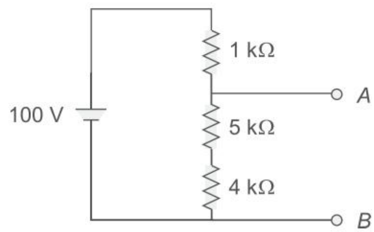


Fig. 1.22

Solution $\text{Voltage across } 9\,\text{k}\Omega = V_9 = V_{AB} = 100 \times \frac{9}{10} = 90\text{ V}$

1.11 POWER IN A SERIES CIRCUIT

LO 8

The total power supplied by the source in any series resistive circuit is equal to the sum of the powers in each resistor in series, i.e.,

$$P_s = P_1 + P_2 + P_3 + \dots + P_m$$

where m is the number of resistors in series, P_S is the total power supplied by source, and P_m is the power in the last resistor in series. The total power in the series circuit is the total voltage applied to a circuit, multiplied by the total current. Expressed mathematically,

$$P_s = V_s I = I^2 R_T = \frac{V_s^2}{R_T}$$

where V_s is the total voltage applied, R_T is the total resistance, and I is the total current.

EXAMPLE 1.12

Determine the total amount of power in the series circuit in Fig. 1.23.

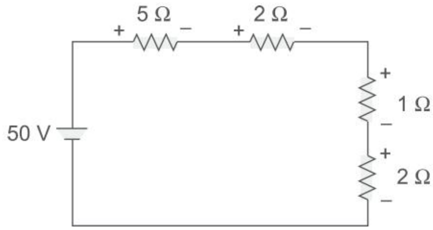


Fig. 1.23

Solution Total resistance = 5 + 2 + 1 + 2= 10 Ω

We know $P_s = \dfrac{V_s^2}{R_T} = \dfrac{(50)^2}{10} = 250 \text{ W}$

We find the power absorbed by each resistor.

$$\text{Current} = \frac{50}{10} = 5 \text{ A}$$

$$P_5 = (5)^2 \times 5 = 125 \text{ W}$$
$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$
$$P_1 = (5)^2 \times 1 = 25 \text{ W}$$
$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$

The sum of these powers gives the total power supplied by the source $P_S = 250$ W.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 8

☆☆★ **1-8.1** What is the voltage V_{AB} across the resistor shown in Fig. Q.1?

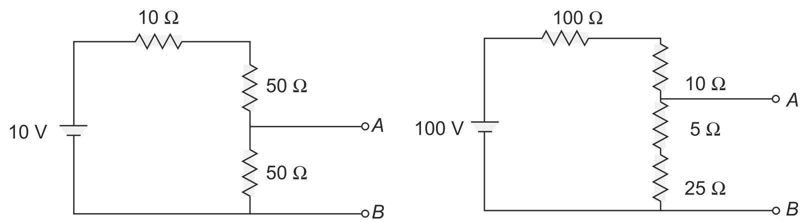


Fig. Q.1

☆☆★ **1-8.2** The source voltage in the circuit shown in Fig. Q.2 is 100 V. How much voltage does each metre read?

☆★★ **1-8.3** Find the node voltages for the network shown in Fig. Q.3 for the case when $k = -2$ using PSpice.

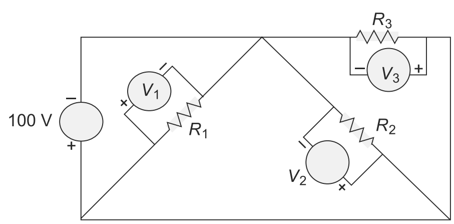


Fig. Q.2

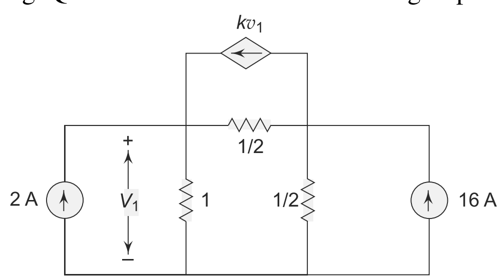


Fig. Q.3

★★★ **1-8.4** If you wish to increase the amount of current in a resistor from 100 mA to 150 mA by changing the 20 V source, by how many volts should you change the source? To what new value should you set it?

☆★★ **1-8.5** The following voltage drops are measured across each of three resistors in series: 5.5 V, 7.2 V, and 12.3 V. What is the value of the source voltage to which these resistors are connected? A fourth resistor is added to the circuit with a source voltage of 30 V. What should be the drop across the fourth resistor?

Frequently Asked Questions linked to LO 8

☆☆★**1-8.1** State and explain Kirchhoff's law. **[AU May/June 2013]**

☆☆★**1-8.2** Find the current *I* and voltage across 30 Ohms of the circuit shown in Fig. Q.2. **[AU May/June 2013]**

☆☆★**1-8.3** Determine current in the circuit shown in Fig. Q.3. **[AU May/June 2014]**

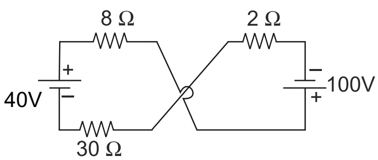


Fig. Q.2

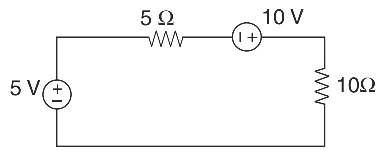


Fig. Q.3

☆☆★**1-8.4** Determine the current through the resistances in the bridge network shown in Fig. Q.4 using Kirchhoff's laws. [AU **May/June 2014]**

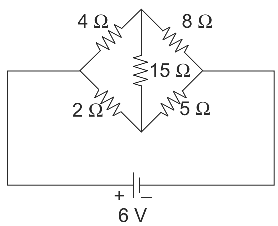


Fig. Q.4

☆☆★**1-8.5** State the voltage division principle for two resistors in series and the current division principle for two resistors in parallel. **[AU May/June 2013]**

☆☆★**1-8.6** Find the equivalent resistance of the circuit shown in Fig. Q.6. **[AU May/June 2014]**

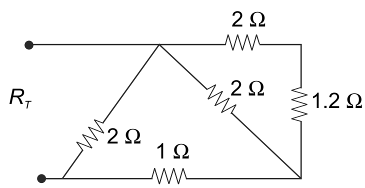


Fig. Q.6

1.12 KIRCHHOFF'S CURRENT LAW

LO 9 Analyse current in a parallel circuit (KCL), current division, and power in a parallel circuit

Kirchhoff's current law states that the sum of the currents entering into any node is equal to the sum of the currents leaving that node. The node may be an interconnection of two or more branches. In any parallel circuit, the node is a junction point of two or more branches. The total current entering into a node is equal to the current leaving that node. For example, consider the circuit shown in Fig. 1.24, which contains two nodes *A* and *B*. The total current I_T entering node A is divided into I_1, I_2 and I_3. These currents flow out of the node *A*. According to Kirchhoff's current law, the current into node *A* is equal to the total current out of the node *A*: that is, $I_T = I_1 + I_2 + I_3$. If we consider the node *B*, all three currents I_1, I_2, I_3 are entering B, and the total current I_T is leaving the node *B*, Kirchhoff's current law formula at this node is therefore the same as at the node *A*.

$$I_1 + I_2 + I_3 = I_T$$

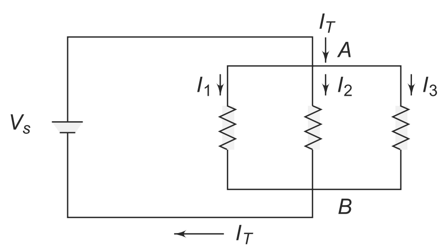


Fig. 1.24

In general, the sum of the currents entering any point or node or junction equal to sum of the currents leaving from that point or node or junction as shown in Fig. 1.25.

$$I_1 + I_2 + I_4 + I_7 = I_3 + I_5 + I_6$$

If all of the terms on the right side are brought over to the left side, their signs change to negative and a zero is left on the right side, i.e.

$$I_1 + I_2 + I_4 + I_7 - I_3 - I_5 - I_6 = 0$$

This means that the algebraic sum of all the currents meeting at a junction is equal to zero.

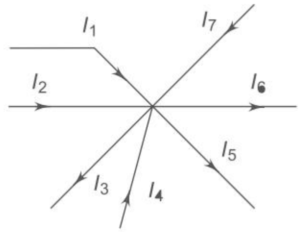


Fig. 1.25

EXAMPLE 1.13

Determine the current in all resistors in the circuit shown in Fig. 1.26.

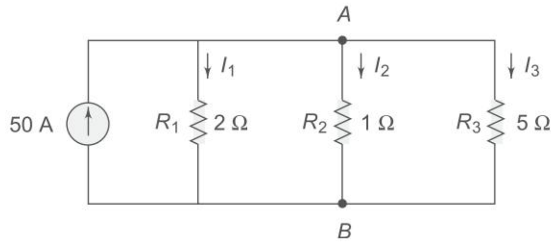


Fig. 1.26

Solution The above circuit contains a single node '*A*' with the reference node '*B*'. Our first step is to assume the voltage V at the node A. In a parallel circuit, the same voltage is applied across each element. According to Ohm's law, the currents passing through each element are $I_1 = V/2, I_2 = V/1, I_3 = V/5$.

By applying Kirchhoff's current law, we have

$$I = I_1 + I_2 + I_3$$

$$I = \frac{V}{2} + \frac{V}{1} + \frac{V}{5}$$

$$50 = V\left[\frac{1}{2} + \frac{1}{1} + \frac{1}{5}\right] = V[0.5 + 1 + 0.2]$$

$$V = \frac{50}{1.7} = \frac{500}{17} = 29.41 \text{ V}$$

Once we know the voltage V at the node A, we can find the current in any element by using Ohm's law.

The current in the 2 Ω resistor is $I_1 = 29.41/2 = 14.705$ A.

$$\text{Similarly} \quad I_2 = \frac{V}{R_2} - \frac{V}{1} = 29.41\,\text{A}$$

$$I_3 = \frac{29.41}{5} = 5.882\,\text{A}$$

$$\therefore \quad I_1 = 14.7\,\text{A}, I_2 = 29.4\,\text{A}, \text{and } I_3 = 5.88\,\text{A}$$

EXAMPLE 1.14

For the circuit shown in Fig. 1.27, find the voltage across the 10 Ω resistor and the current passing through it.

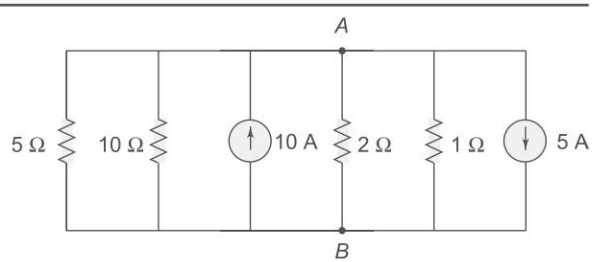


Fig. 1.27

Solution The circuit shown above is a parallel circuit, and consists of a single node A. By assuming voltage V at the node A w.r.t. B, we can find out the current in the 10 Ω branch (See Fig. 1.28).

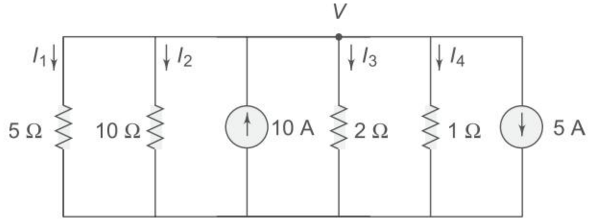


Fig. 1.28

According to Kirchhoff's current law,

$$I_1 + I_2 + I_3 + I_4 + 5 = 10$$

By using Ohm's law, we have

$$I_1 = \frac{V}{5}, I_2 = \frac{V}{10}, I_3 = \frac{V}{2}, I_4 = \frac{V}{1}$$

$$\frac{V}{5} + \frac{V}{10} + \frac{V}{2} + V + 5 = 10$$

$$V\left[\frac{1}{5} + \frac{1}{10} + \frac{1}{2} + 1\right] = 5$$

$$V[0.2 + 0.1 + 0.5 + 1] = 5$$

$$V = \frac{5}{1.8} = 2.78 \text{ V}$$

∴ the voltage across the 10 Ω resistor is 2.78 V and the current passing through it is

$$I_2 = \frac{V}{10} = \frac{2.78}{10} = 0.278 \text{ A}$$

EXAMPLE 1.15

Determine the current through the resistance R_3 in the circuit shown in Fig. 1.29.

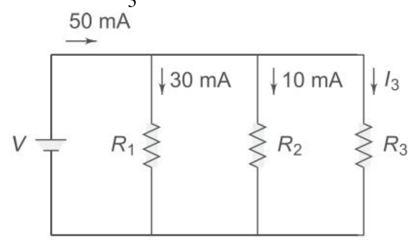


Fig. 1.29

Solution According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3$$

where I_T is the total current and I_1, I_2, and I_3 are the currents in resistances R_1, R_2 and R_3 respectively.

$$\therefore \quad 50 = 30 + 10 + I_3$$

$$\text{or} \quad I_3 = 10 \text{ mA}$$

1.13 PARALLEL RESISTANCE

LO 9

When the circuit is connected in parallel, the total resistance of the circuit decreases as the number of resistors connected in parallel increases. If we consider *m* parallel branches in a circuit as shown in Fig. 1.30, the current equation is

$$I_T = I_1 + I_2 + \dots + I_m$$

The same voltage is applied across each resistor. By applying Ohm's law, the current in each branch is given by

$$I_1 = \frac{V_s}{R_1}, I_2 = \frac{V_s}{R_2}, \dots I_m = \frac{V_s}{R_m}$$

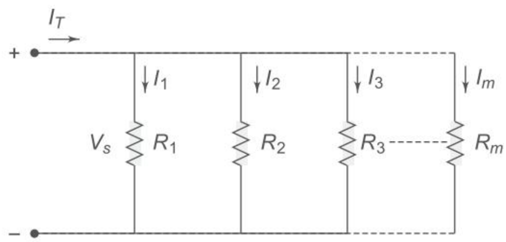


Fig. 1.30

According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3 + \dots + I_m$$

$$\frac{V_s}{R_T} = \frac{V_s}{R_1} + \frac{V_s}{R_2} + \frac{V_s}{R_3} + \dots + \frac{V_s}{R_m}$$

From the above equation, we have

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_m}$$

EXAMPLE 1.16

Determine the parallel resistance between points A and B of the circuit shown in Fig. 1.31.

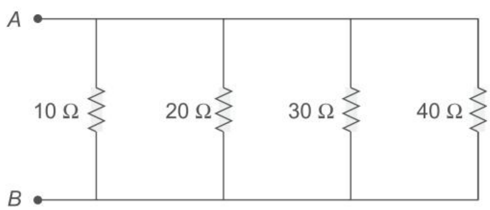


Fig. 1.31

Solution
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

$$\frac{1}{R_T} = \frac{1}{10} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$$

$$= 0.1 + 0.05 + 0.033 + 0.025 = 0.208$$

or $R_T = 4.8\ \Omega$

1.14 CURRENT DIVISION

LO 9

In a parallel circuit, the current divides in all branches. Thus, a parallel circuit acts as a current divider. The total current entering into the parallel branches is divided into the branches currents according to the resistance values. The branch having higher resistance allows lesser current, and the branch with lower resistance allows more current. Let us find the current division in the parallel circuit shown in Fig. 1.32.

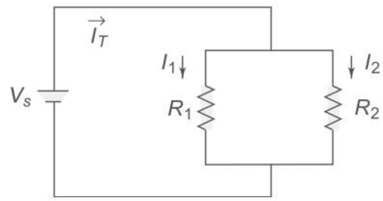


Fig. 1.32

The voltage applied across each resistor is V_s. The current passing through each resistor is given by

$$I_1 = \frac{V_s}{R_1},\ I_2 = \frac{V_s}{R_2}$$

If R_T is the total resistance, which is given by $R_1R_2/(R_1 + R_2)$,

$$\text{Total current} \quad I_T = \frac{V_s}{R_T} = \frac{V_s}{R_1\,R_2}\left(R_1 + R_2\right)$$

$$\text{or} \quad I_T = \frac{I_1\,R_1}{R_1\,R_2}(R_1 + R_2) \text{ since } V_s = I_1R_1$$

$$I_1 = I_T \cdot \frac{R_2}{R_1 + R_2}$$

$$\text{Similarly,} \quad I_2 = I_T \cdot \frac{R_1}{R_1 + R_2}$$

From the above equations, we can conclude that the current in any branch is equal to the ratio of the opposite branch resistance to the total resistance value, multiplied by the total current in the circuit. In general, if the circuit consists of *m* branches, the current in any branch can be determined by

$$I_i = \frac{R_T}{R_i + R_T} I_T$$

where I_i represents the current in the *i*th branch,
R_i is the resistance in the *i*th branch,
R_T is the total parallel resistance to the *i*th branch, and
I_T is the total current entering the circuit.

EXAMPLE 1.17

Determine the current through each resistor in the circuit shown in Fig. 1.33.

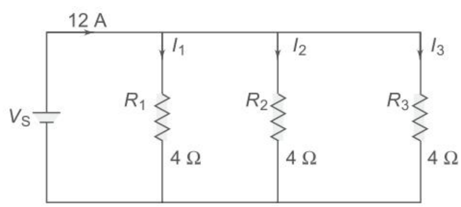


Fig. 1.33

Solution $I_1 = I_T \times \dfrac{R_T}{(R_1 + R_T)}$

$$\text{where} \quad R_T = \frac{R_2\,R_3}{R_2 + R_3} = 2\ \Omega$$

$$\therefore \quad R_1 = 4\ \Omega$$

$$I_T = 12\,\text{A}$$

$$I_1 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

Similarly, $I_2 = 12 \times \frac{2}{2+4} = 4\text{ A}$

and $I_3 = 12 \times \frac{2}{2+4} = 4\text{ A}$

Since all parallel branches have equal values of resistance, they share current equally.

1.15 POWER IN A PARALLEL CIRCUIT

LO 9

The total power supplied by the source in any parallel resistive circuit is equal to the sum of the powers in each resistor in parallel, i.e.,

$$P_s = P_1 + P_2 + P_3 + \ldots + P_m$$

where m is the number of resistors in parallel, P_s is the total power, and P_m is the power in the last resistor.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 9

★★★**1-9.1** Using Ohm's law and Kirchhoff's laws on the circuit given in Fig. Q.1, find all the voltages and currents.

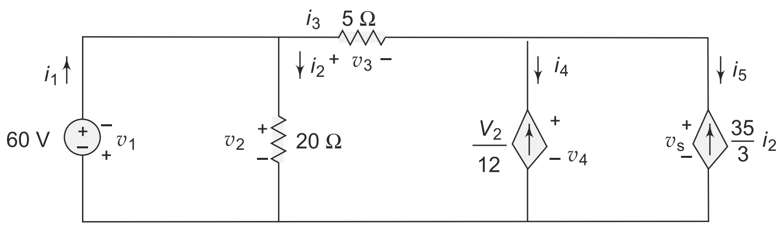


Fig. Q.1

☆★★**1-9.2** Using PSpice, solve for the current I_{ab} in the following circuit. (Fig. Q.2)

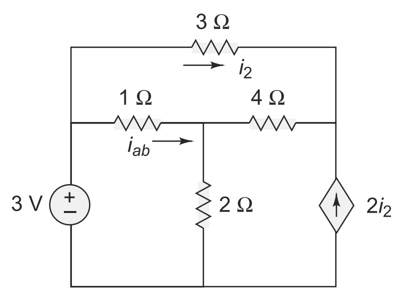


Fig. Q.2

★★★**1-9.3** Find R_{eq} for the resistive network shown in Fig. Q.3.

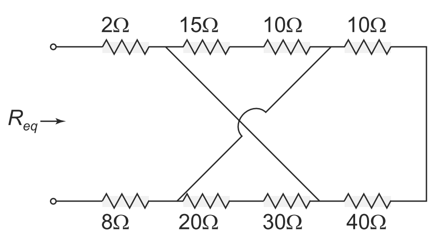


Fig. Q.3

☆☆★ **1-9.4** For the circuit shown in Fig. Q.4, find the total resistance.

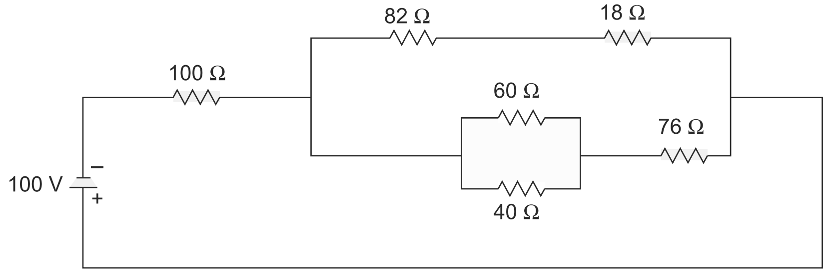


Fig. Q.4

☆☆★ **1-9.5** In the network shown in Fig. Q.5, (a) let $R = 80\ \Omega$, find R_{eq}; (b) find R if $R_{eq} = 80\ \Omega$; (c) find R if $R = R_{eq}$.

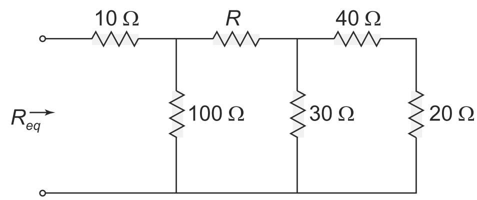


Fig. Q.5

☆★★ **1-9.6** Using the current divider formula, determine the current in each branch of the circuit shown in Fig. Q.6.

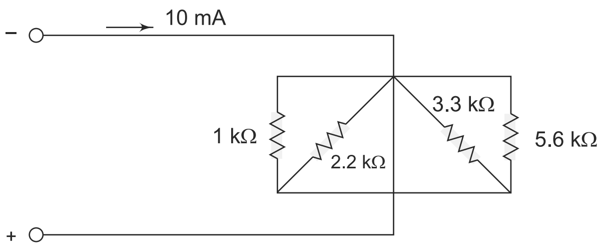


Fig. Q.6

★★★ **1-9.7** Six lightbulbs are connected in parallel across 110 V. Each bulb is rated at 75 W. How much current flows through each bulb, and what is the total current?

Frequently Asked Questions linked to LO 9

☆☆★ **1-9.1** State and explain Kirchhoff's laws. **[AU May/June 2013]**

☆★★ **1-9.2** Determine the voltage across the 20 ohm resistor of the network shown in Fig. Q.2. **[AU May/June 2014]**

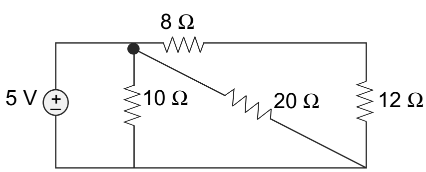


Fig. Q.2

☆☆★ **1-9.3** Determine the current through the 20 V source in the circuit of shown in Fig. Q.3. **[AU May/June 2014]**

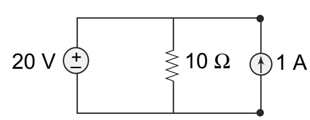


Fig. Q.3

☆☆★**1-9.4** State and explain Kirchhoff's laws, with an example. **[AU April/May 2011]**

☆☆★**1-9.5** State and explain Kirchhoff's laws. **[JNTU Nov. 2012]**

☆☆★**1-9.6** Find the equivalent conductance $G_{eq.}$ of the circuit shown in Fig. Q.6. **[AU Nov./Dec. 2012]**

☆★★**1-9.7** What is the magnitude of current drained from the 10 V source in the circuit shown in Fig. Q.7. **[JNTU Nov. 2012]**

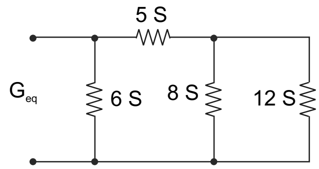


Fig. Q.6

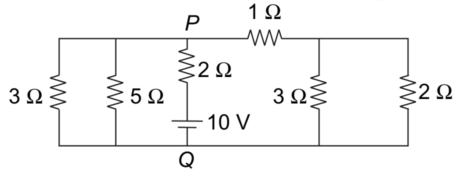


Fig. Q.7

☆★★**1-9.8** Three loads *A,B,* and *C* are connected in parallel to a 240 V source. Load *A* takes 9.6 kW, Load *B* takes 60 A and load C has a resistance of 4.8 Ohms. Calculate (a) R_A and R_B, (b) the total current (c) the total power, and (d) equivalent resistance. **[AU May/June 2013]**

☆☆★**1-9.9** Determine the current in all the resistors of the circuit shown in Fig. Q.9. **[AU May/June 2014]**

☆☆★**1-9.10** Determine the current through each resistor in the circuit shown in Fig. Q.10. **[AU May/June 2014]**

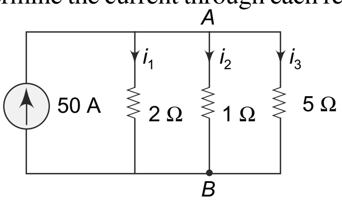


Fig. Q.9

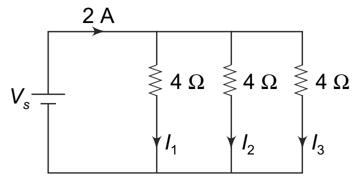


Fig. Q.10

☆☆★**1-9.11** Determine the current through the 2 Ω resistor in the circuit of Fig. Q.11. **[AU May/June 2014]**

★★★**1-9.12** Determine the current delivered by the source in the circuit shown in Fig. Q.12. **[AU April/May 2011]**

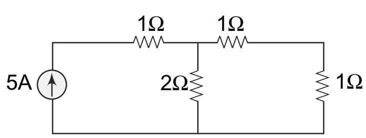


Fig. Q.11

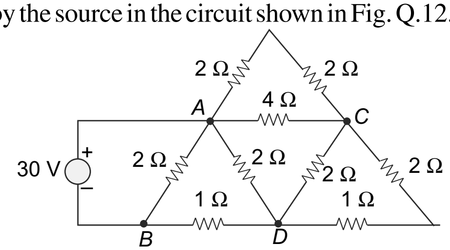


Fig. Q.12

☆★★**1-9.13** Calculate the voltage that is to be connected across terminals *x-y* as shown in Fig. Q. 13 such that the voltage across the 2 Ω resister is 10 V. Also find I_a and I_b. what is the total power loss in the circuit? [JNTU **Nov. 2012]**

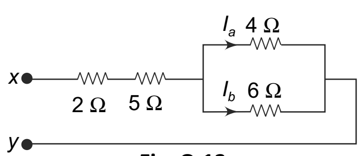


Fig. Q.13

☆☆★**1-9.14** A resistance of 10 ohms is connected across a supply of 200 V. If a resistances *R* is now connected in parallel with a 10-ohm resistance, the current draw from the supply gets doubled. Find the value of unknown resistance. **[PTU 2011-12]**

Additional Solved Problems

PROBLEM 1.1

For the circuit shown in Fig. 1.34, find the total resistance between terminals A and B, the total current drawn from a 6 V source connected from A to B, and the current through 4.7 kΩ; voltage across 3 kΩ.

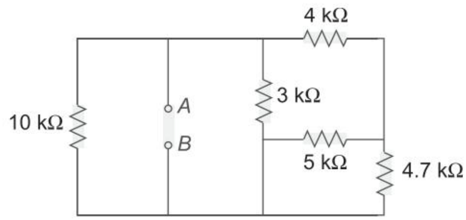


Fig. 1.34

Solution The circuit in Fig. 1.34 can be redrawn as shown in Fig. 1.35 below.

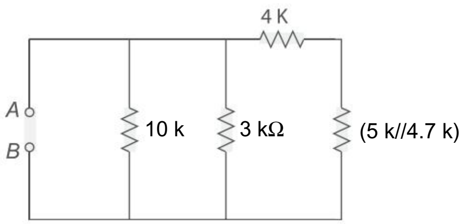


Fig. 1.35

From Fig. 1.35, the total resistance is

$$R_T = 10\text{ k} \parallel 3\text{ k} \parallel [4\text{ k} + 5\text{ k} \parallel 4.7\text{ k}]$$

$$= 1.7\text{ k}\Omega$$

Total current drawn by the circuit is

$$I_T = \frac{6\text{ V}}{1.7\text{ k}\Omega} = 3.53\text{ mA}$$

The current in the 10 kΩ resistor is

$$\therefore \quad I_{10\text{k}} = \frac{6\text{ V}}{10\text{ k}} = 0.6\text{ mA}$$

The current in the 3 kΩ resistor is

$$I_{3\text{k}} = \frac{6\text{ V}}{3\text{k}\,\Omega} = 2\text{ mA}$$

The remaining current blows through the 4 kΩ resistor and the parallel combination of (5 kΩ||4.7 kΩ).

$$I_{4\text{k}} = 3.53\text{ mA} - 2.6\text{ mA} = 0.93\text{ mA}$$

The current in the 4.7 kΩ resistor is

$$I_{4.7\text{k}} = 0.93 \times \frac{5}{5+4.7} = 0.47\text{ mA}$$

The voltage across the 3 kΩ resistor is

$$V_{3\text{k}} = I_{3\text{k}}\, R = 2 \times 10^{-3} \times 3 \times 10^{3} = 6\text{ V}$$

PROBLEM 1.2

A battery of unknown emf is connected across resistances as shown in Fig. 1.36. The voltage drop across the 8 Ω resistor is 20 V. What will be the current reading in the ammeter? What is the emf of the battery?

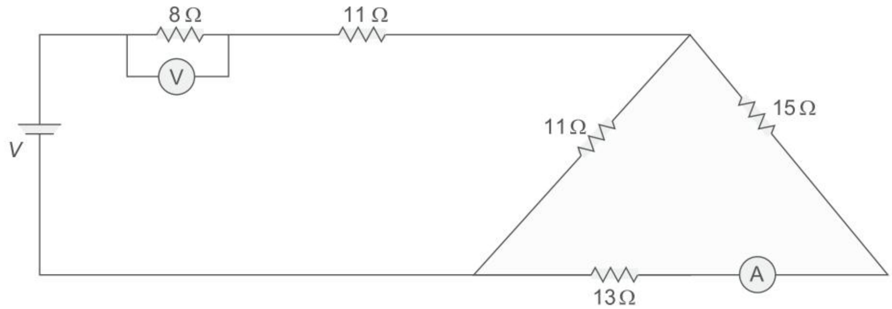


Fig. 1.36

Solution From the circuit shown in Fig. 1.36, the current passing through the 8 Ω resistor is

$$I_8 = \frac{20\,\text{V}}{8\,\Omega} = 2.5\text{ A}$$

The current passing through the 15 Ω resistor is same as the ammeter reading.

∴ the current passing through the 15 Ω resistor is

$$I_{15} = \frac{2.5 \times 11}{11 + 28} = 0.71\text{ A}$$

Reading of the ammeter = 0.71 A

The voltage across the 28 Ω (13 Ω in series with 15 Ω) resistor is

$$V_{28} = 0.71 \times 28 = 19.88\text{ volts}$$

The voltage across the series arm (8 Ω in series with 11 Ω) resistor is

$$V_{19} = 2.5 \times 19 = 47.5\text{ volts}$$

The emf of the battery = 19.88 + 47.5 = 67.38 volts

PROBLEM 1.3

An electric circuit has three terminals A,B,C. Between A and B is connected a 2 Ω resistor, between B and C are connected a 7 Ω resistor and a 5 Ω resistor in parallel, and between A and C is connected a 1 Ω resistor. A battery of 10 V is then connected between terminals A and C calculate (a) total current drawn from the battery, (b) voltage across the 2 Ω resistor, and (c) current passing through the 5 Ω resistor.

Solution The circuit can be drawn as shown in Fig. 1.37 below.

The current passing through the 1 Ω resistor is

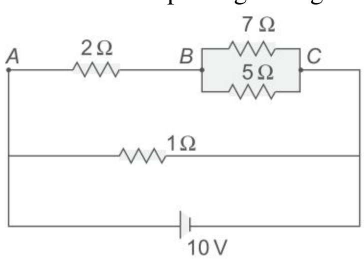


Fig. 1.37

$$I_{1\Omega} = \frac{10}{1} = 10\text{ A}$$

The current passing through the series parallel branch between terminals A and C is

$$I_{2\Omega} = \frac{10}{2 + (7\|5)} = 2\text{ A}$$

Total current drawn from the battery is $I_T = 10 + 2 = 12$ A

Voltage across the 2 Ω resistor is $V_{2\Omega} = 2 \times 2 = 4$ volts

The current passing though the 5 Ω resistor is

$$I_{5\Omega} = \frac{2 \times 7}{5 + 7} = 1.17\text{ A}$$

PROBLEM 1.4

Using Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.38, find V_{in}, V_s, and the power provided by the dependent source.

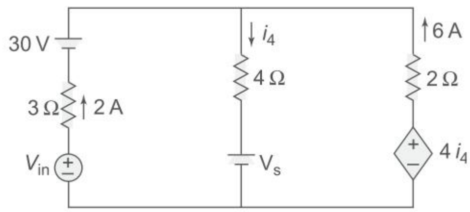


Fig. 1.38

Solution From the circuit shown in Fig. 1.38, applying Kirchhoff's current law, we have

$$i_{4\Omega} = i_{3\Omega} + i_{2\Omega}$$

The current passing through the 4 Ω branch is $i_4 = 2 + 6 = 8$ A

The voltage across the dependent source is

$$V_{4i_4} = (4i_4) = 4 \times 8 = 32\,\text{V}$$

The voltage across the 2 Ω resistor is $V_{2\Omega} = 6 \times 2 = 12$ V

∴ the voltage across each branch is

$$V = V_{4i_4} - V_2 = 32 - 12 = 20\,\text{V}$$

The voltage across the 4 Ω branch is

$$V_4 = 4 \times i_4 = 4 \times 8$$
$$= 32\,\text{V}$$

According to Kirchhoff's voltage law,

$$V - V_4 + V_S = 0$$
$$\therefore\quad 20 - 32 + V_S = 0$$
$$V_S = 12\,\text{V}$$

Similarly, Kirchhoff's voltage law is applied to the 3 Ω branch.

$$V - 30 + V_3 - V_{in} = 0$$
$$20 - 30 + 6 - V_{in} = 0$$

From the above equation, the voltage

$$V_{in} = -4\,\text{V}$$

The power provided by the dependent source is

$$P_{4i_4} = V_{4i_4} \times i_{4i_4}$$
$$P_{4i_4} = 4 \times i_4 \times 6 = 4 \times 8 \times 6$$
$$= 192 \text{ watts}$$

PROBLEM 1.5

Find the power absorbed by each element in the circuit shown in Fig. 1.39.

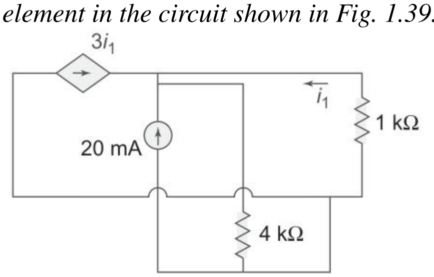


Fig. 1.39

Solution The circuit shown in Fig. 1.39 can be redrawn as shown in Fig. 1.40.

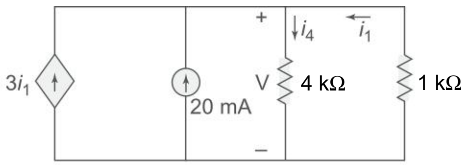


Fig. 1.40

Applying Kirchhoff's current in the single node circuit shown in Fig. 1.40, we have

$$3i_1 + 20\times10^{-3} - \frac{V}{4\times10^3} + i_1 = 0$$

$$i_1 = -V\times10^{-3}$$

$$\therefore \quad -3V\times10^{-3} + 20\times10^{-3} - 0.25\times10^{-3}V - V\times10^{-3} = 0$$

$$-4.25\times10^{-3}V = -20\times10^{-3}$$

$$V = 4.71 \text{ volts}$$

The power absorbed by the $3i_1$ dependent current source is

$$P_{3i_1} = (3i_1)\text{V} = 3\text{V}\times10^{-3}\times V$$

$$= +66.55 \text{ mW}$$

The power absorbed by the 20 mA current source is

$$P_{20} = 20 \times 10^{-3} \times \text{V} = -94.2 \text{ mW}$$

The power absorbed by the 4 kΩ resistor is

$$P_{4\text{k}} = \frac{4.71\times4.71}{4\text{ k}} = 5.55 \text{ mW}$$

The power absorbed by the 1kΩ resistor is

$$P_{1\text{k}} = \frac{4.71\times4.71}{1\text{ k}} = 22.18 \text{ mW}$$

PROBLEM 1.6

Determine the total current in the circuit shown in Fig. 1.41.

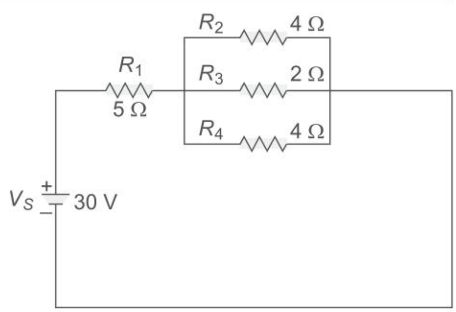


Fig. 1.41

Solution Resistances R_2, R_3 and R_4 are in parallel.

$\therefore$ equivalent resistance $R_5 = R_2 \parallel R_3 \parallel R_4$

$$= \frac{1}{1/R_2 + 1/R_3 + 1/R_4}$$

$\therefore \quad R_5 = 1\ \Omega$

R_1 and R_5 are in series,

$\therefore$ equivalent resistance $R_T = R_1 + R_5 = 5 + 1 = 6\ \Omega$

And the total current $\quad I_T = \dfrac{V_s}{R_T} = \dfrac{30}{6} = 5\text{ A}$

PROBLEM 1.7

Find the current in the 10 Ω resistance, V_1, and source voltage V_s in the circuit shown in Fig. 1.42.

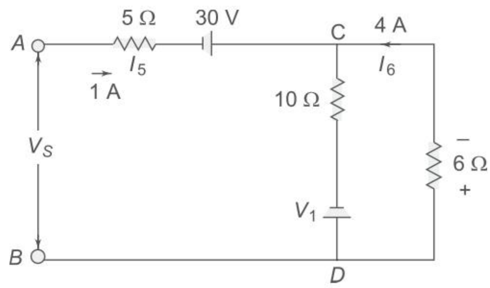


Fig. 1.42

Solution Assume voltage at the node $C = V$

By applying Kirchhoff's current law, we get the current in the 10 Ω resistance as

$$I_{10} = I_5 + I_6$$
$$= 4 + 1 = 5 \text{ A}$$

The voltage across the 6 Ω resistor is $V_6 = 24$ V

∴ voltage at the node C is $V_C = -24$ V.

The voltage across the branch CD is the same as the voltage at the node C.

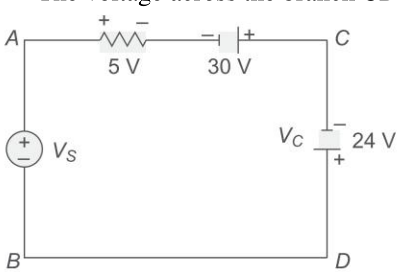


Fig. 1.43

Voltage across 10 Ω only = 10 × 5 = 50 V

So $V_C = V_{10} - V_1$

$$-24 = 50 - V_1$$

$$V_1 = 74 \text{ V}$$

Now, consider the loop CABD shown in Fig. 1.43.

If we apply Kirchhoff's voltage law, we get

$$V_s = 5 - 30 - 24 = -49 \text{ V}$$

PROBLEM 1.8

What is the voltage across A and B in the circuit shown in Fig. 1.44?

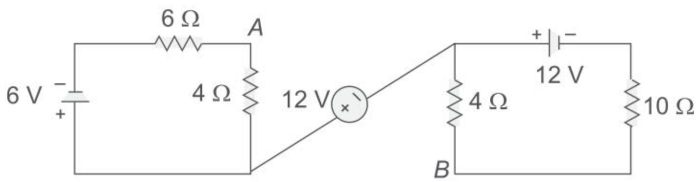


Fig. 1.44

Solution The above circuit can be redrawn as shown in Fig. 1.45.

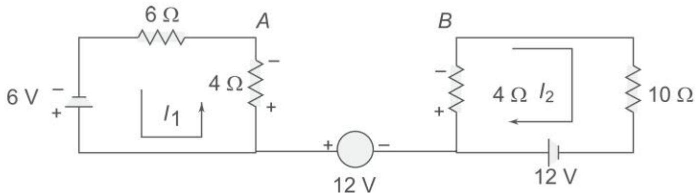


Fig. 1.45

Assume loop currents I_1 and I_2 as shown in Fig. 1.45.

$$I_1 = \frac{6}{10} = 0.6 \text{ A}$$

$$I_2 = \frac{12}{14} = 0.86 \text{ A}$$

V_A = Voltage drop across the 4 Ω resistor = 0.6 × 4 = 2.4 V

V_B = Voltage drop across the 4 Ω resistor = 0.86 × 4 = 3.44 V

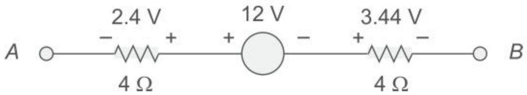


Fig. 1.46

The voltage between points A and B is the sum of voltages as shown in Fig. 1.46.

$$V_{AB} = -2.4 + 12 + 3.44$$
$$= 13.04 \text{ V}$$

PROBLEM 1.9

Determine the current delivered by the source in the circuit shown in Fig. 1.47.

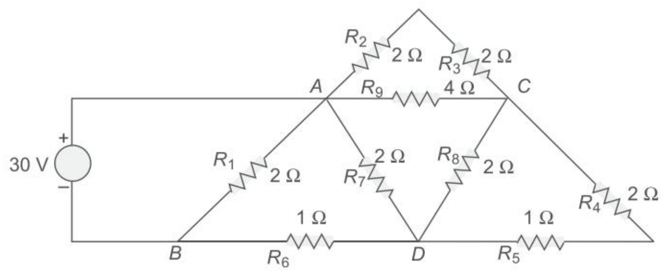


Fig. 1.47

Solution The circuit can be modified as shown in Fig. 1.48, where R_{10} is the series combination of R_2 and R_3.

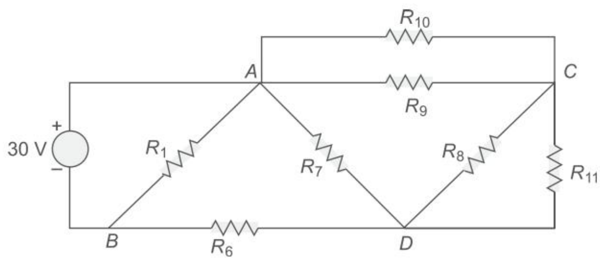


Fig. 1.48

$$R_{10} = R_2 + R_3 = 4\ \Omega$$

R_{11} is the series combination of R_4 and R_5.

$$\therefore \quad R_{11} = R_4 + R_5 = 3\ \Omega$$

Further simplification of the circuit leads to Fig. 1.49 where R_{12} is the parallel combination of R_{10} and R_9.

$$\therefore \quad R_{12} = (R_{10} \parallel R_9) = (4 \parallel 4) = 2\ \Omega$$

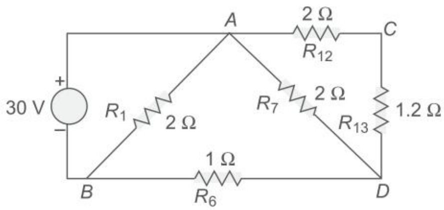


Fig. 1.49

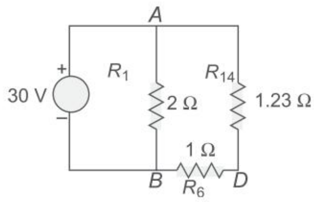


Fig. 1.50

Similarly, R_{13} is the parallel combination of R_{11} and R_8.

$\therefore \quad R_{13} = (R_{11} \parallel R_8) = (3 \parallel 2) = 1.2\ \Omega$

In Fig. 1.49 as shown, R_{12} and R_{13} are in series, which is in parallel with R_7 forming R_{14}. This is shown in Fig. 1.50.

$$\therefore \quad R_{14} = [(R_{12} + R_{13}) \| R_7]$$
$$= [(2 + 1.2) \| 2] = 1.23\ \Omega$$

Further, the resistances R_{14} and R_6 are in series, which is in parallel with R_1 and gives the total resistance

$$R_T = [(R_{14} + R_6) \| R_1]$$
$$= [(1 + 1.23) \| (2)] = 1.05\ \Omega$$

The current delivered by the source = 30/1.05 = 28.57 A

PROBLEM 1.10

Determine the current in the 10 Ω resistance and find V_s in the circuit shown in Fig. 1.51.

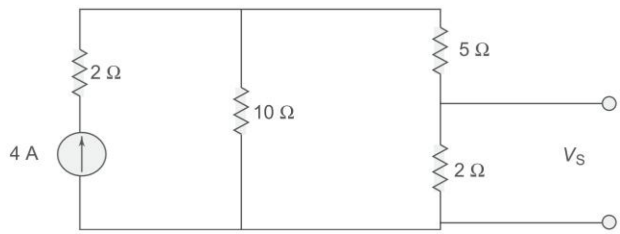


Fig. 1.51

Solution The current in the 10 Ω resistance

$$I_{10} = \text{total current} \times (R_T)/(R_T + R_{10})$$

where R_T is the total parallel resistance.

$$I_{10} = 4 \times \frac{7}{17} = 1.65\ \text{A}$$

Similarly, the current in the resistance R_5 is

$$I_5 = 4 \times \frac{10}{10+7} = 2.35\ \text{A}$$

or $\quad 4 - 1.65 = 2.35$ A

The same current flows through the 2 Ω resistance.

$\therefore \quad$ voltage across the 2 Ω resistance, $V_s = I_5 \times 2$

$$= 2.35 \times 2 = 4.7\ \text{V}$$

PROBLEM 1.11

Determine the value of the resistance R and current in each branch when the total current taken by the circuit shown in Fig. 1.52 is 6 A.

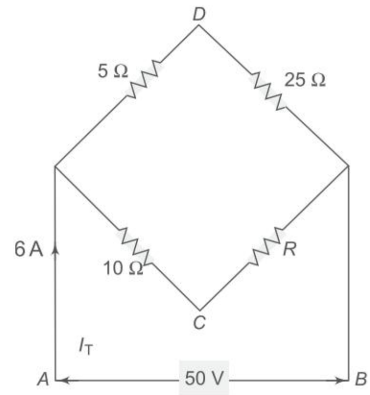


Fig. 1.52

Solution The current in the branch *ADB*

$$I_{30} = 50/(25 + 5) = 1.66 \text{ A}$$

The current in the branch *ACB*, $I_{10+R} = 50/(10 + R)$.
According to Kirchhoff's current law,

$$I_T = I_{30} + I_{10+R}$$

$$6\text{A} = 1.66 \text{ A} + I_{10+R}$$

$$\therefore \quad I_{10+R} = 6 - 1.66 = 4.34 \text{ A}$$

$$\therefore \quad \frac{50}{10+R} = 4.34$$

$$10 + R = \frac{50}{4.34} = 11.52$$

$$R = 1.52\ \Omega$$

PROBLEM 1.12

Find the power delivered by the source in the circuit shown in Fig. 1.53.

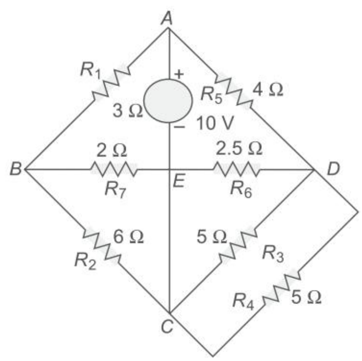


Fig. 1.53

Solution Between points *C*(*E*) and *D*, resistances R_3 and R_4 are in parallel, which gives

$$R_8 = (R_3 \,||\, R_4) = 2.5\ \Omega$$

Between points B and $C(E)$, resistances R_2 and R_7 are in parallel, which gives

$$R_9 = (R_2 \parallel R_7) = 1.5\ \Omega$$

Between points $C(E)$ and D, resistances R_6 and R_8 are in parallel which gives

$$R_{10} = (R_6 \parallel R_8) = 1.25\ \Omega$$

The series combination of R_1 and R_9 gives

$$R_{11} = R_1 + R_9 = 3 + 1.5 = 4.5\ \Omega$$

Similarly, the series combination of R_5 and R_{10} gives

$$R_{12} = R_5 + R_{10} = 5.25\ \Omega$$

The resistances R_{11} and R_{12} are in parallel, which gives

Total resistance $= (R_{11} \parallel R_{12}) = 2.42$ ohms

These reductions are shown in Figs 1.54 (a), (b), (c), and (d).

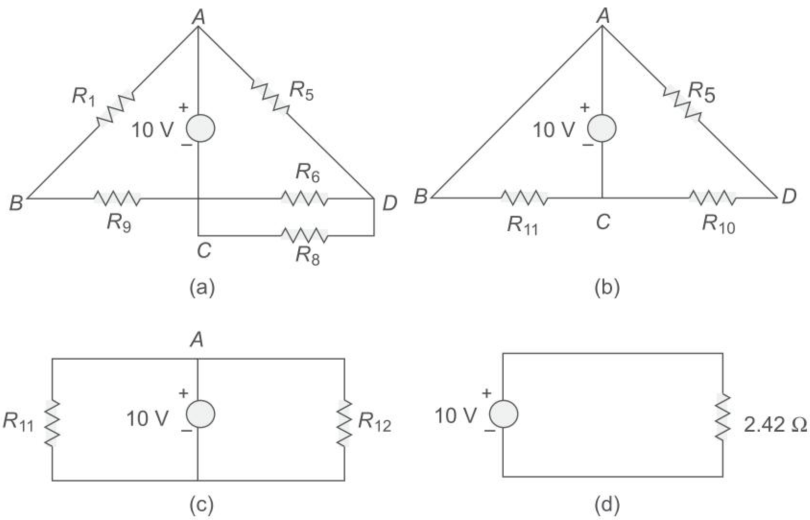


Fig. 1.54

$$\text{Current delivered by the source} = \frac{10}{2.42} = 4.13\ \text{A}$$

$$\begin{aligned}\text{Power delivered by the source} &= VI \\ &= 10 \times 4.13 = 41.3\ \text{W}\end{aligned}$$

PROBLEM 1.13

Determine the voltage drop across the 10 Ω resistance in the circuit as shown in Fig. 1.55.

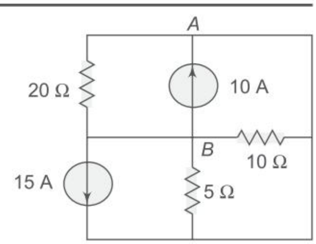


Fig. 1.55

Solution The circuit is redrawn as shown in Fig. 1.56.

This is a single-node pair circuit. Assume voltage V_A at the node A. By applying Kirchhoff's current law at the node A, we have

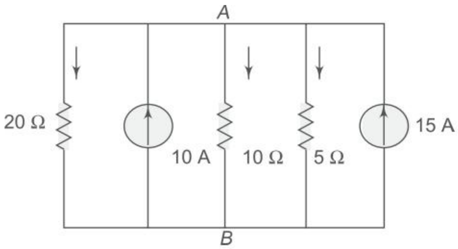


Fig. 1.56

$$\frac{V_A}{20}+\frac{V_A}{10}+\frac{V_A}{5}=10+15$$

$$V_A\left[\frac{1}{20}+\frac{1}{10}+\frac{1}{5}\right]=25\text{ A}$$

$$V_A(0.05+0.1+0.2)=25\text{ A}$$

$$V_A=\frac{25}{0.35}=71.42\text{ V}$$

The voltage across 10 Ω is nothing but the voltage at the node A.

$$V_{10} = V_A = 71.42\text{ V}$$

PROBLEM 1.14

In the circuit shown in Fig. 1.57, what are the values of R_1 and R_2, when the current flowing through R_1 is 1 A and R_2 is 5 A? What is the value of R_2 when the current flowing through R_1 is zero?

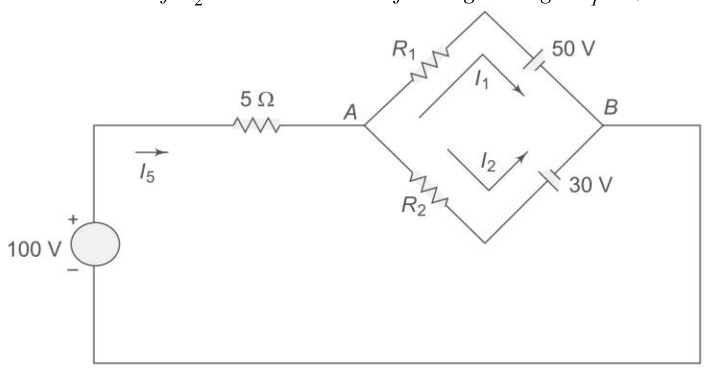


Fig. 1.57

Solution The current in the 5 Ω resistance

$$I_5 = I_1 + I_2 = 1 + 5$$
$$= 6\text{ A}$$

Voltage across the resistance 5 Ω is $V_5 = 5 \times 6 = 30$ V

The voltage at the node A, $V_A = 100 - 30 = 70$ V

$$\therefore \quad I_2=\frac{V_A-30}{R_2}=\frac{70-30}{R_2}$$

$$R_2=\frac{70-30}{I_2}=\frac{40}{5}=8\ \Omega$$

$$\text{Similarly, } R_1=\frac{70-50}{I_1}=\frac{20}{1}=20\ \Omega$$

When $V_A = 50$ V, the current I_1 in the resistance R_1 becomes zero.

$$\therefore \quad I_2 = \frac{50-30}{R_2}$$

where I_2 becomes the total current.

$$\therefore \quad I_2 = \frac{100-V_A}{5} = \frac{100-50}{5} = 10\,\text{A}$$

$$\therefore \quad R_2 = \frac{20}{I_2} = \frac{20}{10} = 2\,\Omega$$

PROBLEM 1.15

Determine the output voltage V_{out} in the circuit shown in Fig. 1.58.

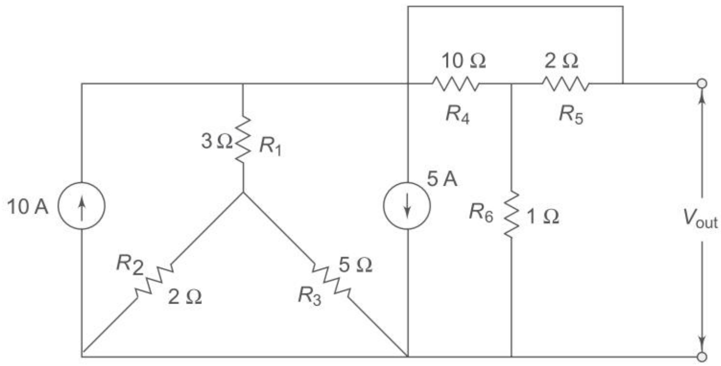


Fig. 1.58

Solution The circuit shown in Fig. 1.58 can be redrawn as shown in Fig. 1.59.

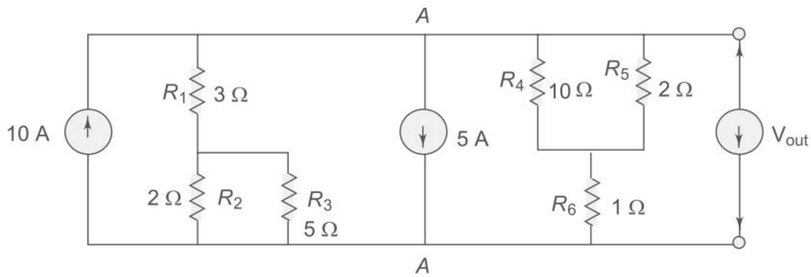


Fig. 1.59

In Fig. 1.59, R_2 and R_3 are in parallel, R_4 and R_5 are in parallel. The complete circuit is a single-node pair circuit. Assuming voltage V_A at the node A and applying Kirchhoff's current law in the circuit, we have

$$10\text{A} - \frac{V_A}{4.43} - 5\text{A} - \frac{V_A}{2.67} = 0$$

$$\therefore \quad V_A\left[\frac{1}{4.43} + \frac{1}{2.67}\right] = 5\,\text{A}$$

$$V_A[0.225+0.375]=5$$

$$\therefore \quad V_A=\frac{5}{0.6}=8.33 \text{ V}$$

PROBLEM 1.16

Determine the voltage V_{AB} in the circuit shown in Fig. 1.60.

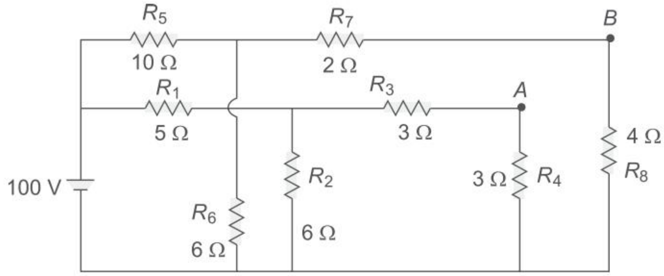


Fig. 1.60

Solution The circuit in Fig. 1.60 can be redrawn as shown in Fig. 1.61 (a).

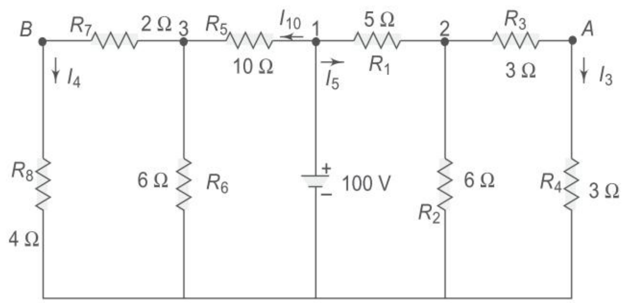


Fig. 1.61 (a)

At the node 3, the series combination of R_7 and R_8 are in parallel with R_6, which gives $R_9 = [(R_7 + R_8) \| R_6] = 3\ \Omega$.

At the node 2, the series combination of R_3 and R_4 are in parallel with R_2, which gives $R_{10} = [(R_3 + R_4) \| R_2] = 3\ \Omega$.

It is further reduced and is shown in Fig. 1.61 (b).

Simplifying further, we draw it as shown in Fig. 1.61 (c).

$$\text{Total current delivered by the source} = \frac{100}{R_T}$$

$$=\frac{100}{(13\,\|\,8)}=20.2 \text{ A}$$

Current in the 8 Ω resistor is $I_8=20.2\times\frac{13}{13+8}=12.5 \text{ A}$

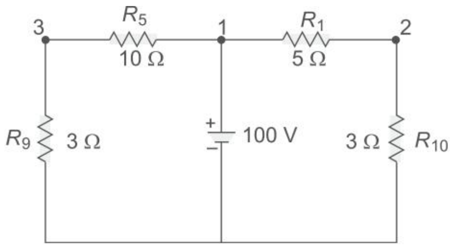


Fig. 1.61 (b)

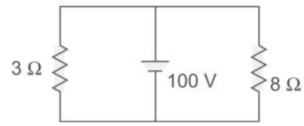


Fig. 1.61 (c)

Current in the 13 Ω resistor is $I_{13} = 20.2 \times \dfrac{8}{13+8} = 7.69$ A

So $I_5 = 12.5$ A, and $I_{10} = 7.69$ A

Current in the 4 Ω resistance, $I_4 = 3.845$ A

Current in the 3 Ω resistance, $I_3 = 6.25$ A

$$V_{AB} = V_A - V_B$$

where $V_A = I_3 \times 3\ \Omega = 6.25 \times 3 = 18.75$ V

$V_B = I_4 \times 4\ \Omega = 3.845 \times 4 = 15.38$ V

$\therefore \quad V_{AB} = 18.75 - 15.38 = 3.37$ V

PROBLEM 1.17

Determine the value of R in the circuit shown in Fig. 1.62, when the current is zero in the branch CD.

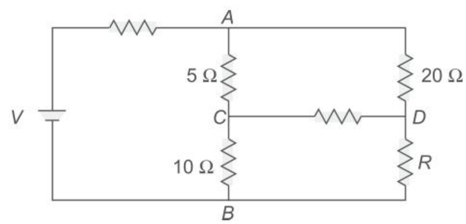


Fig. 1.62

Solution The current in the branch *CD* is zero, if the potential difference across *CD* is zero.

That means, voltage at the point *C* = voltage at the point *D*.

Since no current is flowing, the branch *CD* is open-circuited. So the same voltage is applied across *ACB* and *ADB*.

$$V_{10} = V_A \times \frac{10}{15}$$

$$V_R = V_A \times \frac{R}{20+R}$$

$$\therefore \quad V_{10} = V_R$$

and $\quad V_A \times \dfrac{10}{15} = V_A \times \dfrac{R}{20+R}$

$$\therefore \quad R = 40\ \Omega$$

PROBLEM 1.18

Find the power absorbed by each element in the circuit shown in Fig. 1.63.

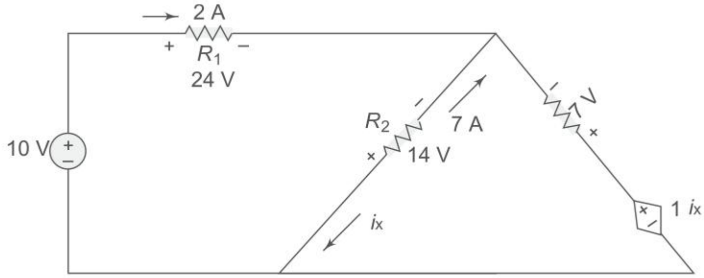


Fig. 1.63

Solution Power absorbed by any element $= VI$

where V is the voltage across the element and I is the current passing through that element.

Here, potential rises are taken as (–) sign.

Power absorbed by the 10 V source $= -10 \times 2 = -20$ W

Power absorbed by the resistor $R_1 = 24 \times 2 = 48$ W

Power absorbed by the resistor $R_2 = 14 \times 7 = 98$ W

Power absorbed by the resistor $R_3 = -7 \times 9 = -63$ W

Power absorbed by dependent voltage source $= (1 \times -7) \times 9 = -63$ W

PROBLEM 1.19

Show that the algebraic sum of the five absorbed power values in Fig. 1.64 is zero.

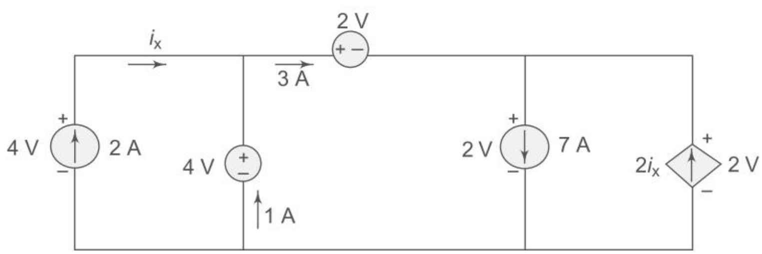


Fig. 1.64

Solution Power absorbed by the 2 A current source $= (-4) \times 2 = -8$ W

Power absorbed by the 4 V voltage source $= (-4) \times 1 = -4$ W

Power absorbed by the 2 V voltage source $= (2) \times 3 = 6$ W

Power absorbed by the 7 A current source $= (7) \times 2 = 14$ W

Power absorbed by the $2i_x$ dependent current source $= (-2) \times 2 \times 2 = -8$ W

Hence, the algebraic sum of the five absorbed power values is zero.

PROBLEM 1.20

For the circuit shown in Fig. 1.65, find the power absorbed by each of the elements.

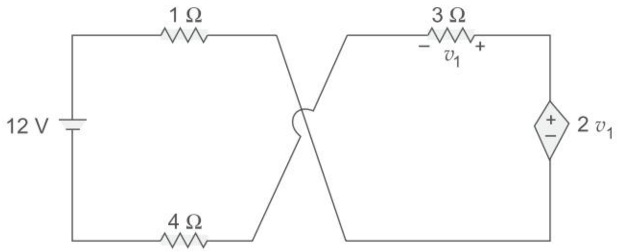


Fig. 1.65

Solution The above circuit can be redrawn as shown in Fig. 1.66.

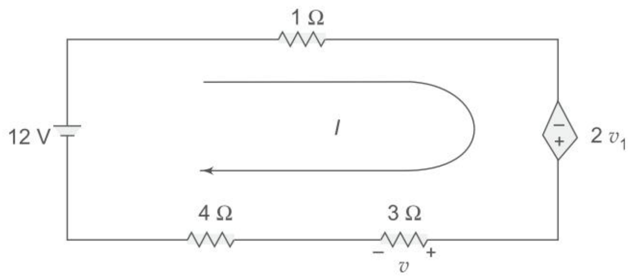


Fig. 1.66

Assume loop current I as shown in Fig. 1.66.
If we apply Kirchhoff's voltage law, we get

$$-12 + I - 2v_1 + v_1 + 4I = 0$$

The voltage across the 3 Ω resistor is $v_1 = 3\,I$
Substituting v_1 in the loop equation, we get $I = 6$ A
Power absorbed by the 12 V source = (–12) × 6 = –72 W
Power absorbed by the 1 Ω resistor = 6 × 6 = 36 W
Power absorbed by $2v_1$ dependent voltage source = $(2v_1)I = 2 \times 3 \times 6 \times 6$
= –216 W
Power absorbed by the 3 Ω resistor = $v_1 \times I = 18 \times 6 = 108$ W
Power absorbed by the 4 Ω resistor = 4 × 6 × 6 = 144 W

PROBLEM 1.21

For the circuit shown in Fig. 1.67, find the power absorbed by each element.

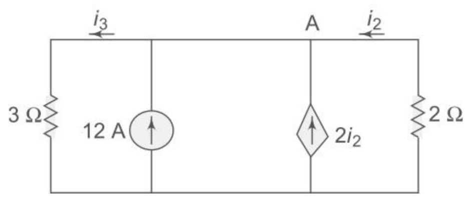


Fig. 1.67

Solution The circuit shown in Fig. 1.67 is a parallel circuit and consists of a single node A. By assuming voltage V at the node A, we can find the current in each element.

According to Kirchhoff's current law,

$$i_3 - 12 - 2i_2 - i_2 = 0$$

By using Ohm's law, we have

$$i_3 = \frac{V}{3},\ i_2 = \frac{-V}{2}$$

$$V\left[\frac{1}{3}+1+\frac{1}{2}\right]=12$$

$$\therefore\quad V=\frac{12}{1.83}=6.56$$

$$i_3=\frac{6.56}{3}=2.187\text{ A};\ i_2=\frac{-6.56}{2}=-3.28\text{ A}$$

Power absorbed by the 3 Ω resistor = (+6.56) (2.187) = 14.35 W
Power absorbed by the 12 A current source = (–6.56) 12 = – 78.72 W
Power absorbed by the $2i_2$ dependent current source
= (–6.56) × 2 × (– 3.28) = 43.03 W
Power absorbed by the 2 Ω resistor = (–6.52) (– 3.28) = 21.51 W

PSpice Problems

PROBLEM 1.1

Determine the total current in the following circuit using PSpice (Fig. 1.68).

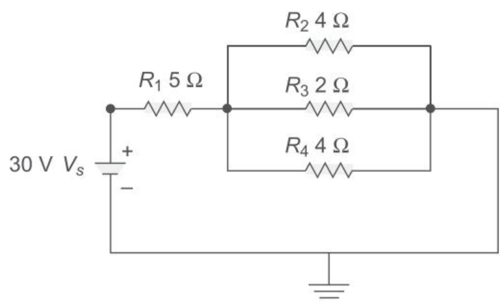


Fig. 1.68

```
* PROGRAM TO CALCULATE TOTAL CURRENT
VS  1  0  DC 30V
R1  1  2  5OHM
R2  2  0  4OHM
R3  2  0  2OHM
R4  2  0  4OHM
.OP
.END
```

OUTPUT

```
**** SMALL SIGNAL BIAS SOLUTION        TEMPERATURE = 27.000 DEG C
```

NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1) 30.0000 (2) 5.0000
VOLTAGE SOURCE CURRENTS
NAME CURRENT
VS –5.000E + 00

Result

Total current in the circuit is –5 A from the node 1 to the node 0.
Total current in the circuit is 5 A. (from the node 0 to the node 1).
.OP statement calculates the DC operating point and displays all the node voltages with respect to ground node with currents through all the voltage sources in the circuit in the output file. The output can be observed by opening the output file.

PROBLEM 1.2

Determine current in the 10 Ω resistance and find V_s in the following circuit (Fig. 1.69).

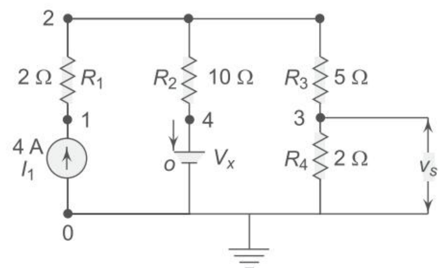


Fig. 1.69

```
* TO DETERMINE THE CURRENT IN A RESISTOR
I1  0  1  DC4A
R1  1  2  2OHM
R2  2  4  10OHM
R3  2  3  5OHM
R4  3  0  2OHM
VX  400 V
.OP
.END
```

OUTPUT
SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
****
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1) 24.4710 (2) 16.4710 (3) 4.7059 (4) 0.0000
VOLTAGE SOURCE CURRENTS
NAME CURRENT
VX 1.647E + 00

Result

Current in the 10 Ω resistance is current through V_x = 1.647 A from 4 to 0 and
V_s is voltage across the node '3' = 4.7059 V.
In order to view the current through a resistor, with bias point calculation (.OP), an additional voltage source of 0 V is inserted in series with the element.

PROBLEM 1.3

What is the voltage across A and B in the circuit shown in Fig. 1.70?

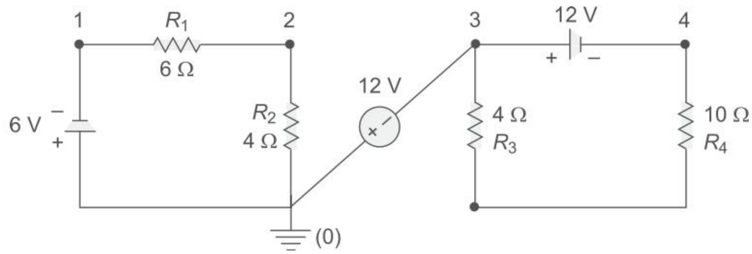


Fig. 1.70

```
* PROGRAM TO CALCULATE VAB
V1  0  1  DC 6V
R1  1  2  6OHM
R2  2  0  4OHM
V2  0  3  DC12V
V3  3  4  12V
R3  3  5  4OHM
R4  4  5  10OHM
.OP
.DC V1 661; DC ANALYSIS FOR V = 6V
.PRINT DC V(2,5); TO PRINT VAB DIRECTLY FROM DC ANALYSIS
.END
```

OUTPUT

```
**** DC TRANSFER CURVES             TEMPERATURE = 27.000 DEG C
******************************************************************
V1                    V(2,5)
6.000E + 00           1.303E + 01
**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
******************************************************************
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1)  – 6.0000    (2)  – 2.4000    (3)  – 12.0000    (4)  – 24.0000    (5)  – 15.4290
VOLTAGE SOURCE CURRENTS
NAME          CURRENT
V1            –6.000E – 01
V2            4.441E – 16
V3            – 8.571E – 01
TOTAL POWER DISSIPATION 1.39E + 01 WATTS
```

Result

$V_{AB} = -V_{(4\Omega)} + 12 + V_{(4\Omega)} = (-0.6X4) + 12 - (4X - 0.857) = 13.028$ V.

PROBLEM 1.4

Determine the output voltage in the following circuit (Fig.1.71).

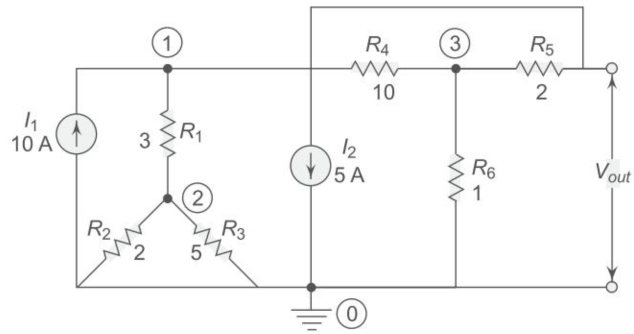


Fig. 1.71

```
* PROGRAM TO CALCULATE OUTPUT VOLTAGE
I1  0  1  10A
R1  1  2  3
R2  2  0  2
R3  2  0  5
I2  1  0  5
R4  3  1  10
R5  3  1  2
R6  3  0  1
.OP
.END
**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
**************************************************************
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1)  8.3221     (2)  2.6846      (3)  3.1208
VOLTAGE SOURCE CURRENTS
NAME CURRENT
```

Result

Output voltage $V_{out} = 8.3221$ V $= V(1)$.

PROBLEM 1.5

Use PSpice to calculate all the voltages and currents in the following circuit (Fig. 1.72).

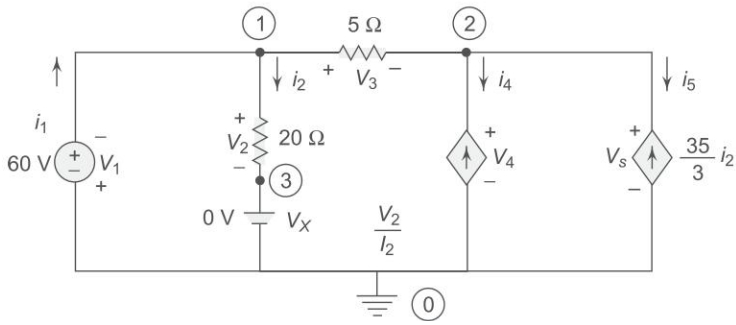


Fig. 1.72

```
* COMPUTATION OF VOLTAGES AND CURRENTS
V1  1  0  DC 60
```

```
R1   1  3   20
R2   1  2   5
VX  3  0   DC 0V
G1   0  2   1     3 0.08333
F1   0  2   VX 11.6667
.OP
.END
```

OUTPUT

```
**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
*************************************************************
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1)  60.0000    (2)  260.0000 (3)  0.0000
VOLTAGE SOURCE CURRENTS
NAME CURRENT
V1        3.700E  + 01
VX        3.000E + 00
```

Result

i_1 = –37 A; i_2 = 3 A; i_5 = –35 A; i_4 = –5 A.

V_1 = –60 V; V_2 = 60 V; V_3 = –200 V; V_4 = V_5 = 260 V.

ANSWERS TO PRACTICE PROBLEMS

1-6.1 (a) 75 A (b) 20 A (c) 2.5 A; 2 S

1-2.4 0.3×10^{-2} J

1-2.6 638 mW

1-2.7 (a) $P_{5A} = -1.389$ kW, $P_{10\text{ mS}} = 771.6$ W, $P_{40\text{ mS}} = 3.08$ kW
$P_{\text{dependent}} = -2.469$ kW
(b) $P_{5A} = -775.9$ W, $P_{10\text{ mS}} = 240.8$ W, $P_{40\text{ mS}} = 963.1$ W
$P_{\text{dependent}} = -428.1$ W

1-2.8 $P_{0.2} = -148.8$ W, $P_{20} = -1090.9$ W, $P_4 = 743.8$ W, $P_6 = 495.9$ V

1-2.9 (a) 0.156 watts (b) 0.14 watts (c) 5 watts

1-4.1 3.33 V

1-6.2 1.5 mF

1-8.2 $V_1 = V_2 = V_3 = 100$ V

1-8.4 10 V; 30 V

1-8.5 25 V; 5 V

1-9.4 150 Ω

1-9.5 (a) 60 Ω; (b) 213.3 Ω; (c) 51.79 Ω

1-9.7 0.682 A; 4.092 A

Objective-Type Questions

☆☆★**1.1** How many coulombs of charge do 50×10^{31} electrons possess?
(a) 80×10^{12} C (b) 50×10^{31} C (c) 0.02×10^{-31} (d) $1/80 \times 10^{12}$ C

☆☆★**1.2** Determine the voltage of 100 J/25 C.
(a) 100 V (b) 25 V (c) 4 V (d) 0.25 V

☆★★**1.3** What is the voltage of a battery that uses 800 J of energy to move 40 C of charge through a resistor?
(a) 800 V (b) 40 V (c) 25 V (d) 20 V

☆☆★**1.4** Determine the current if a 10-coulomb charge passes a point in 0.5 seconds.
(a) 10 A (b) 20 A (c) 0.5 A (d) 2 A

☆☆★**1.5** If a resistor has 5.5 V across it and 3 mA flowing through it, what is the power?
(a) 16.5 mW (b) 15 mW (c) 1.83 mW (d) 16.5 W

☆☆★**1.6** Identify the passive element among the following:
(a) Voltage source (b) Current source (c) Inductor (d) Transistor

☆☆★**1.7** If a resistor is to carry 1 A of current and handle 100 W of power, how many ohms must it be? Assume that voltage can be adjusted to any required value.
(a) 50 Ω (b) 100 Ω (c) 1 Ω (d) 10 Ω

☆☆★**1.8** A 100 Ω resistor is connected across the terminals of a 2.5 V battery. What is the power dissipation in the resistor?
(a) 25 W (b) 100 W (c) 0.4 W (d) 6.25 W

☆☆★**1.9** Determine the total inductance of a parallel combination of 100 mH, 50 mH, and 10 mH.
(a) 7.69 mH (b) 160 mH (c) 60 mH (d) 110 mH

☆☆★**1.10** How much energy is stored by a 100 mH inductance with a current of 1 A?
(a) 100 J (b) 1 J (c) 0.05 J (d) 0.01 J

☆★★**1.11** Five inductors are connected in series. The lowest value is 5 μH. If the value of each inductor is twice that of the preceding one, and if the inductors are connected in order of ascending values, what is the total inductance?
(a) 155 μH (b) 155 H (c) 155 mH (d) 25 μH

☆☆★**1.12** Determine the charge when $C = 0.001$ μF and $v = 1$ kV.
(a) 0.001 C (b) 1 μC (c) 1 C (d) 0.001 C

☆☆★**1.13** If the voltage across a given capacitor is increased, the amount of stored charge
(a) increases (b) decreases (c) remains constant (d) is exactly doubled

☆☆★**1.14** 1 μF, a 2.2 μF, and a 0.05 μF capacitors are connected in series. The total capacitance is less than
(a) 0.07 (b) 3.25 (c) 0.05 (d) 3.2

☆☆★**1.15** How much energy is stored by a 0.05 μF capacitor with a voltage of 100 V?
(a) 0.025 J (b) 0.05 J (c) 5 J (d) 100 J

☆☆★**1.16** Which one of the following is an ideal voltage source?
(a) Voltage independent of current (b) Current independent of voltage
(c) Both (a) and (b) (d) None of the above

☆☆★**1.17** The following voltage drops are measured across each of three resistors in series: 5.2 V, 8.5 V and 12.3 V. What is the value of the source voltage to which these resistors are connected?
(a) 8.2 V (b) 12.3 V (c) 5.2 V (d) 26 V

☆☆★**1.18** A certain series circuit has a 100 Ω, a 270 Ω, and a 330 Ω resistor in series. If the 270 Ω resistor is removed, the current
(a) increases (b) becomes zero (c) decrease (d) remain constant

☆★★**1.19** A series circuit consists of a 4.7 kΩ, 5.6 kΩ, 9 kΩ, and 10 kΩ resistor. Which resistor has the most voltage across it?
(a) 4.7 kΩ (b) 5.6 kΩ (c) 9 kΩ (d) 10 kΩ

☆★★**1.20** The total power in a series circuit is 10 W. There are five equal-value resistors in the circuit. How much power does each resistor dissipate?
(a) 10 W (b) 5 W (c) 2 W (d) 1 W

☆☆★**1.21** When a 1.2 kΩ resistor, 100 Ω resistor, 1 kΩ resistor, and a 50 Ω resistor are in parallel, the total resistance is less than
(a) 100 Ω (b) 50 Ω (c) 1 kΩ (d) 1.2 kΩ

☆☆★**1.22** If a 10 V battery is connected across the parallel resistors of 3 Ω, 5 Ω, 10 Ω, and 20 Ω, how much voltage is there across the 5 Ω resistor?
(a) 10 V (b) 3 V (c) 5 V (d) 20 V

☆★★**1.23** If one of the resistors in a parallel circuit is removed, what happens to the total resistance?
(a) Decreases (b) Increases (c) Remains constant (d) Exactly doubles

☆★★**1.24** The power dissipation in each of three parallel branches is 1 W. What is the total power dissipation of the circuit?
(a) 1 W (b) 4 W (c) 3 W (d) zero

☆★★**1.25** In a four-branch parallel circuit, 10 mA of current flows in each branch. If one of the branch opens, the current in each of the other branches
(a) increases (b) decreases (c) remains unaffected (d) doubles

☆★★**1.26** Four equal-value resistors are connected in parallel. Five volts are applied across the parallel circuit, and 2.5 mA are measured from the source. What is the value of each resistor?
(a) 4 Ω (b) 8 Ω (c) 2.5 Ω (d) 5 Ω

☆★★**1.27** Six lightbulbs are connected in parallel across 110 V. Each bulb is related at 75 W. How much current flows through each bulb?
(a) 0.682 A (b) 0.7 A (c) 75 A (d) 110 A

☆★★**1.28** A 330 Ω resistor is in series with the parallel combination of four 1 kΩ resistors. A 100 V source is connected to the circuit. Which resistor has the most current through it?
(a) 330 Ω resistor (b) Parallel combination of three 1 kΩ resistors
(c) Parallel combination of two 1 kΩ resistors (d) 1 kΩ resistor

☆★★**1.29** The current i_4 in the circuit shown in Fig. 1.73 is equal to
(a) 12 A (b) –12 A (c) 4 A (d) none of the above

☆★★**1.30** The voltage *V* in Fig. 1.74 is equal to
(a) 3 V (b) –3 V (c) 5 V (d) none of the above

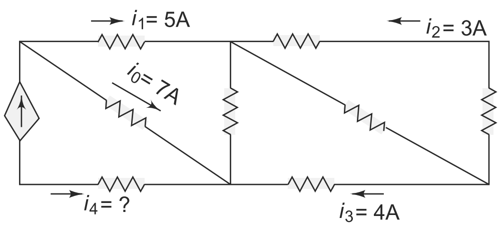


Fig. 1.73

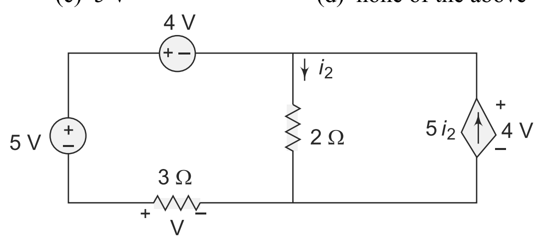


Fig. 1.74

☆★★**1.31** The voltage *V* in Fig. 1.75 is always equal to
(a) 9 V (b) 5 V (c) 1 V (d) none of the above

★★★**1.32** The voltage *V* in Fig. 1.76 is
(a) 10 V (b) 15 V (c) 5 V (d) none of the above

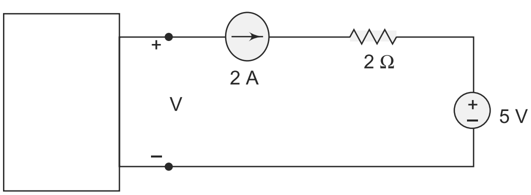


Fig. 1.75

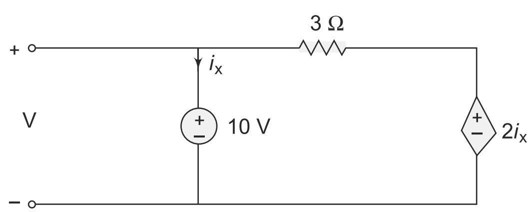


Fig. 1.76

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/259

CHAPTER 2

Methods of Analysing Circuits

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Draw the tree, co-tree, twigs and links for a given network
- LO 2 Describe incidence matrix and its properties; analyse the relationship between KCL and incidence matrix
- LO 3 Describe the link currents and tie-set matrix
- LO 4 Describe cut-set and tree branch voltages
- LO 5 Analyse the network (resistive circuits) using mesh analysis and supermesh analysis and write the mesh equations using inspection method
- LO 6 Analyse the network (resistive circuits) using nodal analysis and supernode analysis and write the nodal equations using inspection method
- LO 7 Analyse the network (resistive circuits) using source transformation technique

2.1 INTRODUCTION

A division of mathematics called topology or graph theory deals with graphs of networks and provides information that helps in the formulation of network equations. In circuit analysis, all the elements in a network must satisfy Kirchhoff's laws, besides their own characteristics. Based on these laws, we can form a number of equations. These equations can be easily written by converting the network into a graph. Certain aspects of network behaviour are brought into better perspective if a graph of the network is drawn. If each element or a branch of a network is represented on a diagram by a line irrespective of the characteristics of the elements, we get a graph. Hence, network topology is network geometry. A network is an interconnection of elements in various branches at different nodes as shown in Fig. 2.1. The corresponding graph is shown in Fig. 2.2 (a).

The graphs shown in Figs 2.2 (b) and (c) are also graphs of the network in Fig. 2.1.

It is interesting to note that the graphs shown in Fig. 2.2 (a), (b) and (c) may appear to be different but they are topologically equivalent. *A branch is represented by a line segment connecting a pair of nodes in the graph of a network. A node is a terminal of a branch, which is represented by a point. Nodes are the end points of branches.* All these graphs have identical relationships between branches and nodes.

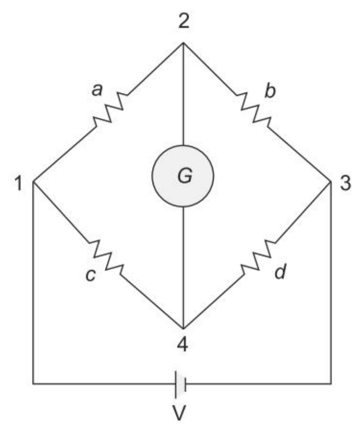


Fig. 2.1

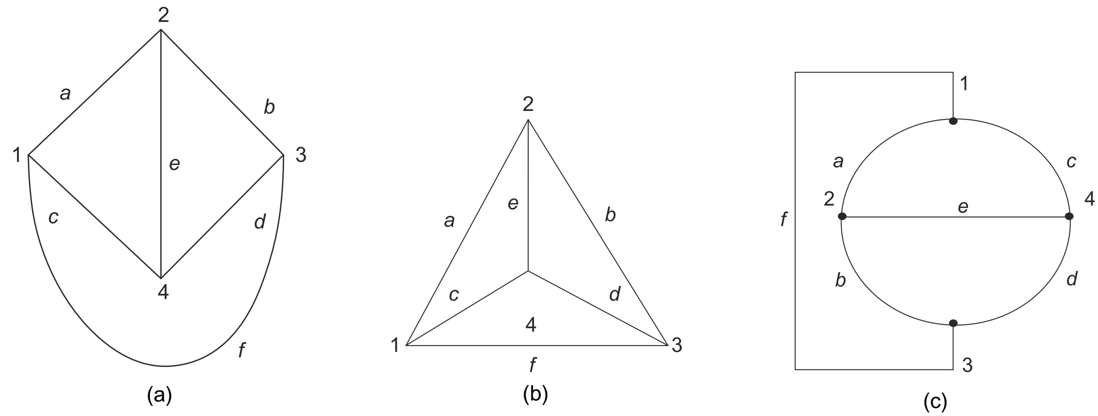


Fig. 2.2

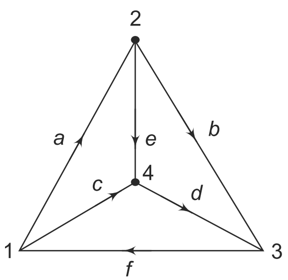


Fig. 2.3

The three graphs in Fig. 2.2 have six branches and four nodes. These graphs are also called undirected. If every branch of a graph has a *direction* as shown in Fig. 2.3, then the graph is called a *directed graph*.

A node and a branch are incident if the node is a terminal of the branch. Nodes can be incident to one or more elements. The number of branches incident at a node of a graph indicates the degree of the node. For example, in Fig. 2.3 the degree of node 1 is three. Similarly, the degree of node 2 is three. If each element of the connected graph is assigned a direction as shown in Fig. 2.3 it is then said to be oriented. A graph is connected if and only if there is a path between every pair of nodes. A path is said to exist between any two nodes, for example 1 and 4 of the graph in Fig. 2.3, if it is possible to reach the node 4 from the node 1 by traversing along any of the branches of the graph. A graph can be drawn if there exists a path between any pair of nodes. A loop exists, if there is more than one path between two nodes.

❑ ***Planar and Non-Planar Graphs*** *A graph is said to be* ***planar*** *if it can be drawn on a plane surface such that no two branches cross each other.* On the other hand, *in a* ***non-planar graph****, there will be branches which are not in the same plane as others, i.e. a non-planar graph cannot be drawn on a plane surface without a crossover.* Figures 2.2 and 2.4 illustrate a planar graph and non-planar graph respectively.

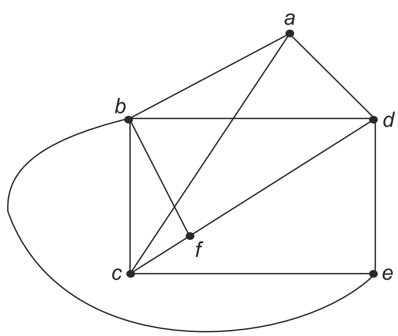


Fig. 2.4

2.2 TREE AND CO-TREE

LO 1 Draw the tree, co-tree, twigs and links for a given network

A tree is a connected subgraph of a network which consists of all the nodes of the original graph but no closed paths. The graph of a network may have a number of trees. The number of nodes in a graph is equal to the number nodes in the tree. The number of branches in a tree is less than the number of branches in a graph. A graph is a tree if there is a unique path between any pair of nodes. Consider a graph with four branches and three nodes as shown in Fig. 2.5.

Five open-ended graphs based on Fig. 2.5 are represented by Figs 2.6 (a) to (e). Since each of these open-ended graphs satisfies all the requirements of a tree, each graph in Fig. 2.6 is a tree corresponding to Fig. 2.5.

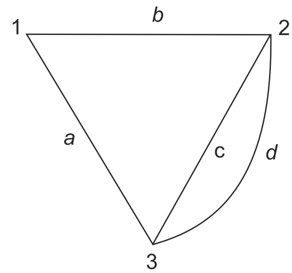


Fig. 2.5

In Fig. 2.6, there is no closed path or loop; the number of nodes $n = 3$ is the same for the graph and its tree, where as the number of branches in the tree is only two. In general, if a tree contains n nodes, then it has $(n - 1)$ branches.

In forming a tree for a given graph, certain branches are removed or opened. The branches thus opened are called links or ***link*** *branches*. The links for Fig. 2.6 (a) for example are a and d and for Fig. 2.6 (b) are b and c. *The set of all links of a given tree is called the* ***co-tree*** *of the graph.* Obviously, the branches a, d are a co-tree for Fig. 2.6 (a). Similarly, for the tree in Fig. 2.6 (b), the branches b, c are the co-tree. Thus the link branches and the tree branches combine to form the graph of the entire network.

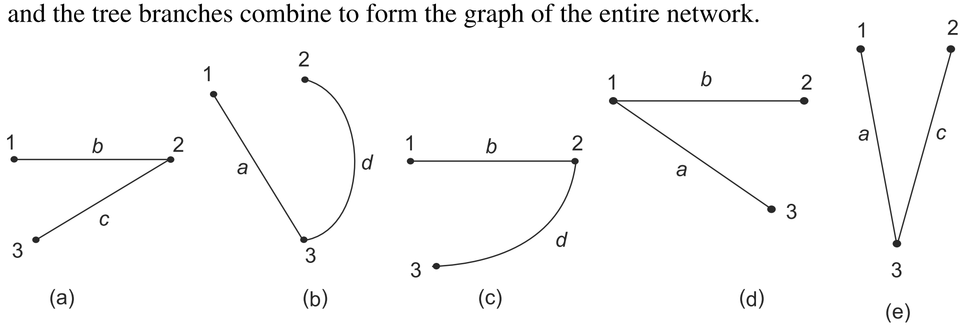


Fig. 2.6

EXAMPLE 2.1

For the given graph shown in Fig. 2.7, draw the number of possible trees.

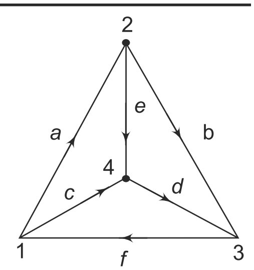


Fig. 2.7

Solution The number of possible trees for Fig. 2.7 are represented by Figs 2.8 (a) – (l).

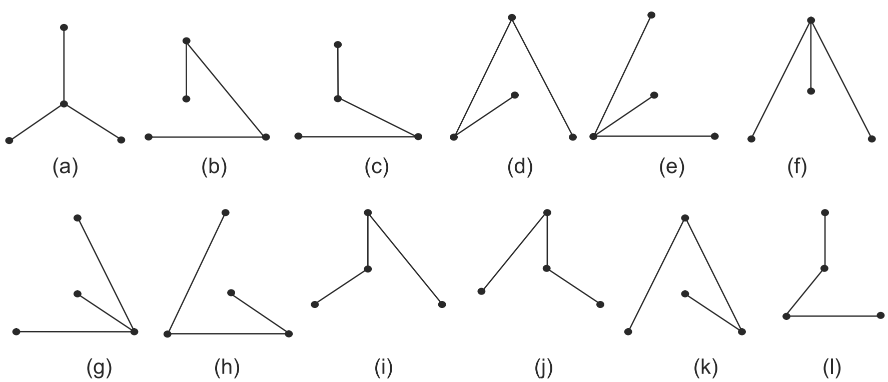


Fig. 2.8

2.3 TWIGS AND LINKS

LO 1

The branches of a tree are called its ***twigs****. For a given graph, the complementary set of branches of the tree is called the co-tree of the graph. The branches of a co-tree are called* ***links****, i.e. those elements of the connected graph that are not included in the tree links and form a subgraph.* For example, the set of branches (b, d, f) represented by dotted lines in Fig. 2.11 form a co-tree of the graph in Fig. 2.9 with respect to the tree in Fig. 2.10.

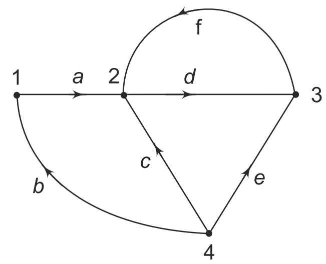


Fig. 2.9

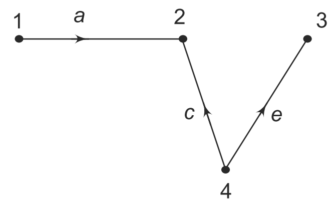


Fig. 2.10

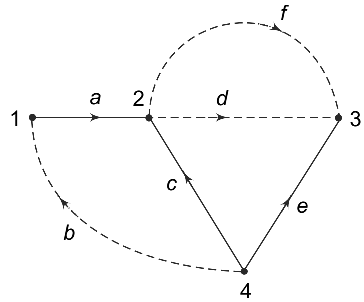


Fig. 2.11

The branches *a*, *c,* and *e* are the twigs while the branches *b*, *d,* and *f* are the links of this tree. It can be seen that for a network with *b* branches and *n* nodes, the number of twigs for a selected tree is $(n - 1)$ and the number of links *I* with respect to this tree is $(b - n + 1)$. *The number of twigs (n – 1) is known as the tree value of the graph. It is also called the* ***rank*** *of the tree. If a link is added to the tree, the resulting graph contains one closed path, called a* ***loop****. The addition of each subsequent link forms one or more additional loops. Loops which contain only one link are independent and are called basic loops.*

Frequently Asked Questions linked to LO1*

☆☆★**2-1.1** Define tree and co-tree. **(PTU 2011-12)**

☆☆★**2-1.2** Define loop. **(PTU 2011-12)**

2.4 INCIDENCE MATRIX (A)

LO 2 Describe incidence matrix and its properties; analyse the relationship between KCL and incidence matrix

The incidence of elements to nodes in a connected graph is shown by the element node ***incidence matrix*** *(A).* Arrows indicated in the branches of a graph result in an oriented or a directed graph. These arrows are the indication for the current flow or voltage rise in the network. It can be easily identified from an oriented graph regarding the incidence of branches to nodes. It is possible to have an analytical description of an oriented-graph in a matrix form. The dimensions of the matrix *A* is $n \times b$ where *n* is the number of nodes and *b* is number of branches. For a graph having *n* nodes and *b* branches, the complete incidence matrix *A* is a rectangular matrix of order $n \times b$.

In the matrix *A* with *n* rows and *b* columns, an entry a_{ij} in the *i*th row and *j*th column has the following values.

$$\left.\begin{aligned} a_{ij} &= 1, \text{ if the } j\text{th branch is incident to and oriented away from the } i\text{th node.} \\ a_{ij} &= -1, \text{ if the } j\text{th branch is incident to and oriented towards the } i\text{th node.} \\ a_{ij} &= 0, \text{ if the } j\text{th branch is not incident to the } i\text{th node.} \end{aligned}\right\} \quad (2.1)$$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

Figure 2.12 shows a directed graph.

Following the above convention, its incidence matrix A is given by

$$\begin{array}{cc} & \text{nodes} \quad\quad \text{branches}\rightarrow \\ & \begin{array}{ccccccc} \downarrow & a & b & c & d & e & f \end{array} \\ A= & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

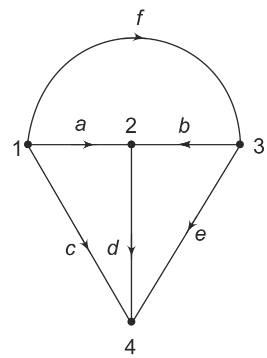


Fig. 2.12

The entries in the first row indicate that three branches a, c, and f are incident to the node 1 and they are oriented away from the node 1 and therefore the entries a_{11}; a_{13} and a_{16} are $+1$. Other entries in the first row are zero as they are not connected to the node 1. Likewise, we can complete the incidence matrix for the remaining nodes 2, 3, and 4.

2.5 PROPERTIES OF INCIDENCE MATRIX *A*

LO 2

The following properties are some of the simple conclusions from the incidence matrix A.

1. Each column representing a branch contains two non-zero entries $+1$ and -1; the rest being zero. The unit entries in a column identify the nodes of the branch between which it is connected.
2. The unit entries in a row identify the branches incident at a node. Their number is called the degree of the node.
3. A degree of 1 for a row means that there is one branch incident at the node. This is commonly possible in a tree.
4. If the degree of a node is two, then it indicates that two branches are incident at the node and these are in series.
5. Columns of A with unit entries in two identical rows correspond to two branches with same end nodes and hence they are in parallel.
6. Given the incidence matrix A, the corresponding graph can be easily constructed since A is a complete mathematical replica of the graph.
7. If one row of A is deleted the resulting $(n - 1) \times b$ matrix is called the reduced incidence matrix A_1. Given A_1, A is easily obtained by using the first property.

It is possible to find the exact number of trees that can be generated from a given graph if the reduced incidence matrix A_1 is known and the number of possible trees is given by Det $(A_1 A_1^T)$ where A_1^T is the transpose of the matrix A_1.

EXAMPLE 2.2

Draw the graph corresponding to the given incidence matrix.

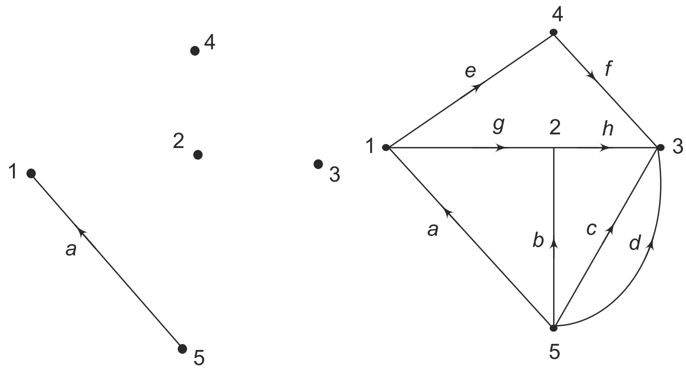
$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution There are five rows and eight columns which indicate that there are five nodes and eight branches. Let us number the columns from a to h and rows as 1 to 5.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Mark the nodes corresponding to the rows 1, 2, 3, 4, and 5 as dots as shown in Fig. 2.13 (a). Examine each column of A and connect the nodes (unit entries) by a branch; label it after marking an arrow.

For example, examine the first column of A. There are two unit entries one in the first row and 2nd in the last row, hence connect branch a between node 1 and 5. The entry of A_{11} is –ve and that of A_{51} is +ve. Hence, the orientation of the branch is away from the node 5 and towards node 1 as per the convention. Proceeding in this manner, we can complete the entire graph as shown in Fig. 2.13 (b).



(a) **Fig. 2.13** (b)

From the incidence matrix A, it can be verified that branches c and d are in parallel (property 5) and branches e and f are in series (property 4).

EXAMPLE 2.3

Obtain the incidence matrix A from the following reduced incidence matrix A_1 and draw its graph.

$$[A_1] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution There are five rows and seven columns in the given reduced incidence matrix $[A_1]$. Therefore, the number of rows in the complete incidence matrix A will be $5 + 1 = 6$. There will be six nodes and seven branches in the graph. The dimensions of matrix A is 6×7. The last row in A, i.e. 6th row for the matrix A can be obtained by using the first property of the incidence matrix. It is seen that the first column of $[A_1]$ has a single non-zero element –1. Hence, the first element in the 6th row will be $+1(-1 + 1 = 0)$. Second column of A_1 has two non-zero elements +1 and –1, hence the second element in the 6th row will be 0. Proceeding in this manner we can obtain the 6th row.

The complete incidence matrix can therefore be written as

$$[A] = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have seen that any one of the rows of a complete incidence matrix can be obtained from the remaining rows. Thus, it is possible to delete any one row from A without loosing any information in A_1. Now the oriented graph can be constructed from the matrix A. The nodes may be placed arbitrarily. The number of nodes to be marked will be six. Taking node 6 as reference node the graph is drawn as shown in Fig. 2.14.

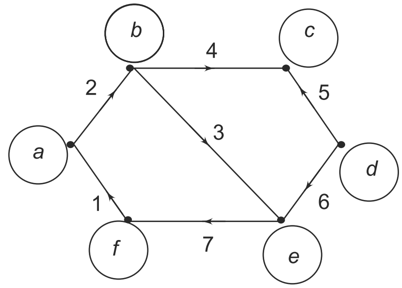


Fig. 2.14

2.6 INCIDENCE MATRIX AND KCL

LO 2

Kirchhoff's Current Law (KCL) of a graph can be expressed in terms of the reduced incidence matrix as A_1 $I = 0$.

A_1, I is the matrix representation of KCL, where I represents branch current vectors $I_1, I_2, \ldots I_6$.

Consider the graph shown in Fig. 2.15. It has four nodes a, b, c, and d.

Let node d be taken as the reference node. The positive reference direction of the branch currents corresponds to the orientation of the graph branches. Let the branch currents be $i_1, i_2, \ldots i_6$. Applying KCL at nodes a, b, and c.

$$-i_1 + i_4 = 0$$

$$-i_2 - i_4 + i_5 = 0$$

$$-i_3 - i_5 - i_6 = 0$$

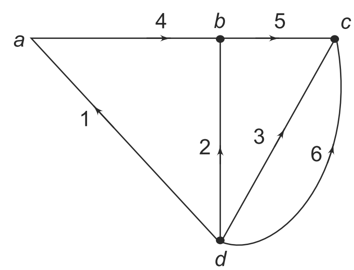


Fig. 2.15

These equations can be written in the matrix form as follows:

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1 I_b = 0 \tag{2.2}$$

Here, I_b represents column matrix or a vector of branch currents.

$$I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$$

A_1 is the reduced incidence matrix of a graph with n nodes and b branches. And it is a $(n-1) \times b$ matrix obtained from the complete incidence matrix of A deleting one of its rows. The node corresponding to the deleted row is called the *reference node* or *datum node*. It is to be noted that $A_1 I_b = 0$ gives a set of $n-1$ linearly independent equations in branch currents $i_1, i_2, \ldots i_6$. Here, $n = 4$. Hence, there are three linearly independent equations.

Frequently Asked Questions linked to LO2

☆☆★**2-2.1** Define tree, co-tree, twig, link , and incidence matrix taking a suitable example. **(PTU 2009-10)**

☆☆★**2-2.2** Define incidence matrix. **(PTU 2009-10)**

☆★★**2-2.3** Explain the formulation of graph, tree, and incidence matrix using suitable examples. Hence, discuss the procedure of forming reduced incidence matrix and its advantages. **[GTU Dec. 2012]**

☆☆★**2-2.4** Explain about linear oriented graph, incidence matrix and circuit matrix. show Kirchhoff's laws in incidence-matrix formulation and circuit-matrix formulation. **[GTU Dec. 2010]**

2.7 LINK CURRENTS: TIE-SET MATRIX

LO 3 Describe the link currents and tie-set matrix

For a given tree of a graph, addition of each link between any two nodes forms a loop called the ***fundamental loop****. In a loop there exists a closed path and a circulating current, which is called the* ***link current****.* The current in any branch of a graph can be found by using link currents.

The fundamental loop formed by one link has a unique path in the tree joining the two nodes of the link. This loop is also called ***f-loop*** *or a* ***tie-set****.*

Consider a connected graph shown in Fig. 2.16 (a). It has four nodes and six branches. One of its trees is arbitrarily chosen and is shown in Fig. 2.16 (b). The twigs of this tree are branches 4, 5, and 6. The links corresponding to this tree are branches 1, 2, and 3. Every link defines a fundamental loop of the network.

Number of nodes $n = 4$

Number of branches $b = 6$

Number of tree branches or twigs $= n - 1 = 3$

Number of link branches $I = b - (n - 1) = 3$

Let $i_1, i_2, \ldots i_6$ be the branch currents with directions as shown in Fig. 2.16 (a). Let us add a link in its proper place to the tree as shown in 2.16(c). It is seen that a loop I_1 is formed by the branches 1, 5, and 6. There is a formation of link current, let this current be I_1. This current passes through the branches 1, 5, and 6. By convention a fundamental loop is given the same orientation as its defining link, i.e. the link current I_1 coincides with the branch current direction i_1 in ab. A tie set can also be defined as the set of branches

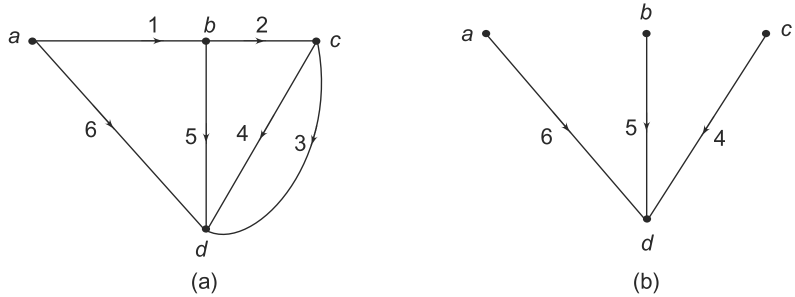


Fig. 2.16

that forms a closed loop in which the link current flows. By adding the other link branches 2 and 3, we can form two more fundamental loops or *f*-loops with link currents I_2 and I_3 respectively as shown in Figs 2.16 (d) and (e).

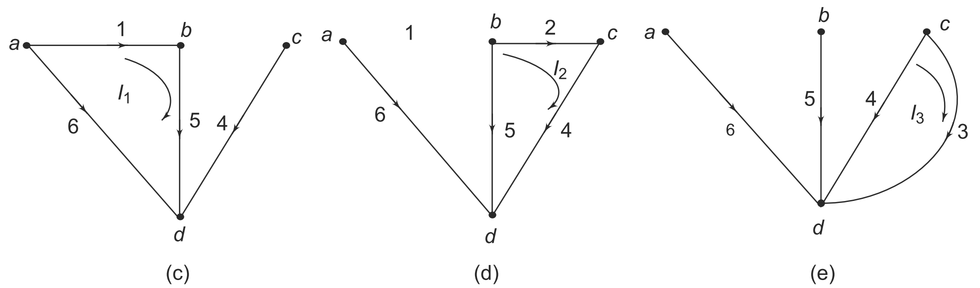


Fig. 2.16 *(continued)*

2.7.1 Tie-Set Matrix

Kirchhoff's voltage law can be applied to the *f*-loops to get a set of linearly independent equations. Consider Fig. 2.17.

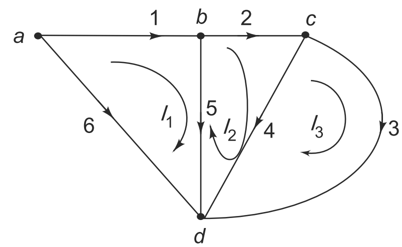


Fig. 2.17

There are three fundamental loops I_1, I_2 and I_3 corresponding to the link branches 1, 2, and 3 respectively. If V_1, V_2, … V_6 are the branch voltages the KVL equations for the three *f*-loops can be written as

$$\left.\begin{aligned} V_1 + V_5 - V_6 &= 0 \\ V_2 + V_4 - V_5 &= 0 \\ V_3 - V_4 &= 0 \end{aligned}\right\} \qquad (2.3)$$

In order to apply KVL to each fundamental loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equation can be written in matrix form as

$$\begin{array}{c} \text{loop} \quad \text{branches} \rightarrow \qquad\qquad 3\times 6\times 6 \\ \begin{array}{cc} \downarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$B\,V_b = 0 \qquad (2.4)$$

where B is an $I \times b$ matrix called the tie-set matrix or fundamental loop matrix and V_b is a column vector of branch voltages.

The tie-set matrix B is written in a compact form as $B\,[b_{ij}]$. (2.5)

The element b_{ij} of B is defined as

$b_{ij} = 1$ when, the branch b_j is in the *f*-loop I_i (loop current) and their reference directions coincide.

$b_{ij} = -1$ when the branch b_j is in the f-loop I_i (loop current) and their reference directions are opposite.

$b_{ij} = 0$ when branch b_j is not in the f-loop I_i.

2.7.2 Tie-set Matrix and Branch Currents

It is possible to express branch currents as a linear combination of link current using the matrix B.

If I_B and I_l represents the branch-current matrix and loop-current matrix respectively and B is the tie-set matrix, then

$$[I_b] = [B^T]\,[I_L] \tag{2.6}$$

where $[B^T]$ is the transpose of the matrix $[B]$. Equation (2.6) is known as *link current transformation equation*.

Consider the tie-set matrix of Fig. 2.17.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The branch-current vector $[I_b]$ is a column vector.

$$[I_b] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

The loop-current vector $[I_L]$ is a column vector.

$$[I_L] = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Therefore, the link-current transformation equation is given by $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are

$$
\begin{aligned}
i_1 &= I_1 \\
i_2 &= I_2 \\
i_3 &= I_3 \\
i_4 &= I_2 - I_3 \\
i_5 &= I_1 - I_2 \\
i_6 &= -I_1
\end{aligned}
$$

EXAMPLE 2.4

For the electrical network shown in Fig. 2.18 (a), draw its topological graph and write its incidence matrix, tie-set matrix, link current transformation equation and branch currents.

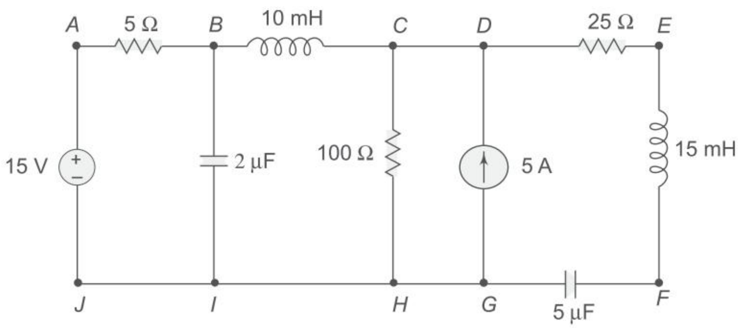


Fig. 2.18 (a)

Solution The voltage source is short-circuited, the current source is open-circuited and the points which are electrically at same potential are combined to form a single node. The graph is shown in Fig. 2.18 (b).

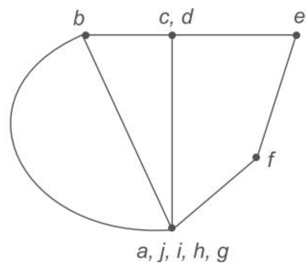


Fig. 2.18 (b)

Combining the simple nodes and arbitrarily selecting the branch current directions, the oriented graph is shown in Fig. 2.18 (c). The simplified graph consists of three nodes. Let them be x, y, and z and five branches 1, 2, 3, 4, and 5. The complete incidence matrix is given by

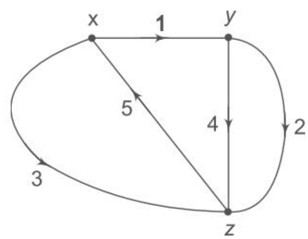


Fig. 2.18 (c)

$$
\begin{array}{c}
\text{nodes} \quad \text{branches} \rightarrow \\
\downarrow \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\end{array}
$$

$$
A = \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}
$$

Let us choose the node z as the reference or datum node for writing the reduced incidence matrix A_1 or we can obtain A_1 by deleting the last row elements in A.

$$
\begin{array}{c}
\text{nodes} \quad \text{branches} \rightarrow \\
\downarrow \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\end{array}
$$

$$
A_1 = \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}
$$

For writing the tie-set matrix, consider the tree in the graph in Fig. 2.18 (c).

Number of nodes $n = 3$

Number of branches = 5

Number of tree branches or twigs $= n - 1 = 2$

Number of link branches $I = b - (n - 1) = 5 - (3 - 1) = 3$

The tree shown in Fig. 2.18 (d) consists of two branches 4 and 5 shown with solid lines and the link branches of the tree are 1, 2, and 3 shown with dashed lines. The tie-set matrix or fundamental loop matrix is given by

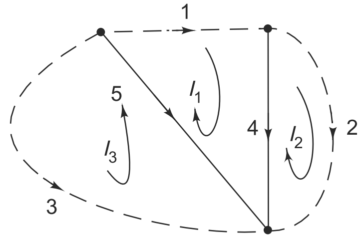


Fig. 2.18 (d)

$$\begin{array}{cc} \text{loop} & \text{branches} \rightarrow \\ \downarrow & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$B = \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

To obtain the link-current transformation equation and thereby branch currents, the transpose of B should be calculated.

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The equation $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are given by

$$i_1 = I_1$$
$$i_2 = I_2$$
$$i_3 = I_3$$
$$i_4 = I_1 - I_2$$
$${}_5 = I_1 + I_3$$

Frequently Asked Questions linked to LO3

☆☆★**2-3.1** Draw a tree of the network in Fig. Q.1 taking the branches denoted by ($b2$), ($b4$), and ($b5$) as tree branches. Give the fundamental loop matrix. Determine the matrix loop equation from the fundamental loop matrix. Branch impedances are in ohms.

(GTU May 2011)

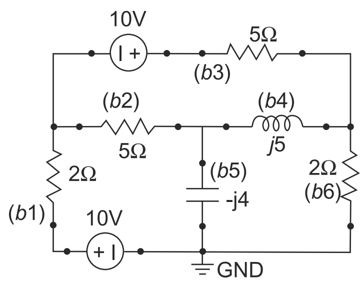


Fig. Q.1

☆☆★**2-3.2** Define the following terms:
i. Node ii. Tree iii. Incidence matrix iv. Basic tie-set

☆★★**2-3.3** Find the branch currents shown in Fig. Q.3 by using the concept of the tie-set matrix.

(JNTU Nov. 2012)

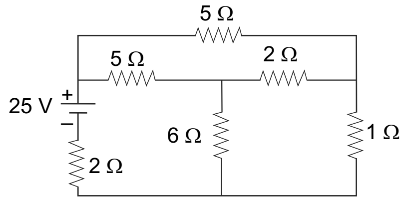


Fig. Q.3

☆☆★**2-3.4** Derive the relationship between fundamental tie-set matrix, impedance matrix, loop current matrix, and loop emf matrix. **(PTU 2009-10)**

☆☆★**2-3.5** For the network shown in Fig. Q.5, write down the tie-set matrix. **(PTU 2009-10)**

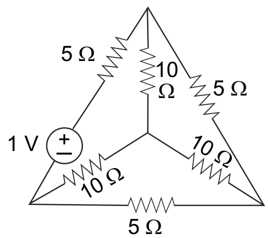


Fig. Q.5

☆☆★**2-3.6** What is loop matrix? **(PTU 2011-12)**

☆☆★**2-3.7** Write down the fundamental loop matrix of the network shown in Fig. Q.7. **(PTU 2011-12)**

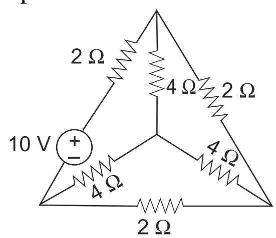


Fig. Q.7

☆★★**2-3.8** Draw the oriented graph of the network shown in Fig. Q.8. Select loop current variables and write the network- equilibrium equation in matrix form. **(PTU 2011-12)**

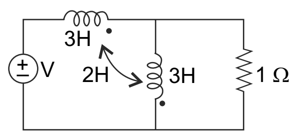


Fig. Q.8

☆☆★**2-3.9** For the network shown in Fig. Q.9 draw the graph and write down the tie-set matrix. **(RGTU Dec. 2013)**

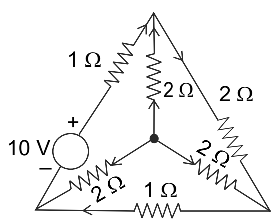


Fig. Q.9

2.8 CUT-SET AND TREE BRANCH VOLTAGES

LO 4 Describe cut-set and tree branch voltages

*A **cut-set** is a minimal set of branches of a connected graph such that the removal of these branches causes the graph to be cut into exactly two parts.* The important property of a cut-set is that by restoring any one of the branches of the cut-set, the graph should become connected. A cut-set consists of one and only one branch of the network tree, together with any links which must be cut to divide the network into two parts.

Consider the graph shown in Fig. 2.19 (a).

If the branches 3, 5, and 8 are removed from the graph, we see that the connected graph of Fig. 2.19 (a) is separated into two distinct parts, each of which is connected as shown in Fig. 2.19 (b). One of the parts is just an isolated node. Now suppose the removed branch 3 is replaced, all others still removed. Figure 2.19 (c) shows the resultant graph. The graph is now connected. Likewise, replacing the removed branches 5 and 8 of the set {3, 5, 8} one at a time, all other ones remaining removed, we obtain the resulting graphs as shown in Figs 2.19 (d) and (e). The set formed by the branches 3, 5, and 8 is called the cut-set of the connected graph of Fig. 2.19 (a).

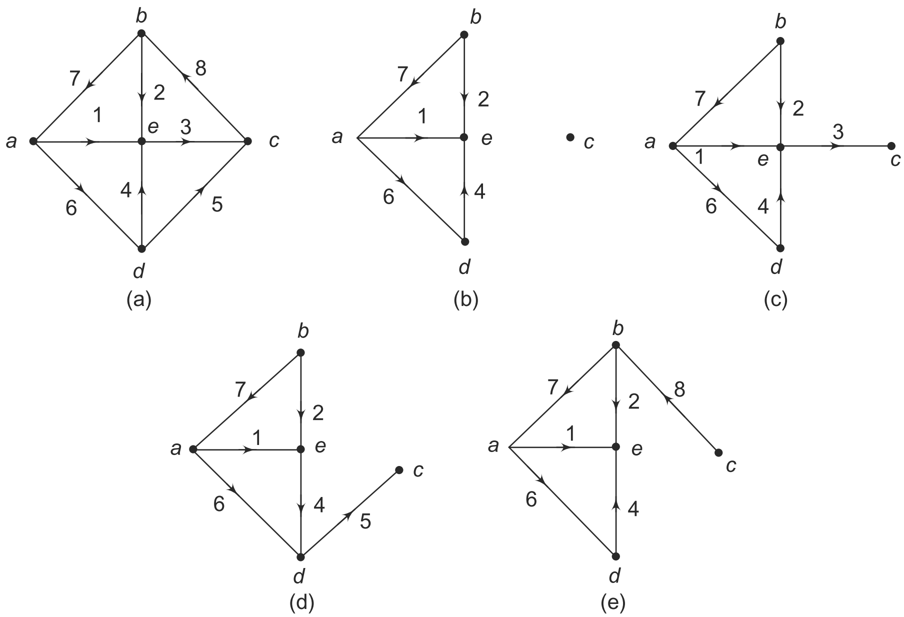


Fig. 2.19

2.8.1 Cut-Set Orientation

A cut-set is oriented by arbitrarily selecting the direction. A cut-set divides a graph into two parts. In the graph shown in Fig. 2.20, the cut-set is {2, 3}. It is represented by a dashed line passing through branches 2 and 3. This cut-set separates the graph into two parts shown as part-1 and part 2. We may take the orientation either from part-1 to part-2 or from part-2 to part-1.

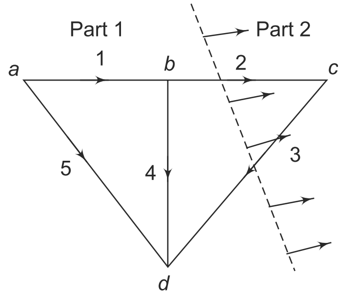


Fig. 2.20

The orientation of some branches of the cut-set may coincide with the orientation of the cut-set while some branches of the cut-set may not coincide. Suppose we choose the orientation of the cut-set {2, 3} from part-1 to part-2 as indicated in Fig. 2.20, then the orientation of the branch 2 coincides with the cut-set, whereas the orientation of the branch 3 is opposite.

2.8.2 Cut-Set Matrix and KCL for Cut-Sets

KCL is also applicable to a cut-set of a network. For any lumped electrical network, the algebraic sum of all the cut-set branch currents is equal to zero. While writing the KCL equation for a cut-set, we assign positive sign for the current in a branch if its direction coincides with the orientation of the cut-set and a negative sign to the current in a branch whose direction is opposite to the orientation of the cut-set. Consider the graph shown in Fig. 2.21. It has five branches and four nodes. The branches have been numbered 1 through 5 and their orientations are also marked. The following six cut-sets are possible as shown in Figs 2.22 (a)-(f).

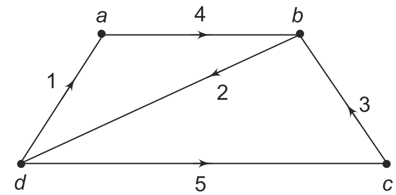


Fig. 2.21

cut-set C_1 : {1, 4}; cut-set C_2 : {4, 2, 3}
cut-set C_3 : {3, 5}; cut-set C_4 : {1, 2, 5}
cut-set C_5 : {4, 2, 5}; cut-set C_6 : {1, 2, 3}

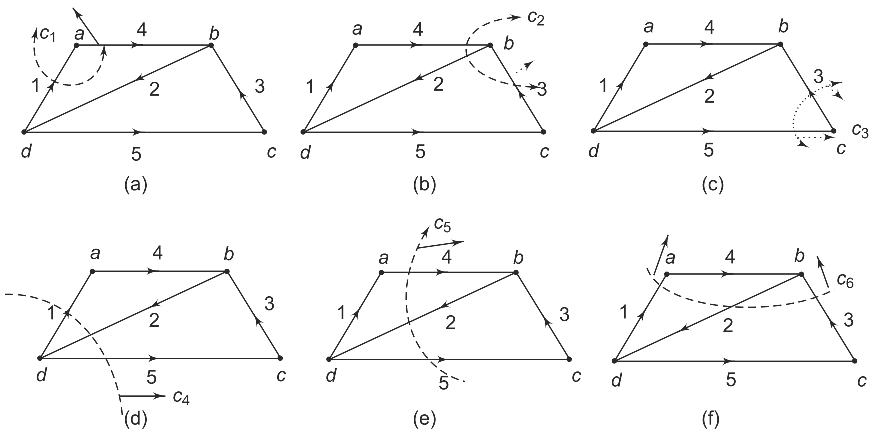


Fig. 2.22

Applying KCL for each of the cut-set, we obtain the following equations. Let i_1, i_2 … i_6 be the branch currents.

$$\left.\begin{aligned}
&C_1 : i_1 - i_4 = 0\\
&C_2 : -i_2 + i_3 + i_4 = 0\\
&C_3 : -i_3 + i_5 = 0\\
&C_4 : i_1 - i_2 + i_5 = 0\\
&C_5 : -i_2 + i_4 + i_5 = 0\\
&C_6 : i_1 - i_2 + i_3 = 0
\end{aligned}\right\} \tag{2.7}$$

These equations can be put into matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$QI_b = 0 \tag{2.8}$$

where the matrix Q is called the *augmented cut-set matrix* of the graph or *all cut-set matrix* of the graph. The matrix I_b is the branch-current vector.

The all cut-set matrix can be written as $Q = [q_{ij}]$.
where q_{ij} is the element in the ith row and jth column. The order of Q is number of cut-sets × number of branch as in the graph.

$$\left.\begin{aligned} q_{ij} &= 1, \text{ if branch } j \text{ in the cut-set } i \text{ and the orientation coincides with each other} \\ q_{ij} &= -1, \text{ if branch } j \text{ is in the cut-set } i \text{ and the orientation is opposite.} \\ q_{ij} &= 0, \text{ if branch } j \text{ is not present in cut-set } i. \end{aligned}\right\} \tag{2.9}$$

EXAMPLE 2.5

For the network-graph shown in Fig. 2.23 (a) with given orientation, obtain the all cut-set (augmented cut-set) matrix.

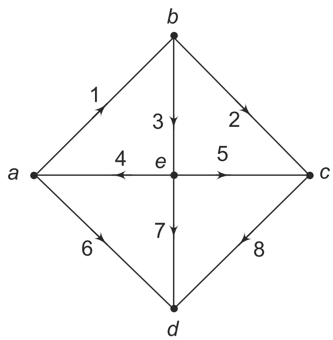


Fig. 2.23 (a)

Solution The graph has four nodes and eight branches. There are in all 12 possible cut-sets as shown with dashed lines in Figs 2.23 (b) and (c). The orientation of the cut-sets has been marked arbitrarily. The cut-sets are

C_1 : {1, 46}; C_2 {1, 2, 3}; C_3: {2, 5, 8}

C_4 : {6, 7, 8}; C_5 {1, 3, 5, 8}; C_6: {1, 4, 7, 8}

C_7 : {2, 5, 6, 7}; C_8 : {2, 3, 4, 6} C_9 : {1, 4, 7, 5, 2}

C_{10} : {2, 3, 4, 7, 8} ; C_{11} : {6, 4, 3, 5, 8}; C_{12} : {1, 3, 5, 7, 6}

Eight cut-sets C_1 to C_8 are shown if Fig. 2.23 (b) and four cut-sets C_9 to C_{11} are shown in Fig. 2.23 (c) for clarity.

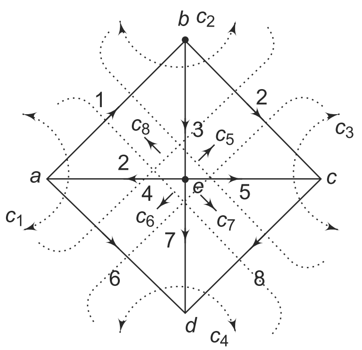


Fig. 2.23 (b)

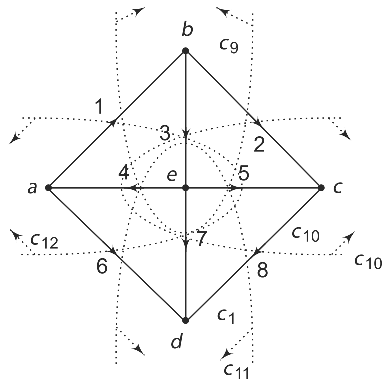


Fig. 2.23 (c)

As explained in Section 2.8.2 with the help of Eq. (2.9), the all cut-set matrix Q is given by

$$\begin{array}{rc}
\text{cut-sets} & \text{branches} \rightarrow \\
\downarrow & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
Q = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{array} &
\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 & -1 & -1 & -1 & 0
\end{bmatrix}
\end{array}$$

Matrix Q is a 12×8 matrix since there are 12 cut-sets and eight branches in the graph.

2.8.3 Fundamental Cut-Sets

Observe the set of Eq. (2.7) in Section 2.8.2 with respect to the graph in Fig. 2.22. Only the first three equations are linearly independent, remaining equations can be obtained as a linear combination of the first three. The concept of fundamental cut-set (f-cut-set) can be used to obtain a set of linearly independent equations in branch current variables. *The f-cut-sets are defined for a given tree of the graph. From a connected graph, first a tree is selected, and then a twig is selected. Removing this twig from the tree separates the tree into two parts. All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree, will constitute a cut-set. This cut-set is called a **fundamental cut-set** or **f-cut-set** of the graph.* Thus, a fundamental cut-set of a graph with respect to a tree is a cut-set that is formed by one twig and a unique set of links. For each branch of the tree, i.e. for each twig, there will be a f-cut-set. So, for a connected graph having n nodes, there will be $(n – 1)$ twigs in a tree, the number of f-cut-sets is also equal to $(n – 1)$.

The fundamental cut-set matrix Q_f is one in which each row represents a cut-set with respect to a given tree of the graph. The rows of Q_1 correspond to the fundamental cut-sets and the columns correspond to the branches of the graph. The procedure for obtaining a fundamental cut-set matrix is illustrated in Example 2.6.

EXAMPLE 2.6

Obtain the fundamental cut-set matrix Q_f for the network graph shown in Fig. 2.23 (a).

Solution A selected tree of the graph is shown in Fig. 2.24 (a).

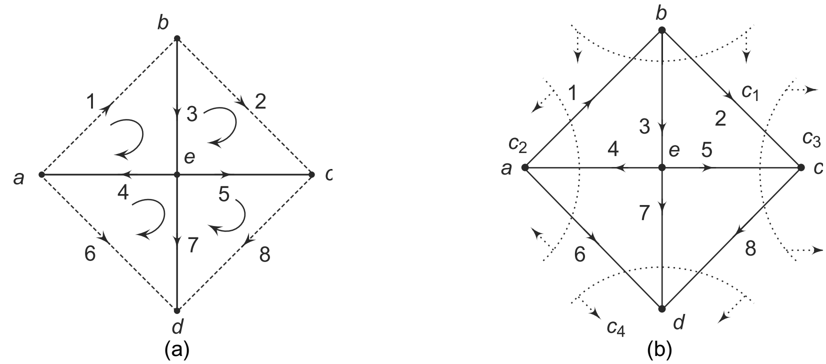


Fig. 2.24

The twigs of the tree are {3, 4, 5, 7}. The remaining branches 1, 2, 6, and 8 are the links, corresponding to the selected tree. Let us consider the twig 3. The minimum number of links that must be added to twig 3 to form a cut-set C_1 is {1, 2}. This set is unique for C_1. Thus, corresponding to the twig 3. The *f*-cut-set C_1 is {1, 2, 3}. This is shown in Fig. 2.24 (b). As a convention the orientation of a cut-set is chosen to coincide with that of its defining twig. Similarly, corresponding to the twig 4, the *f*-cut-set C_2 is {1, 4, 6} corresponding to the twig 5, the *f*-cut-set C_3 is {2, 5, 8} and corresponding to the twig 7, the *f*-cut-set is {6, 7, 8}. Thus, the *f*-cut-set matrix is given by

$$\begin{array}{cc} f\text{-cut-sets} & \text{branches} \rightarrow \\ \downarrow & \end{array}$$

$$Q_f = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{2.10}$$

2.8.4 Tree Branch Voltages and *f*-Cut-Set Matrix

From the cut-set matrix, the branch voltages can be expressed in terms of tree branch voltages. Since all tree branches are connected to all the nodes in the graph, it is possible to trace a path from one node to any other node by traversing through the tree-branches.

Let us consider Example 2.6, there are eight branches. Let the branch voltages be V_1, V_2, … V_8. There are four twigs, let the twig voltages be V_{t3}, V_{t4}, V_{t5} and V_{t7} for twigs 3, 4, 5 and 7 respectively.

We can express each branch voltage in terms of twig voltages as follows.

$$V_1 = -V_3 - V_4 = -V_{t3} - V_{t4}$$

$$V_2 = +V_3 + V_5 = +V_{t3} + V_{t5}$$
$$V_3 = V_{t3}$$
$$V_4 = V_{t4}$$
$$V_5 = V_{t5}$$
$$V_6 = V_7 - V_4 = V_{t7} - V_{t4}$$
$$V_7 = V_{t7}$$
$$V_8 = V_7 - V_5 = V_{t7} - V_{t5}$$

The above equations can be written in matrix form as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_{t_3} \\ V_{t_4} \\ V_{t_5} \\ V_{t_7} \end{bmatrix} \tag{2.11}$$

The first matrix on the right-hand side of Eq. (2.11) is the transpose of the *f*-cut-set matrix Q_f given in Eq. (2.10) in Ex. 2.6. Hence, Eq. (2.11) can be written as

$$V_b = Q_f^T V_t \tag{2.12}$$

where V_b is the column matrix of branch-voltages V_t is the column matrix of twig voltages corresponding to the selected tree and Q_f^T in the transpose of *f*-cut-set matrix.

Equation (2.12) shows that each branch voltage can be expressed as a linear combination of the tree-branch voltages. For this purpose, the fundamental cut-set (*f*-cut-set) matrix can be used without writing loop equations.

Frequently Asked Questions linked to LO4

☆☆★**2-4.1** For the network shown in Fig. Q.1 draw the oriented graph and all possible trees. Also prepare (a) incidence matrix, (b) Fundamental tie-set matrix, (c) Fundamental cut-set matrix. **(GTU Dec. 2012)**

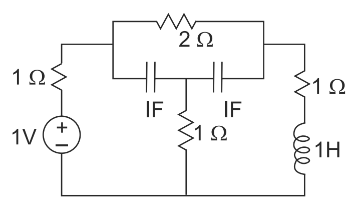


Fig. Q.1

☆☆★**2-4.2** Explain the fundamental cut-set matrix taking a suitable example. **(PTU 2009-10)**

☆☆★**2-4.3** Write relation between branch voltage matrix [V_O], twig voltage matrix [V_T] and node voltage matrix [V_N] in graph theory. **(PTU 2011-12)**

☆☆★**2-4.4** Define basic cut-set. **(PTU 2011-12)**

2.9 MESH ANALYSIS

LO 5 Analyse the network (resistive circuits) using mesh analysis and supermesh analysis and write the mesh equations using inspection method

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit

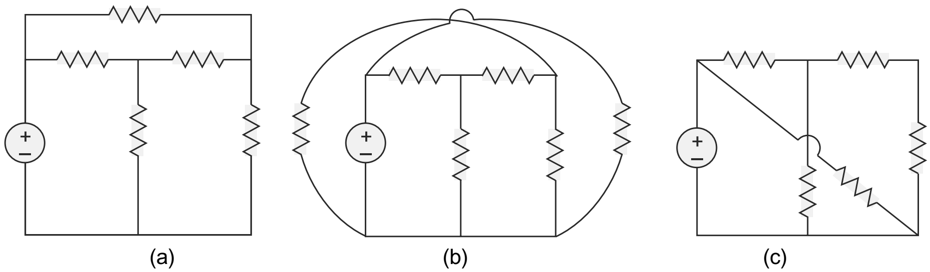


Fig. 2.25

be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

Mesh analysis is applicable only for planar networks. For non-planar circuits, mesh analysis is not applicable. A circuit is said to be planar if it can be drawn on a plane surface without crossovers. A non-planar circuit cannot be drawn on a plane surface without a crossover.

Figure 2.25 (a) is a planar circuit. Figure 2.25 (b) is a non-planar circuit and Fig. 2.25 (c) is a planar circuit which looks like a non-planar circuit. It has already been discussed that a loop is a closed path. A mesh is defined as a loop which does not contain any other loops within it. To apply mesh analysis, our first step is to check whether the circuit is planar or not and the second is to select mesh currents. Finally, writing Kirchhoff's voltage law equations in terms of unknowns and solving them leads to the final solution.

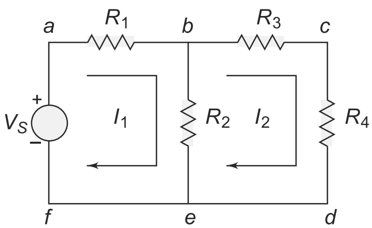


Fig. 2.26

Observation of Fig. 2.26 indicates that there are two loops *abefa*, and *bcdeb* in the network. Let us assume loop currents I_1 and I_2 with directions as indicated in the figure. Considering the loop *abefa* alone, we observe that current I_1 is passing through R_1, and $(I_1 - I_2)$ is passing through R_2. By applying Kirchhoff's voltage law, we can write

$$V_s = I_1R_1 + R_2(I_1 - I_2)$$

Similarly, if we consider the second mesh *bcdeb*, the current I_2 is passing through R_3 and R_4, and $(I_2 - I_1)$ is passing through R_2. By applying Kirchhoff's voltage law around the second mesh, we have

$$R_2(I_2 - I_1) + R_3 I_2 + R_4 I_2 = 0$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(R_1 + R_2) - I_2 R_2 = V_s$$
$$-I_1 R_2 + (R_2 + R_3 + R_4)I_2 = 0$$

By solving the above equations, we can find the currents I_1 and I_2. If we observe Fig. 2.26, the circuit consists of five branches and four nodes, including the reference node. The number of mesh currents is equal to the number of mesh equations.

And the number of equations = branches – (nodes – 1). In Fig. 2.26, the required number of mesh currents would be 5 – (4 – 1) = 2.

In general, if we have B branches and N nodes including the reference node then the number of linearly independent mesh equations $M = B - (N - 1)$.

EXAMPLE 2.7

Write the mesh current equations in the circuit shown in Fig. 2.27, and determine the currents.

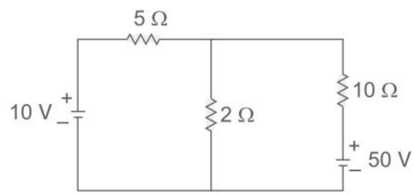


Fig. 2.27

Solution Assume two mesh currents in the direction as indicated in Fig. 2.28.

The mesh current equations are

$$5I_1 + 2(I_1 - I_2) = 10$$

$$10I_2 + 2(I_2 - I_1) + 50 = 0$$

We can rearrange the above equations as

$$7I_1 - 2I_2 = 10$$

$$-2I_1 + 12I_2 = -50$$

By solving the above equations, we have

$I_1 = 0.25\ A$, and $I_2 = -4.125\ A$

Here, the current in the second mesh, I_2, is negative; that is the actual current I_2 flows opposite to the assumed direction of current in the circuit of Fig. 2.28.

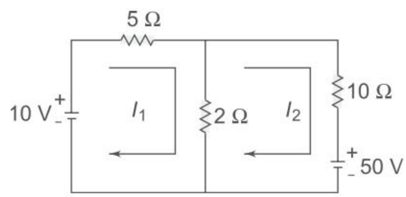


Fig. 2.28

EXAMPLE 2.8

Determine the mesh current I_1 in the circuit shown in Fig. 2.29.

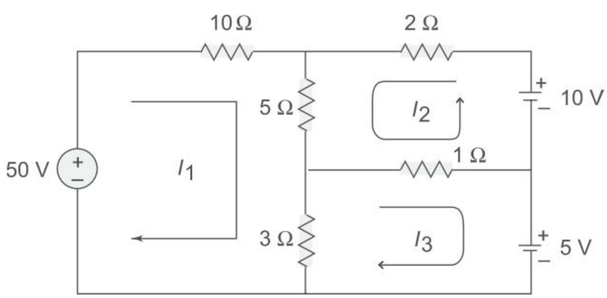


Fig. 2.29

Solution From the circuit, we can form the following three mesh equations:

$$10I_1 + 5(I_1 + I_2) + 3(I_1 - I_3) = 50$$

$$2I_2 + 5(I_2 + I_1) + 1(I_2 + I_3) = 10$$

$$3(I_3 - I_1) + 1(I_3 + I_2) = -5$$

Rearranging the above equations, we get

$$18I_1 + 5I_2 - 3I_3 = 50$$

$$5I_1 + 8I_2 + I_3 = 10$$

$$-3I_1 + I_2 + 4I_3 = -5$$

According to Cramer's rule,

$$I_1 = \left|\frac{\begin{matrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \end{matrix}}{\begin{matrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{matrix}}\right| = \frac{1175}{356}$$

or $I_1 = 3.3$ A

Similarly,

$$I_2 = \left|\frac{\begin{matrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \end{matrix}}{\begin{matrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{matrix}}\right| = \frac{-355}{356}$$

or $I_2 = -0.997$ A

$$I_3 = \left|\frac{\begin{matrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \end{matrix}}{\begin{matrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{matrix}}\right| = \frac{525}{356}$$

or $I_3 = 1.47$ A

$\therefore \quad I_1 = 3.3$ A, $I_2 = -0.997$ A, $I_3 = 1.47$ A

2.10 MESH EQUATIONS BY INSPECTION METHOD

LO 5

The mesh equations for a general planar network can be written by inspection without going through the detailed steps. Consider three mesh networks as shown in Fig. 2.30.

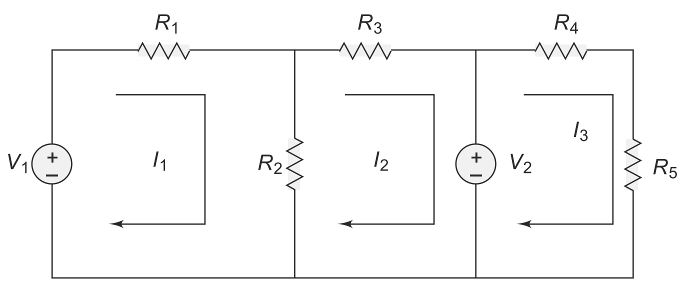


Fig. 2.30

The loop equations are

$$I_1R_1 + R_2(I_1 - I_2) = V_1 \tag{2.13}$$

$$R_2(I_2 - I_1) + I_2R_3 = -V_2 \tag{2.14}$$

$$R_4I_3 + R_5I_3 = V_2 \tag{2.15}$$

Reordering the above equations, we have

$$(R_1 + R_2)I_1 - R_2I_2 = V_1 \tag{2.16}$$

$$-R_2I_1 + (R_2 + R_3)I_2 = -V_2 \tag{2.17}$$

$$(R_4 + R_5)I_3 = V_2 \tag{2.18}$$

The general mesh equations for the three-mesh resistive network can be written as

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{2.19}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{2.20}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{2.21}$$

By comparing equations (2.16), (2.17) and (2.18) with Eqs (2.19), (2.20), and (2.21) respectively, the following observations can be taken into account.

1. The self-resistance in each mesh,
2. The mutual resistances between all pairs of meshes, and
3. The algebraic sum of the voltages in each mesh.

The self-resistance of the loop 1, $R_{11} = R_1 + R_2$, is the sum of the resistances through which I_1 passes.

The mutual resistance of the loop 1, $R_{12} = -R_2$, is the sum of the resistances common to loop currents I_1 and I_2. If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign; and if the directions of the currents passing through the common resistance are opposite then the mutual resistance will have a negative sign.

$V_a = V_1$ is the voltage which drives loop one. Here, the positive sign is used if the direction of the current is the same as the direction of the source. If the current direction is opposite to the direction of the source, then the negative sign is used.

Similarly, $R_{22} = (R_2 + R_3)$ and $R_{33} = R_4 + R_5$ are the self resistances of loops two and three, respectively. The mutual resistances $R_{13} = 0$, $R_{21} = -R_2$, $R_{23} = 0$, $R_{31} = 0$, $R_{32} = 0$ are the sums of the resistances common to the mesh currents indicated in their subscripts.

$V_b = -V_2$, $V_c = V_2$ are the sum of the voltages driving their respective loops.

EXAMPLE 2.9

Write the mesh equations for the circuit shown in Fig. 2.31.

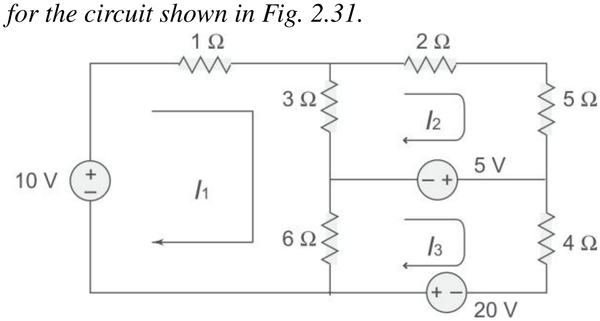


Fig. 2.31

Solution The general equations for the three-mesh network are

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \quad (2.22)$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \quad (2.23)$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \quad (2.24)$$

Consider Eq. (2.22).

R_{11} = self-resistance of the loop 1 = (1 Ω + 3 Ω + 6 Ω) = 10 Ω

R_{12} = the mutual resistance common to loops 1 and 2 = –3 Ω

Here, the negative sign indicates that the currents are in opposite direction

R_{13} = the mutual resistance common to loop 1 and 3 = –6 Ω

V_a = +10 V, the voltage driving the loop 1.

Here, the positive sign indicates the loop current I_1 is in the same direction as the source element.

Therefore, Eq. (2.22) can be written as

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V} \quad (2.25)$$

Consider Eq. (2.23).

R_{21} = mutual resistance common to loops 1 and 2 = –3 Ω

R_{22} = self-resistance of loop 2 = (3 Ω + 2 Ω + 5 Ω) = 10 Ω

R_{23} = 0, there is no common resistance between loops 2 and 3.

V_b = –5 V, the voltage driving the loop 2.

Therefore, Eq. (2.23) can be written as

$$-3I_1 + 10I_2 = -5 \text{ V} \quad (2.26)$$

Consider Eq. (2.24).

R_{31} = mutual resistance common to loops 3 and 1 = –6 Ω

R_{32} = mutual resistance common to loops 3 and 2 = 0

R_{33} = self-resistance of the loop 3 = (6 Ω + 4 Ω) = 10 Ω

V_c = algebraic sum of the voltages driving the loop 3

= (5 V + 20 V) = 25 V

Therefore, Eq. (2.24) can be written as

$$-6I_1 + 10I_3 = 25 \text{ V} \quad (2.27)$$

The three mesh equations are

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V}$$

$$-3I_1 + 10I_2 = -5 \text{ V}$$

$$-6I_1 + 10I_3 = 25 \text{ V}$$

2.11 SUPERMESH ANALYSIS

LO 5

Suppose any of the branches in the network has a current source; then it is slightly difficult to apply mesh analysis straightforward because first we should assume an unknown voltage across the current source, writing mesh equations as before, and then relate the source current to the assigned mesh currents. This is generally a difficult approach. One way to overcome this difficulty is by applying the supermesh technique. Here we have to choose the kind of supermesh. A supermesh is constituted by two adjacent loops that have a common current source. As an example, consider the network shown in Fig. 2.32.

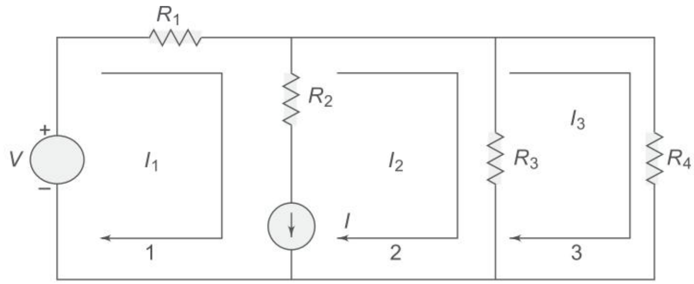


Fig. 2.32

Here, the current source I is in the common boundary for the two meshes 1 and 2. This current source creates a supermesh, which is nothing but a combination of meshes 1 and 2.

$$R_1I_1 + R_3(I_2 - I_3) = \text{V}$$

$$\text{or} \quad R_1I_1 + R_3I_2 - R_4I_3 = \text{V}$$

Considering the mesh 3, we have

$$R_3(I_3 - I_2) + R_4I_3 = 0$$

Finally, the current I from the current source is equal to the difference between the two mesh currents, i.e.

$$I_1 - I_2 = I$$

We have, thus, formed three mesh equations which we can solve for the three unknown currents in the network.

EXAMPLE 2.10

Determine the current in the 5 Ω resistor in the network given in Fig. 2.33.

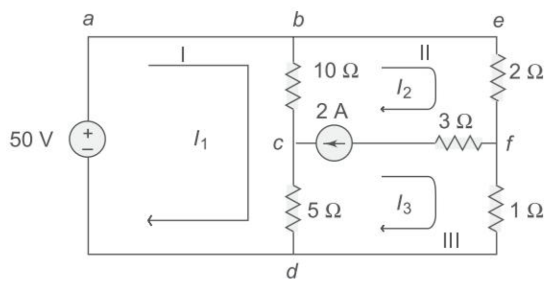


Fig. 2.33

Solution From the first mesh, i.e. *abcda*, we have

$$50 = 10(I_1 - I_2) + 5(I_1 - I_3)$$

$$\text{or} \quad 15I_1 - 10I_2 - 5I_3 = 50 \tag{2.28}$$

From the second and third meshes, we can form a supermesh

$$10(I_2 - I_1) + 2I_2 + I_3 + 5(I_3 - I_1) = 0$$

$$\text{or} \quad -15I_1 + 12I_2 + 6I_3 = 0 \tag{2.29}$$

The current source is equal to the difference between II and III mesh currents,

i.e. $I_2 - I_3 = 2$ A (2.30)

Solving 2.28, 2.29, and 2.30, we have

$$I_1 = 19.99 \text{ A}, I_2 = 17.33 \text{ A, and } I_3 = 15.33 \text{ A}$$

The current in the 5 Ω resistor $= I_1 - I_3$

$$= 19.99 - 15.33 = 4.66 \text{ A}$$

∴ the current in the 5 Ω resistor is 4.66 A.

EXAMPLE 2.11

Write the mesh equations for the circuit shown in Fig. 2.34 and determine the currents, I_1, I_2 and I_3.

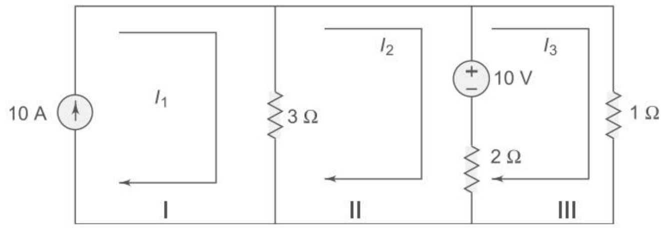


Fig. 2.34

Solution In Fig. 2.34, the current source lies on the perimeter of the circuit, and the first mesh is ignored. Kirchhoff's voltage law is applied only for second and third meshes.

From the second mesh, we have

$$3(I_2 - I_1) + 2(I_2 - I_3) + 10 = 0$$

$$\text{or} \quad -3I_1 + 5I_2 - 2I_3 = -10 \tag{2.31}$$

From the third mesh, we have

$$I_3 + 2(I_3 - I_2) = 10$$

$$\text{or} \quad -2I_2 + 3I_3 = 10 \tag{2.32}$$

From the first mesh,

$$I_1 = 10 \text{ A} \tag{2.33}$$

From the above three equations, we get

$$I_1 = 10 \text{ A}, \quad I_2 = 7.27 \text{ A}, \quad I_3 = 8.18 \text{ A}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO5

☆★★**2-5.1** In the circuit shown in Fig. Q.1, use mesh analysis to find out the power delivered to the 4 Ω resistor. To what voltage should the 100 V battery be changed so that no power is delivered to the 4 Ω resistor?

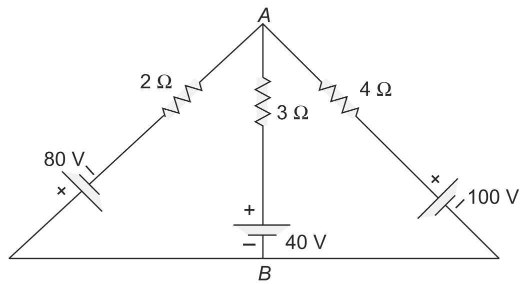


Fig. Q.1

☆★★**2-5.2** Find the voltage between A and B of the circuit shown in Fig. Q.2 by mesh analysis.

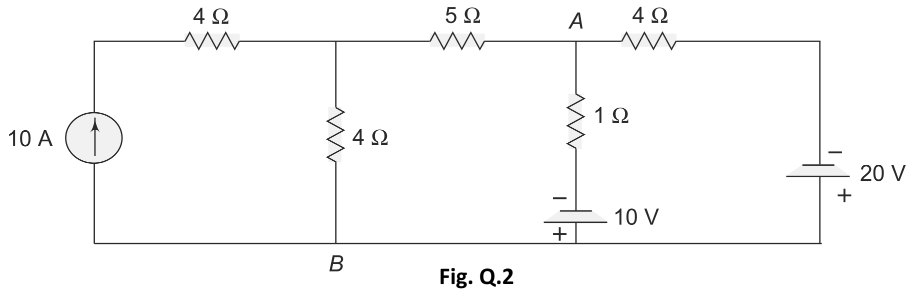


Fig. Q.2

☆★★**2-5.3** Find the value of R_1 and R_2 in the network shown in Fig. Q.3 using mesh analysis.

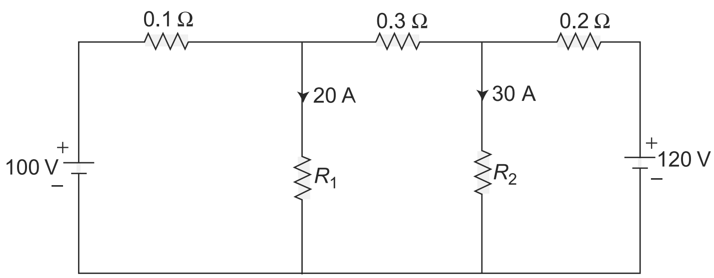


Fig. Q.3

☆★★**2-5.4** Using mesh analysis, determine the voltage across the 10 kΩ resistor at terminals A and B of the circuit shown in Fig. Q.4.

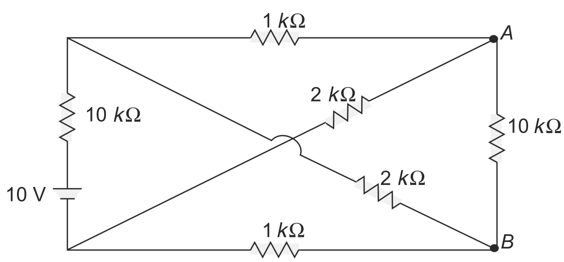


Fig. Q.4

★★★ **2-5.5** In the network shown in Q.5, the resistance R is variable from zero to infinity. The current I through R can be expressed as $I = a + bV$, where V is the voltage across R as shown in the figure, and a and b are constants. Determine a and b.

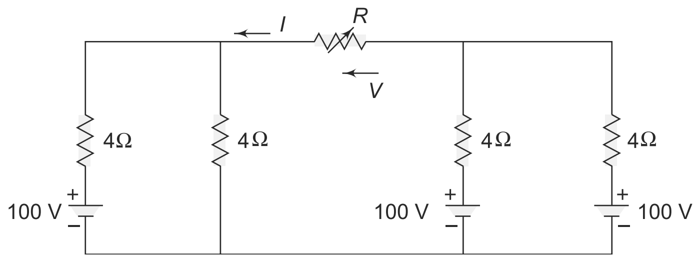


Fig. Q.5

★★★**2-5.6** For the circuit shown in Fig. Q.6, use mesh analysis to find the values of all mesh currents.

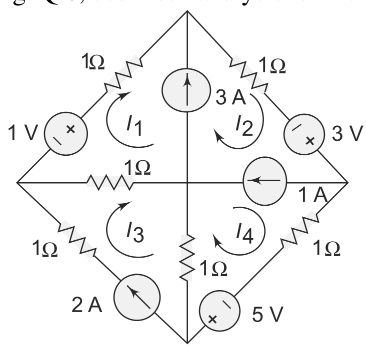


Fig. Q.6

☆★★**2.5.7** Determine the value of current I in the following circuit shown in Fig. Q.7 by using mesh analysis.

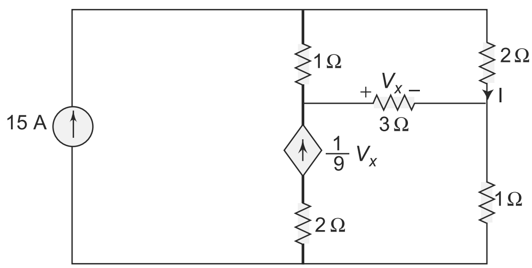


Fig. Q.7

☆★★**2-5.8** Use mesh analysis to find the current supplied by the 31 V source and the current in 4 Ω resistor of the circuit shown in Fig. Q.8.

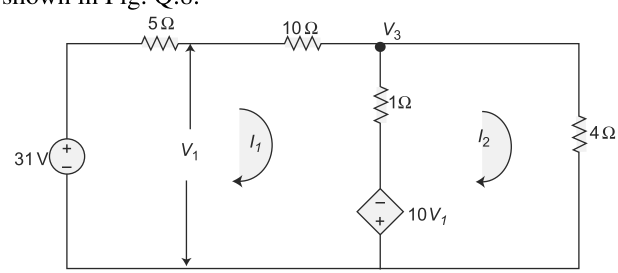


Fig. Q.8

★★★**2-5.9** For the circuit shown in Fig. Q.9, find the value of V_2 that will cause the voltage across 20 Ω to be zero by using mesh analysis.

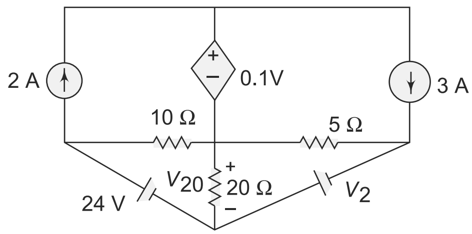


Fig. Q.Q.9

☆★★**2-5.10** Find the mesh currents I_1, I_2, I_3, and I_4 in the following circuit Fig. Q.10, using PSpice

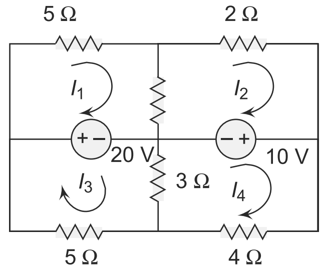


Fig. Q.10

Frequently Asked Questions linked to LO5

☆☆★**2-5.1** Distinguish between mesh and loop of an electric circuit. **(AU May/June 2013)**

☆★★**2-5.2** Using mesh analysis, determine the current through the 1 Ω resistor in the circuit shown in Fig. Q.2 **(AU May/June 2013)**

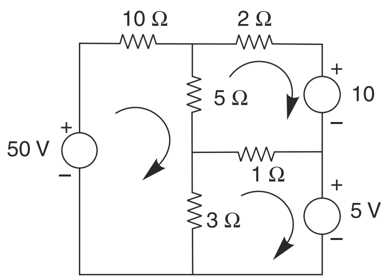


Fig. Q.2

☆★★**2-5.3** Find out the current in each branch of the circuit shown in Fig. Q.3.

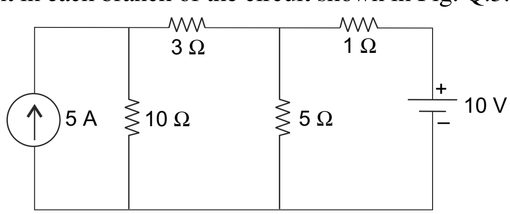


Fig. Q.3

☆★★**2.5.4** Determine current in each mesh of the circuit shown in Fig. Q.4.

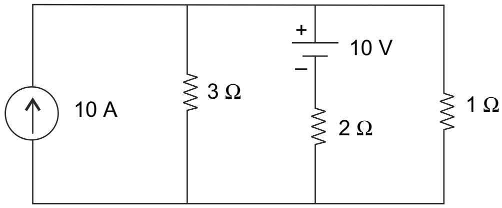


Fig. Q.4

☆☆★**2-5.5** Define mesh analysis of a circuit. **(AU Nov./Dec. 2012)**

☆★★**2-5.6** Determine the current I_2 in the circuit shown in Fig. Q.6 **(AU Nov./Dec. 2012)**

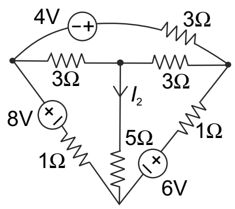


Fig. Q.6

★★★**2-5.7** Using mesh method obtain the loop currents in the network of Fig. Q.7. What is the total power loss? **(BPUT 2007)**

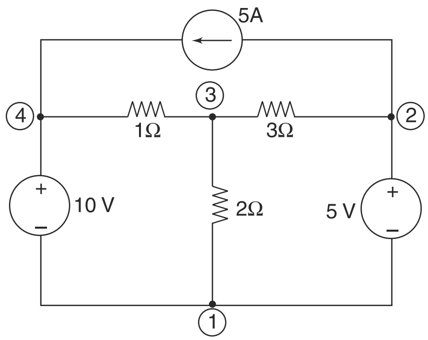


Fig. Q.7

☆★★**2-5.8** Derive a tree of the graph of the network in Fig. Q.8. Determine the node voltage V_1 and V_2 using the mesh analysis. Resistance values are in ohms. **(GTU May 2011)**

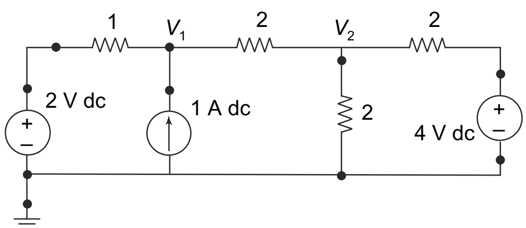


Fig. Q.8

☆☆★**2-5.9** Find mesh current and determine the voltage across each element in the circuit shown in Fig. Q.9. **(JNTU Nov. 2012)**

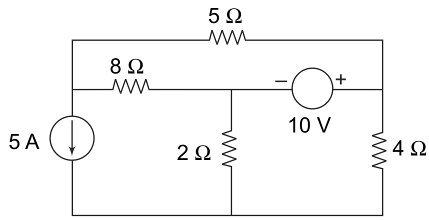


Fig. Q.9

☆★★**2-5.10** Find the power delivered by the two sources to the circuit shown in Fig. Q.10. **(PTU 2009-10)**

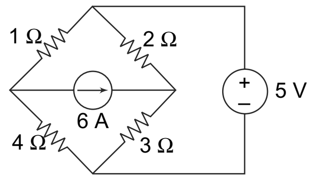


Fig. Q.10

☆★★**2-5.11** Determine the current supplied by each battery in the circuit of Fig. Q.11 using mesh analysis method. **(PTU 2011-12)**

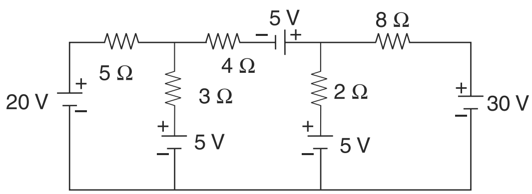


Fig. Q.11

★★★**2-5.12** Find the value of *K* in the circuit shown in Fig. Q.12 such that the power dissipated in the 2 Ω resistor does not exceed 50 watts. **(PU 2012)**

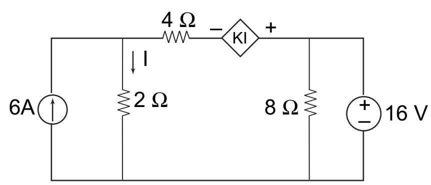


Fig. Q.12

☆★★**2-5.13** Determine the current in the 5 Ω resistor in the network shown in Fig. Q.13.

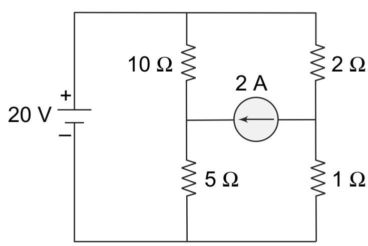


Fig. Q.13

☆★★**2-5.14** In the circuit shown in Fig. Q.14 use loop analysis to find the power delivered to the 4 Ω **resistor**.

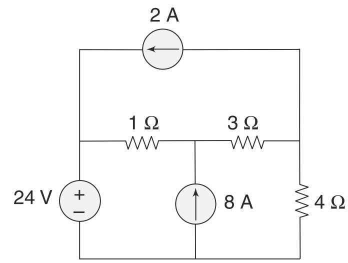


Fig. Q.14

2.12 NODAL ANALYSIS

LO 6 Analyse the network (resistive circuits) using nodal analysis and supernode analysis and write the nodal equations using inspection method

In Chapter 1, we discussed simple circuits containing only two nodes, including the reference node. In general, in an N-node circuit, one of the nodes is choosen as reference or datum node, then it is possible to write $N-1$ nodal equations by assuming $N-1$ node voltages. For example, a 10-node circuit requires nine unknown voltages and nine equations. Each node in a circuit can be assigned a number or a letter. *The node voltage is the voltage of a given node with respect to one particular node, called the* ***reference node****, which we assume at zero potential.* In the circuit shown in Fig. 2.35, the node 3 is assumed as the reference node. The voltage at the node 1 is the voltage at that node with respect to the node 3. Similarly, the voltage at the node 2 is the voltage at that node with respect to the node 3. Applying Kirchhoff's current law at the node 1; the current entering is equal to the current leaving (see Fig. 2.36).

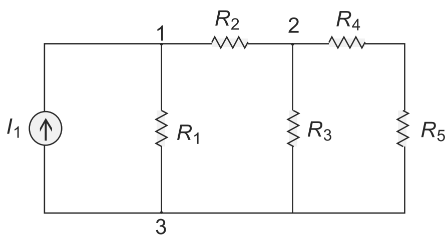


Fig. 2.35

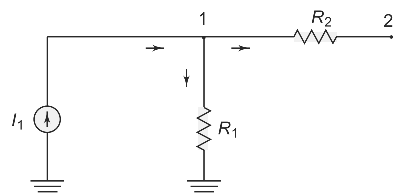


Fig. 2.36

$$I_1 = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

Here, V_1 and V_2 are the voltages at nodes 1 and 2, respectively. Similarly, at the node 2, the current entering is equal to the current leaving as shown in Fig. 2.37.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_2}{R_4 + R_5} = 0$$

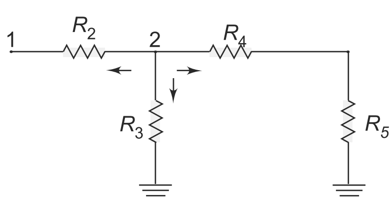


Fig. 2.37

Rearranging the above equations, we have

$$V_1\left[\frac{1}{R_1} + \frac{1}{R_2}\right] - V_2\left[\frac{1}{R_2}\right] = I_1$$

$$-V_1\left[\frac{1}{R_2}\right] + V_2\left[\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4 + R_5}\right] = 0$$

From the above equations, we can find the voltages at each node.

EXAMPLE 2.12

Write the node voltage equations and determine the currents in each branch for the network shown in Fig. 2.38.

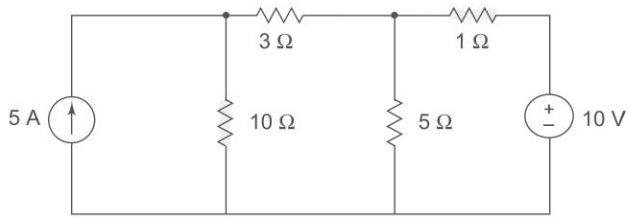


Fig. 2.38

Solution The first step is to assign voltages at each node as shown in Fig. 2.39.

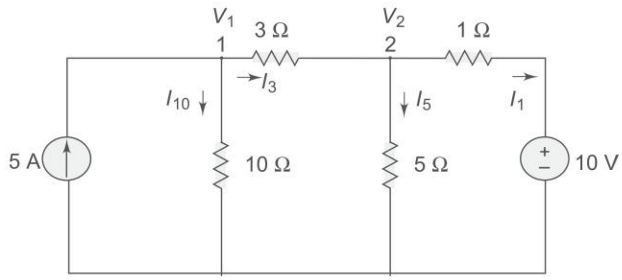


Fig. 2.39

Applying Kirchhoff's current law at the node 1, we have

$$5=\frac{V_1}{10}+\frac{V_1-V_2}{3}$$

$$\text{or}\quad V_1\left[\frac{1}{10}+\frac{1}{3}\right]-V_2\left[\frac{1}{3}\right]=5 \tag{2.34}$$

Applying Kirchhoff's current law at the node 2, we have

$$\frac{V_2-V_1}{3}+\frac{V_2}{5}+\frac{V_2-10}{1}=0$$

$$\text{or}\quad -V_1\left[\frac{1}{3}\right]+V_2\left[\frac{1}{3}+\frac{1}{5}+1\right]=10 \tag{2.35}$$

From Eqs (2.34) and (2.35), we can solve for V_1 and V_2 to get

$$V_1 = 19.85\text{ V},\ V_2 = 10.9\text{ V}$$

$$I_{10}=\frac{V_1}{10}=1.985\text{ A},\ I_3=\frac{V_1-V_2}{3}=\frac{19.85-10.9}{3}=2.98\text{ A}$$

$$I_5=\frac{V_2}{5}=\frac{10.9}{5}=2.18\text{ A},\ I_1=\frac{V_2-10}{1}=0.9\text{ A}$$

EXAMPLE 2.13

Determine the voltages at each node for the circuit shown in Fig. 2.40.

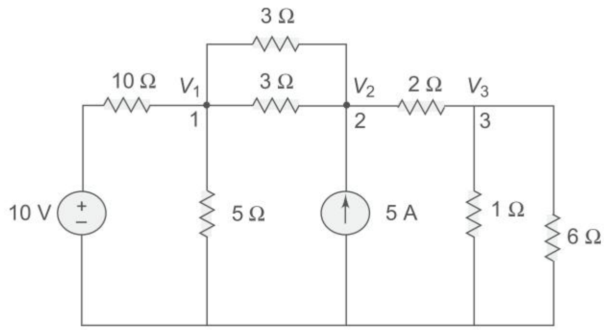


Fig. 2.40

Solution At the node 1, assuming that all currents are leaving, we have

$$\frac{V_1-10}{10}+\frac{V_1-V_2}{3}+\frac{V_1}{5}+\frac{V_1-V_2}{3}=0$$

or $$V_1\left[\frac{1}{10}+\frac{1}{3}+\frac{1}{5}+\frac{1}{3}\right]-V_2\left[\frac{1}{3}+\frac{1}{3}\right]=1$$

$$0.96V_1 - 0.66V_2 = 1 \tag{2.36}$$

At the node 2, assuming that all currents are leaving except the current from the current source, we have

$$\frac{V_2-V_1}{3}+\frac{V_2-V_1}{3}+\frac{V_2-V_3}{2}=5$$

$$-V_1\left[\frac{2}{3}\right]+V_2\left[\frac{1}{3}+\frac{1}{3}+\frac{1}{2}\right]-V_3\left[\frac{1}{2}\right]=5$$

$$-0.66\,V_1 + 1.16\,V_2 - 0.5V_3 = 5 \tag{2.37}$$

At node 3, assuming all currents are leaving, we have

$$\frac{V_3-V_2}{2}+\frac{V_3}{1}+\frac{V_3}{6}=0$$

$$-0.5\,V_2 + 1.66\,V_3 = 0 \tag{2.38}$$

Applying Cramer's rule, we get

$$V_1=\left|\frac{\begin{matrix}1 & -0.66 & 0\\ 5 & 1.16 & -0.5\\ 0 & -0.5 & 1.66\end{matrix}}{\begin{matrix}0.96 & -0.66 & 0\\ -0.66 & 1.16 & -0.5\\ 0 & -0.5 & 1.66\end{matrix}}\right|=\frac{7.154}{0.887}=8.06\,\text{V}$$

Similarly,

$$V_2 = \left| \frac{\begin{matrix} 0.96 & 1 & 0 \\ -0.66 & 5 & -0.5 \\ 0 & 0 & 1.66 \end{matrix}}{\begin{matrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{matrix}} \right| = \frac{9.06}{0.887} = 10.2 \text{ V}$$

$$V_3 = \left| \frac{\begin{matrix} 0.96 & -0.66 & 1 \\ -0.66 & 1.16 & 5 \\ 0 & -0.5 & 0 \end{matrix}}{\begin{matrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{matrix}} \right| = \frac{2.73}{0.887} = 3.07 \text{ V}$$

2.13 NODAL EQUATIONS BY INSPECTION METHOD

LO 6

The nodal equations for a general planar network can also be written by inspection, without going through the detailed steps. Consider a three-node resistive network, including the reference node, as shown in Fig. 2.41.

In Fig. 2.41, the points *a* and *b* are the actual nodes and *c* is the reference node.

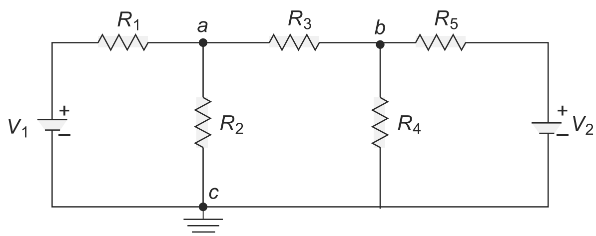


Fig. 2.41

Now consider the nodes *a* and *b* separately as shown in Fig. 2.42 (a) and (b).

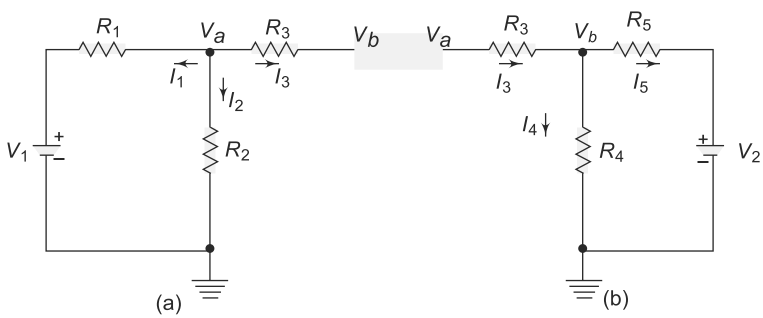


Fig. 2.42

In Fig. 2.42 (a), according to Kirchhoff's current law, we have

$$I_1 + I_2 + I_3 = 0$$

$$\therefore \quad \frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_b}{R_3} = 0 \tag{2.39}$$

In Fig. 2.42 (b), if we apply Kirchhoff's current law, we get

$$I_4 + I_5 = I_3$$

$$\therefore \quad \frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} + \frac{V_b - V_2}{R_5} = 0 \tag{2.40}$$

Rearranging the above equations, we get

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_a - \left(\frac{1}{R_3}\right) V_b = \left(\frac{1}{R_1}\right) V_1 \tag{2.41}$$

$$\left(-\frac{1}{R_3}\right) V_a + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right) V_b = \frac{V_2}{R_5} \tag{2.42}$$

In general, the above equations can be written as

$$G_{aa} V_a + G_{ab} V_b = I_1 \tag{2.43}$$

$$G_{ba} V_a + G_{bb} V_b = I_2 \tag{2.44}$$

By comparing Eqs (2.41), (2.42), and Eqs (2.43), (2.44) we have the self-conductance at node a, $G_{aa} = (1/R_1 + 1/R_2 + 1/R_3)$ is the sum of the conductances connected to node a. Similarly, $G_{bb} = (1/R_3 + 1/R_4 + 1/R_5)$, is the sum of the conductances connected to node b. $G_{ab} = (-1/R_3)$, is the sum of the mutual conductances connected to node a and node b. Here all the mutual conductances have negative signs. Similarly, $G_{ba} = (-1/R_3)$ is also a mutual conductance connected between nodes b and a. I_1 and I_2 are the sum of the source currents at the node a and the node b, respectively. The current which drives into the node has positive sign, while the current that drives away from the node has negative sign.

EXAMPLE 2.14

For the circuit shown in Fig. 2.43, write the node equations by the inspection method.

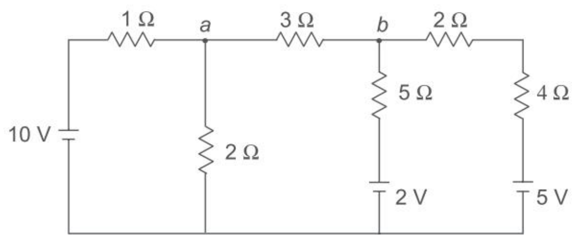


Fig. 2.43

Solution The general equations are

$$G_{aa} V_a + G_{ab} V_b = I_1 \tag{2.45}$$

$$G_{ba} V_a + G_{bb} V_b = I_2 \tag{2.46}$$

Consider Eq. (2.45).

G_{aa} = (1 + 1/2 + 1/3) mho, the self-conductance at the node *a* is the sum of the conductances connected to the node *a*.

G_{bb} = (1/6 + 1/5 + 1/3) mho the self-conductance at the node *b* is the sum of the conductances connected to the node *b*.

G_{ab} = –(1/3) mho, the mutual conductance between nodes *a* and *b* is the sum of the conductances connected between nodes *a* and *b*.

Similarly, G_{ba} = –(1/3), the sum of the mutual conductances between nodes *b* and *a*.

$$I_1 = \frac{10}{1} = 10\text{ A, the source current at the node } a,$$

$$I_2 = \left(\frac{2}{5} + \frac{5}{6}\right) = 1.23\text{ A, the source current at the node } b.$$

Therefore, the nodal equations are

$$1.83\, V_a - 0.33\, V_b = 10 \tag{2.47}$$

$$-0.33\, V_a + 0.7\, V_b = 1.23 \tag{2.48}$$

2.14 SUPERNODE ANALYSIS

LO 6

Suppose any of the branches in the network has a voltage source; then it is slightly difficult to apply nodal analysis. One way to overcome this difficulty is to apply the supernode technique. *In this method, the two adjacent nodes that are connected by a voltage source are reduced to a single node and then the equations are formed by applying Kirchhoff's current law as usual.* This is explained with the help of Fig. 2.44.

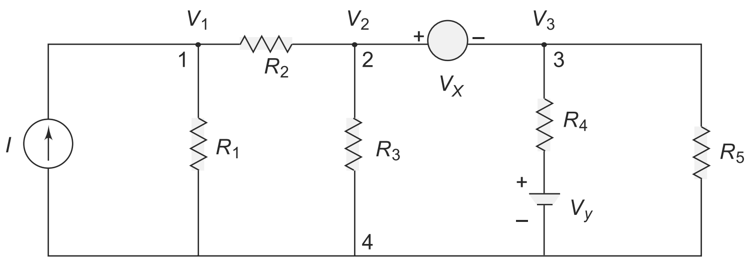


Fig. 2.44

It is clear from Fig. 2.44, that the node 4 is the reference node. Applying Kirchhoff's current law at the node 1, we get

$$I = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

Due to the presence of voltage source V_x in between nodes 2 and 3, it is slightly difficult to find out the current. The supernode technique can be conveniently applied in this case.

Accordingly, we can write the combined equation for nodes 2 and 3 as under.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_3 - V_y}{R_4} + \frac{V_3}{R_5} = 0$$

The other equation is

$$V_2 - V_3 = V_x$$

From the above three equations, we can find the three unknown voltages.

EXAMPLE 2.15

Determine the current in the 5 Ω resistor for the circuit shown in Fig. 2.45.

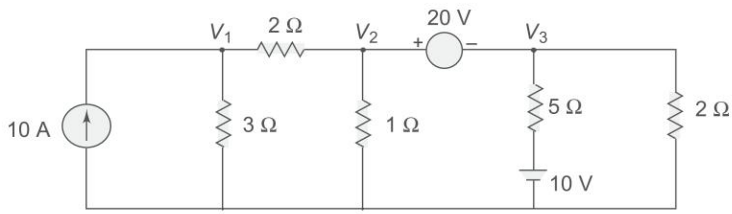


Fig. 2.45

Solution At the node 1,

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

or $$V_1\left[\frac{1}{3} + \frac{1}{2}\right] - \frac{V_2}{2} - 10 = 0$$

$$0.83\, V_1 - 0.5\, V_2 - 10 = 0 \qquad (2.49)$$

At nodes 2 and 3, the supernode equation is

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

or $$\frac{-V_1}{2} + V_2\left[\frac{1}{2} + 1\right] + V_3\left[\frac{1}{5} + \frac{1}{2}\right] = 2$$

$$-0.5\, V_1 + 1.5\, V_2 + 0.7\, V_3 - 2 = 0 \qquad (2.50)$$

The voltage between nodes 2 and 3 is given by

$$V_2 - V_3 = 20 \qquad (2.51)$$

The current in the 5 Ω resistor $I_5 = \dfrac{V_3 - 10}{5}$

Solving Eqs (2.49), (2.50), and (2.51), we obtain

$$V_3 = -8.42\ \text{V}$$

$\therefore$ Current $I_5 = \dfrac{-8.42 - 10}{5} = -3.68$ A (current towards node 3) i.e. the current flows towards the node 3.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆★★**2-6.1** In the circuit shown in Fig. Q.1, use nodal analysis to find out the voltage across 40 Ω and the power supplied by the 5 A source.

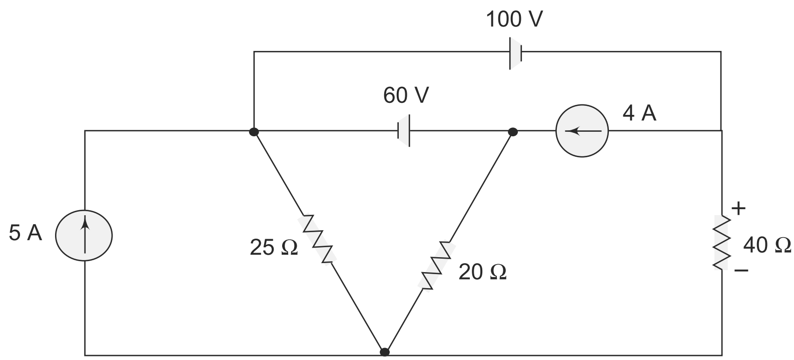


Fig. Q.1

☆★★ **2-6.2** Use nodal analysis in the circuit shown in Fig. Q.2 and determine what value of V will cause $V_{10}=0$.

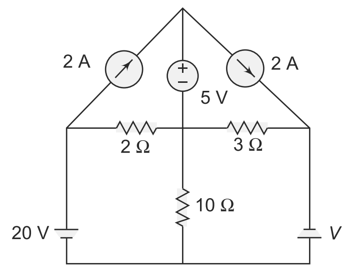


Fig. Q.2

☆☆★ **2-6.3** Find the value of V_0 in the network shown in Fig. Q.3, using PSpice.

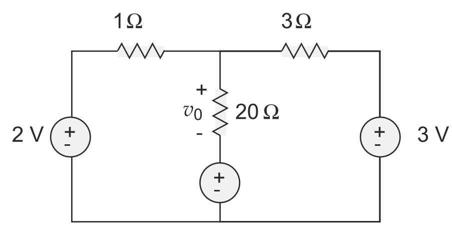


Fig. Q.3

☆★★ **2-6.4** Find the total power dissipation in the following circuit of Fig. Q.4, using PSpice.

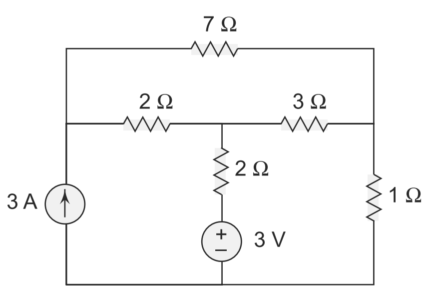


Fig. Q.4

☆☆★**2-6.5** For the circuit shown in Fig. Q.5 find the currents in all branches of the circuit using the node voltage method.

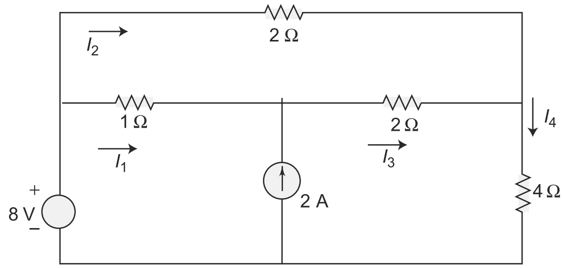


Fig. Q.5

☆☆★**2-6.6** Write nodal equations for the circuit shown in Fig. Q.6, and find the Voltage V_1.

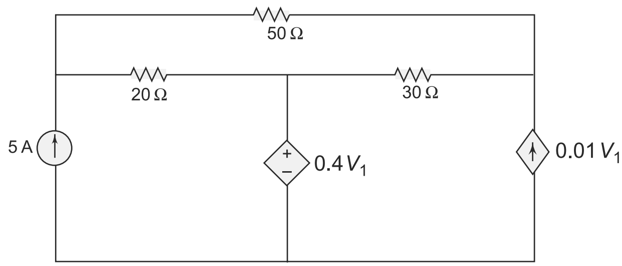


Fig. Q.6

☆★★**2-6.7** Use nodal analysis to find V_2 in the circuit shown in Fig. Q.7.

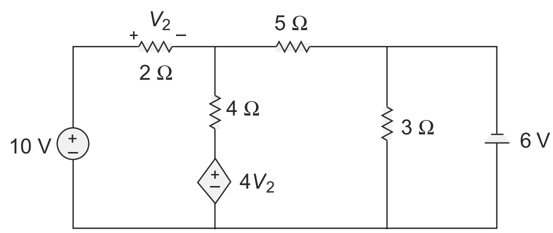


Fig. Q.7

Frequently Asked Questions linked to LO 6

☆★★**2-6.1** Determine the voltage at each node of the circuit shown in Fig. Q.1.

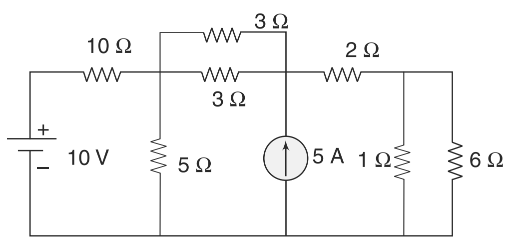


Fig. Q.1

★★★**2-6.2** Using nodal analysis techniques, determine the current '*i*' in the network shown in Fig. Q.2.
(JNTU Nov. 2012)

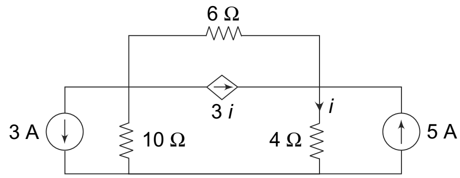


Fig. Q.2

☆☆★**2-6.3** Using graph theory, find node voltages at (*A*) and (*B*) for the network shown in Fig. Q.3.
(PTU 2011-12)

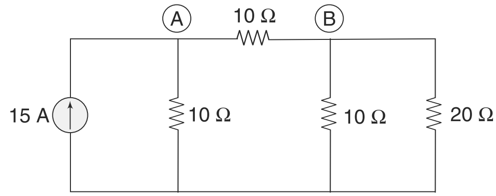


Fig. Q.3

☆☆★**2-6.4** Two batteries having e/m/f of 10 V and 7 V and internal resistance of 2 Ω and 3 Ω respectively are connected across a load resistance of 1 Ω. Calculate
(a) Individual battery currents.
(b) Current through the load.
(c) Voltage across the load using nodal analysis. **(PTU 2011-12)**

☆☆★**2-6.5** Using nodal analysis, find 'I' in the circuit shown in Fig. Q.5 **(PU 2012)**

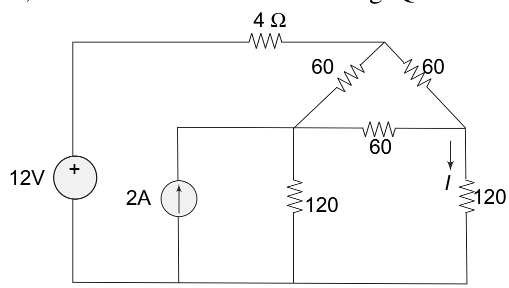


Fig. Q.5

☆★★**2-6.6** Find out the current in the 5 Ω resistance using node voltage analysis and verify the result using mesh analysis. **(RTU Feb. 2011)**

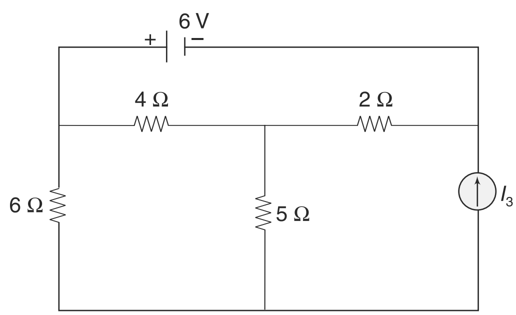


Fig. Q.6

★★★2-6.7 For the circuit shown in Fig. Q.7 find node voltage V_1, V_2, and V_3. **(PU 2012)**

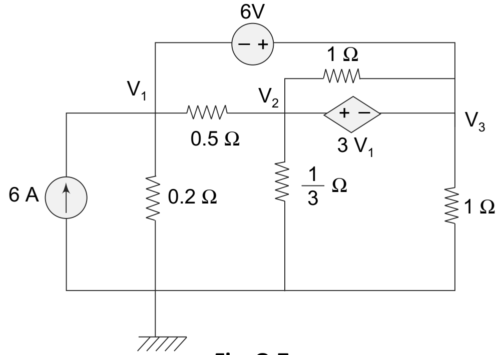


Fig. Q.7

☆☆★2-6.8 Explain the concept of supernode and supermesh. **(PU 2012)**

☆★★2-6.9 Find the current in the 6 Ω resistance in Fig. Q.9 using mesh analysis and verify the result using nodal analysis. **(RTU Feb. 2011)**

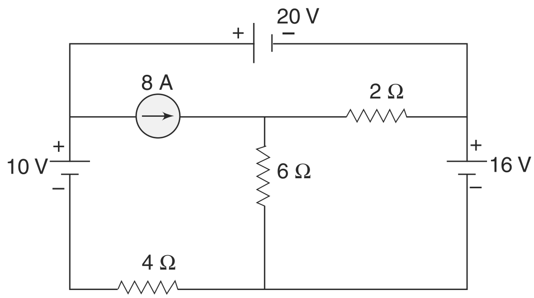


Fig. Q.9

2.15 SOURCE TRANSFORMATION TECHNIQUE

> **LO 7** Analyse the network (resistive circuits) using source transformation technique

In solving networks to find solutions, one may have to deal with energy sources. It has already been discussed in Chapter 1 that basically, energy sources are either voltage sources or current sources. Sometimes it is necessary to convert a voltage source to a current source and vice-versa. Any practical voltage source consists of an ideal voltage source in series with an internal resistance. Similarly, a practical current source consists of an ideal current source in parallel with an internal resistance as shown in Fig. 2.46. R_v and R_i represent the internal resistances of the voltage source V_s, and current source I_s, respectively.

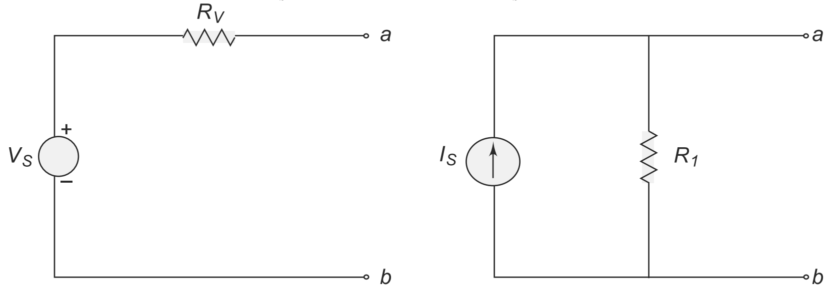


Fig. 2.46

Any source, be it a current source or a voltage source, drives current through its load resistance, and the magnitude of the current depends on the value of the load resistance. Figure 2.47 represents a practical voltage source and a practical current source connected to the same load resistance R_L.

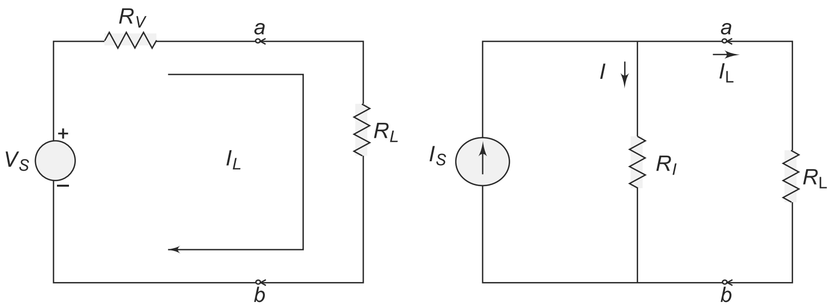


Fig. 2.47

From Fig. 2.47 (a), the load voltage can be calculated by using Kirchhoff's voltage law as

$$V_{ab} = V_s - I_L R_v$$

The open-circuit voltage $V_{OC} = V_s$

The short-circuit current $I_{SC} = \dfrac{V_s}{R_v}$

From Fig. 2.47 (b),

$$I_L = I_s - I = I_S - \frac{V_{ab}}{R_I}$$

The open-circuit voltage $V_{OC} = I_S R_I$

The short-circuit current $I_{SC} = I_S$

The above two sources are said to be equal, if they produce equal amounts of current and voltage when they are connected to identical load resistances. Therefore, by equating the open-circuit voltages and short-circuit currents of the above two sources, we obtain

$$V_{OC} = I_s R_I = V_S$$

$$I_{SC} = I_S = \frac{V_s}{R_v}$$

It follows that $R_1 = R_V = R_s \therefore V_s = I_S R_S$

where R_S is the internal resistance of the voltage or current source. Therefore, any practical voltage source, having an ideal voltage V_S and internal series resistance R_S can be replaced by a current source $I_S = V_S/R_S$ in parallel with an internal resistance R_S. The reverse transformation is also possible. Thus, a practical current source in parallel with an internal resistance R_S can be replaced by a voltage source $V_S = I_s R_s$ in series with an internal resistance R_S.

EXAMPLE 2.16

Determine the equivalent voltage source for the current source shown in Fig. 2.48.

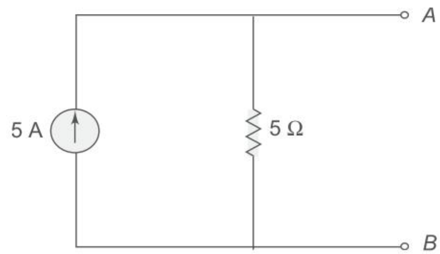


Fig. 2.48

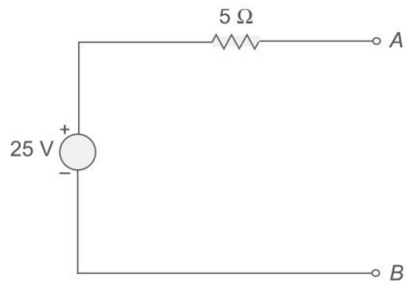


Fig. 2.49

Solution The voltage across terminals A and B is equal to 25 V. Since the internal resistance for the current source is 5 Ω, the internal resistance of the voltage source is also 5 Ω. The equivalent voltage source is shown in Fig. 2.49.

EXAMPLE 2.17

Determine the equivalent current source for the voltage source shown in Fig. 2.50.

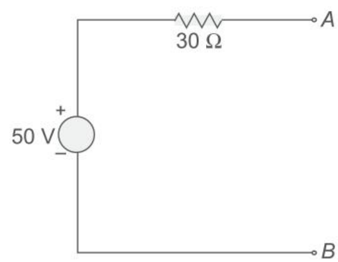


Fig. 2.50

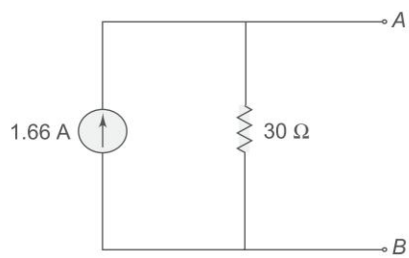


Fig. 2.51

Solution The short-circuit current at terminals A and B is equal to

$$I = \frac{50}{30} = 1.66 \text{ A}$$

Since the internal resistance for the voltage source is 30 Ω, the internal resistance of the current source is also 30 Ω. The equivalent current source is shown in Fig. 2.51.

Frequently Asked Questions linked to LO 7

☆☆★**2-7.1** Explain the source-transformation technique. **(AU May/June 2013)**

☆☆★ **2-7.2** Using source transformation replace the current source in the circuit shown in Fig. Q.2 by a voltage source and find the current delivered by the 50 V voltage source. **(AU Nov./Dec. 2012)**

☆★★ **2-7.3** Use source transformation to find I_0 in the circuit shown in Fig. Q.3. **(AU April/May 2011)**

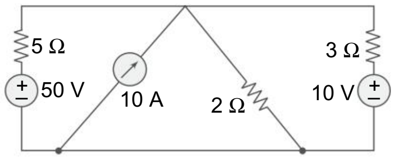


Fig. Q.2

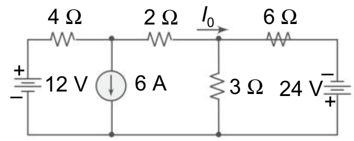


Fig. Q.3

☆★★ **2-7.4** Explain various source-transformation techniques: Using source-transformation techniques, find current I in the network shown in Fig. Q.4. **(GTU Dec. 2012)**

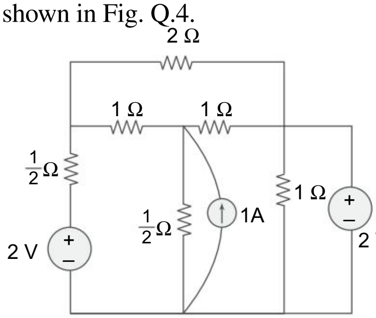


Fig. Q.4

☆☆★ **2-7.5** Explain the source-transformation techniques with suitable circuits. **(JNTU Nov. 2012)**

☆★★ **2-7.6** Obtain V_x using some shifting and source transformation technique. (Fig. Q.6) **(MU 2014)**

☆☆★ **2-7.7** Discuss the properties of an ideal current source and an ideal voltage source. Explain how a voltage source can be converted into an equivalent current source and vice versa. **(RGTU Dec. 2013)**

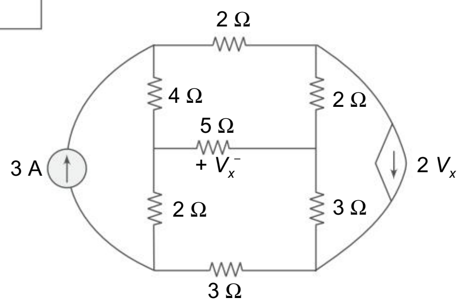


Fig. Q.6

Additional Solved Problems

PROBLEM 2.1

Determine the currents in bridge circuit by using mesh analysis in Fig. 2.52.

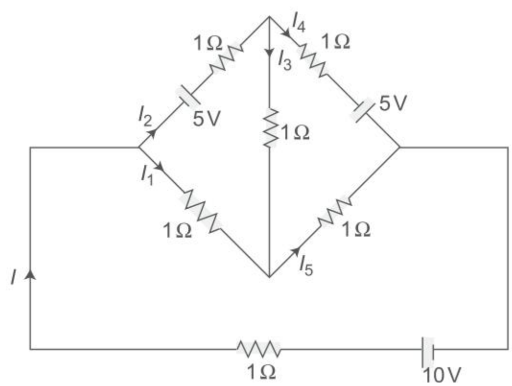


Fig. 2.52

Solution Consider i_a, i_b, and i_c are three loop currents as shown in Fig. 2.53.

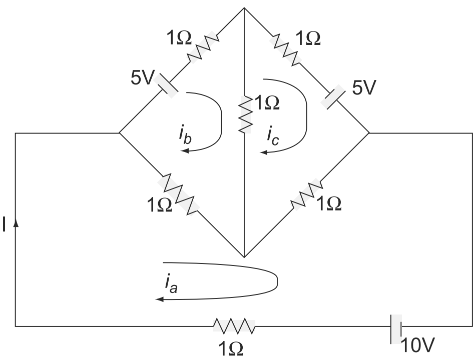


Fig. 2.53

By inspection method, we can find the loop equations

$$3i_a - i_b - i_c = 10 \tag{2.52}$$

$$-i_a + 3i_b - i_c = 5 \tag{2.53}$$

$$-i_a - i_b + 3i_c = 5 \tag{2.54}$$

From the above equations, we have

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$$

The current $i_a = \dfrac{\Delta_1}{\Delta}$

$$\text{where} \quad \Delta_1 = \begin{vmatrix} 10 & -1 & -1 \\ 5 & 3 & -1 \\ 5 & -1 & 3 \end{vmatrix} = 120$$

$$\text{and} \quad \Delta = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

$$\therefore \quad i_a = \frac{\Delta_1}{\Delta} = \frac{120}{16} = 7.5 \text{ A}$$

The current $i_b = \dfrac{\Delta_2}{\Delta}$

$$\text{where} \quad \Delta_2 = \begin{vmatrix} 3 & 10 & -1 \\ -1 & 5 & -1 \\ -1 & 5 & 3 \end{vmatrix} = 100$$

$$\therefore \quad i_b = \frac{100}{16} = 6.25 \text{ A}$$

The current $i_c = \dfrac{\Delta_3}{\Delta}$

$$\text{where} \quad \Delta_3 = \begin{vmatrix} 3 & -1 & 10 \\ -1 & 3 & 5 \\ -1 & -1 & 5 \end{vmatrix} = 100$$

$$\therefore \quad i_c = \frac{100}{16} = 6.25 \text{ A}$$

From the loop currents i_a, i_b, and i_c, we can determine the branch currents.

The branch current $I = i_a = 7.5$ A

The branch current $I_1 = i_a - i_b = 1.25$ A

The branch current $I_2 = i_b = 6.25$ A

The branch current $I_3 = i_b - i_c = 0$ A

The branch current $I_4 = i_c = 6.25$ A

The branch current $I_5 = i_a - i_c = 1.25$ A

PROBLEM 2.2

For the circuit shown in Fig. 2.54, use nodal analysis to find the current delivered by the 24V source.

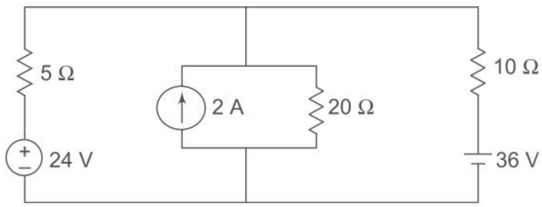


Fig. 2.54

Solution The circuit shown in Fig. 2.54 is a single-node circuit. Consider the node voltage V_1 across each branch in the circuit.

Applying Kirchhoff's current law at the node, we have

$$\frac{V_1-24}{5}-2+\frac{V_1}{20}+\frac{V_1-36}{10}=0 \tag{2.55}$$

$$V_1\left[\frac{1}{5}+\frac{1}{20}+\frac{1}{10}\right]=4.8+2+3.6$$

$$\therefore \quad V_1 = 29.7 \text{ volts}$$

The current delivered by the 24V source is

$$I_{24}=\frac{V_1-24}{5}=1.14\text{A}$$

PROBLEM 2.3

Determine the current I in the circuit by using loop analysis in Fig. 2.55.

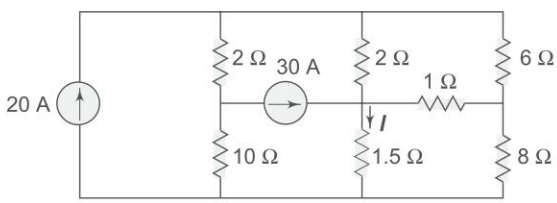


Fig. 2.55

Solution Consider the loop currents as shown in Fig. 2.56.

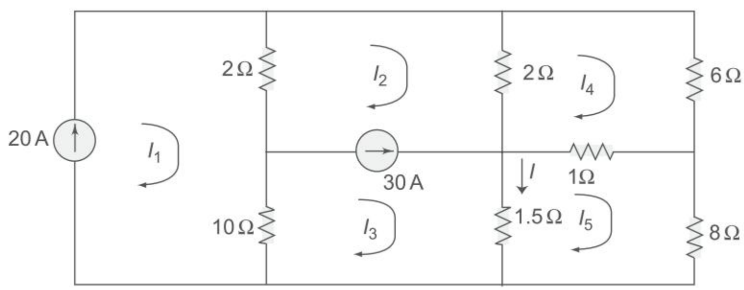


Fig. 2.56

From Fig. 2.56, the supermesh equation can be written as

$$2(I_2 - I_1) + 2(I_2 - I_4) + 10(I_3 - I_1) + 1.5(I_3 - I_5) = 0 \tag{2.56}$$

The other equations are

$$9I_4 - 2I_2 - I_5 = 0 \tag{2.57}$$

$$10.5I_5 - 1.5I_3 - I_4 = 0 \tag{2.58}$$

$$I_3 - I_2 = 30 \tag{2.59}$$

$$I_1 = 20 \tag{2.60}$$

The above equations can be reduced as

$$15.5I_3 - 2I_4 - 1.5I_5 = 360 \tag{2.61}$$

$$-2I_3 + 9I_4 - I_5 = -60 \tag{2.62}$$

$$-1.5I_3 - I_4 + 10.5I_5 = 0 \tag{2.63}$$

From the above equation, we have

$$\begin{bmatrix} 15.5 & -2 & -1.5 \\ -2 & +9 & -1 \\ -1.5 & -1 & 10.5 \end{bmatrix} \begin{bmatrix} I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 360 \\ -60 \\ 0 \end{bmatrix}$$

The branch current $I = I_3 - I_5$

The loop current $I_3 = \dfrac{\Delta_3}{\Delta}$

where, $\Delta = \begin{vmatrix} 15.5 & -2 & -1.5 \\ -2 & +9 & -1 \\ -1.5 & -1 & 10.5 \end{vmatrix} = 1381$

and $\Delta_3 = \begin{vmatrix} 360 & -2 & -1.5 \\ -60 & 9 & -1 \\ 0 & -1 & 10.5 \end{vmatrix} = 32310$

$\therefore$ the current $I_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{32310}{1381} = 23.4\text{A}$

The loop current $I_5 = \dfrac{\Delta_5}{\Delta}$

where, $\Delta_5 = \begin{vmatrix} 15.5 & -2 & 360 \\ -2 & 9 & -60 \\ -1.5 & -1 & 0 \end{vmatrix} = 4470$

$\therefore$ the current $I_5 = \dfrac{\Delta_5}{\Delta} = \dfrac{4470}{1381} = 3.24\text{A}$

The branch current $I = I_3 - I_5 = 23.4 - 3.24 = 20.16$ A

PROBLEM 2.4

Write nodal equations for the circuit shown in Fig. 2.57 and find the power supplied by the 10V source.

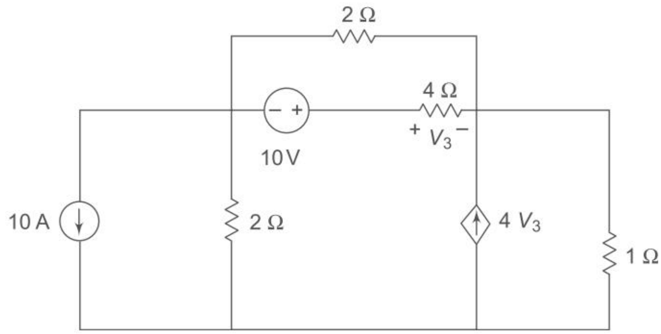


Fig. 2.57

Solution Consider the nodes *a, b* and node voltages V_a and V_b as shown in Fig. 2.58.

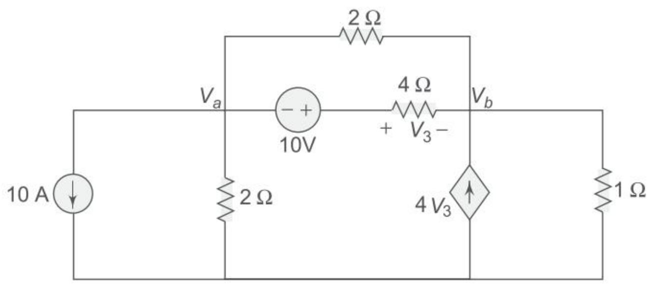


Fig. 2.58

The nodal equation at the node *a*,

$$10+\frac{V_a}{2}+\frac{V_a-V_b}{2}+\frac{V_a+10-V_b}{4}=0 \tag{2.64}$$

From the above equation, we have

$$1.25V_a - 0.75V_b = -12.5 \tag{2.65}$$

The nodal equation at the node *b*,

$$\frac{V_b-10-V_a}{4}+\frac{V_b-V_a}{2}-4V_3+V_b=0 \tag{2.66}$$

From the above equation, we have

$$-4.75V_a + 5.75V_b = 42.5 \tag{2.67}$$

Since $V_3 = V_a + 10 - V_b$

By solving the equations (2.65) and (2.67), we have

$$V_a = -11.03 \text{ volts; } V_b = -1.724 \text{ volts}$$

The current delivered by the 10V source is I_{10}.

$$I_{10}=\frac{V_a-V_b+10}{4}$$

The power supplied by the 10 V source

$$P_{10} = (10)I_{10} = 10\left(\frac{V_a - V_b + 10}{4}\right)$$

$$P_{10} = 1.735 \text{ watts.}$$

PROBLEM 2.5

Use mesh analysis to find V_x in the circuit shown in Fig. 2.59.

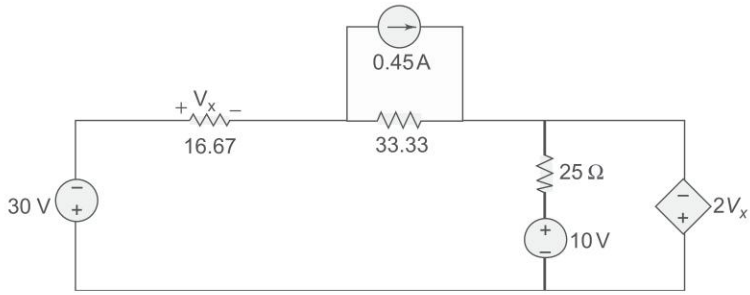


Fig. 2.59

Solution Consider the loop currents I_1, I_2, and I_3 as shown in Fig. 2.60.

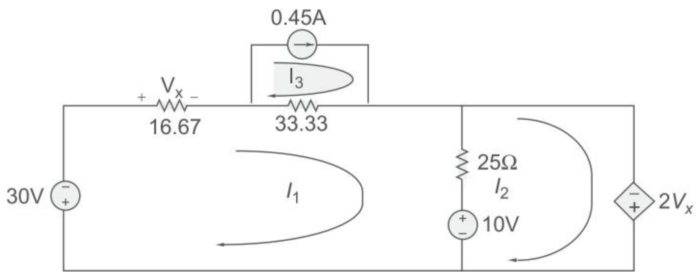


Fig. 2.60

From Fig. 2.60, the loop equations are

$$30 + V_x + 33.33(I_1 - I_3) + 25(I_1 - I_2) + 10 = 0 \tag{2.68}$$

$$-10 + 25(I_2 - I_1) - 2V_x = 0 \tag{2.69}$$

$$I_3 = 0.45 \text{ A}$$

$$\text{and} \quad V_x = 16.67I_1$$

By substituting I_3 and V_x and simplifying Eqs (2.68) and (2.69), we get

$$75I_1 - 25I_2 = 25 \tag{2.70}$$

$$\text{and} \quad -58.35I_1 + 25I_2 = 10 \tag{2.71}$$

By solving Eqs (2.70) and (2.71), we get

$$I_1 = 2.1 \text{ A}$$

The voltage across the 16.67 Ω resistor is

$$V_x = I_1(16.67) = (2.1)(16.67)$$

$$V_x = 35 \text{ volts}$$

PROBLEM 2.6

Determine the voltage ratio V_{out}/V_{in} for the circuit shown in Fig. 2.61 by using nodal analysis.

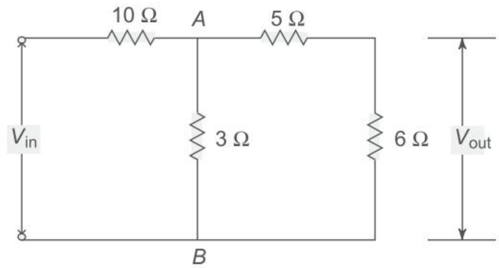


Fig. 2.61

Solution $I_{10} + I_3 + I_{11} = 0$

$$I_{10} = \frac{V_A - V_{in}}{10}$$

$$I_3 = \frac{V_A}{3}$$

$$I_{11} = \frac{V_A}{11}, \text{or} \frac{V_{out}}{6}$$

$$\frac{V_A - V_{in}}{10} + \frac{V_A}{3} + \frac{V_A}{11} = 0$$

Also $\quad \dfrac{V_A}{11} = \dfrac{V_{out}}{6}$

$$\therefore \quad V_A = V_{out} \times 1.83$$

From the above equations, $V_{out}/V_{in} = 1/9.53 = 0.105$

PROBLEM 2.7

Find the voltages V in the circuit shown in Fig. 2.62 which makes the current in the 10 Ω resistor zero by using nodal analysis.

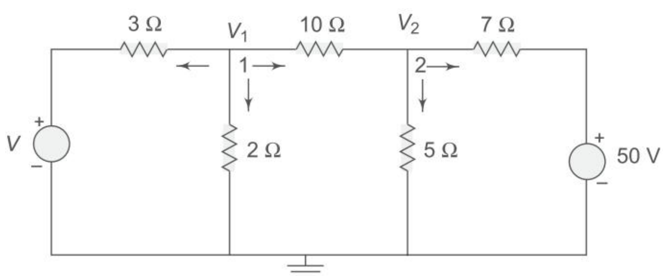


Fig. 2.62

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2.
At the node 1, the current equation in Fig. 2.63 (a) is

$$\frac{V_1 - V}{3} + \frac{V_1}{2} + \frac{V_1 - V_2}{10} = 0$$

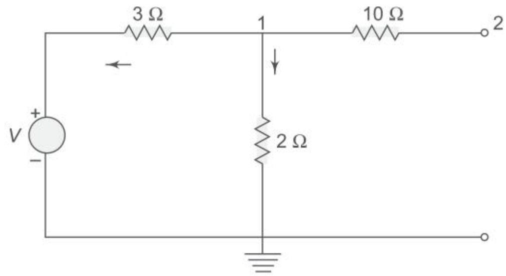


Fig. 2.63 (a)

or $\quad 0.93\ V_1 - 0.1\ V_2 = V/3$

At the node 2, the current equation in Fig. 2.63 (b) is

$$\frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - 50}{7} = 0$$

or $\quad -0.1\ V_1 + 0.443\ V_2 = 7.143$

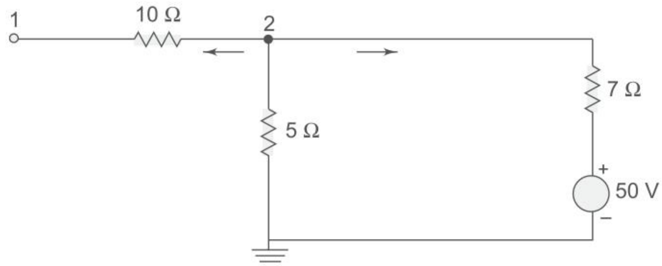


Fig. 2.63 (b)

Since the current in the 10 Ω resistor is zero, the voltage at the node 1 is equal to the voltage at the node 2.

$\therefore \quad V_1 - V_2 = 0$

From the above three equations, we can solve for V.

$V_1 = 20.83$ volts and $V_2 = 20.83$ volts

$\therefore \quad V = 51.87$ V

PROBLEM 2.8

Use nodal analysis to find the power dissipated in the 6 Ω resistor for the circuit shown in Fig. 2.64.

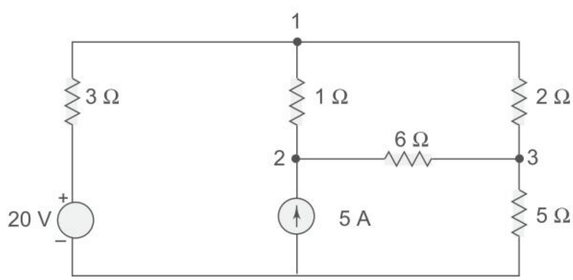


Fig. 2.64

Solution Assume voltage V_1, V_2 , and V_3 at nodes 1, 2, and 3 as shown in Fig. 2.64.

By applying current law at the node 1, we have

$$\frac{V_1 - 20}{3} + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{2} = 0$$

or $\quad 1.83V_1 - V_2 - 0.5V_3 = 6.67$ (2.72)

At the node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{6} = 5\,\text{A}$$

or $\quad -V_1 - 1.167V_2 - 0.167V_3 = 5$ (2.73)

At the node 3,

$$\frac{V_3 - V_1}{2} + \frac{V_3 - V_2}{6} + \frac{V_3}{5} = 0$$

or $\quad -0.5\,V_1 - 0.167\,V_2 + 0.867\,V_3 = 0$ (2.74)

Applying Cramer's rule to Eqs (2.72), (2.73) and (2.74), we have

$$V_2 = \frac{\Delta_2}{\Delta}$$

where $$\Delta = \begin{bmatrix} 1.83 & -1 & -0.5 \\ -1 & -1.167 & -0.167 \\ -0.5 & -0.167 & 0.867 \end{bmatrix} = -2.64$$

$$\Delta_2 = \begin{bmatrix} 1.83 & 6.67 & -0.5 \\ -1 & 5 & -0.167 \\ -0.5 & 0 & 0.867 \end{bmatrix} = 13.02$$

$$\therefore \quad V_2 = \frac{13.02}{-2.64} = -4.93\,\text{V}$$

$$V_2 = -4.93\text{ V}$$

Similarly,

$$V_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta_3 = \begin{bmatrix} 1.83 & -1 & 6.67 \\ -1 & -1.167 & 5 \\ -0.5 & -0.167 & 0 \end{bmatrix} = 1.25$$

$$\therefore \qquad V_3 = \frac{1.25}{-2.64} = -0.47\,\text{V}$$

The current in the 6 Ω resistor is

$$I_6 = \frac{V_2 - V_3}{6}$$

$$= \frac{-4.93 + 0.47}{6} = -0.74\text{ A}$$

The power absorbed or dissipated $= I_6^2 R_6$

$= (0.74)^2 \times 6$

$= 3.29$ W

PROBLEM 2.9

Determine the power dissipated by 5 Ω resistor in the circuit shown in Fig. 2.65.

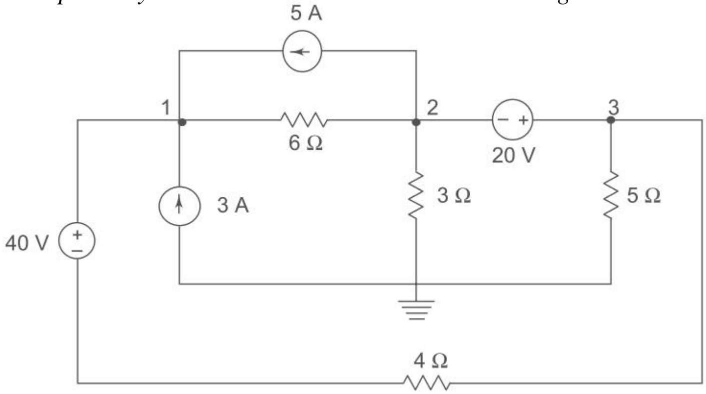


Fig. 2.65

Solution In Fig. 2.65, assume voltages V_1, V_2, and V_3 at nodes 1, 2, and 3. At the node 1, the current law gives

$$\frac{V_1 - 40 - V_3}{4} + \frac{V_1 - V_2}{6} - 3 - 5 = 0$$

or $0.42\ V_1 - 0.167\ V_2 - 0.25\ V_3 = 18$

Applying the supernode technique between nodes 2 and 3, the combined equation at node 2 and 3 is

$$\frac{V_2 - V_1}{6} + 5 + \frac{V_2}{3} + \frac{V_3}{5} + \frac{V_3 + 40 - V_1}{4} = 0$$

or $-0.42\ V_1 + 0.5\ V_2 + 0.45\ V_3 = -15$

Also, $V_3 - V_2 = 20$ V

Solving the above three equations, we get

$V_1 = 52.89$ V, $V_2 = -1.89$ V and

$V_3 = 18.11$ V

$\therefore$ the current in the 5Ω resistor $I_5 = \dfrac{V_3}{5}$

$$= \frac{18.11}{5} = 3.62 \text{ A}$$

The power absorbed by the 5 Ω resistor $P_5 = I_5^2 R_5$

$$= (3.62)^2 \times 5$$

$$= 65.52 \text{ W}$$

PROBLEM 2.10

Find the power delivered by the 5 A current source in the circuit shown in Fig. 2.66 by using the nodal method.

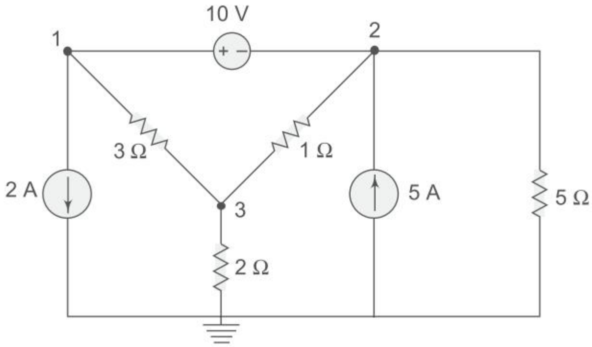


Fig. 2.66

Solution Assume the voltages V_1, V_2, and V_3 at nodes 1, 2, and 3, respectively. Here, the 10 V source is common between nodes 1 and 2. So applying the supernode technique, the combined equation at nodes 1 and 2 is

$$\frac{V_1 - V_3}{3} + 2 + \frac{V_2 - V_3}{1} - 5 + \frac{V_2}{5} = 0$$

or $\quad 0.34\, V_1 + 1.2\, V_2 - 1.34\, V_3 = 3$

At the node 3, $\dfrac{V_3 - V_1}{3} + \dfrac{V_3 - V_2}{1} + \dfrac{V_3}{2} = 0$

or $\quad -0.34\, V_1 - V_2 + 1.83\, V_3 = 0$

Also, $\quad V_1 - V_2 = 10$

Solving the above equations, we get

$$V_1 = 13.72 \text{ V}; V_2 = 3.72 \text{ V}$$

$$V_3 = 4.567 \text{ V}$$

Hence, the power delivered by the source (5 A) $= V_2 \times 5$

$$= 3.72 \times 5 = 18.6 \text{ W}$$

PROBLEM 2.11

Using source transformation, find the power delivered by the 50 V voltage source in the circuit shown in Fig. 2.67.

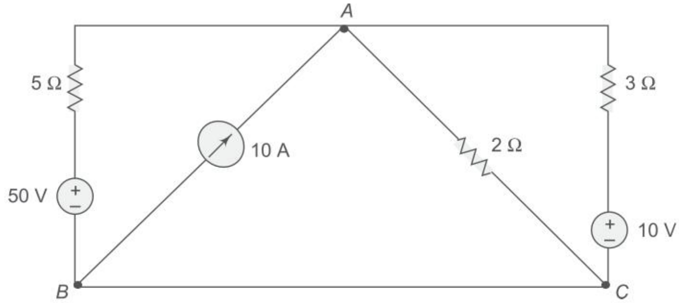


Fig. 2.67

Solution The current source in the circuit in Fig. 2.67 can be replaced by a voltage source as shown in Fig. 2.68.

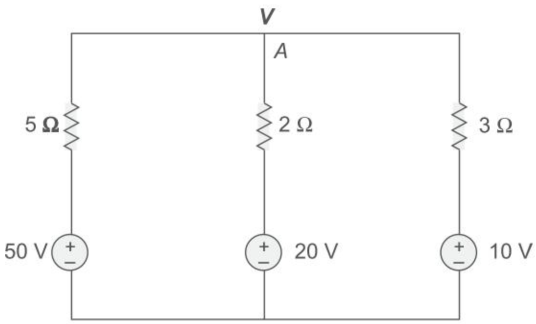


Fig. 2.68

$$\frac{V-50}{5}+\frac{V-20}{2}+\frac{V-10}{3}=0$$

$$V\,[0.2 + 0.5 + 0.33] = 23.33$$

$$\text{or} \quad V=\frac{23.33}{1.03}=22.65\,\text{V}$$

$\therefore$ the current delivered by the 50 V voltage source is $(50 - V)/5$

$$=\frac{50-22.65}{5}=5.47\,\text{A}$$

Hence, the power delivered by the 50 V voltage source = 50 × 5.47 = 273.5 W

PROBLEM 2.12

By using source transformation, source combination and resistance combination convert the circuit shown in Fig. 2.69 into a single voltage source and single resistance.

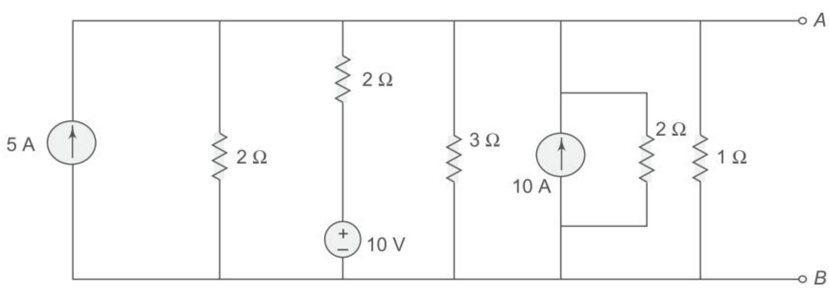


Fig. 2.69

Solution The voltage source in the circuit of Fig. 2.69 can be replaced by a current source as shown in Fig. 2.70 (a).

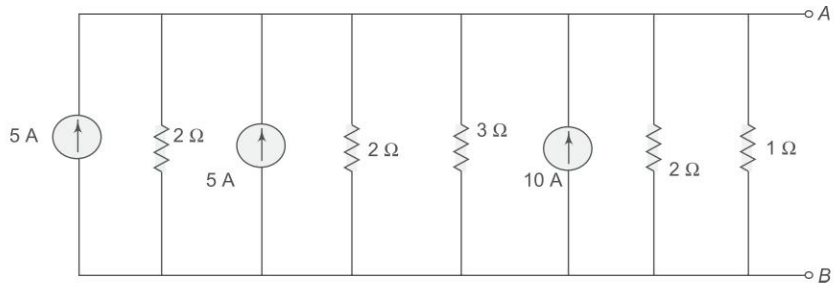


Fig. 2.70 (a)

Here, the current sources can be combined into a single source. Similarly, all the resistances can be combined into a single resistance, as shown in Fig. 2.70 (b).

Figure 2.70 (b) can be replaced by single voltage source and a series resistance as shown in Fig. 2.70 (c).

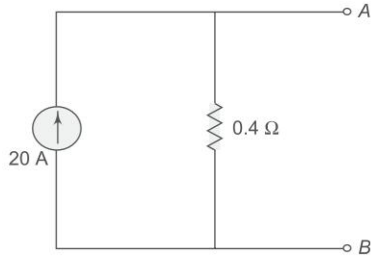


Fig. 2.70 (b)

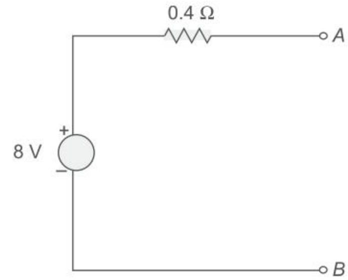


Fig. 2.70 (c)

PROBLEM 2.13

For the circuit shown in Fig. 2.71 find the voltage across the 4 Ω resistor by using nodal analysis.

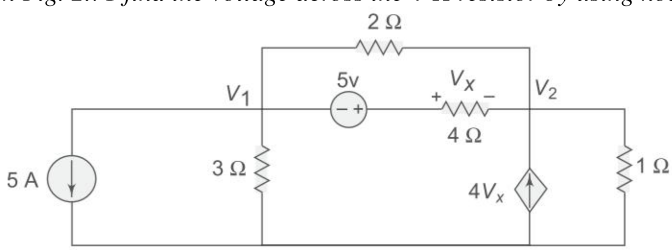


Fig. 2.71

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2. At the node 1, the current equation is

$$5+\frac{V_1}{3}+\frac{V_1+5-V_2}{4}+\frac{V_1-V_2}{2}=0$$

$$\text{or} \quad 1.08\ V_1 - 0.75\ V_2 = -6.25 \tag{2.75}$$

At the node 2, the current equation is

$$\frac{V_2-V_1-5}{4}+\frac{V_2-V_1}{2}-4V_x+\frac{V_2}{1}=0$$

$$V_x = V_1 + 5 - V_2$$

or $-4.75\, V_1 + 5.75\, V_2 = 21.25$ (2.76)

Applying Cramer's rule to Eqs (2.75) and (2.76), we have

$$V_2 = \frac{\Delta_2}{\Delta}$$

where $$\Delta = \begin{vmatrix} 1.08 & -0.75 \\ -4.75 & 5.75 \end{vmatrix} = 2.65$$

$$\Delta_2 = \begin{vmatrix} 1.08 & -6.25 \\ -4.75 & 21.25 \end{vmatrix} = -6.74$$

$$\therefore\ V_2 = \frac{\Delta_2}{\Delta} = \frac{-6.74}{2.65} = -2.54 \text{ V}$$

Similarly, $$V_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} -6.25 & -0.75 \\ 21.25 & 5.75 \end{vmatrix} = -20$$

$$V_1 = \frac{\Delta_1}{\Delta} = \frac{-20}{2.65} = -7.55 \text{ V}$$

The voltage across the 4 Ω resistor is

$$V_x = V_1 + 5 - V_2$$

$$= -7.55 + 5 - (-2.54)$$

$$V_x = -0.01 \text{ volts}$$

PROBLEM 2.14

For the circuit shown in Fig. 2.72, find the current passing through the 5 Ω resistor by using the nodal method.

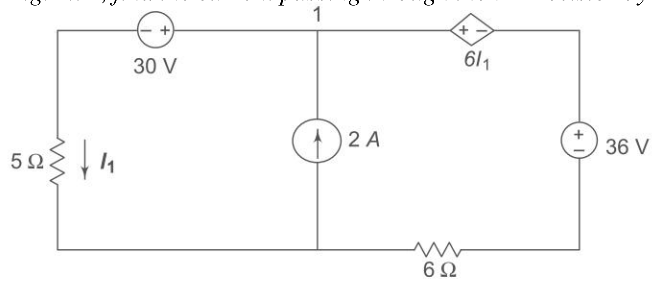


Fig. 2.72

Solution In the circuit shown, assume the voltage V at the node 1.

At the node 1, the current equation is

$$\frac{V-30}{5} - 2 + \frac{V-36-6I_1}{6} = 0$$

where $$I_1 = \frac{V-30}{5}$$

From the above equation

$$V = 48 \text{ V}$$

The current in the 5 Ω resistor is

$$I_1 = \frac{V - 30}{5} = 3.6 \text{ A}$$

PROBLEM 2.15

In the circuit shown in Fig. 2.73, find the power delivered by the 4 V source using mesh analysis and voltage across the 2 Ω resistor.

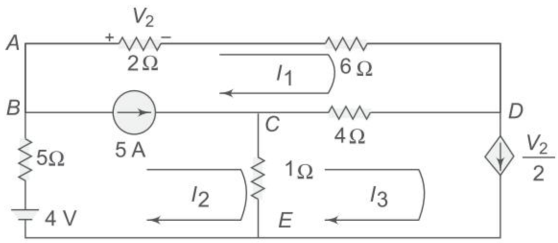


Fig. 2.73

Solution Since branches *BC* and *DE* consist of current sources, we use the supermesh technique.

The combined supermesh equation is

$$2I_1 + 6I_1 + 4(I_1 - I_3) + (I_2 - I_3) - 4 + 5I_2 = 0$$

or

$$12I_1 + 6I_2 - 5I_3 = 4$$

In the branch *BC*, $I_2 - I_1 = 5$

In the branch *DE*, $I_3 = \frac{V_2}{2}$

Solving the above equations,

$$I_1 = -2 \text{ A}; I_2 = 3 \text{ A}$$

The voltage across the 2 Ω resistor $V_2 = 2I_1 = 2 \times (-2) = -4$ V

Power delivered by the 4 V source $P_4 = 4I_2 = 4(3) = 12$ W

PROBLEM 2.16

For the circuit shown in Fig. 2.74, find the current through the 10 Ω resistor by using mesh analysis.

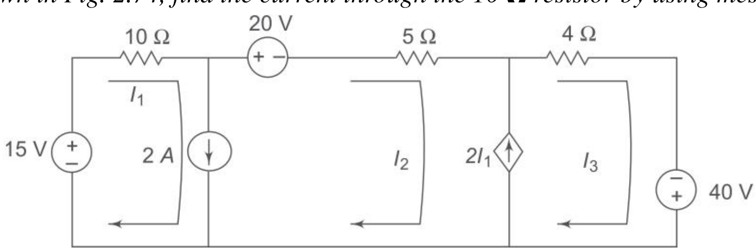


Fig. 2.74

Solution The parallel branches consist of current sources. Here, we use supermesh analysis. The combined supermesh equation is.

or $\quad -15 + 10I_1 + 20 + 5I_2 + 4I_3 - 40 = 0$

and $\quad 10I_1 + 5I_2 + 4I_3 = 35$

$$I_1 - I_2 = 2$$

$$I_3 - I_2 = 2I_1$$

Solving the above equations, we get

$$I_1 = 1.96 \text{ A}$$

The current in the 10 Ω resistor is $I_1 = 1.96$ A

PSpice Problems

PROBLEM 2.1

Determine the currents for the circuit shown in Fig. 2.75.

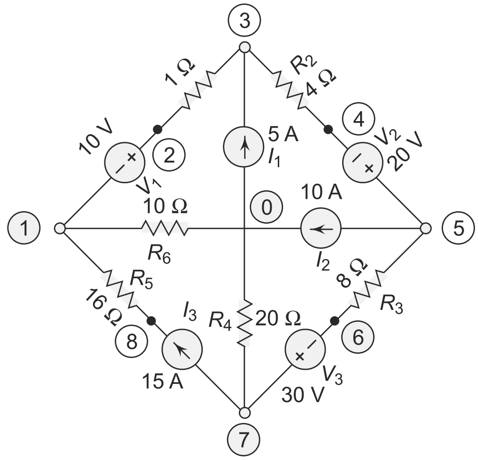


Fig. 2.75

```
*   CURRENTS WITH DC ANALYSIS
V1   2   1   DC   10 V
V2   5   4   DC   20 V
V3   7   6   DC   30 V
I1   0   3   5 A
I2   5   0   10 A
I3   7   8   15 A
R1   2   3          1
R2   3   4   4
R3   5   6   8
R4   7   0   20
R5   8   1   16
R6   1   0   10
.DC LIN V1 10 10 1
.PRINT DC I(R1) I(R2) I(R3) I(R5)
.END
**** DC TRANSFER CURVES TEMPERATURE = 27.000 DEG C
*****************************************************************
      V1            I(R1)          I(R2)          I(R3)          I(R5)
  1.000 E + 01   1.465 E + 01   1.965 E + 01   9.651 E + 00   1.500 E + 01
```

Result

I1 = I(R1) = 14.65 A

I2 = I(R2) = 19.65 A

I3 = I(R5) = 15 A

I4 = I(R3) = 9.65 A

PROBLEM 2.2

Determine the power delivered by the 5 A current source in the circuit shown in Fig. 2.76.

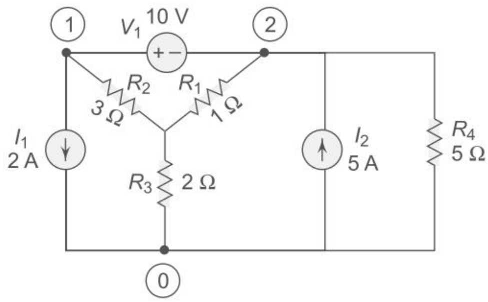


Fig. 2.76

```
*   POWER DELIVERED BY DC SOURCE
I1    1   0   2 A
V1    1   2   10 V
I2    0   2   5 A
R1    2   3   1
R2    1   3   3
R3    3   0   2
R4    2   0   5
.DC LIN V1 10 10 1
.PRINT DC V(1) V(2) V(3)
.END
**** DC TRANSFER CURVES TEMPERATURE = 27.000 DEG C
****************************************************************
          V1               V(1)           V(2)            V(3)
     1.000 E + 01     1.371 E + 01    3.710 E + 00    4.516 E + 00
```

Result

Power delivered by the 5 A source = V_2 X 5 = 3.71 X 5 = 18.55 W

PROBLEM 2.3

Determine I in the Fig. 2.77 using PSpice, with I_1 varying for 5 A, 10 A and 20 A.

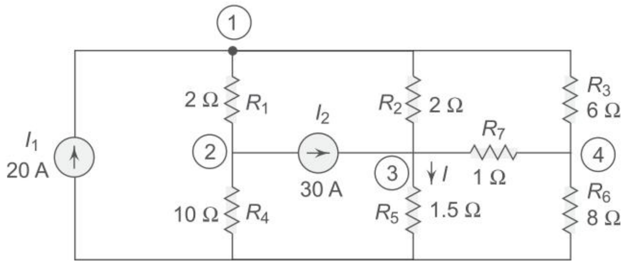


Fig. 2.77

```
*   TO DETERMINE CURRENT IN AN ELEMENT USING DC SWEEP
I1 0 1 20 A
I2 2 3 30
R1 1 2 2
R2 1 3 2
R3 1 4 6
R4 2 0 10
R5 3 0 1.5
R6 4 0 8
R7 3 4 1
```

```
.DC LIN I1 LIST 5 10 20
.PRINT DC I(R5)
.END
```

OUTPUT

```
**** DC TRANSFER CURVES TEMPERATURE = 27.000 DEG C
*****************************************************************
   I1              I(R5)
5.000 E + 00    9.993 E + 00
1.000 E + 01    1.338 E + 01
2.000 E + 01    2.016 E + 01
```

Result

I = I (R5) = 20.16 A

Answers to Practice Problems

2-5.1	2580 W; –32 V	**2-6.1**	–60.9 V; 195.7 W
2-5.3	$R_1 = 4.88\,\Omega$; $R_2 = 3.82\,\Omega$;	**2-6.5**	$I_1 = -0.92$A, $I_2 = 0.615$ A $I_3 = 1.075$A, $I_4 = 1.69$ A
2-5.4	2.65 V	**2-6.6**	$V_1 = 148.1$ volts
2-5.6	1.2 A; 4.2 A; 2 A; 3.2 A		
2-5.7	$I = 11$A		
2-5.8	$I_1 = 5$ A; $I_2 = -11$ A		

Objective-Type Questions

☆☆★ **2.1** A tree has

(a) a closed path (b) no closed paths (c) none

☆☆★ **2.2** The number of branches in a tree is _____ the number of branches in a graph.

(a) less than (b) more than (c) equal to

☆☆★ **2.3** The tie-set schedule gives the relation between

(a) branch currents and link currents (b) branch voltages and link currents
(c) branch currents and link voltages (d) none of the above

☆☆★ **2.4** The cut-set schedule gives the relation between

(a) branch currents and link currents (b) branch voltages and tree branch voltages
(c) branch voltages and link voltages (d) branch current and tree currents

☆☆★ **2.5** Mesh analysis is based on

(a) Kirchhoff's current law (b) Kirchhoff's voltage law
(c) both (d) none

☆☆★ **2.6** If a network contains B branches, and N nodes, then the number of mesh current equations would be

(a) $B - (N - 1)$ (b) $N - (B - 1)$ (c) $B - N - 1$ (d) $(B + N) - 1$

☆★★ **2.7** A network has 10 nodes and 17 branches. The number of different node pair voltages would be

(a) 7 (b) 9 (c) 45 (d) 10

☆☆★ **2.8** A practical voltage source consists of

(a) an ideal voltage source in series with an internal resistance
(b) an ideal voltage source in parallel with an internal resistance
(c) both (a) and (b) are correct
(d) none of the above

☆☆★ **2.9** A practical current source consists of
(a) an ideal current source in series with a resistance
(b) an ideal current source in parallel with a resistance
(c) both are correct
(d) none of the above

☆★★ **2.10** A circuit consists of two resistances, R_1 and R_2, in parallel. The total current passing through the circuit is I_T. The current passing through R_1 is

(a) $\dfrac{I_T R_1}{R_1 + R_2}$ (b) $\dfrac{I_T\left(R_1 + R_2\right)}{R_1}$ (c) $\dfrac{I_T R_2}{R_1 + R_2}$ (d) $\dfrac{I_T R_1 + R_2}{R_2}$

☆☆★ **2.11** A network has seven nodes and five independent loops. The number of branches in the network is
(a) 13 (b) 12 (c) 11 (d) 10

☆☆★ **2.12** The nodal method of circuit analysis is based on
(a) KVL and Ohm's law (b) KCL and Ohm's law
(c) KCL and KVL (d) KCL, KVL, and Ohm's law

☆☆★ **2.13** The number of independent loops for a network with n nodes and b branches is
(a) $n - 1$ (b) $b - n$
(c) $b - n + 1$ (d) independent of the number of nodes

☆★★ **2.14** The two electrical subnetworks N_1 and N_2 are connected through three resistors as shown in Fig. 2.78. The voltage across the 5 Ω resistor and the 1 Ω resistor are given to be 10 V and 5 V respectively. The voltage across the 15 Ω resistor is
(a) – 105 V (b) + 105 V (c) – 15 V (d) + 15 V

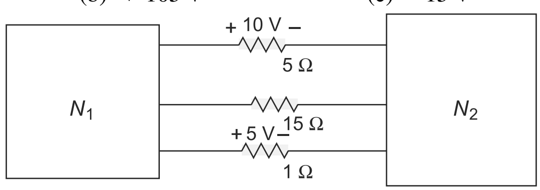


Fig. 2.78

☆☆★ **2.15** Relative to a given fixed tree of a network
(a) link currents form an independent set (b) branch currents form an independent set
(c) link voltages form an independent set (d) branch voltages form an independent set

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/260

CHAPTER 3

Useful Theorems in Circuit Analysis

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Reduce the network by Star-Delta transformation
LO 2 Explain the superposition theorem and apply it to solve the networks
LO 3 Explain Thevenin's theorem and apply it to solve the networks
LO 4 Explain Norton's theorem and apply it to solve the networks
LO 5 Explain the reciprocity theorem and apply it to solve the networks
LO 6 Explain the compensation theorem and apply it to solve the networks
LO 7 State the maximum power transfer theorem and apply it to solve the networks
LO 8 Explain the concept of duals and the principle of duality
LO 9 Explain Tellegen's theorem and apply it to solve the networks
LO 10 State Millman's theorem and apply it for solving the networks

3.1 STAR-DELTA TRANSFORMATION

LO 1 Reduce the network by Star-Delta transformation

In the preceding chapter, a simple technique called the *source transformation technique* was discussed. The star-delta transformation is another technique useful in solving complex networks. Basically, any three circuit elements, i.e. resistive, inductive or capacitive, may be connected in two different ways. One way of connecting these elements is called the star connection, or the Y-connection. The other way of connecting these elements is called the delta (Δ) connection. *The circuit is said to be in star connection, if three elements are connected as shown in Fig. 3.1 (a), when it appears like a star (Y). Similarly, the circuit is said to be in delta connection, if three elements are connected as shown in Fig. 3.1 (b), when it appears like a delta (Δ).*

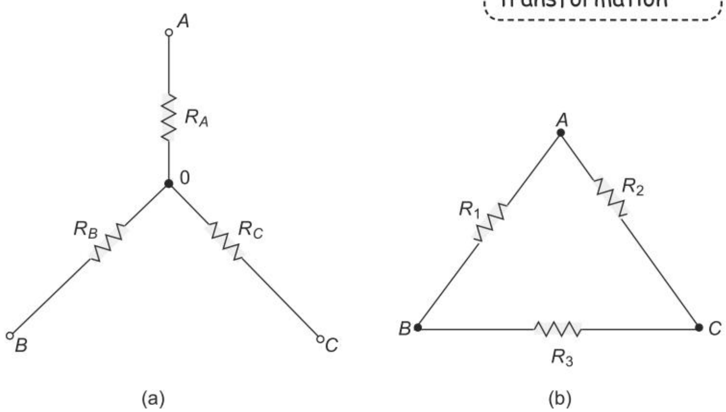


Fig. 3.1

The above two circuits are equal if their respective resistances from the terminals *AB*, *BC,* and *CA* are equal. Consider the star-connected circuit in Fig. 3.1 (a); the resistance from the terminals *AB*, *BC,* and *CA* respectively are

$$R_{AB}(Y) = R_A + R_B$$

$$R_{BC}(Y) = R_B + R_C$$

$$R_{CA}(Y) = R_C + R_A$$

Similarly, in the delta-connected network in Fig. 3.1 (b), the resistances seen from the terminals *AB*, *BC,* and *CA* respectively are

$$R_{AB}(\Delta) = R_1 \| (R_2 + R_3) = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

$$R_{BC}(\Delta) = R_3 \| (R_1 + R_2) = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3}$$

$$R_{CA}(\Delta) = R_2 \| (R_1 + R_3) = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3}$$

Now, if we equate the resistances of star and delta circuits, we get

$$R_A + R_B = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3} \tag{3.1}$$

$$R_B + R_C = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} \tag{3.2}$$

$$R_C + R_A = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3} \tag{3.3}$$

Subtracting Eq. (3.2) from Eq. (3.1), and adding Eq. (3.3) to the resultant, we have

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{3.4}$$

Similarly, $$R_B = \frac{R_1 R_3}{R_1 + R_2 + R_3} \tag{3.5}$$

and $$R_C = \frac{R_2 R_3}{R_1 + R_2 + R_3} \tag{3.6}$$

Thus, a delta connection of R_1, R_2, and R_3 may be replaced by a star connection of R_A, R_B, and R_C as determined from Eqs (3.4), (3.5) and (3.6). Now if we multiply the Eqs (3.4) and (3.5), (3.5) and (3.6), (3.6) and (3.4), and add the three, we get the final equation as under:

$$R_A R_B + R_B R_C + R_C R_A = \frac{R_1^2 R_2 R_3 + R_3^2 R_1 R_2 + R_2^2 R_1 R_3}{(R_1 + R_2 + R_3)^2} \tag{3.7}$$

In Eq. (3.7) dividing the LHS by R_A, gives R_3; dividing it by R_B gives R_2, and doing the same with R_C, gives R_1.

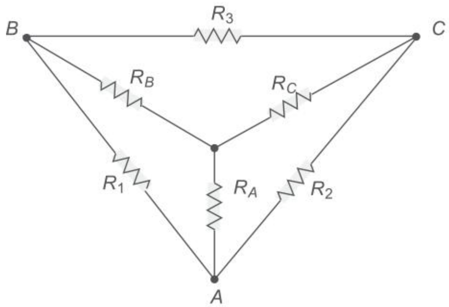


Fig. 3.2

Thus $R_1 = \dfrac{R_A R_B + R_B R_C + R_C R_A}{R_C}$

$$R_2 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_B}$$

and $R_3 = \dfrac{R_A R_B + R_B R_C + R_C R_A}{R_A}$

From the above results, we can say that a star-connected circuit can be transformed into a delta-connected circuit and vice-versa.

From Fig. 3.2 and the above results, we can conclude that any resistance of the delta circuit is equal to the sum of the products of all possible pairs of star resistances divided by the opposite resistance of the star circuit. Similarly, any resistance of the star circuit is equal to the product of two adjacent resistances in the delta-connected circuit divided by the sum of all resistances in delta-connected circuit.

EXAMPLE 3.1

Obtain the star-connected equivalent for the delta-connected circuit shown in Fig. 3.3.

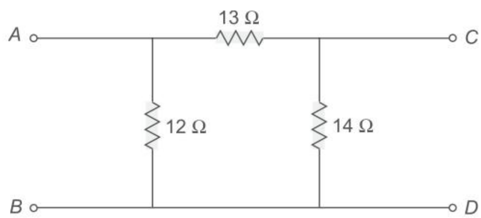


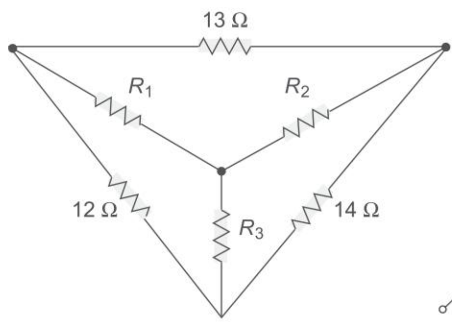
Fig. 3.3

Solution The above circuit can be replaced by a star-connected circuit as shown in Fig. 3.4 (a).

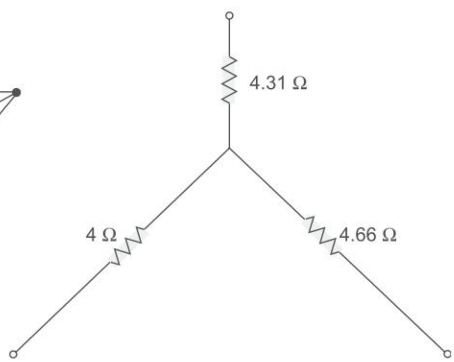
Performing the Δ to Y transformation, we obtain

$$R_1 = \frac{13 \times 12}{14 + 13 + 12}, \; R_2 = \frac{13 \times 14}{14 + 13 + 12}$$

and $R_3 = \dfrac{14 \times 12}{14 + 13 + 12}$



(a)



(b)

Fig. 3.4

$\therefore \quad R_1 = 4\ \Omega, R_2 = 4.66\ \Omega, R_3 = 4.31\ \Omega$

The star-connected equivalent is shown in Fig. 3.4 (b).

EXAMPLE 3.2

Obtain the delta-connected equivalent for the star-connected circuit shown in Fig. 3.5.

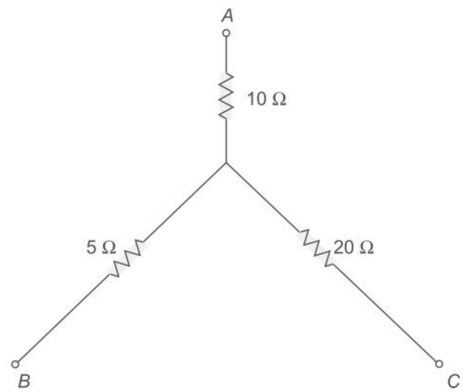


Fig. 3.5

Solution The above circuit can be replaced by a delta-connected circuit as shown in Fig. 3.6 (a).

Performing the Y to Δ transformation, we get from Fig. 3.6 (a),

$$R_1 = \frac{20\times 10 + 20\times 5 + 10\times 5}{20} = 17.5\ \Omega$$

$$R_2 = \frac{20\times 10 + 20\times 5 + 10\times 5}{10} = 35\ \Omega$$

and $$R_3 = \frac{20\times 10 + 20\times 5 + 10\times 5}{5} = 70\ \Omega$$

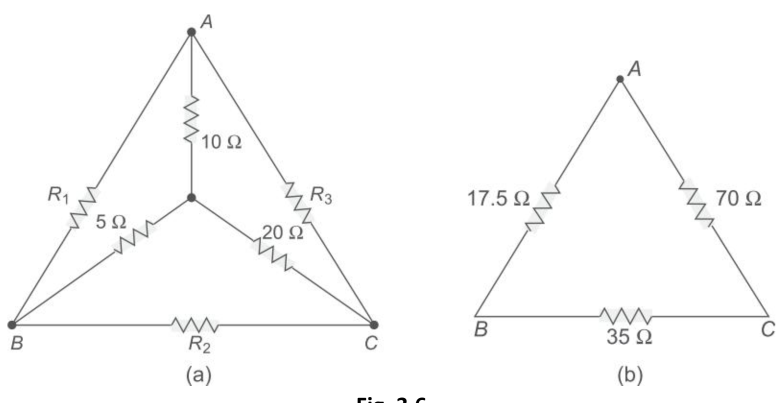


Fig. 3.6

The equivalent delta circuit is shown in Fig. 3.6 (b).

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆★★**3-1.1** For the bridge network shown in Fig. Q.2 (a), determine the total resistance seen from terminals *AB* by using star-delta transformation.

★★★**3-1.2** Find the equivalent resistance across the terminals *A* and *B* of the network shown in Fig. Q.2 (b) using star-delta transformation.

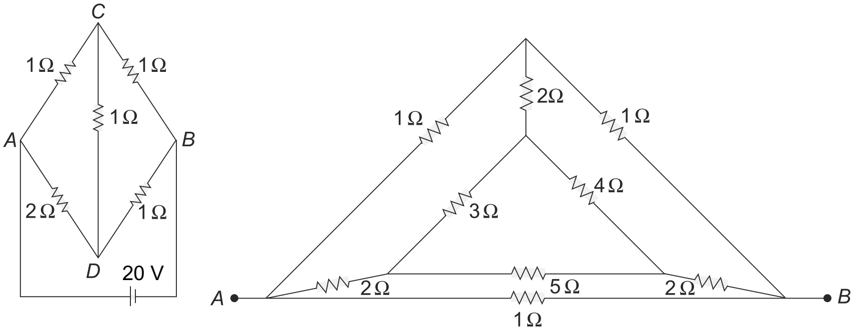


Fig. Q.2

☆☆★**3-1.3** Determine the voltages and currents of the resistances in the circuit shown in Fig. Q.3 using source transformation technique.

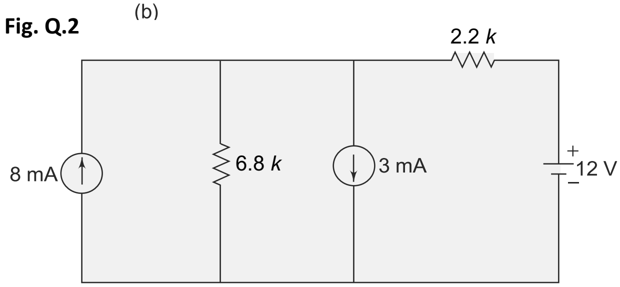


Fig. Q.3

Frequently Asked Questions linked to LO 1*

☆☆★**3-1.1** Derive the expression for star-connected resistances in terms of delta-connected resistances. **[AU May/June 2013]**

☆☆★**3-1.2** Give a delta circuit it having resistors, write the required expressions to transform the circuit to a star circuit. **[AU Nov./Dec. 2012]**

☆★★**3-1.3** Calculate the equivalent resistance *Rab* when all the resistance values are equal to 1 Ω for the circuit shown below. (Fig. Q.3) **[AU Nov./Dec. 2012]**

☆☆★**3-1.4** Transform the circuit shown below, from D to Y. (Fig. Q. 4) **[AU April/May 2011]**

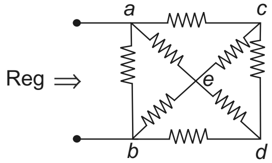


Fig. Q.3

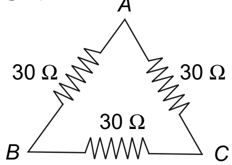


Fig. Q.4

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

☆☆★**3-1.5** Use the technique of D-Y conversion to find the equivalent resistance between terminals A-B of the circuit shown below. **[AU April/May 2011]**

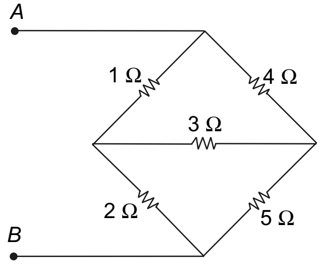


Fig. Q.5

☆☆★**3-1.6** State and explain star-delta conversion in ac systems. **[JNTU Nov. 2012]**

☆☆★**3-1.7** Obtain the expressions for star-delta equivalence of an impedance network. **[JNTU Nov. 2012]**

3.2 SUPERPOSITION THEOREM

LO 2 Explain the superposition theorem and apply it to solve the networks

*The **superposition theorem** states that in any linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone, while the other sources are non-operative;* that is, while considering the effect of individual sources, other ideal voltage sources and ideal current sources in the network are replaced by short circuit and open circuit across their terminals. This theorem is valid only for linear systems. This theorem can be better understood with a numerical example.

Consider the circuit which contains two sources as shown in Fig. 3.7.

Now let us find the current passing through the 3 Ω resistor in the circuit. According to the superposition theorem, the current I_2 due to the 20 V voltage source with 5 A source open circuited = 20/(5 + 3) = 2.5 A (see Fig. 3.8)

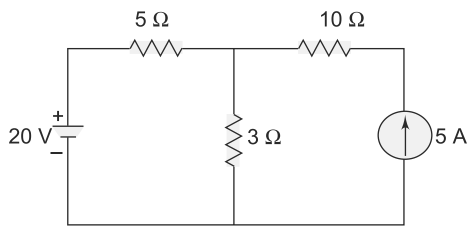


Fig. 3.7

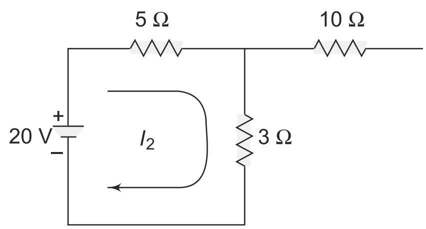


Fig. 3.8

The current I_5 due to the 5 A source with the 20 V source short circuited is

$$I_5 = 5 \times \frac{5}{(3+5)} = 3.125\,\text{A}$$

The total current passing through the 3 Ω resistor is

$$(2.5 + 3.125) = 5.625\text{ A}$$

Let us verify the above result by applying nodal analysis.

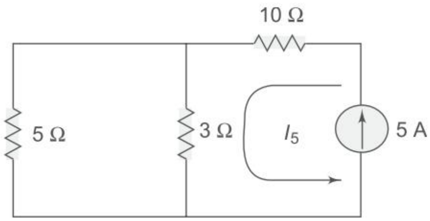


Fig. 3.9

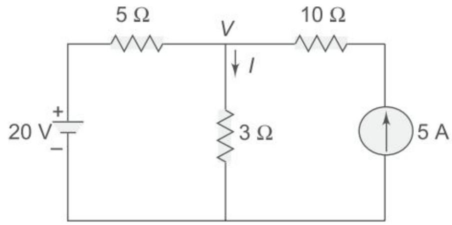


Fig. 3.10

The current passing in the 3 Ω resistor due to both sources should be 5.625 A.

Applying nodal analysis to Fig. 3.10, we have

$$\frac{V-20}{5}+\frac{V}{3}=5$$

$$V\left[\frac{1}{5}+\frac{1}{3}\right]=5+4$$

$$V=9\times\frac{15}{8}=16.875\ \text{V}$$

The current passing through the 3 Ω resistor is equal to $V/3$,

i.e. $I=\dfrac{16.875}{3}=5.625\,\text{A}$

So the superposition theorem is verified.

Let us now examine the power responses.

Power dissipated in the 3 Ω resistor due to the voltage source acting alone

$$P_{20} = (I_2)^2R = (2.5)^2\, 3 = 18.75\ \text{W}$$

Power dissipated in the 3 Ω resistor due to the current source acting alone

$$P_5 = (I_5)^2R = (3.125)^2\, 3 = 29.29\ \text{W}$$

Power dissipated in the 3 Ω resistor when both the sources are acting simultaneously is given by

$$P = (5.625)^2 \times 3 = 94.92\ \text{W}$$

From the above results, the superposition of P_{20} and P_5 gives

$$P_{20} + P_5 = 48.04\ \text{W}$$

which is not equal to $P = 94.92$ W

We can, therefore, state that the superposition theorem is not valid for power responses. It is applicable only for computing voltage and current responses.

EXAMPLE 3.3

Find the voltage across the 2 V resistor in Fig. 3.11 by using the superposition theorem.

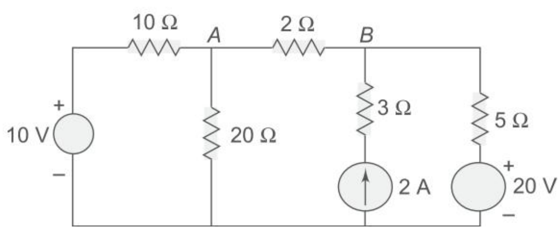


Fig. 3.11

Solution Let us find the voltage across the 2 Ω resistor due to individual sources. The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor.

Our first step is to find the voltage across the 2 Ω resistor due to the 10 V source, while other sources are set equal to zero.

The circuit is redrawn as shown in Fig. 3.12 (a).

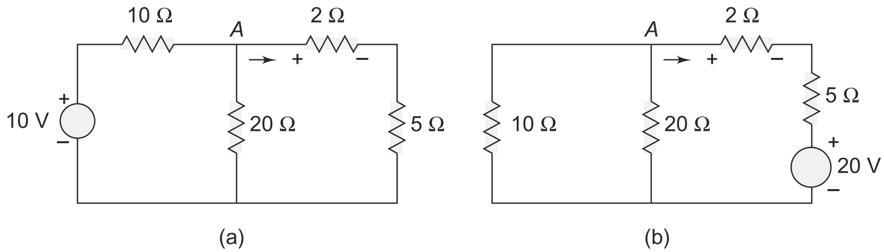


Fig. 3.12

Assuming a voltage V at the node 'A' as shown in Fig. 3.12 (a), the current equation is

$$\frac{V-10}{10}+\frac{V}{20}+\frac{V}{7}=0$$

$$V\,[0.1 + 0.05 + 0.143] = 1$$

$$\text{or}\quad V = 3.41\text{ V}$$

The voltage across the 2 Ω resistor due to the 10 V source is

$$V_2=\frac{V}{7}\times 2=0.97\text{ V}$$

Our second step is to find out the voltage across the 2 Ω resistor due to the 20 V source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (b).

Assuming voltage V at the node A as shown in Fig. 3.12 (b), the current equation is

$$\frac{V-20}{7}+\frac{V}{20}+\frac{V}{10}=0$$

$$V\,[0.143 + 0.05 + 0.1] = 2.86$$

$$\text{or}\quad V=\frac{2.86}{0.293}=9.76\text{ V}$$

The voltage across the 2 Ω resistor due to the 20 V source is

$$V_2=\left(\frac{V-20}{7}\right)\times 2=-2.92\text{ V}$$

The last step is to find the voltage across the 2 Ω resistor due to the 2 A current source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 3.12 (c).

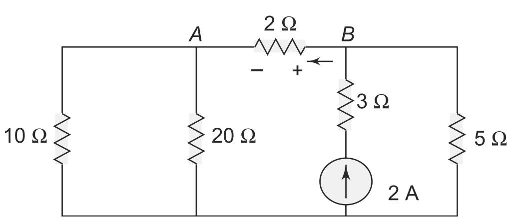


Fig. 3.12 (c)

$$\text{The current in the } 2\ \Omega \text{ resistor} = 2 \times \frac{5}{5+8.67}$$

$$= \frac{10}{13.67} = 0.73\,\text{A}$$

The voltage across the 2 Ω resistor = 0.73 × 2 = 1.46 V

The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor in the network

$V = 0.97 - 2.92 - 1.46 = -3.41$ V

The negative sign of the voltage indicates that the voltage at 'A' is negative.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆☆★**3-2.1** Find the current I in the circuit shown in Fig. Q.1 by using the superposition theorem.

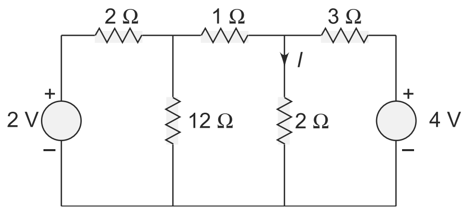


Fig. Q.1

☆★★**3-2.2** Find the current in the 4 Ω resistor of the circuit shown in Fig. Q.2 by using the superposition theorem.

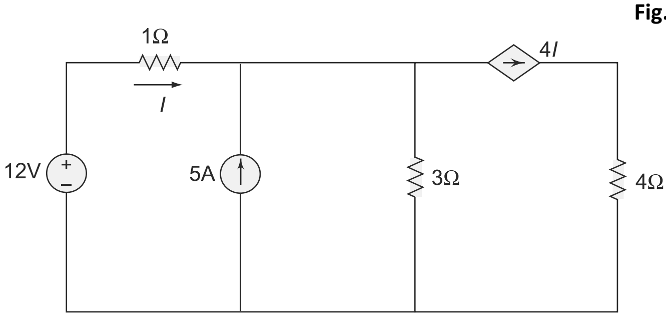


Fig. Q.2

☆★★**3-2.3** Calculate the new current in the circuit shown in Fig. Q.3 when the resistor R_3 is increased by 30%.

★★★**3-2.4** The circuit shown in Fig. Q.4 consists of dependent source. Use the superposition theorem to find the current I in the 3 Ω resistor.

☆★★**3-2.5** Obtain the current passing through the 2 Ω resistor in the circuit shown in Fig. Q.5 by using the superposition theorem.

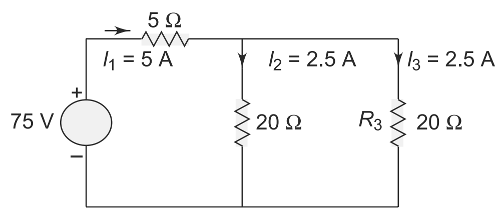


Fig. Q.3

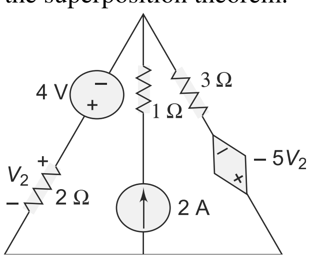


Fig. Q.4

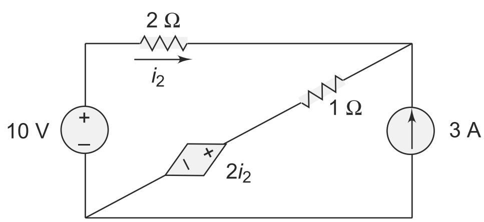


Fig. Q.5

☆★★**3-2.6** Using PSPICE, determine V_0 in the following circuit, shown in Fig. Q.6, using the superposition principle. Calculate the power delivered to the 3 Ω resistor.

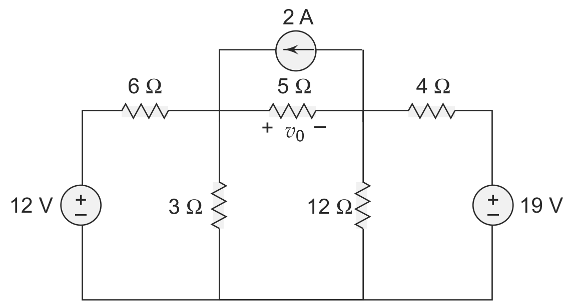


Fig. Q.6

Frequently Asked Questions linked to LO 2

☆☆★**3-2.1** Calculate the current in the 4 Ω resistor of Fig. Q.1 using the superposition theorem. **[AU May/June 2014]**

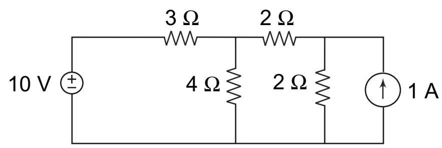


Fig. Q.1

☆☆★**3-2.2** Find the current through various branches of the circuit shown in Fig. Q.2, by employing the superposition theorem. **[AU Nov./Dec. 2012]**

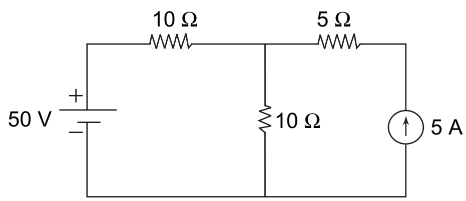


Fig. Q.2

☆★★**3-2.3** Determine the voltage across the 20 Ω resistance in the circuit shown in Fig. Q.3, using the superposition theorem. **[AU April/May 2011]**

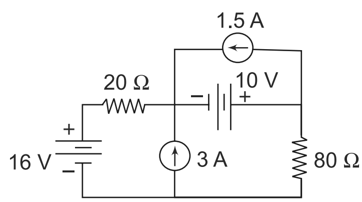


Fig. Q.3

☆☆★**3-2.4** Find the voltage across the 1 k Ω resistor in the circuit shown in Fig. Q.4 using the superposition theorem. [GTU **Dec. 2010]**

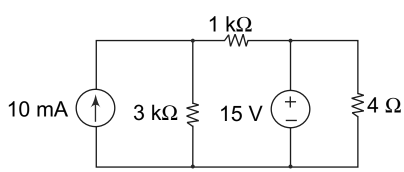


Fig. Q.4

☆★★**3-2.5** Determine the node voltage V_1 and V_2 in the network shown in Fig. Q.5 by applying the superposition theorem. **[GTU Dec. 2010]**

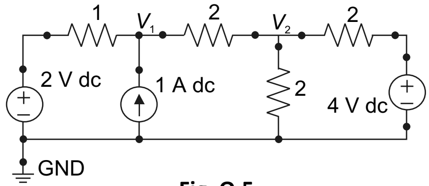


Fig. Q.5

☆☆★**3-2.6** State and explain the superposition theorem. Hence, using this, find V_{ab} in Fig. Q.6. **[GTU Dec. 2012]**

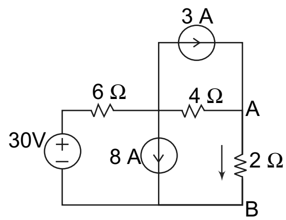


Fig. Q.6

☆☆★**3-2.7** State the superposition theorem. Explain an example. **[PTU 2011-2012]**

★★★**3-2.8** Find the current through R_L = 7.5 Ω using superposition theorem in the network shown in Fig. Q.8. **[PU 2010]**

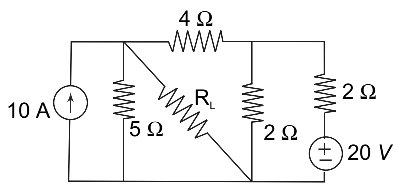


Fig. Q.8

3.3 THEVENIN'S THEOREM

> **LO 3** Explain Thevenin's theorem and apply it to solve the networks

In many practical applications, it is always not necessary to analyse the complete circuit; it requires that the voltage, current, or power in only one resistance of a circuit be found. The use of this theorem provides a simple, equivalent circuit which can be substituted for the original network. ***Thevenin's theorem*** *states that any two terminal linear network having a number of voltage current sources and resistances can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance, where the value of the voltage source is equal to the open-circuit voltage across the two terminals of the network, and resistance is equal to the equivalent resistance measured between the terminals with all the energy sources are replaced by their internal resistances.* According to Thevenin's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.13.

In the circuit, if the 24 Ω load resistance is connected to Thevenin's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experienced in the original circuit. To verify this, let us find the current passing through the 24 Ω resistance due to the original circuit.

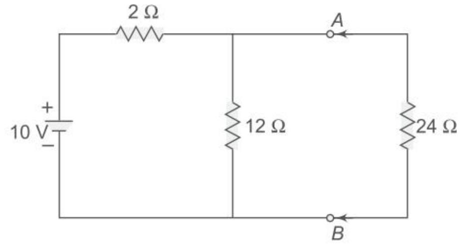


Fig. 3.13

$$I_{24} = I_T \times \frac{12}{12+24}$$

where $I_T = \dfrac{10}{2+(12\|24)} = \dfrac{10}{10} = 1 \text{ A}$

$$\therefore \quad I_{24} = 1 \times \frac{12}{12+24} = 0.33 \text{ A}$$

The voltage across the 24 Ω resistor = 0.33 × 24 = 7.92 V.

Now let us find Thevenin's equivalent circuit.

The Thevenin voltage is equal to the open-circuit voltage across the terminals 'AB', i.e. the voltage across the 12 Ω resistor. When the load resistance is disconnected from the circuit, the Thevenin voltage

$$V_{Th} = 10 \times \frac{12}{14} = 8.57 \text{ V}$$

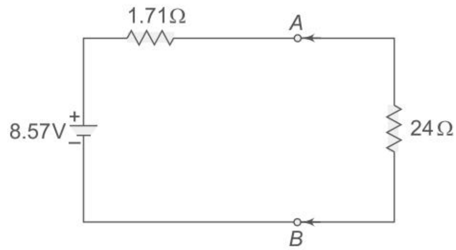


Fig. 3.14

The resistance into the open-circuit terminals is equal to the Thevenin resistance

$$R_{Th} = \frac{12 \times 2}{14} = 1.71 \ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.14.

Now let us find the current passing through the 24 Ω resistance and voltage across it due to Thevenin's equivalent circuit.

$$I_{24} = \frac{8.57}{24+1.71} = 0.33 \text{ A}$$

The voltage across the 24 Ω resistance is equal to 7.92 V. Thus, it is proved that R_L (= 24 Ω) has the same values of current and voltage in both the original circuit and Thevenin's equivalent circuit.

EXAMPLE 3.4

Determine the Thevenin's equivalent circuit across 'AB' for the given circuit shown in Fig. 3.15.

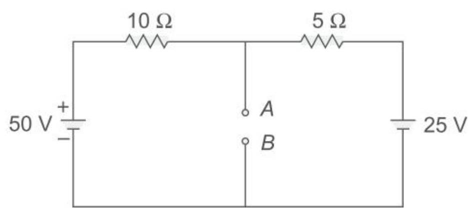


Fig. 3.15

Solution The complete circuit can be replaced by a voltage source in series with a resistance as shown in Fig. 3.16 (a)

where V_{Th} is the voltage across terminals *AB*, and

R_{Th} is the resistance seen into the terminals *AB*.

To solve for V_{Th}, we have to find the voltage drops around the closed path as shown in Fig. 3.16 (b).

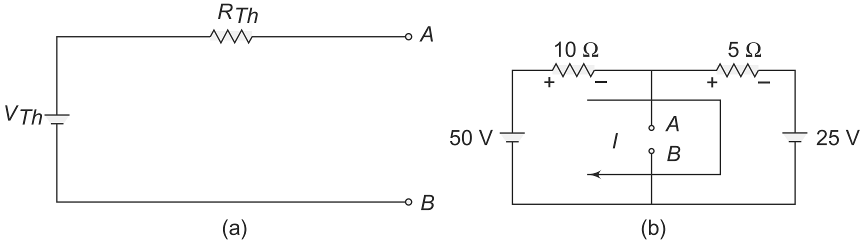


Fig. 3.16

We have $50 - 25 = 10I + 5I$

or $15I = 25$

$$\therefore \quad I = \frac{25}{15} = 1.67\text{ A}$$

Voltage across 10 Ω = 16.7 V

Voltage drop across 5 Ω = 8.35 V

or $V_{Th} = V_{AB} = 50 - V_{10}$

$= 50 - 16.7 = 33.3$ V

To find R_{Th}, the two voltage sources are removed and replaced with short circuit. The resistance at terminals AB then is the parallel combination of the 10 Ω resistor and 5 Ω resistor; or

$$R_{Th} = \frac{10 \times 5}{15} = 3.33\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.16 (c).

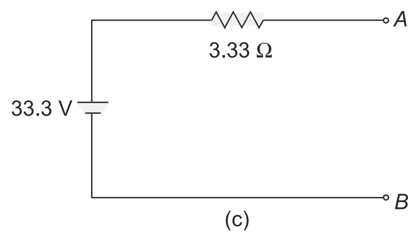


Fig. 3.16 (c)

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★**3-3.1** Determine the current passing through the 2 Ω resistor by using Thevenin's theorem in the circuit shown in Fig. Q.1.

★★★**3-3.2** Find Thevenin's equivalent circuit for the network shown in Fig. Q.2 and hence, find the current passing through the 10 Ω resistor.

☆★★**3-3.3** Find the Thevenin equivalent circuit of the circuit seen from AB using PSPICE (Fig. Q.3).

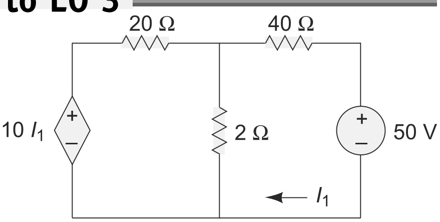


Fig. Q.1

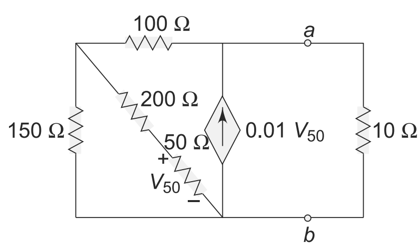


Fig. Q.2

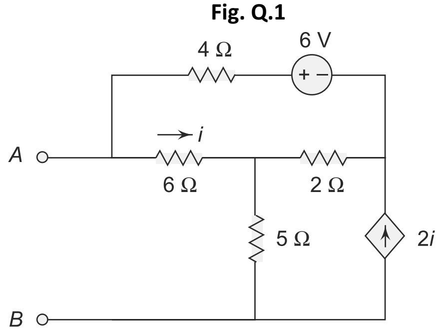


Fig. Q.3

Frequently Asked Questions linked to LO 3

☆☆★**3-3.1** Why do you short circuit the voltage source and open the current source when you find Thevenin's voltage of a network? **[AU May/Jnue 2014]**

☆★★**3-3.2** For the circuit shown in Fig. Q.2 using Thevenin's theorem, the current in the 10-ohm resistor. **[AU May/Jnue 2014]**

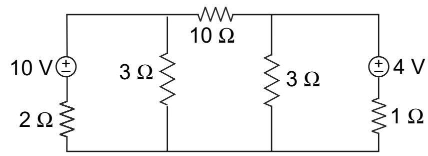


Fig. Q.2

☆☆★**3-3.3** State Thevenin's theorem. **[AU April/May 2011]**

☆★★**3-3.4** State and prove Thevenin's theorem, find R_{th} and V_{th} for the network shown in Fig. Q.4. **[GTU Dec. 2012]**

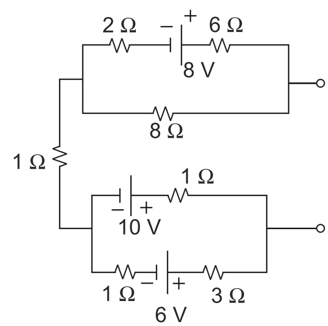


Fig. Q.4

☆★★**3-3.5** Obtain Thevenin's equivalent circuit (Q.5). Hence, find the current flowing through the 10Ω load. **[MU 2014]**

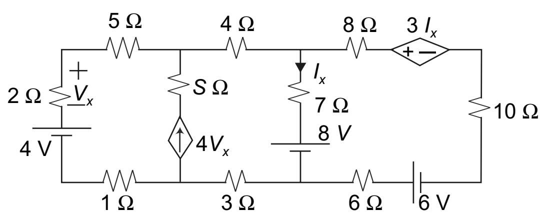


Fig. Q.5

☆★★**3-3.6** Obtain the Thevenin's equivalent circuit for the network shown in Fig. Q.6. **[PTU 2009-10]**

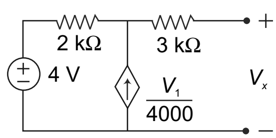


Fig. Q.6

☆☆★**3-3.7** State Thevenin's theorem and find the current through the 4 Ω resistor using Thevenin's theorem for the circuit shown in Fig. Q.7. **[PU 2010]**

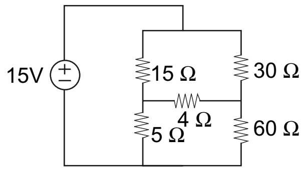


Fig. Q.7

☆☆★**3-3.8** State and prove Thevenin's theorem. Show with an example how theorem can be usefully employed in circuit analysis.

☆★★**3-3.9** Draw the Thevenin's equivalent of the circuit shown in Fig. Q.9. **[RGTU June 2014]**

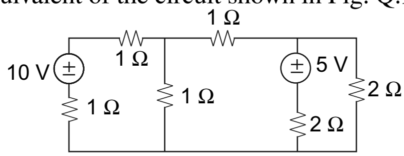


Fig. Q.9

3.4 NORTON'S THEOREM

LO 4 Explain Norton's theorem and apply it to solve the networks

Another method of analysing the circuit is given by Norton's theorem, which states that any two terminal linear network with current sources, voltage sources and resistances can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance. The value of the current source is the short-circuit current between the two terminals of the network and the resistance is the equivalent resistance measured between the terminals of the network with all the energy sources are replaced by their internal resistance.

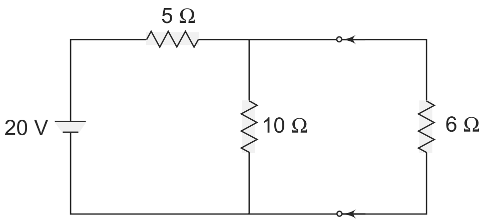


Fig. 3.17

According to Norton's theorem, an equivalent circuit can be found to replace the circuit in Fig. 3.17.

In the circuit, if the load resistance of 6 Ω is connected to Norton's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experiences in the original circuit. To verify this, let us find the current passing through the 6 Ω resistor due to the original circuit.

$$I_6 = I_T \times \frac{10}{10+6}$$

$$\text{where} \quad I_T = \frac{20}{5+(10\|6)} = 2.285\,\text{A}$$

$$\therefore \quad I_6 = 2.285 \times \frac{10}{16} = 1.43\,\text{A}$$

i.e. the voltage across the 6 Ω resistor is 8.58 V. Now let us find Norton's equivalent circuit. The magnitude of the current in the Norton's equivalent circuit is equal to the current passing through short-circuited terminals as shown in Fig. 3.18.

$$\text{Here,} \quad I_N = \frac{20}{5} = 4\,\text{A}$$

Norton's resistance is equal to the parallel combination of both the 5 Ω and 10 Ω resistors.

$$R_N = \frac{5\times10}{15} = 3.33\ \Omega$$

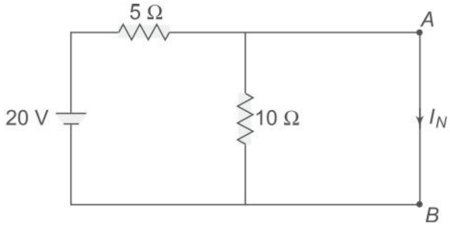


Fig. 3.18

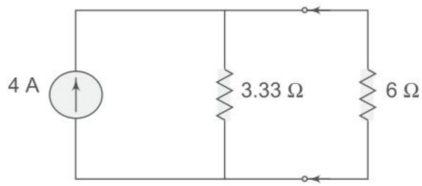


Fig. 3.19

The Norton's equivalent source is shown in Fig. 3.19.

Now let us find the current passing through the 6 Ω resistor and the voltage across it due to Norton's equivalent circuit.

$$I_6 = 4\times\frac{3.33}{6+3.33} = 1.43\ \text{A}$$

The voltage across the 6 Ω resistor = 1.43 × 6 = 8.58 V

Thus, it is proved that R_L (= 6 Ω) has the same values of current and voltage in both the original circuit and Norton's equivalent circuit.

EXAMPLE 3.5

Determine Norton's equivalent circuit at terminals AB for the circuit shown in Fig. 3.20.

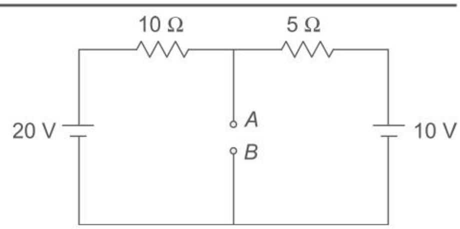


Fig. 3.20

Solution The complete circuit can be replaced by a current source in parallel with a single resistor as shown in Fig. 3.21 (a), where I_N is the current passing through the short circuited output terminals *AB* and R_N is the resistance as seen into the output terminals.

To solve for I_N, we have to find the current passing through the terminals *AB* as shown in Fig. 3.21 (b).

From Fig. 3.21 (b), the current passing through the terminals *AB* is 4 A. The resistance at terminals *AB* is the parallel combination of the 10 Ω resistor and the 5 Ω resistor,

$$\text{or} \quad R_N = \frac{10\times5}{10+5} = 3.33\ \Omega$$

Norton's equivalent circuit is shown in Fig. 3.21 (c).

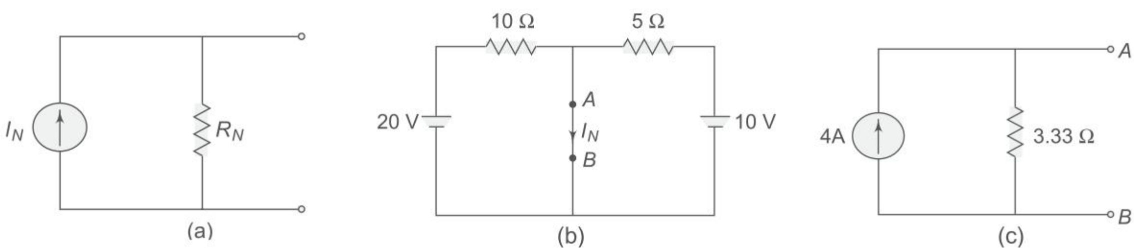


Fig. 3.21

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★**3-4.1** Find the Thevenin's and Norton's equivalent circuits across terminals *ab* for the given circuit shown in Fig. Q.1.

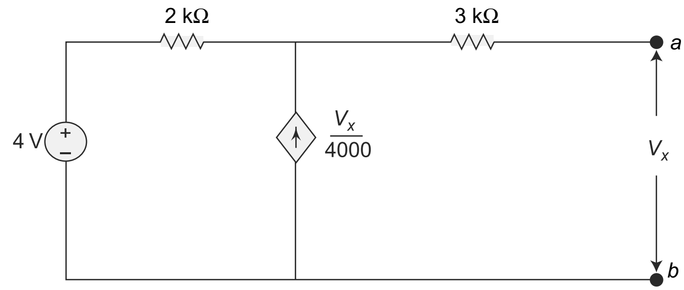


Fig. Q.1

☆★★**3-4.2** Obtain Norton's equivalent circuit of the network shown in Fig. Q.2.

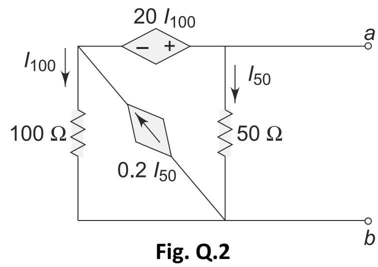


Fig. Q.2

☆☆★**3-4.3** Using PSPICE, for the circuit shown in Fig. Q.3 obtain Norton's equivalent or viewed from terminals

(a) $a-b$ (b) $c-d$

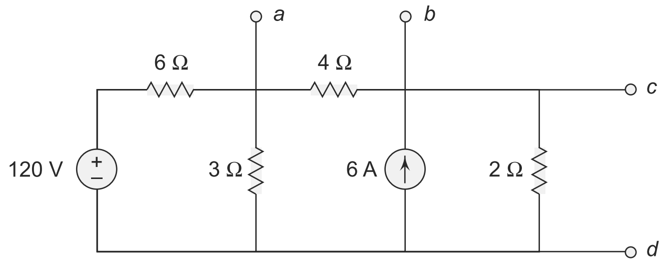


Fig. Q.3

Frequently Asked Questions linked to LO 4

☆★★**3-4.1** Find the voltage drop across the 12 Ω resistance using Norton's for the circuit shown in Q 1.
[AU April/May 2011]

☆☆★**3-4.2** Explain in brief about source transform and Find Norton's equivalent circuit for the network shown in Fig. Q.2 and obtain current in the 10 Ω resistor. **[GTU Dec. 2010]**

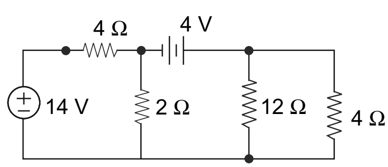


Fig. Q.1

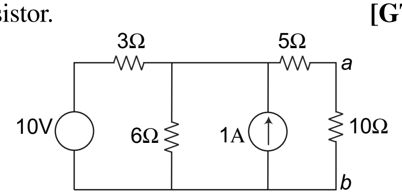


Fig. Q.2

☆★★**3-4.3** In Fig. Q.3 if the 2 V source is replaced by an open circuit then find Thevenin's and Norton's equivalent circuit across V_2 and V_3. Resistance values are in ohms. **[GTU May 2011]**

☆★★**3-4.4** In the circuit shown in Fig. Q.4, find the current through RL connected across A-B terminals by utilising Thevenin's theorem. Verify the results by Norton's theorem. **[JNTU Nov. 2012]**

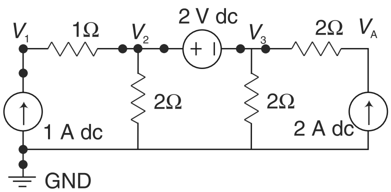


Fig. Q.3

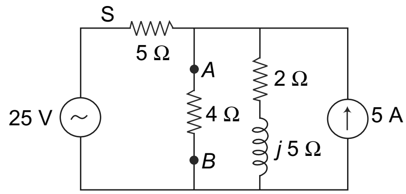


Fig. Q.4

★★★**3-4.5** Determine the current through the 1 Ω resistor if connected across *AB* in the network shown in Fig. Q.5 using Norton's theorem. **[PU 2010]**

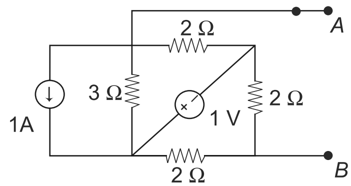


Fig. Q.5

☆★★**3-4.6** Obtain the Thevenin's and Norton's at the terminals *A-B* for the circuit shown in Fig. Q.6. **[PU 2012]**

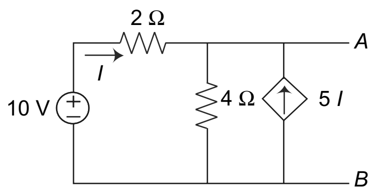


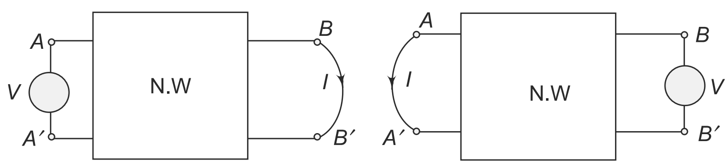
Fig. Q.6

3.5 RECIPROCITY THEOREM

In any linear bilateral network, if a single voltage source V_a in branch 'a' produces a current I_b in branch 'b', then if the voltage source V_a is removed and inserted in branch 'b' will produce a current I_b in branch 'a'. The ratio of response to excitation is same for the two conditions mentioned above. This is called the ***reciprocity theorem****.*

> **LO 5** Explain the reciprocity theorem and apply it to solve the networks

Consider the network shown in Fig. 3.22. *AA*′ denotes input terminals and *BB*′ denotes output terminals.



(a) (b)

Fig. 3.22

The application of voltage V across AA' produces current I at BB'. Now if the positions of the source and responses are interchanged, by connecting the voltage source across BB', the resultant current I will be at terminals AA'. According to the reciprocity theorem, the ratio of response to excitation is the same in both cases.

EXAMPLE 3.6

Verify the reciprocity theorem for the network shown in Fig. 3.23.

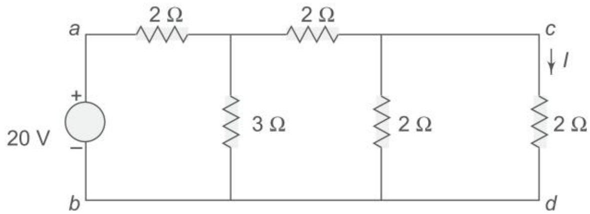


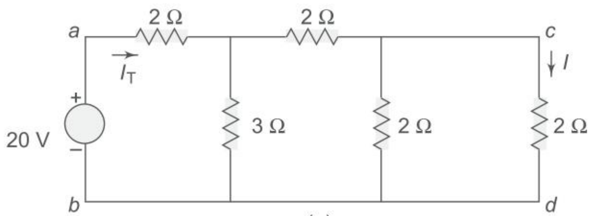
Fig. 3.23

Solution Total resistance in the circuit $= 2 + [3 \parallel (2 + 2 \parallel 2)] = 3.5\ \Omega$

The current drawn by the circuit (See Fig. 3.24 (a)).

$$I_T = \frac{20}{3.5} = 5.71\ \Omega$$

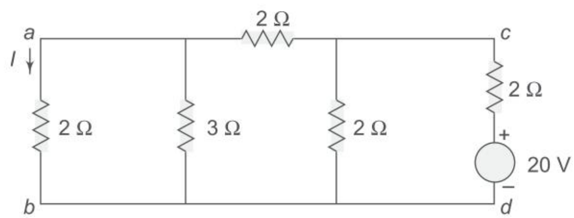
The current in the 2 Ω branch cd is $I = 1.43$ A.



(a)

Fig. 3.24

Applying the reciprocity theorem, by interchanging the source and response, we get Fig. 3.24 (b).



(b)

Fig. 3.24

Total resistance in the circuit $= 3.23\ \Omega$.

Total current drawn by the circuit $= \dfrac{20}{3.23} = 6.19$ A

The current in the branch ab is $I = 1.43$ A

If we compare the results in both cases, the ratio of input to response is the same, i.e. $(20/1.43) = 13.99$

Frequently Asked Questions linked to LO 5

☆☆★**3-5.1** What is the reciprocity theorem? **[AU May/June 2014]**

☆★★**3-5.2** Verify the reciprocity theorem for the circuit shown below. **[AU Nov./Dec. 2012]**

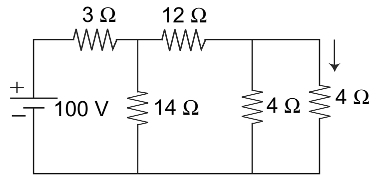


Fig. Q.2

3.6 COMPENSATION THEOREM

> **LO 6** Explain the compensation theorem and apply it to solve the networks

The ***compensation theorem*** *states that any element in the linear, bilateral network, may be replaced by a voltage source of magnitude equal to the current passing through the element multiplied by the value of the element, provided the currents and voltages in other parts of the circuit remain unaltered.* Consider the circuit shown in Fig. 3.25 (a). The element R can be replaced by voltage source V, which is equal to the current I passing through R multiplied by R as shown in Fig. 3.25 (b).

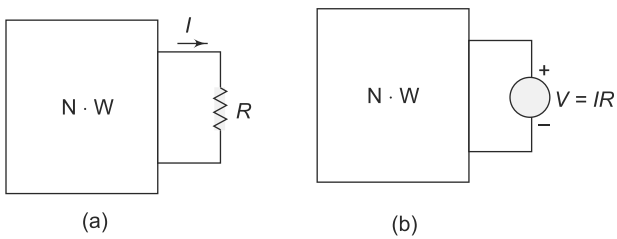


Fig. 3.25

This theorem is useful in finding the changes in current or voltage when the value of resistance is changed in the circuit. Consider the network containing a resistance R shown in Fig. 3.26 (a). A small change in resistance R, that is $(R + \Delta R)$, as shown in Fig. 3.26 (b), causes a change in current in all branches. This current increment in other branches is equal to the current produced by the voltage source of voltage $I.\,\Delta R$ which is placed in series with altered resistance as shown in Fig. 3.26 (c).

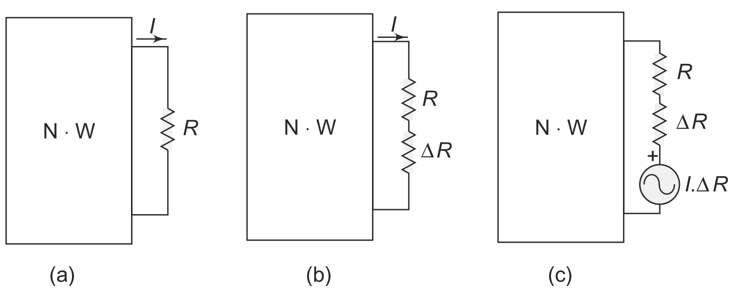


Fig. 3.26

EXAMPLE 3.7

Determine the current flowing in the ammeter having 1 Ω internal resistance connected in series with a 3 Ω resistor as shown in Fig. 3.27.

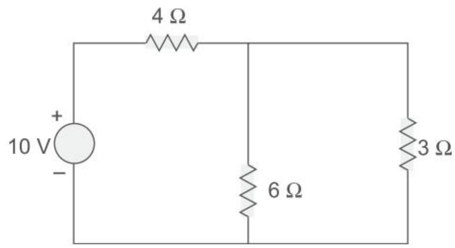


Fig. 3.27

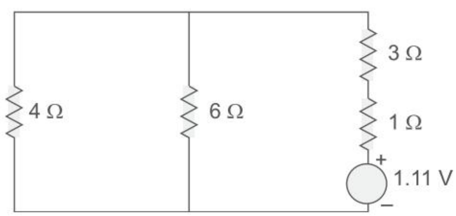


Fig. 3.28

Solution The current flowing through the 3 Ω branch is $I_3 = 1.11$ A. If we connect the ammeter having 1 Ω resistance to the 3 Ω branch, there is a change in resistance. The changes in currents in other branches then result as if a voltage source of voltage $I_3 \Delta R = 1.11 \times 1 = 1.11$ V is inserted in the 3 Ω branch as shown in Fig. 3.28.

Current due to this 1.11 V source is calculated as follows.

Current $I_3' = 0.17$ A

This current is opposite to the current I_3 in the 3 Ω branch.

Hence the ammeter reading = (1.11 – 0.17) = 0.94 A.

Frequently Asked Questions linked to LO 6

☆★★**3-6.1** In Fig. Q.1, if 1 ohm resistance is changed to 1.2 ohms then determine the source-voltage for compensating for the change. **[GTU May 2014]**

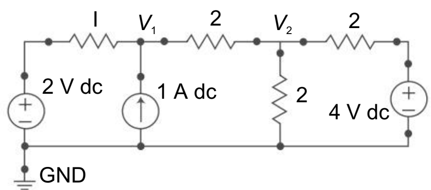


Fig. Q.1

3.7 MAXIMUM POWER TRANSFER THEOREM

LO 7 State the maximum power transfer theorem and apply it to solve the networks

Many circuits basically consist of sources, supplying voltage, current, or power to the load; for example, a radio speaker system, or a microphone supplying the input signals to voltage pre-amplifiers. Sometimes it is necessary to transfer maximum voltage, current or power from the source to the load. In the simple resistive circuit shown in Fig. 3.29, R_s is the source resistance. Our aim is to find the necessary conditions so that the power delivered by the source to the load is maximum.

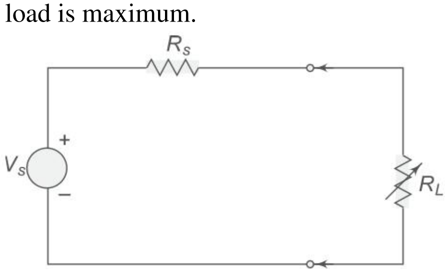


Fig. 3.29

It is a fact that more voltage is delivered to the load when the load resistance is high as compared to the resistance of the source. On the other hand, maximum current is transferred to the load when the load resistance is small compared to the source resistance.

For many applications, an important consideration is the maximum power transfer to the load; for example, maximum power transfer is desirable from the output amplifier to the speaker of an audio sound system. The maximum power transfer theorem states that maximum power is delivered from a source to a load when the load resistance is equal to the source resistance. In Fig. 3.29, assume that the load resistance is variable.

Current in the circuit is $I = V_S/(R_S + R_L)$

Power delivered to the load R_L is $P = I^2R_L = V_S^2 R_L/(R_S + R_L)^2$

To determine the value of R_L for maximum power to be transferred to the load, we have to set the first derivative of the above equation with respect to R_L, i.e. when $\dfrac{dP}{dR_L}$ equals zero.

$$\frac{dP}{dR_L} = \frac{d}{dR_L}\left[\frac{V_S^2}{(R_S+R_L)^2}R_L\right]$$

$$= \frac{V_S^2\left\{(R_S+R_L)^2-(2R_L)(R_S+R_L)\right\}}{(R_S+R_L)^4}$$

$$\therefore \quad (R_S + R_L)^2 - 2R_L(R_S + R_L) = 0$$

$$R^2_S + R^2_L + 2R_SR_L - 2R^2_L - 2R_SR_L = 0$$

$$\therefore \quad R_S = R_L$$

So, maximum power will be transferred to the load when load resistance is equal to the source resistance.

EXAMPLE 3.8

In the circuit shown in Fig. 3.30, determine the value of load resistance when the load resistance draws maximum power. Also find the value of the maximum power.

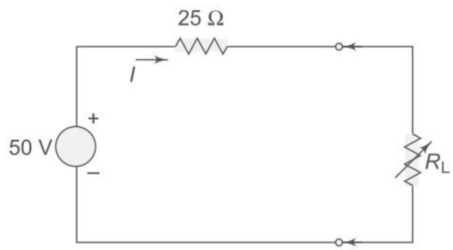


Fig. 3.30

Solution In Fig. 3.30, the source delivers the maximum power when load resistance is equal to the source resistance.

$$R_L = 25\ \Omega$$

The current $I = 50/(25 + R_L) = 50/50 = 1$ A

The maximum power delivered to the load $P = I^2R_L$

$$= 1 \times 25 = 25 \text{ W}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆★★**3-7.1** For the circuit shown in Fig. Q.1, what will be the value of R_L to get the maximum power? What is the maximum power delivered to the load? What is the maximum voltage across the load? What is the maximum current in it?

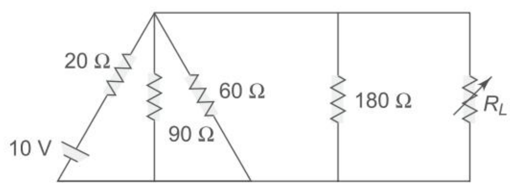


Fig. Q.1

☆☆★**3-7.2** For the circuit shown in Fig. Q.2, determine the value of R_L to get the maximum power. Also find the maximum power transferred to the load.

☆☆★**3-7.3** Using PSPICE, find the maximum power transferred to the resistor R in the following circuit (Fig. Q.3).

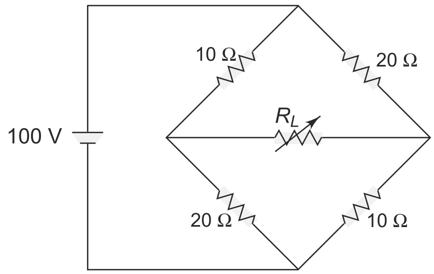


Fig. Q.2

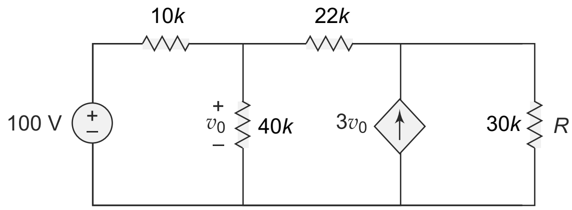


Fig. Q.3

Frequently Asked Questions linked to LO 7

☆☆★**3-7.1** State the maximum power transfer theorem for ac circuits. **[AU May/June 2013]**

☆☆★**3-7.2** In the circuit of Fig. Q.2, find the value if R for maximum power transfer. Also calculate the maximum power. **[AU May/June 2014]**

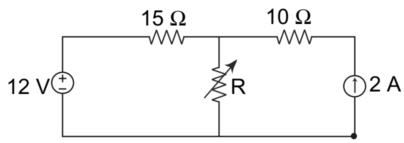


Fig. Q.2

☆☆★**3-7.3** Obtain Thevenin's equivalent circuit for the network shown Fig. Q.3, find the power dissipated in R_L = 5 Ω resistor. Find R_L for maximum power transfer from the source and compute maximum power that can be transferred, i.e. Pmaxi. **[GTU Dec. 2010]**

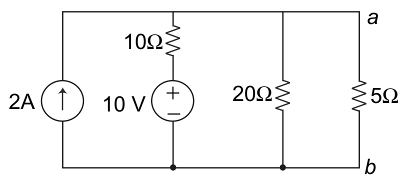


Fig. Q.3

☆☆★**3-7.4** Prove the maximum power transfer theorem for a practical voltage source (V_S). What is the maximum power that can be delivered V_s = 20 V and Rs = 1 ohm? **[GTU May 2011]**

☆☆★**3-7.5** Obtain the equivalent circuit *A*-*B* terminals in Fig. Q.5 and find the value of Z_L to have maximum power. **[JNTU Nov. 2012]**

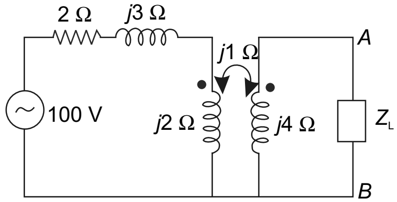


Fig. Q.5

★★★**3-7.6** For the circuit shown in Fig. Q.6, find the value of R that will receive maximum power. Determine this maximum power. **[PU 2012]**

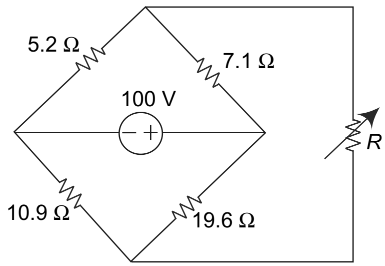


Fig. Q.6

3.8 DUALS AND DUALITY

LO 8 Explain the concept of duals and the principle of duality

In an electrical circuit itself there are pairs of terms which can be interchanged to get new circuits. Such pair of dual terms are given below.

Current — Voltage
Open — Short
L — C
R — G
Series — Parallel
Voltage source — Current source
KCL — KVL

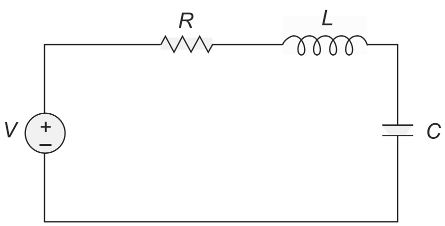


Fig. 3.31

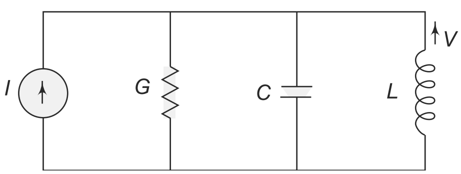


Fig. 3.32

Consider a network containing *R-L-C* elements connected in series, and excited by a voltage source as shown in Fig. 3.31.

The integrodifferential equation for the above network is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = \mathrm{V}$$

Similarly, consider a network containing *R-L-C* elements connected in parallel and driven by a current source as shown in Fig. 3.32.

The integrodifferential equation for the network in Fig. 3.32 is

$$i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int v dt$$

If we observe both the equations, the solutions of these two equations are the same. These two networks are called *duals*.

To draw the dual of any network, the following steps are to be followed.

1. In each loop of a network place a node; and place an extra node, called the *reference node*, outside the network.
2. Draw the lines connecting adjacent nodes passing through each element, and also to the reference node, by placing the dual of each element in the line passing through original elements.

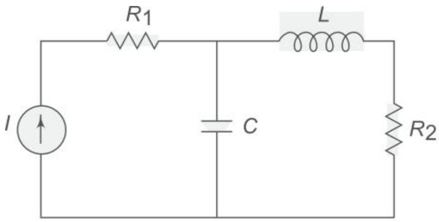


Fig. 3.33

For example, consider the network shown in Fig. 3.33.

Our first step is to place the nodes in each loop and a reference node outside the network.

Drawing the lines connecting the nodes passing through each element, and placing the dual of each element as shown in Fig. 3.34 (a) we get a new circuit as shown in Fig. 3.34 (b).

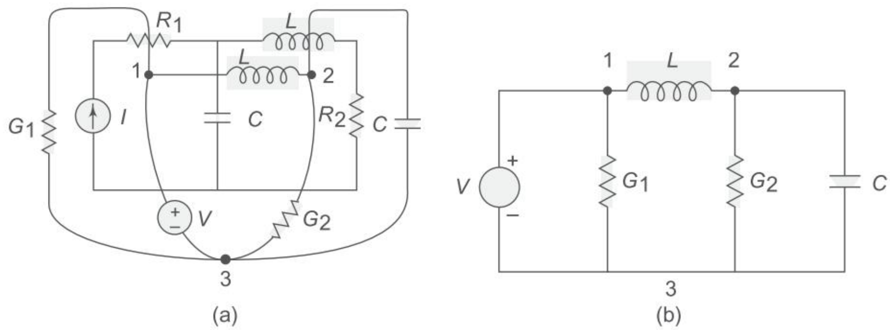


Fig. 3.34

EXAMPLE 3.9

Draw the dual network for the given network shown in Fig. 3.35.

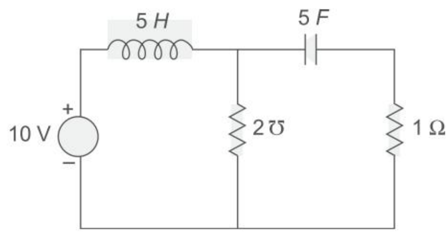


Fig. 3.35

Solution Place nodes in each loop and one reference node outside the circuit. Joining the nodes through each element, and placing the dual of each element in the line, we get the dual circuit as shown in Fig. 3.36 (a).

The dual circuit is redrawn as shown in Fig. 3.36 (b).

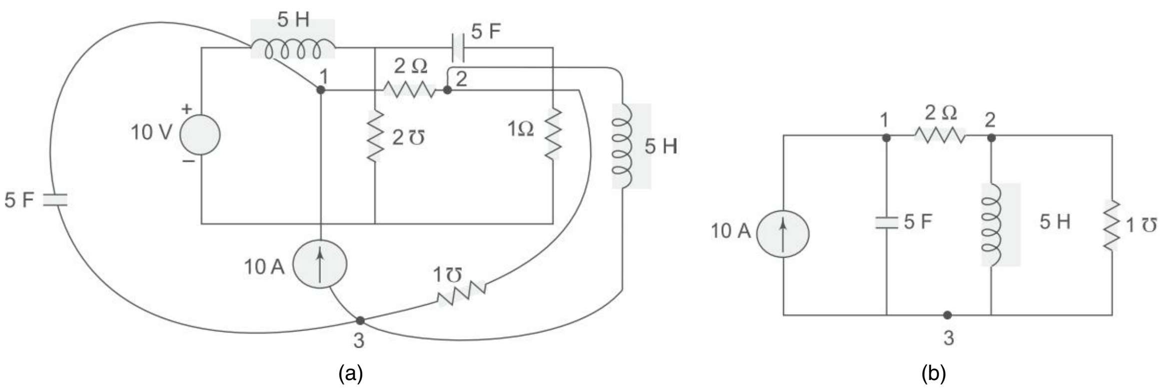


Fig. 3.36

Frequently Asked Questions linked to LO 8

☆☆★**3-8.1** Explain dual with reference to network. **[GTU Dec. 2014]**

☆☆★**3-8.2** Explain the principle of duality. Write a graphical procedure to draw a dual network? **[JNTU Nov. 2012]**

☆☆★**3-8.3** Draw the dual of the network shown in Fig. Q.3 and explain its procedure. **[JNTU Nov. 2012]**

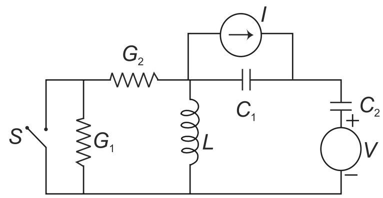


Fig. Q.3

☆★★**3-8.4** Explain the concept of duality. What relationship does duality have with the incidence matrix? **[PTU 2009-10]**

☆☆★**3.8.5** Find the dual of the following network shown in Fig. Q.5. **[PTU 2009-10]**

☆☆★**3.8.6** Draw the dual of the network shown in Fig. Q.6. **[RGTU Dec. 2013]**

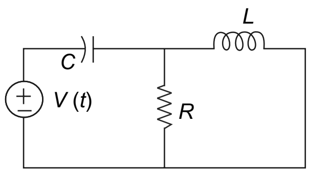


Fig. Q.5

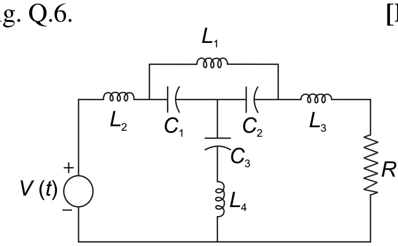


Fig. Q.6

3.9 TELLEGEN'S THEOREM

> **LO 9** Explain Tellegen's theorem and apply it to solve the networks

Tellegen's theorem is valid for any lumped network which may be linear or nonlinear, passive or active, time-varying or time-invarient. *This theorem states that in an arbitrary lumped network, the algebraic sum of the powers in all branches at any instant is zero.* All branch currents and voltages in that network must satisfy Kirchhoff's laws. Otherwise, in a given network, the algebraic sum of the powers delivered by all sources is equal to the algebraic sum of the powers absorbed by all elements. This theorem is based on Kirchhoff's two laws, but not on the type of circuit elements.

Consider two networks N_1 and N_2, having the same graph with different types of elements between the corresponding nodes.

$$\text{Then} \quad \sum_{K=1}^{b} v_{1K} i_{2K} = 0$$

$$\text{and} \quad \sum_{K=1}^{b} v_{2K} i_{1K} = 0$$

To verify Tellegen's theorem, consider two circuits having same graphs as shown in Fig. 3.37.

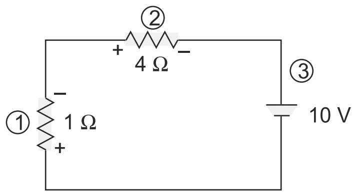


Fig. 3.37 (a)

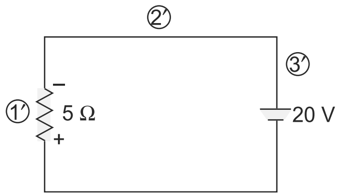


Fig. 3.37 (b)

In Fig. 3.37 (a),

$$i_1 = i_2 = 2\text{ A};\ i_3 = 2\text{ A}$$

and $v_1 = -2\text{ V},\ v_2 = -8\text{ V},\ v_3 = 10\text{ V}$

In Fig. 3.37 (b),

$$i_1^1 = i_2^1 = 4\text{ A}; i_3^1 = 4\text{ A}$$

and $v_1^1 = -20\text{ V}; v_2^1 = 0\text{ V}; v_3^1 = 20\text{ V}$

Now $$\sum_{K=1}^{3} v_K i_K^1 = v_1 i_1^1 + v_2 i_2^1 + v_3 i_3^1$$

$$= (-2)\,(4) + (-8)\,(4) + (10)\,(4) = 0$$

and $$\sum_{K=1}^{3} v_K^1 i_K = v_1^1 i_1 + v_2^1 i_2 + v_3^1 i_3$$

$$= (-20)\,(2) + (0)\,(2) + (20)\,(2) = 0$$

Similarly,

$$\sum_{K=1}^{3} v_K i_K = v_1 i_1 + v_2 i_2 + v_3 i_3$$

$$= (-2)\,(2) + (-8)\,(2) + (10)\,(2) = 0$$

and $$\sum_{K=1}^{3} v_K^1 i_K^1 = (-20)(4) + (0)(4) + (20)(4) = 0$$

This verifies Tellegen's theorem.

Frequently Asked Questions linked to LO 9

★★★**3-9.1** Verify Tellegen's theorem for the network shown in Fig. Q.1. **[PTU 2009-10]**

☆★★**3-9.2** Verify Tellegen's theorem for the pair of networks. **[PTU 2011-12]**

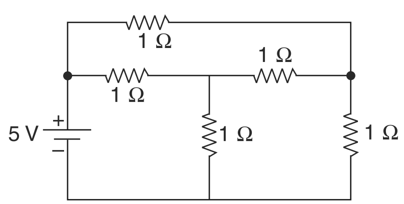


Fig. Q.1

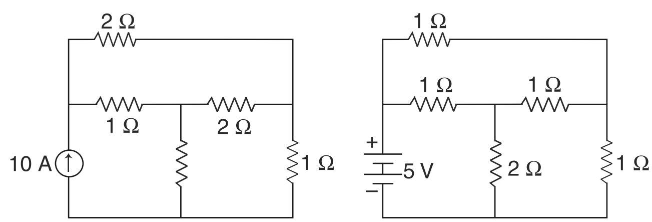


Fig. Q.2

3.10 MILLMAN'S THEOREM

Millman's theorem *states that in any network, if the voltage sources $V_1, V_2, \ldots, V_n$ in series with internal resistances $R_1, R_2, \ldots, R_n$, respectively, are in parallel, then these sources may be replaced by a single voltage source V' in series with* R' as shown in Fig. 3.38.

LO 10 State Millman's theorem and apply it for solving the networks

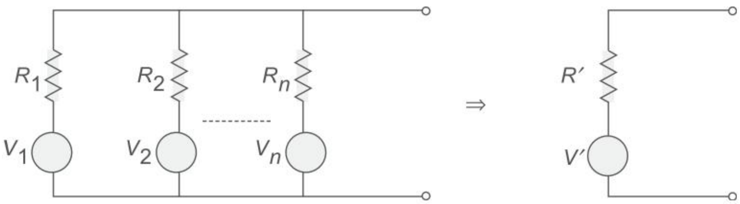


Fig. 3.38

where $V' = \dfrac{V_1G_1 + V_2G_2 + \cdots V_nG_n}{G_1 + G_2 + \cdots + G_n}$

Here, G_n is the conductance of the nth branch,

and $R' = \dfrac{1}{G_1 + G_2 + \cdots + G_n}$

A similar theorem can be stated for n current sources having internal conductances which can be replaced by a single current source I' in parallel with an equivalent conductance.

where $I' = \dfrac{I_1R_1 + I_2R_2 + \cdots I_nR_n}{R_1 + R_2 + \cdots + R_n}$

and $G' = \dfrac{1}{R_1 + R_2 + \cdots + R_n}$

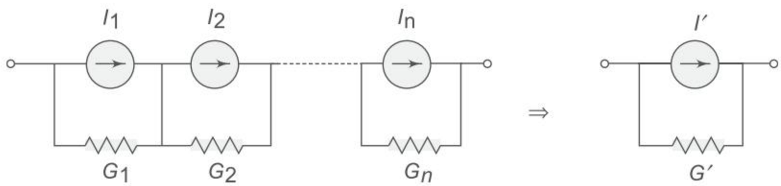


Fig. 3.39

EXAMPLE 3.10

Calculate the current I shown in Fig. 3.40 using Millman's Theorem.

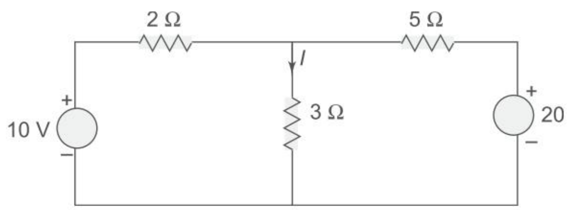


Fig. 3.40

Solution According to Millman's theorem, the two voltage sources can be replaced by a single voltage source in series with resistance as shown in Fig. 3.41.

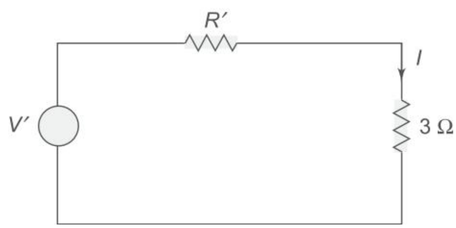


Fig. 3.41

We have $V' = \dfrac{V_1G_1 + V_2G_2}{G_1 + G_2}$

$$= \frac{[10(1/2) + 20(1/5)]}{1/2 + 1/5} = 12.86 \text{ V}$$

and $R' = \dfrac{1}{G_1 + G_2} = \dfrac{1}{1/2 + 1/5} = 1.43\,\Omega$

Therefore, the current passing through the 3 Ω resistor is

$$I = \frac{12.86}{3 + 1.43} = 2.9\ \text{A}$$

Frequently Asked Questions linked to LO 10

☆★★**3-10.1** State Millman's theorem. Obtain the equivalent of a parallel connection of three branches each with a voltage source and a series resistance (2 V, 1 ohm), (3 V, 2 ohms), (5 V, 2 ohms). **[GTU May 2011]**

☆★★**3-10.2** State and explain the following.

(a) Reciprocity theorem

(b) Millman's theorem **(RGTU Dec. 2013)**

Additional Solved Problems

PROBLEM 3.1

Calculate the voltage across AB in the network shown in Fig. 3.42 and indicate the polarity of the voltage using star-delta transformation.

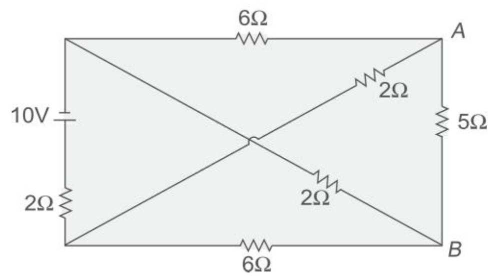


Fig. 3.42

Solution The circuit in Fig. 3.42 can be redrawn as shown in Fig. 3.43.

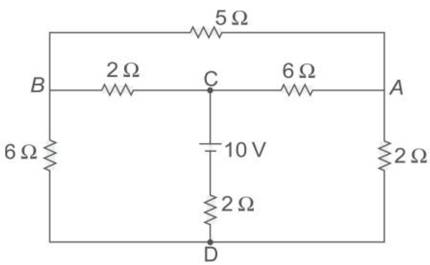


Fig. 3.43

By taking star-delta transformation at *ABC*, we have

The circuit shown in Fig. 3.44 is a single node circuit and *V* is the voltage at mode.

Applying nodal analysis, we have

$$\frac{V}{4.31} + \frac{V}{6.77} + \frac{V - 10}{2.92} = 0 \tag{3.8}$$

From Eq. (3.8), the voltage is $V = 4.744$ volts

The voltage across *AB* is $V_{AB} = V_A - V_B$

$$V_A = V \times \frac{2}{2 + 2.31} = 4.744 \times \frac{2}{2 + 2.31}$$
$$= 2.2 \text{ volts}$$

$$V_B = V \times \frac{6}{6 + 0.77} = 4.744 \times \frac{6}{6.77}$$
$$= 4.2 \text{ volts}$$

∴ the voltage across *AB* is $V_{AB} = V_A - V_B$

$$V_{AB} = 2.2 - 4.2 = -2 \text{ volts}$$

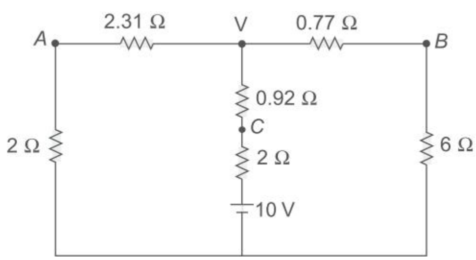


Fig. 3.44

PROBLEM 3.2

Determine the current I in the circuit shown in Fig. 3.45 using the superposition theorem.

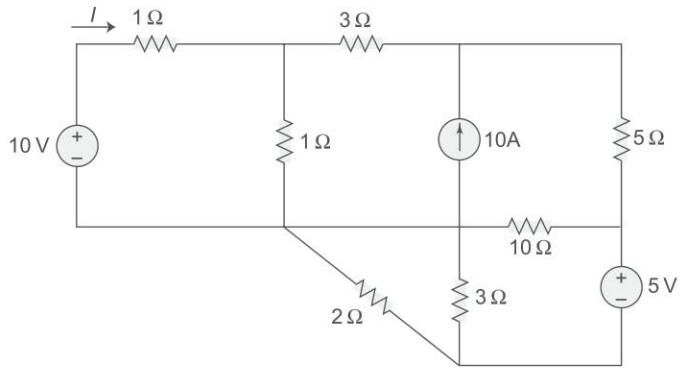


Fig. 3.45

Solution The current I' due to 10 V source, when other sources are zero, is shown in Fig. 3.46.

From the Fig. 3.46 (a), the resistances of $2\,\Omega$, $3\,\Omega$, and $10\,\Omega$ are in parallel and resultant is in series with $5\,\Omega$ and $3\,\Omega$. The equivalent circuit is shown in Fig. 3.46 (b).

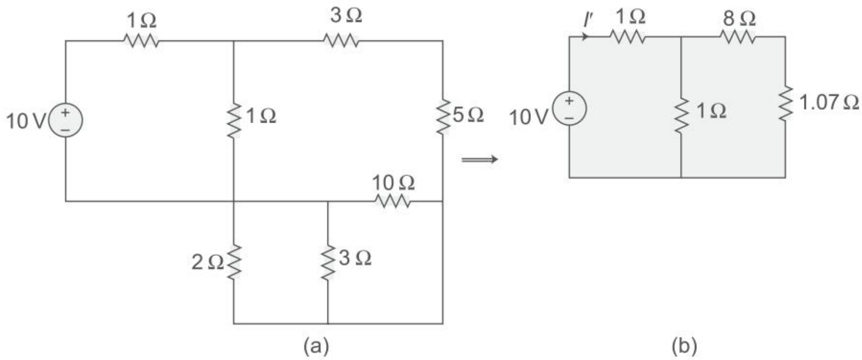


Fig. 3.46

The total equivalent resistance in Fig. 3.46 (b) is

$$R_{eq} = [(8 + 1.07)//1] + 1 = 1.9\,\Omega$$

The current passing through the $1\,\Omega$ resistance is

$$I' = \frac{10}{1.9} = 5.26\,\text{A}$$

The current I'' due to the 5 V source, when other sources are zero, is shown in Fig. 3.47.

The circuit in Fig. 3.47 (a) can be further reduced as shown in Fig. 3.47 (b).

Total current delivered by the 5 V source

$$I_T = \frac{5}{5.8} = 0.86\,\text{A}$$

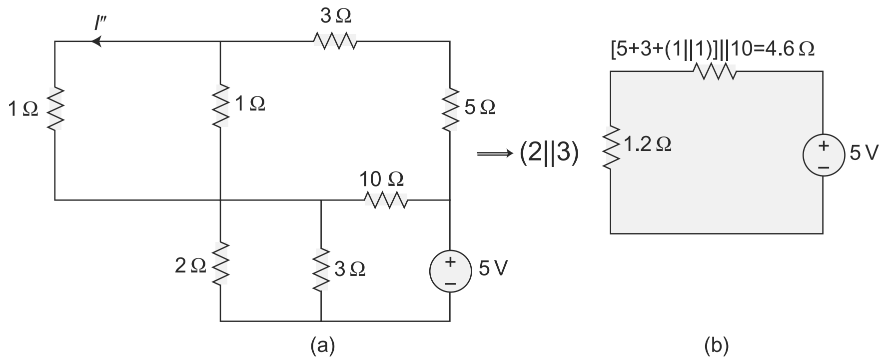


Fig. 3.47

The current passing through the 5 Ω resistor

$$I_5 = \frac{0.86 \times 10}{18.5} = 0.46\,\text{A}$$

The current passing through the 1 Ω resistor

$$I'' = \frac{0.46 \times 1}{2} = 0.23\,\text{A}$$

The current I''' due to the 10A source, when other sources are set to zero as shown in Fig. 3.48.

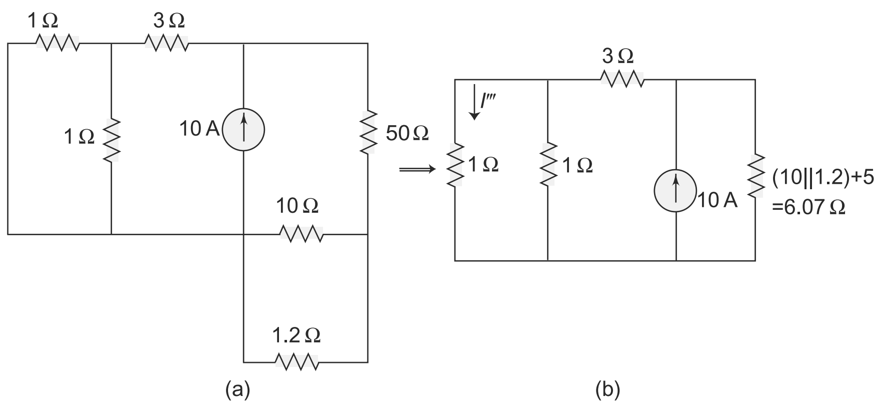


Fig. 3.48

The circuit in Fig. 3.48 (a) can be further simplified as shown in Fig. 3.48 (b).

The current passing through the 3 Ω resistance $= \dfrac{10 \times 6.07}{6.07 + 3.5} = 6.34\,\text{A}$

The current passing through the 1 Ω resistor is

$$I''' = 6.34 \times \frac{1}{2} = 3.17\,\text{A}$$

The current passing through the 1 Ω resistor is

$$\begin{aligned} I &= I' - I'' - I''' \\ &= 5.26 - 0.23 - 3.17 \\ &= 1.86\,\text{A} \end{aligned}$$

PROBLEM 3.3

Find the Thevenin's and Norton's equivalents for the circuit shown in Fig. 3.49 with respect to terminals ab.

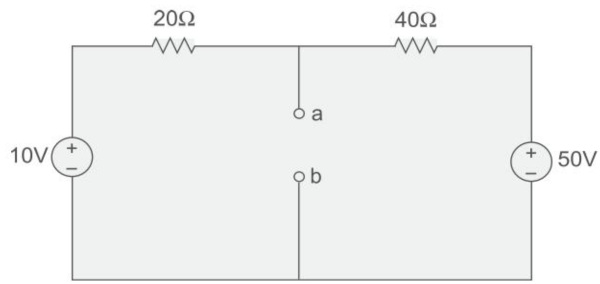


Fig. 3.49

Solution To find the Thevenin's resistance, voltage sources are to be short circuited.

Therefore, the resistance seen into the terminals *ab* is

$$R_{ab} = [20\Omega \| 40\Omega] = 13.33\ \Omega$$

$$\therefore \quad R_{\text{Th}} = R_N = R_{ab} = 13.33\ \Omega$$

In the circuit shown in Fig. 3.50 (a), the open circuit voltage across terminals

$$ab = V_{ab} = 10 + I_{20}$$

$$= 10 + I(20)$$

$$\text{where} \quad I = \frac{50-10}{40+20} = \frac{40}{60} = 0.67\text{A}$$

$\therefore$ Thevenin's voltage $V_{ab} = 10 + (0.67)20 = 23.41$ V

The short-circuit current through terminals *ab* as shown in Fig. 3.50 (b) is

$$I_N = I_1 + I_2$$

$$\text{where} \quad I_1 = \frac{10}{20} = 0.5\,\text{A and } I_2 = \frac{50}{40} = 1.25\,\text{A}$$

$$\therefore \quad I_N = 1.75\,\text{A}$$

$\therefore$ Norton's current $I_N = 1.75$ A

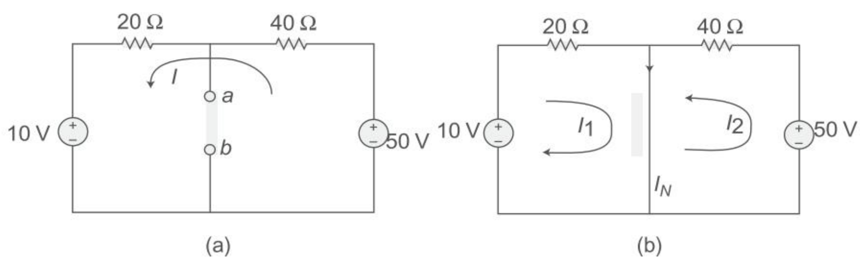


Fig. 3.50

The Thevenin's equivalent circuit is shown in Fig. 3.51 (a) and the Norton's equivalent circuit is shown in Fig. 3.51 (b).

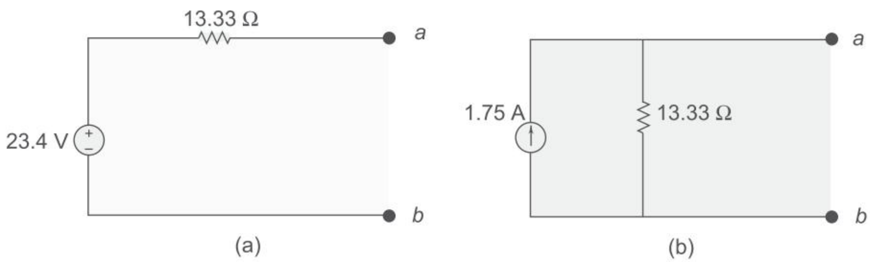


Fig. 3.51

PROBLEM 3.4

Determine the Thevenin's and Norton's equivalent circuits with respect to terminals ab for the circuit shown in Fig. 3.52.

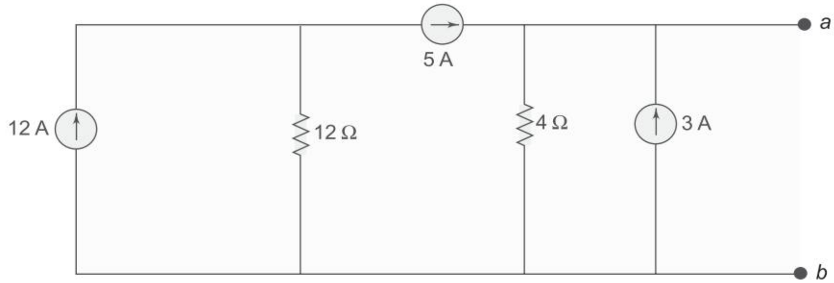


Fig. 3.52

Solution To find out the resistance seen into the terminals, when all the current sources are open circuited.

$$R_{ab} = 4\,\Omega$$

By applying the superposition theorem, we get the voltage across the $4\,\Omega$ resistor is

$$V_{ab} = V_4 = V_{12} + V_5 + V_3$$

The voltage across the $4\,\Omega$ resistor due to the 12A source when other sources are set equal to zero is

$$V_{12} = 0$$

The voltage across the $4\,\Omega$ resistor due to the 5A source when other sources are set equal to zero is

$$V_5 = 5 \times 4 = 20\text{ V}$$

The voltage across $4\,\Omega$ resistor due to 3A source when other sources are set equal to zero

$$V_3 = 4 \times 3 = 12\text{ V}$$

$$\therefore \quad V_{ab} = 0 + 20 + 12 = 32\text{ V}$$

The current in the short-circuited terminals

$$I_{ab} = I_{12} + I_5 + I_3$$

The current due to only the 12A source is $I_{12} = 0\text{ A}$

The current due to only the 5 A source is $I_5 = 5$ A

The current due to only the 3 A source is $I_3 = 3$ A

$\therefore \quad I_{ab} = 0 + 5 + 3 = 8$ A

Therefore, the Thevenin's equivalent circuit is shown in Fig. 3.53 (a) and the Norton's equivalent circuit is shown in Fig. 3.53 (b).

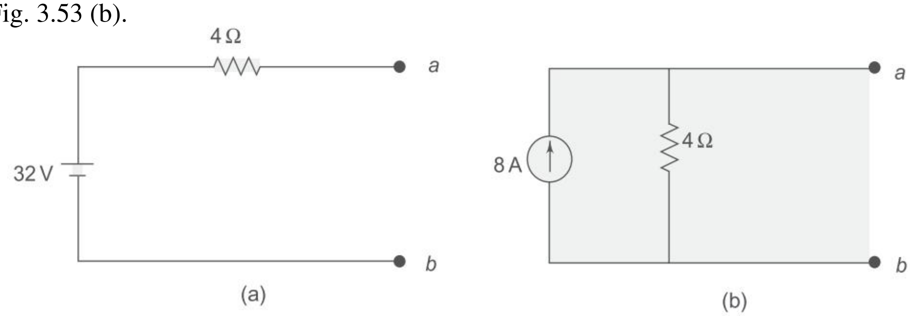


Fig. 3.53

PROBLEM 3.5

By using source transformation or any other technique, replace the circuit shown in Fig. 3.54 between terminals ab with the voltage source in series with a single resistor.

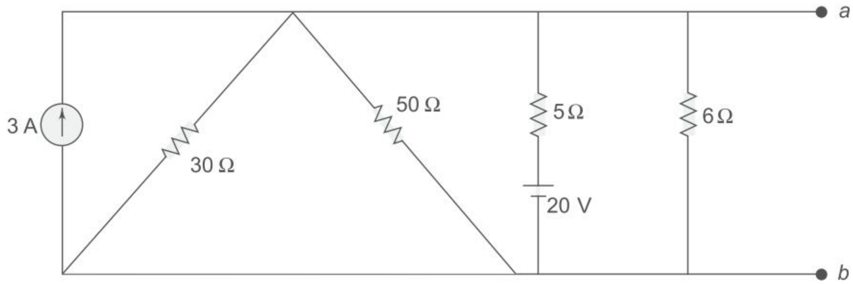


Fig. 3.54

Solution From Fig. 3.54, the resistances of 30 Ω and 50 Ω are in parallel and the resultant resistance (30 Ω ||50 Ω) = 18.75 Ω in parallel with the 3 A current source can be replaced by an equivalent voltage source in series with the resistance as shown in Fig. 3.55 (b).

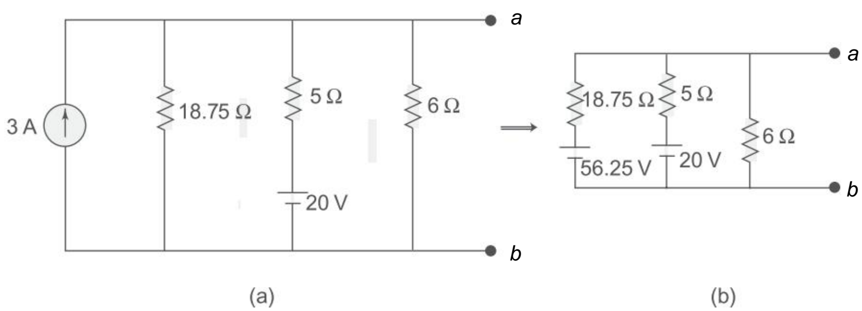


Fig. 3.55

Considering the node voltage V_{ab}, by applying Kirchhoff's current law, we have

$$\frac{V_{ab}-56.25}{18.75}+\frac{V_{ab}-20}{5}+\frac{V_{ab}}{6}=0$$

$$V_{ab}[0.05 + 0.2 + 0.17]–3–4 = 0$$

$\therefore$ Thevenin's voltage V_{ab} = 16.67 V

Resistance seen into the terminals *ab* when the voltage sources are short circuited in Fig. 3.55 (b).

Thevenin's resistance R_{ab} = [18.75||5||6] = 2.38 Ω

$\therefore$ the Thevenin's equivalent circuit is shown in Fig. 3.56.

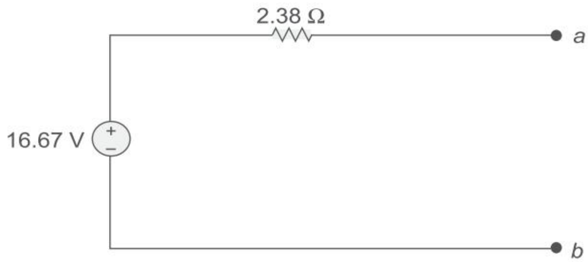


Fig. 3.56

PROBLEM 3.6

Use Thevenin's theorem to find the current through the 5 Ω resistor in Fig. 3.57.

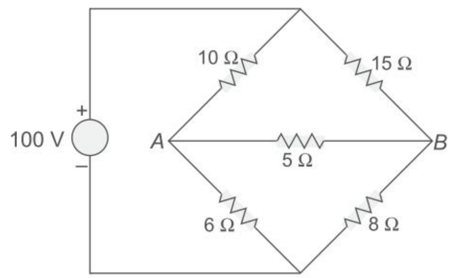


Fig. 3.57

Solution Thevenin's equivalent circuit can be formed by obtaining the voltage across terminals *AB* as shown in Fig. 3.58 (a).

Current in the 6 Ω resistor, $I_6 = \frac{100}{16} = 6.25$ A

Voltage across the 6 Ω resistor, $V_6 = 6 \times 6.25 = 37.5$ V

Current in the 8 Ω resistor, $I_8 = \frac{100}{23} = 4.35$ A

Voltage across the 8 Ω resistor is V_8 = 4.35 × 8 = 34.8 V

Voltage across the terminals *AB* is V_{AB} = 37.5 – 34.8 = 2.7 V

The resistance as seen into the terminals R_{AB}

$$=\frac{6\times10}{6+10}+\frac{8\times15}{8+15}$$

= 3.75 + 5.22 = 8.97 Ω

Thevenin's equivalent circuit is shown in Fig. 3.58 (b).

Current in the 5 Ω resistor, $I_5 = \frac{2.7}{5+8.97} = 0.193$ A

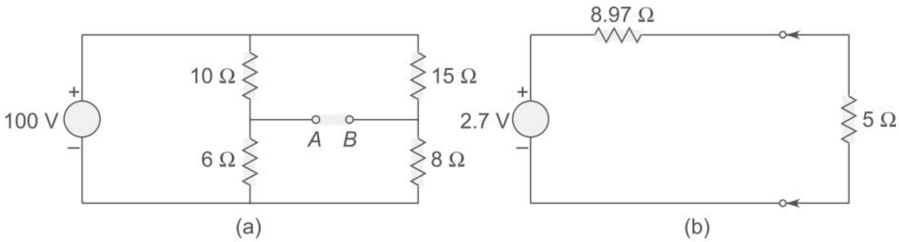


Fig. 3.58

PROBLEM 3.7

Find Thevenin's equivalent circuit for the circuit shown in Fig. 3.59.

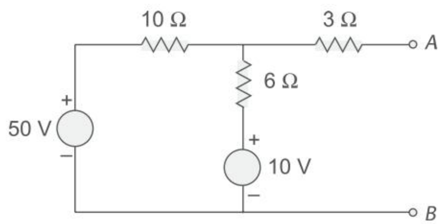


Fig. 3.59

Solution Thevenin's voltage is equal to the voltage across the terminals AB.

$\therefore \quad V_{AB} = V_3 + V_6 + 10$

Here, the current passing through the 3 Ω resistor is zero.

Hence, $\quad V_3 = 0$

By applying Kirchhoff's law, we have

$$50 - 10 = 10I + 6I$$

$$I = \frac{40}{16} = 2.5 \text{ A}$$

The voltage across 6 Ω is V_6 with polarity as shown in Fig. 3.60 (a), and is given by

$$V_6 = 6 \times 2.5 = 15 \text{ V}$$

The voltage across terminals AB is $V_{AB} = 0 + 15 + 10 = 25$ V.

The resistance as seen into the terminals AB

$$R_{AB} = 3 + \frac{10 \times 6}{10 + 6} = 6.75 \ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.60 (b).

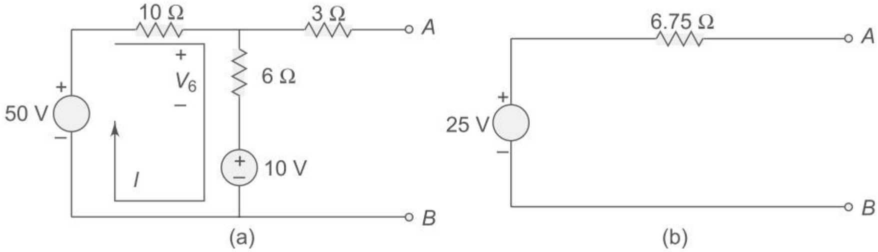


Fig. 3.60

PROBLEM 3.8

Determine the Thevenin's equivalent circuit across terminals AB for the circuit in Fig. 3.61.

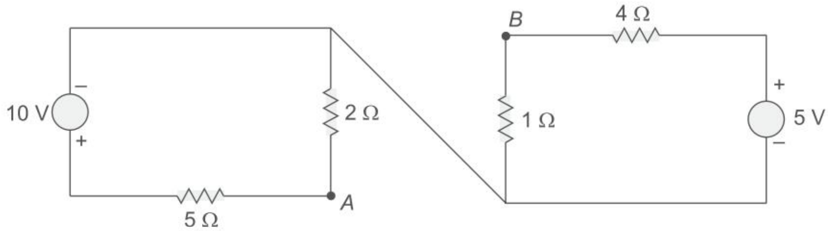
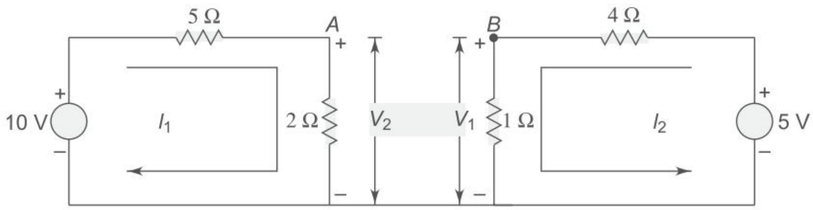


Fig. 3.61

Solution The given circuit is redrawn as shown in Fig. 3.62 (a).

Voltage $V_{AB} = V_2 + V_1$



(a)

Fig. 3.62

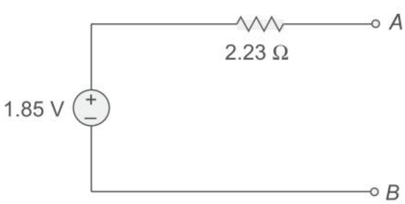
Applying Kirchhoff's voltage law to loops 1 and 2, we have the following:

Voltage across the 2 Ω resistor $V_2 = 2\times\frac{10}{7} = 2.85\,\text{V}$

Voltage across the 1 Ω resistor $V_1 = 1\times\frac{5}{5} = 1\,\text{V}$

$\therefore \quad V_{AB} = V_2 + V_1$

$= 2.85 - 1 = 1.85$ V



(b)

Fig. 3.62

The resistance seen into the terminals *AB*

$R_{AB} = (5 \parallel 2) + (4 \parallel 1)$

$$= \frac{5\times 2}{5+2} + \frac{4\times 1}{4+1}$$

$= 1.43 + 0.8 = 2.23\ \Omega$

Thevenin's equivalent circuit is shown in Fig. 3.62 (b).

PROBLEM 3.9

Determine Norton's equivalent circuit for the circuit shown in Fig. 3.63.

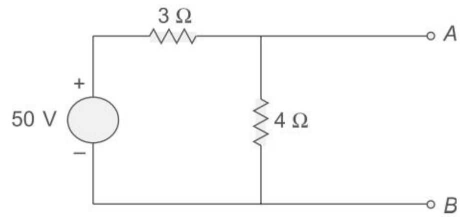


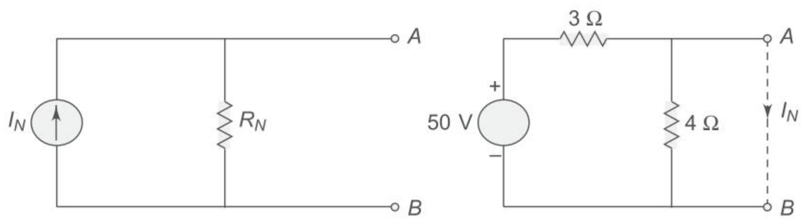
Fig. 3.63

Solution Norton's equivalent circuit is given by Fig. 3.64 (a)

where I_N = short-circuit current at terminals AB

R_N = open-circuit resistance at terminals AB

The current I_N can be found as shown in Fig. 3.64 (b).



(a) (b)

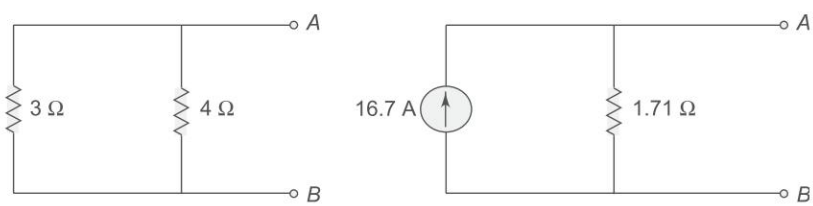
Fig. 3.64

$$I_N = \frac{50}{3} = 16.7\,\text{A}$$

Norton's resistance can be found from Fig. 3.64 (c).

$$R_N = R_{AB} = \frac{3 \times 4}{3 + 4} = 1.71\,\Omega$$

Norton's equivalent circuit for the given circuit is shown in Fig. 3.64 (d).



(c) (d)

Fig. 3.64

PROBLEM 3.10

Determine Norton's equivalent circuit for the given circuit shown in Fig. 3.65.

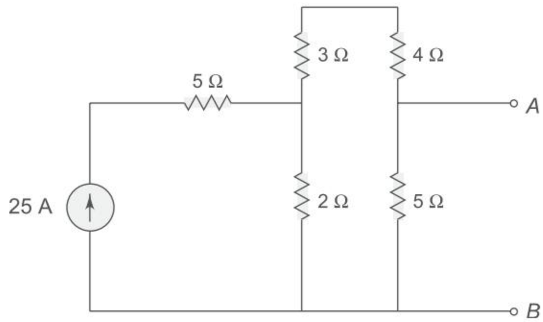
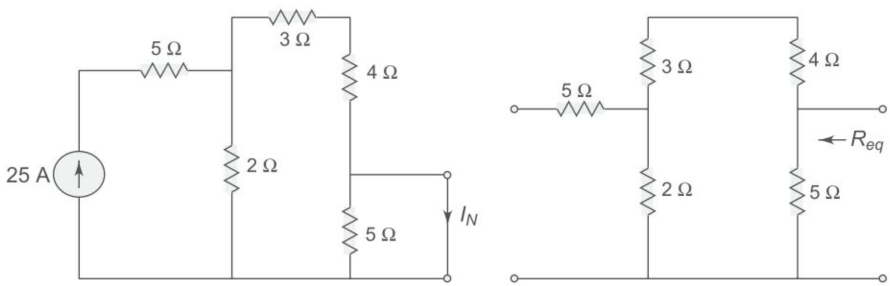


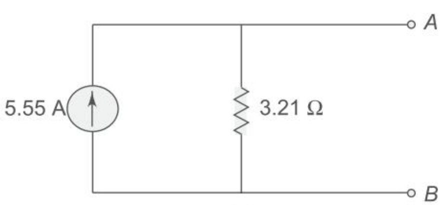
Fig. 3.65

Solution The short-circuit current at terminals AB can be found from Fig. 3.66 (a) and Norton's resistance can be found from Fig. 3.66 (b).

The current I_N is same as the current in the 3 Ω resistor or 4 Ω resistor.

$$I_N = I_3 = 25 \times \frac{2}{7+2} = 5.55\,\text{A}$$





(c)

Fig. 3.66

The resistance as seen into the terminals AB is

$$R_{AB} = 5 \parallel (4 + 3 + 2)$$

$$= \frac{5 \times 9}{5+9} = 3.21\,\Omega$$

Norton's equivalent circuit is shown in Fig. 3.66 (c).

PROBLEM 3.11

Determine the current flowing through the 5 Ω resistor in the circuit shown in Fig. 3.67 by using Norton's theorem.

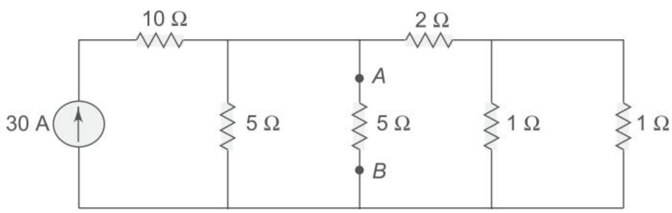


Fig. 3.67

Solution The short-circuit current at terminals *AB* can be found from the circuit as shown in Fig. 3.68 (a). Norton's resistance can be found from Fig. 3.63 (b).

In Fig. 3.68 (a), the current $I_N = 30$ A.

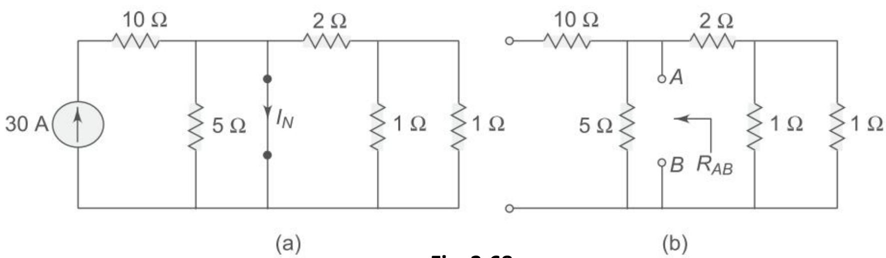


Fig. 3.68

The resistance in Fig. 3.68 (b)

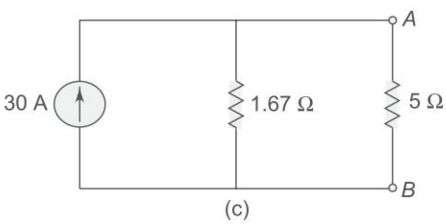


Fig. 3.68

$$R_{AB} = 5 \,||\left(2 + \frac{1 \times 1}{2}\right)$$

$$= 5 \,||\, (2.5) = \frac{5 \times 2.5}{7.5} = 1.67\,\Omega$$

Norton's equivalent circuit is shown in Fig. 3.68 (c).

$\therefore$ the current in the 5 Ω resistor

$$I_5 = 30 \times \frac{1.67}{6.67} = 7.51\,\text{A}$$

PROBLEM 3.12

Replace the given network shown in Fig. 3.69 by a single current source in parallel with a resistance.

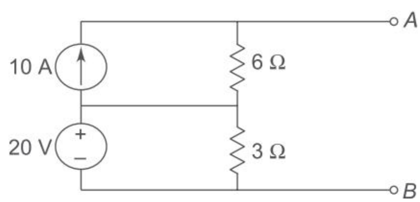


Fig. 3.69

Solution Here, using superposition technique and Norton's theorem, we can convert the given network.

We have to find a short-circuit current at terminals *AB* in Fig. 3.70 (a) as shown.

The current I'_N is due to the 10 A source. $I'_N = 10$ A

The current I''_N is due to the 20 V source (see Fig. 3.70 (b) and (c)).

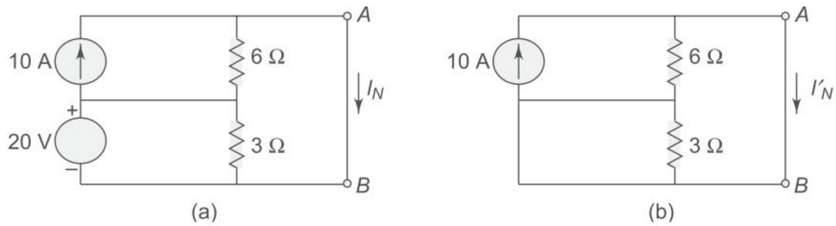


Fig. 3.70

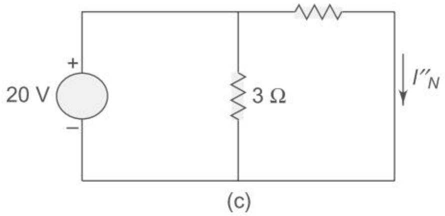


Fig. 3.70

$$I''_N = \frac{20}{6} = 3.33 \text{ A}$$

The current I_N is due to both the sources

$$I_N = I'_N + I''_N$$

$= 10 + 3.33 = 13.33$ A

The resistance as seen from terminals AB

$$R_{AB} = 6\ \Omega \text{ (from Fig. 3.70 (d))}$$

Hence, the required circuit is as shown in Fig. 3.70 (e).

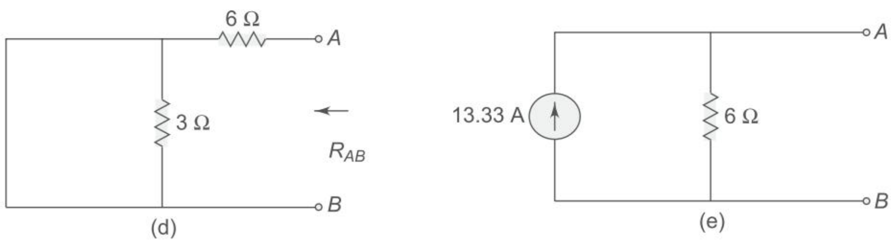


Fig. 3.70

PROBLEM 3.13

Using the compensation theorem, determine the ammeter reading where it is connected to the 6 Ω resistor as shown in Fig. 3.71. The internal resistance of the ammeter is 2 Ω.

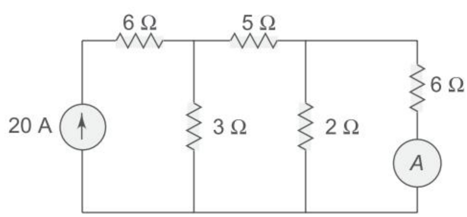


Fig. 3.71

Solution The current flowing through the 5 Ω branch

$$I_5 = 20 \times \frac{3}{3+6.5} = 6.315 \text{ A}$$

So the current in the 6 Ω branch

$$I_6 = 6.315 \times \frac{2}{6+2} = 1.58 \text{ A}$$

If we connect the ammeter having 2 Ω internal resistance to the 6 Ω branch, there is a change in resistance. The changes in currents in other branches results if a voltage source of voltage I_6. $\Delta R = 1.58 \times 2 = 3.16$ V is inserted in the 6 Ω branch as shown in Fig. 3.72.

The current due to this 3.16 V source is calculated.
The total impedance in the circuit

$$R_T = \{[(6 \parallel 3) + 5] \parallel [2]\} + \{6 + 2\}$$
$$= 9.56\ \Omega$$

The current due to the 3.16 V source

$$I_6' = \frac{3.16}{9.56} = 0.33\,\text{A}$$

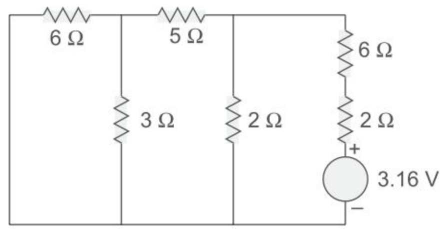


Fig. 3.72

This current is opposite to the current I_6 in the 6 Ω branch.

Hence, the ammeter reading = (1.58 – 0.33)

= 1.25 A

PROBLEM 3.14

Verify the reciprocity theorem for the given circuit shown in Fig. 3.73.

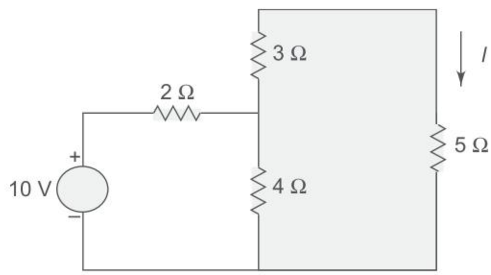


Fig. 3.73

Solution In Fig. 3.73, the current in the 5 Ω resistor is

$$I_5 = I_2 \times \frac{4}{8+4} = 2.14 \times \frac{4}{12}$$
$$= 0.71\text{ A}$$

where $\quad I_2 = \dfrac{10}{R_T}$

and $\quad R_T = 4.67$

$$\therefore \quad I_2 = \frac{10}{4.67} = 2.14\text{ A}$$

We interchange the source and response as shown in Fig. 3.74.

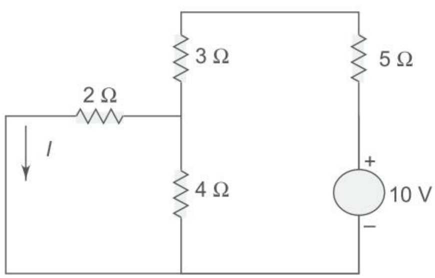


Fig. 3.74

In Fig. 3.74, the current in the 2 Ω resistor is

$$I_2 = I_3 \times \frac{4}{4+2}$$

where $\quad I_3 = \dfrac{10}{R_T}$

and $\quad R_T = 9.33\ \Omega$

$$\therefore I_3 = \frac{10}{9.33} = 1.07\,\text{A}$$

$$I_2 = 1.07 \times \frac{4}{6} = 0.71\,\text{A}$$

In both cases, the ratio of voltage to current is $\dfrac{10}{0.71} = 14.08$.

Hence, the reciprocity theorem is verified.

PROBLEM 3.15

Verify the reciprocity theorem in the circuit shown in Fig. 3.75.

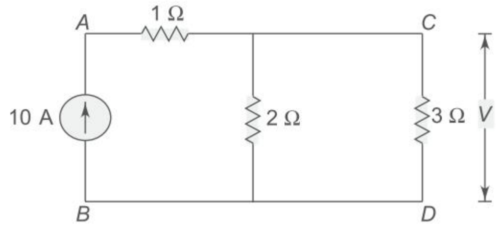


Fig. 3.75

Solution The voltage V across the 3 Ω resistor is

$$V = I_3 \times R$$

where $$I_3 = 10 \times \frac{2}{2+3} = 4\,\text{A}$$

$$\therefore \quad V = 4 \times 3 = 12\,\text{V}$$

We interchange the current source and response as shown in Fig. 3.76.

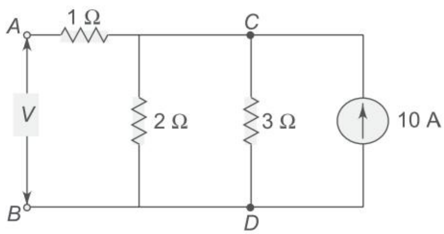


Fig. 3.76

To find the response, we have to find the voltage across the 2 Ω resistor

$$V = I_2 \times R$$

where $$I_2 = 10 \times \frac{3}{5} = 6\,\text{A}$$

$$\therefore \quad V = 6 \times 2 = 12\,\text{V}$$

In both cases, the ratio of current to voltage is the same, i.e. it is equal to 0.833. Hence, the reciprocity theorem is verified.

PROBLEM 3.16

Determine the maximum power delivered to the load in the circuit shown in Fig. 3.77.

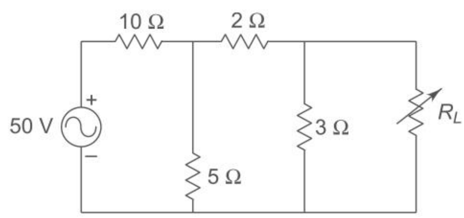


Fig. 3.77

Solution For the given circuit, let us find out the Thevenin's equivalent circuit across AB as shown in Fig. 3.78 (a).

The total resistance is

$$R_T = [\{(3 + 2) \,||\, 5\} + 10]$$

$$= [2.5 + 10] = 12.5\,\Omega$$

Total current drawn by the circuit is

$$I_T = \frac{50}{12.5} = 4\,\text{A}$$

The current in the 3 Ω resistor is

$$I_3 = I_T \times \frac{5}{5+5} = \frac{4 \times 5}{10} = 2\,\text{A}$$

Thevenin's voltage $V_{AB} = V_3 = 3 \times 2 = 6$ V

Thevenin's resistance $R_{\text{Th}} = R_{AB} = [((10 \parallel 5) + 2) \parallel 3]\ \Omega = 1.92\ \Omega$

Thevenin's equivalent circuit is shown in Fig. 3.78 (b).

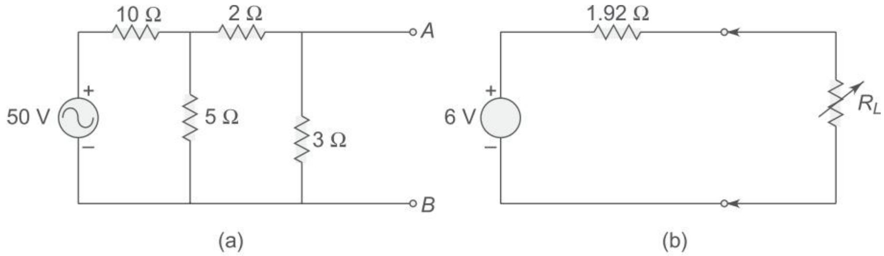


Fig. 3.78

From Fig. 3.78 (b) and maximum power transfer theorem,

$$R_L = 1.92\ \Omega$$

$\therefore$ current drawn by the load resistance R_L

$$I_L = \frac{6}{1.92 + 1.92} = 1.56\ \text{A}$$

Power delivered to the load $= I_L^2 R_L$

$$= (1.56)^2 \times 1.92 = 4.67\ \text{W}$$

PROBLEM 3.17

Determine the load resistance to receive maximum power from the source; also find the maximum power delivered to the load in the circuit shown in Fig. 3.79.

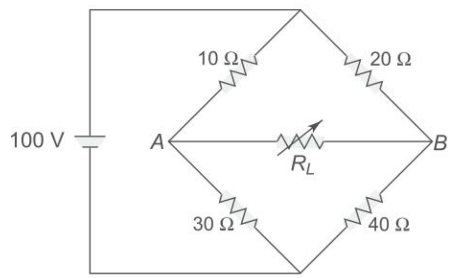


Fig. 3.79

Solution For the given circuit, we find out the Thevenin's equivalent circuit.

Thevenin's voltage across terminals A and B

$$V_{AB} = V_A - V_B$$

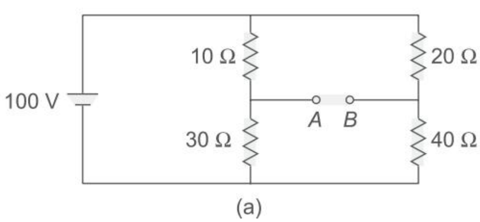


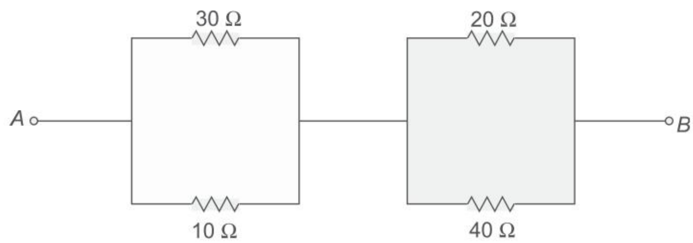
Fig. 3.80

Voltage at the point A is $V_A = 100 \times \dfrac{30}{30+10} = 75$ V

Voltage at the point B is $V_B = 100 \times \dfrac{40}{40+20} = 66.67$ V

$$\therefore \quad V_{AB} = 75 - 66.67 = 8.33\ \text{V}$$

To find Thevenin's resistance, the circuit in Fig. 3.80 (a) can be redrawn as shown in Fig. 3.80 (b).



(b)

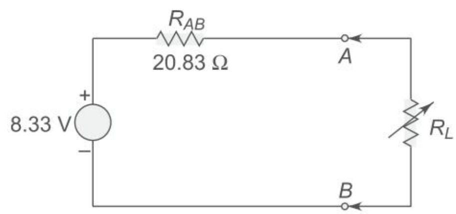
Fig. 3.80

From Fig. 3.80 (b), Thevenin's resistance

$$R_{AB} = [(30 \,\|\, 10) + (20 \,\|\, 40)]$$

$$= [7.5 + 13.33] = 20.83\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 3.80 (c).



(c)

Fig. 3.80

According to maximum power transfer theorem,

$$R_L = 20.83\ \Omega$$

Current drawn by the load resistance

$$I_L = \frac{8.33}{20.83 + 20.83} = 0.2\text{ A}$$

$\therefore$ maximum power delivered to load $= I_L^2 R_L$

$$= (0.2)^2\,(20.83) = 0.833\text{ W}$$

PROBLEM 3.18

Draw the dual circuit for the given circuit shown in Fig. 3.81.

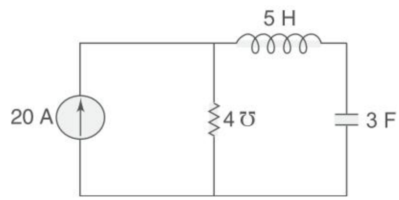
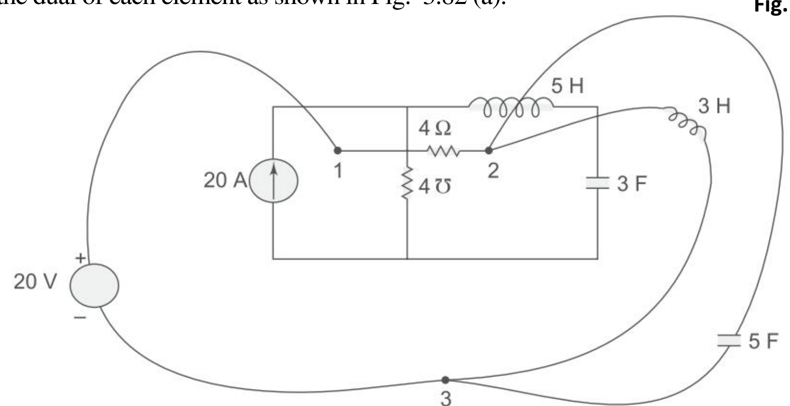


Fig. 3.81

Solution Our first step is to place nodes in each loop, and a reference node outside the circuit.

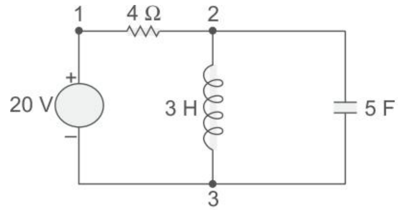
Join the nodes with lines passing through each element and connect these lines with the dual of each element as shown in Fig. 3.82 (a).



(a)

Fig. 3.82

The dual circuit of the given circuit is shown in Fig. 3.82 (b).



(b)

Fig. 3.82

PROBLEM 3.19

Draw the dual circuit of Fig. 3.83 given below.

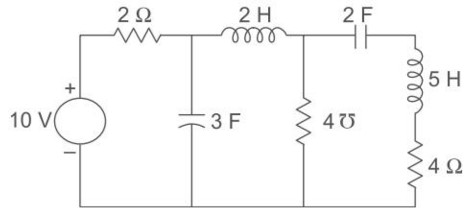
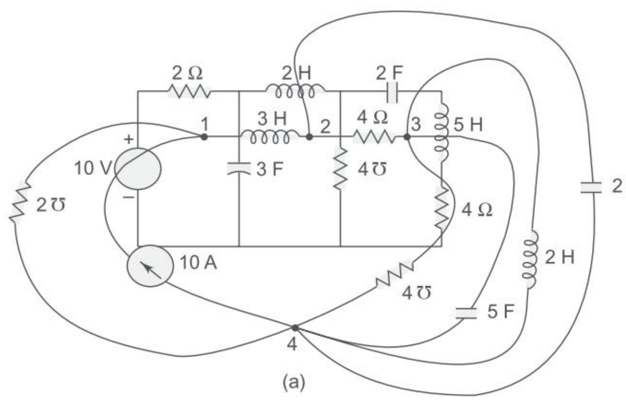


Fig. 3.83

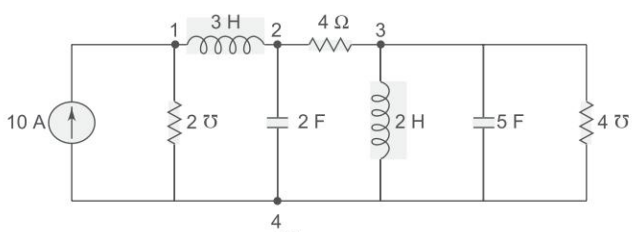
Solution Our first step is to mark nodes in each loop and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with the dual of each element as shown in Fig. 3.84 (a).

The dual circuit of the given circuit is shown in Fig. 3.84 (b).



(a)



(b)

Fig. 3.84

PROBLEM 3.20

For the circuit shown in Fig. 3.85, find the current i_4 using the superposition principle.

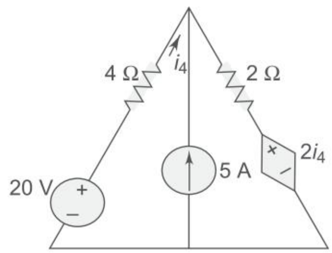


Fig. 3.85

Solution The circuit can be redrawn as shown in Fig. 3.86 (a).

The current i'_4 due to the 20 V source can be found using the circuit shown in Fig. 3.86 (b).

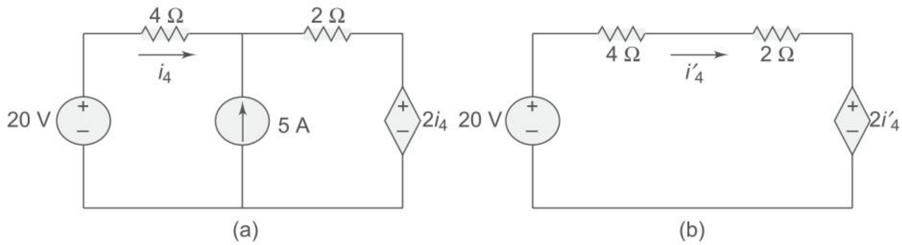


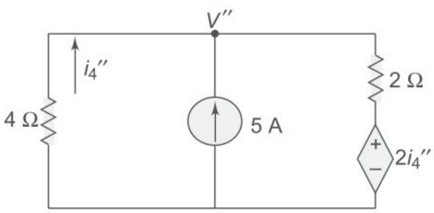
Fig. 3.86

Applying Kirchhoff's voltage law,

$$-20 + 4i'_4 + 2i'_4 + 2i'_4 = 0$$

$$i'_4 = 2.5 \text{ A}$$

The current i''_4 due to the 5 A source can be found using the circuit shown in Fig. 3.86 (c). By assuming V'' at the node shown in Fig. 3.86 (c) and applying Kirchhoff's current law,



(c)

Fig. 3.86

$$\frac{V''}{4} - 5 + \frac{V'' - 2i''_4}{2} = 0$$

$$i''_4 = \frac{-V''}{4}$$

From the above equations,

$$i''_4 = -1.25 \text{ A}$$

$$\therefore \text{ total current } i_4 = i''_4 + i''_4 = 1.25 \text{ A}$$

PROBLEM 3.21

Determine the current through the 2 Ω resistor as shown in Fig. 3.87 by using the superposition theorem.

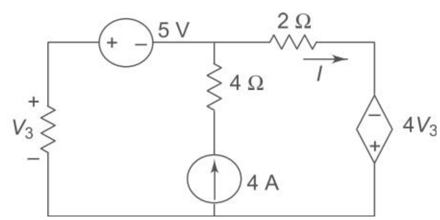


Fig. 3.87

Solution The current I' due to the 5V source can be found using the circuit shown in Fig. 3.88 (a).

By applying Kirchhoff's voltage law, we have

$$3I' + 5 + 2I' - 4V'_3 = 0$$

we know $V_3' = -3I'$

From the above equations,

$I' = -0.294$ A

The current I'' due to the 4 A source can be found using the circuit shown in Fig. 3.88 (b).

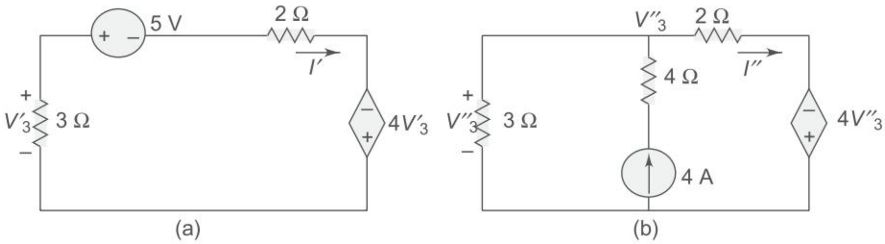


Fig. 3.88

By assuming the node voltage V''_3, we find

$$I'' = \frac{V_3'' + 4V_3''}{2}$$

By applying Kirchhoff's current law at the node, we have

$$\frac{V_3''}{3} - 4 + \frac{V_3'' + 4V_3''}{2} = 0$$

$$V''_3 = 1.55 \text{ V}$$

$$\therefore \quad I'' = \frac{V_3'' + 4V_3''}{2} = 3.875 \text{ A}$$

Total current in the 2 Ω resistor $I = I' + I'' = -0.294 + 3.875$

$\therefore \quad I = 3.581$ A

PROBLEM 3.22

For the circuit shown in Fig. 3.89, obtain Thevenin's equivalent circuit.

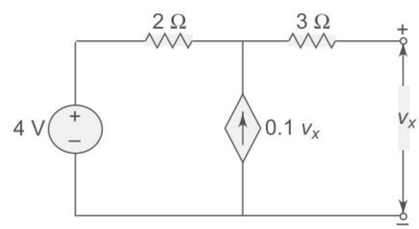


Fig. 3.89

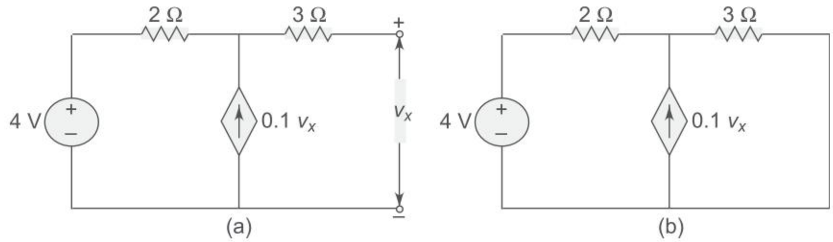
Solution The circuit consists of a dependent source. In the presence of the dependent source, R_{Th} can be determined by finding v_{OC} and i_{SC}

$$\therefore \quad R_{\text{Th}} = \frac{v_{OC}}{i_{SC}}$$

Open-circuit voltage can be found from the circuit shown in Fig. 3.90 (a).
Since the output terminals are open, current passes through the 2 Ω branch only.

$$v_x = 2 \times 0.1\, v_x + 4$$

$$v_x = \frac{4}{0.8} = 5 \text{ V}$$



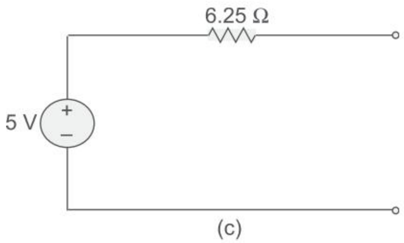


Fig. 3.90

Short-circuit current can be calculated from the circuit shown in Fig. 3.90 (b).

Since $v_x = 0$, the dependent current source is opened.

The current $\quad i_{SC} = \dfrac{4}{2+3} = 0.8 \text{ A}$

$$\therefore \quad R_{\text{Th}} = \frac{v_{OC}}{i_{SC}} = \frac{5}{0.8} = 6.25\ \Omega$$

The Thevenin's equivalent circuit is shown in Fig. 3.90 (c).

PROBLEM 3.23

For the circuit shown in Fig. 3.91, find the current i_2 in the 2 Ω resistor by using Thevenin's theorem.

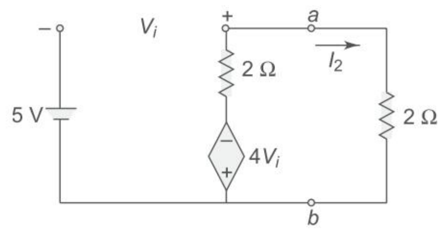


Fig. 3.91

Solution From the circuit, there is open voltage at the terminals *ab* which is

$$V_{OC} = -4V_i$$

where $\quad V_i = -4V_i - 5$

$\therefore \quad V_i = -1$

Thevenin's voltage $V_{OC} = 4$ V

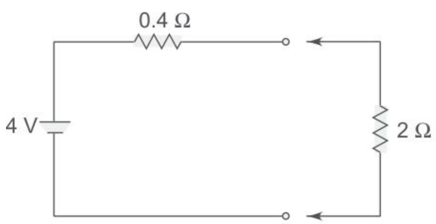


Fig. 3.92

From the circuit, short-circuit current is determined by shorting terminals *a* and *b*.

Applying Kirchhoff's voltage law, we have

$4V_i + 2i_{SC} = 0$

We know $V_i = -5$

Substituting V_i in the above equation, we get

$i_{SC} = 10$ A

$$\therefore \quad R_{\text{Th}} = \frac{V_{OC}}{i_{SC}} = \frac{4}{10} = 0.4\ \Omega$$

The Thevenin's equivalent circuit is as shown in Fig. 3.92.

The current in the 2 Ω resistor $\quad i_2 = \dfrac{4}{2.4} = 1.67 \text{ A}$

PROBLEM 3.24

For the circuit shown in Fig. 3.93, find Norton's equivalent circuit.

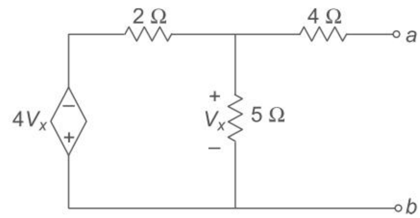


Fig. 3.93

Solution In the case of circuit having only dependent sources (without independent sources), both V_{OC} and i_{SC} are zero. We apply a 1 A source externally and determine the resultant voltage across it, and then find $R_{\text{Th}} = \dfrac{V}{1}$ or we can also apply the 1 V source externally and determine the current through it and then we find $R_{\text{Th}} = 1/i$.

By applying the 1 A source externally as shown in Fig. 3.94 (a) and application of Kirchhoff's current law, we have

$$\frac{V_x}{5} + \frac{V_x + 4V_x}{2} = 1$$

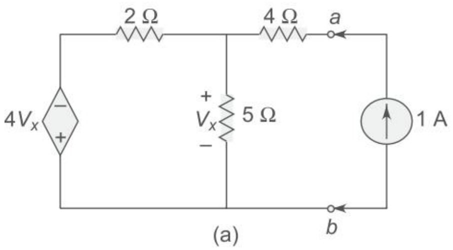
$$V_x = 0.37 \text{ V}$$

The current in the 4 Ω branch is

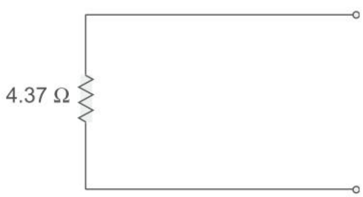
$$\frac{V_x - V}{4} = -1$$

Substituting V_x in the above equation, we get

$$V = 4.37 \text{ V}$$



(a)



(b)

Fig. 3.94

$$\therefore \quad R_{\text{Th}} = \frac{V}{1} = 4.37\ \Omega$$

If we short circuit the terminals *a* and *b,* we have

$$\frac{V_x - 4V_x}{2} = 0$$

$$V_x = 0$$

$$I_{SC} = \frac{V_x}{4} = 0$$

Therefore, Norton's equivalent circuit is as shown in Fig. 3.94 (b).

PSpice Problems

PROBLEM 3.1

Determine the voltage across the terminal AB of the circuit shown in Fig. 3.95.

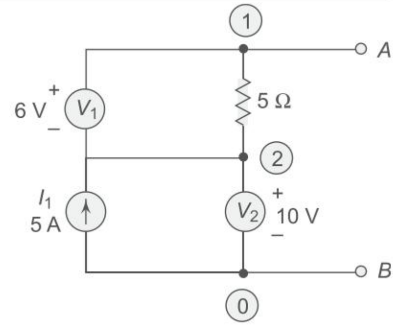


Fig. 3.95

```
* TO DETERMINE THEVENIN VOLTAGE
V1   1    2    6
I1   0    2    5
R1   1    2    5
V2   2    0    10 V
.TF V(1,0) V1 ; TRANSFER FUNCTION ANALYSIS
.END
**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
*************************************************************
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1)   16.0000  (2)   10.0000
VOLTAGE SOURCE CURRENTS
NAME       CURRENT
V1         –1.200E + 00
V2   5.000E + 00
**** SMALL-SIGNAL CHARACTERISTICS
V(1,0)/V1 = 1.000E + 00
INPUT RESISTANCE AT V1 = 5.000E + 00
OUTPUT RESISTANCE AT V(1,0) = 0.000E + 00
```

Result

VAB = V(1,0) = 16 V.

V(1,0) = 1 × V1 = 16 V = V_{th}.

PROBLEM 3.2

Use Thevenin's theorem to find the current through the 5 Ω resistor in Fig. 3.96.

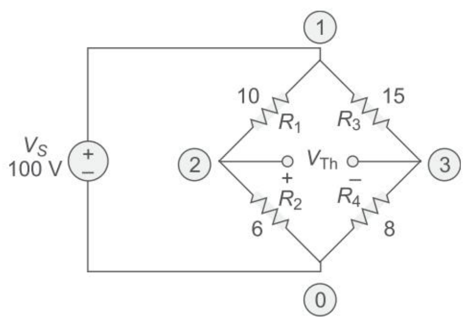


Fig. 3.96

```
* TO DETERMINE THEVENIN CIRCUIT
VS   10   DC   100
R1   1    2    10
R2   2    0    6
R3   1    3    15
R4   3    0    8
.TF V(2, 3) VS
.END
```

OUTPUT

**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
****
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1) 100.0000 (2) 37.5000 (3) 34.7830
VOLTAGE SOURCE CURRENTS
NAME CURRENT
VS –1.060E + 01
**** SMALL-SIGNAL CHARACTERISTICS
V(2,3)/VS = 2.717E – 02
INPUT RESISTANCE AT VS = 9.436E + 00
OUTPUT RESISTANCE AT V(2,3) = 8.967E + 00

Result

VTH = V(2,3)/VS*VS = 2.717E – 02 * 100 = 2.717 V
RH = OUTPUT RESISTANCE AT V(2,3) = 8.967 Ω.
RL = 5 Ω ; IL = VTH/(RTH + RL) = 2.717/(8.967 + 5) = 0.195A.

PROBLEM 3.3

Determine the load resistance to receive maximum power from the source; also find the maximum power delivered to the load in the circuit shown in Fig. 3.97 using PSpice.

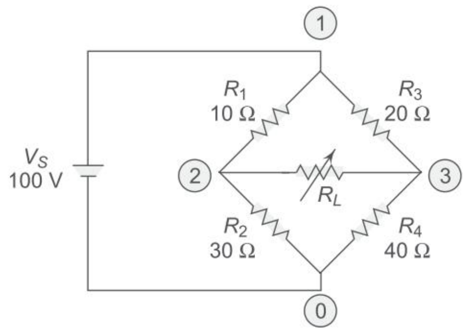


Fig. 3.97

* NETLIST TO FIND MAX.POWER TRANSFER
VS 10 DC 100
R1 1 2 10
R2 2 0 30
R3 1 3 20
R4 3 0 40
RL 23 RLOAD 1
.MODEL RLOAD RES(R = 25)
.DC RES RLOAD(R) 0.001 40 0.01
.TF V(2,3) VS
.PROBE
.END

OUTPUT

**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
****
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1) 100.0000 (2) 73.6360 (3) 69.0910
VOLTAGE SOURCE CURRENTS
NAME CURRENT
VS – 4.182E + 00
TOTAL POWER DISSIPATION 4.18E + 02 WATTS

**** SMALL-SIGNAL CHARACTERISTICS
V(2,3)/VS = 4.545E – 02
INPUT RESISTANCE AT VS = 2.391E + 01
OUTPUT RESISTANCE AT V(2,3) = 1.136E + 01

Result

From the graph shown in Fig. 3.98, which is obtained with .PROBE statement for W(RL), MAX(W(RL)) is found to be 0.833W at RL = 20.81 Ω.

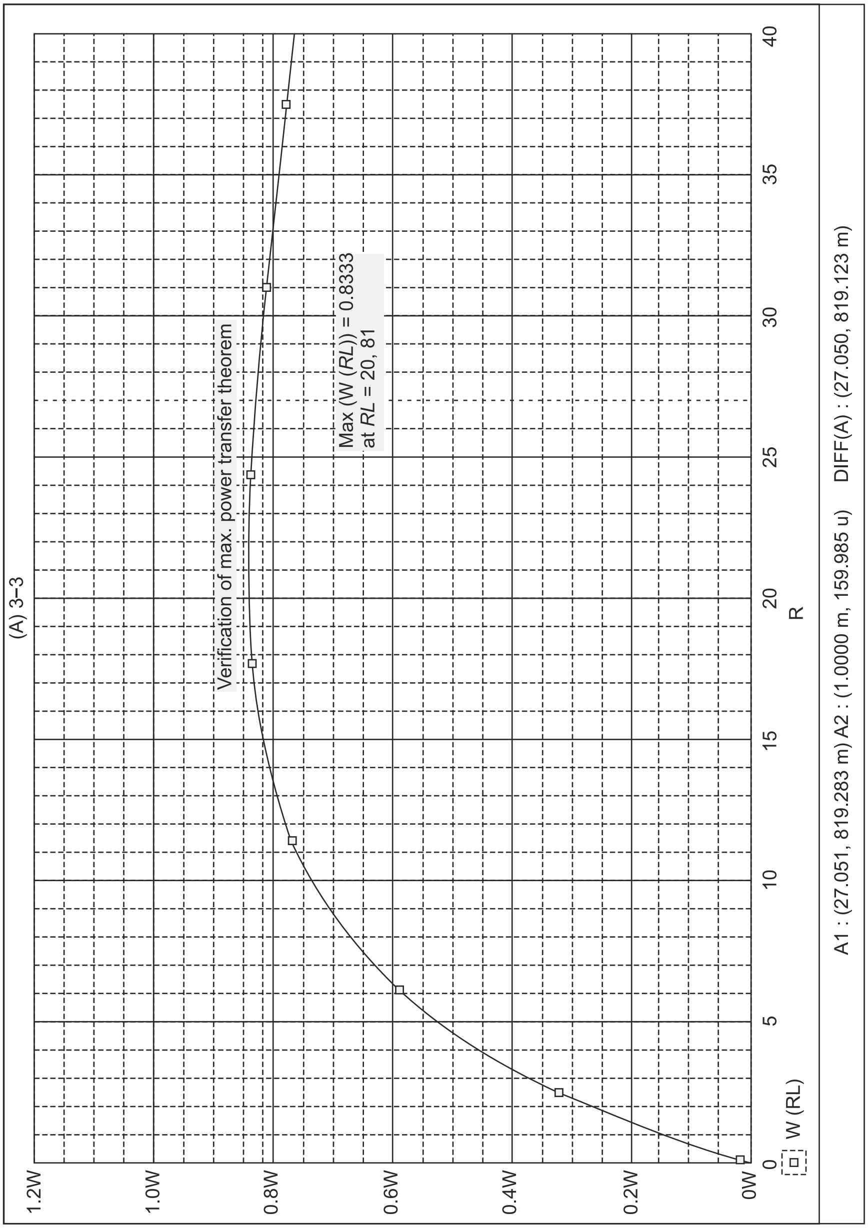


Fig. 3.98

Answers to Practice Problems

3-1.1 1.182 Ω

3-1.2 $R_{eq} = 0.6078$

3-1.3 (i) $I_{6.8k} = 2.55$ mA; $I_{2.2k} = 7.9$ mA
(ii) $V_{6.8k} = 17.34$ V ; $V_{2.2k} = 17.38$ V

3-2.1 0.82 A

3-2.2 $I_{4\Omega} = 1.5\ \Omega$

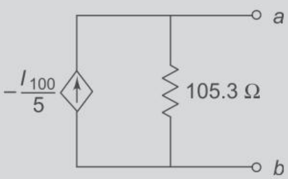
3-2.3 $I_1 = 4.6$ A; $I_2 = 2.6$ A; $I_3 = 2$ A

3-2.4 4 A

3-3.1 0.5 A

3-4.1 (i) 8 V voltage source is the series with 10 kΩ resistance
(ii) 0.8 mA current source is in parallel with 10 kΩ resistance

3-4.2



3-7.1 12 Ω, 0.75 W, 6 V, 0.5 A

3-10.1 $I_{10} = 0.155$ A

Objective-Type Questions

☆☆★**3.1** Three equal resistances of 3 Ω are connected in star. What is the resistance in one of the arms in an equivalent delta circuit?
(a) 10 Ω (b) 3 Ω (c) 9 Ω (d) 27 Ω

☆☆★**3.2** Three equal resistances of 5 Ω are connected in delta. What is the resistance in one of the arms of the equivalent star circuit?
(a) 5 Ω (b) 1.33 Ω
(c) 15 Ω (d) 10 Ω

☆☆★**3.3** Superposition theorem is valid only for
(a) linear circuits (c) both linear and nonlinear
(b) nonlinear circuits (d) neither of the two

☆☆★**3.4** Superposition theorem is not valid for
(a) voltage responses (c) power responses
(b) current responses (d) all the three

☆★★**3.5** Determine the current *I* in the circuit shown in Fig. 3.99. It is
(a) 2.5 A (b) 1 A
(c) 3.5 A (d) 4.5 A

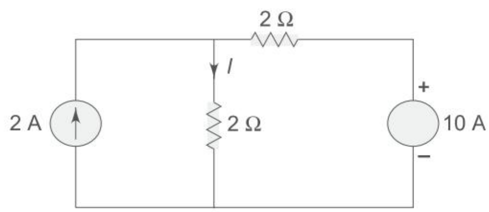


Fig. 3.99

☆☆★**3.6** Reduce the circuit shown in Fig. 3.100 to its Thevenin equivalent circuit as viewed from terminals *A* and *B*.
(a) The circuit consists of 15 V battery in series with 100 kΩ
(b) The circuit consists of 15 V battery in series with 22 kΩ
(c) The circuit consists of 15 V battery in series with parallel combination of 100 kΩ and 22 kΩ
(d) None of the above

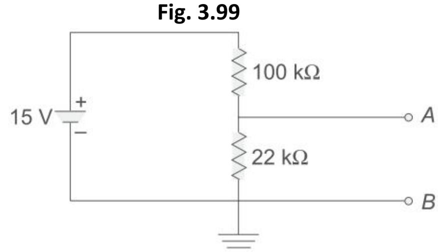


Fig. 3.100

☆☆★**3.7** Norton's equivalent circuit consists of
(a) voltage source in parallel with resistance
(b) voltage source in series with resistance
(c) current source in series with resistance
(d) current source in parallel with resistance

☆☆★**3.8** The reciprocity theorem is applicable to
(a) linear networks only
(c) linear/bilateral networks
(b) bilateral networks only
(d) neither of the two

☆☆★**3.9** Compensation theorem is applicable to
(a) linear networks only
(c) linear and nonlinear networks
(b) nonlinear networks only
(d) neither of the two

☆★★**3.10** Maximum power is transferred when load impedance is
(a) equal to source resistance
(c) equal to zero
(b) equal to half of the source resistance
(d) none of the above

☆☆★**3.11** In the circuit shown in Fig. 3.101, what is the maximum power transferred to the load?
(a) 5 W
(c) 10 W
(b) 2.5 W
(d) 25 W

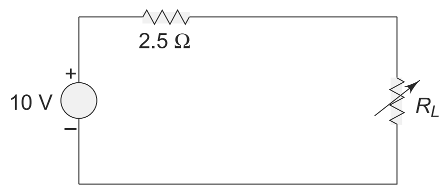


Fig. 3.101

☆☆★**3.12** Indicate the dual of a series network that consists of voltage source, capacitance, inductance in
(a) parallel combination of resistance, capacitance, and inductance
(b) series combination of current source, capacitance, and inductance
(c) parallel combination of current source, inductance, and capacitance
(d) none of the above

☆★★**3.13** When the superposition theorem is applied to any circuit, the dependent voltage source in that circuit is always.
(a) opened (b) shorted (c) active (d) none of the above

☆☆★**3.14** Superposition theorem is not applicable to networks containing.
(a) nonlinear elements
(c) dependent current sources
(b) dependent voltage sources
(d) transformers

☆★★**3.15** Thevenin's voltage in the circuit shown in Fig. 3.102 is
(a) 3 V (b) 2 V (c) 2.5 V (d) 0.1 V

☆★★**3.16** Norton's current in the circuit shown in Fig. 3.103 is
(a) $\frac{2i}{5}$ (b) infinite (c) zero (d) none

☆★★**3.17** A dc circuit shown in Fig. 3.104 has a voltage V, a current source I and several resistors. A particular resistor R dissipates a power of 4 W when V alone is active. The same resistor dissipates a power of 9 W when I alone is active. The power dissipated by R when both sources are active will be
(a) 1 W (b) 13 W (c) 5 W (d) 25 W

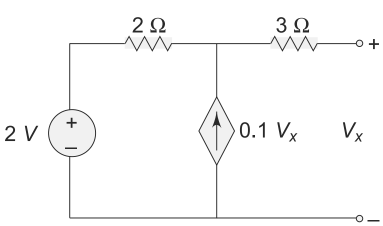


Fig. 3.102

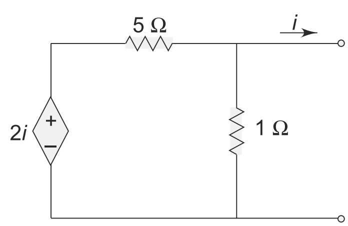


Fig. 3.103

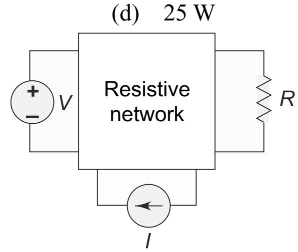


Fig. 3.104

For interactive quiz with answers, scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/261

CHAPTER 4

Introduction to Alternating Currents and Voltages

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- **LO 1** Describe the nature of sine wave
- **LO 2** Analyse the angular relation of a sine wave and represent the sine-wave equation
- **LO 3** Quantify voltage and current values of a sine wave
- **LO 4** Analyse the phase relation in a pure resistor, pure inductor and pure capacitor

4.1 THE SINE WAVE

LO 1 Describe the nature of sine wave

Many a time, alternating voltages and currents are represented by a sinusoidal wave, or simply a sinusoid. It is a very common type of alternating current (ac) and alternating voltage. The sinusoidal wave is generally referred to as a sine wave. Basically, an alternating voltage (current) waveform is defined as the voltage (current) that fluctuates with time periodically, with change in polarity and direction. *In general, the sine wave is more useful than other waveforms, like pulse, sawtooth, square, etc.* There are a number of reasons for this. One of the reasons is that if we take any second- order system, the response of this system is a sinusoid. Secondly, any periodic waveform can be written in terms of sinusoidal function according to Fourier theorem. Another reason is that its derivatives and integrals are also sinusoids. A sinusoidal function is easy to analyse. Lastly, the sinusoidal function is easy to generate, and it is more useful in the power industry. The shape of a sinusoidal waveform is shown in Fig. 4.1.

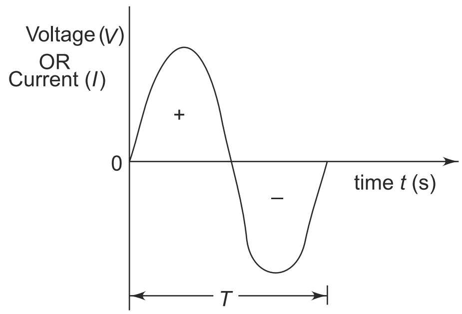


Fig. 4.1

The waveform may be either a current waveform, or a voltage waveform. As seen from Fig. 4.1, the wave changes its magnitude and direction with time. If we start at time $t = 0$, the wave goes to a maximum value and returns to zero, and then decreases to a negative maximum value before returning to zero. The sine wave changes with time in an orderly manner. During the positive portion of voltage, the current flows in one direction; and during the negative portion of voltage, the current flows in the opposite direction. The complete positive and negative portion of the wave is one cycle of the sine wave. Time is designated by t. *The time taken*

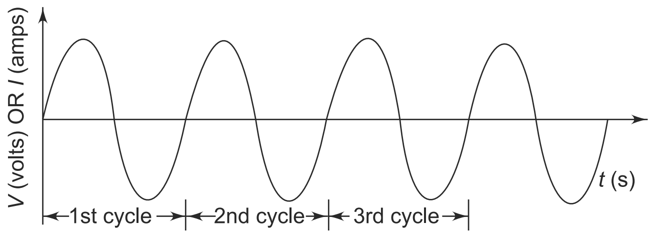


Fig. 4.2

for any wave to complete one full cycle is called the ***period*** *(T).* In general, any periodic wave constitutes a number of such cycles. For example, one cycle of a sine wave repeats a number of times as shown in Fig. 4.2. Mathematically, it can be represented as $f(t) = f(t + T)$ for any t.

The period can be measured in the following different ways (see Fig. 4.3).

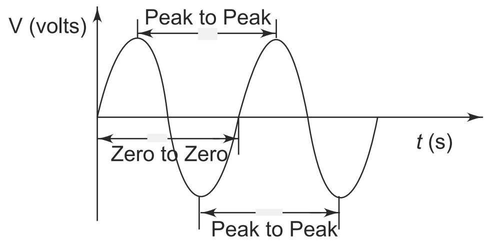


Fig. 4.3

1. From zero crossing of one cycle to zero crossing of the next cycle
2. From positive peak of one cycle to positive peak of the next cycle
3. From negative peak of one cycle to negative peak of the next cycle

The ***frequency of a wave*** *is defined as the number of cycles that a wave completes in one second.*

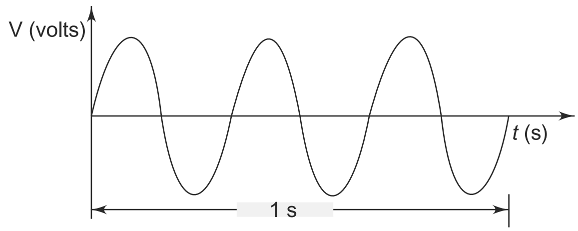


Fig. 4.4

In Fig. 4.4, the sine wave completes three cycles in one second. Frequency is measured in hertz. One hertz is equivalent to one cycle per second, 60 hertz is 60 cycles per second, and so on. In Fig. 4.4, the frequency denoted by f is 3 Hz, that is three cycles per second. The relation between time period and frequency is given by

$$f = \frac{1}{T}$$

A sine wave with a longer period consists of fewer cycles than one with a shorter period.

EXAMPLE 4.1

What is the period of the sine wave shown in Fig. 4.5?

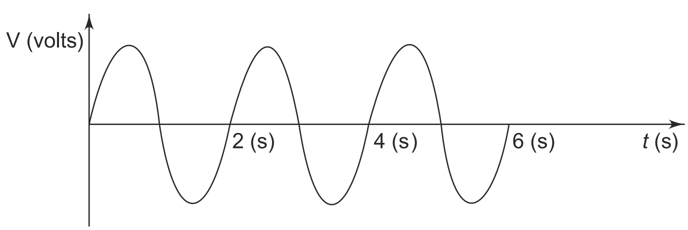


Fig. 4.5

Solution From Fig. 4.5, it can be seen the sine wave takes two seconds to complete one period in each cycle

$$T = 2\,\text{s}$$

EXAMPLE 4.2

The period of a sine wave is 20 milliseconds. What is the frequency?

Solution

$$f = \frac{1}{T}$$

$$= \frac{1}{20\,\text{ms}} = 50\,\text{Hz}$$

EXAMPLE 4.3

The frequency of a sine wave is 30 Hz. What is its period?

Solution

$$T = \frac{1}{f}$$

$$= \frac{1}{30} = 0.03333\ \text{s} \quad = 33.33\,\text{ms}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **4-1.1** Calculate the frequency of the following values of period.
(a) 0.2 s (b) 50 ms (c) 500 μs (d) 10 μs

☆☆★ **4-1.2** Calculate the period for each of the values of frequency.
(a) 60 Hz (b) 500 Hz (c) 1 kHz (d) 200 kHz (e) 5 MHz

4.2 ANGULAR RELATION OF A SINE WAVE

LO 2 Analyse the angular relation of a sine wave and represent the sine-wave equation

A sine wave can be measured along the *X*-axis on a time base which is frequency-dependent. A sine wave can also be expressed in terms of an angular measurement. This angular measurement is expressed in degrees or radians. *A* ***radian*** *is defined as the angular distance measured along the circumference of a circle which is equal to the radius of the circle.* One radian is equal to 57.3°. In a 360° revolution, there are 2π radians. The angular measurement of a sine wave is based on 360° or 2π radians for a complete cycle as shown in Figs 4.6 (a) and (b).

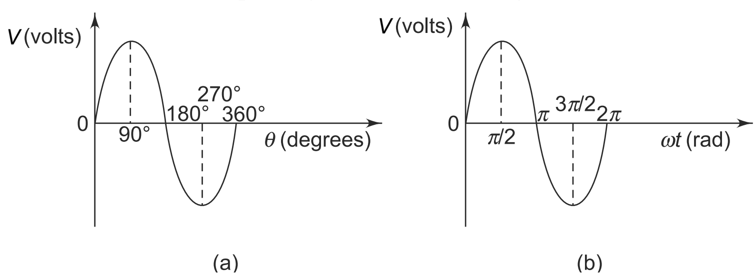


Fig. 4.6

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

A sine wave completes a half cycle in 180° or π radians; a quarter cycle in 90° or π/2 radians, and so on.

❑ ***Phase of a Sine Wave*** The phase of a sine wave is an angular measurement that specifies the position of the sine wave relative to a reference. The wave shown in Fig. 4.7 is taken as the reference wave.

When the sine wave is shifted left or right with reference to the wave shown in Fig. 4.7, there occurs a phase shift. Figure 4.8 shows the phase shifts of a sine wave.

In Fig. 4.8 (a), the sine wave is shifted to the right by 90° (π/2 rad) shown by the dotted lines. There is a phase angle of 90° between *A* and *B*. Here, the waveform *B* is lagging behind waveform *A* by 90°. In other words, the sine wave *A* is leading the waveform *B* by 90°. In Fig. 4.8 (b) the sine wave *A* is lagging behind the waveform *B* by 90°. In both cases, the phase difference is 90°.

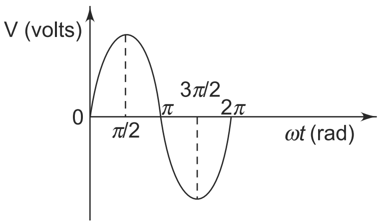


Fig. 4.7

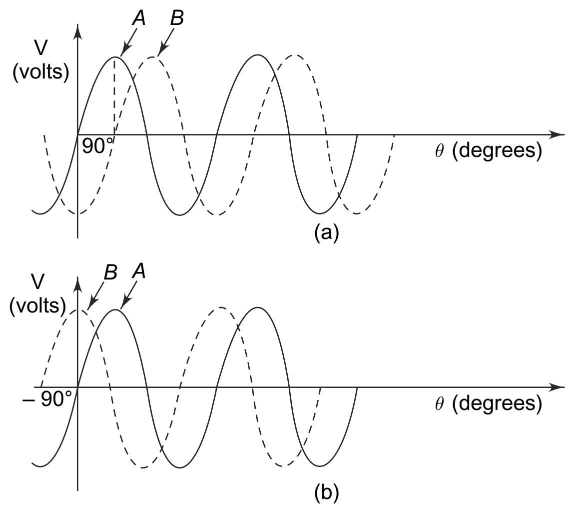


Fig. 4.8

EXAMPLE 4.4

What are the phase angles between the two sine waves shown in Figs 4.9 (a) and (b)?

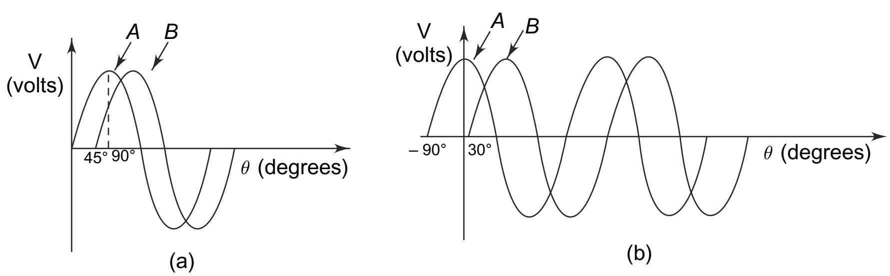


Fig. 4.9

Solution In Fig. 4.9 (a), the sine wave *A* is in phase with the reference wave; the sine wave *B* is out of phase, which lags behind the reference wave by 45°. So we say that the sine wave *B* lags behind the sine wave *A* by 45°.

In Fig. 4.9 (b), the sine wave *A* leads the reference wave by 90°; the sine wave *B* lags behind the reference wave by 30°. So the phase difference between *A* and *B* is 120°, which means that sine wave *B* lags behind sine wave *A* by 120°. In other words, the sine wave *A* leads the sine wave *B* by 120°.

4.3 THE SINE WAVE EQUATION

LO 2

A sine wave is graphically represented as shown in Fig. 4.10 (a). The amplitude of a sine wave is represented on vertical axis. The angular measurement (in degrees or radians) is represented on horizontal axis. Amplitude A is the maximum value of the voltage or current on the Y-axis.

In general, the sine wave is represented by the equation

$$v(t) = V_m \sin \omega t$$

The above equation states that any point on the sine wave represented by an instantaneous value $v(t)$ is equal to the maximum value times the sine of the angular frequency at that point. For example, if a certain sine wave voltage has a peak value of 20 V, the instantaneous voltage at a point $\pi/4$ radians along the horizontal axis can be calculated as

$$v(t) = V_m \sin \omega t$$

$$= 20 \sin\left(\frac{\pi}{4}\right) = 20 \times 0.707 = 14.14 \text{ V}$$

When a sine wave is shifted to the left of the reference wave by a certain angle ϕ, as shown in Fig. 4.10 (b), the general expression can be written as

$$v(t) = V_m \sin(\omega t + \phi)$$

When a sine wave is shifted to the right of the reference wave by a certain angle ϕ, as shown in Fig. 4.10 (c), the general expression is

$$v(t) = V_m \sin(\omega t - \phi)$$

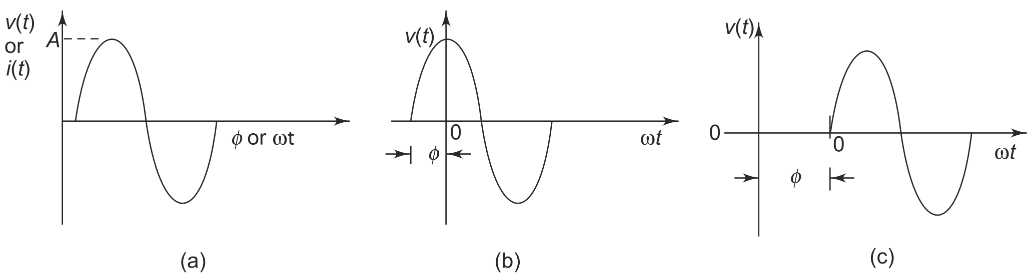


Fig. 4.10

EXAMPLE 4.5

Determine the instantaneous value at the 90° point on the X-axis for each sine wave shown in Fig. 4.11.

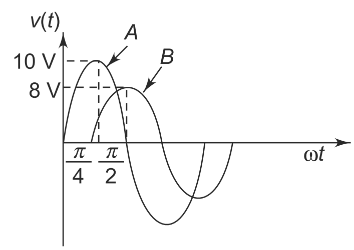


Fig. 4.11

Solution From Fig. 4.11, the equation for the sine wave *A*

$$v(t) = 10 \sin \omega t$$

The value at $\pi/2$ in this wave is

$$v(t) = 10 \sin\frac{\pi}{2} = 10\text{ V}$$

The equation for the sine wave *B*

$$v(t) = 8 \sin(\omega t - \pi/4)$$

At $\omega t = \pi/2$

$$v(t) = 8\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 8\sin 45^\circ$$

$$= 8(0.707)$$

$$= 5.66\text{ V}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆★★**4-2.1** A certain sine wave has a positive going zero crossing at 0° and an *rms* value of 20 V. Calculate its instantaneous value at each of the following angles.

(a) 33° (b) 110° (c) 145° (d) 325°

☆★★**4-2.2** For a particular 0° reference sinusoidal current, the peak value is 200 mA; determine the instantaneous values at each of the following.

(a) 35° (b) 190° (c) 200° (d) 360°

4.4 VOLTAGE AND CURRENT VALUES OF A SINE WAVE

LO 3 Quantify voltage and current values of a sine wave

As the magnitude of the waveform is not constant, the waveform can be measured in different ways. These are instantaneous, peak, peak to peak, root mean square (*rms*) and average values.

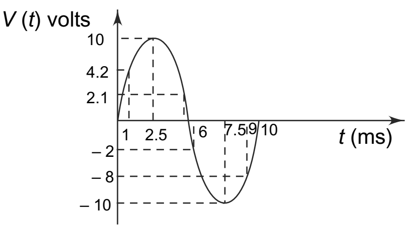


Fig. 4.12

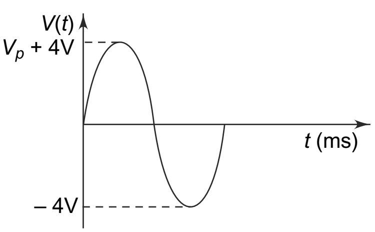


Fig. 4.13

❑ ***Instantaneous Value*** Consider the sine wave shown in Fig. 4.12. At any given time, it has some instantaneous value. This value is different at different points along the waveform.

In Fig. 4.12, during the positive cycle, the instantaneous values are positive and during the negative cycle, the instantaneous values are negative. In Fig. 4.12 shown at time 1 ms, the value is 4.2 V; the value is 10 V at 2.5 ms, –2 V at 6 ms and –10 V at 7.5, and so on.

❑ ***Peak Value*** *The* ***peak value*** *of the sine wave is the maximum value of the wave during positive half cycle, or maximum value of wave during negative half cycle.* Since the values of these two are equal in magnitude, a sine wave is characterised by a single peak value. The peak value of the sine wave is shown in Fig. 4.13; here the peak value of the sine wave is 4 V.

❑ ***Peak-to-Peak Value*** The peak to peak value of a sine wave is the value from the positive to the negative peak as shown in Fig. 4.14. Here, the peak-to-peak value is 8 V.

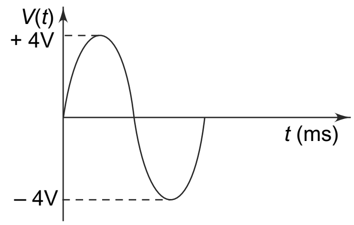


Fig. 4.14

❑ ***Average Value*** In general, the average value of any function $v(t)$, with period T is given by

$$v_{\text{av}} = \frac{1}{T}\int_0^T v(t)dt$$

That means that the average value of a curve in the X-Y plane is the total area under the complete curve divided by the distance of the curve. The average value of a sine wave over one complete cycle is always zero. So the average value of a sine wave is defined over a half-cycle, and not a full cycle period.

The ***average value of the sine wave*** *is the total area under the half-cycle curve divided by the distance of the curve.*

The average value of the sine wave

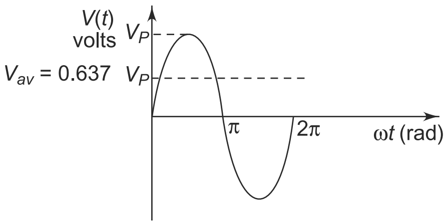


Fig. 4.15

$v(t) = V_P \sin \omega t$ is given by

$$v_{av} = \frac{1}{\pi}\int_0^\pi V_P \sin \omega t \; d(\omega t)$$
$$= \frac{1}{\pi}\left[-V_P \cos \omega t\right]_0^\pi$$
$$= \frac{2V_P}{\pi} = 0.637\ V_P$$

The average value of a sine wave is shown by the dotted line in Fig. 4.15.

EXAMPLE 4.6

Find the average value of a cosine wave f(t) = cos ωt shown in Fig. 4.16.

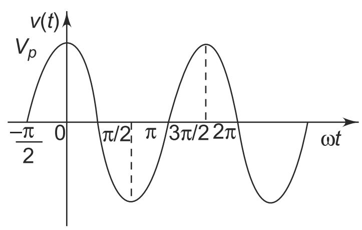


Fig. 4.16

Solution The average value of a cosine wave

$$v(t) = V_P \cos \omega t$$
$$V_{av} = \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} V_P \cos \omega t \; d(\omega t)$$
$$= \frac{1}{\pi} V_P (-\sin \omega t)_{\pi/2}^{3\pi/2}$$
$$= \frac{-V_P}{\pi}[-1-1] = \frac{2V_P}{\pi} = 0.637\ V_P$$

❑ *Root Mean Square Value or Effective Value*

The ***root mean square (rms) value of a sine wave*** *is a measure of the heating effect of the wave.* When a resistor is connected across a dc voltage source as shown in Fig. 4.17(a), a certain amount of heat is produced in the resistor in a given time. A similar resistor is connected across an ac voltage source for the same time as shown

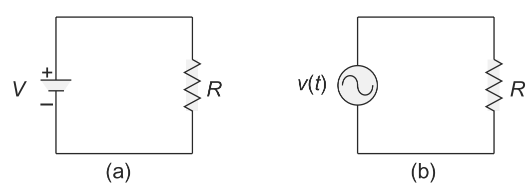


Fig. 4.17

in Fig. 4.17 (b). The value of the ac voltage is adjusted such that the same amount of heat is produced in the resistor as in the case of the dc source. This value is called the rms value.

That means the rms value of a sine wave is equal to the dc voltage that produces the same heating effect. In general, the *rms* value of any function with period T has an effective value given by

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T \overline{v(t)}^2\, dt}$$

Consider a function $v(t) = V_P \sin \omega t$

$$\text{The rms value, } V_{rms} = \sqrt{\frac{1}{T}\int_0^T (V_P \sin \omega t)^2\, d(\omega t)}$$

$$= \sqrt{\frac{1}{T}\int_0^{2\pi} V_P^2 \left[\frac{1-\cos 2\omega t}{2}\right] d(\omega t)}$$

$$= \frac{V_P}{\sqrt{2}} = 0.707\, V_P$$

If the function consists of a number of sinusoidal terms, that is

$$v(t) = V_0 + (V_{c1} \cos\ \omega t + V_{c2} \cos 2\ \omega t + \cdots) + (V_{s1} \sin\ \omega t + V_{s2} \sin 2\ \omega t + \cdots)$$

The rms, or effective value, is given by

$$V_{rms} = \sqrt{V_0^2 + \frac{1}{2}(V_{c1}^2 + V_{c2}^2 + \cdots) + \frac{1}{2}(V_{s1}^2 + V_{s2}^2 + \cdots)}$$

EXAMPLE 4.7

A wire is carrying a direct current of 20 A and a sinusoidal alternating current of peak value 20 A. Find the rms value of the resultant current in the wire.

Solution The rms value of the combined wave

$$= \sqrt{20^2 + \frac{20^2}{2}}$$

$$= \sqrt{400 + 200} = \sqrt{600} = 24.5\,\text{A}$$

❑ ***Peak Factor*** *The **peak factor** of any waveform is defined as the ratio of the peak value of the wave to the rms value of the wave.*

$$\text{Peak factor} = \frac{V_P}{V_{rms}}$$

$$\text{Peak factor of the sinusoidal waveform} = \frac{V_P}{V_P / \sqrt{2}} = \sqrt{2} = 1.414$$

❑ ***Form Factor*** ***Form factor** of a waveform is defined as the ratio of rms value to the average value of the wave.*

$$\text{Form factor} = \frac{V_{rms}}{V_{av}}$$

Form factor of a sinusoidal waveform can be found from the above relation.

For the sinusoidal wave, the form factor $= \dfrac{V_P/\sqrt{2}}{0.637V_P} = 1.11$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★**4-3.1** Sine wave A lags the sine wave B by 30°. Both have peak values of 15 V. Sine wave A is the reference with a positive going crossing at 0°. Determine the instantaneous value of the sine wave B at 30°, 90°, 45°, 180° and 300°.

☆★★**4-3.2** Find the *rms* values of

(a) $v(t) = 25 \cos \omega t + 15 \sin \omega t$

(b) $i(t) = 100 \sin \omega t - 10 \cos 2\omega t$

☆★★**4-3.3** A sawtooth voltage wave increases linearly from 0 to 200 V in the interval from 0 to 2 seconds. At $t_1 = 2$ s, its value drops to zero suddenly. The waveform repeats this pattern. Find the *rms* value of the voltage wave.

☆★★**4-3.4** Determine the V_{rms} of the waveform shown in Fig. Q.4.

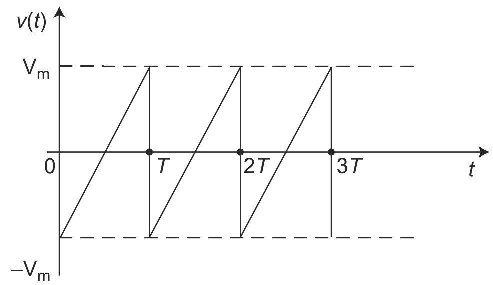


Fig. Q.4

☆☆★**4-3.5** Find the effective value of the resultant current in a wire which carries a direct current of 10 A and a sinusoidal current with a peak value of 15 A.

☆★★**4-3.6** Determine the value of K in the waveform shown in Fig. Q.6 where K is some function of the period T such that the effective value is 2.

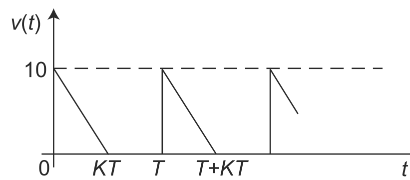


Fig. Q.6

☆★★**4-3.7** Determine the average and *rms* values of the waveform shown in Fig. Q.7, where in the first interval of $v(t) = 20\,e^{-200t}$.

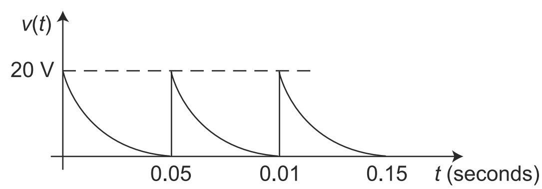


Fig. Q.7

☆★★**4-3.8** Find the effective value of the function $v = 100 + 50 \sin \omega t$.

☆★★**4-3.9** A full-wave rectified sine wave is clipped at 0.707 of its maximum value as shown in Fig. Q.9. Find the average and effective values of the function.

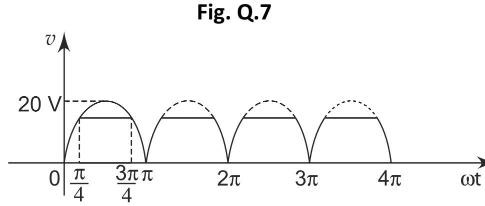


Fig. Q.9

★★★**4-3.10** Find the *rms* value of the function shown in Fig. Q.10 and described as follows:

$0 < t < 0.1 v = 40(1 - e^{-100t})$

$0.1 < t < 0.2 v = 40\, e^{-50(t - 0.1)}$

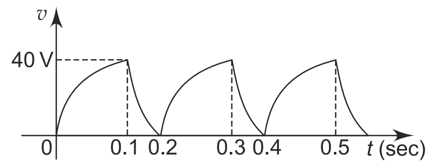


Fig. Q.10

☆★★**4-3.11** Calculate average and effective values of the waveform shown in Fig. Q.11 and, hence, find from factor.

☆★★**4-3.12** A full-wave rectified sine wave is clipped such that the effective value is $0.5\,V_m$ as shown in Fig. Q.12. Determine the amplitude at which the wave form is clipped.

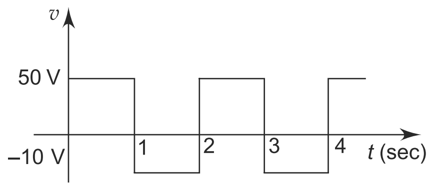


Fig. Q.11

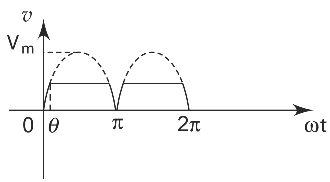


Fig. Q.12

Frequently Asked Questions linked to LO 3

☆☆★**4-3.1** Define RMS voltage. **[AU May/June 2014]**

★★★**4-3.2** A periodic voltage waveform has been shown in Fig. Q.14. Determine the following. **[JNTU Nov. 2012]**

(a) Frequency of the waveform

(b) Wave equation for 0 < t < 100 ms

(c) R.M.S value and

(d) Average value

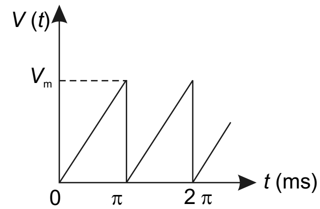


Fig. Q.14

☆★★**4-3.3** A non-alternating periodic waveform has been show in Fig. Q.3. Find its form factor and peak factor. **[JNTU Nov. 2012]**

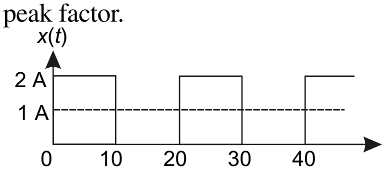


Fig. Q.3

☆☆★**4-3.4** Explain the rms value and average value of alternating quantity. Derive its necessary expressions. **[JNTU Nov. 2012]**

☆☆★**4-3.5** Define the following **[JNTU Nov. 2012]**

(a) Time period (b) Frequency (c) RMS value (d) Average value

4.5 PHASE RELATION IN A PURE RESISTOR

When a sinusoidal voltage of certain magnitude is applied to a resistor, a certain amount of sine wave current passes through it. We know the relation between $v(t)$ and $i(t)$ in the case of a resistor. The voltage/current relation in case of a resistor is linear,

LO 4 Analyse the phase relation in a pure resistor, pure inductor and pure capacitor

i.e. $v(t) = i(t)R$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Consider the function

$$i(t) = I_m \sin \omega t = IM\left[I_m e^{j\omega t}\right] \text{ or } I_m \angle 0^\circ$$

If we substitute this in the above equation, we have

$$v(t) = I_m R \sin \omega t = V_m \sin \omega t$$

$$= IM\left[V_m e^{j\omega t}\right] \text{ or } V_m \angle 0^\circ$$

where $V_m = I_m R$

If we draw the waveform for both voltage and current as shown in Fig. 4.18, there is no phase difference between these two waveforms. The amplitudes of the waveform may differ according to the value of resistance.

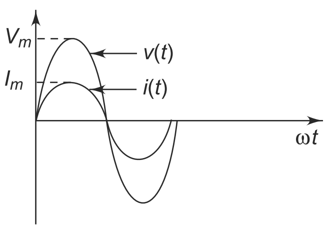


Fig. 4.18

As a result, in pure resistive circuits, the voltages and currents are said to be in phase. Here the term impedance is defined as the ratio of voltage to current function. With ac voltage applied to elements, the ratio of exponential voltage to the corresponding current (impedance) consists of magnitude and phase angles. Since the phase difference is zero in case of a resistor, the phase angle is zero. The impedance in case of resistor consists only of magnitude, i.e.

$$Z = \frac{V_m \angle 0^\circ}{I_m \angle 0^\circ} = R$$

4.6 PHASE RELATION IN A PURE INDUCTOR

LO 4

As discussed earlier in Chapter 1, the voltage current relation in the case of an inductor is given by

$$v(t) = L\frac{di}{dt}$$

Consider the function $i(t) = I_m \sin \omega t = IM\left[I_m e^{j\omega t}\right]$ or $I_m \angle 0^\circ$

$$v(t) = L\frac{d}{dt}(I_m \sin \omega t)$$

$$= L\omega I_m \cos \omega t = \omega L I_m \cos \omega t$$

$$v(t) = V_m \cos \omega t, \quad \text{or} \quad V_m \sin(\omega t + 90^\circ)$$

$$= IM\left[V_m e^{j(\omega t + 90^\circ)}\right] \quad \text{or} \quad V_m \angle 90^\circ$$

where $V_m = \omega L I_m = X_L I_m$

and $e^{j90^\circ} = j = 1\angle 90^\circ$

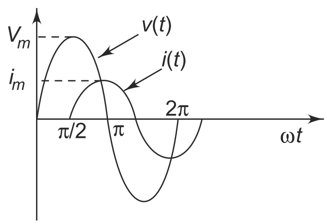


Fig. 4.19

If we draw the waveforms for both, voltage and current, as shown in Fig. 4.19, we can observe the phase difference between these two waveforms.

As a result, in a pure inductor the voltage and current are out of phase. The current lags behind the voltage by 90° in a pure inductor as shown in Fig. 4.20.

The impedance which is the ratio of exponential voltage to the corresponding current, is given by

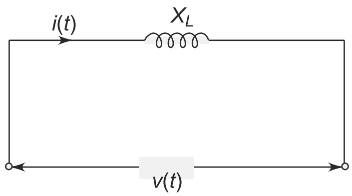


Fig. 4.20

$$Z = \frac{V_m \sin(\omega t + 90^\circ)}{I_m \sin \omega t}$$

where $V_m = \omega L I_m$

$$= \frac{I_m \omega L \sin(\omega t + 90^\circ)}{I_m \sin \omega t}$$

$$= \frac{\omega L I_m \angle 90^\circ}{I_m \angle 0^\circ}$$

$$\therefore \quad Z = j\omega L = jX_L$$

where $X_L = \omega L$ and is called the *inductive reactance*.
Hence, a pure inductor has an impedance whose value is ωL.

4.7 PHASE RELATION IN A PURE CAPACITOR

LO 4

As discussed in Chapter 1, the relation between voltage and current is given by

$$v(t) = \frac{1}{C} \int i(t)\, dt$$

Consider the function $i(t) = I_m \sin \omega t = IM\left[I_m e^{j\omega t}\right]$ or $I_m \angle 0^\circ$

$$v(t) = \frac{1}{C} \int I_m \sin \omega t \, d(t)$$

$$= \frac{1}{\omega C} I_m [-\cos \omega t]$$

$$= \frac{I_m}{\omega C} \sin(\omega t - 90^\circ)$$

$$\therefore \quad v(t) = V_m \sin(\omega t - 90^\circ)$$

$$= IM\left[I_m e^{j(\omega t - 90^\circ)}\right] \text{ or } V_m \angle -90^\circ$$

where $\quad V_m = \dfrac{I_m}{\omega C}$

$$\therefore \quad \frac{V_m \angle -90^\circ}{I_m \angle 0^\circ} = Z = \frac{-j}{\omega C}$$

Hence, the impedance is $Z = \dfrac{-j}{\omega C} = -jX_C$

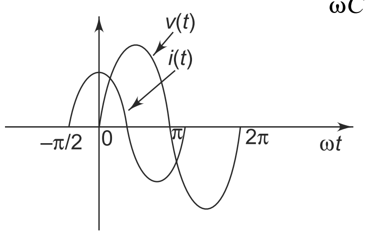


Fig. 4.21

where $X_C = \dfrac{1}{\omega C}$ and is called the capacitive reactance.

If we draw the waveform for both, voltage and current, as shown in Fig. 4.21, there is a phase difference between these two waveforms.

As a result, in a pure capacitor, the current leads the voltage by 90°. The impedance value of a pure capacitor

$$X_C = \frac{1}{\omega C}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★ **4-4.1** A sinusoidal voltage of $v(t) = 50 \sin(500t)$ applied to a capacitive circuit. Determine the capacitive reactance, and the current in the circuit.

Additional Solved Problems

PROBLEM 4.1

Find the average values of the voltages across R_1 and R_2. In Fig. 4.22, values shown are rms.

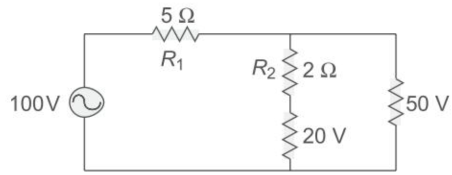


Fig. 4.22

Solution The voltage across the 2 Ω resistor $V_2 = 30$ V

The voltage across the 5 Ω resistor $V_5 = 50$ V

Peak value of the voltage across the 2 Ω resistor $= \sqrt{2} \times 30 = 42.43$ V

Peak value of the voltage across the 5 Ω resistor $= \sqrt{2} \times 50 = 70.71$ V

Average value across the 2 Ω resistor = 42.43 × 0.637 = 27.03 V

Average value across the 5 Ω resistor = 70.71 × 0.637 = 45 V

PROBLEM 4.2

A sinusoidal voltage is applied to the circuit shown in Fig. 4.23, determine rms current, average current, peak current, and peak-to-peak current.

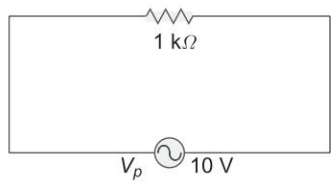


Fig. 4.23

Solution The equation for the applied voltage $v(t) = V_p \sin \omega t$

$= 10 \sin \omega t$

The equation for the current $i(t) = \dfrac{V_p}{1\text{k}\Omega} \sin \omega t = 10 \times 10^{-3} \sin \omega t$

Peak value of the current $i_p = 10$ mA

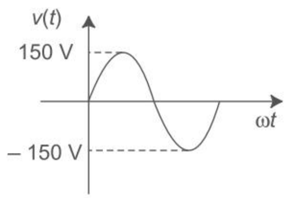
rms value of the current $i_{rms} = \dfrac{10}{\sqrt{2}} = 7.071$ mA

Peak-to-peak value of the current $i_{pp} = 20$ mA

Average value of the current $i_{av} = 10 \times 0.637 = 6.37$ mA.

PROBLEM 4.3

A sinusoidal voltage source in series with a dc source as shown in Fig. 4.24. Sketch the voltage across R_L. Determine the maximum current through R_L and the average voltage across R_L.



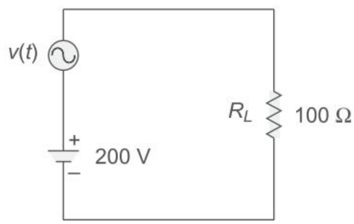


Fig. 4.24

Solution The voltage equation $v(t) = 150 \sin \omega t$

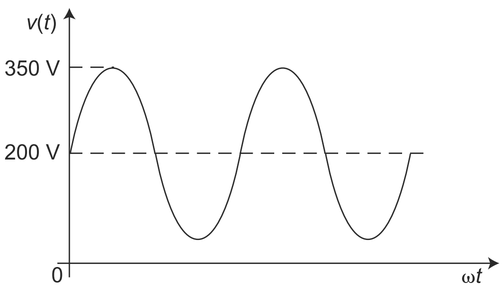


Fig. 4.25

The voltage across resistance R_L is

$v_{R_L}(t) = 200 + 150 \sin \omega t$ V.

Peak-to-peak value of the voltage is 300 V.

The voltage across R_L is shown in Fig. 4.25.

The current through the resistance R_L is

$$i_{R_L}(t) = \frac{1}{100}[200 + 150 \sin \omega t]$$

$$i_{R_L}(t) = 2 + 1.5 \sin \omega t \; A$$

Hence the maximum current is 3.5 A

Average voltage across R_L = 200 V.

PROBLEM 4.4

An alternating current varying sinusoidally, with a frequency of 50 Hz has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (a) 0.0025 s, and (b) 0.0125 s after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A?

Solution The rms value of the current waveform

$$I_{rms} = 20\text{A}$$

Peak of the current waveform

$$I_m = \sqrt{2} \times 20 = 28.28 \text{ volts}$$

The current equation $i(t) = I_m \sin \omega t = 28.28 \sin \omega t$

If it is passing through maximum positive value

$$i(t) = 28.28 \cos \omega t = 28.28 \cos 2\pi ft$$

$$i(t) = 28.28 \cos 100\, \pi t$$

(a) At $t = 0.0025$ s

$i(t) = 28.28 \cos 100\, \pi\, (0.0025) = 20$ A

(b) At $t = 0.0125$ s

$i(t) = 28.28 \cos 100\, \pi\, (0.0125) = -20$ A

At $t = \dfrac{1}{300}$ s, the instantaneous current becomes 14.14 A.

PROBLEM 4.5

Determine the rms value of the voltage defined by

$$v(t) = 5 + 5 \sin\left(314t + \frac{\pi}{6}\right)$$

Solution If the function consists of a number of sinusoidal terms, i.e.

$$v(t) = V_0 + (V_{C1} \cos \omega t + V_{C2} \cos 2\omega t + ...) + (V_{S1} \sin \omega t + V_{S2} \sin 2\omega t + ...)$$

The rms, or effective value is given by

$$V_{rms} = \sqrt{V_0^2 + \frac{1}{2}(V_{C1}^2 + V_{C2}^2 + ...) + (V_{S1}^2 + V_{S2}^2 + ...)}$$

From the equation

$$v(t) = 5 + 5\sin\left(314t + \frac{\pi}{6}\right)$$

The rms value of the waveform is

$$V_{rms} = \sqrt{5^2 + \frac{5^2}{2}} = \sqrt{25 + 12.5} = 6.12 \text{ V}.$$

PROBLEM 4.6

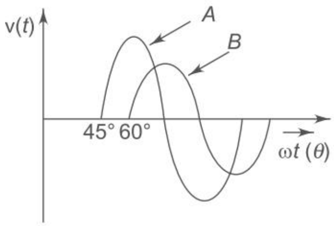


Fig. 4.26

Sine wave 'A' has a positive going zero crossing at 45°. Sine wave 'B' has a positive going zero crossing at 60°. Determine the phase angle between the signals. Which of the signals lags behind the other?

Solution The two signals are shown in Fig. 4.26.

From Fig. 4.26, the signal B lags behind signal A by 15°. In other words, the signal A leads the signal B by 15°.

PROBLEM 4.7

One sine wave has a positive peak at 75°, and another has a positive peak at 100°. How much is each sine wave shifted in phase from the 0° reference? What is the phase angle between them?

Solution The two signals are drawn as shown in Fig. 4.27.

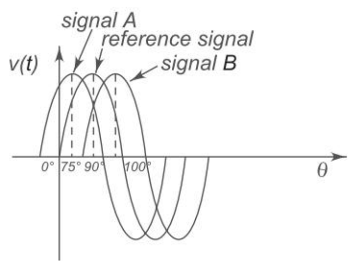


Fig. 4.27

The signal A leads the reference signal by 15°.
The signal B lags behind the reference signal by 10°.
The phase angle between these two signals is 25°.

PROBLEM 4.8

A sinusoidal voltage is applied to the resistive circuit shown in Fig. 4.28. Determine the following values.

(a) I_{rms} *(b)* I_{av}
(c) I_P *(d)* I_{PP}

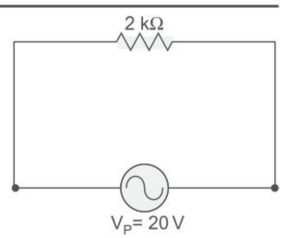


Fig. 4.28

Solution The function given to the circuit shown is

$$v(t) = V_P \sin \omega t = 20 \sin \omega t$$

The current passing through the resistor

$$i(t) = \frac{v(t)}{R}$$

$$i(t) = \frac{20}{2 \times 10^3} \sin \omega t$$

$$= 10 \times 10^{-3} \sin \omega t$$

$$I_P = 10 \times 10^{-3} \text{ A}$$

The peak value $I_P = 10$ mA

Peak-to-peak value $I_{PP} = 20$ mA

rms value I_{rms} $= 0.707\ I_P$

$= 0.707 \times 10 \text{ mA} = 7.07 \text{ mA}$

Average value I_{av} $= (0.637)\ I_P$

$= 0.637 \times 10 \text{ mA} = 6.37 \text{ mA}$

PROBLEM 4.9

A sinusoidal voltage is applied to a capacitor as shown in Fig. 4.29. The frequency of the sine wave is 2 kHz. Determine the capacitive reactance.

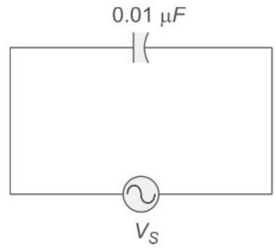


Fig. 4.29

Solution $X_C = \dfrac{1}{2\pi fC}$

$$= \frac{1}{2\pi \times 2 \times 10^3 \times 0.01 \times 10^{-6}}$$

$= 7.96$ kΩ

PROBLEM 4.10

Determine the rms current in the circuit shown in Fig. 4.30.

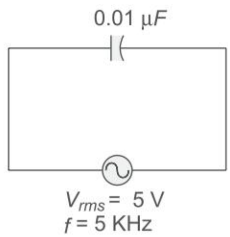


Fig. 4.30

Solution $X_C = \dfrac{1}{2\pi\ fC}$

$$= \frac{1}{2\pi \times 5 \times 10^3 \times 0.01 \times 10^{-6}}$$

$= 3.18$ kΩ

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{5}{3.18K} = 1.57 \text{ mA}$$

PROBLEM 4.11

A sinusoidal voltage is applied to the circuit shown in Fig. 4.31. The frequency is 3 kHz. Determine the inductive reactance.

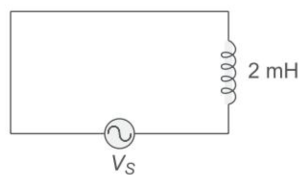


Fig. 4.31

Solution $X_L = 2\pi fL$

$= 2\pi \times 3 \times 10^3 \times 2 \times 10^{-3}$

$= 37.69\ \Omega$

PROBLEM 4.12

Determine the rms current in the circuit shown in Fig. 4.32.

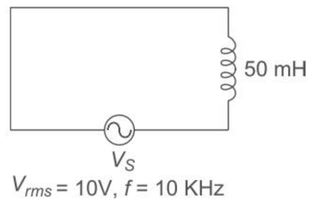


Fig. 4.32

Solution $X_L = 2\pi fL$

$= 2\pi \times 10 \times 10^3 \times 50 \times 10^{-3}$

$X_L = 3.141\ \text{k}\Omega$

$$I_{rms} = \frac{V_{rms}}{X_L}$$

$$= \frac{10}{3.141 \times 10^3} = 3.18\ \text{mA}$$

PROBLEM 4.13

Find the form factor of the half-wave rectified sine wave shown in Fig. 4.33.

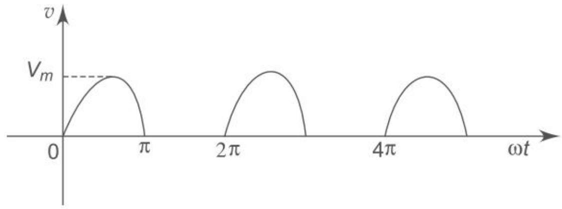


Fig. 4.33

Solution $v = V_m \sin \omega t$, for $0 < \omega t < \pi$

$= 0$, for $\pi < \omega t < 2\pi$

the period is 2π.

Average value
$$V_{av} = \frac{1}{2\pi}\left\{\int_0^{\pi} V_m \sin \omega t \, d(\omega t) + \int_{\pi}^{2\pi} 0 \, d(\omega t)\right\}$$

$$= 0.318 V_m$$

$$V_{rms}^2 = \frac{1}{2\pi}\int_0^{\pi} (V_m \sin \omega t)^2 \, d(\omega t) = \frac{1}{4} V_m^2$$

$$V_{rms} = \frac{1}{2} V_m$$

Form factor
$$= \frac{V_{rms}}{V_{av}}$$

$$= \frac{0.5 V_m}{0.318 V_m}$$

$$= 1.572$$

PROBLEM 4.14

Find the average and effective values of the sawtooth wave-form shown in Fig. 4.34 below.

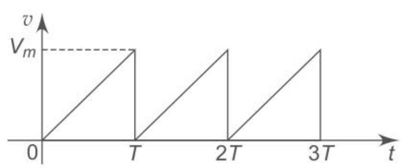


Fig. 4.34

Solution From Fig. 4.34 shown, the period is T.

$$V_{av} = \frac{1}{T}\int_0^{T} \frac{V_m}{T} t \, dt = \frac{1}{T}\frac{V_m}{T}\int_0^{T} t \, dt$$

$$= \frac{V_m}{T^2}\frac{T^2}{2} = \frac{V_m}{2}$$

Effective value $V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2 \; dt}$

$$= \sqrt{\frac{1}{T}\int_0^T \left[\frac{V_m}{T}t\right]^2 \; dt}$$

$$= \frac{V_m}{\sqrt{3}}$$

PROBLEM 4.15

Find the average and rms value of the full-wave rectified sine wave shown in Fig. 4.35.

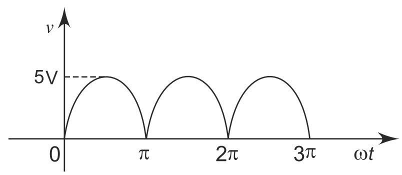


Fig. 4.35

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 5\sin\,\omega t \; d(\omega t)$

$$= 3.185$$

Effective value or rms value $= \sqrt{\frac{1}{\pi}\int_0^{\pi} (5\sin\omega t)^2 \; d(\omega t)}$

$$= \sqrt{\frac{25}{2}} = 3.54$$

PROBLEM 4.16

The full-wave rectified sine wave shown in Fig. 4.36 has a delay angle of 60°. Calculate V_{av} and V_{rms}.

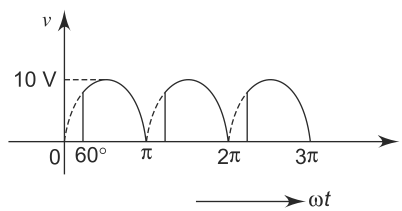


Fig. 4.36

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 10\sin(\omega t)\,d(\omega t)$

$$= \frac{1}{\pi}\int_{60^\circ}^{\pi} 10\sin\omega t\,d(\omega t)$$

$$V_{av} = \frac{10}{\pi}(-\cos\omega t)_{60}^{\pi} = 4.78$$

Effective value $V_{rms} = \sqrt{\frac{1}{\pi}\int_{60^\circ}^{\pi} (10\sin\omega t)^2 \; d(\omega t)}$

$$= \sqrt{\frac{100}{\pi}\int_{60^\circ}^{\pi} \left(\frac{1-\cos 2\omega t}{2}\right) d(\omega t)}$$

$$= 6.33$$

PROBLEM 4.17

Find the form factor of the square wave as shown in Fig. 4.37.

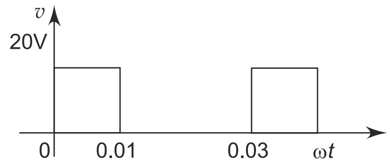


Fig. 4.37

Solution $v = 20$ for $0 < t < 0.01$

$= 0$ for $0.01 < t < 0.03$

The period is 0.03 second.

$$\text{Average value} \quad V_{av} = \frac{1}{0.03}\int_0^{0.01} 20\, dt$$

$$= \frac{20(0.01)}{0.03} = 6.66$$

$$\text{Effective value } V_{eff} = \sqrt{\frac{1}{0.03}\int_0^{0.01} (20)^2\, dt} = 66.6 = 0.816$$

$$\text{Form factor} = \frac{0.816}{6.66} = 0.123$$

Answers to Practice Problems

4-1.1 5 Hz: 20 Hz; 2 kHz; 100 kHz

4-2.1 15.4 V; 26.57 V; 16.22 V; −16.22 V

4-3.1 12.99 V; 12.99 V; 14.49 V; −7.5 V; −7.5 V

4-3.2 V_{rms} = 20.62 V; I_{rms} = 71.06 V

4-3.3 V_{rms} = 115.47 volts

4-3.4 $V_{rms} = \dfrac{V_m}{\sqrt{3}}$

4-3.6 $K = 0.12$

4-3.7 V_{av} = 2 V; V_{rms} = 4.47 V

4-3.10 27.57

4-3.12 0.581 V_m or 35.5°

Objective-Type Questions

☆☆★**4.1** One sine wave has a period of 2 ms, another has a period of 5 ms, and yet other has a period of 10 ms. Which sine wave is changing at a faster rate?
(a) Sine wave with period 2 ms
(b) Sine wave with period of 5 ms
(c) All are at the same rate
(d) Sine wave with period of 10 ms

☆☆★**4.2** How many cycles does a sine wave go through in 10 s when its frequency is 60 Hz?
(a) 10 cycles (b) 60 cycles (c) 600 cycles (d) 6 cycles

☆☆★**4.3** If the peak value of a certain sine wave voltage is 10 V, what is the peak-to- peak value?
(a) 20 V (b) 10 V (c) 5 V (d) 7.07 V

☆☆★**4.4** If the peak value of a certain sine wave voltage is 5 V, what is the rms value?
(a) 0.707 V (b) 3.535 V (c) 5 V (d) 1.17 V

☆☆★**4.5** What is the average value of a sine wave over a full cycle?
(a) V_m (b) $\dfrac{V_m}{\sqrt{2}}$ (c) zero (d) $\sqrt{2}\,V_m$

☆☆★**4.6** A sinusoidal current has peak value of 12 A. What is its average value?
(a) 7.64 A (b) 24 A (c) 8.48 A (d) 12 A

☆☆★**4.7** Sine wave *A* has a positive going zero crossing at 30°. Sine wave *B* has a positive going zero crossing at 45°. What is the phase angle between two signals?
(a) 30° (b) 45° (c) 75° (d) 5°

☆☆★**4.8** A sine wave has a positive going zero crossing at 0° and an rms value of 20 V. What is its instantaneous value at 145°.
(a) 7.32 V (b) 16.22 V (c) 26.57 V (d) 21.66 V

☆☆★**4.9** In a pure resistor, the voltage and current are
(a) out of phase (b) in phase (c) 90° out of phase (d) 45° out of phase

☆☆★**4.10** The rms current through a 10 kΩ resistor is 5 mA. What is the rms voltage drop across the resistor?
(a) 10 V (b) 5 V (c) 50 V (d) zero

☆☆★**4.11** In a pure capacitor, the voltage
(a) is in phase with the current (b) is out of phase with the current
(c) lags behind the current by 90° (d) leads the current by 90°

☆☆★**4.12** A sine-wave voltage is applied across a capacitor; when the frequency of the voltage is increased, the current
(a) increases (b) decreases (c) remains the same (d) is zero

☆☆★**4.13** The current in a pure inductor
(a) lags behind the voltage by 90° (b) leads the voltage by 90°
(c) is in phase with the voltage (d) lags behind the voltage by 45°

☆☆★**4.14** A sine-wave voltage is applied across an inductor; when the frequency of voltage is increased, the current
(a) increases (b) decreases (c) remains the same (d) is zero

☆★★**4.15** The rms value of the voltage for a voltage function $v = 10 + 5\cos(628t + 30°)$ volts through a circuit is
(a) 5 V (b) 10 V (c) 10.6 V (d) 15 V

☆☆★**4.16** For the same peak value, which of the following waves will have the highest rms value?
(a) Sine wave (b) Square wave
(c) Triangular wave (d) Half-wave rectified sine wave

☆☆★**4.17** For 100 volts rms value triangular wave, the peak voltage will be
(a) 100 V (b) 111 V (c) 141 V (d) 173 V

☆☆★**4.18** The form factor of dc voltage is
(a) zero (b) infinite (c) unity (d) 0.5

☆★★**4.19** For the half-wave rectified sine wave shown in Fig. 4.38, the peak factor is
(a) 1.41 (b) 2.0 (c) 2.82 (d) infinite

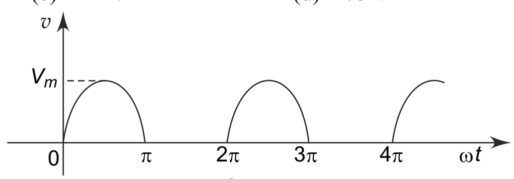


Fig. 4.38

☆★★**4.20** For the square wave shown in Fig. 4.39, the form factor is
(a) 2.0 (b) 1.0 (c) 0.5 (d) zero

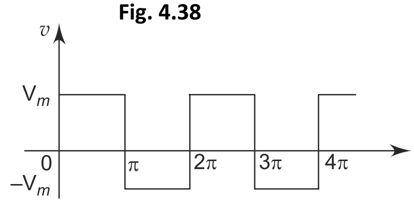


Fig. 4.39

☆★★**4.21** The power consumed in a circuit element will be least when the phase difference between the current and voltage is
(a) 0° (b) 30° (c) 90° (d) 180°

☆★★**4.22** A voltage wave consists of two components: a 50 V dc component and a sinusoidal component with a maximum value of 50 volts. The average value of the resultant will be
(a) zero (b) 86.6 V (c) 50 (d) none of the above

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scan the QR code given here
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http://qrcode.flipick.com/index.php/262

CHAPTER 5

Complex Impedance

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Analyse the circuits using complex impedance
- LO 2 Representation of sine waves in different forms using phasor diagrams
- LO 3 Analyse series circuits (*RL*, *RC*, *RLC*) using impedance diagram; draw phasor diagrams of series circuits (*RL, RC, RLC*)
- LO 4 Analyse parallel circuits (*RC, RL*) using impedance diagrams
- LO 5 Analyse compound circuits using impedance diagrams

5.1 IMPEDANCE DIAGRAM

LO 1 Analyse the circuits using complex impedance

So far, our discussion has been confined to resistive circuits. Resistance restricts the flow of current by opposing free electron movement. Each element has some resistance; for example, an inductor has some resistance; a capacitance also has some resistance. In the resistive element, there is no phase difference between the voltage and the current. In the case of pure inductance, the current lags behind the voltage by 90 degrees, whereas in the case of a pure capacitance, the current leads the voltage by 90 degrees. Almost all electric circuits offer impedance to the flow of current. ***Impedance*** *is a complex quantity having real and imaginary parts; where the real part is the resistance and the imaginary part is the reactance of the circuit.*

Consider the *RL* series circuit shown in Fig. 5.1. If we apply the real function $V_m \cos \omega t$ to the circuit, the response may be $I_m \cos \omega t$. Similarly, if we apply the imaginary function $jV_m \sin \omega t$ to the same circuit, the response is $jI_m \sin \omega t$. If we apply a complex function, which is a combination of real and imaginary functions, we will get a complex response.

This complex function is $V_m e^{j\omega t} = V_m (\cos \omega t + j \sin \omega t)$.

Applying Kirchhoff's law to the circuit shown in Fig. 5.1,

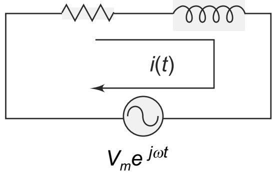


Fig. 5.1

we get $$V_m e^{j\omega t} = Ri(t) + L\frac{di(t)}{dt}$$

The solution of this differential equation is

$i(t) = I_m e^{j\omega t}$

By substituting $i(t)$ in the above equation, we get

$$V_m e^{j\omega t} = RI_m e^{j\omega t} + L\frac{d}{dt}(I_m e^{j\omega t})$$

$$V_m\, e^{j\omega t} = RI_m\, e^{j\omega t} + LI_m\, j\omega e^{j\omega t}$$

$$V_m = (R + j\omega L)I_m$$

Impedance is defined as the ratio of the voltage to current function

$$Z = \frac{V_m e^{j\omega t}}{\dfrac{V_m}{R + j\omega L} e^{j\omega t}} = R + j\omega L$$

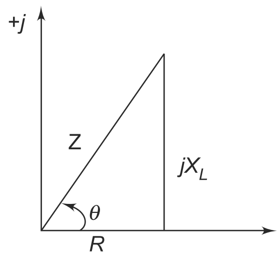


Fig. 5.2

Complex impedance is the total opposition offered by the circuit elements to ac current, and can be displayed on the complex plane. The impedance is denoted by Z. Here the resistance R is the real part of the impedance, and the reactance X_L is the imaginary part of the impedance. The resistance R is located on the real axis. The inductive reactance X_L is located on the positive j axis. The resultant of R and X_L is called the complex impedance.

Figure 5.2 is called the *impedance diagram* for the *RL* circuit. From Fig. 5.2, the impedance $Z = \sqrt{R^2 + (\omega L)^2}$, and angle $\theta = \tan^{-1} \omega L/R$. Here, the impedance is the vector sum of the resistance and inductive reactance. The angle between impedance and resistance is the phase angle between the current and voltage applied to the circuit.

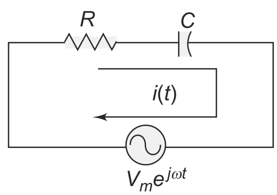


Fig. 5.3

Similarly, if we consider the *RC* series circuit, and apply the complex function $V_m\, e^{j\omega t}$ to the circuit in Fig. 5.3, we get a complex response as follows.

Applying Kirchhoff's law to the above circuit, we get

$$V_m\, e^{j\omega t} = Ri(t) + \frac{1}{C}\int i(t)\, dt$$

Solving this equation, we get

$$i(t) = I_m\, e^{j\omega t}$$

$$V_m\, e^{j\omega t} = R\, I_m\, e^{j\omega t} + \frac{1}{C} I_m \left(\frac{+1}{j\omega}\right) e^{j\omega t}$$

$$= \left[RI_m - \frac{j}{\omega C} I_m\right] e^{j\omega t}$$

$$V_m = \left(R - \frac{j}{\omega C}\right) I_m$$

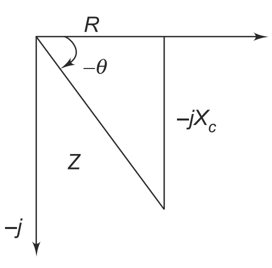


Fig. 5.4

The impedance

$$Z = \frac{V_m\, e^{j\omega t}}{V_m / [R - j/\omega C] e^{j\omega t}}$$

$$= [R - (j/\omega C)]$$

Here, the impedance Z consists of resistance (R), which is the real part, and capacitive reactance ($X_C = 1/\omega C$), which is the imaginary part of the impedance.

The resistance, R, is located on the real axis, and the capacitive reactance X_C is located on the negative j axis in the impedance diagram in Fig. 5.4.

From Fig. 5.4, impedance $Z = \sqrt{R^2 + X_C^2}$ or $\sqrt{R^2 + (1/\omega C)^2}$ and angle $\theta = \tan^{-1}(1/\omega CR)$. Here, the impedance, Z, is the vector sum of resistance and capacitive reactance. The angle between resistance and impedance is the phase angle between the applied voltage and current in the circuit.

5.2 PHASOR DIAGRAM

LO 2 Representation of sine waves in different forms using phasor diagrams

A phasor diagram can be used to represent a sine wave in terms of its magnitude and angular position. Examples of phasor diagrams are shown in Fig. 5.5.

In Fig. 5.5 (a), the length of the arrow represents the magnitude of the sine wave; angle θ represents the angular position of the sine wave. In Fig. 5.5 (b), the magnitude of the sine wave is one and the phase angle is 30°. In Fig. 5.5 (c) and (d), the magnitudes are four and three, and phase angles are 135° and 225°, respectively. The position of a phasor at any instant can be expressed as a positive or negative angle. Positive angles are measured counterclockwise from 0°, whereas negative angles are measured clockwise from 0°. For a given positive angle θ, the corresponding negative angle is θ − 360°. This is shown in Fig. 5.6 (a). In Fig. 5.6 (b), the positive angle 135° of vector A can be represented by a negative angle −225°, (135° − 360°).

A phasor diagram can be used to represent the relation between two or more sine waves of the same frequency. For example, the sine waves shown in Fig. 5.7 (a) can be represented by the phasor diagram shown in Fig. 5.7 (b).

In the above figure, the sine wave B lags behind the sine wave A by 45°; the sine wave C leads the sine wave A by 30°. The length of the phasors can be used to represent peak, rms, or average values.

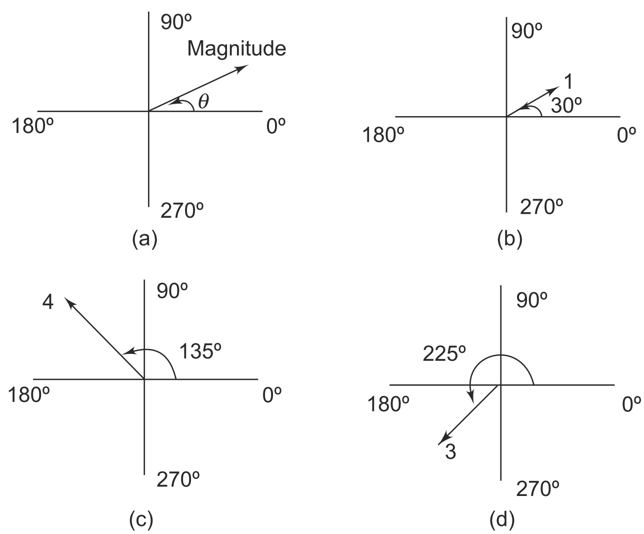


Fig. 5.5

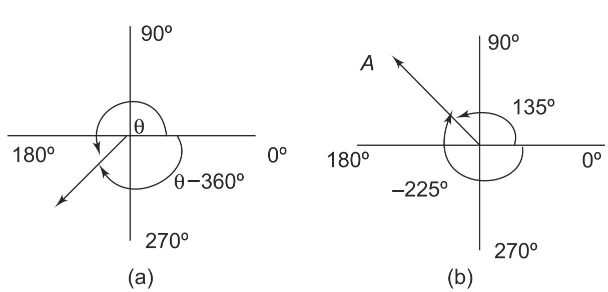
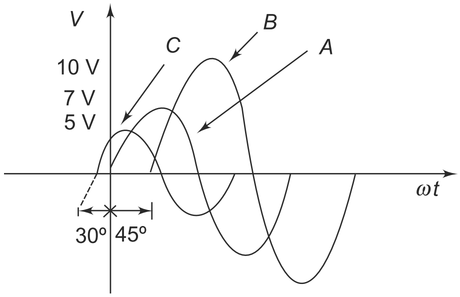
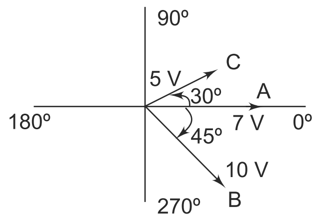


Fig. 5.6



(a)



(b)

Fig. 5.7

EXAMPLE 5.1

Draw the phasor diagram to represent the two sine waves shown in Fig. 5.8.

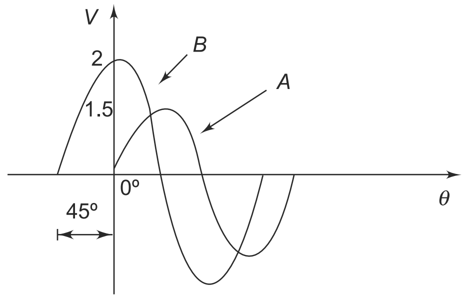


Fig. 5.8

Solution The phasor diagram representing the sine waves is shown in Fig. 5.9. The length of the each phasor represents the peak value of the sine wave.

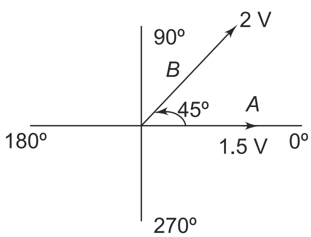


Fig. 5.9

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆★★**5-2.1** A 250 V, 50 Hz voltage is applied to a coil of 5 H inductance and resistance of 2 Ω in series with a capacitance *C*. What value must '*C*' have in order that the voltage across the coil shall be 280 V? Draw the phasor diagram.

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

5.3 SERIES CIRCUITS

The impedance diagram is a useful tool for analysing series ac circuits. Basically we can divide the series circuits as *RL*, *RC*, and *RLC* circuits. In the analysis of series ac circuits, one must draw the impedance diagram. Although the impedance diagram usually is not drawn to scale, it does represent a clear picture of the phase relationships.

> **LO 3** Analyse series circuits (RL, RC, RLC) using impedance diagram; draw phasor diagrams of series circuits (RL, RC, RLC)

5.3.1 Series *RL* Circuit

If we apply a sinusoidal input to an *RL* circuit, the current in the circuit and all voltages across the elements are sinusoidal. In the analysis of the *RL* series circuit, we can find the impedance, current, phase angle and voltage drops. In Fig. 5.10 (a), the resistor voltage (V_R) and current (I) are in phase with each other, but lag behind the source voltage (V_S). The inductor voltage (V_L) leads the source voltage (V_S). The phase angle between current and voltage in a pure inductor is always 90°. The amplitudes of voltages and currents in the circuit are completely dependent on the values of elements (i.e. the resistance and inductive reactance). In the circuit shown, the phase angle is somewhere between zero and 90° because of the series combination of resistance with inductive reactance, which depends on the relative values of R and X_L.

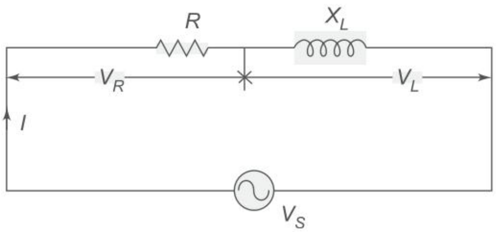


Fig. 5.10 (a)

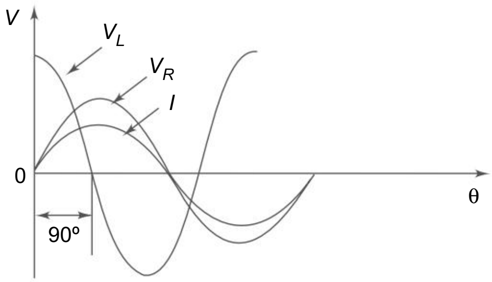


Fig. 5.10 (b)

The phase relation between current and voltages in a series *RL* circuit is shown in Fig. 5.10 (b).

Here, V_R and I are in phase. The amplitudes are arbitrarily chosen. From Kirchhoff's voltage law, the sum of the voltage drops must equal the applied voltage. Therefore, the source voltage V_S is the phasor sum of V_R and V_L.

$$\therefore \quad V_s = \sqrt{V_R^2 + V_L^2}$$

The phase angle between resistor voltage and source voltage is

$$\theta = \tan^{-1}(V_L/V_R)$$

where θ is also the phase angle between the source voltage and the current. The phasor diagram for the series *RL* circuit that represents the waveforms in Fig. 5.10 (c).

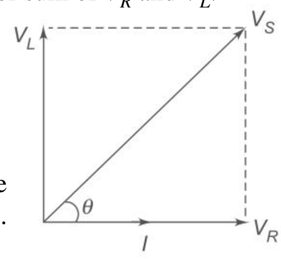


Fig. 5.10 (c)

EXAMPLE 5.2

To the circuit shown in Fig. 5.11, consisting a 1 kΩ resistor connected in series with a 50 mH coil, a 10 V rms, 10 kHz signal is applied. Find impedance Z, current I, phase angle θ, voltage across resistance V_R, and the voltage across inductance V_L.

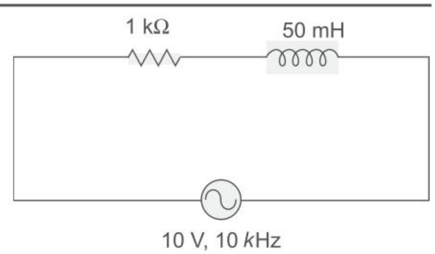


Fig. 5.11

Solution Inductive reactance $X_L = \omega L$

$$= 2\pi f L = (6.28)(10^4)(50 \times 10^{-3}) = 3140\ \Omega$$

In rectangular form,
Total impedance $Z = (1000 + j3140)\ \Omega$

$$= \sqrt{R^2 + X_L^2}$$

$$= \sqrt{(1000)^2 + (3140)^2} = 3295.4\ \Omega$$

Current $I = V_S/Z = 10/3295.4 = 3.03$ mA

Phase angle $\theta = \tan^{-1}(X_L/R) = \tan^{-1}(3140/1000) = 72.33°$

Therefore, in polar form, total impedance $Z = 3295.4\ \angle 72.33°$

Voltage across resistance $V_R = IR$

$$= 3.03 \times 10^{-3} \times 1000 = 3.03\ \text{V}$$

Voltage across inductive reactance $V_L = IX_L$

$$= 3.03 \times 10^{-3} \times 3140 = 9.51\ \text{V}$$

EXAMPLE 5.3

Determine the source voltage and the phase angle, if voltage across the resistance is 70 V and voltage across the inductive reactance is 20 V as shown in Fig. 5.12.

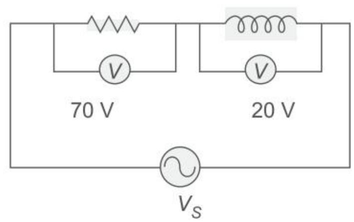


Fig. 5.12

Solution In Fig. 5.12, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_L^2}$$

$$= \sqrt{(70)^2 + (20)^2} = 72.8\ \text{V}$$

The angle between current and source voltage is

$$\theta = \tan^{-1}(V_L/V_R) = \tan^{-1}(20/70) = 15.94°$$

5.3.2 Series *RC* Circuit

When a sinusoidal voltage is applied to an RC series circuit, the current in the circuit and voltages across each of the elements are sinusoidal. The series *RC* circuit is shown in Fig. 5.13 (a).

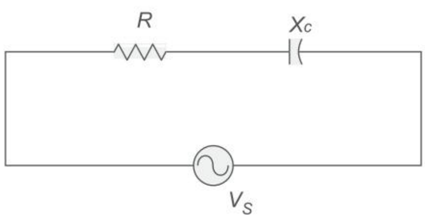


Fig. 5.13 (a)

Here, the resistor voltage and current are in phase with each other. The capacitor voltage lags behind the source voltage. The phase angle between the current and the capacitor voltage is always 90°. The amplitudes and the phase relations between the voltages and current depend on the ohmic values of the resistance and the capacitive reactance. The circuit is a series combination of both resistance and capacitance; and the phase angle between the applied voltage and the total current is somewhere between zero and 90°, depending on the relative values of the resistance and reactance. In a series RC circuit, the current is the same through the resistor and the capacitor. Thus, the resistor voltage is in phase with the current, and the capacitor voltage lags behind the current by 90° as shown in Fig. 5.13 (b).

Here, I leads V_C by 90°. V_R and I are in phase. From Kirchhoff's voltage law, the sum of the voltage drops must be equal to the applied voltage. Therefore, the source voltage is given by

$$\boxed{V_S = \sqrt{V_R^2 + V_C^2}}$$

The phase angle between the resistor voltage and the source voltage is

$$\boxed{\theta = \tan^{-1}(V_C/V_R)}$$

Since the resistor voltage and the current are in phase, θ also represents the phase angle between the source voltage and current. The voltage phasor diagram for the series RC circuit, voltage, and current phasor diagrams represented by the waveforms in Fig. 5.13 (b) are shown in Fig. 5.13 (c).

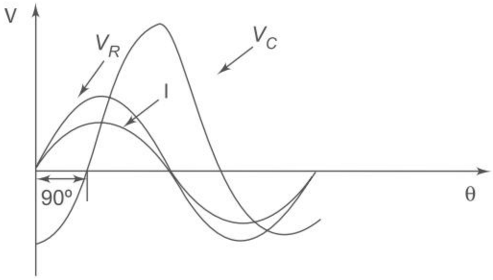


Fig. 5.13 (b)

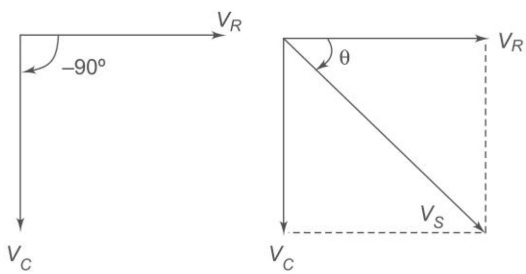


Fig. 5.13 (c)

EXAMPLE 5.4

A sine-wave generator supplies a 500 Hz, 10 V rms signal to a 2 kΩ resistor in series with a 0.1 μF capacitor as shown in Fig. 5.14. Determine the total impedance Z, current I, phase angle θ, capacitive voltage V_C, and resistive voltage V_R.

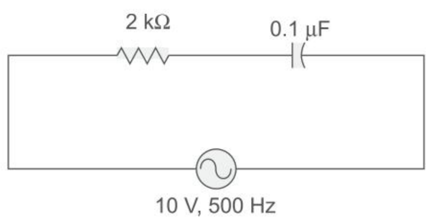


Fig. 5.14

Solution To find the impedance Z, we first solve for X_C

$$X_C = \frac{1}{2\pi f C} = \frac{1}{6.28 \times 500 \times 0.1 \times 10^{-6}}$$

$$= 3184.7\ \Omega$$

In rectangular form,
Total impedance $Z = (2000 - j3184.7)\ \Omega$

$$Z = \sqrt{(2000)^2 + (3184.7)^2}$$

$$= 3760.6\ \Omega$$

Phase angle $\theta = \tan^{-1}(-X_C/R) = \tan^{-1}(-3184.7/2000) = -57.87°$

Current $I = V_S/Z = 10/3760.6 = 2.66$ mA

Capacitive voltage $V_C = IX_C$

$$= 2.66 \times 10^{-3} \times 3184.7 = 8.47 \text{ V}$$

Resistive voltage $V_R = IR$

$$= 2.66 \times 10^{-3} \times 2000 = 5.32 \text{ V}$$

The arithmetic sum of V_C and V_R does not give the applied voltage of 10 volts. In fact, the total applied voltage is a complex quantity. In rectangular form,
Total applied voltage $V_S = 5.32 - j8.47$ V
In polar form,

$$V_S = 10 \angle -57.87° \text{ V}$$

The applied voltage is complex, since it has a phase angle relative to the resistive current.

EXAMPLE 5.5

Determine the source voltage and phase angle when the voltage across the resistor is 20 V and the capacitor is 30 V as shown in Fig. 5.15.

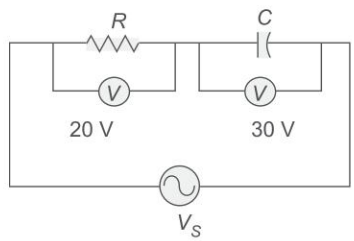


Fig. 5.15

Solution Since V_R and V_C are 90° out of phase, they cannot be added directly. The source voltage is the phasor sum of V_R and V_C.

$$\therefore \ V_S = \sqrt{V_R^2 + V_C^2} = \sqrt{(20)^2 + (30)^2} = 36 \text{ V}$$

The angle between the current and source voltage is

$$\theta = \tan^{-1}(V_C/V_R) = \tan^{-1}(30/20) = 56.3°$$

5.3.3 Series *RLC* Circuit

A series *RLC* circuit is the series combination of resistance, inductance and capacitance. If we observe the impedance diagrams of series *RL* and series RC circuits as shown in Fig. 5.16 (a) and (b), the inductive reactance, X_L, is displayed on the $+j$ axis and the capacitive reactance, X_C, is displayed on the $-j$ axis. These reactance are 180° apart and tend to cancel each other.

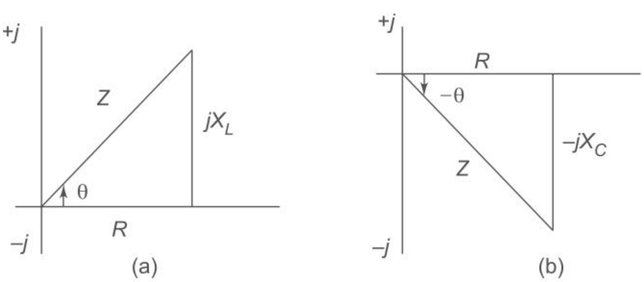


Fig. 5.16

The magnitude and type of reactance in a series *RLC* circuit is the difference of the two reactance. The impedance for an *RLC* series circuit is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. Similarly, the phase angle for an *RLC* circuit is

$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

EXAMPLE 5.6

In the circuit shown in Fig. 5.17, determine the total impedance, current I, phase angle θ, and the voltage across each element.

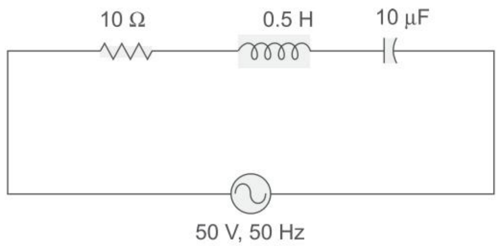


Fig. 5.17

Solution To find impedance Z, we first solve for X_C and X_L

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 50 \times 10 \times 10^{-6}}$$

$$= 318.5\ \Omega$$

$$X_L = 2\pi fL = 6.28 \times 0.5 \times 50 = 157\ \Omega$$

Total impedance in rectangular form

$$Z = (10 + j157 - j\,318.5)\ \Omega$$

$$= 10 + j(157 - 318.5)\ \Omega = 10 - j161.5\ \Omega$$

Here, the capacitive reactance dominates the inductive reactance.

$$Z = \sqrt{(10)^2 + (161.5)^2}$$

$$= \sqrt{100 + 26082.2} = 161.8\ \Omega$$

Current $\quad I = V_S / Z = \dfrac{50}{161.8} = 0.3\ \text{A}$

Phase angle $\theta = \tan^{-1}[(X_L - X_C)/R] = \tan^{-1}(-161.5/10) = -86.45°$

Voltage across the resistor $V_R = IR = 0.3 \times 10 = 3$ V

Voltage across the capacitive reactance $= IX_C = 0.3 \times 318.5 = 95.55$ V

Voltage across the inductive reactance $= IX_L = 0.3 \times 157 = 47.1$ V

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★ **5-3.1** For the circuit shown in Fig. Q.1, determine the impedance, phase angle, and total current.

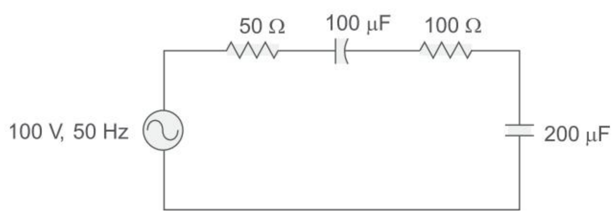


Fig. Q.1

☆☆★**5-3.2** Determine the impedance and phase angle in the circuit shown in Fig. Q.2.

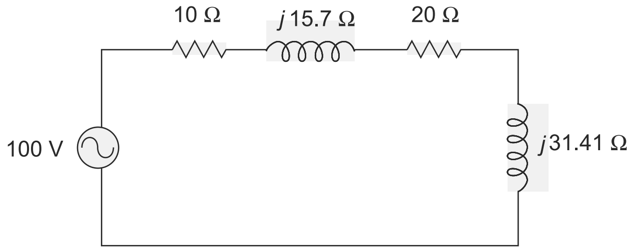


Fig. Q.2

☆☆★**5-3.3** Calculate the impedance at each of the following frequencies; also determine the current at each frequency in the circuit shown in Fig. Q.3.

(a) 100 Hz (b) 3 kHz

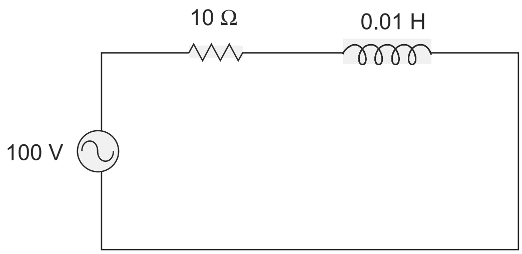


Fig. Q.3

☆★★**5-3.4** Find the values of R and C in the circuit shown in Fig. Q.4 so that $V_b = 4V_a$ and V_a, V_b are in phase quardrature.

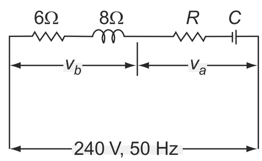


Fig. Q.4

☆☆★**5-3.5** The applied voltage to a series circuit is $v(t) = 50 \sin (2000t - 25°)$ volts and the resultant current through the circuit is $i(t) = 8 \sin (2000t + 5°)$ amperes. Find the circuit elements.

☆★★**5-3.6** The three-element series circuit contains one inductance $L = 0.02$ H. The applied voltage and resulting currents are shown on the phasor diagram in Fig. Q.6. If $\omega = 500$ *rad/sec*, what are the two circuit elements?

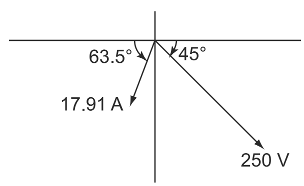


Fig. Q.6

☆☆★**5-3.7** For the circuit shown in Fig. Q.7, the voltage across the inductor is $v_L = 15 \sin 200\, t$. Find the total voltage and the angle by which the current lags the total voltage.

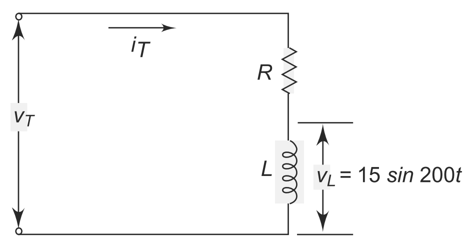


Fig. Q.7

Frequently Asked Questions linked to LO 3

☆☆★ **5-3.1** A coil having a resistance of 10 kΩ and inductance of 50 mH is connected to a 10-volt, 10 kHz power supply. Calculate the impedance. **[AU April/May 2011]**

☆★★ **5-3.2** A two-element series circuit is connected across ac source e(t) = 200 $\sqrt{2}$ sin(wt + 20°)V. The current in the circuit is then found to be i(t) =10 $\sqrt{2}$ cos(wt – 25°) A Determine the parameters of the circuit. **[JNTU Nov. 2012]**

5.4 PARALLEL CIRCUITS

The complex number system simplifies the analysis of parallel ac circuits. In series circuits, the current is the same in all parts of the series circuit. In parallel ac circuits, the voltage is the same across each element.

LO 4 Analyse parallel circuits (RC, RL) using impedance diagrams

5.4.1 Parallel *RC* Circuits

The voltages for an *RC* series circuit can be expressed using complex numbers, where the resistive voltage is the real part of the complex voltage and the capacitive voltage is the imaginary part. For parallel *RC* circuits, the voltage is the same across each component. Here, the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; the capacitive branch current is the imaginary part.

EXAMPLE 5.7

A signal generator supplies a sine wave of 20 V, 5 kHz to the circuit shown in Fig. 5.18. Determine the total current I_T, the phase angle and total impedance in the circuit.

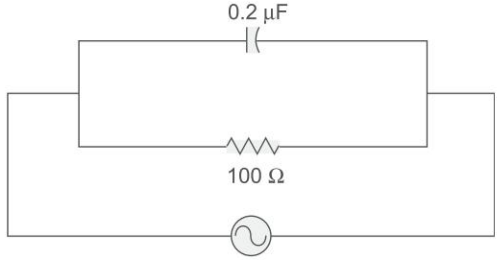


Fig. 5.18

Solution Capacitive reactance

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 5 \times 10^3 \times 0.2 \times 10^{-6}} = 159.2\ \Omega$$

Since the voltage across each element is the same as the applied voltage, we can solve for the two branch currents.

∴ current in the resistance branch

$$I_R = \frac{V_S}{R} = \frac{20}{100} = 0.2\ \text{A}$$

and current in the capacitive branch

$$I_C = \frac{V_S}{X_C} = \frac{20}{159.2} = 0.126\ \text{A}$$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

The total current is the vector sum of the two branch currents.

$\therefore$ total current $I_T = (I_R + jI_C)$ A $= (0.2 + j0.13)$ A

In polar form, $I_T = 0.24\ \angle 33°$

So the phase angle θ between applied voltage and total current is 33°. It indicates that the total line current is 0.24 A and leads the voltage by 33°. Solving for impedance, we get

$$Z = \frac{V_S}{I_T} = \frac{20\angle 0^\circ}{0.24\angle 33^\circ} = 83.3\angle -33^\circ\ \Omega$$

5.4.2 Parallel *RL* Circuits

In a parallel *RL* circuit, the inductive current is imaginary and lies on the $-j$ axis. The current angle is negative when the impedance angle is positive. Here also, the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; and inductive branch current is the imaginary part.

EXAMPLE 5.8

A 50 Ω resistor is connected in parallel with an inductive reactance of 30 Ω. A 20 V signal is applied to the circuit. Find the total impedance and line current in the circuit shown in Fig. 5.19.

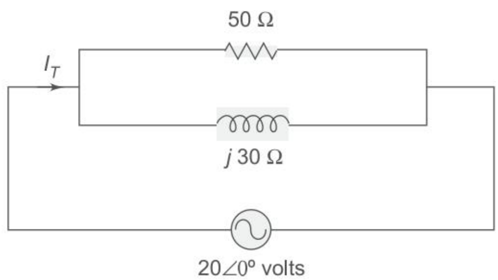


Fig. 5.19

Solution Since the voltage across each element is the same as the applied voltage, current in the resistive branch,

$$I_R = \frac{V_S}{R} = \frac{20\angle 0^\circ}{50\angle 0^\circ} = 0.4\ \text{A}$$

Current in the inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{20\angle 0^\circ}{30\angle 90^\circ} = 0.66\angle -90^\circ$$

Total current is $\quad I_T = 0.4 - j0.66$

In polar form, $\quad I_T = 0.77\ \angle -58.8°$

Here the current lags behind the voltage by 58.8°.

Total impedance $\quad Z = \dfrac{V_S}{I_T}$

$$= \frac{20\angle 0^\circ}{0.77\angle -58.8^\circ} = 25.97\angle 58.8^\circ\ \Omega$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★ **5-4.1** For the parallel circuit shown in Fig. Q.1, find the currents and the total current and construct the phasor diagram.

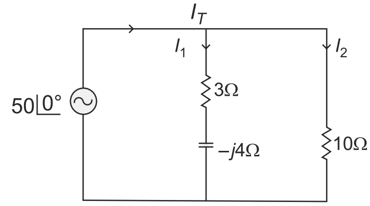


Fig. Q.1

★★★ **5-4.2** Determine the voltage across each element in the circuit shown in Fig. Q.2. Convert the circuit into an equivalent series form. Draw the voltage phasor diagram.

☆☆★ **5-4.3** For the circuit shown in Fig. Q.3, the applied voltage $v = V_m \cos \omega t$. Determine the current in each branch and obtain the total current in terms of the cosine function.

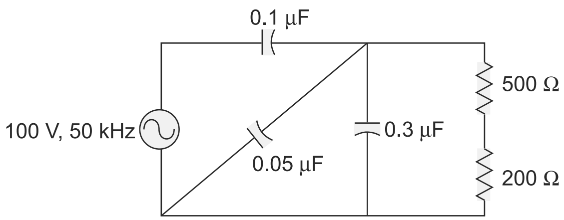


Fig. Q.2

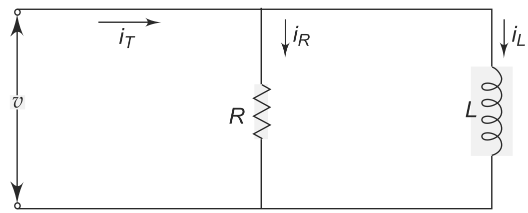


Fig. Q.3

☆☆★ **5-4.4** In a parallel circuit having a resistance $R = 5\ \Omega$ and $L = 0.02$ H, the applied voltage is $v = 100 \sin(1000\, t + 50°)$ volts. Find the total current and the angle by which the current lags the applied voltage.

☆★★ **5-4.5** In the parallel circuit shown in Fig. Q.5, the current in the inductor is five times greater than the current in the capacitor. Find the element values.

☆☆★ **5-4.6** For the circuit shown in Fig. Q.6, find the total current and the magnitude of the impedance.

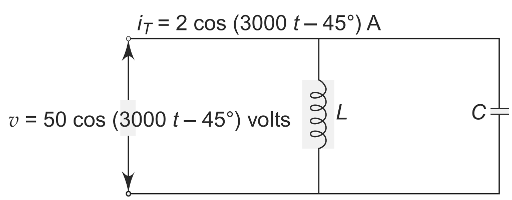


Fig. Q.5

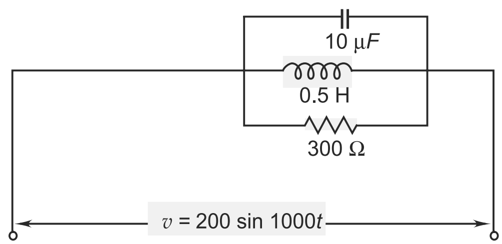


Fig. Q.6

5.5 COMPOUND CIRCUITS

In many cases, ac circuits to be analysed consist of a combination of series and parallel impedances. Circuits of this type are known as series-parallel, or compound circuits. Compound circuits can be simplified in the same manner as a series-parallel dc circuit consisting of pure resistances.

LO 5 Analyse compound circuits using impedance diagrams

EXAMPLE 5.9

Determine the equivalent impedance of Fig. 5.20.

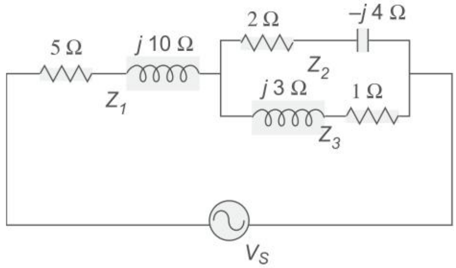


Fig. 5.20

Solution In the circuit, Z_1 is in series with the parallel combination of Z_2 and Z_3.

where $Z_1 = (5 + j10)\ \Omega$

$Z_2 = (2 - j4)\ \Omega$

$Z_3 = (1 + j3)\ \Omega$

The total impedance

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$= (5 + j10) + \frac{(2 - j4)(1 + j3)}{(2 - j4) + (1 + j3)}$$

$$= (5 + j10) + \frac{4.47\angle -63.4° \times 3.16\angle +71.5°}{3 - j1}$$

$$= (5 + j10) + \frac{14.12\angle 81°}{3 - j1}$$

$$= (5 + j10) + \frac{14.12\angle 81°}{3.16\angle -18°}$$

$$= 5 + j10 + 4.46\ \angle 26.1°$$

$$= 5 + j10 + 4 + j1.96$$

$$= 9 + j11.96$$

The equivalent circuit for the compound circuit shown in Fig. 5.20 is a series circuit containing 9 Ω of resistance and 11.96 Ω of inductive reactance. In polar form,

$$Z = 14.96\ \angle 53.03°$$

The phase angle between current and applied voltage is

$$\theta = 53.03°$$

EXAMPLE 5.10

In the circuit of Fig. 5.21, determine the values of the following (a) Z_T (b) I_T (c) θ.

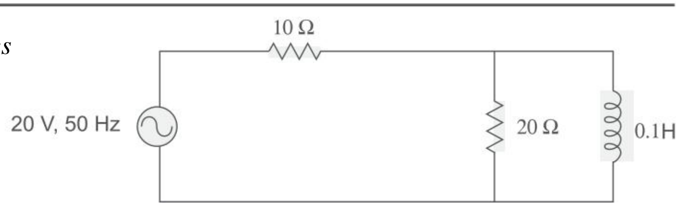


Fig. 5.21

Solution First, the inductive reactance is calculated.

$$X_L = 2\pi f L$$
$$= 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

In Fig. 5.22, the 10 Ω resistance is in series with the parallel combination of 20 Ω and $j31.42\ \Omega$.

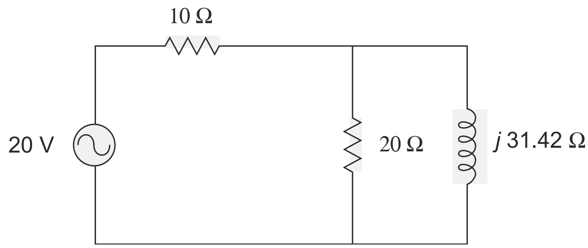


Fig. 5.22

$$\therefore\ Z_T = 10 + \frac{(20)\,(j31.42)}{(20 + j31.42)}$$

$$= 10 + \frac{628.4\angle 90^{\circ}}{37.24\angle 57.52^{\circ}} = 10 + 16.87\angle 32.48^{\circ}$$

$$= 10 + 14.23 + j9.06 = 24.23 + j\,9.06$$

In polar form, $Z_T = 25.87\ \angle 20.5°$

Here, the current lags behind the applied voltage by 20.5°.

Total current $I_T = \dfrac{V_S}{Z_T}$

$$= \frac{20}{25.87\angle 20.5^{\circ}} = 0.77\angle -20.5^{\circ}$$

The phase angle between voltage and current is

$$\theta = 20.5°$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆☆★**5-5.1** Determine the total impedance Z_T, the total current I_T, phase angle θ, voltage across inductor L, and voltage across resistor R_3 in the circuit shown in Fig. Q.1.

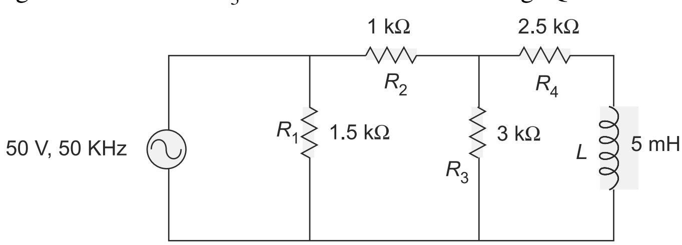


Fig. Q.1

☆☆★**5-5.2** For the circuit shown in Fig. Q.2, determine the total current I_T, phase angle θ and voltage across each element.

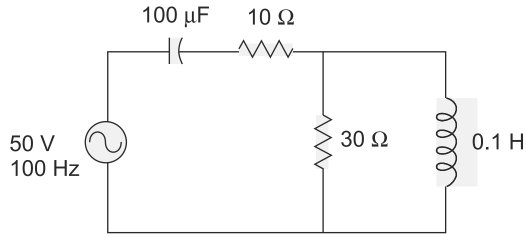


Fig. Q.2

☆☆★**5-5.3** In the parallel circuit shown in Fig. Q.3, the applied voltage is $v = 100 \sin 5000t$ V. Find the currents in each branch and also the total current in the circuit.

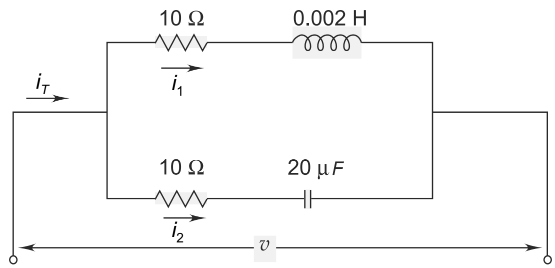


Fig. Q.3

☆★★**5-5.4** Solve for current I using PSpice in the circuit shown in Fig. Q.4.

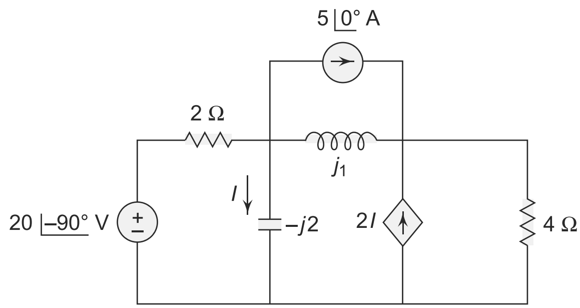


Fig. Q.4

☆★★ **5-5.5** Using PSpice, find i_0 for the circuit shown in Fig. Q.5.

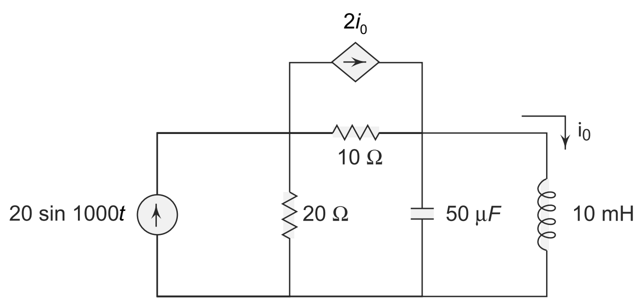


Fig. Q.5

☆★★**5-5.6** Using PSpice, find Thevenin equivalent circuit of the following Fig. Q.6 from terminals $a-b$ and $c-d$.

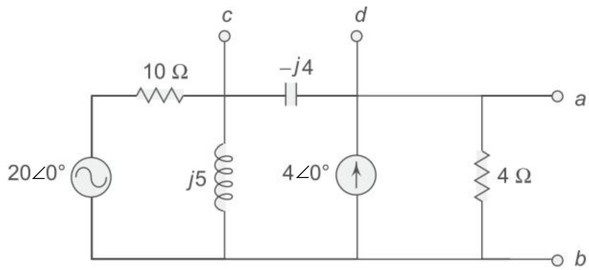


Fig. Q.6

☆★★**5-5.7** With PSpice, obtain i_0 using superposition principle (Fig. Q.7).

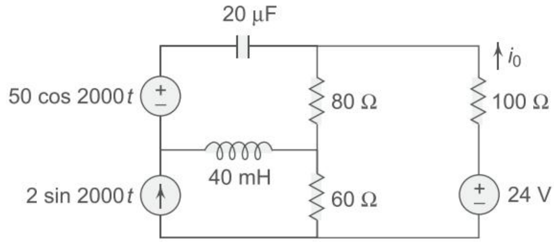


Fig. Q.7

Frequently Asked Questions linked to LO 5

☆☆★**5-5.1** For the circuit shown in Fig. Q.1, determine the total current IT, phase angle, and power factor. **[AU Nov./Dec. 2012]**

☆★★**5-5.2** In the two-mesh network shown in Fig. Q.2, determine the **[JNTU Nov. 2012]**
a) mesh current, b) power supplied by the source, and c) power dissipated in each resistor.

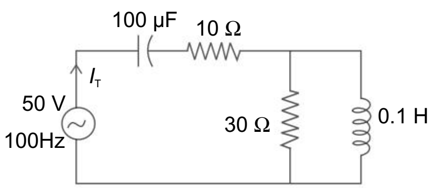


Fig. Q.1

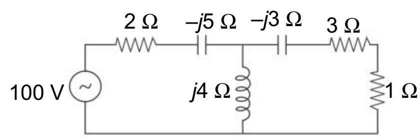


Fig. Q.2

Additional Solved Problems

PROBLEM 5.1

Calculate the total current in the circuit in Fig. 5.23, and determine the voltage across resistor V_R, and across capacitor V_C.

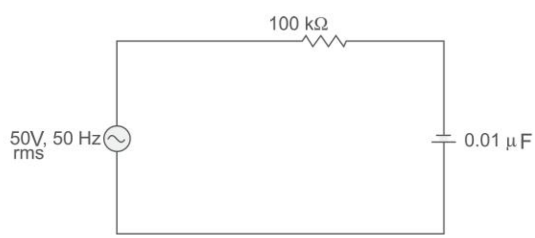


Fig. 5.23

Solution Total impedance in the circuit shown in Fig. 5.23 is

$$Z = 100\times10^3 - j\frac{1}{100\pi\times0.01\times10^{-6}}$$

$$Z = 333.65\times10^3\angle{-72.56^\circ}\ \Omega$$

The current in the circuit is

$$I = \frac{V}{Z} = \frac{50}{333.65\times10^3\angle{-72.56^\circ}}$$

$$= 0.15\times10^{-3}\angle{72.56^\circ}\ \text{A}$$

The voltage across resistor

$$V_R = IR = 0.15\times10^{-3}\times100\times10^3\angle{72.56^\circ}$$

$$= 15\angle{72.56^\circ}\ \text{V}$$

The voltage across capacitor

$$V_C = IX_C = 0.15\times10^{-3}\angle{72.56^\circ}\times\frac{1\angle{-90^\circ}}{100\pi\times0.01\times10^{-6}}$$

$$= 47.75\angle{-17.44^\circ}\ \text{V}$$

PROBLEM 5.2

A signal generator supplies a sine wave of 10 V, 10 kHz, to the circuit shown in Fig. 5.24(a). Calculate the total current in the circuit. Determine the phase angle θ for the circuit. If the total inductance in the circuit is doubled, does θ increase or decrease, and by how many degrees?

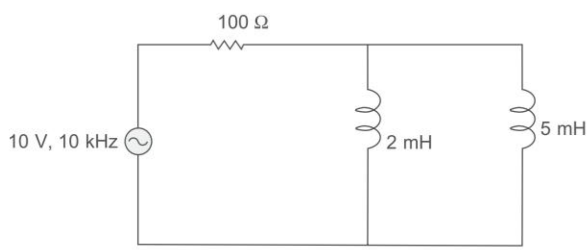


Fig. 5.24(a)

Solution In the circuit shown in Fig. 5.24 (a), the inductances in mH are converted into ohms as shown in Fig. 5.24 (b).

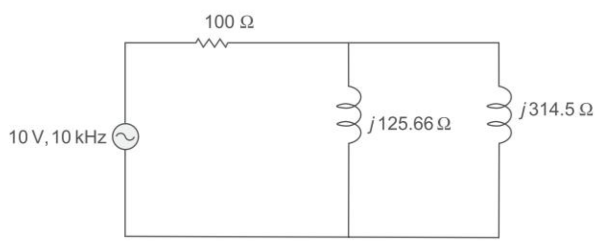


Fig. 5.24 (b)

The total impedance in the circuit

$$Z = 100 + (j125.66 \parallel j314.15)\ \Omega$$

$$= 100 + j89.85 = 134.43\angle{41.94^\circ}\ \Omega$$

The total current in the circuit

$$I = \frac{10}{134.43}\angle{-41.94^\circ} = 0.074\angle{-41.94^\circ}$$

If the inductance value is doubled, the impedance

$$Z = 100 + j179.7 = 205.65\angle{60.90^\circ}\ \Omega$$

Hence, the θ increased by 19°.

PROBLEM 5.3

For the circuit shown in Fig. 5.25, determine the voltage across each element. Is the circuit predominantly resistive or inductive? Find the current in each branch and the total current.

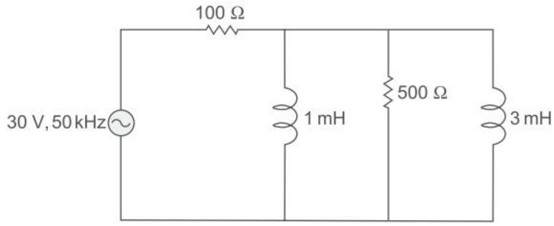


Fig. 5.25

Solution The circuit in Fig. 5.25 is converted as shown in Fig. 5.26.

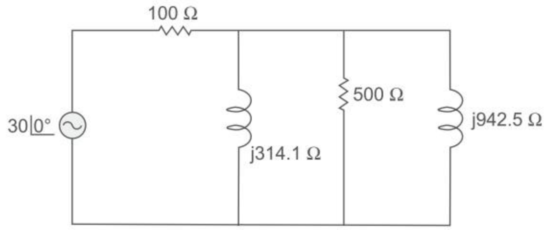


Fig. 5.26

Consider the node voltage V_1 at the node 1.
By applying Kirchhoff's current law,

$$\frac{V_1 - 30\angle 0^\circ}{100} + \frac{V_1}{314.1\angle 90^\circ} + \frac{V_1}{500} + \frac{V_1}{942.5\angle 90^\circ} = 0 \tag{5.1}$$

From Eq. (5.1), we get the node voltage

$$V_1 = 23.1\angle 19.71^\circ = 21.75 + j\,7.8$$

The voltage across the 100 Ω resistor $= 30\angle 0^\circ - \mathrm{V}_1$

$$= 30 - 21.75 - j\,7.8$$
$$= 8.25 - j\,7.8$$
$$= 11.35\angle -43.4^\circ \text{ V}$$

The voltage across remaining branch element is

$$V_1 = 23.1\angle 19.71^\circ \text{ V}$$

The current through the 100 Ω resistance $= \dfrac{V_{100}}{100} = \dfrac{11.35\angle -43.4^\circ}{100}$

$$I_{100} = 0.113\angle -43.4^\circ \text{ A}$$

The current through the 500 Ω resistance

$$I_{500} = \frac{V_1}{500} = \frac{23.1\angle 19.71^\circ}{500} = 0.046\angle 19.71^\circ \text{ A}$$

The current through the 1 mH inductor $= \dfrac{23.1\angle 19.71^\circ}{314.1\angle 90^\circ} = 0.073\angle -70.29^\circ$ A

The current through the 3 mH inductor $= \dfrac{23.1\angle 19.71^\circ}{942.5\angle 90^\circ} = 0.024\angle -70.29^\circ$ A

Since the total current lags, the circuit is predominantly inductive.

Total current $I = 0.113\angle -43.4^\circ$ A.

PROBLEM 5.4

For the circuit shown in Fig. 5.27, determine the value of frequency of supply voltage when a 100 V, 50 A current is supplied to the circuit.

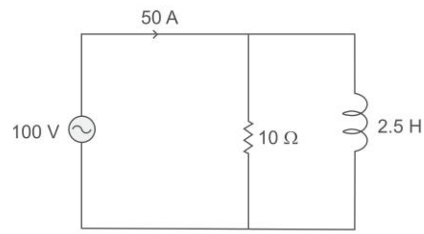


Fig. 5.27

Solution Magnitude of the total current = 50 A

Total impedance $Z_T = \dfrac{10(j\omega L)}{10 + j\omega L}$

Total current in the circuit $I_T = \dfrac{V}{Z_T}$

$$I_T = \frac{100(10 + j\omega L)}{j10\omega L}$$

$$\therefore I_T = 10 - j\frac{100}{\omega L}$$

The magnitude of the current $= \sqrt{(10)^2 + \left(\dfrac{100}{\omega L}\right)^2} = 50$

From the above equation, we have

ω = 0.816 and f = 0.125 Hz.

PROBLEM 5.5

A sine-wave generator supplies a signal of 100 V, 50 Hz to the circuit shown in Fig. 5.28. Find the current in each branch, and total current. Determine the voltage across each element.

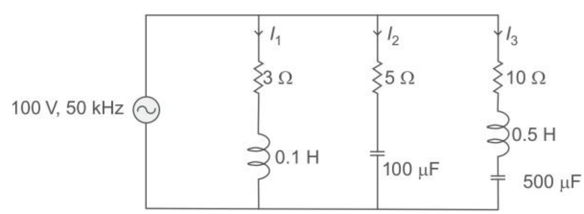


Fig. 5.28

Solution The inductance and capacitance values are converted into ohms as shown in Fig. 5.29.

The node voltage $V = 100\angle 0^\circ$

The current in (3 + j31.41) Ω is

$$I_1 = \frac{100\angle 0^\circ}{3 + j31.41} = (0.3 - j3.15)\text{ A}$$

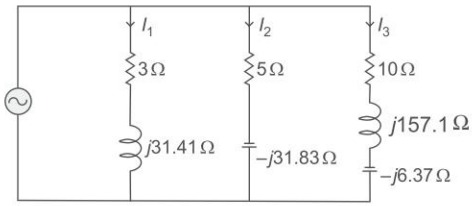


Fig. 5.29

The current in (5 − j31.83) Ω is

$$I_2 = \frac{100\angle 0^\circ}{5 - j31.83} = (0.48 + j3.1)\text{ A}$$

The current in (10 + j150.73) Ω is

$$I_3 = \frac{100\angle 0^\circ}{10 + j150.73} = (0.044 - j\,0.66)\text{ A}$$

The total current $I = I_1 + I_2 + I_3$

$$I = (0.824 - j0.71)\text{ A}$$

The voltage across the 3 Ω resistor $V_3 = 3(0.3 - j3.15) = 9.51\angle -84.5^\circ$ V

The voltage across the 5 Ω resistor $V_5 = 5(0.48 + j3.1) = 15.49\angle 81.1^\circ$ V

The voltage across the 10 Ω resistor $V_{10} = 10(0.044 - j0.66) = 6.6\angle -86.2^\circ$ V

The voltage across the 0.1 H inductance

$$V_{0.1\text{ H}} = (0.3 - j3.15)\,j(31.41) = 99.35\angle 5.44^\circ\text{ V}$$

The voltage across the 100 μF capacitance

$$V_{100\,\mu\text{F}} = (0.48 + j3.1) \times (-j31.83) = 99.62\angle -8.8^\circ\text{ V}$$

The voltage across the 0.5 H inductance

$$V_{0.5\text{ H}} = (0.044 - j0.66)\,(j157.81) = 103.84\angle 3.8^\circ\text{ V}$$

The voltage across the 500 μF capacitance

$$V_{500\,\mu\text{F}} = (0.044 - j0.66)\,(-j6.37) = 4.21\angle -176.2^\circ\text{ V}$$

PROBLEM 5.6

Determine the impedance and phase angle in the circuit shown in Fig. 5.30.

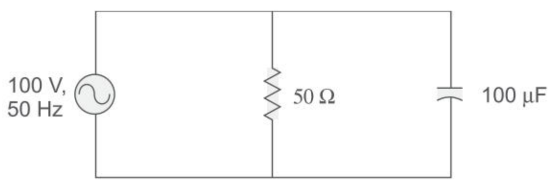


Fig. 5.30

Solution

$$\text{Capacitive reactance } X_C = \frac{1}{2\pi f C}$$

$$= \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

Capacitive susceptance $B_C = \dfrac{1}{X_C}$

$$= \frac{1}{31.83} = 0.031 \text{ S}$$

Conductance $G = \dfrac{1}{R} = \dfrac{1}{50} = 0.02 \text{ S}$

Total admittance $Y = \sqrt{G^2 + B_C^2}$

$$= \sqrt{(0.02)^2 + (0.031)^2}$$

$$= 0.037 \text{ S}$$

Total impedance $Z = \dfrac{1}{Y} = \dfrac{1}{0.037} = 27.02\ \Omega$

Phase angle $\theta = \tan^{-1}\left(\dfrac{R}{X_C}\right)$

$$= \tan^{-1}\left(\frac{50}{31.83}\right)$$

$$\theta = 57.52°$$

PROBLEM 5.7

For the parallel circuit in Fig. 5.31, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current?

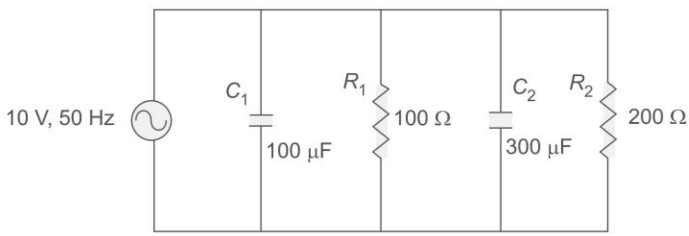


Fig. 5.31

Solution First let us find the capacitive reactances

$$X_{C1} = \frac{1}{2\pi f C_1}$$

$$= \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

$$X_{C2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}}$$

$$= 10.61\ \Omega$$

Here, the voltage across each element is the same as the applied voltage.

Current in the 100 μF capacitor, $I_{C_1} = \dfrac{V_S}{X_{C_1}}$

$$= \frac{10\angle 0^{\circ}}{31.83\angle -90^{\circ}} = 0.31\angle 90^{\circ}\text{ A}$$

Current in the 300 μF capacitor, $I_{C_2} = \dfrac{V_S}{X_{C_2}}$

$$= \frac{10\angle 0^{\circ}}{10.61\angle -90^{\circ}} = 0.94\angle 90^{\circ}\text{ A}$$

Current in the 100 Ω resistor is $I_{R_1} = \dfrac{V_S}{R_1} = \dfrac{10}{100} = 0.1\text{ A}$

Current in the 200 Ω resistor is $I_{R_2} = \dfrac{V_S}{R_2} = \dfrac{10}{200} = 0.05\text{ A}$

Total current $I_T = I_{R_1} + I_{R_2} + j(I_{C_1} + I_{C_2})$

$$= 0.1 + 0.05 + j(0.31 + 0.94) = 1.26\angle 83.2^{\circ}\text{ A}$$

The circuit shown in Fig. 5.31 can be simplified into a single parallel *RC* circuit as shown in Fig. 5.32.

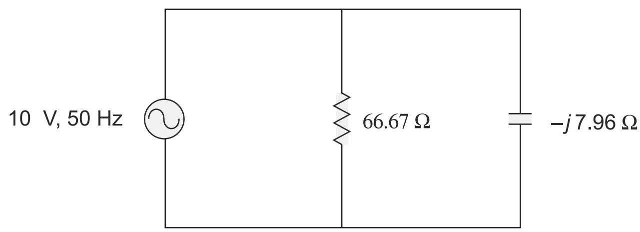


Fig. 5.32

In Fig. 5.31, the two resistances are in parallel and can be combined into a single resistance. Similarly, the two capacitive reactances are in parallel and can be combined into a single capacitive reactance.

$$R = \frac{R_1 R_2}{R_1 + R_2} = 66.67\ \Omega$$

$$X_C = \frac{X_{C_1} X_{C_2}}{X_{C_1} + X_{C_2}} = 7.96\ \Omega$$

Phase angle θ between voltage and current is

$$\theta = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}\left(\frac{66.67}{7.96}\right) = 83.19^{\circ}$$

Here, the current leads the applied voltage by 83.19°.

PROBLEM 5.8

For the circuit shown in Fig. 5.33, determine the total impedance, total current, and phase angle.

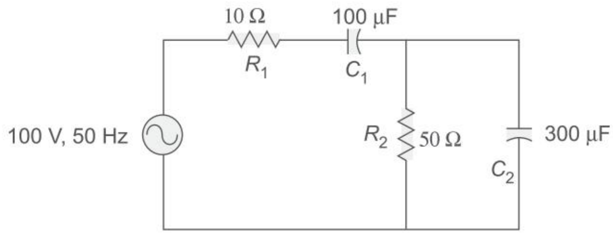


Fig. 5.33

Solution First, we calculate the magnitudes of the capacitive reactances.

$$X_{C_1} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

$$X_{C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}} = 10.61\ \Omega$$

We find the impedance of the parallel portion by finding the admittance.

$$G_2 = \frac{1}{R_2} = \frac{1}{50} = 0.02\ \text{S}$$

$$B_{C_2} = \frac{1}{X_{C_2}} = \frac{1}{10.61} = 0.094\ \text{S}$$

$$Y_2 = \sqrt{G_2^2 + B_{C_2}^2} = \sqrt{(0.02)^2 + (0.094)^2} = 0.096\ \text{S}$$

$$Z_2 = \frac{1}{Y_2} = \frac{1}{0.096} = 10.42\ \Omega$$

The phase angle associated with the parallel portion of the circuit

$$\theta_P = \tan^{-1}(R_2/X_{C_2}) = \tan^{-1}(50/10.61) = 78.02°$$

The series equivalent values for the parallel portion are

$$R_{eq} = Z_2 \cos\theta_P = 10.42 \cos(78.02°) = 2.16\ \Omega$$

$$X_{C(eq)} = Z_2 \sin\theta_P = 10.42 \sin(78.02°) = 10.19\ \Omega$$

The total resistance

$$R_T = R_1 + R_{eq}$$
$$= (10 + 2.16) = 12.16\ \Omega$$

$$X_{C_T} = X_{C_1} + X_{C(eq)}$$
$$= (31.83 + 10.19) = 42.02\ \Omega$$

Total impedance

$$Z_T = \sqrt{R_T^2 + X_{C_T}^2}$$

$$= \sqrt{(12.16)^2 + (42.02)^2} = 43.74\ \Omega$$

We can also find the total current by using Ohm's law.

$$I_T = \frac{V_S}{Z_T} = \frac{100}{43.74} = 2.29\ \text{A}$$

The phase angle

$$\theta = \tan^{-1}\left(\frac{X_{CT}}{R_T}\right)$$

$$= \tan^{-1}\left(\frac{42.02}{12.16}\right) = 73.86°$$

PROBLEM 5.9

Determine the voltage across each element of the circuit shown in Fig. 5.34 and draw the voltage phasor diagram.

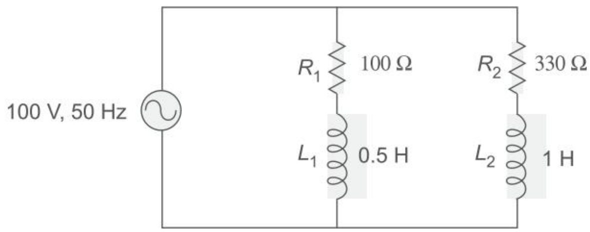


Fig. 5.34

Solution First we calculate X_{L_1} and X_{L_2}

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 0.5 = 157.08\ \Omega$$

$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 1.0 = 314.16\ \Omega$$

Now we determine the impedance of each branch

$$Z_1 = \sqrt{R_1^2 + X_{L_1}^2} = \sqrt{(100)^2 + (157.08)^2} = 186.2\ \Omega$$

$$Z_2 = \sqrt{R_2^2 + X_{L_2}^2} = \sqrt{(330)^2 + (314.16)^2} = 455.63\ \Omega$$

The current in each branch

$$I_1 = \frac{V_S}{Z_1} = \frac{100}{186.2} = 0.537\ \text{A}$$

and $$I_2 = \frac{V_S}{Z_2} = \frac{100}{455.63} = 0.219\ \text{A}$$

The voltage across each element

$$V_{R_1} = I_1 R_1 = 0.537 \times 100 = 53.7 \text{ V}$$
$$V_{L_1} = I_1 X_{L_1} = 0.537 \times 157.08 = 84.35 \text{ V}$$
$$V_{R_2} = I_2 R_2 = 0.219 \times 330 = 72.27 \text{ V}$$
$$V_{L_2} = I_2 X_{L_2} = 0.219 \times 314.16 = 68.8 \text{ V}$$

The angles associated with each parallel branch are now determined.

$$\theta_1 = \tan^{-1}\left(\frac{X_{L_1}}{R_1}\right) = \tan^{-1}\left(\frac{157.08}{100}\right)$$
$$= 57.52°$$

$$\theta_2 = \tan^{-1}\left(\frac{X_{L_2}}{R_2}\right) = \tan^{-1}\left(\frac{314.16}{330}\right)$$
$$= 43.59°$$

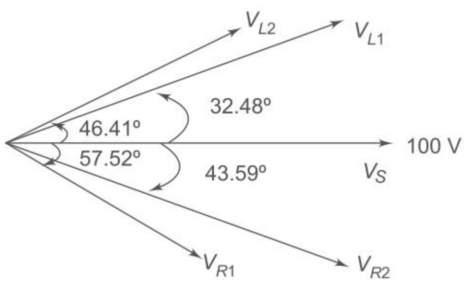


Fig. 5.35

i.e. I_1 lags behind V_S by 57.52° and I_2 lags behind V_S by 43.59°.

Here, V_{R_1} and I_1 are in phase and therefore, lag behind V_S by 57.52°.

V_{R_2} and I_2 are in phase, and therefore lag behind V_S by 43.59°.

V_{L_1} leads I_1 by 90°, so its angle is 90° – 57.52° = 32.48°.

V_{L_2} leads I_2 by 90°, so its angle is 90° – 43.59° = 46.41°.

The phase relations are shown in Fig. 5.35.

PROBLEM 5.10

In the series–parallel circuit shown in Fig. 5.36, the effective value of voltage across the parallel parts of the circuits is 50 V. Determine the corresponding magnitude of V.

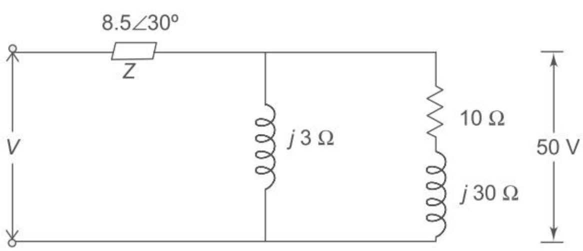


Fig. 5.36

Solution Here, we can determine the current in each branch of the parallel part.

Current in the $j3\ \Omega$ branch, $I_1 = \dfrac{50}{3} = 16.67 \text{ A}$

Current in $(10 + j30)\ \Omega$ branch, $I_2 = \dfrac{50}{31.62} = 1.58\ \text{A}$

Total current $I_T = 16.67 + 1.58 = 18.25$ A

$$\text{Total impedance } Z_T = 8.5\angle 30^\circ + \frac{3\angle 90^\circ \times (10 + j30)}{(10 + j30) + 3\angle 90^\circ}$$

$$= 8.5\angle 30^\circ + \frac{3\angle 90^\circ \times 31.62\angle 71.57^\circ}{10 + j33}$$

$$= 7.36 + j4.25 + \frac{94.86\angle 161.57^\circ}{34.48\angle 73.14^\circ}$$

$$= 7.36 + j4.25 + 2.75\angle 88.43^\circ$$

$$= 7.36 + j4.25 + 0.075 + j\,2.75$$

$$= (7.435 + j7)\ \Omega$$

$$= 10.21\ \angle 43.27^\circ$$

In polar form, total impedance is $Z_T = 10.21\ \angle 43.27^\circ$

The magnitude of applied voltage $V = I \times Z_T = 18.25 \times 10.21 = 186.33$ V.

PROBLEM 5.11

For the series parallel circuit shown in Fig. 5.37, determine (a) the total impedance between the terminals a, b and state if it is inductive or capacitive, (b) the voltage across in the parallel branch, and (c) the phase angle.

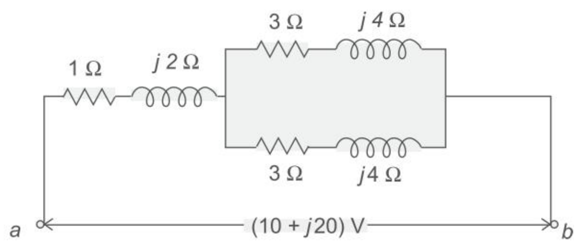


Fig. 5.37

Solution Here, the parallel branch can be combined into a single branch

$$Z_P = (3 + j4)\ \|\ (3 + j4) = (1.5 + j2)\ \Omega$$

Total impedance $Z_T = 1 + j2 + 1.5 + j2 = (2.5 + j4)\ \Omega$

Hence, the total impedance in the circuit is inductive
Total current in the circuit

$$I_T = \frac{V_S}{Z_T} = \frac{10 + j20}{2.5 + j4}$$

$$= \frac{22.36\angle 63.43^\circ}{4.72\angle 57.99^\circ}$$

$\therefore \qquad I_T = 4.74\angle 5.44°\text{ A}$

i.e. the current lags behind the voltage by 57.99°.

Phase angle $\theta = 57.99°$

Voltage across in the parallel branch

$$V_P = (1.5 + j2)\ 4.74\ \angle 5.44°$$
$$= 2.5 \times 4.74\ \angle(5.44° + 53.13°)$$
$$= 11.85\ \angle 58.57°\text{ V}$$

PROBLEM 5.12

In the series parallel circuit shown in Fig. 5.38, the two parallel branches A and B are in series with C. The impedances are $Z_A = 10 + j8$, $Z_B = 9 - j6$, $Z_C = 3 + j2$ and the voltage across the circuit is $(100 + j0)$ V. Find the currents I_A, I_B and the phase angle between them.

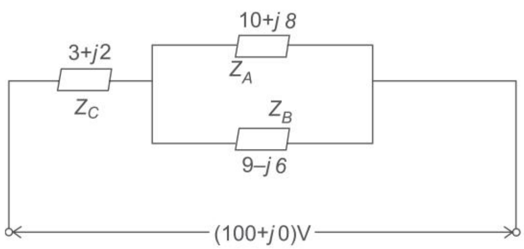


Fig. 5.38

Solution Total parallel branch impedance,

$$Z_P = \frac{Z_A Z_B}{Z_A + Z_B}$$
$$= \frac{(10 + j8)(9 - j6)}{19 + j2}$$
$$= \frac{12.8\angle 38.66° \times 10.82\angle -33.7°}{19.1\angle 6°} = 7.25\angle -1.04°$$

In rectangular form,

Total parallel impedance $Z_P = 7.25 - j0.13$

This parallel impedance is in series with Z_C

Total impedance in the circuit

$$Z_T = Z_P + Z_C = 7.25 - j0.13 + 3 + j2 = (10.25 + j1.87)\ \Omega$$

Total current $\quad I_T = \dfrac{V_S}{Z_T}$

$$= \frac{(100 + j0)}{(10.25 + j1.87)} = \frac{100\angle 0°}{10.42\angle 10.34°}$$
$$= 9.6\angle -10.34°$$

The current lags behind the applied voltage by 10.34°.

Current in the branch A is

$$I_A = I_T \frac{Z_B}{Z_A + Z_B}$$

$$= 9.6\angle{-10.34°} \times \frac{(9 - j6)}{19 + j2}$$

$$= \frac{9.6\angle{-10.34°} \times 10.82\angle{-33.7°}}{19.1\angle 6°}$$

$$= 5.44\angle{-50.04°}\text{ A}$$

Current in the branch B is I_B

$$I_B = I_T \times \frac{Z_A}{Z_A + Z_B}$$

$$= 9.6\angle{-10.34°} \times \frac{10 + j8}{19 + j2}$$

$$= \frac{9.6\angle{-10.34°} \times 12.8\angle 38.66°}{19.1\angle 6°}$$

$$= 6.43\angle 22.32°\text{ A}$$

The angle between I_A and I_B,

$$\theta = (50.04° + 22.32°) = 72.36°$$

PROBLEM 5.13

A series circuit of two pure elements has the following applied voltage and resulting current.

V ∇ 15 cos (200 t – 30°) volts

I ∇ 8.5 cos (200 t ⩽ 15) volts

Find the elements comprising the circuit.

Solution By inspection, the current leads the voltage by 30° + 15° = 45°. Hence, the circuit must contain resistance and capacitance.

$$\tan 45° = \frac{1}{\omega CR}$$

$$1 = \frac{1}{\omega CR}, \quad \therefore \quad \frac{1}{\omega C} = R$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{R^2 + R^2} = \sqrt{2}$$

$$\therefore \qquad R = \frac{15}{8.5 \times \sqrt{2}} = 1.248\ \Omega$$

$$\frac{1}{\omega C} = 1.248\ \Omega$$

$$\text{and} \quad C = \frac{1}{200 \times 1.248} = 4 \times 10^{-3}\ \text{F}$$

PROBLEM 5.14

A resistor having a resistance of $R = 10\ \Omega$ and an unknown capacitor are in series. The voltage across the resistor is $V_R = 50 \sin (1000\, t + 45°)$ volts. If the current leads the applied voltage by 60° what is the unknown capacitance C?

Solution Here, the current leads the applied voltage by 60°.

$$\tan 60° = \frac{1}{\omega CR}$$

Since $$R = 10\ \Omega$$

$$\omega = 1000 \text{ radians}$$

$$\tan 60° = \frac{1}{\omega CR}$$

$$C = \frac{1}{\tan 60 \times 1000 \times 10} = 57.7\ \mu\text{F}$$

PROBLEM 5.15

A series circuit consists of two pure elements has the following current and voltage.

$\omega \nabla 100 \sin (2000\, t \leqslant 50°)\ V$

$i \nabla 20 \cos (2000\, t \leqslant 20°)\ A$

Find the elements in the circuit.

Solution We can write $i = 20 \sin (2000\, t + 20° + 90°)$

Since $\cos \theta = \sin (\theta + 90°)$

Current $i = 20 \sin (2000\, t + 110°)$ A

The current leads the voltage by 110° – 50° = 60°

and the circuit must consist of resistance and capacitance.

$$\tan \theta = \frac{1}{\omega CR}$$

$$\frac{1}{\omega C} = R \tan 60° = 1.73\, R$$

$$\frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \frac{100}{20}$$

$$R\sqrt{1 + (1.73)^2} = \frac{100}{20}$$

$$R(1.99) = 5$$

$$R = 2.5\ \Omega$$

and $$C = \frac{1}{\omega (1.73 \text{R})} = 115.6\ \mu\text{F}$$

PROBLEM 5.16

A two-branch parallel circuit with one branch of R = 100 Ω and a single unknown element in the other branch has the following applied voltage and total current.

$v = 2000 \cos (1000\, t + 45°)$ V

$i_T = 45 \sin (1000\, t + 135°)$ A

Find the unknown element.

Solution Here, the voltage applied is same for both elements.

Current passing through resistor is $i_R = \dfrac{v}{R}$

$$\therefore \quad i_R = 20 \cos (1000\, t + 45°)$$

Total current $i_T = i_R + i_X$

where i_x is the current in unknown element.

$$\begin{aligned} i_x &= i_T - i_R \\ &= 45 \sin (1000\, t + 135°) - 20 \cos (1000\, t + 45°) \\ &= 45 \sin (1000\, t + 135°) - 20 \sin (1000\, t + 135°) \end{aligned}$$

Current passing through the unknown element.

$$i_x = 25 \sin (1000\, t + 135°)$$

Since the current and voltage are in phase, the element is a resistor.
And the value of the resistor

$$R = \frac{v}{i_X} = \frac{2000}{25} = 80\ \Omega$$

PROBLEM 5.17

Find the total current to the parallel circuit with L = 0.05 H and C = 0.667 μF with an applied voltage of v = 200 sin 5000 t V.

Solution Current in the inductor $i_L = \dfrac{1}{L}\displaystyle\int v dt$

$$\therefore \quad i_L = \frac{1}{0.05}\int 200 \sin 5000t$$

$$= \frac{-200 \cos 5000t}{0.05 \times 5000}$$

$$i_L = -0.8 \cos 5000t$$

Current in the capacitor $i_C = C\dfrac{dv}{dt}$

$$\therefore \quad i_C = 0.667 \times 10^{-6} \frac{d}{dt}(200 \sin 5000t)$$

$$i_C = 0.667 \cos 5000\, t$$

Total current $i_T = i_L + i_C$

$$= 0.667 \cos 5000\, t - 0.8 \cos 5000\, t$$

$$= -0.133 \cos (5000\, t)$$

Total current $i_T = 0.133 \sin (5000\, t - 90°)$ A

PSpice Problems

PROBLEM 5.1

For the parallel circuit shown in Fig. 5.39, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current.

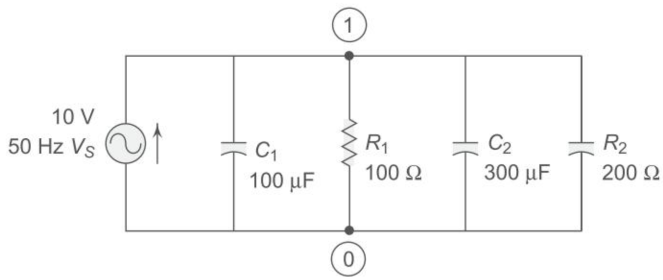


Fig. 5.39

```
*   AC ANALYSIS
VS  1  0  AC 10 V0
C1  1  0  100 U
R1  1  0  100
C2  1  0  300 U
R2  1  0  200
.AC LIN 1 50 100
.PRINT AC IM(VS) IP(VS) IM(C1) IP(C1) IM(C2) IP(C2) IM(R1) + IP(R1) IM(R2) IP(R2)
.END
OUTPUT
**** AC ANALYSIS TEMPERATURE = 27.000 DEG C
*******************************************************************
   FREQ       IM(VS)      IP(VS)      IM(C1)      IP(C1)                    IM(C2)
 5.000E+01  1.266E+00  -9.681E+01  3.142E-01  9.000E+01  9.425E-01
   FREQ       IP(C2)      IM(R1)      IP(R1)      IM(R2)      IP(R2)
 5.000E+01  9.000E+01  1.000E-01  0.000E+00  5.000E-02  0.000E+00
```

Result

Frequency = 50 Hz;

Total current It = I(VS) = $1.266\angle{-96.81}$ A
$I_{c1} = 0.3142\angle 90$ A
$I_{c2} = 0.9425\angle 90$ A
$I_{R1} = 0.1\angle 0$ A
$I_{R2} = 0.05\angle 0$ A

PROBLEM 5.2

Determine voltage across each element of circuit shown in Fig. 5.40.

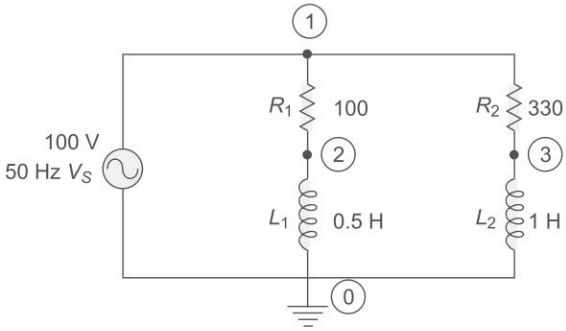


Fig. 5.40

```
*   AC ANALYSIS
VS 1 0 AC 100V  0
R1    1    2    100
L1    2    0    0.5
R2    1    3    330 OHM
L2    3    0    1H
.AC LIN 1 50 100
.PRINT AC V(R1) VP(R1) VM(L1) VP(L1) + V(R2) VP(R2) V(L2) VP(L2)
.END
**** AC ANALYSIS TEMPERATURE = 27.000 DEG C
****************************************************************
```

FREQ	V(R1)	VP(R1)	VM(L1)	VP(L1)	V(R2)
5.000E+01	5.370E+01	−5.752E+01	8.436E+01	3.248E+01	7.243E+01

FREQ	VP(R2)	V(L2)	VP(L2)
5.000E+01	– 4.359E+01	6.895E+01	4.641E+01

Result

$V_{R1} = 53.7\ \angle{-57.32}$ V
$V_{L1} = 84.36\ \angle 32.48$ V
$V_{R2} = 72.43\ \angle{-43.59}$ V
$V_{L2} = 68.95\ \angle 46.41$ V

PROBLEM 5.3

For the series–parallel circuit shown in Fig. 5.41 determine (a) total impedance between a, b, (b) voltage across each parallel branch, and (c) phase angle.

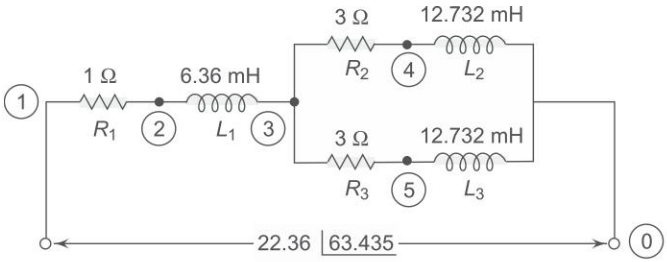


Fig. 5.41

```
*   AC ANALYSIS OF SERIES PARALLEL CIRCUIT
VS        1    0    AC   22.36   63.435
R1        1    2    1
L1        2    3    6.366 M
R2        3    4    3
L2        4    0    12.732 M
R3        3    5    3
L3        5    0    12.732 M
.AC LIN   1    50   100
.PRINT AC IM(R1) IP(R1) VM(3,0) VP(3,0)
.END
```

OUTPUT
**** AC ANALYSIS TEMPERATURE = 27.000 DEG C
****

FREQ	IM(R1)	IP(R1)	VM(3,0)	VP(3,0)
5.000E + 01	4.740E + 00	5.441E + 00	1.185E + 01	5.857E + 01

Result

(a) Total impedance between *a* and *b* : ZT = VS/IT = 10 + j20/11.85∠58.57° = 4.72 ∠57.99°
(b) Voltage across each parallel branch = 11.85 ∠58.77º
(c) Phase angle = 63.435° – 5.441° = 57.994°

Answers to Practice Problems

5-2.1 $C = 19.2\ \mu F$

5-3.1 157.4 ∠– 17.6°; 17.6° lead, 0.635 A

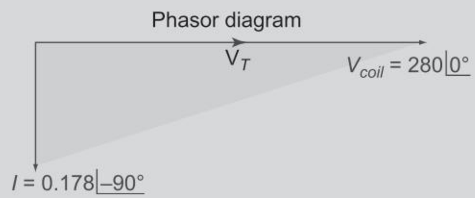
5-3.2 55.85 ∠–57.5°; 57.5°

5-3.4 $R = 2\ \Omega$; $C = 2.12 \times 10^{-3}$ F

5-3.5 $R = 5.412\ \Omega$; $C = 160 \times 10^{-6}$ F

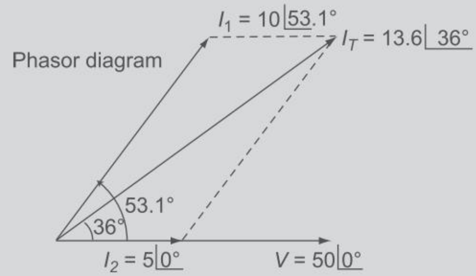
5-3.6 L_1=0.02 H; L_2=0.04 H

5-3.7 $V_T = \sqrt{R^2 + (\omega L)^2}\, I_m \sin\left(\omega t + \tan^{-1}\frac{\omega L}{R}\right)$



$\theta = \tan^{-1}\dfrac{\omega L}{R}$, where $\omega = 200$ rad/s

5-4.1 $I_1 = 10\underline{|53.1°}$ A; $I_2 = 5\underline{|0°}$ A; $I_T = 13.6\underline{|36°}$ A



5-4.5 $L = 6.67$ mH; $C = 3.33$ μF

5-4.6 $i_T = 1.74 \sin(100t + 67.4°)$ A

$\theta = 67.4°$; $Z = 115\ \Omega$

5-5.1 944.2 Ω; 0.053 A; 3.67°; 16.3 V; 30.7 V

5-5.2 1.44 A; 7.05°; $V_{100\ \mu F} = 22.9$ V;

$V_{10\ \Omega} = 14.4$ V

$V_{30\ \Omega} = 38.93$ V; $V_{0.1H} = 38.93$ V

Objective-Type Questions

☆☆★**5.1** A 1 kHz sinusoidal voltage is applied to an *RL* circuit, what is the frequency of the resulting current?
(a) 1 kHz (b) 0.1 kHz (c) 100 kHz (d) 2 kHz

☆☆★**5.2** A series *RL* circuit has a resistance of 33 kΩ, and an inductive reactance of 50 kΩ. What is its impedance and phase angle?
(a) 56.58 W, 59.9° (b) 59.9 kW, 56.58° (c) 59.9 Ω, 56.58° (d) 5.99 Ω, 56.58°

☆☆★**5.3** In a certain *RL* circuit, $V_R = 2$ V and $V_L = 3$ V. What is the magnitude of the total voltage?
(a) 2 V (b) 3 V (c) 5 V (d) 3.61 V

☆☆★**5.4** When the frequency of applied voltage in a series *RL* circuit is increased what happens to the inductive reactance?
(a) Decreases (b) Remains the same (c) Increases (d) Becomes zero

☆★★**5.5** In a certain parallel *RL* circuit, $R = 0\ \Omega$, and $X_L = 75\ \Omega$. What is the admittance?
(a) 0.024 S (b) 75 S (c) 50 S (d) 1.5 S

☆★★**5.6** What is the phase angle between the inductor current and the applied voltage in a parallel *RL* circuit?
(a) 0° (b) 45° (c) 90° (d) 30°

☆★★**5.7** When the resistance in an *RC* circuit is greater than the capacitive reactance, the phase angle between the applied voltage and the total current is closer to
(a) 90° (b) 0° (c) 45° (d) 120°

☆☆★**5.8** A series *RC* circuit has a resistance of 33 kΩ, and a capacitive reactance of 50 kΩ. What is the value of the impedance?
(a) 50 kΩ (b) 33 kΩ (c) 20 kΩ (d) 59.91 Ω

☆☆★**5.9** In a certain series *RC* circuit, $V_R = 4$ V and $V_C = 6$ V. What is the magnitude of the total voltage?
(a) 7.2 V (b) 4 V (c) 6 V (d) 52 V

☆☆★**5.10** When the frequency of the applied voltage in a series *RC* circuit is increased what happens to the capacitive reactance?
(a) It increases (b) It decreases (c) It is zero (d) Remains the same

☆☆★**5.11** In a certain parallel *RC* circuit, $R = 50\ \Omega$ and $X_C = 75\ \Omega$. What is *Y*?
(a) 0.01 S (b) 0.02 S (c) 50 S (d) 75 S

☆☆★**5.12** The admittance of an *RC* circuit is 0.0035 S, and the applied voltage is 6 V. What is the total current?
(a) 6 mA (b) 20 mA (c) 21 mA (d) 5 mA

☆☆★**5.13** What is the phase angle between the capacitor current and the applied voltage in a parallel *RC* circuit?

(a) 90° (b) 0° (c) 45° (d) 180°

☆☆★**5.14** In a given series *RLC* circuit, X_C is 150 Ω, and X_L is 80 Ω, what is the total reactance? What is the type of reactance?

(a) 70 Ω, inductive (b) 70 Ω, capacitive (c) 70 Ω, resistive (d) 150 Ω, capacitive

☆☆★**5.15** In a certain series *RLC* circuit, $V_R = 24$ V, $V_L = 15$ V, and $V_C = 45$ V. What is the source voltage?

(a) 38.42 V (b) 45 V (c) 15 V (d) 24 V

☆★★**5.16** When $R = 10\ \Omega$, $X_C = 18\ \Omega$ and $X_L = 12\ \Omega$, the current

(a) leads the applied voltage (b) lags behind the applied voltage

(c) is in phase with the voltage (d) is none of the above

☆★★**5.17** A current $i = A \sin 500\,t$ A passes through the circuit shown in Fig. 5.42. The total voltage applied will be

(a) $B \sin 500\,t$ (b) $B \sin (500\,\mathrm{t} - \theta°)$ (c) $B \sin (500\,t + \theta°)$ (d) $B \cos (200\,t + \theta°)$

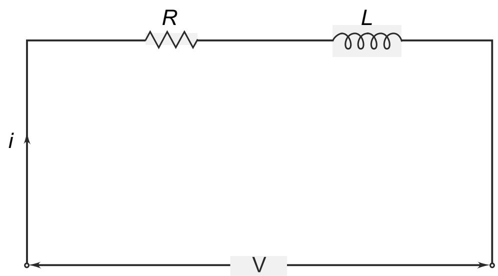


Fig. 5.42

☆★★**5.18** A current of 100 mA through an inductive reactance of 100 Ω produces a voltage drop of

(a) 1 V (b) 6.28 V (c) 10 V (d) 100 V

☆★★**5.19** When a voltage $v = 100 \sin 5000\,t$ volts is applied to a series circuit of $L = 0.05$ H and unknown capacitance, the resulting current is $i = 2 \sin (5000\,t + 90°)$ amperes. The value of capacitance is

(a) 66.7 pF (b) 6.67 pF (c) 0.667 μF (d) 6.67 μF

☆★★**5.20** A series circuit consists of two elements has the following current and applied voltage.

$$i = 4 \cos (2000\,t + 11.32°)\ \mathrm{A}$$
$$v = 200 \sin (2000\,t + 50°)\ \mathrm{V}$$

The circuit elements are

(a) resistance and capacitance (b) capacitance and inductance

(c) inductance and resistance (d) both resistances

☆★★**5.21** A pure capacitor of $C = 35$ μF is in parallel with another single circuit element. The applied voltage and resulting current are

$$v = 150 \sin 300\,t\ \mathrm{V}$$
$$i = 16.5 \sin (3000\,t + 72.4°)\ \mathrm{A}$$

The other element is

(a) capacitor of 30 μF (b) inductor of 30 mH (c) resistor of 30 Ω (d) none of the above

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CHAPTER 6

Power and Power Factor

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Describe instantaneous power
- LO 2 Explain the concept of average power
- LO 3 Explain the concept of apparent power and define power factor
- LO 4 Describe reactive power
- LO 5 Illustrate power triangle

6.1 INSTANTANEOUS POWER

LO 1 Describe instantaneous power

In a purely resistive circuit, all the energy delivered by the source is dissipated in the form of heat by the resistance. In a purely reactive(inductive or capacitive) circuit, all the energy delivered by the source is stored by the inductor or capacitor in its magnetic or electric field during a portion of the voltage cycle, and then is returned to the source during another portion of the cycle, so that no net energy is transferred. When there is complex impedance in a circuit, a part of the energy is alternately stored and returned by the reactive part, and part of it is dissipated by the resistance. The amount of energy dissipated is determined by the relative values of resistance and reactance.

Consider a circuit having complex impedance. Let $v(t) = V_m \cos \omega t$ be the voltage applied to the circuit and let $i(t) = I_m \cos(\omega t + \theta)$ be the corresponding current flowing through the circuit. Then the power at any instant of time is

$$P(t) = v(t)\, i(t)$$

$$= V_m \cos \omega t\, I_m \cos(\omega t + \theta) \tag{6.1}$$

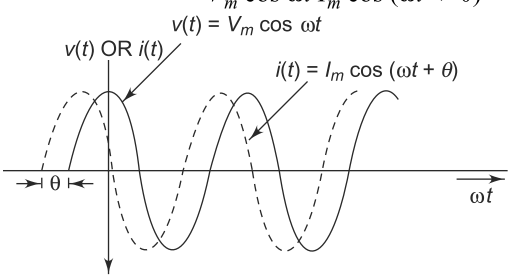


Fig. 6.1

From Eq. (6.1), we get

$$P(t) = \frac{V_m I_m}{2}\left[\cos(2\omega t + \theta) + \cos\theta\right] \tag{6.2}$$

Equation (6.2) represents *instantaneous power*. It consists of two parts. One is a fixed part, and the other is time-varying which has a frequency twice that of the voltage or current waveforms. The voltage, current and power waveforms are shown in Figs. 6.1 and 6.2.

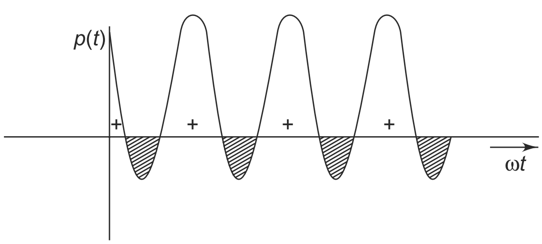
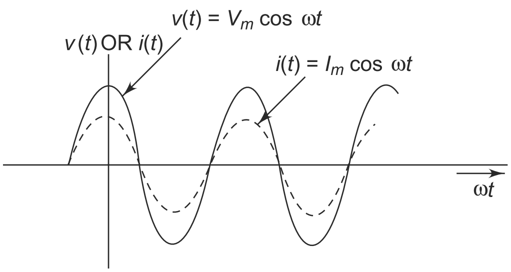


Fig. 6.2

Here, the negative portion (hatched) of the power cycle represents the power returned to the source. Figure 6.2 shows that the instantaneous power is negative whenever the voltage and current are of opposite sign. In Fig. 6.2, the positive portion of the power is greater than the negative portion of the power; hence, the average power is always positive, which is almost equal to the constant part of the instantaneous power (Eq. 6.2). The positive portion of the power cycle varies with the phase angle between the voltage and current waveforms. If the circuit is pure resistive, the phase angle between voltage and current is zero; then there is no negative cycle in the $P(t)$ curve. Hence, all the power delivered by the source is completely dissipated in the resistance.



If θ becomes zero in Eq. (6.1), we get

$P(t) = v(t)\, i(t)$

$= V_m I_m \cos^2 \omega t$

$$= \frac{V_m I_m}{2}(1+\cos 2\omega t) \tag{6.3}$$

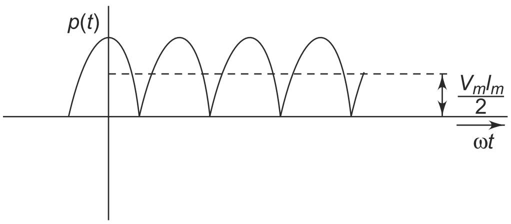


Fig. 6.3

The waveform for Eq. (6.3), is shown in Fig. 6.3, where the power wave has a frequency twice that of the voltage or current. Here, the average value of power is $V_m I_m/2$.

When phase angle θ is increased, the negative portion of the power cycle increases and lesser power is dissipated. When θ becomes π/2, the positive and negative portions of the power cycle are equal. At this instant, the power dissipated in the circuit is zero, i.e. the power delivered to the load is returned to the source.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **6.1.1** Using PSpice, find the instantaneous power on each of the elements in the circuit of Fig. Q.1.

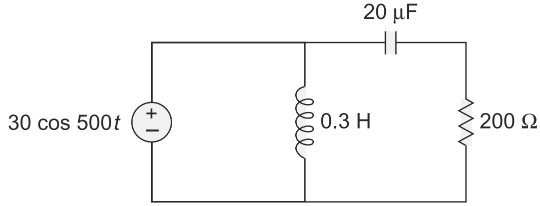


Fig. Q.1

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

6.2 AVERAGE POWER

To find the average value of any power function, we have to take a particular time interval from t_1 to t_2; by integrating the function from t_1 to t_2 and dividing the result by the time interval $t_2 - t_1$, we get the average power.

LO 2 Explain the concept of average power

$$\text{Average power } P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P(t)\, dt \tag{6.4}$$

In general, the average value over one cycle is

$$P_{av} = \frac{1}{T} \int_0^T P(t)\, dt \tag{6.5}$$

By integrating the instantaneous power $P(t)$ in Eq. (6.5) over one cycle, we get average power

$$P_{av} = \frac{1}{T} \int_0^T \left\{ \frac{V_m I_m}{2} [\cos(2\omega t + \theta) + \cos\theta]\, dt \right\}$$

$$= \frac{1}{T} \int_0^T \frac{V_m I_m}{2} [\cos(2\omega t + \theta)]\, dt + \frac{1}{T} \int_0^T \frac{V_m I_m}{2} \cos\theta\, dt \tag{6.6}$$

In Eq. (6.6), the first term becomes zero, and the second term remains. The average power is therefore

$$P_{av} = \frac{V_m I_m}{2} \cos\theta \text{ W} \tag{6.7}$$

We can write Eq. (6.7) as

$$P_{av} = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right) \cos\theta \tag{6.8}$$

In Eq. (6.8), $V_m/\sqrt{2}$ and $I_m/\sqrt{2}$ are the effective values of both voltage and current.

$$\therefore \quad P_{av} = V_{eff} I_{eff} \cos\theta$$

To get average power, we have to take the product of the effective values of both voltage and current multiplied by cosine of the phase angle between voltage and the current.

If we consider a purely resistive circuit, the phase angle between voltage and current is zero. Hence, the average power is

$$P_{av} = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R$$

If we consider a purely reactive circuit (i.e. purely capacitive or purely inductive), the phase angle between voltage and current is 90°. Hence, the average power is zero or $P_{av} = 0$.

If the circuit contains complex impedance, the average power is the power dissipated in the resistive part only.

EXAMPLE 6.1

A voltage of $v(t) = 100 \sin \omega t$ is applied to a circuit. The current flowing through the circuit is $i(t) = 15 \sin (\omega t - 30°)$. Determine the average power delivered to the circuit.

Solution The phase angle between voltage and current is 30°.

Effective value of the voltage $V_{eff} = \dfrac{100}{\sqrt{2}}$

Effective value of the current $I_{eff} = \dfrac{15}{\sqrt{2}}$

Average power $P_{av} = V_{eff} I_{eff} \cos\theta$

$$= \frac{100}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \cos 30°$$

$$= \frac{100 \times 15}{2} \times 0.866 = 649.5 \text{ W}$$

EXAMPLE 6.2

Determine the average power delivered to the circuit consisting of an impedance Z = 5 + j8 when the current flowing through the circuit is I = 5 < 30°.

Solution The average power is the power dissipated in the resistive part only.

or $\quad P_{av} = \dfrac{I_m^2}{2} R$

Current $\quad I_m = 5\text{A}$

$\therefore \quad P_{av} = \dfrac{5^2}{2} \times 5 = 62.5\text{W}$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

★★★**6-2.1** Find the average power dissipated by the 500 Ω resistor shown in Fig. Q.1.

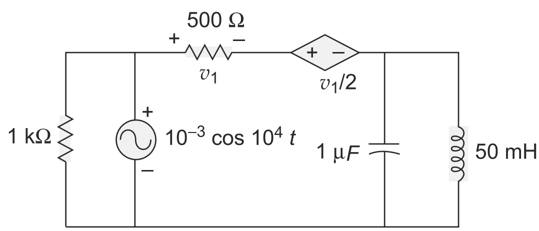


Fig. Q.1

★★★**6-2.2** Find the power delivered by current source shown in Fig. Q.2.

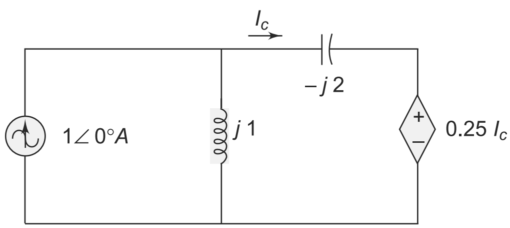


Fig. Q.2

☆☆★ **6-2.3** Using PSpice, find the average power absorbed by the 10 Ω resistor (Fig. Q.3).

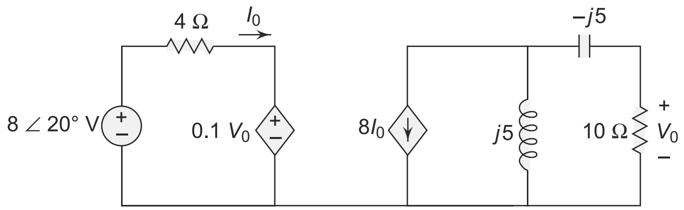


Fig. Q.3

6.3 APPARENT POWER AND POWER FACTOR

LO 3 Explain the concept of apparent power and define power factor

The power factor is useful in determining useful power (true power) transferred to a load. The highest power factor is 1, which indicates that the current to a load is in phase with the voltage across it (i.e. in the case of resistive load). When the power factor is 0, the current to a load is 90° out of phase with the voltage (i.e. in case of reactive load).

Consider the following equation:

$$P_{av} = \frac{V_m I_m}{2} \cos\theta \text{ W} \tag{6.9}$$

In terms of effective values,

$$P_{av} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos\theta$$

$$= V_{eff} I_{eff} \cos\theta \text{ W} \tag{6.10}$$

The average power is expressed in watts. It means the useful power transferred from the source to the load, which is also called true power. If we consider a dc source applied to the network, true power is given by the product of the voltage and the current. In case of sinusoidal voltage applied to the circuit, the product of voltage and current is not the true power or average power. This product is called *apparent power*. The apparent power is expressed in volt amperes, or simply VA.

∴ ***Apparent power*** $= V_{eff} I_{eff}$

In Eq. (6.10), the average power depends on the value of cos θ; this is called the *power factor* of the circuit.

$$\therefore \quad \text{Power factor (pf)} = \cos\theta = \frac{P_{av}}{V_{eff}\, I_{eff}}$$

Therefore, power factor is defined as the ratio of average power to the apparent power, whereas apparent power is the product of the effective values of the current and the voltage. ***Power factor*** *is also defined as the factor with which the volt amperes are to be multiplied to get true power in the circuit.*

In the case of sinusoidal sources, the power factor is the cosine of the phase angle between voltage and current

$$pf = \cos\theta$$

As the phase angle between voltage and total current increases, the power factor decreases. The smaller the power factor, the smaller the power dissipation. The power factor varies from 0 to 1. For purely resistive circuits, the phase angle between voltage and current is zero, and hence the power factor is unity. For purely reactive circuits, the phase angle between voltage and current is 90°, and hence the power factor is zero. In an *RC* circuit, the power factor is referred to as *leading* power factor because the current leads the voltage. In an *RL* circuit, the power factor is referred to as lagging power factor because the current lags behind the voltage.

EXAMPLE 6.3

A sinusoidal voltage $v = 50 \sin \omega t$ is applied to a series RL circuit. The current in the circuit is given by $i = 25 \sin (\omega t - 53°)$. Determine (a) apparent power, (b) power factor, and (c) average power.

Solution (a) Apparent power $P = V_{eff} I_{eff}$

$$= \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

$$= \frac{50 \times 25}{2} = 625 \text{ VA}$$

(b) Power factor = $\cos \theta$

where θ is the angle between voltage and current

$\theta = 53°$

$\therefore$ Power factor = $\cos \theta = \cos 53° = 0.6$

(c) Average power $P_{av} = V_{eff} I_{eff} \cos \theta$

$= 625 \times 0.6 = 375$ W

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★ **6-3.1** For the circuit shown in Fig. Q.1, a voltage of $250 \sin \omega t$ is applied. Determine the power factor of the circuit, if the voltmeter readings are $V_1 = 100$ V, $V_2 = 125$ V, $V_3 = 150$ V and the ammeter reading is 5 A.

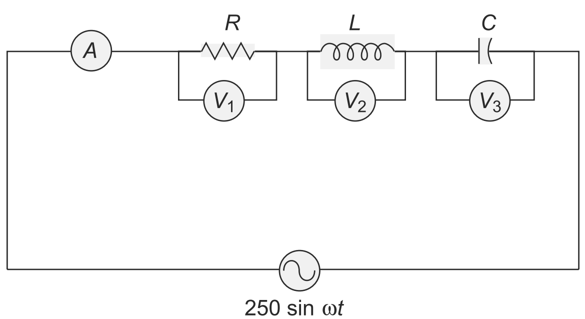


Fig. Q.1

☆★★ **6-3.2** A series *RL* circuit draws a current of $i(t) = 8 \sin (50t + 45°)$ from the source. Determine the circuit constants, if the power delivered by the source is 100 W and there is a lagging power factor of 0.707.

☆☆★ **6-3.3** The current in a circuit lags the voltage by 30°. If the input power be 400 W and the supply voltage be $v = 100 \sin (377t + 10°)$, find the complex power in voltamperes.

☆★★ **6-3.4** For the circuit shown in Fig. Q.4, determine the power dissipated and the power factor of the circuit.

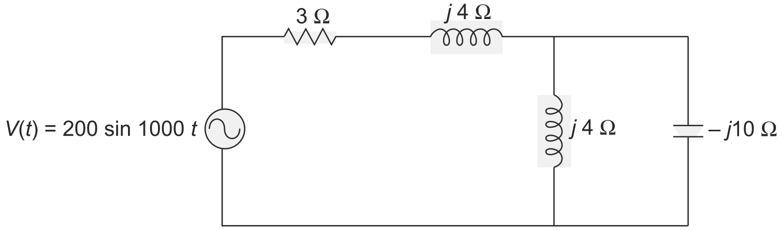


Fig. Q.4

☆☆★**6-3.5** For the circuit shown in Fig. Q.5, determine the power factor and the power dissipated in the circuit.

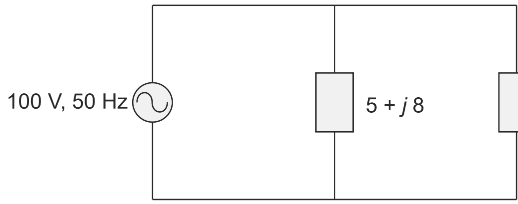


Fig. Q.5

★★★**6-3.6** In the circuit shown in Fig. Q.6, the total effective current is 30 amperes. Determine the three powers.

☆★★**6-3.7** In the parallel circuit shown in Fig. Q.7, the total power is 1100 watts. Find the power in each resistor and the reading on the ammeter.

☆☆★**6-3.8** For the circuit shown in Fig. Q.8, determine the true power, reactive power, and apparent power in each branch. What is the power factor of the total circuit?

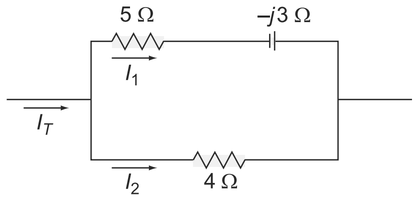


Fig. Q.6

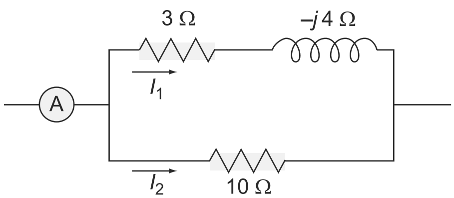


Fig. Q.7

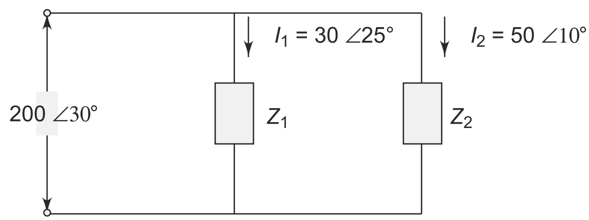


Fig. Q.8

★★★**6-3.9** Determine the value of the voltage source, and the power factor in the network shown in Fig. Q.9 if it delivers a power of 500 W to the circuit shown in Fig. Q.9. Also find the reactive power drawn from the source.

★★★**6-3.10** Find the power dissipated by the voltage source shown in Fig. Q.10.

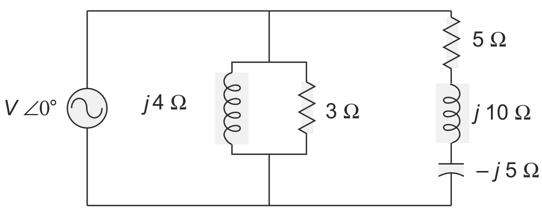


Fig. Q.9

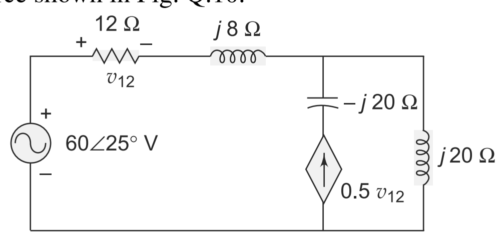


Fig. Q.10

☆★★**6-3.11** For the circuit shown in Fig. Q.11, find:

(a) real power dissipated by each element

(b) the total apparant power supplied by the circuit

(c) the power factor of the circuit.

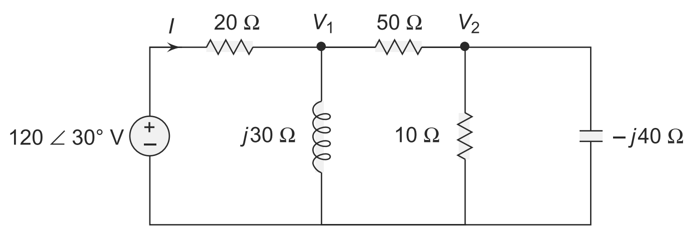


Fig. Q.11

Frequently Asked Questions linked to LO 3*

☆★★**6-3.1** Write the effect of power factor in energy-consumption billing. **[AU May/June 2014]**

☆☆★**6-3.2** (a) Determine the currents in all the branches
(b) Calculate the power and power factor of the source
(c) Show that power delivered by the source is equal to power consumed by the 2-ohm resistor. **[AU May/June 2014]**

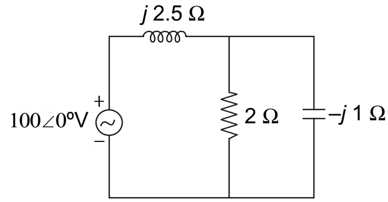


Fig. Q.2

☆☆★**6-3.3** Calculate the power factor if v(t) = Vm sin wt and i(t) = Im sin (wt – 45°). **[AU April/May 2011]**

☆☆★**6-3.4** Define power factor. **[JNTU Nov. 2012]**

6.4 REACTIVE POWER

LO 4 Describe reactive power

We know that the average power dissipated is

$$P_{av} = V_{eff}[I_{eff}\cos\theta] \tag{6.11}$$

From the impedance triangle shown in Fig. 6.4,

$$\cos\theta = \frac{R}{|Z|} \tag{6.12}$$

$$\text{and } V_{eff} = I_{eff}Z \tag{6.13}$$

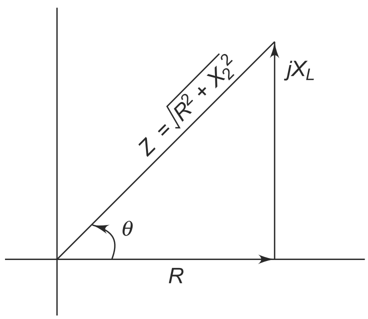


Fig. 6.4

If we substitute Eqs (6.12) and (6.13) in Eq. (6.11), we get

$$P_{av} = I_{eff}\, Z\left[I_{eff}\,\frac{R}{Z}\right]$$

$$= I_{eff}^2 R \text{ watts} \tag{6.14}$$

This gives the average power dissipated in a resistive circuit.

If we consider a circuit consisting of a pure inductor, the power in the inductor

$$P_r = iv_L \tag{6.15}$$

$$= iL\frac{di}{dt}$$

Consider $\quad i = I_m \sin(\omega t + \theta)$

Then $\quad P_r = I_m^2 \sin(\omega t + \theta)\, L\omega \cos(\omega t + \theta)$

$$= \frac{I_m^2}{2}(\omega L)\,\sin 2(\omega t + \theta)$$

$$\therefore \quad P_r = I_{eff}^2(\omega L)\sin 2(\omega t + \theta) \tag{6.16}$$

From the above equation, we can say that the average power delivered to the circuit is zero. This is called *reactive* power. It is expressed in volt-amperes reactive (VAR).

$$P_r = I_{eff}^2 X_L \text{ VAR} \tag{6.17}$$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

From Fig. 6.4, we have

$$X_L = Z \sin \theta \tag{6.18}$$

Substituting Eq. (6.18) in Eq. (6.17), we get

$$P_r = I_{eff}^2 Z \sin \theta$$
$$= (I_{eff} Z) I_{eff} \sin \theta$$
$$\boxed{= V_{eff} I_{eff} \sin \theta \text{ VAR}}$$

6.5 THE POWER TRIANGLE

LO 5 Illustrate power triangle

A generalised impedance phase diagram is shown in Fig. 6.5. A phasor relation for power can also be represented by a similar diagram because of the fact that true power P_{av} and reactive power P_r differ from R and X by a factor I_{eff}^2, as shown in Fig. 6.5.

The resultant power phasor $I_{eff}^2 Z$, represents the apparent power P_a.

At any instant in time, P_a is the total power that appears to be transferred between the source and reactive circuit. Part of the apparent power is true power and part of it is reactive power.

$$\therefore \quad P_a = I_{eff}^2 Z$$

The power triangle is shown in Fig. 6.6.

From Fig. 6.6, we can write

$$P_{true} = P_a \cos \theta$$

or average power $P_{av} = P_a \cos \theta$

and reactive power $P_r = P_a \sin \theta$

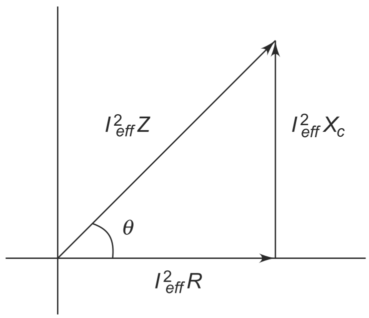


Fig. 6.5

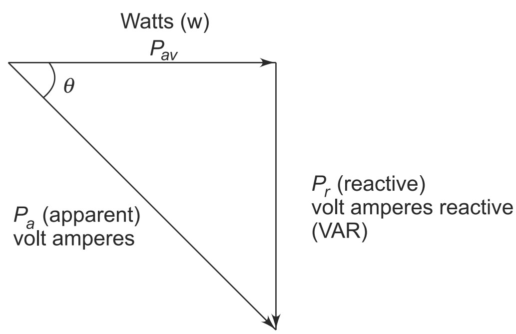


Fig. 6.6

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆☆★ **6-5.1** Determine the branch and total real and reactive powers in the parallel circuit shown in Fig. Q.1. Use j notation.

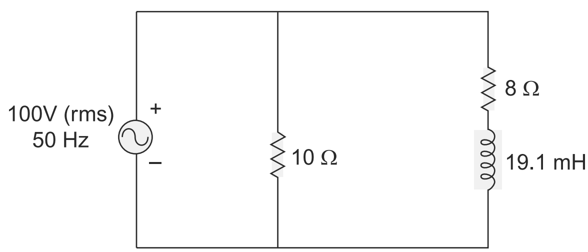


Fig. Q.1

☆☆★**6-5.2** Two impedances $Z_1 = (4 + j5)\,\Omega$ and $Z_2 = (8 + j10)\,\Omega$ are connected in parallel across a 230V, 50Hz supply. Find (a) total admittance, (b) current drawn from supply and power factor, and (c) value of capacitance to be connected in parallel with the above admittances to raise the power factor unity.

☆★★**6-5.3** A voltage of $v(t) = 100 \sin 500t$ is applied across a series R-L-C circuit where $R = 10\ \Omega$, $L = 0.05$ H, and $C = 20\ \mu F$. Determine the power supplied by the source, the reactive power supplied by the source, the reactive power of the capacitor, the reactive power of the inductor, and the power factor of the circuit.

☆★★**6-5.4** For the circuit shown in Fig. Q.4, determine the power factor, active power, reactive power, and apparent power.

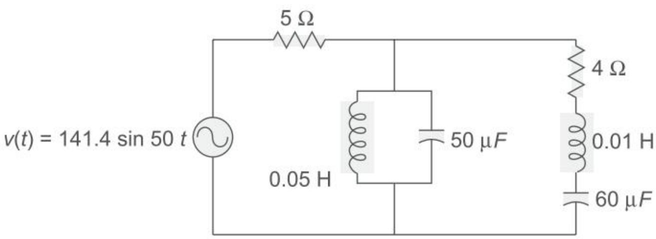


Fig. Q.4

★★★**6-5.5** For the parallel circuit shown in Fig. Q.5, the total power dissipated is 1000 W. Determine the apparent power, the reactive power, and the power factor.

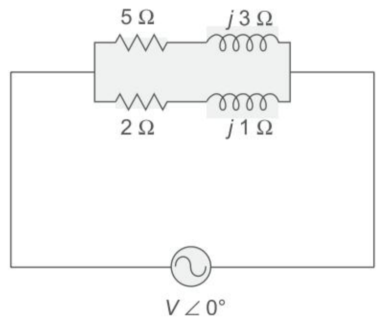


Fig. Q.5

★★★**6-5.6** For the circuit shown in Fig. Q.6, determine the power factor, active power, reactive power and apparent power.

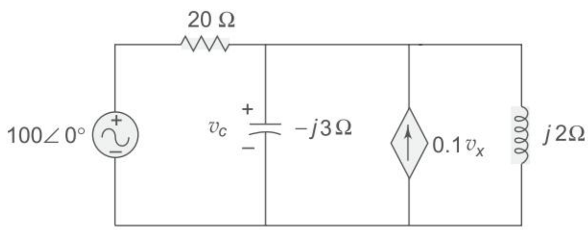


Fig. Q.6

Additional Solved Problems

PROBLEM 6.1

For the circuit shown in Fig. 6.7, a voltage v(t) is applied and the resulting current in the circuit i(t) = 15 sin (ωt + 30°) amperes. Determine the active power, reactive power, power factor, and the apparent power.

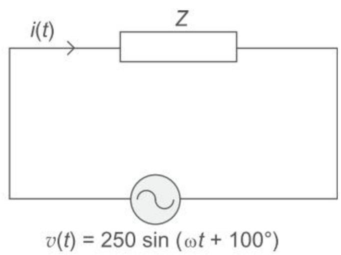


Fig. 6.7

Solution The voltage applied to the circuit

$$v(t) = 250\ \sin(\omega t + 100°)$$
$$V = 250\underline{|100°}$$

The current in the circuit

$$i(t) = 15\sin(\omega t + 30°)$$
$$I = 15\angle 30°$$
$$\text{Phase angle } \theta = 100 - 30 = 70°$$
$$\text{Power factor } \cos\theta = \cos 70° = 0.342$$
$$\text{Active power } = VI\cos\theta = \frac{250}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \times 0.342$$
$$= 641.25\,\text{watts}$$
$$\text{Reactive power } = VI\sin\theta = \frac{250}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \times \sin 70°$$
$$= 1762.5\,\text{VAR}$$
$$\text{Apparent power } = VI = \frac{250}{\sqrt{2}} \times \frac{15}{\sqrt{2}} = 1875\,\text{VA}$$

PROBLEM 6.2

Two impedances, $Z_1 = 10\angle -60°\ \Omega$ and $Z_2 = 16\angle 70°\ \Omega$ are in series and pass an effective current of 5A. Determine the active power, reactive power, apparent power and power factor.

Solution $Z_1 = 10\angle -60°$ and $Z_2 = 16\angle 70°$

$$\text{Total Impedance } Z = Z_1 + Z_2$$
$$= 10\angle -60° + 16\angle 70° = (10.47 + j\,6.37)\,\Omega$$
$$\text{The power factor } \cos\theta = \frac{R}{Z} = \frac{10.47}{12.25} = 0.855$$
$$\text{Active power } = VI\cos\theta = I^2R = (5)^2 \times 10.47 = 261.75 \text{ watts}$$
$$\text{Apparent power } = I^2Z = (5)^2 \times 12.25 = 306.25 \text{ VA}$$
$$\text{Reactive power } = I^2X_L = (5)^2 \times 6.37 = 159.25 \text{ VAR}$$

PROBLEM 6.3

For the circuit shown in Fig. 6.8, determine the value of the impedance if the source delivers a power of 200 W and there is a lagging power factor of 0.707. Also find the apparent power.

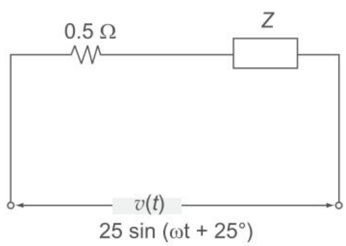


Fig. 6.8

Solution The lagging power factor

$$\cos\theta = \frac{R}{Z} = 0.707$$
$$R = Z(0.707)$$
$$\text{Active power } = VI\cos\theta = 200 \text{ watts}$$
$$\text{where } \quad V = \frac{25}{\sqrt{2}} = 17.68$$

$$\frac{25}{\sqrt{2}} \times I\cos\theta = 200$$

$$I = 16\text{A}$$

The total impedance $Z_T = \dfrac{V}{I} = \dfrac{17.68\angle 25^\circ}{16\angle -20^\circ} = 1.105\angle 45^\circ$

$$Z_T = 0.78 + j\,0.78$$

The impedance $Z = Z_T - R = (0.28 + j\,0.78)\,\Omega$

Apparent power $= VI = 17.68 \times 16 = 282.7$ VA.

PROBLEM 6.4

In the parallel circuit shown in Fig. 6.9, the power in the $5\,\Omega$ resistor is 600 W and the total circuit takes 3000 VA at a leading power factor of 0.707. Find the value of Impedance Z.

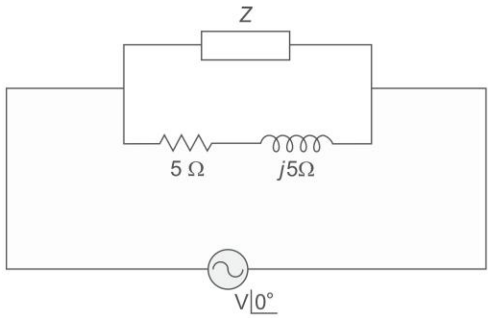


Fig. 6.9

Solution Power in the $5\,\Omega$ resistor

$$I_5^2 R = 600 \text{ W}$$

$$I_5^2 \times 5 = 600 \text{ W}$$

Current in the $5\,\Omega$ resistor $I_5 = 10.95$ A

$$I_5 = \frac{V\angle 0^\circ}{5 + j5}\text{ A}$$

The magnitude of the current $I_5 = \dfrac{V}{\sqrt{5^2 + 5^2}}$ A

$\therefore$ $$V = 10.95 \times \sqrt{50} = 77.42 \text{ volts}$$

Apparent power $P_{ap} = 3000$ VA

Total current $I_T = \dfrac{P_{ap}}{V} = \dfrac{3000}{77.42} = 38.75\,\text{A}$

$$I_T = 38.75\angle 45^\circ\text{A}$$

$$I_5 = 10.95\angle -45^\circ\text{A}$$

The current in $Z = I_Z = I_T - I_5$

$$= 27.4 + j27.4 - (7.74 - j7.74)$$

$$= 19.66 + j35.14 = 40.26\angle 60.77^\circ$$

The value of impedance $= Z = \dfrac{V\angle 0^\circ}{I_Z}$

$$Z = \frac{77.42\angle 0^\circ}{40.26\angle 60.77^\circ}$$

The value of impedance $Z = 1.923\angle -60.77^\circ\ \Omega$

PROBLEM 6.5

A voltage source $v(t) = 150 \sin \omega t$ in series with $5\,\Omega$ resistance is supplying two loads in parallel, $Z_A = 60\angle 30$ and $Z_B = 50\angle -25$. Find the average power delivered to Z_A, the average power delivered to Z_B the average power dissipated in the circuit, and the power factor of the circuit.

Solution The circuit shown in Fig. 6.10 indicates the voltage source $v(t)$ in series with the $5\,\Omega$ resistor provides supply to two parallel impedances Z_A and Z_B.

where $Z_A = 60\angle 30^\circ\,\Omega$; $\quad Z_B = 50\angle -25^\circ\,\Omega$

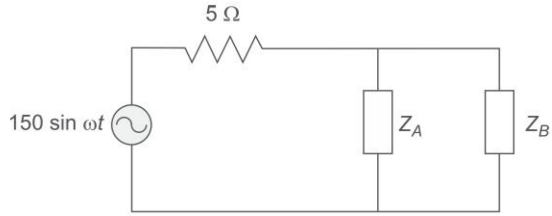


Fig. 6.10

Total impedance

$$Z_T = 5 + \frac{Z_A Z_B}{Z_A + Z_B} = 5 + \frac{60\angle 30^\circ \times 50\angle -25^\circ}{60\angle 30^\circ + 50\angle -25^\circ}$$

$$= 35.709 - 0.112j$$

$$Z_T = 35.71\angle -0.179^\circ\,\Omega$$

Power factor = cos(0.179) = 0.999

The total current $I = \dfrac{150}{\sqrt{2}Z_T} = 2.97\angle 0.179^\circ$ A

The current in impedance Z_A is

$$I_A = I \times \frac{Z_B}{(Z_A + Z_B)} = \frac{2.97\angle 0.179^\circ \times 50\angle -25^\circ}{60\angle 30^\circ + 50\angle -25^\circ}$$

$$I_A = 1.52\angle -30.03^\circ \text{A}$$

The current in impedance Z_B is

$$I_B = I \times \frac{Z_A}{(Z_A + Z_B)} = \frac{2.97\angle 0.179^\circ \times 60\angle 30^\circ}{60\angle 30^\circ + 50\angle -25^\circ}$$

$$I_B = 1.82\angle 24.96^\circ \text{ A}$$

Average power delivered to $Z_A = I_A^2 R_A = 120$ watts

Average power delivered to $Z_B = I_B^2 R_B = 150$ watts

Average power delivered to circuit $= I^2 R = 314.98$ watts

PROBLEM 6.6

A sine wave of $v(t) = 200 \sin 50\,t$ is applied to a $10\,\Omega$ resistor in series with a coil. The reading of a voltmeter across the resistor is 120 V and across the coil, 75 V. Calculate the power and reactive volt-amperes in the coil and the power factor of the circuit.

Solution The rms value of the sine wave

$$V = \frac{200}{\sqrt{2}} = 141.4 \text{ V}$$

Voltage across the resistor, $V_R = 120$ V
Voltage across the coil, $V_L = 75$ V

$$\therefore \quad IR = 120 \text{ V}$$

The current resistor, $I = \dfrac{120}{10} = 12$ A

Since $IX_L = 5$ V

$$\therefore \quad X_L = \frac{75}{12} = 6.25\,\Omega$$

Power factor, $pf = \cos\theta = \dfrac{R}{Z}$

where $Z = 10 + j\,6.25 = 11.8\angle 32°$

$$\therefore \quad \cos\theta = \frac{R}{Z} = \frac{10}{11.8} = 0.85$$

True power $P_{true} = I^2 R = (12)^2 \times 10 = 1440$ W
Reactive power $P_r = I^2 X_L = (12)^2 \times 6.25 = 900$ VAR

PROBLEM 6.7

For the circuit shown in Fig. 6.11, determine the true power, reactive power, and apparent power in each branch. What is the power factor of the total circuit?

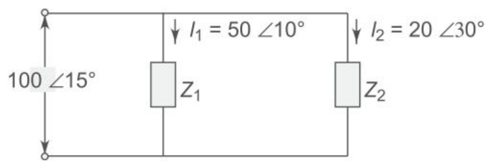


Fig. 6.11

Solution In the circuit shown in Fig. 6.11, we can calculate Z_1 and Z_2.

Impedance $Z_1 = \dfrac{100\angle 15°}{50\angle 10°} = 2\angle 5° = (1.99 + j0.174)\,\Omega$

Impedance $Z_2 = \dfrac{100\angle 15°}{20\angle 30°} = 5\angle -15° = (4.83 - j1.29)\,\Omega$

True power in branch Z_1 is $P_{t_1} = I^2{}_1 R = (50)^2 \times 1.99 = 4975$ W

Reactive power in branch Z_1, $P_{r_1} = I^2{}_1 X_L$
$= (50)^2 \times 0.174 = 435$ VAR

Apparent power in branch Z_1, $P_{a_1} = I^2{}_1 Z_1$
$= (50)^2 \times 2$
$= 2500 \times 2 = 5000$ VA

True power in branch Z_2, $P_{t_2} = I^2{}_2 R$
$= (20)^2 \times 4.83 = 1932$ W

Reactive power in branch Z_2, $P_{r_2} = I^2{}_2 X_C$
$= (20)^2 \times 1.29 = 516$ VAR

Apparent power in branch Z_2, $P_{a_2} = I^2{}_2\, Z_2$

$$= (20)^2 \times 5 = 2000 \text{ VA}$$

Total impedance of the circuit, $Z = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$

$$= \frac{2\angle 5^\circ \times 5 \times \angle -15^\circ}{1.99 + j0.174 + 4.83 - j1.29}$$

$$= \frac{10\angle -10^\circ}{6.82 - j1.116}$$

$$= \frac{10\angle -10^\circ}{6.9\angle -9.29} = 1.45\angle -0.71^\circ$$

The phase angle between voltage and current, $\theta = 0.71°$

$\therefore$ Power factor $pf = \cos\theta$

$$= \cos 0.71° = 0.99 \text{ leading}$$

PROBLEM 6.8

A voltage of $v(t) = 141.4 \sin \omega t$ is applied to the circuit shown in Fig. 6.12. The circuit dissipates 450 W at a lagging power factor, when the voltmeter and ammeter readings are 100 V and 6 A, respectively. Calculate the circuit constants.

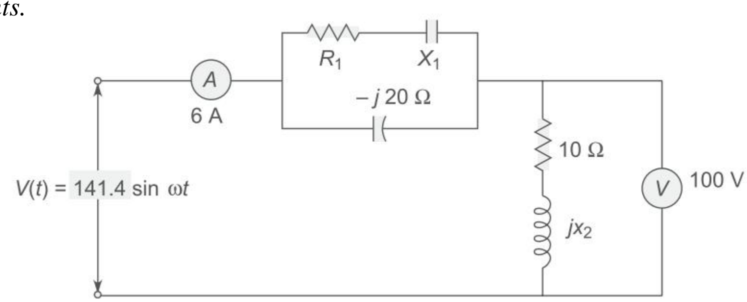


Fig. 6.12

Solution The magnitude of the current passing through $(10 + jX_2)\ \Omega$ is

$I = 6$ A

The magnitude of the voltage across the $(10 + jX_2)$ ohms, $V = 100$ V. The magnitude of impedance $(10 + jX_2)$ is V/I.

Hence $$\sqrt{10^2 + X_2^2} = \frac{100}{6} = 16.67\,\Omega$$

$\therefore$ $$X_2 = \sqrt{(16.67)^2 - (10)^2} = 13.33\,\Omega$$

Total power dissipated in the circuit $= VI \cos\theta = 450$ W

$\therefore$ $$V = \frac{141.4}{\sqrt{2}} = 100 \text{ V}$$

$$I = 6 \text{ A}$$

$$100 \times 6 \times \cos\theta = 450$$

The power factor $$pf = \cos\theta = \frac{450}{600} = 0.75$$

$$\theta = 41.4°$$

The current lags behind the voltage by 41.4°

The current passing through the circuit, $I = 6\angle -41.4°$

The voltage across (10 + j13.33) Ω, $V = 6\angle -41.4° \times 16.66\angle 53.1°$

$= 100\angle 11.7°$

The voltage across parallel branch, $V_1 = 100\angle 0° - 100\angle 11.7°$

$= 100 - 97.9 - j20.27$

$= (2.1 - j20.27)\text{V} = 20.38\angle -84.08°$

The current in ($-j$20) branch, $I_2 = \dfrac{20.38\angle -84.08°}{20\angle -90°} = 1.02\angle +5.92°$

The current in $(R_1 - jX_1)$ branch, I_1

$= 6\angle -41.4° - 1.02\angle 5.92° = 4.5 - j3.97 - 1.01 - j0.1$

$= 3.49 - j4.07 = 5.36\angle -49.39°$

The impedance $Z_1 = \dfrac{V_1}{I_1} = \dfrac{20.38\angle -84.08°}{5.36\angle -49.39°}$

$= 3.8\angle -34.69° = (3.12 - j2.16)\ \Omega$

Since $R_1 - jX_1 = (3.12 - j2.16)\ \Omega$

$R_1 = 3.12\ \Omega$

$X_1 = 2.16\ \Omega$

PROBLEM 6.9

Determine the value of the voltage source and power factor in the following network if it delivers a power of 100 W to the circuit shown in Fig. 6.13. Find also the reactive power drawn from the source.

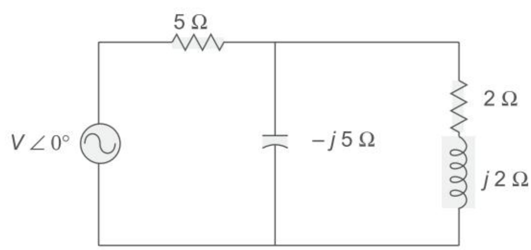


Fig. 6.13

Solution Total impedance in the circuit,

$$Z_{eq} = 5 + \frac{(2+j2)(-j5)}{2+j2-j5}$$

$$= 5 + \frac{10-j10}{2-j3} = 5 + \frac{14.14\angle -45°}{3.6\angle -56.3°} = 5 + 3.93\angle 11.3°$$

$$= 5 + 3.85 + j0.77 = 8.85 + j0.77 = 8.88\angle 4.97°$$

Power delivered to the circuit, $P_T = I^2 R_T = 100$ W

$\therefore \quad I^2 \times 8.85 = 100$

Current in the circuit, $I = \sqrt{\dfrac{100}{8.85}} = 3.36\ \text{A}$

Power factor $pf = \cos\theta = \dfrac{R}{Z}$

$$= \frac{8.85}{8.88} = 0.99$$

Since $VI \cos\theta = 100$ W

$$V \times 3.36 \times 0.99 = 100$$

$$\therefore \quad V = \frac{100}{3.36 \times 0.99} = 30.06 \text{ V}$$

The value of the voltage source, $V = 30.06$ V

Reactive power $P_r = VI \sin\theta$

$$= 30.06 \times 3.36 \times \sin(4.97°)$$
$$= 30.06 \times 3.36 \times 0.087$$
$$= 8.8 \text{ VAR}$$

PROBLEM 6.10

For the circuit shown in Fig. 6.14, determine the circuit constants when a voltage of 100 V is applied to the circuit, and the total power absorbed is 600 W. The circuit constants are adjusted such that the currents in the parallel branches are equal and the voltage across the inductance is equal and in quadrature with the voltage across the parallel branch.

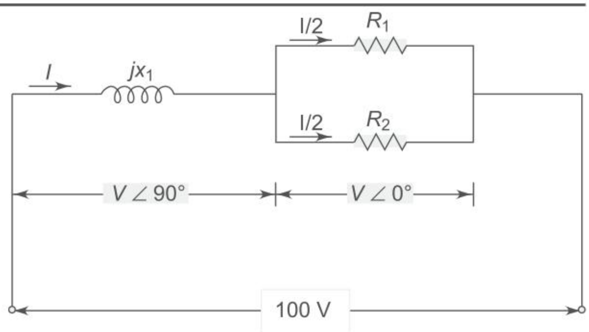


Fig. 6.14

Solution Since the voltages across the parallel branch and the inductance are in quadrature, the total voltage becomes $100\angle 45°$ as shown in Fig. 6.15.

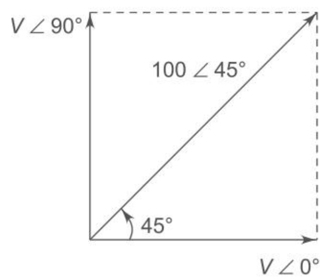


Fig. 6.15

Total voltage is $100\angle 45° = V + j0 + 0 + jV$

From the above result, $70.7 + j70.7 = V + jV$

$\therefore \quad V = 70.7$

If we take current as the reference, then current passing through the circuit is $I\angle 0°$. Total power absorbed by the circuit $= VI \cos\theta = 600$ W

or $\quad 100 \times I \times \cos 45° = 600$ W

$\therefore \quad I = 8.48$ A

Hence, the inductance, $X_1 = \dfrac{V\angle 90°}{I\angle 0°} = \dfrac{70.7\angle 90°}{8.48} = 8.33\angle 90°$

$$\therefore \quad X_1 = 8.33\ \Omega$$

Current through the parallel branch R_1 is $I/2 = 4.24$ A

Resistance, $R_1 = \dfrac{V\angle 0}{I/2\angle 0} = \dfrac{70.7}{4.24} = 16.67\,\Omega$

Current through parallel branch R_2 is $I/2 = 4.24$ A

Resistance is $R_2 = \dfrac{70.7}{4.24} = 16.67\,\Omega$

PROBLEM 6.11

Determine the average power delivered by the 500∠0° voltage source in Fig. 6.16 and also the dependent source.

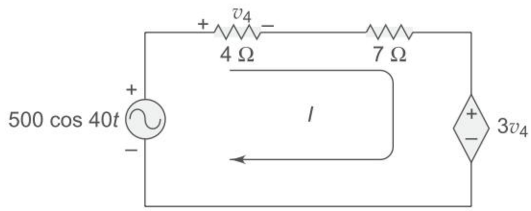


Fig. 6.16

Solution The current I can be determined by using Kirchhoff's voltage law.

$$I = \frac{500\angle 0° - 3v_4}{7+4}$$

where $v_4 = 4I$

$$I = \frac{500\angle 0°}{11} - \frac{12I}{11}$$

$$I = 21.73\angle 0°$$

Power delivered by the 500∠0° voltage source $= \dfrac{500 \times 21.73}{2} = 5.432$ kW

Power delivered by the dependent voltage source $= \dfrac{3v_4 \times I}{2} = \dfrac{3 \times 4I \times I}{2} = 2.833$ kW

PROBLEM 6.12

Find the average power delivered by the dependent voltage source in the circuit shown in Fig. 6.17.

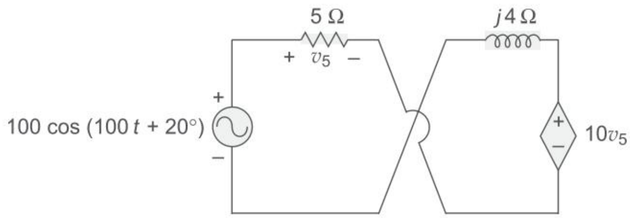


Fig. 6.17

Solution The circuit is redrawn as shown in Fig. 6.18.

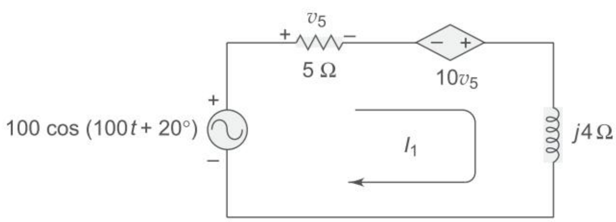


Fig. 6.18

Assume current I_1 flowing in the circuit.
The current I_1 can be determined by using Kirchhoff's voltage law.

$$I_1 = \frac{100\angle 20° + 10 \times 5I_1}{5 + j4}$$

$$I_1 - \frac{50I_1}{5 + j4} = \frac{100\angle 20°}{5 + j4}$$

$$I_1 = 2.213\angle -154.9°$$

Average power delivered by the dependent source

$$= \frac{V_m I_m}{2} = \cos\theta$$

$$= \frac{10V_5 I_1}{2}\cos\theta$$

$$= \frac{50 \times (2.213)^2}{2} = 122.43 \text{ W}$$

PROBLEM 6.13

For the circuit shown in Fig. 6.19, find the average power delivered by the voltage source.

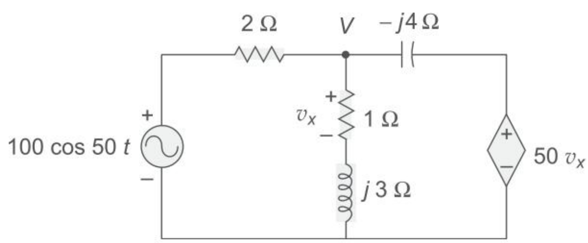


Fig. 6.19

Solution Applying Kirchhoff's current law at the node,

$$\frac{V - 100\angle 0°}{2} + \frac{V}{1 + j3} + \frac{V - 50V_x}{-j4} = 0$$

$$V_x = \frac{V}{1 + j3} \text{ volts}$$

Substituting in the above equation, we get

$$\frac{V - 100\angle 0°}{2} + \frac{V}{1 + j3} + \frac{V}{-j4} - \frac{50\text{V}}{(1 + j3)(-j4)} = 0$$

$$V = 14.705\ \angle 157.5°$$

$$I = \frac{V - 100\ \angle 0°}{2} = \frac{14.705\angle 157.5° - 100\ \angle 0°}{2} = 56.865\angle 177.18°$$

$$\text{Power delivered by the source } = \frac{100 \times 56.865 \cos 177.18°}{2}$$

$$= 2.834 \text{ kW}$$

PROBLEM 6.14

For the circuit shown in Fig. 6.20, find the average power delivered by the dependent current source.

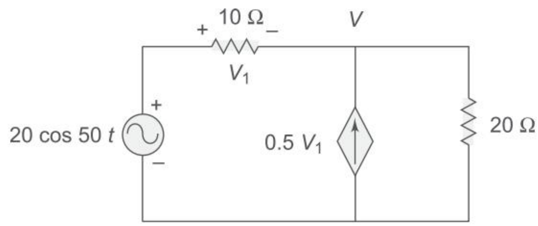


Fig. 6.20

Solution Applying Kirchhoff's current law at the node,

$$\frac{V - 20\angle 0^\circ}{10} - 0.5V_1 + \frac{V}{20} = 0$$

$$\text{where} \quad V_1 = 20\angle 0^\circ - V$$

Substituting V_1 in the above equation, we get

$$V = 18.46 \angle 0^\circ$$

$$V_1 = 1.54 \angle 0^\circ$$

Average power delivered by the dependent source

$$\frac{V_m I_m \cos\theta}{2} = \frac{18.46 \times 0.5 \times 1.54}{2} = 7.107 \text{ W}$$

PSpice Problems

PROBLEM 6.1

Determine the power factor, true power, reactive power, and apparent power using PSpice in the circuit shown in Fig. 6.21.

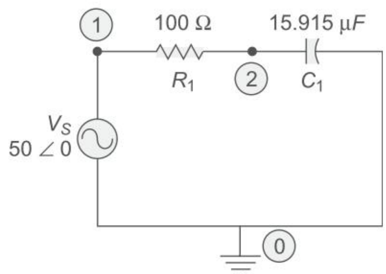


Fig. 6.21

$$f = 50 \text{ Hz}$$

$$c = \frac{1}{2\pi f \times c} = 15.915\,\mu F$$

```
* PROGRAM FOR OBTAINING PF, S, P, AND Q
VS        1     0     AC 50 0
R1        1     2     100
C1        2     0     15.915 U
.AC LIN   1     50    100
.PRINT AC IM(R1)IP(R1)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*****************************************************************
```

FREQ	IM(R1)	IP(R1)
5.000E + 01	2.236E – 01	6.344E + 01

Result

Power factor = COS(63.44) = 0.4471 Lead

True power = VICOSϕ = 50 × 0.2236 × COS(63.44)
= 5 W

Reactive power = VISINϕ = 50 × 0.2236 × SIN(63.44)
= 10 VAr

Apparent power = VI = 50 × 0.2236 = 11.18 VA

PROBLEM 6.2

For the circuit shown in Fig. 6.22, determine the average power delivered by the 50∠0° voltage source and dependent source using PSpice.

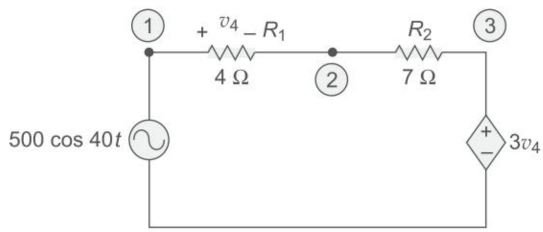


Fig. 6.22

$$V_s = \frac{500}{\sqrt{2}} = \text{RMS Value}$$

$$= 353.55\text{ V}$$

$$\omega = 40\text{ rad / sec}$$

$$\Rightarrow f = \frac{\omega}{2\pi} = 6.366\text{ Hz}$$

```
* PROGRAM TO FIND AVERAGE POWER
VS   1   0   AC 353.55 0
R1   1   2   4
R2   2   3   7
E3   0   1   2   3
.AC LIN 1 6.366 10
.PRINT AC I(VS), IP(VS), VM(R1), VP(R1)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*********************************************************************
```

FREQ	I(VS)	IP(VS)	VM(R1)	VP(R1)
6.366E + 00	1.537E + 01	1.800E + 02	6.149E + 01	0.000E + 00

Result It is a pure resistive circuit.

Power delivered by the 500∠0° source = $V_S \times I$ = 353.55 × 15.37
= 54.34 kW

Power delivered by dependent source = $3V_4 \times I$ = 3 × 61.49 × 15.37
= 2.835 kW

Answers to Practice Problems

6-2.1 0.0812 mW

6-2.2 – 0.114 W

6-3.1 0.97

6-3.2 3.12 Ω, 9.93 H

6-3.3 $P_{\text{complex}} = 461.8\,\text{VA}$

6-3.4 486.5; 0.27

6-3.6 $P_{\text{active}} = 2160$ watts; $P_{\text{reactive}} = 483$ VAR leading
$P_{\text{apparent}} = 2210$ VA

6-3.7 $P_{10} = 500\,\text{W}$; $P_3 = 600\,\text{W}$;
Ammeter reading = $19.25\underline{|-36^\circ}$ A

6-3.8 15.396 kW; 3944 VAR; 15.87 KVA; 0.97

6-5.1 $P_{\text{T real}} = 1800$ w; $P_{\text{T reactive}} = 600$ VAR lagging
$P_{10\text{ real}} = 100$ w; $P_{10\text{ reactive}} = 0$; $P_{(8+j6)\text{ real}} = 800$ w
$P_{(8+j6)\text{reactive}} = 600$ VAR

6-5.2 (i) Total admittance = $0.234\underline{|-51.4^\circ}$ ℧
(ii) Current drawn = 53.825 A
Power factor = 0.6238
(iii) $C = 5.79 \times 10^{-4}$ F

6-5.4 0.891; 1587.7 W; 806.2 VAR; 1781.9 VA

6-5.5 1136.36 VA; 529.6 VAR; 0.88

Objective-Type Questions

☆☆★**6.1** The phasor combination of resistive power and reactive power is called
(a) true power (c) reactive power (b) apparent power (d) average power

☆☆★**6.2** Apparent power is expressed in
(a) volt-amperes (b) watts (c) volt-amperes or watts (d) VAR

☆☆★**6.3** A power factor of ‘1’ indicates
(a) purely resistive circuit (c) combination of both (a) and (b)
(b) purely reactive circuit (d) none of these

☆☆★**6.4** A power factor of ‘0’ indicates
(a) purely resistive element (c) combination of both (a) and (b)
(b) purely reactive element (d) none of the above

☆☆★**6.5** For a certain load, the true power is 100 W and the reactive power is 100 VAR. What is the apparent power?
(a) 200 VA (b) 100 VA (c) 141.4 VA (d) 120 VA

☆☆★**6.6** If a load is purely resistive and the true power is 5 W, what is the apparent power?
(a) 10 VA (b) 5 VA (c) 25 VA (d) $\sqrt{50}$ VA

☆☆★**6.7** True power is defined as
(a) $VI\cos\theta$ (b) VI (c) $VI\sin\theta$ (d) none of these

☆☆★**6.8** In a certain series *RC* circuit, the true power is 2 W, and the reactive power is 3.5 VAR. What is the apparent power?
(a) 3.5 VA (b) 2 VA (c) 4.03 VA (d) 3 VA

☆☆★**6.9** If the phase angle θ is 45°, what is the power factor?
(a) cos 45° (b) sin 45° (c) tan 45° (d) none of these

☆☆★**6.10** To which component in an RC circuit is the power dissipation due?
(a) Capacitance (b) Resistance (c) Both (d) None

☆★★**6.11** A two element series circuit with an instantaneous current $I = 4.24 \sin(5000t + 45°)$ A has a power of 180 watts and a power factor of 0.8 lagging. The inductance of the circuit must have the value.

(a) 3 H (c) 3 mH
(b) 0.3 H (d) 0.3 mH

☆★★**6.12** In the circuit shown in Fig. 6.23, if branch A takes 8 KVAR, the power of the circuit will be

(a) 2 kW (c) 6 kW
(b) 4 kW (d) 8 kW

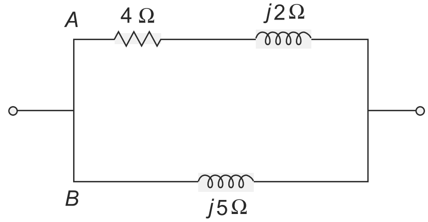


Fig. 6.23

☆★★**6.13** In the circuit shown in Fig. 6.24, the voltage across 30 Ω resistor is 45volts. The reading of the ammeter A will be

(a) 10 A (c) 22.4 A
(b) 19.4 A (d) 28 A

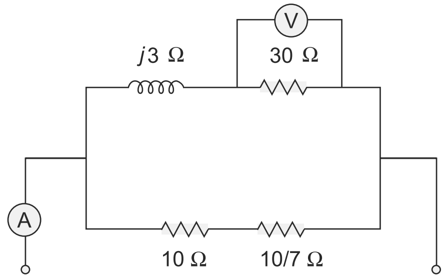


Fig. 6.24

★★★**6.14** In the circuit shown in Fig. 6.25, V_1 and V_2 are two identical sources of $10\angle 90°$. The power supplied by V_1 is

(a) 6 W (c) 11 W
(b) 8.8 W (d) 16 W

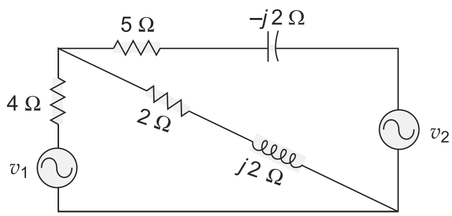


Fig. 6.25

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/264

CHAPTER 7

Steady-State AC Analysis

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Analyse ac circuits using mesh analysis and write the mesh equations using inspection method

LO 2 Analyse ac circuits using nodal analysis and write the nodal equations using inspection method

LO 3 Comprehend the superposition theorem and apply it to solve the ac circuits

LO 4 Comprehend Thèvenin's theorem and apply it to solve the ac circuits

LO 5 Comprehend Norton's theorem and apply it to solve the ac circuits

LO 6 Comprehend the maximum power transfer theorem and apply it to solve the ac circuits

7.1 MESH ANALYSIS

LO 1 Analyse ac circuits using mesh analysis and write the mesh equations using inspection method

We have earlier discussed mesh analysis but have applied it only to resistive circuits. Some of the ac circuits presented in this chapter can also be solved by using mesh analysis. In Chapter 2, the two basic techniques for writing network equations for mesh analysis and node analysis were presented. These concepts can also be used for sinusoidal steady-state condition. In the sinusoidal steady-state analysis, we use voltage phasors, current phasors, impedances and admittances to write branch equations, KVL and KCL equations. For ac circuits, the method of writing loop equations is modified slightly. The voltages and currents in ac circuits change polarity at regular intervals. At a given time, the instantaneous voltages are driving in either the positive or negative direction. If the impedances are complex, the sum of their voltages is found by vector addition. We shall illustrate the method of writing network mesh equations with the following example.

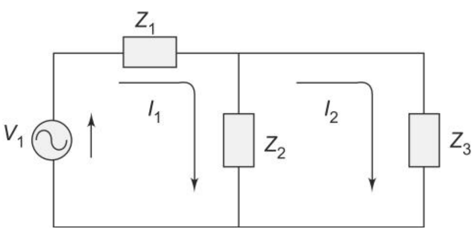


Fig. 7.1

Consider the circuit shown in Fig. 7.1, containing a voltage source and impedances.

The current in impedance Z_1 is I_1, and the current in Z_2, (assuming a positive direction downwards through the impedance) is $I_1 - I_2$. Similarly, the current in the impedance Z_3 is I_2. By applying Kirchhoff's voltage law for each loop, we can get two equations. The voltage across any element is the product of the phasor current in the element and the complex impedance.

Equation for the loop 1 is

$$I_1Z_1 + (I_1 - I_2)Z_2 = V_1 \tag{7.1}$$

Equation for the loop 2, which contains no source is

$$Z_2(I_2 - I_1) + Z_3I_2 = 0 \tag{7.2}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(Z_1 + Z_2) - I_2Z_2 = V_1 \tag{7.3}$$

$$-I_1Z_2 + I_2(Z_2 + Z_3) = 0 \tag{7.4}$$

By solving the above equations, we can find out currents I_1 and I_2. In general, if we have M meshes, B branches and N nodes including the reference node, we assume M branch currents and write M independent equations; then the number of mesh currents is given by $M = B - (N - 1)$.

EXAMPLE 7.1

Write the mesh current equations in the circuit shown in Fig. 7.2, and determine the currents.

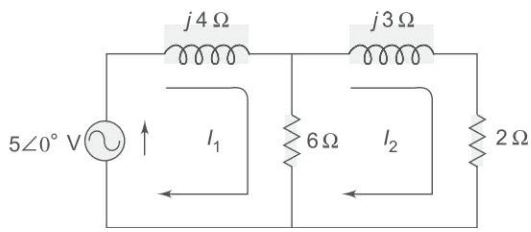


Fig. 7.2

Solution The equation for the loop 1 is

$$I_1(j4) + 6(I_1 - I_2) = 5\angle 0° \tag{7.5}$$

The equation for the loop 2 is

$$6(I_2 - I_1) + (j_3)I_2 + (2)I_2 = 0 \tag{7.6}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(6 + j4) - 6I_2 = 5\angle 0° \tag{7.7}$$

$$-6I_1 + (8 + j3)I_2 = 0 \tag{7.8}$$

Solving the above equations, we have

$$I_1 = \left[\frac{(8+j3)}{6}\right] I_2$$

$$\left[\frac{(8+j3)(6+j4)}{6}\right] I_2 - 6I_2 = 5\angle 0°$$

$$I_2\left[\frac{(8+j3)(6+j4)}{6} - 6\right] = 5\angle 0°$$

$$I_2\,[10.26\angle 54.2° - 6\angle 0°] = 5\angle 0°$$

$$I_2\,[(6 + j8.32) - 6] = 5\angle 0°$$

$$I_2 = \frac{5\angle 0°}{8.32\angle 90°} = 0.6\angle{-90°}$$

$$I_1 = \frac{8.54\angle 20.5°}{6} \times 0.6\angle{-90°}$$

$I_1 = 0.855 \angle -69.5°$

Current in loop 1, $I_1 = 0.855 = \angle -69.5°$

Current in loop 2, $I_2 = 0.6 \angle -90°$

7.2 MESH EQUATIONS BY INSPECTION

LO 1

In general, mesh equations can be written by observing any network. Consider the three-mesh network shown in Fig. 7.3.

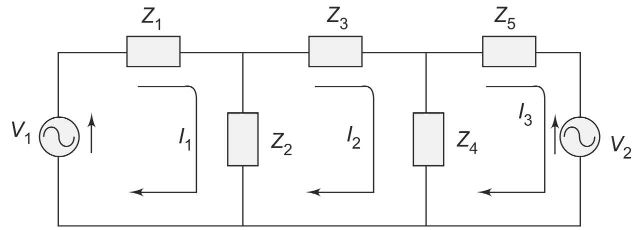


Fig. 7.3

The loop equations are

$$I_1 Z_1 + Z_2(I_1 - I_2) = V_1 \tag{7.9}$$

$$Z_2(I_2 - I_1) + Z_3 I_2 + Z_4(I_2 - I_3) = 0 \tag{7.10}$$

$$Z_4(I_3 - I_2) + Z_5 I_3 = -V_2 \tag{7.11}$$

By rearranging the above equations, we have

$$(Z_1 + Z_2)I_1 - Z_2 I_2 = V_1 \tag{7.12}$$

$$-Z_2 I_1 + (Z_2 + Z_3 + Z_4)I_2 - Z_4 I_3 = 0 \tag{7.13}$$

$$-Z_4 I_2 + (Z_4 + Z_5)I_3 = -V_2 \tag{7.14}$$

In general, the above equations can be written as

$$Z_{11} I_1 \pm Z_{12} I_2 + Z_{13} I_3 = V_a \tag{7.15}$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \tag{7.16}$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \tag{7.17}$$

If we compare the general equations with the circuit equations, we get the self-impedance of the loop 1

$$Z_{11} = Z_1 + Z_2$$

i.e. the sum of the impedances through which I_1 passes. Similarly, $Z_{22} = (Z_2 + Z_3 + Z_4)$, and $Z_{33} = (Z_4 + Z_5)$ are the self-impedances of loops 2 and 3. This is equal to the sum of the impedances in their respective loops, through which I_2 and I_3 passes, respectively.

Z_{12} is the sum of the impedances common to loop currents I_1 and I_2. Similarly Z_{21} is the sum of the impedances common to loop currents I_2 and I_1. In the circuit shown in Fig. 7.3, $Z_{12} = -Z_2$, and $Z_{21} = -Z_2$. Here, the positive sign is used if both currents passing through the common impedance are in the same direction; and the negative sign is used if the currents are in opposite directions. Similarly, Z_{13}, Z_{23}, Z_{31}, Z_{32} are the sums of the impedances common to the mesh currents indicated in their subscripts. V_a, V_b, and V_c are sums of the voltages driving their respective loops. Positive sign is used, if the direction of the loop current is the same as the direction of the source current. In Fig. 7.3, $V_b = 0$ because no source is driving the loop 2. Since the source, V_2 drives against the loop current I_3, $V_c = -V_2$.

EXAMPLE 7.2

For the circuit shown in Fig. 7.4, write the mesh equations using the inspection method.

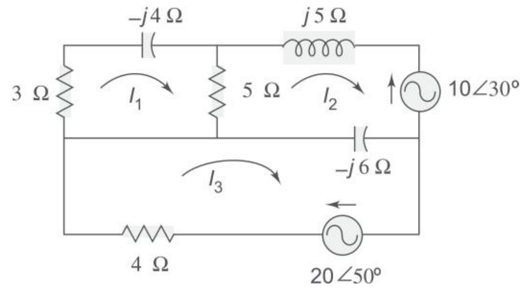


Fig. 7.4

Solution The general equations are

$$Z_{11} I_1 \pm Z_{12} I_2 \pm Z_{13} I_3 = V_a \qquad (7.18)$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \qquad (7.19)$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \qquad (7.20)$$

Consider Eq. (7.18)

Z_{11} = the self-impedance of the loop 1 = $(5 + 3 - j4)$ Ω

Z_{12} = the impedance common to both loops 1 and 2 = -5 Ω

The negative sign is used because the currents are in opposite directions.

$Z_{13} = 0$, because there is no common impedance between loop 1 and loop 3.

$V_a = 0$, because no source is driving the loop 1.

∴ Eq. (7.18) can be written as

$$(8 - j4)I_1 - 5I_2 = 0 \qquad (7.21)$$

Now, consider Eq. (7.19).

$Z_{21} = -5$, the impedance common to loops 1 and 2.

$Z_{22} = (5 + j5 - j6)$, the self-impedance of the loop 2.

$Z_{23} = -(-j6)$, the impedance common to loops 2 and 3.

$V_b = -10\angle 30°$, the source driving the loop 2.

The negative sign indicates that the source is driving against the loop current, I_2.

Hence, Eq. (7.19) can be written as

$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10\angle 30° \qquad (7.22)$$

Consider Eq. (7.20).

$Z_{31} = 0$, there is no common impedance between loops 3 and 1

$Z_{32} = -(-j6)$, the impedance common to loops 2 and 3

$Z_{33} = (4 - j6)$, the self impedance of the loop 3

$V_b = 20\angle 50°$, the source driving the loop 3

The positive sign is used because the source is driving in the same direction as the loop current 3. Hence, the equation can be written as

$$(j6)I_2 + (4 - j6)I_3 = 20\angle 50° \qquad (7.23)$$

The three mesh equations are

$$(8 - j4)I_1 - 5I_2 = 0$$

$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10\angle 30°$$

$$(j6)I_2 + (4 - j6)I_3 = 20\angle 50$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **7-1.1** For the circuit shown in Fig. Q.1, determine the value of the current I_x in the impedance $Z = 4 + j5$ between nodes a and b.

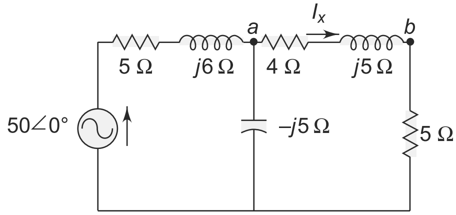


Fig. Q.1

☆☆★ **7-1.2** For the given network shown in Fig. Q.2, find the current through $(2 + j3)\,\Omega$ impedance using mesh analysis.

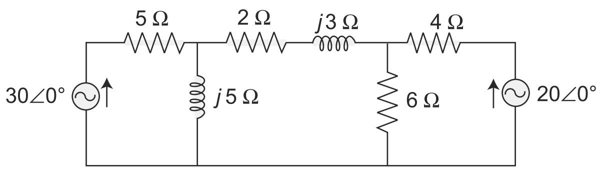


Fig. Q.2

☆☆★ **7-1.3** For the circuit shown in Fig. Q.3, find the voltage across the dependent source branch by using mesh analysis.

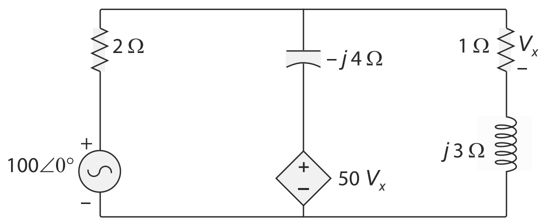


Fig. Q.3

☆★★ **7-1.4** By application of Thèvenin's theorem, find the current I in the circuit shown in Fig. Q.4. The voltage source shown is sinusoidal having a 10 V rms value and frequency is such that the inductor has an impedance of 20 Ω magnitude and each capacitor has an impedance of 10 Ω magnitude.

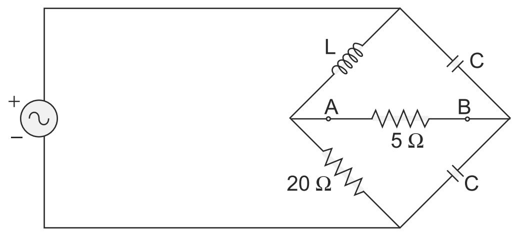


Fig. Q.4

Frequently Asked Questions linked to LO 1*

☆☆★ **7-1.1** Define 'mesh analysis' of a circuit. **[AU April/May 2011]**

☆★★ **7-1.2** For the network shown in Fig. Q.2, obtain the current ratio (I1/I3) using mesh analysis.

[AU April/May 2011]

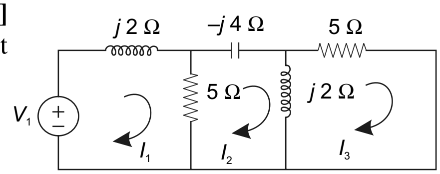


Fig. Q.2

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

☆★★**7-1.3** Using mesh analysis, determine the current the 1 Ω resistor of the network shown in Fig. Q.3. **[BPUT 2008]**

★★★**7-1.4** For the circuit shown in Fig. Q.4, determine the value of V_2 such that the current through (3+*j*4) Ω impedance is zero. **(AU Nov./Dec. 2012)**

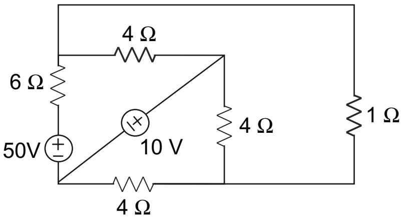


Fig. Q.3

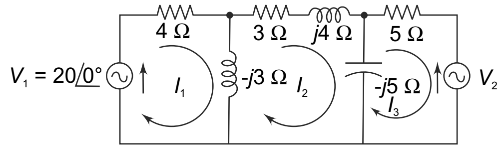


Fig. Q.4

7.3 NODAL ANALYSIS

> **LO 2** Analyse ac circuits using nodal analysis and write the nodal equations using inspection method

The node-voltage method can also be used with networks containing complex impedances and excited by sinusoidal voltage sources. In general, in an *N*-node network, we can choose any node as the reference or datum node. In many circuits, this reference is most conveniently choosen as the common terminal or ground terminal. Then it is possible to write $(N - 1)$ nodal equations using KCL. We shall illustrate nodal analysis with the following example.

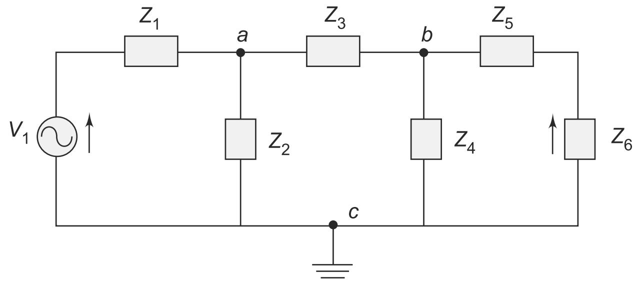


Fig. 7.5

Consider the circuit shown in Fig. 7.5.

Let us take *a* and *b* as nodes, and *c* as reference node. V_a is the voltage between nodes *a* and *c*. V_b is the voltage between nodes *b* and *c*. Applying Kirchhoff's current law at each node, the unknowns V_a and V_b are obtained.

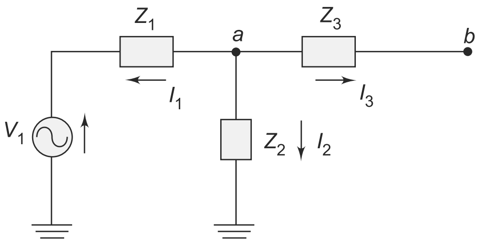


Fig. 7.6

In Fig. 7.6, node *a* is redrawn with all its branches, assuming that all currents are leaving the node *a*.

In Fig. 7.6, the sum of the currents leaving the node *a* is zero.

$$\therefore \quad I_1 + I_2 + I_3 = 0 \tag{7.24}$$

$$\text{where} \quad I_1 = \frac{V_a - V_1}{Z_1}, I_2 = \frac{V_a}{Z_2}, I_3 = \frac{V_a - V_b}{Z_3}$$

Substituting I_1, I_2 and I_3 in Eq. (7.24), we get

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \tag{7.25}$$

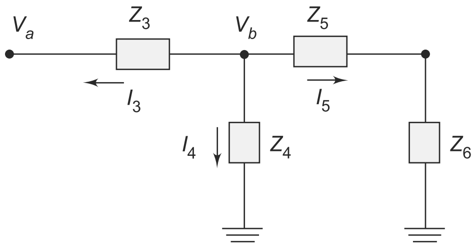


Fig. 7.7

Similarly, in Fig. 7.7, the node *b* is redrawn with all its branches, assuming that all currents are leaving the node *b*.

In Fig. 7.7, the sum of the currents leaving the node b is zero.

$$\therefore \quad I_3 + I_4 + I_5 = 0 \tag{7.26}$$

where $I_3 = \dfrac{V_b - V_a}{Z_3}, I_4 = \dfrac{V_b}{Z_4}, I_5 = \dfrac{V_b}{Z_5 + Z_6}$

Substituting I_3, I_4 and I_5 in Eq. (7.26)

we get $$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b}{Z_5 + Z_6} = 0 \tag{7.27}$$

Rearranging Eqs 7.25 and 7.27, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)V_a - \left(\frac{1}{Z_3}\right)V_b = \left(\frac{1}{Z_1}\right)V_1 \tag{7.28}$$

$$-\left(\frac{1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5 + Z_6}\right)V_b = 0 \tag{7.29}$$

From Eqs 7.28 and 7.29, we can find the unknown voltages V_a and V_b.

EXAMPLE 7.3

In the network shown in Fig. 7.8, determine V_a and V_b.

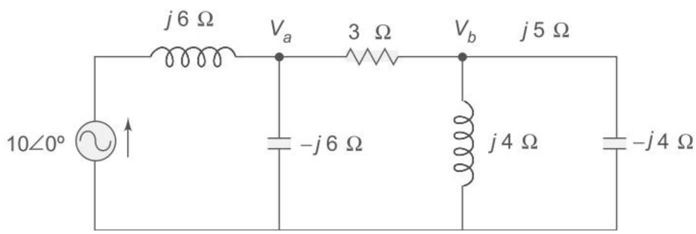


Fig. 7.8

Solution To obtain the voltage V_a at a, consider the branch currents leaving the node a as shown in Fig. 7.9 (a).

In Fig. 7.9 (a), $I_1 = \dfrac{V_a - 10\angle 0^\circ}{j6}, I_2 = \dfrac{V_a}{-j6}, I_3 = \dfrac{V_a - V_b}{3}$

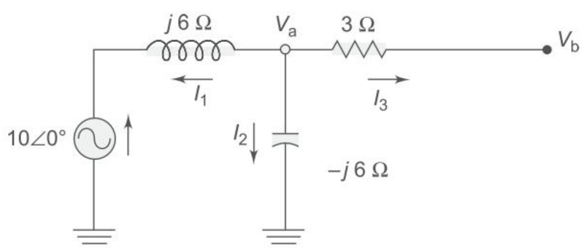


Fig. 7.9(a)

Since the sum of the currents leaving the node a is zero,

$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_a - 10\angle 0^\circ}{j6} + \frac{V_a}{-j6} + \frac{V_a - V_b}{3} = 0 \tag{7.30}$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)V_a - \frac{1}{3}V_b = \frac{10\angle 0^\circ}{j6}$$

$$\therefore \quad \frac{1}{3}V_a - \frac{1}{3}V_b = \frac{10\angle 0^\circ}{j6} \tag{7.31}$$

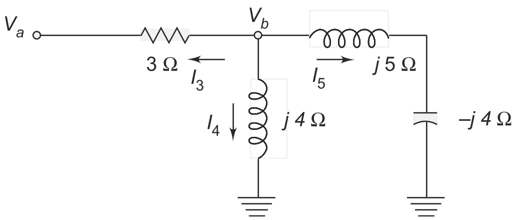


Fig. 7.9(b)

To obtain the voltage V_b at b, consider the branch currents leaving the node b as shown in Fig. 7.9 (b).

In Fig. 7.9 (b), $I_3 = \dfrac{V_b - V_a}{3}, I_4 = \dfrac{V_b}{j4}, I_5 = \dfrac{V_b}{(j5 - j4)}$

Since the sum of the currents leaving the node b is zero,

$$I_3 + I_4 + I_5 = 0$$

$$\frac{V_b - V_a}{3} + \frac{V_b}{j4} + \frac{V_b}{j1} = 0 \tag{7.32}$$

$$-\frac{1}{3}V_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)V_b = 0 \tag{7.33}$$

From Eqs (7.31) and (7.33), we can solve for V_a and V_b,

$$0.33V_a - 0.33V_b = 1.67\angle -90^\circ \tag{7.34}$$

$$-0.33V_a + (0.33 - 0.25j - j)V_b = 0 \tag{7.35}$$

Adding Eqs (7.34) and (7.35), we get $(-1.25j)V_b = 1.67\angle -90^\circ$

$$-1.25\angle 90^\circ\ V_b = 1.67\angle -90^\circ$$

$$V_b = \frac{1.67\angle -90^\circ}{-1.25\angle 90^\circ}$$

$$= -1.34\angle -180^\circ$$

Substituting V_b in Eq. (7.34), we get

$$0.33V_a - (0.33)(-1.34\angle -180^\circ) = 1.67\angle -90^\circ$$

$$V_a = \frac{1.67\angle -90^\circ}{0.33} = -1.31\text{V}$$

$$V_a = 5.22\angle -104.5^\circ\ \text{V}$$

Voltages V_a and V_b are $5.22\angle -104.5^\circ$ V and $-1.34\angle -180^\circ$ V respectively.

7.4 NODAL EQUATIONS BY INSPECTION

LO 2

In general, nodal equations can also be written by observing the network. Consider a four-node network including a reference node as shown in Fig. 7.10.

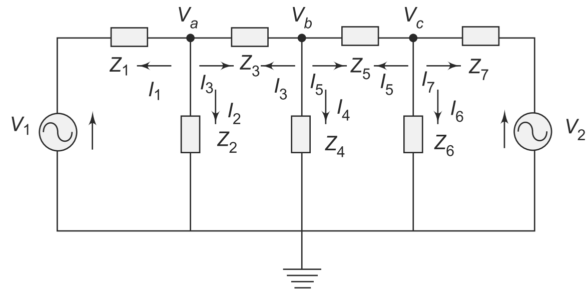


Fig. 7.10

Consider nodes a, b, and c separately as shown in Figs 7.11 (a), (b), and (c).

Assuming that all the currents are leaving the nodes, the nodal equations at a, b and c are

$$I_1 + I_2 + I_3 = 0$$

$$I_3 + I_4 + I_5 = 0$$

$$I_5 + I_6 + I_7 = 0$$

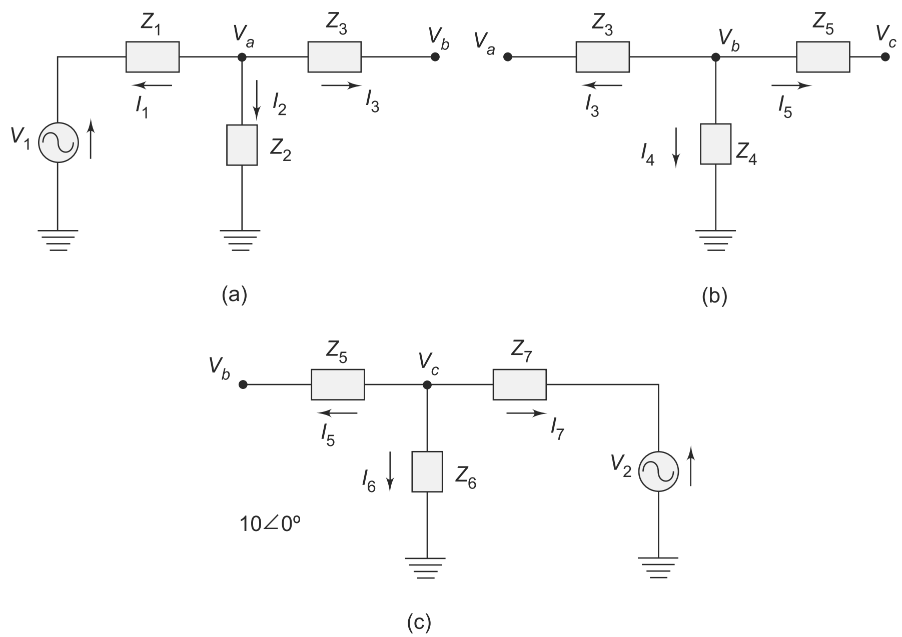


Fig. 7.11

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \tag{7.36}$$

$$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b - V_c}{Z_5} = 0 \tag{7.37}$$

$$\frac{V_c - V_b}{Z_5} + \frac{V_c}{Z_6} + \frac{V_c - V_2}{Z_7} = 0 \tag{7.38}$$

Rearranging the above equations, we get

$$\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)V_a-\left(\frac{1}{Z_3}\right)V_b=\left(\frac{1}{Z_1}\right)V_1 \tag{7.39}$$

$$\left(\frac{-1}{Z_3}\right)V_a+\left(\frac{1}{Z_3}+\frac{1}{Z_4}+\frac{1}{Z_5}\right)V_b-\left(\frac{1}{Z_5}\right)V_c=0 \tag{7.40}$$

$$\left(\frac{-1}{Z_5}\right)V_b+\left(\frac{1}{Z_5}+\frac{1}{Z_6}+\frac{1}{Z_7}\right)V_c=\left(\frac{1}{Z_7}\right)V_2 \tag{7.41}$$

In general, the above equations can be written as

$$Y_{aa}V_a + Y_{ab}V_b + Y_{ac}V_c = I_1$$

$$Y_{ba}V_a + Y_{bb}V_b + Y_{bc}V_c = I_2$$

$$Y_{ca}V_a + Y_{cb}V_b + Y_{cc}V_c = I_3$$

If we compare the general equations with the circuit equations, the self-admittance at the node a is

$$Y_{aa}=\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}$$

which is the sum of the admittances connected to the node a.

Similarly, $Y_{bb}=\frac{1}{Z_3}+\frac{1}{Z_4}+\frac{1}{Z_5}$, and $Y_{cc}=\frac{1}{Z_5}+\frac{1}{Z_6}+\frac{1}{Z_7}$

are the self admittances at nodes b and c, respectively. Y_{ab} is the mutual admittance between nodes a and b, i.e. it is the sum of all the admittances connecting nodes a and b. $Y_{ab} = -1/Z_3$ has a negative sign. All the mutual admittances have negative signs. Similarly, Y_{ac}, Y_{ba}, Y_{bc}, Y_{ca}, and Y_{cb} are also mutual admittances. These are equal to the sums of the admittances connecting to nodes indicated in their subscripts. I_1 is the sum of all the source currents at node a. The current which drives into the node has a positive sign, while the current driving away from the node has a negative sign.

EXAMPLE 7.4

For the circuit shown in Fig. 7.12, write the node equations by the inspection method.

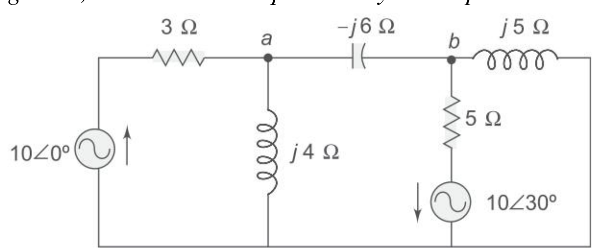


Fig. 7.12

Solution The general equations are

$$Y_{aa}\,V_a + Y_{ab}\,V_b = I_1 \tag{7.42}$$

$$Y_{ba}\,V_a + Y_{bb}\,V_b = I_2 \tag{7.43}$$

Consider Eq. (7.42)

$$Y_{aa} = \frac{1}{3} + \frac{1}{j4} + \frac{1}{-j6}$$

The self-admittance at the node a is the sum of admittances connected to the node a.

$$Y_{bb} = \frac{1}{-j6} + \frac{1}{5} + \frac{1}{j5}$$

The self-admittance at the node b is the sum of admittances connected to the node b.

$$Y_{ab} = -\left(\frac{1}{-j6}\right)$$

The mutual admittance between nodes a and b is the sum of admittances connected between nodes a and b. Similarly, $Y_{ba} = -(-1/j6)$, the mutual admittance between nodes b and a is the sum of the admittances connected between nodes b and a.

$$I_1 = \frac{10\angle 0^\circ}{3}$$

The source current at the node a

$$I_2 = \frac{-10\angle 30^\circ}{5} = \text{source current leaving at the node } b.$$

Therefore, the nodal equations are

$$\left(\frac{1}{3} + \frac{1}{j4} - \frac{1}{j6}\right)V_a - \left(\frac{-1}{j6}\right)V_b = \frac{10\angle 0^\circ}{3} \tag{7.44}$$

$$-\left(\frac{-1}{j6}\right)V_a + \left(\frac{1}{5} + \frac{1}{j5} - \frac{1}{j6}\right)V_b = \frac{-10\angle 30^\circ}{5} \tag{7.45}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆☆★ **7-2.1** Determine the value of source currents by loop analysis for the circuit shown in Fig. Q.1 and verify the results by using node analysis.

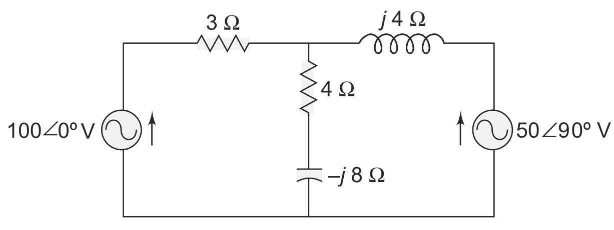


Fig. Q.1

☆☆★ **7-2.2** Determine the power out of the source in the circuit shown in Fig. Q.2 by nodal analysis and verify the results by using loop analysis.

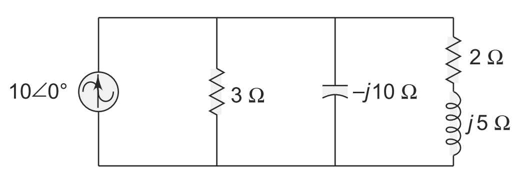


Fig. Q.2

★★★**7-2.3** For the circuit shown in Fig. Q.3, obtain the voltage across 500 kΩ resistor.

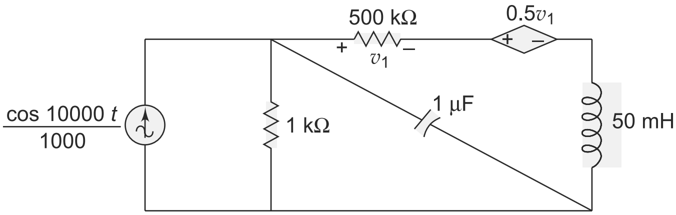


Fig. Q.3

Frequently Asked Questions linked to LO 2

☆★★**7-2.1** In the network system shown in Fig. Q.1, find the current through Z_3 using nodal method. The values of voltages are given volts and the impedances in ohms. **[BPUT 2008]**

☆☆★**7-2.2** Using nodal analysis find voltage V_1 and V_2 for the circuit shown in Fig. Q.2. **[GTU Dec. 2010]**

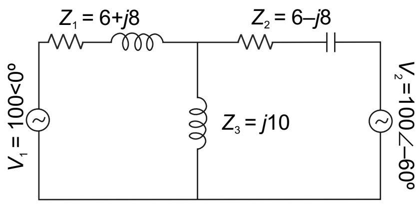


Fig. Q.1

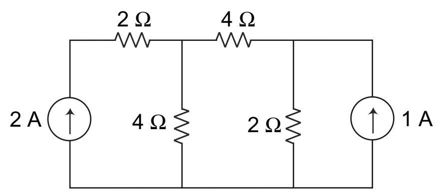


Fig. Q.2

☆☆★**7-2.3** Solve for the nodal voltage V_1, V_2, V_3, and V_4 as shown in the network in Fig. Q.3, using the nodal analysis. **[GTU May 2011]**

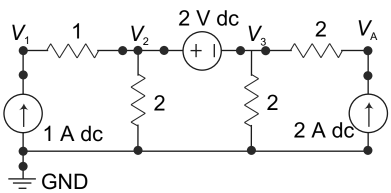


Fig. Q.3

☆☆★**7-2.4** Find the current through the 2 V source in Fig. Q.4, using node voltage analysis. **[GTU Dec. 2012]**

☆★★**7-2.5** Find the source voltage V_s by using nodal technique, assume $I = 5\angle 45°$ A. **[JNTU Nov. 2012]**

☆★★**7-2.6** By applying nodal analysis for the circuit shown Fig. Q.6, determine the power output of the source and the power in each resistor of the circuit. **[AU May/June 2013]**

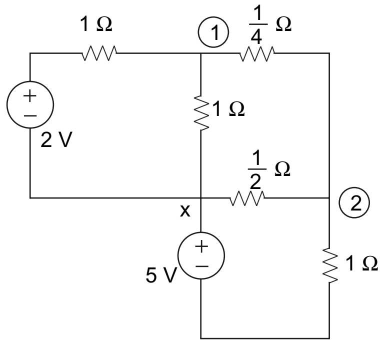


Fig. Q.4

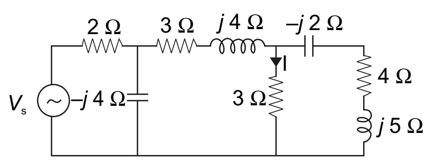


Fig. Q.5

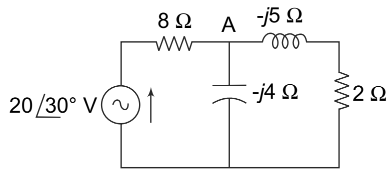


Fig. Q.6

7.5 SUPERPOSITION THEOREM

LO 3 Comprehend the superposition theorem and apply it to solve the ac circuits

The superposition theorem also can be used to analyse ac circuits containing more than one source. *The* ***superposition theorem*** *states that the response in any element in a circuit is the vector sum of the responses that can be expected to flow if each source acts independently of other sources.* As each source is considered, all of the other sources are replaced by their internal impedances, which are mostly short circuits in the case of a voltage source, and open circuits in the case of a current source. This theorem is valid only for linear systems. In a network containing complex impedance, all quantities must be treated as complex numbers.

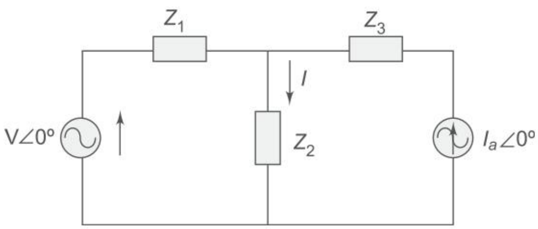


Fig. 7.13

Consider a circuit which contains two sources as shown in Fig. 7.13.

Now let us find the current I passing through the impedance Z_2 in the circuit. According to the superposition theorem, the current due to voltage source V $\angle 0°$ V is I_1 with current source $I_a \angle 0°$ A open-circuited.

$$I_1 = \frac{V\angle 0°}{Z_1 + Z_2}$$

The current due to $I_a \angle 0°$ A is I_2 with voltage source $V \angle 0°$ short circuited.

$$\therefore \quad I_2 = I_a \angle 0° \times \frac{Z_1}{Z_1 + Z_2}$$

The total current passing through the impedance Z_2 is

$$I = I_1 + I_2$$

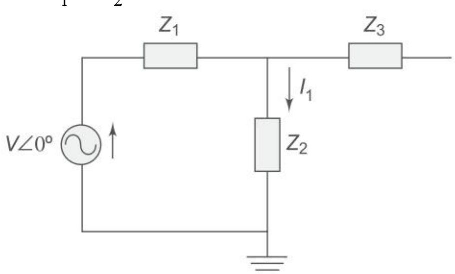


Fig. 7.14

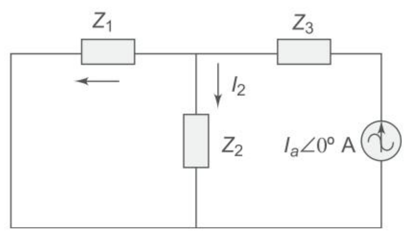


Fig. 7.15

EXAMPLE 7.5

Determine the voltage across the (2 + j5) Ω impedance as shown in Fig. 7.16 by using the superposition theorem.

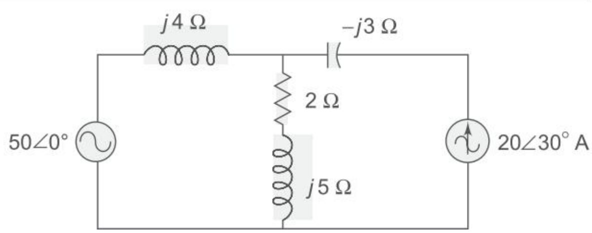


Fig. 7.16

Solution According to the superposition theorem, the current due to the 50 $\angle 0°$ V voltage source is I_1 as shown in Fig. 7.17 with current source 20 $\angle 30°$ A open-circuited.

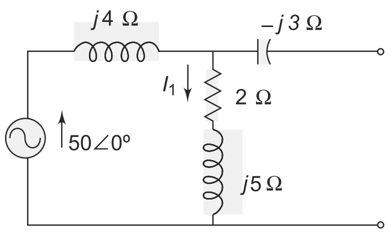


Fig. 7.17

$$\text{Current } I_1 = \frac{50\angle 0°}{2+j4+j5} = \frac{50\angle 0°}{(2+j9)}$$

$$= \frac{50\angle 0°}{9.22\angle 77.47} = 5.42\angle -77.47° \text{ A}$$

Voltage across (2 + j5) Ω due to the current I_1 is

$$V_1 = 5.42 \angle - 77.47° \,(2 + j5)$$
$$= (5.38)\,(5.42) \angle -77.47° + 68.19°$$
$$= 29.16 \angle -9.28°$$

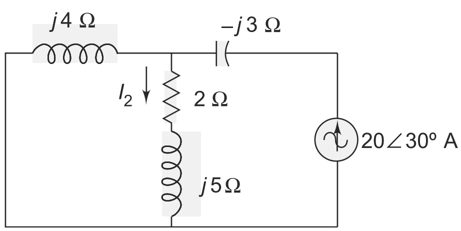


Fig. 7.18

The current due to the 20 $\angle 30°$ A current source is I_2 as shown in Fig. 7.18, with the voltage source 50 $\angle 0°$ V short-circuited.

$$\text{Current } I_2 = 20\angle 30° \times \frac{(j4)\,\Omega}{(2+j9)\,\Omega}$$

$$= \frac{20\angle 30° \times 4\angle 90°}{9.22\angle 77.47°}$$

$$\therefore \quad I_2 = 8.68 \angle 120° - 77.47° = 8.68 \angle 42.53°$$

Voltage across (2 + j5) Ω due to the current I_2 is

$$V_2 = 8.68 \angle 42.53° \,(2 + j5)$$
$$= (8.68)\,(5.38) \angle 42.53° + 68.19°$$
$$= 46.69 \angle 110.72°$$

Voltage across (2 + j5) Ω due to both sources is

$$V = V_1 + V_2$$
$$= 29.16 \angle -9.28° + 46.69 \angle 110.72°$$
$$= 28.78 - j4.7 - 16.52 + j43.67$$
$$= (12.26 + j38.97) \text{ V}$$

Voltage across (2 + j5) Ω is $V = 40.85 \angle 72.53°$.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★**7.3.1** Apply the superposition theorem to the circuit shown in Fig. Q.1 and find the current I.

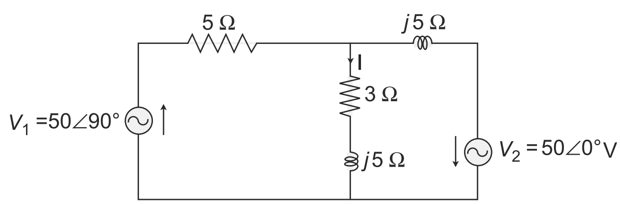


Fig. Q.1

Frequently Asked Questions linked to LO 3

☆★★**7-3.1** Use the superposition theorem to find the current through the 4 Ω resistor in the circuit shown in Fig. Q.1. [AU **May/June 2013]**

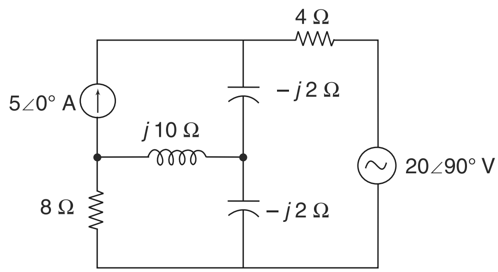


Fig. Q.1

☆★★**7-3.2** Find the current through the capacitor of $-j\,5\,\Omega$ reactance as shown in Fig. Q.2 using the superposition theorem. **[JNTU Nov. 2012]**

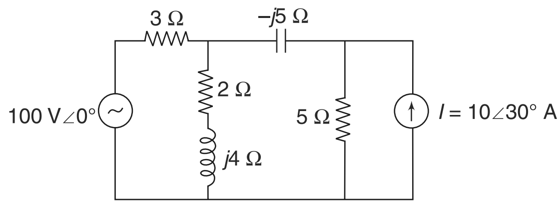


Fig. Q.2

☆★★**7-3.3** By the superposition theorem, calculate the current the $(2 + j3)\,\Omega$ impedance branch of the circuit in Fig. Q.3. **[PTU 2011-12]**

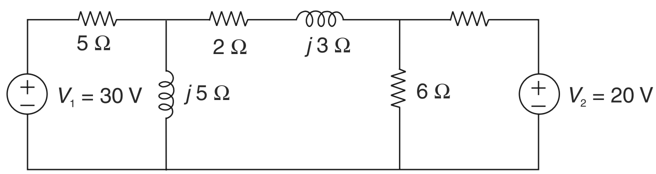


Fig. Q.3

7.6 THÈVENIN'S THEOREM

> **LO 4** Comprehend Thèvenin's theorem and apply it to solve the ac circuits

Thèvenin's theorem gives us a method for simplifying a given circuit. The Thèvenin equivalent form of any complex impedance circuit consists of an equivalent voltage source V_{Th}, and an equivalent impedance Z_{Th}, arranged as shown in Fig. 7.19. The values of equivalent voltage and impedance depend on the values in the original circuit.

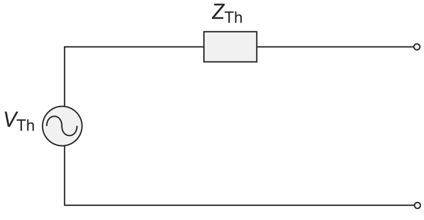


Fig. 7.19

Though the Thèvenin equivalent circuit is not the same as its original circuit, the output voltage and output current are the same in both cases. Here, the Thèvenin voltage is equal to the open-

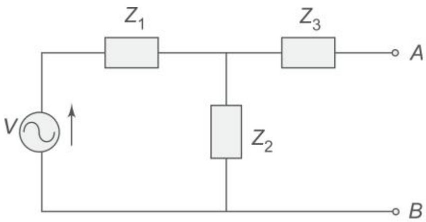


Fig. 7.20

circuit voltage across the output terminals, and impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.20.

Thèvenin equivalent for the circuit shown in Fig. 7.20 between points A and B is found as follows.

The voltage across points A and B is the Thèvenin equivalent voltage. In the circuit shown in Fig. 7.20, the voltage across A and B is the same as the voltage across Z_2 because there is no current through Z_3.

$$\therefore \quad V_{\text{Th}} = V\left(\frac{Z_2}{Z_1 + Z_2}\right)$$

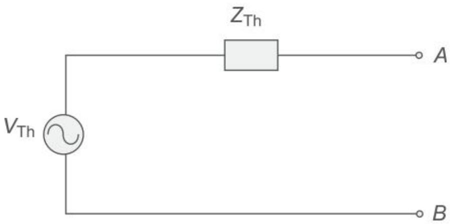


Fig. 7.21

The impedance between points A and B with the source replaced by short circuit is the Thèvenin equivalent impedance. In Fig. 7.20, the impedance from A to B is Z_3 in series with the parallel combination of Z_1 and Z_2.

$$\therefore \quad Z_{\text{Th}} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}$$

The Thèvenin equivalent circuit is shown in Fig. 7.21.

EXAMPLE 7.6

For the circuit shown in Fig. 7.22, determine Thèvenin's equivalent between the output terminals.

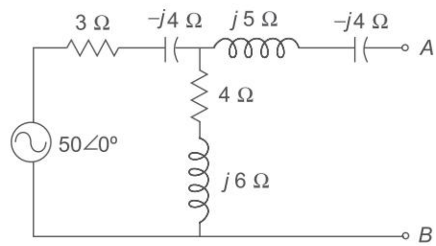


Fig. 7.22

Solution The Thèvenin voltage, V_{Th}, is equal to the voltage across the $(4 + j6)\ \Omega$ impedance. The voltage across $(4 + j6)\ \Omega$ is

$$V = 50\angle 0° \times \frac{(4 + j6)}{(4 + j6) + (3 - j4)}$$

$$= 50\angle 0° \times \frac{4 + j6}{7 + j2}$$

$$= 50\angle 0° \times \frac{7.21\angle 56.3°}{7.28\angle 15.95°}$$

$$= 50\ \angle 0° \times 0.99\ \angle 40.35°$$

$$= 49.5\ \angle 40.35°\ \text{V}$$

The impedance seen from terminals A and B is

$$Z_{\text{Th}} = (j5 - j4) + \frac{(3 - j4)(4 + j6)}{3 - j4 + 4 + j6}$$

$$= j1 + \frac{5\angle 53.13° \times 7.21\angle 56.3°}{7.28\angle 15.95°}$$

$$= j1 + 4.95\ \angle -12.78° = j1 + 4.83 - j1.095$$

$$= 4.83 - j0.095$$

$$\therefore \quad Z_{\text{Th}} = 4.83\ \angle -1.13°\ \Omega$$

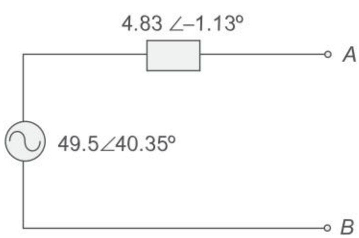


Fig. 7.23

The Thèvenin equivalent circuit is shown in Fig. 7.23.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★ **7-4.1** Find the current in the 15 Ω resistor in the network shown in Fig. Q.1 by Thèvenin's theorem.

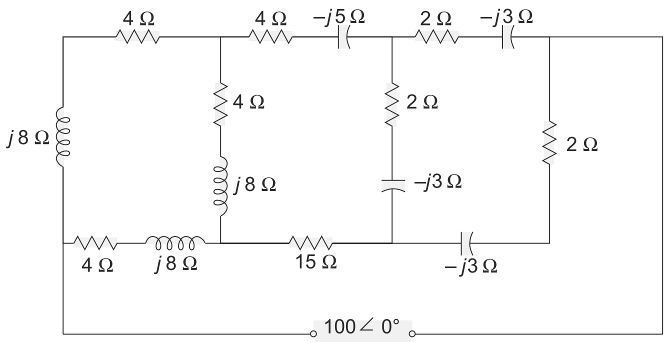


Fig. Q.1

☆★★ **7-4.2** Use Thèvenin's theorem to find the current through the (5 + *j*4) Ω impedance in Fig. Q.2. Verify the results using Norton's theorem.

☆★★ **7-4.3** Find Thèvenin's equivalent for the network shown in Fig. Q.3.

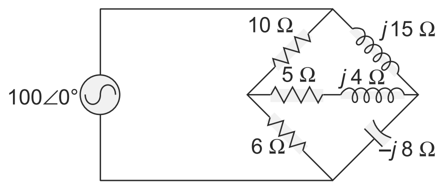


Fig. Q.2

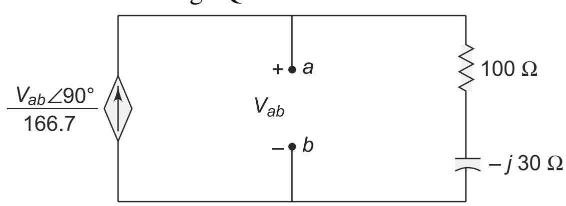


Fig. Q.3

☆★★ **7-4.4** For the circuit shown in Fig. Q.4, obtain the Thèvenin's equivalent circuit at terminals *ab*.

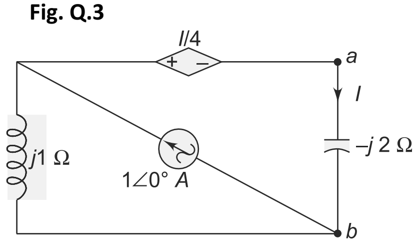


Fig. Q.4

Frequently Asked Questions linked to LO 4

☆☆★ **7-4.1** Find the current through the branch *a-b* of the network shown in Fig. Q.1 using Thevenin's theorem. **[AU May/June 2013]**

☆★★ **7-4.2** Find the current in the (1 + *j*1) Ω resistor across *A*, *B* of the network shown in Fig. Q.2 using Thevenin's theorem. **[PTU 2009-10]**

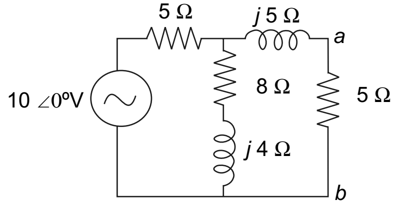


Fig. Q.1

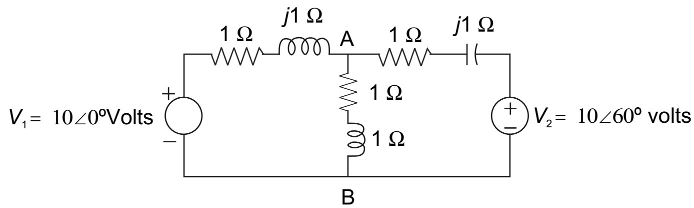


Fig. Q.2

7.7 NORTON'S THEOREM

LO 5 Comprehend Norton's theorem and apply it to solve the ac circuits

Another method of analysing a complex impedance circuit is given by Norton's theorem. The Norton equivalent form of any complex impedance circuit consists of an equivalent current source I_N and an equivalent impedance Z_N, arranged as shown in Fig. 7.24. The values of equivalent current and impedance depend on the values in the original circuit.

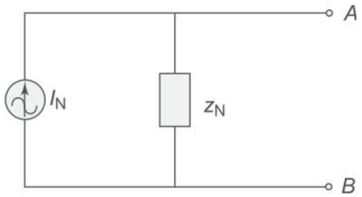


Fig. 7.24

Though Norton's equivalent circuit is not the same as its original circuit, the output voltage and current are the same in both cases; Norton's current is equal to the current passing through the short-circuited output terminals and the value of impedance is equal to the impedance seen into the network across the output terminals.

Consider the circuit shown in Fig. 7.25.

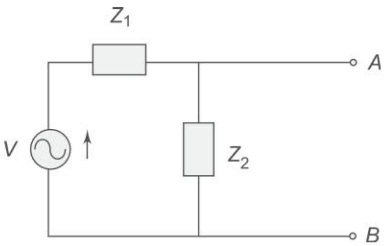


Fig. 7.25

Norton's equivalent for the circuit shown in Fig. 7.25 between points A and B is found as follows. The current passing through points A and B when it is short-circuited is the Norton's equivalent current, as shown in Fig. 7.26.

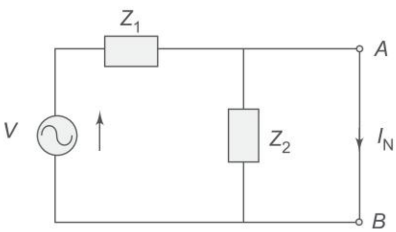


Fig. 7.26

Norton's current $I_N = V/Z_1$

The impedance between points A and B, with the source replaced by a short circuit, is Norton's equivalent impedance.

In Fig. 7.25, the impedance from A to B,

Z_2 is in parallel with Z_1.

$$\therefore \quad Z_N = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Norton's equivalent circuit is shown in Fig. 7.27.

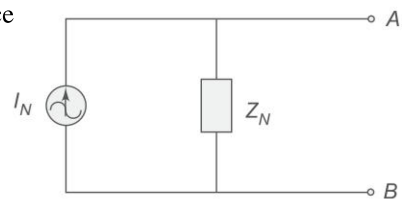


Fig. 7.27

EXAMPLE 7.7

For the circuit shown in Fig. 7.28, determine Norton's equivalent circuit between the output terminals, AB.

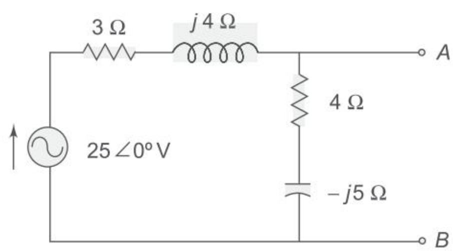


Fig. 7.28

Solution Norton's current I_N is equal to the current passing through the short-circuited terminals AB as shown in Fig. 7.29.

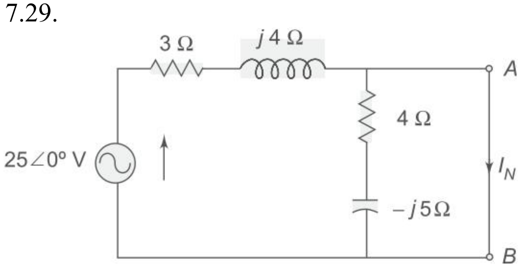


Fig. 7.29

The current through terminals AB is

$$I_N = \frac{25\angle 0^\circ}{3+j4} = \frac{25\angle 0^\circ}{5\angle 53.13^\circ}$$
$$= 5\angle -53.13^\circ$$

The impedance seen from terminals AB is

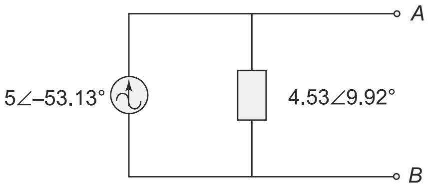


Fig. 7.30

$$Z_N = \frac{(3+j4)(4-j5)}{(3+j4)+(4-j5)}$$
$$= \frac{5\angle 53.13^\circ \times 6.4\angle -51.34^\circ}{7.07\angle -8.13^\circ}$$
$$= 4.53\angle 9.92^\circ$$

Norton's equivalent circuit is shown in Fig. 7.30.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆★★ **7-5.1** Determine Thèvenin's and Norton's equivalent circuits across terminals AB, in Fig. Q.1.

☆☆★ **7-5.2** Determine Norton's and Thèvenin's equivalent circuits for the circuit shown in Fig. Q.2.

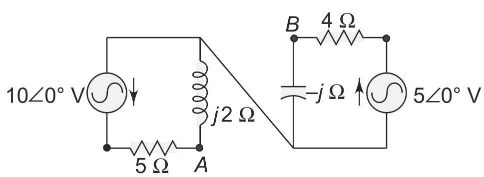


Fig. Q.1

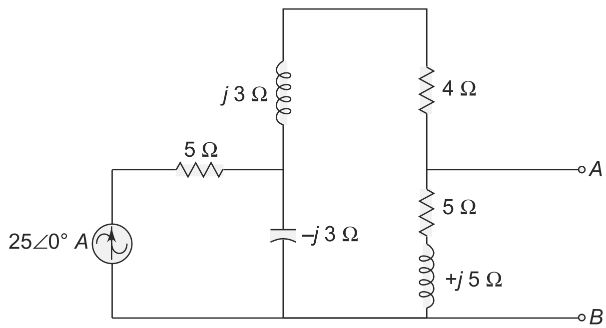


Fig. Q.2

Frequently Asked Questions linked to LO 5

☆★★ **7-5.1** Find the current through the 10-ohm resistor in the following circuit using Norton's theorem. (Fig. Q.1)

[JNTU Nov. 2012]

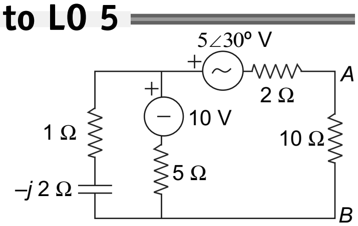


Fig. Q.1

7.8 MAXIMUM POWER TRANSFER THEOREM

LO 6 Comprehend the maximum power transfer theorem and apply it to solve the ac circuits

In Chapter 3, the maximum power transfer theorem has been discussed for resistive loads. The maximum power transfer theorem states that the maximum power is delivered from a source to its load when the load resistance is equal to the source resistance. It is for this reason that the ability to obtain impedance matching between circuits is so important. For example, the audio output transformer must match the high impedance of the audio power amplifier output to the low input impedance of the speaker. Maximum power transfer is not always desirable, since the transfer occurs at a 50 per cent efficiency. In many situations, a maximum voltage transfer is desired which means that unmatched impedances are necessary. If maximum power transfer is required, the load resistance should equal the given source resistance. *The maximum power transfer theorem can be applied to complex impedance circuits. If the source impedance is complex, then the maximum power transfer occurs when the load impedance is the complex conjugate of the source impedance.*

Consider the circuit shown in Fig. 7.31, consisting of a source impedance delivering power to a complex load.

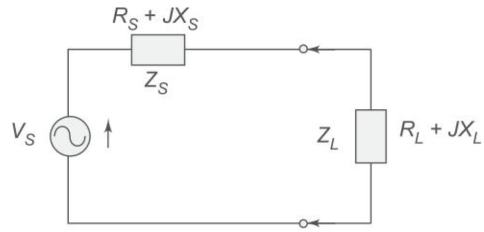


Fig. 7.31

Current passing through the circuit shown

$$I = \frac{V_s}{(R_s + jX_s) + (R_L + jX_L)}$$

Magnitude of current $I = |I| = \dfrac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$

Power delivered to the circuit is

$$P = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

In the above equation, if R_L is fixed, the value of P is maximum when

$$X_s = -X_L$$

Then the power $P = \dfrac{V_s^2 R_L}{(R_s + R_L)^2}$

Let us assume that R_L is variable. In this case, the maximum power is transferred when the load resistance is equal to the source resistance (already discussed in Chapter 3). If $R_L = R_s$ and $X_L = -X_s$, then $Z_L = Z^*_s$. This means that *the maximum power transfer occurs when the load impedance is equal to the complex conjugate of source impedance Z_s.*

EXAMPLE 7.8

For the circuit shown in Fig. 7.32, find the value of load impedance for which the source delivers maximum power. Calculate the value of the maximum power.

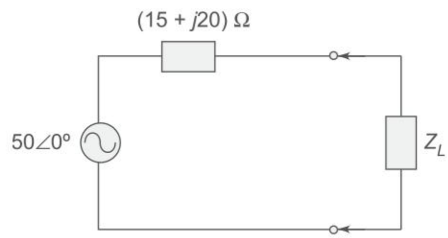


Fig. 7.32

Solution In the circuit shown in Fig. 7.32, the maximum power transfer occurs when the load impedance is complex conjugate of the source impedance.

$$\therefore \quad Z_L = Z^*_s = 15 - j20$$

When $Z_L = 15 - j20$, the current passing through circuit is

$$I = \frac{V_s}{R_s + R_L} = \frac{50\angle 0^\circ}{15 + j20 + 15 - j20} = \frac{50\angle 0^\circ}{30} = 1.66\angle 0^\circ$$

The maximum power delivered to the load is

$$P = I^2 R_L = (1.66)^2 \times 15 = 41.33 \text{ W}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆★★**7-6.1** For the circuit shown in Fig. Q.1, find the value of Z that will receive the maximum power. Also determine this power.

☆☆★**7-6.2** Determine the power output of the voltage source by loop analysis for the network shown in Fig. Q.2. Also determine the power extended in the resistors.

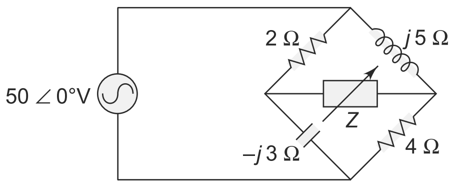


Fig. Q.1

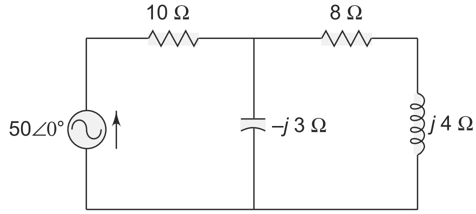


Fig. Q.2

★★★**7-6.3** Obtain the Thèvenin's equivalent circuit at terminals AB shown in Fig. Q.3.

☆★★**7-6.4** In the circuit shown in Fig. Q.4, the resistance R_g is variable between 2 and 55 ohms. What value of R_g results in maximum power transfer across terminals AB?

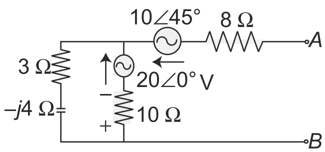


Fig. Q.3

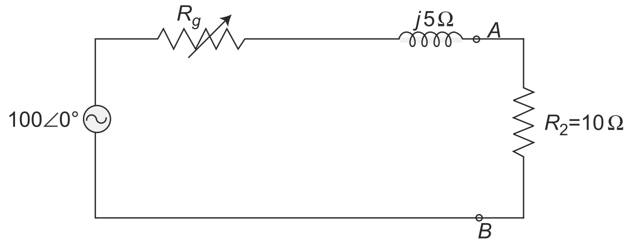


Fig. Q.4

☆★★**7-6.5** For the power transmission system, shown in Fig. Q.5, $V_S = 240\angle 0°$. Using PSpice, find the average power absorbed by the load.

☆★★**7-6.6** For the circuit shown in Fig. Q.6, determine the value of Z that will result in the maximum power being delivered to Z. Calculate the value of maximum power using PSpice.

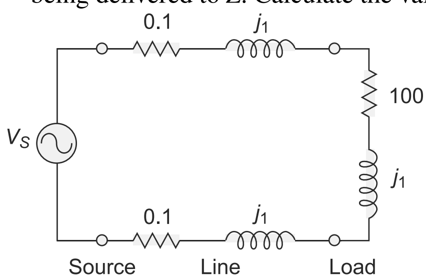


Fig. Q.5

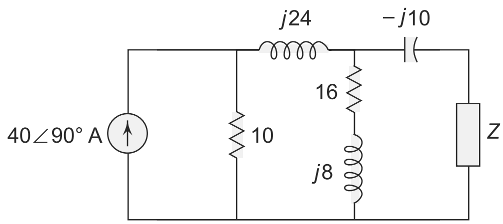


Fig. Q.6

☆★★**7-6.7** For the circuit shown in Fig. Q.7, the load resistance R_L is adjusted until it absorbs the maximum average power. Calculate the value of R_L and the maximum average power. Use PSpice.

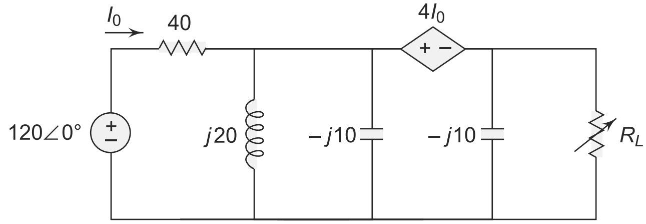


Fig. Q.7

Frequently Asked Questions linked to LO 6

☆☆★**7-6.1** In the circuit shown below, find the value of the load impedance Z_L for maximum power transfer to the load. **[AU Nov./Dec. 2012]**

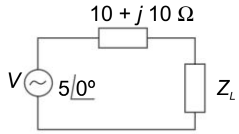


Fig. Q.1

Additional Solved Problems

PROBLEM 7.1

Determine (a) the equivalent voltage generator, and (b) the equivalent current generator which may be used to represent the given network in Fig. 7.33 at the terminals AB.

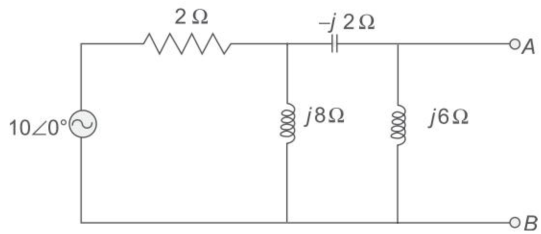


Fig. 7.33

Solution The impedance seen into the terminals when the voltage source is short- circuited

$$Z_{AB} = \{[(2 \,||\, j8) - j2] \,||\, j6\} = 2.99\underline{|-16.32^\circ}$$

$$Z_{AB} = (2.87 - j0.84)\,\Omega$$

Considering the node voltage V across $j8\,\Omega$ and applying nodal analysis at the node, we have

$$\frac{V - 10\underline{|0^\circ}}{2} + \frac{V}{j8} + \frac{V}{j4} = 0$$

$$V\left[0.5 + \frac{1}{8\underline{|90^\circ}} + \frac{1}{4\underline{|90^\circ}}\right] = 5\underline{|0^\circ}$$

$$\therefore \quad V = 8\underline{|36.87^\circ}\text{ V}$$

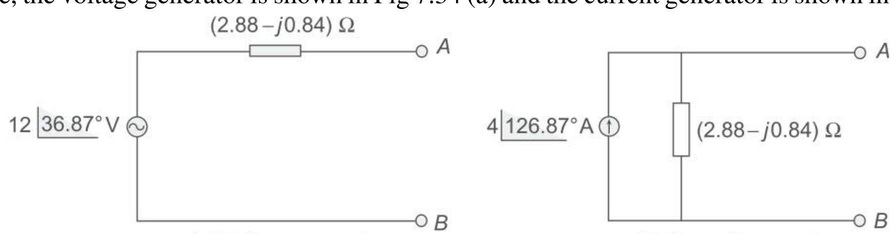
The voltage across AB is

$$V_{AB} = V \cdot \frac{j6}{j6 - j2} = 8\underline{|36.87^\circ} \times \frac{j6}{j4} = 12\underline{|36.87^\circ}\text{ V}$$

The current in the short-circuited terminals AB

$$I_A = \frac{V}{-j2} = \frac{8\underline{|36.87^\circ}}{2\underline{|-90^\circ}} = 4\underline{|126.87^\circ}\text{ A}$$

Therefore, the voltage generator is shown in Fig 7.34 (a) and the current generator is shown in Fig 7.34 (b).



(a) Voltage generator (b) Current generator

Fig. 7.34

PROBLEM 7.2

Determine the voltage V_{ab} and V_{bc} in the network shown in Fig 7.35 by loop analysis, where source voltage $e(t) = 100 \cos (314t + 45°)$.

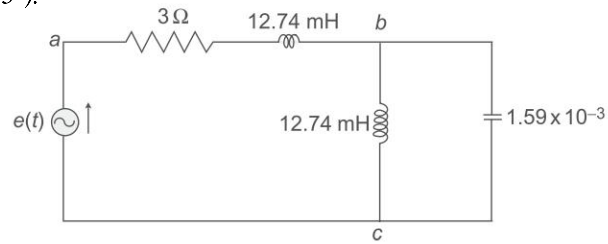


Fig. 7.35

Solution The circuit is redrawn as shown in Fig. 7.36.

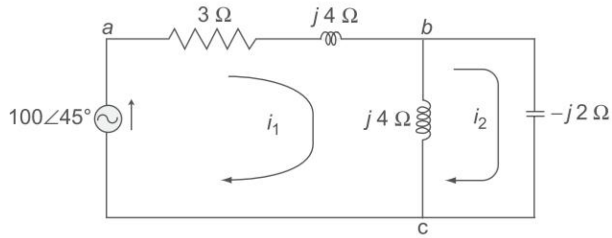


Fig. 7.36

From Fig. 7.36 shown, the loop-current equations are given by

$$(3+j8)i_1-(j4)i_2=100\angle 45° \tag{7.46}$$

$$-j4i_1+(j2)i_2=0 \tag{7.47}$$

From the above equations, we have

$$\begin{bmatrix} 3+j8 & -j4 \\ -j4 & j2 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}=\begin{bmatrix} 100\angle 45° \\ 0 \end{bmatrix}$$

The loop current $i_1=\dfrac{\Delta_1}{\Delta}$

$$\text{Where } \Delta=\begin{vmatrix} 3+j8 & -j4 \\ -j4 & j2 \end{vmatrix}=j6$$

$$\Delta_1=\begin{vmatrix} 100\angle 45° & -j4 \\ 0 & j2 \end{vmatrix}=200\angle 135°$$

$$\therefore \quad i_1=\frac{\Delta_1}{\Delta}=33.33\angle 45° \text{ A}$$

The loop current $i_2=\dfrac{\Delta_2}{\Delta}$

$$\text{where } \Delta_2=\begin{vmatrix} 3+j8 & 100\angle 45° \\ -j4 & 0 \end{vmatrix}=400\angle 135°$$

$$\therefore \quad i_2=\frac{\Delta_2}{\Delta}=66.67\angle 45° \text{ A}$$

The voltage across *ab* is $V_{ab} = i_1(3 + j4) = 166.65\angle 98.13^\circ$ V

The voltage across *bc* is $V_{bc} = (i_1 - i_2) j_4 = 133.4\angle -45^\circ$ V

PROBLEM 7.3

In the circuit shown in Fig 7.37, determine the power in the impedance (2 + j5) Ω connected between A and B using Norton's theorem.

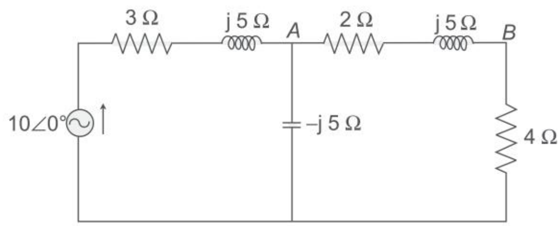


Fig. 7.37

Solution To find the current in the short-circuited terminals *AB*, the circuit is redrawn as shown in Fig. 7.38.

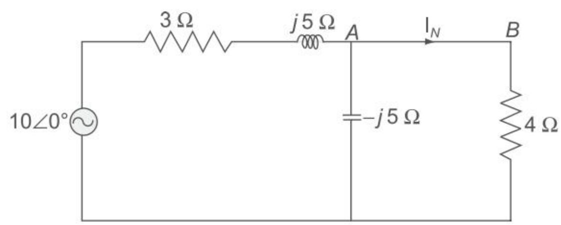


Fig. 7.38

The total impedance in the circuit shown in Fig. 7.38,

$$Z_T = [(4 \parallel -j5) + (3 + j5)]\ \Omega = 6.24\angle 29.26^\circ$$

The total current $I_T = \dfrac{10\angle 0^\circ}{6.24\angle 29.26^\circ} = 1.6\angle -29.26^\circ$ A

The current $I_N = I_T \times \dfrac{-j5}{4 - j5} = 1.24\angle -67.92^\circ$ A

Open-circuit impedance seen into the terminals *AB,*

$$Z_N = 4 + \frac{(3 + j5)(-j5)}{3} = (12.33 - j5)\ \Omega$$

The Norton's equivalent circuit is shown in Fig. 7.39.

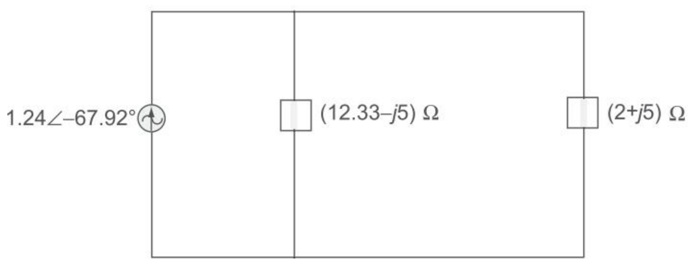


Fig. 7.39

$\therefore$ the current in $(2 + j5)\Omega$ is

$$I_{(2+j5)\Omega} = I_N \times \frac{12.33 - j5}{12.33 - j5 + 2 + j5}$$

$$= 1.16\angle -90.02^\circ \text{ A}$$

The power in the $(2 + j5)\ \Omega$ impedance,

$$P_{2+j5} = (1.16)^2 \times 2 = 2.69 \text{ watts}$$

PROBLEM 7.4

For the circuit shown in Fig. 7.40, find the current in each resistor using the superposition theorem.

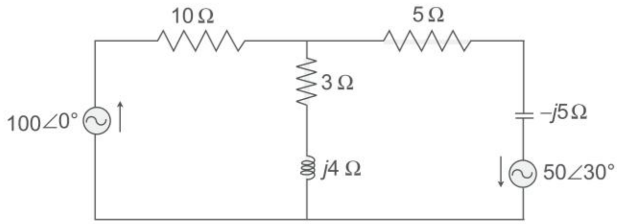


Fig. 7.40

Solution Consider the currents I_1, I_2, and I_3 are flowing in the branches $10\,\Omega$, $(3 + j4)\,\Omega$ and $(5 - j5)\,\Omega$ respectively.

The branch currents due to the $100\angle 0^\circ$ voltage source can be determined by the circuit shown in Fig. 7.41, where the $50\angle 30^\circ$ source is short-circuited.

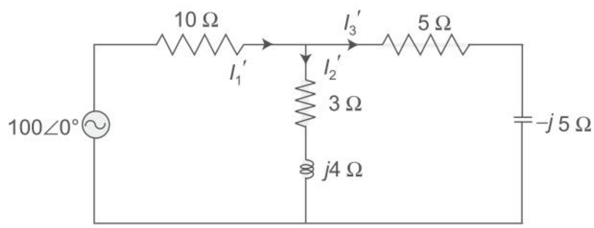


Fig. 7.41

Total impedance $Z_T = [\{(5 - j5) \parallel (3 + j4)\} + 10] = 14.28\angle 4.64^\circ\ \Omega$

$$I_1' = \frac{100\angle 0^\circ}{14.28\angle 4.64^\circ} = (6.97 - j0.57) \text{ A}$$

$$I_2' = I_1^1 \times \frac{(5 - j5)\Omega}{5 - j5 + 3 + j4} = (4.53 - j4.15) \text{ A}$$

$$I_3' = I_1^1 \times \frac{(3 + j4)}{8 - j1} = (2.45 + j3.5) \text{ A}$$

The branch current due to $50\angle 30^\circ$ V source can be determined by the circuit shown in Fig. 7.42.

Total impedance in the circuit shown in Fig 7.42.

$$Z_T = [(10 \| (3 + j4)) + 5 - j5] = 8.46\underline{|-19.6^\circ}\ \Omega$$

$$\text{The current } I_3'' = \frac{50\underline{|30^\circ}}{8.46\underline{|-19.6^\circ}} = (3.83 + j4.5)\ \text{A}$$

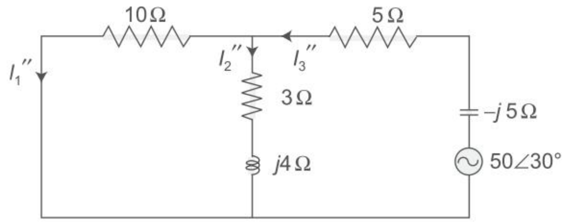


Fig. 7.42

$$I_2'' = I_3'' \times \frac{10}{10 + 3 + j4} = (3.66 + j2.33)\ \text{A}$$

$$I_1'' = I_3'' \times \frac{3 + j4}{10 + 3 + j4} = (0.16 + j2.16)\ \text{A}$$

The current in the 10 Ω resistor

$$I_1 = I_1' - I_1'' = 7.34\underline{|-21.84^\circ}\ \text{A}$$

The current in the (3 + j4) Ω branch

$$I_2 = I_2' + I_2'' = 8.39\underline{|-12.53^\circ}\ \text{A}$$

The current in the (5−j5) Ω branch

$$I_3 = I_3'' - I_3' = 1.7\underline{|35.9^\circ}\ \text{A}$$

PROBLEM 7.5

Determine the maximum power delivered to the load in the circuit shown in Fig. 7.43.

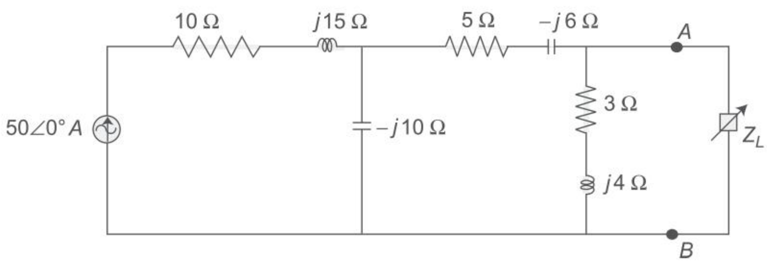


Fig. 7.43

Solution The circuit is replaced by Thèvenin's equivalent circuit in series with Z_L as shown in Fig. 7.44

where $V_{AB} = I_{(3+j4)\Omega} \times (3 + j4)$ volts

$$I_{(3+j4)\Omega} = \frac{50\underline{|0^\circ} \times (-j10)}{5 - j6 + 3 + j4 - j10} = 34.67\underline{|-33.7^\circ}\ \text{A}$$

Voltage across AB is $V_{AB} = 34.67\underline{|-33.7^\circ} \times 5\underline{|53.13^\circ}$

$$= 173.35\underline{|19.43^\circ}\ \text{V}$$

Impedance across terminals AB is

$$Z_{AB} = \{[(10 + j15) \| (-j10)] + (5 - j6)\} \| (3 + j4$$

$$Z_{AB} = 5.27\underline{|41.16^\circ} = 4 + j3.5$$

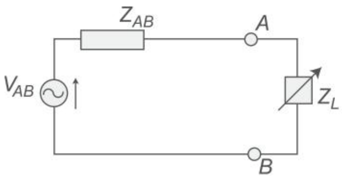


Fig. 7.44

To get the maximum power delivered to the load impedance, the load impedance must be equal to complex conjugate of source impedance. Therefore, the total impedance in the circuit shown in Fig. 7.44 is 8 Ω. The current in the circuit is

$$I_2 = \frac{V_{AB}}{8} = \frac{173.35}{8} = 21.66\text{ A}$$

The maximum power transferred to the load is

$$P = I_L^2 R_L = (21.66)^2 \times 4 = 1874.9\text{ watts}$$

PROBLEM 7.6

For the circuit shown in Fig. 7.45, determine the power output of the source and the power in each resistor of the circuit.

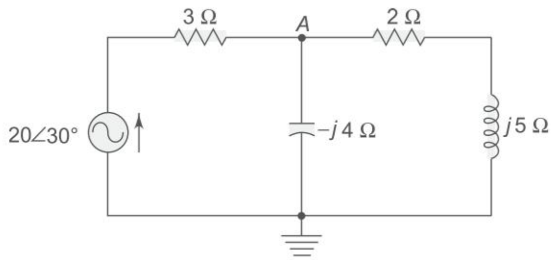


Fig. 7.45

Solution Assume that the voltage at the node A is V_A.

By applying nodal analysis, we have

$$\frac{V_A - 20\angle 30^\circ}{3} + \frac{V_A}{-j4} + \frac{V_A}{2+j5} = 0$$

$$V_A\left[\frac{1}{3} + \frac{1}{2+j5} - \frac{1}{j4}\right] = \frac{20\angle 30^\circ}{3}$$

$$V_A\,[0.33 + 0.068 + j0.078] = 6.67\angle 30^\circ$$

$$\therefore \quad V_A = \frac{6.67\angle 30^\circ}{0.41\angle 11.09^\circ} = 16.27\angle 18.91^\circ$$

Current in the 2 Ω resistor

$$I_2 = \frac{V_A}{2+j5} = \frac{16.27\angle 18.91^\circ}{5.38\angle 68.19^\circ}$$

$$\therefore \quad I_2 = 3.02\angle -49.28^\circ$$

Power dissipated in the 2 Ω resistor

$$P_2 = I_2^2 R = (3.02)^2 \times 2 = 18.24\text{ W}$$

Current in the 3 Ω resistor

$$I_3 = \frac{-20\angle 30^\circ + 16.27\angle 18.91^\circ}{3}$$

$$= -6.67\angle 30^\circ + 5.42\angle 18.91^\circ$$

$$= -5.78 - j3.34 + 5.13 + j1.76 = -0.65 - j1.58$$

$$I_3 = 1.71\angle -112^\circ$$

Power dissipated in the 3 Ω resistor

$$= (1.71)^2 \times 3 = 8.77 \text{ W}$$

Total power delivered by the source

$$= VI \cos \phi = 20 \times 1.71 \cos 142° = 26.95 \text{ W}$$

PROBLEM 7.7

For the circuit shown in Fig. 7.46, determine the voltage V_{AB} using the superposition theorem.

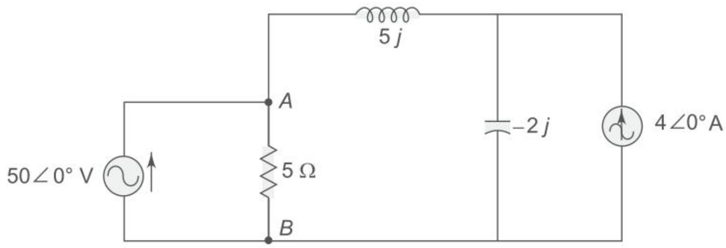


Fig. 7.46

Solution Let the source 50 ∠0° V act on the circuit and set the source 4 ∠0° *A* equal to zero. If the current source is zero, it becomes open-circuited. Then the voltage across *AB* is $V_{AB} = 50 \angle 0°$.

Now set the voltage source 50 ∠0° V at zero, and it is short-circuited, or the voltage drop across *AB* is zero. The total voltage is the sum of the two voltages.

$$\therefore \quad V_T = 50 \angle 0°$$

PROBLEM 7.8

For the circuit shown in Fig. 7.47, determine the current in (2 + j3) Ω by using the superposition theorem.

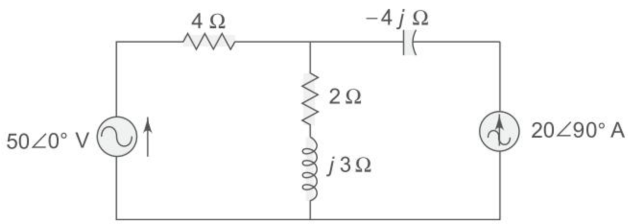


Fig. 7.47

Solution The current in (2 + *j*3) Ω, when the voltage source 50∠0° is acting alone is

$$I_1 = \frac{50\angle 0°}{(6+j3)} = \frac{50\angle 0°}{6.7\angle 26.56°}$$

$$\therefore \quad I_1 = 7.46 \angle -26.56° \text{ A}$$

Current in (2 + *j*3) Ω, when the current source 20∠90° A is acting alone is

$$I_2 = 20\angle 90° \times \frac{4}{(6+j3)}$$

$$= \frac{80\angle 90°}{6.7\angle 26.56°} = 11.94\angle 63.44° \text{ A}$$

Total current in $(2 + j3)\ \Omega$ due to both sources is

$$I = I_1 + I_2$$
$$= 7.46\ \angle -26.56° + 11.94\ \angle 63.44°$$
$$= 6.67 - j3.33 + 5.34 + j10.68$$
$$= 12.01 + j7.35 = 14.08\ \angle 31.46°$$

Total current in $(2 + j3)\ \Omega$ is $I = 14.08\ \angle 31.46°$

PROBLEM 7.9

For the circuit shown in Fig. 7.48, determine the load current by applying Thèvenin's theorem.

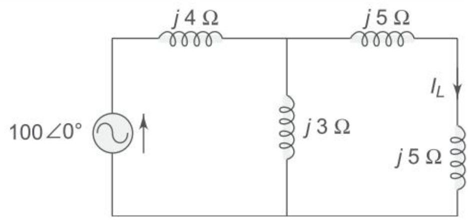


Fig. 7.48

Solution Let us find the Thèvenin equivalent circuit for the circuit shown in Fig. 7.49 (a).

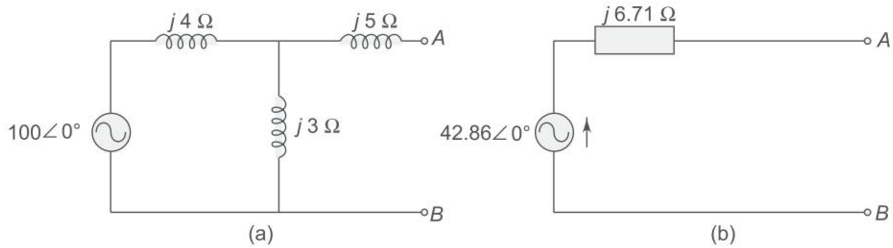


Fig. 7.49

Voltage across AB is the voltage across $(j3)\ \Omega$

$$\therefore \quad V_{AB} = 100\ \angle 0° \times \frac{(j3)}{(j3)+(j4)}$$

$$= 100\ \angle 0° \frac{(j3)}{j7} = 42.86 \angle 0°$$

Impedance seen from terminals AB

$$Z_{AB} = (j5) + \frac{(j4)(j3)}{j7}$$

$$= j5 + j1.71 = j6.71\ \Omega$$

Thèvenin's equivalent circuit is shown in Fig. 7.49 (b).

If we connect a load to Fig. 7.49 (b), the current passing through the $(j5)\ \Omega$ impedance is

$$I_L = \frac{42.86\angle 0°}{(j6.71 + j5)} = \frac{42.86\angle 0°}{11.71\angle 90°} = 3.66\angle -90°$$

PROBLEM 7.10

For the circuit shown in Fig. 7.50, determine the Thèvenin's equivalent circuit.

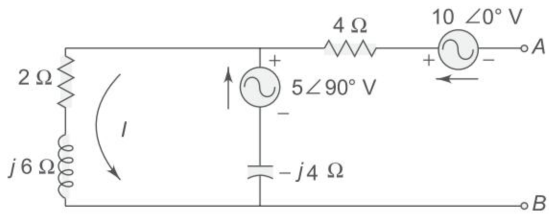


Fig. 7.50

Solution Voltage across $(-j4)\ \Omega$ is

$$V_{-j4} = \frac{5\angle 90^\circ}{(2+j2)}(-j4)$$

$$= \frac{20\angle 0^\circ}{2.83\angle 45^\circ} = 7.07\angle -45^\circ$$

Voltage across *AB* is $V_{AB} = -V_{10} + V_5 - V_{-j4}$

$$= -10\ \angle 0^\circ + 5\ \angle 90^\circ - 7.07\ \angle -45^\circ$$

$$= j5 - 10 - 4.99 + j4.99$$

$$= -14.99 + j9.99$$

$$V_{AB} = 18\ \angle 146.31^\circ$$

The impedance seen from terminals *AB*, when all voltage sources are short- circuited is

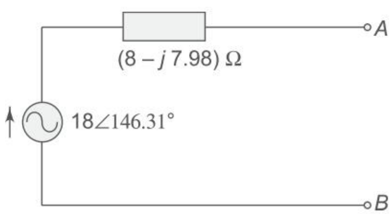


Fig. 7.51

$$Z_{AB} = 4 + \frac{(2+j6)\,(-j4)}{2+j2}$$

$$= 4 + \frac{6.32\angle 71.56^\circ \times 4\angle -90^\circ}{2.83\angle 45^\circ}$$

$$= 4 + 8.93\ \angle -63.44^\circ$$

$$= 4 + 4 - j\,7.98 = (8 - j7.98)\ \Omega$$

Thèvenin's equivalent circuit is shown in Fig. 7.51.

PROBLEM 7.11

For the circuit shown in Fig. 7.52, determine the load current I_L by using Norton's theorem.

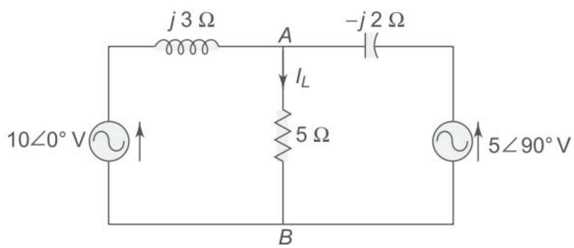


Fig. 7.52

Solution Norton's impedance seen from terminals *AB* is

$$Z_{AB} = \frac{(j3)\,(-j2)}{(j3)-(j2)} = \frac{6}{j1}$$

$$\therefore\quad Z_{AB} = 6\ \angle -90^\circ$$

Current passing through AB, when it is shorted

$$I_N = \frac{10\angle 0^\circ}{3\angle 90^\circ} + \frac{5\angle 90^\circ}{2\angle -90^\circ}$$

$$\therefore \quad I_N = 3.33\ \angle -90^\circ + 2.5\ \angle 180^\circ$$

$$= -j3.33 - 2.5$$

$$I_N = 4.16\ \angle -126.8^\circ$$

Norton's equivalent circuit is shown in Fig. 7.53.

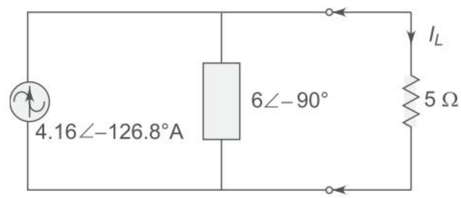


Fig. 7.53

$$\text{Load current is } I_L = I_N \times \frac{6\angle -90^\circ}{5 + 6\angle -90^\circ}$$

$$= 4.16\angle -126.8^\circ \times \frac{6\angle -90^\circ}{5 - j6}$$

$$= \frac{4.16 \times 6\angle -216.8^\circ}{7.81\angle -50.19^\circ}$$

$$= 3.19\ \angle -166.61^\circ$$

PROBLEM 7.12

For the circuit shown in Fig. 7.54, determine Norton's equivalent circuit.

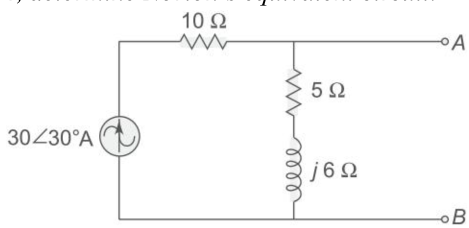


Fig. 7.54

Solution The impedance seen from the terminals when the source is reduced to zero,

$Z_{AB} = (5 + j6)\ \Omega$

Current passing through the short circuited terminals, A and B, is

$I_N = 30\ \angle 30^\circ$ A

Norton's equivalent circuit is shown in Fig. 7.55.

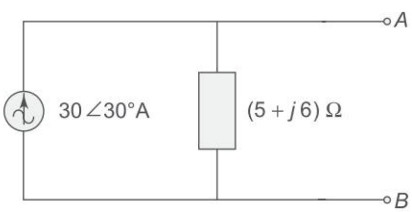


Fig. 7.55

PROBLEM 7.13

Convert the active network shown in Fig. 7.56 by a single voltage source in series with impedance.

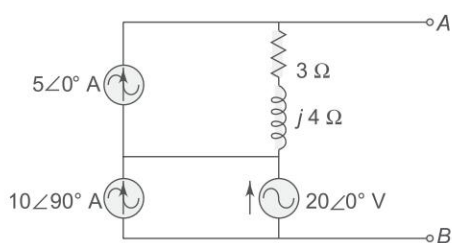


Fig. 7.56

Solution Using the superposition theorem, we can find Thèvenin's equivalent circuit. The voltage across AB, with 20 $\angle 0^\circ$ V source acting alone, is V'_{AB}, and can be calculated from Fig. 7.57 (a).

Since no current is passing through the $(3 + j4)\ \Omega$ impedance, the voltage

$$V'_{AB} = 20\ \angle 0^\circ$$

The voltage across *AB*, with $5\ \angle 0^\circ$ A source acting alone, is V'_{AB}, and can be calculated from Fig. 7.57 (b).

$$V''_{AB} = 5\ \angle 0^\circ\ (3 + j4) = 5\ \angle 0^\circ \times 5\ \angle 53.13^\circ = 25\ \angle 53.13^\circ \text{ V}$$

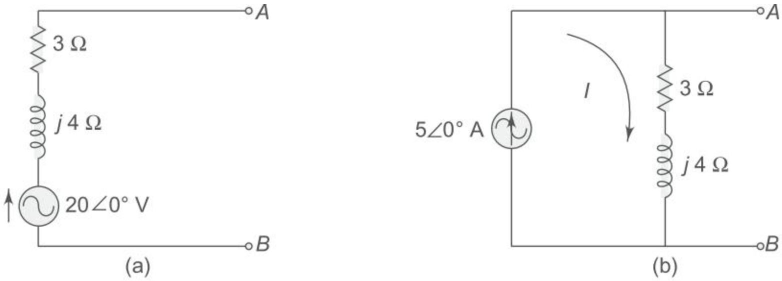


Fig. 7.57

The voltage across *AB*, with the $10\ \angle 90^\circ$ A source acting alone, is V'''_{AB}, and can be calculated from Fig. 7.57 (c).

$$V'''_{AB} = 0$$

According to the superposition theorem, the voltage across *AB* due to all sources is

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB}$$

$$V_{AB} = 20\ \angle 0^\circ + 25\ \angle 53.13^\circ = 20 + 15 + j19.99$$

$$= (35 + j19.99) \text{ V} = 40.3\ \angle 29.73^\circ \text{ V}$$

The impedance seen from terminals *AB*

$$Z_{\text{Th}} = Z_{AB} = (3 + j4)\ \Omega$$

$\therefore$ the required Thèvenin circuit is shown in Fig. 7.57 (d).

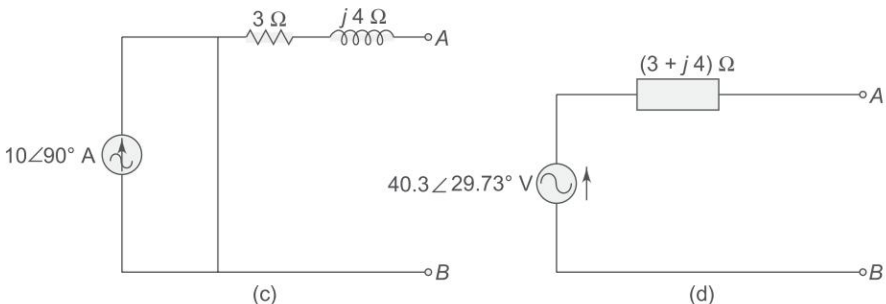


Fig. 7.57

PROBLEM 7.14

For the circuit shown in Fig. 7.58, find the value of Z that will receive maximum power; also determine this power.

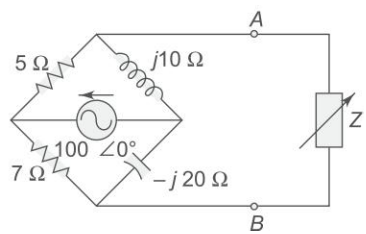


Fig. 7.58

Solution The equivalent impedance at terminals AB with the source set equal to zero is

$$Z_{AB} = \frac{5(j10)}{5+j10} + \frac{7(-j20)}{(7-j20)}$$

$$= \frac{50\angle 90^\circ}{11.18\angle 63.43^\circ} + \frac{140\angle -90^\circ}{21.19\angle -70.7^\circ}$$

$$= 4.47\ \angle 26.57^\circ + 6.6\ \angle -19.3^\circ$$

$$= 3.99 + j1.99 + 6.23 - j2.18$$

$$= 10.22 - j0.19$$

The Thèvenin equivalent circuit is shown in Fig. 7.59 (a).

The circuit in Fig. 7.59 (a) is redrawn as shown in Fig. 7.59 (b).

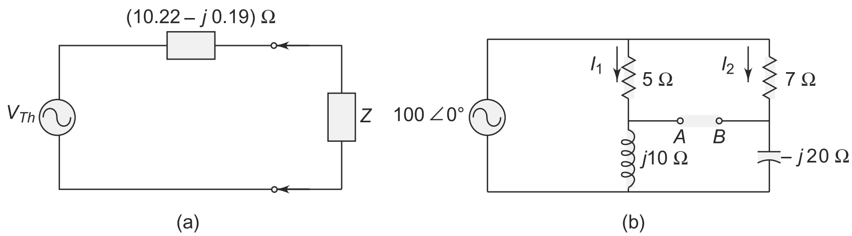


Fig. 7.59

Current $I_1 = \dfrac{100\angle 0^\circ}{5+j10}$

$$= \frac{100\angle 0^\circ}{11.18\angle 63.43^\circ} = 8.94\angle -63.43^\circ \text{A}$$

Current $I_2 = \dfrac{100\angle 0^\circ}{7-j20} = \dfrac{100\angle 0^\circ}{21.19\angle -70.7^\circ} = 4.72\angle 70.7^\circ$

Voltage at A, $V_A = 8.94\ \angle -63.43^\circ \times j10 = 89.4\ \angle 26.57^\circ$

Voltage at B, $V_B = 4.72\ \angle 70.7^\circ \times -j20 = 94.4\ \angle -19.3^\circ$

Voltage across terminals AB

$$V_{AB} = V_A - V_B$$

$$= 89.4\ \angle 26.57^\circ - 94.4\ \angle -19.3^\circ$$

$$= 79.96 + j39.98 - 89.09 + j31.2$$

$$= -9.13 + j71.18$$

$$V_{\text{Th}} = V_{AB} = 71.76\ \angle 97.3^\circ \text{ V}$$

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ load $Z = 10.22 + j0.19$

Current passing through the load Z

$$I = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z} = \frac{71.76\angle 97.3^\circ}{20.44} = 3.51\angle 97.3^\circ$$

Maximum power delivered to the load is

$$= (3.51)^2 \times 10.22 = 125.91 \text{ W}$$

PROBLEM 7.15

For the circuit shown in Fig. 7.60, the resistance R_s is variable from 2 Ω to 50 Ω. What value of R_s results in maximum power transfer across the terminals AB?

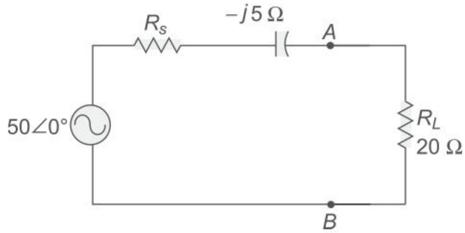


Fig. 7.60

Solution In the circuit shown, the resistance R_L is fixed. Here, the maximum power transfer theorem does not apply. Maximum current flows in the circuit when R_s is minimum. For the maximum current,

$$R_s = 2$$

But $Z_T = R_s - j5 + R_L = 2 - j5 + 20 = (22 - j5) = 22.56\ \angle -12.8^\circ$

$$\therefore \quad I = \frac{V_s}{Z_T} = -\frac{50\angle 0^\circ}{22.56\angle -12.8^\circ} = 2.22\angle 12.8^\circ$$

Maximum power $P = I^2R = (2.22)^2 \times 20 = 98.6$ W

PROBLEM 7.16

Determine the voltage V which results in a zero current through the $2 + j3\ \Omega$ impedance in the circuit shown in Fig. 7.61.

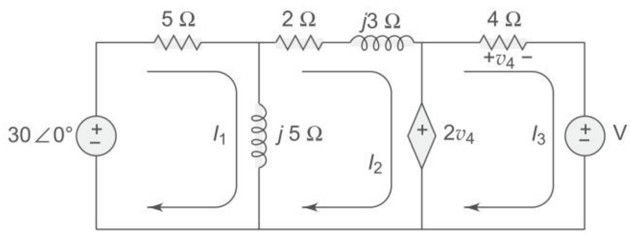


Fig. 7.61

Solution Choosing mesh currents as shown in Fig. 7.61, the three loop equations are

$$(5 + j5)\, I_1 - j5\, I_2 = 30\ \angle 0^\circ$$

$$-j5\, I_1 + (2 + j8)\, I_2 = -2V_4$$

$$-2V_4 + V_4 + V = 0$$

$$V_4 = V$$

Since the current in $(2 + j3)\ \Omega$ is zero,

$$I_2 = \frac{\Delta_2}{\Delta} = 0$$

where $\Delta_2 = \begin{vmatrix} 5+j5 & 30\angle 0° \\ -j5 & -2V \end{vmatrix} = 0$

$$(5 + j5)(-2V) + (j5)\,30\angle 0° = 0$$

$$V = \frac{30\angle 0°(j5)}{2(5+j5)} = \frac{150\angle 90°}{14.14\angle 45°}$$

$$V = 10.608\angle 45° \text{ volts}$$

PROBLEM 7.17

Find the value of the voltage V which results in $V_0 = 5\angle 0°$ V in the circuit shown in Fig. 7.62.

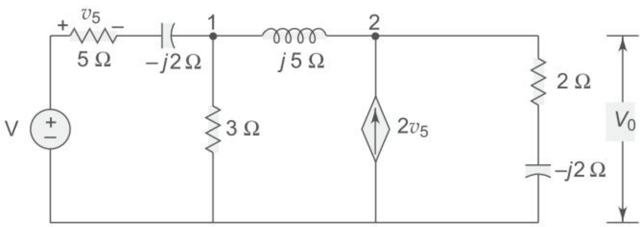


Fig. 7.62

Solution Assuming all currents are leaving the nodes, the nodal equations are

$$V_1\left[\frac{1}{5-j2}+\frac{1}{3}+\frac{1}{j5}\right]-V_2\left[\frac{1}{j5}\right]=\frac{V}{5-j2}$$

$$-V_1\left[\frac{1}{j5}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=2V_5$$

where $V_5 = 5\left(\dfrac{V_1-V}{5-j2}\right)$

The second equation becomes

$$V_1\left[\frac{-1}{j5}-\frac{10}{5-j2}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=\frac{-10\,V}{5-j2}$$

$$V_0 = V_2 = \frac{\Delta_2}{\Delta} = 5\angle 0°$$

$$\frac{\begin{vmatrix} \dfrac{1}{5-j2}+\dfrac{1}{3}+\dfrac{1}{j5} & \dfrac{V}{5-j2} \\[2ex] \dfrac{-1}{j5}-\dfrac{10}{5-j2} & \dfrac{-10\,V}{5-j2} \end{vmatrix}}{\begin{vmatrix} \dfrac{1}{5-j2}+\dfrac{1}{3}+\dfrac{1}{j5} & \dfrac{-1}{j5} \\[2ex] \dfrac{-1}{j5}-\dfrac{10}{5-j2} & \dfrac{1}{j5}+\dfrac{1}{2-j2} \end{vmatrix}} = 5 < 0°$$

The source voltage $V = 2.428\angle -88.74°$ volts.

PROBLEM 7.18

For the circuit shown in Fig. 7.63, find the current in the $j2\ \Omega$ inductance by using Thèvenin's theorem.

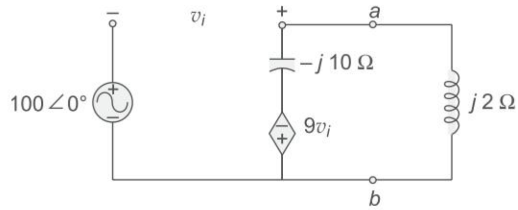


Fig. 7.63

Solution From the circuit shown in Fig. 7.63, the open-circuit voltage at terminals a and b is

$$V_{oc} = -9\ V_i$$

where $V_i = -9V_i - 100\ \angle 0°$

$$10V_i = -100\ \angle 0°$$

$$V_i = -10\ \angle 0°$$

Thèvenin's voltage $V_{oc} = 90\ \angle 0°$

From the circuit, short-circuit current is determined by shorting terminals a and b. Applying Kirchhoff's voltage law, we have

$$9V_i - j10\ i_{sc} = 0$$

$$i_{sc} = 9\angle 90°$$

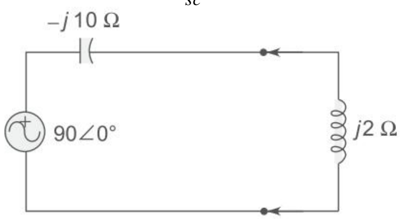


Fig. 7.64

$$\therefore\ Z_{\text{Th}} = \frac{V_{oc}}{I_{sc}} = \frac{90\angle 0°}{9\angle 90°} = 10\angle -90°$$

$$Z_{Th} = -\ j\ 10\ \Omega$$

The Thèvenin's equivalent circuit is shown in Fig. 7.64.

The current in the $j2\ \Omega$ inductor is $\dfrac{90\angle 0°}{j8}$

$$= 11.25\ \angle 90°$$

PROBLEM 7.19

For the circuit shown in Fig. 7.65, find the value of Z that will receive maximum power; also determine this power.

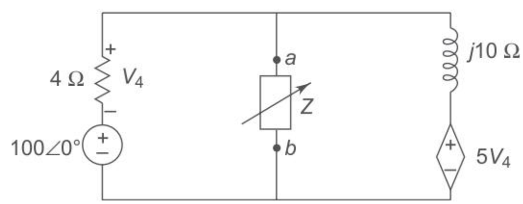


Fig. 7.65

Solution The equivalent impedance can be obtained by finding V_{oc} and i_{sc} at terminals $a\ b$. Assume that current i is passing in the circuit.

$$i = \frac{100\angle 0° - 5V_4}{4 + j10}$$

$$= \frac{100\angle 0°}{4 + j10} - \frac{5 \times 4i}{4 + j10}$$

$$i = 3.85\ \angle -22.62°$$

$$V_{oc} = 100\ \angle 0° - 4 \times 3.85\ \angle -22.62°$$

$$= 86\ \angle 3.94°$$

$$i_{sc} = 25 + j50 = 56\ \angle 63.44°$$

Thèvenin's equivalent impedance

$$Z_{\text{Th}} = \frac{V_{oc}}{i_{sc}} = 1.54\angle -59.5°$$

$$= 0.78 - j1.33$$

The circuit is drawn as shown in Fig. 7.66.

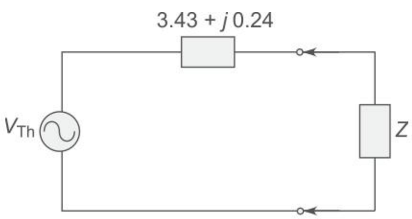


Fig. 7.66

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ load $Z = 0.78 + j\,1.33$

Current passing through the load Z

$$I = \frac{V_{Th}}{Z_{Th} \quad Z} \frac{86\angle 3.94^\circ}{1.56} = 55.13\angle 3.94^\circ$$

Maximum power delivered to the load is $(55.13)^2 \times (0.78) = 2370.7$ W.

PSpice Problems

PROBLEM 7.1

For the circuit shown in Fig. 7.67, determine the power output of the source and the power in each resistor.

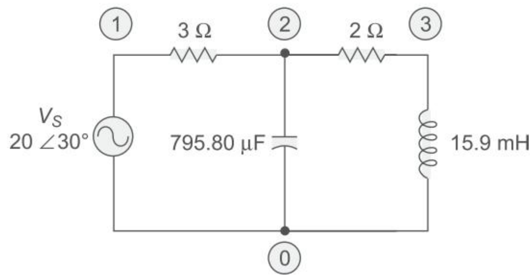


Fig. 7.67

```
* AC ANALYSIS
VS   1   0   AC 20 30
R1   1   2   3
R2   2   3   2
C1   2   0   795.77 U
L1   3   0   15.9 M
.AC LIN 1 50 100
.PRINT AC IM(VS) IP(VS) IM(R1) IP(R1)
+ VM(R1) VP(R1) IM(R2) IP(R2) VM(R2)
+ VP(R2)
.END
```

OUTPUT

**** AC ANALYSIS TEMPERATURE = 27.000 DEG C

****

FREQ	IM(VS)	IP(VS)	IM(R1)	IP(R1)	VM(R1)
5.000E + 01	1.688E + 00	− 1.126E + 02	1.688E + 00	6.738E + 01	5.065E + 00

FREQ	VP(R1)	IM(R2)	IP(R2)	VM(R2)	VP(R2)
5.000E + 01	6.738E + 01	3.023E + 00	−4.908E + 01	6.047E + 00	− 4.908E + 01

Result Power dissipated in R1 = Re{$5.065\angle 67.38 \times 1.688\angle -67.35$}

= Re{V(R1) × I(R1)*} = 8.55 W

Power dissipated in R2 = Re{$6.049\angle -49.08 \times 3.023\angle 49.08$} = Re{V(R2) × I(R2)*}.
= 18.28 W

Power output of source = Re{$20\angle 30^\circ \times 1.688\angle 112.6^\circ$} = 26.82 W = Re {Vs × Is*}.

PROBLEM 7.2

Using PSpice, for the circuit shown in Fig. 7.68, determine Thèvenin's equivalent circuit.

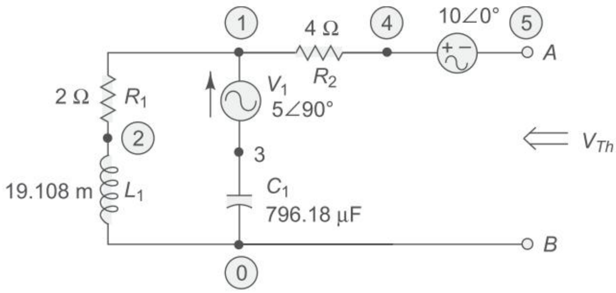


Fig. 7.68

```
* NETLIST TO FIND VTH
V1   1   3   AC 5 90
R1   1   2   2
L1   2   0   19.108 MH
C1   3   0   796.18 UF
R2   1   4   4
V2   4   5   AC 10 0
.AC LIN 1 50 50
.PRINT AC VM(5,0) VP(5,0) IM(V2) IP(V2)
.END
* NETLIST TO FIND ZTH
```

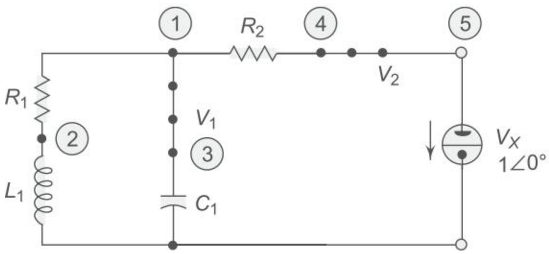


Fig. 7.69

```
V1   1   3   AC 0 0
R1   1   2   2
L1   2   0   19.108 MH
C1   3   0   796.18 UF
R2   1   4   4
V2   4   5   AC 0 0
VX   0   5   AC 1 0
.AC LIN 1 50 50
.PRINT AC IM(VX) IP(VX)
.END
```

```
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
******************************************************************
FREQ          VM(5,0)       VP(5,0)       IM(V2)        IP(V2)
5.000E + 01   1.801E + 01   1.463E + 02   1.000E − 30   0.000E + 00
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
******************************************************************
FREQ          IM(VX)          IP(VX)
5.000E + 01   8.850E − 02     −1.350E + 02
```

Result

Vth = V(5) = 180.1∠146.3°

Zth = − 1/I(VX) = 1/0.0850∠135° = 8.318∠– 45°

Answers to Practice Problems

7-1.1 3.39 $\angle -97.3°$

7-1.2 $I_{2+j3} = 1.74\underline{|40.1°}$ A

7-1.4 I = 0.25 A

7-3.1 I = OA

7-4.1 4.37 A

7-4.3 $(-0.18 - j0.6)V_1$ volts in series with $(100 - j30)\ \Omega$

7-4.4 0.894 $\angle -63.4°$ in series with $(0.4 + j1.25)\ \Omega$

7-5.1 $(1.1 + j4.7)$ V in series with $(0.93 + j0.75)\ \Omega$
$(3.2 + j2.4)$ A in parallel with $(0.93 + j0.75)\ \Omega$

7-6.1 $(3.82 - j1.03)\ \Omega$; 15.11 W

7-6.3 The voltage source $11.39\underline{|264.4°}$ V is in series with impedance $(10.97 - j2.16)\ \Omega$

7-6.4 P_{AB} = 593 watts

Objective-Type Questions

☆☆★**7.1** The superposition theorem is valid
(a) only for ac circuits (b) only for dc circuits
(c) For both, ac and dc circuits (d) neither of the two

☆☆★**7.2** When applying the superposition theorem to any circuit,
(a) the voltage source is shorted, the current source is opened
(b) the voltage source is opened, the current source is shorted
(c) both are opened
(d) both are shorted

☆☆★**7.3** While applying Thèvenin's theorem, the Thèvenin's voltage is equal to
(a) short-circuit voltage at the terminals
(b) open-circuit voltage at the terminals
(c) voltage of the source
(d) total voltage available in the circuit

☆☆★**7.4** Thèvenin impedance Z_{Th} is found
(a) by short-circuiting the given two terminals
(b) between any two open terminals
(c) by removing voltage sources along with the internal resistances
(d) between same open terminals as for V_{Th}

☆☆★**7.5** Thèvenin impedance of the circuit at its terminals A and B in Fig. 7.70 is
(a) 5 H (b) 2 Ω
(c) 1.4 Ω (d) 7 H

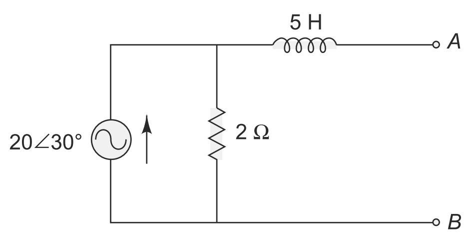


Fig. 7.70

☆☆★**7.6** Norton's equivalent form in any complex impedance circuit consists of
(a) an equivalent current source in parallel with an equivalent resistance
(b) an equivalent voltage source in series with an equivalent conductance

(c) an equivalent current source in parallel with an equivalent impedance
(d) None of the above

☆☆★ **7.7** The maximum power transfer theorem can be applied

(a) only to dc circuits
(b) only to ac circuits
(c) to both dc and ac circuits
(d) neither of the two

☆☆★ **7.8** In a complex impedance circuit, the maximum power transfer occurs when the load impedance is equal to

(a) complex conjugate of source impedance
(b) source impedance
(c) source resistance
(d) none of the above

☆☆★ **7.9** Maximum power transfer occurs at a

(a) 100% efficiency
(b) 50% efficiency
(c) 25% efficiency
(d) 75% efficiency

☆★★ **7.10** In the circuit shown in Fig. 7.71, the power supplied by the 10 V source is

(a) 6.6 W
(b) 21.7 W
(c) 30 W
(d) 36.7 W

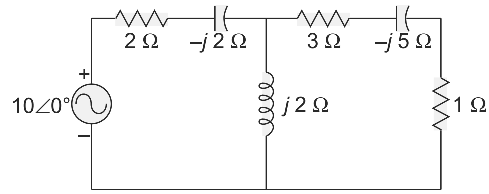


Fig. 7.71

☆★★ **7.11** The Thèvenin equivalent impedance of the circuit in Fig. 7.72 is

(a) $(1 + j5)\ \Omega$
(b) $(2.5 + j25)\ \Omega$
(c) $(6.25 + j6.25)\ \Omega$
(d) $(2.5 + j6.25)\ \Omega$

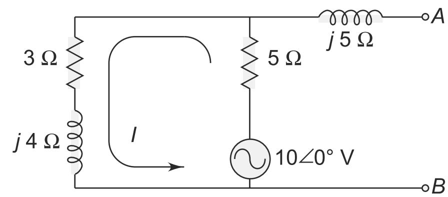


Fig. 7.72

☆★★ **7.12** A source has an emf of 10 V and an impedance of $500 + j100\ \Omega$. The amount of maximum power transferred to the load will be

(a) 0.5 mW
(b) 0.05 mW
(c) 0.05 W
(d) 0.5 W

☆★★ **7.13** For the circuit shown in Fig. 7.73, find the voltage across the dependent source.

(a) $8\ \angle 0°$
(b) $4\ \angle 0°$
(c) $4\ \angle 90°$
(d) $8\ \angle -90°$

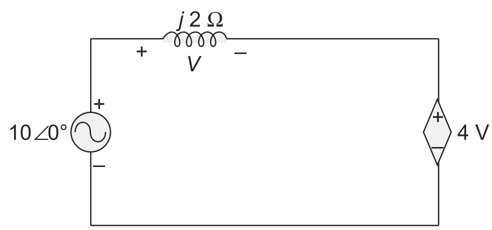


Fig. 7.73

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/265

CHAPTER 8

Resonance

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- **LO 1** Identify the importance of resonance and find the condition for series resonance
- **LO 2** Determine the variation of impedance and phase angle of a series resonant circuit with frequency
- **LO 3** Explain bandwidth, quality factor; identify magnification in a series resonance circuit and its importance
- **LO 4** Explain parallel resonance and resonant frequency for tank circuit
- **LO 5** Analyse the change in impedance with frequency in a parallel resonant circuit; also explain bandwidth, *Q*-factor and magnification in parallel resonance circuits
- **LO 6** Draw the locus diagrams for series and parallel *R*,*L* and *C* circuits

8.1 SERIES RESONANCE

LO 1 Identify the importance of resonance and find the condition for series resonance

In many electrical circuits, resonance is a very important phenomenon. The study of resonance is very useful, particularly in the area of communications. For example, the ability of a radio receiver to select a certain frequency, transmitted by a station and to eliminate frequencies from other stations is based on the principle of resonance. In a series *RLC* circuit, the current lags behind, or leads the applied voltage depending upon the values of X_L and X_C. X_L causes the total current to lag behind the applied voltage, while X_C causes the total current to lead the applied voltage. When $X_L > X_C$, the circuit is predominantly inductive, and when $X_C > X_L$, the circuit is predominantly capacitive. However, if one of the parameters of the series *RLC* circuits is varied in such a way that the current in the circuit is in phase with the applied voltage, then the circuit is said to be in resonance.

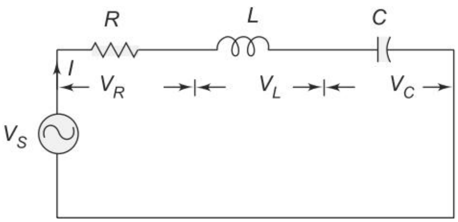


Fig. 8.1

Consider the series *RLC* circuit shown in Fig. 8.1.

The total impedance for the series *RLC* circuit is

$$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

It is clear from the circuit that the current $I = V_S/Z$.

The circuit is said to be in resonance if the current is in phase with the applied voltage. In a series *RLC* circuit, series resonance occurs when $X_L = X_C$. The frequency at which the resonance occurs is called the *resonant frequency*.

Since $X_L = X_C$, the impedance in a series *RLC* circuit is purely resistive. At the resonant frequency, f_r, the voltages across capacitance and inductance are equal in magnitude. Since they are 180° out of phase with each other, they cancel each other and, hence zero voltage appears across the *LC* combination.

At resonance,

$$X_L = X_C, \quad \text{i.e.,} \quad \omega L = \frac{1}{\omega C}$$

Solving for resonant frequency, we get

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$f_r^2 = \frac{1}{4\pi^2 LC}$$

$$\therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

In a series *RLC* circuit, resonance may be produced by varying the frequency, keeping *L* and *C* constant; otherwise, resonance may be produced by varying either *L* or *C* for a fixed frequency.

EXAMPLE 8.1

For the circuit shown in Fig. 8.2, determine the value of capacitive reactance and impedance at resonance.

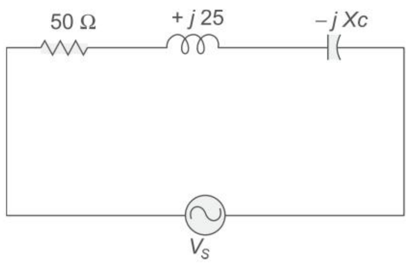


Fig. 8.2

Solution At resonance,

$$X_L = X_C$$

Since $X_L = 25\ \Omega$

$$X_C = 25\ \Omega \qquad \therefore \frac{1}{\omega C} = 25$$

The value of impedance at resonance is

$$Z = R$$

$$\therefore \quad Z = 50\ \Omega$$

EXAMPLE 8.2

Determine the resonant frequency for the circuit shown in Fig. 8.3.

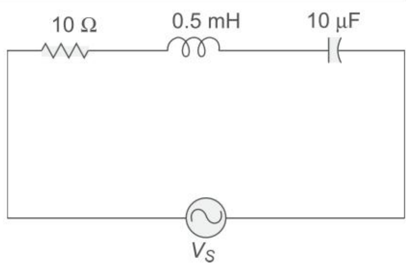


Fig. 8.3

Solution The resonant frequency is

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{10\times10^{-6}\times0.5\times10^{-3}}}$$

$$f_r = 2.25\ \text{kHz}$$

Frequently Asked Questions linked to LO 1*

☆☆★**8-1.1** When do you say that a given ac circuit is at resonance? **[AU April/May 2011]**
☆☆★**8-1.2** What do you understand by resonance? **[RGTU June 2014]**

8.2 IMPEDANCE AND PHASE ANGLE OF A SERIES RESONANT CIRCUIT

LO 2 Determine the variation of impedance and phase angle of a series resonant circuit with frequency

The impedance of a series *RLC* circuit is

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The variation of X_C and X_L with frequency is shown in Fig. 8.4.

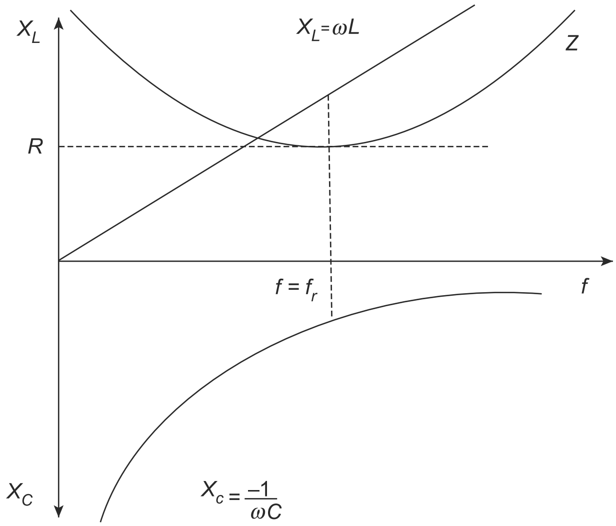


Fig. 8.4

At zero frequency, both X_C and Z are infinitely large, and X_L is zero because at zero frequency, the capacitor acts as an open circuit and the inductor acts as a short circuit. As the frequency increases, X_C decreases and X_L increases. Since X_C is larger than X_L, at frequencies below the resonant frequency f_r, Z decreases along with X_C. At resonant frequency f_r, $X_C = X_L$, and $Z = R$. At frequencies above the resonant frequency f_r, X_L is larger than X_C, causing Z to increase. The phase angle as a function of frequency is shown in Fig. 8.5.

At a frequency below the resonant frequency, current leads the source voltage because the capacitive reactance is greater than the inductive reactance. The phase angle decreases as the frequency approaches the resonant value, and is 0° at resonance. At frequencies above resonance, the current lags behind the source voltage, because the inductive reactance is greater than capacitive reactance. As the frequency goes higher, the phase angle approaches 90°.

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

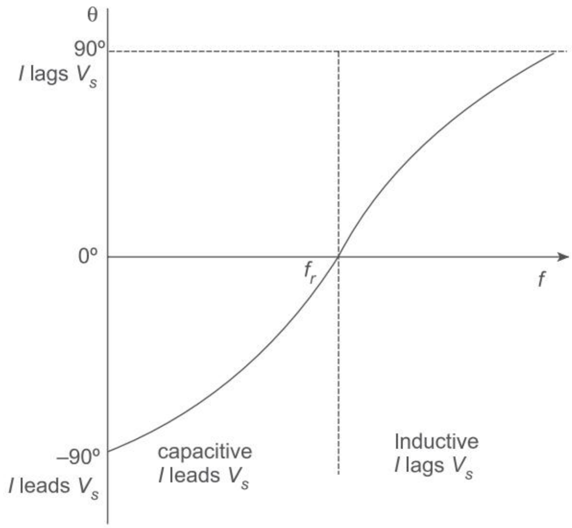


Fig. 8.5

EXAMPLE 8.3

For the circuit shown in Fig. 8.6, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.

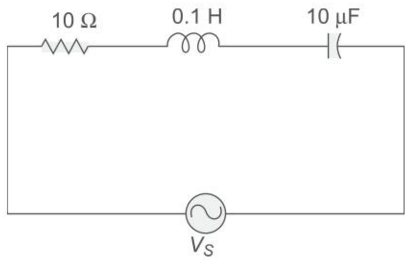


Fig. 8.6

Solution Resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{0.1\times 10\times 10^{-6}}} = 159.2 \text{ Hz}$$

At 10 Hz below f_r = 159.2 − 10 = 149.2 Hz

At 10 Hz above f_r = 159.2 + 10 = 169.2 Hz

Impedance at resonance is equal to R

$\therefore \quad Z = 10\ \Omega$

Capacitive reactance at 149.2 Hz is

$$X_{C_1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi\times 149.2\times 10^{-6}\times 10}$$

$\therefore \quad X_{C_1} = 106.6\ \Omega$

Capacitive reactance at 169.2 Hz is

$$X_{C_2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi\times 169.2\times 10\times 10^{-6}}$$

$\therefore \quad X_{C_2} = 94.06\ \Omega$

Inductive reactance at 149.2 Hz is

$$X_{L_1} = \omega_2 L = 2\pi \times 149.2 \times 0.1 = 93.75\ \Omega$$

Inductive reactance at 169.2 Hz is

$$X_{L_2} = \omega_2 L = 2\pi \times 169.2 \times 0.1 = 106.31\ \Omega$$

Impedance at 149.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_1} - X_{C_1})^2}$$
$$= \sqrt{(10)^2 + (93.75 - 106.6)^2}$$
$$= 16.28\ \Omega$$

Here X_{C_1} is greater than X_{L_1}, so Z is capacitive.
Impedance at 169.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_2} - X_{C_2})^2}$$
$$= \sqrt{(10)^2 + (106.31 - 94.06)^2}$$
$$= 15.81\ \Omega$$

Here, X_{L_1} is greater than X_{C_1}, so Z is inductive.

Frequently Asked Questions linked to LO 2

☆★★**8-2.1** For the circuit shown in Fig. Q.1, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.
[AU May/June 2014]

☆☆★**8-2.2** Define the following: **[JNTU Nov. 2012]**
(a) Impedance (b) Phase angle

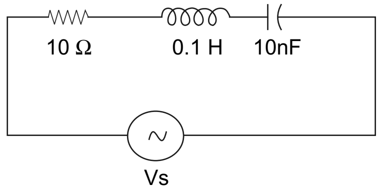


Fig. Q.1

8.3 VOLTAGES AND CURRENTS IN A SERIES RESONANT CIRCUIT

LO 3 Explain bandwidth, quality factor; identify magnification in a series resonance circuit and its importance

The variation of impedance and current with frequency is shown in Fig. 8.7.

At resonant frequency, the capacitive reactance is equal to inductive reactance, and hence the impedance is minimum. Because of minimum impedance, maximum current flows through the circuit. The current variation with frequency is plotted.

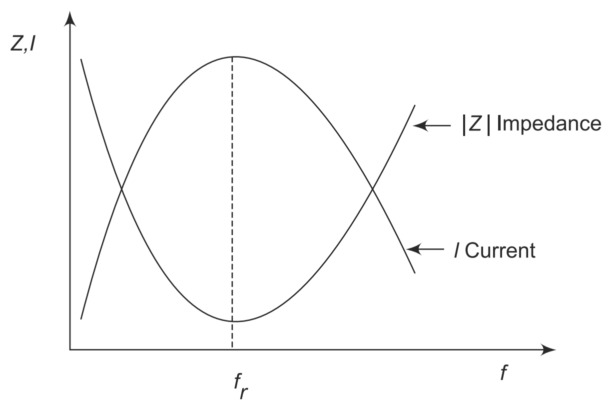


Fig. 8.7

The voltage drop across resistance, inductance and capacitance also varies with frequency. At $f = 0$, the capacitor acts as an open circuit and blocks current. The complete source voltage appears across the capacitor. As the frequency increases, X_C decreases and X_L increases, causing total reactance $X_C - X_L$ to decrease. As a result, the impedance decreases and the current increases. As the current increases, V_R also increases, and both V_C and V_L increase.

When the frequency reaches its resonant value f_r, the impedance is equal to R, and hence, the current reaches its maximum value, and V_R is at its maximum value.

As the frequency is increased above resonance, X_L continues to increase and X_C continues to decrease, causing the total reactance, $X_L - X_C$ to increase. As a result there is an increase in impedance and a decrease in current. As the current decreases, V_R also decreases, and both V_C and V_L decrease. As the frequency becomes very high, the current approaches zero, both V_R and V_C approach zero, and V_L approaches V_s.

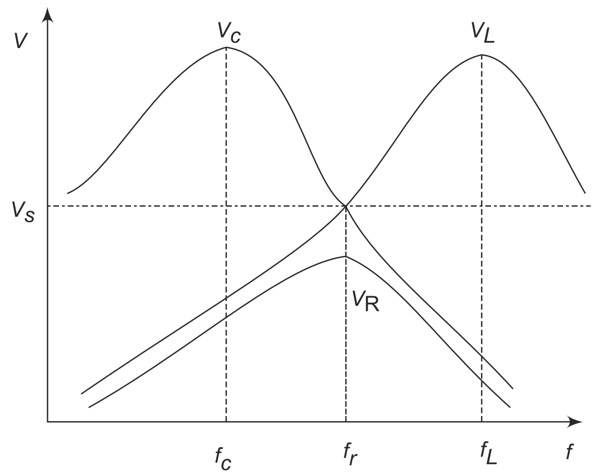


Fig. 8.8

The response of different voltages with frequency is shown in Fig. 8.8.

The drop across the resistance reaches its maximum when $f = f_r$. The maximum voltage across the capacitor occurs at $f = f_c$. Similarly, the maximum voltage across the inductor occurs at $f = f_L$.

The voltage drop across the inductor is

$$V_L = IX_L$$

$$\text{where} \quad I = \frac{V}{Z}$$

$$\therefore \quad V_L = \frac{\omega L V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

To obtain the condition for maximum voltage across the inductor, we have to take the derivative of the above equation with respect to frequency, and make it equal to zero.

$$\therefore \quad \frac{dV_L}{d\omega} = 0$$

If we solve for ω, we obtain the value of ω when V_L is maximum.

$$\frac{dV_L}{d\omega} = \frac{d}{d\omega}\left\{\omega L V \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\right\}$$

$$LV\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)^{-1/2}$$

$$\frac{-\frac{\omega L V}{2}\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)\left(2\omega L^2 - \frac{2}{\omega^3 C^2}\right)}{R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}} = 0$$

From this,

$$R^2 - \frac{2L}{C} + 2/\omega^2 C^2 = 0$$

$$\therefore \quad \omega L = \sqrt{\frac{2}{2LC - R^2C^2}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{2}{2 - \dfrac{R^2C}{L}}}$$

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{1 - \dfrac{R^2C}{2L}}}$$

Similarly, the voltage across the capacitor is

$$V_C = IX_C = \frac{1}{\omega C}$$

$$\therefore \quad V_C = \frac{V}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \times \frac{1}{\omega C}$$

To get maximum value $\dfrac{dV_C}{d\omega} = 0$

If we solve for ω, we obtain the value of ω when V_C is maximum.

$$\frac{dV_C}{d\omega} = \omega C \frac{1}{2}\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\left[2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)\right]$$

$$+ \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, C = 0$$

From this,

$$\omega_C^2 = \frac{1}{LC} - \frac{R^2}{2L}$$

$$\omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$\therefore \quad f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

The maximum voltage across the capacitor occurs below the resonant frequency; and the maximum voltage across the inductor occurs above the resonant frequency.

EXAMPLE 8.4

A series circuit with $R = 10\ \Omega$, $L = 0.1$ H and $C = 50\ \mu F$ has an applied voltage $V = 50\angle 0°$ with a variable frequency. Find the resonant frequency, the value of frequency at which maximum voltage occurs across the inductor and the value of frequency at which maximum voltage occurs across the capacitor.

Solution The frequency at which maximum voltage occurs across the inductor is

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{\left(1-\frac{R^2C}{2L}\right)}}$$

$$= \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}}\sqrt{\frac{1}{1-\left(\frac{(10)^2\times 50\times 10^{-6}}{2\times 0.1}\right)}}$$

$$= 72.08 \text{ Hz}$$

Similarly, $$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{2L}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times 50\times 10^{-6}}-\frac{(10)^2}{2\times 0.1}}$$

$$= \frac{1}{2\pi}\sqrt{200000-500}$$

$$= 71.08 \text{ Hz}$$

Resonant frequency $$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}} = 71.18 \text{ Hz}$$

It is clear that the maximum voltage across the capacitor occurs below the resonant frequency and the maximum inductor voltage occurs above the resonant frequency.

8.4 BANDWIDTH OF AN *RLC* CIRCUIT

LO 3

The bandwidth of any system is the range of frequencies for which the current or output voltage is equal to 70.7% of its value at the resonant frequency, and it is denoted by *BW*. Figure 8.9 shows the response of a series *RLC* circuit.

*The frequency f_1 is the frequency at which the current is 0.707 times the current at resonant value, and it is called the **lower cut-off frequency**. The frequency f_2 is the frequency at which the current is 0.707 times the current at resonant value (i.e. maximum value), and is called the **upper cut-off frequency**. The **bandwidth**, or BW, is defined as the frequency difference between f_2 and f_1.*

$\therefore \quad BW = f_2 - f_1$

The unit of *BW* is hertz (Hz).

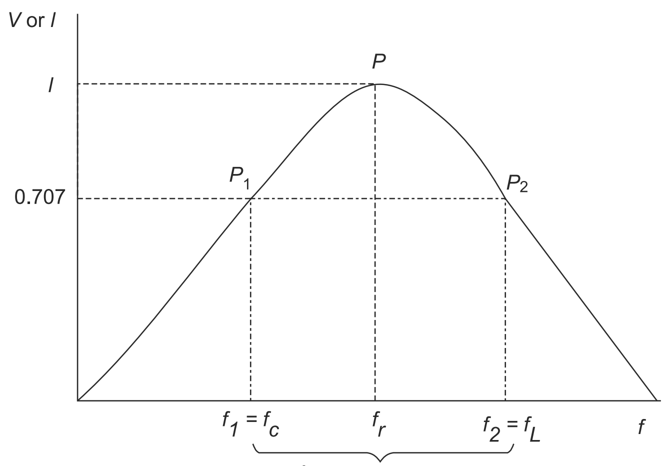


Fig. 8.9

If the current at P_1 is $0.707 I_{max}$, the impedance of the circuit at this point is $\sqrt{2}R$, and hence

$$\frac{1}{\omega_1 C} - \omega_1 L = R \tag{8.1}$$

Similarly, $$\omega_2 L - \frac{1}{\omega_2 C} = R \tag{8.2}$$

If we equate both the above equations, we get

$$\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}$$

$$L(\omega_1 + \omega_2) = \frac{1}{C}\left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right) \tag{8.3}$$

From Eq. (8.3), we get

$$\omega_1 \omega_2 = \frac{1}{LC}$$

we have $$\omega_r^2 = \frac{1}{LC}$$

$$\therefore \quad \omega^2{}_r = \omega_1 \omega_2 \tag{8.4}$$

If we add Eqs (8.1) and (8.2), we get

$$\frac{1}{\omega_1 C} - \omega_1 L + \omega_2 L - \frac{1}{\omega_2 C} = 2R$$

$$(\omega_2 - \omega_1)L + \frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right) = 2R \tag{8.5}$$

Since $$C = \frac{1}{\omega_r^2 L}$$

and $$\omega_1 \omega_2 = \omega^2{}_r$$

$$(\omega_2 - \omega_1)L + \frac{\omega_r^2 L(\omega_2 - \omega_1)}{\omega_r^2} = 2R \tag{8.6}$$

From Eq. (8.6), we have

$$\omega_2 - \omega_1 = \frac{R}{L} \tag{8.7}$$

$$\therefore \quad f_2 - f_1 = \frac{R}{2\pi L} \tag{8.8}$$

or $$BW = \frac{R}{2\pi L}$$

From Eq. (8.8), we have

$$f_2 - f_1 = \frac{R}{2\pi L}$$

$$\therefore \quad f_r - f_1 = \frac{R}{4\pi L}$$

$$f_2 - f_r = \frac{R}{4\pi L}$$

The lower frequency limit $\boxed{f_1 = f_r - \frac{R}{4\pi L}}$ (8.9)

The upper frequency limit $\boxed{f_2 = f_r + \frac{R}{4\pi L}}$ (8.10)

If we divide the equation on both sides by f_r, we get

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} \tag{8.11}$$

Here, an important property of a coil is defined. It is the ratio of the reactance of the coil to its resistance. This ratio is defined as the Q of the coil. Q is known as a *figure of merit*, it is also called *quality factor* and is an indication of the quality of a coil.

$$\boxed{Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R}} \tag{8.12}$$

If we substitute Eq. (8.11) in Eq. (8.12), we get

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} \tag{8.13}$$

The upper and lower cut-off frequencies are sometimes called the *half-power frequencies*. At these frequencies, the power from the source is half of the power delivered at the resonant frequency.

At resonant frequency, the power is

$$P_{\max} = I^2_{\max} R$$

At frequency f_1, the power is $P_1 = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 R = \frac{I^2_{\max} R}{2}$

Similarly, at frequency f_2, the power is

$$P_2 = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 R$$

$$= \frac{I^2_{\max} R}{2}$$

The response curve in Fig. 8.9 is also called the *selectivity curve* of the circuit. Selectivity indicates how well a resonant circuit responds to a certain frequency and eliminates all other frequencies. The narrower the bandwidth, the greater the selectivity.

EXAMPLE 8.5

Determine the quality factor of a coil for the series circuit consisting of $R = 10\ \Omega$, $L = 0.1$ H and $C = 10\ \mu$F.

Solution Quality factor $Q = \dfrac{f_r}{BW}$

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2 \text{ Hz}$$

At lower half-power frequency, $X_C > X_L$

$$\frac{1}{2\pi f_1 C} - 2\pi f_1 L = R$$

From which $f_1 = \dfrac{-R + \sqrt{R^2 + 4L/C}}{4\pi L}$

At upper half-power frequency, $X_L > X_C$

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = R$$

From which $f_2 = \dfrac{+R + \sqrt{R^2 + 4L/C}}{4\pi L}$

Bandwidth $BW = f_2 - f_1 = \dfrac{R}{2\pi L}$

Hence $Q_0 = \dfrac{f_r}{BW} = \dfrac{2\pi f_r L}{R} = \dfrac{2 \times \pi \times 159.2 \times 0.1}{10}$

$$Q_0 = \frac{f_r}{BW} = 10$$

8.5 THE QUALITY FACTOR (Q) AND ITS EFFECT ON BANDWIDTH

LO 3

*The **quality factor**, Q, is the ratio of the reactive power in the inductor or capacitor to the true power in the resistance in series with the coil or capacitor.*

The quality factor

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}$$

In an inductor, the maximum energy stored is given by $\dfrac{LI^2}{2}$

$$\text{Energy dissipated per cycle} = \left(\frac{I}{\sqrt{2}}\right)^2 R \times T = \frac{I^2 RT}{2}$$

$$\therefore \quad \text{Quality factor of the coil } Q = 2\pi \times \frac{\frac{1}{2} LI^2}{\frac{I^2 R}{2} \times \frac{1}{f}}$$

$$= \frac{2\pi f L}{R} = \frac{\omega L}{2}$$

Similarly, in a capacitor, the maximum energy stored is given by $\dfrac{CV^2}{2}$

The energy dissipated per cycle $= (I/\sqrt{2})^2 \; R \times T$

The quality factor of the capacitance circuit

$$Q = \frac{2\pi \frac{1}{2} C \left(\frac{1}{\omega C} \right)^2}{\frac{I^2}{2} R \times \frac{1}{f}} = \frac{1}{\omega CR}$$

In series circuits, the quality factor $Q = \frac{\omega L}{R} = \frac{1}{\omega CR}$

We have already discussed the relation between bandwidth and quality factor, which is $Q = \frac{f_r}{BW}$.

A higher value of the circuit Q results in a smaller bandwidth. A lower value of Q causes a larger bandwidth.

EXAMPLE 8.6

For the circuit shown in Fig. 8.10, determine the value of Q at resonance and bandwidth of the circuit.

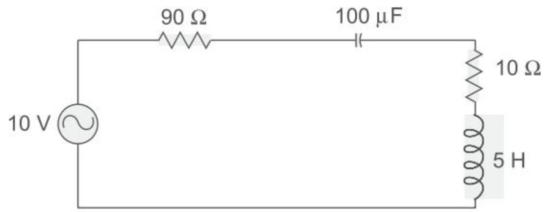


Fig. 8.10

Solution The resonant frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{5 \times 100 \times 10^{-6}}}$$

$$= 7.12 \text{ Hz}$$

Quality factor $Q = X_L/R = 2\pi f_r L/R$

$$= \frac{2\pi \times 7.12 \times 5}{100} = 2.24$$

Bandwidth of the circuit is $BW = \frac{f_r}{Q} = \frac{7.12}{2.24} = 3.178 \text{ Hz}$

8.6 MAGNIFICATION IN RESONANCE

LO 3

If we assume that the voltage applied to the series RLC circuit is V, and the current at resonance is I, then the voltage across L is $V_L = IX_L = (V/R)\,\omega_r L$

Similarly, the voltage across C

$$V_C = IX_C = \frac{V}{R\omega_r C}$$

Since $Q = 1/\omega_r CR = \omega_r L/R$

where ω_r is the frequency at resonance.

Therefore, $V_L = VQ$

$V_C = VQ$

The ratio of voltage across either L or C to the voltage applied at resonance can be defined as magnification.

$\therefore$ magnification $= Q = V_L / V$ or V_C / V

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★**8-3.1** A voltage $v(t) = 50 \sin \omega t$ is applied to a series *RLC* circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 400 V. The bandwidth is known to be 500 rad/sec and the impedance at resonance is 100 Ω. Find the resonant frequency, and compute the upper and lower limits of the bandwidth. Determine the values of *L* and *C* of the circuit.

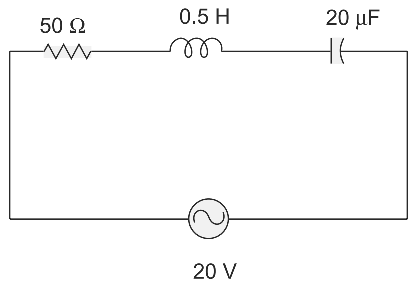


Fig. Q.2

☆☆★**8-3.2** For the circuit shown in Fig. Q.2, determine the frequency at which the circuit resonates. Also find the voltage across the capacitor at resonance, and the *Q*-factor of the circuit.

☆☆★**8-3.3** A series *RLC* circuit has a quality factor of 10 at 200 rad/s. The current flowing through the circuit at resonance is 0.5 A and the supply voltage is 10 V. The total impedance of the circuit is 40 Ω. Find the circuit constants.

☆★★**8-3.4** An *RLC* series circuit is to be chosen to produce a magnification of 10 at 100 rad/s. The source can supply a maximum current of 10 A and the supply voltage is 100 V. The power frequency impedance of the circuit should not be more than 14.14 Ω. Find the values of *R*, *L*, and *C*.

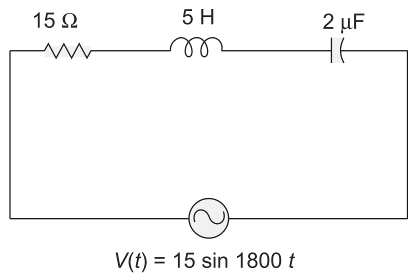


Fig. Q.5

☆☆★**8-3.5** For the circuit shown in Fig. Q.5, the applied voltage $v(t) = 15 \sin 1800t$. Determine the resonant frequency. Calculate the quality factor and bandwidth. Compute the lower and upper limits of the bandwidth.

☆★★**8-3.6** In the circuit shown in Fig. Q.6, the current is at its maximum value with inductor value $L = 0.5$ H, and 0.707 times of its maximum value with $L = 0.2$ H. Find the value of *Q* at $\omega = 200$ rad/s and circuit constants.

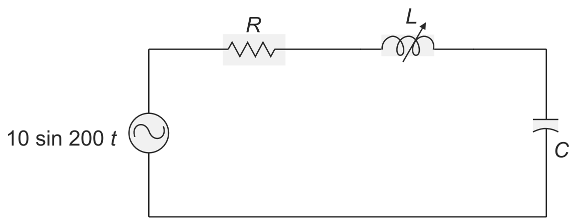


Fig. Q.6

☆☆★**8-3.7** In a series *RLC* circuit, if the applied voltage is 10 V, what is the maximum voltage across the inductor, if the resonance frequency is 1 kHz, and *Q*-factor is 10.

☆☆★**8-3.8** Obtain the expression for the frequency at which the maximum voltage occurs across the capacitor in series resonance circuit in terms of *Q*-factor and resonance frequency.

☆☆★**8-3.9** The voltage applied to the series *RLC* circuit is 5 V. The Q of the coil is 25 and the value of the capacitor is 200 PF. The resonant frequency of the circuit is 200 kHz. Find the value of inductance, the circuit current and the voltage across the capacitor.

☆★★**8-3.10** The resonance frequency of a series *RLC* circuit is 1 kH; the quality factor is 10, the source voltage is 10 V. (a) Find the voltage across the conductor at resonance (b) Also, find the frequency for which the voltage across the inductor is maximum.

Frequently Asked Questions linked to LO 3

☆☆★**8-3.1** Draw the frequency response of an *RLC* series circuit.

☆☆★**8-3.2** Define bandwidth of a resonant circuit. **[AU May/June 2013]**

☆☆★**8-3.3** An *RLC* series circuit consists of $R = 16\ \Omega$, $L = 5$ mH, and $C = 2$ mF. Calculate the quality factor at resonance, bandwidth, and half-power frequencies. **[AU May/June 2014]**

☆☆★**8-3.4** State the concept of bandwidth of a series *RLC* circuit. **[AU Nov./Dec. 2012]**

☆☆★**8-3.5** Calculate the halfpower frequencies of a series resonant circuit where the resonance frequency is 250×10^3 Hz and bandwidth is 150 kHz. **[BPUT 2007]**

☆☆★**8-3.6** Under what condition will the power in a series *RLC* circuit will half that at resonance? **[BPTU 2008]**

☆☆★**8-3.7** Derive the expression for bandwidth of a series *RLC* circuit. **[JNTU Nov. 2012]**

☆★★**8-3.8** A series *RLC* circuit has the following parameters: $R = 15$ ohms, $L = 2$ H, $C = 100$ micro F. Calculate the resonant frequency. Under resonant condition, calculate current, power, and voltage drops across various elements if the applied voltage is 100 V. **[JNTU Nov. 2012]**

☆★★**8-3.9** In a series resonancetype bandpass filter, $L = 60$ mHz, $C = 150$ nF, and $R = 70$ W. **[PTU 2011-12]**
Determine
(a) Resonance frequency in Hz, (b) Bandwidth, (c) Cut-off frequencies
Assume the load resistance to be 600 W.

★★★**8-3.10** A capacitor of 400 pF is in series with a coil resonant at 1 MHz. The halfpower frequencies are 0.9 MHz. Specify values of R and L. Also find second halfpower frequency. **[PU 2010]**

☆☆★**8-3.11** Derive the expression for bandwidth of a series resonant circuit. **[PU 2012]**

☆☆★**8-3.12** Define quality factor in the resonant circuit. **[AU May/June 2014]**

☆☆★**8-3.13** Determine the quality factor of a coil for series resonant circuit of $R = 10$ ohms, $L = 0.1$ H, and $C = 10$ microfarads. **[AU May/June 2014]**

☆☆★**8-3.14** Determine the quality factor of a coil for the series circuit consisting of $R = 10\ \Omega$, $L = 0.1$ H, and $C = 10\ \mu$F. Derive the formula used. **[AU April/May 2011]**

☆☆★**8-3.15** Derive the expression for the impedance of a series resonating circuit in the terms of Q_0 and δ. **[PU 2010]**

☆☆★**8-3.16** A series *RLC* circuit consists of a resistance of 1 kΩ and an inductance of 100 mH in series with a capacitance of 10 pF. If 100 V is applied at input across the combination, determine **[PU 2010]**
(a) Resonant frequency (b) Maximum current in the circuit
(c) Quality factor of the circuit (d) Half-power frequencies.

☆★★**8-3.17** A series circuit with $R = 10\ \Omega$, $L = 0.1$ H, and $C = 50\ \mu$F has an applied voltage $V = 50\angle 0^\circ$ V with a variable frequency. Find (a) the resonant frequency, (b) the value of the frequency at which maximum voltage occurs across the inductor, (c) the value of the frequency at which maximum voltage occurs across the capacitor, (d) the quality factor of the coil. **[AU May/June 2013]**

☆★★**8-3.18** A series resonant circuit has a bandwidth of 20 kHz and a quality factor of 40. The resistor value is 10 kΩ. Find the value of L of this circuit. **[AU Nov./Dec. 2012]**

☆☆★**8-3.19** A series *RLC* circuit consists of 50 Ω resistance, 0.2 H inductance, and 10 μF capacitance with the applied voltage of 20 V. Determine the resonant frequency, the Q-factor, the lower and upper frequency limits, and the bandwidth of the circuit. **[AU Nov./Dec. 2012]**

☆☆★**8-3.20** A series *RLC* circuit has $R = 5\ \Omega$, $L = 10$ mH, and $C = 15\ \mu$F. Calculate (a) Q-factor of the circuit (b) the bandwidth (c) the resonant frequency, and (d) the half-power frequency f_1 and f_2. **[BPUT 2007]**

☆☆★**8-3.21** An inductive coil having a resistance of 50 Ω and an inductance of 0.05 H is connected in series with a 0.02 μF capacitor. Find **[PU 2012]**
(a) Q-factor of the coil, (b) Resonant frequencies, (c) Half-power frequencies.

☆☆★**8-3.22** Derive the expression for the impedance of a series resonant circuit in terms of Q_0 and δ. **[PU 2012]**

8.7 PARALLEL RESONANCE

LO 4 Explain parallel resonance and resonant frequency for tank circuit

Basically, parallel resonance occurs when $X_C = X_L$. The frequency at which resonance occurs is called the *resonant* frequency. When $X_C = X_L$, the two branch currents are equal in magnitude and 180° out of phase with each other. Therefore, the two currents cancel each other out, and the total current is zero. Consider the circuit shown in Fig. 8.11. The condition for resonance occurs when $X_L = X_C$.

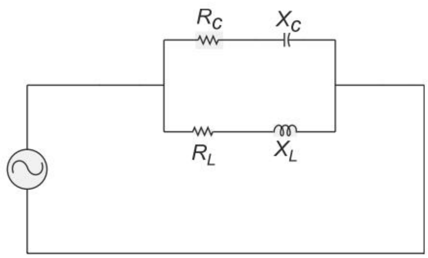


Fig. 8.11

In Fig. 8.11, the total admittance

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$

$$= \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + (j/\omega C)}{R_C^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \dfrac{1}{\omega^2 C^2}} + j\left\{\left[\frac{1/\omega C}{R_C^2 + \dfrac{1}{\omega^2 C^2}}\right] - \left[\frac{\omega L}{R_L^2 + \omega^2 L^2}\right]\right\} \tag{8.14}$$

At resonance, the susceptance part becomes zero.

$$\therefore \quad \frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\dfrac{1}{\omega_r C}}{R_C^2 + \dfrac{1}{\omega_r^2 C^2}} \tag{8.15}$$

$$\omega_r L\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{\omega_r C}\left[R_L^2 + \omega_r^2 L^2\right]$$

$$\omega_r^2\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{LC}\left[R_L^2 + \omega_r^2 L^2\right]$$

$$\omega_r^2 R_C^2 - \frac{\omega_r^2 L}{C} = \frac{1}{LC} R_L^2 - \frac{1}{C^2}$$

$$\omega_r^2\left[R_C^2 - \frac{L}{C}\right] = \frac{1}{LC}\left[R_L^2 - \frac{L}{C}\right]$$

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2-(L/C)}{R_C^2-(L/C)}} \tag{8.16}$$

The condition for resonant frequency is given by Eq. (8.16.) As a special case, if $R_L = R_C$, then Eq. (8.16) becomes

$$\omega_r = \frac{1}{\sqrt{LC}}$$

Therefore $$f_r = \frac{1}{2\pi\sqrt{LC}}$$

EXAMPLE 8.7

Find the resonant frequency in the ideal parallel LC circuit shown in Fig. 8.12.

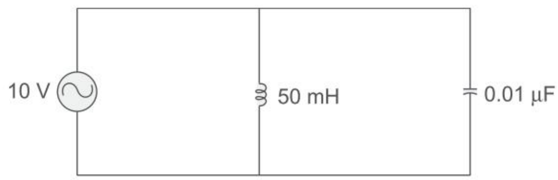


Fig. 8.12

Solution
$$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$= \frac{1}{2\pi\sqrt{50\times10^{-3}\times0.01\times10^{-6}}}$$
$$= 7117.6 \text{ Hz}$$

8.8 RESONANT FREQUENCY FOR A TANK CIRCUIT

LO 4

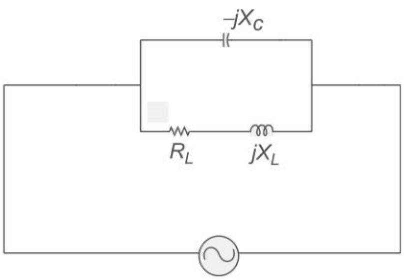


Fig. 8.13

The parallel resonant circuit is generally called a tank circuit because of the fact that the circuit stores energy in the magnetic field of the coil and in the electric field of the capacitor. The stored energy is transferred back and forth between the capacitor and coil and vice-versa. The tank circuit is shown in Fig. 8.13. The circuit is said to be in resonant condition when the susceptance part of admittance is zero.

The total admittance is

$$Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C} \tag{8.17}$$

Simplifying Eq. (8.17), we have

$$Y = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}$$
$$= \frac{R_L}{R_L^2 + X_L^2} + j\left[\frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2}\right]$$

To satisfy the condition for resonance, the susceptance part is zero.

$$\therefore \quad \frac{1}{X_C} = \frac{X_L}{R_L^2 + X_L^2} \tag{8.18}$$

$$\omega C = \frac{\omega L}{R_L^2 + \omega^2 L^2} \tag{8.19}$$

From Eq. (8.19), we get

$$R_L^2 + \omega^2 L^2 = \frac{L}{C}$$

$$\omega^2 L^2 = \frac{L}{C} - R_L^2$$

$$\omega^2 = \frac{1}{LC} - \frac{R_L^2}{L^2}$$

$$\therefore \quad \omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \tag{8.20}$$

The resonant frequency for the tank circuit is

$$\boxed{f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}} \tag{8.21}$$

EXAMPLE 8.8

For the tank circuit shown in Fig. 8.14, find the resonant frequency.

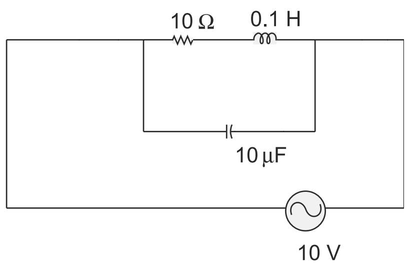


Fig. 8.14

Solution The resonant frequency

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times 10\times 10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$

$$= \frac{1}{2\pi}\sqrt{(10)^6 - (10)^2} = \frac{1}{2\pi}(994.98) = 158.35 \text{ Hz}$$

Frequently Asked Questions linked to LO 4

☆☆★**8-4.1** Derive the resonance frequency 'f_r' for the circuit shown in Fig. Q.1. **[AU May/June 2013]**

☆☆★**8-4.2** Derive the resonance frequency 'f_r' for the circuit shown in Fig. Q.2. **[AU Nov./Dec. 2012]**

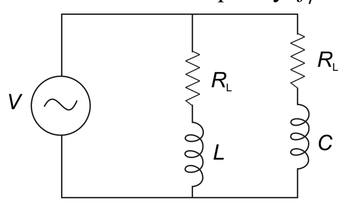


Fig. Q.1

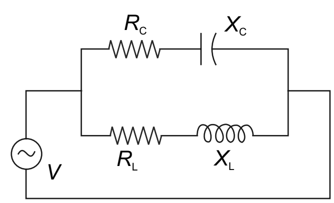


Fig. Q.2

☆★★**8-4.3** Calculate are values at R in the circuit shown in Fig. Q.3 to nodal resonance. **[BPUT 2007]**

☆★★**8-4.4** For the circuit shown in Fig. Q.4, find the value of X_c in ohms at which the circuit under resonance. **[JNTU Nov. 2012]**

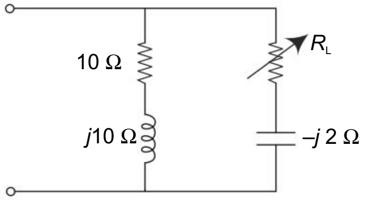


Fig. Q.3

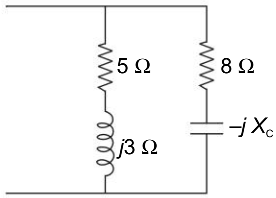


Fig. Q.4

☆☆★**8-4.5** Explain series and parallel resonance. What are their similarities and dissimilarities? **[RGTU Dec. 2013]**

☆☆★**8-4.6** Discuss the condition of resonance for parallel circuit. **[RGTU June 2014]**

☆☆★**8-4.7** For the tank circuit shown in Fig. Q.7, find the resonance frequency, f_r. **[AU April/ May 2011]**

☆★★**8-4.8** Find the value of L for which the circuit shown in Fig. Q.8 is resonance at a frequency of $W_0 = 1000$ rad/s. **[PU 2010]**

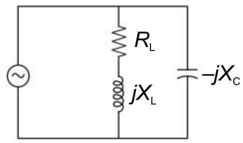


Fig. Q.7

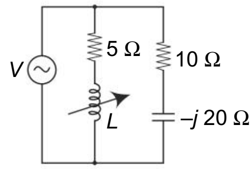


Fig. Q.8

☆☆★**8-4.9** In the circuit shown in Fig. Q.9, the capacitive reactance of resonant frequency is 1 Ω. Find the value of inductive reactance at resonant frequency. **[PU 2010]**

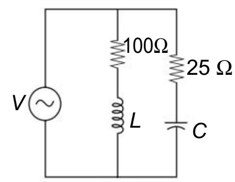


Fig. Q.9

☆☆★**8-4.10** Discuss the applications and properties of parallel resonant circuits. **[PU 2012]**

☆★★**8-4.11** A parallel resonant circuit has a coil of 100 mH with a Q-factor of 50. The coil is resonant at a frequency of 900 kHz. Find
(a) Value of the capacitor, (b) Resistance in series with the coil
(c) Circuit impedance at resonance. **[PU 2012]**

8.9 VARIATION OF IMPEDANCE WITH FREQUENCY

> **LO 5** Analyse the change in impedance with frequency in a parallel resonant circuit; also explain bandwidth, Q-factor and magnification in parallel resonance circuits

The impedance of a parallel resonant circuit is maximum at the resonant frequency and decreases at lower and higher frequencies as shown in Fig. 8.15.

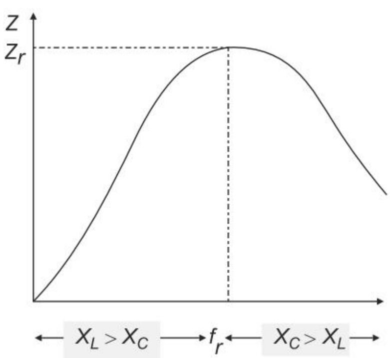


Fig. 8.15

At very low frequencies, X_L is very small and X_C is very large, so the total impedance is essentially inductive. As the frequency increases, the impedance also increases, and the inductive reactance dominates until the resonant frequency is reached. At this point $X_L = X_C$, and the impedance is at its maximum. As the frequency goes above resonance, capacitive reactance dominates and the impedance decreases.

8.10 Q-FACTOR OF PARALLEL RESONANCE

LO 5

Consider the parallel *RLC* circuit shown in Fig. 8.16.

In the circuit shown, the condition for resonance occurs when the susceptance part is zero.

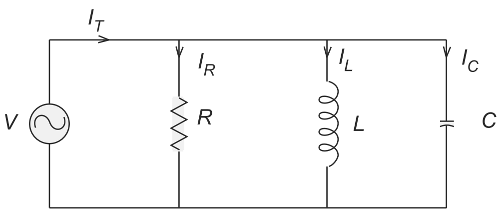


Fig. 8.16

Admittance $\quad Y = G + jB \qquad (8.22)$

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \qquad (8.23)$$

The frequency at which resonance occurs is

$$\omega_r C - \frac{1}{\omega_r L} = 0 \qquad (8.24)$$

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad (8.25)$$

The voltage and current variation with frequency is shown in Fig. 8.17. At resonant frequency, the current is minimum.

The bandwidth, $\quad BW = f_2 - f_1$

For parallel circuit, to obtain the lower half-power frequency,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R} \qquad (8.26)$$

From Eq. (8.26), we have

$$\omega_1^2 + \frac{\omega_1}{RC} - \frac{1}{LC} = 0 \qquad (8.27)$$

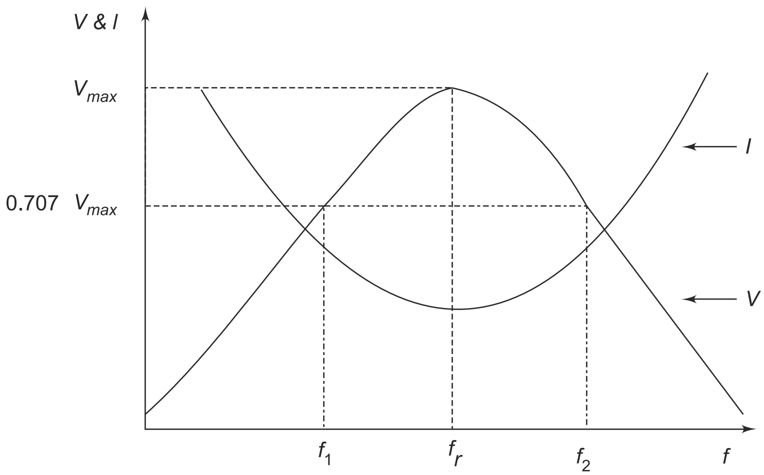


Fig. 8.17

If we simplify Eq. (8.27), we get

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{8.28}$$

Similarly, to obtain the upper half-power frequency

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R} \tag{8.29}$$

From Eq. (8.29), we have

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{8.30}$$

Bandwidth $\boxed{BW = \omega_2 - \omega_1 = \frac{1}{RC}}$

The quality factor is defined as $Q_r = \frac{\omega_r}{\omega_2 - \omega_1}$

$$Q_r = \frac{\omega_r}{1/RC} = \omega_r RC$$

In other words,

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{Energy dissipated /cycle}}$$

In the case of an inductor,

The maximum energy stored $= \frac{1}{2} LI^2$

Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 \times R \times T$

The quality factor $Q = 2\pi \times \frac{1/2(LI^2)}{\frac{I^2}{2} R \times \frac{1}{f}}$

$$\therefore \quad Q = 2\pi \times \frac{\frac{1}{2} L \left(\frac{V}{\omega L}\right)^2 R}{\frac{V^2}{2} \times \frac{1}{f}}$$

$$= \frac{2\pi f LR}{\omega^2 L^2} = \frac{R}{\omega L}$$

For a capacitor, maximum energy stored $= 1/2\ (CV^2)$

Energy dissipated per cycle $= P \times T = \frac{V^2}{2 \times R} \times \frac{1}{f}$

The quality factor $$\boxed{\begin{aligned}Q &= 2\pi\times\frac{1/2(CV^2)}{\dfrac{V^2}{2R}\times\dfrac{1}{f}}\\ &= 2\pi fCR = \omega CR\end{aligned}}$$

8.11 MAGNIFICATION

LO 5

Current magnification occurs in a parallel resonant circuit. The voltage applied to the parallel circuit, $V = IR$

Since $I_L = \dfrac{V}{\omega_r L} = \dfrac{IR}{\omega_r L} = IQ_r$

For the capacitor, $I_C = \dfrac{V}{1/\omega_r C} = IR\omega_r C = IQ_r$

Therefore, the quality factor $Q_r = I_L/I$ or I_C/I

8.12 REACTANCE CURVES IN PARALLEL RESONANCE

LO 5

The effect of variation of frequency on the reactance of the parallel circuit is shown in Fig. 8.18.

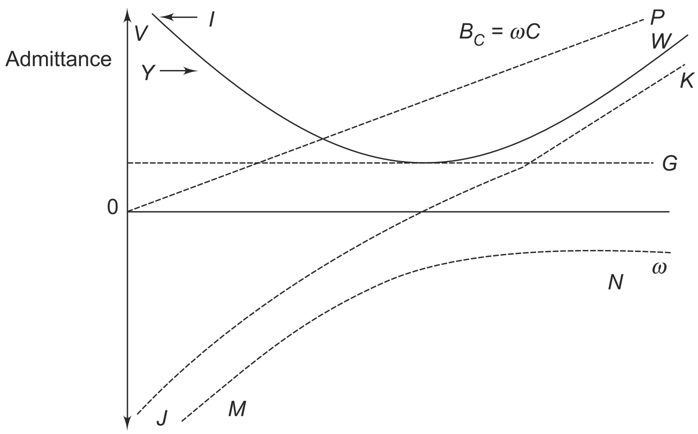


Fig. 8.18

The effect of inductive susceptance,

$$B_L = \frac{-1}{2\pi f L}$$

Inductive susceptance is inversely proportional to the frequency or ω. Hence, it is represented by a rectangular hyperbola, *MN*. It is drawn in fourth quadrant, since B_L is negative. Capacitive susceptance, $B_C = 2\pi fC$. It is directly proportional to the frequency f or ω. Hence it is represented by *OP*, passing through the origin. Net susceptance $B = B_C - B_L$. It is represented by the curve *JK*, which is a hyperbola. At the point ω_r, the total susceptance is zero, and resonance takes place. The variation of the admittance *Y* and the current *I* is represented by curve *VW*. The current will be minimum at resonant frequency.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆★★**8-5.1** The impedance $Z_1 = (5 + j3)\ \Omega$ and $Z_2 = (10 - j30)\ \Omega$ are connected in parallel as shown in Fig. Q.1. Find the value of X_3 which will produce resonance at the terminals a and b.

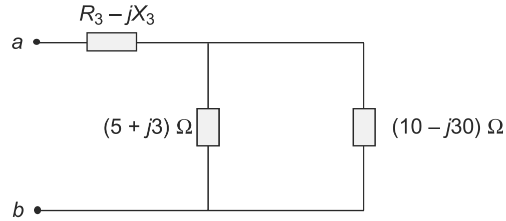


Fig. Q.1

☆☆★**8-5.2** A current source is applied to the parallel arrangement of R, L, and C where $R = 12\ \Omega$, $L = 2$ H and $C = 3\ \mu$F. Compute the resonant frequency in rad/s. Find the quality factor. Calculate the value of bandwidth. Compute the lower and upper frequency of the bandwidth. Compute the voltage appearing across the parallel elements when the input signal is $i(t) = 10 \sin 1800\, t$.

☆☆★**8-5.3** For the circuit shown in Fig. Q.3, determine the value of R_C for which the given circuit resonates.

★★★**8-5.4** Using PSpice, obtain the frequency response of the circuit shown in Fig. Q.4. Use a linear frequency sweep. Consider $1 < f < 1000$ with 100 points.

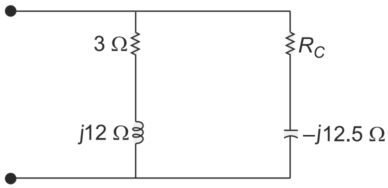


Fig. Q.3

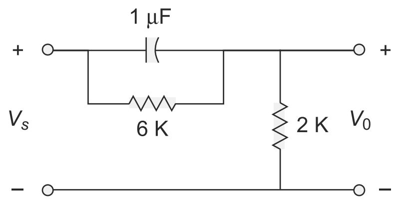


Fig. Q.4

☆★★**8-5.5** Using PSpice, obtain bodeplots for V_0 over a frequency from 1 kHz to 100 kHz for the circuit shown in Fig. Q.5, using 20 points per decade.

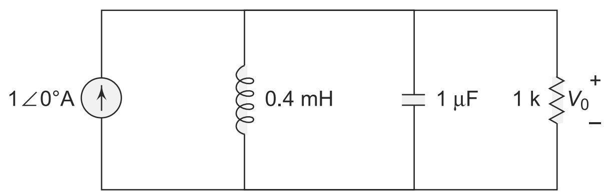


Fig. Q.5

☆☆★**8-5.6** In a parallel resonance circuit shown in Fig. Q.6 find the (a) resonance frequency, (b) dynamic resistance, and (c) bandwidth.

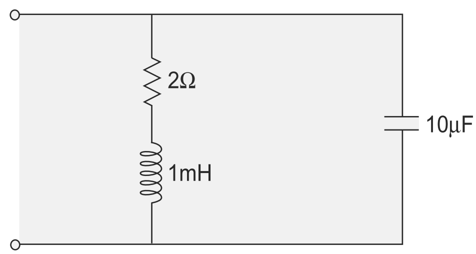


Fig. Q.6

Frequently Asked Questions linked to LO 5

☆☆★**8-5.1** Explain how to derive Q-factor of parallel resonance. **[AU May/June 2014]**

☆☆★**8-5.2** Obtain the expression for selective frequency and bandwidth of an antiresonant circuit. **[PU 2012]**

☆★★**8-5.3** Determine the value of *RL* for resonance in the network shown in Fig. 3 **[AU May/June 2014]**

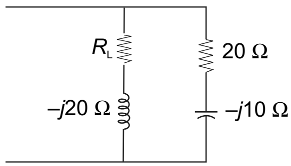


Fig. Q.3

☆★★**8-5.4** A parallel resonant circuit is a current amplifier. Justify. **[PU 2010]**

8.13 LOCUS DIAGRAMS

LO 6 Draw the locus diagrams for series and parallel R,L and C circuits

A phasor diagram may be drawn and is expanded to develop a curve; known as a locus. ***Locus diagrams*** *are useful in determining the behaviour or response of an RLC circuit when one of its parameters is varied while the frequency and voltage kept constant.* The magnitude and phase of the current vector in the circuit depends upon the values of *R*, *L*, and *C* and frequency at the fixed source voltage. The path traced by the terminus of the current vector when the parameters *R*, *L,* or *C* are varied while *f* and *v* are kept constant is called the current locus.

The term *circle diagram* identifies locus plots that are either circular or semicircular loci of the terminus (the tip of the arrow) of a current phasor or voltage phasor. Circle diagrams are often employed as aids in analysing the operating characteristics of circuits like equivalent circuit of transmission lines and some types of ac machines.

Locus diagrams can be also drawn for reactance, impedance, susceptance, and admittance when frequency is variable. Loci of these parameters furnish important information for use in circuit analysis. Such plots are particularly useful in the design of electric wave filters.

8.13.1 Series Circuits

To discuss the basis of representing a series circuit by means of a circle diagram, consider the circuit shown in Fig. 8.19 (a). The analytical procedure is greatly simplified by assuming that inductance elements have no resistance and that capacitors have no leakage current.

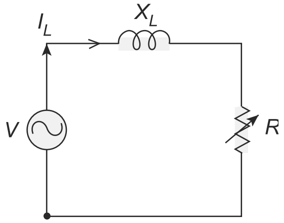


Fig. 8.19 (a)

The circuit under consideration has constant reactance but variable resistance. The applied voltage will be assumed with constant rms voltage V. The power factor angle is designated by θ. If $R = 0$, I_L is obviously equal to $\dfrac{V}{X_L}$ and has maximum value. Also, I lags V by 90°. This is shown in Fig. 8.19 (b). If R is increased from zero value, the magnitude of I becomes less than $\dfrac{V}{X_L}$ and θ becomes less than 90° and finally when the limit is reached, i.e. when R equals to infinity, I equals to zero and θ equals to zero. It is observed that the tip of the current vector represents a semicircle as indicated in Fig. 8.19 (b).

In general,

$$I_L = \frac{V}{Z}$$

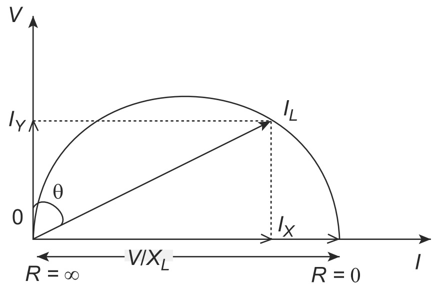


Fig. 8.19 (b)

$$\sin\theta = \frac{X}{Z}$$

or $$Z = \frac{X_L}{\sin\theta}$$

$$I = \frac{V}{X_L}\sin\theta \tag{8.31}$$

For constant V and X, Eq. (8.31) is the polar equation of a circle with diameter $\dfrac{V}{X_L}$. Figure 8.19 (b) shows the plot of Eq. (8.31) with respect to V as reference.

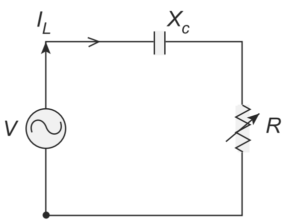


Fig. 8.19 (c)

The active component of the current I_L in Fig. 8.19 (b) is OI_L cos θ which is proportional to the power consumed in the *RL* circuit. In a similar way we can draw the loci of current if the inductive reactance is replaced by a capacitive reactance as shown in Fig. 8.19 (c). The current semicircle for the RC circuit with variable R will be on the left-hand side of the voltage vector OV with diameter $\dfrac{V}{X_L}$ as shown in Fig. 8.19 (d). The current vector OI_C leads V by θ°. The active component of the current I_CX in Fig. 8.19 (d) is OI_C cos θ which is proportional to the power consumed in the *RC* circuit.

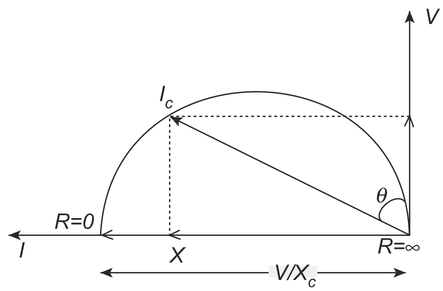


Fig. 8.19 (d)

❑ *Circle Equations for an RL Circuit*

Fixed reactance and variable resistance. The X-co-ordinate and Y-co-ordinate of I_L in Fig. 8.19 (b) respectively are $I_X = I_L \sin\theta$; $I_y = I_L \cos\theta$,

$$\text{Where}\quad I_L = \frac{V}{Z}; \sin\theta = \frac{X_L}{Z}; \cos\theta = \frac{R}{Z}; Z = \sqrt{R^2 + X_L^2}$$

$$\therefore\quad I_X = \frac{V}{Z}\cdot\frac{X_L}{Z} = V\cdot\frac{X_L}{Z^2} \tag{8.32}$$

$$I_Y = \frac{V}{Z}\cdot\frac{R}{Z} = V\cdot\frac{R}{Z^2} \tag{8.33}$$

Squaring and adding Eqs (8.32) and (8.33), we obtain

$$I_X^2 + I_Y^2 = \frac{V^2}{R^2 + X_L^2} \tag{8.34}$$

From Eq. (8.32),

$$Z^2 = R^2 + X_L^2 = V\cdot\frac{X_L}{I_X}$$

∴ Eq. (8.34) can be written as $$I_X^2 + I_Y^2 = \frac{V}{X_L}\cdot I_X$$

or $$I_X^2 + I_Y^2 - \frac{V}{X_L}\cdot I_X = 0$$

Adding $\left(\dfrac{V}{2X_L}\right)^2$ to both sides the above equation can be written as

$$\left(I_X - \frac{V}{2X_L}\right)^2 + I_Y^2 = \left(\frac{V}{2X_L}\right)^2 \tag{8.35}$$

Equation (8.35) represents a circle whose radius is $\dfrac{V}{2X_L}$ and the coordinates of the centre are $\dfrac{V}{2X_L}, 0$.

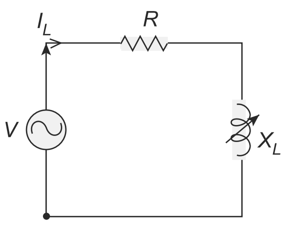
In a similar way, we can prove that for a series *RC* circuit as in Fig. 8.19 (c), with variable *R*, the locus of the tip of the current vector is a semicircle and is given by

$$\left(I_X + \frac{V}{2X_C}\right)^2 + I_Y^2 = \frac{V^2}{4X_C^2} \tag{8.36}$$

The centre has co-ordinates of $-\dfrac{V}{2X_L}$, 0 and radius as $\dfrac{V}{2X_L}$.

❑ ***Fixed Resistance, Variable Reactance*** Consider the series *RL* circuit with constant resistance *R* but variable reactance X_L as shown in Fig. 8.20 (a).

When $X_L = 0$; I_L assumes maximum value of $\dfrac{V}{R}$ and $\theta = 0$, the power factor of the circuit becomes unity; as the value X_L is increased from zero, I_L is reduced and finally when X_L is α, current becomes zero and θ will be lagging behind the voltage by 90° as shown in Fig. 8.20 (b). The current vector describes a semicircle with diameter $\dfrac{V}{R}$ and lies in the right-hand side of voltage vector *OV*. The active component of the current $OI_L \cos\theta$ is proportional to the power consumed in the *RL* circuit.



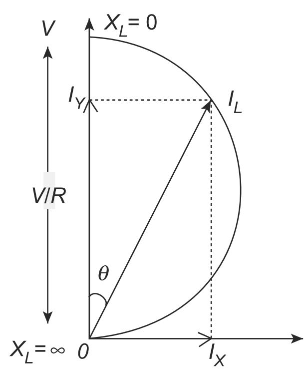


Fig. 8.20

❑ ***Equation of Circle***

Consider Eq. (8.34) $\quad I_X^2 + I_Y^2 = \dfrac{V^2}{R^2 + X_L^2}$

From Eq. (8.33) $\quad Z^2 = R^2 + X_L^2 = \dfrac{VR}{I_Y} \quad$ (8.37)

Substituting Eq. (8.37) in Eq. (8.34),

$$I_X^2 + I_Y^2 = \frac{V}{R} I_Y \tag{8.38}$$

$$I_X^2 + I_Y^2 - \frac{V}{R} I_Y = 0$$

Adding $\left(\frac{V}{2R}\right)^2$ to both sides the above equation can be written as

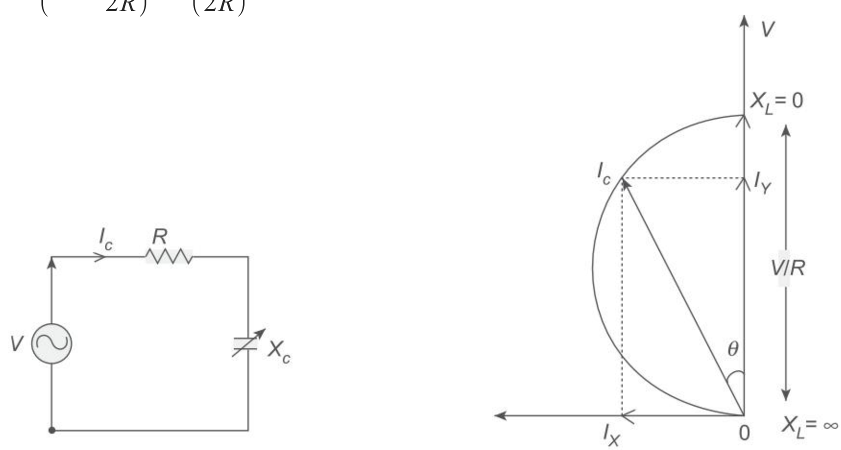
$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2 \tag{8.39}$$

Equation (8.39) represents a circle whose radius is $\frac{V}{2R}$ and the coordinates of the centre are 0; $\frac{V}{2R}$.

Let the inductive reactance in Fig. 8.20 (a) be replaced by a capacitive reactance as shown in Fig. 8.21 (a).

The current semicircle of a RC circuit with variable X_C will be on the left-hand side of the voltage vector OV with diameter $\frac{V}{R}$. The current vector OI_C leads V by θ°. As before, it may be proved that the equation of the circle shown in Fig. 8.21 (b) is

$$I_X^2 + \left(I_Y - \frac{V}{2R}\right)^2 = \left(\frac{V}{2R}\right)^2$$



(a) **Fig. 8.21** (b)

EXAMPLE 8.9

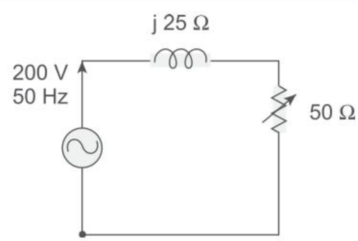


Fig. 8.22 (a)

For the circuit shown in Fig. 8.22 (a), plot the locus of the current, mark the range of I for maximum and minimum values of R, and the maximum power consumed in the circuit. Assume X_L = 25 Ω and R = 50 Ω. The voltage is 200 V; 50 Hz.

Solution Maximum value of current

$$I_{\max} = \frac{200}{25} = 8\,\text{A};\ \theta = 90°$$

Minimum value of current

$$I_{\min} = \frac{200}{\sqrt{(50)^2 + (25)^2}} = 3.777\,\text{A};\ \theta = 27.76°$$

The locus of the current is shown in Fig. 8.22 (b).

Power consumed in the circuit is proportional to $I\cos\theta$ for constant V. The maximum ordinate possible in the semicircle (AB in Fig. 8.22 (b)) represents the maximum power consumed in the circuit. This is possible when $\theta = 45°$, under the condition power factor $\cos\theta = \cos 45° = \frac{1}{\sqrt{2}}$.

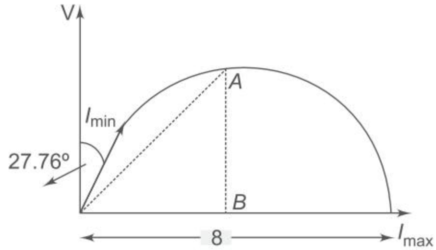


Fig. 8.22 (b)

Hence, the maximum power consumed in the circuit $= V \times AB = V \times \frac{I_{max}}{L}$

$$I_{max} = \frac{V}{X_L} = 84\text{ A}$$

$$P_{max} = \frac{V^2}{2X_L} = \frac{(200)^2}{2\times 25} = 800\text{ W}$$

EXAMPLE 8.10

For the circuit shown in Fig. 8.22 (a), if the reactance is variable, plot the range of I for maximum and minimum values of X_L and maximum power consumed in the circuit.

Solution

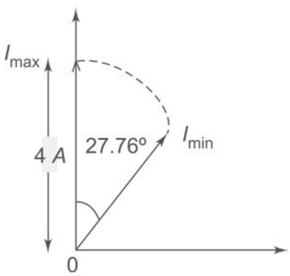


Fig. 8.23

$$\text{Maximum value of current } I_{max} = \frac{200}{50} = 4\text{ A}; \theta = 0$$

$$\text{Minimum value of current } I_{min} = \frac{200}{\sqrt{(50)^2+(25)^2}}$$

$$= 3.777\text{ A}; \theta = 27.76°$$

The locus of current is shown in Fig. 8.23.

Maximum power will be when $I = 4$ A

Hence, $P_{max} = 4 \times 200 = 800$ W

EXAMPLE 8.11

For the circuit shown in Fig. 8.24 (a), draw the locus of the current. Mark the range of I for maximum and minimum values. Assume $X_C = 50\ \Omega$; $R = 10\ \Omega$; $V = 400$ V.

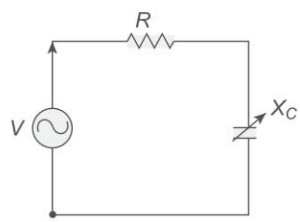


Fig. 8.24 (a)

Solution

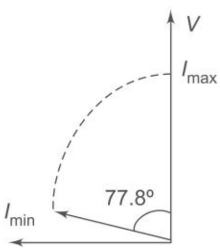


Fig. 8.24 (b)

$$I_{max} = \frac{400}{10} = 40\text{ A}; \theta = 0°$$

$$I_{min} = \frac{400}{\sqrt{(50)^2+(10)^2}} = 7.716\text{ A}. \theta = \tan^{-1} 5 = 77.8°$$

The locus of the current is shown in Fig. 8.24 (b).

8.13.2 Parallel Circuits

❑ ***Variable X_L*** Locus plots are drawn for parallel branches containing *RLC* elements in a similar way as for series circuits. Here we have more than one current locus. Consider the parallel circuit shown in Fig. 8.25 (a). The quantities that may be varied are X_L, X_C, R_L and R_C for a given voltage and frequency.

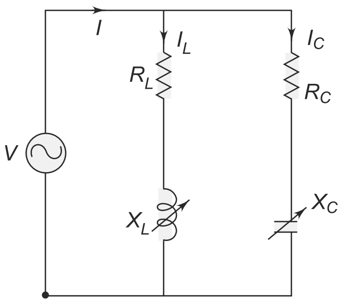


Fig. 8.25 (a)

Let us consider the variation of X_L from zero to ∞. Let *OV* shown in Fig. 8.25 (b), be the voltage vector, taken as reference. A current, I_C, will flow in the condenser branch whose parameters are held constant and leads *V* by an angle $\theta_C = \tan^{-1}\left(\frac{X_C}{R_C}\right)$, when $X_L = 0$, the current I_L, through the inductive branch is maximum and is given by $\frac{V}{R_L}$ and it is in phase with the applied voltage. When X_L is increased from zero, the current through the inductive branch I_L decreases and lags *V* by $\theta_L = \tan^{-1}\frac{X_L}{R_L}$ as shown in Fig. 8.25 (b). For any value of I_L, the $I_L R_L$ drop and $I_L X_L$ drop must add at right angles to give the applied voltage. These drops are shown as *OA* and *AV* respectively. The locus of I_L is a semicircle, and the locus of $I_L R_L$ drop is also a semicircle. When $X_L = 0$, i.e. I_L is maximum, I_L coincides with the diameter of its semicircle and $I_L R_L$ drop also coincides with the diameter of its semi-circle as shown in the figure; both these semicircles are shown with dotted circles as OI_LB and *OAV* respectively.

Since the total current is $I_C + I_L$. For example, for a particular value of I_C and I_L, the total current is represented by *OC* on the total current semicircle. As X_L is varied, the locus of the resultant current is therefore, the circle $I_C CB$ as shown with thick line in the Fig. 8.25 (b).

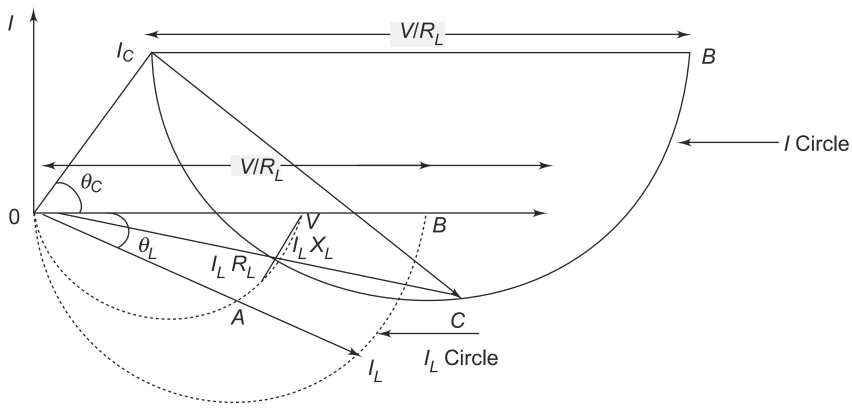


Fig. 8.25 (b)

❑ ***Variable X_C*** A similar procedure can be adopted as outlined above to draw the locus plots of I_l and *I* when X_C is varying while R_L, R_C, X_L, *V* and *f* are held constant. The curves are shown in Fig. 8.25 (c).

OV presents the voltage vector, *OB* is the maximum current through *RC* branch when $X_L = 0$; OI_L is the current through the R_L branch lagging *OV* by an angle $\theta_L = \tan^{-1}\frac{C_L}{R_L}$. As X_C is increased from zero, the current through the capacitive branch I_C decreases and leads *V* by $\theta_C = \tan^{-1}\frac{X_C}{R_C}$. For a particular I_C, the resultant current $I = I_L + I_C$ and is given by OC. The dotted semicircle OI_CB is the locus of the I_C, thick circle I_LCB is the locus of the resultant current.

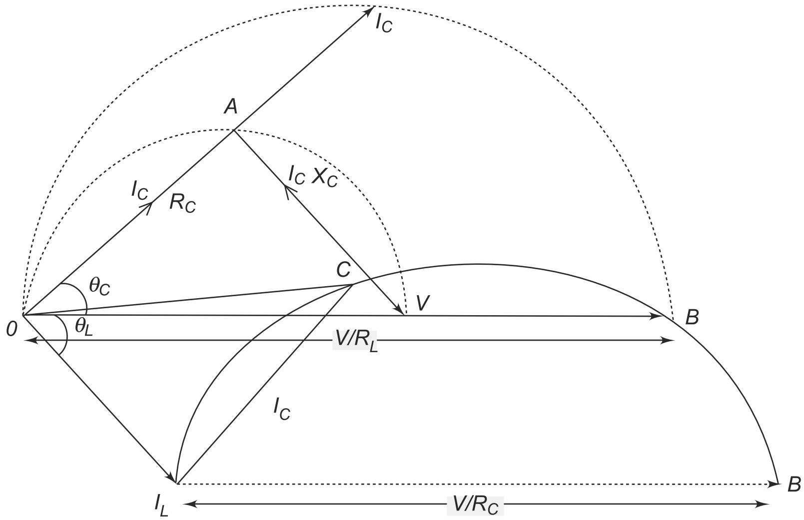


Fig. 8.25 (c)

❑ ***Variable R_L*** The locus of current for the variation of R_L in Fig. 8.26 (a) is shown in Fig. 8.26 (b). OV represents the reference voltage, OI_LB represents the locus of I_L and I_CCB represents the resultant current locus. Maximum $I_L = \dfrac{V}{X_L}$ is represented by OB.

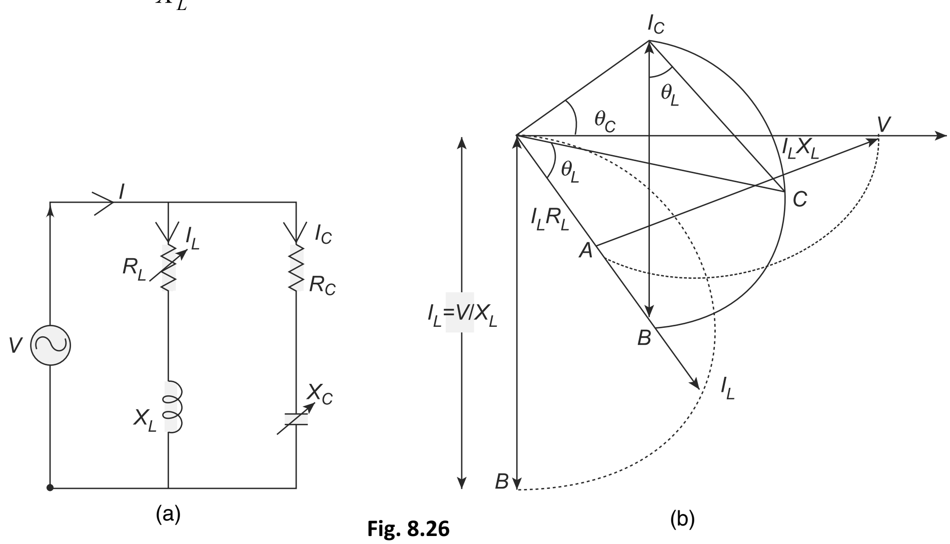


Fig. 8.26

❑ ***Variable R_C*** The locus of currents for the variation of R_C in Fig. 8.27 (a) is plotted in Fig. 8.27 (b) where OV is the source voltage and semicircle OAB represents the locus of I_C. The resultant current locus is given by BCI_L.

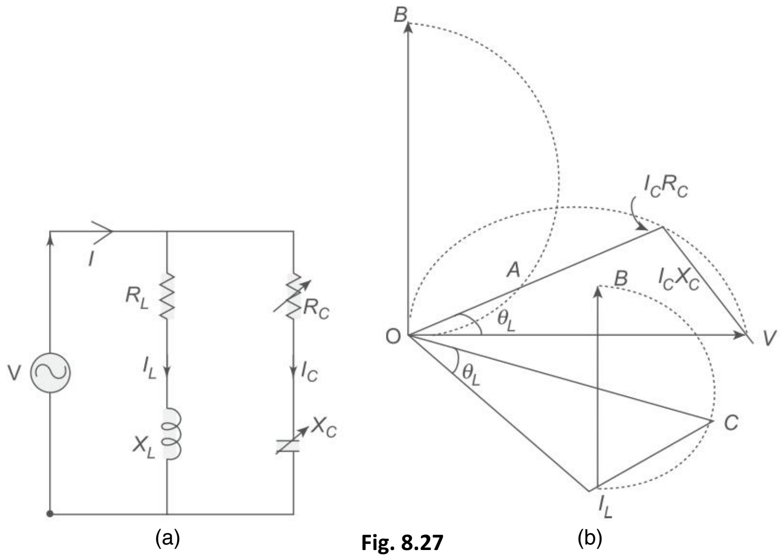


Fig. 8.27

EXAMPLE 8.12

For the parallel circuit shown in Fig. 8.28 (a), draw the locus of I_1 and I. Mark the range of values for R_1 between 10 Ω and 100 Ω. Assume $X_L = 25\ \Omega$ and $R_2 = 25\ \Omega$. The supply voltage is 200 V and frequency is 50 Hz, both held constant.

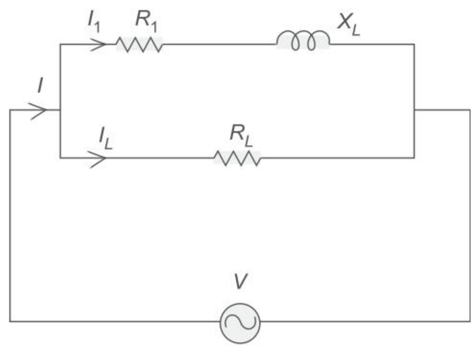


Fig. 8.28 (a)

Solution Let us take voltage as reference; on the positive X-axis. I_2 is given by $I_2 = \dfrac{200}{25} = 8\text{ A}$ and is in phase with V.

$$\text{When} \quad R_1 = 10\ \Omega \qquad I_1 = \frac{200}{\sqrt{(100+625)}} = 7.42\text{ A};\ \theta_1 = \tan^{-1}\frac{25}{10} = 68.19°$$

$$\text{When} \quad R_2 = 100\ \Omega \qquad I_1 = \frac{200}{\sqrt{(10000+625)}} = 1.94\text{ A};\ \theta_2 = \tan^{-1}\frac{25}{100} = 14.0°$$

The variation of I_1 and I are shown in Fig. 8.28 (b).

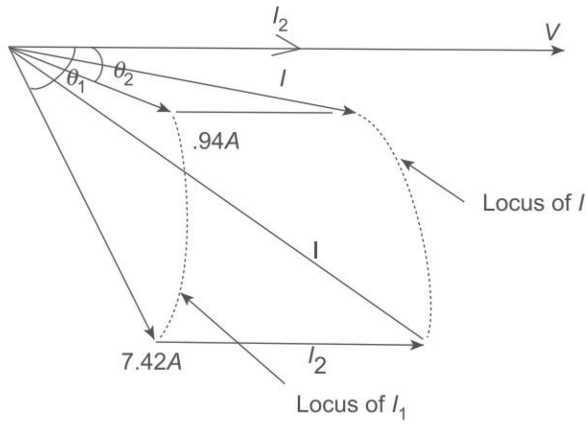


Fig. 8.28 (b)

EXAMPLE 8.13

Draw the locus of I_2 and I for the parallel circuit shown in Fig. 8.29 (a).

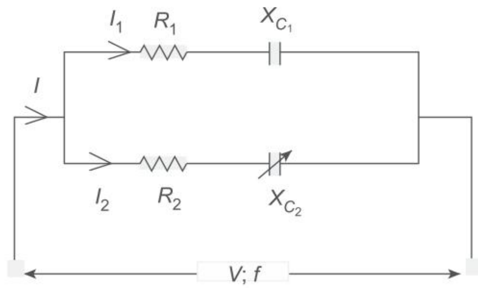


Fig. 8.29 (a)

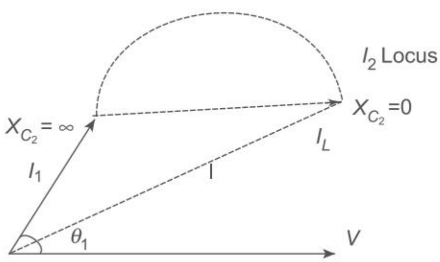


Fig. 8.29 (b)

Solution I_1

leads the voltage by a fixed angle θ_1 given by $\tan^{-1}\dfrac{X_C}{R_1}$.

I_2 varies according to the value of X_{C_2}.

I_2 is maximum when $X_{C_2} = 0$ and is in phase with V.

I_2 is zero when $X_{C_2} = \infty$ as shown in Fig. 8.29 (b).

EXAMPLE 8.14

For a parallel circuit shown in Fig. 8.30 (a), plot the locus of currents.

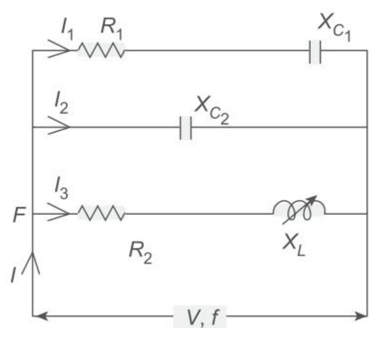


Fig. 8.30 (a)

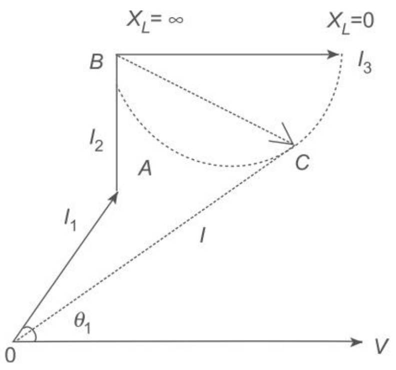


Fig. 8.30 (b)

Solution Current I_1 leads the voltage by a fixed angle θ_1 given by $\tan^{-1}\dfrac{X_C}{R_1}$, current I_2 leads the voltage by 90°. I_3 varies according to the value of X_L, when $X_L = 0$, I_3 is maximum and is given by $\dfrac{V}{R_L}$; is in phase with V; when $X_L = \infty$, I_3 is zero. Both these extremities are shown in Fig. 8.30 (b). For a particular value of I_3 the total current I is given by $I_1 + I_2 + I_3 = OA + AB + BC$.

Additional Solved Problems

PROBLEM 8.1

For the circuit shown in Fig. 8.31, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and the Q-factor of the circuit.

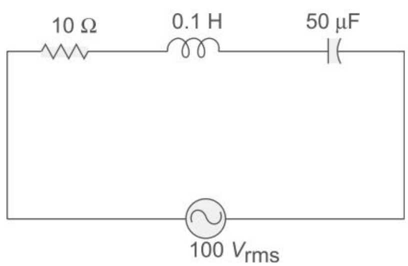


Fig. 8.31

Solution The frequency of resonance occurs when $X_L = X_C$

$$\omega L = \frac{1}{\omega C}$$

$$\therefore \quad \omega = \frac{1}{\sqrt{LC}} \text{ rad/sec}$$

$$= \frac{1}{\sqrt{0.1 \times 50 \times 10^{-6}}} = 447.2 \text{ rad/sec}$$

$$f_r = \frac{1}{2\pi}(447.2) = 71.17 \text{ Hz}$$

The current passing through the circuit at resonance,

$$I = \frac{V}{R} = \frac{100}{10} = 10 \text{ A}$$

The voltage drop across the inductor

$$V_L = IX_L = I\omega L$$

$$= 10 \times 447.2 \times 0.1 = 447.2 \text{ V}$$

The quality factor $\quad Q = \dfrac{\omega L}{R} = \dfrac{447.2 \times 0.1}{10} = 4.47$

PROBLEM 8.2

A series RLC circuit has a quality factor of 5 at 50 rad/s. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. The total impedance of the circuit is 20 Ω. Find the circuit constants.

Solution The quality factor $Q = 5$

At resonance, the impedance becomes resistance.

The current at resonance is $I = \dfrac{V}{R}$

$$10 = \frac{100}{R}$$

$$\therefore \quad R = 10\ \Omega$$

$$Q = \frac{\omega L}{R}$$

Since $Q = 5, R = 10$

$$\omega L = 50$$

$$\therefore \quad L = \frac{50}{\omega} = 1\ \text{H}$$

$$Q = \frac{1}{\omega CR}$$

$$\therefore \quad C = \frac{1}{Q\ \ R}$$

$$= \frac{1}{5 \times 50 \times 10}$$

$$C = 400\ \mu\text{F}$$

PROBLEM 8.3

A voltage v(t) = 10 sin ωt is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 500 V. Moreover, the bandwidth is known to be 400 rad/s and the impedance at resonance is 100 Ω. Find the resonant frequency. Also find the values of L and C of the circuit.

Solution The applied voltage to the circuit is

$$V_{\text{max}} = 10\ \text{V}$$

$$V_{\text{rms}} = \frac{10}{\sqrt{2}} = 7.07\ \text{V}$$

The voltage across capacitor $V_C = 500$ V

The magnification factor $Q = \dfrac{V_C}{V} = \dfrac{500}{7.07} = 70.7$

The bandwidth $BW = 400$ rad/s

$$\omega_2 - \omega_1 = 400\ \text{rad/s}$$

The impedance at resonance $Z = R = 100\ \Omega$

$$\text{Since} \quad Q = \frac{\omega_r}{\omega_2 - \omega_1}$$

$$\omega_r = Q(\omega_2 - \omega_1) = 28280 \text{ rad/s}$$

$$f_r = \frac{28280}{2\pi} = 4499 \text{ Hz}$$

$$\text{The bandwidth} \quad \omega_2 - \omega_1 = \frac{R}{L}$$

$$\therefore \quad L = \frac{R}{\omega_2 - \omega_1} = \frac{100}{400} = 0.25 \text{ H}$$

$$\text{Since} \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$C = \frac{1}{(2\pi f_r)^2 \times L} = \frac{1}{2\pi \times (4499)^2 \times 0.25} = 5\ n\text{F}$$

PROBLEM 8.4

Find the value of L at which the circuit resonates at a frequency of 1000 rad/s in the circuit shown in Fig. 8.32.

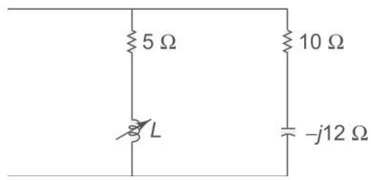


Fig. 8.32

Solution $$Y = \frac{1}{10 - j12} + \frac{1}{5 + jX_L}$$

$$Y = \frac{10 + j12}{10^2 + 12^2} + \frac{5 - jX_L}{25 + X_L^2}$$

$$= \frac{10}{10^2 + 12^2} + \frac{5}{25 + X_L^2} + j\left[\frac{12}{10^2 + 12^2} - \frac{X_L}{25 + X_L^2}\right]$$

At resonance, the susceptance becomes zero.

$$\text{Then} \quad \frac{X_L}{25 + X_L^2} = \frac{12}{10^2 + 12^2}$$

$$12X_L^2 - 244\,X_L + 300 = 0$$

From the above equation,

$$X_L^2 - 20.3\,X_L + 25 = 0$$

$$X_L = \frac{+20.3 \pm \sqrt{(20.3)^2 - 4 \times 25}}{2}$$

$$= \frac{20.3 + \sqrt{412 - 100}}{2} \quad \text{or} \quad \frac{20.3 - \sqrt{412 - 100}}{2}$$

$$= 18.98\ \Omega \text{ or } 1.32\ \Omega$$

$$\therefore \quad X_L = \omega L = 18.98 \text{ or } 1.32\ \Omega$$

$$L = \frac{18.98}{1000} \quad \text{or} \quad \frac{1.32}{1000}$$

$$L = 18.98 \text{ mH or } 1.32 \text{ mH}$$

PROBLEM 8.5

Two impedances $Z_1 = 20 + j10$ and $Z_2 = 10 - j30$ are connected in parallel and this combination is connected in series with $Z_3 = 30 + jX$. Find the value of X which will produce resonance.

Solution Total impedance is

$$Z = Z_3 + (Z_1 \parallel Z_2)$$

$$= (30 + jX) + \left\{\frac{(20 + j10)(10 - j30)}{20 + j10 + 10 - j30}\right\}$$

$$= (30 + jX) + \frac{200 - j600 + j100 + 300}{30 - j20}$$

$$= 30 + jX + \left(\frac{500 - j500}{30 - j20}\right)$$

$$= 30 + jX + \left[\frac{500(1 - j)(30 + j20)}{(30)^2 + (20)^2}\right]$$

$$= (30 + jX) + \left[\frac{500(30 + j20 - j30 + 20)}{900 + 400}\right]$$

$$= 30 + jX + \frac{5}{13}(50 - j10)$$

$$= \left(30 + \frac{5}{13} \times 50\right) + j\left(X - \frac{5}{13} \times 10\right)$$

At resonance, the imaginary part is zero

$$\therefore \quad X - \frac{50}{13} = 0$$

$$X = \frac{50}{13} = 3.85\ \Omega$$

PROBLEM 8.6

A 50 Ω resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 8.33. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.

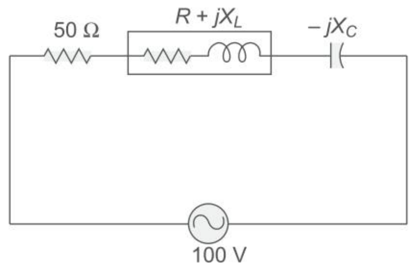


Fig. 8.33

Solution At resonance, current in the circuit is maximum.

$$I = 0.7 \text{ A}$$

Voltage across capacitor is $V_C = IX_C$

Since $V_C = 200, I = 0.7$

$$X_C = \frac{1}{\omega C}$$

$$\omega C = \frac{0.7}{200}$$

$$\therefore \quad C = \frac{0.7}{200 \times 2\pi \times 200} = 2.785\ \mu\text{F}$$

At resonance,

$$X_L - X_C = 0$$

$$\therefore \quad X_L = X_C$$

Since $X_C = \dfrac{1}{\omega C} = \dfrac{200}{0.7} = 285.7\ \Omega$

$$X_L = \omega L = 285.7\ \Omega$$

$$\therefore \quad L = \frac{285.7}{2\pi \times 200} = 0.23\ \text{H}$$

At resonance, the total impedance

$$Z = R + 50$$

$$\therefore \quad R + 50 = \frac{V}{I} = \frac{100}{0.7}$$

$$R + 50 = 142.86\ \Omega$$

$$\therefore \quad R = 92.86\ \Omega$$

PROBLEM 8.7

In the circuit shown in Fig. 8.34, a maximum current of 0.1 A flows through the circuit when the capacitor is at 5 μF with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, the quality factor Q and the value of resistance at resonant frequency.

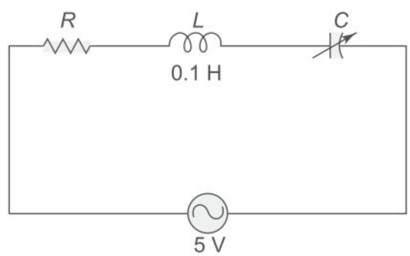


Fig. 8.34

Solution At resonance, the current is maximum in the circuits.

$$I = \frac{V}{R}$$

$$\therefore \quad R = \frac{V}{I} = \frac{5}{0.1} = 50\ \Omega$$

The resonant frequency is

$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{0.1 \times 5 \times 10^{-6}}} = 1414.2\ \text{rad/sec}$$

$$f_r = \frac{1414.2}{2\pi} = 225\ \text{Hz}$$

The quality factor is $Q = \frac{\omega L}{R} = \frac{1414.2 \times 0.1}{50} = 28$

Since $\frac{f_r}{BW} = Q$

The bandwidth $BW = \frac{f_r}{Q} = \frac{225}{2.8} = 80.36$ Hz

PROBLEM 8.8

In the circuit shown in Fig. 8.35, determine the circuit constants when the circuit draws a maximum current at 10 μF with a 10 V, 100 Hz supply. When the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/sec. Also find the maximum current that flows through the circuit.

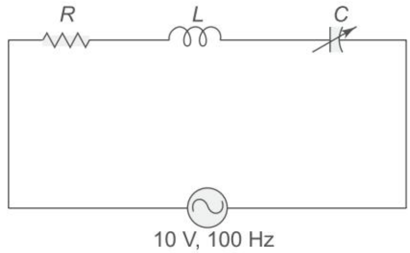


Fig. 8.35

Solution At resonant frequency, the circuit draws maximum current. So, the resonant frequency $f_r = 100$ Hz

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{C \times (2\pi f_r)^2}$$

$$= \frac{1}{10 \times 10^{-6}\,(2\pi \times 100)^2} = 0.25 \text{ H}$$

We have $\omega L - \frac{1}{\omega C} = R$

$$900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}} = R$$

$$\therefore \quad R = 132.4\ \Omega$$

The quality factor $Q = \frac{\omega L}{R} = \frac{900 \times 0.25}{132.4} = 1.69$

The maximum current in the circuit is $I = \frac{10}{132.4} = 0.075$ A

PROBLEM 8.9

In the circuit shown in Fig. 8.36, the current is at its maximum value with capacitor value C = 20 μF and 0.707 times its maximum value with C = 30 μF. Find the value of Q at ω = 500 rad/s, and circuit constants.

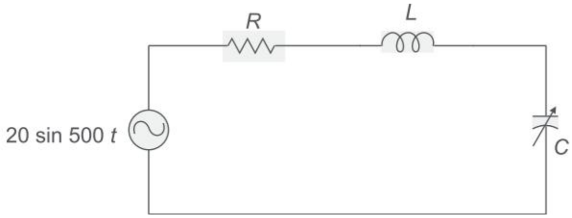


Fig. 8.36

Solution The voltage applied to the circuit is $V = 20$ V. At resonance, the current in the circuit is maximum. The resonant frequency $\omega_r = 500$ rad/s.

$$\text{Since} \quad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad X_L = \frac{1}{\omega_r^2 C} = \frac{1}{(500)^2 \times 20 \times 10^{-6}} = 0.2 \text{ H}$$

$$\text{Since we have} \quad \omega L - \frac{1}{\omega C} = R$$

$$500 \times 0.2 - \frac{1}{500 \times 30 \times 10^{-6}} = R$$

$$\therefore \quad R = 100 - 66.6 = 33.4$$

$$\text{The quality factor is} \quad Q = \frac{\omega L}{R} = \frac{500 \times 0.2}{33.4} = 2.99$$

PROBLEM 8.10

In the circuit shown in Fig. 8.37, an inductance of 0.1 H having a Q of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at resonant frequency of 500 rad/s.

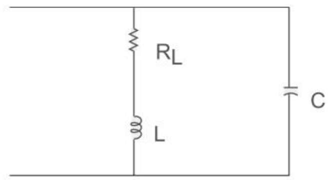


Fig. 8.37

Solution The quality factor $Q = \dfrac{\omega_r L}{R}$

$$\text{Since} \quad L = 0.1 \text{ H}, Q = 5 \text{ and } \omega_r = 500 \text{ rad/s}$$

$$Q = \frac{500 \times 0.1}{R}$$

$$\therefore \quad R = \frac{500 \times 0.1}{5} = 10\ \Omega$$

$$\text{Since} \quad \omega_r^2 = \frac{1}{LC}$$

$$(500)^2 = \frac{1}{0.1 \times C}$$

$$\therefore \text{The capacitance value } C = \frac{1}{0.1 \times (500)^2} = 40\ \mu\text{F}$$

PROBLEM 8.11

A series RLC circuit consists of a 50 Ω resistance, 0.2 H inductance, and 10 μF capacitor with an applied voltage of 20 V. Determine the resonant frequency. Find the Q-factor of the circuit. Compute the lower and upper frequency limits and also find the bandwidth of the circuit.

Solution Resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}} = 112.5 \text{ Hz}$$

$$\text{Quality factor} \quad Q = \frac{\omega L}{R} = \frac{2\pi \times 112.5 \times 0.2}{50} = 2.83$$

Lower frequency limit

$$f_1 = f_r - \frac{R}{4\pi L} = 112.5 - \frac{50}{4 \times \pi \times 0.2} = 92.6 \text{ Hz}$$

Upper frequency limit

$$f_2 = f_r + \frac{R}{4\pi L} = 112.5 + \frac{50}{4\pi \times 0.2} = 112.5 + 19.89 = 132.39 \text{ Hz}$$

Bandwidth of the circuit

$$BW = f_2 - f_1 = 132.39 - 92.6 = 39.79 \text{ Hz}$$

PROBLEM 8.12

A tumed circuit consists of a coil having an inductance of 200 μH and a resistance of 15 Ω in parallel with a series combination of a variable capacitance and resistor of 80 Ω. It is supplied by a 60 V source. If the supply frequency is 1 MHz. what is the value of C to give resonance?

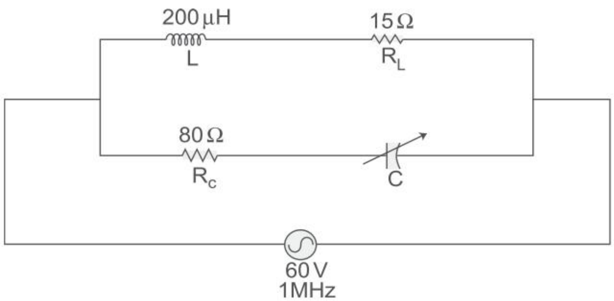


Fig. 8.38

Solution Total admittance of the circuit $y = \dfrac{1}{R_L + j\omega L} + \dfrac{1}{R_c - \dfrac{j}{\omega c}}$

Rationalising the above equation,

$$y = \frac{R_L - j\omega L}{{R_L}^2 + \omega^2 L^2} + \frac{R_c + j/\omega c}{R_c^2 + \dfrac{1}{\omega^2 c^2}}$$

Separating the real and imaginary parts,

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_c}{R_c^2 + \dfrac{1}{\omega^2 c^2}} + j\left[\frac{1/\omega c}{R_c^2 + \dfrac{1}{\omega^2 c^2}} - \frac{\omega L}{R_L^2 + \omega^2 L}\right]$$

At resonance, susceptance part $= 0$

$$\omega = \omega_r$$

Hence $$\frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\dfrac{1}{\omega_r C}}{R_C^2 + \dfrac{1}{\omega_r^2 C^2}}$$

$$\omega_r L\left[R_C^2+\frac{1}{\omega_r^2 c^2}\right]=\frac{1}{\omega_r c}\left[R_L^2+\omega_r^2 L^2\right]$$

$$\omega_r^2\left[R_C^2+\frac{1}{\omega_r^2 c^2}\right]=\frac{1}{LC}\left[R_L^2+\omega_r^2 L^2\right]$$

$$\omega_r=\frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2-L/C}{R_C^2-L/C}}$$

$$\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2-L/C}{R_C^2-L/C}}\quad\left(\because\quad \omega_r=\frac{1}{\sqrt{LC}}\right)$$

$$R_C^2-\frac{L}{C}=R_L^2-\frac{L}{C}$$

$$R_C=R_L$$

$$(15+j1256)\cdot\left(80-j\frac{1}{2\pi C\times10^6}\right)$$

Imaginary part = 0 at resonance

$$1256\times80=\frac{15}{2\pi C\times10^6}$$

$$C=23.76\text{ PF}$$

PROBLEM 8.13

For the parallel circuit shown in Fig. 8.39; $V = 200$ V; $R_2 = 50\ \Omega$; $X_1 = 25\ \Omega$. R_1 is varied from $10\ \Omega$ to $50\ \Omega$, draw the locus diagram. Find maximum and minimum values of source current.

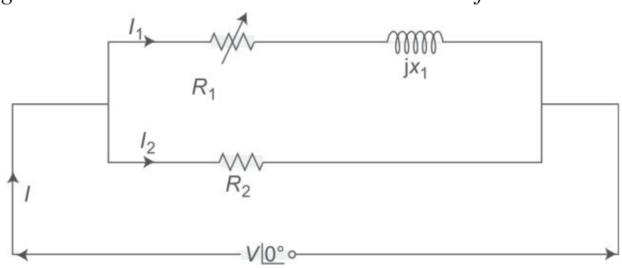


Fig. 8.39

Solution $I=I_1+I_2\ ;\ I_2=\dfrac{V}{R_2}\ ;\ I_1=\dfrac{V}{\sqrt{R_1^2+X_1^2}\,\angle\tan^{-1}\left(\dfrac{X_1}{R_1}\right)}$

$$I_2=\frac{200\angle0°}{50}=4\text{ A}$$

When $R_1=10\ \Omega\ ;\ I_1=\dfrac{200\angle0°}{\sqrt{100+625}\,\angle\tan^{-1}\left(\dfrac{25}{10}\right)}=7.42\angle-68.2°$

When $R_l = 50\ \Omega\ ;\ I_1 = \dfrac{200\underline{|0^\circ}}{\sqrt{2500+625}\underline{\left|\tan^{-1}\left(\dfrac{25}{50}\right)\right.}} = 3.57\underline{|-26.6^\circ}$

$$\begin{aligned}\text{Maximum value of}\quad I &= 4+7.42\underline{|-68.2^\circ}\\ &= 4+2.76-j6.9\\ &= 6.76-j6.9 = 9.66\underline{|-45.58^\circ}\end{aligned}$$

$$\begin{aligned}\text{Minimum value of}\quad I &= 4+3.57\,\underline{|-26.6^\circ}\\ &= 4+3.192-j1.6\\ &= 7.192-j1.6 = 7.36\underline{|-12.54^\circ}\end{aligned}$$

The locus diagram is shown in Fig. 8.40.

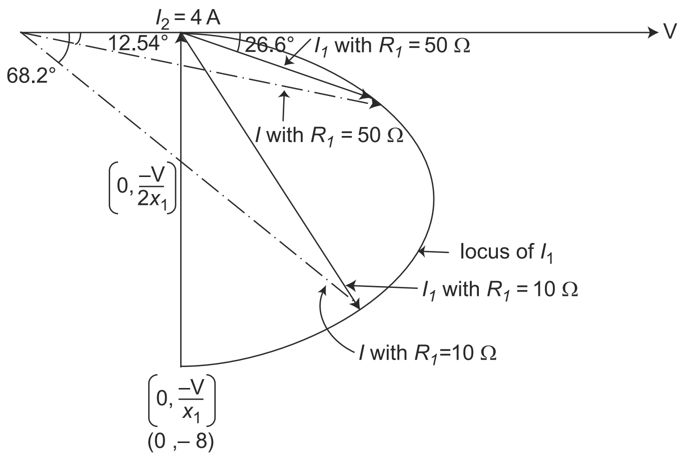


Fig. 8.40

PROBLEM 8.14

A series circuit has a resonance frequency of 120 kHz, a bandwidth of 50 kHz, and Q=2. Determine the cut-off frequencies.

Solution Since $Q < 10$, we cannot use the formulas $f_2 = f_r + \dfrac{B\cdot W}{2}$

$$f_1 = f_r - \frac{B\cdot W}{2}$$

$$B\cdot W = f_2 - f_1$$

$$50{,}000 = f_2 - f_1$$

$$f_2 = 50{,}000 + f_1$$

$$\text{also}\quad f_r = \sqrt{f_1 f_2}$$

$$1,20,000 = \sqrt{f_1 f_2}$$

$$14.4 \times 10^9 = f_1 f_2$$

$$14.4 \times 10^9 = f_1 (50,000 + f_1)$$

$$f_1^2 + 50,000 f_1 - 14.4 \times 10^9 = 0$$

$$f_1 = \frac{-50,000 \pm \sqrt{(50,000)^2 + 4(14.4 \times 10^9)}}{2}$$

$$= \frac{-50,000 \pm 24.5 \times 10^4}{2}$$

$$f_1 = -147.5 \text{ kHz} ; 195 \text{ kHz}$$

Since $-$ve frequency does not exist,

$$f_1 = 195 \text{ kHz}$$

$$50 = f_2 - 195 \Rightarrow f_2 = 245 \text{ kHz}$$

PROBLEM 8.15

A series RLC circuit is supplied at 220 V : 50 Hz. At resonance, the voltage across the capacitor = 550 V, I = 1 A. Determine R, L, and C.

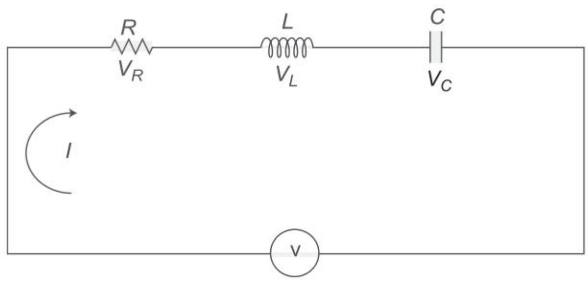


Fig. 8.41

Solution At resonance, $X_L = X_c$

Current at resonance,

$$I = \frac{V}{R + j(X_L - X_C)} = \frac{V}{R} = \frac{220}{R}$$

$$\therefore \quad I = \frac{220}{R} \Rightarrow R = 220\ \Omega$$

$$V_C = I_o X_C$$

$$550 = 1 \cdot \frac{1}{\omega_o C}$$

$$C = \frac{1}{550 \times 2\pi \times 50} \cdot 5.78\ \mu\text{F}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

$$LC = \left(\frac{1}{2\pi f_o}\right)^2 \Rightarrow L = \frac{1}{C}\left(\frac{1}{2\pi f_o}\right)^2$$

$$L = \frac{1}{5.78 \times 10^{-6}}\left(\frac{1}{100\pi}\right)^2 = 1.75 \text{ H}$$

Hence, the elements are $R = 220\ \Omega$

$L = 1.75$ H

$C = 5.78\ \mu$F

PSpice Problems

PROBLEM 8.1

Using PSpice, for the circuit shown in Fig 8.42, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and Q-factor of the circuit.

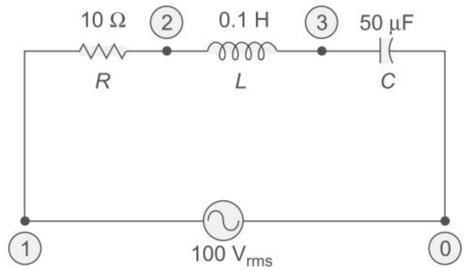


Fig. 8.42

```
*    TO OBTAIN RESONANT FREQUENCY
VS   1    0    AC    100    0
R    1    2    10
L    2    3    0.1
C    3    0    50 U
.AC LIN 1000 1 100
.PROBE I(L)
.PLOT AC VM(L) VP(L)
.END
*** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*************************************************************
LEGEND:
* : VM(L)
+: VP(L)
FREQ        VM(L)
(*)------  1.0000E–02  1.0000E+00  1.0000E+02  1.0000E+04  1.0000E+06
(+)------  0.0000E+00  5.0000E+01  1.0000E+02  1.5000E+02  2.0000E+02
          _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _
1.000E + 00  1.974E – 02 .   *    .    .    .    +    .
1.099E + 00  2.385E – 02 .   *    .    .    .    +    .
1.198E + 00  2.835E – 02 .   *    .    .    .    +    .
1.297E + 00  3.323E – 02 .   *    .    .    .    +    .
1.396E + 00  3.850E – 02 .   *    .    .    .    +    .
1.496E + 00  4.417E – 02 .   *    .    .    .    +    .
1.595E + 00  5.022E – 02 .   *    .    .    .    +    .
1.694E + 00  5.666E – 02 .   *    .    .    .    +    .
1.793E + 00  6.348E – 02 .   *    .    .    .    +    .
9.960E + 01  1.943E + 02 .    +   .    . *  .    .
9.970E + 01  1.939E + 02 .    +   .    . *  .    .
```

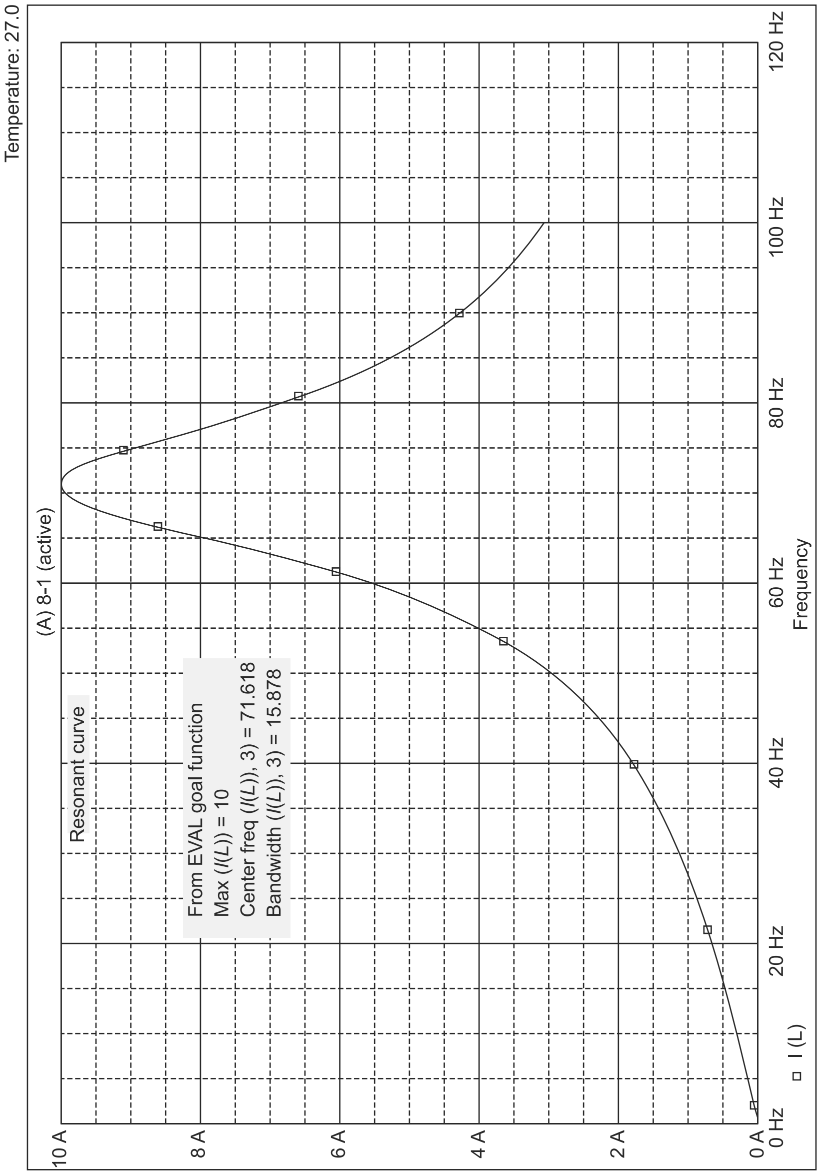


Fig. 8.43

```
9.980E + 01   1.936E + 02 .     +        .         . *        .        .
9.990E + 01   1.932E + 02 .     +        .         . *        .        .
1.000E + 02   1.929E + 02 .     +        .         . *        .        .
1.001E + 02   1.926E + 02 .     +        .         . *        .        .
              - - - - - - - - - - - - - - - - - - - - - - - - -
```

JOB CONCLUDED

Result

Q- Factor = ω_r/BW = 71.618/15.878 = 4.51, Refer Fig. 8.43.

PROBLEM 8.2

For circuit shown in Fig. 8.44, the applied voltage is $v(t) = 15 \sin 1800t$. Determine the resonant frequency. Calculate the quality factor and bandwidth. Compute the lower and upper limits of the bandwidth.

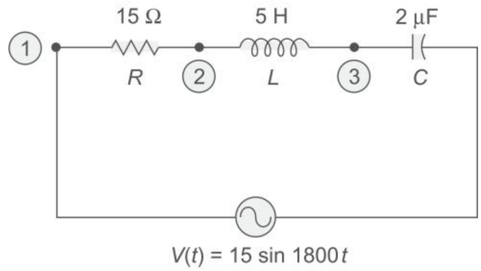


Fig. 8.44

$$f = \frac{\omega}{2\pi} = \frac{1800}{2\pi}$$

$$V_{\text{rms}} = \frac{15}{\sqrt{2}} = 10.607 \text{ V}$$

```
*
VS   1    0    AC 10.607 0
R    1    2    15
L    2    3    5
C    3    0    2U
.AC LIN 100 40 60
.PROBE I(VS)
.PLOT AC IM(VS) IP(VS)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
******************************************************************
LEGEND:
*: IM(VS)
+: IP(VS)
      FREQ    IM(VS)
(*)-----   1.0000E-02    1.0000E-01  1.0000E+00  1.0000E+01  1.0000E+02
(+)----- -2.0000E+02  -1.0000E+02 0.0000E+00  1.0000E+02  2.0000E+02
      - - - - - - - - - - - - - - - - - - - - - - - - - - - -
4.000E + 01   1.447E - 02 . *    . +       .        .        .
4.020E + 01   1.480E - 02 . *    . +       .        .        .
4.040E + 01   1.514E - 02 . *    . +       .        .        .
4.061E + 01   1.550E - 02 . *    . +       .        .        .
4.869E + 01   1.001E - 01 .        +*      .        .        .
4.889E + 01   1.140E - 01 .        +*      .        .        .
4.909E + 01   1.323E - 01 .        + *     .        .        .
4.929E + 01   1.572E - 01 .        + *     .        .        .
4.950E + 01   1.930E - 01 .        +. *    .        .        .
4.970E + 01   2.484E - 01 .        +.  *   .        .        .
4.990E + 01   3.420E - 01 .       + .   *  .        .        .
5.010E + 01   5.106E - 01 .    +    .    * .        .        .
5.030E + 01   7.029E - 01 . +       .     *.        .        .
5.051E + 01   5.697E - 01 .         .    * .        .   +    .
5.071E + 01   3.787E - 01 .         .    * .        . +      .
```

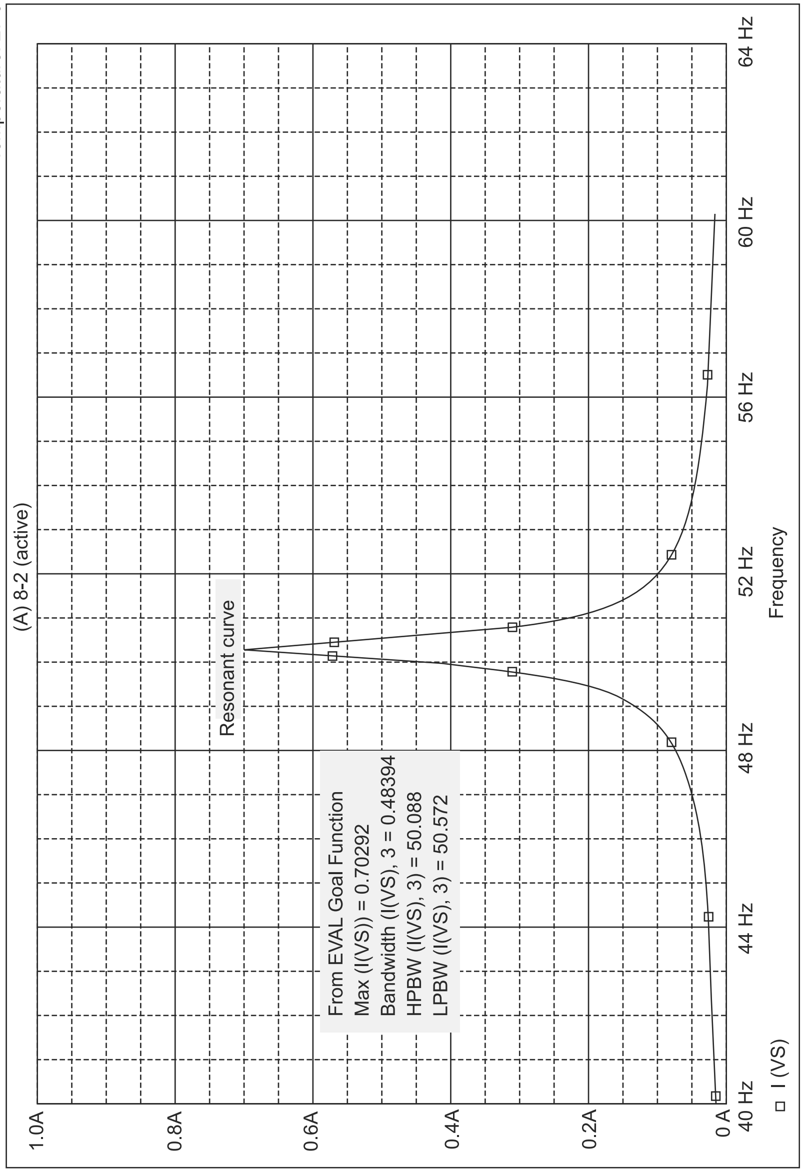


Fig. 8.45

```
5.091E + 01   2.705E – 01 .          .  *   .            . +        .
5.111E + 01   2.079E – 01 .          .  *   .            .+         .
5.131E + 01   1.683E – 01 .          . *    .            +          .
5.152E + 01   1.411E – 01 .          .*     .            +          .
5.172E + 01   1.215E – 01 .          .*     .            +          .
5.192E + 01   1.066E – 01 .          .*     .            +          .
5.212E + 01   9.498E – 02 .          *      .            +          .
```

Result

Q-FACTOR = 105.35, Refer Fig. 8.45.

Answers to Practice Problems

8-3.1 875.35 Hz; 914.42 Hz; 836.28 Hz; 0.2 H; 0.165 μF

8-3.2 50.3 Hz; 63.2 V; 3 (approx.)

8-3.6 $Q = 1$; $R = 60\ \Omega$; $C = 50\ \mu F$

8-3.7 100 V

8-3.8

$$f_c = \frac{f_r}{Q}\left[\sqrt{\frac{L}{CR^2} - \frac{1}{2}}\right]$$

where $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$; and $f_r = \frac{1}{2\pi\sqrt{LC}}$

8-3.10 (i) 100 V (ii) 1002.5 Hz

8-5.1 2.07 Ω

8-5.3 1.77

8-5.6 (i) $fr = 1559.4$ Hz (ii) 50 Ω (iii) 318.31 Hz

Objective-Type Questions

☆☆★**8.1** What is the total reactance of a series *RLC* circuit at resonance?
(a) equal to X_L (b) equal to X_C (c) equal to R (d) zero

☆☆★**8.2** What is the phase angle of a series *RLC* circuit at resonance?
(a) zero (b) 90° (c) 45° (d) 30°

☆☆★**8.3** In a series circuit of $L = 15$ mH, $C = 0.015\ \mu F$, and $R = 80\ \Omega$, what is the impedance at the resonant frequency?
(a) (15 mH) ω (c) 80 Ω (b) (0.015 F) ω (d) $1/(\omega \times (0.015))$

☆☆★**8.4** In a series *RLC* circuit operating below the resonant frequency, the current
(a) I leads V_S (b) I lags behind V_S (c) I is in phase with V_S

☆☆★**8.5** In a series *RLC* circuit, if *C* is increased, what happens to the resonant frequency?
(a) It increases (c) It remains the same (b) It decreases (d) It is zero

☆★★**8.6** In a certain series resonant circuit, $V_C = 150$ V, $V_L = 150$ V, and $V_R = 50$ V. What is the value of the source voltage?
(a) Zero (b) 50 V (c) 150 V (d) 200 V

☆☆★**8.7** A certain series resonant circuit has a bandwidth of 1000 Hz. If the existing coil is replaced by a coil with a lower *Q*, what happens to the bandwidth?
(a) It increases (c) It is zero (b) It decreases (d) It remains the same

☆☆★**8.8** In a parallel resonance circuit, why does the current lag behind the source voltage at frequencies below resonance?
(a) Because the circuit is predominantly resistive (b) Because the circuit is predominantly inductive
(c) Because the circuit is predominantly capacitive (d) None of the above

☆☆★**8.9** In order to tune a parallel resonant circuit to a lower frequency, the capacitance must
(a) be increased (c) be zero (b) be decreased (d) remain the same

☆☆★**8.10** What is the impedance of an ideal parallel resonant circuit without resistance in either branch?
(a) Zero (b) Inductive (c) Capacitive (d) I0nfinite

☆☆★**8.11** If the lower cut-off frequency is 2400 Hz and the upper cut-off frequency is 2800 Hz, what is the bandwidth?
(a) 400 Hz (b) 2800 Hz (c) 2400 Hz (d) 5200 Hz

☆☆★**8.12** What values of *L* and *C* should be used in a tank circuit to obtain a resonant frequency of 8 kHz? The bandwidth must be 800 Hz. The winding resistance of the coil is 10 Ω.
(a) 2 mH, 1 μF (c) 1.99 mH, 0.2 μF (b) 10 H, 0.2 μF (d) 1.99 mH, 10 μF

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/266

CHAPTER 9

Polyphase Circuits

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 List the advantages of a three-phase system

LO 2 Illustrate the generation of three-phase voltages

LO 3 Define phase sequence

LO 4 Describe the interconnection of three-phase sources and loads

LO 5 Discuss star-to-delta and delta-to-star transformations

LO 6 Determine voltage, current and power in a star-connected system

LO 7 Illustrate voltage, current and power in a delta-connected system

LO 8 Analyse three-phase balanced and unbalanced circuits

LO 9 Execute power measurement in three-phase circuits using single, two- and three-wattmeter methods

LO 10 Analyse the effects of harmonics in wye and delta connections

LO 11 Analyse the effects of phase-sequence

LO 12 Calculate power factor of an unbalanced system

9.1 POLYPHASE SYSTEM

In an ac system, it is possible to connect two or more individual circuits to a common polyphase source. Though it is possible to have any number of sources in a polyphase system, the increase in the available power is not significant beyond the three-phase system. The power generated by the same machine increases 41.4 per cent from single phase to two-phase, and the increase in the power is 50 per cent from single phase to three-phase. Beyond three-phase, the maximum possible increase is only seven per cent, but the complications are many. So, an increase beyond three-phase does not justify the extra complications. In view of this, it is only in exceptional cases where more than three phases are used. Circuits supplied by six, twelve, and more phases are used in high power radio transmitter stations. Two-phase systems are used to supply two-phase servo motors in feedback control systems.

In general, a three-phase system of voltages (currents) is merely a combination of three single-phase systems of voltages (currents) of which the three voltages (currents) differ in phase by 120 electrical degrees from each other in a particular sequence. One such three-phase system of sinusoidal voltages is shown in Fig. 9.1.

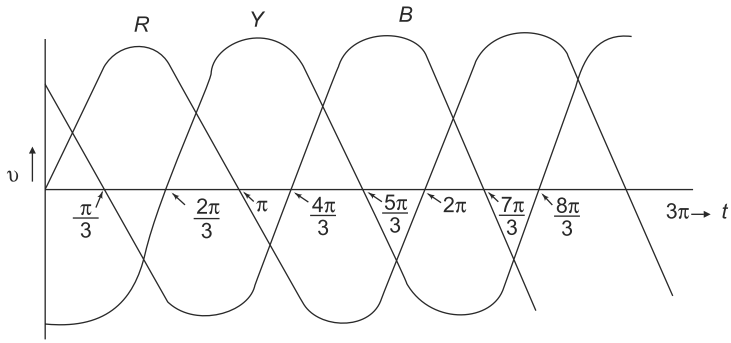


Fig. 9.1

9.2 ADVANTAGES OF A THREE-PHASE SYSTEM

It is observed that the polyphase, especially three-phase, system has many advantages over the single-phase system, both from the utility point of view as well as from the consumer point of view. Some of the advantages are as under.

LO 1 List the advantages of a three-phase system

1. The power in a single-phase circuit is pulsating. When the power factor of the circuit is unity, the power becomes zero 100 times in a second in a 50 Hz supply. Therefore, single-phase motors have a pulsating torque. Although the power supplied by each phase is pulsating, the total three-phase power supplied to a balanced three-phase circuit is constant at every instant of time. Because of this, three-phase motors have an absolutely uniform torque.
2. To transmit a given amount of power over a given length, a three-phase transmission circuit requires less conductor material than a single-phase circuit.
3. In a given frame size, a three-phase motor or a three-phase generator produces more output than its single-phase counterpart.
4. Three-phase motors are more easily started than single-phase motors. Single phase motors are not self-starting, whereas three-phase motors are.

In general, we can conclude that the operating characteristics of a three-phase apparatus are superior than those of a similar single-phase apparatus. All three-phase machines are superior in performance. Their control equipment are smaller, cheaper, lighter in weight and more efficient. Therefore, the study of three-phase circuits is of great importance.

Frequently Asked Questions linked to LO1*

☆☆★**9-1.1** What are the advantages of a three-phase system? **[AU May/June 2013]**

9.3 GENERATION OF THREE-PHASE VOLTAGES

Three-phase voltages can be generated in a stationary armature with a rotating field structure, or in a rotating armature with a stationary field as shown in Fig. 9.2 (a) and (b).

LO 2 Illustrate the generation of three-phase voltages

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

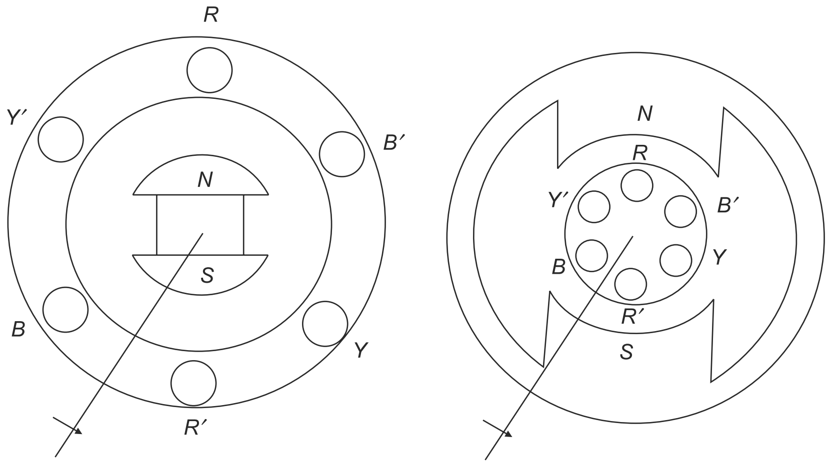


Fig. 9.2

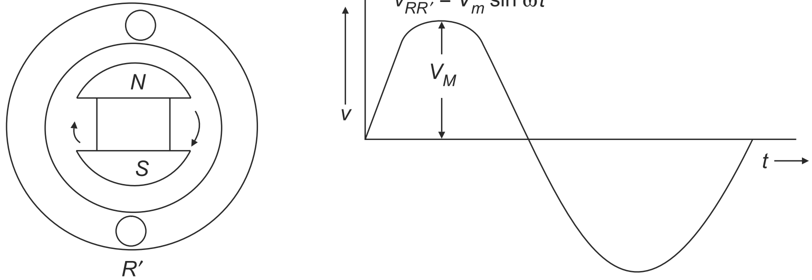
Single-phase voltages and currents are generated by single-phase generators as shown in Fig. 9.3 (a). The armature (here a stationary armature) of such a generator has only one winding, or one set of coils. In a two-phase generator, the armature has two distinct windings, or two sets of coils that are displaced 90° (electrical degrees) apart, so that the generated voltages in the two phases have 90° phase displacement as shown in Fig. 9.3 (b). Similarly, three-phase voltages are generated in three separate but identical sets of windings or coils that are displaced by 120 electrical degrees in the armature, so that the voltages generated in them are 120° apart in time phase. This arrangement is shown in Fig. 9.3 (c). Here, RR' constitutes one coil (R-phase); YY' another coil (Y-phase), and BB' constitutes the third phase (B-phase). The field magnets are assumed in clockwise rotation.

The voltages generated by a three-phase alternator is shown in Fig. 9.3 (d). The three voltages are of the same magnitude and frequency, but are displaced from one another by 120°. Assuming the voltages to be sinusoidal, we can write the equations for the instantaneous values of the voltages of the three phases. Counting the time from the instant when the voltage in phase R is zero.

The equations are

$$v_{RR'} = V_m \sin \omega t$$
$$v_{YY'} = V_m \sin (\omega t - 120°)$$
$$v_{BB'} = V_m \sin (\omega t - 240°)$$

At any given instant, the algebraic sum of the three voltages must be zero.



(a)

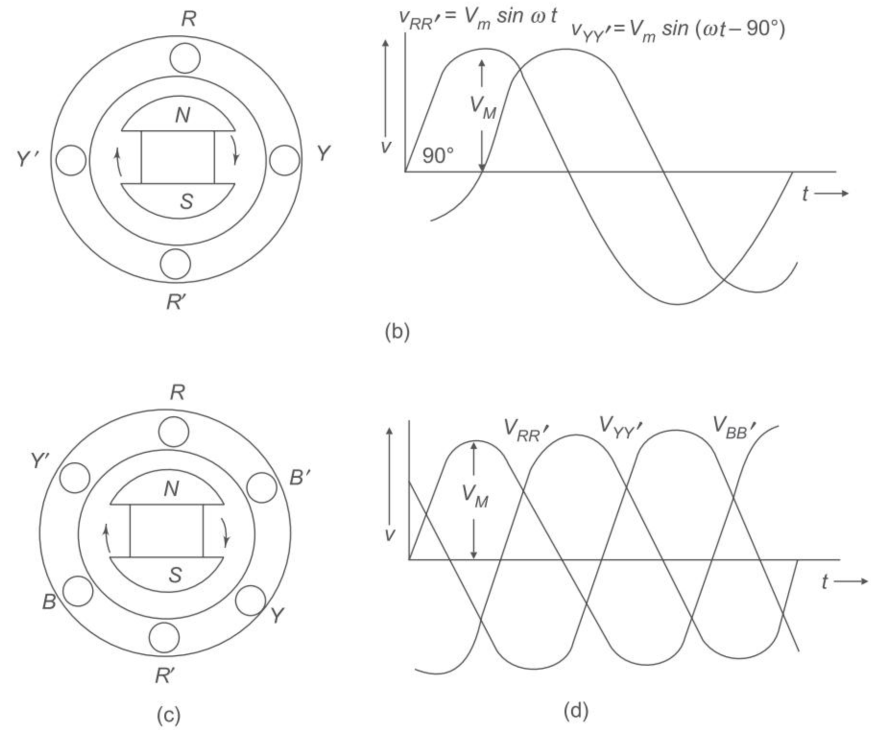


Fig. 9.3

9.4 PHASE SEQUENCE

LO 3 Define phase sequence

*The sequence of voltages in the three phases are in the order $v_{RR'} - v_{YY'} - v_{BB'}$, and they undergo changes one after the other in the above mentioned order. This is called the **phase sequence***. It can be observed that this sequence depends on the rotation of the field. If the field system is rotated in the anticlockwise direction, then the sequence of the voltages in the three-phases are in the order $v_{RR'} - v_{BB'} - v_{YY'}$; briefly we say that the sequence is *RBY*. Now the equations can be written as

$$v_{RR'} = V_m \sin \omega t$$
$$v_{BB'} = V_m \sin (\omega t - 120°)$$
$$v_{YY'} = V_m \sin (\omega t - 240°)$$

EXAMPLE 9.1

What is the phase sequence of the voltages induced in the three coils of an alternator shown in Fig. 9.4? Write the equations for the three voltages.

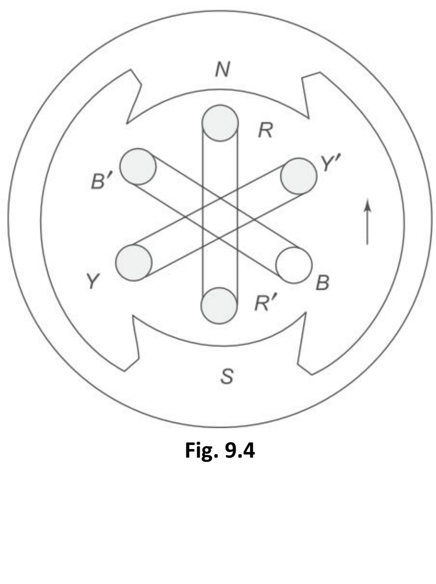


Fig. 9.4

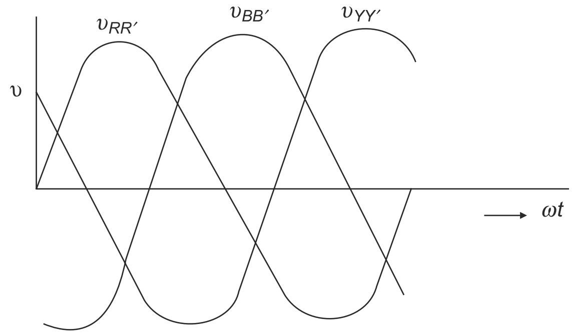


Fig. 9.5

Solution Here, the field system is stationary and the three coils, RR', YY' and BB', are rotating in the anticlockwise direction, so the sequence of voltages is RBY, and the induced voltages are as shown in Fig. 9.4.

$$v_{RR'} = V_m \sin \omega t$$
$$v_{BB'} = V_m \sin (\omega t - 120°)$$
$$v_{YY'} = V_m \sin (\omega t - 240°) \text{ or } V_m \text{ s in } (\omega t + 120°)$$

EXAMPLE 9.2

What is the possible number of phase sequences in Fig. 9.4? What are they?

Solution There are only two possible phase sequences; they are *RBY* and *RYB*.

Frequently Asked Questions linked to LO3

☆☆★**9-3.1** What is phase sequence of a 3-phase system? **[AU May/June 2013]**

9.5 INTERCONNECTION OF THREE-PHASE SOURCES AND LOADS

9.5.1 Interconnection of Three-phase Sources

LO 4 Describe the interconnection of three-phase sources and loads

In a three-phase alternator, there are three independent phase windings or coils. Each phase or coil has two terminals, viz. *start* and *finish*. The end connections of the three sets of the coils may be brought out of the machine, to form three separate single phase sources to feed three individual circuits as shown in Fig. 9.6 (a) and (b).

The coils are interconnected to form a wye (Y) or delta (Δ) connected three-phase system to achieve economy and to reduce the number of conductors, and thereby, the complexity in the circuit. The three-phase sources so obtained serve all the functions of the three separate single phase sources.

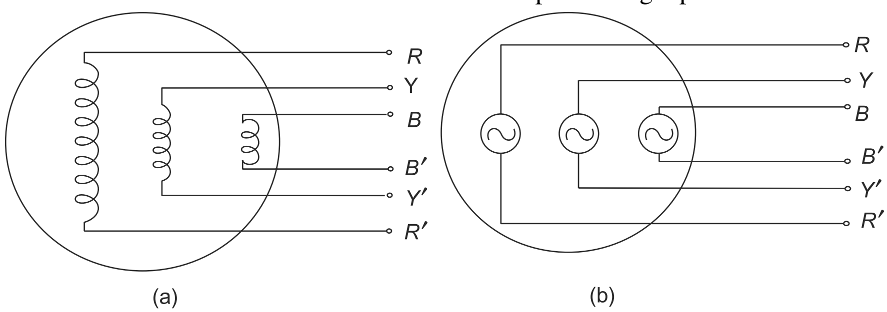


Fig. 9.6

9.5.2 Wye or Star-Connection

In this connection, similar ends (*start* or *finish*) of the three phases are joined together within the alternator as shown in Fig. 9.7. The common terminal so formed is referred to as the neutral point (N), or neutral terminal. Three lines are run from the other free ends (R, Y, B) to feed power to the three-phase load.

Figure 9.7 represents a three-phase, four-wire, star-connected system. The terminals R, Y, and B are called the *line terminals* of the source. The voltage between any line and the neutral point is called the *phase voltage* (V_{RN}, V_{YN}, and V_{BN}), while the voltage between any two lines is called the *line voltage* (V_{RY}, V_{YB}, and V_{BR}). The currents flowing through the phases are called the phase currents, while those flowing in the lines are called the line currents. *If the neutral wire is not available for external connection, the system is called a* ***three-phase, three-wire, star-connected system***. The system so formed will supply equal line voltages displaced 120° from one another and acting simultaneously in the circuit like three independent single phase sources in the same frame of a three-phase alternator.

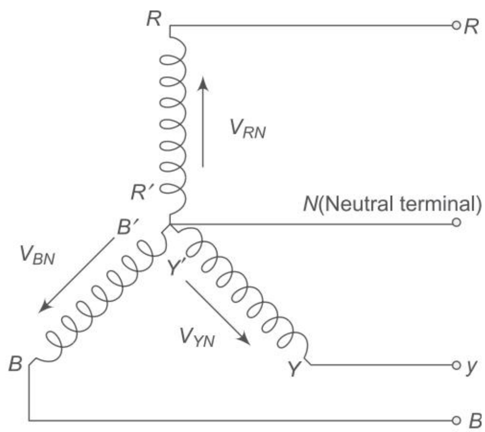


Fig. 9.7

EXAMPLE 9.3

Figure 9.8 represents three phases of an alternator. Arrange the possible number of three-phase star connections and indicate phase voltages and line voltages in each case. ($V_{RR'} = V_{YY'} = V_{BB'}$)

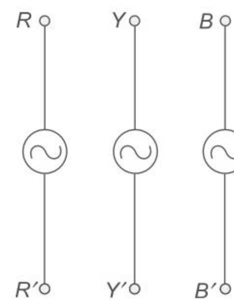


Fig. 9.8

Solution There are two possible star-connections and they can be arranged as shown in Fig. 9.9 (a).

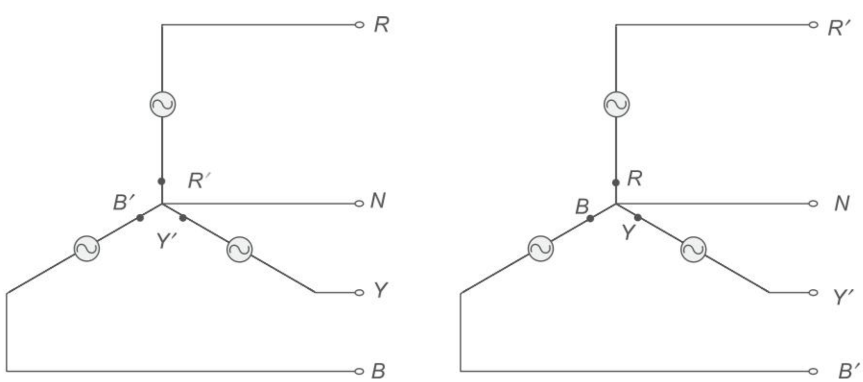
The phase voltages are

$$V_{RN}, V_{YN}, V_{BN} \text{ and } V_{R'N}, V_{Y'N}, V_{B'N}$$

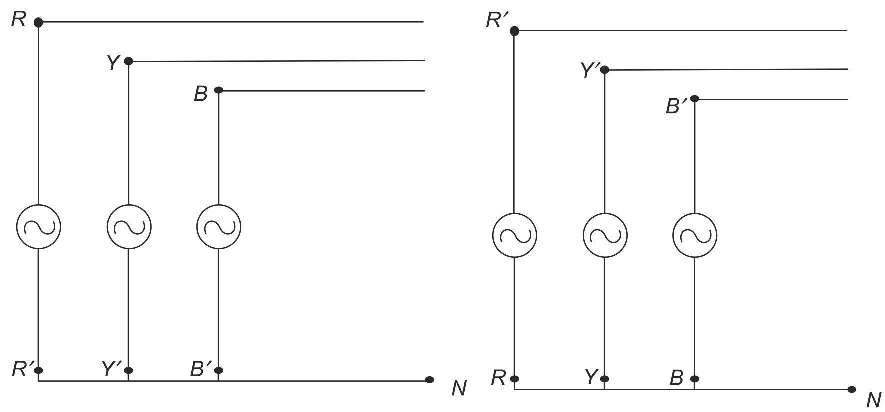
The line voltages are

$$V_{RY}, V_{YB}, V_{BR} \text{ and } V_{R'Y'}, V_{Y'B'}, V_{B'R'}$$

Note The phases can also be arranged as shown in Fig. 9.9 (b), in which case they do not look like a star; so the name star or wye-connection is only a convention.



(a)



(b)

Fig. 9.9

9.5.3 Delta or Mesh-Connection

In this method of connection, the dissimilar ends of the windings are joined together, i.e. R' is connected to Y, Y' to B and B' to R as shown in Fig. 9.10.

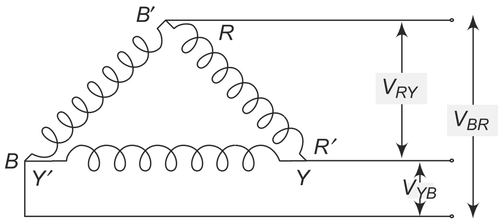


Fig. 9.10

The three line conductors are taken from the three junctions of the mesh or delta connection to feed the three-phase load. This constitutes a three-phase, three-wire, delta-connected system. Here there is no common terminal; only three line voltages V_{RY}, V_{YB} and V_{BR} are available.

These line voltages are also referred to as *phase voltages* in the delta-connected system. When the sources are connected in delta, loads can be connected only across the three line terminals, R, Y, and B. In general, a three-phase source, star or delta, can be either balanced or unbalanced. A balanced three-phase source is one in which the three individual sources have equal magnitude, with 120° phase difference as shown in Fig. 9.3 (d).

EXAMPLE 9.4

Figure 9.11 represents three phases of an alternator. Arrange the possible number of three-phase, delta connections and indicate phase voltages and line voltages in each case (Note $V_{RR'} = V_{YY'} = V_{BB'}$).

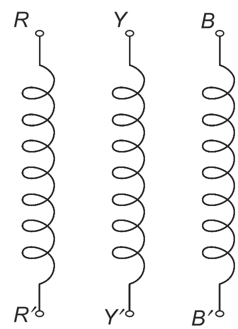


Fig. 9.11

Solution There are two possible delta connections which are shown as follows.

$$V_{\text{phase}} = V_{\text{line}}$$

The line voltages are

V_{RY}, V_{YB}, and V_{BR} from Fig. 9.12 (a) and V_{RB}, V_{BY}, and V_{YR} from Fig. 9.12 (b).

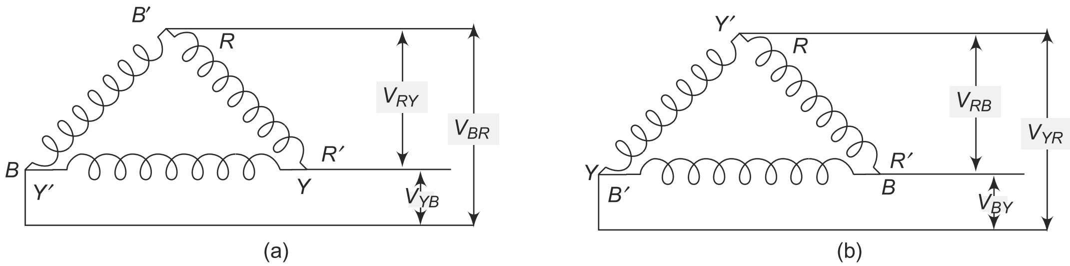


Fig. 9.12

9.5.4 Interconnection of Loads

The primary question in a star or delta-connected three-phase supply is how to apply the load to the three-phase supply. *An impedance, or load, connected across any two terminals of an active network (source) will draw power from the source, and is called a* ***single-phase load***. Like alternator phase windings, a load can also be connected across any two terminals, or between line and neutral terminal (if the source is star-connected). Usually, the three-phase load impedances are connected in star or delta formation, and then connected to the three-phase source as shown in Fig. 9.13.

Figure 9.13 (a) represents the typical interconnections of loads and sources in a three-phase star system, and is of a three-phase four-wire system. A three-phase star- connected load is connected to a three-phase star-connected source, terminal to terminal, and both the neutrals are joined with a fourth wire. Figure 9.13 (b) is a three-phase, three-wire system. A three-phase, delta-connected load is connected to a three-phase star-connected source, terminal to terminal, as shown in Fig. 9.13 (b). When either source or load, or both are connected in delta, only three wires will suffice to connect the load to source.

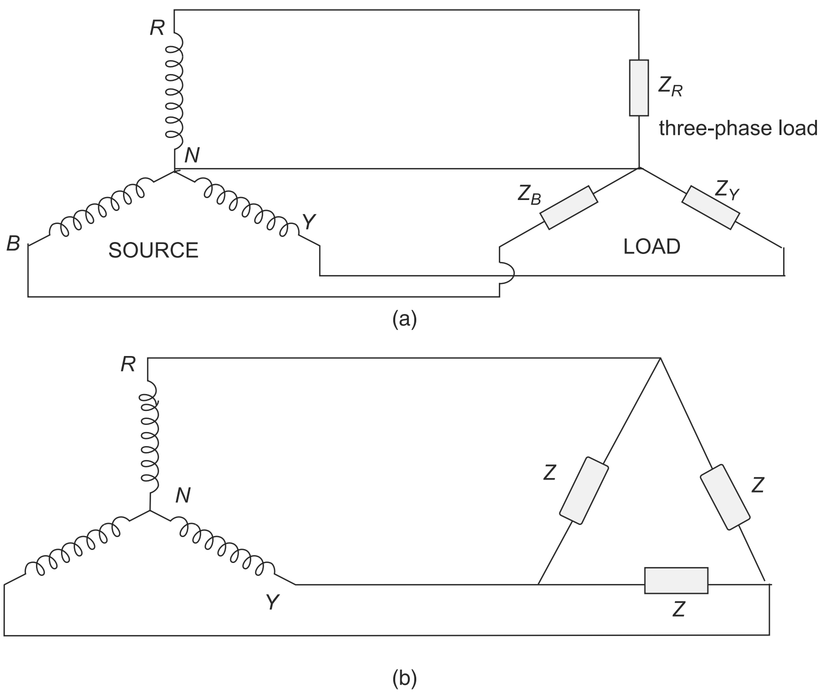


Fig. 9.13

Just as in the case of a three-phase source, a three-phase load can be either balanced or unbalanced. A balanced three-phase load is one in which all the branches have identical impedances, i.e. each impedance has the same magnitude and phase angle. The resistive and reactive components of each phase are equal. Any load which does not satisfy the above requirements is said to be an unbalanced load.

EXAMPLE 9.5

Draw the interconnection between a three-phase, delta-connected source and a star-connected load.

Solution When either source or load, or both are connected in delta, only three wires are required to connect, the load to source, and the system is said to be a three-phase, three-wire system. The connection diagram is shown in Fig. 9.14.

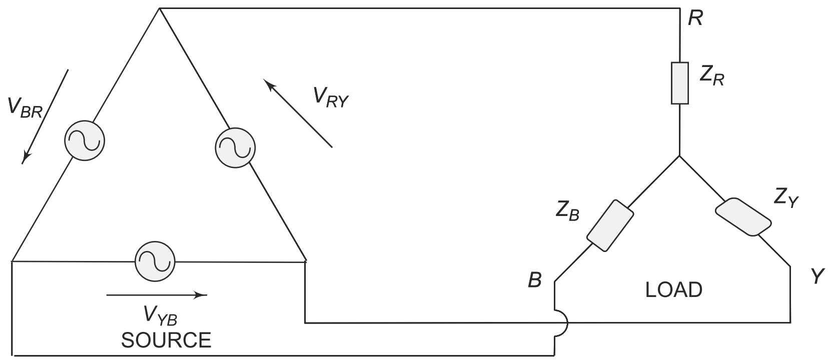


Fig. 9.14

The three line voltages are V_{RY}, V_{YB}, and V_{BR}.

EXAMPLE 9.6

Draw the interconnection between a three-phase, delta-connected source and delta-connected load.

Solution Since the source and load are connected in delta, it is a three-wire system. The connection diagram is shown in Fig. 9.15.

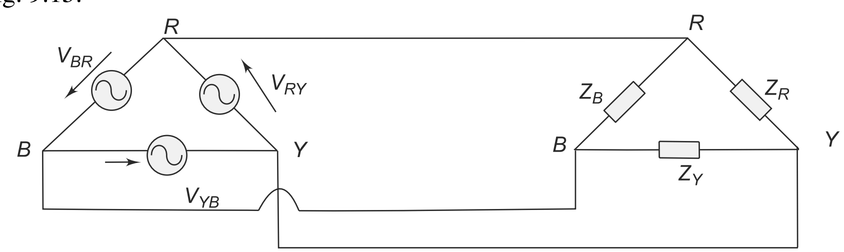


Fig. 9.15

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★**9-4.1** A three-phase, four-wire symmetrical 440 V; *RYB* system supplies a star-connected load in which $Z_R = 10\angle 0°\ \Omega$, $Z_Y = 10\angle 26.8°\ \Omega$ and $Z_B = 10\ \angle -26.8°\ \Omega$. Find the line currents, the neutral current and the load power.

☆☆★**9-4.2** Three impedances of $(7 + j4)\ \Omega$; $(3 + j2)\ \Omega$ and $(9 + j2)\ \Omega$ are connected between neutral and the red, yellow and blue phases, respectively of a three-phase, four-wire system; the line voltage is 440 V. Calculate (a) the current in each line, and (b) the current in the neutral wire.

☆★★**9-4.3** A symmetrical 3-phase, 3-wire, 440 V is connected to a star-connected load. The impedances in each branch are $Z_1 = (2 + j3)\ \Omega$, $Z_2 = (1 - j2)\ \Omega$, $Z_3 = (3 - j4)\ \Omega$. Find its equivalent delta connected load. Hence find the phase and line currents and the total power consumed in the circuit.

☆☆★**9-4.4** Three impedances of $(7 + j4)\ \Omega$; $(3 + j2)\ \Omega$, and $(9 + j2)\ \Omega$ are connected between neutral and *R*,*Y*, and *B* phases. The line voltage of a 3-phase, four-wire system is 440 V; calculate the active power in each phase and the total power drawn by the circuit.

Frequently Asked Questions linked to LO 4

☆☆★**9-4.1** Three inductive coils each having a resistance of 16 Ω and a reactance of j 12 Ω are connected in star across a 400 V, 3 ϕ, 50 Hz supply. Calculate phase voltage. **[AU Nov./Dec. 2012]**

9.6 STAR-TO-DELTA AND DELTA-TO-STAR TRANSFORMATION

LO 5 Discuss star-to-delta and delta-to-star transformations

While dealing with currents and voltages in loads, it is often necessary to convert a star load to delta load, and vice versa. It has already been shown in Chapter 3 that delta (Δ) connection of resistances can be replaced by an equivalent star (Y) connection and vice versa. Similar methods can be applied in the case of networks containing general impedances in complex form. So also with ac, where the same formulae hold good, except that resistances are replaced by the impedances. These formulae can be applied even if the loads are unbalanced. Thus, considering Fig. 9.16 (a), star load can be replaced by an equivalent delta-load with branch impedances as shown.

Delta impedances, in terms of star impedances, are

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$

and $$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

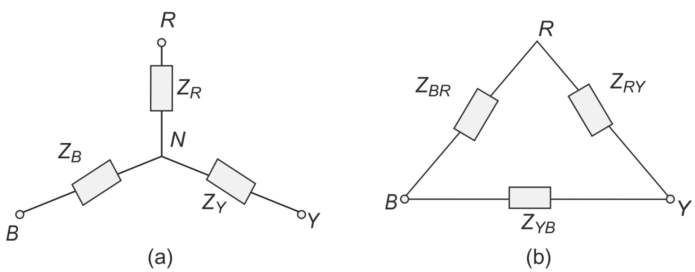


Fig. 9.16

The converted network is shown in Fig. 9.16 (b). Similarly, we can replace the delta load of Fig. 9.16 (b) by an equivalent star load with branch impedances as

$$Z_R = \frac{Z_{RY} Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$Z_Y = \frac{Z_{RY} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

and $$Z_B = \frac{Z_{BR} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

It should be noted that all impedances are to be expressed in their complex form.

EXAMPLE 9.7

A symmetrical three-phase, three-wire 440 V supply is connected to a star-connected load as shown in Fig. 9.17 (a). The impedances in each branch are $Z_R = (2 + j3)\ \Omega$, $Z_Y = (1 - j2)\ \Omega$ and $Z_B = (3 + j4)\ \Omega$. Find its equivalent delta-connected load. The phase sequence is RYB.

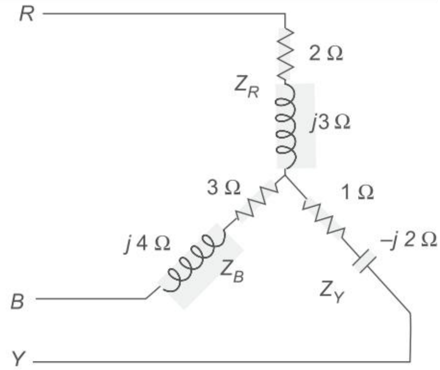


Fig. 9.17 (a)

Solution The equivalent delta network is shown in Fig. 9.17 (b). From Section 9.6, we can write the equations to find Z_{RY}, Z_{YB}, and Z_{BR}. We have

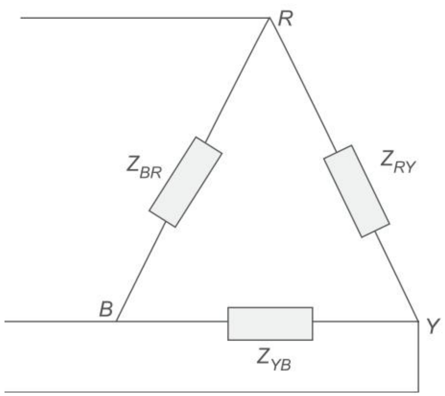


Fig. 9.17 (b)

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$

$$Z_R = 2 + j3 = 3.61 \angle 56.3°$$

$$Z_Y = 1 - j2 = 2.23 \angle -63.4°$$

$$Z_B = 3 + j4 = 5 \angle 53.13°$$

$$Z_R Z_Y + Z_Y Z_B + Z_B Z_R = (3.61 \angle 56.3°)$$
$$(2.23 \angle -63.4°) + (2.23 \angle -63.4°)$$
$$(5 \angle 53.13°) + (5 \angle 53.13°)(3.61 \angle 56.3°)$$
$$= 8.05 \angle -7.1° + 11.15 \angle -10.27° + 18.05 \angle 109.43°$$
$$= 12.95 + j14.04 = 19.10 \angle 47.3°$$

$$Z_{RY} = \frac{19.10 \angle 47.3°}{5 \angle 53.13°} = 3.82 \angle -5.83° = 3.8 - j0.38$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$
$$= \frac{19.10 \angle 47.3°}{3.61 \angle 56.3°} = 5.29 \angle -9° = 5.22 - j0.82$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

$$= \frac{19.10 \angle 47.3°}{2.23 \angle -63.4°} = 8.56 \angle 110.7° = -3.02 + j8$$

The equivalent delta impedances are

$$Z_{RY} = (3.8 - j0.38)\ \Omega$$
$$Z_{YB} = (5.22 - j0.82)\ \Omega$$
$$Z_{BR} = (-3.02 + j8)\ \Omega$$

EXAMPLE 9.8

A symmetrical three-phase, three-wire 400 V, supply is connected to a delta-connected load as shown in Fig. 9.18 (a). Impedances in each branch are $Z_{RY} = 10\angle 30°\ \Omega$; $Z_{YB} = 10\angle -45°\ \Omega$ and $Z_{BR} = 2.5\angle 60°\ \Omega$. Find its equivalent star-connected load; the phase sequence is RYB.

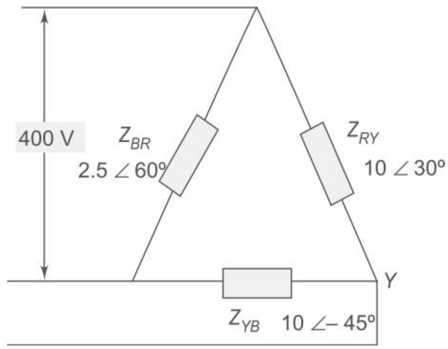


Fig. 9.18 (a)

Solution The equivalent star network is shown in Fig. 9.18 (b). From Section 9.6, we can write the equations to find Z_R, Z_Y, and Z_B as

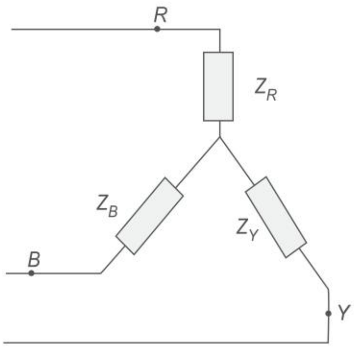


Fig. 9.18 (b)

$$Z_R = \frac{Z_{RY}\,Z_{BR}}{Z_{RY}+Z_{YB}+Z_{BR}}$$

$$Z_{RY}+Z_{YB}+Z_{BR} = 10\angle 30° + 10\angle -45° + 2.5\angle 60°$$

$$= (8.66 + j5) + (7.07 - j7.07) + (1.25 + j2.17)$$

$$= 16.98 + j0.1 = 16.98\angle 0.33°\ \Omega$$

$$Z_R = \frac{(10-30)(2.5-60)}{16.98-0.33} = 1.47\angle 89.67°$$

$$= (0.008 + j1.47)\ \Omega$$

$$Z_Y = \frac{Z_{RY}\ Z_{YB}}{Z_{RY}+Z_{YB}+Z_{BR}}$$

$$= \frac{(10-30)(10-45)}{16.98-0.33} = 5.89\angle -15.33°\ \Omega$$

$$Z_B = \frac{Z_{BR}Z_{YB}}{Z_{RY}+Z_{YB}+Z_{BR}}$$

$$= \frac{(2.5-60)(10-45)}{16.98-0.33} = 1.47\angle 14.67°\ \Omega$$

The equivalent star impedances are

$$Z_R = 1.47\angle 89.67°\ \Omega,\ Z_Y = 5.89\angle -15.33°\ \Omega \text{ and } Z_B = 1.47\angle 14.67°\ \Omega$$

Balanced Star-Delta and Delta-Star Conversion

If the three-phase load is balanced, then the conversion formulae in Section 9.6 get simplified. Consider a balanced star-connected load having an impedance Z_1 in each phase as shown in Fig. 9.19 (a).

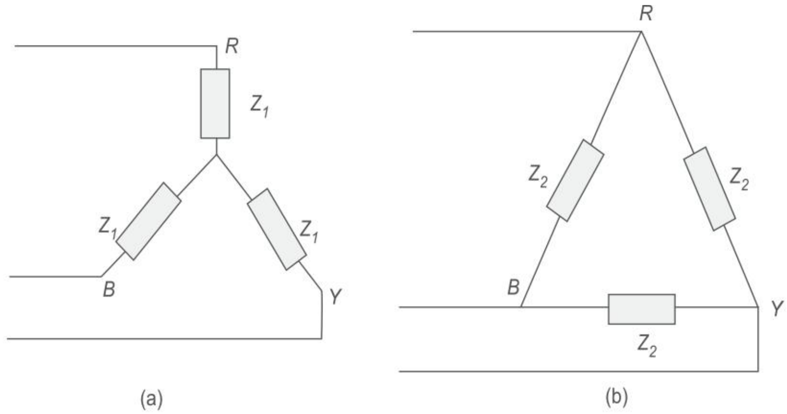


Fig. 9.19

Let the equivalent delta-connected load have an impedance of Z_2 in each phase as shown in Fig. 9.19 (b). Applying the conversion formulae from Section 9.6 for delta impedances in terms of star impedances, we have

$$Z_2 = 3Z_1$$

Similarly, we can express star impedances in terms of delta as $Z_1 = Z_2/3$.

EXAMPLE 9.9

Three identical impedances are connected in delta as shown in Fig. 9.20 (a). Find an equivalent star network such that the line current is the same when connected to the same supply.

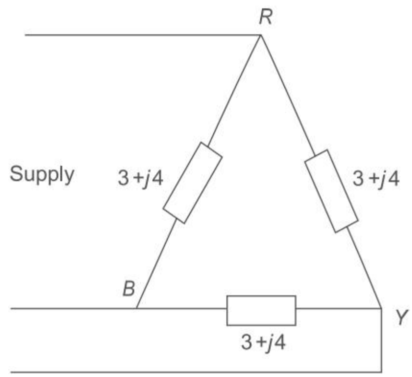


Fig. 9.20 (a)

Solution The equivalent star network is shown in Fig. 9.20 (b). From Section 9.6.1, we can write the equation to find $Z_1 = Z_2/3$

$$Z_2 = 3 + j4$$
$$= 5 \angle 53.13° \, \Omega$$
$$\therefore Z_1 = \frac{5}{3} \angle 53.13°$$
$$= 1.66 \angle 53.15°$$
$$= (1.0 + j1.33) \, \Omega$$

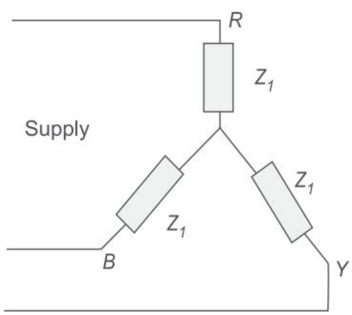


Fig. 9.20 (b)

Frequently Asked Questions linked to LO 5

☆☆★**9-5.1** A symmetrical three-phase; threewire 440 V supply goes to a star-connected load. The impedances in each branch are $Z_R = 2 + j3\ \Omega$, $Z_Y = 1 - j2\ \Omega$ and $Z_B = 3 + j4\Omega$. Find its equivalent delta connected load. **[AU May/June 2014]**

☆☆★**9-5.2** A asymmetrical three-phase, three-wire 400 V supply is connected to a delta-connected load. Impedances in each branch are $Z_{RY} = 10\ \angle 30°\ \Omega$, $Z_{YB} = 10\angle 45°\ \Omega$, and $Z_{BR} = 2.5\angle 60°\ \Omega$. Find its equivalent star-connected load. **[AU May/June 2014]**

9.7 VOLTAGE, CURRENT, AND POWER IN A STAR CONNECTED SYSTEM

9.7.1 Star-Connected System

LO 6 Determine voltage, current and power in a star-connected system

Figure 9.21 shows a balanced three-phase, Y-connected system. *The voltage induced in each winding is called the **phase voltage** (V_{Ph}).* Likewise V_{RN}, V_{YN} , and V_{BN} represent the rms values of the induced voltages in each phase. *The voltage available between any pair of terminals is called the **line voltage** (V_L).* Likewise V_{RY}, V_{YB} and V_{BR} are known as *line voltages*. The double subscript notation is purposefully used to represent voltages and currents in polyphase circuits. Thus, V_{RY} indicates a voltage V between points R and Y, with R being positive with respect to the point Y during its positive half cycle.

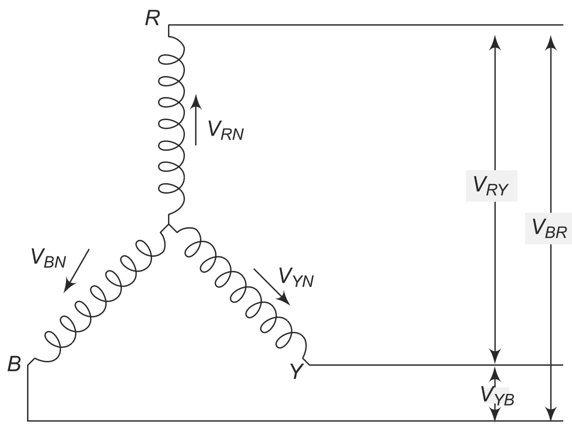


Fig. 9.21

Similarly, V_{YB} means that Y is positive with respect to the point B during its positive half cycle; it also means that $V_{RY} = -V_{YR}$.

❑ ***Voltage Relations*** The phasors corresponding to the phase voltages constituting a three-phase system can be represented by a phasor diagram as shown in Fig. 9.22.

From Fig. 9.22, considering the lines R, Y, and B, the line voltage V_{RY} is equal to the phasor sum of V_{RN} and V_{NY} which is also equal to the phasor difference of V_{RN} and V_{YN} ($V_{NY} = -V_{YN}$). Hence, V_{RY} is found by compounding V_{RN} and V_{YN} reversed. To subtract V_{YN} from V_{RN}, we reverse the phasor V_{YN} and find its phasor sum with V_{RN} as shown in Fig. 9.22. The two phasors, V_{RN} and $-V_{YN}$, are equal in length and are 60° apart.

$$|V_{RN}| = -|V_{YN}| = V_{Ph}$$

$$\therefore \quad V_{RY} = 2V_{Ph}\cos 60/2 = \sqrt{3}\ V_{Ph}$$

Similarly, the line voltage V_{YB} is equal to the phasor difference of V_{YN} and V_{BN}, and is equal to $\sqrt{3}\ V_{Ph}$. The line voltage V_{BR} is equal to the phasor difference of V_{BN} and V_{RN}, and is equal to $\sqrt{3}\ V_{Ph}$. Hence, in a balanced star-connected system

(i) Line voltage = $\sqrt{3}\ V_{Ph}$,
(ii) All line voltages are equal in magnitude and are displaced by 120°, and
(iii) All line voltages are 30° ahead of their respective phase voltages (from Fig. 9.22).

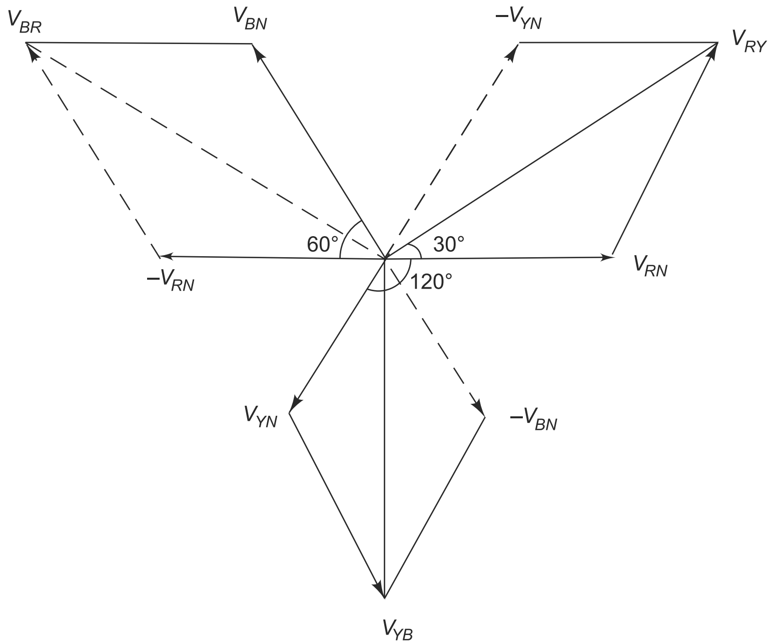


Fig. 9.22

EXAMPLE 9.10

A symmetrical star-connected system is shown in Fig. 9.23 (a). Calculate the three line voltages, given $V_{RN} = 230 \angle 0°$. The phase sequence is RYB.

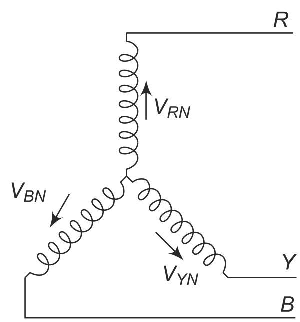


Fig. 9.23 (a)

Solution Since the system is a balanced system, all the phase voltages are equal in magnitude, but displaced by 120° as shown in Fig. 9.23 (b).

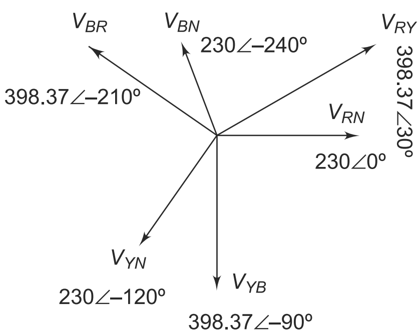


Fig. 9.23 (b)

$$\therefore \quad V_{RN} = 230 \angle 0° \text{ V}$$

$$V_{YN} = 230 \angle -120° \text{ V}$$

$$V_{BN} = 230 \angle -240° \text{ V}$$

Corresponding line voltages are equal to $\sqrt{3}$ times the phase voltages, and are 30° ahead of the respective phase voltages.

$$\therefore \quad V_{RY} = \sqrt{3} \times 230 \angle 0 + 30° \text{ V} = 398.37 \angle 30° \text{ V}$$

$$V_{YB} = \sqrt{3} \times 230 \angle -120° + 30° \text{ V} = 398.37 \angle -90° \text{ V}$$

$$V_{BR} = \sqrt{3} \times 230 \angle -240° + 30° \text{ V} = 398.37 \angle -210° \text{ V}$$

❑ ***Current Relations*** Figure 9.24 (a) shows a balanced three-phase, wye-connected system indicating phase currents and line currents. The arrows placed alongside the currents I_R, I_Y, and I_B flowing in the three phases indicate the directions of currents when they are assumed to be positive and not the directions at that particular instant. The phasor diagram for phase currents with respect to their phase voltages is shown in Fig. 9.24 (b). All the phase currents are displaced by 120° with respect to each other, 'ϕ' is the phase angle between phase voltage and phase current (lagging load is assumed). For a balanced load, all the phase currents are equal in magnitude. It can be observed from Fig. 9.24 (a) that each line conductor is connected in series with its individual phase winding Therefore, the current in a line conductor is the same as that in the phase to which the line conductor is connected.

$\therefore \quad I_L = I_{Ph} = I_R = I_Y = I_B$

It can be observed from Fig. 9.24 (b) that the angle between the line (phase) current and the corresponding line voltage is $(30 + \phi)°$ for a lagging load. Consequently, if the load is leading, then the angle between the line (phase) current and corresponding line voltage will be $(30 - \phi)°$.

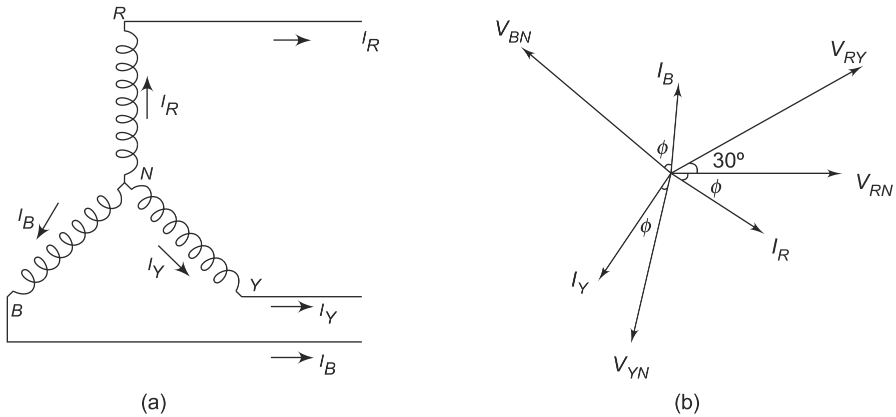


Fig. 9.24

EXAMPLE 9.11

In Fig. 9.24 (a), the value of the current in phase R is $I_R = 10 \angle 20°$ A. Calculate the values of the three line currents. Assume an RYB phase sequence.

Solution In a balanced star-connected system $I_L = I_{Ph}$, and is displaced by 120°. Therefore the three line currents are

$$I_R = 10 \angle 20° \text{ A}$$
$$I_Y = 10 \angle 20° - 120° \text{ A} = 10 \angle -100° \text{ A}$$
$$I_B = 10 \angle 20° - 240° \text{ A} = 10 \angle -220° \text{ A}$$

❑ ***Power in the Star-Connected Network*** The total active power or true power in the three-phase load is the sum of the powers in the three phases. For a balanced load, the power in each load is the same; hence, total power = 3 × power in each phase

or $\quad P = 3 \times V_{Ph} \times I_{Ph} \cos\phi$

It is the usual practice to express the three-phase power in terms of line quantities as follows.

$$V_L = \sqrt{3}\, V_{Ph}, I_L = I_{Ph}$$
$$P = \sqrt{3}\, V_L I_L \cos\phi \text{W}$$

or $\sqrt{3}\, V_L I_L \cos\phi$ is the active power in the circuit.

Total reactive power is given by

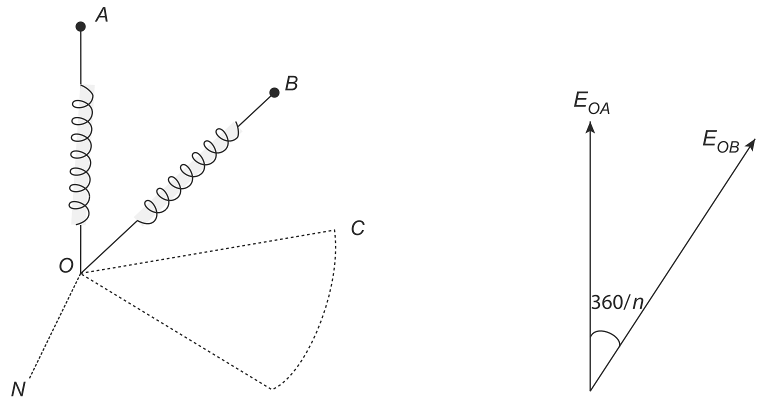
$$Q = \sqrt{3}\, V_L I_L \sin\phi \text{VAR}$$

Total apparent power or volt-amperes

$$= \sqrt{3}\, V_L I_L \text{ VA}$$

9.7.2 *n*-Phase Star System

It is to be noted that star and mesh are general terms applicable to any number of phases; but wye and delta are special cases of star and mesh when the system is a three-phase system. Consider an n-phase balanced star system with two adjacent phases as shown in Fig. 9.25 (a). Its vector diagram is shown in Fig. 9.25 (b).



(a) (b)

Fig. 9.25

The angle of phase difference between adjacent phase voltages is 360°/n. Let E_{Ph} be the voltage of each phase. The line voltage, i.e. the voltage between A and B is equal to $E_{AB} = E_L = E_{AO} + E_{OB}$. The vector addition is shown in Fig. 9.25 (c). It is evident that the line current and phase current are same.

$$E_{AB} = E_{AO} + E_{OB}$$

Consider the parallelogram $OABC$.

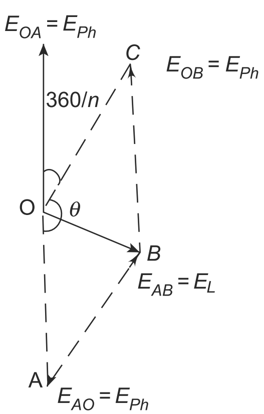


Fig. 9.25 (c)

$$OB = \sqrt{OC^2 + OA^2 + 2 \times OA \times OC \times \cos\theta}$$

$$= \sqrt{E_{ph}^2 + E_{ph}^2 + 2E_{ph}^2 \cos\left(180° - \frac{360°}{n}\right)}$$

$$= \sqrt{2E_{ph}^2 \quad 2E_{ph}^2 \cos\frac{360}{n}}$$

$$= \sqrt{2}\ E_{ph}\sqrt{\left[1 - \cos 2\left(\frac{180°}{n}\right)\right]}$$

$$= \sqrt{2}\ E_{ph}\sqrt{2\sin^2\left(\frac{180°}{n}\right)}$$

$$E_L = 2E_{ph}\sin\frac{180}{n}$$

The above equation is a general equation for line voltage, for example, for a three-phase system, $n = 3$; $E_L = 2\,E_{ph}\sin 60° = \sqrt{3}\,E_{Ph}$.

EXAMPLE 9.12

A balanced star-connected load of (4 + J3) Ω per phase is connected to a balanced 3-phase 400 V supply. The phase current is 12 A. Find (a) the total active power, (b) reactive power, and (c) total apparent power.

Solution The voltage given in the data is always the rms value of the line voltage unless otherwise specified.

$$\therefore \quad Z_{Ph} = \sqrt{4^2 + 3^2} = 5\ \Omega$$

$$\text{PF} = \cos\phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{4}{5} = 0.8$$

$$\sin\phi = 0.6$$

(a) Active power $= \sqrt{3}\ V_L I_L \cos\phi$ W

$= \sqrt{3} \times 400 \times 12 \times 0.8 = 6651$ W

(b) Reactive power $= \sqrt{3}\ V_L\, I_L \sin\phi$ VAR

$= \sqrt{3} \times 400 \times 12 \times 0.6 = 4988.36$ VAR

(c) Apparent power $= \sqrt{3}\ V_L I_L$

$= \sqrt{3} \times 400 \times 12 = 8313.84$ VA

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆☆★**9-6.1** Three non-reactive resistors of 5 Ω, 10 Ω and 15 Ω are star-connected to *R*, *Y* and *B* phase of a 440 V symmetrical system. Determine the current and power in each resistor and the voltage between star point and neutral; assume the phase sequence as *RYB*.

☆☆★**9-6.2** A three-phase, three-wire symmetrical 440 V source is supplying power to an unbalanced, delta-connected load in which $Z_{RY} = 20\angle 30°\ \Omega$, $Z_{YB} = 20\angle 0°\ \Omega$ and $Z_{BR} = 20\angle -30°\ \Omega$. If the phase sequence is *RYB*, calculate the line currents.

☆☆★**9-6.3** Three non-inductive resistances of 25 Ω, 10 Ω and 15 Ω are connected in star to a 400 V symmetrical supply. Calculate the line currents and the voltage across the each load phase.

☆★★**9-6.4** Three impedances $Z_1 = (10 + j0)\ \Omega$; $Z_2 = (3 + j4)\ \Omega$ and $(0 - j10)\ \Omega$ are connected in star across a balanced line voltage of 100 volts. Find the neutral shift voltage between supply and load. Use Millsman's theorem.

Frequently Asked Questions linked to LO 6

☆★★**9-6.1** A balanced star-connected load having an impedance $15 + j20\ \Omega$ per phase is connected to 3ϕ, 440 V, 50 H Z. Find the line current and power absorbed by the load.

[AU May/June 2014]

☆☆★**9-6.2** A star-connected balanced load draws a current of 35 A per phase when connected to a 440 V supply. Determine the apparent power. **[AU May/June 2014]**

☆★★**9-6.3** For the circuit shown in Q.3, calculate the line current, the power and the power factor. The values of *R*, *L*, and *C* in each phase are 10 Ω, 1 H and 100 μF respectively.

[AU Nov./Dec. 2012]

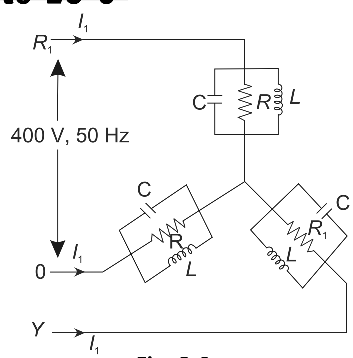


Fig. Q.3

9.8 VOLTAGE, CURRENT, AND POWER IN A DELTA-CONNECTED SYSTEM

9.8.1 Delta-Connected System

Figure 9.26 shows a balanced three-phase, three-wire, delta-connected system. This arrangement is referred to as mesh connection because it forms a closed circuit. It is also known as delta connection because the three branches in the circuit can also be arranged in the shape of delta (Δ).

LO 7 Illustrate voltage, current and power in a delta-connected system

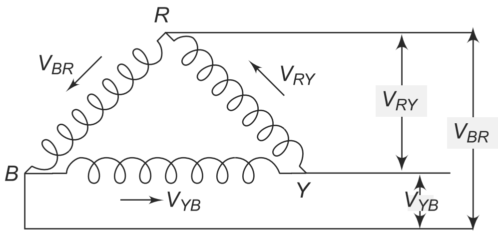


Fig. 9.26

From the manner of interconnection of the three phases in the circuit, it may appear that the three phases are short-circuited among themselves. However, this is not the case. Since the system is balanced, the sum of the three voltages round the closed mesh is zero; consequently, no current can flow around the mesh when the terminals are open.

The arrows placed alongside the voltages, V_{RY}, V_{YB} and V_{BR}, of the three phases indicate that the terminals *R*, *Y* and *B* are positive with respect to *Y*, *B* and *R*, respectively, during their respective positive half cycles.

❑ ***Voltage Relations*** From Fig. 9.27, we notice that only one phase is connected between any two lines. Hence, the voltage between any two lines (V_L) is equal to the phase voltage (V_{Ph}).

$$\therefore \quad V_{RY} = V_L = V_{Ph}$$

Since the system is balanced, all the phase voltages are equal, but displaced by 120° from one another as shown in the phasor diagram in Fig. 9.27. The phase sequence *RYB* is assumed.

$$\therefore \quad |V_{RY}| = |V_{YB}| = |V_{BR}| = V_L = V_{Ph}$$

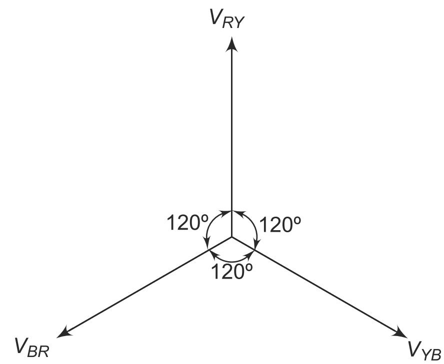


Fig. 9.27

EXAMPLE 9.13

In Fig. 9.27, the voltage across the terminals R and Y is 400 ∠0°. Calculate the values of the three line voltages. Assume RYB phase sequence.

Solution In a balanced delta-connected system, $|V_L| = |V_{Ph}|$, and is displaced by 120°; therefore, the three line voltages are

$$V_{RY} = 400 \angle 0° \text{ V}$$
$$V_{YB} = 400 \angle -120° \text{ V}$$
$$V_{BR} = 400 \angle -240° \text{ V}$$

❑ ***Current Relations*** In Fig. 9.28, we notice that since the system is balanced, the three phase currents (I_{Ph}), i.e. I_R, I_Y, I_B are equal in magnitude but displaced by 120° from one another as shown in Fig. 9.28 (b). I_1, I_2, and I_3 are the line currents (I_L), i.e. I_1 is the line current in the line 1 connected to the common point of *R*. Similarly, I_2 and I_3 are the line currents in lines 2 and 3, connected to common points *Y* and *B*, respectively. Though here all the line currents are directed outwards, at no instant will all the three

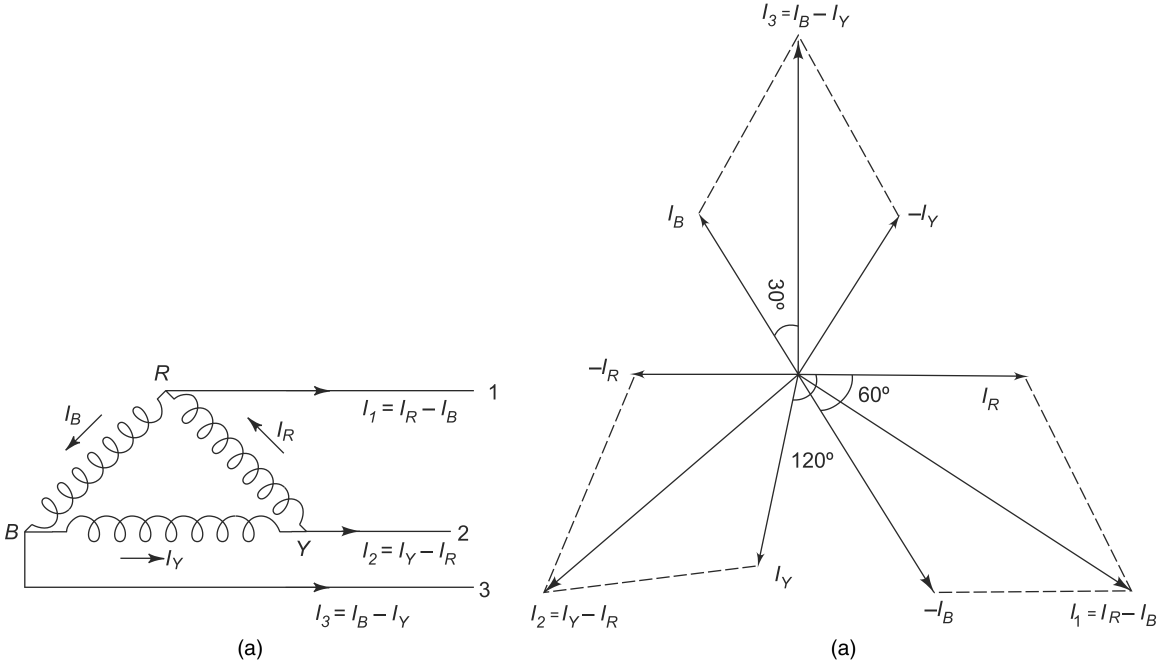


Fig. 9.28

line currents flow in the same direction, either outwards or inwards. Because the three line currents are displaced 120° from one another, when one is positive, the other two might both be negative, or one positive and one negative. Also it is to be noted that arrows placed alongside phase currents in Fig. 9.28 (a), indicate the direction of currents when they are assumed to be positive and not their actual direction at a particular instant. We can easily determine the line currents in Fig. 9.28 (a), I_1, I_2, and I_3 by applying KCL at the three terminals R, Y and B, respectively. Thus, the current in line 1, $I_1 = I_R - I_B$; i.e. the current in any line is equal to the phasor difference of the currents in the two phases attached to that line. Similarly, the current in the line 2, $I_2 = I_Y - I_R$, and the current in the line 3, $I_3 = I_B - I_Y$.

The phasor addition of these currents is shown in Fig. 9.28 (b). From the figure,

$$I_1 = I_R - I_B$$

$$I_1 = \sqrt{I_R^2 + I_B^2 + 2I_R I_B \cos 60°}$$

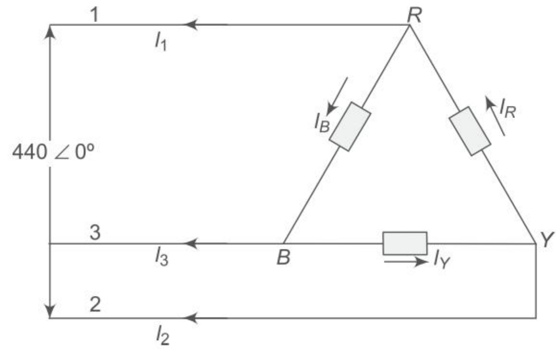
$$I_1 = \sqrt{3}\ I_{Ph}, \text{ since } I_R = I_B = I_{Ph}$$

Similarly, the remaining two line currents, I_2 and I_3, are also equal to $\sqrt{3}$ times the phase currents; i.e. $I_L = \sqrt{3}\ I_{Ph}$.

As can be seen from Fig. 9.28 (b), all the line currents are equal in magnitude but displaced by 120° from one another; and the line currents are 30° behind the respective phase currents.

EXAMPLE 9.14

Three identical loads are connected in delta to a three-phase supply of 440 ∠0° V as shown in Fig. 9.29 (a). If the phase current I_R is 15 ∠0° A, calculate the three line currents.



(a)

Solution All the line currents are equal and 30° behind their respective phase currents, and $\sqrt{3}$ times their phase values, displaced by 120° from one another, assuming *RYB* phase sequence.

Let the line currents in line 1, 2, and 3 be I_1, I_2 and I_3, respectively.

$$I_1 = \sqrt{3} \times I_R \angle(\phi - 30°)$$

$$= \sqrt{3} \times 15 \angle -30° = 25.98 \angle -30° \text{ A}$$

$$I_2 = \sqrt{3} \times 15 \angle(-30 - 120)° = 25.98 \angle -150° \text{ A}$$

$$I_3 = \sqrt{3} \times 15 \angle(-30 - 240)° = 25.98 \angle -270° \text{ A}$$

The phasor diagram is shown in Fig. 9.29 (b).

❑ ***Power in the Delta-Connected System*** Obviously, the total power in the delta circuit is the sum of the powers in the three phases. Since the load is balanced, the power consumed in each phase is the same. Total power is equal to three times the power in each phase.

Power per phase = $V_{Ph} I_{Ph} \cos \phi$

where ϕ is the phase angle between phase voltage and phase current.

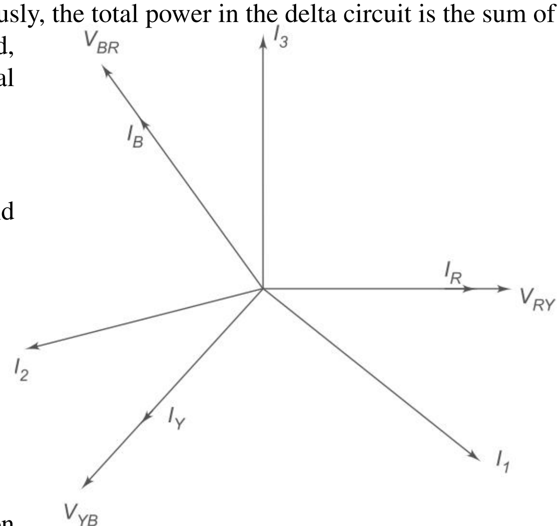
Total power $P = 3 \times V_{Ph} I_{Ph} \cos \phi$

In terms of line quantities,

$$P = \sqrt{3}\ V_L I_L \cos \phi \text{ W}$$

Since $V_{Ph} = V_L$ and $I_{Ph} = \dfrac{I_L}{\sqrt{3}}$

for a balanced system, whether star or delta, the expression for the total power is the same.



(b)

Fig. 9.29

EXAMPLE 9.15

A balanced delta-connected load of (2 + j3) Ω per phase is connected to a balanced three-phase 440 V supply. The phase current is 10 A. Find the (a) total active power (b) reactive power, and (c) apparent power in the circuit.

Solution $$Z_{Ph} = \sqrt{(2)^2 + (3)^2} = 3.6 \angle 56.3° \ \Omega$$

$$\cos\phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{2}{3.6} = 0.55$$

So, $\sin\phi = 0.83$

$$I_L = \sqrt{3} \times I_{Ph} = \sqrt{3} \times 10 = 17.32 \text{ A}$$

(a) Active power $= \sqrt{3}\ V_L I_L \cos\phi$

$= \sqrt{3} \times 440 \times 17.32 \times 0.55 = 7259.78$ W

(b) Reactive power $= \sqrt{3}\ V_L I_L \sin\phi$

$= \sqrt{3} \times 440 \times 17.32 \times 0.83 = 10955.67$ VAR

(c) Apparent power $= \sqrt{3}\ V_L I_L$

$= \sqrt{3} \times 440 \times 17.32 = 13199.61$ VA

9.8.2 *n*-Phase Mesh System

Figure 9.30 (a) shows part of an *n*-phase balanced mesh system. Its vector diagram is shown in Fig. 9.30 (b).

Let the current in line BB' be I_L. This is same in all the remaining lines of the *n*-phase system. I_{AB}, I_{BC} are the phase currents in *AB* and *BC* phases respectively. The vector addition of the line current is shown in Fig. 9.30 (c). It is evident from the Fig. 9.30 (b) that the line and phase voltages are equal.

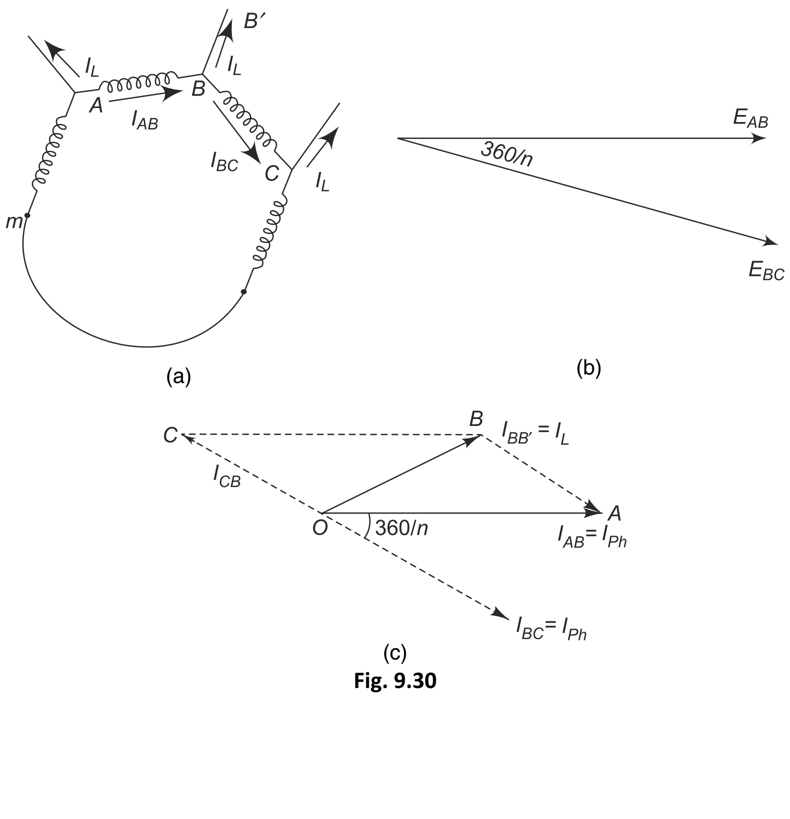


Fig. 9.30

$$I_{BB} = I_L \; = I_{AB} \; + I_{CB}$$
$$= I_{AB} - I_{BC}$$

Consider the parallelogram *OABC*.

$$OB = \sqrt{OA^2 + OC^2 + 2 \times OA \times OC \times \cos\left(180 - \frac{360}{n}\right)}$$

$$= \sqrt{I_{Ph}^2 \quad I_{Ph}^2 \quad 2I_{Ph}^2 \cos\frac{360}{n}}$$

$$= \sqrt{2}\ I_{Ph}\sqrt{1 \quad \cos 2\left(\frac{180}{n}\right)}$$

$$= \sqrt{2}\ I_{Ph}\sqrt{2\sin^2\frac{180}{n}}$$

$$I_L = 2I_{Ph}\sin\frac{180}{n}$$

The above equation is a general equation for the line current in a balanced n-phase mesh system.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆☆★**9-7.1** Three equal resistances connected in star across a three-phase balanced supply consume 1000 W. If the same three resistors were reconnected in delta across the same supply, determine the power consumed.

☆★★**9-7.2** The currents in R_Y, Y_B, and B_R branches of a mesh connected system with symmetrical voltages are 20 A at 0.7 lagging power factor, 20 A at 0.8 leading power factor, and 10 A at UPF respectively. Determine the current in each line. Phase sequence is *RYB*. Draw a phasor diagram.

★★★**9-7.3** Three identical impedances 10 $\angle 30°\ \Omega$ in a delta connection, and three identical impedances 5 $\angle 35°\ \Omega$ in a star connection are on the same three-phase, three-wire 173 V system. Find the line currents and the total power.

☆★★**9-7.4** Three capacitors, each of 100 μF are connected in delta to a 440 V, three -phase, 50 Hz supply. What will be the capacitance of each of the three capacitors if the same three capacitors are connected in star across the same supply to draw the same line current.

★★★**9-7.5** Three impedances, $Z_R = (3 + j2)\ \Omega$; $Z_Y = j9\ \Omega$ and $Z_B = 3\ \Omega$ are connected in star across a 400 V, 3-wire system. Find the loads on the equivalent delta-connected system phase-sequence *RYB*.

Frequently Asked Questions linked to LO 7

☆☆★**9-7.1** A delta-connected load has $(30 + j40)\ \Omega$ impedance per phase. Determine the phase current if it is connected to a 415 V, 3-phase, 50 Hz supply. **[AU May/June 2013]**

☆☆★**9-7.2** Prove that the total instantaneous power in a balanced three-phase system is constant and is equal to the average power whether the load is star or delta-connected. **[AU May/June 2013]**

☆★★**9-7.3** A three-phase balanced delta-connected load of $4 + j8\ \Omega$ is connected across a 400 V, 3 ϕ balanced supply. Determine the phase currents and line currents. (Phase sequence is RYB). **[AU May/June 2014]**

☆☆★**9-7.4** Write the relation the line and phase value of voltage and current in a balanced delta-connected system. **[AU May/June 2014]**

☆★★**9-7.5** A 3-phase, 3-wire 120 V *RYB* system feeds a Δ-connected load whose phase impedance is $30\angle 45°\ \Omega$. Find the phase and line current in this system and draw the phasor diagram. **[AU Nov./Dec. 2012]**

9.9 THREE-PHASE BALANCED CIRCUITS

LO 8 Analyse three-phase balanced and unbalanced circuits

The analysis of three-phase balanced systems is presented in this section. It is no way different from the analysis of *ac* systems in general. The relation between voltages, currents and power in delta-connected and star-connected systems has already been discussed in the previous sections. We can make use of those relations and expressions while solving the circuits.

9.9.1 Balanced Three-Phase System-Delta Load

Figure 9.31 (a) shows a three-phase, three-wire, balanced system supplying power to a balanced three-phase delta load. The phase sequence is *RYB*. We are required to find out the currents in all branches and lines.

Let us assume the line voltage $V_{RY} = V\angle 0^\circ$ as the reference phasor. Then the three source voltages are given by

$$V_{RY} = V\angle 0^\circ \text{ V}$$
$$V_{YB} = V\angle -120^\circ \text{ V}$$
$$V_{BR} = V\angle -240^\circ \text{ V}$$

These voltages are represented by phasors in Fig. 9.31 (b). Since the load is delta-connected, the line voltage of the source is equal to the phase voltage of the load. The current in phase *RY*, I_R will lag (lead) behind (ahead of) the phase voltage V_{RY} by an angle ϕ as dictated by the nature of the load impedance. The angle of lag of I_Y with respect to V_{YB}, as well as the angle of lag of I_B with respect to V_{BR} will be ϕ as the load is balanced. All these quantities are represented in Fig. 9.31 (b).

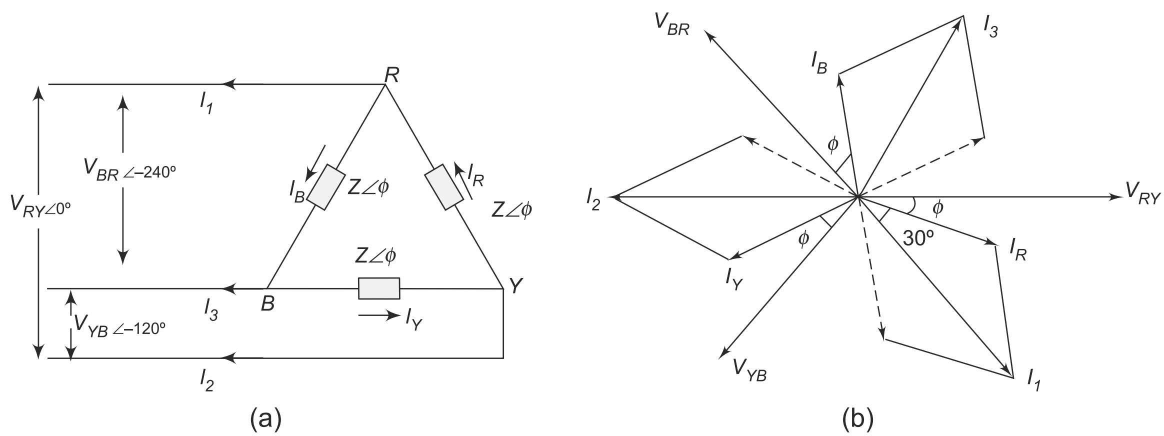


Fig. 9.31

If the load impedance is $Z\angle\phi$, the current flowing in the three load impedances are then

$$I_R = \frac{V_{RY} - 0}{Z - \phi} = \frac{V}{Z}\angle -\phi$$

$$I_Y = \frac{V_{YB} - 120}{Z - \phi} = \frac{V}{Z}\angle -120^\circ - \phi$$

$$I_B = \frac{V_{BR} - 240}{Z - \phi} = \frac{V}{Z}\angle -240^\circ - \phi$$

The line currents are $\sqrt{3}$ times the phase currents and are 30° behind their respective phase currents.

$\therefore$ current in the line 1 is given by

$$I_1 = \sqrt{3}\left|\frac{V}{Z}\right|\angle(-\phi - 30°), \quad \text{or} \quad I_R - I_B \text{ (phasor difference)}$$

Similarly, the current in the line 2

$$I_2 = \sqrt{3}\left|\frac{V}{Z}\right|\angle(-120 - \phi - 30°),$$

or $\quad I_Y - I_R\,(\text{phasor difference}) = \sqrt{3}\left|\frac{V}{Z}\right|\angle(-\phi - 150°)$, and

$$I_3 = \sqrt{3}\left|\frac{V}{Z}\right|\angle(-240 - \phi - 30°), \quad \text{or } I_B - I_Y \text{ (phasor difference)}$$

$$= \sqrt{3}\left|\frac{V}{Z}\right| - (-270 - \phi)$$

To draw all these quantities vectorially, $V_{RY} = V\angle 0°$ is taken as the reference vector.

EXAMPLE 9.16

A three-phase, balanced delta-connected load of (4 + j8) Ω is connected across a 400 V, 3 – φ balanced supply. Determine the phase currents and line currents. Assume the phase sequence to be RYB. Also calculate the power drawn by the load.

Solution Refering to Fig. 9.31 (a), taking the line voltage $V_{RY} = \text{V}\angle 0°$ as reference $V_{RY} = 400\ \angle 0°$ V; $V_{YB} = 400\ \angle{-120°}$ V, $V_{BR} = 400\ \angle{-240°}$ V

Impedance per phase = $(4 + j8)\ \Omega = 8.94\ \angle 63.4°\ \Omega$

Phase currents are: $I_R = \dfrac{400\angle 0°}{8.94\angle 63.4°} = 44.74\angle -63.4°$ A

$$I_Y = \frac{400 - 120}{8.94 - 63.4} = 44.74\angle -183.4° \text{ A}$$

$$I_B = \frac{400 - 240}{8.94 - 63.4} = 44.74\angle -303.4° \text{ A}$$

The three line currents are

$$I_1 = I_R - I_B = (44.74\ \angle{-63.4°} - 44.74\ \angle{-303.4°})$$
$$= (20.03 - j40) - (24.62 + j37.35) = (-4.59 - j77.35) \text{ A}$$
$$= 77.49\ \angle 266.6° \text{ A}$$

or the line current I_1 is equal to the $\sqrt{3}$ times the phase current and 30° behind its respective phase current

$$I_1 = \sqrt{3} \times 44.74\ \angle{-63.4°} - 30° = 77.49\ \angle{-93.4°}$$

or $\quad = 77.49\ \angle 266.6°$ A

Similarly, $\quad I_2 = I_Y - I_R$

$$= \sqrt{3} \times 44.74\ \angle{-183.4°} - 30° = 77.49 - 213.4° \text{ A}$$

$$= 77.49 \angle 146.6° \text{ A}$$

$$I_3 = I_B - I_Y$$

$$= \sqrt{3} \times 44.74 \angle -303.4° -30° = 77.49 -333.4° \text{ A}$$

$$= 77.49 \angle 26.6° \text{ A}$$

Power drawn by the load is $P = 3V_{Ph} I_{Ph} \cos \phi$

or $\sqrt{3} \times V_L \times I_L \cos 63.4° = 24.039$ kW

9.9.2 Balanced Three Phase System-Star Connected Load

Figure 9.32 (a) shows a three-phase, three-wire system supplying power to a balanced three phase star connected load. The phase sequence *RYB* is assumed.

In star connection, whatever current is flowing in the phase is also flowing in the line. The three line (phase) currents are I_R, I_Y, and I_B.

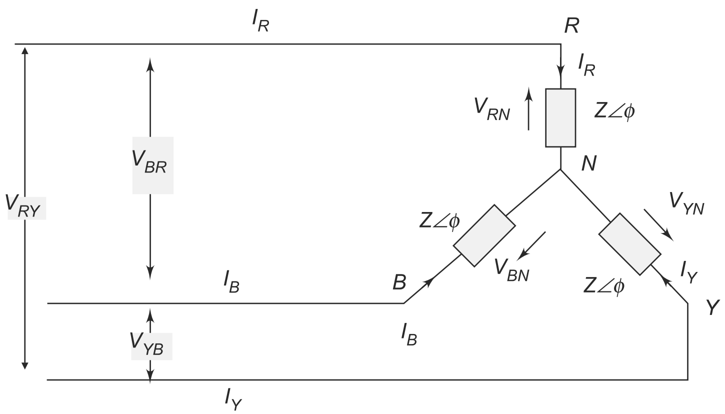


Fig. 9.32 (a)

V_{RN}, V_{YN}, and V_{BN} represent three phase voltages of the network, i.e. the voltage between any line and neutral. Let us assume the voltage $V_{RN} = V\angle 0°$ as the reference phasor. Consequently, the phase voltage

$$V_{RN} = V \angle 0°$$

$$V_{YN} = V \angle -120°$$

$$V_{BN} = V \angle -240°$$

Hence $I_R = \dfrac{V_{RN}}{Z\angle\phi} = \dfrac{V - 0}{Z - \phi} = \left|\dfrac{V}{Z}\right| \angle - \phi$

$$I_Y = \frac{V_{YN}}{Z - \phi} = \frac{V - 120}{Z - \phi} = \left|\frac{V}{Z}\right| \angle -120° - \phi$$

$$I_B = \frac{V_{BN}}{Z - \phi} = \frac{V - 240}{Z - \phi} = \left|\frac{V}{Z}\right| \angle -240° - \phi$$

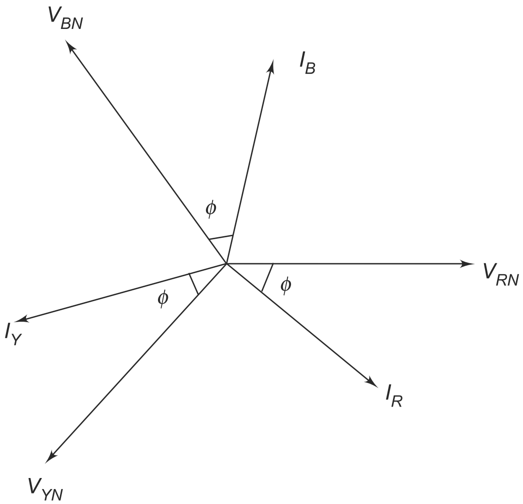


Fig. 9.32 (b)

As seen from the above expressions, the currents, I_R, I_Y, and I_B, are equal in magnitude and have a 120° phase difference. The disposition of these vectors is shown in Fig. 9.32 (b). Sometimes, a fourth wire, called *neutral wire* is run from the neutral point, if the source is also star-connected. This gives three-phase, four-wire star-connected system. However, if the three line currents are balanced, the current in the fourth wire is zero; removing this connecting wire between the source neutral and load neutral is, therefore, not going to make any change in the condition of the system. The availability of the neutral wire makes it possible to use all the three phase voltages, as well as the three line voltages. Usually, the neutral is grounded for safety and for the design of insulation.

It makes no difference to the current flowing in the load phases, as well as to the line currents, whether the sources have been connected in star or in delta, provided the voltage across each phase of the delta connected source is $\sqrt{3}$ times the voltage across each phase of the star-connected source.

EXAMPLE 9.17

A balanced star-connected load having an impedance (15 + j20) Ω per phase is connected to a three-phase, 440 V; 50 Hz supply. Find the line currents and the power absorbed by the load. Assume RYB phase sequence.

Solution Referring to Fig. 9.32 (a), taking V_{RN} as the reference voltage, we have

$$V_{RN} = \frac{440\angle 0°}{\sqrt{3}} = 254\angle 0°$$

$$V_{YN} = 254\ \angle{-120°}$$

$$V_{BN} = 254\ \angle{-240°}$$

Impedance per phase, $Z_{Ph} = 15 + j20 = 25\ \angle 53.13°\ \Omega$

The phase currents are $I_R = \dfrac{V_{RN}}{Z_{Ph}} = \dfrac{254 - 0}{25 - 53.13} = 10.16\angle{-53.13°}\text{ A}$

$$I_Y = \frac{V_{YN}}{Z_{Ph}} = \frac{254 - 120}{25 - 53.13} = 10.16\angle{-173.13°}\text{ A}$$

$$I_B = \frac{V_{BN}}{Z_{Ph}} = \frac{254 - 240}{25 - 53.13} = 10.16\angle{-293.13°}\text{ A}$$

The three phase currents are equal in magnitude and displaced by 120° from one another. Since the load is star-connected, these currents also represents line currents.

The power absorbed by the load (P)

$$= \sqrt{3} \times V_{Ph} \times I_{Ph} \cos\phi$$

or $$= \sqrt{3} \times V_L \times I_L \cos\phi$$

$$= \sqrt{3} \times 440 \times 10.16 \times \cos 53.13° = 4645.78\text{ W}$$

9.10 THREE-PHASE-UNBALANCED CIRCUITS

LO 8

9.10.1 Types of Unbalanced Loads

An unbalance exists in a circuit when the impedances in one or more phases differ from the impedances of the other phases. In such a case, line or phase currents are different and are displaced from one another by unequal angles. So far, we have considered balanced loads connected to balanced systems. It is enough to solve problems, considering one phase only on balanced loads; the conditions on other two phases being similar. Problems on unbalanced three-phase loads are difficult to handle because conditions in the three phases are different. However, the source voltages are assumed to be balanced. If the system is a three-wire system, the currents flowing towards the load in the three lines must add to zero at any given instant. If the system is a four-wire system, the sum of the three outgoing line currents is equal to the return current in the neutral wire. We will now consider different methods to handle unbalanced star-connected and delta-connected loads. In practice, we may come across the following unbalanced loads:

1. Unbalanced delta-connected load,
2. Unbalanced three-wire star-connected load, and
3. Unbalanced four-wire star-connected load.

❑ ***Unbalanced Delta-connected Load*** Figure 9.33 shows an unbalanced delta-load connected to a balanced three-phase supply.

The unbalanced delta-connected load supplied from a balanced three-phase supply does not present any new problems because the voltage across the load phase is fixed. It is independent of the nature of the load and is equal to the line voltage of the supply. The current in each load phase is equal to the line voltage divided by the impedance of that phase. The line current will be the phasor difference of the corresponding phase currents, taking V_{RY} as the reference phasor.

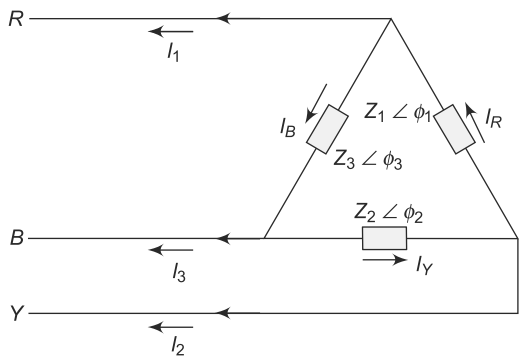


Fig. 9.33

Assuming *RYB* phase sequence, we have

$$V_{RY} = V \angle 0° \text{ V}, V_{YB} = V \angle{-120°} \text{ V}, V_{BR} = V \angle{-240°} \text{ V}$$

Phase currents are

$$I_R = \frac{V_{RY}}{Z_1 \angle \phi} = \frac{V \angle 0°}{Z_1 \angle \phi_1} = \left|\frac{V}{Z_1}\right| \angle -\phi_1 \text{ A}$$

$$I_Y = \frac{V_{YB}}{Z_2 \angle \phi_2} = \frac{V \angle -120°}{Z_2 \angle \phi_2} = \left|\frac{V}{Z_2}\right| \angle -120° - \phi_2 \text{ A}$$

$$I_B = \frac{V_{BR}}{Z_3 \angle \phi_3} = \frac{V \angle -240°}{Z_3 \angle \phi_3} = \frac{V}{Z_3} \angle -240° - \phi_3 \text{ A}$$

The three line currents are

$I_1 = I_R - I_B$ phasor difference

$I_2 = I_Y - I_R$ phasor difference

$I_3 = I_B - I_Y$ phasor difference

EXAMPLE 9.18

Three impedances $Z_1 = 20\angle 30°\ \Omega$, $Z_2 = 40\angle 60°\ \Omega$ and $Z_3 = 10\angle -90°\ \Omega$ are delta-connected to a 400 V, 3 – ϕ system as shown in Fig. 9.34. Determine the (a) phase currents, (b) line currents, and (c) total power consumed by the load.

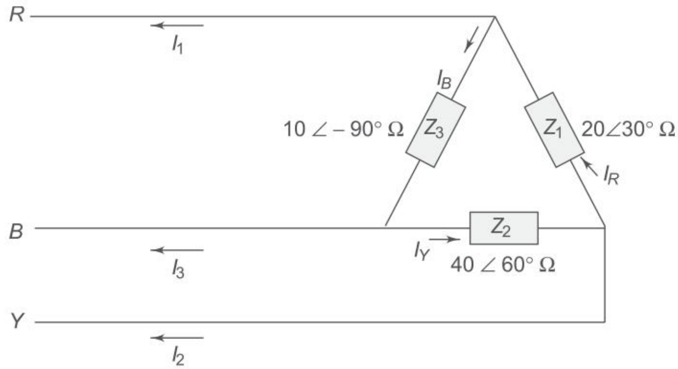


Fig. 9.34

Solution The three phase currents are I_R, I_Y, and I_B, and the three line currents are I_1, I_2 and I_3. Taking $V_{RY} = V\angle 0°$ V as reference phasor, and assuming *RYB* phase sequence, we have

$$V_{RY} = 400\angle 0°\text{ V},\ V_{YB} = 400\angle -120°\text{ V},$$

$$V_{BR} = 400\angle -240°\text{ V}$$

$$Z_1 = 20\angle 30°\ \Omega = (17.32 + j10)\ \Omega;$$

$$Z_2 = 40\angle 60°\ \Omega = (20 + j34.64)\ \Omega;$$

$$Z_3 = 10\angle -90°\ \Omega = (0 - j10)\ \Omega$$

(a) $$I_R = \frac{V_{RY}}{Z_1\angle\phi_1} = \frac{400\angle 0°}{20\angle 30°}\text{ A} = 20\angle -30°\text{ A}$$

$$= (17.32 - j10)\text{ A}$$

$$I_Y = \frac{V_{YB}}{Z_2\angle\phi_2} = \frac{400\angle -120°}{40\angle 60°}\text{ A} = 10\angle -180°\text{ A}$$

$$= (-10 + j0)\text{ A}$$

$$I_B = \frac{V_{BR}}{Z_3\angle\phi_3} = \frac{400\angle -240°}{10\angle -90°}\text{ A} = 40\angle -150°\text{ A}$$

$$= (-34.64 - j20)\text{ A}$$

(b) Now, the three line currents are

$$I_1 = I_R - I_B = [(17.32 - j10) - (-34.64 - j20)]$$

$$= (51.96 + j10)\text{ A} = 52.91\angle 10.89°\text{ A}$$

$$I_2 = I_Y - I_R = [(-10 + j0) - (17.32 - j10)]$$

$$= (-27.32 + j10)\text{ A} = 29.09\angle 159.89°\text{ A}$$

$$I_3 = I_B - I_Y = [(-34.64 - j20) - (-10 + j0)]$$

$$= (-24.64 - j20)\text{ A} = 31.73\angle -140.94°\text{ A}$$

(c) To calculate the total power, first the powers in the individual phases are to be calculated, then they are added up to get the total power in the unbalanced load.

Power in R phase $= I^2{}_R \times R_R = (20)^2 \times 17.32 = 6928$ W

Power in Y phase $= I^2{}_Y \times R_Y = (10)^2 \times 20 = 2000$ W

Power in B phase $= I^2{}_B \times R_B = (40)^2 \times 0 = 0$

$\therefore$ total power in the load = 6928 + 2000 = 8928 W

❑ ***Unbalanced Four-Wire Star-Connected Load*** Figure 9.35 shows an unbalanced star load connected to a balanced 3-phase, 4-wire supply.

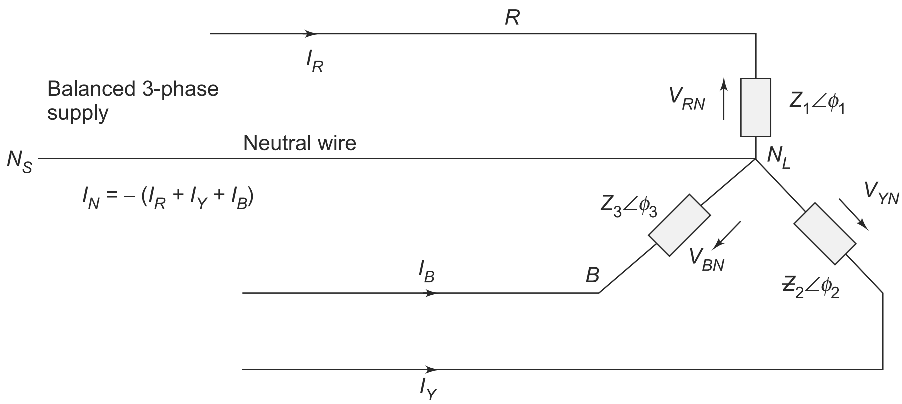


Fig. 9.35

The star point, N_L, of the load is connected to the star point, N_S of the supply. It is the simplest case of an unbalanced load because of the presence of the neutral wire; the star points of the supply N_S (generator) and the load N_L are at the same potential. It means that the voltage across each load impedance is equal to the phase voltage of the supply (generator), i.e. the voltages across the three load impedances are equalised even though load impedances are unequal. However, the current in each phase (or line) will be different. Obviously, the vector sum of the currents in the three lines is not zero, but is equal to neutral current. Phase currents can be calculated in similar way as that followed in an unbalanced delta-connected load.

Taking the phase voltage $V_{RN} = V\angle 0°$ V as reference, and assuming RYB phase sequences, we have the three phase voltages as follows:

$$V_{RN} = V\angle 0° \text{ V},\ V_{YN} = V\angle -120° \text{ V},\ V_{BN} = V\angle -240° \text{ V}$$

The phase currents are

$$I_R = \frac{V_{RN}}{Z_1} = \frac{V\angle 0°}{Z_1\angle\phi_1} A = \left|\frac{V}{Z_1}\right| \angle -\phi_1 \text{ A}$$

$$I_Y = \frac{V_{YN}}{Z_2} = \frac{V\angle 120°}{Z_2\angle\phi_2} A = \left|\frac{V}{Z_2}\right| \angle -120° - \phi_2 \text{ A}$$

$$I_B = \frac{V_{BN}}{Z_3} = \frac{V\angle 240°}{Z_3\angle\phi_3} A = \left|\frac{V}{Z_3}\right| \angle -240° - \phi_3 \text{ A}$$

Incidentally, I_R, I_Y and I_B are also the line currents; the current in the neutral wire is the vector sum of the three line currents.

EXAMPLE 9.19

An unbalanced four-wire, star-connected load has a balanced voltage of 400 V, the loads are

$$Z_1 = (4 + j8)\ \Omega;\ Z_2 = (3 + j4)\ \Omega;\ Z_3 = (15 + j20)\ \Omega$$

Calculate the (a) line currents, (b) current in the neutral wire, and (c) the total power.

Solution $Z_1 = (4 + j8)\ \Omega;\ Z_2 = (3 + j4)\ \Omega;\ Z_3 = (15 + j20)\ \Omega$

$Z_1 = 8.94 \angle 63.40°\ \Omega;\ Z_2 = 5 \angle 53.1°\ \Omega;\ Z_3 = 25 \angle 53.13°\ \Omega$

Let us assume *RYB* phase sequence.

$$\text{The phase voltage } \mathrm{V_{RN}} = \frac{400}{\sqrt{3}} = 230.94\,\mathrm{V}$$

Taking V_{RN} as the reference phasor, we have

$V_{RN} = 230.94 \angle 0°$ V, $V_{YN} = 230.94 \angle –120°$ V

$V_{BN} = 230.94 \angle –240°$ V

The three line currents are

(a) $$I_R = \frac{V_{RN}}{Z_1} = \frac{230.94 \angle 0°}{8.94 \angle 63.4°} \mathrm{A} = 25.83 \angle -63.4°\ \mathrm{A}$$

$$I_Y = \frac{V_{YN}}{Z_2} = \frac{230.94 \angle -120°}{5 \angle 53.1} \mathrm{A} = 46.188 \angle -173.1°\ \mathrm{A}$$

$$I_B = \frac{V_{BN}}{Z_3} = \frac{230.94 \angle -120°}{25 \angle 53.13} \mathrm{A} = 9.23 \angle -293.13°\ \mathrm{A}$$

(b) To find the neutral current, we must add the three line currents. The neutral current must then be equal and opposite to this sum.

Thus, $I_N = -(I_R + I_Y + I_B)$

$= -(25.83 \angle –63.4° + 46.188 \angle –173.1° + 9.23 \angle –293.13°)$ A

$I_N = -[(11.56 - j23.09) + (-45.85 - j5.54) + (3.62 + j8.48)]$ A

$I_N = -[(-30.67 - j20.15)]$ A $= (30.67 + j20.15)$ A

$I_N = 36.69 \angle 33.30°$ A

Its phase with respect to V_{RN} is 33.3°, the disposition of all the currents is shown in Fig. 9.36.

(c) Power in *R* phase $= I^2_R \times R_R = (25.83)^2 \times 4 = 2668.75$ W

Power in *Y* phase $= I^2_Y \times R_Y = (46.18)^2 \times 3 = 6397.77$ W

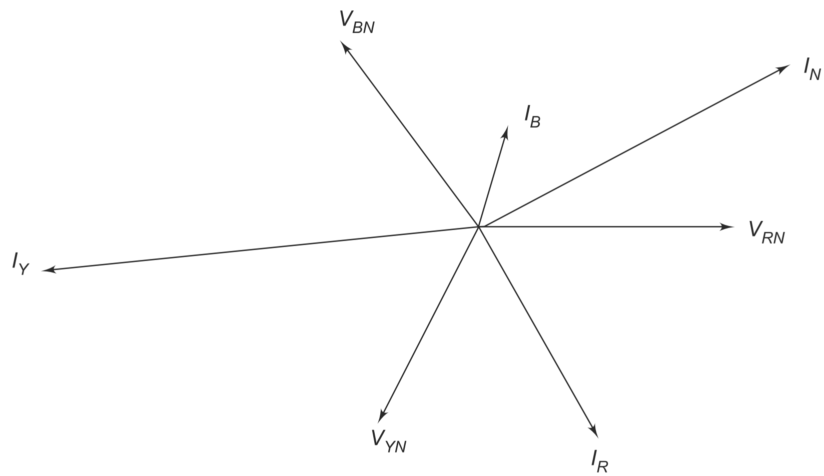


Fig. 9.36

Power in B phase $= I^2_B \times R_B = (9.23)^2 \times 15 = 1277.89$ W

Total power absorbed by the load

$= 2668.75 + 6397.77 + 1277.89 = 10344.41$ W

❑ ***Unbalanced Three-Wire Star-Connected Load*** In a three-phase, four-wire system if the connection between supply neutral and load neutral is broken, it would result in an unbalanced three-wire star-load. This type of load is rarely found in practice, because all the three-wire star loads are balanced. Such as system is shown in Fig. 9.37. Note that the supply star point (N_S) is isolated from the load star point (N_L). The potential of the load star point is different from that of the supply star point. The result is that the load phase voltages is not equal to the supply phase voltage; and they are not only unequal in magnitude, but also subtend angles other than 120° with one another. The magnitude of each phase voltage depends upon the individual phase loads. The potential of the load neutral point changes according to changes in the impedances of the phases, that is why sometimes the load neutral is also called a floating neutral point. All star-connected, unbalanced loads supplied from polyphase systems without a neutral wire have floating neutral point. The phasor sum of the three unbalanced line currents is zero. The phase voltage of the load is not $1/\sqrt{3}$ of the line voltage. The

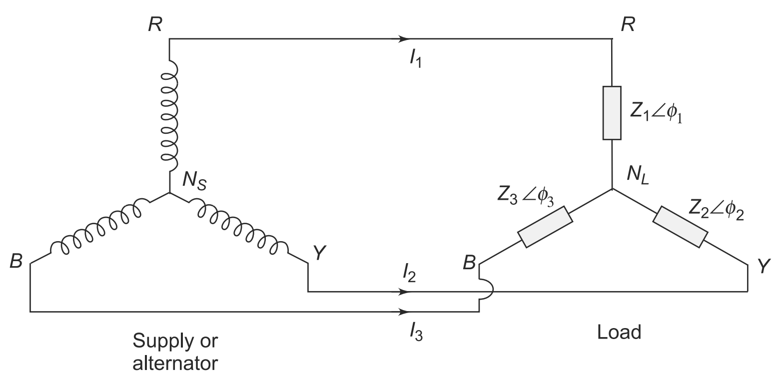


Fig. 9.37

unbalanced three-wire star load is difficult to deal with. It is because load phase voltages cannot be determined directly from the given supply line voltages. There are many methods to solve such unbalanced *Y*-connected loads. Two frequently used methods are presented here. They are

1. Star-delta conversion method, and
2. The application of Millman's theorem

9.10.2 Star-Delta Method of Solving Unbalanced Load

Figure 9.38 (a) shows an unbalanced wye-connected load. It has already been shown in Section 9.6 that a three-phase star-connected load can be replaced by an equivalent delta-connected load. Thus, the star load of Fig. 9.38 (a) can be replaced by equivalent delta as shown in Fig. 9.38 (b), where the impedances in each phase is given by

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

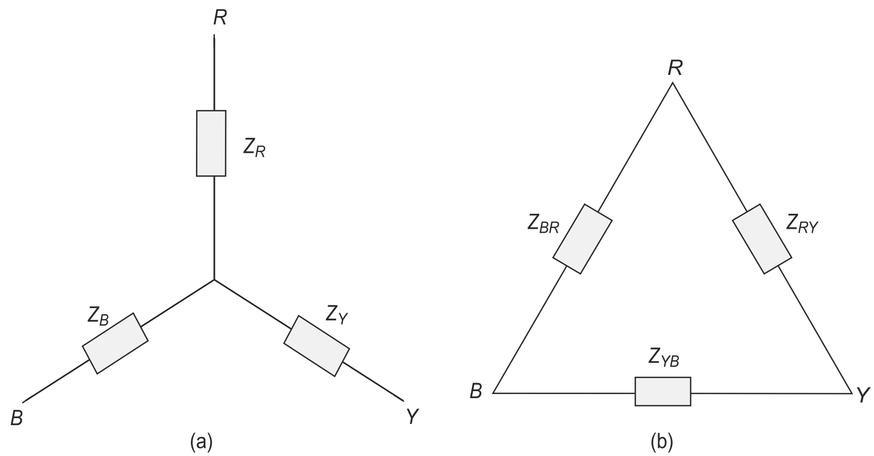


Fig. 9.38

The problem is then solved as an unbalanced delta-connected system. The line currents so calculated are equal in magnitude and phase to those taken by the original unbalanced wye (*Y*) connected load.

EXAMPLE 9.20

A 400 V, three-phase supply feeds an unbalanced three-wire, star-connected load. The branch impedances of the load are $Z_R = (4 + j8)\ \Omega$; $Z_y = (3 + j4)\ \Omega$ and $Z_B = (15 + j20)\ \Omega$. Find the line currents and voltage across each phase impedance. Assume RYB phase sequence.

Solution The unbalanced star load and its equivalent delta (Δ) is shown in Fig. 9.39 (a) and (b) respectively.

$$Z_R = (4 + j8)\ \Omega = 8.944\ \angle 63.4°\ \Omega$$

$$Z_Y = (3 + j4)\ \Omega = 5\ \angle 53.1°\ \Omega$$

$$Z_B = (15 + j20)\ \Omega = 25\ \angle 53.1°\ \Omega$$

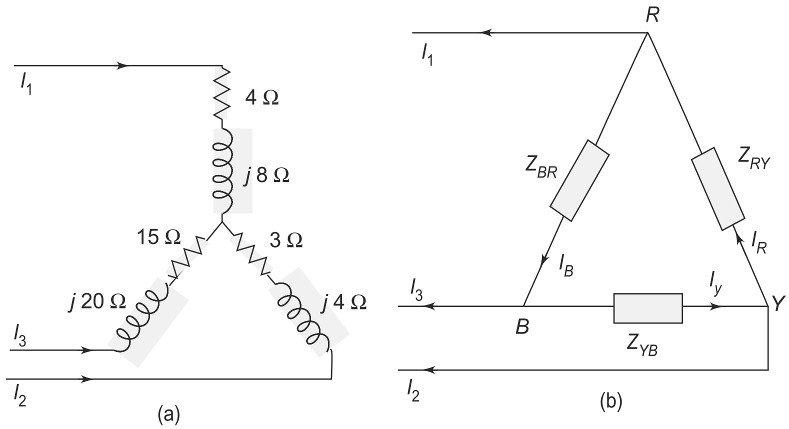


Fig. 9.39

Using the expression in Section 9.10.2, we can calculate Z_{RY}, Z_{YB} and Z_{BR}

$$
\begin{aligned}
& Z_R Z_Y + Z_Y Z_B + Z_B Z_R \\
& \quad = (8.94\ \angle 63.4°)\ (5\ \angle 53.1°) + (5\ \angle 53.1°)\ (25\ \angle 53.1°) \\
& \qquad + (25\ \angle 53.1°)\ (8.94\ \angle 63.4°) \\
& \quad = 391.80\ \angle 113.23°
\end{aligned}
$$

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B} = \frac{391.80 \angle 113.23°}{25 \angle 53.1°} = 15.67\ \angle 60.13°$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R} = \frac{391.80 \angle 113.23°}{8.94 \angle 63.4°} = 43.83\ \angle 49.83°$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y} = \frac{391.80 \angle 113.23°}{5 \angle 53.1°} = 78.36\ \angle 60.13°$$

Taking V_{RY} as reference, $V_{RY} = 400\ \angle 0$

$$V_{YB} = 400\ \angle -120°;\ V_{BR} = 400\ \angle -240°$$

$$I_R = \frac{V_{RY}}{Z_{RY}} = \frac{400 \angle 0°}{15.67 \angle 60.13°} = 25.52\ \angle -60.13°$$

$$I_Y = \frac{V_{YB}}{Z_{YB}} = \frac{400 \angle -120°}{43.83 \angle 49.83°} = 9.12\ \angle -169.83°$$

$$I_B = \frac{V_{BR}}{Z_{BR}} = \frac{400 \angle -240°}{78.36 \angle 60.13°} = 5.10\ \angle -300.13°$$

The various line currents in the delta load are

$$
\begin{aligned}
I_1 &= I_R - I_B = 25.52\ \angle -60.13° - 5.1\ \angle -300.13° \\
&= 28.41\ \angle -69.07°\ \text{A}
\end{aligned}
$$

$$I_2 = I_Y - I_R = 9.12 \angle{-169.83°} - 5.52 \angle{-60.13°}$$
$$= 29.85 \angle 136.58° \text{ A}$$
$$I_3 = I_B - I_Y = 5.1 \angle{-300.13°} - 9.12 \angle{-169.83°}$$
$$= 13 \angle 27.60° \text{ A}$$

These line currents are also equal to the line (phase) currents of the original star-connected load. The voltage drop across each star-connected load will be as follows.

Voltage drop across $Z_R = I_1 Z_R$

$$= (28.41 \angle{-69.070°})\,(8.94 \angle 63.4°) = 253.89 \angle{-5.67°} \text{ V}$$

Voltage drop across $Z_Y = I_2 Z_Y$

$$= (29.85 \angle 136.58°)\,(5 \angle 53.1°) = 149.2 \angle 189.68° \text{ V}$$

Voltage drop across $Z_B = I_3 Z_B$

$$= (13 \angle 27.60°)\,(25 \angle 53.1°) = 325 \angle 80.70° \text{ V}$$

9.10.3 Millman's Method of Solving Unbalanced Load

One method of solving an unbalanced three-wire star-connected load by star-delta conversion is described in Section 9.10.2. But this method is laborious and involves lengthy calculations. By using Millman's theorem, we can solve this type of problems in a much easier way. Consider an unbalanced wye (Y) load connected to a balanced three-phase supply as shown in Fig. 9.40 (a). V_{RO}, V_{YO} and V_{BO} are the phase voltages of the supply. They are equal in magnitude, but displaced by 120° from one another. $V_{RO'}$, $V_{YO'}$ and $V_{BO'}$ are the load phase voltages; they are unequal in magnitude as well as differ in phase by unequal angles. Z_R, Z_Y and Z_B are the impedances of the branches of the unbalanced wye (Y) connected load. Figure 9.40 (b) shows the triangular phasor diagram of the complete system. Distances *RY*, *YB,* and *BR* represent the line voltages of the supply as well as load. They are equal in magnitude, but displaced by 120°. Here, O is the star-point of the supply and is located at the centre of the equilateral triangle *RYB*. O' is the load star point. The star point of the supply which is at the zero potential is different from that of the star point at the load, due to the load being unbalanced. O' has some potential with respect to O and is shifted away from the centre of the triangle. Distance $O'O$ represents the voltage of the load star point with respect to the star point of the supply $V_{o'o}$.

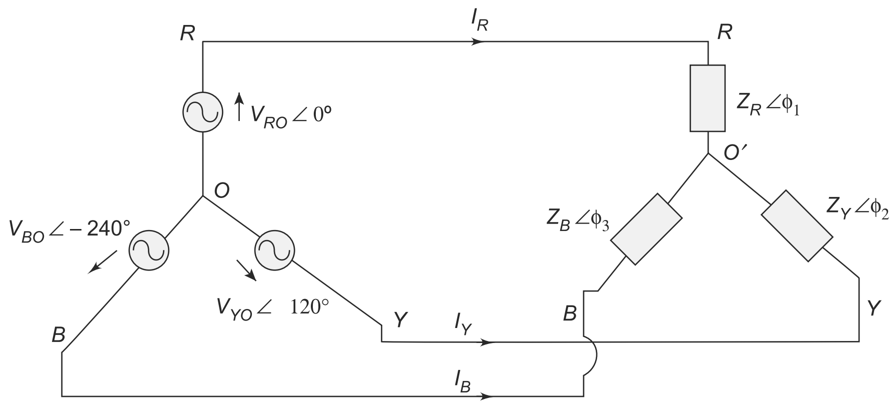


Fig. 9.40 (a)

$V_{o'o}$ is calculated using Millman's theorem. If $V_{o'o}$ is known, the load phase voltages and corresponding currents in the unbalanced wye load can be easily determined.

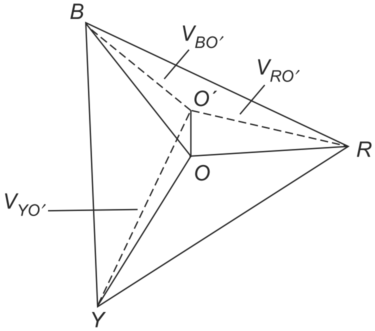


Fig. 9.40 (b)

According to Millman's theorem, $V_{o'o}$ is given by

$$V_{o'o} = \frac{V_{Ro}Y_R + V_{Yo}Y_Y + V_{Bo}Y_B}{Y_R + Y_Y + Y_B}$$

where the parameters Y_R, Y_Y, and Y_B are the admittances of the branches of the unbalanced wye connected load. From Fig. 9.40 (a), we can write the equation

$$V_{Ro} = V_{Ro'} + V_{o'o}$$

or the load phase voltage

$$V_{Ro'} = V_{Ro} - V_{o'o}$$

Similarly, $V_{Yo'} = V_{Yo} - V_{o'o}$ and $V_{Bo'} = V_{Bo} - V_{o'o}$ can be calculated. The line currents in the load are

$$I_R = \frac{V_{Ro'}}{Z_R} = (V_{Ro} - V_{o'o})Y_R$$

$$I_Y = \frac{V_{Yo'}}{Z_Y} = (V_{Yo} - V_{o'o})Y_Y$$

$$I_B = \frac{V_{Bo'}}{Z_B} = (V_{Bo} - V_{o'o})Y_B$$

The unbalanced three-wire star-connected loads can also be determined by using Kirchhoff's laws, and Maxwell's mesh or loop equation. In general, any method which gives quick results in a particular case should be used.

EXAMPLE 9.21

To illustrate the application of Millman's method to unbalanced loads, let us take the problem in example given in Section 9.10.2.

Solution The circuit diagram is shown in Fig. 9.41.

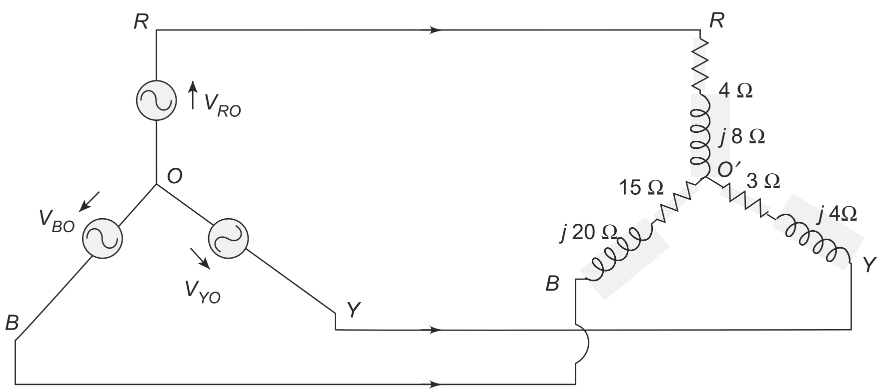


Fig. 9.41

Taking V_{RY} as reference line voltage = 400 $\angle 0°$, phase voltages lag 30° behind their respective line voltages. Therefore, the three phase voltages are

$$V_{Ro} = \frac{400}{\sqrt{3}} \angle -30° \text{ V}$$

$$V_{Yo} = \frac{400}{\sqrt{3}} \angle -150° \text{ V}$$

$$V_{Bo} = \frac{400}{\sqrt{3}} \angle -270° \text{V}$$

The admittances of the branches of the wye load are

$$Y_R = \frac{1}{Z_R} = \frac{1}{8.94 \angle 63.4°} = 0.11 \angle -63.40° \text{ ℧}$$

$$Y_Y = \frac{1}{Z_Y} = \frac{1}{5 \angle 53.1°} = 0.2 \angle -53.1° \text{ ℧}$$

$$Y_B = \frac{1}{Z_B} = \frac{1}{25 \angle 53.1°} = 0.04 \angle -53.1° \text{ ℧}$$

$$\begin{aligned} V_{Ro}Y_R + V_{Yo}Y_Y + V_{Bo}Y_B &= (230.94 \angle -30°)(0.11 \angle -63.40°) \\ &\quad + (230.94 \angle -150°)(0.2 \angle -53.1°) \\ &\quad + (230.94 \angle -270°)(0.04 \angle -53.1°) = 36.68 \angle 182.66° \end{aligned}$$

$$\begin{aligned} Y_R + Y_Y + Y_B &= 0.11 \angle -63.4° + 0.2 \angle -53.1° + 0.04 \angle -53.1° \\ &= 0.35 \angle -56.2° \end{aligned}$$

Substituting the above values in the Millman's theorem, we have

$$V_{o'o} = \frac{V_{Ro}\ Y_R + V_{Yo}\ Y_Y + V_{Bo}\ Y_B}{Y_R + Y_Y + Y_B}$$

$$= \frac{36.68 \angle 182.66°}{0.35 \angle -56.2°} = 104.8 \angle 238.86°$$

The three load phase voltages are

$$\begin{aligned} V_{Ro'} &= V_{Ro} - V_{o'o} \\ &= 230.94 \angle -30° - 104.8 \angle 238.86° = 253.89 \angle -5.67° \text{ V} \end{aligned}$$

$$\begin{aligned} V_{Yo'} &= V_{Yo} - V_{o'o} \\ &= 230.94 \angle -150° - 104.8 \angle 238.86° = 149.2 \angle 189.68° \text{ V} \end{aligned}$$

$$\begin{aligned} V_{Bo'} &= V_{Bo} - V_{o'o} \\ &= 230.94 \angle -270° - 104.8 \angle 238.86° = 325 \angle 80.7° \text{ V} \end{aligned}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 8

☆☆★**9-8-1** A balanced three-phase, star-connected voltage source has $V_{RN} = 230\ \angle 60°\ \Omega\ V_{rms}$ with RYB phase sequence, and it supplies a balanced delta-connected three-phase load. The total power drawn by the load is 15 kW at 0.8 lagging power factor. Find the line currents, load and phase currents.

☆★★**9-8.2** A 400 V, three-phase supply feeds an unbalanced three-wire, star-connected load, consisting of impedances $Z_R = 7\ \angle 10°\ \Omega$, $Z_Y = 8\ \angle 30°\ \Omega$ and $Z_B = 8\ \ \angle 50°\ \Omega$. The phase sequence is RYB. Determine the line currents and total power taken by the load.

☆☆★**9-8.3** The circuit shown in Fig. Q.3 is supplied from a 3-ϕ balanced supply. Use PSpice to calculate rms magnitudes and phase angles of currents: I_a, I_b, I_c, and I_N.

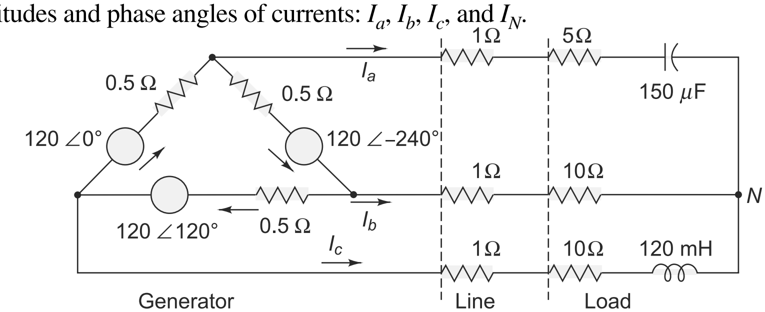


Fig. Q.3

☆★★**9-8.4** A balanced star-connected load is supplied from a symmetrical 3-phase, 440 V; 50 Hz supply. The current in phase is 20 A and lags its respective phase voltage by an angle 40°. Calculate (a) load parameters, (b) total power, and (c) readings of two wattmeters in load current to measure total power.

☆★★**9-8.5** Consider the unbalanced Δ – Δ circuit shown in Fig. Q.5. Use PSpice to find generator currents, the line currents and phase currents.

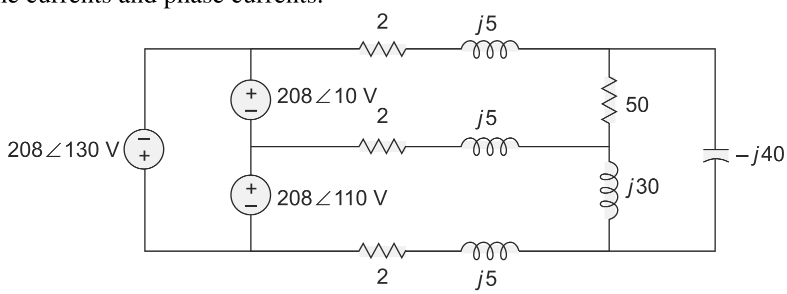


Fig. Q.5

☆★★**9-8.6** For the unbalanced circuit shown in Fig. Q.6, calculate
(a) the line currents,
(b) the real power absorbed by the load, and
(c) the total complex power supplied by the source. Use PSpice.

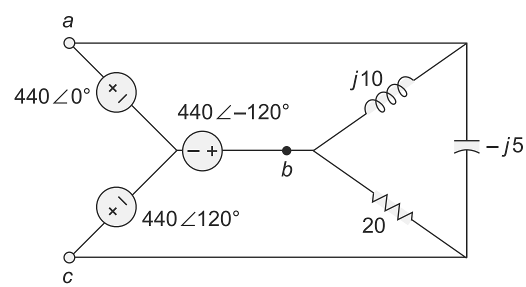


Fig. Q.6

Frequently Asked Questions linked to LO8

☆★★**9-8.1** A three-phase motor can be regarded as a balanced Y-load. A three-phase motor draws 5.6 kW when the line voltage is 220 V and the line current is 18.2 A. Determine the power factor of the motor. **[AU Nov./Dec. 2012]**

☆☆★**9-8.2** In a three-phase balanced delta system the voltage across *R* and *Y* is $400\angle 0°$V. What will be the voltage across *Y* and *B*? Assume *RYB* phase sequence. **[AU April/May 2011]**

☆★★**9-8.3** A balanced Δ-connected load has one phase current $I_{BC} = 2\angle -90°$ A. Find the other phase current and the three line currents if the system is an *ABC* system. If the line voltage is 100 V, what is the load impedance? **[AU April/May 2011]**

☆★★**9-8.4** The power consumed in a three-phase, balanced star-connected load is 2 kW at a power factor of 0.8 lagging. The supply voltage is 400 V, 50 Hz. Calculate the resistance and reactance of each phase. **[AU April/May 2011]**

☆☆★**9-8.5** Give examples for balanced networks. Why are they called so? **[PTU 2011-12]**

☆★★**9-8.6** An unbalanced star-connected load has balanced voltage of 100 V and *RBY* phase sequence. Calculate the line currents and the neutral current. **[AU May/June 2013]**
Take: $Z_A = 15\ \Omega$, $Z_B = (10 + j5)\ \Omega$, $Z_C = (6 - j8)\ \Omega$.

☆☆★**9-8.7** Distinguish between unbalanced source and unbalanced load. **[AU May/June 2013]**

☆★★**9-8.8** A symmetrical 3-phase, 100 V, 3-wire supply feeds an unbalanced star-connected load with impedances of the load as $Z_R = 5\angle 0°\ \Omega$, $Z_Y = 2\angle 90°\ \Omega$, and $Z_B = 4\angle -90°\Omega$. Find the line currents, voltage across the impedances and draw the phasor diagram. Also calculate the power consumed by the load. **[AU May/June 2014]**

☆★★**9-8.9** A three-phase four-wire 120 V *ABC* system feeds an unbalanced Y-connected load with $Z_A = 5\angle 0°$ Ω, $Z_B = 10\angle 30°\ \Omega$, and $Z_C = 20\angle 60°\ \Omega$, Obtain the four line currents. **[AU Nov./Dec. 2012]**

★★★**9-8.10** Three impedances $Z_1 = (17.32 + j10)$, $Z_2 = (20 + j34.64)$, and $Z_3 = (0 - j10)$ ohms are delta-connected to a 400 V, three-phase system. Determine the phase currents, line current, and total power consumed by the load. **[AU Nov./Dec. 2012]**

9.11 POWER MEASUREMENT IN THREE-PHASE CIRCUITS

9.11.1 Power Measurement in a Single-Phase Circuit by Wattmeter

LO 9 Execute power measurement in three-phase circuits using single, two- and three-wattmeter methods

Wattmeters are generally used to measure power in the circuits. A wattmeter principally consists of two coils, one coil is called the current coil, and the other the pressure or voltage coil. A diagramatric representation of a wattmeter connected to measure power in a single-phase circuit is shown in Fig. 9.42.

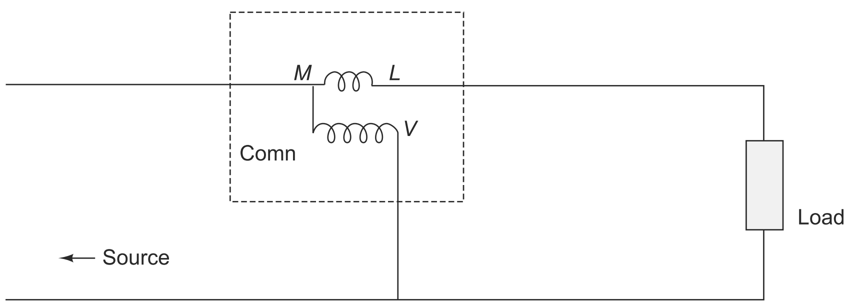


Fig. 9.42

The coil represented with less number of turns between M and L is the current coil, which carries the current in the load and has very low impedance. The coil with more number of turns between the common terminal (comn) and V is the pressure coil, which is connected across the load and has high impedance. The load voltage is impressed across the pressure coil. The terminal M denotes the mains side, L denotes load side, *common* denotes the common point of current coil and pressure coil, and V denotes the second terminal of the pressure coil, usually selected as per the range of the load voltage in the circuit. From the figure, it is clear that a wattmeter has four terminals, two for current coil and two for potential coil. When the current flow through the two coils, they set up magnetic fields in space. An electromagnetic torque is produced by the interaction of the two magnetic fields. Under the influence of the torque, one of the coils (which is movable) moves on a calibrated scale against the action of a spring. The instantaneous torque produced by electromagnetic action is proportional to the product of the instantaneous values of the currents in the two coils. The small current in the pressure coil is equal to the input voltage divided by the impedance of the pressure coil. The inertia of the moving system does not permit it to follow the instantaneous fluctuations in torque. The wattmeter deflection is therefore, proportional to the average power ($VI \cos \phi$) delivered to the circuit. Sometimes, a wattmeter connected in the circuit to measure power gives downscale reading or backward deflection. This is due to improper connection of the current coil and pressure coil.

To obtain up scale reading, the terminal marked as 'Comn' of the pressure coil is connected to one of the terminals of the current coil as shown in Fig. 9.43. Note that the connection between the current coil terminal and pressure coil terminal is not inherent, but has to be made externally. Even with proper connections, sometimes the wattmeter will give downscale reading whenever the phase angle between the voltage across the pressure coil and the current through the current coil is more than 90°. In such a case, connection of either the current coil or the pressure coil must be reversed.

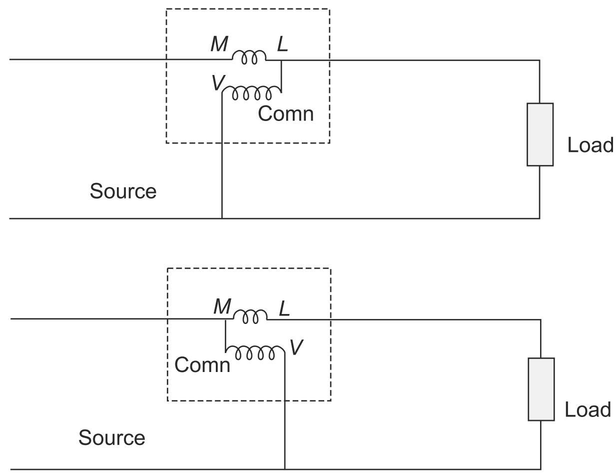


Fig. 9.43

9.11.2 Power in Three-Phase Circuits

Measurement of power by a wattmeter in a single-phase circuit can be extended to measure power in a three-phase circuit. From Section 9.11.1, it is clear that we require three wattmeters, one in each phase to measure the power consumed in a three-phase system. Obviously, the total power is the algebraic sum of the readings of the three wattmeters. In this way we can measure power in balanced and unbalanced loads. In a balanced case it would be necessary to measure power only in one phase and the reading is multiplied by three to get the total power in all the three phases. This is true in principle, but presents a few difficulties in practice. To verify this fact let us examine the circuit diagram in Figs 9.44 (a) and (b).

Observation of Figs 9.44 (a) and (b) reveals that for a star-connected load, the neutral must be available for connecting the pressure coil terminals. The current coils must be inserted in each phase for a delta-connected load. Such connections sometimes may not be practicable, because the neutral terminal is not available all the time in a star-connected load, and the phases of the delta-connected load are not accessible for connecting the current coils of the wattmeter. In most of the commercially available practical three-phase loads, only three line terminals are available. We, therefore, require a method where we can measure power in the three-phases

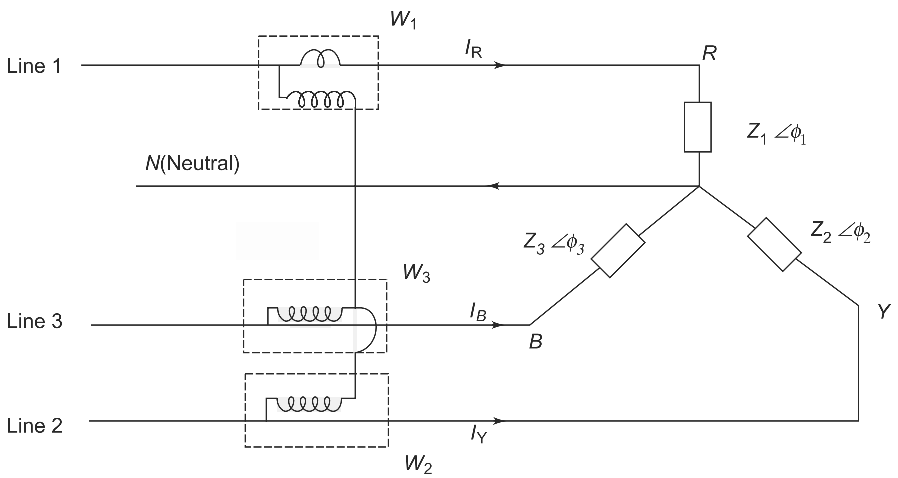


Fig. 9.44 (a)

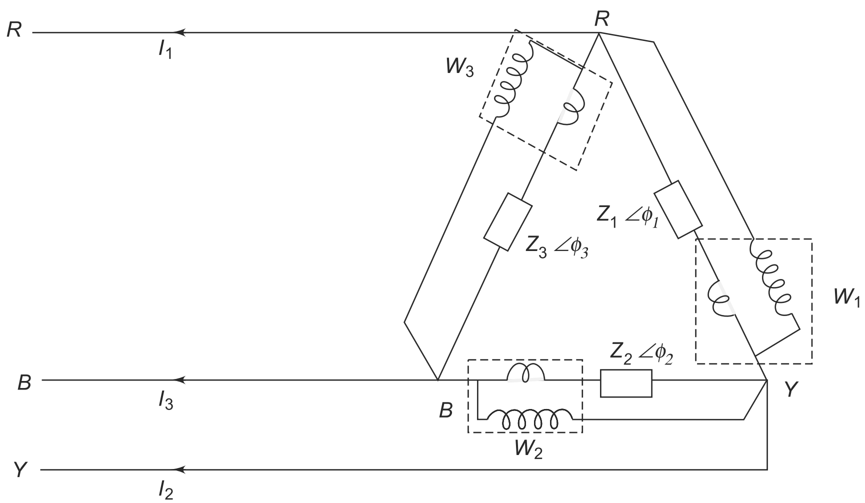


Fig. 9.44 (b)

with an access to the three lines connecting the source to the load. Two such methods are discussed here.

9.11.3 Three-Wattmeter and Two-Wattmeter Methods

In this method, the three wattmeters are connected in the three lines as shown in Fig. 9.45, i.e. the current coils of the three wattmeters are introduced in the three lines, and one terminal of each potential coil is connected to one terminal of the corresponding current coil, the other three being connected to some common point which forms an effective neutral *n*.

The load may be either star-connected or delta-connected. Let us assume a star-connected load, and let the neutral of this load be denoted by *N*. Now the reading on the wattmeter W_R will correspond to the average value of the product of the instantaneous value of the current I_R flowing in line 1, with the voltage drop V_{Rn}, where V_{Rn} is the voltage between points *R* and *n*. This can be written as $V_{Rn} = V_{RN} + V_{Nn}$, where V_{RN} is the load phase voltage and V_{Nn} is the voltage between load neutral, *N*, and the common point, *n*. Similarly, $V_{Yn} = V_{YN} + V_{Nn}$,

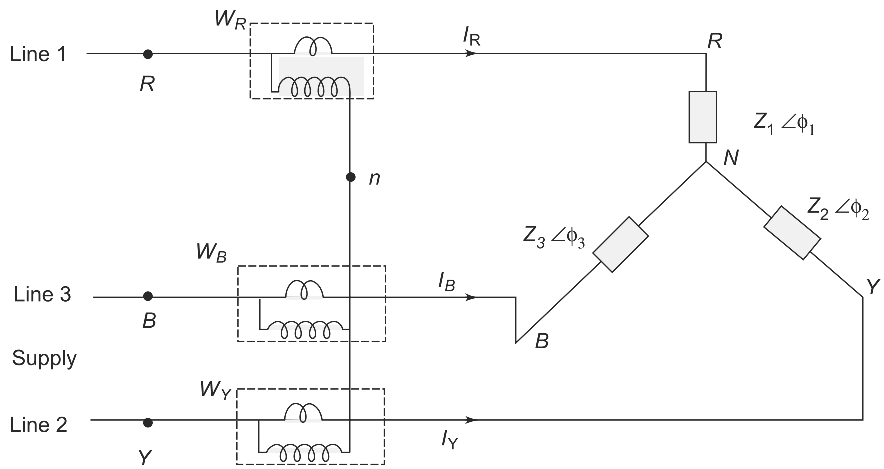


Fig. 9.45

and $V_{Bn} = V_{BN} + V_{Nn}$. Therefore, the average power, W_R indicated by the wattmeter is given by

$$W_R = \frac{1}{T}\int_o^T V_{Rn} I_R \, dt$$

where T is the time period of the voltage wave

$$W_R = \frac{1}{T}\int_o^T (V_{RN} + V_{Nn}) I_R \, dt$$

Similarly, $$W_Y = \frac{1}{T}\int_o^T V_{Yn} \, I_Y \, dt$$

$$= \frac{1}{T}\int_o^T (V_{YN} + V_{Nn}) I_Y \, dt$$

and $$W_B = \frac{1}{T}\int_o^T V_{Bn} \, I_B \, dt$$

$$= \frac{1}{T}\int_o^T (V_{BN} + V_{Nn}) I_B \, dt$$

Total average power $= W_R + W_Y + W_B$

$$= \frac{1}{T}\int_o^T (V_{RN} \, I_R + V_{YN} \, I_Y + V_{BN} \, I_B) dt + \frac{1}{T}\int_o^T V_{Nn} (I_R + I_Y + I_B) dt$$

Since the system in the problem is a three-wire system, the sum of the three currents I_R, I_Y, and I_B at any given instant is zero. Hence, the power read by the three wattmeters is given by

$$W_R + W_Y + W_B = \frac{1}{T}\int_o^T (V_{RN} \, I_R + V_{YN} \, I_Y + V_{BN} \, I_B) dt$$

If the system has a fourth wire, i.e. if the neutral wire is available, then the common point, n is to be connected to the system neutral, N. In that case, V_{Nn} would be zero, and the above equation for power would still be valid. In other words, whatever be the value of V_{Nn}, the algebraic sum of the three currents I_R, I_Y, and I_B is zero. Hence, the term V_{Nn} $(I_R + I_Y + I_B)$ would be zero. Keeping this advantage in mind, suppose the common point, n, in Fig. 9.45 is connected to line B. In such case, $V_{Nn} = V_{NB}$; then the voltage across the potential coil of wattmeter W_B will be zero and this wattmeter will read zero. Hence, this can be removed from the circuit. The total power is read by the remaining two wattmeters, W_R and W_Y.

$\therefore$ total power $= W_R + W_Y$

Let us verify this fact from Fig. 9.46.

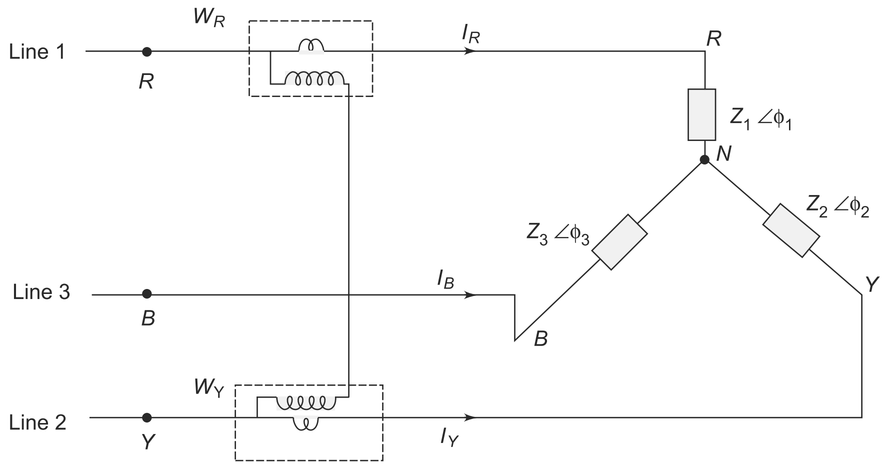


Fig. 9.46

The average power indicated by the wattmeter W_R is

$$W_R = \frac{1}{T}\int_o^T V_{RB}\, I_R\, dt$$

and that by $$W_Y = \frac{1}{T}\int_o^T V_{YB}\quad I_Y\, dt$$

Also $$V_{RB} = V_{RN} + V_{NB}$$

$$V_{YB} = V_{YN} + V_{NB}$$

$$W_R + W_Y = \frac{1}{T}\int_o^T \left(V_{RB}\cdot I_R + V_{YB} I_Y\right) dt$$

$$= \frac{1}{T}\int_o^T \left\{\left(V_{RN} + V_{NB}\right) I_R + \left(V_{YN} + V_{NB}\right) I_Y\right\} dt$$

$$= \frac{1}{T}\int_o^T \left\{\left(V_{RN} I_R + V_{YN} I_Y\right) + \left(I_R + I_Y\right) V_{NB}\right\} dt$$

We know that $\quad I_R + I_Y + I_B = 0$

$$I_R + I_Y = -I_B$$

Substituting this value in the above equation, we get

$$W_R + W_Y = \frac{1}{T}\int_o^T \left\{\left(V_{RN} I_R + V_{YN} I_Y\right) + \left(-I_B\right) V_{NB}\right\} dt$$

$$V_{NB} = -V_{BN}$$

$$W_R + W_Y = \frac{1}{T}\int_o^T \left\{\left(V_{RN} I_R + V_{YN} \cdot I_Y + V_{BN} \cdot I_B\right)\right\} dt$$

which indicates the total power in the load.

From the above discussion, it is clear that the power in a three-phase load, whether balanced or unbalanced, star-connected or delta-connected, three-wire or four-wire, can be measured with only two wattmeters as shown in Fig. 9.46. In fact, the two wattmeter method of measuring power in three-phase loads has become a universal method. If neutral wire is available in this method it should not carry any current, or the neutral of the load should be isolated from the neutral of the source.

The current flowing through the current coil of each wattmeter is the line current, and the voltage across the pressure coil is the line voltage. In case the phase angle between line voltage and current is greater than 90°, the corresponding wattmeter would indicate downscale reading. To obtain upscale reading, the connections of either the current coil, or the pressure coil has to be interchanged. Reading obtained after reversal of coil connection should be taken as negative. Then, the algebraic sum of the two wattmeter readings gives the total power.

9.11.4 Power Factor by Two-Wattmeter Method

When we talk about the power factor in three-phase circuits, it applies only to balanced circuits, since the power factor in a balanced load is the power factor of any phase. We cannot strictly define the power factor in three-phase unbalanced circuits, as every phase has a separate power factor. The two-wattmeter method, when applied to measure power in a three-phase balanced circuits, provides information that help us to calculate the power factor of the load.

Figure 9.47 shows the vector diagram of the circuit shown in Fig. 9.46. Since the load is assumed to be balanced, we can take $Z_1 \angle\phi_1 = Z_2 \angle\phi_2 = Z_3 \angle\phi_3 = Z \angle\phi$ for the star-connected load. Assuming *RYB* phase sequence, the three rms load phase voltages are V_{RN}, V_{YN} and $V_{BN} \cdot I_R$, I_Y and I_B are the rms line (phase) currents. These currents will lag behind their respective phase voltages by an angle ϕ. (An inductive load is considered).

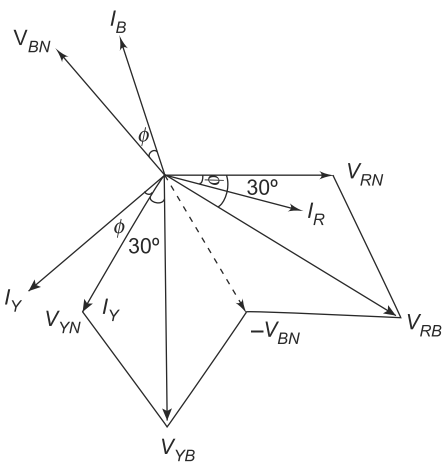


Fig. 9.47

Now consider the readings of the two wattmeters in Fig. 9.46. W_R measures the product of effective value of the current through its current coil I_R, effective value of the voltage across its pressure coil V_{RB} and the cosine of the angle between the phasors I_R and V_{RB}. The voltage across the pressure coil of W_R is given as follows.

$V_{RB} = V_{RN} - V_{BN}$ phasor difference

It is clear from the phasor diagram that the phase angle between

V_{RB} and I_R is $(30° - \phi)$

$\therefore \qquad W_R = V_{RB} I_R \cos(30° - \phi)$

Similarly, W_Y measures the product of effective value of the current through its current coil I_Y, the effective value of the voltage across its pressure coil, V_{YB} and the cosine of the angle between the phasors V_{YB} and I_Y.

$$V_{YB} = V_{YN} - V_{BN}$$

From Fig. 9.47, it is clear that the phase angle between V_{YB} and I_Y is $(30° + \phi)$.

$$\therefore \qquad W_Y = V_{YB}\, I_Y \cos (30° + \phi)$$

Since the load is balanced, the line voltage $V_{RB} = V_{YB} = V_L$ and the line current $I_R = I_Y = I_L$

$$W_R = V_L \cdot I_L \cos (30° - \phi)$$

$$W_Y = V_L I_L \cos (30° + \phi)$$

Adding W_R and W_Y gives total power in the circuit, thus

$$W_R + W_Y = \sqrt{3}\, V_L I_L \cos\phi$$

From the two wattmeter readings, it is clear that for the same load angle ϕ, wattmeter W_R registers more power when the load is inductive. It is also connected in the leading phase as the phase sequence is *RYB*. Therefore, W_R is higher reading wattmeter in the circuit of Fig. 9.46. In other words, if the load is capacitive, the wattmeter connected in the leading phase reads less for the same load angle. So, if we know the nature of the load, we can easily identify the phase sequence of the system. The higher reading wattmeter always reads positive. By proper manipulation of two wattmeter readings, we can obtain the power factor of the load.

$$W_R = V_L I_L \cos (30° - \phi) \qquad \text{(Higher reading)}$$

$$W_Y = V_L I_L \cos (30° + \phi) \qquad \text{(Lower reading)}$$

$$W_R + W_Y = \sqrt{3}\, V_L I_L \cos\phi$$

$$W_R - W_Y = V_L I_L \sin \phi$$

Taking the ratio of the above two values, we get

$$\frac{W_R - W_Y}{W_R + W_Y} = \frac{\tan\phi}{\sqrt{3}}$$

or

$$\tan\phi = \sqrt{3}\left[\frac{W_R - W_Y}{W_R + W_Y}\right]$$

$$\phi = \tan^{-1} \sqrt{3}\left[\frac{W_R - W_Y}{W_R + W_Y}\right]$$

Thereafter, we can find $\cos \phi$.

EXAMPLE 9.22

The two-wattmeter method is used to measure power in a three-phase load. The wattmeter readings are 400 W and –35 W. Calculate (a) total active power, (b) power factor, and (c) reactive power.

Solution From the given data, the two wattmeter readings $W_R = 400$ W (Higher reading wattmeter) $W_Y = -35$ W (Lower reading wattmeter).

(a) Total active power $= W_1 + W_2$

$$= 400 + (-35) = 365 \text{ W}$$

(b) $$\tan\phi = \sqrt{3}\,\frac{W_R \quad W_Y}{W_R \quad W_Y} = \sqrt{3}\,\frac{400-(-35)}{400+(-35)} = \sqrt{3}\times\frac{435}{365} = 2.064$$

$$\phi = \tan^{-1} 2.064 = 64.15°;\ \text{P.F} = 0.43$$

(c) Reactive power $= \sqrt{3}\, V_L I_L \sin\phi$

We know that $W_R - W_Y = V_L I_L \sin \phi$

$$\therefore \quad W_R - W_Y = 400 - (-35) = 435$$

$$\text{Reactive power} = \sqrt{3} \times 435 = 753.44 \text{ VAR}$$

EXAMPLE 9.23

The input power to a three-phase load is 10 kW at 0.8 Pf. Two wattmeters are connected to measure the power, find the individual readings of the wattmeters.

Solution Let W_R be the higher reading wattmeter and W_Y the lower reading wattmeter

$$W_R + W_Y = 10 \text{ kW} \tag{9.1}$$

$$\phi = \cos^{-1} 0.8 = 36.8°$$

$$\tan\phi = 0.75 = \sqrt{3}\frac{W_R - W_Y}{W_R + W_Y}$$

or $$W_R - W_Y = \frac{(0.75)}{\sqrt{3}}(W_R + W_Y) = \frac{0.75}{\sqrt{3}} \times 10 \text{ kW}$$

$$= 4.33 \text{ kW} \tag{9.2}$$

From Eqs (9.1) and (9.2), we get

$$W_R + W_Y = 10 \text{ kW}$$

$$\frac{W_R - W_Y - 4.33 \text{ kW}}{2W_R - 14.33 \text{ kW}}$$

or $$W_R = 7.165 \text{ kW}$$

$$W_Y = 2.835 \text{ kW}$$

9.11.5 Variation in Wattmeter Readings with Load Power Factor

It is useful to study the effect of the power factor on the readings of the wattmeter. In Section 9.11.4, we have proved that the readings of the two wattmeters depend on the load power factor angle ϕ, such that

$$W_R = V_L I_L \cos (30 - \phi)°$$

$$W_Y = V_L I_L \cos (30 + \phi)°$$

We can, therefore, make the following deductions

1. When ϕ is zero, i.e. power factor is unity, from the above expressions we can conclude that the two wattmeters indicate equal and positive values.
2. When ϕ rises from 0 to 60°, i.e. upto power factor 0.5, the wattmeter W_R reads positive (since it is connected in the leading phase); whereas wattmeter W_Y reads positive, but less than W_R. When $\phi = 60°$, $W_Y = 0$ and the total power is being measured only by wattmeter W_R.
3. If the power factor is further reduced from 0.5, i.e. when ϕ is greater than 60°, W_R indicates positive value, whereas W_Y reads down scale reading in such case. As already explained in Section 9.11.3,

the connections of either the current coil, or the pressure coil of the corresponding wattmeter have to be interchanged to obtain an upscale reading, and the reading thus obtained must be given a negative sign. Then the total power in the circuit would be $W_R + (-W_Y) = W_R - W_Y$. Wattmeter W_Y reads downscale for the phase angle between 60° and 90°. When the power factor is zero (i.e. $\phi = 90°$), the two wattmeters will read equal and opposite values.

i.e. $$W_R = V_L I_L \cos(30-90)° = 0.5\, V_L I_L$$
$$W_Y = V_L I_L \cos(30+90)° = -0.5\, V_L I_L$$

9.11.6 Leading Power Factor Load

The above observations are made considering the lagging power factor. Suppose the load in Fig. 9.46 (a) is capacitive, the wattmeter connected in the leading phase would read less value. In that case, W_R will be the lower reading wattmeter, and W_Y will be the higher reading wattmeter. Figure 9.48 shows the phasor diagram for the leading power factor.

As the power factor is leading, the phase currents, I_R, I_Y and I_B are leading their respective phase voltage by an angle ϕ. From Fig. 9.48, the reading of the wattmeter connected in the leading phase is given by

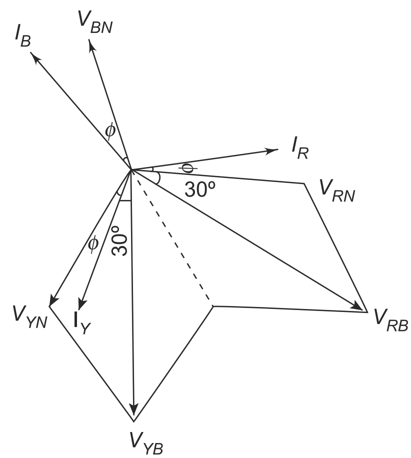


Fig. 9.48

$$W_R = V_{RB} \,.\, I_R \cos(30+\phi)°$$
$$= V_L I_L \cos(30+\phi)° \text{ (lower reading wattmeter)}$$

Similarly, the reading of the wattmeter connected in the lagging phase is given by

$$W_Y = V_{YB}\, I_Y \cos(30-\phi)°$$
$$= V_L I_L \cos(30-\phi)° \text{ (higher reading wattmeter)}$$

Again, the total power is given by

$$W_R + W_Y = \sqrt{3}\, V_L I_L \cos\phi$$
$$W_Y - W_R = V_L I_L \sin\phi$$

Hence $$\tan\phi = \sqrt{3}\,\frac{W_Y - W_R}{W_Y + W_R}$$

A comparison of this expression with that of lagging power factor reveals the fact that the two wattmeter readings are interchanged, i.e. for lagging power factor, W_R is the higher reading wattmeter, and W_Y is the lower reading wattmeter; where as for leading power factor, W_R is the lower reading wattmeter and W_Y is the higher reading wattmeter. While using the expression for power factor, whatever may be the nature of the load, the lower reading is to be subtracted from the higher reading in the numerator. The variation in the wattmeter reading with the capacitive load follows the same sequence as in inductive load, with a change in the roles of wattmeters.

EXAMPLE 9.24

The readings of the two wattmeters used to measure power in a capacitive load are –3000 W and 8000 W, respectively. Calculate (a) the input power, and (b) the power factor at the load. Assume RYB sequence.

Solution (a) Total power $= W_R + W_Y = -3000 + 8000 = 5000$ W

(b) As the load is capacitive, the wattmeter connected in the leading phase gives less value.

$\therefore \quad W_R = -3000$

Consequently, $W_Y = 8000$

$$\tan\phi = \sqrt{3}\,\frac{W_Y - W_R}{W_Y + W_R} = \sqrt{3}\,\frac{(8000-(-3000))}{5000} = 3.81$$

$\therefore \quad \phi = 75.29°$ (lead); $\cos\phi = 0.25$

9.11.7 Reactive Power with Wattmeter

We have already seen in the preceding section that the difference between higher reading wattmeter and lower reading wattmeter yields $V_L I_L \sin\phi$. So, the total reactive power $= \sqrt{3}\,V_L I_L \sin\phi$. Reactive power in a balanced three-phase load can also be calculated by using a single wattmeter.

As shown in Fig. 9.49 (a), the current coil of the wattmeters is connected in any one line (R in this case), and the pressure coil across the other two lines (between Y and B in this case). Assuming phase sequence RYB and an inductive load of angle ϕ, the phasor diagram for the circuit in Fig. 9.49 (a) is shown in Fig. 9.49 (b).

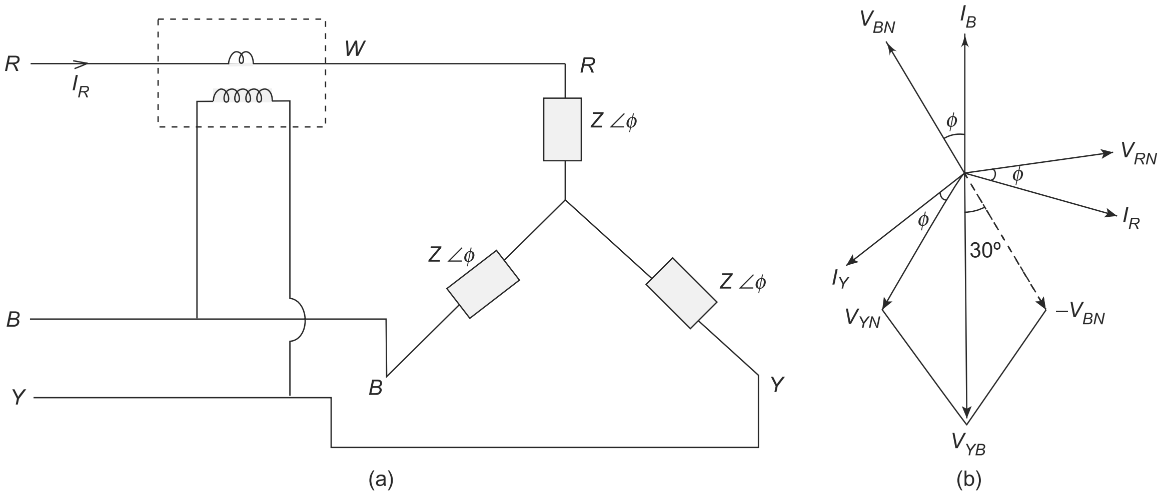


Fig. 9.49

From Fig. 9.49 (a), it is clear that the wattmeter power is proportional to the product of current through its current coil, I_R, voltage across its pressure coil, V_{YB}, and cosine of the angle between V_{YB} and I_R.

or $\quad V_{YB} = V_{YN} - V_{BN} = V_L$

From the vector diagram the angle between V_{YB} and I_R is $(90 - \phi)°$

$\therefore$ wattmeter reading $= V_{YB} I_R \cos(90-\phi)°$

$= V_L I_L \sin\phi$ VAR

If the above expression is multiplied by $\sqrt{3}$, we get the total reactive power in the load.

EXAMPLE 9.25

A single wattmeter is connected to measure reactive power of a three-phase, three-wire balanced load as shown in Fig. 9.49 (a). The line current is 17 A and the line voltage is 440 V. Calculate the power factor of the load if the reading of the wattmeter is 4488 VAR.

Solution We know that wattmeter reading is equal to $V_L I_L \sin\phi$.

$\therefore$ $4488 = 440 \times 17 \sin\phi$

$\sin\phi = 0.6$

Power factor $= \cos\phi = 0.8$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 9

☆☆★**9-9.1** In the two-wattmeter method of power measurement, the power registered by one wattmeter is 3500 W, while the other reads down scale. After reversing the later, it reads 300 W. Determine the total power in the circuit and the power factor.

☆☆★**9-9.2** Three impedances $20\underline{|0^\circ}\ \Omega$; $16\underline{|20^\circ}\ \Omega$ and $25\underline{|90^\circ}\ \Omega$ are connected in delta across a balanced supply of $250\underline{|0^\circ}$ V. Find phase currents; active power, and Reactive power in each phase and total apparent power.

Frequently Asked Questions linked to LO9

☆★★**9-9.1** The two-wattmeter method produces wattmeter readings $P_1 = 1560$ W and $P_2 = 2100$ W when connected to a delta-connected load. If the line voltage is 220 V, calculate (a) the per-phase average power, (b) the per-phase reactive power, (c) the power factor, and (d) the phase impedance. **[AU May/June 2013]**

★★★**9-9.2** The two-wattmeter method is used to measure power in a three-phase delta-connected load. The dlta-connected load consists of $Z_{RY} = 10\,j\,10\ \Omega$, $Z_{YB} = 15 - j15\ \Omega$, and $Z_{BR} = 20 + j\,10\ \Omega$ and it is connected to a 400 V, three-phase supply of phase sequence *RYB*. Calculate the readings of the wattmeter with current coil in lines *R* and *B*. **[AU May/June 2014]**

☆☆★**9-9.3** Show that two wattmeters are sufficient to measure power in a balanced or un-balanced three-phase load connected to a balanced supply. **[AU April/May 2011]**

★★★**9-9.4** A three-phase, 220 V, 50 Hz, 11.2 kW induction motor has a full-load efficiency of 88% and draws a line current of 38 A under full load, when connected to a three-phase, 220 V supply. Find the readings on two wattmeters connected in the circuit to measure the input to the motor. Determine also the power factor at which the motor is operating. **[AU April/May 2011]**

9.12 EFFECTS OF HARMONICS

LO 10 Analyse the effects of harmonics in wye and delta connections

The relationship between line and phase quantities for wye and delta connections as derived earlier are strictly valid only if the source voltage is purely sinusoidal. Such a waveform is an ideal one. Modern alternations are designed to give a terminal voltage which is almost sinusoidal. But it is nearly impossible to realise an ideal waveform in practice. All sinusoidally varying alternating waveforms deviate to a greater or lesser degree, from an ideal sinusoidal shape. Due to non-uniform distribution of the field flux and armature reaction in ac. machines, the current and voltage waves may get distorted. Such waveforms are referred to as non-sinusoidal or complex waveforms. In modern machines this distortion is relatively small. All non-sinusoidal waves can be broken up into a series of sinusoidal waves whose frequencies are integral multiples of the frequency of the fundamental wave. The sinusoidal components of a complex wave are called *harmonics*. It is therefore necessary to consider the effect of certain harmonics on currents and voltages in the phase of three-phase wye and delta systems in effecting the line and phase quantities.

The fundamental wave is called the basic wave or first harmonic. The second harmonic has a frequency of twice the fundamental, the third harmonic frequency is three times the fundamental frequency, and so on. Each harmonic is a pure sinusoid. Waves having $2f$, $4f$, $6f$, etc. are called *even harmonics* and those having frequencies $3f$, $5f$, $7f$, etc. are called *odd harmonics*. Since the negative half of the wave is a reproduction of the positive half, the even harmonics are absent. Therefore, a complex wave can be represented as a sum of fundamental and odd harmonics.

9.12.1 Harmonic Effect in Wye

Let us consider a wye connected generator winding, whose arrangement is shown diagrammatically in Fig. 9.50.

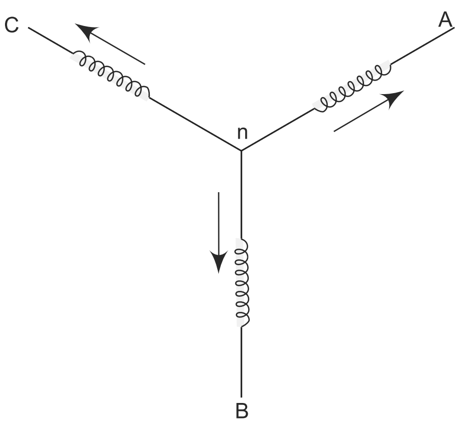


Fig. 9.50

The voltage induced in phase a of the three-phase symmetrical system, including odd harmonics is given by

$$V_{na} = E_{m_1}\sin(\omega t + \theta_1) + E_{m_3}\sin(3\omega t + \theta_3) + E_{m_5}\sin(5\omega t + \theta_5) + \ldots \quad (9.3)$$

where E_{m_1}, E_{m_3}, E_{m_5}, etc. are the peak values of the fundamental and other harmonics and θ_1, θ_3, θ_5, etc. are phase angles. Assuming *abc* phase sequence. The voltage in phase *b* will be

$$\begin{aligned} V_{nb} &= E_{m_1}\sin(\omega t + \theta_1 - 120°) + E_{m_3}\sin(3\omega t - 360° + \theta_3) + E_{m_5}\sin(5\omega t - 600° + \theta_5) + \ldots \\ &= E_{m_1}\sin(\omega t + \theta_1 - 120°) + E_{m_3}\sin(3\omega t + \theta_3) + E_{m_5}\sin(5\omega t + \theta_5 - 240°) + \ldots \end{aligned} \quad (9.4)$$

The voltage in phase *c* will be

$$\begin{aligned} V_{nc} &= E_{m_1}\sin(\omega t + \theta_1 - 240°) + E_{m_3}\sin(3\omega t + \theta_3 - 720°) + E_{m_5}\sin(5\omega t + \theta_5 - 1200°) + \ldots \\ &= E_{m_1}\sin(\omega t + \theta_1 - 240°) + E_{m_3}\sin(3\omega t + \theta_3) + E_{m_5}\sin(5\omega t + \theta_5 - 120°) + \ldots \end{aligned} \quad (9.5)$$

Equations (9.3), (9.4), and (9.5) show that all third harmonics are in time phase with each other in all the three phases as shown in Fig. 9.51 (a). The same applies to the, ninth, fifteenth, twenty first… harmonics, i.e. all odd multiples of 3. Other than odd multiples of 3, the fifth, seventh, eleventh… and all other harmonics are displaced 120° in time phase mutually with either the same phase sequence or opposite phase sequence compared with that of the fundamentals. Fifth harmonics and seventh harmonic sequences are shown in Figs 9.5 (c) and 9.5 (d) respectively.

Summarising the above facts, the fundamental and all those harmonics obtained by adding a multiple of 6, i.e. 1, 7, 13, 19,…, etc. will have the same sequence. Similarly, the fifths and all harmonics obtained by adding a multiple of 6, i.e. 5, 11, 17, 23,…, etc. will have sequence opposite to that of the fundamental.

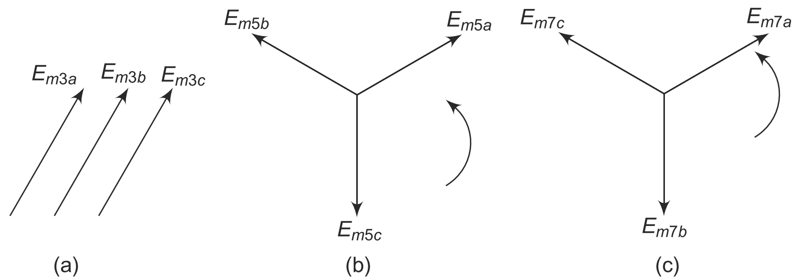


Fig. 9.51

❑ ***Voltage Relations*** The voltage between lines ab in the wye connected winding in Fig. 9.50 may be written as

$$e_{ab} = e_{an} + e_{nb}$$

Adding of each harmonic separately is shown in Fig. 9.52.

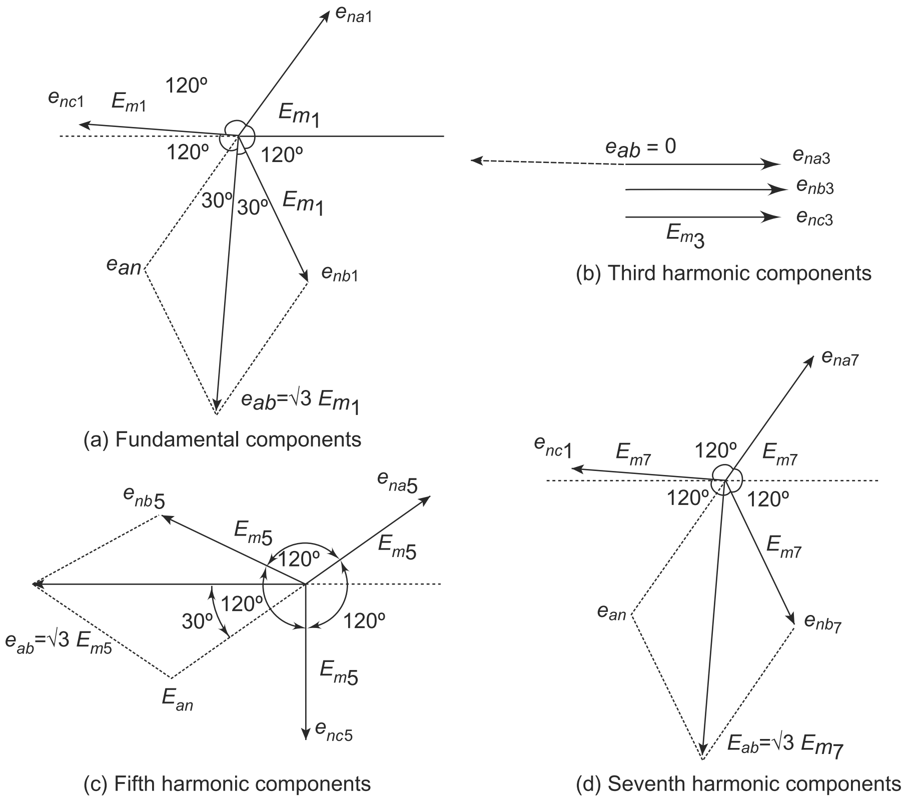


Fig. 9.52

It is seen from the vector diagrams of Fig. 9.52 that there is no third harmonic component in the line voltage. Hence, the rms value of the line voltage is given by

$$E_{ab} = \sqrt{3}\sqrt{\frac{E_{m_1}^2 + E_{m_5}^2 + E_{m_7}^2 + \ldots}{2}} \tag{9.6}$$

and the rms value of the phase voltage is

$$E_{na} = \sqrt{\frac{E_{m_1}^2 + E_{m_3}^2 + E_{m_5}^2 + E_{m_7}^2 + \ldots}{2}} \tag{9.7}$$

It is seen from the above equations that in a wye-connected system, the line voltage is not equal to $\sqrt{3}$ times the phase voltage if harmonics are present. This is true only when the third harmonics are absent.

❑ ***Current Relations*** Similar to the complex voltage wave, the instantaneous value of the complex current wave can be written as

$$i = I_{m_1} \sin(\omega t + \phi_1) + I_{m_3} \sin(3\omega t + \phi_3) + I_{m_5} \sin(5\omega t + \phi_5) \tag{9.8}$$

where I_{m_1}, I_{m_3}, I_{m_5}, etc. are the peak values of fundamental and other harmonics; $(\phi_1 - \phi_1)$ is the phase difference between fundamental component of the harmonic voltage and current and $(\phi_3 - \phi_3)$ is the phase difference between 3rd harmonics and so on. Applying *KCL* for the three phase wye connected winding in Fig. 9.50.

$$i_{na} + i_{nb} + i_{nc} = 0 \tag{9.9}$$

The equations for i_{na}, i_{nb} and i_{nc} can be obtained by replacing currents in the place of voltages in Eqs (9.3), (9.4) and (9.5) under balanced conditions the sum of the three currents is equal to zero, only when they have equal magnitudes and displaced by 120° apart in time phase in the three phases. All harmonics fulfil the above condition except the third harmonics and their odd multiples as they are in the same phase as shown in Fig. 9.51 (a) or 9.52 (b). Hence, the resultant of $i_{na} + i_{nb} + i_{nc}$ consists of the arithmetic sum of the third harmonics in the three phases. Hence, there must be a neutral wire or fourth wire to provide a return path for the third harmonic. We can summarise the above facts as follows. In a balanced three-wire wye connection, all harmonics are present except third and its odd multiples. In a four-wire wye connection, i.e. with a neutral wire, all harmonics will exist.

9.12.2 Harmonic Effect in Delta

Let the three windings of the generator be delta-connected as shown in Fig. 9.53. Let v_{na}, v_{nb} and v_{nc} be the phase emfs and v_{na}, v_{nb} and v_{nc} be the terminal voltages of the three phases a, b and c respectively. According to KVL, the algebraic sum of the three terminal voltages in the closed loop is given by

$$\begin{aligned} &v_{na} + v_{nb} + v_{nc} \\ &\quad = v_{ca} + v_{ab} + v_{bc} = 0 \end{aligned} \tag{9.10}$$

There will be a circulating current in the closed loop due to the resultant third harmonic and their multiple induced emfs. This resultant emf is dropped in the closed loop impedance. Hence, the third harmonic voltage does not appear across the terminals of the delta. Hence, the terminal voltages in delta connection v_{ca}, v_{ab} and v_{bc} are given by equations 1, 2, and 3 respectively without third harmonic terms.

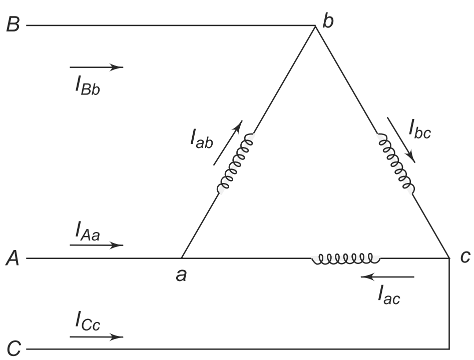


Fig. 9.53

❑ ***Current Relations*** The three phase windings in Fig. 9.53, carry all the harmonics internally and are given by

$$i_{na} = i_{ca} = I_{m_1} \sin(\omega t + \theta_1) + I_{m_2} \sin(3\omega t + \theta_3) + I_{m_5} \sin(5\omega t + \theta_5) + \dots \tag{9.11}$$

$$\begin{aligned} i_{nb} = i_{ab} &= I_{m_1} \sin(\omega t + \theta_1 - 120°) + I_{m_3} \sin(3\omega t + \theta_3 - 360°) \\ &\quad + I_{m_5} \sin(5\omega t + \theta_5 - 600°) + \dots \\ &= I_{m_1} \sin(\omega t + \theta_1 - 120°) + I_{m_3} \sin(3\omega t + \theta_3) + I_{m_5} \sin(5\omega t + \theta_5 - 240°) + \dots \end{aligned} \tag{9.12}$$

$$\begin{aligned} i_{nc} = i_{bc} &= I_{m_1} \sin(\omega t + \theta_1 - 240°) + I_{m_3} \sin(3\omega t + \theta_3 - 720°) + I_{m_5} \sin(5\omega t + \theta_5 - 1200°) \\ &= I_{m_1} \sin(\omega t + \theta_1 - 240°) + I_{m_3} \sin(3\omega t + \theta_3) + I_{m_5} \sin(5\omega t + \theta_5 - 120°) \end{aligned} \tag{9.13}$$

Equations (9.11), (9.12) and (9.13) represent the phase currents in the delta connection. The line currents I_{Aa} I_{Bb} and I_{Cc} can be obtained by applying KCL at the three junctions of the delta in Fig. 9.53. The current vector diagrams are similar to the voltage vector diagrams shown in Fig. 9.52 except that the voltages are to be replaced with currents. The line currents can be obtained in terms of phase currents by applying KCL at three junctions as follows:

$$\begin{aligned} i_{Aa} &= i_{ab} - i_{Ca} \\ &= I_{m_1} \sin(\omega t + \theta_1 - 120°) + I_{m_5} \sin(5\omega t + \theta_5 - 240°) \\ &\quad - I_{m_1} \sin(\omega t + \theta_1) - I_{m_5} \sin(5\omega t + \theta_5) \end{aligned} \tag{9.14}$$

$$\begin{aligned} i_{Bb} &= i_{bc} - i_{ab} \\ &= I_{m_1} \sin(\omega t + \theta_1 - 240°) + I_{m_5} \sin(5\omega t + \theta_5 - 120°) \\ &\quad - I_{m_1} \sin(\omega t + \theta_1 - 120°) - I_{m_5} \sin(5\omega t + \theta_5 - 240°) \end{aligned} \tag{9.15}$$

$$\begin{aligned} i_{Cc} &= i_{ca} - i_{bc} \\ &= I_{m_1} \sin(\omega t + \theta_1) + I_{m_5} \sin(5\omega t + \theta_5) - I_{m_1} \sin(\omega t + \theta_1 - 240°) \\ &\quad - I_{m_5} \sin(5\omega t + \theta_5 - 120°) \end{aligned} \tag{9.16}$$

Equations (9.14), (9.15) and (9.16) indicate that no third harmonic currents can exist in the line currents of a delta connection.

The rms value of the line current from the above equation is

$$I_L = \sqrt{3}\sqrt{\frac{I_{m_1}^2 + I_{m_5}^2 + \dots}{2}} \tag{9.17}$$

The rms value of the phase current from Eqs (9.11), (9.12) and (9.13) is given by

$$I_{ph} = \sqrt{\frac{I_{m_1}^2 + I_{m_3}^2 + I_{m_5}^2 + \dots}{2}} \tag{9.18}$$

It is seen from Eqs (9.17) and (9.18), that in a delta-connected system the line current is not equal to $\sqrt{3}$ times the phase current. It is only true when there are no third harmonic currents in the system.

9.13 EFFECTS OF PHASE-SEQUENCE

LO 11 Analyse the effects of phase-sequence

The effects of phase sequence of the source voltages are not of considerable importance for applications like lighting, heating, etc. but in case of a three-phase induction motor, reversal of sequence results in the reversal of its direction. In the case of an unbalanced polyphase system, a reversal of the voltage phase sequence will, in general, cause certain branch currents to change in magnitude as well as in phase position. Even though the system is balanced, the readings of the two-wattmeters in the two wattmeter method of measuring power interchange when subjected to a reversal of phase sequence when two or more three-phase generators are running parallel to supply a common load, reversing the phase sequence of any one machine cause severe damage to the entire system. Hence, when working on such systems, it is very important to consider the phase sequence of the system. Unless otherwise stated, the term "phase sequence" refers to voltage phase sequence. The line currents and phase currents follow the same sequence as the system voltage. The phase sequence of a given system, is a small meter with three long connecting leads in side which it has a circular disc. The rotation of which previously been checked against a known phase sequence. In three-phase systems, only two different phase sequences are possible. The three leads are connected to the three lines whose sequence is to be determined, the rotation of the disc can be used as an indicator of phase sequence.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 11

☆★★**9-11.1** The power taken by a 440 V, 50 Hz, three-phase induction motor on full load is measured by two wattmeters, which indicate 250 W and 1000 W, respectively. Calculate (a) the input, (b) the power factor, (c) the current, and (d) the motor output, if the efficiency is 80%.

9.14 POWER FACTOR OF AN UNBALANCED SYSTEM

LO 12 Calculate power factor of an unbalanced system

The concept of power factor in three-phase balanced circuits has been discussed in Section 9.11.4. It is the ratio of the phase watts to the phase volt-amperes of any one of the three phases. We cannot strictly define the power factor in three-phase unbalanced circuits, as each phase has a separate power factor. Generally, for three-phase unbalanced loads, the ratio of total watts $\left(\sqrt{3}V_L I_L \cos\theta\right)$ to total volt-amperes $\left(\sqrt{3}V_L I_L\right)$ is a good general indication of the power factor.

Another recognised definition for an unbalanced polyphase system is called the vector power factor, given by

$$\text{Power factor} = \frac{\sum VI\cos\theta}{\sum VI}$$

where $\sum VI\cos\theta$ is the algebraic sum of the active powers of all individual phases given by

$$\sum VI\cos\theta = V_a I_a \cos\theta_a + V_b I_b \cos\theta_b + V_c I_c \cos\theta_c + \ldots$$

$$\text{and} \quad \sum VI = \sqrt{\left(\sum VI\cos\theta\right)^2 + \left(\sum VI\sin\theta\right)^2}$$

$\sum VI\sin\theta$ is the algebraic sum of the individual phase reactive volt-amperes. The following example illustrates the application of vector power factor for unbalanced loads.

Consider Example 9.19 where the phase voltage and currents have been already calculated. Here $V_{RN} = 230.94\angle 0°$ V, $V_{YN} = 230.94\angle -120°$ V, $V_{BN} = 230.94\angle -240°$ V

$I_R = 24.83\angle -63.4°$ A; $I_Y = 46.188\angle -173.1°$ A; $I_B = 9.23\angle -293.13°$ A

Active power of phase $R = 230.94 \times 25.83 \times \cos 63.4° = 2.6709$ kW

Active power of phase $Y = 230.94 \times 46.188 \times \cos 53.1° = 6.4044$ kW

Active power of phase $B = 230.94 \times 9.23 \times \cos 53.13° \quad = 1.2789$ kW

$\overline{10.3542 \text{ kW}}$

$$\sum VI\cos\theta = 10.3542 \text{ kW}$$

Reactive power of phase $R = 230.94 \times 25.83 \times \sin 63.4° \quad = 5.3197$ KVAR
Reactive power of phase $Y = 230.94 \times 46.188 \times \sin 53.1° \quad = 8.5299$ KVAR
Reactive power of phase $B = 230.94 \times 9.23 \times \sin 53.13° \quad = 1.7052$ KVAR

$\overline{15.5548 \text{ KVAR}}$

$$\sum VI\sin\theta = 15.5548 \text{ KVAR}$$

$$\text{Power factor} = \frac{10.3542}{\sqrt{(15.5548)^2 + (10.3542)^2}} = 0.5541$$

Additional Solved Problems

PROBLEM 9.1

The phase voltage of a star-connected three-phase ac generator is 230 V. Calculate the (a) line voltage, (b) active power output if the line current of the system is 15 A at a power factor of 0.7, and (c) active and reactive components of the phase currents.

Solution The supply voltage (generator) is always assumed to be balanced

$\therefore \quad V_{Ph} = 230$ V; $I_L = I_{Ph} = 15$ A, $\cos\phi = 0.7$, $\sin\phi = 0.71$

(a) In a star-connected system $V_L = \sqrt{3}\ V_{Ph} = 398.37$ V

(b) Power output $= \sqrt{3}\ V_L I_L \cos\phi$

$$= \sqrt{3} \times 398.37 \times 15 \times 0.7 = 7244.96 \text{ W}$$

(c) Active component of the current $= I_{Ph}\cos\phi$

$$= 15 \times 0.7 = 10.5 \text{ A}$$

Reactive component of the current $= I_{Ph}\sin\phi$

$$= 15 \times 0.71 = 10.65 \text{ A}$$

PROBLEM 9.2

A three-phase delta-connected RYB system with an effective voltage of 400 V, has a balanced load with impedances 3 + j4 Ω. Calculate the (a) phase currents, (b) line currents, and (c) power in each phase.

Solution

$$V_L = V_{Ph} = 400 \text{ V}$$

Assuming *RYB* phase sequence, we have

$$V_{RY} = 400\angle 0°;\ V_{YB} = 400\angle{-120°};\ V_{BR} = 400\angle{-240°}$$

$$Z = 3 + j4 = 5\angle 53.1°$$

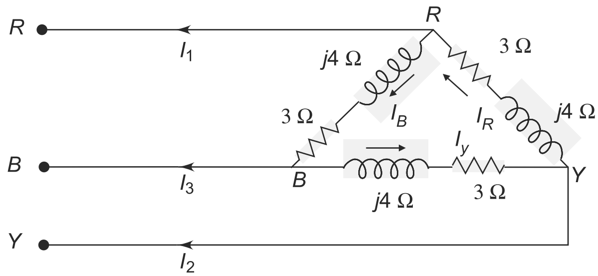


Fig. 9.54

(a) The three phase currents are

$$I_R = \frac{V_{RY}}{Z} = \frac{400\angle 0^\circ}{5\angle 53.1^\circ} = 80\angle -53.1^\circ$$

$$I_Y = \frac{V_{YB}}{Z} = \frac{400\angle -120^\circ}{5\angle 53.1^\circ} = 80\angle -173.1^\circ$$

$$I_B = \frac{V_{YB}}{Z} = \frac{400\angle -240^\circ}{5\angle 53.1^\circ} = 80\angle -293.1^\circ$$

$$I_R = 80\ \angle -53.1^\circ = 48.03 - j63.97$$

$$I_Y = 80\ \angle -173.1^\circ = -79.42 - j9.61$$

$$I_B = 80\ \angle -293.1^\circ = 31.38 + j73.58$$

(b) The three line currents are

$$I_1 = I_R - I_B = 138.55\ \angle -83.09^\circ$$

$$I_2 = I_Y - I_R = 138.55\ \angle 156.9^\circ$$

$$I_3 = I_B - I_Y = 138.55\ \angle 36.89^\circ$$

$$\cos\phi = \frac{R_{Ph}}{Z_{Ph}} = \frac{3}{5} = 0.6$$

(c) Power consumed in each phase $= V_{Ph}\, I_{Ph} \cos\phi$

$$= 400 \times 80 \times 0.6 = 19200 \text{ W}$$

$$\text{Total power} = 3 \times 19200 = 57600 \text{ W}$$

PROBLEM 9.3

The load in Problem 9.2 is connected in star with the same phase sequence across the same system. Calculate (a) the phase and line currents (b) the total power in the circuit, and (c) phasor sum of the three line currents.

Solution The circuit is shown in Fig. 9.55.

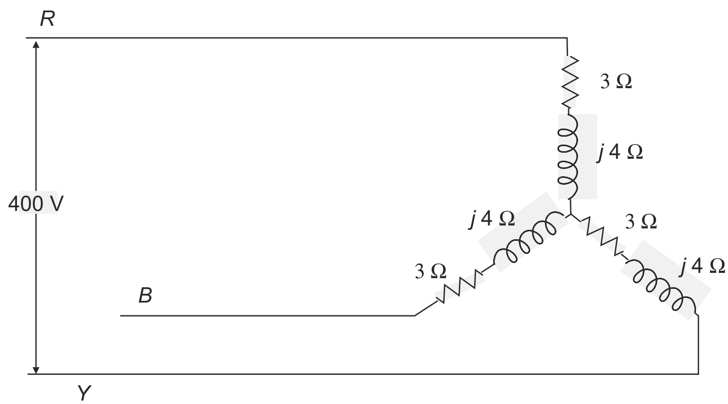


Fig. 9.55

Assuming *RYB* phase sequence, since

$$V_L = 400 \text{ V}$$

$$V_{Ph} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

Taking V_{RN} as reference, the three phase voltages are $V_{RN} = 230.94\angle 0°$; $V_{YN} = 230.94\ \angle{-120°}$; and $V_{BN} = 230.9\ \angle{-240°}$.

The three line voltages, V_{RY}, V_{YB} and V_{BR} are 30° ahead of their respective phase voltages.

$$I_{Ph} = I_L;\ Z_{Ph} = 3 + j4 = 5\ \angle 53.1°$$

(a) The three phase currents are

$$I_R = \frac{V_{RN}}{Z_{Ph}} = \frac{230.94\angle 0°}{5\angle 53.1°} = 46.18\angle -53.1°$$

$$I_Y = \frac{V_{YN}}{Z_{Ph}} = \frac{230.09\angle -120°}{5\angle 53.1°} = 46.18\angle -173.1°$$

$$I_B = \frac{V_{BN}}{Z_{Ph}} = \frac{230.09\angle -240°}{5\angle 53.1°} = 46.18\angle -293.1°$$

(b) Total power $= \sqrt{3}\ V_L I_L \cos\phi$

$$= \sqrt{3} \times 400 \times 46.18 \times 0.6 = 19196.6 \text{ W}$$

Thus, it can be observed that the power consumed in a delta load will be three times more than that in the star connection

(c) Phasor sum of the three line currents

$$= I_R + I_Y + I_B$$

$$= 46.18\ \angle{-53.1°} + 46.18\ \angle{-173.1°} + 46.18\ \angle{-293.1°} = 0$$

PROBLEM 9.4

A three-phase balanced delta-connected load with line voltage of 200 V, has line currents as $I_1 = 10\angle 90°$; $I_2 = 10\angle -150°$ and $I_3 = 10\angle -30°$. (a) What is the phase sequence? (b) What are the impedances?

Solution Figure 9.56 (a) represents all the three line currents in the phasor diagram.

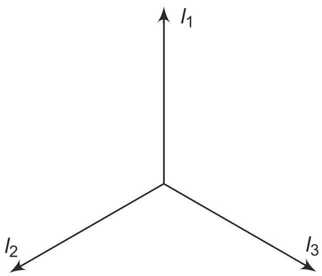


Fig. 9.56 (a)

(a) It can be observed from Figs 9.56 (a) and (b) that the current flowing in the line *B*, i.e. I_3 lags behind I_1 by 120°, and the current flowing in line *Y*, i.e. I_2 lags behind i_3 by 120°. ∴ The phase sequence is *RBY*.

(b) $V_{Ph} = V_L = 200$

$$I_{Ph} = \frac{I_L}{\sqrt{3}} = \frac{10}{\sqrt{3}}$$

$$Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{200\sqrt{3}}{10} = 34.64\ \Omega$$

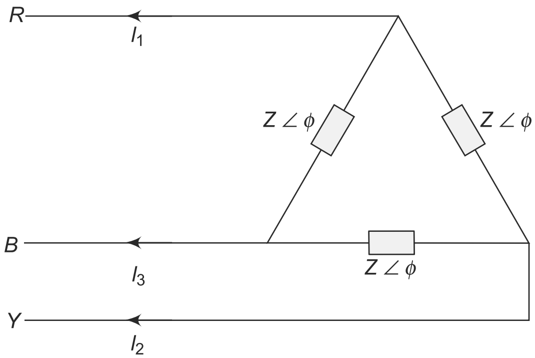


Fig. 9.56 (b)

PROBLEM 9.5

Three equal inductors connected in star take 5 kW at 0.7 Pf when connected to a 400 V, 50 Hz three-phase, three-wire supply. Calculate the line currents (a) if one of the inductors is disconnected, and (b) If one of the inductors is short circuited.

Solution Total power when they are connected to the 400 V supply

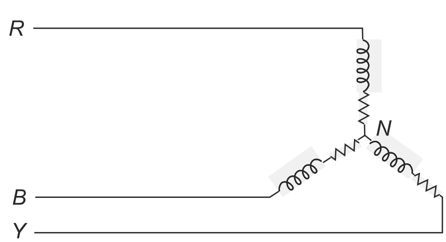


Fig. 9.57 (a)

$$P = \sqrt{3}\ V_L I_L \cos\phi = 5000\text{ W}$$

$$I_{Ph} = I_L = \frac{5000}{\sqrt{3}\times 400\times 0.7} = 10.31\text{A}$$

$$\text{Impedance/phase} = \frac{V_{Ph}}{I_{Ph}} = \frac{400}{\sqrt{3}\times 10.31} = 22.4\ \Omega$$

$$R_{Ph} = Z_{Ph}\cos\phi = 22.4 \times 0.7 = 15.68\ \Omega$$

$$X_{Ph} = Z_{Ph}\sin\phi = 22.4 \times 0.714 = 16\ \Omega$$

(a) If the phase Y is disconnected from the circuit, the other two inductors are connected in series across the line voltage of 400 V as shown in Fig. 9.57 (a).

$$I_R = I_B = \frac{400}{2 \times Z_{Ph}} = 8.928\,\text{A}$$

$$I_Y = 0$$

(b) If phases Y and N are short circuited as shown in Fig. 9.57 (b), the phase voltages V_{RN} and V_{BN} will be equal to the line voltage 400 V.

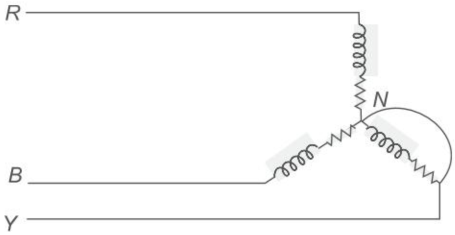


Fig. 9.57 (b)

$$I_{Ph} = I_R = I_B = \frac{400}{Z_{Ph}} = \frac{400}{22.4} = 17.85\,\text{A}$$

The current in the Y phase is equal to the phasor sum of the R and B.

$$\therefore \quad I_Y = 2 \times I_{Ph} \cos\left(\frac{60}{2}\right) = 30.91\,\text{A}$$

PROBLEM 9.6

For the circuit shown in Fig. 9.58, calculate the line current, the power and the power factor. The value of R, L, and C in each phase are 10 ohms, 1 H and 100 μF, respectively.

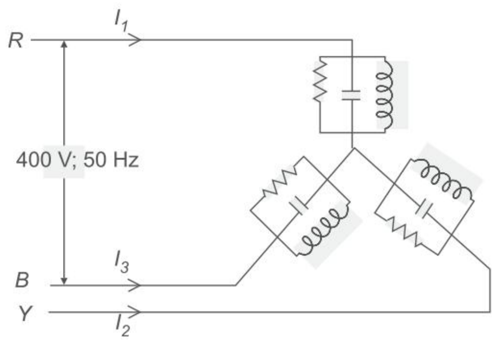


Fig. 9.58

Solution Let us assume *RYB* sequence.

$$V_{RN} = \frac{400}{\sqrt{3}}\angle 0° = 231\angle 0°; V_{YN} = 231\angle -120°; V_{BN} = 231\angle -240°$$

$$\text{Admittance of each phase } Y_{Ph} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$$

$$= \frac{1}{10} + \frac{1}{j314} + j314 \times 100 \times 10^{-6}$$

$$Y_{Ph} = 0.1 + j28.22 \times 10^{-3}$$

$$= 0.103\angle 15.75° \; ℧$$

$$I_{Ph} = V_{Ph}\, Y_{Ph}$$

$$= (231\ \angle 0°)\ (0.103\ \angle 15.75°)$$

$$= 23.8\ \angle 15.75°\ \text{A}$$

$$\text{Power} = \sqrt{3}\ V_L I_L \cos\phi$$

$$= \sqrt{3} \times 400 \times 23.8 \cos 15.75°$$

$$= 15869.57\ \text{W}$$

$$\text{Power factor} = \cos 15.75° = 0.96 \text{ leading}$$

PROBLEM 9.7

For the circuit shown in Fig. 9.59, an impedance is connected across YB, and a coil of resistance 3 Ω and inductive reactance of 4 Ω is connected across RY. Find the value of R and X of the impedance across YB such that $I_2 = 0$. Assume a balanced three-phase supply with RYB sequence.

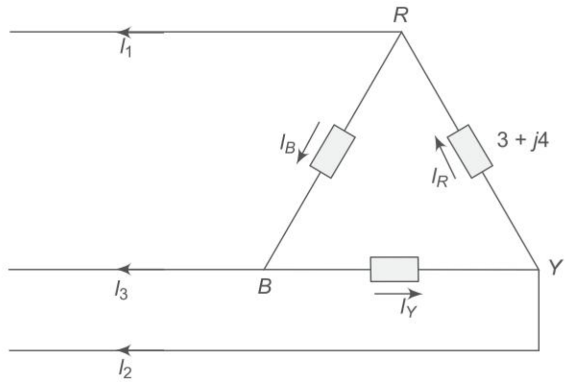


Fig. 9.59

Solution As usual I_R, I_Y, and I_B are phase currents, and I_1, I_2, and I_3 are line currents.

Applying KCL at the node Y, we have

$$I_2 = I_Y - I_R$$

$$I_2 = I_Y - I_R$$

Since $I_2 = 0$

$$I_Y = I_R$$

$$\therefore \quad I_R = \frac{V_{RY}}{3+j4}, I_Y = \frac{V_{YB}}{Z_{YB}}$$

$$V_{RY} = V\angle 0°, V_{YB} = V\angle -120°$$

$$\frac{V\angle 0°}{3+j4} = \frac{V\angle -120°}{Z_{YB}}$$

$$Z_{YB} = \frac{V\angle -120°}{V\angle 0°}(3+j4)$$

$$= 1.96 - j4.6$$

$\therefore$ $R = 1.96\ \Omega$; $X = 4.6\ \Omega$ (capacitive reactance)

PROBLEM 9.8

A symmetrical three-phase 440 V system supplies a balanced delta-connected load. The branch current is 10 A at a phase angle of 30°, lagging. Find (a) the line current, (b) the total active power, and (c) the total reactive power. Draw the phasor diagram.

Solution (a) In a balanced delta-connected system

$$I_L = \sqrt{3}\, I_{Ph} = \sqrt{3} \times 10 = 17.32 \text{ A}$$

(b) Total active power

$$= \sqrt{3}\, V_L I_L \cos\phi$$

$$= \sqrt{3} \times 440 \times 17.32 \times \cos 30° = 11.431 \text{ kW}$$

(c) Total reactive power

$$= \sqrt{3}\, V_L I_L \sin\phi$$

$= \sqrt{3} \times 440 \times 17.32 \times \sin 30° = 6.5998$ KVAR

The phasor diagram is as under.

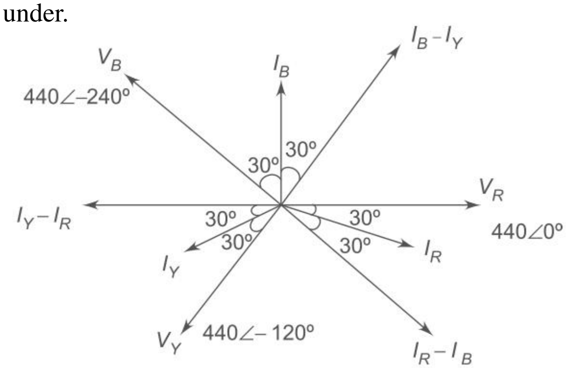


Fig. 9.60

V_R, V_Y, and V_B are phase voltages, and are equal to the line values. I_R, I_Y, and I_B are the phase currents, and lag behind their respective phase voltages by 30°. Line currents $(I_R - I_B)$, $(I_Y - I_R)$ and $(I_B - I_Y)$ lag behind their respective phase currents by 30°.

PROBLEM 9.9

Find the line currents and the total power consumed by the unbalanced delta-connected load shown in Fig. 9.61.

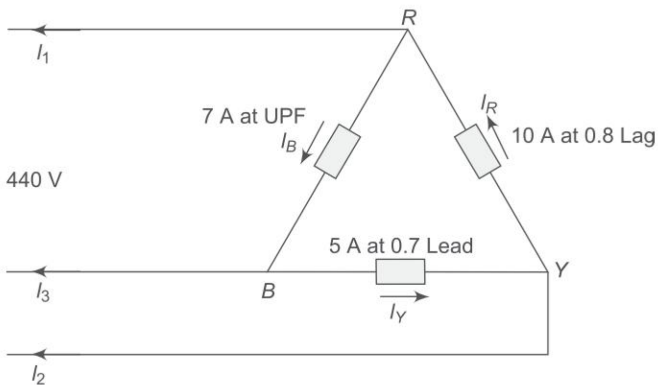


Fig. 9.61

Solution Assuming *RYB* phase sequence, from the given data,

$$I_R = 10 \angle{-36.88°};\ I_Y = 5 \angle 45.57°;\ I_B = 7 \angle 0°$$

Line currents are

$$I_1 = I_R - I_B = 6.08 \angle{-80°}$$
$$I_2 = I_Y - I_R = 10.57 \angle 11.518°$$
$$I_3 = I_B - I_Y = 5 \angle{-45.56°}$$

Total power is the sum of the powers consumed in all the three phases.

$\therefore$ Power in $RY = V_{RY} \times I_R \times 0.8$

$$= 400 \times 10 \times 0.8 = 3200 \text{ W}$$

$$\text{Power in } YB = V_{YB} \times I_Y \times 0.7$$
$$= 400 \times 5 \times 0.7 = 1400 \text{ W}$$
$$\text{Power in } BY = V_{BR} \times I_B \times 1$$
$$= 400 \times 7 \times 1 = 2800 \text{ W}$$
$$\text{Total power} = 3200 + 1400 + 2800 = 7400 \text{ W}$$

PROBLEM 9.10

A delta-connected three-phase load has 10 Ω between R and Y, 6.36 mH between Y and B, and 636 μF between B and R. The supply voltage is 400 V, 50 Hz. Calculate the line currents for RBY phase sequence.

Solution $Z_{RY} = 10 + j0 = 10 \angle 0°$; $Z_{YB} = 0 + jX_L = 0 + jX_L$

$$= 0 + j2\pi fL = 2 \angle 90°3$$

$$Z_{BR} = 0 - jX_C = 0 - \frac{j}{2\pi fC} = 5\angle -90°$$

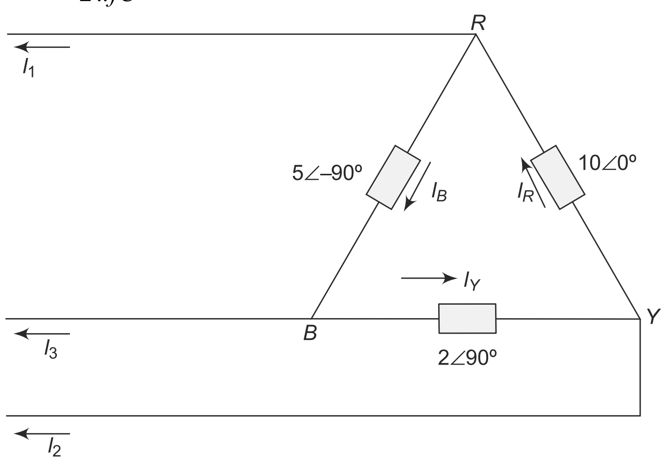


Fig. 9.62

Since the phase sequence is *RBY*, taking V_{RY} as reference voltage, we have

$$V_{RY} = 400 \angle 0°;\ V_{BR} = 400 \angle -120°;\ V_{YB} = 400 \angle -240°$$

$$I_R = \frac{V_{RY}}{10\angle 0°} = \frac{400\angle -0°}{10\angle 0°} = 40\angle 0°$$

$$I_Y = \frac{V_{YB}}{2\angle 90°} = \frac{400\angle -240°}{2\angle 90°} = 200\angle -330°$$

$$I_B = \frac{V_{BR}}{5\angle -90°} = \frac{400\angle -120°}{5\angle -90°} = 80\angle -30°$$

The three line currents are

$$I_1 = I_R - I_B = 40 \angle 0° - 80 \angle -30° = 49.57 \angle 126.2°$$
$$I_2 = I_Y - I_R = 200 \angle -300° \angle -40 \angle 0° = 166.56 \angle 36.89°$$
$$I_3 = I_B - I_Y = 80 \angle - 30° - 200 \angle -330° = 174.35 \angle 233.41°$$

PROBLEM 9.11

The power consumed in a three-phase balanced star-connected load is 2 kW at a power factor of 0.8 lagging. The supply voltage is 400 V, 50 Hz. Calculate the resistance and reactance of each phase.

Solution Phase voltage $= \dfrac{400}{\sqrt{3}}$

Power consumed $= 2000 \text{ W} = \sqrt{3}\, V_L I_L \cos\phi$

Phase current or line current $I_L = \dfrac{2000}{\sqrt{3}\times 400\times 0.8} = 3.6\text{ A}$

Impedance of each phase

$$Z_{Ph} = \frac{V_{Ph}}{I_{Ph}} = \frac{400}{\sqrt{3}\times 3.6} = 64.15\text{ A}$$

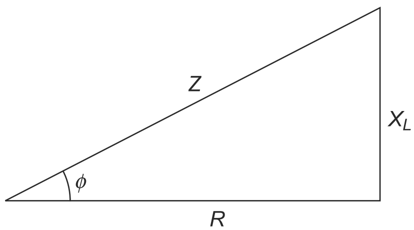


Fig. 9.63

Since the power factor of the load is lagging, the reactance is inductive reactance. From the impedance triangle shown in Fig. 9.63, we have

Resistance of each phase $R_{Ph} = Z_{Ph} \cos\phi$

$= 64.15 \times 0.8 = 51.32\ \Omega$

Reactance of each phase $X_{Ph} = Z_{Ph} \sin\phi$

$= 64.15 \times 0.6 = 38.5\ \Omega$

PROBLEM 9.12

A symmetrical three-phase 100 V; three-wire supply feeds an unbalanced star-connected load, with impedances of the load as $Z_R = 5\angle 0°\ \Omega$, $Z_Y = 2\angle 90°\ \Omega$ and $Z_B = 4\angle -90°\ \Omega$. Find the (a) line currents, (b) voltage across the impedances, and (c) the displacement neutral voltage.

Solution As explained earlier, this type of unbalanced *Y*-connected three-wire load can be solved either by star-delta conversion method or by applying Millman's theorem.

(a) *Star-Delta Conversion Method*

The unbalanced star-connected load and its equivalent delta load are shown in Figs 9.64 (a) and (b).

$$Z_R Z_Y + Z_Y Z_B + Z_B Z_R = (5\angle 0°)(2\angle 90°) + (2\angle 90°)(4\angle -90°)$$

$$+ (4\angle -90°)(5\angle 0°) = 8 - j10$$

$$= 12.8\angle -51.34°$$

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B} = \frac{12.8\angle -51.34°}{4\angle -90°} = 3.2\angle 38.66°$$

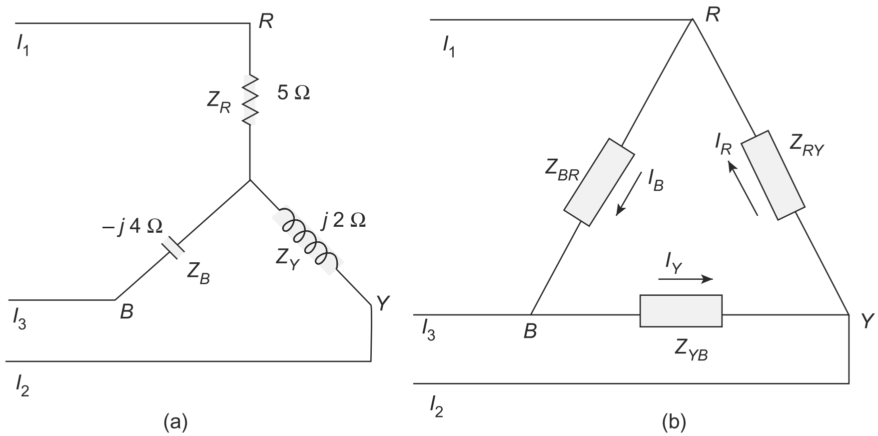


Fig. 9.64

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R} = \frac{12.8\angle{-51.34}}{5\angle 0} = 2.56\angle -51.34$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y} = \frac{12.8\angle{-51.34^\circ}}{2\angle 90^\circ} = 6.4\angle -141.34^\circ$$

Taking V_{RY} as the reference, we have

$$V_{RY} = 100\angle 0^\circ;\ V_{YB} = 100\angle{-120^\circ};\ V_{BR} = 100\angle{-240^\circ}$$

The three phase currents in the equivalent delta load are

$$I_R = \frac{V_{RY}}{Z_{RY}} = \frac{100\angle 0^\circ}{3.2\angle 38.66^\circ} = 31.25\angle{-38.66^\circ}$$

$$I_Y = \frac{V_{YB}}{Z_{YB}} = \frac{100\angle{-120^\circ}}{2.56\angle{-51.34^\circ}} = 39.06\angle{-68.66^\circ}$$

$$I_B = \frac{V_{BR}}{Z_{BR}} = \frac{100\angle{-240^\circ}}{6.4\angle{-141.34^\circ}} = 15.62\angle{-98.66^\circ}$$

The line currents are

$$\begin{aligned}
I_1 &= I_R - I_B = 31.25\angle{-38.66^\circ} - 15.62\angle{-98.66^\circ} \\
&= (24.4 - j19.52) - (-2.35 + j15.44) = (26.75 - j4.08) \\
&= 27.06\angle{-8.671^\circ} \\
I_2 &= I_Y - I_R = 39.06\angle{-68.66^\circ} - 31.25\angle{-38.66^\circ} \\
&= (14.21 - j36.38) - (24.4 - j19.52) = (-10.19 - j16.86) \\
&= 19.7\angle 238.85^\circ \\
I_3 &= I_B - I_Y = 15.62\angle{-98.66^\circ} - 39.06\angle{-98.66^\circ} \\
&= (-2.35 - j15.44) - (14.21 - j36.38) = (-16.56 + j20.94) \\
&= 26.7\angle 128.33^\circ
\end{aligned}$$

These line currents are also equal to the line (phase) currents of the original star connected load.

(b) Voltage drop across each star-connected load will be as under.

Voltage across $Z_R = I_1 \times Z_R$

$= (27.06 \angle -8.671°)\ (5 \angle 0°) = 135.3° \angle -8.67°$

Voltage across $Z_Y = I_2 \times Z_Y$

$= (19.7 \angle 238.85°)\ (2 \angle 90°) = 39.4° \angle 328.85°$

Voltage across $Z_B = I_3 \times Z_B$

$= (26.7 \angle 128.33°)\ (4 \angle -90°) = 106.8° \angle 38.33°$

(c) *By Applying Millman's Theorem*

Consider Fig. 9.64 (c), taking V_{RY} as reference line voltage $= 100 \angle 0°$.

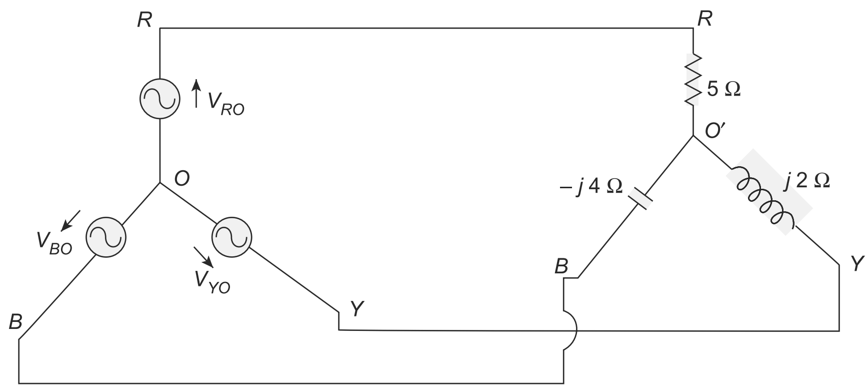


Fig. 9.64 (c)

Phase voltages log 30° behind their respective line voltages. Therefore, the three phase voltages are

$$V_{RO} = \frac{100}{\sqrt{3}} \angle -30°$$

$$V_{YO} = \frac{100}{\sqrt{3}} \angle -150°$$

$$V_{BO} = \frac{100}{\sqrt{3}} \angle -270°$$

$$Y_R = \frac{1}{Z_R} = \frac{1}{5\angle 0°} = 0.2\angle 0°$$

$$Y_Y = \frac{1}{Z_Y} = \frac{1}{2\angle 90°} = 0.5\angle -90°$$

$$Y_B = \frac{1}{Z_B} = \frac{1}{4\angle -90°} = 0.25\angle 90°$$

$V_{RO}Y_R + V_{YO}Y_Y + Y_{BO}Y_B = (57.73 \angle -30°)\ (0.2 \angle 0°)$

$$+ (57.73 \angle{-150°}) (0.5 \angle{-90°})$$
$$+ (57.73 \angle{-270°}) (0.25 \angle 90°)$$
$$= 11.54 \angle{-30°} + 28.86 \angle{-240°} + 14.43 \angle{-180°}$$
$$= (10 - j5.77) + (-14.43 + j25) + (-14.43 + j0)$$
$$= -18.86 + j19.23 = 26.93 \angle 134.44°$$
$$Y_R + Y_Y + Y_B = 0.2 + 0.5 \angle{-90°} + 0.25 \angle 90°$$
$$= 0.32 \angle{-51.34°}$$
$$V_{O'O} = \frac{V_{RO}\,Y_R + V_{YO}\,Y_Y + V_{BO}\,Y_B}{Y_R + Y_Y + Y_B} = \frac{26.93\angle 134.44°}{0.32\angle -51.34°}$$
$$= 84.15 \angle 185.78°$$

The three load phase voltages are

$$V_{RO'} = V_{RO} - V_{O'O}$$
$$= 57.73 \angle{-30°} - 84.15 \angle 185.78°$$
$$= (50 - j28.86) - (-83.72 - j8.47)$$
$$= (133.72 - j20.4) = 135.26 \angle{-8.67°}$$
$$V_{YO'} = V_{YO} - V_{O'O}$$
$$= 57.73 \angle{-150°} - 84.15 \angle 185.78°$$
$$= (-50 - j28.86) - (-83.72 - j8.47)$$
$$= 33.72 - j20.4 = 39.4 \angle{-31.17°} \text{ or } 39.4 \angle 328.8°$$
$$V_{BO'} = V_{BO} - V_{O'O}$$
$$= 57.73 \angle{-270°} - 84.15 \angle 185.78°$$
$$= 0 + j57.73 + 83.72 + j8.47$$
$$= 83.72 + j66.2 = 106.73 \angle 38.33°$$
$$I_R = \frac{135.26\angle -8.67°}{5\angle 0°} = 20.06\angle -8.67°$$
$$I_Y = \frac{39.4\angle 328.80°}{2\angle 90°} = 19.7\angle 238.8°$$
$$I_B = \frac{106.73\angle 38.33°}{4\angle -90°}$$
$$= 26.68 \angle 128.33°$$

PROBLEM 9.13

A three-phase three-wire unbalanced load is star-connected. The phase voltages of two of the arms are

$$V_R = 100\ \angle{-10°};\ V_Y = 150\ \angle 100°$$

Calculate voltage between star point of the load and the supply neutral.

Solution As shown in Fig. 9.65,

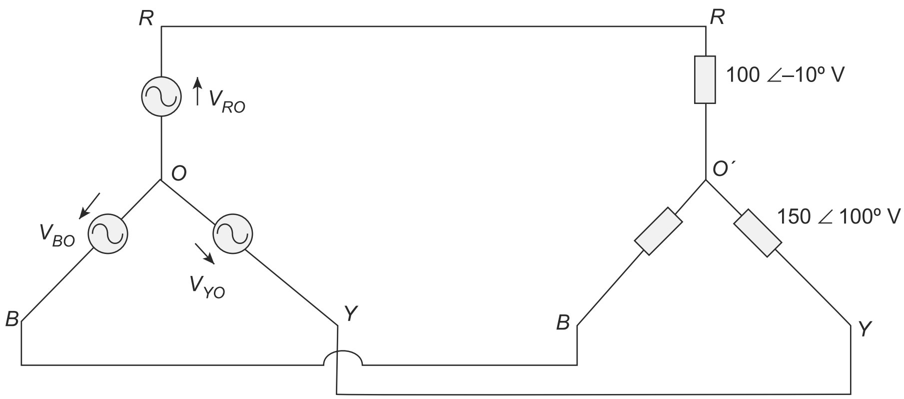


Fig. 9.65

$$V_{RO} = V_{RO'} + V_{O'O}$$

or $$V_{O'O} = V_{RO} - V_{RO'} \tag{9.19}$$

Also $$V_{O'O} = V_{YO} - V_{YO'} \tag{9.20}$$

Let $$V_{RO} = V\ \angle 0°$$

Assuming *RYB* phase sequence,

$$V_{YO} = V\ \angle{-120°}$$

Substituting in Eqs (9.19) and (9.20), we have

$$V_{O'O} = V\ \angle 0° - 100\ \angle{-10°} \tag{9.21}$$

$$V_{O'O} = V\ \angle{-\ 120°} - \ 50\ \angle 100° \tag{9.22}$$

Subtracting Eq. (9.22) from Eq. (9.21), we get

$$O = [(V + jO) - (98.48 - j17.36)] - [(0.5V + j0.866V\,) - (-26.04 + j147.72)]$$

$$O = 1.5V - j0.8666V - 124.52 + j165.08$$

$$= V\,(1.5 - j0.866) = 124.52 - j165.08$$

$$V = \frac{124.52 - j165.08°}{1.5 - j0.866} = \frac{206.77\angle -52.97°}{1.732\angle -30°}$$

$$V = 119.38\ \angle{-22.97°}$$

Voltage between $O'O = V_{RO} - V_{RO'}$

$$V_{O'O} = 119.38\ \angle{-22.97°} - 100\ \angle{-10°}$$
$$= 109.91 - j46.58 - 98.48 + j17.36$$
$$= 11.43 - j29.22 = 31.37\ \angle{-\ 68.63°}$$

PROBLEM 9.14

Find the reading of a wattmeter in the circuit shown in Fig. 9.66 (a). Assume a symmetrical 400 V supply with RYB phase sequence and draw the vector diagram.

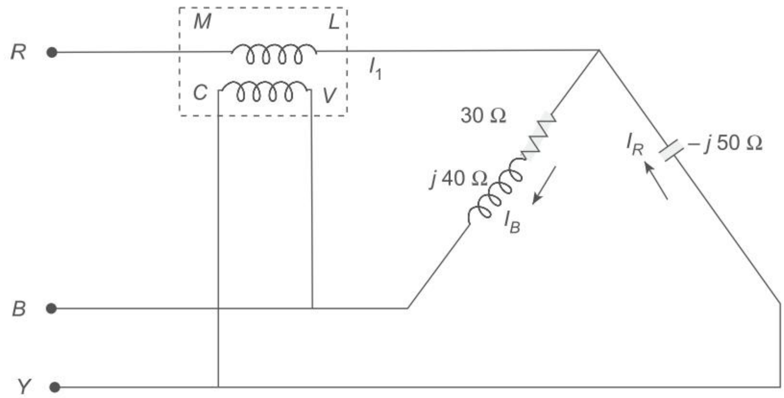


Fig. 9.66 (a)

Solution The reading in the wattmeter is equal to the product of the current through the current coil I_1 voltage across its pressure coil V_{YB} and cos of the angle between the V_{YB} and I_1.

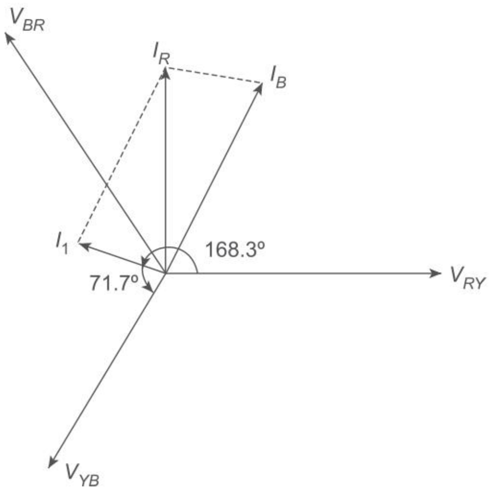


Fig. 9.66 (b)

$$I_R = \frac{V_{RY}}{-j50} = \frac{400\angle 0°}{50\angle -90°} = 8\angle 90°$$

$$I_B = \frac{V_{BR}}{30 + j40} = \frac{400\angle -240°}{50\angle 53.13°}$$

$$= 8\angle -293.13° \quad \text{or} \quad 8\angle 66.87°$$

Line current

$$I_1 = I_R - I_B$$

$$= 8\angle 90° - 8\angle -293.13°$$

$$= 0 + j8 - 3.14 - j7.35 = -3.14 + j0.65 = 3.2\angle 168.3°$$

From the vector diagram in Fig. 9.66 (b), it is clear that the angle between V_{YB} and I_1 is 71.7°.

$\therefore$ Wattmeter reading is equal to $V_{YB} \times I_1 \cos 71.7°$

$$= 400 \times 3.2 \times \cos 71.7 = 402 \text{ W}$$

PROBLEM 9.15

Calculate the total power input and readings of the two wattmeters connected to measure power in a three-phase balanced load, if the reactive power input is 15 kVAR, and the load pf is 0.8.

Solution Let W_1 be the lower reading wattmeter and W_2 the higher reading wattmeter.

$$\cos\phi = 0.8$$

$$\phi = 36.86°$$

$$\tan\phi = \frac{\text{Reactive power}}{\text{Active power}}$$

or $$= \sqrt{3}\frac{W_2 - W_1}{W_2 + W_1}$$

$$\text{Reactive power} = \sqrt{3}(W_2 - W_1) = 15000$$

$$= W_2 - W_1 = 8660.508 \text{ W} \qquad (9.23)$$

$$\therefore \quad 0.75 = \sqrt{3}\frac{15000}{W_2 + W_1}$$

or total power input $W_2 + W_1 = 34641.01$ W (9.24)

From Eqs (9.23) and (9.24), we get

$$W_2 = 21650.76 \text{ W}$$

$$W_1 = 12990.24 \text{ W}$$

PROBLEM 9.16

Two wattmeters are connected to measure power in a three-phase circuit. The reading of one of the meters is 5 kW when the load power factor is unity. If the power factor of the load is changed to 0.707 lagging, without changing the total input power, calculate the readings of the two wattmeters.

Solution Both wattmeters indicate equal values when the power factor is unity

$\therefore$ $W_1 + W_2 = 10$ kW (Total power input) (9.25)

Let W_2 be the higher reading wattmeter.

Then W_1 is the lower reading wattmeter.

$$\cos\phi = 0.707 \quad \therefore \phi = 45°$$

$$\tan\phi = \sqrt{3}\frac{W_2 - W_1}{W_2 + W_1}\ 1 = \sqrt{3}\frac{W_2 - W_1}{10}$$

$$\therefore \quad W_2 - W_1 = \frac{10}{\sqrt{3}} = 5.773 \text{ kW} \qquad (9.26)$$

From Eqs (9.25) and (9.26),

$$W_2 = 7.886 \text{ kW}$$
$$W_1 = 2.113 \text{ kW}$$

PROBLEM 9.17

The line currents in a balanced six-phase mesh connected generator are 35.35 A. What is the magnitude of the phase current?

Solution From Section 9.8.2,

$$I_L = 2I_{Ph} \sin \frac{180°}{n}$$

$$I_{Ph} = \frac{35.35}{2\sin \frac{180°}{6}} = 35.35$$

PROBLEM 9.18

Find the voltage between the adjacent lines of a balanced six-phase star-connected system with a phase voltage of 132.8 volts.

Solution From Section 9.7.2, $E_L = 2E_{Ph} \sin \frac{180°}{n}$

$$E_L = 2 \times 132.8 \times \sin \frac{180°}{6} = 132.8 \text{ V}$$

PROBLEM 9.19

In the wye- connected system shown in Fig. 9.67, it is assumed that only fundamental and third harmonic voltages are present when the voltages are measured with a voltmeter between na and ba. They are given by 230 and 340 volts respectively. Calculate the magnitude of the third harmonic voltages in the system.

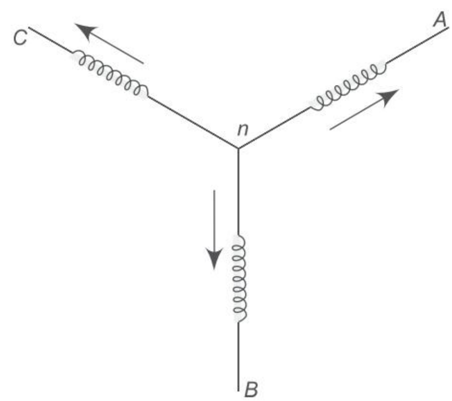


Fig. 9.67

Solution Only phase voltage V_{na} of the system shown in Fig. 9.67 contains third harmonic whereas line voltage V_{ba} contains only first harmonic. Hence,

$$\text{Fundamental component of the phase} = \frac{340}{\sqrt{3}}$$

$$\text{Third harmonic component} = \sqrt{220^2 - \left(\frac{340}{\sqrt{3}}\right)^2} = 99.33 \text{ V}$$

PROBLEM 9.20

Illustrate the effect of reversal of voltage sequence up on the magnitudes of the currents in the system shown in Example 9.20.

Solution The line currents for *RYB* sequence have already been calculated. $I_1 = 28.41 \angle{-69.07°}$, $I_2 = 29.85 \angle 136.58°$, and $I_3 = 13 \angle 27.60°$ A.

If the phase sequence is reversed by *RBY* then

$$I_R = \frac{V_{RY}}{Z_{RY}} = \frac{400\angle 0^\circ}{15.67\angle 60.13^\circ} = 25.52\angle -60.13^\circ \text{ A}$$

$$I_Y = \frac{V_{YB}}{Z_{YB}} = \frac{400\angle -240^\circ}{43.83\angle 49.83^\circ} = 9.12\angle -289.83^\circ \text{ A}$$

$$I_B = \frac{V_{BR}}{Z_{BR}} = \frac{400\angle -120^\circ}{78.36\angle 60.13^\circ} = 5.1\angle -180.13^\circ \text{ A}$$

Various line currents are given by

$$I_1 = I_R - I_B = 25.52\ \angle -60.13^\circ - 5.1\ \angle -180.13^\circ = 28.41\ \angle -51.189^\circ \text{ A}$$

$$I_2 = I_Y - I_R = 9.12\ \angle -289.83^\circ - 25.52\ \angle -60.13^\circ = 32.175\ \angle 107.37^\circ \text{ A}$$

$$I_3 = I_B - I_Y = 5.1\ \angle -180.13^\circ - 9.12\ \angle -289.83^\circ = 11.85\ \angle 46.26^\circ \text{ A}$$

From the above calculations, it can be verified that the magnitudes of the line currents are not same when the phase sequence is changed.

PROBLEM 9.21

A balanced delta load is supplied from a symmetrical 3-phase, 400 V, 50 Hz supply system. The current in each phase is 20 A and lags behind its phase voltage by an angle of 40°. Calculate

(a) The line current
(b) Total power
(c) Also draw the phasor diagram
(d) The wattmeter readings if two wattmeters are used

Solution

(a) $I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 20 = 34.64$ A

(b) Total power $= \sqrt{3}\, V_L I_L \cos\phi$

$$= \sqrt{3} \times 400 \times 34.64 \times \cos 40^\circ$$

$$= 18.384 \text{ kW}$$

(c) Phasor diagram is shown in Fig. 9.68 (a).

(d) Total active power = $W_1 + W_2$

$$= \sqrt{3}\, V_L I_L \cos\phi$$

$$= \sqrt{3} \times 400 \times 34.64 \times \cos 40^\circ$$

$$= 18.384 \text{ kW} \tag{9.27}$$

$$W_1 - W_2 = V_L I_L \sin\phi$$

$$= 400 \times 34.64 \times \sin 40^\circ$$

$$= 8.906 \text{ kW} \tag{9.28}$$

On solving, $W_1 = 13.645$ kW

$W_2 = 4.739$ kW

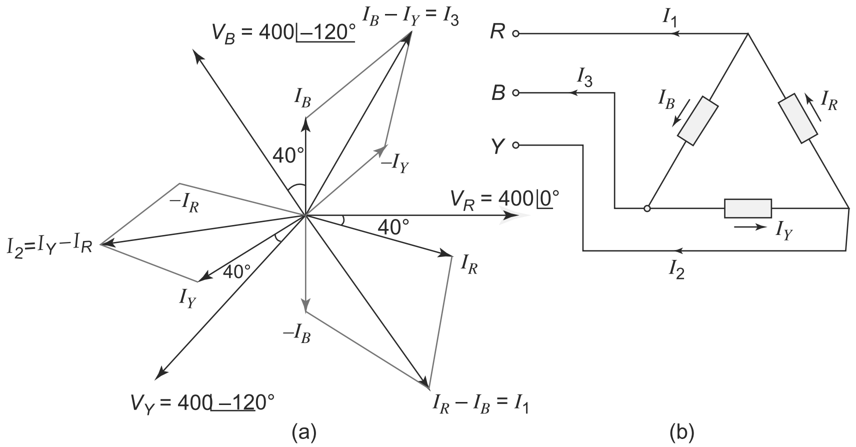


Fig. 9.68

PROBLEM 9.22

Three identical impedances are star connected across a balanced 440 V; 50 Hz supply. The three line currents are $I_R = 20\underline{|-40°}$, $I_Y = 20\underline{|-160°}$ *and* $I_B = 20\underline{|80°}$; *Find the values of the elements. Total power and the readings of wattmeters to measure the power.*

Solution

$$V_{R\ ph} = \frac{440\ \underline{|0°}}{\sqrt{3}} = 254\underline{|0°};\ V_{Y\ ph} = \frac{440\ \underline{|-120°}}{\sqrt{3}} = 254\ \underline{|-120°}$$

$$V_{B\ ph} = \frac{440\ \underline{|-240°}}{\sqrt{3}} = 254\ \underline{|-240°}$$

Given $\quad I_R = 20\ \underline{|-40°};\ I_Y = 20\ \underline{|-160°};\ I_B = 20\ \underline{|80°}$

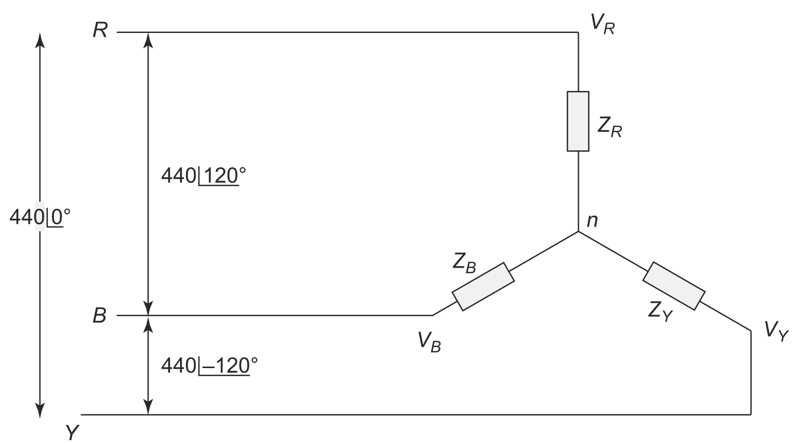


Fig. 9.69

$$Z_R = \frac{254\,\underline{|0^\circ}}{20\,\underline{|-40^\circ}} = 12.7\,\underline{|40^\circ}\ \Omega = (9.728 + j8.16)\ \Omega$$

$$Z_Y = \frac{254\,\underline{|-120^\circ}}{20\,\underline{|-160^\circ}} = 12.7\,\underline{|40^\circ}\ \Omega = (9.728 + j8.16)\ \Omega$$

$$Z_B = \frac{254\,\underline{|-240^\circ}}{20\,\underline{|80^\circ}} = 12.7\,\underline{|40^\circ}\ \Omega = (9.728 + j8.16)\ \Omega$$

$$\text{Power consumed} = 3 \times I^2 R_{ph}$$
$$= 3 \times (20)^2 \times 9.728$$
$$= 11673\ \text{W}$$

Wattmeter readings

$$W_1 = V_L I_L \cos(30^\circ - \phi)$$
$$W_2 = V_L I_L \cos(30^\circ + \phi)$$

where ϕ is the power factor angle between V_{ph} and I_{ph} i.e, $\phi = 40^\circ$

$$W_1 = 440 \times 20 \times \cos(30\text{-}40)^\circ = 8666.3\ \text{W}$$
$$W_2 = 440 \times 20 \times \cos(30+40)^\circ = 3009.7\ \text{W}$$

$$\text{Total power} = W_1 + W_2$$
$$= 8666.3 + 3009.7$$
$$P_T = 11676\ \text{W}$$

PROBLEM 9.23

A 3-phase, 3-wire, 208 V, CBA system has a star-connected load with $Z_A = 5\underline{|0^\circ}\ \Omega$; $Z_B = 5\underline{|30^\circ}\ \Omega$; $Z_C = 10\underline{|-60^\circ}\ \Omega$. Find line currents and voltage across each load impedance. Draw the phasor diagram.

Solution

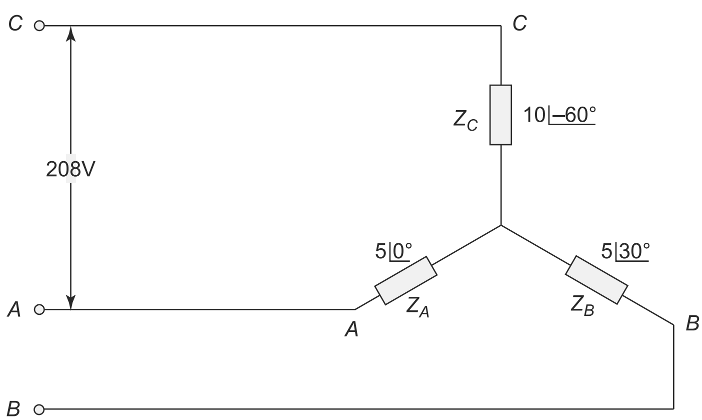


Fig. 9.70

By converting Y-network into Δ,

$$Z_{BC} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_A} = \frac{105.85\underline{|-31.81^\circ}}{5\underline{|0^\circ}} = 21.17\underline{|-31.81^\circ}\ \Omega$$

$$Z_{CA} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_B} = \frac{105.85\underline{|-31.81^\circ}}{5\underline{|30^\circ}} = 21.17\underline{|-61.81^\circ}\ \Omega$$

$$Z_{BA} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_C} = \frac{105.85\underline{|-31.81^\circ}}{10\underline{|-60^\circ}} = 10.5\underline{|-61.81^\circ}\ \Omega$$

Phase sequence is CBA.

$$\therefore \quad V_{CB} = 208\underline{|0^\circ}; \quad V_{BA} = 208\underline{|-120^\circ}; \quad V_{AC} = 208\underline{|-240^\circ}$$

Phase currents

$$I_C = \frac{V_{CB}}{Z_{BC}} = \frac{208\underline{|0^\circ}}{21.17\underline{|-31.81^\circ}} = 9.82\underline{|31.81^\circ}$$

$$I_B = \frac{V_{BA}}{Z_{AB}} = \frac{208\underline{|-120^\circ}}{10.58\underline{|28.19^\circ}} = 19.65\underline{|-148^\circ}$$

$$I_A = \frac{V_{AC}}{Z_{AC}} = \frac{208\underline{|-240^\circ}}{21.17\underline{|-61.81^\circ}} = 9.825\underline{|-179^\circ}$$

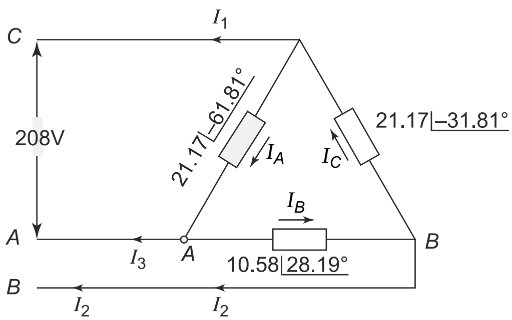


Fig. 9.71

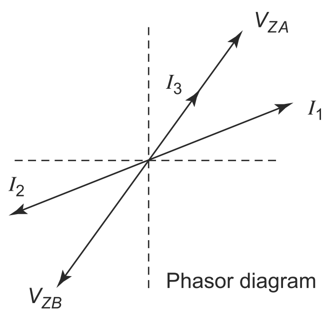


Fig. 9.72

Line currents are

$$I_1 = I_C - I_A = 9.82\underline{|31.81^\circ} - 9.825\underline{|-179.19^\circ}$$

$$= 18.93\underline{|16.30^\circ}\ \text{A}$$

$$I_2 = I_B - I_C = 19.65\underline{|-148.19^\circ} - 9.82\underline{|31.81^\circ}$$

$$= 29.47\underline{|-148.19^\circ}\ \text{A}$$

$$I_3 = I_A - I_B = 9.825\underline{|-179.19^\circ} - 19.65\underline{|-148.19^\circ}$$

$$= 12.31\underline{|56.06^\circ}\ \text{A}$$

Voltages across each load impedance are

$$V_{ZC} = I_1 \times Z_C = (18.93\angle 16.3^\circ)(10\angle -60^\circ)$$

$$= 189.3\angle -43.7^\circ \text{ V}$$

$$V_{ZB} = I_2 \times Z_B = (29.47\angle -148.19^\circ)(5\angle 30^\circ)$$

$$= 147.35\angle -118.19^\circ \text{ V}$$

$$V_{ZA} = I_3 \times Z_C = (12.31\angle 56.06^\circ)(5\angle 0^\circ)$$

$$= 61.55\angle 56.06^\circ \text{ V}$$

PROBLEM 9.24

A 3-phase, 3-wire star-connected unbalanced load is connected as shown. Find the line currents by node voltage method. Assume the line voltage as 415 V and abc phase sequence.

Solution

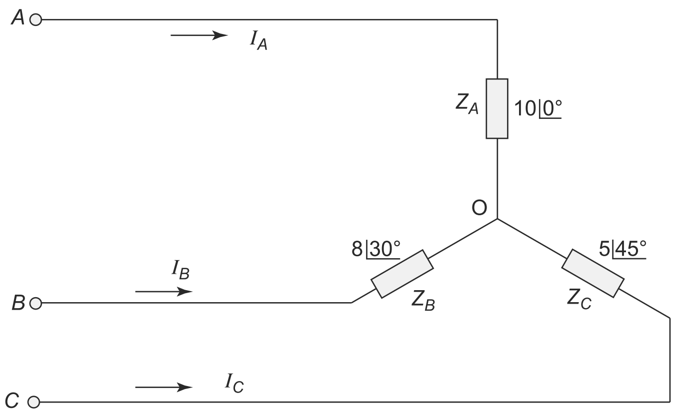


Fig. 9.73

Taking B as reference, the circuit can be redrawn as shown below.

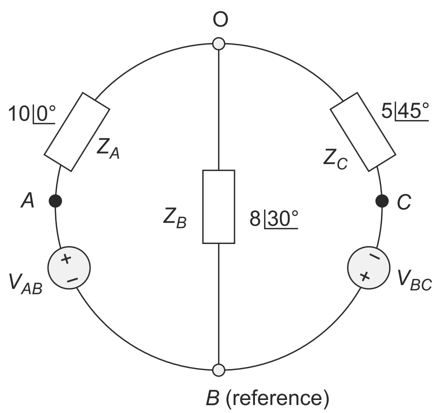


Fig. 9.74

If the voltage at o is V_o w.r.t the reference node B.

Then the single-node path equation can be written as

$$\frac{V_o - V_{AB}}{Z_A} + \frac{V_o}{Z_B} + \frac{V_o + V_{BC}}{Z_C} = 0$$

$$V_o\left[\frac{1}{Z_A} + \frac{1}{Z_B} + \frac{1}{Z_C}\right] - \frac{V_{AB}}{Z_A} + \frac{V_{BC}}{Z_C} = 0$$

$$V_o\left[\frac{1}{10\angle 0^\circ} + \frac{1}{8\angle 30^\circ} + \frac{1}{5\angle 45^\circ}\right] = \frac{415\angle 0^\circ}{10\angle 0^\circ} - \frac{415\angle -120^\circ}{5\angle 45^\circ}$$

$$V_o(0.1\angle 0^\circ + 0.125\angle -30^\circ + 0.2\angle -45^\circ) = 41.5\angle 0^\circ - 83\angle -165^\circ$$

$$V_o(0.404\angle -30.26^\circ) = 123.55\angle 10^\circ$$

$$V_o = 305.8\angle 40.26^\circ$$

$$V_{OA} = V_o - V_{AB} = 305.8\angle 40.26^\circ - 415\angle 0^\circ = 268.4\angle 132^\circ$$

$$V_{OC} = V_o + V_{BC} = 305.8\angle 40.26^\circ + 415\angle -120^\circ = 163.83\angle -80.9^\circ$$

Line currents/phase currents are given by

$$I_A = \frac{-V_{OA}}{Z_A} = \frac{-268.4\angle 132^\circ}{10\angle 0^\circ} = -26.8\angle 132^\circ \text{ A}$$

$$I_B = \frac{-V_{OB}}{Z_B} = \frac{-305.8\angle 40.26^\circ}{8\angle 30^\circ} = -38.225\angle 10.26^\circ \text{ A}$$

$$I_C = \frac{-V_{OC}}{Z_C} = \frac{-163.83\angle -80.9^\circ}{5\angle 45^\circ} = -32.766\angle -125.9^\circ \text{ A}$$

PROBLEM 9.25

A 3-phase, 4-wire, 380 V supply is connected to an unbalanced load having phase impedances of $Z_R = 4 + j3$; $Z_Y = 4 - j3$; and $Z_B = 2\ \Omega$. Impedance of the neutral wire is $Z_n = (1 + j2)\ \Omega$. Find the phase currents and voltages of the load using Millman's theorem.

Solution

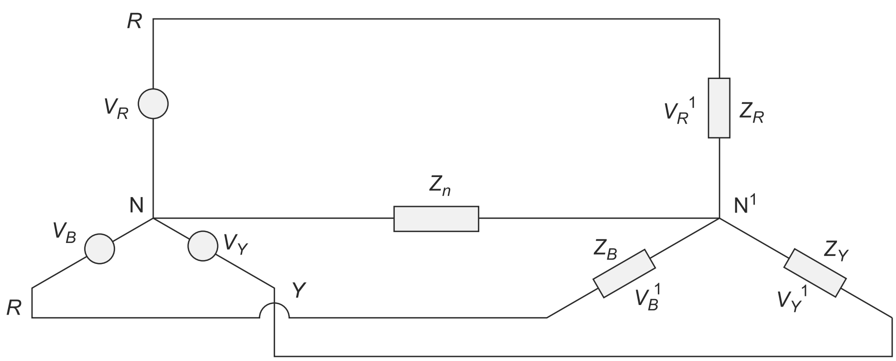


Fig. 9.75

$$Y_R = \frac{1}{Z_R} = \frac{1}{4 + j3} = 0.16 - j0.12\ \mho = 0.2\angle{-36.86^\circ}$$

$$Y_Y = \frac{1}{Z_Y} = \frac{1}{4 - j3} = 0.16 + j0.12\ \mho = 0.2\angle{36.86^\circ}$$

$$Y_B = \frac{1}{Z_B} = \frac{1}{2} = 0.5\ \mho = 0.5\angle{0^\circ}$$

$$Y_N = \frac{1}{1 + j2} = 0.2 - j0.4 = 0.447\angle{-63.43^\circ}$$

Assuming *RYB* sequence, $V_R = \dfrac{380}{\sqrt{3}}\angle{0^\circ} = 219.4\angle{0^\circ}$

Similarly, $V_Y = 219.4\angle{-12^\circ}$, $V_B = 219.4\angle{-240^\circ}$

According to Millman's theorem, the voltage between two nodes is given by

$$V_{N'N} = \frac{V_R Y_R + V_Y Y_Y + V_B Y_B}{Y_R + Y_Y + Y_B + Y_N}$$

$$V_{N'N} = \frac{(219.4\angle{0^\circ})(0.2\angle{-36.86^\circ}) + (219.4\angle{-120^\circ})(0.2\angle{-36.86^\circ}) + (219.4\angle{-240^\circ})(0.5)}{0.2\angle{-36.86^\circ} + 0.2\angle{36.86^\circ} + 0.5\angle{0^\circ} + 0.447\angle{-63.43^\circ}}$$

$$= \frac{43.88\angle{-36.86^\circ} + 43.88\angle{-83.14^\circ} + 109.7\angle{-240^\circ}}{1.02 - j0.4}$$

$$= \frac{-14.51 + j25.12}{1.02 - j0.4}$$

$$= \frac{29\angle{120^\circ}}{1.09\angle{-21.4^\circ}}$$

$$= 26.6\angle{141.4^\circ}$$

$$V_{N'N} = 26.6\angle{141.4^\circ}$$

$$= -20.93 + j16.41$$

The three phase voltages can be found as

$$V'_R = V_R - V_{N'N}$$

$$= 219.4 - (-20.93 + j16.41)$$

$$= 2409\angle{-39^\circ}$$

$$V'_Y = V_Y - V_{N'N}$$

$$= (-110 + j190) - (-20.93 + j16.41)$$

$$= 256.3\angle{246.65^\circ}$$

$$V_B' = V_B - V_{N'N}$$

$$= (-110 + j190) - (-20.93 + j16.41)$$

$$= 195.1\underline{|117.16^\circ}$$

Phase currents are given by

$$I_R = V_R'Y_Y = (290.9\underline{|-3.9^\circ})(0.2\underline{|-36.86^\circ}) = 48.18\underline{|-40.76^\circ}$$

$$I_Y = V_Y'Y_Y = (256.3\underline{|246.65^\circ})(0.2\underline{|36.86^\circ}) = 51.26\underline{|283.51^\circ}$$

$$I_B = V_B'Y_B = (195.1\underline{|117.16^\circ})(0.5\underline{|0^\circ}) = 97.55\underline{|117.10^\circ}$$

$$I_N = V_N'Y_N = (26.5\underline{|141.9^\circ})(0.447\underline{|-63.43^\circ}) = 11.89\underline{|78.4^\circ}$$

PSpice Problems

PROBLEM 9.1

A 3-phase, Δ-connected RYB system is shown in Fig. 9.76 with an effective voltage of 400 V, has a balanced load with impedances 3 + j4 Ω. Using PSpice, calculate (a) phase currents, (b) line currents, and (c) power in each phase.

L = 12.73 mH

R = 3 Ω

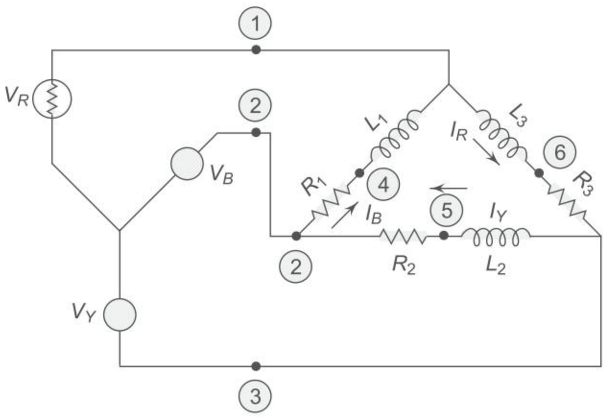


Fig. 9.76 (a)

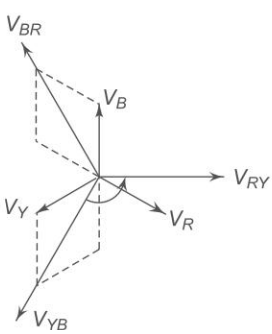


Fig. 9.76 (b)

```
*   3 PHASE BALANCED ANALYSIS
VR   1   0   AC 230.94 -30
VY   3   0   AC 230.94 -150
VB   2   0   AC 230.94 90
R1   2   4   3
L1   1   4   12.732 M
R2   2   5   3
L2   5   3   12.732 M
R3   3   6   3
L3   1   6   12.732 M
```

```
.AC LIN 1 50 50
.PRINT AC IM(VR) IP(VR) IM(VY) IP(VY) IM(VB) IP(VB)
+ IM(R1) IP(R1) IM(R2) IP(R2) IM(R3) IP(R3)
.PROBE
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*****************************************************************
   FREQ       IM(VR)       IP(VR)       IM(VY)       IP(VY)       IM(VB)
5.000E+01 1.386E+02ty  9.687E+01   1.386E+02  -2.313E+01  1.386E+02
   FREQ       IP(VB)       IM(R1)       IP(R1)       IM(R2)       IP(R2)
5.000E+01  -1.431E+02  8.000E+01  6.687E+01  8.000E+01  6.871E+00
   FREQ       IM(R3)          IP(R3)
5.000E+01    8.000E+01       1.269E+02
```

Result

IR = –I(R3) = 48.03 – *J*63.97

IY = –I(R2) = –79.42 – *J*9.61

IB = I(R1) = 31.38 + *J*73.58

I1 = IR – IB = 138.55∠–83.09

I2 = IY – IR = 138.55∠156.9

I3 = IB – IY = 138.55∠36.89

POWER CONSUMED IN EACH PHASE = VPH IPH COS ϕ = 400 × 80 × 0.6 = 19.2 kW

TOTAL POWER = 3 × 19.2 K = 57.6 kW

PROBLEM 9.2

The ac circuit of Fig. 9.77 is supplied from a three-phase balanced supply. Use PSpice to calculate RMS magnitudes and phase angles of currents, I_R, I_Y, I_B, and I_N.

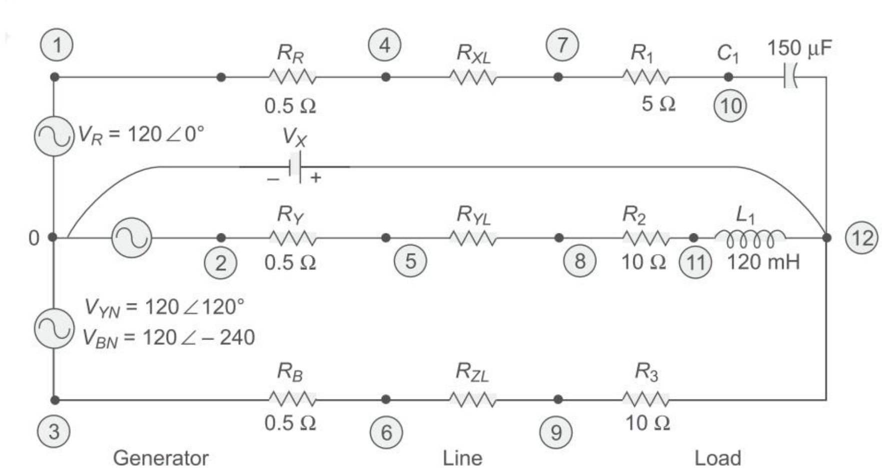


Fig. 9.77 (a)

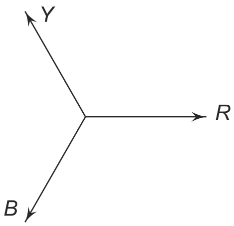


Fig. 9.77 (b)

```
9.2   THREE PHASE CIRCUIT
VRN   1    0    AC 120 0
VYN   2    0    AC 120 120
VBN   3    0    AC 120 -120
RR    1    4    0.5
RY    2    5    0.5
RB    3    6    0.5
RXL   4    7    1
RYL   5    8    1
RZL   6    9    1
R1    7    10   5
R2    8    11   10
R3    9    12   10
C1    10   12   150 UF
L2    11   12   120 MH
VX    12   0    DC 0V
.AC LIN 1 50 100
.PRINT AC IM(RR) IP(RR) IM(RY) IP(RY) IM(RB) IP(RB)
+ VM(7,12) VP(7,12) VM(8,12) VP(8,12) VM(9,12) VP(9,12)
.PRINT AC IM(VX) IP(VX)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*************************************************************
   FREQ        IM(RR)       IP(RR)       IM(RY)       IP(RY)       IM(RB)
5.000E+01   5.407E+00   7.297E+01   3.045E+00   4.696E+01   1.043E+01
   FREQ        IP(RB)       VM(7,12)     VP(7,12)     VM(8,12)     VP(8,12)
5.000E+01   –1.200E+02  1.179E+02   –3.772E+00  1.187E+02   1.221E+02
   FREQ        VM(9,12)     VP(9,12)
5.000E+01   1.043E+02   –1.200E+02
   FREQ        IM(VX)       IP(VX)
5.000E+01   2.262E+00   –1.335E+02
```

Answers to Practice Problems

9-4.1 Taking V_{RN} reference
$i_R = 25.4 \angle 0^\circ$; $i_Y = 25.4 \angle -146.8^\circ$; $i_B = 25.4 \angle 146.8^\circ$
$i_N = 17.1 \angle 0^\circ$; Power = 17967.7 W

9-4.3 Delta impedances:
$Z_{RY} = 3.822\angle{-5.97^\circ}\ \Omega$; $Z_{YB} = 5.3\angle{-9.15^\circ}\ \Omega$ and $Z_{BR} = 8.546\angle{-110.59^\circ}\ \Omega$
Line or phase currents in star load:
$63.83\angle{-146.55^\circ}$ A; $169.66\angle{-118.12^\circ}$ A; $117.58\angle{-103.13^\circ}$ A
Phase currents in delta load: $115.12\angle{35.97^\circ}$; $83.01\angle{-80.85^\circ}$; $51.486\angle{-320.59^\circ}$
Power consumed = 78.4 kW

9-4.4 Powers $P_R = 8007.69$ W; $P_Y = 17{,}923.37$ W; $P_B = 6998$ W
Total power = 32.929 kW

9-6.1 $i_R = 50.8 \angle 0°$; $i_Y = 25.4 \angle -120°$; $i_B = 16.936 \angle 120°$
$P_R = 12903.2$ W; $P_Y = 6451.6$ W; $P_B = 4302.4$ W
(Taking R-phase voltage reference)

9-6.3 Taking V_{RY} reference
$i_R = 11.25 \angle -23.42°$; $i_Y = 18.06 \angle 218.25°$; $i_B = 16.12 \angle 76°$
$281.32 \angle -23.42°$; $180.6 \angle 218.25°$; $241.8 \angle 76°$

9-6.4 $V_{ON} = 87.43 \angle -142.98°$ V

9-7.1 3000 W

9-7.3 $i_R = 50 \angle -62°$; $i_Y = 50 \angle -182°$; $i_B = 50 \angle 58°$; Power = 12705 W

9-7.4 173 μF

9-8.4 $R = 9.72\ \Omega$; L = 25.9 mH; $P = 23.352$ kW; W_1=3 kW; W_2=8.6 kW

9-9.2 Phase currents : $10\angle -90°; 15.625\angle -140°; 12.5\angle 120°$
Active powers : 0 W; 3662 W; 3125 W
Reactive powers : 2500 VAR; 1333 VAR; 0 VAR
Apparent power : $7794.5\angle 29°$ VA

9-11.1 1250 W; 0.693; 2.36 A; 1000 W

Objective-Type Questions

☆☆★**9.1** The resultant voltage in a closed balanced delta circuit is given by

(a) three times the phase voltage
(b) $\sqrt{3}$ times the phase voltage
(c) zero

☆☆★**9.2** Three coils A, B, C, displaced by 120° from each other are mounted on the same axis and rotated in a uniform magnetic field in clockwise direction. If the instantaneous value of emf in coil A is $E_{max} \sin \omega t$, the instantaneous value of emf in B and C coils will be

(a) $E_{max} \sin\left(\omega t - \frac{2\pi}{3}\right)$; $E_{max} \sin\left(\omega t - \frac{4\pi}{3}\right)$

(b) $E_{max} \sin\left(\omega t + \frac{2\pi}{3}\right)$; $E_{max} \sin\left(\omega t + \frac{4\pi}{3}\right)$

(c) $E_{max} \sin\left(\omega t - \frac{2\pi}{3}\right)$; $E_{max} \sin\left(\omega t + \frac{4\pi}{3}\right)$

☆☆★**9.3** The current in the neutral wire of a balanced three-phase, four-wire star connected load is given by
(a) zero
(b) $\sqrt{3}$ times the current in each phase
(c) 3 times the current in each phase

☆☆★**9.4** In a three-phase system, the volt ampere rating is given by
(a) $3V_L I_L$
(b) $\sqrt{3}\ V_L I_L$
(c) $V_L I_L$

☆☆★**9.5** In a three-phase balanced star connected system, the phase relation between the line voltages and their respective phase voltage is given as under

(a) the line voltages lead their respective phase voltages by 30°
(b) the phase voltages lead their respective line voltage by 30°
(c) the line voltages and their respective phase voltages are in phase

☆☆★**9.6** In a three-phase balanced delta connected system, the phase relation between the line currents and their respective phase currents is given by

(a) the line currents lag behind their respective phase currents by 30°
(b) the phase currents lag behind their respective line currents by 30°
(c) the line currents and their respective phase currents are in phase

☆☆★**9.7** In a three-phase unbalanced, four-wire star-connected system, the current in the neutral wire is given by

(a) zero
(b) three times the current in individual phases
(c) the vector sum of the currents in the three lines

☆☆★**9.8** In a three-phase unbalanced star-connected system, the vector sum of the currents in the three lines is

(a) zero
(b) not zero
(c) three times the current in the each phase

☆☆★**9.9** Wattmeter deflection in ac circuit is proportional to the

(a) maximum power in the circuit
(b) instantaneous power in the circuit
(c) average power in the circuit

☆☆★**9.10** The three-wattmeter method of power measurement can be used to measure power in

(a) balanced circuits
(b) unbalanced circuits
(c) both balanced and unbalanced circuits

☆☆★**9.11** The two-wattmeter method of power measurement can be used to measure power in

(a) balanced circuits
(b) unbalanced circuits
(c) both balanced and unbalanced circuits

☆☆★**9.12** In the two-wattmeter method of power measurements, when the pf is 0.5,

(a) the readings of the two wattmeters are equal and positive
(b) the readings of the two wattmeters are equal and opposite
(c) the total power is measured by only one wattmeter

☆☆★**9.13** The reading of the wattmeter connected to measure the reactive power in a three-phase circuit is given by zero, the line voltage is 400 V and line current 15 A; then the pf of the circuit is

(a) zero (b) unity (c) 0.8

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CHAPTER 10

Coupled Circuits

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Understand conductively coupled circuit and mutual impedance
- LO 2 Understand the concept of mutual inductance
- LO 3 Establish the choice of correct sign for mutually induced voltages using the dot-convention technique
- LO 4 Estimate the amount of coupling in terms of coefficient of coupling
- LO 5 Analyse ideal transformer circuits
- LO 6 Analyse multi-winding coupled circuits
- LO 7 Analyse series-connected coupled inductors
- LO 8 Analyse parallel-connected coupled coils
- LO 9 Explain single-tuned and double-tuned circuits and their applications
- LO 10 Analyse magnetic circuits
- LO 11 Compare electric and magnetic circuits
- LO 12 Explain magnetic leakage and fringing
- LO 13 Analyse composite series circuit and parallel magnetic circuit

10.1 INTRODUCTION

Two circuits are said to be 'coupled' when energy transfer takes place from one circuit to the other when one of the circuits is energised. There are many types of couplings like conductive coupling as shown by the potential divider in Fig. 10.1 (a), inductive or magnetic coupling as shown by a two-winding transformer in Fig. 10.1 (b), or conductive and inductive coupling as shown by an auto transformer in Fig. 10.1 (c). A majority of the electrical circuits in practice are conductively or electromagnetically coupled. Certain coupled elements are frequently used in network analysis and synthesis. Transformer, transistor, and electronic pots,

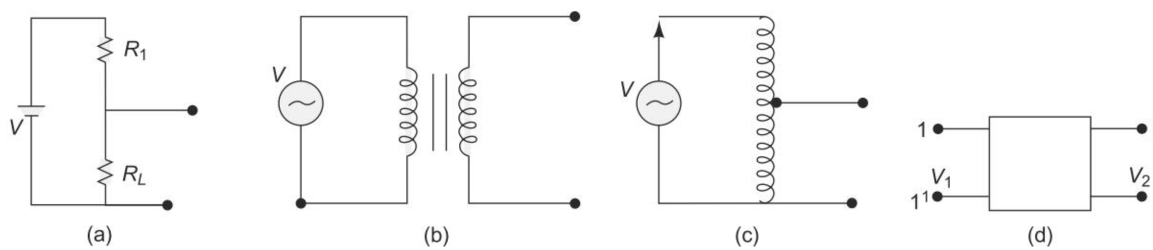


Fig. 10.1

etc., are some among these circuits. Each of these elements may be represented as a two-port network as shown in Fig. 10.1 (d).

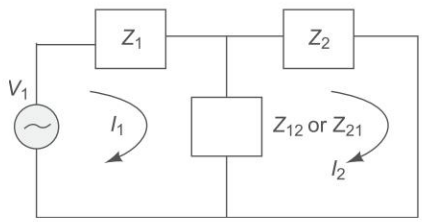
10.2 CONDUCTIVELY COUPLED CIRCUIT AND MUTUAL IMPEDANCE

A conductively coupled circuit which does not involve magnetic coupling is shown in Fig. 10.2 (a).

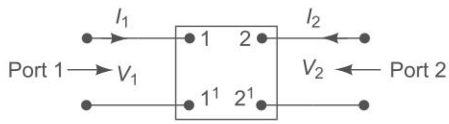
LO 1 Understand conductively coupled circuit and mutual impedance

In the circuit shown, the impedance Z_{12} or Z_{21} common to loops 1 and 2 is called *mutual impedance*. It may consists of a pure resistance, a pure inductance, a pure capacitance, or a combination of any of these elements. Mesh analysis, nodal analysis or Kirchhoff's laws can be used to solve these type of circuits as described in Chapter 7.

The general definition of mutual impedance is explained with the help of Fig. 10.2 (b).



(a)



(b)

Fig. 10.2

The network in the box may be of any configuration of circuit elements with two ports having two pairs of terminals 1-1′ and 2-2′ available for measurement. The mutual impedance between port 1 and 2 can be measured at 1-1′ or 2-2′. If it is measured at 2-2′. It can be defined as the voltage developed (V_2) at 2-2′ per unit current (I_1) at port 1-1′. If the box contains linear bilateral elements, then the mutual impedance measured at 2-2′ is same as the impedance measured at 1-1′ and is defined as the voltage developed (V_1) at 1-1′ per unit current (I_2) at port 2-2′.

EXAMPLE 10.1

Find the mutual impedance for the circuit shown in Fig. 10.3.

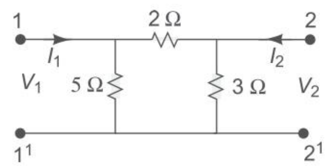


Fig. 10.3

Solution Mutual impedance is given by

$$\frac{V_2}{I_1} \text{ or } \frac{V_1}{I_2}$$

$$V_2 = \frac{3}{2} I_1 \text{ or } \frac{V_2}{I_1} = 1.5\ \Omega$$

or $$V_1 = 5 \times I_2 \times \frac{3}{10} \text{ or } \frac{V_2}{I_2} = 1.5\ \Omega$$

10.3 MUTUAL INDUCTANCE

The property of inductance of a coil was introduced in Section 1.6. A voltage is induced in a coil when there is a time rate of change of current through it. The inductance parameter L, is defined in terms of the voltage across it and the time rate of change of current through it $v(t) = L\dfrac{di(t)}{dt}$, where $v(t)$ is the voltage

LO 2 Understand the concept of mutual inductance across the coil, $I(t)$ is the current through the coil and L is the inductance of the coil. Strictly speaking, this definition is of self-inductance and this is considered as a circuit element with a pair of terminals. Whereas in a circuit element "mutual inductor" does not exist. Mutual inductance is a property associated with two or more coils or inductors which are in close proximity and the presence of common magnetic flux which links the coils. A transformer is such a device whose operation is based on mutual inductance.

Let us consider two coils, L_1 and L_2, as shown in Fig. 10.4 (a), which are sufficiently close together, so that the flux produced by i_1 in the coil L_1 also link the coil L_2. We assume that the coils do not move with respect to one another, and the medium in which the flux is established has a constant permeability. The two coils may be also arranged on a common magnetic core, as shown in Fig. 10.4 (b). The two coils are said to be magnetically coupled, but act as a separate circuits. It is possible to relate the voltage induced in one coil to the time rate of change of current in the other coil. When a voltage v_1 is applied across L_1, a current i_1 will start flowing in this coil, and produce a flux ϕ. This flux also links the coil L_2. If i_1 were to change with respect to time, the flux 'ϕ' would also change with respect to time. The time-varying flux surrounding the second coil, L_2 induces an emf, or voltage, across the terminals of L_2; this voltage is proportional to the time rate of change of current flowing through the first coil L_1. The two coils, or circuits, are said to be inductively coupled, because of this property they are called coupled elements or coupled circuits and the induced voltage, or emf is called the voltage/emf of mutual induction and is given by $v_2(t) = M_1 \dfrac{di_1(t)}{dt}$ volts, where v_2 is the voltage induced in coil L_2 and M_1 is the coefficient of proportionality, and is called the *coefficient of mutual inductance*, or *simple mutual inductance*.

If current i_2 is made to pass through coil L_2 as shown in Fig. 10.4 (c) with coil L_1 open, a change of i_2 would cause a voltage v_1 in coil L_1, given by $v_1(t) = M_2 \dfrac{di_2(t)}{dt}$.

In the above equation, another coefficient of proportionality M_2 is involved. Though it appears that two mutual inductances are involved in determining the mutually induced voltages in the two coils, it can be shown from energy considerations that the two coefficients are equal and, therefore, need not be represented by two different letters. Thus, $M_1 = M_2 = M$.

$$\therefore \quad v_2(t) = M \frac{di_1(t)}{dt} \text{ volts}$$

$$v_1(t) = M \frac{di_2(t)}{dt} \text{ volts}$$

In general, in a pair of linear time invariant coupled coils or inductors, a non-zero current in each of the two coils produces a mutual voltage in each coil due to the flow of current in the other coil. This mutual voltage is present independently of, and in addition to, the voltage due to self induction. Mutual inductance is also measured

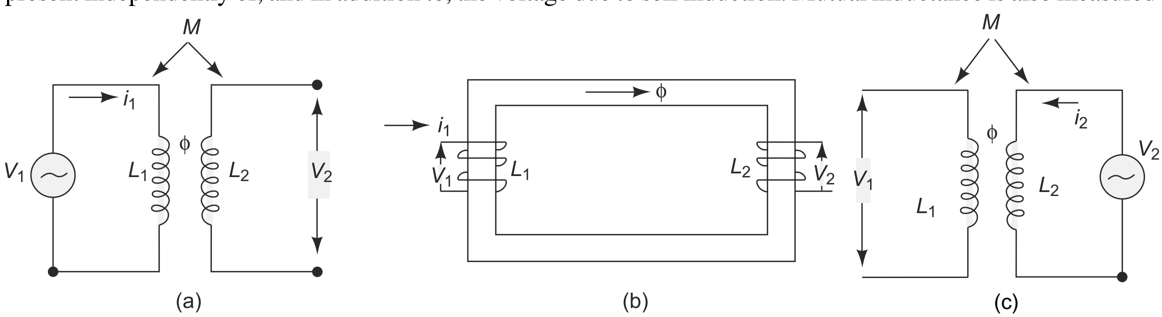


Fig. 10.4

in henries and is positive, but the mutually induced voltage, $M\dfrac{di}{dt}$ may be either positive or negative, depending on the physical construction of the coil and reference directions. To determine the polarity of the mutually induced voltage (i.e. the sign to be used for the mutual inductance), the dot convention is used.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2*

☆☆★ **10-2.1** Two inductively coupled coils have self-inductances $L_1 = 40$ mH and $L_2 = 150$ mH. If the coefficient of coupling is 0.7, (a) find the value of mutual inductance between the coils, and (b) the maximum possible mutual inductance.

☆☆★ **10-2.2** Two coils connected in series have an equivalent inductance of 0.8 H when connected in aiding, and an equivalent inductance of 0.5 H when the connection is opposing. Calculate the mutual inductance of the coils.

☆★★ **10-2.3** Calculate the effective inductance of the circuit shown in Fig. Q.3 across *XY*.

☆★★ **10-2.4** Calculate the effective inductance of the circuit shown in Fig. Q.4.

☆★★ **10-2.5** Find the equivalent inductance between the terminals across *ab* for the coupled circuit shown, $M = 0.5$ H. All the coils are coupled. (Fig. Q.5)

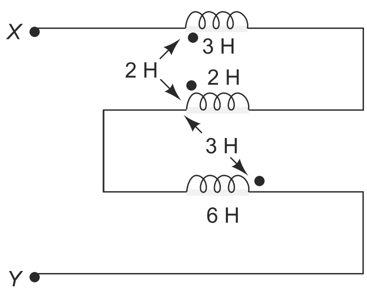


Fig. Q.3

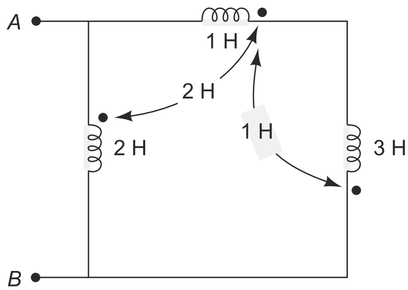


Fig. Q.4

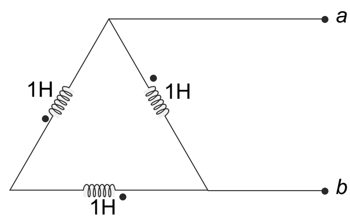


Fig. Q.5

Frequently Asked Questions linked to LO 2*

☆★★ **10-2.1** Obtain a conductively coupled equivalent circuit for the magnetically coupled circuit shown in Fig. Q.1. **[AU Nov./Dec. 2012]**

☆★★ **10-2.2** In the coupled circuit shown in Fig. Q.2, find the voltage across the 5 Ω resistor. **[AU April/May 2011]**

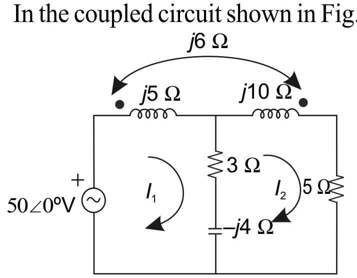


Fig. Q.1

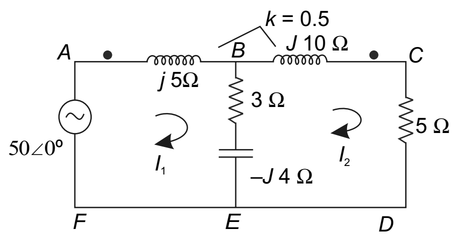


Fig. Q.2

☆☆★ **10-2.3** Write the expression which relates the self and mutual inductance. **[AU May/June 2014]**

☆☆★ **10-2.4** Define mutual inductance. **[AU Nov./Dec. 2012]**

☆☆★ **10-2.5** Two coupled coils have self-inductances of L_1 = 100 mH and L_2 = 400 mH. The coupling coefficient is 0.8. Find M. If N_1 is 1000 turns, what is the values of N_2? If a current $i_1 = 2 \sin(500t)$ A through the coil 1, find the flux ϕ_1 and the mutually induced voltage V_{2M}. **[AU Nov./Dec. 2012]**

☆★★ **10-2.6** Two inductively coupled coil have self-inductances $L_1 = 50$ mH and $L_2 = 200$ mH. If the coefficient of coupling is 0.5, compute the value of mutual inductance between the coils. **[AU April/May 2011]**

☆☆★ **10-2.7** For the network shown in Fig. Q.7, find the voltage across the load resistance R_L. **[JNTU Nov. 2012]**

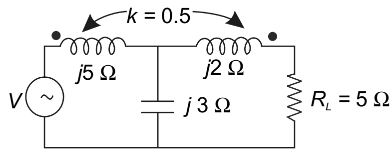


Fig. Q.7

10.4 DOT CONVENTION

LO 3 Establish the choice of correct sign for mutually induced voltages using the dot-convention technique

Dot convention is used to establish the choice of correct sign for the mutually induced voltages in coupled circuits.

Circular dot marks and/or special symbols are placed at one end of each of two coils which are mutually coupled to simplify the diagrammatic representation of the windings around its core.

Let us consider Fig. 10.5, which shows a pair of linear, time-invariant, coupled inductors with self- inductances L_1 and L_2 and a mutual inductance M. If these inductors form a portion of a network, currents i_1 and i_2 are shown, each arbitrarily assumed entering at the dotted terminals, and voltages v_1 and v_2 are developed across the inductors. The voltage across L_1 is, thus composed of two parts and is given by

$$v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt}$$

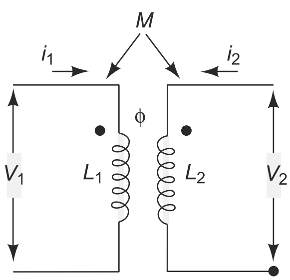


Fig. 10.5

The first term on the RHS of the above equation is the self induced voltage due to i_1, and the second term represents the mutually induced voltage due to i_2.

Similarly, $$v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt}$$

Although the self-induced voltages are designated with positive sign, mutually induced voltages can be either positive or negative depending on the direction of the winding of the coil and can be decided by the presence of the *dots* placed at one end of each of the two coils. The convention is as follows.

If two terminals belonging to different coils in a coupled circuit are marked identically with dots then for the same direction of current relative to like terminals, the magnetic flux of self- and mutual induction in each coil add together. The physical basis of the dot convention can be verified by examining Fig. 10.6. Two coils *ab* and *cd* are shown wound on a common iron core.

It is evident from Fig. 10.6 that the direction of the winding of the coil *ab* is clock-wise around the core as viewed at X, and that of *cd* is anti-clockwise as viewed at Y. Let the direction of the current i_1 in the first coil be from a to b, and increasing with time. The flux produced by i_1 in the core has a direction which may be

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

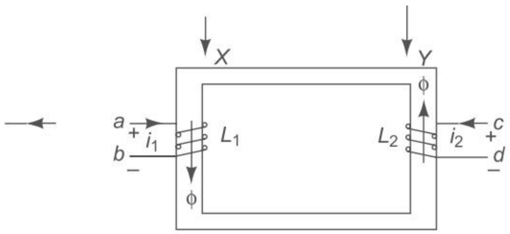


Fig. 10.6

found by right hand rule, and which is downwards in the left limb of the core. The flux also increases with time in the direction shown at *X*. Now suppose that the current i_2 in the second coil is from *c* to *d*, and increasing with time. The application of the right hand rule indicates that the flux produced by i_2 in the core has an upward direction in the right limb of the core. The flux also increases with time in the direction shown at *Y*. The assumed currents i_1 and i_2 produce flux in the core that are additive. The terminals *a* and *c* of the two coils attain similar polarities simultaneously. The two simultaneously positive terminals are identified by two dots by the side of the terminals as shown in Fig. 10.7.

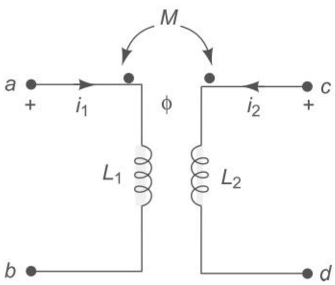


Fig. 10.7

The other possible location of the dots is the other ends of the coil to get additive fluxes in the core, i.e. at *b* and *d*. It can be concluded that the mutually induced voltage is positive when currents i_1 and i_2 both enter (or leave) the windings by the dotted terminals. If the current in one winding enters at the dot-marked terminals and the current in the other winding leaves at the dot-marked terminal, the voltages due to self and mutual induction in any coil have opposite signs.

EXAMPLE 10.2

Using dot convention, write voltage equations for the coils shown in Fig. 10.8.

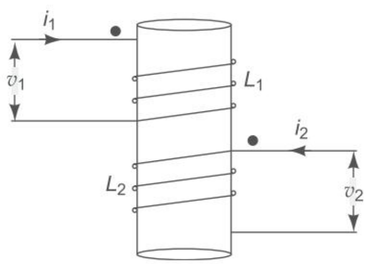


Fig. 10.8

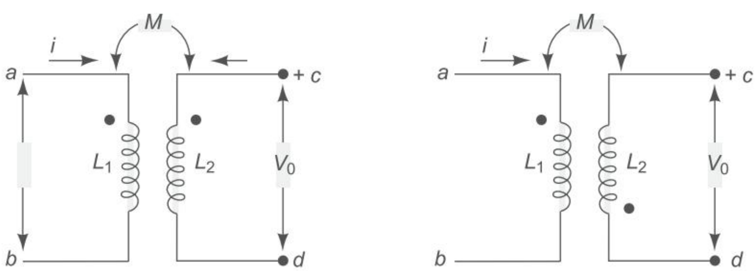
Solution Since the currents are entering at the dot-marked terminals, the mutually induced voltages or the sign of the mutual inductance is positive; using the sign convention for the self inductance, the equations for the voltages are

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

EXAMPLE 10.3

Write the equation for voltage v_0 for the circuits shown in Fig. 10.9.



(a) (b)

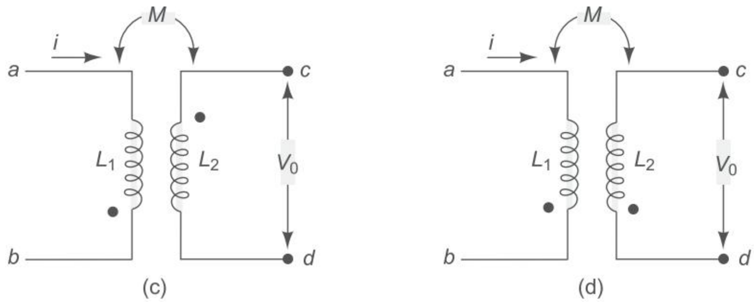


Fig. 10.9

Solution v_0 is assumed positive with respect to the terminal C and the equation is given by

(a) $v_0 = M\dfrac{di}{dt}$ (b) $v_0 = -M\dfrac{di}{dt}$ (c) $v_0 = -M\dfrac{di}{dt}$ (d) $v_0 = M\dfrac{di}{dt}$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★**10-3.1** Using the dot convention, write the voltage equations for the coils shown in Fig. Q.3.

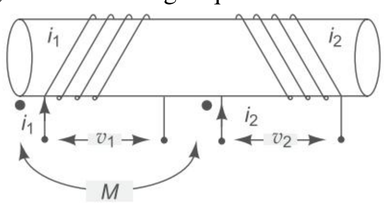


Fig. Q.3

Frequently Asked Questions linked to LO 3

☆☆★**10-3.1** What is dot convention in coupled circuit? **[BPUT 2007]**

☆★★**10-3.2** Explain the dot convention rule, for the magnetically coupled network using network shown in Fig.Q.2. Also formulate KVL equations. **[GTU Dec. 2012]**

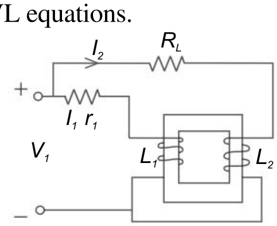


Fig. Q.2

10.5 COEFFICIENT OF COUPLING

LO 4 Estimate the amount of coupling in terms of coefficient of coupling

The amount of coupling between the inductively coupled coils is expressed in terms of the coefficient of coupling, which is defined as $K = M/\sqrt{L_1L_2}$

where M = mutual inductance between the coils,
L_1 = self-inductance of the first coil, and
L_2 = self-inductance of the second coil.

Coefficient of coupling is always less than unity, and has a maximum value of 1 (or 100%). This case, for which $K = 1$, is called *perfect coupling*, when the entire flux of one coil links the other. The greater the coefficient of coupling between the two coils, the greater the mutual inductance between them, and vice versa. It can be expressed as the fraction of the magnetic flux produced by the current in one coil that links the other coil.

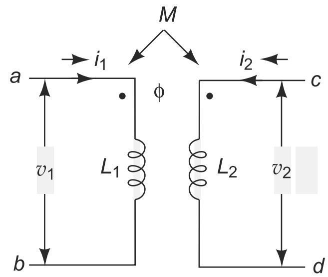


Fig. 10.10

For a pair of mutually coupled circuits shown in Fig. 10.10, let us assume initially that i_1, i_2 are zero at $t = 0$.

Then $\quad v_1(t) = L_1 \dfrac{di_1(t)}{dt} + M \dfrac{di_2(t)}{dt}$

and $\quad v_2(t) = L_2 \dfrac{di_2(t)}{dt} + M \dfrac{di_1(t)}{dt}$

Initial energy in the coupled circuit at $t = 0$ is also zero. The net energy input to the system shown in Fig. 10.10 at time t is given by

$$W(t) = \int_0^t \left[v_1(t)\, i_1(t) + v_2(t)\, i_2(t)\right] dt$$

Substituting the values of $v_1(t)$ and $v_2(t)$ in the above equation yields

$$W(t) = \int_0^t \left[L_1 i_1(t) \frac{di_1(t)}{dt} + L_2 i_2(t) \frac{di_2(t)}{dt} + M\left(i_1(t)\right) \frac{di_2(t)}{dt} + i_2(t) \frac{di_1(t)}{dt}\right] dt$$

from which, we get

$$W(t) = \frac{1}{2} L_1 \left[i_1(t)\right]^2 + \frac{1}{2} L_2 \left[i_2(t)\right]^2 + M\left[i_1(t)\, i_2(t)\right]$$

If one current enters a dot-marked terminal while the other leaves a dot-marked terminal, the above equation becomes

$$W(t) = \frac{1}{2} L_1 \left[i_1(t)\right]^2 + \frac{1}{2} L_2 \left[i_2(t)\right]^2 - M\left[i_1(t)\, i_2(t)\right]$$

According to the definition of passivity, the net electrical energy input to the system is non-negative. $W(t)$ represents the energy stored within a passive network, it cannot be negative.

$\therefore \quad W(t) \geq 0$ for all values of i_1, i_2; L_1, L_2 or M

The statement can be proved in the following way. If i_1 and i_2 are both positive or negative, $W(t)$ is positive. The other condition where the energy equation could be negative is

$$W(t) = \frac{1}{2} L_1 \left[i_1(t)\right]^2 + \frac{1}{2} L_2 \left[i_2(t)\right]^2 - M\left[i_1(t)\, i_2(t)\right]$$

The above equation can be rearranged as

$$W(t) = \frac{1}{2}\left(\sqrt{L_1} i_1 - \frac{M}{\sqrt{L_1}} i_2\right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right) i_2^2$$

The first term in the parenthesis of the right side of the above equation is positive for all values of i_1 and i_2, and, thus, the last term cannot be negative; hence,

$$L_2 - \frac{M^2}{L_1} \geq 0$$

$$\frac{L_1 L_2 - M^2}{L_1} \geq 0$$

$$L_1 L_2 - M^2 \geq 0$$

$$\sqrt{L_1 L_2} \geq M$$

$$M \leq \sqrt{L_1 L_2}$$

Obviously, the maximum value of the mutual inductance is $\sqrt{L_1\, L_2}$. Thus, we define the coefficient of coupling for the coupled circuit as

$$K = \frac{M}{\sqrt{L_1\, L_2}}$$

The coefficient, K, is a non-negative number and is independent of the reference directions of the currents in the coils. If the two coils are a great distance apart in space, the mutual inductance is very small, and K is also very small. For iron-core coupled circuits, the value of K may be as high as 0.99, for air-core coupled circuits, K varies between 0.4 to 0.8.

EXAMPLE 10.4

Two inductively coupled coils have self inductances $L_1 = 50$ mH and $L_2 = 200$ mH. If the coefficient of coupling is 0.5 (a), find the value of mutual inductance between the coils, and (b) what is the maximum possible mutual inductance?

Solution (a) $M = K\sqrt{L_1 L_2}$

$$= 0.5\sqrt{50\times10^{-3}\times200\times10^{-3}} = 50\times10^{-3}\text{ H}$$

(b) Maximum value of the inductance when $K = 1$,

$$M = \sqrt{L_1 L_2} = 100\text{ mH}$$

Frequently Asked Questions linked to LO 4

☆☆★ **10-4.1** Derive the expression for coefficient of coupling in terms of mutual and self-inductances of the coils. **[AU May/June 2013]**

☆☆★ **10-4.2** Determine the coefficient of coupling of two magnetically coupled coil of turns N_1 and N_2. **[JNTU Nov. 2012]**

☆★★ **10-4.3** The following data refers to two coupled coils 1 and 2, as shown in Fig. Q.3.
$\phi_{11} = 0.5 \times 10^{-3}$ Wb; $\phi_{12} = 0.3 \times 10^{-3}$ Wb: $N_1 = 100$ turns; $N_2 = 500$ turns; $i_1 = 1$A.
Find, K, the coefficient of coupled, the inductances L_1 and L_2 and M, the mutual inductance. **[RGTU Dec. 2013]**

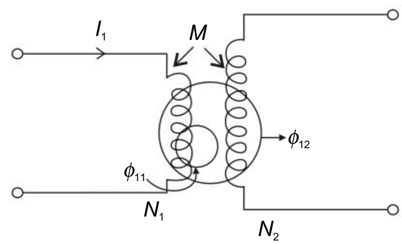


Fig. Q.3

10.6 IDEAL TRANSFORMER

LO 5 Analyse ideal transformer circuits

Transfer of energy from one circuit to another circuit through mutual induction is widely utilised in power systems. This purpose is served by transformers. Most often, they transform energy at one voltage (or current) into energy at some other voltage (or current).

*A **transformer** is a static piece of apparatus, having two or more windings or coils arranged on a common magnetic core. The transformer winding to which the supply source is connected is called the primary, while the winding connected to load is called the secondary*. Accordingly, the voltage across the primary is called the primary voltage, and that across the secondary, the secondary voltage. Correspondingly, i_1 and i_2 are the currents in the primary and secondary windings. One such transformer is shown in Fig. 10.11 (a). In circuit diagrams, ideal transformers are represented by Fig. 10.11 (b).

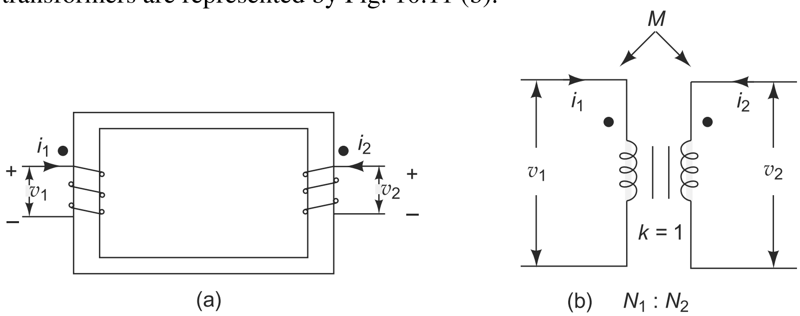


Fig. 10.11

The vertical lines between the coils represent the iron core; the currents are assumed such that the mutual inductance is positive. An ideal transformer is characterised by assuming (i) zero power dissipation in the primary and secondary windings, i.e. resistances in the coils are assumed to be zero, (ii) the self inductances of the primary and secondary are extremely large in comparison with the load impedance, and (iii) the coefficient of coupling is equal to unity, i.e. the coils are tightly coupled without having any leakage flux. If the flux produced by the current flowing in a coil links all the turns, the self inductance of either the primary or secondary coil is proportional to the square of the number of turns of the coil. This can be verified from the following results.

The magnitude of the self induced emf is given by

$$v = L\frac{di}{dt}$$

If the flux linkages of the coil with N turns and current are known, then the self- induced emf can be expressed as

$$v = N\frac{d\phi}{dt}$$

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

$$L = N\frac{d\phi}{di}$$

But $\quad \phi = \dfrac{Ni}{\text{reluctance}}$

$$\therefore \quad L = N\frac{d}{di}\left(\frac{Ni}{\text{reluctance}}\right)$$

$$L = \frac{N^2}{\text{reluctance}}$$

$$L \propto N^2$$

From the above relation, it follows that

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2$$

where $a = N_2/N_1$ is called the *turns ratio* of the transformer. The turns ratio, a, can also be expressed in terms of primary and secondary voltages. If the magnetic permeability of the core is infinitely large then the flux would be confined to the core. If ϕ is the flux through a single turn coil on the core and N_1, N_2 are the number of turns of the primary and secondary, respectively, then the total flux through windings 1 and 2, respectively, are

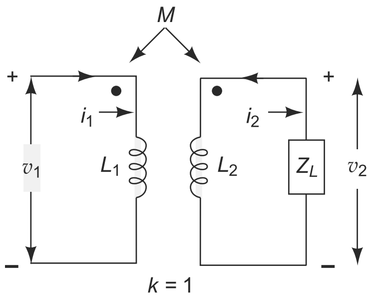


Fig. 10.12

$$\phi_1 = N_1\,\phi;\ \phi_2 = N_2\,\phi$$

Also, we have $v_1 = \dfrac{d\phi_1}{dt}$, and $v_2 = \dfrac{d\phi_2}{dt}$

so that $$\boxed{\frac{v_2}{v_1} = \frac{N_2\dfrac{d\phi}{dt}}{N_1\dfrac{d\phi}{dt}} = \frac{N_2}{N_1}}$$

Figure 10.12 shows an ideal transformer to which the secondary is connected to a load impedance Z_L. The turns ratio $\dfrac{N_2}{N_1} = a$.

The ideal transformer is a very useful model for circuit calculations, because with few additional elements like R, L, and C, the actual behaviour of the physical transformer can be accurately represented. Let us analyse this transformer with sinusoidal excitations. When the excitations are sinusoidal voltages or currents, the steady state response will also be sinusoidal. We can use phasors for representing these voltages and currents. The input impedance of the transformer can be determined by writing mesh equations for the circuit shown in Fig. 10.12.

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \tag{10.1}$$

$$0 = -j\omega M I_1 + (Z_L + j\omega L_2)I_2 \tag{10.2}$$

where V_1, V_2 are the voltage phasors, and I_1, I_2 are the current phasors in the two windings. $j\omega L_1$ and $j\omega L_2$ are the impedances of the self inductances and $j\omega M$ is the impedance of the mutual inductance, ω is the angular frequency.

From Eq. (10.2), $$I_2 = \frac{j\omega M I_1}{(Z_L + j\omega L_2)}$$

Substituting in Eq. (10.1), we have

$$V_1 = I_1 j\omega L_1 + \frac{I_1\omega^2 M^2}{Z_L + j\omega L_2}$$

The input impedance $Z_{in} = \dfrac{V_1}{I_1}$

$$\therefore \quad Z_{in} = j\omega L_1 + \frac{\omega^2 M^2}{(Z_L + j\omega L_2)}$$

When the coefficient of coupling is assumed to be equal to unity,

$$M = \sqrt{L_1 L_2}$$

$$\therefore \quad Z_{in} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{(Z_L + j\omega L_2)}$$

We have already established that $\dfrac{L_2}{L_1} = a^2$

where a is the turns ratio N_2 / N_1.

$$\therefore \quad Z_{in} = j\omega L_1 + \frac{\omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

Further simplication leads to

$$Z_{in} = \frac{(Z_L + j\omega L_2)\, j\omega L_1 + \omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

$$Z_{in} = \frac{j\omega L_1 Z_L}{(Z_L + j\omega L_2)}$$

As L_2 is assumed to be infinitely large compared to Z_L,

$$Z_{in} = \frac{j\omega L_1 Z_L}{j\omega a^2 L_1} = \frac{Z_L}{a^2} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

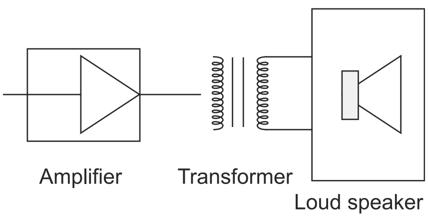


Fig. 10.13

The above result has an interesting interpretation, that is the ideal transformers change the impedance of a load, and can be used to match circuits with different impedances in order to achieve maximum power transfer. For example, the input impedance of a loudspeaker is usually very small, say 3 Ω to 12 Ω, for connecting directly to an amplifier. The transformer with proper turns ratio can be placed between the output of the amplifier and the input of the loudspeaker to match the impedances as shown in Fig. 10.13.

EXAMPLE 10.5

An ideal transformer has $N_1 = 10$ turns, and $N_2 = 100$ turns. What is the value of the impedance referred to as the primary, if a 1000 Ω resistor is placed across the secondary?

Solution The turns ratio $a = \dfrac{100}{10} = 10$

$$Z_{in} = \frac{Z_L}{a^2} = \frac{1000}{100} = 10\ \Omega$$

The primary and secondary currents can also be expressed in terms of turns ratio. From Eq. (10.2), we have

$$I_1 \, j\omega M = I_2(Z_L + j\omega L_2)$$

$$\frac{I_1}{I_2} = \frac{Z_L + j\omega L_2}{j\omega M}$$

When L_2 is very large compared to Z_L,

$$\frac{I_1}{I_2} = \frac{j\omega L_2}{j\omega M} = \frac{L_2}{M}$$

Substituting the value of $M = \sqrt{L_1 L_2}$ in the above equation, $\dfrac{I_1}{I_2} = \dfrac{L_2}{M}$

$$\frac{I_1}{I_2} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}}$$

$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}} = a = \frac{N_2}{N_1}$$

EXAMPLE 10.6

An amplifier with an output impedance of 1936 Ω is to feed a loudspeaker with an impedance of 4 Ω.

(a) Calculate the desired turns ratio for an ideal transformer to connect the two systems.

(b) An rms current of 20 mA at 500 Hz is flowing in the primary. Calculate the rms value of current in the secondary at 500 Hz.

(c) What is the power delivered to the load?

Solution (a) To have maximum power transfer, the output impedance of the amplifier $= \dfrac{\text{Load impedance}}{a^2}$

$$\therefore \quad 1936 = \frac{4}{a^2}$$

$$\therefore \quad a = \sqrt{\frac{4}{1936}} = \frac{1}{22}$$

$$\text{or} \quad \frac{N_2}{N_1} = \frac{1}{22}$$

(b) $I_1 = 20$ mA

We have $\dfrac{I_1}{I_2} = a$

RMS value of the current in the secondary winding

$$= \frac{I_1}{a} = \frac{20 \times 10^{-3}}{1/22} = 0.44 \text{ A}$$

(c) The power delivered to the load (speaker)

$$= (0.44)^2 \times 4 = 0.774 \text{ W}$$

The impedance changing properties of an ideal transformer may be utilised to simplify circuits. Using this property, we can transfer all the parameters of the primary side of the transformer to the secondary side, and vice versa. Consider the coupled circuit shown in Fig. 10.14 (a).

To transfer the secondary side load and voltage to the primary side, the secondary voltage is to be divided by the ratio, a, and the load impedance is to be divided by a^2. The simplified equivalent circuits is shown in Fig. 10.14 (b).

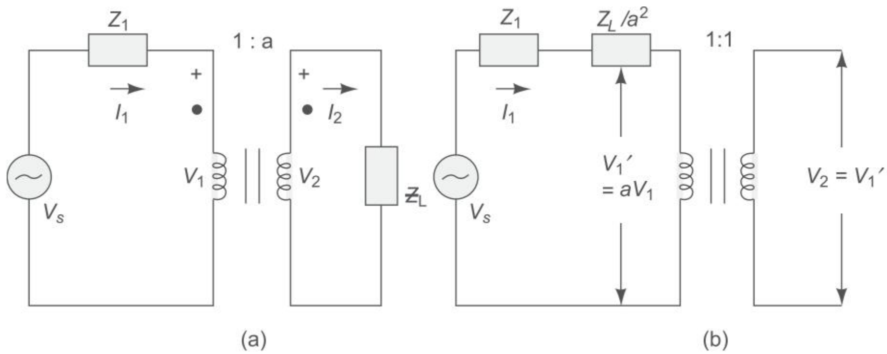


Fig. 10.14

EXAMPLE 10.7

For the circuit shown in Fig. 10.15 with turns ratio, $a = 5$, draw the equivalent circuit referring (a) to primary, and (b) secondary. Take source resistance as 10 Ω.

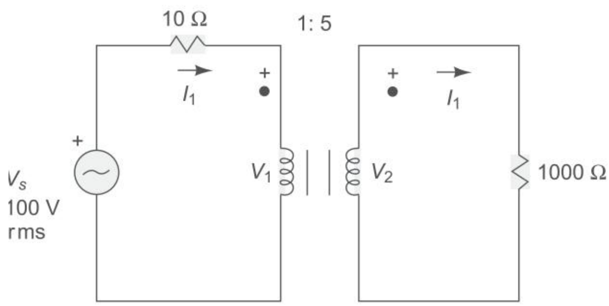


Fig. 10.15

Solution (a) Equivalent circuit referred to primary is as shown in Fig. 10.16 (a).

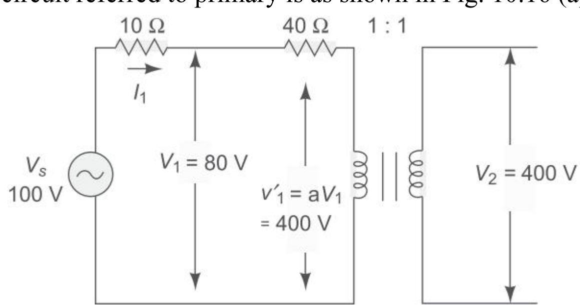


Fig. 10.16 (a)

(b) Equivalent circuit referred to secondary is as shown in Fig. 10.16 (b).

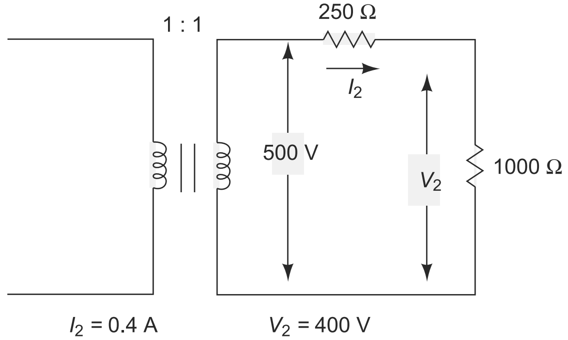


Fig. 10.16 (b)

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆★★**10-5.1** In Fig. Q.1, $L_1 = 2$ H; $L_2 = 6$ H; $K = 0.5$; $i_1 = 4 \sin (40t - 30°)$ A; $i_2 = 2 \sin (40t - 30°)$ A. Find the values of (a) v_1, and (b) v_2.

☆☆★**10-5.2** For the circuit shown in Fig. Q.2, write the mesh equations.

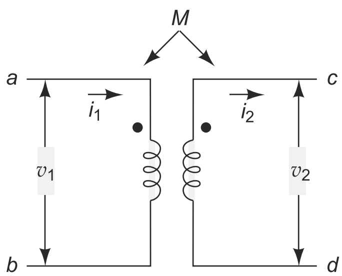


Fig. Q.1

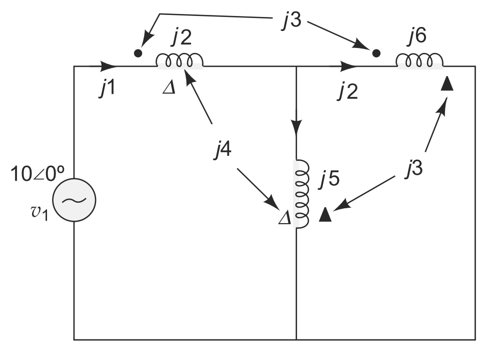


Fig. Q.2

☆☆★**10-5.3** Write the mesh equations for the network shown in Fig. Q.3.

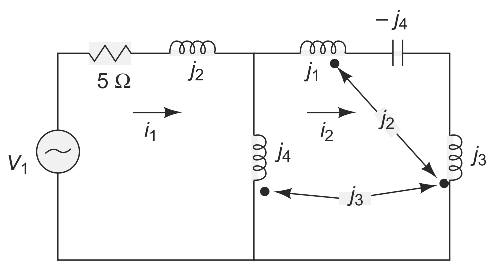


Fig. Q.3

Frequently Asked Questions linked to LO 5

☆★★**10-5.1** A voltage of 100 V at a frequency of $10^6/2\pi$ Hz is applied to the primary of coupled circuit. Calculate primary and secondary currents. **[RTU Feb. 2011]**

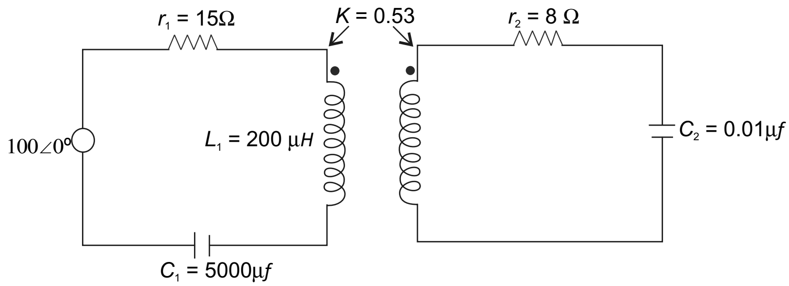


Fig. Q. 1

OR

☆★★**10-5.2** In the figure, two coils A ($R_1 = 5\ \Omega$, $L_1 = 0.01$ H) and B ($R_2 = 100\ \Omega$, $L_2 = 5$ H) have coefficient of coupling 0.8. Calculating the percentage change in effective resistance of the coil A at a frequency of 50 Hz when resistance connected across terminals of the coil B becomes 0 ohms. **[RTU Feb. 2011]**

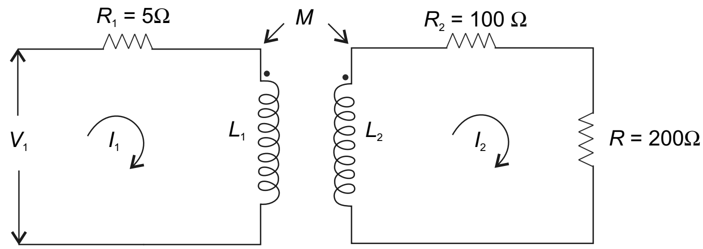


Fig. Q.2

10.7 ANALYSIS OF MULTI-WINDING COUPLED CIRCUITS

LO 6 *Analyse multi-winding coupled circuits*

Inductively coupled multi-mesh circuits can be analysed using Kirchhoff's laws and by loop current methods. Consider Fig. 10.17, where three coils are inductively coupled. For such a system of inductors, we can define a inductance matrix L as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

where L_{11}, L_{22} and L_{33} are self-inductances of the coupled circuits, and $L_{12} = L_{21}$; $L_{23} = L_{32}$ and $L_{13} = L_{31}$ are mutual inductances. More precisely, L_{12} is the mutual inductance between coils 1 and 2, L_{13} is the mutual inductance between coils 1 and 3, and L_{23} is the mutual inductance between coils 2 and 3. The inductance matrix has its order equal to the number of inductors and is symmetric. In terms of voltages across the coils, we have a voltage vector related to i by

$$[v] = [L]\left[\frac{di}{dt}\right]$$

where v and i are the vectors of the branch voltages and currents, respectively. Thus, the branch volt-ampere relationships of the three inductors are given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

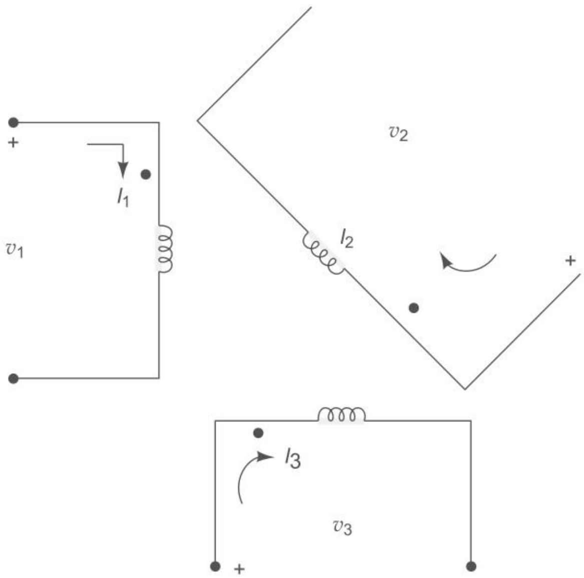


Fig. 10.17

Using KVL and KCL, the effective inductances can be calculated. The polarity for the inductances can be determined by using passivity criteria, whereas the signs of the mutual inductances can be determined by using the dot convention.

EXAMPLE 10.8

For the circuit shown in Fig. 10.18, write the inductance matrix.

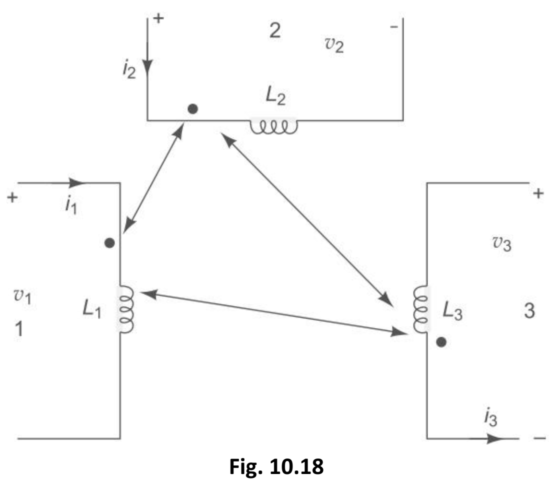


Fig. 10.18

Solution Let L_1, L_2, and L_3 be the self-inductances, and $L_{12} = L_{21}$, $L_{23} = L_{32}$ and $L_{13} = L_{31}$ be the mutual inductances between coils, 1, 2, 2, 3 and 1, 3, respectively.

$L_{12} = L_{21}$ is positive, as both the currents are entering at dot marked terminals, whereas $L_{13} = L_{31}$ and $L_{23} = L_{32}$ are negative.

$$\therefore \quad \text{The inductance matrix is} \quad L = \begin{bmatrix} L_1 & L_{12} & -L_{13} \\ L_{21} & L_2 & -L_{23} \\ -L_{31} & -L_{32} & L_3 \end{bmatrix}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆☆★**10-6.1** For the circuit shown in Fig. Q.1, write the inductance matrix.

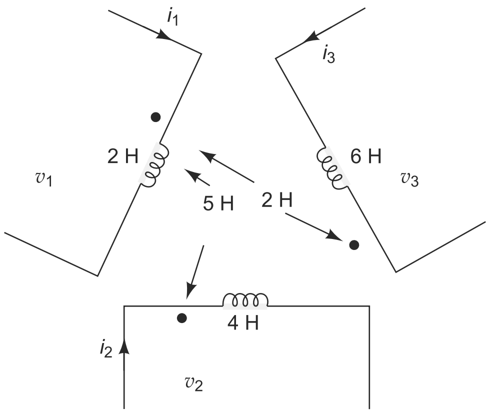


Fig. Q.1

10.8 SERIES CONNECTION OF COUPLED INDUCTORS

LO 7 Analyse series-connected coupled inductors

Let there be two inductors connected in series, with self-inductances L_1 and L_2 and mutual inductance of M. Two kinds of series connections are possible; series aiding as in Fig. 10.19 (a), and series opposition as in Fig. 10.19 (b).

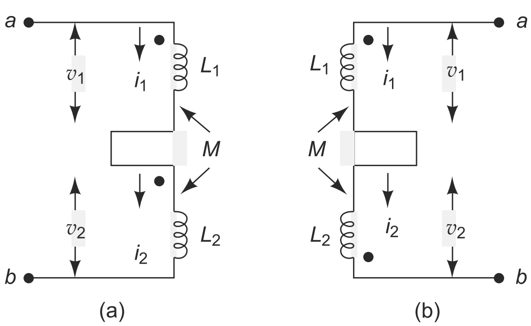


Fig. 10.19

In the case of series-aiding connection, the currents in both inductors at any instant of time are in the same direction relative to like terminals as shown in Fig. 10.19 (a). For this reason, the magnetic fluxes of self-induction and of mutual induction linking with each element add together.

In the case of series-opposition connection, the currents in the two inductors at any instant of time are in opposite direction relative to like terminals as shown in Fig. 10.19 (b). The inductance of an element is given by $L = \phi/i$, where ϕ is the flux produced by the inductor.

$$\therefore \qquad \phi = Li$$

For the series-aiding circuit, if ϕ_1 and ϕ_2 are the flux produced by the coils 1 and 2, respectively, then the total flux

$$\phi = \phi_1 + \phi_2$$

where $\phi_1 = L_1 i_1 + M i_2$

$\phi_2 = L_2 i_2 + M i_1$

$\therefore \quad \phi = Li = L_1 i_1 + M i_2 + L_2 i_2 + M i_1$

Since $i_1 = i_2 = i$

$L = L_1 + L_2 + 2M$

Similarly, for the series opposition,

$\phi = \phi_1 + \phi_2$

where $\phi_1 = L_1 i_1 - M i_2$

$\phi_2 = L_2 i_2 - M i_1$

$\phi = Li = L_1 i_1 - M i_2 + L_2 i_2 - M i_1$

Since $i_1 = i_2 = i$

$L = L_1 + L_2 - 2M$

In general, the inductance of two inductively coupled elements in series is given by $L = L_1 + L_2 \pm 2M$. Positive sign is applied to the series aiding connection, and negative sign to the series opposition connection.

EXAMPLE 10.9

Two coils connected in series have an equivalent inductance of 0.4 H when connected in aiding, and an equivalent inductance 0.2 H when the connection is opposing. Calculate the mutual inductance of the coils.

Solution When the coils are arranged in aiding connection, the inductance of the combination is $L_1 + L_2 + 2M = 0.4$; and for opposing connection, it is $L_1 + L_2 - 2M = 0.2$. Solving the two equations, we get

$4M = 0.2$ H

$M = 0.05$ H

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆★★**10-7.1** The inductance matrix for the circuit of a three series connected coupled coils is given below. Find the inductances and indicate the dots for the coils.

$$L = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 4 & -6 \\ 1 & -6 & 6 \end{bmatrix}$$

Frequently Asked Questions linked to LO 7

☆★★**10-7.1** What is the expression for the total induce. lance of the three series connected coupled coils shown in Fig. Q.1. **[BPUT 2007]**

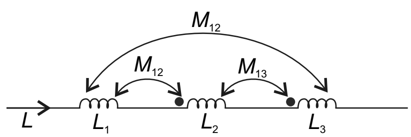


Fig. Q.1

10.9 PARALLEL CONNECTION OF COUPLED COILS

LO 8 Analyse parallel-connected coupled coils

Consider two inductors with self-inductances L_1 and L_2 connected parallel which are mutually coupled with mutual inductance M as shown in Fig. 10.20.

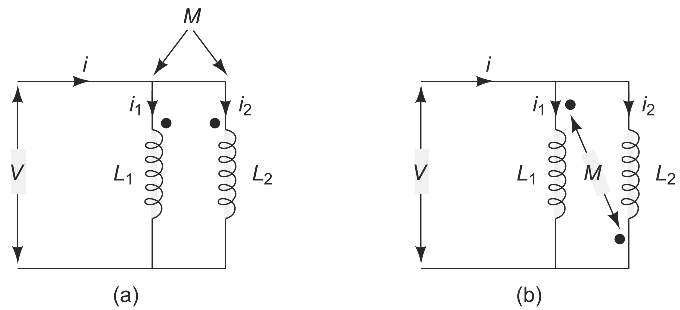


Fig. 10.20

Let us consider Fig. 10.20 (a) where the self-induced emf in each coil assists the mutually induced emf as shown by the dot convention.

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \tag{10.3}$$

The voltage across the parallel branch is given by

$$v = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \text{ or } L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Also, $$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\frac{di_1}{dt}(L_1 - M) = \frac{di_2}{dt}(L_2 - M)$$

$$\therefore \quad \frac{di_1}{dt} = \frac{di_2}{dt} \frac{(L_2 - M)}{(L_1 - M)} \tag{10.4}$$

Substituting Eq. (10.4) in Eq. (10.3), we get

$$\frac{di}{dt} = \frac{di_2}{dt} \frac{(L_2 - M)}{(L_1 - M)} + \frac{di_2}{dt}\left[\frac{(L_2 - M)}{L_1 - M} + 1\right] \tag{10.5}$$

If L_{eq} is the equivalent inductance of the parallel circuit in Fig. 10.20 (a) then v is given by

$$v = L_{eq} \frac{di}{dt}$$

$$L_{eq} \frac{di}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L_{eq}}\left[L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}\right]$$

Substituting Eq. (10.4) in the above equation, we get

$$\frac{di}{dt} = \frac{1}{L_{eq}}\left[L_1 \frac{di_2 (L_2 - M)}{dt (L_2 - M)} + M \frac{di_2}{dt}\right]$$

$$= \frac{1}{L_{eq}}\left[L_1 \frac{(L_2 - M)}{(L_1 - M)} + M\right]\frac{di_2}{dt} \tag{10.6}$$

Equating Eq. (10.6) and Eq. (10.5), we get

$$\frac{L_2 - M}{L_2 - M} + 1 = \frac{1}{L_{eq}}\left[L_1 \left(\frac{L_2 - M}{L_1 - M}\right) + M\right]$$

Rearranging and simplifying the above equation results in

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

If the voltage induced due to mutual inductance opposes the self-induced emf in each coil as shown by the dot convention in Fig. 10.20 (b), the equivalent inductance is given by

$$\boxed{L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 8

☆★★**10-8.1** Find the source voltage if the voltage across 100 ohms is 50 V for the network in Fig. Q.1.

☆★★**10-8.2** Using PSpice, find V_0 in the circuit shown in Fig. Q.2.

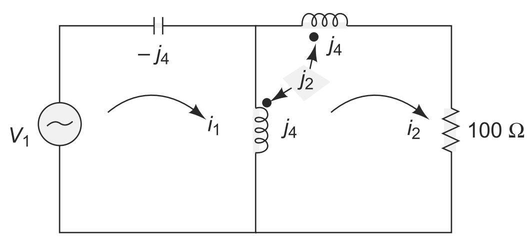


Fig. Q.1

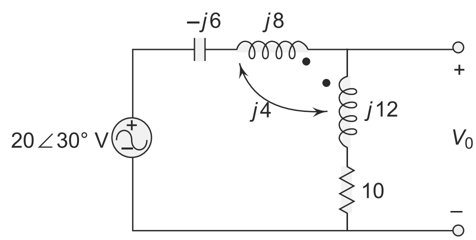


Fig. Q.2

Frequently Asked Questions linked to LO 8

☆★★**10-8.1** A coil having an inductance of 100 mH is magnetically coupled to another coil having an inductance of 900 mH. The coefficient of coupled between the coil is 0.45. Calculate the equivalent inductance if the two coil are connected in (a) series opposing, and (b) parallel opposing. **[AU May/June 2014]**

☆★★**10-8.2** Find the voltage drop across the resistance "r" in the network shown in Fig. Q.2. **[BPUT 2007]**

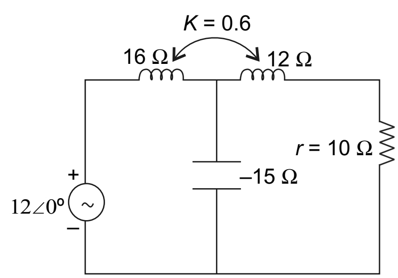


Fig. Q.2

☆☆★**10-8.3** Two coils in differential connection have self-inductances of 2 mH and 4 mH and a mutual inductance of 0.15 mH.

What is the equivalent inductance? **[BPUT 2008]**

☆☆★**10-8.4** Find the equivalent inductance for the series and the parallel connections of L_1 and L_2 if their mutual inductance is M. **[GTU May 2011]**

10.10 TUNED CIRCUITS

Tuned circuits are, in general, single-tuned and double-tuned. Double-tuned circuits are used in radio receivers to produce uniform response to modulated signals over a specified bandwidth; double-tuned circuits are very useful in a communication system.

LO 9 Explain single-tuned and double-tuned circuits and their applications

10.10.1 Single Tuned Circuit

Consider the circuit in Fig. 10.21. A tank circuit (i.e. a parallel resonant circuit) on the secondary side is inductively coupled to the coil 1 which is excited by a source, v_i. Let R_s be the source resistance and R_1, R_2 be the resistances of coils 1 and 2, respectively. Also let L_1, L_2 be the self inductances of the coils, 1 and 2, respectively.

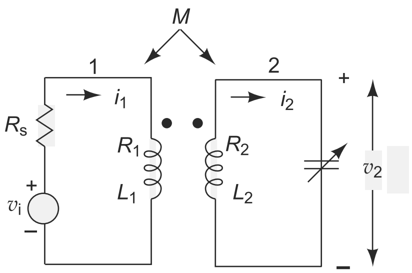


Fig. 10.21

Let $R_s + R_1 + j\omega L_1 = R_s$

with the assumption that $R_s >> R_1 >> j\omega L_1$ The mesh equations for the circuit shown in Fig. 10.21 are

$$i_1 R_s - j\omega M i_2 = v_i$$

$$-j\omega M i_1 + \left(R_2 + j\omega L_2 - \frac{j}{\omega C}\right) i_2 = 0$$

$$i_2 = \begin{vmatrix} R_s & v_i \\ -j\omega M & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} R_s & (-j\omega M) \\ (-j\omega M) & \left(R_2 + j\omega L_2 - \dfrac{j}{\omega C}\right) \end{vmatrix}$$

or $$i_2 = \frac{j v_i \omega M}{R_s\left(R_2 + j\omega L_2 - \dfrac{j}{\omega C}\right) + \omega^2 M^2}$$

The output voltage $$v_o = i_2 \cdot \frac{1}{j\omega C}$$

$$v_o = \frac{j v_i \omega M}{j\omega C\left\{R_s\left[R_2 + \left(j\omega L_2 - \dfrac{1}{\omega C}\right)\right] + \omega^2 M^2\right\}}$$

The voltage transfer function, or voltage amplification, is given by

$$\frac{v_o}{v_i} = A = \frac{M}{C\left\{R_s\left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C}\right)\right] + \omega^2 M^2\right\}}$$

When the secondary side is tuned, i.e. when the value of the frequency ω is such that $\omega L_2 = 1/\omega C$, or at resonance frequency ω_r, the amplification is given by

$$A = \frac{v_o}{v_i} = \frac{M}{C\left[R_s R_2 + \omega_r^2 M^2\right]}$$

The current i_2 at resonance $i_2 = \dfrac{jv_i\,\omega_r\,M}{R_s R_2 + \omega_r^2 M^2}$

Thus, it can be observed that the output voltage, current, and amplification depend on the mutual inductance M at resonance frequency, when $M = K\sqrt{L_1 L_2}$. The maximum output voltage or the maximum amplification depends on M. To get the condition for maximum output voltage, make $dv_o/dM = 0$.

$$\frac{dv_o}{dM} = \frac{d}{dM}\left[\frac{v_i M}{C\left[R_s R_2 + \omega_r^2 M^2\right]}\right]$$

$$= 1 - 2M^2\,\omega^2{}_r\,[R_s R_2 + \omega^2{}_r M^2]^{-1} = 0$$

From which, $R_s R_2 = \omega^2{}_r\,M^2$

or $\quad M = \sqrt{\dfrac{R_s R_2}{\omega_r}}$

From the above value of M, we can calculate the maximum output voltage. Thus,

$$v_{oM} = \frac{v_i}{2\omega_r C\sqrt{R_s R_2}},$$

or the maximum amplification is given by

$$A_m = \frac{1}{2\omega_r C\sqrt{R_s R_2}} \text{ and } i_2 = \frac{jv_i}{2\sqrt{R_s R_2}}$$

The variation of the amplification factor or output voltage with the coefficient of coupling is shown in Fig. 10.22.

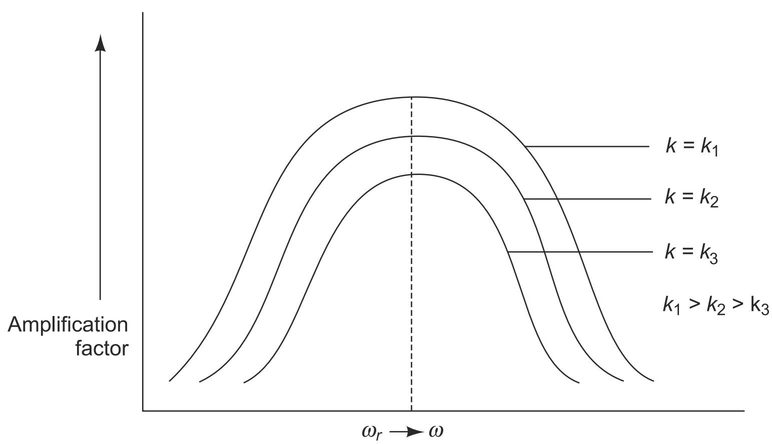


Fig. 10.22

EXAMPLE 10.10

Consider the single-tuned circuit shown in Fig. 10.23 and determine (a) the resonant frequency, (b) the output voltage at resonance, and (c) the maximum output voltage. Assume $R_s >> \omega_r L_1$, and $K = 0.9$.

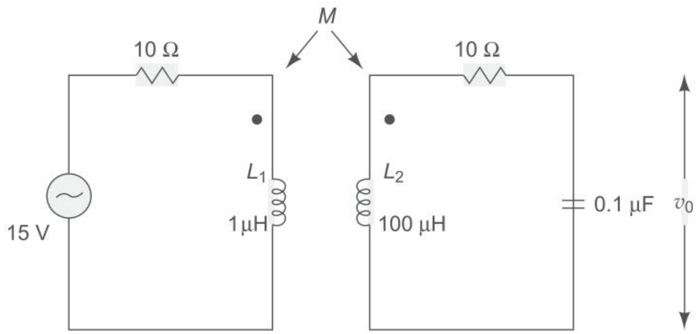


Fig. 10.23

Solution

$$M = K\sqrt{L_1 L_2}$$
$$= 0.9\sqrt{1\times10^{-6}\times100\times10^{-6}}$$
$$= 9\ \mu\text{H}$$

(a) Resonance frequency

$$\omega_r = \frac{1}{\sqrt{L_2 C}} = \frac{1}{\sqrt{100\times10^{-6}\times0.1\times10^{-6}}}$$
$$= \frac{10^6}{\sqrt{10}}\ \text{rad/sec.}$$

or $\quad f_r = 50.292$ kHz

The value of $\quad \omega_r L_1 = \dfrac{10^6}{\sqrt{10}} 1\times10^{-6} = 0.316$

Thus, the assumption that $\omega_r L_1 < R_s$ is justified.

(b) Output voltage

$$v_o = \frac{Mv_i}{C\left[R_s R_2 + \omega_r^2 M\right]}$$
$$= \frac{9\times10^{-6}\times15}{0.1\times10^{-6}\left[10\times10+\left(\frac{10^6}{\sqrt{10}}\right)^2\times9\times10^{-6}\right]}$$
$$= 1.5\ \text{mV}$$

(c) Maximum value of output voltage

$$v_{oM} = \frac{v_i}{2\omega_r C\sqrt{R_s R_2}}$$

$$= \frac{15}{2 \times \frac{10^6}{\sqrt{10}} \times 0.1 \times 10^{-6} \sqrt{100}}$$

$$v_{oM} = 23.7 \text{ V}$$

10.10.2 Double-Tuned Coupled Circuits

Figure 10.24 shows a double-tuned transformer circuit involving two series resonant circuits.

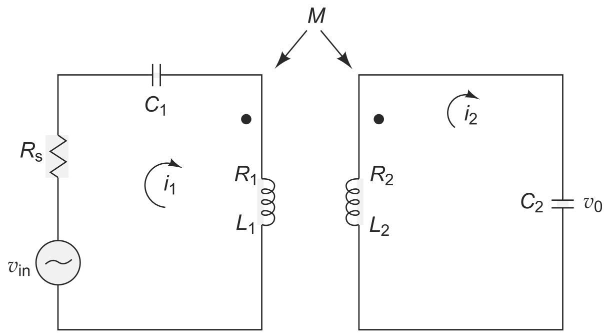


Fig. 10.24

For the circuit shown in the figure, a special case where the primary and secondary resonate at the same frequency ω_r, is considered here,

i.e $\omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$

The two mesh equations for the circuit are

$$v_{in} = i_1 \left(R_s + R_1 + j\omega L_1 - \frac{j}{\omega C_1} \right) - i_2 j\omega M$$

$$0 = -j\omega M i_1 + i_2 \left(R_2 + j\omega L_2 - \frac{j}{\omega C_2} \right)$$

from which,

$$i_2 = \frac{v_{in} j\omega M}{\left[(R_s + R_1) + j\left(\omega L_1 - \frac{1}{\omega C_1} \right) \right] \left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2} \right) \right] + \omega^2 M^2}$$

Also, $\omega_r = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$ at resonance

$$v_o = \frac{v_{in} M}{C_2 \left[(R_s + R_1) R_2 + \omega_r^2 M^2 \right]}$$

or $\quad v_o = A v_{in}$

where A is the amplification factor given by

$$A = \frac{M}{C_2\left[(R_1 + R_s)R_2 + \omega_r^2 M^2\right]}$$

The maximum amplification or the maximum output voltage can be obtained by taking the first derivative of v_o with respect to M, and equating it to zero.

$$\therefore \quad \frac{dv_o}{dM} = 0, \text{ or } \frac{dA}{dM} = 0$$

$$\frac{dA}{dM} = (R_1 + R_s)R_2 + \omega_r^2 M^2 - 2M^2\omega_r^2 = 0$$

$$\omega_r^2 M^2 = R_2(R_1 + R_s)$$

$$M_c = \frac{\sqrt{R_2(R_1 + R_s)}}{\omega_r}$$

where M_c is the critical value of mutual inductance. Substituting the value of M_c in the equation of v_o, we obtain the maximum output voltage as

$$|v_o| = \frac{v_{in}}{2\omega_r^2 C_2 M_c} = \frac{v_{in}}{2\omega_r C_2\sqrt{R_2(R_1 + R_s)}}$$

and $$|i_2| = \frac{v_{in}}{2\omega_r M_c} = \frac{v_{in}}{2\sqrt{R_2(R_1 + R_s)}}$$

By definition, $M = K\sqrt{L_1 L_2}$, the coefficient of coupling, K at $M = M_c$ is called the *critical coefficient of coupling*, and is given by $K_c = M_c/\sqrt{L_2 L_1}$.

The critical coupling causes the secondary current to have the maximum possible value. At resonance, the maximum value of amplification is obtained by changing M, or by changing the coupling coefficient for a given value of L_1 and L_2. The variation of output voltage with frequency for different coupling coefficients is shown in Fig. 10.25.

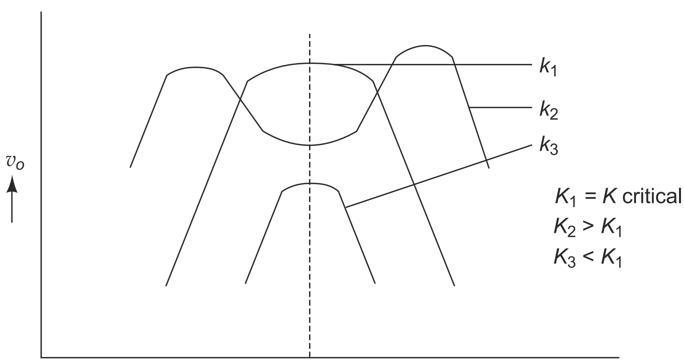


Fig. 10.25

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 9

☆★★ **10-9.1** For the circuit shown in Fig. Q.1, find the ratio of output voltage to the input voltage.

★★★**10-9.2** Using PSpice, find I_1 and I_2 in the circuit shown in Fig. Q.2. Also calculate the power absorbed by the 4 Ω resistor.

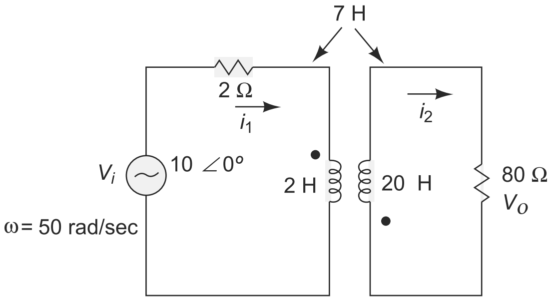


Fig. Q.1

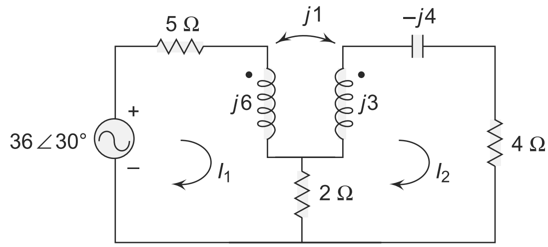


Fig. Q.2

☆★★ **10-9.3** Using PSpice, find the Thevenin equivalent circuit for the circuit shown in Fig. Q.3 at $a-b$.

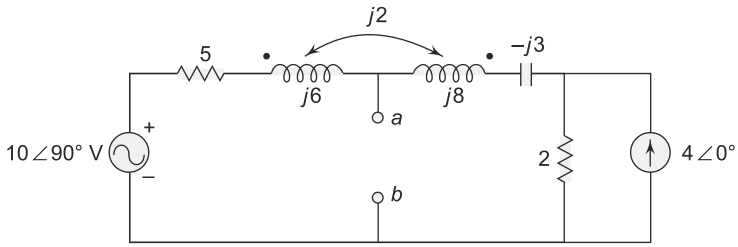


Fig. Q.3

★★★**10-9.4** Solve for the currents I_1 and I_2 in the circuit shown. Also find the ratio of $\frac{V_2}{V_1}$.

☆★★ **10-9.5** For the coupled circuit shown in Fig. Q.5. Find L_{eq} across xy.

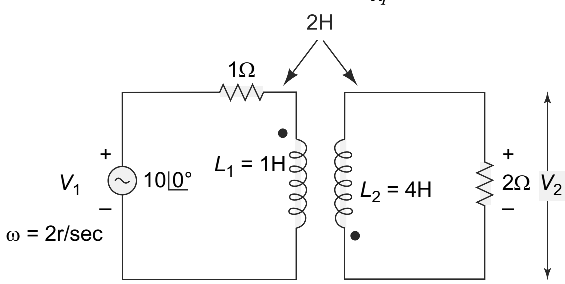


Fig. Q.5

Frequently Asked Questions linked to LO 9

☆☆★ **10-9.1** Give the applications of tuned circuits. **[AU May/June 2013]**

☆★★ **10-9.2** For the circuit shown in Fig. Q.2 determine the voltage ratio V_1/V_2. Which will make the current I_1 equal to zero. **[AU May/June 2014]**

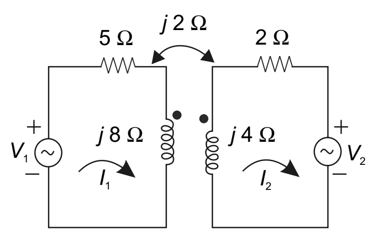


Fig. Q.2

☆☆★ **10-9.3** Derive the expressions for maximum output voltage and maximum amplification of a single-tuned circuit. **[AU April/May 2011]**

☆★★ **10-9.4** Explain in detail the following: **[RGTU June 2014]**

(a) Double-tuned circuit

(b) Single-tuned air-core transformer

10.11 ANALYSIS OF MAGNETIC CIRCUITS

LO 10 Analyse magnetic circuits

The presence of charges in space or in a medium creates an electric field; similarly, the flow of current in a conductor sets up a magnetic field. Electric field is represented by electric flux lines, and magnetic flux lines are used to describe the magnetic field. The path of the magnetic flux lines is called the *magnetic circuit*. Just as a flow of current in the electric circuit requires the presence of an electromotive force, so the production of magnetic flux requires the presence of magneto-motive force (mmf). We now discuss some properties related to magnetic flux.

❑ ***Flux Density (B)*** *The magnetic flux lines start and end in such a way that they form closed loops. Weber (Wb) is the unit of magnetic flux (ϕ).* ***Flux density*** *(B) is the flux per unit area.* Tesla (T) or Wb/m^2 is the unit of flux density.

$$B = \frac{\phi}{A}\,\text{Wb/m}^2 \quad \text{or Tesla}$$

where B is a quantity called magnetic flux density in teslas, ϕ is the total flux in webers, and A is the area perpendicular to the lines in m^2.

❑ ***Magneto-motive force, MMF ($\Im$)*** A measure of the ability of a coil to produce a flux is called the *magneto-motive force*. It may be considered as a magnetic pressure, just as emf is considered as an electric pressure. A coil with N turns which is carrying a current of I amperes constitutes a magnetic circuit and produces an mmf of NI ampere turns. The source of flux (ϕ) in the magnetic circuit is the mmf. The flux produced in the circuit depends on mmf and the length of the circuit.

❑ ***Magnetic Field Strength (H)*** The magnetic field strength of a circuit is given by the mmf per unit length.

$$H = \frac{\Im}{l} = \frac{NI}{l}\ \text{AT/ms}$$

The magnetic flux density (B) and its intensity (field strength) in a medium can be related by the following equation

$$B = \mu H$$

where $\mu = \mu_0\,\mu_r$ is the permeability of the medium in Henrys/metre (H/m),

μ_0 = absolute permeability of free space and is equal to $4\pi \times 10^{-7}$ H/m, and

and μ_r = relative permeability of the medium.

Relative permeability is a nondimensional numeric which indicates the degree to which the medium is a better conductor of magnetic flux as compared to free space. The value of $\mu_r = 1$ for air and nonmagnetic materials. It varies from 1,000 to 10,000 for some types of ferromagnetic materials.

❑ ***Reluctance ($\Re$)*** *It is the property of the medium which opposes the passage of magnetic flux. The magnetic reluctance is analogous to resistance in the electric circuit.* Its unit is AT/Wb. Air has a much higher reluctance than does iron or steel. For this reason, magnetic circuits used in electrical machines are designed with very small air gaps.

According to definition, reluctance $= \dfrac{\text{mmf}}{\text{flux}}$

The reciprocal of reluctance is known as *permeance* $\frac{1}{\Re} = \frac{\phi}{\Im}$

Thus, reluctance is a measure of the opposition offered by a magnetic circuit to the setting up of the flux. The reluctance of the magnetic circuit is given by $\Re = \frac{1}{\mu}\frac{l}{a}$.

where l is the length, a is the cross-sectional area of the magnetic circuit and μ is the permeability of the medium.

From the above equations,

$$\frac{1}{\mu}\cdot\frac{l}{a} = \frac{\Im}{\phi}$$

$$\text{or}\quad \frac{\Im}{1} = \frac{1}{\mu}\cdot\frac{\phi}{a}$$

$$\frac{NI}{l} = \frac{1}{\mu}\cdot B$$

$$H = \frac{1}{\mu}\cdot B$$

$$\text{or}\quad B = \mu H$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 10

☆★★**10-10.1** An iron ring has a mean diameter of 25 cm and a cross-sectional area of 4 cm^2. It is wound with a coil of 1200 turns. An air gap of 1.5 mm width is cut in the ring. Determine the current required in the coil to produce a flux of 0.48 mwb in the air gap. The relative permeability of iron under this condition is 800. Neglect leakage.

Frequently Asked Questions linked to LO 10

☆★★**10-10.1** Find the net impedance the central mesh and then find the net input impedance of the circuit shown in Fig. Q.1. **[BPUT 2007]**

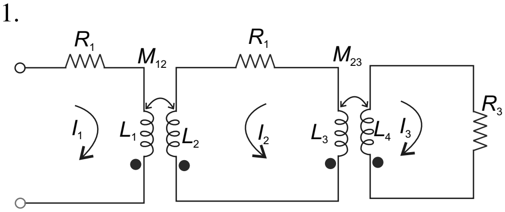


Fig. Q.1

10.12 SERIES MAGNETIC CIRCUIT

LO 10

A series magnetic circuit is analogous to a series electric circuit. Kirchhoff's laws are applicable to magnetic circuits also. Consider a ring specimen having a magnetic path of l metres, area of cross section (A)m^2 with a mean radius of R metres having a coil of N turns carrying I amperes wound uniformly as shown in Fig. 10.26. The mmf is responsible for the establishment of flux in the magnetic medium. This mmf acts along the magnetic lines of force.

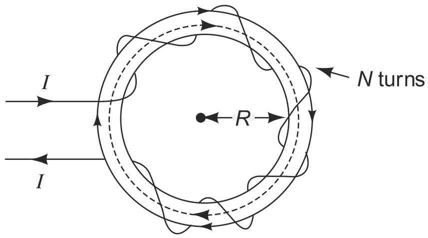


Fig. 10.26

The flux produced by the circuit is given by

$$\phi = \frac{\text{mmf}}{\text{reluctance}}$$

The magnetic field intensity of the ring is given by

$$H = \frac{\text{mmf}}{l} = \frac{NI}{l} = \text{AT/m}$$

where l is the mean length of the magnetic path and is given by $2\pi R$.

Flux density $B = \mu_0 \mu_r H = \mu_0 \mu_r \dfrac{NI}{l}\ \text{Wb/m}^2$

$$\text{Flux } \phi = \mu \text{HA webers}$$

$$= \mu_0 \mu_r \frac{NI}{l} \times A \text{ Wb}$$

$$\phi = \frac{NI}{l/\mu_0 \mu_r A} \text{Wb}$$

NI is the mmf of the magnetic circuit, which is analogous to emf in electric circuit. $l/\mu_0 \mu_r A$ is the reluctance of the magnetic circuit which is analogous to resistance in electric circuit.

10.13 COMPARISON OF ELECTRIC AND MAGNETIC CIRCUITS

LO 11 Compare electric and magnetic circuits

Series electric and magnetic circuits are shown in Figs. 10.27 (a) and (b) respectively.

Figure 10.27 (a) represents an electric circuit with three resistances connected in series, the dc source E drives the current I through all the three resistances whose voltage drops are V_1, V_2 and V_3. Hence, $E = V_1 + V_2 + V_3$, also $E = I(R_1 + R_2 + R_3)$. We also know that $R = \dfrac{\rho l}{2}$, where ρ is the specific resistance of the material, l is the length of the wire of the resistive material and a is the area of cross section of the wire.

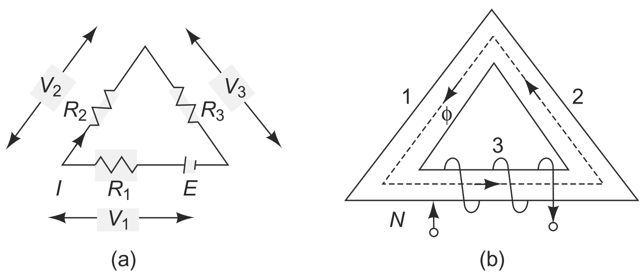


Fig. 10.27

The drop across each resistor $V = RI = \rho l \dfrac{I}{a}$

$$\text{or} \quad \frac{V}{l} = \rho \frac{I}{a}$$

Voltage drop per unit length = Specific resistance × Current density.

Let us consider the magnetic circuit in Fig. 10.27 (b). The MMF of the circuit is given by $\Im = NI$, drives the flux ϕ around the three parts of the circuit which are in series. Each part has a reluctance $\Re = \frac{1}{\mu} \cdot \frac{l}{a}$, where l is the length and a is the area of cross section of each arm. The mmf of the magnetic circuit is given by $\Im = \Im_1 + \Im_2 + \Im_3$. $\Im = \phi(\Re_1 + \Re_2 + \Re_3)$ where $\Re_1$, $\Re_2$ and $\Re_3$ are the reluctances of the portions 1, 2, and 3 respectively.

Also
$$\Im = \frac{1}{\mu} \cdot \frac{l}{a} \cdot \phi$$
$$\frac{\Im}{l} = \frac{1}{\mu} \cdot \frac{\phi}{a}$$
$$H = \frac{1}{\mu} \cdot B.$$

$\frac{1}{\mu}$ can be termed the *reluctance* of a cubic metre of magnetic material from which, the above equation gives the mmf per unit length (intensity) which is analogous to the voltage per unit length. Parallels between electric-circuit and magnetic-circuit quantities are shown in Table 10.1.

Thus, it is seen that the magnetic reluctance is analogous to resistance, mmf is analogous to emf, and flux is analogous to current. These analogies are useful in magnetic-circuit calculations. Though we can draw many parallels between the two circuits, the following differences do exists.

The electric current is a true flow but there is no flow in a magnetic flux. For a given temperature, ρ is independent of the strength of the current, but μ is not independent of the flux.

In an electric circuit, energy is expended so long as the current flows, but in a magnetic circuit energy is expended only in creating the flux, and not in maintaining it. Parallels between the quantities are shown in Table 10.1.

Table 10.1 Analogy between magnetic and electric circuits

Electric circuit	*Magnetic circuit*
Exciting force = emf in volts	mmf in AT
Response = current in amps	flux in webers
Voltage drop = VI volts	mmf drop = $\Re\phi$ AT
Electric field density = $\frac{V}{l}$ volt/m	Magnetic field Intensity = $\frac{\Im}{1}$ AT/m
$\text{Current}(I) = \frac{E}{R}$ A	$\text{Flux}(\phi) = \frac{\Im}{R}$ Web
$\text{Current density}(J) = \frac{I}{a}$ Amp/m^2	Flux density $(B) = \frac{\phi}{A}$ Web/m^2
Resistance $(R) = \frac{\rho l}{a}$ ohm	Reluctance $(\Re) = \frac{1}{\mu} \cdot \frac{l}{a}$ AT/Web
Conductance $(G) = \frac{1}{R}$ Mho	Permeance = $\frac{1}{\Re} = \frac{\mu a}{\mu} \cdot \frac{l}{a}$ Web/AT

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 11

☆★★**10.11.1** A steel ring of 25 cm mean diameter and of circular section of 3 cm in diameter has an air gap of 1.5 mm length. It is wound uniformly with 700 turns of were carrying a current of 2A. Calculate (a) magnetomotive force, (b) flux density, and (c) magnetic flux

Frequently Asked Questions linked to LO 11

☆☆★**10-11.1** Contrast between magnetic circuits and electrical circuits. **[JNTU Nov. 2012]**

10.14 MAGNETIC LEAKAGE AND FRINGING

LO 12 Explain magnetic leakage and fringing

Figure 10.28 shows a magnetised iron ring with a narrow air gap, and the flux which crosses the gap can be regarded as useful flux. *Some of the total flux produced by the ring does not cross the air gap, but instead takes a shorter route as shown in Fig. 10.28 and is known as* ***leakage flux****. The flux while crossing the air gap bulges outwards due to variation in reluctance. This is known as* ***fringing***. This is because the lines of force repel each other when passing through the air as a result the flux density in the air gap decreases. For the purpose of calculation, it is assumed that the iron carries the whole of the total flux throughout its length. The ratio of total flux to useful flux is called the *leakage coefficient* or leakage factor.

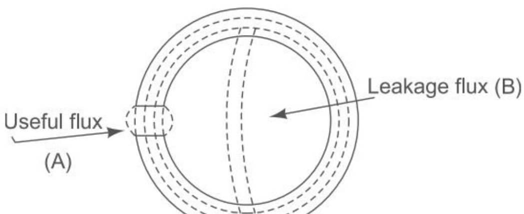


Fig. 10.28

Leakage factor = Total flux/Useful flux.

EXAMPLE 10.11

A coil of 100 turns is wound uniformly over a insulator ring with a mean circumference of 2 m and a uniform sectional area of 0.025 cm^2. If the coil is carrying a current of 2 A. Calculate (a) the mmf of the circuit, (b) magnetic field intensity, (c) flux density, and (d) the total flux.

Solution (a) mmf $= NI = 100 \times 2 = 2000$ AT

(b) $$H = \frac{\text{mmf}}{I} = \frac{2000}{2} = 1000 \text{ AT/m}$$

(c) $B = \mu_0 H = 4\pi \times 10^{-7} \times 1000 = 1.2565$ mWb/m^2

(d) $\phi = B \times A = 1.2565 \times 10^{-3} \times 0.025 \times 10^{-4} = 0.00314 \times 10^{-6}$ Wb

EXAMPLE 10.12

Calculate the mmf required to produce a flux of 5 mWb across an air gap of 2.5 mm of length having an effective area of 100 cm^2 of a cast steel ring of mean iron path of 0.5 m and cross-sectional area of 150 cm^2 as shown in Fig. 10.29. The relative permeability of the cast steel is 800. Neglect leakage flux.

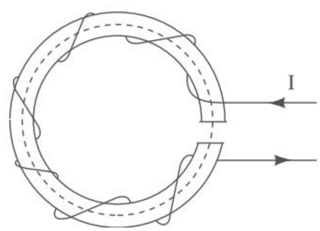


Fig. 10.29

Solution Area of the gap $= 100 \times 10^{-4}$ m^2

Flux density of the gap $= \dfrac{5\times10^{-3}\times10^{4}}{100} = 0.5$ T

H of the gap $= \dfrac{B}{\mu_0} = \dfrac{0.5}{4\pi\times10^{-7}}$

$= 0.39 \times 10^6$ A/m

Length of the gap $= 2.5 \times 10^{-3}$ m

mmf required for the gap $= 0.39 \times 10^6 \times 2.5 \times 10^{-3} = 975$ AT

Flux density in the cast steel ring is $= \dfrac{\phi}{\text{Area}}$

$$= \frac{5\times10^{-3}\times10^{4}}{100}$$

$$= 0.333 \text{ T}$$

$$\therefore \quad H = \frac{B}{\mu_0\mu_r} = \frac{0.333}{4\pi10^{-7}\times800} = 332 \text{ A T/m}$$

Length of the cast steel path $= 0.5$ m

The required mmf for the cast steel to produce the necessary flux $= 0.5 \times 332 = 166$ AT

Therefore, total mmf $= 975 + 166 = 1141$ AT

10.15 COMPOSITE SERIES CIRCUIT

LO 13 Analyse composite series circuit and parallel magnetic circuit

Consider a toroid composed of three different magnetic materials of different permeabilities, areas and lengths excited by a coil of N turns.

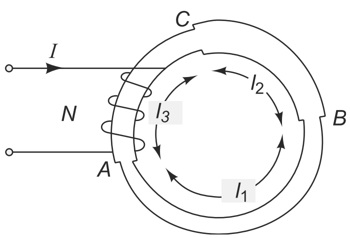


Fig. 10.30

With a current of I amperes as shown in Fig. 10.30. The lengths of sections AB, BC and CA are l_1, l_2 and l_3 respectively. Each section will have its own reluctance and permeability. Since all of them are joined in series, the total reluctance of the combined magnetic circuit is given by

$$\Re_{\text{Total}} = \frac{1}{\mu A}$$

$$= \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3}$$

The flux produced in the circuit is given by $\phi = \dfrac{\text{mmf}}{\text{Total reluctance}}$ Wb

$$\therefore \quad \phi = \left[\frac{NI}{\dfrac{l_1}{\mu_1 A_1} + \dfrac{l_2}{\mu_2 A_2} + \dfrac{l_3}{\mu_3 A_3}}\right] \text{Wb}$$

10.16 PARALLEL MAGNETIC CIRCUIT

LO 13

We have seen that a series magnetic circuit carries the same flux and the total mmf required to produce a given quantity of flux is the sum of the mmf's for the separate parts. In a parallel magnetic circuit, different parts of the circuit are in parallel. For such circuits the Kirchhoff's laws, in their analogous magnetic form can be applied for the analysis. Consider an iron core having three limbs *A*, *B,* and *C* as shown in Fig. 10.31 (a). *A* coil with *N* turns is arranged around limb *A* which carries a current *I* amperes. The flux is produced by the coil in the limb *A*. ϕ_A is divided between limbs *B* and *C* and each equal to $\phi_A/2$. The reluctance offered by the two parallel paths is equal to the half the reluctance of each path (Assuming equal lengths and cross-sectional areas). Similar to Kirchhoff's current law in an electric circuit, the total magnetic flux directed towards a junction in a magnetic circuit is equal to the sum of the magnetic fluxes directed away from that junction. Accordingly $\phi_A = \phi_B + \phi_C$ or $\phi_A - \phi_B - \phi_C = 0$. The electrical equivalent of the above circuit is shown in Fig. 10.31 (b). Similar to Kirchhoff's second law, in a closed magnetic circuit, the resultant mmf is equal to the algebraic sum of the products of field strength and the length of each part in the closed path. Thus applying the law to the first loop in Fig. 10.31 (a), we get

$$NI = H_A l_A + H_B l_B$$

or $$NI = \phi_A \Re_A + \phi_B \Re_B$$

The mmf across the two parallel paths is identical.
Therefore, *NI* is also equal to

$$NI = \phi_A \Re_A + \phi_C \Re_C$$

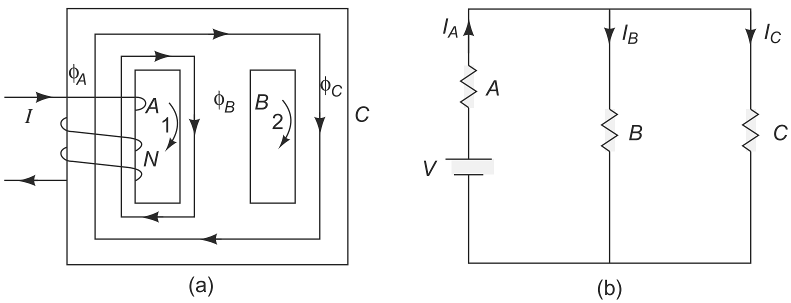


Fig. 10.31

Frequently Asked Questions linked to LO 13

★★★**10-13.1** Two coil with 300 turns and 700 turns are wound side by side on a closed magnetic circuit of area of 400 cm^2 cross section and 80 cm mean length. The magnetic circuit has a relative permeability of 4000. Determine the mutual inductance, self-induced emf, and mutually induced emf when the current in the coil with 300 turns grows from zero to 25 A in a time of 0.3 second. **[JNTU Nov. 2012]**

Additional Solved Problems

PROBLEM 10.1

In the circuit shown in Fig. 10.32, write the equation for the voltages across the coils ab and cd; also mention the polarities of the terminals.

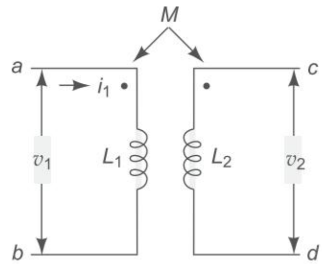


Fig. 10.32

Solution Current i_1 is only flowing in coil *ab*, whereas coil *cd* is open. Therefore, there is no current in coil *cd*. The emf due to self induction is zero on coil *cd*.

$$\therefore \quad v_2(t) = M\frac{di_1(t)}{dt} \quad \text{with } C \text{ being positive}$$

Similarly, the emf due to mutual induction in coil *ab* is zero.

$$\therefore \quad v_1(t) = L\frac{di_1(t)}{dt}$$

PROBLEM 10.2

In the circuit shown in Fig. 10.33, write the equation for the voltages v_1 and v_2. L_1 and L_2 are the coefficients of self-inductances of coils 1 and 2, respectively, and M is the mutual inductance.

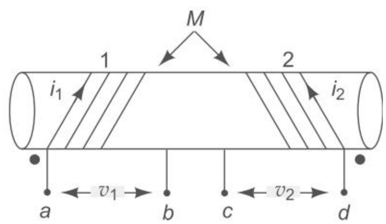


Fig. 10.33

Solution In the figure, *a* and *d* are like terminals.

Currents i_1 and i_2 are entering at dot-marked terminals.

$$v_1 = L_1\frac{di_1(t)}{dt} + \frac{M\,di_2(t)}{dt}$$

$$v_2 = L_2\frac{di_2(t)}{dt} + \frac{M\,di_1(t)}{dt}$$

PROBLEM 10.3

In Fig. 10.34, $L_1 = 4\,H$; $L_2 = 9\,H$, $K = 0.5$, $i_1 = 5\cos(50t - 30°)\,A$, $i_2 = 2\cos(50t - 30°)\,A$. Find the values of (a) v_1; (b) v_2, and (c) the total energy stored in the system at $t = 0$.

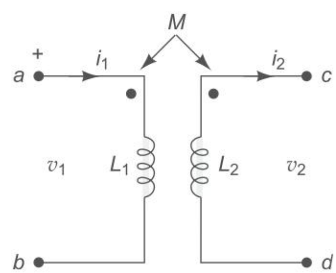


Fig. 10.34

Solution Since the current in the coil *ab* is entering at the dot marked terminal, whereas in coil *cd* the current is leaving, we can write the equations as

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

$$M = K\sqrt{L_1\,L_2} = 0.5\sqrt{36} = 3$$

(a) $$v_1 = 4\frac{d}{dt}\left[5\cos(50t - 30°) - 3\frac{d}{dt}2\cos(50t - 30°)\right]$$

$$v_1 = 20\,[-\sin(50t - 30°)\times 50] - 6\,[-\sin(50t - 30°)50]$$

at $t = 0$

$$v_1 = 500 - 150 = 350\text{ V}$$

(b) $$v_2 = -3\frac{d}{dt}\left[5\cos(50t - 30°)\right] + 9\frac{d}{dt}\left[2\cos(50t - 30°)\right]$$

$$v_2 = -15\,[-\sin(50t - 30°)\times 50] - 18\,[-\sin(50t - 30°)50]$$

at $t = 0$

$$v_2 = -375 + 450 = 75\text{ V}$$

(c) The total energy stored in the system

$$W(t) = \frac{1}{2}L_1\left[i_1(t)\right]^2 + \frac{1}{2}L_2\left[i_2(t)\right]^2 - M\left[i_1(t)i_2(t)\right]$$

$$= \frac{1}{2}\times 4\left[5\cos(50t - 30°)\right]^2 + \frac{1}{2}\times 9\left[2\cos(50t - 30°)\right]^2$$

$$-3\,[5\cos(50t - 30°)\times 2\cos(50t - 30°)]$$

at $t = 0$ $\quad W(t) = 28.5$ j

PROBLEM 10.4

For the circuit shown in Fig. 10.35, write the mesh equations.

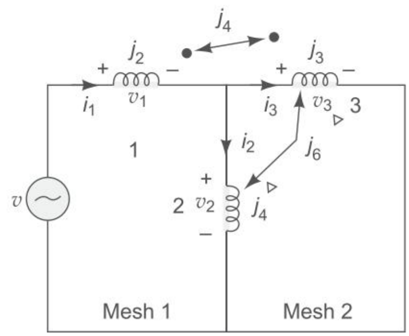


Fig. 10.35

Solution There exists mutual coupling between coils 1 and 3, and 2 and 3. Assuming branch currents i_1, i_2, and i_3 in coils 1, 2, and 3, respectively, the equation for mesh 1 is

$$v = v_1 + v_2$$

$$v = i_1 j_2 - i_3 j_4 + i_2 j_4 - i_3 j_6 \tag{10.7}$$

$j_4 i_3$ is the mutual inductance drop between coils 1 and 3, and is considered negative according to dot convention and $i_3 j_6$ is the mutual inductance drop between coils 2 and 3.

For the second mesh, $0 = -v_2 + v_3$

$$= -(j_4 i_2 - j_6 i_3) + j_3 i_3 - j_6 i_2 - j_4 i_1 \quad (10.8)$$

$$= -j_4 i_1 - j_{10} i_2 + j_9 i_3 \quad (10.9)$$

$$i_1 = i_3 + i_2$$

PROBLEM 10.5

Calculate the effective inductance of the circuit shown in Fig. 10.36 across terminals a and b.

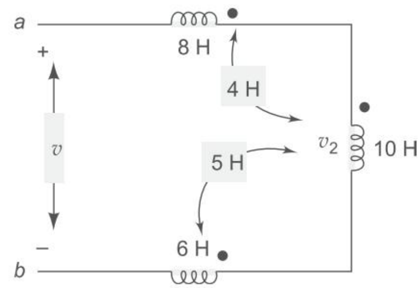


Fig. 10.36

Solution Let the current in the circuit be *i*.

$$v = 8\frac{di}{dt} - 4\frac{di}{dt} + 10\frac{di}{dt} - 4\frac{di}{dt}$$

$$+5\frac{di}{dt} + 6\frac{di}{dt} + 5\frac{di}{dt}$$

or $$\frac{di}{dt}[34 - 8] = 26\frac{di}{dt} = v$$

Let L be the effective inductance of the circuit across *ab*. Then the voltage across $ab = v = L\frac{di}{dt} = 26\frac{di}{dt}$ Hence, the equivalent inductance of the circuit is given by 26 H.

PROBLEM 10.6

For the circuit shown in Fig. 10.37, find the ratio of output voltage to the source voltage.

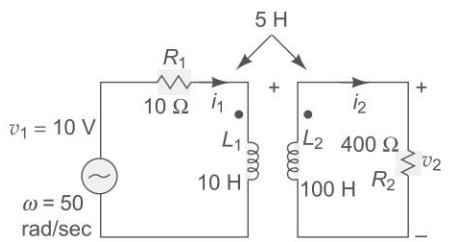


Fig. 10.37

Solution Let us consider i_1 and i_2 as mesh currents in the primary and secondary windings.

As the current i_1 is entering at the dot-marked terminal, and current i_2 is leaving the dot-marked terminal, the sign of the mutual inductance is to be negative. Using Kirchhoff's voltage law, the voltage equation for the first mesh is

$$i_1(R_1 + j\omega L_1) - i_2\, j\omega M = v_1$$

$$i_1(10 + j500) - i_2\, j250 = 10 \quad (10.10)$$

Similarly, for the second mesh,

$$i_2(R_2 + j\omega L_2) - i_1 j\omega M = 0$$

$$i_2(400 + j5000) - i_1 j250 = 0 \quad (10.11)$$

$$i_2 = \frac{\begin{vmatrix} (10 + j500) & 10 \\ -j250 & 0 \end{vmatrix}}{\begin{vmatrix} (10 + j500) & -j250 \\ -j250 & (400 + j5000) \end{vmatrix}}$$

$$i_2 = 0.00102 \angle -84.13°$$

$$v_2 = i_2 \times R_2$$

$$= 0.00102 \angle -84.13° \times 400$$

$$= 0.408 \angle -84.13°$$

$$\frac{v_2}{v_1} = \frac{0.408}{10} \angle -84.13°$$

$$\frac{v_2}{v_1} = 40.8 \times 10^{-3} \angle -84.13°$$

PROBLEM 10.7

Calculate the effective inductance of the circuit shown in Fig. 10.38 across AB.

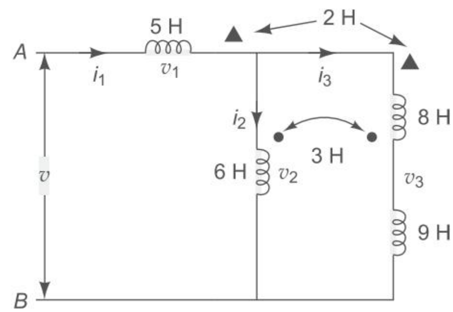


Fig. 10.38

Solution The inductance matrix is

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix}$$

From KVL, $v = v_1 + v_2$ (10.12)

and $v_2 = v_3$ (10.13)

from KCL, $i_1 = i_2 + i_3$ (10.14)

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

$$v_1 = 5\frac{di_1}{dt} - 2\frac{di_3}{dt} \quad (10.15)$$

and $$v_2 = 6\frac{di_2}{dt} - 3\frac{di_3}{dt} \quad (10.16)$$

$$v_3 = -2\frac{di_1}{dt} - 3\frac{di_2}{dt} + 17\frac{di_3}{dt} \quad (10.17)$$

From Eq. (10.12), we have

$$v = v_1 + v_2$$

$$= 5\frac{di_1}{dt} - 2\frac{di_3}{dt} + 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

$$v = 5\frac{di_1}{dt} + 6\frac{di_2}{dt} - 5\frac{di_3}{dt} \tag{10.18}$$

From Eq. (10.14),

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{10.19}$$

Substituting Eq. (10.19) in Eq. (10.17), we have

$$v_3 = -2\left[\frac{di_2}{dt} + \frac{di_3}{dt}\right] - 3\left[\frac{di_2}{dt}\right] + 17\left[\frac{di_3}{dt}\right]$$

or $$-5\frac{di_2}{dt} + 15\frac{di_3}{dt} = v_3 \tag{10.20}$$

Multiplying Eq. (10.16) by 5, we get

$$30\frac{di_2}{dt} - 15\frac{di_3}{dt} = 5v_2 \tag{10.21}$$

Adding Eqs (10.20) and (10.21), we get

$$25\frac{di_2}{dt} = v_3 + 5v_2$$

$$25\frac{di_2}{dt} = 6v_2$$

$$= 6v_3, \text{ since } v_2 = v_3$$

or $$v_2 = \frac{25}{6}\frac{di_2}{dt}$$

From Eq. (10.16),

$$\frac{25}{6}\frac{di_2}{dt} = 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

from which $$\frac{di_2}{dt} = \frac{18}{11}\frac{di_3}{dt}$$

From Eq. (10.19),

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{11}{18}\frac{di_2}{dt} = \frac{29}{18}\frac{di_2}{dt}$$

Substituting the values of $\frac{di_2}{dt}$ and $\frac{di_3}{dt}$ in Eq. (10.18) yields

$$v = 5\frac{di_1}{dt} + 6\frac{18}{29}\frac{di_1}{dt} - 5\frac{11}{18}\frac{di_2}{dt}$$

$$= 5\frac{di_1}{dt} + \frac{108}{29}\frac{di_1}{dt} - \frac{55}{18}\frac{18}{29}\frac{di_1}{dt}$$

$$v = \frac{198}{29}\frac{di_1}{dt} = 6.827\frac{di_1}{dt}$$

$\therefore$ equivalent inductance across AB = 6.827 H

PROBLEM 10.8

Write the mesh equations for the network shown in Fig. 10.39.

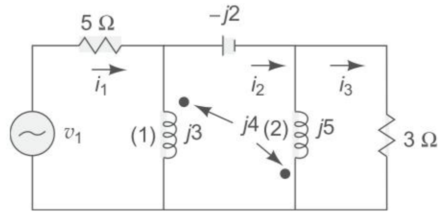


Fig. 10.39

Solution The circuit contains three meshes. Let us assume three loop currents i_1, i_2 and i_3.

For the first mesh,

$$5i_1 + j3(i_1 - i_2) + j4(i_3 - i_2) = v_1 \qquad (10.22)$$

The drop due to self-inductance is $j3(i_1 - i_2)$ is written by considering the current $(i_1 - i_2)$ entering at dot-marked terminal in the first coil, $j4(i_3 - i_2)$ is the mutually induced voltage in coil 1 due to current $(i_3 - i_2)$ entering at dot-marked terminal of the coil 2.

Similarly, for the second mesh,

$$j3(i_2 - i_1) + j5(i_2 - i_3) - j2i_2 + j_4(i_2 - i_3) + j4(i_2 - i_1) = 0 \qquad (10.23)$$

$j4(i_2 - i_1)$ is the mutually induced voltage in the coil 2 due to the current in the coil 1, and $j4(i_2 - i_3)$ is the mutually induced voltage in the coil 1 due to the current in the coil 2.

For the third mesh,

$$3i_3 + j5(i_3 - i_2) + j4(i_1 - i_2) = 0 \qquad (10.24)$$

Further simplification of Eqs (10.22), (10.23), and (10.24) leads to

$$(5 + j3)i_1 - j7i_2 + j4i_3 = v_1 \qquad (10.25)$$
$$- j7i_1 + j14i_2 - j9i_3 = 0 \qquad (10.26)$$
$$j4i_1 - j9i_2 + (3 + j5)i_3 = 0 \qquad (10.27)$$

PROBLEM 10.9

The inductance matrix for the circuit of three series connected coupled coils is given in Fig. 10.40. Find the inductances, and indicate the dots for the coils.

$$L = \begin{bmatrix} 4 & -4 & 1 \\ -4 & 2 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

All elements are in henries.

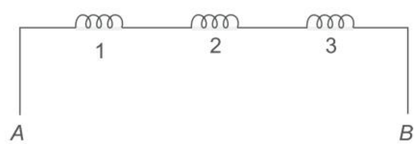


Fig. 10.40

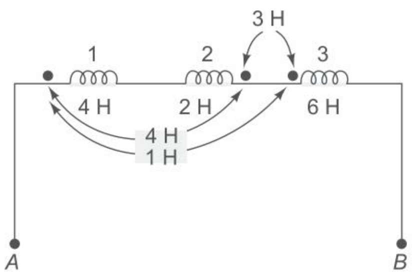


Fig. 10.41

Solution The diagonal elements (4, 2, 6) in the matrix represent the self-inductances of the three coils 1, 2, and 3, respectively. The second element in the first row (−4) is the mutual inductance between coils 1 and 2, the negative sign indicates that the current in the first coil enters the dotted terminal, and the current in the second coil enters at the undotted terminal. Similarly, the remaining elements are fixed. The values of inductances and the dot convention is shown in Fig. 10.41.

PROBLEM 10.10

Find the voltage across the 10 Ω resistor for the network shown in Fig. 10.42.

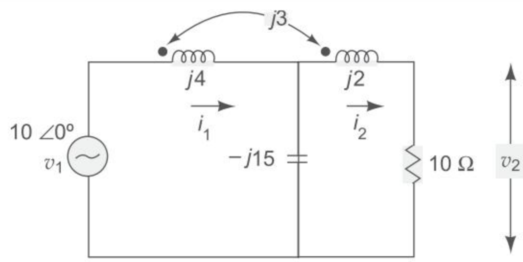


Fig. 10.42

Solution From Fig. 10.42, it is clear that

$$v_2 = i_2\, 10 \tag{10.28}$$

Mesh equation for the first mesh is

$$j4i_1 - j15\,(i_1 - i_2) + j3i_2 = 10\angle 0^\circ$$

$$-\,j11i_1 + j18i_2 = 10\angle 0^\circ \tag{10.29}$$

Mesh equation for the second mesh is

$$j2i_2 + 10i_2 - j15(i_2 - i_1) + j3i_1 = 0$$

$$j18i_1 - j13i_2 + 10i_2 = 0$$

$$j18i_1 + i_2(10 - j13) = 0 \tag{10.30}$$

Solving for i_2 from Eqs (10.29) and (10.30), we get

$$i_2 = \begin{bmatrix} -j11 & 10\angle 0^\circ \\ j18 & 0 \end{bmatrix} \Big/ \begin{bmatrix} -j11 & j18 \\ j18 & 10 - j3 \end{bmatrix}$$

$$= \frac{-180\angle 90^\circ}{291 - j110}$$

$$= \frac{-180\angle 90^\circ}{311\angle 20.70^\circ} = -0.578\angle 110.7^\circ$$

$$\therefore \qquad v_2 = i_2\, 10 = -5.78\,\angle 110.7^\circ$$

$$|v_2| = 5.78$$

PROBLEM 10.11

The resonant frequency of the tuned circuit shown in Fig. 10.43 is 1000 rad/sec. Calculate the self- inductances of the two coils and the optimum value of the mutual inductance.

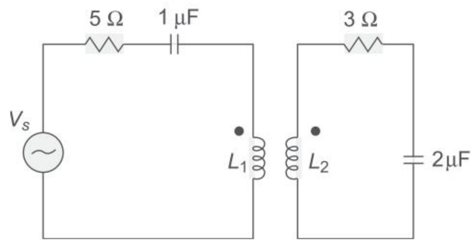


Fig. 10.43

Solution From Section 10.7, we know that

$$\omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$

$$L_1 = \frac{1}{\omega_r^2 C_1} = \frac{1}{(1000)^2 1\times 10^{-6}} = 1 \text{ H}$$

$$L_2 = \frac{1}{\omega_r^2 C_2} = \frac{1}{(1000)^2 \times 2\times 10^{-6}} = 0.5 \text{ H}$$

Optimum value of the mutual inductance is given by

$$M_{\text{optimum}} = \frac{\sqrt{R_1 R_2}}{\omega_r}$$

where R_1 and R_2 are the resistances of the primary and secondary coils

$$M = \frac{\sqrt{15}}{1000} = 3.87 \text{ mH}$$

PROBLEM 10.12

The tuned frequency of a double-tuned circuit shown in Fig. 10.44 is 10^4 rad/s. If the source voltage is 2 V and has a resistance of 0.1 Ω, calculate the maximum output voltage at resonance if $R_1 = 0.01\,\Omega$, $L_1 = 2\,\mu H$; $R_2 = 0.1\,\Omega$, and $L_2 = 25\,\mu H$.

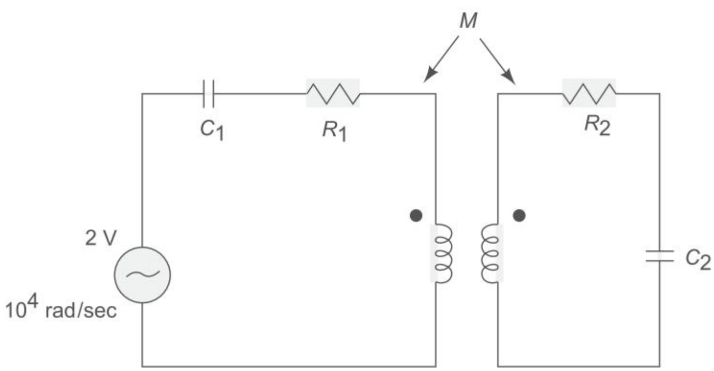


Fig. 10.44

Solution The maximum output voltage $v_0 = \dfrac{v_i}{2\omega_r^2 C_2 M_c}$

where M_c is the critical value of the mutual inductance given by

$$M_c = \frac{\sqrt{R_2(R_1 + R_s)}}{\omega_r}$$

$$M_c = \frac{\sqrt{0.1(0.01+0.1)}}{10^4} = 10.48\ \mu\text{H}$$

At resonance,
$$\omega_r^2 = \frac{1}{L_2 C_2}$$

$$C_2 = \frac{1}{\omega_r^2 L_2} = \frac{1}{\left(10^4\right)^2 \times 25\times10^{-6}} = 0.4\times10^{-3}\ \text{F}$$

$$v_0 = \frac{2}{2\left(10^4\right)^2 \times 0.4\times10^{-3}\times10.48\times10^{-6}}$$

$$= 2.385\ \text{V}$$

PROBLEM 10.13

An iron ring of 10 cm diameter and 15 cm² cross section is wound with 250 turns of wire for a flux density of 1.5 Web/m² and permeability 500. Find the exciting current, the inductance and stored energy. Find corresponding quantities when there is a 2 mm air gap.

Solution (a) *Without air gap*

Length of the flux path = $\pi D = \pi \times 10 = 31.41$ cm
$= 0.3141$ m

Area of flux path = 15 cm² = 15×10^{-4} m²

mmf = A.T

$$A = \frac{\text{mmf}}{T}$$

$$H = \frac{B}{\mu_0\mu_r} = \frac{1.5}{4\pi\times10^{-7}\times500} = 2387$$

mmf = $H \times l = 2387 \times 0.3141 = 750$ AT

$$\text{Exciting current} = \frac{\text{mmf}}{T} = \frac{750}{250} = 3\ \text{A}$$

$$\text{Reluctance} = \frac{1}{\mu_0\mu_r A} = \frac{0.3141}{4\pi10^{-7}\times500\times15\times10^{-4}}$$

$$= 333270$$

$$\text{Self-inductance} = \frac{N^2}{\text{Reluctance}} = \frac{(250)^2}{333270} = 0.1875\ \text{H}$$

$$\text{Energy} = \frac{1}{2}LI^2 = \frac{1}{2}\times0.1875\times(3)^2$$

$$= 0.843\ \text{joules}$$

(b) *With air gap*

$$\text{Reluctance of the gap} = \frac{1}{\mu_0 A} = \frac{2\times10^{-3}}{4\pi\times10^{-7}\times15\times10^{-4}}$$

$$= 1.06\times10^{6} \text{ A/Wb}$$

Total reluctance = $(0.333 + 1.06)\ 10^6 = 1.393 \times 10^6$ A/Wb

mmf = $\phi \times$ reluctance

$= 1.5 \times 15 \times 10^{-4} \times 1.393 \times 10^6$

$= 3134$ AT

$$\text{Exciting current} = \frac{3134}{250} = 12.536\text{ A}$$

$$L = \frac{N^2}{\Re} = \frac{(250)^2}{1.393\times10^6} = 44.8\text{ mH}$$

$$\text{Energy} = \frac{1}{2}LI^2$$

$$= \frac{1}{2}\times44.8\times10^{-3}\times(12.536)^2$$

$$= 3.52 \text{ joules}$$

PROBLEM 10.14

A 700-turn coil is wound on the central limb of the cast steel frame as shown in Fig. 10.45. A total flux of 1.8 m Wb is required in the gap. What is the current required? Assume that the gap density is uniform and that all lines pass straight across the gap. All dimensions are in centimeters. Assume μ_r as 600.

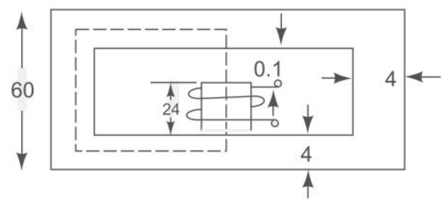


Fig. 10.45

Solution Each of the side limbs carry half the total flux as their reluctances are equal.

Total mmf required is equal to the sum of the mmf required for gap, central limb and side limb.

Reluctance of gap and central limb are in series and they carry the same flux.

Air Gap

$\phi_g = 1.8 \times 10^{-3}$ Wb

$A_g = 4 \times 4 \times 10^{-4}$ m²

$$B_g = \frac{1.8\times10^{-3}}{16\times10^{-4}} = 1.125 \text{ Wb/m}^2$$

$$H_g = \frac{B_g}{\mu_0} = \frac{1.125}{4\pi\times10^{-7}} = 8.95\times10^5 \text{ AT/m}$$

Required mmf for the gap = $H_g l_g$

$= 8.95 \times 10^5 \times 0.001 = 895$ AT

Central Limb

$$\phi_c = 1.8 \times 10^{-3} \text{ Wb}$$
$$A_c = 4 \times 4 \times 10^{-4} \text{ m}^2$$
$$B_c = 1.125 \text{ Wb/m}^2$$

$$H_c = \frac{B_c}{\mu_0 \mu_r} = \frac{1.125}{4\pi \times 10^{-7} \times 600} = 1492 \text{ AT/m}$$

$$\text{Required mmf for central limb} = H_c l_c$$
$$= 1492 \times 0.24 = 358 \text{ AT}$$

Side Limb

$$\phi_s = \frac{1}{2} \times \text{flux in central limb}$$
$$= \frac{1}{2} \times 1.8 \times 10^{-3} = 0.9 \times 10^{-3} \text{ Wb}$$

$$B_s = \frac{0.9 \times 10^{-3}}{16 \times 10^{-4}} = 0.5625 \text{ Wb/m}^2$$
$$H_s = \frac{B_s}{\mu_0 \mu_r} = \frac{0.5625}{4\pi \times 10^{-7} \times 600} = 746 \text{ AT/m}$$

$$\text{Required mmf for side limb} = H_s l_s$$
$$= 746 \times 0.6 = 447.6 \cong 448$$

$$\text{Total mmf} = 895 + 358 + 448 = 1701 \text{ AT}$$

$$\text{Required current} = \frac{1701}{700} = 2.43 \text{ A}$$

PROBLEM 10.15

Determine $i_1(t)$ and $i_2(t)$ in the circuit shown in Fig. 10.46. if $L_1 = 0.4$ H; $L_2 = 0.4$ H; $v_1 = 15 \sin t$ and $M = 0.2$ H.

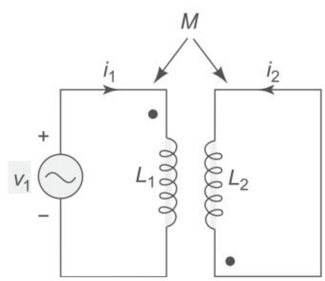


Fig. 10.46

Solution

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$
$$15 \sin t = 0.4 \frac{di_1}{dt} - 0.2 \frac{di_2}{dt}$$
$$0 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$
$$0.2 \frac{di_1}{dt} = 0.4 \frac{di_2}{dt}$$
$$\frac{di_1}{dt} = 2 \frac{di_2}{dt}$$
$$15 \sin t = 0.4 \times 2 \frac{di_2}{dt} - 0.2 \frac{di_2}{dt}$$

$$15\sin t = 0.6\frac{di_2}{dt}$$
$$\frac{di_2}{dt} = 25\ \sin t$$
$$i_2(t) = \int_0^t 25\sin t\,dt = -25\cos t]_0^t$$
$$i_2(t) = 25(1-\cos t)$$
$$\frac{di_1}{dt} = 2\frac{di_2(t)}{dt} \Rightarrow i_1(t) = 2i_2(t) = 50(1-\cos t)$$

PROBLEM 10.16

Obtain $V_1(t)$ and $V_2(t)$ at $t = 2$ s for the circuit shown in Fig. 10.47.
If $L_1 = 0.5$ H; $L_2 = 0.125$ H; $M = 0.2$ H.

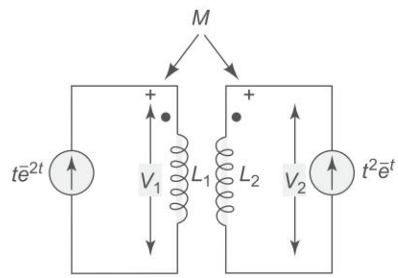


Fig. 10.47

Solution
$$V_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$
$$= 0.5\frac{d}{dt}(te^{-2t}) + 0.2\frac{d}{dt}(t^2e^{-t})$$
$$= 0.5\left[t(-2)e^{-2t} + e^{-2t}\right] + 0.2\left[t^2(-1)e^{-t} + 2te^{-t}\right]$$

At $t = 2$ s,

$$V_1(t) = 0.5\left[2(-2)e^{-4} + e^{-4}\right] + 0.2\left[2^2(-1)e^{-2} + 2(2)e^{-2}\right]$$
$$= 0.4056 \text{ volts}$$

Similarly, $$V_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$
$$= 0.125\frac{d}{dt}(t^2e^{-t}) + 0.2\frac{d}{dt}(te^{-2t})$$

$$V_2(t) = -0.125\left[2te^{-t} - t^2e^{-t}\right] + 0.2\left[e^{-2t} - 2te^{-2t}\right]$$

At $t = 2$ s,

$$V_2(2) = -0.125\left[2(2)e^{-2} - 2^2e^{-2}\right] + 0.2\left[e^{-2(2)} - 2(2)e^{-2(2)}\right]$$

$$= -0.01098 \text{ volts}$$

PROBLEM 10.17

For the coupled circuit shown in Fig. 10.48, obtain $i_1(t)$, $i_2(t)$ when the switch is closed at $t = 0$ using Laplace transform.

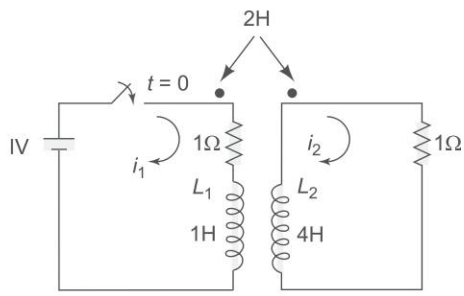


Fig. 10.48

Solution Applying KVL for the first loop when the switch is closed at $t = 0$,

$$1 = Ri_1(t) + L_1\frac{di_1(t)}{dt} - M\frac{di_2(t)}{dt}$$

Taking Laplace transform of the above equation with zero initial conditions,

$$\frac{1}{S} = I_1(S)[R + L_1S] - MSI_2(S)$$

$$\frac{1}{S} = I_1(S)[1+S] - 2SI_2(S) \tag{10.31}$$

KVL for the second loop

$$0 = Ri_2(t) + L_2\frac{di_2(t)}{dt} - M\frac{di_1(t)}{dt}$$

Taking Laplace transform,

$$= RI_2(S) + L_2SI_2(S) - MSI_1(S)$$
$$= -MSI_1(S) + (R + L_2S)I_2(S)$$

Substituting the values,

$$0 = -2SI_1(S) + (1+4S)I_2(S)$$
$$I_1(S) = I_2(S)\frac{1+4S}{2S} \tag{10.32}$$

Substituting in Eq. (10.31),

$$\frac{1}{S} = (1+S)\frac{(1+4S)}{2S}I_2(S) - 2SI_2(S)$$

$$\frac{1}{S} = I_2(S)\left[\frac{(1+S)}{2S}(1+4S) - 2S\right]$$

$$\frac{1}{S} = I_2(S)\left[\frac{1+4S+S+4S^2-4S^2}{2S}\right]$$

from which $I_2(S) = \dfrac{2}{1+5S}$

and $I_1(S) = I_2(S)\dfrac{(1+4S)}{2S}$

$$= \frac{2}{(1+5S)}\frac{(1+4S)}{2S}$$

$$I_1(S) = \frac{1+4S}{S(1+5S)} = \frac{1}{S} - \frac{1}{1+5S} = \frac{1}{S} - \frac{1}{5\left(S+\frac{1}{5}\right)}$$

Taking inverse Laplace transform of the above equation,

$$i_1(t) = t - \frac{1}{5}e^{-t/5}$$

$$I_2(S) = \frac{2}{1+5S}$$

$$= \frac{2/5}{\left(S+\frac{1}{5}\right)}$$

$$i_2(t) = \frac{2}{5}\cdot e^{-\frac{t}{5}}$$

PROBLEM 10.18

An iron ring has a mean circumferential length of 60 cm and a uniform winding of 300 turns. An air gap has been made by a saw cut across the section of the ring. When a current of 1 A flows through the coil, the fluxdensity in the air gap is found to be 0.126 m web/m². How long is the air gap? Assume iron has a relative permeability of 300. Also calculate the reluctance.

Solution Mean circumferential length = 60 cm

= 0.6 m

$2\pi r = 0.6$ m

$r = 0.09549$ m

Given $N = 300$; $I = 1$ A; AT = 300

Cross-sectional area = $\pi r^2 = 0.02864$ m²

Reluctance $R = \dfrac{l}{\mu a} = \dfrac{0.6}{4\pi\times10^{-7}\times300\times0.02864}$

$= 55570.86$ AT/wb

Total ampere turns = Ampere turns of gap + Ampere turns of iron path

$$AT = AT_g + AT_i$$

$$300 = \frac{B_g l_g}{\mu_0} + \frac{B_i l_i}{\mu_0 \mu_r} \qquad \text{where } I_g \text{-length of air gap}$$

$$300 = \frac{0.126\times10^{-3} l_g}{4\pi\times10^{-7}} + \frac{0.126\times10^{-3}\left(0.6-l_g\right)}{4\pi\times10^{-7}\times300}$$

from which $l_g = \dfrac{90.345}{299} = 0.3021$ metres

PROBLEM 10.19

Calculate the current to be passed through the coil 'C' having 500 turns so that a flux of 1 m is produced in the air gap in the Fig. 10.49 shown the core is of square cross section over the entire length and has permeability of 800.

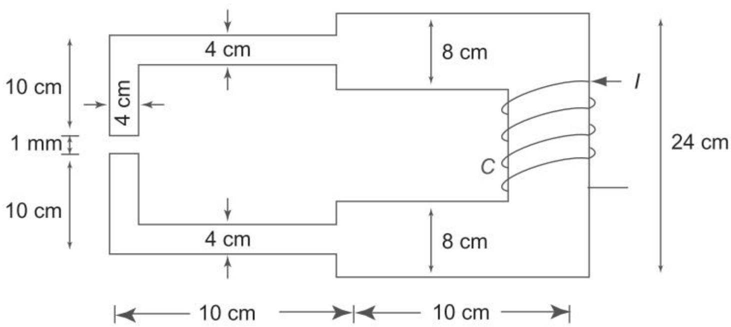


Fig. 10.49

Solution Total flux $= \dfrac{\text{total mmf}}{\text{total reluctance}}$

Given figure can be considered as different sections

$$\therefore \quad \phi = \frac{\text{mmf}}{\dfrac{l_1}{\mu_1 A_1} + \dfrac{l_2}{\mu_2 A_2} + \dfrac{l_3}{\mu_3 A_3}}$$

$$1\times10^{-3} = \frac{I\times500}{\dfrac{(20+20)\times10^{-2}}{800\times4\pi\times10^{-7}\times16\times10^{-4}} + \dfrac{(20+8)\times10^{-2}}{800\times4\pi\times10^{-7}\times64\times10^{-4}} + \dfrac{1\times10^{-3}}{4\pi\times10^{-7}\times64\times10^{-4}}}$$

$$= \frac{I\times500\times4\pi\times10^{-7}}{0.3125+0.05459+2.5}$$

$$= \frac{I\times6283\times10^{-7}}{2.867}$$

$$I = 4.563 \text{ A}$$

PSpice Problems

PROBLEM 10.1

Using PSpice, calculate the effective inductance of the circuit shown in Fig. 10.50 across AB.

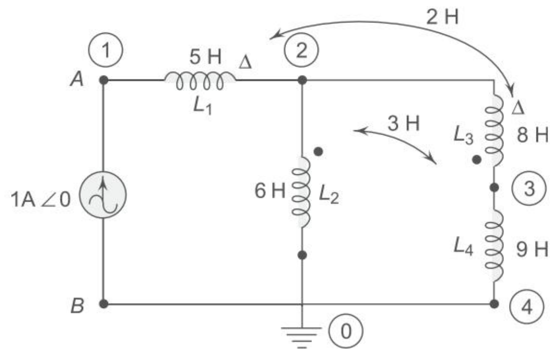


Fig. 10.50

```
* CALCULATION OF EFFECTIVE INDUCTANCE
IS    0    1    AC    1 0
L1    1
           2
                5
L2    2    0    6
L3    2    3    8
L4    3    4    9
R4  01  UOHM
K23  L2  L3  −0.433
K13  L1  L3  −0.316
.AC  LIN  1  50  50
.PRINT  AC  VM(1)  VP(1)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
*******************************************************************
FREQ          VM(1)          VP(1)
5.000E + 01   2.145E + 03    9.000E + 01
```

Result

$Z_{EFF} = X_{EFF} = V(1)/IS = 2145\angle 90/1\angle 0 = j2145$

$L_{EFF} = j2145/(2\pi f) = 6.828$ H

Inject $1\angle 0$ current to AB.

After obtaining the driving point impedance across *AB*, L_{eff} can be calculated.

PSpice will not allow a pure inductor to connect across a voltage source. Hence, inductor voltage source loop is to be breaked with a small negligible resistance.

PROBLEM 10.2

Using PSpice, for the circuit shown in Fig. 10.51, find the ratio of output voltage to input voltage.

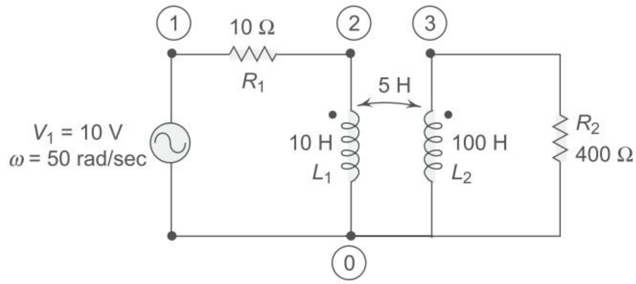


Fig. 10.51

$$f_z = \frac{\omega}{2\pi}$$
$$= 7.958 \text{ Hz}$$

$$K_{12} = \frac{M}{\sqrt{L_1 L_2}} = \frac{5}{\sqrt{10\times100}} = 0.158 \text{ H.}$$

```
* CALCULATION OF VO / VI
VS   1   0   AC 10 0
R1   1   2   10
L1   2   0   10 H
L2   3   0   100 H
K12  L1  L2  0.158H
R2   3   0   400
.AC  LIN 1   7.958   7.958
.PRINT  AC  VM(R2)  VP(R2)
.END
**** AC ANALYSIS    TEMPERATURE = 27.000 DEG C
***************************************************************
   FREQ          VM(R2)         VP(R2)
7.958E + 00   4.085E − 01   −8.413E + 01
```

Result

$V_2 / V_1 = 0.4085 \angle{-84.13}/10 \angle 0 = 0.04085 \angle{-84.13} = 40.85\text{m} \angle{-84.13}$

ANSWERS TO PRACTICE PROBLEMS

10-2.3 $L = 13$ H

10-2.4 $L = \dfrac{2}{3}$ M

10-2.5 $L_{eq} = \dfrac{1}{3}$ H

10-3.1 $v_1 = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt}$; $v_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt}$

10-5.1 $v_1 = 181.44 \cos(40t - 30°)$

$v_2 = 202.88 \cos(40t - 30°)$

10-6.1 $\begin{bmatrix} 2 & 5 & -2 \\ 5 & 4 & 0 \\ -2 & 0 & 6 \end{bmatrix}$

10-8.1 $1 \angle -90$ V

10-9.4 $I_1 = 6.75 - j0.54$; $I_2 = -3.243 - j0.54$

$\dfrac{V_2}{V_1} = 0.657\angle -170.537°$

10-9.5 $L_{eq} = \left(L_1 - \dfrac{M^2}{L_2} \right)$ H

10-10.1 299.6 A

10-11.1 (i) MMR = 1400 AT (ii) $B = 1.120$ AT (iii) $\phi = 0.784$ mweb

Objective-Type Questions

☆☆★**10.1** Mutual inductance is a property associated with

(a) only one coil
(b) two or more coils
(c) two or more coils with magnetic coupling

☆☆★**10.2** Dot convention in coupled circuits is used

(a) to measure the mutual inductance
(b) to determine the polarity of the mutually induced voltage in coils
(c) to determine the polarity of the self induced voltage in coils

☆☆★**10.3** Mutually induced voltage is present independently of, and in addition to, the voltage due to self-induction.

(a) true (b) false

☆★★**10.4** Two terminals belonging to different coils are marked identically with dots, if for the different direction of current relative to like terminals the magnetic flux of self and mutual induction in each circuit add together.

(a) true (b) false

☆☆★**10.5** The maximum value of the coefficient of coupling is

(a) 100% (b) more than 100% (c) 90%

☆☆★**10.6** The case for which the coefficient of coupling $K = 1$ is called perfect coupling is

(a) true (b) false

☆☆★**10.7** The maximum possible mutual inductance of two inductively coupled coils with self-inductances $L_1 = 25$ mH and $L_2 = 100$ mH is given by

(a) 125 mH (b) 75 mH (c) 50 mH

☆☆★ **10.8** The value of the coefficient of coupling is more for aircored coupled circuits compared to the iron core coupled circuits.

(a) true (b) false

☆★★ **10.9** Two inductors are connected as shown in Fig. 10.52. What is the value of the effective inductance of the combination?

(a) 8 H (b) 10 H (c) 4 H

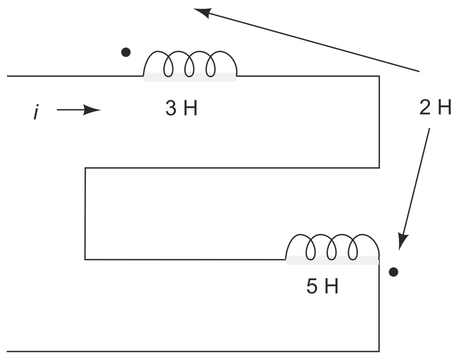


Fig. 10.52

☆☆★ **10.10** Two coils connected in series have an equivalent inductance of 3 H when connected in aiding. If the self-inductance of the first coil is 1 H, what is the self inductance of the second coil (Assume $M = 0.5\,\text{H}$)

(a) 1 H (b) 2 H (c) 3 H

☆★★ **10.11** For Fig. 10.53 shown below, the inductance matrix is given by

(a) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

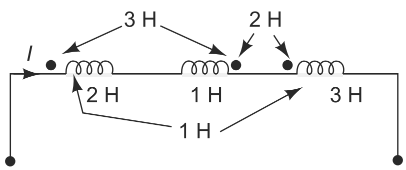


Fig. 10.53

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/268

CHAPTER 11

Transients

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Explain steady-state and transient behaviour of the circuit
LO 2 Analyse the dc response for *R-L* circuits, *R-C* circuits, and *R-L-C* circuits
LO 3 Analyse sinusoidal response for *R-L* circuits, *R-C* circuits, and *R-L-C* circuits

11.1 STEADY STATE AND TRANSIENT RESPONSE

LO 1 Explain steady-state and transient behaviour of the circuit

A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency, sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to another state. *The behaviour of the voltage or current when it is changed from one state to another is called the* ***transient state****. The time taken for the circuit to change from one steady state to another steady state is called the* ***transient time****. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic, equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the* ***natural response****. Storage elements deliver their energy to the resistances. Hence, the response changes with time, gets saturated after some time, and is referred to as the* ***transient response****. When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called* ***forced response***. In other words, the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after a short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **11-1.1** What do you understand by transient and steady-state parts of response? How can they be identified in a general solution?

Frequently Asked Questions linked to LO 1*

☆☆★ **11-1.1** Distinguish between natural and forced response. **[AU May/June 2013]**
☆☆★ **11-1.2** What is free and forced response? **[AU May/June 2014]**
☆☆★ **11-1.3** What is natural response? **[RGTU June 2014]**
☆☆★ **11-1.4** What do you mean by forced response? **[RGTU June 2014]**

11.2 DC RESPONSE OF AN *R-L* CIRCUIT

LO 2 Analyse the dc response for R-L circuits, R-C circuits, and R-L-C circuits

Consider a circuit consisting of a resistance and inductance as shown in Fig. 11.1. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

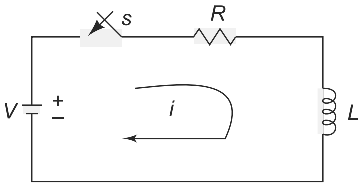


Fig. 11.1

$$V = Ri + L\frac{di}{dt} \tag{11.1}$$

$$\text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \tag{11.2}$$

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order. Comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K \tag{11.3}$$

whose solution is

$$x = e^{-pt}\int Ke^{+Pt}dt + ce^{-Pt} \tag{11.4}$$

where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = ce^{-(R/L)t} + e^{-(R/L)t}\int \frac{V}{L}e^{(R/L)t}dt$$

$$\therefore \quad i = ce^{-(R/L)t} + \frac{V}{R} \tag{11.5}$$

To determine the value of c in Eq. (11.5), we use the initial conditions. In the circuit shown in Fig. 11.1, the switch S is closed at $t = 0$. At $t = 0^-$, i.e. just before closing the switch S, the current in the inductor is

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

zero. Since the inductor does not allow sudden changes in currents, at $t = 0^+$ just after the switch is closed, the current remains zero.

Thus, at $\quad t = 0, i = 0$

Substituting the above condition in Eq. (11.5), we have

$$0 = c + \frac{V}{R}$$

Hence, $\quad c = -\frac{V}{R}$

Substituting the value of c in Eq. (11.5), we get

$$i = \frac{V}{R} - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)$$

$$i = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right) \tag{11.6}$$

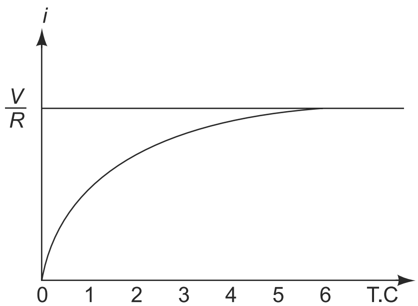


Fig. 11.2

Equation (11.6) consists of two parts, the steady state part V/R, and the transient part $(V/R)e^{-(R/L)t}$. When the switch S is closed, the response reaches a steady-state value after a time interval as shown in Fig. 11.2.

Here, the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R. The time constant of a function $\frac{V}{R}e^{-\left(\frac{R}{L}\right)t}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the *time constant* and is denoted by τ

$$\therefore \quad \tau = \frac{L}{R} \text{ sec}$$

$\therefore$ the transient part of the solution is

$$i = -\frac{V}{R}\exp\left(-\frac{R}{L}t\right) = \frac{V}{R}e^{-t/\tau}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135\,\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498\,\frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067\,\frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its final value.

In Fig. 11.1, we can find out the voltages and powers across each element by using the current.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

$$\therefore \quad v_R = V\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

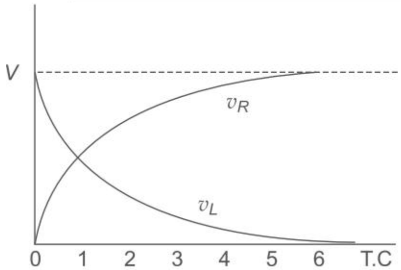


Fig. 11.3

Similarly, the voltage across the inductance is

$$v_L = L\frac{di}{dt}$$

$$= L\frac{V}{R} \times \frac{R}{L}\exp\left(-\frac{R}{L}t\right) = V\exp\left(-\frac{R}{L}t\right)$$

The responses are shown in Fig. 11.3.

Power in the resistor is

$$p_R = v_R i = V\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\frac{V}{R}$$

$$= \frac{V^2}{R}\left(1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right)\right)$$

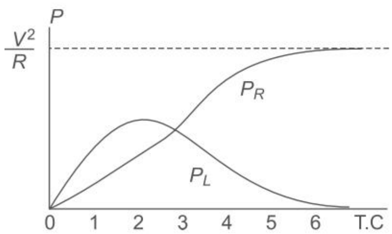


Fig. 11.4

Power in the inductor is

$$p_L = v_L i = V\exp\left(-\frac{R}{L}t\right) \times \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$

$$= \frac{V^2}{R}\left(\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right)$$

The responses are shown in Fig. 11.4.

EXAMPLE 11.1

A series RL circuit with R = 30 Ω and L = 15 H has a constant voltage V = 60 V applied at t = 0 as shown in Fig. 11.5. Determine the current i, the voltage across resistor and the voltage across the inductor.

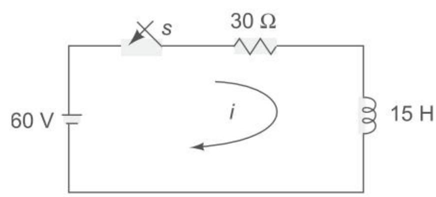


Fig. 11.5

Solution By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\therefore \quad \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt}\int Ke^{Pt}\,dt$$

where $P = 2$, $K = 4$

$$\therefore \quad i = ce^{-2t} + e^{-2t}\int 4e^{2t}\,dt$$

$$\therefore \quad i = ce^{-2t} + 2$$

At $t = 0$, the switch S is closed.

Since the inductor never allows sudden changes in currents, at $t = 0^+$, the current in the circuit is zero.

Therefore, at $t = 0^+$, $i = 0$

$$\therefore \quad 0 = c + 2$$

$$\therefore \quad c = -2$$

Substituting the value of c in the current equation, we have

$$i = 2(1 - e^{-2t})\text{ A}$$

Voltage across the resistor $v_R = iR$

$$= 2(1 - e^{-2t}) \times 30 = 60\,(1 - e^{-2t})\text{ V}$$

Voltage across the inductor $v_L = L\dfrac{di}{dt}$

$$= 15 \times \frac{d}{dt} 2(1 - e^{-2t}) = 30 \times 2e^{-2t} = 60e^{-2t}\text{ V}$$

11.3 DC RESPONSE OF AN *R-C* CIRCUIT

LO 2

Consider a circuit consisting of resistance and capacitance as shown in Fig. 11.6. The capacitor in the circuit is initially uncharged, and is in series with a resistor. When the switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.

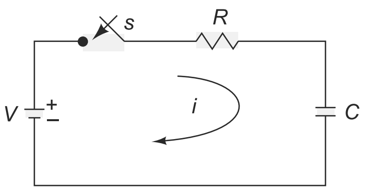


Fig. 11.6

$$V = Ri + \frac{1}{C}\int i\,dt \tag{11.7}$$

By differentiating the above equation, we get

$$0 = R\frac{di}{dt} + \frac{i}{C} \tag{11.8}$$

or

$$\frac{di}{dt} + \frac{1}{RC}i = 0 \tag{11.9}$$

Equation (11.9) is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = ce^{-t/RC} \tag{11.10}$$

Here, to find the value of c, we use the initial conditions.

In the circuit shown in Fig. 11.6, the switch S is closed at $t = 0$. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t = 0^+$. So, the current in the circuit at $t = 0^+$ is V/R

$\therefore$ At $t = 0$, the current $i = \dfrac{V}{R}$

Substituting this current in Eq. (11.10), we get

$$\frac{V}{R} = c$$

$\therefore$ the current equation becomes

$$i = \frac{V}{R} e^{-t/RC} \tag{11.11}$$

When the switch S is closed, the response decays with time as shown in Fig. 11.7.

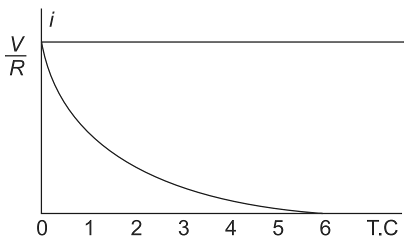


Fig. 11.7

In the solution, the quantity RC is the time constant, and is denoted by τ,

where $\tau = RC$ seconds

After 5 TC, the curve reaches 99 percent of its final value. In Fig. 11.6, we can find out the voltage across each element by using the current equation.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} e^{-(1/RC)t}; v_R = Ve^{-t/RC}$$

Similarly, voltage across the capacitor is

$$\begin{aligned} v_C &= \frac{1}{C}\int i\, dt \\ &= \frac{1}{C}\int \frac{V}{R} e^{-t/RC} dt \\ &= -\left(\frac{V}{RC} \times RC\, e^{-t/RC}\right) + c = -Ve^{-t/RC} + c \end{aligned}$$

At $t = 0$, voltage across capacitor is zero

$$\therefore \quad c = V$$

$$\therefore \quad v_C = V(1 - e^{-t/RC})$$

The responses are shown in Fig. 11.8.

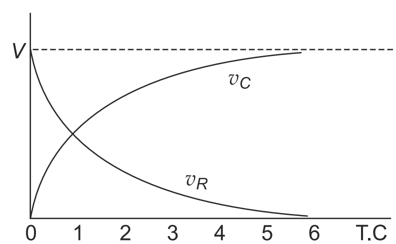


Fig. 11.8

Power in the resistor

$$p_R = v_R i = Ve^{-t/RC} \times \frac{V}{R} e^{-t/RC} = \frac{V^2}{R} e^{-2t/RC}$$

Power in the capacitor

$$\begin{aligned} p_C &= v_C i = V(1 - e^{-t/RC}) \frac{V}{R} e^{-t/RC} \\ &= \frac{V^2}{R}(e^{-t/RC} - e^{-2t/RC}) \end{aligned}$$

The responses are shown in Fig. 11.9.

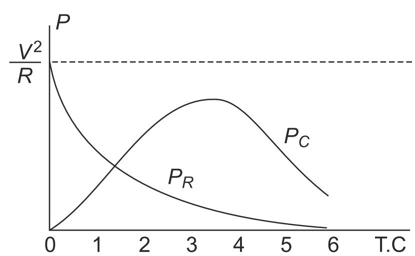


Fig. 11.9

EXAMPLE 11.2

A series RC circuit consists of a resistor of 10 Ω and a capacitor of 0.1 F as shown in Fig. 11.10. A constant voltage of 20 V is applied to the circuit at t = 0. Obtain the current equation. Determine the voltages across the resistor and the capacitor.

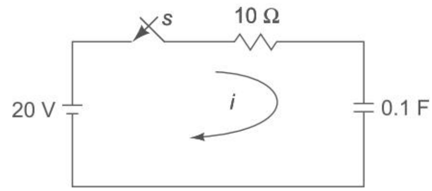


Fig. 11.10

Solution By applying Kirchhoff's law, we get

$$10i + \frac{1}{0.1}\int i\,dt = 20$$

Differentiating with respect to t, we get

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \quad \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t = 0$, the switch S is closed. Since the capacitor does not allow sudden changes in the voltage, the current in the circuit is $i = V/R = 20/10 = 2$ A.

At $t = 0$, $i = 2$ A.

$\therefore$ the current equation $i = 2e^{-t}$

Voltage across the resistor is $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}$ V

Voltage across the capacitor is $v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$

$$= 20(1 - e^{-t})\text{ V}$$

11.4 DC RESPONSE OF AN *R-L-C* CIRCUIT

LO 2

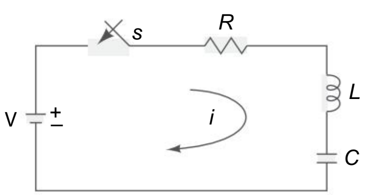


Fig. 11.11

Consider a circuit consisting of resistance, inductance, and capacitance as shown in Fig. 11.11. The capacitor and inductor are initially uncharged, and are in series with a resistor. When the switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \tag{11.12}$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i \tag{11.13}$$

or $$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0 \tag{11.14}$$

The above equation is a second-order linear differential equation, with only complementary function. The particular solution for the above equation is zero. Characteristic equation for the above differential equation is

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)=0 \tag{11.15}$$

The roots of Eq. (11.15) are

$$D_1, D_2=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

By assuming $K_1=-\frac{R}{2L}$ and $K_2=\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$

$$D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

Here, K_2 may be positive, negative or zero.

K_2 is positive, when $\left(\frac{R}{2L}\right)^2 > 1/LC$

The roots are real and unequal, and give the over damped response as shown in Fig. 11.12. Then Eq. (11.14) becomes

$$[D-(K_1 + K_2)]\,[D-(K_1-K_2)]\,i = 0$$

The solution for the above equation is

$$i = c_1e^{(K_1 + K_2)t} + c_2\, e^{(K_1 - K_2)t}$$

The current curve for the overdamped case is shown in Fig. 11.12.

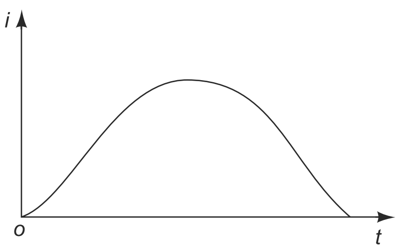


Fig. 11.12

K_2 is negative, when $(R/2L)^2 < 1/LC$

The roots are complex conjugate, and give the underdamped response as shown in Fig. 11.13. Then Eq. (11.14) becomes

$$[D-(K_1 + jK_2)]\,[D-(K_1-jK_2)]i = 0$$

The solution for the above equation is

$$i = e^{K_1t}\,[c_1 \cos K_2t + c_2 \sin K_2t]$$

The current curve for the underdamped case is shown in Fig. 11.13.

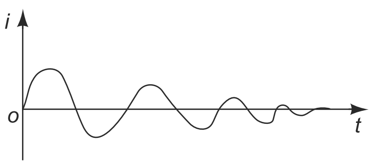


Fig. 11.13

K_2 is zero, when $(R/2L)^2 = 1/LC$

The roots are equal, and give the critically damped response as shown in Fig. 11.14. Then Eq. (11.14) becomes

$$(D-K_1)\,(D-K_1)i = 0$$

The solution for the above equation is

$$i = e^{K_1t}(c_1 + c_2t)$$

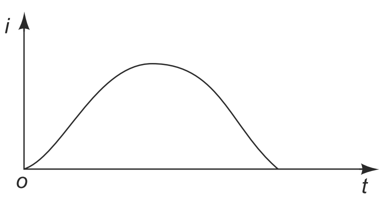


Fig. 11.14

The current curve for the critically damped case is shown in Fig. 11.14.

EXAMPLE 11.3

The circuit shown in Fig. 11.15 consists of resistance, inductance, and capacitance in series with a 100 V constant source when the switch is closed at t = 0. Find the current transient.

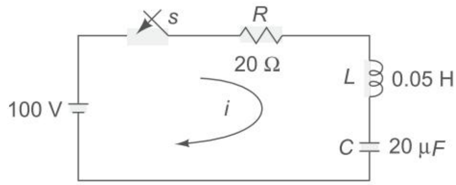


Fig. 11.15

Solution At the $t = 0$, the switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20\times10^{-6}}\int i\,dt \tag{11.16}$$

Differentiating Eq. (11.16), we get

$$0.05\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20\times10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 400\frac{di}{dt} + 10^6 i = 0$$

$$\therefore \quad (D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore, the current

$$i = e^{+k_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

$$i = e^{-200t}\,[c_1 \cos 979.8t + c_2 \sin 979.8t] \text{ A}$$

At $t = 0$, the current flowing through the circuit is zero.

$$i = 0 = (1)\,[c_1 \cos 0 + c_2 \sin 0]$$

$$\therefore \quad c_1 = 0$$

$$\therefore \quad i = e^{-200t}\,c_2 \sin 979.8t \text{ A}$$

Differentiating, we have

$$\frac{di}{dt} = c_2\left[e^{-200t}\,979.8\cos 979.8t + e^{-200t}(-200)\sin 979.8t\right]$$

At $t = 0$, the voltage across the inductor is 100 V.

$$\therefore \quad L\frac{di}{dt} = 100$$

$$\text{or} \quad \frac{di}{dt} = 2000$$

$$\text{At } t = 0 \quad \frac{di}{dt} = 2000 = c_2 979.8 \cos 0$$

$$\therefore \quad c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t}(2.04 \sin 979.8t) \text{ A}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆☆★**11-2.1** Obtain an expression for the current $i(t)$ from the differential equation

$$\frac{d^2 i(t)}{dt^2} + 10\frac{di(t)}{dt} + 25 i(t) = 0$$

with initial conditions

$$i(0^+) = 2 \frac{di(0^+)}{dt} = 0$$

★★★**11-2.2** In the given circuit shown in Fig. Q.2, the switch K is closed at time $t = 0$, the steady-state condition having reached previously. Obtain expression for the current in the circuit at any time t. If $R_1 = R_2 = 100$ ohms, $V = 10$ volts, and $L = 1$ henry. Calculate at time $t = 5$ ms, (a) current i, (b) voltage drop across R_2, and (c) voltage across L.

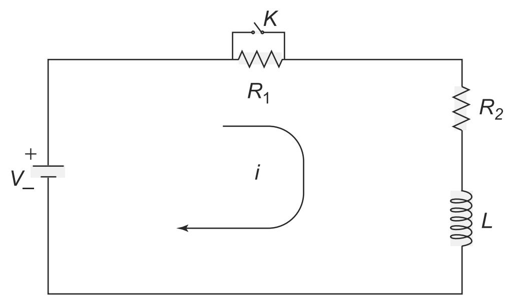


Fig. Q.2

☆★★**11-2.3** In the circuit shown in Fig. Q.3, the capacitor C has an initial voltage $v_c(-0) = 10$ volts and at the same instant, the current in the inductor is zero. Switch K is closed at time $t = 0$. Obtain the expression for the voltage $v(t)$ across the inductor L.

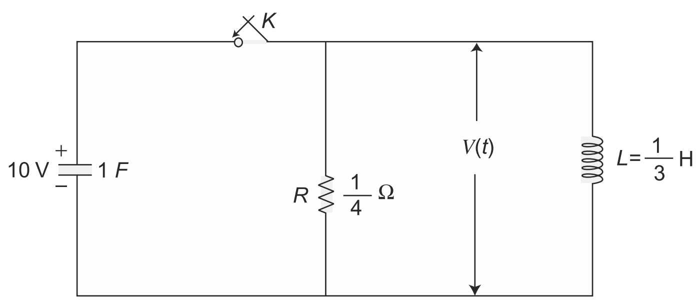


Fig. Q.3

☆★★**11-2.4** The network shown in Fig. Q.4 is initially under steady-state condition with the switch in the position 1. The switch is moved from the position 1 to the position 2 at $t \neq 0$. Calculate the current $i(t)$ through R_1 after switching.

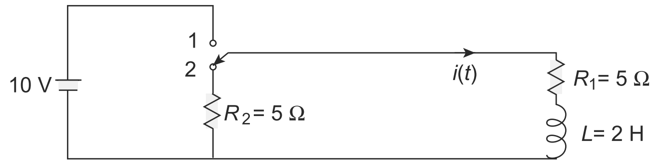


Fig. Q.4

☆★★ **11-2.5** In the network shown in Fig. Q.5, the capacitor c_1 is charged to a voltage of 100V and the switch S is closed at $t = 0$. Determine the current expressions i_1 and i_2.

☆☆★ **11-2.6** In the circuit shown in Fig. Q.6, the switch K is closed at $t = 0$. The current waveform is observed with CRO. The initial value of the current is measured to be 0.0 1amp. The transient appears to disappear in 0.1 second. Find the value of R and C.

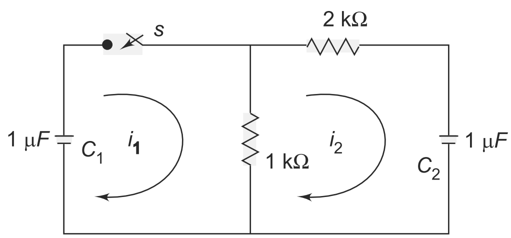


Fig. Q.5

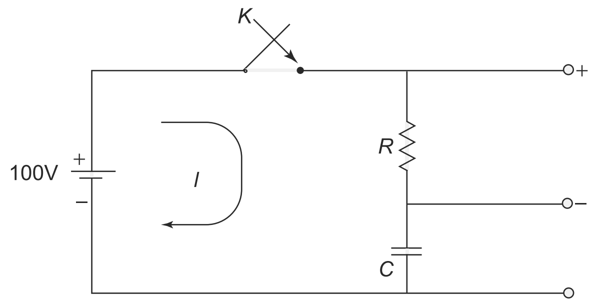


Fig. Q.6

☆★★ **11-2.7** In the circuit shown in Fig. Q.7, steady-state conditions are reached with the switch K in the position 1. At $t = 0$, the switch is changed over to the position 2. Using time domain methods, determine the current through the inductor $i(t)$ for all $t \geq 0^+$.

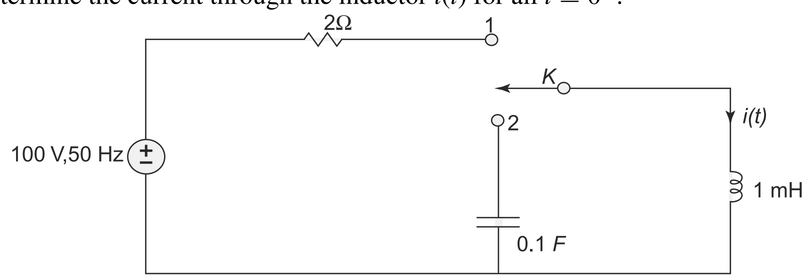


Fig. Q.7

☆★★ **11-2.8** In the circuit shown in Fig. Q.8, the initial current in the inductance is 2 A and its direction is as shown in the figure. The initial charge on the capacitor is 200 C with polarity as shown when the switch is closed. Determine the current expression in the inductance.

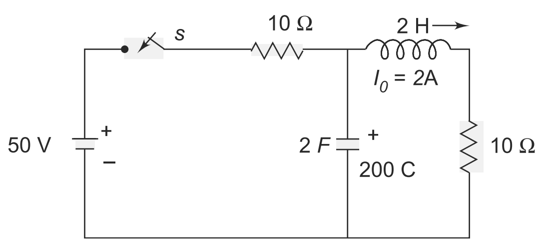


Fig. Q.8

☆☆★ **11-2.9** In the circuit shown in Fig. Q.9, the switch is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Determine V_1 and V_2 at $t = 0^+$.

☆☆★ **11-2.10** In the network shown in Fig. Q.10, determine the current expression for $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$. The network has no initial energy.

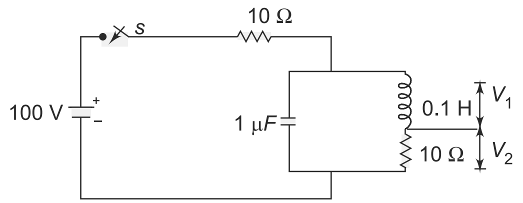


Fig. Q.9

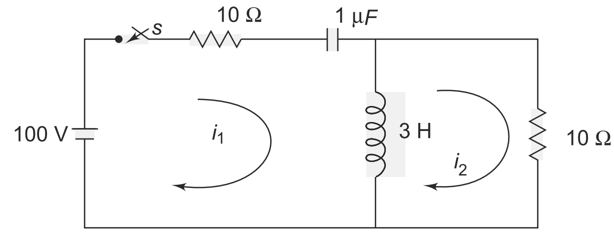


Fig. Q.10

☆★★ **11-2.11** In the network shown in Fig. Q.11, the switch is moved from the position 1 to the position 2 at $t = 0$. Determine the current expression.

☆★★ **11-2.12** Calculate the voltage $v_1(t)$ across the inductance for $t > 0$ in the circuit shown in Fig. Q.12.

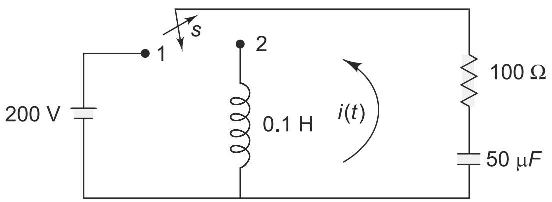


Fig. Q.11

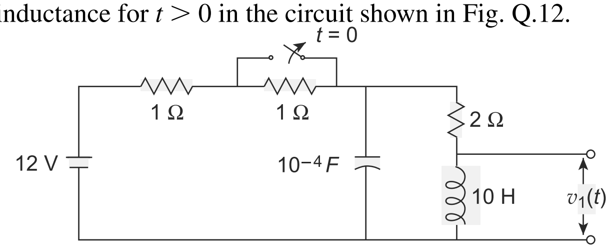


Fig. Q.12

Frequently Asked Questions linked to LO 2

☆☆★ **11-2.1** Find the time constant of RL circuit having $R = 10\ \Omega$ and $L = 0.1$ mH. **[AU May/June 2013]**

☆★★ **11-2.2** A series RL circuit with $R = 30\Omega$ and $L = 15H$ has a constant voltage $V = 60\ V$ applied at $t = 0$ as shown in Fig Q.2 Determine the current i, the voltage across resistor and the voltage across the inductor. **[AU May/June 2014]**

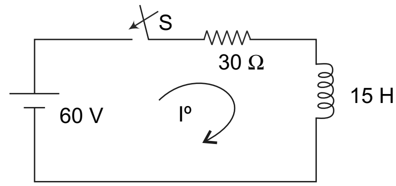


Fig. Q. 2

☆☆★ **11-2.3** What is the time constant of an RL circuit with $R = 10$ ohms and $L = 20$ mH? **[AU May/June 2014]**

☆☆★ **11-2.4** Derive the transient response of a series R-L circuit with dc input. Sketch the variation of current and of the voltage across the inductor. **[AU Nov./Dec. 2012]**

☆☆★ **11-2.5** Solve for i and V as functions of time in the circuit shown in Fig. Q.5, when the switch is closed at time $t = 0$. **[AU Nov./Dec. 2012]**

☆☆★ **11-2.6** Define the term 'Time constant' of a circuit, in general. **[AU Nov./Dec. 2012]**

☆☆★ **11-2.7** In the circuit shown in Fig. Q.7, find the expression for the transient current. The initial current is as shown in the figure. **[AU April/May 2011]**

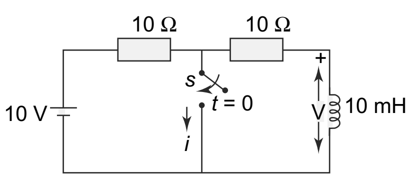


Fig. Q.5

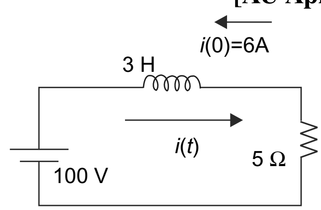


Fig. Q.7

★★★**11-2.8** In the network shown in Fig. Q.8, the switch is closed at t = 0. Find the value of current in each loop. **[JNTU Nov. 2012]**

☆★★**11-2.9** In the circuit of Fig. Q.9 the switch is closed at $t = 0$. Determine the mesh currents $i_1(t)$ and $i_2(t)$. **[AU May/June 2014]**

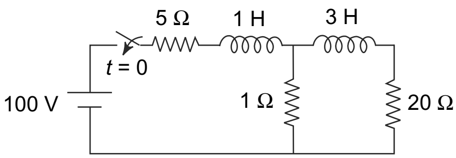


Fig. Q.8

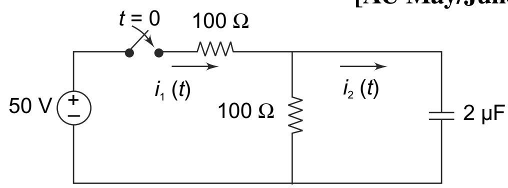


Fig. Q.9

☆★★**11-2.10** For the circuit shown in Fig. Q.10 the switch "*S*" is at position "*I*" and the steady-state condition is reached. The switch is moved to a position "2" at $t = 0$. Find the current $i(t)$ in both the cases, i.e., with switch at the position *I* and switch at the position 2. **[GTU Dec. 2010]**

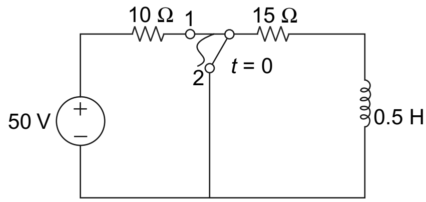


Fig. Q.10

☆☆★**11-2.11** Explain how to determine the initial conditions in an *RL* network and the current $i(t)$ based on these conditions. **[GTU May 2011]**

☆☆★**11-2.12** Using Laplace transformation technique, find current in each loop at $t = 0^+$ following switching at $t = 0$ of the switch *K* as shown in Fig. Q.12. Assume the network previously de-energized. **[JNTU Nov. 2012]**

☆★★**11-2.13** In the network shown in Fig. Q.13, find the current through the inductor for all values of '*t*'. **[JNTU Nov. 2012]**

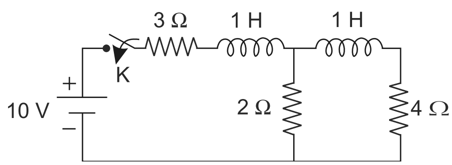


Fig. Q.12

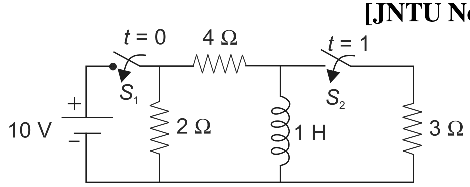


Fig. Q. 13

☆☆★**11-2.14** Find $i(t)$ in the circuit of fig. Q.14 when the switch is moved from the position 1 to 2. Initially, it was under a steady state. **[PU 2010]**

☆★★**11-2.15** In the circuit shown in Fig. Q.15 the switch is closed at $t = 0$, here steady state is reached before $t = 0$, determine current through the inductor of 3 H. **[PU 2010]**

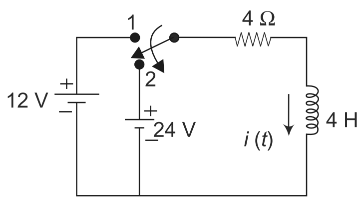


Fig. Q.14

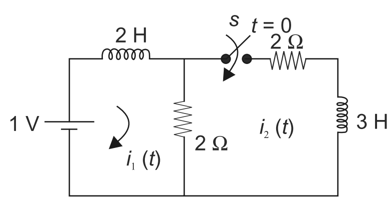


Fig. Q.15

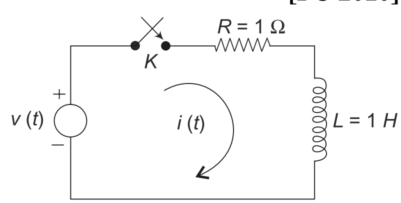


Fig. Q.16

☆★★**11-2.16** In the network shown in Fig. Q.16, the voltage source follows the law $v(t) = Ve^{-as}$, where α is a constant. The switch is closed at $t = 0$. **[RGTU Dec. 2013]**

a) Solve for the current assuming that $\alpha = \dfrac{R}{L}$, **b)** Solve for the current when $\alpha = \dfrac{R}{L}$

☆☆★ **11-2.17** Derive the equation for decay of current in *R-L* circuits. Discuss the role of time constant. **[RGTU June 2014]**

☆☆★ **11-2.18** Derive the step responses of *RL* and *RC* circuit. Compare their performances. **[AU May/June 2013]**

☆☆★ **11-2.19** What is the time constant for *RL* and *RC* circuit? **[AU May/June 2014]**

☆☆★ **11-2.20** Obtain the response $V_c(t)$ and $i_L(t)$ for the source-free *RC* and *RL* circuits respectively. Assume initial voltage V_o and initial current I_o respectively. **[GTU Dec. 2010]**

☆☆★ **11-2.21** Define the time constant of *RL* and *RC* networks and explain the significance of the time constant. **[GTU May 2011]**

☆★★ **11-2.22** Write down voltage and current relationships in resistor, inductor, and capacitor. Also mention the initial and final condition for *R, L* and *C* components in the different cases. **[GTU Dec. 2012]**

★★★ **11-2.23** In the figure Q.23, the initial voltage in the capacitor is 1 V as the polarity shown. Find the voltage appearing across with application of the step voltage. **[JNTU Nov. 2012]**

☆★★ **11-2.24** Find $U_C(t)$, $i_1(t)$ for $t > 0$, if the switch is closed at $t = 0$ after being open for a long time. Refer Fig. Q.24. **[PU 2010]**

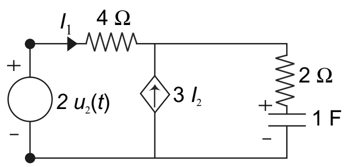


Fig. Q.23

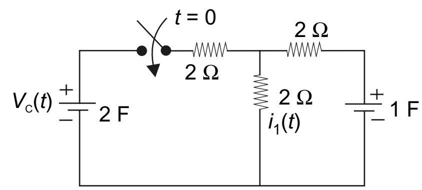


Fig. Q.24

☆☆★ **11-2.25** What are initial conditions in a network? **[PU 2010]**

☆☆★ **11-2.26** For the network shown in Fig. Q.26 the switch *K* is closed at $t = 0$, with the capacitor uncharged. Find the values of i, di/dt, d^2i/dt^2 at $t = 0^+$. **[PU 2012]**

☆☆★ **11-2.27** Discuss the initial conditions in a network. Outline the procedure for evaluating the initial conditions in network problems. **[RGTU Dec. 2013]**

☆☆★ **11-2.28** For the circuit shown in the following Fig. Q.28, find the current equation when the switch *S* is opened at $t = 0^+$. **[RGTU June 2014]**

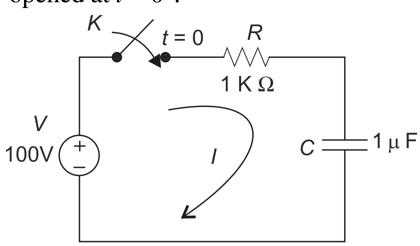


Fig. Q.26

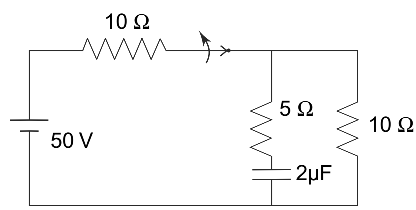


Fig. Q.28

☆★★ **11-2.29** An *RLC* series circuit has $R = 10\ \Omega$, $L = 2$ H. What value of capacitance will make the circuit critically damped? **[AU May/June 2013]**

☆☆★ **11-2.31** The circuit shown in Fig. Q.31 consists of resistance, inductance, and capacitance in series with 100 V dc when the switch is closed at $t = 0$. Find the current transient. **[AU May/June 2013] [BPUT 2008]**

☆☆★ **11-2.32** In the circuit shown in Fig. Q.32, the capacitor *C* has an initial voltage $V_C = 10$ V and at the same instant when current through the inductor *L* is zero, the switch *K* is closed at time $t = 0$. Find out the expression for the voltage $V(t)$ across the inductor *L*. **[BPUT 2008]**

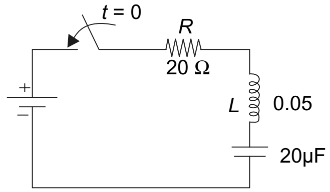


Fig. Q.31

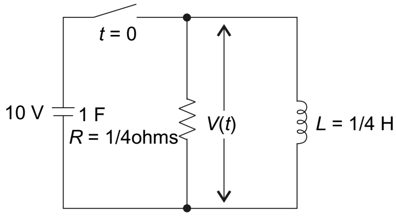


Fig. Q.32

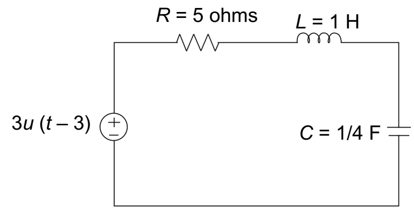


Fig. Q.33

☆★★**11-2.33** A step voltage $3u(t-3)$ is applied to a series *RLC* circuit comprising a resistor $R = 5\Omega$, inductor $L = 1$ H, and capacitor $C = \frac{1}{4}$ F. Find the expression for the current $i(t)$ in the circuit. (Fig. Q.33) **[BPUT 2008]**

☆☆★**11-2.34** How can volt classify that the given circuit is of first order or second order? Obtain second-order circuit models for series *RLC* and parallel *RLC* circuit in time domain and in "*s*" domain. **[GTU Dec. 2010]**

☆★★**11-2.35** In the circuit shown in Fig. Q.35, S_1 is closed at $t = 0$, and S_2 is opened at $t = 4$ msec. Determine $i(t)$ for $t > 0$, Assume the inductor is initially de-energized. **[PUT 2011-12]**

☆☆★**11-2.36** In the circuit shown in Fig. Q.36, the switch *S* is closed at $t = 0$. Determine the initial value of i, di/dt, d^2i/dt^2. **[PU 2010]**

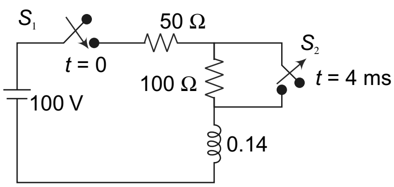


Fig. Q.35

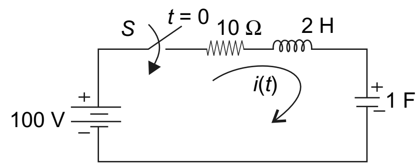


Fig. Q.36

★★★**11-2.37** In the circuit shown in Fig. Q.37 switch *S* is closed at $t = 0$. Determine the initial value of i, di/dt, d^2i/dt^2. **[PU 2012]**

★★★**11-2.38** In the network shown in Fig. Q.38, the switch *K* is closed at $t = 0$, with zero capacitor voltage and zero inductor current. **[RGTU Dec. 2013]**

Solve for (a) v_1 and v_2 at $t = 0$, (b) v_1 and v_2 at $t = \infty$ (c) $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0$

☆★★**11-2.39** Find the current $i(t)$ in a series *RLC* circuit comprising $R = 3$ ohms, $L = 1$ H, and $C = 0.5$ F, when a ramp voltage of 10 volts is applied. Assume initial condition as zero. **[RGTU Dec. 2013]**

☆★★**11-2.40** Figure Q.40 shows a parallel *RLC* circuit. The switch is suddenly opened at $t = 0$. Assuming no, charge on the capacitor and no current in the inductor before switching, find the voltage across the switch. [RTU **Feb. 2011]**

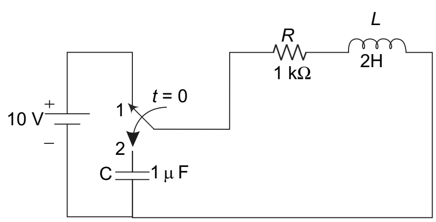


Fig. Q.37

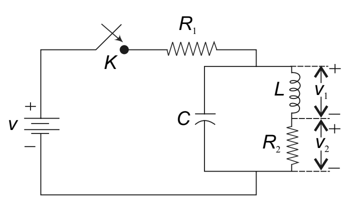


Fig. Q.38

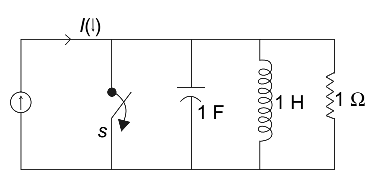


Fig. Q.40

11.5 SINUSOIDAL RESPONSE OF AN *R-L* CIRCUIT

LO 3 *Analyse sinusoidal response for R-L circuits, R-C circuits, and R-L-C circuits*

Consider a circuit consisting of resistance and inductance as shown in Fig. 11.16. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the series R-L circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

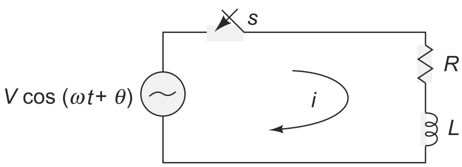


Fig. 11.16

$$V\cos(\omega t+\theta) = Ri + L\frac{di}{dt} \tag{11.17}$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t+\theta)$$

The corresponding characteristic equation is

$$\left(D+\frac{R}{L}\right)i = \frac{V}{L}\cos(\omega t+\theta) \tag{11.18}$$

For the above equation, the solution consists of two parts, viz., complementary function and particular integral.

The complementary function of the solution i is

$$i_c = ce^{-t(R/L)} \tag{11.19}$$

The particular solution can be obtained by using undetermined coefficients.

By assuming $i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta)$ (11.20)

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \tag{11.21}$$

Substituting Eqs (11.20) and (11.21) in Eq. (11.18), we have

$$\{-A\omega\sin(\omega t+\theta)+B\omega\cos(\omega t+\theta)\} + \frac{R}{L}\{A\cos(\omega t+\theta)$$

$$+ B\sin(\omega t+\theta)\} = \frac{V}{L}\cos(\omega t+\theta)$$

or $$\left(-A\omega+\frac{BR}{L}\right)\sin(\omega t+\theta) + \left(B\omega+\frac{AR}{L}\right)\cos(\omega t+\theta) = \frac{V}{L}\cos(\omega t+\theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$

$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

$$A = V\frac{R}{R^2+(\omega L)^2}$$

$$B = V\frac{\omega L}{R^2+(\omega L)^2}$$

Substituting the values of A and B in Eq. (11.20), we get

$$i_p = V\frac{R}{R^2+(\omega L)^2}\cos(\omega t+\theta) + V\frac{\omega L}{R^2+(\omega L)^2}\sin(\omega t+\theta) \tag{11.22}$$

Putting $\quad M\cos\phi = \dfrac{VR}{R^2+(\omega L)^2}$

and $\quad M\sin\phi = V\dfrac{\omega L}{R^2+(\omega L)^2}$

To find M and ϕ, we divide one equation by the other.

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{R^2+(\omega L)^2}$$

$$\text{or} \quad M = \frac{V}{\sqrt{R^2+(\omega L)^2}}$$

$\therefore$ the particular current becomes

$$i_p = \frac{V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\omega t+\theta-\tan^{-1}\frac{\omega L}{R}\right) \tag{11.23}$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\omega t+\theta-\tan^{-1}\frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents, at $t = 0$, $i = 0$

$$\therefore \quad c = -\frac{V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\theta-\tan^{-1}\frac{\omega L}{R}\right)$$

The complete solution for the current is

$$i = e^{-(R/L)t}\left[\frac{-V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\theta-\tan^{-1}\frac{\omega L}{R}\right)\right] + \frac{V}{\sqrt{R^2+(\omega L)^2}}\cos\left(\omega t+\theta-\tan^{-1}\frac{\omega L}{R}\right)$$

EXAMPLE 11.4

In the circuit shown in Fig. 11.17, determine the complete solution for the current, when switch S is closed at t = 0. Applied voltage is v(t) = 100 cos (10^3t + π/2). Resistance R = 20 Ω and inductance L = 0.1 H.

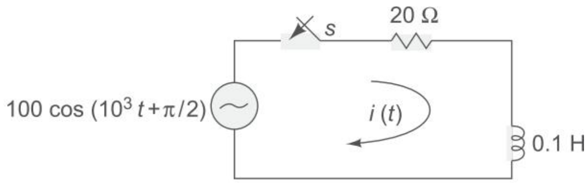


Fig. 11.17

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1\frac{di}{dt} = 100\ \cos\,(10^3 t + \pi/2)$$

$$\frac{di}{dt} + 200i = 1000\ \cos\,(1000t + \pi/2)$$

$$(D + 200)i = 1000 \cos\,(1000t + \pi/2)$$

The complementary function $i_c = ce^{-200t}$

By assuming particular integral as

$$i_p = A \cos\,(\omega t + \theta) + B \sin\,(\omega t + \theta)$$

we get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

where $\omega = 1000$ rad/s, $V = 100$ V

$$\theta = \pi/2$$

$$L = 0.1 \text{ H}, R = 20\ \Omega$$

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{(20)^2 + (1000 \times 0.1)^2}} \cos\left(1000t + \frac{\pi}{2} - \tan^{-1}\frac{100}{20}\right)$$

$$= \frac{100}{101.9} \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

$$= 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

The complete solution is

$$i = ce^{-200t} + 0.98\ \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

At $t = 0$, the current flowing through the circuit is zero, i.e., $i = 0$

$$\therefore\quad c = -0.98 \cos\left(\frac{\pi}{2} - 78.6°\right)$$

$\therefore$ the complete solution is

$$i = \left[-0.98\ \cos\left(\frac{\pi}{2} - 78.6°\right)\right] e^{-200t} + 0.98\ \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

11.6 SINUSOIDAL RESPONSE OF AN *R-C* CIRCUIT

LO 3

Consider a circuit consisting of resistance and capacitance in series as shown in Fig. 11.18. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the R-C circuit, where V is the amplitude of the wave and θ is the phase angle. Applying Kirchhoff's voltage law to the circuit results in the following differential equation.

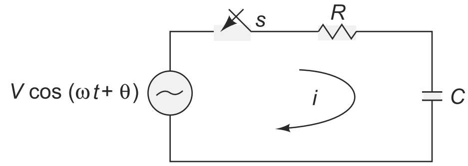


Fig. 11.18

$$V \cos(\omega t + \theta) = Ri + \frac{1}{C}\int i\,dt \tag{11.24}$$

$$R\frac{di}{dt} + \frac{i}{C} = -V\omega \sin(\omega t + \theta)$$

$$\left(D + \frac{1}{RC}\right)i = -\frac{V\omega}{R}\sin(\omega t + \theta) \tag{11.25}$$

The complementary function $i_C = ce^{-t/RC}$ (11.26)

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \tag{11.27}$$

$$i_P' = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \tag{11.28}$$

Substituting Eqs (11.27) and (11.28) in Eq. (11.25), we get

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\} + \frac{1}{RC}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\}$$

$$= -\frac{V\omega}{R}\sin(\omega t + \theta)$$

Comparing both sides, $-A\omega + \dfrac{B}{RC} = -\dfrac{V\omega}{R}$

$$B\omega + \frac{A}{RC} = 0$$

From which,

$$A = \frac{VR}{R^2 + \left(\frac{1}{\omega c}\right)^2}$$

and
$$B = \frac{-V}{\omega C\left[R^2 + \left(\frac{1}{\omega c}\right)^2\right]}$$

Substituting the values of A and B in Eq. (11.27), we have

$$i_p = \frac{VR}{R^2 + \left(\frac{1}{\omega c}\right)^2}\cos(\omega t + \theta) + \frac{-V}{\omega C\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}\sin(\omega t + \theta)$$

Putting $$M\cos\phi = \frac{VR}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

and $$M\sin\phi = \frac{V}{\omega C\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

To find M and ϕ, we divide one equation by the other.

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{1}{\omega CR}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

$$\therefore \quad M = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \tag{11.29}$$

The complete solution for the current $i = i_c + i_p$

$$\therefore \quad i = ce^{-(t/RC)} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \tag{11.30}$$

Since the capacitor does not allow sudden changes in voltages at $t = 0$, $i = \frac{V}{R}\cos\theta$

$$\therefore \quad \frac{V}{R}\cos\theta = c + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$c = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$i = e^{-(t/RC)}\left[\frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\cos\left(\theta+\tan^{-1}\frac{1}{\omega CR}\right)\right]$$

$$+\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t+\theta+\tan^{-1}\frac{1}{\omega CR}\right) \tag{11.31}$$

EXAMPLE 11.5

In the circuit shown in Fig. 11.19, determine the complete solution for the current when the switch S is closed at t = 0. Applied voltage is $v(t) = 50\cos\left(10^2 t + \frac{\pi}{4}\right)$. *Resistance* $R = 10\,\Omega$ *and capacitance* $C = 1\,\mu F$.

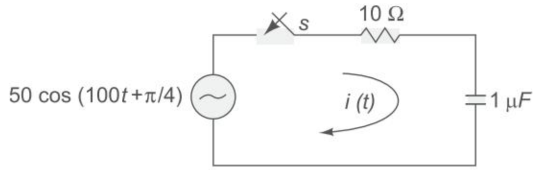


Fig. 11.19

Solution By applying Kirchhoff's voltage law to the circuit, we have

$$10i + \frac{1}{1\times 10^{-6}}\int i\,dt = 50\cos\left(100t+\frac{\pi}{4}\right)$$

$$10\frac{di}{dt} + \frac{i}{1\times 10^{-6}} = -5(10)^3\sin\left(100t+\frac{\pi}{4}\right)$$

$$\frac{di}{dt} + \frac{i}{10^{-5}} = -500\sin\left(100t+\frac{\pi}{4}\right)$$

$$\left(D+\frac{1}{10^{-5}}\right)i = -500\sin\left(100t+\frac{\pi}{4}\right)$$

The complementary function is $i_C = ce^{-t/10^{-5}}$. By assuming the particular integral as $i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta)$,

we get $$i_p = \frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t+\theta+\tan^{-1}\frac{1}{\omega CR}\right)$$

where $\omega = 100$ rad/s $\quad \theta = \pi/4$

$C = 1\,\mu F \qquad R = 10\,\Omega$

Substituting the values in the above equation, we have

$$i_p = \frac{50}{\sqrt{(10)^2 + \left(\dfrac{1}{100\times10^{-6}}\right)^2}} \cos\left(\omega t + \frac{\pi}{4} + \tan^{-1}\frac{1}{100\times10^{-6}\times10}\right)$$

$$i_p = 4.99\times10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94°\right)$$

At $t = 0$, the current flowing through the circuit is

$$\frac{V}{R}\cos\theta = \frac{50}{10}\cos\pi/4 = 3.53\,\text{A}$$

$$i = \frac{V}{R}\cos\theta = 3.53\,\text{A}$$

$$\therefore \quad i = ce^{-t/10^{-5}} + 4.99\times10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94°\right)$$

At $t = 0$,

$$c = 3.53 - 4.99\times10^{-3}\cos\left(\frac{\pi}{4} + 89.94°\right)$$

Hence, the complete solution is

$$i = \left[3.53 - 4.99\times10^{-3}\cos\left(\frac{\pi}{4} + 89.94°\right)\right]e^{-(t/10^{-5})}$$

$$+ 4.99\times10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94°\right)$$

11.7 SINUSOIDAL RESPONSE OF AN *R-L-C* CIRCUIT

LO 3

Consider a circuit consisting of resistance, inductance, and capacitance in series as shown in Fig. 11.20. Switch S is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V\cos(\omega t + \theta)$ is applied to the RLC series circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

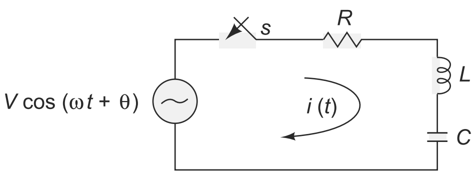


Fig. 11.20

$$V\cos(\omega t + \theta) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{11.32}$$

Differentiating the above equation, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + i/C = -V\omega\sin(\omega t + \theta)$$

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = -\frac{V\omega}{L}\sin(\omega t + \theta) \tag{11.33}$$

The particular solution can be obtained by using undetermined coefficients. By assuming

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \tag{11.34}$$

$$i_p' = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \tag{11.35}$$

$$i_p'' = -A\omega^2\cos(\omega t + \theta) - B\omega^2\sin(\omega t + \theta) \tag{11.36}$$

Substituting i_p, i_p' and i_p'' in Eq. (11.33), we have

$$\left\{-A\omega^2\cos(\omega t+\theta) - B\omega^2\sin(\omega t+\theta)\right\} + \frac{R}{L}\left\{-A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta)\right\}$$
$$+\frac{1}{LC}\left\{A\cos(\omega t+\theta) + B\sin(\omega t+\theta)\right\} = -\frac{V\omega}{L}\sin(\omega t+\theta) \tag{11.37}$$

Comparing both sides, we have

Sine coefficients

$$-B\omega^2 - A\frac{\omega R}{L} + \frac{B}{LC} = -\frac{V\omega}{L}$$

$$A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = \frac{V\omega}{L} \tag{11.38}$$

Cosine coefficients

$$-A\omega^2 + B\frac{\omega R}{L} + \frac{A}{LC} = 0$$

$$A\left(\omega^2 - \frac{1}{LC}\right) - B\left(\frac{\omega R}{L}\right) = 0 \tag{11.39}$$

Solving Eqs (11.38) and (11.39), we get

$$A = \frac{V \times \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

$$B = \frac{\left(\omega^2 - \dfrac{1}{LC}\right)V\omega}{L\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

Substituting the values of A and B in Eq. (11.34), we get

$$i_p = \frac{V\dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}\cos(\omega t + \theta)$$

$$+\frac{\left(\omega^2-\frac{1}{LC}\right)V\omega}{L\left[\left(\frac{\omega R}{L}\right)^2-\left(\omega^2-\frac{1}{LC}\right)^2\right]}\sin(\omega t+\theta) \tag{11.40}$$

Putting $$M\cos\phi=\frac{V\frac{\omega^2 R}{L^2}}{\left(\frac{\omega R}{L}\right)^2-\left(\omega^2-\frac{1}{LC}\right)^2}$$

and $$M\sin\phi=\frac{V\left(\omega^2-\frac{1}{LC}\right)\omega}{L\left[\left(\frac{\omega R}{L}\right)^2-\left(\omega^2-\frac{1}{LC}\right)^2\right]}$$

To find M and ϕ, we divide one equation by the other.

or $$\frac{M\sin\phi}{M\cos\phi}=\tan\phi=\frac{\left(\omega L-\frac{1}{\omega C}\right)}{R}$$

$$\phi=\tan^{-1}\left[\left(\omega L-\frac{1}{\omega C}\right)\Big/R\right]$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi+M^2\sin^2\phi=\frac{V^2}{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}$$

$$\therefore\quad M=\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}$$

The particular current becomes

$$i_p=\frac{V}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\cos\left[\omega t+\theta+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right] \tag{11.41}$$

The complementary function is similar to that of a dc series RLC circuit.
To find out the complementary function, we have the characteristic equation

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)=0 \tag{11.42}$$

The roots of Eq. (11.42), are

$$D_1, D_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

$$\therefore \quad D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

K_2 becomes positive, when $(R/2L)^2 > 1/LC$

The roots are real and unequal, which gives an overdamped response. Then Eq. (11.42) becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]i = 0$$

The complementary function for the above equation is

$$i_c = c_1 e^{(K_1+K_2)t} + c_2 e^{(K_1-K_2)t}$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$i_c = c_1 e^{(K_1+K_2)t} + c_2 e^{(K_1-K_2)t} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes negative, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

Then the roots are complex conjugate, which gives an underdamped response. Equation (11.42) becomes

$$[D - (K_1 + jK_2)]\,[D - (K_1 - jK_2)]i = 0$$

The solution for the above equation is

$$i_c = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

K_2 becomes zero, when $\left(\frac{R}{2L}\right)^2 = 1/LC$

Then the roots are equal which gives critically damped response. Then, Eq. (11.42) becomes $(D - K_1)(D - K_1)i = 0$.

The complementary function for the above equation is

$$i_c = e^{K_1 t}(c_1 + c_2 t)$$

Therefore, the complete solution is $i = i_c + i_p$

$$\therefore \quad i = e^{K_1 t}[c_1 + c_2 t] + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

EXAMPLE 11.6

In the circuit shown in Fig. 11.21, determine the complete solution for the current, when the switch is closed at t = 0. Applied voltage is $v(t) = 400\cos\left(500t + \frac{\pi}{4}\right)$. *Resistance R = 15 Ω, inductance L = 0.2 H, and capacitance C = 3 μF.*

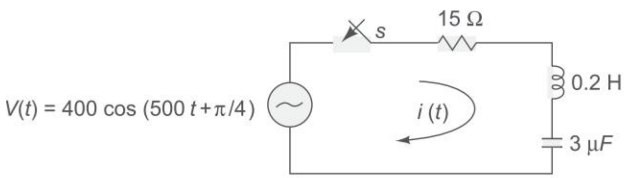


Fig. 11.21

Solution By applying Kirchhoff's voltage law to the circuit,

$$15i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3\times 10^{-6}}\int i(t)dt = 400\cos\left(500t + \frac{\pi}{4}\right)$$

Differentiating the above equation once, we get

$$15\frac{di}{dt} + 0.2\frac{d^2 i}{dt} + \frac{i}{3\times 10^{-6}} = -2\times 10^5 \sin\left(500t + \frac{\pi}{4}\right)$$

$$(D^2 + 75D + 16.7\times 10^5)i = \frac{-2\times 10^5}{0.2}\sin\left(500t + \frac{\pi}{4}\right)$$

The roots of the characteristic equation are

$$D_1 = -37.5 + j1290 \text{ and } D_2 = -37.5 - j1290$$

The complementary current

$$i_c = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t)$$

Particular solution is

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

$$\therefore \quad i_p = 0.71\cos\left(500t + \frac{\pi}{4} + 88.5°\right)$$

The complete solution is

$$i = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t) + 0.71 \cos (500t + 45° + 88.5°)$$

At $t = 0$, $i_0 = 0$

$\therefore \quad c_1 = -0.71 \cos (133.5°) = +0.49$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-37.5t}(-1290c_1 \sin 1290t + c_2 1290 \cos 1290t)$$

$$-37.5e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t)$$

$$-0.71 \times 500 \sin (500t + 45° + 88.5°)$$

At $t = 0, \dfrac{di}{dt} = 1414$

$\therefore \quad 1414 = 1290c_2 - 37.5 \times 0.49 - 0.71 \times 500 \sin (133.5°)$

$1414 = 1290c_2 - 18.38 - 257.5$

$\therefore \quad c_2 = 1.31$

The complete solution is

$$i = e^{-37.5t}(0.49 \cos 1290t + 1.31 \sin 1290t) + 0.71 \cos (500t + 133.5°)$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★ **11-3.1** In the network shown in Fig. Q.1, find $i_2(t)$ for $t > 0$, if $i_1(0) = 5$ A.

☆★★ **11-3.2** The switch in Fig. Q.2 was open for a long time but closed at $t = 0$. If $i(0) = 10$A, find $i(t)$ for $t > 0$ by hand and also PSpice.

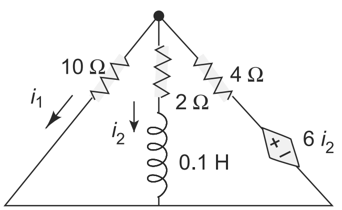


Fig. Q.1

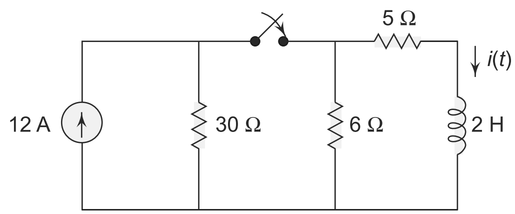


Fig. Q.2

☆☆★ **11-3.3** Using PSpice, find $V(t)$ for $t < 0$ and $t > 0$ in the circuit shown in Fig. Q.3.

☆☆★ **11-3.4** In the network shown in Fig. Q.4, the switch is closed at $t = 0$ and there is no initial charge on either of the capacitances. Find the resulting current $i(t)$.

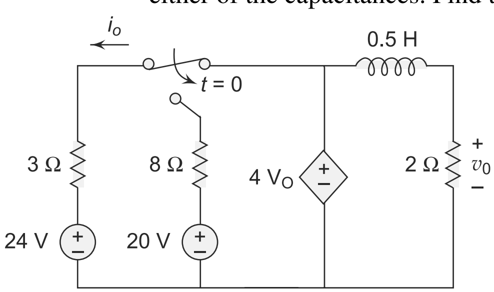


Fig. Q.3

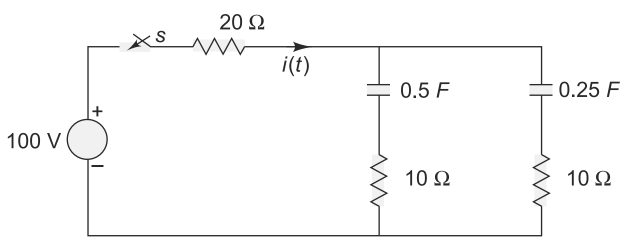


Fig. Q.4

☆★★**11-3.5** In the *RC* circuit shown in Fig. Q.5, the capacitor has an initial charge $q_0 = 25 \times 10^{-6}$ C with polarity as shown. A sinusoidal voltage $v = 100 \sin (200t + \phi)$ is applied to the circuit at a time corresponding to $\phi = 30°$. Determine the expression for the current $i(t)$.

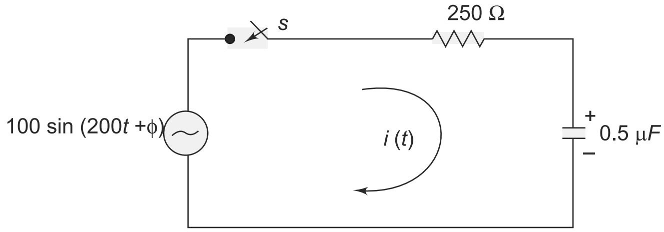


Fig. Q.5

★★★**11-3.6** For the circuit shown in Fig. Q.6, find v_5, if the switch is opened for $t > 0$.

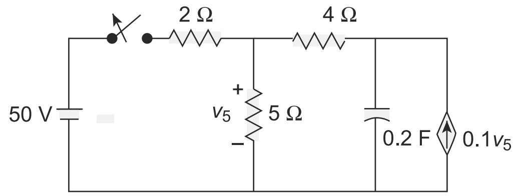


Fig. Q.6

☆☆★**11-3.7** Determine the response $V(t)$ using PSpice for the circuit shown in Fig. Q.7.

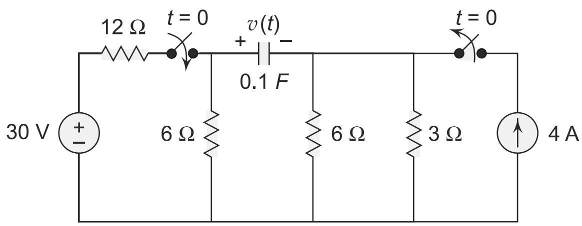


Fig. Q.7

☆☆★**11-3.8** In the network shown in Fig. Q.8, the values of *R*, *L*, and *C* are $\frac{1}{4}$ ohm, $\frac{1}{4}$ H and 1 *F* respectively. If $I = 5$ amp and the switch *K* is opened at $t = 0$. Obtain an expression for voltage of the node *a* for $t \geq 0$.

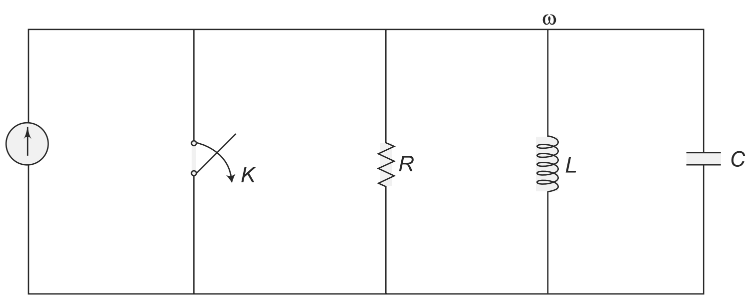


Fig. Q.8

Frequently Asked Questions linked to LO 3

☆☆★11-3.1 An *RL* series circuit is excited by a sinusoidal source $e(t) = 10 \sin 100t$ volts, by closing the switch at $t = 0$. Take $R = 10\ \Omega$ and $L = 0.1$ H. Determine the current $i(t)$ flowing through the *RL* circuit. **[AU May/June 2014]**

☆★★11-3.2 Obtain the current at $t > 0$, if ac voltage V is applied when the switch K is moved to 2 from 1 at $t = 0$. Assume a steady-state current of 1 A in LR circuit when the switch was at the position 1. **[RTU Feb. 2011]**

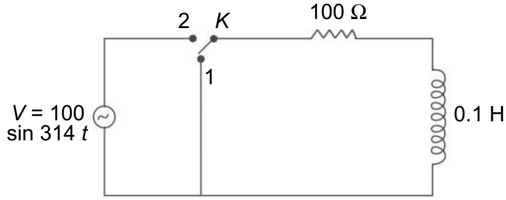


Fig. Q.2

☆★★11-3.3 Derive the expression for the complete solution of the current response of an *RC* series circuit with an excitation of $V \cos(\omega t + \phi)$. Briefly explain the significance of phase angle in the solution. **[AU Nov./Dec. 2012]**

☆★★11-3.4 Derive the expression for the complete solution of the current response of an *RC* series circuit with an excitation of $V \cos(\omega t + \phi)$. Briefly explain the significance of phase angle in the solution. **[AU April/May 2011]**

☆☆★11-3.5 Write down the condition for critically damped response of a series *RLC* circuit excited by a sinusoidal ac source. **[AU Nov./Dec. 2012]**

★★★11-3.6 In the network shown in Fig. Q.6, the switch k is closed at $t = 0$, connecting voltage Vo sin ωt to the parallel *RL*- *RC* circuit. Find (a) di_1/dt and (b) di_2/dt at $t = 0$. **[JNTU Nov. 2012]**

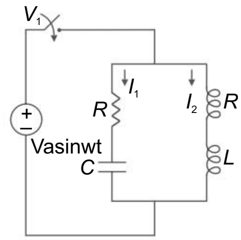


Fig. Q.6

☆★★11-3.7 Plot the response of *RLC* circuit to sinusoidal input. **[RGTU June 2014]**

☆★★11-3.8 Derive an expression for the current response of an *RLC* series circuit with sinusoidal excitation. Assume that the circuit is working in critical damping condition. **[AU May/June 2013]**

Additional Solved Problems

PROBLEM 11.1

A series circuit shown in Fig. 11.22 comprising of a resistance of 10 Ω and an inductance of 0.5 H, is connected to a 100 V source at t = 0. Determine the complete expression for the current i(t).

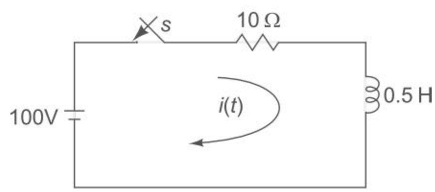


Fig. 11.22

Solution The current equation for the circuit is shown in Fig. 11.22 by using Kirchhoff's voltage law when the switch is closed.

$$10i(t) + 0.5\frac{di(t)}{dt} = 100$$

$$\frac{di(t)}{dt} + 20\,i(t) = 200$$

The general solution for a linear differential equation is

$$i(t) = Ce^{-pt} + e^{-pt}\int Ke^{pt}\,dt$$

where $p = 20$, $K = 200$

$$\therefore \quad i(t) = Ce^{-20t} + e^{-20t}\int 200\, e^{20t}\, dt$$

$$i(t) = Ce^{-20t} + 10$$

At $t = 0$, the switch S is closed.

Since the inductor never allows sudden changes in currents, At $t = 0^+$, the current in the circuit is zero.

$\therefore$ At $t = 0^+$, $i = 0$

$\therefore$ $0 = C + 10$

$\therefore$ $C = -10$

Substituting the value of C in the current equation, we have

$$i = 10(1 - e^{-20t})\text{ A}$$

PROBLEM 11.2

A series RLC circuit shown in Fig 11.23, comprising $R = 10\,\Omega$, $L = 0.5\,H$, and $C = 1\,\mu F$, is excited by a constant voltage source of 100 v. Obtain the expression for the current. Assume that the circuit is relaxed initially.

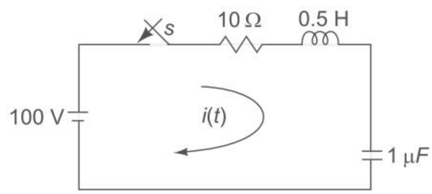


Fig. 11.23

Solution At $t = 0$, the switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation:

$$100 = 10i + 0.5\frac{di}{dt} + \frac{1}{1\times 10^{-6}}\int i dt$$

Differentiating the above equation, we get

$$0.5\frac{d^2 i}{dt^2} + 10\frac{di}{dt} + \frac{1}{1\times 10^{-6}} i = 0$$

$$\frac{d^2 i}{dt^2} + 20\frac{di}{dt} + 2\times 10^6 i = 0$$

$$(D^2 + 20D + 2\times 10^6)i = 0$$

$$D_1, D_2 = \frac{-20}{2} \pm \sqrt{\left(\frac{20}{2}\right)^2 - 2\times 10^6}$$

$$= -10 \pm \sqrt{(10)^2 - 2\times 10^6}$$

$$D_1 = -10 + j1414.2$$

$$D_2 = -10 - j1414.2$$

Therefore, the current

$$i = e^{k_1 t}\left[c_1 \cos K_2 t + c_2 \sin K_2 t\right]$$

$$i = e^{-10t}\left[c_1 \cos 1414.2t + c_2 \sin 1414.2t\right]\text{A}$$

At $t = 0$, the current flowing through the circuit is zero.

$$i = 0 = (1)[c_1 \cos 0 + c_2 \sin 0]$$
$$c_1 = 0$$
$$\therefore \quad i = e^{-10t} c_2 \sin 1414.2t \text{ A}$$

Differentiating, we have

$$\frac{di}{dt} = c_2 \left[e^{-10t} 1414.2 \cos 1414.2t + e^{-10t}(-10) \sin 1414.2t\right]$$

At $t = 0$, the voltage across the inductor is 100 V.

$$L\frac{di}{dt} = 100$$

$$\text{or} \quad \frac{di}{dt} = \frac{100}{0.5} = 200$$

$$\text{At} \quad t = 0 \quad \frac{di}{dt} = 200 = c_2 \times 1414.2 \cos 0$$

$$c_2 = \frac{200}{1414.2} = 0.1414$$

The current equation is

$$i = e^{-10t}(0.1414 \sin 1414.2t)\text{A}$$

PROBLEM 11.3

In the network shown in Fig. 11.24, the switch is moved from the position 1 to the position 2 at t = 0. The switch is in position 1 for a long time. Determine the current expression i(t).

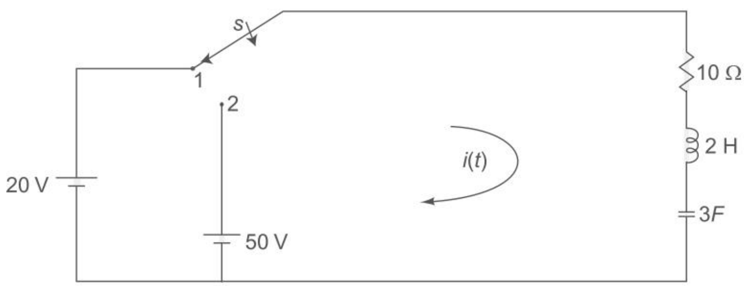


Fig. 11.24

Solution At $t = 0$, the switch S is moved to the position 2 and the 50 V source is applied to the circuit and results in the following differential Eq. (11.43).

$$50 = 10i + 2\frac{di}{dt} + \frac{1}{3}\int i dt + \frac{q(0)}{3} \tag{11.43}$$

Differentiating Eq. (11.43), we get

$$2\frac{d^2i}{dt}+10\frac{di}{dt}+\frac{1}{3}i=0$$

$$\frac{d^2i}{dt}+5\frac{di}{dt}+\frac{1}{6}i=0$$

$$\left(D^2+5D+\frac{1}{6}\right)i=0$$

$$D_1,D_2=\frac{-5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-\left(\frac{1}{6}\right)}$$

$$=-2.5\pm\sqrt{6.25-0.167}$$

$$=-2.5\pm 2.47$$

$$[D-(-0.03)][D-(-4.97)]i=0$$

The solution for the above equation is

$$i=c_1e^{-0.03t}+c_2e^{-4.97t}$$

At $t = 0$, $i = 0$

$\therefore \quad c_1 + c_2 = 0$

$$\frac{di}{dt}=-0.03c_1e^{-0.03t}-4.97c_2e^{-4.97t} \tag{11.44}$$

At $t = 0$, the voltage across the inductor is 50 V.

$$L\frac{di}{dt}=50$$

$$\frac{di}{dt}=25$$

At $t = 0$, Eq. (11.44) becomes

$$25=-0.03\,c_1-4.97\,c_2$$

$$\therefore \quad c_1=5.06, \quad c_2=-5.06$$

The current equation is

$$i=5.06\,e^{-0.03t}-5.06\,e^{-4.97t}$$

PROBLEM 11.4

In the network in Fig. 11.25, the switch is moved from the position 1 to the position 2 at $t = 0$. The switch is in the position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression i(t), when $\omega = 1000$ rad/s.

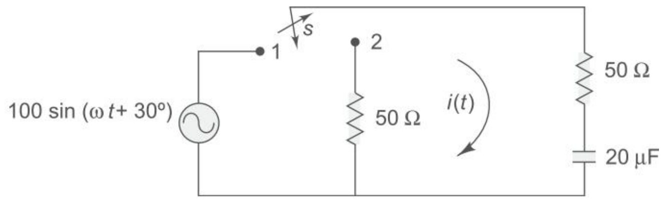


Fig. 11.25

Solution When the switch is at the position 2, by applying Kirchhoff's law, the differential equation is

$$100i + \frac{1}{20\times10^{-6}}\int_0^t i\,dt = 0$$

$$100i + \frac{1}{20\times10^{-6}}\int i\,dt + v(0) = 0$$

Where $v(0)$ = initial voltage across capacitor

$$100i + \frac{1}{20\times10^{-6}}\int i\,dt + \frac{7\times10^{-4}}{20\times10^{-6}} = 0 \tag{11.45}$$

Differentiating Eq. (11.45), we get

$$100\frac{di}{dt} + \frac{1}{20\times10^{-6}}i = 0$$

$$\left(D + \frac{1}{20\times10^{-4}}\right)i = 0$$

The transient current is

$$i = c\,e^{\frac{-1}{20\times10^{-4}}t}$$

$$i = c\,e^{-50t}$$

At $t = 0$, the switch is moved from the position 1 to the position 2. Hence, the current passing through the circuit is same as the steady-state current passing through the circuit, when the switch is in the position 1.

$$i = \frac{v}{z} = \frac{100\angle 30^\circ}{R - j/\omega c} = \frac{100\angle 30^\circ}{50 - j50}$$

$$i = \frac{100\angle 30^\circ}{70.71\angle -45^\circ}$$

$$i = 1.414\angle 75^\circ\,\text{A}$$

Therefore, the steady-state current passing through the circuit, when the switch is in the position 1 is

$$i = 1.414\sin(1000t + 75^\circ)\,\text{A}$$

At $t = 0;\ i = 1.414\sin 75^\circ = 1.365\,\text{A}$

Therefore, the current equation is $i = 1.365\,e^{-50t}$ A

PROBLEM 11.5

The switch in the circuit shown in Fig. 11.26 is closed at $t = 0$. Find $v_2(t)$ for all $t \geq 0$ by time domain method. Assume zero initial current in the inductance.

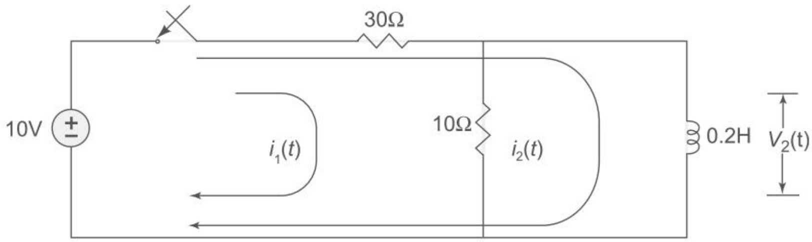


Fig. 11.26

Solution By applying KVL to the loop 1, we get

$$10 = 30[i_1(t) + i_2(t)] + 10\,i_1(t)$$

$$40\,i_1(t) + 30\,i_2(t) = 10 \tag{11.46}$$

By applying KVL to the loop 2, we get

$$30[i_1(t) + i_2(t)] + 0.2\frac{di_2(t)}{dt} = 10 \tag{11.47}$$

From Eq. (11.46),

$$i(t) = 0.25 - 0.75 i_2(t) \tag{11.48}$$

Substituting Eq. (11.48) into Eq. (11.47), we get

$$\frac{di_2(t)}{dt} - 37.5\,i_2(t) = 2.5 \tag{11.49}$$

$$(D - 37.5)\,i_2 = 2.5$$

$$i_2(t) = 0.066 + c\,e^{-37.5t}$$

At $t = 0$; $i_2(t) = 0$

$$c = -0.066$$

$\therefore$ $$i_2(t) = 0.066[1 - e^{-37.5t}] \text{ A}$$

$$v_2(t) = L\frac{di_2(t)}{dt}$$

$$= 0.2\frac{d}{dt}\left[0.066(1 - e^{-37.5t})\right]$$

$$v_2(t) = 0.495\,e^{-37.5t} \text{ V}$$

PROBLEM 11.6

The circuit shown in Fig. 11.27, consists of series RL elements with $R = 150\ \Omega$ and $L = 0.5$ H. The switch is closed when $\phi = 30°$. Determine the resultant current when voltage $V = 50 \cos(100t + \phi)$ is applied to the circuit at $\phi = 30°$.

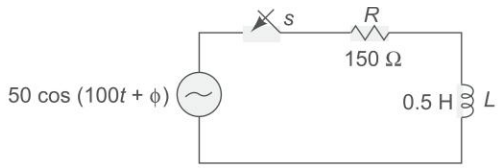


Fig. 11.27

Solution By using Kirchhoff's laws, the differential equation, when the switch is closed at $\phi = 30°$, is

$$150i + 0.5\frac{di}{dt} = 50\cos(100t + \phi)$$

$$0.5Di + 150i = 50\cos(100t + 30°)$$

$$(D + 300)i = 100\cos(100t + 30°)$$

The complementary current $i_c = ce^{-300t}$

To determine the particular current, first we assume a particular current.

$$i_p = A\cos(100t + 30°) + B\sin(100t + 30°)$$

Then $\quad i_p' = -100A\sin(100t + 30°) + 100B\cos(100t + 30°)$

Substituting i_p and i_p' in the differential equation and equating the coefficients, we get

$$-100A\sin(100t + 30°) + 100B\cos(100t + 30°) + 300A\cos(100t + 30°)$$
$$+ 300B\sin(100t + 30°) = 100\cos(100t + 30°)$$

$$-100A + 300B = 0$$

$$300A + 100B = 100$$

From the above equation, we get

$$A = 0.3 \text{ and } B = 0.1$$

The particular current is

$$i_p = 0.3\cos(100t + 30°) + 0.1\sin(100t + 30°)$$

$$\therefore \quad i_p = 0.316\cos(100t + 11.57°) \text{ A}$$

The complete equation for the current is $i = i_p + i_c$

$$\therefore \quad i = ce^{-300t} + 0.316\cos(100t + 11.57°)$$

At $\quad t = 0$, the current $i_0 = 0$

$$\therefore \quad c = -0.316\cos(11.57°) = -0.309$$

Therefore, the complete solution for the current is

$$i = -0.309e^{-300t} + 0.316\cos(100t + 11.57°) \text{ A}$$

PROBLEM 11.7

The circuit shown in Fig. 11.28, consists of series RC elements with $R = 15\ \Omega$ and $C = 100\ \mu F$. A sinusoidal voltage $v = 100\sin(500t + \phi)$ volts is applied to the circuit at time corresponding to $\phi = 45°$. Obtain the current transient.

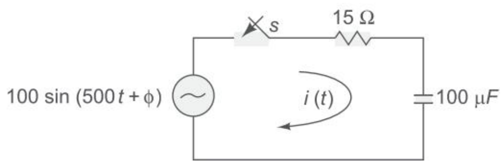


Fig. 11.28

Solution By using Kirchhoff's laws, the differential equation is

$$15i + \frac{1}{100 \times 10^{-6}} \int i dt = 100 \sin(500t + \phi)$$

Differentiating once, we have

$$15 \frac{di}{dt} + \frac{1}{100 \times 10^{-6}} i = (100)\ (500)\ \cos(500t + \phi)$$

$$\left(D + \frac{1}{1500 \times 10^{-6}}\right) i = 3333.3 \cos(500t + \phi)$$

$$(D + 666.67)i = 3333.3 \cos (500t + \phi)$$

The complementary function $i_c = ce^{-666.67t}$

To determine the particular current, first we assume a particular current.

$$i_p = A \cos (500t + 45°) + B \sin (500t + 45°)$$

$$i_p' = -500 A \sin (500t + 45°) + 500 B \cos (500t + 45°)$$

Substituting i_p and i_p' in the differential equation, we get

$$-500 A \sin (500t + 45°) + 500 B \cos (500t + 45°)$$
$$+ 666.67 A \cos (500t + 45°) + 666.67 B \sin (500t + 45°)$$
$$= 3333.3 \cos (500t + \phi)$$

By equating coefficients, we get

$$500 B + 666.67 A = 3333.3$$

$$666.67 B - 500 A = 0$$

From which, the coefficients

$$A = 3.2; B = 2.4$$

Therefore, the particular current is

$$i_p = 3.2 \cos (500t + 45°) + 2.4 \sin (500t + 45°)$$

$$i_p = 4 \sin (500t + 98.13°)$$

The complete equation for the current is

$$i = i_c + i_p$$

$$i = ce^{-666.67t} + 4 \sin (500t + 98.13°)$$

At $t = 0$, the differential equation becomes

$$15i = 100 \sin 45°$$

$$i = \frac{100}{15} \sin 45° = 4.71 \text{ A}$$

$\therefore$ at $t = 0$,

$$4.71 = c + 4 \sin (98.13°)$$

$$\therefore \quad c = 0.75$$

The complete current is

$$i = 0.75\ e^{-666.67t} + 4 \sin (500t + 98.13°)$$

PROBLEM 11.8

The circuit shown in Fig. 11.29 consisting of series RLC elements with $R = 10\ \Omega$, $L = 0.5$ H and $C = 200\,\mu F$ has a sinusoidal voltage $v = 150 \sin (200t + \phi)$. If the switch is closed when $\phi = 30°$, determine the current equation.

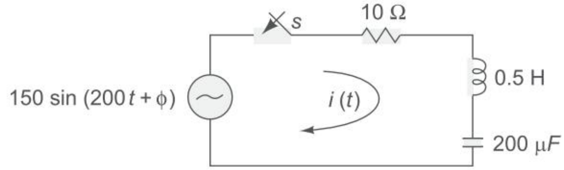


Fig. 11.29

Solution By using Kirchhoff's laws, the differential equation is

$$10i + 0.5\frac{di}{dt} + \frac{1}{200\times 10^{-6}}\int i\,dt = 150 \sin (200t + \phi)$$

Differentiating once, we have

$$(D^2 + 20D + 10^4)i = 60000 \cos (200t + \phi)$$

The roots of the characteristics equation are

$$D_1 = -10 + j99.49 \text{ and } D_2 = -10 - j99.49$$

The complementary function is

$$i_c = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49)$$

We can find the particular current by using the undetermined coefficient method.
Let us assume

$$i_p = A \cos (200t + 30°) + B \sin (200t + 30°)$$

$$i_p' = -200 A \sin (200t + 30°) + 200 B \cos (200t + 30°)$$

$$i_p'' = -(200)^2 A \cos (200t + 30°) - (200)^2 B \sin (200t + 30°)$$

Substituting these values in the equation, and equating the coefficients, we get

$$A = 0.1\ B = 0.067$$

Therefore, the particular current is

$$i_p = 1.98 \cos (200t - 52.4°) \text{ A}$$

The complete current is

$$i = e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t) + 1.98 \cos (200t - 52.4°) \text{ A}$$

From the differential equation at $t = 0$, $i_0 = 0$ and $\dfrac{di}{dt} = 300$

$\therefore$ At $t = 0$

$$c_1 = -1.98 \cos (-52.4°) = -1.21$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-10t}(-99.49c_1 \sin 99.49t + 99.49c_2 \cos 99.49t)$$

$$-200\,(1.98) \sin (200t - 52.4°) - 10e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t)$$

At $t = 0, \dfrac{di}{dt} = 300$ and $c_1 = -1.21$

$$300 = 99.49\, c_2 - 396 \sin(-52.4°) - 10(-1.21)$$

$$300 = 99.49\, c_2 + 313.7 + 12.1$$

$$c_2 = -25.8$$

Therefore, the complete current equation is

$i = e^{-10t}(0.07 \cos 99.49t - 25.8 \sin 99.49t) + 1.98 \cos(200t - 52.4°)$ A

PROBLEM 11.9

For the circuit shown in Fig. 11.30, determine the transient current when the switch is moved from the position 1 to the position 2 at t = 0. The circuit is in steady state with the switch in position 1. The voltage applied to the circuit is v = 150 cos (200t + 30°) V.

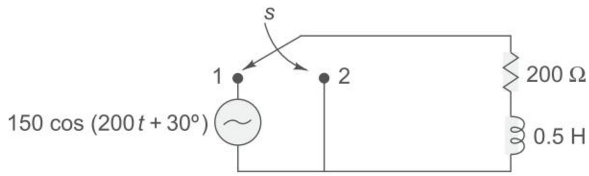


Fig. 11.30

Solution When the switch is at the position 2, by applying Kirchhoff's law, the differential equation is

$$200i + 0.5\frac{di}{dt} = 0$$

$$(D + 400)i = 0$$

$\therefore$ the transient current is

$$i = ce^{-400t}$$

At $t = 0$, the switch is moved from the position 1 to the position 2. Hence, the current passing through the circuit is the same as the steady state-current passing through the circuit when the switch is in position 1.

When the switch is in the position 1, the current passing through the circuit is

$$i = \frac{v}{z} = \frac{150\angle 30°}{R + j\omega L}$$

$$= \frac{150\angle 30°}{200 + j(200)(0.5)} = \frac{150\angle 30°}{223.6\angle 26.56°} = 0.67\angle 3.44°$$

Therefore, the steady-state current passing through the circuit when the switch is in the position 1 is

$$i = 0.67 \cos(200t + 3.44°)$$

Now substituting this equation in the transient current equation, we get

$$0.67 \cos(200t + 3.44°) = ce^{-400t}$$

At $t = 0$; $c = 0.67 \cos(3.44°) = 0.66$

Therefore, the current equation is $i = 0.66e^{-400t}$

PROBLEM 11.10

In the circuit shown in Fig. 11.31, determine the current equations for i_1 and i_2 when the switch is closed at $t = 0$.

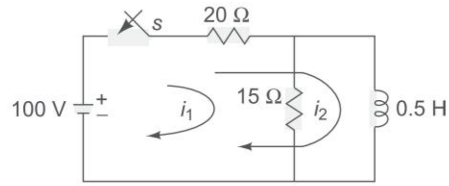


Fig. 11.31

Solution By applying Kirchhoff's laws, we get two equations:

$$35i_1 + 20i_2 = 100 \tag{11.50}$$

$$20i_1 + 20i_2 + 0.5\frac{di_2}{dt} = 100 \tag{11.51}$$

From Eq. (11.50), we have

$$35i_1 = 100 - 20i_2$$

$$i_1 = \frac{100}{35} - \frac{20}{35}i_2$$

Substituting i_1 in Eq. (11.51), we get

$$20\left(\frac{100}{35} - \frac{20}{35}i_2\right) + 20i_2 + 0.5\frac{di_2}{dt} = 100 \tag{11.52}$$

$$57.14 - 11.43i_2 + 20i_2 + 0.5\frac{di_2}{dt} = 100$$

$$(D + 17.14)i_2 = 85.72$$

From the above equation,

$$i_2 = ce^{-17.14t} + 5$$

Loop current i_2 passes through inductor and must be zero at $t = 0$

At $t = 0$, $i_2 = 0$

$\therefore \quad c = -5$

$\therefore \quad i_2 = 5(1 - e^{-17.14t})$ A

and the current $i_1 = 2.86 - \{0.57 \times 5(1 - e^{-17.14t})\}$

$= (0.01 + 2.85\, e^{-17.14t})$ A

PROBLEM 11.11

For the circuit shown in Fig. 11.32, find the current equation when the switch is changed from the position 1 to the position 2 at $t = 0$.

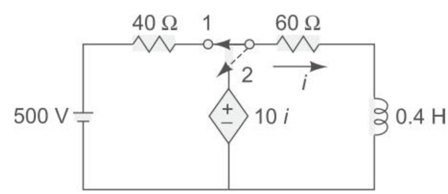


Fig. 11.32

Solution By using Kirchhoff's voltage law, the current equation is given by

$$60i + 0.4\frac{di}{dt} = 10i$$

At $t = 0^-$, the switch is at the position 1, the current passing through the circuit is

$$i(0^-) = \frac{500}{100} = 5\text{A}$$

$$0.4\frac{di}{dt} + 50i = 0$$

$$\left(D + \frac{50}{0.4}\right)i = 0$$

$$i = ce^{-125t}$$

At $t = 0$, the initial current passing through the circuit is same as the current passing through the circuit when the switch is at the position 1.

At $t = 0$, $i(0) = i(0^-) = 5$ A

At $t = 0$, $c = 5$ A

$\therefore$ the current $I = 5e^{-125t}$

PROBLEM 11.12

For the circuit shown in Fig. 11.33, find the current equation when the switch S is opened at t = 0.

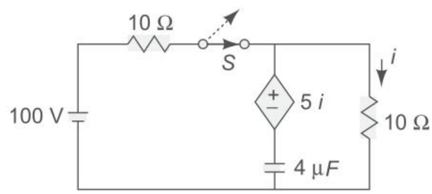


Fig. 11.33

Solution When the switch is closed for a long time,

$$\text{At } t = 0^-, \text{the current } i(0^-) = \frac{100}{20} = 5\text{A}$$

When the switch is opened at $t = 0$, the current equation by using Kirchhoff's voltage law is given by

$$\frac{1}{4\times10^{-6}}\int i\,dt + 10i = 5i$$

$$\frac{1}{4\times10^{-6}}\int i\,dt + 5i = 0$$

Differentiating the above equation,

$$5\frac{di}{dt} + \frac{1}{4\times10^{-6}}i = 0$$

$$\left(D + \frac{1}{20\times10^{-6}}\right)i = 0$$

$$\therefore \qquad i = ce^{\frac{-1}{20\times10^{-6}}t}$$

At $t = 0^-$, just before the switch S is opened, the current passing through the $10\,\Omega$ resistor is 5A. The same current passes through $10\,\Omega$ at $t = 0$.

$\therefore$ At $t = 0$, $i(0) = 5$A

At $t = 0$, $c_1 = 5$A

The current equation is $i = 5e^{\frac{-t}{20\times10^{-6}}}$

PROBLEM 11.13

For the circuit shown in Fig. 11.34, find the current in the 20 Ω when the switch is opened at t = 0.

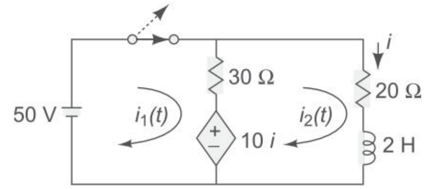


Fig. 11.34

Solution When the switch is closed, the loop currents i_1 and i_2 are flowing in the circuit.

The loop equations are $30(i_1 - i_2) + 10i_2 = 50$

$$30(i_2 - i_1) + 20i_2 = 10i_2$$

From the above equations, the current in the 20 Ω resistor $i_2 = 2.5$ A.

The same initial current is flowing when the switch is opened at $t = 0$.

When the switch is opened, the current equations

$$30i + 20i + 2\frac{di}{dt} = 10i$$

$$40i + \frac{2di}{dt} = 0$$

$$(D + 20)i = 0$$

$$i = ce^{-20t}$$

At $t = 0$, the current $i(0) = 2.5$ A

$\therefore$ At $t = 0$, $c = 2.5$

The current in the 20 Ω resistor is $i = 2.5\,e^{-20t}$

PROBLEM 11.14

For the circuit shown in Fig. 11.35, find the current equation when the switch is opened at t = 0.

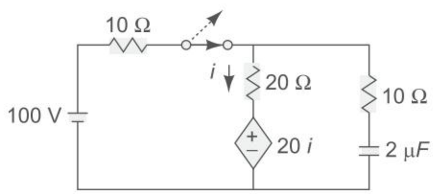


Fig. 11.35

Solution When the switch is closed, the current in the 20 Ω resistor i can be obtained using Kirchhoff's voltage law.

$$10i + 20i + 20i = 100$$

$$50i = 100, \quad \therefore i = 2\text{ A}$$

The same initial current passes through the 20 Ω resistor when the switch is opened at $t = 0$.

The current equation is

$$20i + 10i + \frac{1}{2\times 10^{-6}}\int i\,dt = 20i$$

$$10i + \frac{1}{2\times 10^{-6}}\int i\,dt = 0$$

Differentiating the above equation, we get

$$10\frac{di}{dt}+\frac{1}{2\times10^{-6}}i=0$$

$$\left(D+\frac{1}{20\times10^{-6}}\right)i=0$$

The solution for the above equation is

$$i=ce^{\frac{-1}{20\times10^{-6}}t}$$

At $t = 0$, $i(0) = i(0^-) = 2$ A

$\therefore$ At $t = 0$, $c = 2$ A

The current equation is

$$i=2e^{\frac{-1}{20\times10^{-6}}t}$$

PSpice Problems

PROBLEM 11.1

Using PSpice, for the circuit shown in Fig. 11.36, find the complete expression for the current when the switch is closed at t = 0.

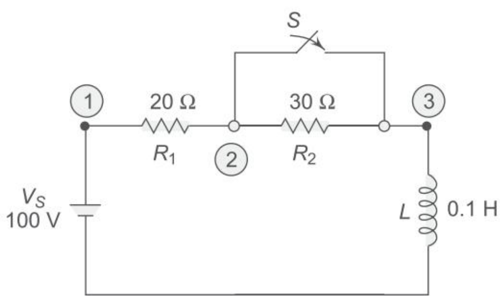


Fig. 11.36

RL TRANSIENT WITH SWITCH

V_S 1 0 PWL(0 100 1N 100 1U 100 1 100)

R_1 1 2 20

R_2 2 3 30

L_3 0 0.1H IC = 2

W1 2 3 VS SMOD

.MODEL SMOD ISWITCH(ION = 0 IOFF = 2)

.TRAN 1U 50M UIC

.PROBE

.END

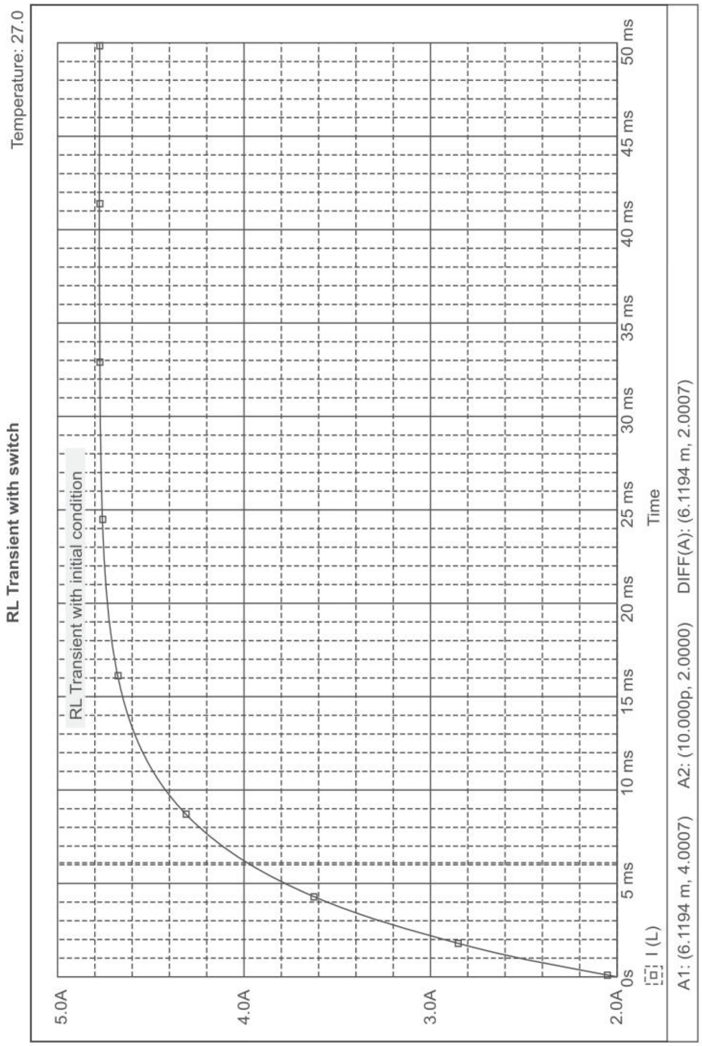


Fig. 11.37

Result $\quad \mathrm{I}(t) = 5 + (2-5)^{e^{-t/5^m}}$

$$= 5 - 3e^{-200t}$$

PROBLEM 11.2

A series RLC circuit shown in Fig. 11.38, comprising $R = 10\,\Omega$, $L = 0.5\,H$, and $C = 1\,\mu F$, is excited by a constant voltage source of 100 V, using PSpice, obtain the expression for current.

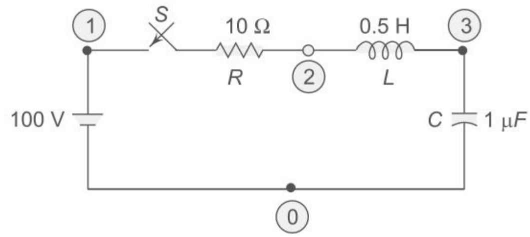


Fig. 11.38

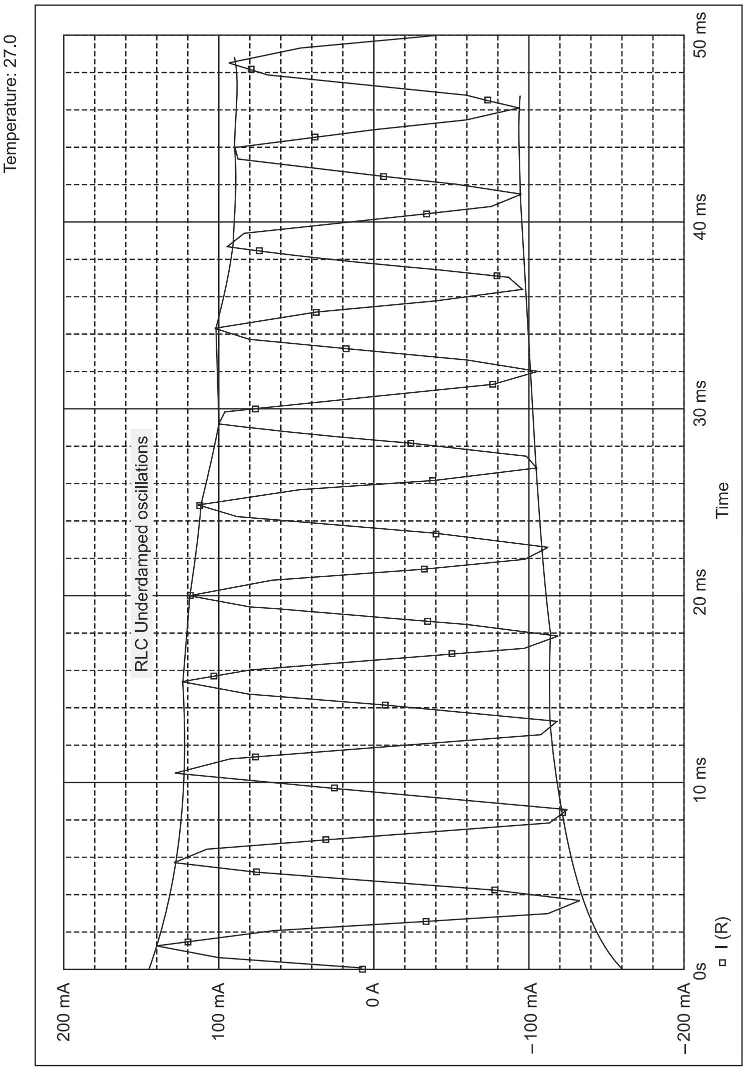


Fig. 11.39

* RLC UNDERDAMPED OSCILLATIONS

V1	1	0	PWL(0 0 0.1N 100 1 100)
R	1	2	10 OHM
L	2	3	0.5 H
C	3	0	1 UF

.TRAN 10 U 50 M
.PROBE
.END
FROM EVAL GOAL FUNCTION
PEIROD(I(C)) = 4.4715m Sec
Wd = 1405.16 rad/sec

Answers to Practice Problems

11-2.1 $i(t) = (2 + 10t)e^{-5t}$

11-2.2 $i = 69.7\text{mA}; V_{R_2} = 6.97\text{volts}; V_L = 3.03\text{volts}$

11-2.3 $v(t) = -5e^{-t} + 15e^{-3t}$ volts

11-2.5 $i_1(t) = 9.99 - 8.49\, e^{-5 \times 10^4\, t}; i_2(t) = 5e^{-5 \times 10^4\, t}$

11-2.6 $R = 10$ K ; $c = 2.5\,\mu$F

11-2.7 $i(t) = 5 \cos 100t$

11-2.8 $i(t) = 101.2 + 30.9\, e^{-0.1t} - 52.11\, e^{-4.94t}$

11-2.12 $V_1(t) = -4e^{-0.4t} + 4e^{-4999.8t}$

11-3.1 $5e^{-5.71t}$

11-3.4 $i(t) = 3.8 + e^{-0.05t} + 0.12\, e^{-0.31t}$

11-3.8 $v(t) = 5t\, e^{-2t}$ volts

Objective-Type Questions

☆☆★ **11.1** Transient behaviour occurs in any circuit when
(a) there are sudden changes of applied voltage
(b) the voltage source is shorted
(c) the circuit is connected or disconnected from the supply
(d) all of the above happen

☆☆★ **11.2** The transient response occurs
(a) only in resistive circuits
(b) only in inductive circuits
(c) only in capacitive circuits
(d) both in (b) and (c)

☆☆★ **11.3** Inductor does not allow sudden changes
(a) in currents
(b) in voltages
(c) in both (a) and (b)
(d) in none of the above

☆★★ **11.4** When a series *RL* circuit is connected to a voltage source *V* at $t = 0$, the current passing through the inductor *L* at $t = 0^+$ is
(a) $\dfrac{V}{R}$ (b) infinite (c) zero (d) $\dfrac{V}{L}$

☆☆★ **11.5** The time constant of a series *RL* circuit is
(a) LR (b) $\dfrac{L}{R}$ (c) $\dfrac{R}{L}$ (d) $e^{-R/L}$

☆☆★ **11.6** A capacitor does not allow sudden changes
(a) in currents
(b) in voltages
(c) in both currents and voltages
(d) in neither of the two

☆☆★ **11.7** When a series *RC* circuit is connected to a constant voltage at $t = 0$, the current passing through the circuit at $t = 0^+$ is
(a) infinite (b) zero (c) $\dfrac{V}{R}$ (d) $\dfrac{V}{\omega C}$

☆☆★ **11.8** The time constant of a series *RC* circuit is
(a) $\dfrac{1}{RC}$ (b) $\dfrac{R}{C}$ (c) RC (d) e^{-RC}

☆☆★ **11.9** The transient current in a lossfree *LC* circuit when excited from an ac source is an ______ sine wave.
(a) undamped
(b) overdamped
(c) underdamped
(d) critically damped

☆★★**11.10** Transient current in an *RLC* circuit is oscillatory when

(a) $R = 2\sqrt{L/C}$ (b) $R = 0$ (c) $R > 2\sqrt{L/C}$ (d) $R < 2\sqrt{L/C}$

☆★★**11.11** The initial current in the circuit shown in Fig. 11.40 when the switch is opened for $t > 0$ is

(a) 1.67 A (c) 0 A
(b) 3 A (d) 2 A

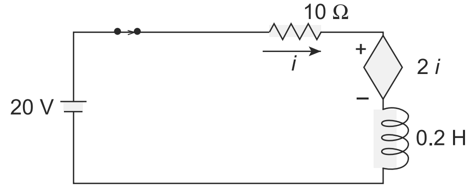


Fig. 11.40

☆★★**11.12** The initial current in the circuit shown in Fig. 11.41 below when the switch is opened for $t > 0$ is

(a) 1.5 A (c) 2 A
(b) 0 A (d) 10 A

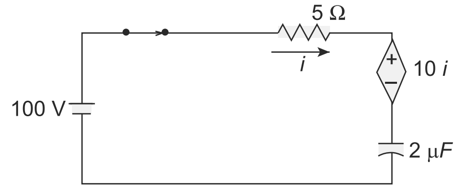


Fig. 11.41

☆★★**11.13** For the circuit shown in Fig. 11.42 the current in the 10 Ω resistor when the switch is changed from 1 to 2 is

(a) $5\ e^{+20t}$ (c) $20\ e^{+5t}$
(b) $5\ e^{-20t}$ (d) $20e^{-5t}$

☆★★**11.14** For the circuit shown in Fig. 11.43, the current in the 5 Ω resistor when the switch is changed from 1 to 2 is

(a) $2.5e^{\frac{1}{2\times10^{-6}}}$ (c) $2.5\ e^{-10t}$ (b) 0 (d) $5e^{-5t}$

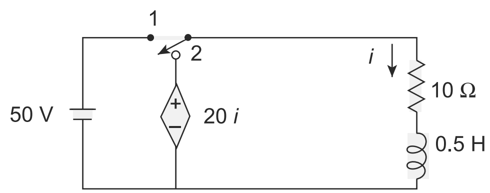


Fig. 11.42

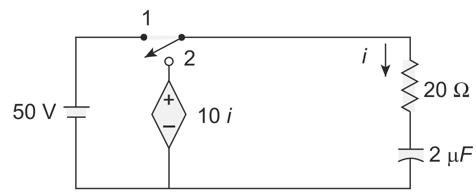


Fig. 11.43

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/269

CHAPTER 12

Fourier Method of Waveform Analysis

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- **LO 1** Representation of Periodic Signals using Trigonometric Series and Compact Trigonometric Series
- **LO 2** Analyse the Fourier series with a period of 2T
- **LO 3** Analyse the complex form of complex Fourier series and, therefore, examine the amplitude, phase, and frequency spectrum of the Fourier series
- **LO 4** Explain Fourier transform and Parseval's theorem
- **LO 5** Analyse power signals and periodic signals using Fourier transform
- **LO 6** List the properties of the Fourier transform
- **LO 7** Describe the applications of Fourier series in circuit analysis

12.1 INTRODUCTION TO FOURIER ANALYSIS

Fourier series are series of cosine and sine terms and are used to represent general periodic signals. The ideas and techniques of Fourier series can be extended to non-periodic signals. The signal representation for such non-periodic signals are given by Fourier integrals and Fourier transforms.

Fourier analysis deals with Fourier series and Fourier transforms and has several applications in mathematics, science and engineering, particularly in the area of communications and signal processing. Any periodic signal can be represented as the sum of a finite or infinite number of sinusoidal functions, the responses of linear systems to nonsinusoidal excitations can be determined by applying the superposition integral. The Fourier analysis provides the ways of solving the above problem.

12.2 TRIGONOMETRIC SERIES

LO 1 Representation of Periodic Signals using Trigonometric Series and Compact Trigonometric Series

Periodic signals can be represented by the sum of sinusoids whose frequencies are harmonics or integer multiples of fundamental frequency. The Fourier series representation of a periodic signal will be of the form

$$x(t) = a_0 + a_1 \cos \omega t + a_1 \cos 2\omega t + ... + b_1 \sin \omega t + b_2 \sin 2\omega t + ...$$

where a_0, a_1, a_2 ...., b_1, b_2 .... are real constants. Such a series is called trigonometric series and a_n and b_n are called the coefficients of the series.

Using summation, we may write this series as

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega t + b_n \sin n\,\omega t)$$

Let us assume $x(t)$ is a periodic function of period $\frac{2\pi}{\omega}$ and is integrable over a period $\frac{2\pi}{\omega}$. Assume $x(t)$ can be represented by a trigonometric series.

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega t + b_n \sin n\,\omega t) \tag{12.1}$$

We assume that this series converges and has $x(t)$ as its sum. From the function $x(t)$, we can determine the coefficients a_n and b_n.

12.2.1 Determination of Constant Term a_0

Integrating both sides of Eq. (12.1) from $\frac{-\pi}{\omega}$ to $\frac{\pi}{\omega}$,

we get $$\int_{-\pi/\omega}^{\pi/\omega} x(t)dt = \int_{-\pi/\omega}^{\pi/\omega}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t\right]dt \tag{12.2}$$

The term-by-term integration gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t)dt = a_0 \int_{-\pi/\omega}^{\pi/\omega} dt + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi/\omega}^{\pi/\omega} \cos n\omega t\, dt + b_n \int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t\, dt\right] \tag{12.3}$$

The first term, on the right equals $\frac{2\pi}{\omega} a_0$. All the other integrals on the right are zero.

$$\int_0^{2\pi/\omega} x(t)dt = \frac{2\pi}{\omega} a_0$$

$$a_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)dt$$

An alternate form of the evaluation integral with the variable ωt and the corresponding period of 2π radians is

$$\boxed{a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, d(\omega t)} \tag{12.4}$$

12.2.2 Determination of the Coefficients a_n of the Cosine Terms

We multiply Eq. (12.1) by cos $m\omega t$, where m is any fixed positive integer and integrate from $\frac{-\pi}{\omega}$ to $\frac{\pi}{\omega}$.

$$\int_{-\pi/\omega}^{\pi/\omega} x(t)\cos m\omega t\; dt = \int_{-\pi/\omega}^{\pi/\omega} \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)\right]\cos m\omega t\; dt \tag{12.5}$$

Integrating term-by-term, the right-hand side becomes,

$$a_0 \int_{-\pi/\omega}^{\pi/\omega} \cos m\omega t\; dt + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi/\omega}^{\pi/\omega} \cos n\omega t \cos m\omega t\; dt + b_n \int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t \cos m\omega t\; dt\right] \tag{12.6}$$

The first and third integrals are zero. The second integral becomes $\frac{a_m \pi}{\omega}$ when $n = m$

$$\therefore \quad a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} x(t)\cos n\omega t\; dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(t)\cos n\omega t\; d(\omega t) \tag{12.7}$$

12.2.3 Determination of the Coefficients b_n of the Sine Terms

We multiply Eq. (12.1) by sin $m\omega t$, where m is the fixed positive integer and integrate $\frac{-\pi}{\omega}$ to $\frac{\pi}{\omega}$

$$\int_{-\pi/\omega}^{\pi/\omega} x(t)\sin m\omega t\; dt = \int_{-\pi/\omega}^{\pi/\omega} \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)\right]\sin m\omega t\; dt \tag{12.8}$$

Integrating term-by-term, the right hand-side becomes,

$$a_0 \int_{-\pi/\omega}^{\pi/\omega} \sin m\omega t\; dt + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi/\omega}^{\pi/\omega} \cos n\omega t \sin m\omega t\; dt + b_n \int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t \sin m\omega t\; dt\right] \tag{12.9}$$

The first two integrals are zero and the last integral becomes $\frac{b_n \pi}{\omega}$ when $n = m$

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} x(t)\sin n\omega t\; dt$$

or

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin n\omega t\; d(\omega t) \tag{12.10}$$

The coefficients a_0, a_n, and b_n are called the Fourier coefficients of $x(t)$.

12.3 COMPACT TRIGONOMETRIC FOURIER SERIES

LO 1

The sine and cosine terms of the same frequency can be combined as single sine or cosine with a phase angle.

$$a_n \cos n\omega t + b_n \sin n\omega t = c_n \cos(n\omega t + \theta_n)$$

or

$$a_n \cos n\omega t + b_n \sin n\omega t = c_n \sin(n\omega t + \phi_n) \tag{12.11}$$

where

$$c_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \text{ and } \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

The Fourier series in Eq. (12.1) can be expressed in the compact form

$$x(t) = c_o + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \theta_n) \tag{12.12}$$

where $c_0 = a_0$ is the average value of $x(t)$ and c_n; θ_n are computed from a_n and b_n.

12.3.1 Existence of the Fourier Series, Dirichlet Conditions

Any periodic function $x(t) = x(t + T)$ can be expressed by a Fourier series provided the following conditions are satisfied.

The function $x(t)$ is absolutely integrable over one period.

$$\int_T |x(t)|\,dt < \infty$$

where T is the period.

The function $x(t)$ has only a finite number of positive and negative maxima in one period.

The function $x(t)$ has a finite number of discontinuities in one period.

The above conditions are known as Dirichlet conditions and hence possess a convergent Fourier series.

EXAMPLE 12.1

Find the Fourier series for the waveform shown in Fig. 12.1.

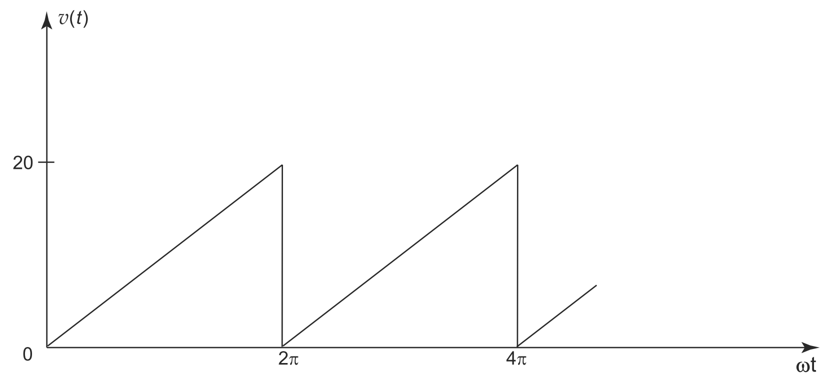


Fig. 12.1

Solution The waveform equation is given by

$$v(t) = \frac{20}{2\pi}\omega t$$

The average value of the waveform is

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{20}{2\pi}\omega t\, d(\omega t) = \left[20 \times \frac{(\omega t)^2}{2}\right]_0^{2\pi} = 10$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{20}{2\pi}\right)\omega t \cos n\omega t\ d(\omega t)$$

$$= \frac{20}{2\pi^2}\left[\frac{\omega t}{n}\sin n\omega t + \frac{1}{n^2}\cos n\omega t\right]_0^{2\pi}$$

$$= \frac{20}{2\pi^2 n^2}[\cos n2\pi - \cos 0]$$

$= 0$ for all integer values of n

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{20}{2\pi}\right)\omega t \sin n\omega t\ d(\omega t)$$

$$= \frac{20}{2\pi^2}\left[\frac{-\omega t}{n}\cos n\omega t + \frac{1}{n^2}\sin n\omega t\right]_0^{2\pi} = \frac{-20}{\pi n}$$

Using these sine term coefficients and the average term, the series is

$$v(t) = 10 - \frac{20}{\pi}\sin\omega t - \frac{20}{2\pi}\sin 2\omega t - \frac{20}{3\pi}\sin 3\omega t$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **12-1.1** Find the trigonometric Fourier series for sawtooth wave shown in Fig. Q.1 and plot the spectrum.

☆★★ **12-1.2** Find the trigonometric Fourier series for the waveform shown in Fig. Q.2.

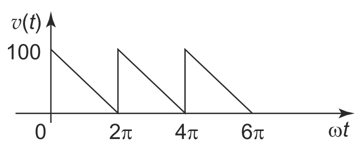


Fig. Q.1

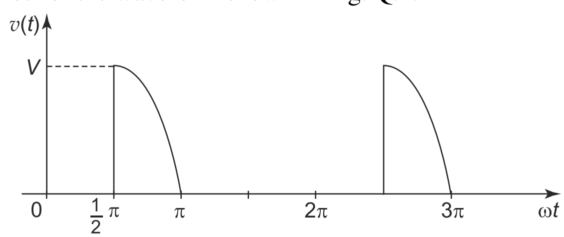


Fig. Q.2

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

☆☆★ **12-1.3** Find the trigonometric Fourier series for the waveform shown in Fig. Q.3.

☆★★ **12-1.4** Find the exponential Fourier series for the waveform shown is Fig. Q.4 and plot the spectrum. Concert the coefficients obtained into the trigonometric series coefficients.

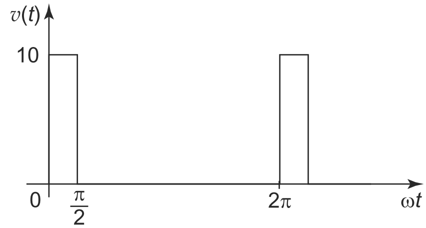


Fig. Q.3

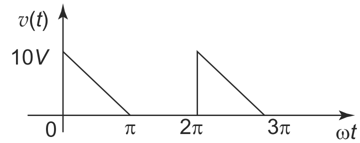


Fig. Q.4

Frequently Asked Questions linked to LO 1*

☆☆★ **12-1.1** Find the trigonometric form of Fourier series for the waveform shown in Fig. Q.1. **[BPUT 2007]**

☆☆★ **12-1.2** Explain time invariant. **[GTU Dec. 2010]**

☆★★ **12-1.3** Discuss the effect of symmetry for a periodic function to determine the trigonometric Fourier series coefficients. **[RGTU Dec. 2013]**

☆★★ **12-1.4** Find the Fourier coefficients for the waveform shown in Fig. Q.4. **[RGTU Dec. 2010]**

★★★ **12-1.5** Obtain trigonometric Fourier series of the signal shown in Fig. Q. 5. **[RTU Feb. 2011]**

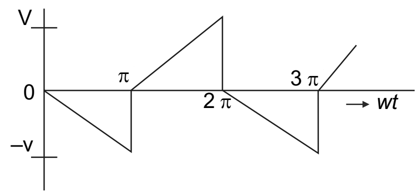


Fig. Q. 1

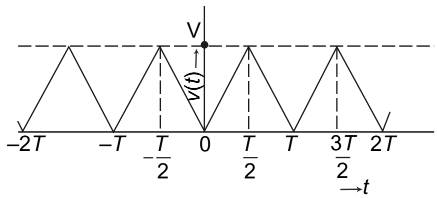


Fig. Q. 4

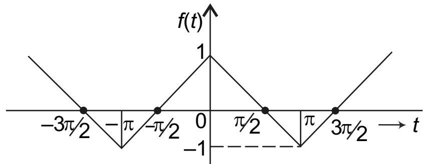


Fig. Q.5

12.4 FUNCTIONS OF ANY PERIOD 2*T*

> **LO 2** Analyse the Fourier series with a period of 2T

The functions considered so far had period 2π. In several applications, periodic functions will generally have other periods.

If a functions $x(t)$ of period $2T$ has a Fourier serie, then this series is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{T}\omega t + b_n \sin\frac{n\pi}{T}\omega t\right) \tag{12.13}$$

With the Fourier coefficients of $x(t)$ given by

$$a_0 = \frac{1}{2T}\int_{-T}^{T} x(t)\, d(\omega t)$$

$$a_n = \frac{1}{T}\int_{-T}^{T} x(t)\cos\left(\frac{n\pi\omega t}{T}\right) d(\omega t) \text{ for } n = 1, 2....$$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

$$b_n = \frac{1}{T}\int_{-T}^{T} x(t)\sin\left(\frac{n\pi\omega t}{T}\right) d(\omega t) \text{ for } n = 1, 2....$$

EXAMPLE 12.2

Find the Fourier series of the function shown in Fig. 12.2.

$$v(t) = \begin{cases} 0 & \text{if} \quad -2\pi < \omega t < -\pi \\ k & \text{if} \quad -\pi < \omega t < \pi \\ 0 & \text{if} \quad \pi < \omega t < 2\pi \end{cases} \tag{12.14}$$

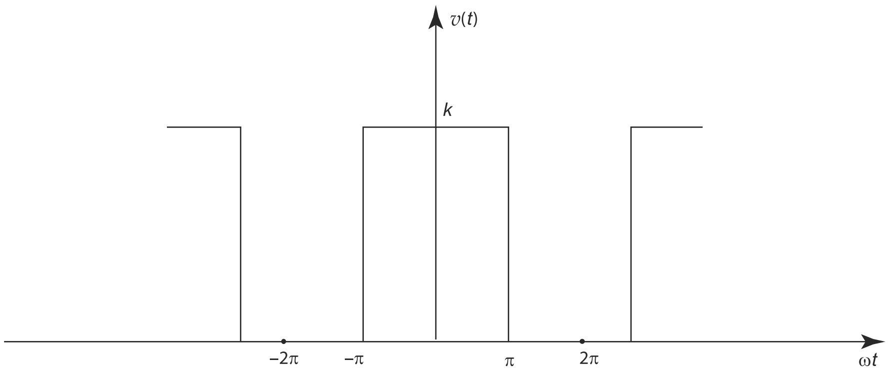


Fig. 12.2

Solution From Eq. (12.14), the Fourier coefficients are

$$a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} v(t)\, d(\omega t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} k\, d(\omega t) = \frac{k}{2} \tag{12.15}$$

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} v(t)\cos\frac{n\pi\omega t}{2}\, d(\omega t) = \frac{1}{2}\int_{-\pi}^{\pi} k\cos\frac{n\pi\omega t}{2}\, d(\omega t)$$

$$= \frac{2k}{n\pi}\sin\frac{n\pi}{2} \tag{12.16}$$

Thus, $a_n = 0$ if n is even and

$$a_n = \frac{2k}{n\pi} \quad \text{if} \quad n = 1, 5, 9, ..., a_n = \frac{-2k}{n\pi} \quad \text{if} \quad n = 3, 7, 11....$$

we also find that $b_n = 0$ for $n = 1,2...$ Hence, the result is

$$v(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi}{2}\omega t - \frac{1}{3}\cos\frac{3\pi}{2}\omega t + \frac{1}{5}\cos\frac{5\pi}{2}\omega t + ...\right] \tag{12.17}$$

12.4.1 Even and Odd Functions

A function $x(t)$ is even if $x(-t) = x(t)$. The waveform of such a function is symmetric with respect to the y-axis. A function $x(t)$ is odd if $x(-t) = -x(t)$. The function $\cos n\omega t$ is even and $\sin n\omega t$ is odd. The even and odd functions are shown in Figs 12.3 (a and b).

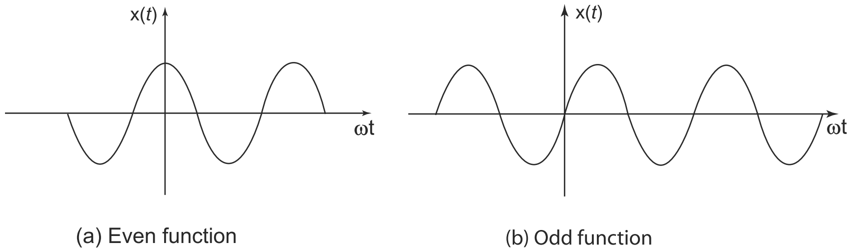


Fig. 12.3

The sum of two or more even functions is an even function and with the addition of a constant, the even nature of the function remains. The sum of two or more odd functions is an odd function, but the addition of a constant removes the odd nature of the function. The product of two odd functions is an even function.

The Fourier series of an even function of period $2T$ is a Fourier cosine series.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{T} \omega t \tag{12.18}$$

with coefficients

$$a_0 = \frac{1}{T}\int_0^T x(t)\, d(\omega t);$$

$$a_n = \frac{2}{T}\int_0^T x(t)\ \cos \frac{n\pi\omega t}{T}\ d(\omega t)$$

The Fourier series of an odd function of period $2T$ is a Fourier sine series.

$$x(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T}\ (\omega t) \tag{12.19}$$

with coefficients

$$b_n = \frac{2}{T}\int_0^T x(t)\ \sin \frac{n\pi\omega t}{T}\ d(\omega t)$$

A periodic function $x(t)$ is said to have half-wave symmetry if $x(t) = -x\left(t + \frac{T}{2}\right)$ where T is the period. If the waveform has half-wave symmetry, only odd harmonics are present in the series. This series will contain both sine and cosine terms unless the function is also odd or even. In any case, a_n and b_n are equal to zero for $n = 2,4,6 \ldots$ for any waveform with half-wave symmetry.

EXAMPLE 12.3

Find the trigonometric Fourier series for the triangular even waveform shown in Fig. 12.4.

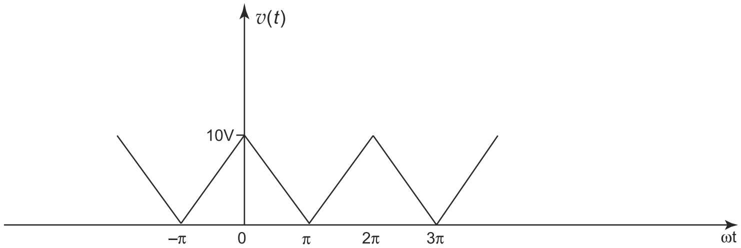


Fig. 12.4

Solution The waveform shown in Fig. 12.4 is an even function

since $v(t) = v(-t)$

By observation, the average value of the wave

$$a_0 = 5$$

The waveform has half-wave symmetry

$$v(t) = -v\left(t + \tfrac{T}{2}\right)$$

The equation for the waveform

$$v(t) = 10 + (10/\pi)\omega t \text{ for } -\pi < \omega t < 0$$

$$= 10 - (10/\pi)\omega t \text{ for } 0 < \omega t < \pi \qquad (12.20)$$

Since even waveforms have only cosine terms,

$$b_n = 0 \text{ for all } n$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}\left[10 + \left(\frac{10}{\pi}\right)\omega t\right]\cos n\omega t \; d(\omega t) + \frac{1}{\pi}\int_{0}^{\pi}\left[10 - \left(\frac{10}{\pi}\right)\omega t\right]\cos n\omega t \; d(\omega t)$$

$$= \frac{10}{\pi}\left\{\int_{-\pi}^{\pi}\cos n\omega t \; d(\omega t) + \int_{-\pi}^{0}\frac{\omega t}{\pi}\cos n\omega t \; d(\omega t) - \int_{0}^{\pi}\frac{\omega t}{\pi}\cos n\omega t \; d(\omega t)\right\}$$

$$= \frac{10}{\pi^2}\left\{\left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{\pi}\sin n\omega t\right]_{-\pi}^{0} - \left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{\pi}\sin n\omega t\right]_{0}^{\pi}\right\}$$

$$= \frac{10}{\pi^2 n^2}\{\cos 0° - \cos(-n\pi) - \cos n\pi + \cos 0°\}$$

$$a_n = \frac{20}{\pi^2 n^2}(1 - \cos n\pi)$$

$$a_n = 0 \text{ for } n = 2,4,6\ldots$$

$$a_n = \frac{40}{\pi^2 n^2} \text{ for } n = 1, 3, 5...$$

The Fourier series is

$$v(t) = 5 + \frac{40}{\pi^2}\cos\omega t + \frac{40}{(3\pi)^2}\cos 3\omega t + \frac{40}{(5\pi)^2}\cos 5\omega t + \cdots$$

EXAMPLE 12.4

Find the trigonometric Fourier series for the waveform shown in Fig. 12.5.

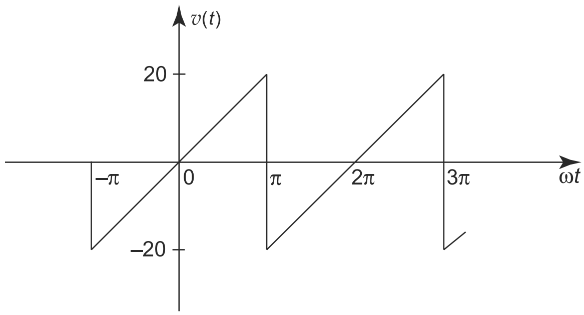


Fig. 12.5

Solution By inspection, the average value of the waveform $a_n = 0$. The waveform is odd and contains only sine terms. The expression of the waveform

$$v(t) = \left(\frac{20}{\pi}\right)\omega t \quad \text{for} -\pi < \omega t < \pi \tag{12.21}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{20}{\pi}\right)\omega t \quad \sin n\omega t \; d(\omega t)$$

$$= \frac{20}{\pi^2}\left[\frac{1}{n^2}\sin n\omega t - \frac{\omega t}{n}\cos n\omega t\right]_{-\pi}^{\pi} = \frac{-40}{n\pi}\cos n\pi \tag{12.22}$$

The Fourier series is

$$v(t) = \frac{40}{\pi}\left\{\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t + \cdots\right\} \tag{12.23}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆★★**12-2.1** Find the trigonometric Fourier series for the waveform shown in Fig. Q.1 and plot the line spectrum.

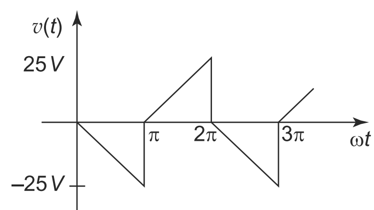


Fig. Q.1

☆★★**12-2.2** Find the trigonometric Fourier series for the half-wave rectified-wave shown in Fig. Q.2 and plot the spectrum.

☆★★**12-2.3** Find the exponential Fourier series for the square wave shown in Fig. Q.3 and plot the spectrum.

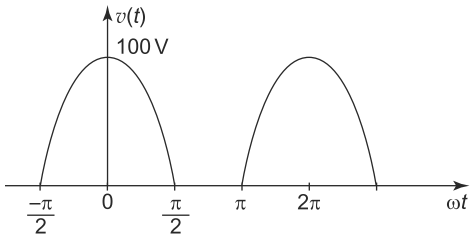


Fig. Q.2

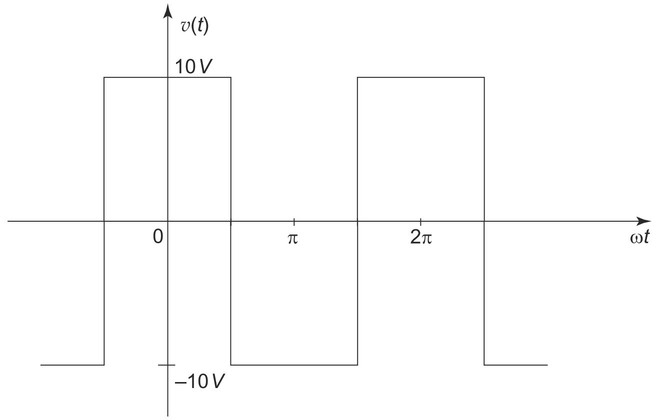


Fig. Q.3

Frequently Asked Questions linked to LO 2

☆★★**12-2.1** Find the Fourier series of the waveform shown in Fig. Q.1 and also find the line spectrum.

[RTU Feb. 2011]

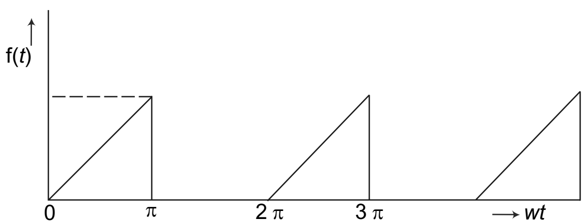


Fig. Q.1

12.5 COMPLEX FOURIER SERIES

> **LO 3** Analyse the complex form of complex Fourier series and, therefore, examine the amplitude, phase, and frequency spectrum of the Fourier series

The Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{12.24}$$

can be written in the complex form

We know that

$$e^{jn\omega t} = \cos n\omega t + j \sin n\omega t \tag{12.25}$$

$$e^{-jn\omega t} = \cos n\omega t - j \sin n\omega t \tag{12.26}$$

By adding the Eqs (12.25) and (12.26), we get

$$\cos n\omega t = \frac{1}{2}(e^{jn\omega t} + e^{-jn\omega t})$$

Subtracting Eq. (12.26) from Eq. (12.25) and dividing by $2j$, we get

$$\sin n\omega t = \frac{1}{2j}(e^{jn\omega t} - e^{-jn\omega t})$$

$$a_n \cos n\omega t + b_n \sin n\omega t = \frac{1}{2}a_n(e^{jn\omega t} + e^{-jn\omega t}) + \frac{1}{2j}b_n(e^{jn\omega t} - e^{-jn\omega t})$$

$$= \frac{1}{2}(a_n - jb_n)\, e^{jn\omega t} + \frac{1}{2}(a_n + jb_n)\, e^{-jn\omega t}$$

Consider $a_0 = c_0$

$$\frac{1}{2}(a_n - jb_n) = c_n$$

and $\frac{1}{2}(a_n + jb_n) = c_{-n}$

Equation (12.24) becomes

$$x(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{jn\omega t} + c_{-n}e^{-jn\omega t}) \tag{12.27}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)(\cos n\omega t - j\sin n\omega t)d(\omega t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)e^{-jn\omega t}d(\omega t)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)(\cos n\omega t + j\sin n\omega t)d(\omega t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)e^{jn\omega t}d(\omega t)$$

By combining the two formulas and writing $c_n = c_{-n}$, we get

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \tag{12.28}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jn\omega t}d(\omega t)$ for $n = 0, \pm 1, \pm 2 \cdots$

This is called complex form of Fourier series or complex Fourier series of $x(t)$. The c_n are called the complex Fourier coefficients of $x(t)$.

EXAMPLE 12.5

Find the complex Fourier series for the square wave shown in Fig. 12.6.

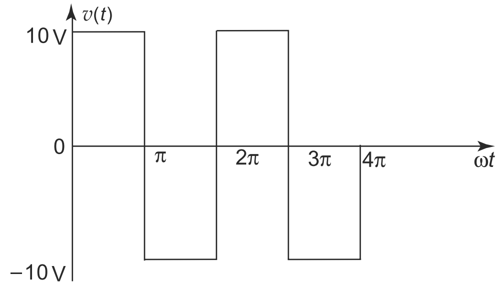


Fig. 12.6

Solution The expression of the waveform

$$v(t) = -10 \text{ for } -\pi < \omega t < 0$$
$$= 10 \text{ for } 0 < \omega t < \pi$$

The average value of the wave is zero

$$c_n = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}(-10)e^{-jn\omega t}\,d(\omega t) + \int_{0}^{\pi}(10)e^{-jn\omega t}\,d(\omega t)\right\}$$

$$= \frac{10}{2\pi}\left\{-\left[\frac{1}{-jn}e^{-jn\omega t}\right]_{-\pi}^{0} + \left[\frac{1}{-jn}e^{-jn\omega t}\right]_{0}^{\pi}\right\}$$

$$= \frac{10}{-j2\pi n}\left(-e^{0} + e^{jn\pi} + e^{-jn\pi} - e^{0}\right)$$

$$= \frac{j10}{n\pi}\left(e^{jn\pi} - 1\right) \tag{12.29}$$

For n even, $e^{jn\pi} = 1$ and $c_n = 0$

For n odd, $e^{jn\pi} = -1$ and $c_n = \dfrac{-j(20)}{n\pi}$

The Fourier series is

$$x(t) = \mathrm{n} + j\frac{20}{3\pi}e^{-j3\omega t} - j\frac{20}{\pi}e^{-j\omega t} - j\frac{20}{\pi}e^{j\omega t} - j\frac{20}{3\pi}e^{j3\omega t} \tag{12.30}$$

12.6 AMPLITUDE AND PHASE SPECTRUMS

LO 3

Fourier coefficient c_n of the exponential form is a complex quantity and can be represented by

$$c_n = \mathrm{Re}[c_n] + j\,Im[c_n] \tag{12.31}$$

The real part of c_n is

$$\mathrm{Re}[c_n] = \frac{1}{2\pi}\int_{0}^{2\pi} x(t)\cos n\omega t \tag{12.32}$$

The imaginary part of c_n is

$$Im[c_n] = \frac{1}{2\pi}\int_{0}^{2\pi} x(t)\sin n\omega t \tag{12.33}$$

$\text{Re}[c_n]$ is an even function of n, where as $Im\,[c_n]$ is an odd function of n. Therefore, the amplitude spectrum of the Fourier series is given by

$$|c_n| = \{\text{Re}^2[c_n] + Im^2[c_n]\}^{1/2} \tag{12.34}$$

while the phase spectrum is given by

$$\theta_n = \tan^{-1}\left[\frac{Im[c_n]}{\text{Re}[c_n]}\right] \tag{12.35}$$

12.7 THE FREQUENCY SPECTRUM

LO 3

Consider the Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

Let $a_n = c_n \cos \alpha_n$

and $b_n = c_n \sin \alpha_n$ (12.36)

Therefore, $x(t) = a_0 + \sum_{n=1}^{\infty}(c_n \cos\alpha_n \cos n\omega t + c_n \sin\alpha_n \sin n\omega t)$

$$= a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t - \alpha_n) \tag{12.37}$$

From Eq. (12.36), we have

$$\tan\alpha_n = \frac{b_n}{a_n}$$

$$\alpha_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \tag{12.38}$$

Also, we have

$$c_n = \sqrt{a_n^2 + b_n^2} \tag{12.39}$$

The magnitude of C_n is plotted against $n\omega$, the graph obtained is called the frequency spectrum of the given waveform $x(t)$. The amplitudes decreases rapidly for waves with rapidly convergent series. Waves with discontinuities such as sawtooth and square waves have spectra with slowly decreasing amplitudes since their series have strong high harmonics.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★**12-3.1** Find the exponential Fourier series for the full-wave rectified sine wave shown in Fig. Q.1 and plot the spectrum.

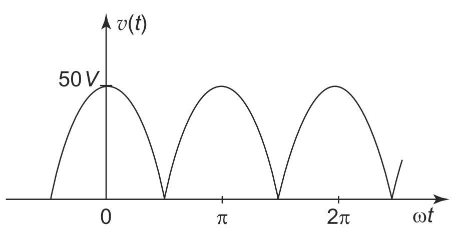


Fig. Q.1

☆★★**12-3.2** Find the exponential Fourier series for the waveform shown in Fig. Q.2 and plot the line spectrum.

☆★★**12-3.3** Find the exponential Fourier series for the waveform shown in Fig. Q.3 and plot the spectrum.

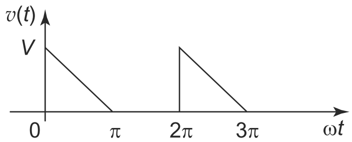


Fig. Q.2

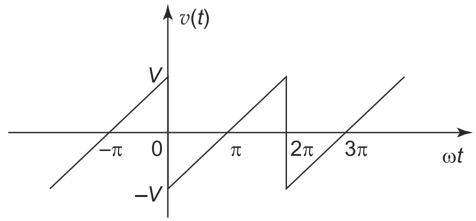


Fig. Q.3

☆★★**12-3.4** Find and sketch the frequency spectrum of rectangular pulse shown in Fig.Q.4.

☆★★**12-3.5** Find and sketch the Fourier transform of the modulated signal $x(t)$ cos $10t$ in which $x(t)$ is shown in Fig. Q.5.

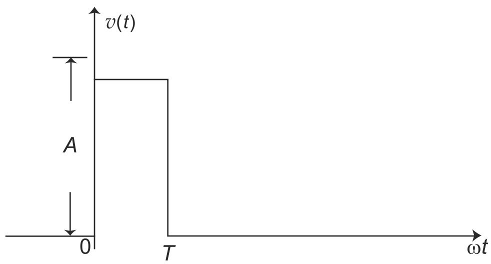


Fig. Q.4

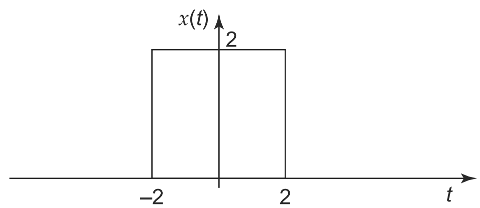


Fig. Q.5

Frequently Asked Questions linked to LO 3

☆★★**12-3.1** Expand the square-wave voltage signal, as shown in the following figure into a Fourier series.

[RGTU June 2014]

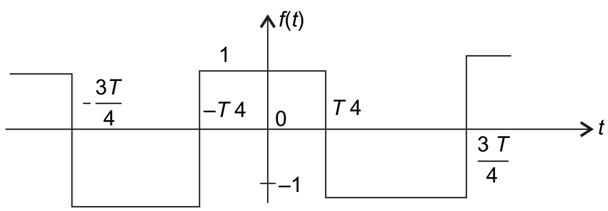


Fig. Q.1

12.8 FOURIER TRANSFORM

Fourier series are powerful tools in dealing various problems involving periodic functions. In several applications, many practical problems involve non-periodic functions. The method of Fourier series can be extended to such non-periodic functions.

LO 4 Explain Fourier transform and Parseval's theorem

The exponential form of Fourier series for periodic signal is given below for convenience.

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \tag{12.40}$$

$$X_n = \frac{1}{T_0} \int_{\frac{-T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jn\omega_0 t}\, dt$$

or
$$X_n = \frac{1}{T_0} \int_{\frac{-T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jn2\pi f_0 t}\, dt \tag{12.41}$$

where f_0 is the fundamental frequency of the periodic signal $x(t)$.

If we define non-periodic signal

$$x(t) = \tilde{x}(t) \quad \text{for} \quad |t| < \frac{T_0}{2} \tag{12.42}$$

$$= 0 \text{ otherwise}$$

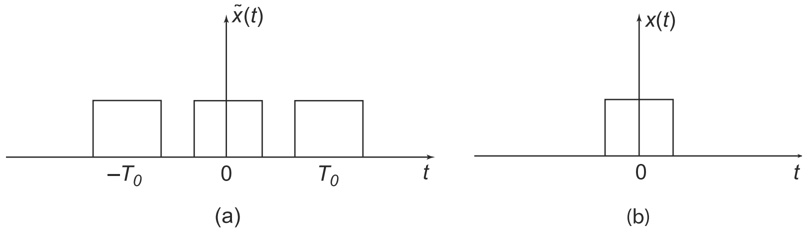


Fig. 12.7

$$X_n = \frac{1}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-jn2\pi f_0 t}\, dt \tag{12.43}$$

$$T_0 X_n = \int_{-\infty}^{\infty} x(t)\, e^{-jn2\pi f_0 t}\, dt \tag{12.44}$$

We can define the envelope $T_0\, X_n$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{12.45}$$

where $\omega = 2\pi\, n\, f_0$

Therefore, the coefficient X_n becomes

$$X_n = \frac{1}{T_0} X(\omega)$$

or
$$X_n = \frac{1}{T_0} X(n\omega_0) \tag{12.46}$$

Substituting Eq. (12.46) in Eq. (12.45), we get

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} X(n\omega_0)\right] e^{jn\omega_0 t}$$

or $$\tilde{x}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}$$

Substituting $\omega_0 = \dfrac{2\pi}{T_0}$

$$\tilde{x}(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t} \omega_0 \tag{12.47}$$

In the above complex sum, let the frequency ω_0 approach to zero, the index n approach to infinity such that the product $n\omega_0$ approaches a continuous frequency variable ω and the discrete sum is replaced by a continuous integral.

Therefore, $\tilde{x}(t) \to x(t)$ as ω_0 approaches zero.

Then Eq. (12.45) becomes

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{12.48}$$

and Eq. (12.47) becomes

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{12.49}$$

These expressions define a Fourier transform pair for the signal $x(t)$ and are denoted by the notation

$$x(t) \leftrightarrow X(\omega)$$

The integral Eq. (12.44) is referred to as Fourier transform of $x(t)$ and is denoted by

$$X(\omega) = F[x(t)]$$

The integral Eq. (12.49) is referred to as the inverse Fourier transform and is often denoted by the symbol $x(t) = F^{-1}[X(\omega)]$.

Fourier transform *is essentially a transformation from a function of the time variable t to the frequency variable* ω. Not all functions can be transformed but satisfying the Dirichlet conditions in any finite interval and

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

are sufficient conditions for the existence of Fourier transforms.

Fourier transform may be represented by writing $X(\omega)$ in terms of magnitude and phase as

$$X(\omega) = |X(\omega)| e^{-j\theta(\omega)} \tag{12.50}$$

Plots of $|X(\omega)|$ and $\theta(\omega) = \underline{/X(\omega)}$ versus frequency f are referred to as the amplitude and phase spectra of $x(t)$ respectively.

12.9 THE ENERGY SPECTRUM

LO 4

The signal is defined as an energy signal if it has finite energy over the interval $(-\infty,\infty)$. So that power is zero. The energy of a signal can be expressed in the frequency domain.

The normalised energy for a signal is defined

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \tag{12.51}$$

Using the Fourier transform representation for $x(t)$, that is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{12.52}$$

we can write the energy as

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\; X(-\omega)\, d\omega \tag{12.53}$$

Therefore, the energy

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, X^*(\omega)\, d\omega \quad \text{since} \quad X(-\omega) = X^*(\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(\omega)\,|^2 \, d\omega \tag{12.54}$$

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(\omega)\,|^2 \, d\omega \tag{12.55}$$

This is referred to as Parseval's theorem for Fourier transforms. Thus, $|X(\omega)|^2$ has the units of energy density with frequency and is called energy density spectrum of $x(t)$.

$$G(\omega) = |\, X(\omega)\,|^2$$

Integration of $G(\omega)$ over all frequencies gives the total energy contained in a signal. Similarly, integration of $G(\omega)$ over a finite range of frequencies gives the energy contained in a signal within the limits of integration.

Frequently Asked Questions linked to LO 4

☆★★**12-4.1** Find the exponential form of Fourier series for the waveform shown in Fig. Q.1. **[BPUT 2007]**

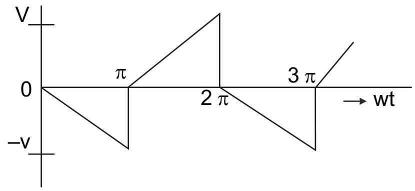


Fig. Q.1

12.10 FOURIER TRANSFORM OF POWER SIGNALS

LO 5 Analyse power signals and periodic signals using Fourier transform

The signal is defined as power signal if and only if it has finite power and infinite energy. The power signals are not absolutely integrable. Power signals do not satisfy the condition.

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

This class of signals are called power signals because the signal energy is infinite over the entire internal, but the power is finite. Therefore, the power

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)\,dt < \infty \tag{12.56}$$

EXAMPLE 12.6

Determine the Fourier transform of a signal given in Fig. 12.8.

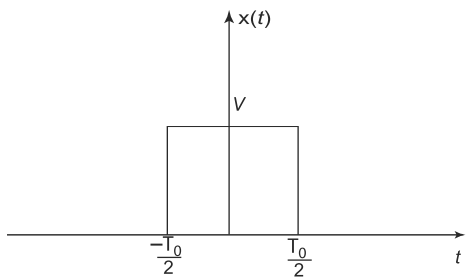


Fig. 12.8

Solution From the waveform shown in Fig. 12.8,

$$x(t) = V \quad \text{for} \quad \frac{-T_0}{2} < t < \frac{T_0}{2}$$
$$= 0 \text{ otherwise} \tag{12.57}$$

The Fourier transform of the expression

$$X(\omega) = \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} V\, e^{-j\omega t}\,dt$$

$$= \frac{V}{j\omega}\left[e^{j\omega\frac{T_0}{2}} - e^{-j\omega\frac{T_0}{2}}\right] \tag{12.58}$$

$$X(\omega) = VT_0 \left[\frac{e^{j\omega\frac{T_0}{2}} - e^{-j\omega\frac{T_0}{2}}}{j2\omega\frac{T_0}{2}} \right]$$

$$= VT_0 \frac{\sin\left(\omega\frac{T_0}{2}\right)}{\omega\frac{T_0}{2}} \tag{12.59}$$

The Fourier transform of the signal is illustrated in Fig. 12.9.

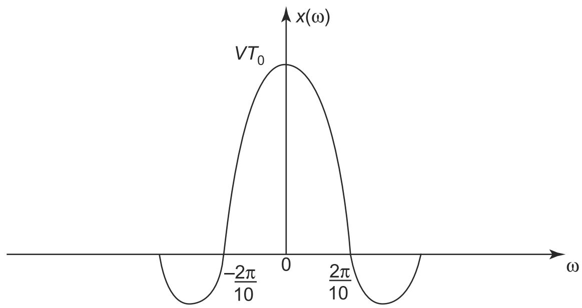


Fig. 12.9

EXAMPLE 12.7

Determine the Fourier transform of the general impulse, $x(t) = A\delta(t)$.

Solution The impulse function for the given equation is shown in Fig. 12.10.

The transform of the above function is

$$X(\omega) = \int_{-\infty}^{\infty} A\,\delta(t)\,e^{-j\omega t}\,dt \tag{12.60}$$

$$X(\omega) = A$$

The transform of the impulse function is shown in Fig. 12.11.

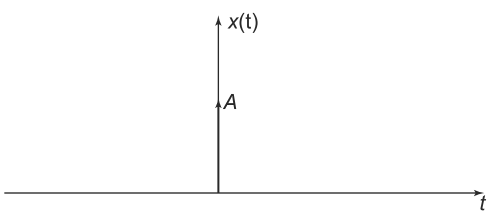


Fig. 12.10

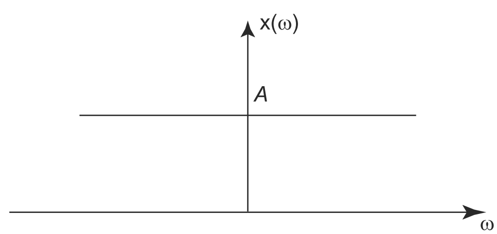


Fig. 12.11

EXAMPLE 12.8

Determine the Fourier transform of the signal shown in Fig. 12.12.

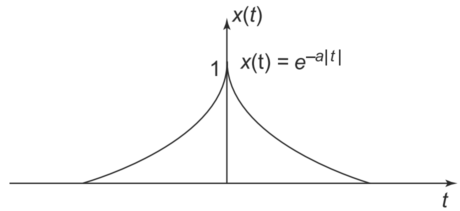


Fig. 12.12

Solution This waveform does not satisfy the condition

$$\int_{-\infty}^{\infty} |X(t)|\,dt < \infty \tag{12.61}$$

However, the transform of $x(t)$ in Fig. 12.12 can be found, if a is allowed to tend to zero. Therefore, Fig. 12.12 can be represented by Fig. 12.13.

The transform of the signal in Fig. 12.13 can be determined.

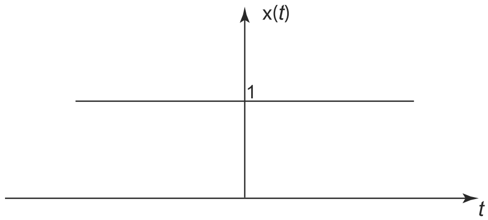


Fig. 12.13

$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\,dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2} \tag{12.62}$$

As $a \to 0$, $X(\omega)$ is zero except when $\omega = 0$

where $X(\omega) = \lim_{a \to 0} \frac{2}{2a} = \infty$

Thus, the transform is an impulse whose strength may be obtained by integrating $X(\omega)$ over the frequency range.

Therefore,
$$\int_{-\infty}^{\infty} X(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2}\,d\omega = 2\pi \tag{12.63}$$

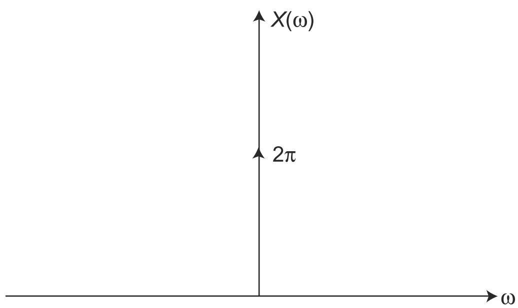


Fig. 12.14

The transform of the signal is shown in Fig. 12.14

EXAMPLE 12.9

Determine the Fourier transform of the signum function shown is Fig. 12.15.

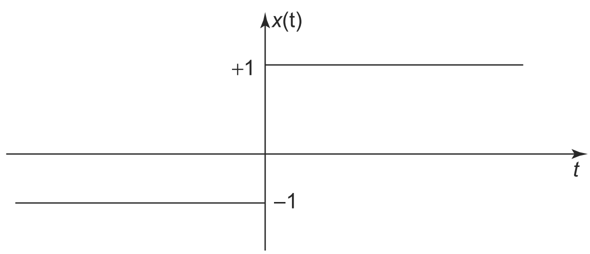


Fig. 12.15

Solution The function is defined as

$$\begin{aligned} \text{sign}(t) &= 1 \quad \text{for } t > 0 \\ &= 0 \quad \text{for } t = 0 \\ &= -1 \quad \text{for } t < 0 \end{aligned} \tag{12.64}$$

The Fourier transform of the function

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} (-1)e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-j\omega t}\,dt \tag{12.65}$$

$$X(\omega) = \frac{+1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega} \tag{12.66}$$

The Fourier transform of signum function is shown in Fig. 12.16.

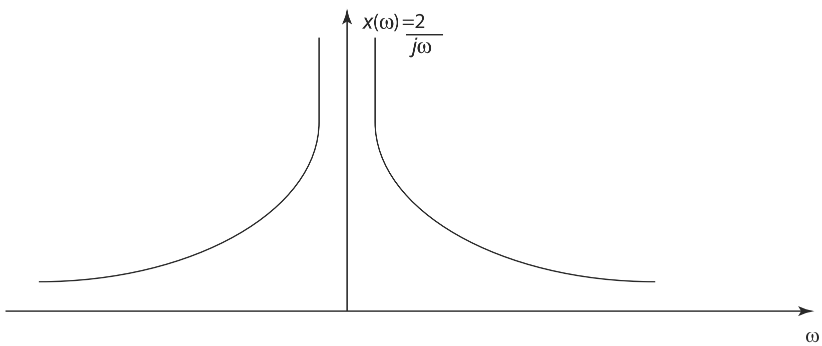


Fig. 12.16

EXAMPLE 12.10

Determine the Fourier transform of unit step function shown in Fig. 12.17.

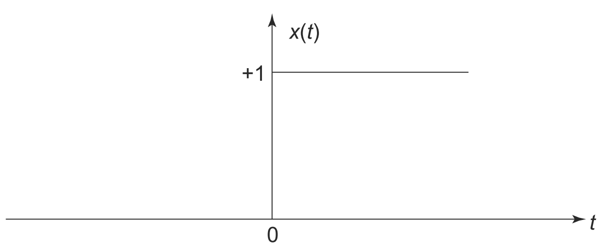


Fig. 12.17

Solution The function is defined as

$$\begin{aligned} u(t) &= 1 \text{ for } t > 0 \\ &= 0 \text{ for } t < 0 \end{aligned} \tag{12.67}$$

This function is obtained by the sum of two functions shown in Fig. 12.14 and in Fig. 12.16 and dividing by 2. The transform is half the sum of transforms of individual time functions.

$$X(\omega)=\frac{1}{2}\left[\frac{2}{j\omega}+2\pi\delta(\omega)\right]=\frac{1}{j\omega}+\pi\delta(\omega) \tag{12.68}$$

The Fourier transform is given below.

$$X(\omega)=\frac{1}{j\omega}+\pi\delta(\omega) \tag{12.69}$$

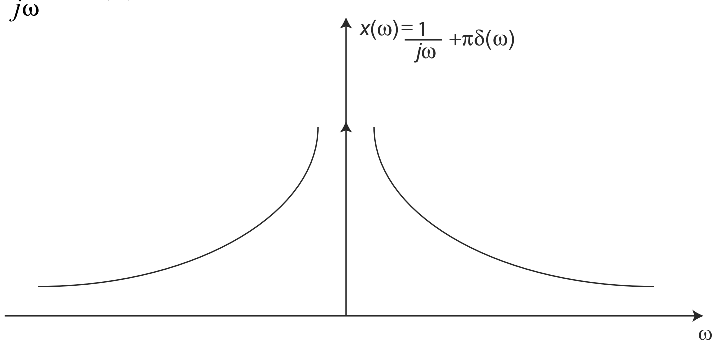


Fig. 12.18

12.11 FOURIER TRANSFORM OF PERIODIC SIGNALS

LO 5

In the previous section, Fourier transform representation for non-periodic signals have been discussed. Fourier transform of periodic signals have been discussed in this section. In general, Fourier transform of periodic signals are not absolutely integrable and have infinite discontinuities. Here, we obtain the Fourier transform of a periodic signal by Fourier transforming its complex Fourier series term-by-term. The Fourier transform consists of a train of impulses in the frequency domain. The area of these impulses is directly proportional to the Fourier series coefficients.

Consider a signal $x(t)$ with Fourier transform $X(\omega)$.

$X(\omega)=2\pi\delta(\omega-\omega_0)$; an impulse at $\omega = \omega_0$. To determine the signal $x(t)$ for the Fourier transform $X(\omega)$, we use inverse Fourier transform relation

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \tag{12.70}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega \tag{12.71}$$

$$x(t)=e^{j\omega_0 t}$$

Any periodic signal is represented by a linear combination of impulses equally spaced in frequency.

$$X(\omega)=\sum_{n=-\infty}^{\infty} 2\pi\, X_n\,\delta(\omega-n\omega_0) \tag{12.72}$$

From the Fourier inverse transform,

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega \tag{12.73}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\omega - n\omega_0)\right] e^{j\omega t}\, d\omega \tag{12.74}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \tag{12.75}$$

Equation (12.75) gives the exponential form of the Fourier series.

EXAMPLE 12.11

Find the Fourier transform of a periodic unit impulse train shown in Fig. 12.19.

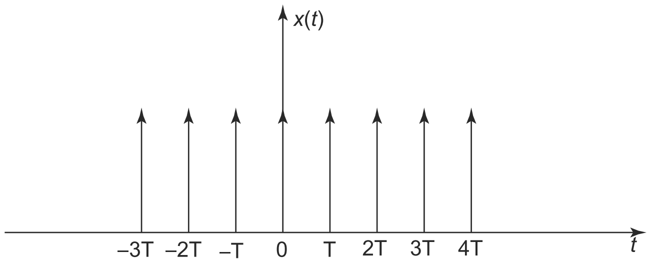


Fig. 12.19

Solution The periodic signal is represented by

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{12.76}$$

By expanding $x(t)$ in a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \tag{12.77}$$

where $$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}\, dt$$

$$X_n = \frac{1}{T} \tag{12.78}$$

Each of the X_n gives the same constant $\frac{1}{T}$

$\therefore$ Fourier series representation of the unit impulse train is

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{12.79}$$

If we take the Fourier transform on both sides, we get

$$X(\omega) = \{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\}$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

where $$\omega_0 = \frac{2\pi}{T} \tag{12.80}$$

Therefore, a unit impulse train in the time domain is transformed into a impulse train in the frequency domain with an area of an each impulse as ω_0.

Practice Problems linked to LO 5

☆★★**12-5.1** Find the Fourier transform of the signal shown in Fig. Q.1.

☆☆★**12-5.2** Find the Fourier transform of the signal shown in Fig. Q.2.

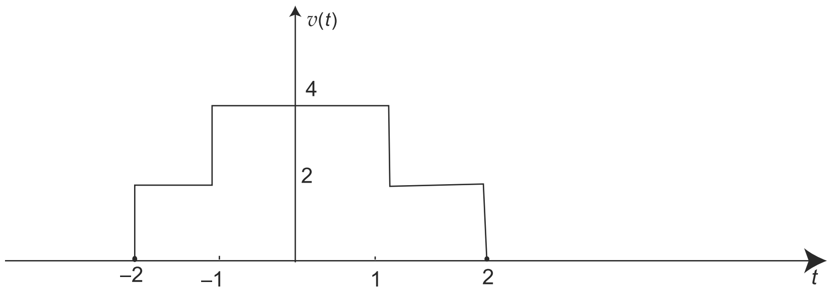


Fig. Q.1

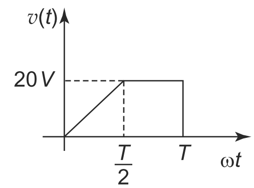


Fig. Q.2

★★★**12-5.3** Find the Fourier transform of the functions.

(i) $v(t) = e^{-t}\, u(t)$ (ii) $v(t) = e^{-|t|}\, u(-t)$

☆☆★**12-5.4** Find the Fourier transform of the functions given below.

(i) $x(t) = e^{-at} \cos bt$ (ii) $x(t) = \sin(\omega_c t + \theta)$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

12.12 PROPERTIES OF THE FOURIER TRANSFORM

Properties of the Fourier transform facilitate the transformation from the time domain to the frequency domain and vice versa.

LO 6 List the properties of the Fourier transform

12.2.1 Linearity

The Fourier transform satisfies linearity and principle of superposition. Consider two signals $x_1(t)$ and $x_2(t)$.

If $x_1(t) \leftrightarrow X_1(\omega)$

$x_2(t) \leftrightarrow X_2(\omega)$

then $$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega) \tag{12.81}$$

Proof: $F[a_1 x_1(t) + a_2 x_2(t)]$

$$= a_1 F[x_1(t)] + a_2 F[x_2(t)]$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega) \tag{12.82}$$

Therefore, the linearity is proved.

12.12.2 Scaling

Consider the Fourier transform of $x(t)$ is $X(\omega)$.

If $x(t) \leftrightarrow X(\omega)$

then for real constant a

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{12.83}$$

Proof: Assume $a > 0$. Then Fourier transform of $x(at)$ is

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t} dt \quad (12.84)$$

Let $\qquad m = at$

Then $\qquad dm = adt$

$$\therefore \qquad F\{x(at)\} = \int_{-\infty}^{\infty} x(m) e^{-\left(j\omega\frac{m}{a}\right)} \cdot \frac{dm}{a} = \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad (12.85)$$

If $a < 0$ then

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t} dt \quad (12.86)$$

Now, $\qquad -m = +at$

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(-m) e^{-j\left(\frac{\omega}{a}\right)(-m)} \left(-\frac{dm}{a}\right)$$

$$= \frac{-1}{a} \int_{-\infty}^{\infty} x(-m) e^{-j\left(\frac{-m}{a}\right)\omega} dm$$

$$= \frac{-1}{a} X(\omega) \quad (12.87)$$

Combining these two values, we get

$$x(at) \leftrightarrow \frac{1}{|a|} X(\omega) \quad (12.88)$$

12.12.3 Symmetry

If $\qquad x(t) \leftrightarrow X(\omega)$

then

$X(t) \leftrightarrow 2\pi\, X(-\omega)$

Proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{+j\omega t} d\omega \quad (12.89)$

Then replacing the dummy variable ω by ω',

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega') e^{j\omega' t} d\omega' \quad (12.90)$$

Now replace t by ω

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(\omega') e^{j\omega' \omega} d\omega' \quad (12.91)$$

Finally, replace ω' by t, thus

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{j\omega t} dt = F\{X(t)\} \quad (12.92)$$

$X(t) \leftrightarrow 2\pi\, X(-\omega)$

12.12.4 Convolution

Fourier transform makes the convolution of two signals into the product of their Fourier transforms. There are two types of convolution properties, one for the time domain and one for the frequency domain.

❑ ***Time Convolution***

If $x_1(t) \leftrightarrow X_1(\omega)$

$x_2(t) \leftrightarrow X_2(\omega)$

then

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \leftrightarrow Y(\omega) \tag{12.93}$$

$$= X_1(\omega)\, X_2(\omega)$$

Proof: $$F\{y(t)\} = Y(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]dt \tag{12.94}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dt\right]d\tau \tag{12.95}$$

Now, let $k = t-\tau$, then $dk = dt$ and $t = k + \tau$

$$\therefore \quad Y(\omega) = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(k)e^{-j\omega(k+\tau)}dt\right]d\tau \tag{12.96}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} \int_{-\infty}^{\infty} x_2(k)\, e^{-j\omega k} dk \tag{12.97}$$

$$Y(\omega) = X_1(\omega)\, X_2(\omega) \tag{12.98}$$

❑ ***Frequency Convolution***

If $x_1(t) \leftrightarrow X_1(\omega)$

$x_2(t) \leftrightarrow X_2(\omega)$

then

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \tag{12.99}$$

Proof: Considering the inverse transform of $[X_1(\omega) * X_2(\omega)]/2\pi$, we have

$$F^{-1}\left\{\frac{X_1(\omega) * X_2(\omega)}{2\pi}\right\} = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{j\omega t} \int_{-\infty}^{\infty} X_1(u)X_2(\omega-u)\, du \cdot d\omega$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} X_1(u) \int_{-\infty}^{\infty} X_2(\omega-u)\, e^{j\omega t} d\omega \cdot du \tag{12.100}$$

Let $m = \omega - u$ then $dm = d\omega$ and $\omega = m + u$

Thus,

$$F^{-1}\left\{\frac{x_1(\omega) * x_2(\omega)}{2\pi}\right\} = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} X_1(u) \int_{-\infty}^{\infty} X_2(m) e^{j(m+u)t}\, dm \cdot du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(u)\, e^{jut}\, dt \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} x_2(m)\, e^{jmt}\, dm$$

$$= x_1(t)\cdot x_2(t) \tag{12.101}$$

Therefore, $x_1(t)\cdot x_2(t) \leftrightarrow \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$ (12.102)

12.12.5 Frequency Differentiation and Integration

If $x(t) \leftrightarrow X(\omega)$

then $(-jt)\; x(t) \leftrightarrow \dfrac{dx(\omega)}{d\omega}$

Proof: Consider the Fourier transform $x(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{12.103}$$

Differentiating with respect to ω,

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} (-jt)\, x(t) e^{-j\omega t}\, dt \tag{12.104}$$

Fourier transform of $(-jt)\; x(t)$ is $\dfrac{dX(\omega)}{d\omega}$

$$(-jt)\; x(t) \leftrightarrow \frac{dX(\omega)}{d\omega} \tag{12.105}$$

In general,

$$(-jt)^n\; x(t) \leftrightarrow \frac{d^n X(\omega)}{d\omega^n} \tag{12.106}$$

Similarly, in the frequency integration,

If $x(t) \leftrightarrow X(\omega)$

then $\dfrac{x(t)}{-jt} \leftrightarrow \displaystyle\int_0^{\omega} X(\omega)\, d\omega$ (12.107)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\; dt \tag{12.108}$$

$$\int_0^{\omega} X(\omega)\, d\omega = \int_{-\infty}^{\infty} x(t)\, dt \int_0^{\omega} e^{-j\omega t}\; d\omega$$

$$= \int_{-\infty}^{\infty} x(t)\, dt\, \frac{e^{-j\omega t}}{-jt} = \frac{1}{-jt}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{x(t)}{-jt} \leftrightarrow \int_0^{\omega} X(\omega)\, d\omega \tag{12.109}$$

12.12.6 Time Shifting

If $x(t) \leftrightarrow X(\omega)$

Then $x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

From the above, we can say that shift in time domain implies shift in phase in the transform.

$$\textit{Proof}: \; F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t} dt \tag{12.110}$$

Let $t - t_0 = m$ or $t = m + t_0 : dt = dm$

$$\text{Therefore,} \quad F[x(t-t_0)] = \int_{-\infty}^{\infty} x(m)\, e^{-j\omega(t_0+m)} dm \tag{12.111}$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(m) e^{-j\omega m} dm = e^{-j\omega t_0} X(\omega)$$

$$\therefore \quad x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega) \tag{12.112}$$

12.12.7 Frequency Shifting

If $x(t) \leftrightarrow X(\omega)$

$$\text{then} \quad x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \tag{12.113}$$

Frequency shifting or translation has an important significance in communication engineering. This process is known as *modulated signal*.

$$\textit{Proof}: \; F\{x(t)\, e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\; e^{-j\omega t} dt \tag{12.114}$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t} dt$$

$$= X(\omega - \omega_0)$$

$$\text{Therefore,} \quad x(t)\, e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \tag{12.115}$$

Table 12.1 *Properties of the Fourier Transform*

1.	Transformation	$x(t) \leftrightarrow X(\omega)$
2.	Linearity	$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$
3.	Scaling	$x(at) \leftrightarrow \frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
4.	Symmetry	$X(jt) \leftrightarrow 2\pi\, X(-\omega)$

5.	Time shifting	$x(t-t_0) \leftrightarrow e^{-j\omega t_0}\ X(\omega)$
6.	Frequency shifting	$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega-\omega_0)$
7.	Time convolution	$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)X_2(\omega)$
8.	Frequency convolution	$x_1(t)x_2(t) \leftrightarrow \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$
9.	Reversal	$x(-t) \leftrightarrow X(-\omega)$
10.	Time differentiation	$\dfrac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)$
11.	Time integration	$\displaystyle\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
12.	Frequency differentiation	$-jt\ x(t) \leftrightarrow \dfrac{dX(\omega)}{d\omega}$
13.	Frequency Integration	$\dfrac{x(t)}{-jt} \leftrightarrow \displaystyle\int_{0}^{\omega} X(\omega)\, d\omega$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

★★★**12-6.1** Find the Inverse Fourier transform shown in Fig. Q.1. Use properties of Fourier transform to help to determine the time function.

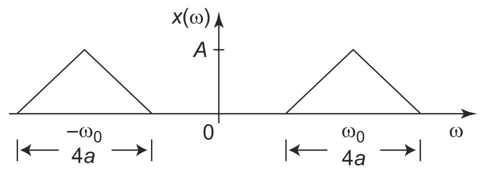


Fig. Q.1

Frequently Asked Questions linked to LO 6

☆☆★**12-6.1** What is waveform symmetry? **[RGTU June 2014]**

12.13 APPLICATIONS IN CIRCUIT ANALYSIS

LO 7 Describe the applications of Fourier series in circuit analysis

A voltage $v(t)$ represented by Fourier series can be applied to a linear circuit to obtain the corresponding harmonic terms of the current series. This result is obtained by superposition principle. We consider each term of the Fourier series representing the voltage as a single source as shown in Fig. 12.20. The equivalent impedance of the network at each harmonic frequency is used to compute the current at that harmonic. The sum of these individual responses is the total response i in series form due to the applied voltage.

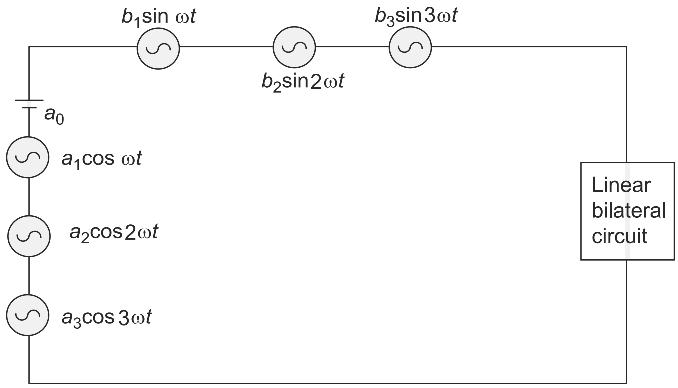


Fig. 12.20

The non-sinusoidal voltage $v(t)$ is represented by a Fourier series

$$v(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{12.116}$$

The effective value of a voltage waveform is given by

$$V_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T |v(t)|^2 \, dt}$$

From Eq. (12.116), we have

$$V_{\text{rms}} = V_{\text{eff}} = \left\{ \frac{1}{T} \int_0^T \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \right]^2 dt \right\}^{1/2}$$

$$= \left\{ a_0^2 + \frac{1}{2} \left[a_1^2 + a_2^2 + a_3^2 + \cdots + b_1^2 + b_2^2 + b_3^2 + \cdots \right] \right\}^{1/2}$$

$$= \left[A_0^2 + \frac{1}{2} \left(A_1^2 + A_2^2 + A_3^2 + \cdots \right) \right]^{1/2} \tag{12.117}$$

where $A_n^2 = a_n^2 + b_n^2$, A_0 is the average value and $A_1, A_2, A_3 \ldots$ are the amplitudes of the harmonics

If $\quad v(t) = V_0 + \sum V_n \sin(n\omega t + \phi_n)$

and $\quad i(t) = I_0 + \sum I_n \sin(n\omega t + \theta_n)$

Then their effective values are given by

$$V_{\text{rms}} = \left[V_0^2 + \frac{1}{2} \left(V_1^2 + V_2^2 + V_3^2 + \cdots \right) \right]^{1/2}$$

$$I_{\text{rms}} = \left[I_0^2 + \frac{1}{2} \left(I_1^2 + I_2^2 + I_3^2 + \cdots \right) \right]^{1/2}$$

or

$$V_{\text{rms}} = \left[V_0^2 + V_{\text{rms}1}^2 + V_{\text{rms}2}^2 + V_{\text{rms}3}^2 + \cdots \right]^{1/2} \tag{12.118}$$

$$I_{\text{rms}} = \left[I_0^2 + I_{\text{rms1}}^2 + I_{\text{rms2}}^2 + I_{\text{rms3}}^2 + \cdots\right]^{1/2} \tag{12.119}$$

The average power is given by

$$P = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt$$
$$= \frac{1}{T}\int_0^T \left[V_0 + \sum V_n \sin\left(n\omega t + \phi_n\right)\right]\left[I_0 + \sum I_n \sin\left(n\omega t + \theta_n\right)\right] dt$$

After simplification, we get

$$P = \frac{1}{T}\int_0^T V_0 I_0 dt + \frac{1}{T}\int_0^T V_n \sin\left(n\omega t + \phi_n\right) \mathrm{I}_n \sin\left(n\omega t + \theta_n\right) dt \tag{12.120}$$
$$= V_0 I_0 + \sum \frac{1}{2} V_n I_n \cos\left(\phi_n - \theta_n\right)$$
$$= V_0 I_0 + \sum V_{\text{rms}n}\, I_{\text{rms}n} \cos\psi_n \tag{12.121}$$

where $\psi_n = \phi_n - \theta_n$

and $V_{\text{rms}n} = \dfrac{V_n}{\sqrt{2}}$; $I_{\text{rms}n} = \dfrac{I_n}{\sqrt{2}}$

The power factor is the ratio of average power to the apparent power.

Apparent power $V_{\text{rms}}\, I_{\text{rms}} = \left(\sqrt{V_0^2 + V_{\text{rms1}}^2 + V_{\text{rms2}}^2 + \cdots}\right) \times \left(\sqrt{I_0^2 + I_{\text{rms1}}^2 + I_{\text{rms2}}^2 + \cdots}\right)$

$$\text{The power factor p.f} = \frac{V_0 I_0 + \left(V_{\text{rms}n}\, I_{\text{rms}n} \cos\phi_n\right)}{\sqrt{\left(V_0^2 + V_{\text{rms1}}^2 + V_{\text{rms2}}^2 + \cdots\right)\left(I_0^2 + I_{\text{rms1}}^2 + I_{\text{rms2}}^2 + \cdots\right)}}$$

EXAMPLE 12.12

For the circuit shown in Fig. 12.21, find the current and the average power where v(t) = 100 + 50 sin ωt + 25 sin3 ωt.

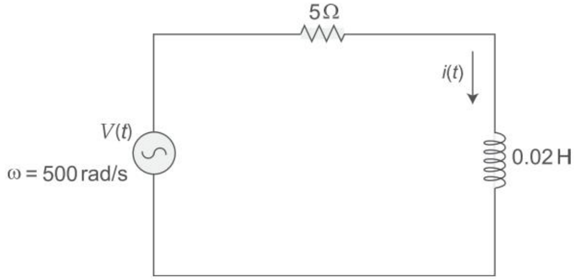


Fig. 12.21

Solution At each frequency, we have to calculate the equivalent impedance of the circuit.

At $\omega = 0$, impedance $Z = 5\,\Omega$

and $I_0 = \dfrac{V_0}{R} = \dfrac{100}{5} = 20$ A

At ω = 500 rad/s, impedance $Z_1 = 5 + j\,(500)(0.02)$

$= 5 + j\,10\,\Omega$

$$i_1(t) = \frac{V_1}{|Z_1|}\sin(\omega t - \theta_1) = \frac{50}{11.15}\sin(\omega t - 63.4°)$$

$$i_1(t) = 4.48\sin(\omega t - 63.4°)\text{ A}$$

At ω = 1500 rad/s, impedance $Z_3 = 5 + j\,(1500)(0.02)$

$= 5 + j\,30$

$$i_3(t) = \frac{V_3}{|Z_3|}\sin(3\omega t - \theta_3) = \frac{25}{30.4}\sin(3\omega t - 80.54°)$$

$$i_3(t) = 0.823\sin(3\omega t - 80.54°)\text{ A}$$

The total current

$$i(t) = 20 + 4.48\sin(\omega t - 63.4°) + 0.823\sin(3\omega t - 80.54°)\text{ A}$$

The effective value of the current $i(t)$

$$I_{\text{rms}} = \sqrt{20^2 + \frac{4.48^2}{2} + \frac{0.823^2}{2}} = 20.25\text{ A}$$

The average power

$$P = I_{\text{rms}}^2\,R$$

$$= (410.6)5 = 2053\text{ W}$$

Additional Solved Problems

PROBLEM 12.1

Find the Fourier series of the periodic function v(t) as shown in Fig. 12.22.

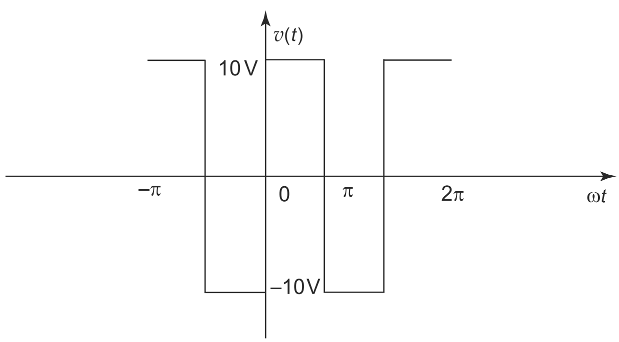


Fig. 12.22

Solution The equation for Fig. 12.22 is given by

$$\begin{aligned} v(t) &= -10 \quad \text{for} \quad -\pi < \omega t < 0 \\ &= \;\;10 \quad \text{for} \quad 0 < \omega t < \pi \end{aligned} \tag{12.122}$$

We obtain $a_0 = 0$, since the area under the curve of $v(t)$ between $-\pi$ and π is zero.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(t)\cos n\omega t \; d(\omega t)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-10)\cos n\omega t \; d(\omega t) + \int_{0}^{\pi}(10)\cos n\omega t \; d(\omega t)\right]$$

$$= \frac{1}{\pi}\left[-10\frac{\sin n\omega t}{n}\bigg)_{-\pi}^{0} + 10\frac{\sin n\omega t}{n}\bigg)_{0}^{\pi}\right]$$

$= 0$ because $\sin \omega nt = 0$ at $-\pi$, 0 and π for $n = 1, 2...$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(t)\sin n\omega t \; d(\omega t)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-10)\sin n\omega t \; d(\omega t) + \int_{0}^{\pi}(10)\sin n\omega t \; d(\omega t)\right]$$

$$= \frac{1}{\pi}\left[(10)\frac{\cos n\omega t}{n}\bigg)_{-\pi}^{0} - 10\frac{\cos n\omega t}{n}\bigg)_{0}^{\pi}\right]$$

$$b_n = \frac{10}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right]$$

$$= \frac{20}{n\pi}(1 - \cos n\pi)\left(\because \quad \cos\pi = -1, \cos 2\pi = 1, \cos 3\pi = -1\right)$$

$1 - \cos n\pi = 2$ for odd n

$= 0$ for even n

The Fourier series of $v(t)$ is

$$v(t) = \frac{40}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right) \tag{12.123}$$

PROBLEM 12.2

Find the trigonometric Fourier series for the waveform shown in Fig. 12.23 and plot the spectrum.

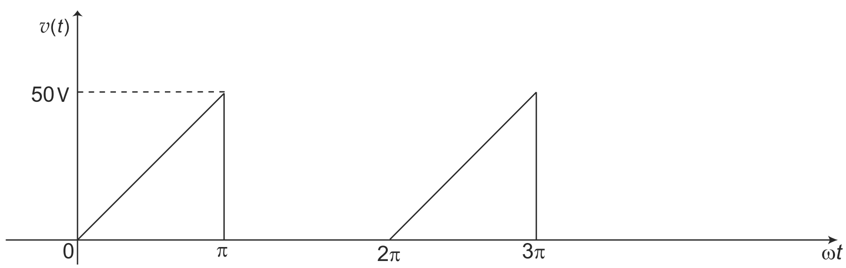


Fig. 12.23

Solution The equation for Fig. 12.23 is given by

$$v(t) = \frac{50}{\pi}\omega t \quad 0 < \omega t < \pi \tag{12.124}$$

$$= 0 \qquad \pi < \omega t < 2\pi$$

The average value of the wave is

$$a_v = \frac{1}{2\pi}\int_0^{\pi} \frac{50}{\pi}\,\omega t\; d(\omega t)$$

$$= \frac{1}{2\pi}\left[\frac{50}{\pi}\left(\frac{\omega t}{2}\right)_0^{\pi}\right] = \frac{50}{2\pi^2}\left[\frac{\pi}{2} - 0\right] = \frac{25}{4\pi} = \frac{12.5}{\pi}$$

Since the wave is neither even nor odd, the series contains both sine and cosine terms.

$$a_n = \frac{1}{\pi}\int_0^{\pi}\left(\frac{50}{\pi}\right)\omega t\,\cos n\omega t\; d(\omega t)$$

$$= \frac{50}{\pi^2}\left[\frac{1}{n^2}\cos n\omega t + \frac{\omega t}{n}\sin n\omega t\right]_0^{\pi}$$

$$= \frac{50}{\pi^2 n^2}[\cos n\pi - 1]$$

$$a_n = 0 \text{ for } n \text{ is even } (\because \quad \cos n\pi - 1 = 0)$$

$$a_n = \frac{-100}{\pi^2 n^2} \qquad \text{for } n \text{ odd}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\left(\frac{50}{\pi}\right)\omega t\,\sin n\omega t\; d(\omega t)$$

$$= \frac{50}{\pi^2}\left[\frac{1}{n^2}\sin \omega t - \frac{\omega t}{n}\cos n\omega t\right]_0^{\pi} = \frac{-50}{\pi n}\cos n\pi$$

$$b_n = \frac{-50}{\pi n} \quad \text{for } n \text{ even}$$

$$b_n = \frac{-50}{\pi n} \quad \text{for } n \text{ odd}$$

The Fourier series of the waveform is

$$v(t) = 12.5 - \frac{100}{\pi^2}\cos\omega t - \frac{100}{(3\pi)^2}\cos 3\omega t - \frac{100}{(5\pi)^5}\cos 5\omega t - \ldots$$

$$+ \frac{50}{\pi}\sin\omega t - \frac{50}{2\pi}\sin 2\omega t + \frac{50}{3\pi}\sin 3\omega t \tag{12.125}$$

The spectrum is given in Fig. 12.24.

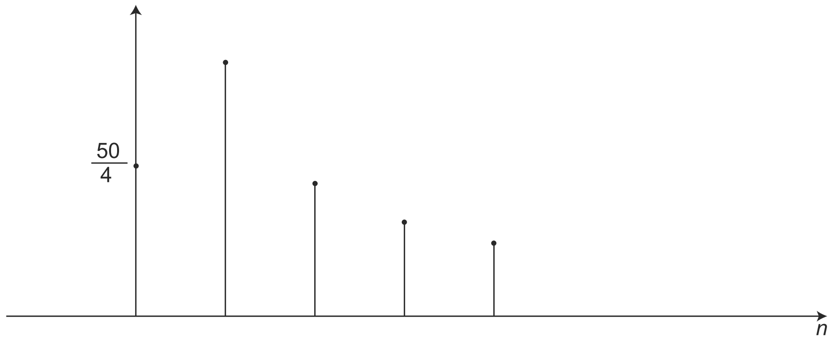


Fig. 12.24

PROBLEM 12.3

Find the exponential Fourier series for the waveform shown in Fig. 12.25 and sketch the spectrum.

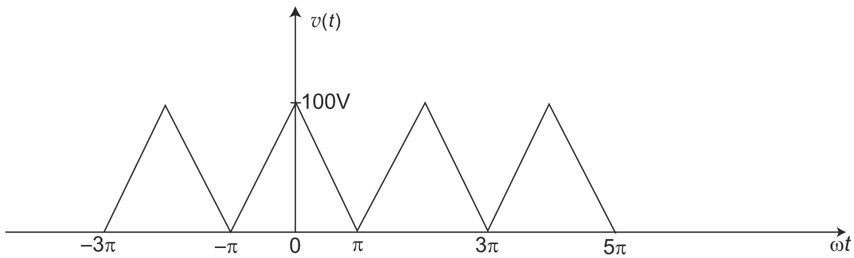


Fig. 12.25

Solution The equation for the waveform shown in Fig. 12.25 is given by

$$v(t) = 100 + \left(\frac{100}{\pi}\right)\omega t \quad \text{for } 0 < \omega t < \pi$$

$$v(t) = 100 - \left(\frac{100}{\pi}\right)\omega t \quad \text{for } -\pi < \omega t < 0 \tag{12.126}$$

The waveform is even and the coefficients of A_n are pure real. By inspection, the average value is

$$C_0 = \frac{100}{2} = 50\,\text{V}$$

$$C_n = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}\left[\left(100 + \frac{100}{\pi}\right)\omega t\right]e^{-jn\omega t}\,d(\omega t) + \int_{0}^{\pi}\left[\left(100 - \frac{100}{\pi}\right)\omega t\right]e^{-jn\omega t}\,d(\omega t)\right\}$$

$$= \frac{100}{2\pi^2}\left\{\int_{-\pi}^{0}\omega t\, e^{-jn\omega t}\,d(\omega t) + \int_{0}^{\pi}(-\omega t)e^{-jn\omega t}\,d(\omega t) + \int_{-\pi}^{\pi}\pi e^{-jn\omega t}\,d(\omega t)\right\}$$

$$= \frac{100}{2\pi^2}\left\{\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_{-\pi}^{0} - \left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_{0}^{\pi}\right\}$$

$$= \frac{100}{\pi^2 n^2}\left[1 - e^{jn\pi}\right]$$

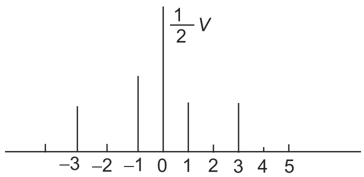


Fig. 12.26

$$C_n = 0 \qquad \text{for } n \text{ even}$$

$$C_n = \frac{200}{\pi^2 n^2} \qquad \text{for } n \text{ odd}$$

The exponential Fourier series is

$$v(t) = \cdots + \frac{200}{(-3\pi)^2} e^{-j3\omega t} + \frac{200}{(-\pi)^2} e^{-j\omega t} + \frac{100}{2}$$

$$+ \frac{200}{(\pi)^2} e^{j\omega t} + \frac{200}{(3\pi)^2} e^{j\omega t} + \ldots\ldots \qquad (12.127)$$

The spectrum is shown in Fig. 12.26.

PROBLEM 12.4

Find the exponential Fourier series of the triangular waveform shown in Fig. 12.27.

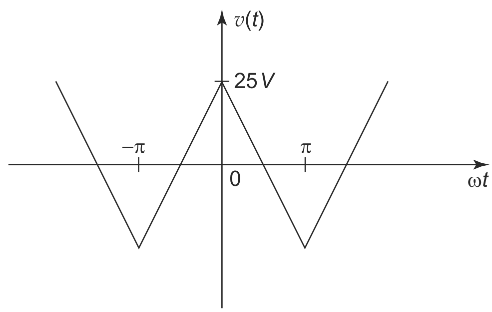


Fig. 12.27

Solution The voltage equation for the waveform is shown in Fig. 12.27.

$$v(t) = 25 + \left(\frac{50}{\pi}\right)\omega t \quad \text{for} \quad -\pi < \omega t < 0$$

$$= 25 - \left(\frac{50}{\pi}\right)\omega t \quad \text{for} \quad 0 < \omega t < \pi \qquad (12.128)$$

The coefficients of the exponential series

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{0}\left[25 + \left(\frac{50}{\pi}\right)\omega t\right] e^{-jn\omega t}\, d(\omega t) + \frac{1}{2\pi}\int_{0}^{\pi}\left[25 - \left(\frac{50}{\pi}\right)\omega t\right] e^{-jn\omega t}\, d(\omega t)$$

$$C_n = \frac{4 \times 25}{\pi^2 n^2} \quad \text{for } n \text{ odd}$$

$$C_n = 0 \quad \text{for } n \text{ even}$$

PROBLEM 12.5

Find the trigonometric Fourier series for the half-wave rectified sine wave shown in Fig. 12.28 and sketch the spectrum.

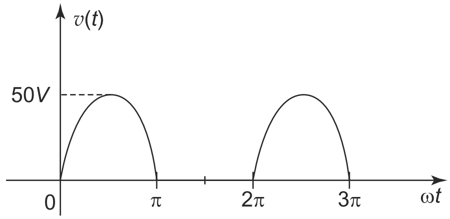


Fig. 12.28

The average value of the waveform

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 50\sin\omega t \; d(\omega t) = \frac{50}{2\pi}[-\cos\omega t]_o^{\pi} = \frac{50}{\pi}$$

The series contains both sine terms and cosine terms

$$a_n = \frac{1}{\pi}\int_0^{\pi} 50\sin\omega t \; \cos n\omega t \; d(\omega t)$$

$$= \frac{50}{\pi}\left[\frac{-n\sin\omega t \; \sin n\omega t - \cos n\omega t \; \cos\omega t}{-n^2+1}\right]_0^{\pi}$$

$$= \frac{50}{\pi(1-n^2)}(\cos n\pi + 1)$$

$$a_n = \frac{100}{\pi(1-n^2)} \quad \text{for } n \text{ even}$$

$$a_n = 0 \qquad\qquad \text{for } n \text{ odd}$$

However, this expression is indeterminate for $n = 1$ and, therefore, we must integrate seperately for a_1,

$$a_1 = \frac{1}{\pi}\int_0^{\pi} 50\sin\omega t \; \cos\omega t \; d(\omega t) = \frac{50}{\pi}\int_0^{\pi}\frac{1}{2}\sin 2\omega t \; d(\omega t) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 50\sin\omega t \; \sin n\omega t \; d(\omega t) = \frac{50}{\pi}\left[\frac{n\sin\omega t\cos n\omega t - \sin n\omega t\cos\omega t}{-n^2+1}\right]_0^{\pi} = 0$$

Here again, the expression is indeterminate for $n = 1$, and b_1 is evaluated separately.

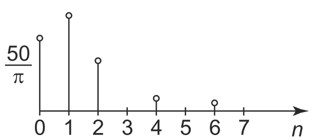


Fig. 12.29

$$b_1 = \frac{1}{\pi}\int_0^{\pi} 50\sin^2\omega t \; d(\omega t) = \frac{50}{\pi}\left[\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right]_0^{\pi} = 25$$

Then the Fourier series is

$$v(t) = \frac{50}{\pi}\left\{1 + \frac{\pi}{2}\sin\omega t - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \ldots\right\} \quad (12.129)$$

The spectrum is shown in Fig. 12.29.

PROBLEM 12.6

Find the Fourier transform of a single triangular pulse shown in Fig. 12.30 and draw its spectrum.

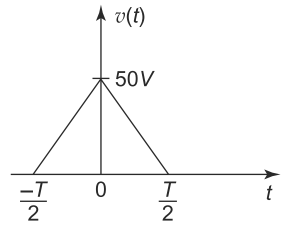


Fig. 12.30

Solution The equation for the waveform is

$$v(t) = 50\left(1 - \frac{2}{T}|t|\right) \tag{12.130}$$

$$v(\omega) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t} dt$$

$$= \int_{\frac{-T}{2}}^{\frac{T}{2}} 50\left[1 - \frac{2}{T}|t|\right] e^{-j\omega t} dt$$

$$= 50\int_{\frac{-T}{2}}^{\frac{T}{2}} e^{-j\omega t} dt - \frac{100}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} |t|\, e^{-j\omega t} dt \quad = \frac{8\times 50}{\omega^2 T}\sin^2\left(\frac{\omega T}{4}\right)$$

$$= \frac{8\times 50}{\omega^2 T}\times\frac{\omega^2 T^2}{16}\,\frac{\sin^2\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2}$$

$$= 25T\,\frac{\sin^2\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)^2} \tag{12.131}$$

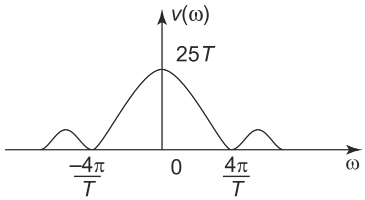


Fig. 12.31

The spectrum is shown in Fig. 12.31.

PROBLEM 12.7

Determine the Fourier transform of the pulse shown in Fig. 12.32.

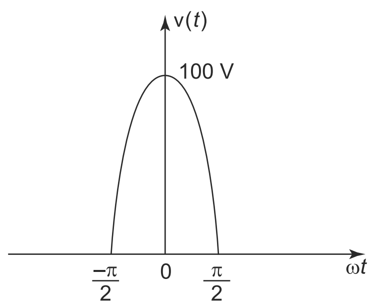


Fig. 12.32

Solution The equation of the waveform in Fig. 12.32 is given by

$$v(t) = 100\cos\omega t \quad \text{for} \quad \frac{-\pi}{2} < \omega t < \frac{\pi}{2} \tag{12.132}$$

$$V(\omega) = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 100\cos\omega t\cdot e^{-j\omega t} d(\omega t) \tag{12.133}$$

$$= 200\int_0^{\frac{\pi}{2}} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) e^{-j\omega t} d(\omega t)$$

$$= 200\int_0^{\frac{\pi}{2}} \frac{1}{2} d(\omega t) + 200\int_0^{\frac{\pi}{2}} \frac{1}{2} e^{-2j\omega t} d(\omega t)$$

$$= 100(\omega t)_0^{\pi/2} + 100\left(\frac{e^{-2j\omega t}}{-2j}\right)_0^{\frac{\pi}{2}}$$

$$= 100 \times \frac{\pi}{2} + \left(\frac{-50}{j}\right)\left[e^{-j2\frac{\pi}{2}} - e^0\right]$$

$$= 50\pi + \frac{50}{j} - \frac{50}{j} e^{-j\pi}$$

$$= 50\pi - j50 + j50e^{-j\pi}$$

$$V(\omega) = 50\left[\pi - j\left(1 - e^{-j\pi}\right)\right] \tag{12.134}$$

PROBLEM 12.8

Obtain the Fourier transform of the function shown in Fig. 12.33.

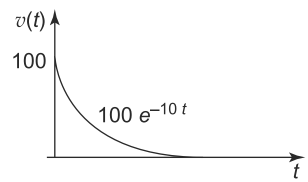


Fig. 12.33

Solution The Fourier transform of the waveform shown in Fig. 12.33 is

$$V(\omega) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t} dt \tag{12.135}$$

$$= \int_0^{\infty} 100\, e^{-10t}\; e^{-j\omega t} dt$$

$$= \int_0^{\infty} 100\, e^{-(10+j\omega)t} dt$$

$$V(\omega) = \frac{-100}{10 + j\omega}\left[e^{-(10+j\omega)t}\right]_0^{\infty}$$

$$V(\omega) = \frac{100}{10 + j\omega} \tag{12.136}$$

PROBLEM 12.9

Obtain the magnitude and phase spectrum of the waveform shown in Fig. 12.34.

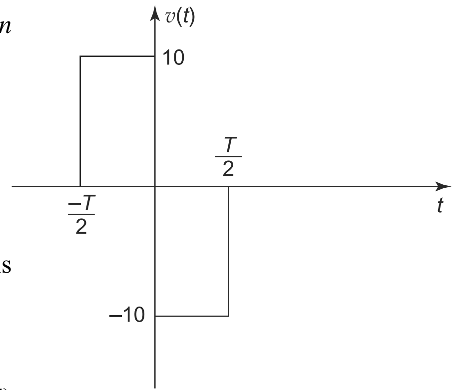


Fig. 12.34

Solution The equation of the voltage waveform in Fig. 12.34 is given by

$$v(t) = \begin{cases} 10 & \text{for } \dfrac{-T}{2} \le t \le 0 \\ -10 & \text{for } 0 \le t < \dfrac{T}{2} \\ 0 & \text{otherwise} \end{cases} \tag{12.137}$$

$$V(\omega) = F[v(t)] = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt$$

$$= \int_{\frac{-T}{2}}^{0} v(t)\, e^{-j\omega t}\, dt + \int_{0}^{\frac{T}{2}} v(t)\, e^{j\omega t}\, dt$$

$$= \int_{\frac{-T}{2}}^{0} 10\, e^{-j\omega t}\, dt + \int_{0}^{\frac{T}{2}} (-10)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\frac{T}{2}} 10\, e^{j\omega t}\, dt - \int_{0}^{\frac{T}{2}} (+10)\, e^{-j\omega t}\, dt$$

$$= 10 \int_{0}^{\frac{T}{2}} \left(e^{j\omega t} - e^{-j\omega t}\right) dt = 20\, j \int_{0}^{\frac{T}{2}} \sin \omega t\; dt$$

$$= 20\, j \left[\frac{-\cos \omega t}{\omega}\right]_{0}^{\frac{T}{2}}$$

$$V(\omega) = \frac{20\, j}{\omega}\left[1 - \cos \frac{\omega T}{2}\right] \tag{12.138}$$

The magnitude of the spectrum is

$$|V(\omega)| = \frac{20}{\omega}\left[1 - \cos \frac{\omega T}{2}\right] \tag{12.139}$$

Phase spectrum is

$$\phi(\omega) = \tan^{-1}\left\{\frac{\mathrm{Im}(V(\omega))}{\mathrm{Re}(V(\omega))}\right\}$$

$$= \tan^{-1}\left\{\frac{\frac{20}{\omega}\left(1-\cos\frac{\omega T}{2}\right)}{0}\right\}$$

$$= \begin{cases} +\dfrac{\pi}{2} & \text{for } \omega \text{ positive} \\ -\dfrac{\pi}{2} & \text{for } \omega \text{ negative} \end{cases} \tag{12.140}$$

The magnitude and phase spectrum are shown in Fig. (12.35).

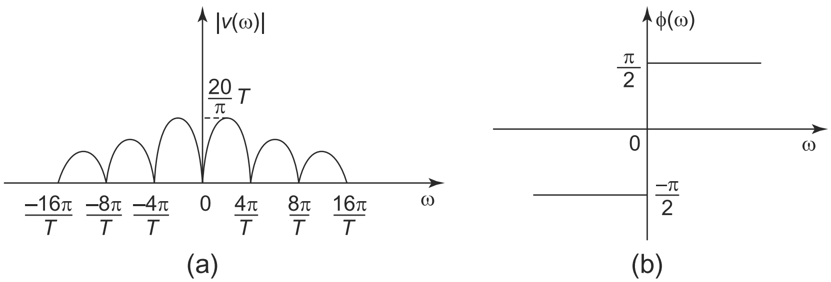


Fig. 12.35

PROBLEM 12.10

Determine the Fourier transform of the sinc function shown in Fig. 12.36.
x(t) = 50 sinc (2Tt)

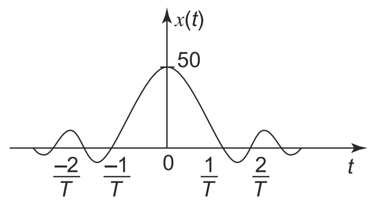


Fig. 12.36

Solution The transform pair is given by

$$A\,x\left(\frac{t}{T}\right) \leftrightarrow AT\,\mathrm{sinc}(\omega T)$$

Applying duality and scaling property,

$$50\sin(2Tt) \leftrightarrow \frac{50}{2T}X\left(\frac{-\omega}{2T}\right) \tag{12.141}$$

Rectangular function is an even function, we get

$$50\sin(2Tt) \leftrightarrow \frac{50}{2T}X\left(\frac{\omega}{2T}\right) \tag{12.142}$$

The spectrum of sinc function is shown in Fig. 12.37.

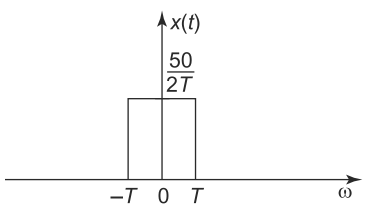


Fig. 12.37

PROBLEM 12.11

Find the Fourier transform of exponentially damped sinusoidal waveform shown in Fig. 12.38.

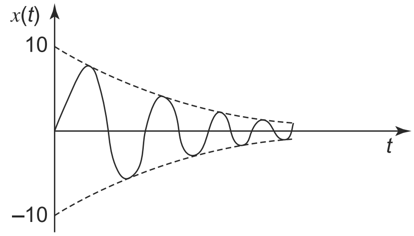


Fig. 12.38

Solution The equation for the damped sinusoidal waveform in Fig. 12.38 is given by

$$x(t) = e^{-10t} \sin \omega_c t\, u(t) \tag{12.143}$$

The above equation can be written as

$$x(t) = e^{-10t}\left[\frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j}\right]u(t) \tag{12.144}$$

From the transform pair,

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega} \tag{12.145}$$

Applying the frequency shifting property to the equation of $x(t)$,

$$X(\omega) = \frac{1}{2j}\left[\frac{1}{10 + j(\omega - \omega_c)} - \frac{1}{10 + j(\omega - \omega_c)}\right]$$

$$= \frac{\omega_c}{(10+\omega)^2 + \omega_c^2} \tag{12.146}$$

PROBLEM 12.12

Find the Fourier transform of the normalised Gaussian pulse shown in Fig. 12.39 is given by

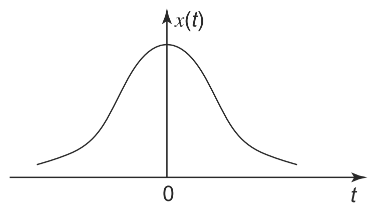


Fig. 12.39

Solution Let $x(t) \leftrightarrow X(\omega)$

Differentiating $x(t)$ with respect to t, we get

$$\frac{dx(t)}{dt} = -2\pi t\, e^{-\pi t^2} \tag{12.147}$$

$$j\frac{dx(t)}{dt} = -j2\pi t\, e^{-\pi t^2} = -j2\pi t\, x(t) \tag{12.148}$$

Taking Fourier transform both sides,

$$jF\left[\frac{dx(t)}{dt}\right] = F[-j2\pi t\, x(t)]$$

Using differentiation property, we get

$$jj\,\omega X(\omega) = 2\pi \frac{d}{d\omega} X(\omega) \tag{12.149}$$

$$\frac{1}{X(\omega)}\frac{d}{d\omega}[X(\omega)]=\frac{-\omega}{2\pi}$$

$$2\pi\frac{dX(\omega)}{d\omega}=-\omega\,d\omega \tag{12.150}$$

Integrating with respect to ω on both sides,

$$\ln X(\omega)=\frac{-\omega^2}{4\pi}+C \tag{12.151}$$

where C is the constant of integration.

$$\text{At } \omega = 0;\ C = \ln\{x(0)\} \tag{12.152}$$

By definition,

$$X(\omega)=\int_{-\infty}^{\infty} e^{-\pi t^2}e^{-j\omega t}\,dt \tag{12.153}$$

$$X(0)=\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt \tag{12.154}$$

The area under the curve is one $X(0) = 1$

If we substitute in Eq. (12.152), we get

$$C = 0$$

If we substitute in Eq. (12.151), we get

$$\ln X(\omega)=\frac{-\omega^2}{4\pi}$$

$$\text{or}\quad X(\omega)=e^{\frac{-\omega^2}{4\pi}}$$

$$\text{Thus}\quad e^{-\pi t^2}\leftrightarrow e^{\frac{-\omega^2}{4\pi}} \tag{12.155}$$

The spectrum of the normalized Gaussian pulse is shown in Fig. 12.40.

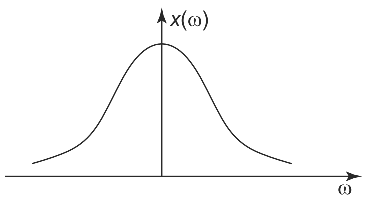


Fig. 12.40

PROBLEM 12.13

Find the Fourier transform of the trapezoidal pulse shown in Fig. 12.41.

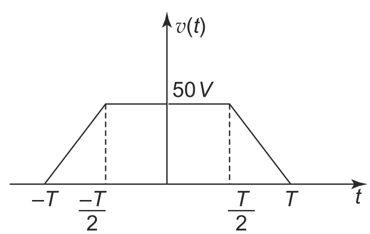


Fig. 12.41

Solution By differentiating twice, we get

$$\frac{d^2x(t)}{dt^2}=\frac{50}{T-\dfrac{T}{2}}\left[\delta(t+T)-\delta\left(t+\frac{T}{2}\right)-\delta(t-T)+\delta\left(t-\frac{T}{2}\right)\right] \tag{12.156}$$

Using the time-shifting property, we get

$$F\left[\frac{d^2x(t)}{dt^2}\right]=\frac{50}{T/2}\left[e^{j\omega T}-e^{j\omega\frac{T}{2}}-e^{-j\omega T}+e^{-j\omega\frac{T}{2}}\right] \tag{12.157}$$

Using time differentiation, we get

$$(j\omega)^2\,X(\omega)=\frac{100}{T}\left[e^{j\omega T}-e^{j\omega\frac{T}{2}}-e^{-j\omega T}+e^{-j\omega\frac{T}{2}}\right] \tag{12.158}$$

By simplyfying Eq. (12.158),

$$X(\omega)=\frac{200}{T}\left[\frac{\cos\omega T-\cos\frac{\omega T}{2}}{-\omega^2}\right]$$

or $$X(\omega)=\frac{200}{T}\left[\frac{\cos\frac{\omega T}{2}-\cos\omega T}{\omega^2}\right] \tag{12.159}$$

PROBLEM 12.14

Find the Fourier transform of a periodic pulse train shown in Fig. 12.42.

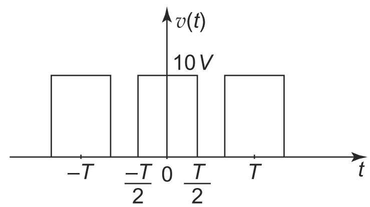


Fig. 12.42

Solution The Fourier series representation of $v(t)$ is given

$$v(t)=\sum_{n=-\infty}^{\infty}V_n\,e^{-j\omega nt}\ ;\ \omega_0=\frac{2\pi}{T} \tag{12.160}$$

where

$$V_n=\frac{1}{T}\int_{-T/2}^{T/2}v(t)\,e^{-jn\omega_0 t}dt=\frac{1}{T}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}}10\,e^{-jn\omega_0 t}dt \tag{12.161}$$

$$=\frac{1}{T}\left[\frac{1}{-jn\omega_0}\,e^{-jn\omega_0 t}\right]_{\frac{-T_0}{2}}^{\frac{T_0}{2}}$$

$$=\frac{1}{T}\times\frac{1}{-jn\omega_0}\left[e^{-jn\frac{\omega_0T_0}{2}}-e^{+jn\frac{\omega_0T_0}{2}}\right]$$

$$=\frac{2}{M\omega_0T}\left[\frac{e^{jn\frac{\omega_0T_0}{2}}-e^{-jn\frac{\omega_0T_0}{2}}}{2j}\right]$$

$$=\frac{2}{n\frac{2\pi}{T}T}\sin\left(n\frac{\omega_0T_0}{2}\right)$$

$$V_n = \left[\frac{\sin\left(\frac{n\omega_0 T_0}{2}\right)}{n\pi}\right] \quad \text{for } n \neq 0 \tag{12.162}$$

Therefore, the Fourier transform of the continuous function is

$$F[v(t)] = \sum_{n=-\infty}^{\infty} V_n 2\pi\delta(\omega - n\omega_0)$$

$$V(\omega) = \sum_{n=-\infty}^{\infty} 2\pi V_n\, \delta(\omega - n\omega_0)$$

$$V(\omega) = \sum_{n=-\infty}^{\infty} 2\pi\left[\frac{\sin\frac{n\omega_0 T_0}{2}}{n\pi}\right]\delta(\omega - n\omega_0) \tag{12.163}$$

PROBLEM 12.15

Find the Fourier transform of the train of unit impulses shown in Fig. 12.43.

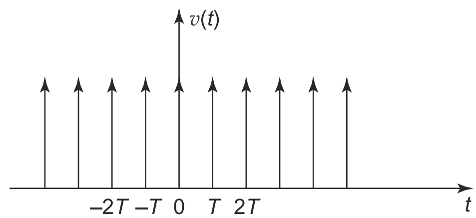


Fig. 12.43

Solution The periodic function is given by

$$v(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{12.164}$$

$v(t)$ can be expanded in Fourier series

$$v(t) = \sum_{n=-\infty}^{\infty} V_n\, e^{jn\omega_0 t} \tag{12.165}$$

where $$V_n = \frac{1}{T}\int_{-T/2}^{T/2} v(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}\, dt \tag{12.166}$$

$$V_n = \frac{1}{T} \tag{12.167}$$

Therefore, $$v(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{12.168}$$

Taking Fourier transform on both sides,

$$V(\omega) = \left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\right]$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)\,;\ \omega_0 = \frac{2\pi}{T}$$

$$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \tag{12.169}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} \delta(t-nT) \leftrightarrow \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \tag{12.170}$$

The unit impulse train in the time domain has a transform of an impulse train is the frequency domain.

PROBLEM 12.16

A waveform shown in Fig. 12.44 (a) is applied to the network shown in Fig. 12.44(b). Calculate the current through the resistor. Assume ω = 1 rad / s.

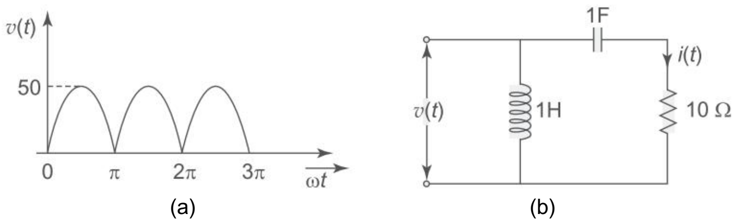


Fig. 12.44

Solution From Fig. 12.44 (a), the voltage waveform

$$\begin{aligned} v(t) &= 50 \sin \omega t \qquad && 0 \le \omega t \le \pi \\ &= -50 \sin \omega t && \omega t \le 2\pi \end{aligned}$$

The function $v(t)$ is an even function and hence, $b_n = 0$ for all values of n.

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\cos n\omega t\, d(\omega t)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 50 \sin \omega t \cos n\omega t d(\omega t) + \int_{\pi}^{2\pi} -50 \sin \omega t \cos n\omega t d(\omega t)\right]$$

$$= \frac{50}{\pi}\left[\int_0^{\pi} \{\sin(n+1)\omega t - \sin(n-1)\omega t\}\, d(\omega t) - \int_{\pi}^{2\pi} \{\sin(n+1)\omega t - \sin(n-1)\omega t\}\, d(\omega t)\right]$$

$$= \frac{50}{\pi}\left\{\left[\frac{\cos(n+1)\omega t}{n+1} - \frac{\cos(n-1)\omega t}{n-1}\right]_0^{\pi} + \left[\frac{\cos(n+1)\omega t}{n+1} - \frac{\cos(n-1)\omega t}{n-1}\right]_{\pi}^{0}\right\}$$

$$= 100\left[1 - \frac{2}{\pi}\sum_{n=2,4,6}^{\infty} \frac{\cos n\omega t}{n^2-1}\right] \tag{12.171}$$

The average current will be zero, since the capacitor does not allow dc current. Taking transform of the given circuit, we have

$$\text{where} \quad V(t) = \frac{-200 \cos n\omega t}{\pi(n^2-1)}$$

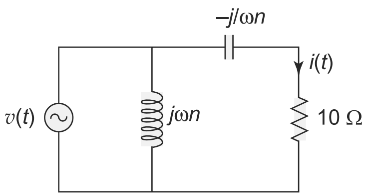


Fig. 12.45

From the circuit shown in Fig. 12.45, the current

$$i(t) = \frac{v(t)}{10 - \dfrac{j}{\omega n}}$$

$$= \frac{v(t)\omega n}{\sqrt{100\omega^2 n^2 + 1}} \angle \tan^{-1}\left(\frac{1}{10\omega n}\right)$$

By substituting the value of $v(t)$ and $\omega = 1$ rad/s, we get

$$i(t) = \sum_{n=2,4,6}^{\infty} \frac{-200n}{\sqrt{(1+100n^2)}} \angle \tan^{-1}\left(\frac{1}{10n}\right) \tag{12.172}$$

PROBLEM 12.17

Find the value of R if the average power dissipated in the resistor is 1000 watts if the voltage has the following Fourier series.

$v(t) = 200 \sin \omega t + 100 \sin 3\,\omega t + 50 \sin 5\,\omega t$

Solution The current through resistance R is

$$i(t) = \frac{v(t)}{R} = \frac{1}{R}[200 \sin \omega t + 100 \sin 3\omega t + 50 \sin 5\omega t]$$

The power dissipated in resistance R is

$$= \frac{1}{2}\left[\left(200 \times \frac{200}{R}\right) + \left(100 \times \frac{100}{R}\right) + \left(50 \times \frac{50}{R}\right)\right]$$

$$= \frac{1}{2R}[40000 + 10000 + 2500] = \frac{52500}{2R}$$

$$P_R = \frac{52500}{2R} = 1000\,\omega$$

$$\therefore \quad R = 26\,\Omega$$

PROBLEM 12.18

Find the average power supplied to a network if the applied voltage and resulting current are given by

$v(t)$ 5 $100 + 25 \sin 30t + 80 \sin 60t + 40 \sin 90t$ volts.

$i(t)$ 5 $12 \sin (30t + 65°) + 20 \sin (60t + 45°) + 15 \sin (90t + 25°)$ amperes.

Solution The total average power is the sum of the harmonic power

$$P = \frac{1}{2}[100 \times 12 \cos 65° + 80 \times 20 \cos 45° + 40 \times 15 \cos 25°]$$

$$= \frac{1}{2}[1200 \times 0.423 + 1600 \times 0.707 + 600 \times 0.906]$$

$$= \frac{1}{2}[507.6 + 1131.2 + 543.6] = 1091.2 \text{ watts.}$$

PROBLEM 12.19

For the circuit shown in Fig. 12.46, find the output voltage $v_0(t)$ by using the Fourier transform method.

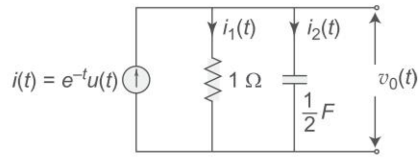


Fig. 12.46

Solution According to Kirchhoff's law,

$$i(t) = i_1(t) + i_2(t) \tag{12.173}$$

and $v_0(t) = i_2(t)$

$$e^{-t}u(t) = v_0(t) + \left(\frac{1}{2}\right)v_0(t) \tag{12.174}$$

Converting Eq. (12.174) into Fourier transform, we have

$$\frac{1}{j\omega+1} = V_0(j\omega) + \frac{1}{2}(j\omega)V_0(j\omega)$$

$$\frac{1}{j\omega+1} = V_0(j\omega)\left[\frac{2+j\omega}{2}\right]$$

$$V_0(j\omega) = \frac{2}{(j\omega+1)(j\omega+2)}$$

Taking partial fractions,

$$V_0(j\omega) = \frac{2}{j\omega+1} - \frac{2}{j\omega+2}$$

By taking inverse Fourier transform, we have

$$v_0(t) = 2\,e^{-t}u(t) - 2\,e^{-2t}\,u(t)$$

ANSWERS TO PRACTICE PROBLEMS

12-1.1 $v(t) = \dfrac{-200}{\pi}\left(\sin\omega t + \dfrac{1}{2}\sin 2\omega t + \dfrac{1}{3}\sin 3\omega t + \cdots\right)$

12-1.3 $v(t) = \dfrac{5}{12} + \sum\limits_{n=1}^{\infty}\left\{\dfrac{10}{n\pi}\left(\sin\dfrac{n\pi}{12}\right)\cos n\omega t + \dfrac{10}{n\pi}\left(1-\cos\dfrac{n\pi}{12}\right)\sin n\omega t \cdots\right\}$

12-2.2 $v(t) = \dfrac{100}{\pi}\left\{1 + \dfrac{\pi}{2}\cos\omega t + \dfrac{2}{3}\cos 2\omega t - \dfrac{2}{15}\cos 4\omega t + \dfrac{2}{35}\cos 6\omega t - \cdots\right\}$

12-2.3 $v(t) = \dfrac{20}{\pi}\left\{\cdots + \dfrac{1}{5}e^{-j5\omega t} - \dfrac{1}{3}e^{-j3\omega t} + e^{-j\omega t} + e^{j\omega t} - \dfrac{1}{3}e^{+j\omega t} + \dfrac{1}{5}e^{j5\omega t} - \cdots\right\}$

12-3.2 $v(t) = V\left\{\cdots + \left(\dfrac{1}{9\pi^2} + j\dfrac{1}{6\pi}\right)e^{-j3\omega t} + j\dfrac{1}{4\pi}e^{-j2\omega t} + \left(\dfrac{1}{\pi^2} + j\dfrac{1}{2\pi}\right)e^{-j\omega t}\right.$

$\left. + \dfrac{1}{4} + \left(\dfrac{1}{\pi^2} - j\dfrac{1}{2\pi}\right)e^{j\omega t} - j\dfrac{1}{4\pi}e^{j2\omega t} + \left(\dfrac{1}{9\pi^2} - j\dfrac{1}{6\pi}\right)e^{j3\omega t} + \cdots\right\}$

12-3.4 $V(\omega) = 2T\left|\dfrac{\sin\left(\dfrac{\omega T}{2}\right)}{\dfrac{\omega T}{2}}\right|$

12-3.5 $X(\omega) = 2\sin c[2(\omega+10)] + 2\sin c[2(\omega-10)]$

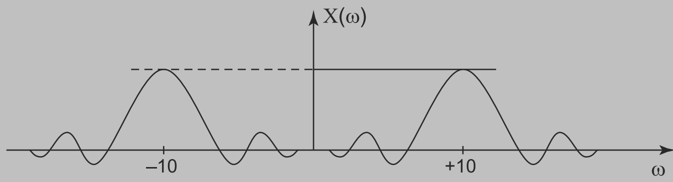


Fig. 12.62

12-5.1 $V(\omega) = 2\sin\omega + 4\sin 2\omega$

12-6.1 $x(t) = \dfrac{2A}{\pi t}\sin at\,\cos\omega_0 t$

Objective-Type Questions

☆☆★ **12.1** In a periodic signal, the period T_0 is doubled, the fundamental frequency ω_0 in the spectrum becomes
(a) doubled (c) increased 4 times (b) halved (d) no change

☆☆★ **12.2** Any periodic function can be expressed by a Fourier series when the function has
(a) infinite number of finite discontinuities in a period
(b) finite number of infinite discontinuities in a period
(c) finite number of finite discontinuities in a period
(d) infinite number of infinite discontinuities

☆☆★ **12.3** A function $x(t)$ is said to be even, if $x(t)$ is
(a) $x(-t)$ (b) $-x(t)$ (c) $x(2t)$ (d) $x(t)$

☆☆★ **12.4** A function $x(t)$ is said to be odd, if $x(-t)$ is
(a) $x(-t)$ (b) $x(2t)$ (c) $x(-2t)$ (d) $-x(t)$

☆★★ **12.5** A periodic function $x(t)$ is said to have half-wave symmetry if $x(t)$ is
(a) $-x\left(t+\dfrac{T}{2}\right)$ (b) $x\left(t+\dfrac{T}{2}\right)$ (c) $x\left(t-\dfrac{T}{2}\right)$ (d) $-x\left(t-\dfrac{T}{2}\right)$

☆☆★ **12.6** Fourier transform for the signal $e^{-at}\,u(t)$ does not exist if
(a) $a > 0$ (b) $a = 0$ (c) $a = 1$ (d) $a < 0$

☆☆★ **12.7** The Fourier transform
(a) satisfies linearity (b) does not satisfy linearity

☆★★ **12.8** The Fourier transform exists, if the following condition is satisfied.
(a) $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt > K$ (c) $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt = 0$ (b) $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$ (d) none

☆☆★ **12.9** Fourier transform of the unit impulse $\delta(t)$ is

(a) 0 (b) π (c) 1 (d) $\delta(\omega)$

☆☆★ **12.10** What is the spectrum of a dc signal?

(a) 0 (b) 1 (c) 2π (d) $2\pi\delta(\omega)$

☆★★ **12.11** Inverse Fourier transform of $\delta(\omega - \omega_0)$ is

(a) $\frac{1}{2\pi} e^{j\omega_0 t}$ (b) $\frac{1}{2\pi}$ (c) $e^{-j\omega_0 t}$ (d) $e^{j\omega_0 t}$

☆☆★ **12.12** The Fourier transform of the signal $x(t)$ is

(a) $-X(\omega)$ (b) $X(-\omega)$ (c) $-X(-\omega)$ (d) $X(\omega)$

☆☆★ **12.13** Time convolution property states that

(a) $f_1(t) * f_2(t)$ (c) $F_1(\omega) * F_1{}^*(\omega)$ (b) $f_1(t) f_2(t)$ (d) $F_1(\omega) / F_2(\omega)$

☆☆★ **12.14** Frequency convolution property states that

(a) $f_1(t) * f_2(t)$ (c) $F_1(\omega) F_2(\omega)$ (b) $F_1(\omega) * F_2(\omega)$ (d) $F_1(\omega) / F_2(\omega)$

☆☆★ **12.15** Fourier transform of the sgn(t) function is

(a) $\frac{2}{j\omega}$ (b) $\frac{1}{j\omega}$ (c) $j\omega$ (d) $2j\omega$

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CHAPTER 13

Introduction to the Laplace Transform

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Explain Laplace transform and inverse Laplace transform
- LO 2 Determine Laplace transform for time-domain functions
- LO 3 Determine Laplace transform of periodic functions
- LO 4 Determine inverse Laplace transform for frequency-domain functions
- LO 5 Explain initial-value and final-value theorems

13.1 DEFINITION OF THE LAPLACE TRANSFORM

LO 1 Explain Laplace transform and inverse Laplace transform

The Laplace transform is a powerful analytical technique that is widely used to study the behaviour of linear, lumped parameter circuits. Laplace transforms are useful in engineering, particularly when the driving function has discontinuities and appears for a short period only.

In circuit analysis, the input and output functions do not exist forever in time. For causal functions, the function can be defined as $f(t)\ u(t)$. The integral for the Laplace transform is taken with the lower limit at $t = 0$ in order to include the effect of any discontinuity at $t = 0$.

Consider a function $f(t)$ which is to be continuous and defined for values of $t \geq 0$. The Laplace transform is then

$$\mathcal{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\ u(t)\ dt = \int_{0}^{\infty} f(t) e^{-st}\ dt \tag{13.1}$$

$f(t)$ is a continuous function for $t \geq 0$ multiplied by e^{-st} which is integrated with respect to t between the limits 0 and ∞. The resultant function of the variables is called Laplace transform of $f(t)$. Laplace transform is a function of independent variable s corresponding to the complex variable in the exponent of e^{-st}. The complex variable S is, in general, of the form $S = \sigma + j\omega$ and σ and ω being the real and imaginary parts respectively. For a function to have a Laplace transform, it must satisfy the condition $\int_0^{\infty} f(t) e^{-st} dt < \infty$.

Laplace transform changes the time-domain function $f(t)$ to the frequency-domain function $F(s)$. Similarly, the inverse Laplace transformation converts frequency-domain function $F(s)$ to the time-domain function $f(t)$ as follows.

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{-j}^{+j} F(s)\ e^{st}\ ds \tag{13.2}$$

Here, the inverse transform involves a complex integration. $f(t)$ can be represented as a weighted integral of complex exponentials. We will denote the transform relationship between $f(t)$ and $F(s)$ as

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

In Eq. (13.1), if the lower limit is 0 then the transform is referred to as one-sided, or unilateral, Laplace transform. In the two-sided, or bilateral, Laplace transform, the lower limit is $-\infty$.

In the following discussion, we divide the Laplace transforms into two types: functional transforms and operational transforms. A functional transform is the Laplace transform of a specific function, such as $\sin\omega t$, t, e^{-at}, and so on. An operational transform defines a general mathematical property of the Laplace transform, such as binding the transform of the derivative of $f(t)$. Before considering functional and operational transforms, we used to introduce the step and impulse functions.

Frequently Asked Questions linked to LO 1

☆☆★**13-1.1** Why do we use Laplace transform in circuit analysis? **[RGTU June 2014]**

13.2 STEP FUNCTION

LO 2 Determine Laplace transform for time-domain functions

In switching operations, abrupt changes may occur in current and voltages. On some functions, discontinuity may appear at the origin. We accommodate these discontinuities mathematically by introducing the step and impulse functions.

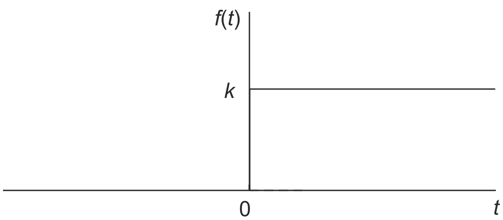

Fig. 13.1

Figure 13.1 shows the step function. It is zero for $t < 0$. It is denoted by $k\,u(t)$.

Mathematically, it is defined as

$$k\,u(t) = 0,\ t < 0$$

$$k\,u(t) = k,\ t > 0 \tag{13.3}$$

If k is 1, the function defined by Eq. (13.3) is the unit step. The step function is not defined at $t = 0$. In situations where we need to define the transition between 0^- and 0^+, we assume that it is linear and that

$$k\,u(0) = 0.5\text{ K} \tag{13.4}$$

Figure 13.2 shows the linear transition from 0^- to 0^+.

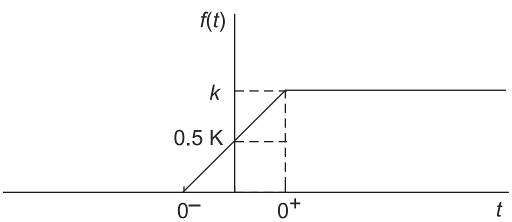

Fig. 13.2

A discontinuity may occur at some time other than $t = 0$, for example, in sequential switching. The step function occuring at $t = a$ when $a > 0$ is shown in Fig. 13.3. A step occurs at $t = a$ is expressed as $k\,u(t - a)$. Thus,

$$k\,u(t-a) = 0,\ t < a$$

$$k\,u(t-a) = k,\ t > a \tag{13.5}$$

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

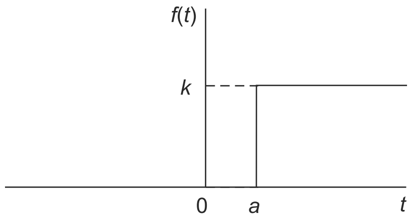

Fig. 13.3

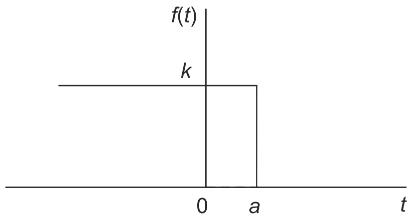

Fig. 13.4

If $a > 0$, the step occurs to the right of the origin, and if $a < 0$, the step occurs to the left of the origin. Step function is 0 when the argument $t - a$ is negative, and it is k when the argument is positive.

A step function equal to k for $t < a$ is written as $k\,u(a - t)$. Thus,

$$k\,u(a - t) = k, t < a$$

$$k\,u(a - t) = 0, t > 0 \tag{13.6}$$

The discontinuity is to the left of the origin when $a < 0$. A step function $k\,u(a - t)$ for $a > 0$ is shown in Fig. 13.4.

Step function is useful to define a finite-width pulse, by adding two step functions. For example, the function $k[u(t - 1) - u(t - 3)]$ has the value k for $1 < t < 3$ and the value 0 everywhere else, so it is a finite-width pulse of height k initiated at $t = 1$ and terminated at $t = 3$. Here, $u(t - 1)$ is a function "turning on" the constant value k at $t = 1$, and the step function $- u(t = 3)$ as "turning off" the constant value k at $t = 3$. We use step functions to turn on and turn off linear functions.

EXAMPLE 13.1

Use step functions to write an expression for the function shown in Fig. 13.5.

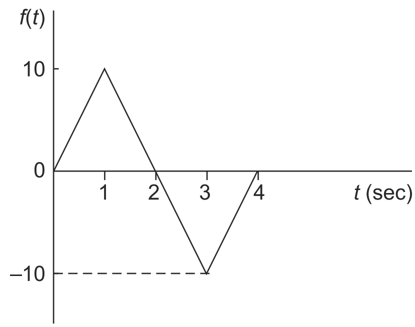

Fig. 13.5

Solution The function shown in Fig. 13.5 is made up of linear segments with break points at 0, 1, 3, and 4 seconds. Figure 13.5 consists of three linear segments as shown in Fig. 13.6.

$$\begin{aligned}
&\text{(i)} \quad f_1(t) = 10t && \text{for} \quad 0 < t < 1\\
&\text{(ii)} \quad f_2(t) = -10t + 20 && \text{for} \quad 1 < t < 3\\
&\text{(iii)} \quad f_3(t) = 20t - 40 && \text{for} \quad 3 < t < 4
\end{aligned} \tag{13.7}$$

We use the step function to initiate and terminate these linear segments at the proper times.

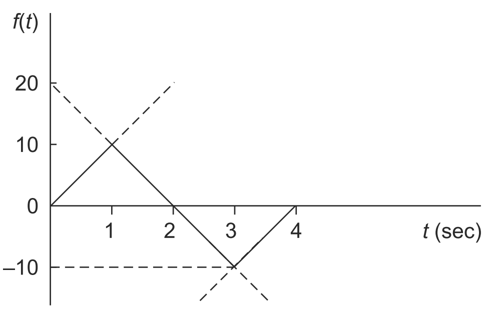

Fig. 13.6

(i) $f_1(t) = 10t[u(t) - u(t - 1)]$, this function turns on at $t = 0$, turns off at $t = 1$.

(ii) $f_2(t) = (-10t + 20)\,[u(t - 1) - u(t - 3)]$, this function turns on at $t = 1$, turns off at $t = 3$.

(iii) $f_3(t) = (20t - 40)\,[u(t - 3) - u(t - 4)]$, this function turns on at $t = 3$, turns off at $t = 4$.

The expression for $f(t)$ is

$$f(t) = 10t[u(t) - u(t - 1)] + (-10t + 20)\,[u(t - 1) - u(t - 3)] + (20t - 40)\,[u(t - 3) - u(t - 4)] \tag{13.8}$$

EXAMPLE 13.2

Use step function to write the expression for the following function.

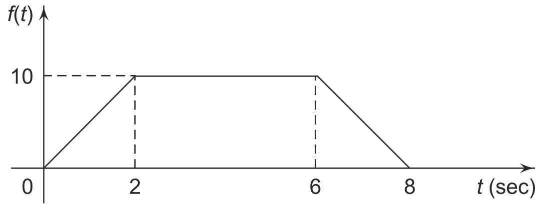

Fig. 13.7

Solution The function shown in Fig. 13.7 is a combination of linear segments at break points 0, 2, 6, 8. To construct this function, we must add and subtract linear functions of the proper slope. We use the step function to start and terminate these linear segments at the proper times.

Figure 13.7 consists of the three linear segments with the following equations.

$$\begin{aligned} f_1(t) &= 5t && \text{for} \quad 0 < t < 2 \\ f_2(t) &= 10 && \text{for} \quad 2 < t < 6 \\ f_3(t) &= -5t + 40 && \text{for} \quad 6 < t < 8 \end{aligned} \tag{13.9}$$

Using step function, the above equations can be written as

$$\begin{aligned} f_1(t) &= 5t\,[u(t) - u(t-2)] \\ f_2(t) &= 10\,[u(t-2) - u(t-6)] \\ f_3(t) &= (-5t + 40)\,[u(t-6) - u(t-8)] \end{aligned} \tag{13.10}$$

The expression for $f(t)$ is

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$\begin{aligned} f(t) = {} & 5t\,[u(t) - u(t-2)] + 10\,[u(t-2) - u(t-6)] \\ & + (-5t + 40)\,[u(t-6) - u(t-8)] \end{aligned} \tag{13.11}$$

EXAMPLE 13.3

Use step function to write the expression for the following waveform.

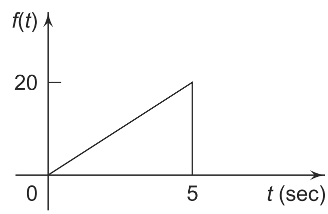

Fig. 13.8

Solution The waveform shown in Fig. 13.8 starts at $t = 0$ and ends at $t = 5$ seconds. The equation for the above waveform is $f(t) = 4t$. In terms of unit function, the waveform can be expressed as

$$f(t) = 4t\,[u(t) - u(t-5)] \tag{13.12}$$

EXAMPLE 13.4

Use step function to write the expression for the following sinusoidal waveform.

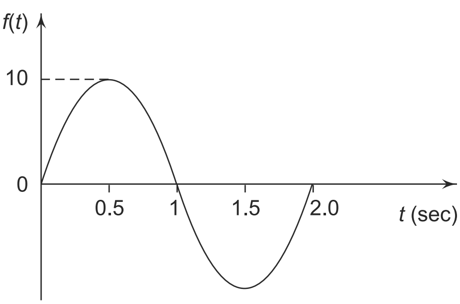

Fig. 13.9

Solution The sine wave shown in Fig. 13.9 originates at $t = 0$ and ends at $t = 2$ seconds. The wave equation

$$f(t) = 10 \sin \omega t \text{ for } 0 < t < 2$$

In terms of unit step functions, the equation

$$f(t) = 10 \sin \omega t\,[u(t) - u(t-2)] \tag{13.13}$$

EXAMPLE 13.5

Use step function to write the expression for the function shown in Fig. 13.10.

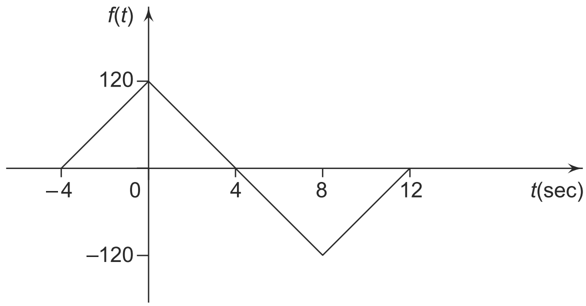

Fig. 13.10

Solution The waveform in Fig. 13.10 consists of three linear segments. The function $f(t)$ is defined as follows.

$$f_1(t) = 80t + 120 \quad \text{for} \quad -4 < t < 0$$
$$f_2(t) = -30t + 120 \quad \text{for} \quad 0 < t < 8$$
$$f_3(t) = 30t - 360 \quad \text{for} \quad 8 < t < 12 \tag{13.14}$$

In terms of unit step function

$$f_1(t) = (80t + 120)\,[u(t + 4) - u(t)]$$
$$f_2(t) = (-30t + 120)\,[u(t) - u(t - 8)]$$
$$f_3(t) = (30t - 360)\,[u(t - 8) - u(t - 12)] \tag{13.15}$$

The expression for $f(t)$ is

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$
$$f(t) = (80t + 120)\,[u(t + 4) - u(t)] + (-30t + 120)\,[u(t) - u(t - 8)]$$
$$+(30t - 360)\,[u(t - 8) - u(t - 12)] \tag{13.16}$$

13.3 IMPULSE FUNCTION

LO 2

An impulse is a signal of infinite amplitude and zero duration. In general, an impulse signal doesn't exist in nature, but some circuit signals come very close to approximating this definition. Due to switching operations, impulsive voltages and currents occur in circuit analysis. The impulse function enables us to define the derivative at a discontinuity, and thus to define the Laplace transform of that derivative.

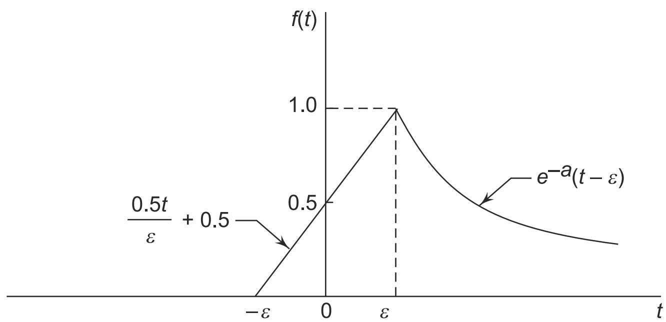

Fig. 13.11

To define derivative of a function at a discontinuity, consider that the function varies linearly across the discontinuity as shown in Fig. 13.11.

In Fig. 13.11 shown as $\varepsilon \to 0$, an abrupt discontinuity occurs at the origin. When we differentiate the function, the derivative between $-\varepsilon$ and $+\varepsilon$ is constant at a value of $\frac{1}{2\varepsilon}$. For $t > \varepsilon$, the derivative is $-ae^{-a(t-\varepsilon)}$. The derivative of the function shown in Fig. 13.11 is shown in Fig. 13.12.

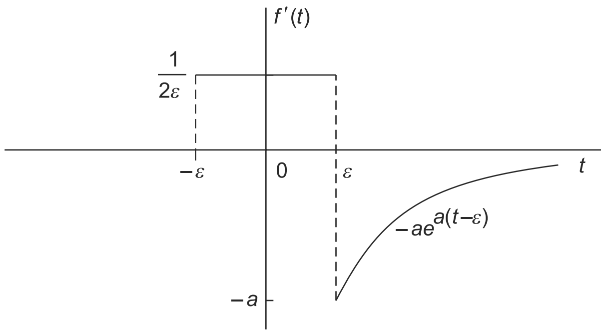

Fig. 13.12

As ε approaches zero, the value of $f'(t)$ between $\pm\varepsilon$ approaches infinity. At the same time, the duration of this large value is approaching zero. Furthermore, the area under $f'(t)$ between $\pm\varepsilon$ remains constant as $\varepsilon \to 0$. In this example, the area is unity. As ε approaches zero, we say that the function between $\pm\varepsilon$ approaches a unit impulse function; denoted $\delta(t)$. Thus, the derivative of $f(t)$ at the origin approaches a unit impulse function as ε approaches zero, or

$$f'(0) \to \delta(t) \text{ as } \varepsilon \to 0$$

If the area under the impulse function curve is other than unity, the impulse function is denoted by $K\,\delta(t)$, where K is the area. K is often referred to as the strength of the impulse function.

Mathematically, the impulse function is defined

$$\int_{-\infty}^{\infty} K\,\delta(t)\,dt = k \tag{13.17}$$

$$\delta(t) = 0,\ t \neq 0 \tag{13.18}$$

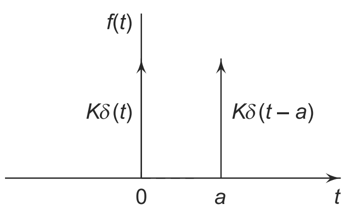

Fig. 13.13

Equation (13.17) states that the area under the impulse function is constant. This area represents the strength of the impulse. Equation (13.18) states that the impulse is zero everywhere except at $t = 0$. An impulse that occurs at $t = a$ is denoted by $K\,\delta(t - a)$. The graphical symbol is shown in Fig. 13.13. The impulse $K\,\delta(t - a)$ is also shown in Fig. 13.13.

An important property of the impulse function is the shifting property, which is expressed as

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a) \tag{13.19}$$

Equation (13.19) shows that the impulse function shifts out everything except the value of $f(t)$ at $t = a$. The value of $\delta(t-a)$ is zero everywhere except at $t = a$, and, hence, the integral can be written

$$I = \int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = \int_{a-\epsilon}^{a+\epsilon} f(t)\,\delta(t-a)\,dt \tag{13.20}$$

But because $f(t)$ is continuous at a, it takes on the value $f(a)$ as $t \to a$, so

$$I = \int_{a-\epsilon}^{a+\epsilon} f(a)\,\delta(t-a)\,dt = f(a)\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)\,dt = f(a) \tag{13.21}$$

We use the shifting property of the impulse function to find its Laplace transform.

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) dt = 1 \tag{13.22}$$

Which is important Laplace transform pair that we make good use of the circuit analysis.

We can also define the derivatives of the impulse function and the Laplace transform of these derivatives.

The function illustrated in Fig. 13.14 (a) generates an impulse function as $\varepsilon \to 0$. Figure 13.14 (b) shows the derivative of the impulse generating function, which is defined as the derivative of the impulse $[\delta'(t)]$ as $\epsilon \to 0$. The derivative of the impulse function sometimes is referred to as a moment function, or unit doublet.

To find the Laplace transform of $\delta'(t)$, we simply apply the defining integral to the function shown in Fig. 13.14 (b) and, after integrating, let $\varepsilon \to 0$. Then,

$$\begin{aligned} L\{\delta'(t)\} &= \lim_{\varepsilon\to 0}\left[\int_{-\varepsilon}^{0^-} \frac{1}{\varepsilon^2} e^{-st} dt + \int_{0^+}^{\varepsilon} \left(\frac{-1}{\varepsilon^2}\right) e^{-st} dt\right] \\ &= \lim_{\varepsilon\to 0} \frac{e^{s\varepsilon} + e^{-s\varepsilon} - 2}{s\varepsilon^2} \\ &= \lim_{\varepsilon\to 0} \frac{se^{s\varepsilon} - se^{-s\varepsilon}}{2\varepsilon s} \\ &= \lim_{\varepsilon\to 0} \frac{s^2 e^{s\varepsilon} + s^2 e^{-s\varepsilon}}{2s} \\ &= s \end{aligned} \tag{13.23}$$

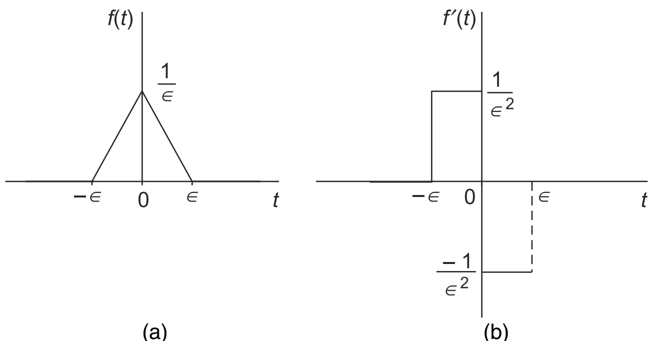

Fig. 13.14

For the nth derivative of the impulse function, we find that its Laplace transform simply is s^n; that is,

$$\mathcal{L}\left\{\delta^{n}(t)\right\} = s^n \tag{13.24}$$

An impulse function can be thought of as a derivative of a step function, that is,

$$\delta(t) = \frac{du(t)}{dt} \tag{13.25}$$

Figure 13.15 (a) approaches a unit step function as $\varepsilon \to 0$. The function shown in Fig. 13.15 (b), the derivative of the function in 13.15 (b), approaches a unit impulse as $\varepsilon \to 0$.

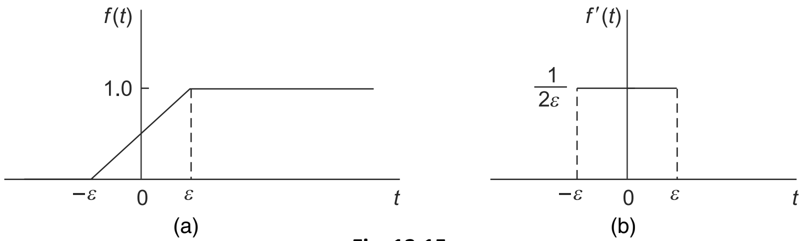

Fig. 13.15

The impulse function is an extremely useful concept in circuit analysis where discontinuities occur at the origin.

EXAMPLE 13.6

(a) Find the area under the function shown in Fig. 13.16. (b) What is the duration of the function when ε = 0? (c) What is the magnitude of f(0) when ε = 0?

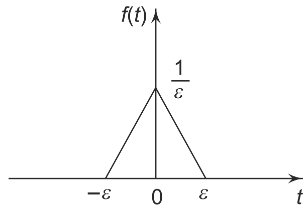

Fig. 13.16

Solution (a) Area under the function is

$$A = \int_{-\varepsilon}^{\varepsilon} f(t)\, dt \tag{13.26}$$

$$= \int_{-\varepsilon}^{0} f(t)\, dt + \int_{0}^{\varepsilon} f(t)\, dt$$

$$= \int_{-\varepsilon}^{0} \left(\frac{t}{\varepsilon^2} + \frac{1}{\varepsilon}\right) dt + \int_{0}^{\varepsilon} \left(\frac{-1}{\varepsilon^2} t + \frac{1}{\varepsilon}\right) dt$$

$$= \left[\frac{t^2}{2\varepsilon^2} + \frac{t}{\varepsilon}\right]_{-\varepsilon}^{0} + \left[\frac{-1}{\varepsilon}\frac{t^2}{2} + \frac{t}{\varepsilon}\right]_{0}^{\varepsilon} = 1 \tag{13.27}$$

(b) As $\varepsilon \to 0$, the above function shown in Fig. 13.16 becomes an impulse function. The duration of the function becomes zero.

(c) For an impulse function, the magnitude becomes infinite. Therefore, as $\varepsilon \to 0$, the magnitude of the above function becomes infinite.

13.4 FUNCTIONAL TRANSFORMS

LO 2

A functional transform is simply the Laplace transform of a specified function of t. Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

❑ *Unit Step Function* $f(t) = u(t)$ (13.28)

$$\text{where } u(t) = 1 \quad \text{for} \quad t > 0$$
$$= 0 \quad \text{for} \quad t < 0$$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty 1e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

$$\boxed{\mathcal{L}[u(t)] = \frac{1}{s}} \tag{13.29}$$

❑ ***Exponential Function*** $f(t) = e^{-at}$ (13.30)

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-at}\cdot e^{-st}dt$$

$$= \int_0^\infty e^{-(s+a)t} = \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = \frac{1}{s+a}$$

$$\boxed{\mathcal{L}\left[e^{-at}\right] = \frac{1}{s+a}} \tag{13.31}$$

❑ ***Cosine Function*** $\cos \omega t$ (13.32)

$$\mathcal{L}(\cos\omega t) = \int_0^\infty \cos\omega t\, e^{-st}dt$$

$$= \int_0^\infty e^{-st}\left[\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right]dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t}dt + \int_0^\infty e^{-(s+j\omega)t}dt\right]$$

$$= \frac{1}{2}\left[-\frac{e^{-(s-j\omega)t}}{s-j\omega}\right]_0^\infty + \frac{1}{2}\left[-\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

$$\therefore \qquad \boxed{\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}} \tag{13.33}$$

❑ ***Sine Function*** $\sin \omega t$ (13.34)

$$\mathcal{L}(\sin\omega t) = \int_0^\infty \sin\omega t\, e^{-st}dt$$

$$= \int_0^\infty e^{-st}\frac{\left[e^{j\omega t}-e^{-j\omega t}\right]}{2j}dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}dt - \int_0^\infty e^{-(s+j\omega)t}dt\right]$$

$$= \frac{1}{2j}\left\{\left[-\frac{e^{(s-j\omega)t}}{s-j\omega}\right]_0^{\infty} + \left[\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

$$\therefore \qquad \boxed{\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}} \tag{13.35}$$

❑ Function t^n where n is a positive Integer

$$\mathcal{L}(t^n) = \int_0^{\infty} t^n \cdot e^{-st} dt \tag{13.36}$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} n t^{n-1} dt$$

$$= \frac{n}{s}\int_0^{\infty} e^{-st} t^{n-1} dt$$

$$= \frac{n}{s}\mathcal{L}(t^{n-1}) \tag{13.37}$$

Similarly, $\mathcal{L}(t^{n-1}) = \dfrac{n-1}{s}\mathcal{L}(t^{n-2})$

By taking Laplace transformations of t^{n-2}, t^{n-3} ... and substituting in the above equation, we get

$$\mathcal{L}\,(t^n) = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\ldots\ldots\frac{2}{s}\frac{1}{s}\mathcal{L}(t^{n-m})$$

$$= \frac{\angle n}{s^n}\mathcal{L}\,(t^{\circ}) = \frac{\angle n}{s^n}\times\frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

$$\therefore \qquad \mathcal{L}\,(t) = \frac{1}{s^2} \tag{13.38}$$

❑ *Hyperbolic Sine and Cosine Functions*

$$\mathcal{L}(\cosh at) = \int_0^{\infty} \cosh at\, e^{-st} dt \tag{13.39}$$

$$= \int_0^{\infty}\left[\frac{e^{at}+e^{-at}}{2}\right] e^{-st} dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s-a)t} dt + \frac{1}{2}\int_0^{\infty} e^{-(s+a)t} dt$$

$$\boxed{= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}} \tag{13.40}$$

Similarly,

$$\mathcal{L}(\sinh at) = \int_0^{\infty} \sinh(at) e^{-st} dt \tag{13.41}$$

$$= \int_0^\infty \left[\frac{e^{at} - e^{-at}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2} \tag{13.42}$$

Table 13.1 *List of Laplace transform pairs*

Type	*f(t)*	*F(s)*
Impulse	$\delta(t)$	1
Step	$U(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$
Hyperbolic cosine	$\cosh at$	$\frac{s}{s^2-a^2}$
Damped ramp	te^{-at}	$\frac{1}{(s+a)^2}$
Damped sine	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

13.5 OPERATIONAL TRANSFORMS

LO 2

Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain. The operations of primary interest are

(1) multiplication by a constant
(2) addition (subtraction)
(3) differentiation
(4) integration
(5) translation in the time domain
(6) translation in the frequency domain
(7) scale charging

13.5.1 Multiplication by a Constant

From the defining integral, if

$$\mathcal{L}[f(t)] = F(s),$$

then $$\mathcal{L}\{K f(t)\} = K F(s) \tag{13.43}$$

Consider a function $f(t)$ multiplied by a constant K.

The Laplace transform of this function is given by

$$\mathcal{L}[Kf(t)] = \int_0^\infty Kf(t)e^{-st}\,dt \tag{13.44}$$

$$= K\int_0^\infty f(t)e^{-st}\,dt = KF(s) \tag{13.45}$$

This property is called *linearity property*.

13.5.2 Addition (Subtraction)

Addition (subtraction) in the time domain translates into addition (subtraction) in the frequency domain.

Thus, if

$f_1(t) \xleftrightarrow{\mathcal{L}} F_1(s)$ and

$f_2(t) \xleftrightarrow{\mathcal{L}} F_2(s)$, then

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \tag{13.46}$$

Consider two functions $f_1(t)$ and $f_2(t)$. The Laplace transform of the sum or difference of these two functions is given by

$$\mathcal{L}\{f_1(t) \pm f_2(t)\} = \int_0^\infty \{f_1(t) \pm f_2(t)\}e^{-st}\,dt$$

$$= \int_0^\infty f_1(t)e^{-st}\,dt \pm \int_0^\infty f_2(t)e^{-st}\,dt$$

$$= F_1(s) \pm F_2(s)$$

$$\therefore \quad \mathcal{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s) \tag{13.47}$$

The Laplace transform of the sum of the two or more functions is equal to the sum of transforms of the individual function. This is called *superposition property*.

If we can use the linearity and superposition properties jointly, we have

$$\mathcal{L}[K_1 f_1(t) + K_2 f_2(t)] = K_1\,\mathcal{L}[f_1(t)] + K_2\,\mathcal{L}[F_2(t)]$$

$$= K_1 F_1(s) + K_2 F_2(s) \tag{13.48}$$

EXAMPLE 13.7

Find the Laplace transform of the function.

$$f(t) = 4t^3 + t^2 - 6t + 7 \tag{13.49}$$

Solution $\mathcal{L}(4t^3 + t^2 - 6t + 7) = 4\mathcal{L}(t^3) + \mathcal{L}(t^2) - 6\mathcal{L}(t) + 7\mathcal{L}(1)$

$$= 4\times\frac{\angle 3}{s^4}+\frac{\angle 2}{s^3}-6\frac{\angle 1}{s^2}+7\frac{1}{s}$$

$$= \frac{24}{s^4}+\frac{2}{s^3}-\frac{6}{s^2}+\frac{7}{s} \tag{13.50}$$

EXAMPLE 13.8

Find the Laplace transform of the function.

$$f(t) = \cos^2 t \tag{13.51}$$

Solution $\mathcal{L}(\cos^2 t) = \mathcal{L}\left(\frac{1+\cos 2t}{2}\right)$

$$= \mathcal{L}\left(\frac{1}{2}\right)+\mathcal{L}\left(\frac{\cos 2t}{2}\right)=\frac{1}{2}[\mathcal{L}(1)+\mathcal{L}(\cos 2t)]$$

$$= \frac{1}{2s}+\frac{s}{2(s^2+4)}=\frac{2s^2+4}{2s(s^2+4)} \tag{13.52}$$

EXAMPLE 13.9

Find the Laplace transform of the function.

$$f(t) = 3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t \tag{13.53}$$

Solution $\mathcal{L}(3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$

$$= 3\,\mathcal{L}(t^4) - 2\,\mathcal{L}(t^3) + 4\,\mathcal{L}(e^{-3t}) - 2\,\mathcal{L}(\sin 5t) + 3\,\mathcal{L}(\cos 2t)$$

$$= 3\frac{\angle 4}{s^5}-2\frac{\angle 3}{s^4}+4\frac{1}{s+3}-2\times\frac{5}{s^2+25}+3\times\frac{s}{s^2+4}$$

$$= \frac{72}{s^5}-\frac{12}{s^4}+\frac{4}{s+3}-\frac{10}{s^2+25}+\frac{3s}{s^2+4} \tag{13.54}$$

13.5.3 Differentiation

If a function $f(t)$ is piecewise continuous then the Laplace transform of its derivative $\frac{d}{dt}[f(t)]$ is given by

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = SF(s) - f(0) \tag{13.55}$$

By definition,

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \left[\frac{df(t)}{dt}\right]e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,d\{f(t)\} \tag{13.56}$$

Integrating by parts, we get

$$= [e^{-st} f(t)]_0^{\infty} + \int_0^{\infty} se^{-st} f(t)\, dt$$

$$= -f(0) + SF(s) \tag{13.57}$$

Hence, we have

$$\mathcal{L}[f'(t)] = SF(s) - f(0) \tag{13.58}$$

This is applicable to higher order derivatives also. The Laplace transform of the second derivative of $f(t)$ is

$$\mathcal{L}[f''(t)] = \mathcal{L}\left[\frac{d}{dt}(f'(t))\right]$$

$$= S\mathcal{L}[f'(t)] - f'(0) = S\{SF(s) - f(0)\} - f'(0)$$

$$= S^2 F(s) - Sf(0) - f'(0) \tag{13.59}$$

where $f'(0)$ is the initial value of the first derivative of $f(t)$. We find the Laplace transform of the nth derivative by successively applying the proceeding process, which leads to the general result.

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = S^n F(s) - S^{n-1} f(0^-) - S^{n-2} \frac{dt(0)}{dt}$$

$$-S^{n-3} \frac{d^2 f(0^-)}{dt^2} - \text{K} - \frac{d^{n-1}}{dt^{n-1}} f(0^-) \tag{13.60}$$

EXAMPLE 13.10

Using the formula for Laplace transform of derivatives, obtain the Laplace transform of (a) sin 3t (b) t^3.

Solution (a) Let $f(t) = \sin 3t$

$$f'(t) = 3 \cos 3t$$

$$f''(t) = -9 \sin 3t$$

$$\mathcal{L}[f''(t)] = s^2[\mathcal{L} f(t)] - sf(0) - f'(0) \tag{13.61}$$

$$f(0) = 0, f'(0) = 3$$

$$\mathcal{L}[f''(t)] = \mathcal{L}[-9 \sin 3t]$$

Substituting in Eq. (13.61), we get

$$\mathcal{L}[-9 \sin 3t] = s^2 \mathcal{L}[f(t)] - 3$$

$$\mathcal{L}[-9 \sin 3t] - s^2 [\mathcal{L}(\sin 3t)] = -3$$

$$\mathcal{L}[(s^2 + 9)\sin 3t] = 3 \qquad \therefore \mathcal{L}(\sin 3t) = \frac{3}{s^2 + 9} \tag{13.62}$$

(b) Let $f(t) = t^3$ (13.63)

Differentiating successively, we get

$$f'(t) = 3t^2, f''(t) = 6t, f'''(t) = 6$$

By using the differentiation theorem, we get

$$\mathcal{L}\,[f'''(t)] = s^3\,\mathcal{L}\,[f(t)] - s^2 f(0) - sf'(0) - f''(0) \tag{13.64}$$

Substituting all initial conditions, we get

$$\mathcal{L}\,[f'''(t)] = s^3\,\mathcal{L}\,[f(t)]$$

$$\mathcal{L}\,[6] = s^3\,\mathcal{L}\,[f(t)]$$

$$\frac{6}{s} = s^3\mathcal{L}\,[f(t)]$$

$$F(s) = \mathcal{L}[f(t)] = \frac{6}{s^4} \tag{13.65}$$

13.5.4 Integration

If a function $f(t)$ is continuous then the Laplace transform of its integral $\int f(t)\,dt$ is given by

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s) \tag{13.66}$$

By definition,

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty\left[\int_0^t f(t)dt\right]e^{-st}dt \tag{13.67}$$

Integrating by parts, we get

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(t)dt\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt \tag{13.68}$$

Since the first term is zero, we have

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}\mathcal{L}\,[f(t)] = \frac{F(s)}{s} \tag{13.69}$$

EXAMPLE 13.11

Find the Laplace transform of the ramp function r(t) = t.

Solution We know that $\int_0^t u(t) = r(t) = t$ (13.70)

Integration of unit step function gives the ramp function.

$$\mathcal{L}[r(t)] = \mathcal{L}\left[\int_0^t u(t)dt\right]$$

Using the integration theorem, we get

$$\mathcal{L}\left[\int_0^t u(t)dt\right] = \frac{1}{s}\mathcal{L}\,[u(t)] = \frac{1}{s^2}$$

since $\mathcal{L}\,[u(t)] = \frac{1}{s}$ (13.71)

13.5.5 Differentiation of Transforms

If the Laplace transform of the function $f(t)$ exists then the derivative of the corresponding transform with respect to s in the frequency domain is equal to its multiplication by t in the time domain.

i.e. $$\mathcal{L}\,[t\,f(t)] = \frac{-d}{ds}F(s) \tag{13.72}$$

By definition,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^{\infty} f(t)e^{-st}\,dt \tag{13.73}$$

Since s and t are independent variables, and the limits 0, ∞ are constants not depending on s, we can differentiate partially with respect to s within the integration and then integrate the function obtained with respect to t.

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^{\infty}[f(t)e^{-st}]\,dt \\ &= \int_0^{\infty} f(t)[-te^{-st}]\,dt \\ &= -\int_0^{\infty}\{tf(t)\}\,e^{-st}\,dt = -\mathcal{L}\,[tf(t)]\end{aligned}$$

Hence, $$\mathcal{L}\,[tf(t)] = \frac{-d}{ds}F(s) \tag{13.74}$$

EXAMPLE 13.12

Find the Laplace transform of function.

$$f(t) = t \sin 2t \tag{13.75}$$

Solution Let $f_1(t) = \sin 2t$

$$\mathcal{L}\,[f_1(t)] = \mathcal{L}\,[\sin 2t] = F_1(s)$$

where $$F_1(s) = \frac{2}{s^2+4}$$

$$\mathcal{L}[tf_1(t)] = \mathcal{L}[t\sin 2t] = \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] = +\frac{4s}{(s^2+4)^2} \tag{13.76}$$

13.5.6 Integration of Transforms

If the Laplace transform of the function $f(t)$ exists then the integral of corresponding transform with respect to s in the complex frequency domain is equal to its division by t in the time domain.

i.e. $$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\,ds \tag{13.77}$$

i.e. $f(t) \leftrightarrow F(s)$

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt \tag{13.78}$$

Integrating both sides from s to ∞,

$$\int_s^{\infty} F(s)\ ds = \int_s^{\infty}\left[\int_0^{\infty} f(t)e^{-st}dt\right]ds \tag{13.79}$$

By changing the order of integration, we get

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}ds\right]dt \tag{13.80}$$

$$= \int_0^{\infty} f(t)\left(\frac{e^{-st}}{t}\right)dt$$

$$= \int_0^{\infty}\left[\frac{f(t)}{t}\right]e^{-st}dt = \mathcal{L}\left[\frac{f(t)}{t}\right] \tag{13.81}$$

$$\int_0^{\infty} F(s)\ ds = \mathcal{L}\left[\frac{f(t)}{t}\right] \tag{13.82}$$

EXAMPLE 13.13

Find the Laplace transform of the function

$$f(t) = \frac{2-2e^{-2t}}{t}$$

Solution Let $f_1(t) = 2 - 2e^{-2t}$. Then

$$\mathcal{L}\,[f_1(t)] = \mathcal{L}\,[2-2e^{-2t}] \tag{13.83}$$

$$= \mathcal{L}(2) - \mathcal{L}\,(2e^{-2t}) = \frac{2}{s} - \frac{2}{s+2}$$

$$= \frac{2s+4-2s}{s(s+2)} = \frac{4}{s(s+2)}$$

Hence $\mathcal{L}\left[\frac{2-2e^{-2t}}{t}\right] = \int_0^{\infty} F_1(s)\ ds$

$$= \int_s^{\infty} \frac{4}{s(s+2)}\ ds \tag{13.84}$$

By taking the partial fraction expansion (discussed in later section), we get

$$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{2}{s} - \frac{2}{s+2}$$

$$\therefore \qquad \mathcal{L}\left[\frac{2-e^{-2t}}{t}\right] = \int_s^{\infty} \mathcal{L}\,[2-2e^{-2t}]\ ds$$

$$= \int_s^\infty \frac{2}{s}\,ds - \int_s^\infty \frac{2}{s+2}\,ds$$

$$= [2\log s - 2\log(s+2)]_s^\infty$$

$$= \left[2\log\frac{1}{1+\frac{2}{s}}\right]_s^\infty = -2\log\left(\frac{s}{s+2}\right)$$

$$\mathcal{L}\left[\frac{2-2e^{-2t}}{t}\right] = 2\log\left(\frac{s+2}{s}\right) \qquad (13.85)$$

13.5.7 Translation in the Time Domain

If the function $f(t)$ has the transform $F(s)$ then the Laplace transform of $f(t-a)\,u(t-a)$ is $e^{-as}\,F(s)$. By definition,

$$\mathcal{L}\,[f(t-a)\,u(t-a)] = \int_0^\infty [f(t-a)\,u(t-a)]\,e^{-st}\,dt \qquad (13.86)$$

Since $f(t-a)\,u(t-a) = 0$ for $t < a$

$= f(t-a)$ for $t > a$

$$\therefore\quad \mathcal{L}\,[f(t-a)\,u(t-a)] = \int_\omega^\infty f(t-a)\,e^{-st}\,dt \qquad (13.87)$$

Put $t - a = \tau$ then $\tau + a = t$

$dt = d\tau$

Therefore, the above becomes

$$\mathcal{L}\,[f(t-a)\,u(t-a)] = \int_0^\infty f(t)e^{-s(\tau+a)}d\tau \qquad (13.88)$$

$$= e^{-as}\int_0^\infty f(\tau)\,e^{-s\tau}d\tau = e^{-as}F(s)$$

$$\therefore\quad \mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}F(s) \qquad (13.89)$$

Translation in the time domain corresponds to multiplication by an exponential in the frequency domain.

EXAMPLE 13.14

If $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$, determine the Laplace transform of $[u(t) - u(t-a)]$.

Solution The function $f(t) = u(t) - u(t-a)$ is shown in Fig. 13.17.

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[u(t) - u(t-a)] \qquad (13.90)$$

$$= \mathcal{L}\,[u(t)] - \mathcal{L}\,[u(t-a)]$$

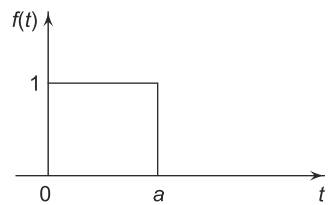

Fig. 13.17

$$= \frac{1}{s} - e^{-as}\frac{1}{s} = \frac{1}{s}(1 - e^{-as})$$

$$\mathcal{L}[f(t) = \frac{1}{s}(1 - e^{-as}) \tag{13.91}$$

13.5.8 Translation in the Frequency Domain

If the function $f(t)$ has the transform $F(s)$ then the Laplace transform of $e^{-at} f(t)$ is $F(s + a)$.

By definition, $$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{13.92}$$

and therefore, $$F(s+a) = \int_0^\infty f(t)\, e^{-(s+a)t}\, dt \tag{13.93}$$

$$= \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt = \mathcal{L}\,[e^{-at} f(t)] \tag{13.94}$$

$\therefore$ $$F(s + a) = \mathcal{L}\,[e^{-at} f(t)] \tag{13.95}$$

Similarly, we have

$$\mathcal{L}\,[e^{at} f(t)] = F(s - a) \tag{13.96}$$

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain.

EXAMPLE 13.15

Find the Laplace transform of $e^{at} \sin bt$.

Solution Let $f(t) = \sin bt$ (13.97)

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[\sin bt] = \frac{b}{s^2 + b^2}$$

Since $\mathcal{L}\,[e^{at} f(t)] = F(s - a)$

$$\mathcal{L}\,[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2} \tag{13.98}$$

EXAMPLE 13.16

Find the Laplace transform of $(t + 2)^2 e^t$.

Solution Let $f(t) = (t + 2)^2 = t^2 + 2t + 4$ (13.99)

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[t^2 + 2t + 4] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}$$

Since $\mathcal{L}\,[e^{at} f(t)] = F(s - a)$

$$\mathcal{L}\,[e^t f(t)] = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{4}{s-1} \tag{13.100}$$

Table 13.2 List of operational transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$K f(t)$	$K F(s)$
Addition/Subtraction	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
First derivative (time)	$\frac{df(t)}{dt}$	$SF(s) - f(0)$
Second derivative (time)	$\frac{d^2 f(t)}{dt^2}$	$S^2 F(s) - Sf(0) - \frac{df(0)}{dt}$
*n*th derivative (time)	$\frac{d^n f(t)}{dt^n}$	$S^n F(s) - S^{n-1} f(0) - S^{n-2} f'(0) - S^{n-3} f''(0) \ldots f_{(0)}{}^{n-1}$
Operation	$f(t)$	$F(s)$
Time integral	$\int_0^t f(t)\, dt$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)\, u(t-a),\ a > 0$	$e^{-as} F(s)$
Translation in frequency	$e^{-at} f(t)$	$F(s+a)$
Scale changing	$f(at),\ a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
First derivative (s)	$t f(t)$	$-\frac{dF(s)}{ds}$
*n*th derivative (s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
S integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\, du$

13.5.9 Scale Changing

The scale-change property gives the relationship between $f(t)$ and $F(s)$ when the time variable is multiplied by a positive constant.

$$\mathcal{L}\left\{f(at)\right\} = \frac{1}{a} F\left(\frac{s}{a}\right), a > 0 \tag{13.101}$$

By definition,

$$\mathcal{L}\,[f(at)] = \int_0^\infty f(at)\, e^{-st} dt \tag{13.102}$$

Put $\quad at = \tau$

$$dt = \frac{1}{a} d\tau$$

$$\mathcal{L}[f(at)] = \int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau} \cdot \frac{1}{a} d\tau$$

$$= \frac{1}{a}\int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau} d\tau$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right) \tag{13.103}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2*

☆★★**13-2.1** Use step functions to write the expression for the function shown in Fig. Q.1.

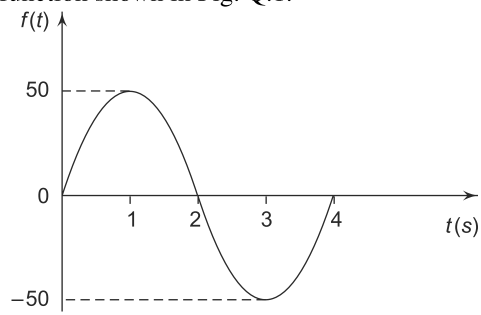

Fig. Q.1

★★★**13-2.2** Step functions can be used to define a window function. Thus, $u(t - 1) - u(t - 4)$ defines a window 1 unit high and 3 units wide located on the time axis between 1 and 4.

A function $f(t)$ is defined as follows:

$f(t) = 0, t \leq 0$

$= 30t, 0 \leq t \leq 2s$

$= 60, 2s \leq t \leq 4s$

$= 60, \cos\left(\frac{\pi}{4}t - \pi\right), 4s \leq t \leq 8s$

$= 30t - 300, 8s \leq t \leq 10s$

$= 0, 10s \leq t \leq \infty$

Sketch $f(t)$ over the internal $-2s \leq t \leq 12s$.

☆★★**13-2.3** Evaluate the following integrals

(a) $I = \int_{-1}^{3} (t^3 + 2)\ [\delta(t) + 8\delta(t-1)]\ dt$

(b) $I = \int_{-1}^{2} t^2[\delta(t) + \delta(t+1.5) + \delta(t-3)]\ dt$

☆★★**13-2.4** Explain why the following function generates an impulse function as $\varepsilon \to 0$

$$f(t) = \frac{\varepsilon/\pi}{\varepsilon^2 + t^2}, \qquad -\infty \leq t \leq \infty$$

☆★★**13-2.5** Find the Laplace transform of the signal $t(0.5)^t\, u(t)$.

☆★★**13-2.6** Make a sketch of $f(t)$ for $-25s \leq t \leq 25s$ when $f(t)$ is given by the following expression:

$f(t) = -(20t + 400)\,u(t + 20) + (40t + 400)\,u(t + 10)$
$\quad + (400 - 40t)\,u(t - 10) + (20t - 400)\,u(t - 20)$

☆☆★**13-2.7** Find the Laplace transform of each of the following functions:

(a) te^{-at} (b) $\sin \omega t$ (c) $\sin(\omega t + \theta)$

(d) $\cosh t$ (e) $\cosh(t + \theta)$

*__Note:__ ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

☆★★ **13-2.8** Use the appropriate operational transform to find the Laplace transform of each function

(a) $t^2 e^{-at}$ (b) $\dfrac{d}{dt}(e^{-at}\sinh\beta t)$ (c) $t\cos\omega t$

☆★★ **13-2.9** Find the Laplace transforms for (a) and (b):

(a) $f(t) = \dfrac{d}{dt}(e^{-at}\sin\omega t)$

(b) $f(t) = \displaystyle\int_{0-}^{t} e^{-ax}\cos\omega x\, dx$

(c) Verify the results obtained in (a) and (b) by first carrying out the mathematical operation and the finding the Laplace transform.

☆★★ **13-2.10** (a) Show that $\mathcal{L}[f(at)] = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

(b) Show that if $F(s) = \mathcal{L}[f(t)]$ and $\left[\dfrac{f(t)}{t}\right]$ is Laplace transformable then

$$\int_{s}^{\infty} F(u)\, du = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Hint: Use the determing integral to write

$$\int_{s}^{\infty} F(u)\, du = \int_{s}^{\infty}\left(\int_{0}^{\infty} f(t)e^{-ut}dt\right) du$$

and then reverse the order of integration.

☆★★ **13-2.11** Find the signal $y(t)$, the Laplace transform of signal which is

$$Y(s) = \frac{s^3 + 7s^2 + 18s + 20}{s^2 + 5s + 6}$$

☆☆★ **13-2.12** Find $f(t)$ if $F(s) = \dfrac{10(s^2 + 119)}{(s+5)(s^2 + 10s + 169)}$

☆☆★ **13-2.13** Find $f(t)$ for each of the following functions:

(a) $F(s) = \dfrac{18s^2 + 66s + 54}{(s+1)(s+2)(s+3)}$ (b) $F(s) = \dfrac{11s^2 + 172s + 700}{(s+2)(s^2 + 12s + 100)}$

(c) $F(s) = \dfrac{56s^2 + 112s + 5000}{s(s^2 + 14s + 625)}$

☆★★ **13-2.14** Find $f(t)$ of the following functions:

(a) $F(s) = \dfrac{40}{(s^2 + 4s + 5)^2}$ (b) $F(s) = \dfrac{5s^2 + 29s + 32}{(s+2)(s+4)}$

(c) $F(s) = \dfrac{2s^3 + 8s^2 + 2s - 4}{s^2 + 5s + 4}$

Frequently Asked Questions linked to LO 2

☆☆★ **13-2.1** What are the Laplace transforms of the following voltage waveform shown in Fig. Q.1? **[BPTU 2008]**

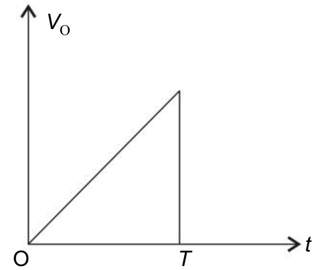

Fig. Q.1

☆☆★ **13-2.2** Give Laplace transform of a unit step function shifted by $+a$ and $-a$ units in time with the diagram. **[PTU 2011-12]**

☆★★ **13-2.3** What is an impulse function? for the network function $H(s)$ given, find the impulse response $h(t)$. **[GTU Dec. 2010]**

$$H(s) = \frac{1}{s^2 + 4s + 1}$$

☆☆★ **13-2.4** What is the relation between unit step, unit ramp, and unit impulse input? **[PTU 2011-12]**

☆☆★ **13-2.5** What is a unit doublet function? **[PTU 2011-12]**

☆☆★ **13-2.6** Define 'unit impulse function' and derive its Laplace transform. **[RGTU Dec. 2013]**

☆★★ **13-2.7** Using Laplace transform, obtain the expression for i_1 and i_2 in the circuit shown below, when dc voltage source is applied suddenly. Assume that the initial energy stored in the circuit is zero. **[AU April/May 2011]**

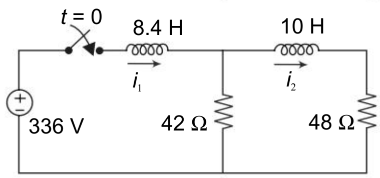

Fig. Q.7

☆☆★ **13-2.8** Derive an expression for the Laplace transform of the derivative of a function. **[RGTU June 2014]**

13.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

LO 3 Determine Laplace transform of periodic functions

Periodic functions appear in many practical problems. Let the function $f(t)$ be a periodic function which satisfies the condition $f(t) = f(t + T)$ for all $t > 0$ where T is the period of the function.

$$\mathcal{L}[f(t)] = \int_0^T f(t)\, e^{-st}\, dt + \int_T^{2T} f(t)\, e^{-st}\, dt + \ldots + \int_{nT}^{(n+1)T} f(t)\, e^{-st}\, dt + \ldots \tag{13.104}$$

$$= \int_0^T f(t)\, e^{-st}\, dt + \int_0^T f(t)\, e^{-st} e^{-sT}\, dt + \ldots + \int_0^T f(t)\, e^{-st} e^{-nsT}\, dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots + e^{-nsT} + \ldots)\int_0^T f(t)\, e^{-st}\, dt \tag{13.105}$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt \tag{13.106}$$

EXAMPLE 13.17

Find the transform of the waveform shown in Fig. 13.18.

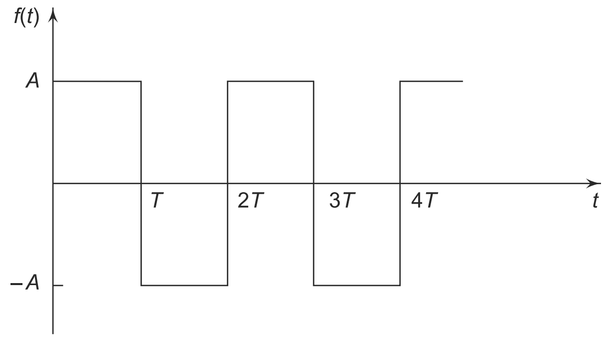

Fig. 13.18

Solution Here,. the period is $2T$

$$\mathcal{L}\,[f(t)] = \frac{1}{1-e^{-2sT}} \int_0^{2T} f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2sT}} \left[\int_0^{T} A e^{-st}\, dt + \int_T^{2T} (-A) e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2sT}} \left[-\frac{A}{s} e^{-st} \Big|_0^T + \frac{A}{s} e^{-st} \Big|_T^{2T} \right] \tag{13.107}$$

$$= \frac{1}{1-e^{-2sT}} \left[-\frac{A}{s}(e^{-sT} - 1) + \frac{A}{s}(e^{-2sT} - e^{-sT}) \right]$$

$$= \frac{1}{1-e^{-2sT}} \left[\frac{A}{s}(1-e^{-sT})^2 \right] = \frac{A}{s} \left(\frac{1-e^{-sT}}{1+e^{-sT}} \right)$$

$$\therefore \qquad \mathcal{L}\,[f(t)] = \frac{A}{s} \left(\frac{1-e^{-sT}}{1+e^{-sT}} \right) \tag{13.108}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆★★**13-3.1** Find the Laplace transform of the waveform shown in Fig. Q.1.

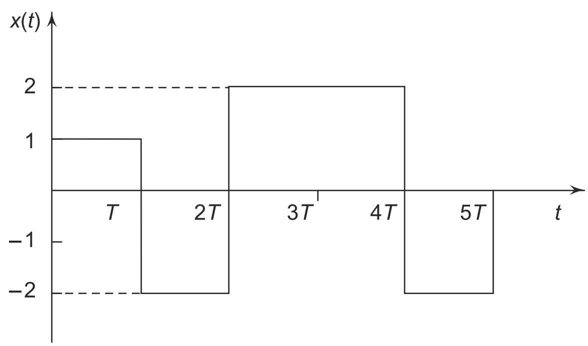

Fig. Q.1

☆★★ **13-3.2** Find the Laplace transform of the periodic waveform shown in Fig. Q.2.

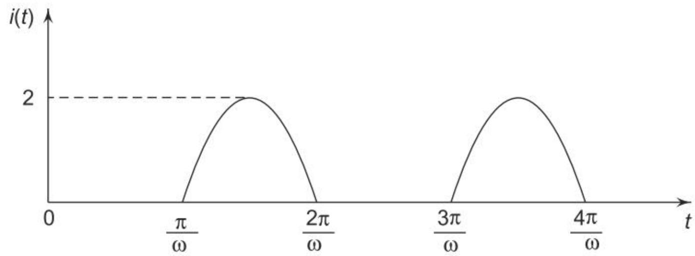

Fig. Q.2

Frequently Asked Questions linked to LO 3

☆☆★ **13-3.1** Determine the Laplace transform for the unit step function $u(t)$. **[AU Nov./Dec. 2012]**

☆☆★ **13-3.2** Obtain the Laplace transform for $f_1(t) = t$ and $f_2(t) = te^{-at}$ **[GTU Dec. 2010]**

☆☆★ **13-3.3** Explain the Laplace transformation method. Find Laplace transforms of unit step and exponential functions. **[GTU Dec. 2012]**

☆★★ **13-3.4** Find the particular solution for the current using Laplace transformation in the network shown in Fig. Q.4. The switch k is closed at $t = 0$. Assume zero initial conditions in the elements. **[GTU Dec. 2012]**

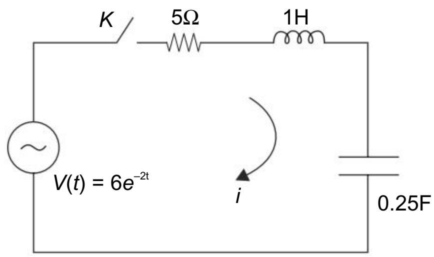

Fig. Q.4

☆☆★ **13-3.5** Obtain the s-domain (Laplace transform) equivalent circuit diagram of an inductor and capacitor with initial conditions. **[MU 2014]**

☆☆★ **13-3.6** What is Laplace transform? Define its applications. **[PTU 2011-12]**

☆☆★ **13-3.7** Find the Laplace transforms of the following functions? **[PTU 2011-12]**
(a) $\cos \omega t$, (b) $e^{-at} \sin \omega t$, (c) $\sin h\, at$

☆★★ **13-3.8** Solve the following differential equations using Laplace transform: **[PU 2012]**

(a) $\dfrac{d^2 i}{dt^2} + \dfrac{di}{dt} = t^2 + 2t;\ i(0^-) = 4;\ \dfrac{di}{dt}(0^-) = -2$

(b) $\dfrac{di}{dt} + 4i = \sin t - \cos 2t;\ i(0^-) = 0;\ \dfrac{di}{dt}(0^-) = 0.$

13.7 INVERSE TRANSFORMS

LO 4 Determine inverse Laplace transform for frequency-domain functions

So far, we have discussed Laplace transform of a functions $f(t)$. If the function is a rational function of s, which can be expressed in the form of a ratio of two polynomials in s such that no non-integral powers of s appear in the polynomials. In fact, for linear, lumped-parameter circuits whose component values are constant, the s-domain expressions for the unknown voltages and currents are always rational functions of s. If we can inverse transform rational functions of s, we can solve for the time domain expressions for the voltages and currents.

In general, we need to find the inverse transform of a function that has the form.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0} \tag{13.109}$$

The coefficients a and b are real constants, and the exponents m and n are positive integers. The ratio $\frac{N(s)}{D(s)}$ is called a *proper rational function* if $m > n$, and an improper *rational function* if $m \leq n$. Only a proper rational function can be expanded as a sum of partial fractions.

13.7.1 Partial Fraction Expansion: Proper Rational Functions

A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$. Thus, $D(s)$ must be in factored form before we can make a partial fraction expansion. The roots of $D(s)$ are either (1) real and distinct, (2) complex and distinct, (3) real and repeated, or (H) complex and repeated.

❑ *When the Roots are Real and Distinct*

In this case, $$F(s) = \frac{N(s)}{D(s)} \tag{13.110}$$

where $$D(s) = (s - a)(s - b)(s - c) \tag{13.111}$$

Expanding $F(s)$ into partial fractions, we get

$$F(s) = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} \tag{13.112}$$

To obtain the constant A, multiplying Eq. (13.112) with $(s - a)$ and putting $s = a$, we get

$$F(s)\,(s-a)\big|_{s=a} = A$$

Similarly, we can get the other constants.

$$B = (s-b)\,F(s)\big|_{s=b}$$

$$C = (s-c)\,F(s)\big|_{s=c}$$

EXAMPLE 13.18

Determine the partial fraction expansion for

$$F(s) = \frac{s^2+s+1}{s(s+5)(s+3)}$$

Solution $$F(s) = \frac{s^2+s+1}{s(s+5)(s+3)} \tag{13.113}$$

$$\frac{s^2+s+1}{s(s+5)(s+3)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s+3} \tag{13.114}$$

$$A = sF(s)\big|_{s=0} = \left.\frac{s^2+s+1}{(s+5)(s+3)}\right|_{s=0} = \frac{1}{15}$$

$$B = (s+5)\,F(s)\big|_{s=-5} = \left.\frac{s^2+s+1}{s(s+3)}\right|_{s=-5} = 2.1$$

$$C = (s+3)\,F(s)\big|_{s=-3} = \left.\frac{s^2+s+1}{s(s+5)}\right|_{s=-3} = -1.17$$

$$\frac{s^2+s+1}{s(s+5)(s+3)} = \frac{1}{15s} + \frac{2.1}{s+5} - \frac{1.17}{s+3} \tag{13.115}$$

❑ *When Roots are Real and Repeated*

In this case, $F(s) = \dfrac{N(s)}{D(s)}$

where $D(s) = (s - a)^n D_1(s)$

The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{A_0}{(s-a)^n} + \frac{A_1}{(s-a)^{n-1}} + \ldots + \frac{A_{n-1}}{s-a} + \frac{N_1(s)}{D_1(s)} \tag{13.116}$$

where $\dfrac{N_1(s)}{D_1(s)}$ represents the remainder terms of expansion.

To obtain the constant $A_0, A_1, \ldots A_{n-1}$, let us multiply both sides of Eq. (13.116) by $(s - a)^n$. Thus,

$$\begin{aligned}(s-a)^n F(s) = F_1(s) &= A_0 + A_1(s-a) + A_2(s-a)^2 + \ldots \\ &\quad + A_{n-1}(s-a)^{n-1} + R(s)(s-a)^n \end{aligned} \tag{13.117}$$

where $R(s)$ indicates the remainder terms

Putting $s = a$, we get

$$A_0 = (s-a)^n F(s)\big|_{s=a}$$

Differentiating Eq. (13.117) with respect to s, and putting $s = a$, we get

$$A_1 = \frac{d}{ds} F_1(s)\Big|_{s=a}$$

Similarly, $$A_2 = \frac{1}{2!}\frac{d^2}{ds^2} F_1(s)\Big|_{s=a}$$

In general, $$A_n = \frac{1}{n!}\frac{d^n F_1(s)}{ds^n}\Bigg|_{s=a} \tag{13.118}$$

EXAMPLE 13.19

Determine the partial fraction expansion for

$$F(s) = \frac{s-5}{s(s+2)^2}.$$

Solution $$F(s) = \frac{s-5}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{B_1}{s+2}$$

$$A = F(s)S\big|_{s=0} = \frac{s-5}{(s+2)^2}\bigg|_{s=0} = -\frac{5}{4} = -1.25$$

$$N_1(s) = (s+2)^2 F(s) = \frac{s-5}{s}$$

$$B_0 = F(s)(s+2)^2 \big|_{s=2} = \left.\frac{s-5}{s}\right|_{s=-2} = 3.5$$

$$B_1 = \frac{d}{ds} F_1(s)\Big|_{s=-2}$$

$$= \frac{d}{ds}\left(1-\frac{5}{3}\right)\bigg|_{s=-2} = \left.\frac{5}{s^2}\right|_{s=-2} = \frac{5}{4} = 1.25$$

❑ *When Roots are Distinct Complex Roots of D(s)*

Consider a function $F(s) = \dfrac{N(s)}{D(s)(s-\alpha+j\beta)(s-\alpha-j\beta)}$ (13.119)

The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta} + \frac{N_1(s)}{D_1(s)} \tag{13.120}$$

where $\dfrac{N_1(s)}{D_1(s)}$ is the remainder term.

Multiplying Eq. (13.120) by $(s - \alpha - j\beta)$ and putting

$S = \alpha + j\beta$,

we get $A = \dfrac{N(\alpha+j\beta)}{D_1(\alpha+j\beta)(+2j\beta)}$ (13.121)

Similarly $B = \dfrac{N(\alpha-j\beta)}{(-2j\beta)D_1(\alpha-j\beta)}$ (13.122)

In general, $B = A^*$ where A^* is complex conjugate of A.
If we denote the inverse transform of the complex conjugate terms as $f(t)$

$$f(t) = \mathcal{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{A^*}{s-\alpha+j\beta}\right] \tag{13.123}$$

where A and A^* are conjugate terms.

If we denote $A = C + jD$, then

$$B = C - jD = A^*$$

$\therefore$ $f(t) = e^{\alpha t}(Ae^{j\beta t} + A^* e^{-j\beta t})$ (13.124)

EXAMPLE 13.20

Find the inverse transform of the function.

$$F(s) = \frac{s+5}{s(s^2+2s+5)}$$

Solution $$F(s) = \frac{s+5}{s(s^2+2s+5)} \tag{13.125}$$

By taking partial fractions, we have

$$F(s) = \frac{s+5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2} \tag{13.126}$$

$$A = F(s)\,s\Big|_{s=0} = \frac{s+5}{s^2+2s+5} = 1$$

$$B = F(s)(s+1-j2)\Big|_{s=-1+j2} = \frac{s+5}{s(s+1+j2)}\Big|_{s=-1+j2}$$

$$= \frac{4+j2}{(-1+j2)j4} = \frac{2+j}{2j(-1+j2)} = \frac{2+j}{-2j-4} = \frac{-1}{2}$$

$$B^* = F(s)(s+1+j2)\Big|_{s=-1-j2} = \frac{s+5}{s(s+1-j2)}\Big|_{s=-1-j2}$$

$$= \frac{-1-j2+5}{(-1-j2)(-1-j2+1-j2)}$$

$$= \frac{4-j2}{-(1+j2)(j4)} = \frac{4-j2}{4j-8} = \frac{2(2-j)}{-4(2-j)} = \frac{-1}{2}$$

$$\therefore \quad F(s) = \frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)} \tag{13.127}$$

The inverse transform of $F(s)$ is $f(t)$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+1-j2)}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1+j2}\right]$$

$$= 1 - \frac{1}{2}e^{(-1+j2)t} - \frac{1}{2}e^{(-1-j2)t} \tag{13.128}$$

❑ ***When Roots are Repeated and Complex of D(S)*** The complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugate, so that only half the *Ks* need to be evolved.

Consider the function $$F(s) = \frac{768}{(s^2+6s+2s)^2} \tag{13.129}$$

By factoring the denominator polynomial, we have

$$F(s) = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$= \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{s+3-j4}$$

$$+ \frac{K_1^*}{(s+3+j4)^2} + \frac{K_2^*}{s+3+j4} \tag{13.130}$$

Now we need to evaluate only K_1 and K_2, because K_1^* and K_2^* are conjugate values.
The value of K_1, is

$$K_1 = \frac{768}{(s+3+j4)^2}\bigg|_{s=-3+j4} = \frac{768}{(j8)^2} = -12 \tag{13.131}$$

The value of K_2 is

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right]_{s=-3+j4}$$

$$= -\frac{2(768)}{(s+3+j4)^3}\bigg|_{s=-3+j4} = -\frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\angle -90° \tag{13.132}$$

From Eqs (13.131) and (13.132)

$$K_1^* = -12,\ K_2^* = j3 = 3\angle 90° \tag{13.133}$$

We now group the partial fraction expansion by conjugate terms to obtain

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2}\right]$$

$$+\left(\frac{3\angle -90°}{s+3-j4} + \frac{3\angle 90°}{s+3+j4}\right) \tag{13.134}$$

Inverse transform of the above function is

$$f(t) = [-24t\,e^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90°)]\,u(t) \tag{13.135}$$

Table 13.3 Useful transform pairs

Nature of roots	***F(s)***	***f(t)***
Distinct real	$\frac{k}{s+a}$	$Ke^{-at}\,u(t)$
Repeated real	$\frac{k}{(s+a)^2}$	$Kt\,e^{-at}\,u(t)$
Distinct complex	$\frac{k}{s+\alpha-j\beta} + \frac{k^*}{s+\alpha+j\beta}$	$2\lvert k\rvert e^{-\alpha t}\cos(\beta t + \theta)\,u(t)$
Repeated complex	$\frac{k}{(s+\alpha-j\beta)^2} + \frac{k^*}{(s+\alpha+j\beta)^2}$	$2t\lvert k\rvert e^{-\alpha t}\cos(\beta t + \theta)\,u(t)$

13.7.2 Partial Fraction Expansion: Improper Rational Function

An improper rational function can always be expanded into a polynomial plus a proper rational function. The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions.

Consider a function

$$F(s)=\frac{s^4+13s^3+66s^2+200s+300}{s^2+9s+20} \tag{13.136}$$

Dividing the denominator into the numerator until the remainder is a proper rational function gives

$$F(s)=s^2+4s+10+\frac{30s+100}{s^2+9s+20} \tag{13.137}$$

Now we expand the proper rational function into a sum of partial fractions.

$$\frac{30s+100}{s^2+9s+20}=\frac{30s+100}{(s+4)(s+5)}=\frac{-20}{s+4}+\frac{50}{s+5} \tag{13.138}$$

Substituting Eq. (13.138) into Eq. (13.137) yields

$$F(s)=s^2+4s+10-\frac{20}{s+4}+\frac{50}{s+5} \tag{13.139}$$

By taking inverse transform, we get

$$f(t)=\frac{d^2\delta(t)}{dt^2}+4\frac{d\delta(t)}{dt}+10\ \delta(t)-(20e^{-4t}-50e^{-5t})u(t) \tag{13.140}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★ **13-4.1** Find f(t) if $f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}F(\omega)e^{jtw}d\omega$

$$F(\omega)=\frac{4+j\omega}{9+j\omega}\pi\, d(\omega)$$

☆☆★ **13-4.2** Find the inverse transforms of the following functions

(a) $\dfrac{1}{s^2+9}$ (b) $\dfrac{2\pi}{s+\pi}$

(c) $\dfrac{8}{(s+3)(s+5)}$ (d) $\dfrac{5}{s^2+9}$

(e) $\dfrac{K_1}{s}+\dfrac{K_2}{s^2}+\dfrac{K_3}{s^3}$

☆★★ **13-4.3** Find the inverse Laplace transform of

$$F(s)=\frac{1}{(s+2)^2}$$

13.8 INITIAL AND FINAL VALUE THEOREMS

LO 5 Explain initial-value and final-value theorems

The initial- and final-value theorems are useful because they enable us to determine from $F(s)$ the behaviour of $f(t)$ at 0 and ∞. Hence, we can check the initial and final values of $f(t)$ to see if they conform with known circuit behaviour, before actually finding the inverse transform of $F(s)$.

The initial-value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} SF(s) \tag{13.141}$$

and the final-value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s) \tag{13.142}$$

The initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions.

To prove the initial-value theorem, we start with the operational transform of the first derivative.

$$\mathcal{L}\left[\frac{df}{dt}\right] = SF(s) - f(0)$$

$$= \int_0^\infty \frac{df}{dt} e^{-st}\, dt \tag{13.143}$$

Now, we take the limit as $s \to \infty$

$$\lim_{s\to\infty} [SF(s) - f(0)] = \lim_{s\to\infty} \int_0^\infty \frac{df}{dt} e^{-st}\, dt \tag{13.144}$$

The right-hand side of the above equation becomes zero as $s \to \infty$

$$\therefore \quad \lim_{s\to\infty} [SF(s) - f(0)] = 0$$

$$\lim_{s\to\infty} SF(s) = f(0) = \lim_{t\to 0} f(t) \tag{13.145}$$

The proof of the final-value theorem also starts with Eq. (13.143). Here, we take the limit as $s \to 0$.

$$\lim_{s\to\infty} [SF(s) - f(0)] = \lim_{s\to 0}\left(\int_0^\infty \frac{df}{dt} e^{-st}\, dt\right) \tag{13.146}$$

$$\lim_{s\to 0} [SF(s) - f(0)] = [f(t)]_0^\infty \tag{13.147}$$

$$\lim_{s\to 0} SF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides.

$$\therefore \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s) \tag{13.148}$$

The final-value theorem is useful only if $f(\infty)$ exists.

The Application of Initial- and Final-Value Theorems

Consider the transform pair given by

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} \leftrightarrow [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13°)]u(t) \tag{13.149}$$

The initial-value theorem gives

$$\lim_{s\to\infty} [SF(s) = \lim_{s\to\infty} \frac{100s^2\left(1+\dfrac{3}{s}\right)}{s^3\left[1+\dfrac{6}{s}\right]\left[1+\dfrac{6}{s}+\dfrac{25}{s^2}\right]} = 0 \tag{13.150}$$

$$\lim_{t\to 0} f(t) = [-12+20\cos(-53.13°)](1) = -12+12 = 0 \tag{13.151}$$

The final-value theorem gives

$$\lim_{s\to 0} SF(s) = \lim_{s\to 0} \frac{100s(s+3)}{(s+6)\,(s^2+6s+25)} = 0 \tag{13.152}$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [-12e^{-6t} + 20e^{-3t}\cos(4t-53.13°)]u(t) = 0 \tag{13.153}$$

EXAMPLE 13.21

Verify the initial-value theorem for the following functions.

(a) $5e^{-4t}$ *(b)* $2 - e^{5t}$

Solution (a) Let $f(t) = 5e^{-4t}$ (13.154)

then $$F(s) = \frac{5}{s+4}$$

$$SF(s) = \frac{5s}{s+4}$$

$$\lim_{s\to\infty} SF(s) = \lim_{s\to\infty} \frac{5}{1+4/s} = 5$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} 5e^{-4t} = 5 \tag{13.155}$$

Hence, the initial-value theorem is proved.

(b) Let $f(t) = 2 - e^{5t}$ (13.156)

Then $F(s) = \mathcal{L}(2 - e^{5t}) = \mathcal{L}(2) - \mathcal{L}(e^{5t})$

$$= \frac{2}{s} - \frac{1}{s-5} = \frac{s-10}{s(s-5)}$$

$$SF(s) = \frac{s-10}{s-5}$$

$$\lim_{s\to\infty} SF(s) = \frac{\left(1-\dfrac{10}{s}\right)}{\left(1-\dfrac{5}{s}\right)} = 1$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} (2-e^{5t}) = 1 \tag{13.157}$$

Hence, the initial-value theorem is proved.

EXAMPLE 13.22

Verify the final-value theorem for the following functions.

(a) $2 + e^{-3t} \cos 2t$ *(b)* $6(1 - e^{-t})$

Solution (a) Let $f(t) = 2 + e^{-3t} \cos 2t$ (13.158)

then $$F(s) = \frac{2}{s} + \frac{s+3}{(s+3)^2 + 4}$$

$$SF(s) = 2 + \frac{s^2}{(s+3)^2+4} + \frac{3s}{(s+3)^2+4}$$

$$\lim_{s\to 0} SF(s) = \lim_{s\to 0}\left[2 + \frac{s(s+3)}{(s+3)^2+4}\right] = 2$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[2 + e^{-3t}\cos 2t] = 2 \tag{13.159}$$

Hence, the final-value theorem is proved.

(b) Let $f(t) = 6(1 - e^{-t})$ (13.160)

then $$F(s) = \frac{6}{s} - \frac{6}{s+1} = \frac{6}{s(s+1)}$$

$$SF(s) = \frac{6}{s+1}$$

$$\lim_{s\to 0} SF(s) = 6$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 6(1 - e^{-t}) = 6 \tag{13.161}$$

Hence, the final value theorem is proved.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆☆★ **13-5.1** Apply the initial- and final-value theorems to each transform pair in Problem 13.2.13 (refer page 584).

☆☆★ **13-5.2** Use the initial- and final-value theorems to find the initial- and final-values of $f(t)$ for the following functions.

(a) $F(s) = \dfrac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)}$ (b) $F(s) = \dfrac{4s^2 + 7s + 1}{s(s+1)^2}$

(c) $F(s) = \dfrac{40}{(s^2 + 4s + 5)^2}$

☆★★ **13-5.3** Using the initial-value theorem, find the initial value of the signal corresponding to the Laplace transform.

$$Y(s) = \frac{s+1}{s(s+2)}$$

Verity that the answer obtained is correct.

Frequently Asked Questions linked to LO 5

☆★★**13-5.1** State the final-value theorem of Laplace transform and find the final value of the function $f(t) = 5u(t) + 10e^{-t}$ using the final-value theorem. Under what conditions the final *value* theorem cannot be used? give one example. **[GTU Dec. 2010]**

☆☆★**13-5.2** State and explain the initial and final-value theorems. **[GTU Dec. 2012]**

Additional Solved Problems

PROBLEM 13.1

Find the Laplace transforms of the following functions.

(a) $t^3 + at^2 + bt + 3$ *(b)* $\sin^2 5t$ *(c)* e^{5t+6} *(d)* $\cosh^2 3t$

Solution (a) $t^3 + at^2 + bt + 3$

Taking Laplace transforms,

$$\mathcal{L}[t^3 + at^2 + bt + 3] = \frac{3!}{s^4} + \frac{2!a}{s^3} + \frac{b}{s^2} + \frac{3}{s}$$

$$= \frac{6}{s^4} + \frac{2a}{s^3} + \frac{b}{s^2} + \frac{3}{s}$$

$$= \frac{6 + 2as + bs^2 + 3s^3}{s^4}$$

(b) $$\mathcal{L}[\sin^2 5t] = \mathcal{L}\left[\frac{1-\cos 10t}{2}\right] = \mathcal{L}\left[\frac{1}{2}\right] - \mathcal{L}\left[\frac{1}{2}\cos 10t\right]$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+100)}$$

$$= \frac{100}{2s(s^2+100)}$$

(c) $$\mathcal{L}[e^{5t+6}] = \mathcal{L}[e^6 . e^{5t}] = \frac{e^6}{s-5}$$

(d) $$\mathcal{L}[\cosh^2 3t] = \mathcal{L}\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right)^2\right]$$

$$= \mathcal{L}\left[\frac{e^{6t}+e^{-6t}+2}{4}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$$

$$= \frac{s^2-18}{s(s-6)(s+6)}$$

PROBLEM 13.2

Find the inverse transforms of the following functions.

(a) $\dfrac{5s+4}{(s-1)(s^2+2s+5)}$ (b) $\dfrac{4s+2}{s^2+2s+5}$

(c) $\dfrac{s}{s^2-2s+5}$ (d) $\dfrac{s(s+1)}{s^2+4s+5}$

Solution (a) $\mathcal{L}^{-1}\left[\dfrac{5s+4}{(s-1)(s^2+2s+5)}\right]$

By taking partial fraction of the given function,

$$F(s)=\frac{5s+4}{(s-1)(s^2+2s+5)}=\frac{A}{s-1}+\frac{BS+C}{S^2+2S+5}$$

Finding the values of *A, B, C,* we have

$$A=\frac{9}{8};\quad B=\frac{-9}{8};\quad C=\frac{11}{8}$$

$$\therefore\quad \mathcal{L}^{-1}\left[\frac{5s+4}{(s-1)(s^2+2s+5)}\right]=\mathcal{L}^{-1}\left[\frac{9}{8(s-1)}-\frac{\frac{9}{8}s}{s^2+2s+5}+\frac{11/8}{s^2+2s+5}\right]$$

$$=\frac{9}{8}e^{t}-\frac{9}{8}e^{-t}\cos 2t-\frac{11}{8}e^{-t}\sin 2t$$

(b) $$\mathcal{L}^{-1}\left[\frac{4s+2}{s^2+2s+5}\right]=\mathcal{L}^{-1}\left[\frac{4s+4}{(s+1)^2+2^2}-\frac{2}{(s+1)^2+2^2}\right]$$

$$=e^{-t}[4\cos 2t+\sin 2t]$$

(c) $$\mathcal{L}^{-1}\left[\frac{s}{s^2-2s+5}\right]=\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+2^2}-\frac{1}{(s-1)^2+2^2}\right]$$

$$=\frac{e^{t}}{2}[2\cos 2t+\sin 2t]$$

PROBLEM 13.3

Find the transforms of the following functions.

(a) $t e^{-2t}\sin 2t+\dfrac{\cos 2t}{t}$ (b) $\log\left[\dfrac{s^2-1}{s(s+1)}\right]$

(c) $(1+2t\,e^{-5t})^3$ (d) $\dfrac{s+4}{(s^2+5s+12)^2}$

Solution (a) $\mathcal{L}\left[t\,e^{-2t}\sin 2t+\dfrac{\cos 2t}{t}\right]$

$$= \mathcal{L}[t\,e^{-2t}\sin 2t]+\mathcal{L}\left[\frac{\cos 2t}{t}\right]$$

$$= \frac{4(s+2)}{[(s+2)^2+4]^2}+\int_s^{\infty} L(\cos 2t)ds$$

$$= \frac{4(s+2)}{[(s+2)^2+4]^2}+\log\left(\frac{1}{\sqrt{s^2+4}}\right)$$

(b) By taking inverse transform,

$$\mathcal{L}^{-1}\left[\log\frac{s^2-1}{s(s+1)}\right]=\mathcal{L}^{-1}\left[\log\left(\frac{s-1}{s}\right)\right]$$

$$=\frac{-1}{t}\left[\mathcal{L}^{-1}\frac{d}{ds}\left(\frac{s-1}{s}\right)\right]$$

$$=\frac{-1}{t}\left[\mathcal{L}^{-1}\left\{\frac{1}{s-1}-\frac{1}{s}\right\}\right]=\frac{-1}{t}[e^{+t}-1]=\frac{1}{t}(1-e^{t})$$

(c) $\mathcal{L}[1+2t\,e^{-5t}]^3=\mathcal{L}[1+8t^3e^{-15t}+6t\,e^{-5t}+12t^2e^{-10t}]$

$$=\frac{1}{s}+\frac{8\underline{|3}}{(s+15)^4}+\frac{6}{(s+5)^2}+12.\frac{2}{(s+10)^3}$$

$$=\frac{1}{s}+\frac{48}{(s+15)^4}+\frac{6}{(s+5)^2}+\frac{24}{(s+10)^3}$$

(d) $\mathcal{L}^{-1}\left[\dfrac{s+4}{(s^2+5s+12)^2}\right]=\mathcal{L}^{-1}\left[\dfrac{s+4}{\left(s+\frac{5}{2}\right)^2+\frac{23}{4}}\right]$

$$=\mathcal{L}^{-1}\left[\frac{s+\frac{5}{2}+\frac{3}{2}}{\left(s+\frac{5}{2}\right)^2+\left(\frac{\sqrt{23}}{2}\right)^2}\right]$$

$$=\mathcal{L}^{-1}\left[\frac{s+\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\left(\frac{\sqrt{23}}{2}\right)^2}\right]+\frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{5}{2}\right)^2+\left(\frac{\sqrt{23}}{2}\right)^2}\right]$$

$$=e^{\frac{-5}{2}t}\cos\left(\frac{\sqrt{23}}{2}\right)t+\frac{3}{2}\frac{2}{\sqrt{23}}e^{-\frac{5}{2}t}\sin\frac{\sqrt{23}}{2}t$$

$$=e^{\frac{-5}{2}t}\left[\cos\left(\frac{\sqrt{23}}{2}\right)t+\frac{3}{\sqrt{23}}\sin\left(\frac{\sqrt{23}}{2}\right)t\right]$$

PROBLEM 13.4

Find the Laplace transform of a sawtooth waveform f(t) which is periodic, with period equal to unity, and is given by f(t) = at for 0 < t < 1.

Solution Laplace transform of a periodic function $f(t)$ is

$$F(s) = \frac{1}{1-e^{-ST}} \int_0^T f(t)e^{-st}\,dt$$

Here, $f(t) = at$ for $0 < t < 1$

$$\therefore \quad F(s) = \frac{1}{1-e^{-ST}} \int_0^1 at\; e^{-st}\,dt$$

$$= \frac{a}{1-e^{-s}}\left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^1$$

$$= \frac{a}{1-e^{-s}}\left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

PROBLEM 13.5

For the given function $f(t) = 3u(t) + 2e^{-t}$, find its final value $f(\infty)$ using final-value theorem.

Solution The final-value theorem is given by

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s)$$

$$f(t) = 3u(t) + 2e^{-t}$$

By taking Laplace transform,

$$F(s) = \frac{3}{s} + \frac{2}{s+1}$$

$$SF(s) = 3 + \frac{2s}{s+1}$$

$$\lim_{s\to 0} SF(s) = \lim_{s\to 0}\left[3 + \frac{2s}{s+}\right] = 3$$

$$\therefore \quad \lim_{t\to\infty} f(t) = f(\infty) = 3$$

PROBLEM 13.6

Determine the inverse Laplace transform of the function.

$$\left\{\frac{4}{s^2+64}\right\}$$

Solution
$$\frac{4}{s^2+64} = \frac{4}{s^2+8^2} \tag{13.162}$$

$$= \frac{(4)}{8}\left\{\frac{8}{s^2+8^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{8}{s^2+8^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+64}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{2}\right)\frac{8}{s^2+8^2}\right\} \tag{13.163}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{8}{s^2+8^2}\right\}$$

$$= \frac{1}{2}\sin 8t \tag{13.164}$$

PROBLEM 13.7

Determine the inverse Laplace transform of the function.

$$F(s) = \frac{s-3}{s^s+4s+13}$$

Solution $$F(s) = \frac{s-3}{s^2+4s+13} = \frac{s-3}{(s+2)^2+9} = \frac{(s+2)-5}{(s+2)^2+9} \tag{13.165}$$

We can write the above equation as

$$\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

By taking the inverse transform, we get

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - \mathcal{L}^{-1}\left[\frac{5}{(s+2)^2+9}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

$$= \frac{e^{-2t}}{3}[3\cos 3t - 5\sin 3t] \tag{13.166}$$

PROBLEM 13.8

Find the inverse transform of the following.

(a) $\log\left(\dfrac{s+5}{s+6}\right)$ *(b)* $\dfrac{1}{(s^2+5^2)^2}$

Solution (a) Let $$F(s) = \log\left(\frac{s+5}{s+6}\right) \tag{13.167}$$

Then $$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\log\left(\frac{s+5}{s+6}\right)\right] = \frac{1}{s+5} - \frac{1}{s+6}$$

We know that

$$\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = -tf(t) \tag{13.168}$$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s+5} - \frac{1}{s+6}\right] = e^{-5t} - e^{-6t}$$

Hence, $\quad -tf(t) = e^{-5t} - e^{-6t}$

$$f(t) = \frac{e^{-6t} - e^{-5t}}{t} \tag{13.169}$$

(b) $\quad$ Let $F(s) = \dfrac{1}{(s^2+5^2)^2}$ $\quad$ (13.170)

$$\frac{1}{(s^2+5^2)^2} = \frac{1}{s}\cdot\frac{s}{(s^2+5^2)^2}$$

Therefore $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+5^2)^2}\right] = \mathcal{L}^{-1}\left[\dfrac{1}{s}\dfrac{s}{(s^2+5^2)^2}\right]$ $\quad$ (13.171)

According to integration theorem,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+5^2)^2}\right] = \int_0^t \left[\mathcal{L}^{-1}\frac{s}{(s^2+5^2)}\right] dt \tag{13.172}$$

if $\quad \mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty F(s)\,ds$

Here, $\quad \displaystyle\int_s^\infty \frac{s}{(s^2+5^2)^2}\,ds = \frac{-1}{2}\left[\frac{1}{s^2+5^2}\right]_s^\infty = \frac{1}{2}\cdot\frac{1}{s^2+5^2}$

Therefore, $\quad \dfrac{f(t)}{t} = \mathcal{L}^{-1}\left(\dfrac{1}{2}\cdot\dfrac{1}{s^2+5^2}\right) = \dfrac{1}{10}\sin 5t$

$\therefore \quad f(t) = \dfrac{t}{10}\sin 5t$

or $\quad \mathcal{L}^{-1}\left[\dfrac{1}{s}\dfrac{s}{(s^2+5^2)^2}\right] = \displaystyle\int_0^t \frac{t\sin 5t}{10}\,dt$ $\quad$ (13.173)

$$= \frac{1}{10}\left[t\left(\frac{-\cos 5t}{5}\right) + \frac{\sin 5t}{25}\right]_0^t$$

$$= \frac{1}{250}\left[\sin 5t - 5t\cos 5t\right] \tag{13.174}$$

PROBLEM 13.9

Find the Laplace transform of the full-wave rectified output as shown in Fig. 13.19.

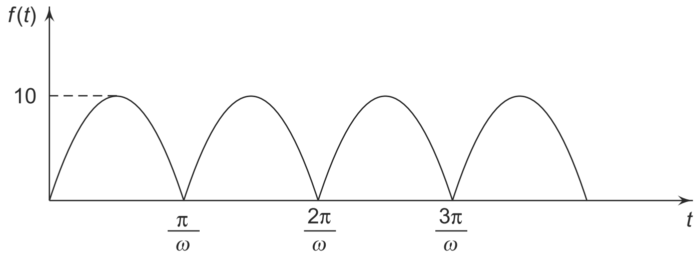

Fig. 13.19

Solution We have

$$f(t) = 10\sin\omega t \ \textit{for}\ 0 < t < \frac{\pi}{\omega} \tag{13.175}$$

Hence,
$$\mathcal{L}[f(t)] = \int_0^{\pi/\omega} \frac{(e^{-st} f(t))dt}{1-e^{\frac{-s\pi}{w}}} \tag{13.176}$$

$$= \int_0^{\pi/w} \frac{(e^{-st}10\sin\omega t)dt}{1-e^{\frac{-s\pi}{\omega}}}$$

$$= \frac{10}{1-e^{\frac{-s\pi}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{10}{\left(1-e^{\frac{-s\pi}{\omega}}\right)(s^2+\omega^2)}\left[\omega e^{\frac{-s\pi}{\omega}} + \omega\right]$$

$$= \frac{10\omega}{s^2+\omega^2}\frac{\left(1+e^{\frac{-s\pi}{\omega}}\right)}{\left(1-e^{\frac{-s\pi}{\omega}}\right)}$$

$$= \frac{10\omega}{s^2+\omega^2}\frac{e^{\frac{s\pi}{2\omega}}+e^{\frac{-s\pi}{2\omega}}}{e^{\frac{s\pi}{2\omega}}-e^{\frac{-s\pi}{2\omega}}}$$

$$= \frac{10\omega}{s^2+\omega^2}\cosh\left(\frac{s\pi}{2\omega}\right) \tag{13.177}$$

PROBLEM 13.10

Find the Laplace transform of the square wave shown in Fig. 13.20.

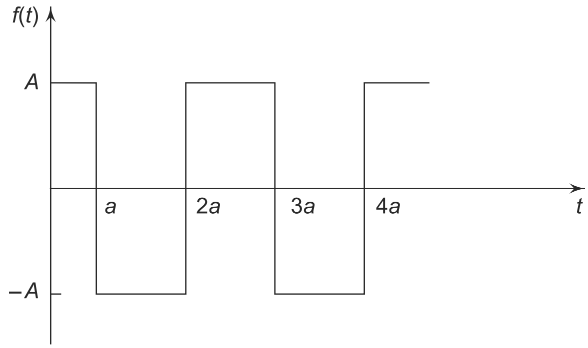

Fig. 13.20

Solution We have

$$f(t) = A, \qquad 0 < t < a$$
$$= -A, \qquad a < t < 2a \tag{13.178}$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a Ae^{-st}\,dt + \int_a^{2a}(-A)e^{-st}\,dt\right]$$

$$= \frac{A}{s}\frac{(1-2e^{-as}+e^{-2as})}{1-e^{-2as}}$$

$$= \frac{A}{s}\frac{(1-e^{-as})^2}{(1+e^{-as})(1-e^{-as})} = \frac{A}{s}\tanh\left(\frac{as}{2}\right) \tag{13.179}$$

PROBLEM 13.11

Determine the form of the partial fraction expansion for the proper fraction.

$$\frac{s-1}{(s+9)^2(s+4)(s^2+3s+2)(s+7)^2}$$

Solution

$$\frac{s-1}{(s+9)^2(s+4)(s^2+3s+2)(s+7)^2} = \frac{K_1}{(s+9)^2} + \frac{K_2}{s+9} + \frac{K_3}{s+4}$$
$$+ \frac{K_4 s + K_5}{s^2+3s+2} + \frac{K_6}{(s+7)^2} + \frac{K_7}{s+7} \tag{13.180}$$

Alternatively, $(s^2 + 3s + 2)$ can be factored and written as $(s + 2)$ $(s + 1)$, and the resulting partial fraction expansion can be written as

$$\frac{s-1}{(s+9)^2(s+4)(s+2)(s+1)(s+7)^2} = \frac{K_1}{(s+9)^2} + \frac{K_2}{(s+9)} + \frac{K_3}{(s+4)} + \frac{K_4}{s+2}$$
$$+ \frac{K_5}{s+1} + \frac{K_6}{(s+7)^2} + \frac{K_7}{s+7} \tag{13.181}$$

PROBLEM 13.12

Determine the form for a partial fraction expansion for the improper fraction.

$$\frac{6s^3+100s^2+85s+52}{s^3+7s^2+14s+8}$$

Solution Because the expression is not a proper fraction, it cannot be expanded into partial fractions. However, if the denominator is divided into the numerator, a part of the expression becomes a proper fraction, and that part can be expanded.

$$\frac{6s^3+100s^2+85s+52}{s^3+7s^2+14s+8}=6+\frac{58s^2+s+4}{s^3+7s^2+14s+8}$$

$$=6+\frac{58s^2+s+4}{(s+1)(s+2)(s+4)} \tag{13.182}$$

$$\therefore \quad \frac{6s^3+100s^2+85s+52}{s^3+7s^2+14s+8}=6+\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+4} \tag{13.183}$$

PROBLEM 13.13

Determine the inverse Laplace transform of F(s), where

$$F(s)=\frac{s}{(s+1)^2(s+4)}.$$

Solution From the rules given for expanding proper fractions,

$$\frac{s}{(s+1)^2(s+4)}=\frac{K_1}{(s+1)^2}+\frac{K_2}{s+1}+\frac{K_3}{s+4} \tag{13.184}$$

$$K_1=F(s)(s+1)^2\,\big|_{s=1}$$

$$=\frac{s}{s+4}\Big|_{s=-1}=\frac{-1}{3}$$

$$K_3=F(s)(s+4)\big|_{s=-4}$$

$$=\frac{s}{(s+1)^2}=\frac{-4}{9}$$

We can determine K_2 by putting the right side of the equation on a common denominator and equating numerators.

$$\frac{s}{(s+1)^2(s+4)}=\frac{K_1(s+4)+K_2(s+1)(s+4)+K_3(s+1)^2}{(s+1)^2(s+4)} \tag{13.185}$$

Since $K_1=\dfrac{-1}{3}$ and $K_3=\dfrac{-4}{9}$

Equating numerators results in

$$s=\left(\frac{-11}{9}+5K_2\right)s+\left(K_2-\frac{4}{9}\right)s^2+\left(\frac{-16}{9}+4K_2\right)$$

Equating coefficients results in

$$\frac{-11}{9} + 5K_2 = 1$$

$$K_2 = \frac{4}{9}$$

$$F(s) = \frac{s}{(s+1)^2(s+4)} = \frac{\left(\frac{-1}{3}\right)}{(s+1)^2} + \frac{(4/9)}{s+1} + \frac{(-4/9)}{s+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{-1}{3}}{(s+1)^2} + \frac{(4/9)}{s+1} + \frac{(-4/9)}{s+4}\right\}$$

From the rules,

$$f(t) = \frac{-1}{3}t\,e^{-t} + \frac{4}{9}e^{-t} - \frac{4}{9}e^{-4t} \tag{13.186}$$

PROBLEM 13.14

Expand the following proper fraction into partial fraction.

$$F(s) = \frac{1}{(s^2+3s+2)(s+4)}$$

Solution By expanding proper fractions,

$$\frac{1}{(s^2+3s+2)(s+4)} = \frac{K_1 s + K_2}{(s^2+3s+2)} + \frac{K_3}{s+4} \tag{13.187}$$

$$K_3 = F(s)(s+4)\big|_{s=-4}$$

$$= \frac{1}{(s^2+3s+2)}\Big|_{s=-4}$$

$$K_3 = \frac{1}{6}$$

Putting the right-hand side of the expanded equation over a common denominator to determine K_1 and K_2 results in

$$\frac{1}{(s^2+3s+2)(s+4)} = \frac{K_1 s + K_2}{(s^2+3s+2)} + \frac{\frac{1}{6}}{s+4}$$

$$= \frac{K_1 s^2 + 4s K_1 + K_2 s + 4K_2 + \left(\frac{1}{6}\right)s^2 + \left(\frac{1}{2}\right)s + 1/3}{(s^2+3s+2)(s+4)}$$

$$= \frac{\left(K_1 + \frac{1}{6}\right)s^2 + \left(4K_1 + 4K_2 + \frac{1}{2}\right)s + (4K_2 + 1/3)}{(s^2+3s+2)(s+4)} \tag{13.188}$$

Equating numerators, we get

$$K_1 + \frac{1}{6} = 0$$

$$4K_1 + K_2 + \frac{1}{2} = 0$$

$$4K_2 + \frac{1}{3} = 1$$

From the above equations, we get

$$K_1 = \frac{-1}{6}, K_2 = \frac{1}{6}$$

The required partial fraction expansion is

$$F(s) = \frac{\left(\frac{-1}{6}\right)s + \left(\frac{1}{6}\right)}{s^2 + 3s + 2} + \frac{\left(\frac{1}{6}\right)}{s + 4} \tag{13.189}$$

PROBLEM 13.15

Determine the inverse Laplace transform of the following function.

$$F(s) = \frac{96(s^2 + 17s + 60)}{s^3 + 14s^2 + 48s}$$

Solution The function $F(s)$ can be factorised as given.

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \tag{13.190}$$

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

To find K_1, we multiply both sides by s and then put $s = 0$

$$K_1 = F(s)s\big|_{s=0} = \frac{96(s+5)(s+12)}{(s+8)(s+6)}\Big|_{s=0} = 120$$

To find the value of K_2, we multiply both sides by $s + 8$ and then evaluate both sides at $s = -8$

$$K_2 = F(s)(s+8)\big|_{s=-8} = \frac{96(s+5)(s+12)}{s(s+6)}\Big|_{s=-8} = -72$$

Then K_3 is

$$\frac{96(s+5)(s+12)}{s(s+8)}\Big|_{s=-6} = K_3 = 48$$

Therefore,

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8} \tag{13.191}$$

By taking inverse transform of the above function, we get

$$\mathcal{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{120}{s}+\frac{48}{s+6}-\frac{72}{s+8}\right\}$$

$$= 120 + 48e^{-6t} - 72e^{-8t} \tag{13.192}$$

PROBLEM 13.16

Determine the inverse Laplace transform of the following function.

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

Solution By factoring denominator, we have

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{K_1}{s+6}+\frac{K_2}{s+3-j4}+\frac{K_3}{s+3+j4} \tag{13.193}$$

To find K_1, K_2, and K_3, we use the same process as before:

$$K_1 = \frac{100(s+3)}{s^2+6s+25}\bigg|_{s=-6} = \frac{100(-3)}{25} = -12$$

$$K_2 = \frac{100(s+3)}{(s+6)(s^2+3+j4)}\bigg|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$K_3 = \frac{100(s+3)}{(s+6)(s^2+3-j4)}\bigg|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

Then

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6}+\frac{10\angle-53.13°}{s+3-j4}+\frac{10\angle 53.13°}{s+3+j4} \tag{13.194}$$

By taking inverse Laplace transform, we get

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = -12e^{-6t}+10e^{-j53.13°}\cdot e^{-(3-j4)t}$$

$$+ 10e^{j53.13°}\cdot e^{-(3+j4)t}$$

By simplifying, we get

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = \left\{-12e^{-6t}+20e^{-3t}\cos(4t-53.13°)\right\} \tag{13.195}$$

PROBLEM 13.17

Obtain inverse Laplace transform of the following function.

$$F(s)=\frac{100(s+25)}{s(s+5)^3}$$

Solution By factorising the denominator, we have

$$\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5} \tag{13.196}$$

We find K_1, as

$$K_1=\frac{100(s+25)}{(s+5)^3}\Big|_{s=0}=20$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at –5.

$$\frac{100(s+25)}{s}\Big|_{s=-5}=\frac{K_1(s+5)^3}{s}\Big|_{s=-5}+K_2+K_3(s+5)\big|_{s=-5}+K_4(s+5)^2\big|_{s=-5}$$

$$\therefore \qquad K_2=-400$$

To find K_3, we first multiply both sides by $(s+5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s=-5$.

$$\begin{aligned}\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5}&=\frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5}+\frac{d}{ds}[K_2]_{s=-5}\\&\quad+\frac{d}{ds}[K_3(s+5)]_{s=-5}+\frac{d}{ds}[K_4(s+5)^2]_{s=-5}\\100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5}&=K_3=-100\end{aligned}$$

To find K_4, we first multiply both sides by $(s+5)^3$. Next we differentiate both sides twice with respect to s and then evaluate both sides at $s=-5$. After simplifying the first derivative, the second derivative becomes

$$\begin{aligned}100\frac{d}{ds}\left[\frac{-25}{s^2}\right]_{s=-5}&=K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5}+0+\frac{d}{ds}[K_3]_{s=-5}\\&\quad+\frac{d}{ds}[2K_4(s+5)]_{s=-5}\end{aligned}$$

or

$$\begin{aligned}-40&=2K_4\\K_4&=-20\end{aligned}$$

Then

$$\frac{100(s+25)}{s(s+5)^3}=\frac{20}{s}-\frac{400}{(s+5)^3}-\frac{100}{(s+5)^2}-\frac{20}{s+5} \tag{13.197}$$

By taking inverse transform, we get

$$\mathcal{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = 20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t} \tag{13.198}$$

PROBLEM 13.18

Verify the initial- and final-value theorems for the function.

$$f(t) = e^{-t}(\sin 3t + \cos 5t)$$

Solution $f(t) = e^{-t}(\sin 3t + \cos 5t)$ (13.199)

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[e^{-t}(\sin 3t + \cos 5t)] \tag{13.200}$$

Since $\mathcal{L}(e^{-t}\sin 3t) = \dfrac{3}{(s+1)^2+3^2}$

and $\mathcal{L}(e^{-t}\cos 5t) = \dfrac{s+1}{(s+1)^2+5^2}$

$$\therefore \quad F(s) = \mathcal{L}[f(t)] = \frac{3}{(s+1)^2+3^2} + \frac{s+1}{(s+1)^2+5^2} \tag{13.201}$$

According to initial-value theorem,

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, SF(s)$$

$$F(s) = \frac{3}{s^2+2s+10} + \frac{s+1}{s^2 2s+26}$$

$$SF(s) = \frac{3s}{s^2\left(1+\frac{2}{s}+\frac{10}{s^2}\right)} + \frac{s^2+s}{s^2\left(1+\frac{2}{s}+\frac{26}{s^2}\right)}$$

$$= \frac{3}{s\left(1+\frac{2}{s}+\frac{10}{s^2}\right)} + \frac{1}{1+\frac{2}{8}+\frac{26}{s^2}} + \frac{1}{s\left(1+\frac{2}{s}+\frac{26}{s^2}\right)} \tag{13.202}$$

$$\lim_{s\to\infty} SF(s) = 1$$

$$f(t) = e^{-t}(\sin 3t + \cos 5t)$$

$$\lim_{t\to 0} f(t) = 1$$

Hence, the initial-value theorem is proved.
According to the final-value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} SF(s)$$

$$\lim_{s\to 0} SF(s) = 0$$

$$\lim_{t\to\infty} f(t) = 0$$

Hence, the final-value theorem is proved.

PROBLEM 13.19

Find the value of i(0+) using the initial-value theorem for the function given

$$I(s) = \frac{2s+3}{(s+1)(s+3)}$$

Verify the result by solving it for i(t).

Solution The initial-value theorem is given by

$$\lim_{t\to 0} i(t) = \lim_{s\to\infty} SI(s)$$
$$= \lim_{s\to\infty} \frac{s(2s+3)}{(s+1)(s+3)} \tag{13.203}$$

Taking S common and putting $S = \infty$, we get

$$\lim_{s\to\infty} \frac{s^2\left(2+\dfrac{3}{s}\right)}{s^2\left(1+\dfrac{1}{s}\right)\left(1+\dfrac{3}{s}\right)} = 2 \tag{13.204}$$

To verify the result, we solve for $i(t)$ and put $t \to \infty$.
Taking partial fractions,

$$I(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

where
$$A = (s+1)\frac{2s+3}{(s+1)(s+3)}\bigg|_{s=-1} = \frac{1}{2}$$
$$B = (s+3)\frac{2s+3}{(s+1)(s+3)}\bigg|_{s=-3} = \frac{3}{2}$$

$\therefore$
$$I(s) = \frac{1}{2(s+1)} + \frac{3}{2(s+3)} \tag{13.205}$$

Taking inverse transform, we get

$$i(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \tag{13.206}$$

Answers to Practice Problems

13-2.1 $f(t) = \sin\dfrac{\pi t}{2}\ [u(t) - u(t-4)]$

13-2.3 (a) $I = 26$ (b) $I = 0$ (c) $I = 0$

13-2.5 $\dfrac{-e^{-s}}{(-0.5e^{-s})[\log(0.5e^{-s})]^2}$

13-2.8 (a) $\dfrac{2}{(s+a)^3}$ (b) $\dfrac{S\beta}{(s+a)^2-\beta^2}$ (c) $\dfrac{S^2-\omega^2}{(s^2+\omega^2)^2}$

13-2.11 $y(t)=\dfrac{d}{dt}\delta(t)+2\delta(t)-2e^{-3t}u(t)+4e^{-2t}u(t)$

13-2.12 $f(t) = e^{-5t}\left(10-\dfrac{25}{3}\sin 12t\right)$

13-2.14 (a) $f(t) = -20e^{-2t}(t\cos t - \sin t)$

(b) $f(t) = 5\,\delta(t) - 3e^{-2t} + 2e^{-4t}$

(c) $f(t) = 2\,\dfrac{d}{dt}\,\delta(t) - 2\delta(t) + 4e^{-4t}$

13-3.1 $X(s)=\dfrac{1}{s}[1-3e^{-ST}+4e^{-2ST}-4e^{-4ST}+2e^{-5ST}]$

13-4.3 $f(t) = t\,e^{-2t}\,u(t)$

13-5.2 (a) $f(0) = 7, f(\infty) = 0$

(b) $f(0) = 4, f(\infty) = 1$

(c) $f(0) = 0, f(\infty) = 0$

13-5.3 $\underset{s\to\infty}{Lt}\ SY(s)=1$

Objective-Type Questions

☆☆★ **13.1** Laplace transform analysis gives

(a) time-domain response only (b) frequency-domain response only
(c) both (a) and (b) (d) none

☆☆★ **13.2** The Laplace transform of a unit step function is

(a) $\dfrac{1}{s}$ (b) 1 (c) $\dfrac{1}{s^2}$ (d) $\dfrac{1}{s+a}$

☆☆★ **13.3** The Laplace transform of the first derivative of a function $f(t)$ is

(a) $F(s)/s$ (b) $SF(s) - f(0)$ (c) $F(s) - f(0)$ (d) $f(0)$

☆☆★ **13.4** The Laplace transform of the integral of function $f(t)$ is

(a) $\dfrac{1}{s}F(s)$ (b) $SF(s) - f(0)$ (c) $F(s) - f(0)$ (d) $f'(0)$

☆★★ **13.5** The Laplace transform of $e^{5t} f(t)$ is

(a) $F(s)$ (b) $F(s - 1)$ (c) $F\left(\dfrac{s}{5}\right)$ (d) $F(s - 5)$

☆★★ **13.6** The inverse Laplace transform of $\dfrac{1}{s}\,(1 - e^{-as})$ is

(a) $u(t) - u(t - a)$ (b) $u(t)$ (c) $u(t - a)$ (d) zero

☆☆★ **13.7** The inverse transform of $\dfrac{6}{S^4}$ is

(a) 3 (b) t^2 (c) t^3 (d) $3t$

☆★★ **13.8** The inverse transform of $2\ \log\left(\dfrac{s+2}{s}\right)$ is

(a) $\dfrac{2-e^{-2t}}{t}$ (b) $\dfrac{e^{-2t}}{t}$ (c) $\dfrac{2}{t}$ (d) $\dfrac{2+e^{-2t}}{t}$

☆★★ **13.9** The Laplace transform of a square wave with amplitude of peak value A and period T is

(a) $\dfrac{1+e^{-sT}}{1-e^{-sT}}$ (b) $\dfrac{A}{s}\left(\dfrac{1-e^{-sT}}{1+e^{-sT}}\right)$ (c) $\dfrac{A}{s}\left(\dfrac{1+e^{sT}}{1-e^{sT}}\right)$ (d) $\dfrac{A}{s}\left(\dfrac{1-e^{sT}}{1+e^{sT}}\right)$

☆☆★ **13.10** The inverse Laplace transform of the function $\dfrac{s+5}{(s+1)(s+3)}$ is

(a) $2e^{t} - e^{-3t}$ (b) $2e^{-t} + e^{-3t}$ (c) $e^{-t} - 2e^{-3t}$ (d) $e^{-t} + 2e^{-3t}$

☆★★ **13.11** The Laplace transform of a unit ramp function at $t = a$ is

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{e^{-as}}{(s+a)^2}$ (c) $\dfrac{e^{-as}}{s^2}$ (d) $\dfrac{a}{s^2}$

☆☆★ **13.12** The initial value of $\dfrac{2s+1}{s^4+8s^3+16s^2+s}$ is

(a) 2 (b) infinite (c) zero (d) 1

☆☆★ **13.13** The initial value of $20 - 10t - e^{25t}$ is

(a) 20 (b) 19 (c) 10 (d) 25

☆☆★ **13.14** $\mathcal{L}[f(t)] = \dfrac{2(s+1)}{s^2+2s+5}$, then $f(0+)$ and $f(\infty)$ are given by

(a) 0, 2 respectively (b) 2, 0 respectively

(c) 0, 1 respectively (d) $\dfrac{2}{5}$, 0 respectively

☆☆★ **13.15** The final-value theorem is used to find the

(a) steady-state value of the system output (b) initial value of the system output

(c) transient behaviour of the system output (d) none of these

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CHAPTER 14

Application of the Laplace Transform in Circuit Analysis

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Analyse various circuit elements in S-domain
LO 2 Determine the transient behaviour of lumped parameter circuits using Laplace transform
LO 3 Discuss the transfer function and its application in circuit analysis
LO 4 Analyse the relationship of transfer function with convolution integral and steady-state sinusoidal response
LO 5 Discuss the impulse function and its application in circuit analysis

The Laplace transform is an attractive tool in circuit analysis. It transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations. It automatically introduces into the polynomial equations the initial values of the current and voltage variables. In the circuit analysis, we can develop the *s*-domain circuit models for various elements and *s*-domain equations can be written directly.

14.1 CIRCUIT ELEMENTS IN THE S-DOMAIN

LO 1 Analyse various circuit elements in S-domain

For any element, we write the time-domain equation that relates the terminal voltage to the terminal current. Then, we take the Laplace transform of the time-domain equation. This gives an algebraic relation between the *s*-domain current and voltage. The dimensions of a transformed voltage is volt-seconds, and the dimension of a transformed current is ampere-seconds. A voltage-to-current ratio in the *s*-domain carries the dimension of volts per ampere. An impedance in the *s*-domain is measured in ohms, and admittance is measured in Siemens.

14.1.1 A Resistor in the s-Domain

Consider the resistive element shown in Fig. 14.1 From Ohm's law,

$$v = Ri \tag{14.1}$$

The Laplace transform of Eq. (14.1) is

$$V = RI \tag{14.2}$$

where $V = \mathcal{L}[v]$ and $I = \mathcal{L}[i]$

Equation (14.2) states that the s-domain equivalent circuit of a resistor is simply a resistance of R ohms that carries a current of I ampere seconds and has a terminal voltage of V volt-seconds.

Figures 14.1 (a) and (b) show the time- and frequency-domain circuits of the resistor respectively.

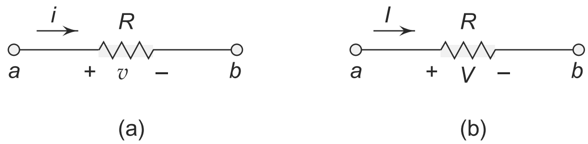

Fig. 14.1

The resistance element does not change while going from the time domain to the frequency domain.

14.1.2 An Inductor in the s-Domain

Consider an inductor shown in Fig. 14.2 with an initial current of I_0 amperes.

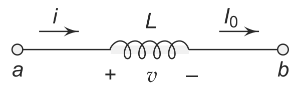

Fig. 14.2

The time domain relation between voltage and current is

$$v = L\frac{di}{dt} \tag{14.3}$$

The Laplace transform of Eq. (14.3) gives

$$V = L\,[SI - i(0)] = SLI - LI_0 \tag{14.4}$$

The above equation satisfies two circuits. The first consists of an impedance of SL ohms in series with an independent voltage source of LI_0 volt-seconds as shown in Fig. 14.3 (a).

The second s-domain equivalent circuit that satisfies Eq. (14.4) consists of an impedance of SL ohms in parallel with an independent current source of I_0/S ampere- seconds, as shown in Fig. 14.3 (b).

By solving Eq. (14.4) for the current I, we can construct the circuit shown in Fig. 14.3 (b).

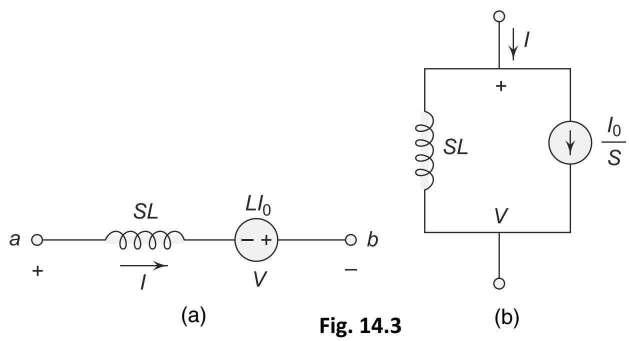

Fig. 14.3

$$I = \frac{V + LI_0}{SL} = \frac{V}{SL} + \frac{I_0}{S} \tag{14.5}$$

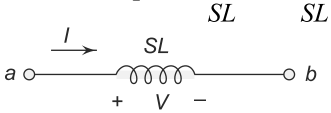

Fig. 14.4

If the initial energy stored in the inductor is zero, i.e., if $I_0 = 0$, the s-domain equivalent circuit of the inductor reduces to an inductor with an impedance of SL ohms as shown in Fig. 14.4.

14.1.3 A Capacitor in the s-domain

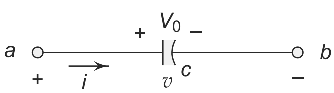

Fig. 14.5

Consider an initially charged capacitor shown in Fig. 14.5. The initial voltage on the capacitor is V_0 volts.

The voltage–current relation in the time domain is

$$i = c\frac{dv}{dt} \tag{14.6}$$

By taking Laplace transforms both sides, we get

$$I = C\,[SV - v(0)]$$

$$I = SCV - CV_0 \tag{14.7}$$

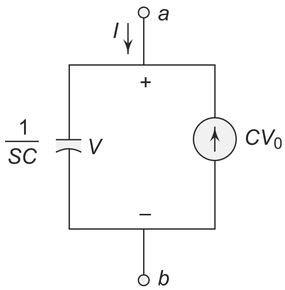

Fig. 14.6

The Equation 14.7 represents two circuits. First, the parallel equivalent circuit for capacitor initially charged to V_0 volts is shown in Fig. 14.6.

Secondly, the series equivalent circuit can be derived for the charged capacitor by solving Eq. (14.7) for V.

$$V = \left(\frac{1}{SC}\right) I + \frac{V_0}{S} \tag{14.8}$$

Figure 14.7(a) shows the circuit that satisfies Eq. (14.8).

The s-domain circuit for a capacitor when the initial voltage is zero is shown in Fig. 14.7 (b).

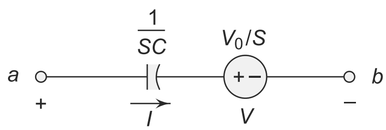

Fig. 14.7 (a)

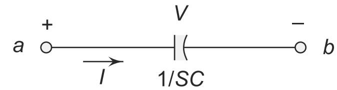

Fig. 14.7 (b)

Frequently Asked Questions linked to LO 1*

☆☆★**14-1.1** Write down the voltage and current relationships in a resistor, inductor, and capacitor. Obtain these relationships in the s-domain also. State assumptions if any in obtaining the relationships. **[GTU Dec. 2010]**

14.2 APPLICATIONS

In this section, we illustrate how to use the Laplace transform to determine the transient behaviour of several linear lumped-parameter circuits. In analysis of familiar circuits, the Laplace transform approach yields the same results like the time-domain analysis. In all the examples, the ease of manipulating algebraic equations instead of differential equations should be apparent.

LO 2 Determine the transient behaviour of lumped parameter circuits using Laplace transform

14.2.1 Natural Response of an *RC* Circuit

In this section, we find the natural response of an *RC* circuit through Laplace transform techniques. Consider the capacitor discharge circuit shown in Fig. 14.8. Assume the capacitor is initially charged to V_0 volts. The series equivalent s-domain circuit is shown in Fig. 14.9.

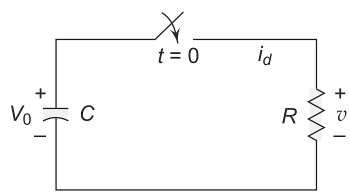

Fig. 14.8

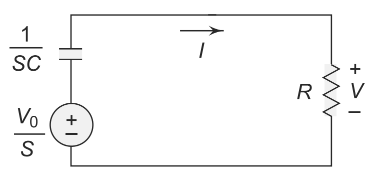

Fig. 14.9

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

From the circuit shown in Fig. 14.9(b), applying Kirchhoff's voltage law around the loop, we have

$$\frac{V_0}{S} = \frac{1}{SC} I + RI \tag{14.9}$$

Solving for the above equation yields

$$I = \frac{CV_0}{RCS+1} = \frac{V_0 / R}{S + \left[\frac{1}{RC}\right]} \tag{14.10}$$

By taking the inverse transform of Eq. (14.10), we get

$$i = \frac{V_0}{R} e^{\frac{-t}{RC}} \tag{14.11}$$

We can determine v by simply applying Ohm's law from the circuit

$$v = Ri = V_0\ e^{\frac{-t}{RC}} \tag{14.12}$$

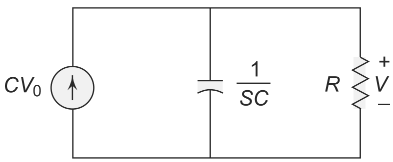

Fig. 14.10

Now we can use the parallel equivalent circuit of Fig. 14.9 (a). Figure 14.10 shows the new s-domain equivalent circuit.

By taking node voltage equation, we get

$$\frac{V}{R} + SCV = CV_0 \tag{14.13}$$

Solving Eq. (14.13) for V gives

$$V = \frac{V_0}{S + \frac{1}{RC}} \tag{14.14}$$

By taking inverse transform, we get

$$v = V_0 e^{\frac{-t}{RC}} = V_0 e^{\frac{-t}{\tau}} \tag{14.15}$$

where τ is the time constant $\tau = RC$

14.2.2 Step Response of a Parallel Circuit

Consider the parallel RLC circuit shown in Fig. 14.11. We can find the expression for i_L after the constant current source is switched across the parallel elements. The initial energy stored in the circuit is zero.

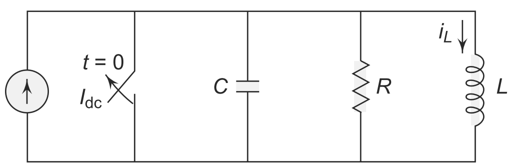

Fig. 14.11

The s-domain equivalent circuit is shown in Fig. 14.12. Here, an independent source can be transformed easily from the time domain to the frequency domain. Opening the switch results in a step change in the current applied to the circuit.

By applying Kirchhoff's current law, we get

$$SCR + \frac{V}{R} + \frac{V}{SL} = \frac{I_{dc}}{S}. \tag{14.16}$$

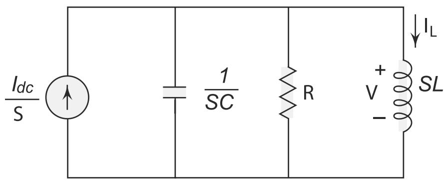

Fig. 14.12

Solving Eq. (14.16) for V gives

$$V = \frac{I_{\text{dc}} / C}{S^2 + \left(\frac{1}{RC}\right)S + \frac{1}{LC}} \tag{14.17}$$

We know the current in inductor I_L

$$I_L = \frac{V}{SL} \tag{14.18}$$

Substituting Eq. (14.17) into Eq. (14.18) gives

$$I_L = \frac{I_{\text{dc}} / LC}{S\left[S^2 + \left(\frac{1}{RC}\right)S + \frac{1}{LC}\right]} \tag{14.19}$$

By taking the inverse transform, we can obtain I_L.

14.2.3 Transient Response of a Parallel *RLC* Circuit

The transient behaviour of a circuit arises from replacing the dc current source in the circuit shown in Fig. 14.11 with a sinusoidal current source. The new current source is

$$i_g = I_m \cos \omega t \tag{14.20}$$

The s-domain expression for the source current is

$$I_g = \frac{SI_M}{S^2 + \omega^2} \tag{14.21}$$

The voltage across the parallel elements is

$$V = \frac{(I_g / C)S}{S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)} \tag{14.22}$$

Substituting Eq. (14.21) into Eq. (14.22) results in

$$V = \frac{(I_m / C)S^2}{(S^2 + \omega^2)\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{14.23}$$

from which

$$I_L = \frac{V}{SL} = \frac{(I_m / LC)S}{(S^2 + \omega^2)\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{14.24}$$

14.2.4 Use of Thevenin's Equivalent

In this section, we show how to use Thevenin's equivalent in the s-domain. Consider a circuit shown in Fig. 14.13. We find the capacitor current that results from closing the switch. The energy stored in the circuit prior to closing is zero.

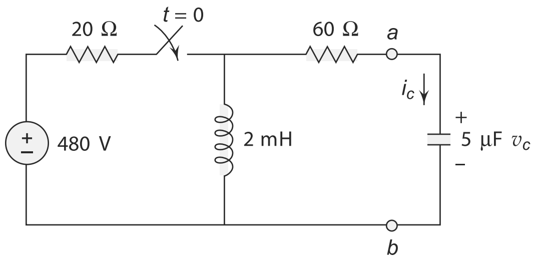

Fig. 14.13

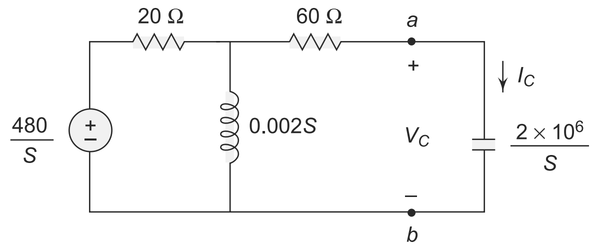

Fig. 14.14

To find i_c, we first construct the *s*-domain equivalent circuit and the find the Thevenin equivalent of this circuit with respect to the terminals of the capacitor. Figure 14.14 shows the *s*-domain circuit.

The open-circuit voltage across terminals *a*, *b* is

$$V_{\text{Th}} = \frac{\left(\frac{480}{S}\right)(0.0025)}{20 + 0.002S} = \frac{480}{S + 10^4} \tag{14.25}$$

The Thevenin impedance seen from terminals *a* and *b* equals the 60 Ω resistor in series with the parallel combination of the 20 Ω resistor and the 2 mH inductor.

Thus, $$Z_{\text{Th}} = 60 + \frac{0.002S(20)}{20 + 0.002S} = \frac{80(S + 7500)}{S + 10^4} \tag{14.26}$$

A simplified version of the Thevenin equivalent circuit is shown in Fig. 14.15.

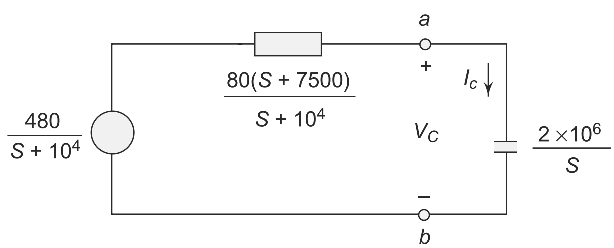

Fig. 14.15

Thus, the capacitor current I_C equals the Thevenin voltage divided by the total series impedance. Thus,

$$I_C = \frac{480/(S + 10^4)}{\left[80(S + 7500)/(S + 10^4) + \left[(2 \times 10^5)/S\right]\right]} \tag{14.27}$$

We simplify Eq. (14.27) to

$$\begin{aligned} I_C &= \frac{6S}{S^2 + 10{,}000S + 25 \times 10^6} \\ &= \frac{6S}{(S + 5000)^2} \end{aligned} \tag{14.28}$$

By taking partial fraction expansion, we get

$$I_C = \frac{-3000}{(S+5000)^2} + \frac{6}{(S+5000)} \tag{14.29}$$

By taking inverse transform, we get

$$i_c = (-30{,}000t\, e^{-5000t} + 6\, e^{-5000t})\ \text{A} \tag{14.30}$$

Now the voltage across capacitor is

$$V_C = \frac{I_C}{SC} = \frac{2\times 10^5}{S}\frac{6S}{(S+5000)^2} = \frac{12\times 10^5}{(S+5000)^2} \tag{14.31}$$

By taking inverse transform, we get

$$v_c = 12 \times 10^5\, t\, e^{-5000t} \tag{14.32}$$

14.2.5 Circuit with Mutual Inductance

In this section, we illustrate an example how to use the Laplace transform to analyze the transient response of a circuit that contains mutual inductance as shown in Fig. 14.16.

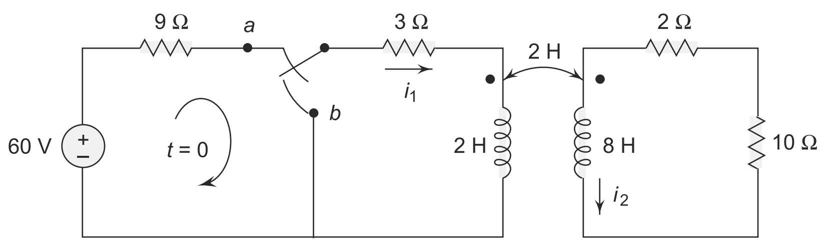

Fig. 14.16

To make-before break switch has been in position '*a*' for a long time. At $t = 0$, the switch moves instantaneously to the position *b*. The problem is to derive the time-domain expression for i_2.

We begin by redrawing the circuit in Fig. 14.16, with the switch in the position *b* and the magnetically coupled coils replaced with a *T* equivalent circuit as shown in Fig. 14.17.

The *s*-domain equivalent circuit for the circuit of Fig. 14.17 is shown in Fig. 14.18.

The initial currents are

$$i_1(0) = \frac{60}{12} = 5\ \text{A} \tag{14.33}$$

$$i_2(0) = 0 \tag{14.34}$$

The initial value of the current in the 2 H inductor is

$$i_1(0) + i_2(0) = 5\text{A} \tag{14.35}$$

The *s*-domain mesh equations in Fig. 14.18 are

$$(3 + 2S)\, I_1 + 2SI_2 = 0 \tag{14.36}$$

$$2SI_1 + (12 + 8S)I_2 = 10 \tag{14.37}$$

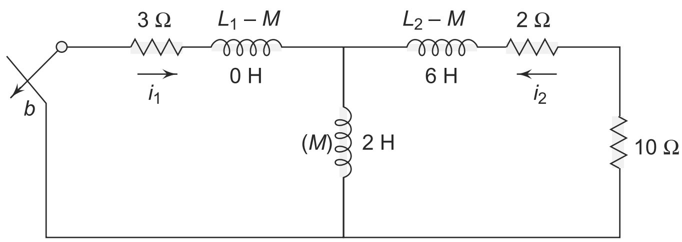

Fig. 14.17

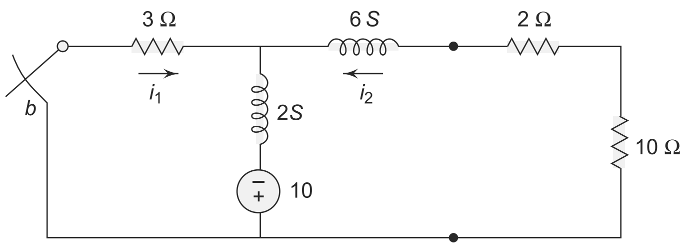

Fig. 14.18

Solving for I_2 yields

$$I_2 = \frac{2.5}{(S+1)(S+3)} \tag{14.38}$$

By taking partial fraction expansion gives

$$I_2 = \frac{1.25}{S+1} - \frac{1.25}{S+3} \tag{14.39}$$

By taking inverse transform of Eq. (14.39) gives

$$i_2 = (1.25\, e^{-t} - 1.25\, e^{-3t})\text{ A} \tag{14.40}$$

14.2.6 Use of Superposition

Consider a circuit shown in Fig. 14.19 having two sources and the inductor is carrying and initial current $i_L(0)$ amperes and the capacitor is carrying an initial voltage of $v_c(0)$ volts. The desired response of the circuit is the voltage across the resistor R_2, labeled v_2.

The *s*-domain equivalent circuit for Fig. 14.19 is shown in Fig. 14.20. Here, we have taken parallel equivalents for L and C into consideration. Now we find V_2 using node-voltage method.

To find V_2 by superposition, we calculate the voltage V_2 resulting from each source acting alone, and then we sum the voltages. We consider V_g acting alone by setting the other three current sources equal to zero. Figure 14.21 shows the resulting circuit.

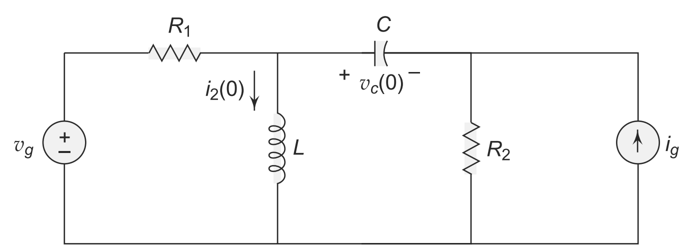

Fig. 14.19

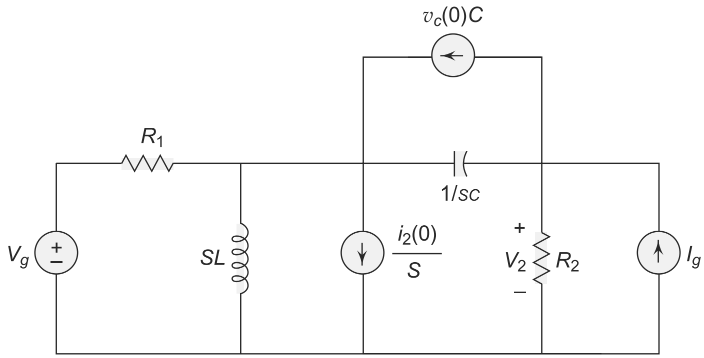

Fig. 14.20

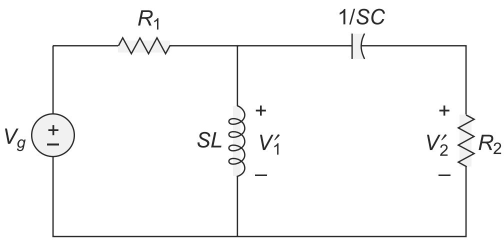

Fig. 14.21

V_1' and V_2' are the voltages across the inductor and resistor when V_g acting alone.

The two equations described in the circuit in Fig. 14.21 are

$$\left(\frac{1}{R_1}+\frac{1}{SL}+SC\right)V_1' - SCV_2' = \frac{V_g}{R_1} \tag{14.41}$$

$$-SC\,V_1' + \left(\frac{1}{R_2}+SC\right)V_2' = 0 \tag{14.42}$$

The above equations can be written as

$$Y_{11}\,V_1' + Y_{12}\,V_2' = \frac{V_g}{R_1} \tag{14.43}$$

$$Y_{21}\,V_1' + Y_{22}\,V_2' = 0 \tag{14.44}$$

where $Y_{11} = \frac{1}{R_1}+\frac{1}{SL}+SC$

$Y_{12} = -SC = Y_{21}$

$Y_{22} = \frac{1}{R_2}+SC$

Solving Eqs (14.43) and (14.44) for V_2' gives

$$V_2' = \frac{-Y_{12}/R_1}{Y_{11}\,Y_{22}-Y_{12}^2}V_g \tag{14.45}$$

With the current source I_g acting alone, the circuit shown in Fig. 14.20 reduces to the one shown in Fig. 14.22.

The two node-voltage equations are given by

$$Y_{11}\,V_1'' + Y_{12}\,V_2'' = 0 \tag{14.46}$$

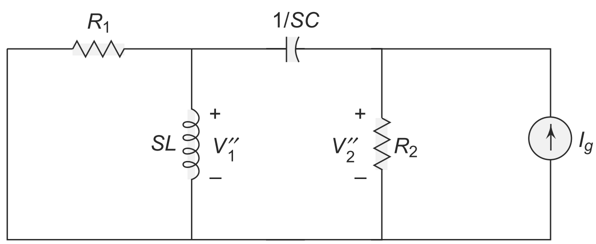

Fig. 14.22

$$Y_{21}\,V_1'' + Y_{22}\,V_2'' = I_g \tag{14.47}$$

Solving for V_2'' yields

$$V_2'' = \frac{Y_{11}}{Y_{11}\,Y_{22} - Y_{12}^2}\,I_g \tag{14.48}$$

The circuit shown in Fig. 14.23 gives when the energised inductor acting alone on the circuit of Fig. 14.20.

The two node-voltage equations are given by

$$Y_{11}V_1''' + Y_{12}V_2''' = -\frac{i_L(0)}{S} \tag{14.49}$$

$$Y_{21}V_1''' + Y_{22}V_2''' = 0 \tag{14.50}$$

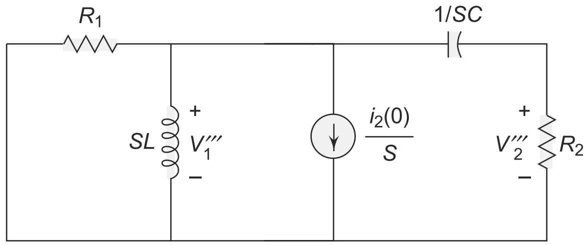

Fig. 14.23

Thus,

$$V_2''' = \frac{Y_{12}/S}{Y_{11}\,Y_{22} - Y_{12}^2} - i_L(0) \tag{14.51}$$

The circuit shown in Fig. 14.24 gives when the energy stored in the capacitor acting alone.

The node-voltage equations describing this circuit are

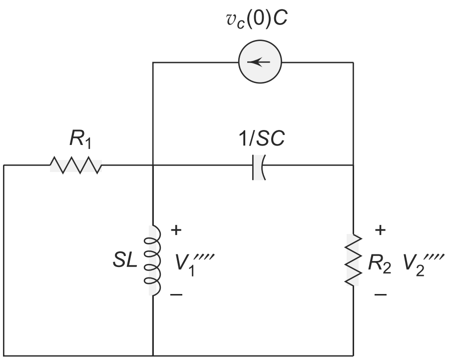

Fig. 14.24

$$Y_{11}\,V_1'''' + Y_{12}\,V_2'''' = v_c(0)C \tag{14.52}$$

$$Y_{21}\,V_1'''' + Y_{22}\,V_2'''' = -\,v_c(0)C \tag{14.53}$$

Solving for V_2'''' yields

$$V'''' = \frac{-(Y_{11} + Y_{12})C}{Y_{11}\,Y_{22} - Y_{12}^2}\,v_c(0) \tag{14.54}$$

The expression for V_2 is

$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

$$= \frac{-(Y_{12}/R_1)}{Y_{11}\,Y_{22} - Y_{12}^2}V_g + \frac{Y_{11}}{Y_{11}\,Y_{22} - Y_{12}^2}I_g$$

$$+\frac{Y_{12}/S}{Y_{11}\,Y_{22} - Y_{12}^2}\,i_L(0) + \frac{-C(Y_{11} + Y_{12})}{Y_{11}\,Y_{22} - Y_{12}^2}\,v_c(0) \tag{14.55}$$

By taking inverse transform, we can obtain time domain voltage across the resistor R_2.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2*

☆☆★ **14-2.1** A 500 Ω resistor, a 16 mH inductor, and a 25 nF capacitor are connected in parallel which is placed in series with a 2000 Ω resistor. Express the impedance of this series combination as a rational function of *s*.

☆☆★ **14-2.2** A 1 kΩ resistor is in series with a 500 mH inductor. This series combination is in parallel with a 0.4 μF capacitor. Express the equivalent *s*-domain impedance of these parallel branches as a

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

rational function.

☆☆★ **14-2.3** The energy stored in the circuit shown is zero at the time when the switch is closed.

(a) Find the *s*-domain expression for *I*.

(b) Find the *s*-domain expression for *i* when $t > 0$.

(c) Find the *s*-domain expression for *V*.

(d) Find the time-domain expression for *v* when $t > 0$.

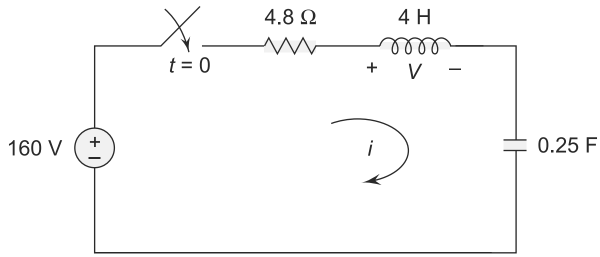

Fig. Q.3

☆★★ **14-2.4** The dc current and voltage sources are applied simultaneously to the circuit shown. No energy is stored in the circuit at the instant of application.

(a) Derive the *s*-domain expressions for V_1 and V_2.

(b) For $t > 0$, derive the time-domain expressions for v_1 and v_2.

(c) Calculate $v_1(0+)$ and $v_2(0+)$.

(d) Compute the steady state value of v_1 and v_2.

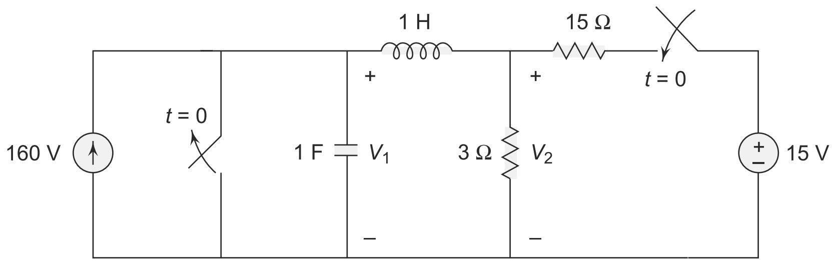

Fig. Q.4

☆★★ **14-2.5** The energy stored in the circuit shown is zero at the instant the two sources are turned on.

(a) Find the component of *v* for $t > 0$ owing to the voltage source.

(b) Find the component of *v* for $t > 0$ owing to the current source.

(c) Find the expression for *v* when $t > 0$.

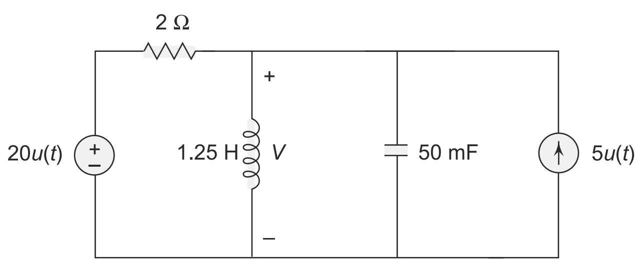

Fig. Q.5

☆★★ **14-2.6** In the circuit shown in Fig. Q.6, there is no energy stored at the time the current source turns on. Given that $i_g = 100\ u(t)$ A;

(a) Find $I_0(s)$.
(b) Use the initial- and final value theorems to find $i_0(0+)$ and $i_0(\infty)$.
(c) Determine if the results obtained in (b) agree with known circuit behaviour.
(d) Find $i_0(t)$.

☆★★ **14-2.7** There is no energy stored in the circuit seen in Fig. Q.7 at the time the two sources are energised.
(a) Use the principle of superposition to find V_0. (b) Find v_0 for $t > 0$.

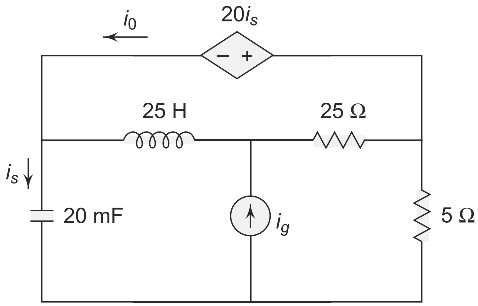

Fig. Q.6

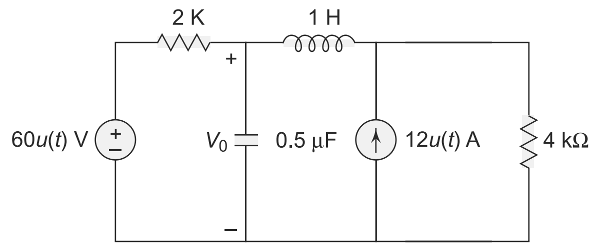

Fig. Q.7

☆☆★ **14-2.8** Find (a) the unit step, and (b) the unit impulse response of the circuit shown in Fig. Q.8.

☆☆★ **14-2.9** There is no energy stored in the circuit shown in Fig. Q.9 at the time the impulse voltage is applied. Find $v_0(t)$ for $t \geq 0$.

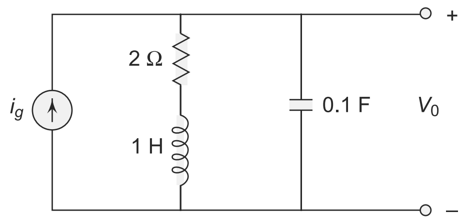

Fig. Q.8

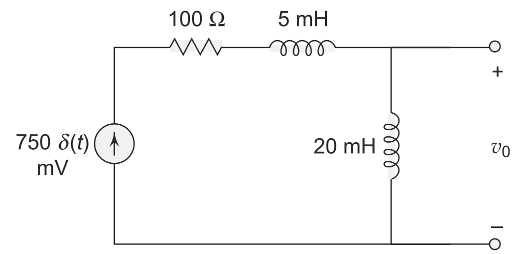

Fig. Q.9

★★★ **14-2.10** The switch in the circuit shown in Fig. Q.10 has been in position a for a long time. At $t = 0$, the switch moves to the position b. Compute (a) $v_1(0)$ (b) $v_1(0-)$ (c) $v_3(0-)$ (d) $i(t)$ (e) v_1 $(0+)$ (f) v_2 $(0+)$ (g) v_3 $(0+)$.

☆☆★ **14-2.11** Find v_0 (t) in the circuit shown in Fig. Q.11, $i_s = 5\,u(t)$ A, Using PSpice.

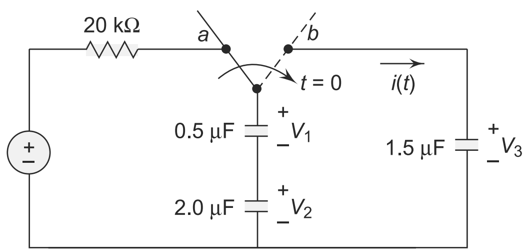

Fig. Q.10

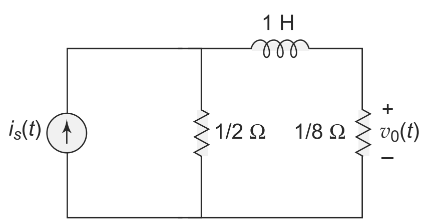

Fig. Q.11

☆☆★ **14-2.12** For the RLC circuit shown in Fig. Q.12, find the complete response if v (0) = 2 V when the switch is closed. Use PSpice.

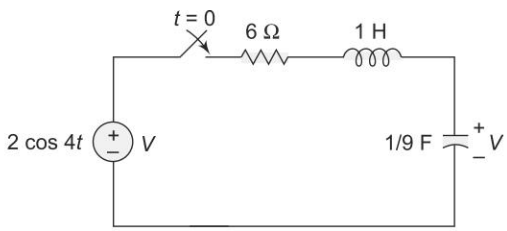

Fig. Q.12

Frequently Asked Questions linked to LO 2

☆★★**14-2.1** In the network shown in Fig. Q.1, the switch is moved from a to b at $t = 0$. Determine $i(t)$ and $V_C(t)$ using Laplace transform. **[PU 2012]**

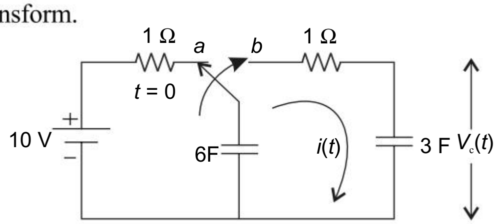

Fig. Q.1

☆★★**14-2.2** In the network shown in Fig. Q.2, the switch is in position 'a' until a steady state is reached. At $t = 0$, the switch is moved to position 'b'. Using that condition, determine the transform of the voltage across the $\frac{1}{2}$ F capacitor using Thevenin's theorem. **[RGTU Dec. 2013]**

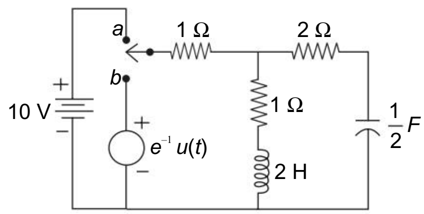

Fig. Q.2

☆★★**14-2.3** For the network shown in Fig. Q.3, determine the voltage ratio transfer function, $\frac{V_2(s)}{V_1(s)}$. **[RGTU Dec. 2013]**

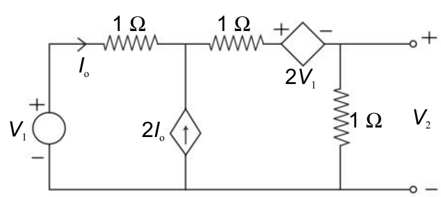

Fig. Q.3

14.3 TRANSFER FUNCTION

*The **transfer function** is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source).* In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero. If a circuit has a multiple independent sources, we can find the transfer

LO 3 Discuss the transfer function and its application in circuit analysis function for each source and use superposition to find the response to all sources.

The transfer function is

$$H(S) = \frac{Y(S)}{X(S)} \tag{14.56}$$

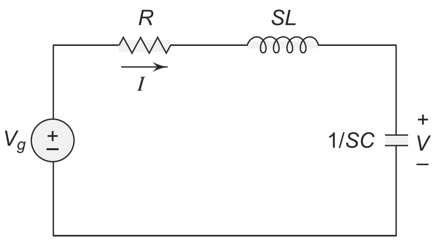

Fig. 14.25

where $Y(S)$ is the Laplace transform of the output signal, and $X(S)$ is the Laplace transform of the input signal. Note that the transfer function depends on what is defined as the output signal. Consider a series circuit shown in Fig. 14.25.

If the current is defined as the response signal of the circuit then the transfer function

$$H(S) = \frac{I}{V_g} = \frac{1}{R + SL + \dfrac{1}{SC}} = \frac{SC}{S^2LC + RCS + 1} \tag{14.57}$$

In the above equation, we recognised that I corresponds to the output $Y(S)$ and V_g corresponds to the input $X(S)$. If the voltage across the capacitor is defined as the output signal of the circuit in Fig. 14.25, the transfer function is

$$H(S) = \frac{V}{V_g} = \frac{1/SC}{R + SL + \dfrac{1}{SC}} = \frac{1}{S^2LC + RCS + 1} \tag{14.58}$$

Thus, because circuits may have multiple sources and because the definition of the output signal of interest can vary, a single circuit can generate many transfer functions. When multiple sources are involved, no single transfer function represent the total output-transfer functions associated with each source must be combined using superposition to yield the total response. We can write the circuit output as the product of the transfer function and the driving function

$$Y(S) = H(S)\,X(S) \tag{14.59}$$

$H(S)$ is a rational function of S and $X(S)$ is also a rational function of S for the excitation functions of most interest in circuit analysis. We can expand the right-hand side of Eq. (14.58) into a sum of partial fractions.

14.4 USE OF TRANSFER FUNCTION IN CIRCUIT ANALYSIS

LO 3

Consider the response of the circuit to a delayed input. If the input is delayed by a seconds

$$\mathcal{L}\,[x(t-a)\,u(t-a)] = e^{-as}-X(S) \tag{14.60}$$

The response becomes

$$Y(S) = H(S)\,X(S)e^{-as} \tag{14.61}$$

If $y(t) = \mathcal{L}^{-1}\,[H(S)\,X(S)]$, then from Eq. 14.61,

$$y(t-a)\,u(t-a) = \mathcal{L}^{-1}\,[H(S)\,X(S)\,e^{-as}] \tag{14.62}$$

Therefore, delaying the input by a seconds simply delays the response function by a seconds. A circuit that exhibits this characteristic is said to be *time invariant*.

If a unit impulse source drives the circuit, the response of the circuit equals the inverse transform of the transfer function. Thus, if

$$x(t) = \delta(t), \text{ then } X(S) = 1$$

and $\quad Y(S) = H(S) \qquad (14.63)$

Hence, $\quad y(t) = h(t) \qquad (14.64)$

where the inverse transform of the transfer function equals the unit impulse response of the circuit. The unit impulse response of the circuit $h(t)$ contains enough information to compute the response to any source that drives the circuit.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★ **14-3.1** Derive the numerical expression for the transfer function v_0/I_g for the circuit shown in Fig. Q.1.

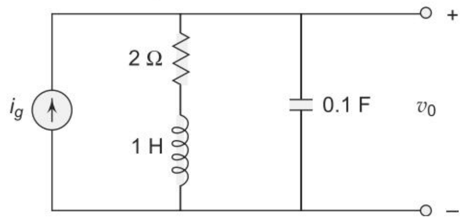

Fig. Q.1

☆☆★ **14-3.2** The unit impulse response of a circuit is

$$v_0(t) = 10{,}000\, e^{-70t} \cos(240\,t + \theta)\, u(t)\,\text{V}$$

where $\quad \tan\theta = \dfrac{7}{24}$

(a) Find the transfer function of the circuit.
(b) Find the unit step response of the circuit.

Frequently Asked Questions linked to LO 3

☆★★ **14-3.1** Show that the voltage transfer function of the network shown in Fig. Q.1 can be written as **[PTU 2009-10]**

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{R_1C_2}\,\frac{S}{S^2 + R_1C_2 + \dfrac{R_1C_2 + R_2(C_1 + C_2)}{R_1R_2C_1C_2} + \dfrac{1}{R_1R_2C_1C_2}}$$

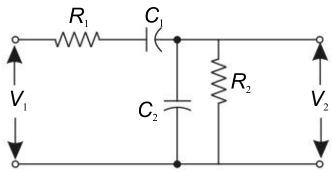

Fig. Q.1

☆☆★ **14-3.2** Show that the voltage transfer function of the network shown in Fig. Q.2, can be written as

$$\frac{V_2(s)}{V_1(s)} = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}$$

where $a_3 = b_3 = R_1\, R_2\, R_3\, C_1\, C_2\, C_3$

$a_2 = R_3\,[R_1\, C_3\,(C_1 + C_2) + (R_1 + R_2)C_1\, C_2] + R_1\, R_2\, C_2\, C_3$

$b_2 = R_3\,(R_1 + R_2)\, C_1\, C_2$

$a_1 = R_3\,(C_1 + C_2) + R_2C_2 + (C_2 + C_3)\, R_1$

$b_1 = R_3(C_1 + C_2)$

$a_0 = b_0 = 1$

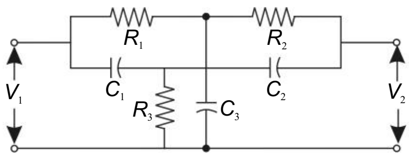

Fig. Q.2

[PTU 2011-12]

14.5 THE TRANSFER FUNCTION AND THE CONVOLUTION INTEGRAL

LO 4 Analyse the relationship of transfer function with convolution integral and steady-state sinusoidal response

The convolution integral relates the output $y(t)$ of a linear time invariant circuit to the input $x(t)$ of the circuit and the circuits impulse response $h(t)$. The convolution integral is defined as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau \qquad (14.65)$$

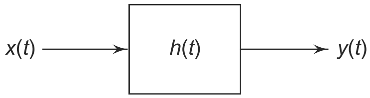

Fig. 14.26

The above equation is based on the assumption that the circuit is linear and time invariant. Because the circuit is linear, the principle of superposition is valid, and because it is time invariant, the amount of the response delay is exactly the same as that of the input delay. Consider block diagram of a general circuit shown in Fig. 14.26 in which $h(t)$ represents any linear time-invariant circuit whose impulse response is known, $x(t)$ represents the excitation signal and $y(t)$ represents the desired output signal.

We assume that $x(t)$ is the general excitation signal shown in Fig. 14.27 (a). Also assume that $x(t) = 0$ for $t < 0$.

Now we see approximate $x(t)$ by a series of rectangular pulses of uniform width $\Delta\tau$ as shown in Fig. 14.27 (b). Thus,

$$x(t) = x_0(t) + x_1(t) + ... + x_i(t) + ... \tag{14.66}$$

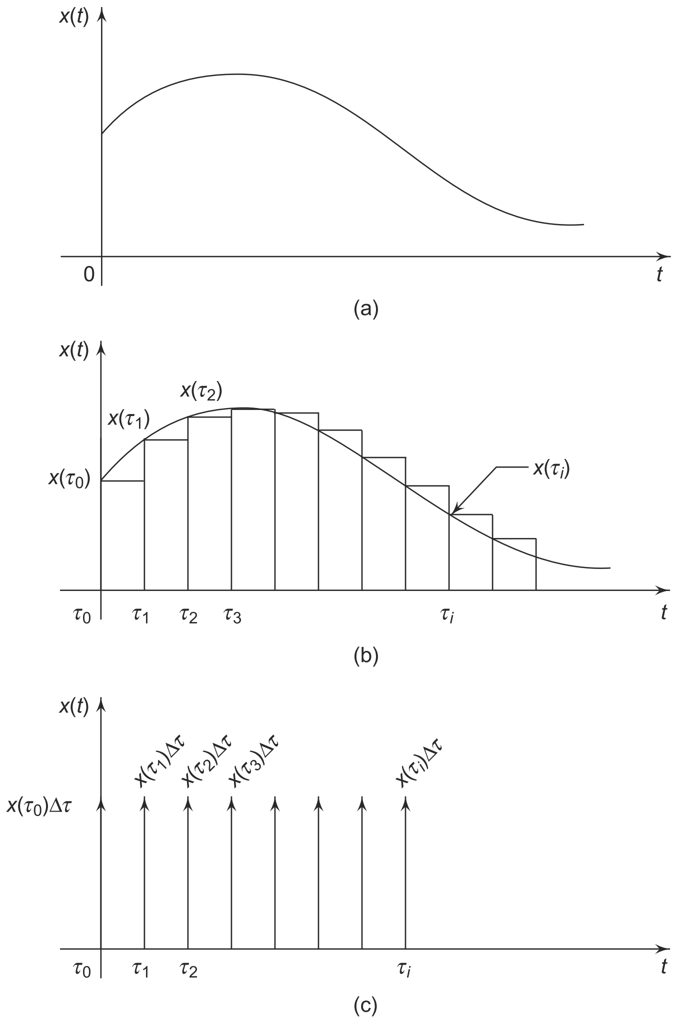

Fig. 14.27

where $x_i(t)$ is a rectangular pulse that equals $x(\tau_i)$ between τ_i and τ_{i+1} and is zero elsewhere. Note that the ith pulse can be expressed in terms of step functions; that is

$$x_i(t) = x(\tau_i)\,[u(t-\tau_i) - u(t-(\tau_i+\Delta\tau))] \tag{14.67}$$

The next step in the approximation of $x(t)$ is to make $\Delta\tau$ small enough that the ith component can be approximated by an impulse function of strength $x(\tau_i)\,\Delta\tau$. Figure 14.27 (c) shows the impulse representation, with the strength of each impulse shown in brackets beside each arrow. The impulse representation of $x(t)$ is

$$x(t) = x\,(\tau_0)\,\Delta\tau\,\delta\,(t-\tau_0) + x\,(\tau_1)\,\Delta\tau\,\delta\,(t-\tau_1) + ...$$
$$+\, x\,(\tau_i)\,\Delta\tau\,\delta\,(t-\tau_i) + ... \tag{14.68}$$

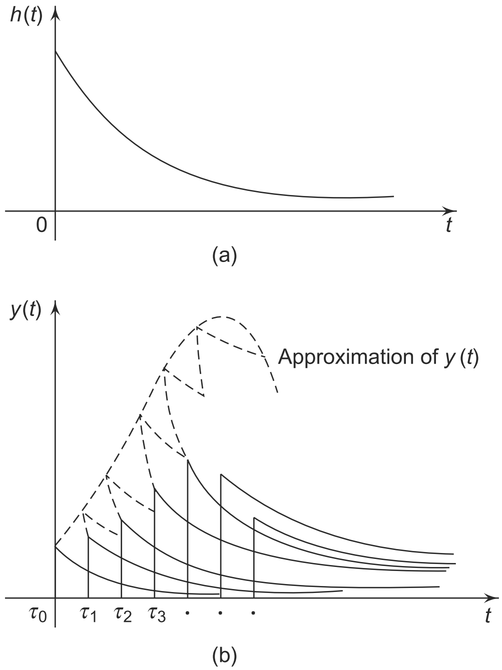

Fig. 14.28

Now when $x(t)$ is represented by a series of impulse functions, the response function $y\,(t)$ consists of the sum of a series of uniformly delayed impulse responses. The strength of each response depends on the strength of the impulse driving the circuit. For example, let's assume that the unit impulse response of the circuit contained within the box in Fig. 14.26 is the exponential decay function shown in Fig. 14.28 (a). Then the approximation of $y\,(t)$ is the sum of the impulse responses shown in Fig. 14.28 (b).

Analytically, the expression for $y(t)$ is

$$\begin{aligned} y(t) \quad &= x(\tau_0)\,\Delta\tau\,h(t-\tau_0) + x(\tau_1)\,\Delta\tau\,h(t-\tau_1) \\ &\quad + x(\tau_2)\,\Delta\tau\,h(t-\tau_2) + ... \\ &\quad + x(\tau_i)\,\Delta\tau\,h(t-\tau_i) + ... \end{aligned} \tag{14.69}$$

As $\Delta\tau \to 0$, the summation in Eq. (14.69) approaches a continuous integration, or

$$\sum_{i=0}^{\infty} x(\tau_i)\,h(t-\tau_i)\,\Delta\tau \to \int_0^{\infty} x(\tau)\,h(t-\tau)d\tau \tag{14.70}$$

Therefore,

$$y(t) = \int_0^{\infty} x(\tau)\,h(t-\tau)\,d\tau \tag{14.71}$$

If $x(t)$ exists over all time, then the lower limit on Eq. (14.71) becomes $-\infty$, thus, in general

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \tag{14.72}$$

The integral relation between $y(t)$, $h(t)$ and $x(t)$ is written in a shorthand notation

$$y(t) = \; h(t) * x(t) = x(t) * h(t) \tag{14.73}$$

Thus, $h(t) * x(t)$ is read as "$h(t)$ is convolved with $x(t)$" and implies that

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (14.74)$$

The above integral gives the most general relation for the convolution of two functions. However, in our applications, we can change the lower limit to zero and the upper limit to t. Then the above equation can be written as

$$\begin{aligned} y(t) &= \int_0^t x(\tau)\, h(t-\tau)\, d\tau \\ &= \int_0^t h(\tau)\, x(t-\tau)\, d\tau \qquad (14.75) \end{aligned}$$

For physically realizable circuits, $h(t)$ is zero for $t < 0$. In other words, there can be no impulse response before an impulse is applied. We start measuring time at the instant the excitation $x(t)$ is turned on, therefore $x(t) = 0$ for $t < 0$.

A graphical interpretation of the convolution integrals contained in Eq. (14.75) is important in the use of integral as a computational tool. Consider the impulse response of our circuit is the exponential decay function shown in Fig. 14.29 (a) and the excitation function has the waveform shown in Fig. 14.29 (b).

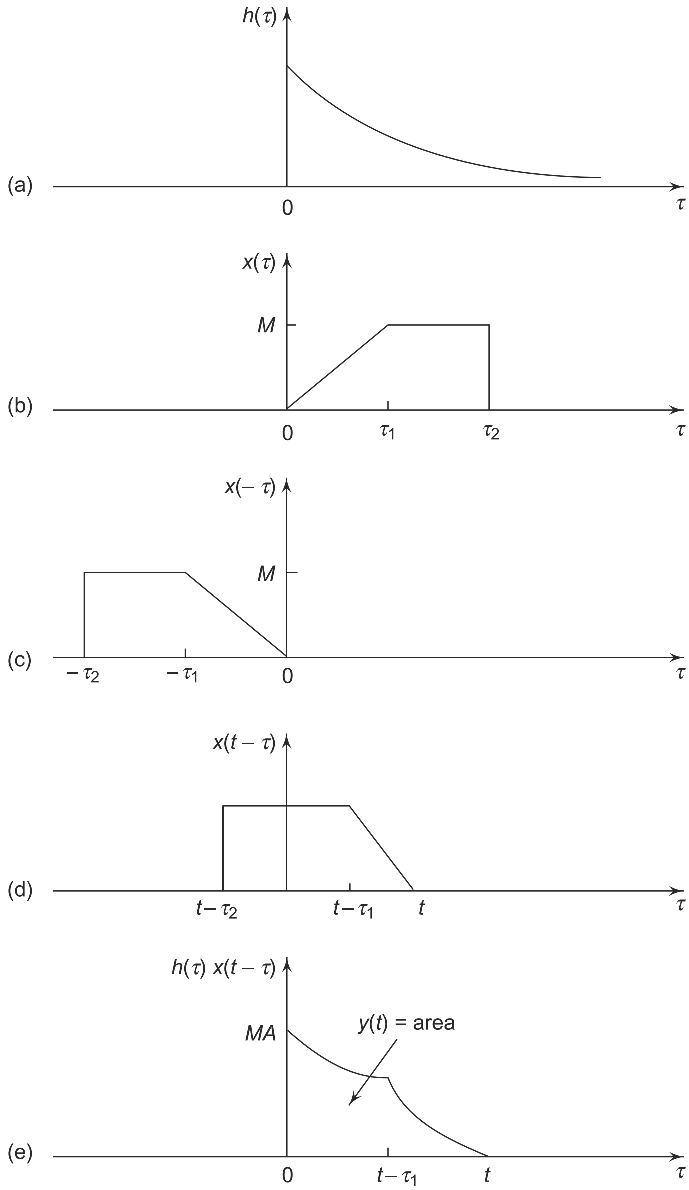

Fig. 14.29

Replacing τ with –τ simply folds the excitation function over the vertical axis and replacing – λ with τ –λ slides the folded function to the right. This folding and sliding operation gives rise to the term *convolution*. At any specified value of t, the response function $y(t)$ is the area under the product function $\lambda(\tau)\, x(t - \tau)$ as shown in Fig. 14.29 (c). For $\tau < 0$, the product $\lambda(\tau)\, x(t - \tau)$ is zero because $h(\tau)$ is zero. For $\tau > t$, the product $h(\tau)\, x(t - \tau)$ is zero because $x(t - \tau)$ is zero.

14.6 THE TRANSFER FUNCTION AND THE STEADY-STATE SINUSOIDAL RESPONSE LO 4

We use the transfer function to relate the steady-state response to the excitation source. First we assume that

$$x(t) = A\cos(\omega t + \phi) \tag{14.76}$$

and we use the equation

$$Y(S) = H(S)\,X(S) \tag{14.77}$$

to find the steady-state solution of y(t).

To find the Laplace transform of $x(t)$, we first write $x(t)$ as

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi \tag{14.78}$$

from which

$$\begin{aligned} X(S) &= \frac{A\cos\phi\, S}{S^2+\omega^2} - \frac{A\sin\phi\,\omega}{S^2+\omega^2} \\ &= \frac{A(S\cos\phi - \omega\sin\phi)}{S^2+\omega^2} \end{aligned} \tag{14.79}$$

Substituting Eq. (14.79) into Eq. (14.77) gives the s-domain expression for the response

$$Y(S) = H(s)\frac{A(S\cos\phi - \omega\sin\phi)}{S^2+\omega^2} \tag{14.80}$$

By taking partial fractions

$$\begin{aligned} Y(S) &= \frac{k_1}{S-j\omega} + \frac{k_1^*}{S+j\omega} \\ &\quad + \Sigma \text{ terms generated by the poles of } H(S) \end{aligned} \tag{14.81}$$

In Eq. (14.81), the first two terms result from the complex conjugate poles of the deriving source. However, the terms generated by the poles of $H(s)$ do not contribute to the steady-state response of $y(t)$, because all these poles lie in the left half of the s-plane, consequently, the corresponding time-domain terms approach zero as t increases.

Thus, the first two terms on the right-hand side of Eq. (14.81) determine the steady-state response. Now K_1 can be determined.

$$\begin{aligned} k_1 &= \left.\frac{H(S)\,A(S\cos\phi - \omega\sin\phi)}{S+j\omega}\right|_{S=j\omega} \\ &= \frac{H(j\omega)\,A(j\omega\cos\phi - \omega\sin\phi)}{2j\omega} \\ &= H(j\omega)\frac{A(\cos\phi + j\sin\phi)}{2} = \frac{1}{2}H(j\omega)Ae^{j\phi} \end{aligned} \tag{14.82}$$

In general, $H(j\omega)$ is a complex quantity, thus

$$H(j\omega) = |H(j\omega)|\, e^{j\theta(\omega)} \tag{14.83}$$

where $|H(j\omega)|$ is the magnitude, and phase angle is $\theta(\omega)$ of the transfer function vary with the frequency ω, the expression for K_1 becomes

$$K_1 = \frac{A}{2}|H(j\omega)|\, e^{j[\theta(\omega)+\phi]} \tag{14.84}$$

We obtain the steady-state solution for $y(t)$ by taking inverse transform of Eq. (14.81) ignoring the terms generated by the poles of $H(S)$. Thus,

$$y_{ss}(t) \;= A|H(j\omega)| \cos\,[\omega t + \phi + \theta(\omega)] \tag{14.85}$$

which indicates how to use the transfer function to find the steady-state sinusoidal response of a circuit.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★ **14-4.1** A rectangular voltage pulse $v_i = [u(t) - u(t-1)]$ V is applied to the circuit shown in Fig. Q.1. Use convolution to find v_0.

☆★★ **14-4.2** Interchange the inductor and resistor in Problem 14.4.1 and again use the convolution integral to find v_0.

☆☆★ **14-4.3** The current source in the circuit shown is delivering 10 cos 4t A. Use the transfer function to compute the steady-state expression for v_0.

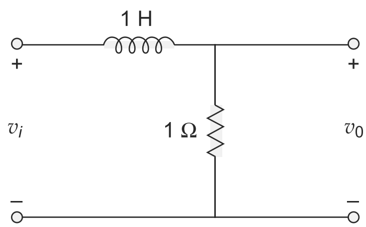

Fig. Q.1

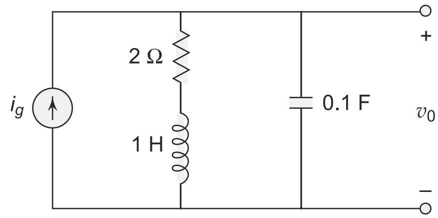

Fig. Q.3

14.7 THE IMPULSE FUNCTION IN CIRCUIT ANALYSIS

Impulse functions occur in circuit analysis either because of a switching operation or because a circuit is excited by an impulse source. The Laplace transform can be used to predict the impulsive currents and voltages created during switching and the response of a circuit to an impulsive source.

LO 5 Discuss the impulse function and its application in circuit analysis

14.7.1 Switching Operation

We use two different circuits to illustrate how an impulse function can be created with a switching operation: a capacitor circuit and a series inductor circuit.

❑ ***Capacitor Circuit*** In the circuit shown in Fig. 14.30, the capacitor C_1 is charged to an initial voltage of V_0 at the time the switch is closed.

In the circuit, the initial charge on C_2 is zero. Figure 14.31 shows the s-domain equivalent circuit.

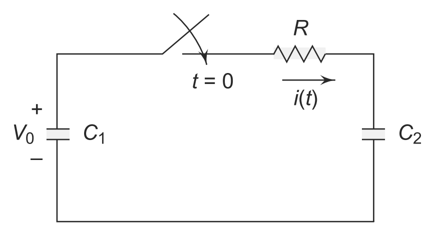

Fig. 14.30

From Fig. 14.31,

$$I = \frac{V_0/S}{R + \left(\frac{1}{SC_1}\right) + \left(\frac{1}{SC_2}\right)}$$

$$= \frac{V_0/R}{S + \left(\frac{1}{RC_e}\right)} \quad (14.86)$$

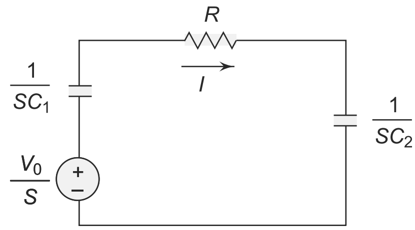

Fig. 14.31

where the equivalent capacitance $\frac{C_1 C_2}{C_1 + C_2}$ is replaced by C_e.

By taking inverse transform of Eq. (14.86), we obtain

$$i = \frac{V_0}{R} e^{-t/RC_e} \quad (14.87)$$

which indicates that as R decreases, the initial current $\left(\frac{V_0}{R}\right)$ increases and the time constant (RC_e) decreases. Thus, R gets smaller, the current starts from a larger initial value and then dropped off more rapidly. Figure 14.32 shows these characteristics of i.

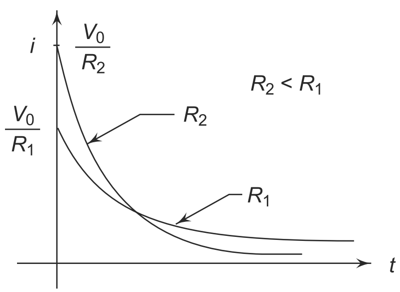

Fig. 14.32

The characteristics show, i is approaching an impulse function as R approaching to zero because the initial value of i is approaching infinity and time duration of i is approaching zero. We still have to determine whether the area under the current function is independent of R. Physically, the total area under the i versus t curve represents the total charge transferred to C_2 after the switch is closed. Thus,

$$\text{Area} = q = \int_{0-}^{\infty} \frac{V_0}{R} e^{-t/RC_e} dt = V_0 C_e \quad (14.88)$$

which says that the total charge transferred to C_2 is independent of time and equals $V_0 C_e$ coulombs. Thus, as R approaches zero, the current approaches an impulse strength $V_0 C_e$.

$$i \to V_0 C_e \, \delta(t) \quad (14.89)$$

when $R = 0$, a finite amount of charge is transferred to C_2 instantaneously. When the switch is closed, the voltage across C_2 does not jump to V_0 but its final value of

$$v_2 = \frac{C_1 V_0}{C_1 + C_2} \quad (14.90)$$

If we set R equal to zero, the Laplace transform analysis will predict the impulsive current response. Thus,

$$I = \frac{V_0/S}{\left(\frac{1}{SC_1}\right) + \left(\frac{1}{SC_2}\right)} = \frac{C_1 C_2 V_0}{C_1 + C_2} = C_e V_0 \quad (14.91)$$

The inverse transform of the above equation is

$$i = C_e V_0 \, \delta(t) \quad (14.92)$$

❑ ***Series Inductor Circuit*** The circuit shown in Fig. 14.33 illustrates a second switching operation that produces an impulsive response. The problem is to find the time-domain expression for v_0– after the switch has been opened. Note that opening the switch forces an instantaneous change in the current of L_2, which causes v_0 to contain an impulsive component.

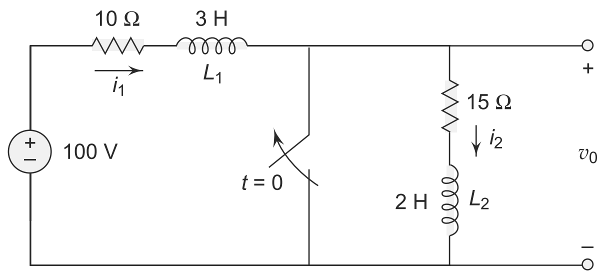

Fig. 14.33

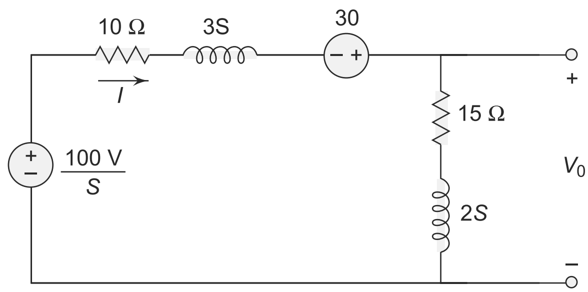

Fig. 14.34

Figure 14.34 shows the *s*-domain equivalent with the switch open. The current in the 3 H inductor at $t = 0$ is 10 A, and the current in 2 H inductor at $t = 0$ is zero.

Applying Kirchhoff's current law, we get

$$\frac{V_0}{2S+15}+\frac{V_0-[(100/S)+30]}{3S+10}=0 \qquad (14.93)$$

Solving for V_0 yields

$$V_0=\frac{40(S+7.5)}{S(S+5)}+\frac{12(S+7.5)}{S+5} \qquad (14.94)$$

By taking partial fractions, we get

$$V_0=\frac{60}{S}-\frac{20}{S+5}+12+\frac{30}{S+5}$$
$$=12+\frac{60}{S}+\frac{10}{S+5} \qquad (14.95)$$

By taking inverse transform, we have

$$v_0 = 12\,\delta(t) + (60 + 10e^{-5t})\,u(t) \text{ volts} \qquad (14.96)$$

Let us derive the expression for the current when $t > 0$. After the switch has been opened, the current in L_1 is the same as the current in L_2. The current equation is

$$I=\frac{\left(\frac{100}{S}\right)+30}{5S+25}=\frac{20}{S(S+5)}+\frac{6}{S+5}$$
$$=\frac{4}{S}-\frac{4}{S+5}+\frac{6}{S+5}$$
$$=\frac{4}{S}+\frac{2}{S+5} \qquad (14.97)$$

By taking inverse transform gives

$$i = (4 + 2e^{-5t})\,u(t) \text{ A} \qquad (14.98)$$

Before the switch is opened, the current in L_1 is 10 A, and the current in L_2 is 0A. We know that at $t = 0$, the current in L_1 and in L_2 is 6A. Then, the current in L_1 changes instantaneously from 10A to 6A,

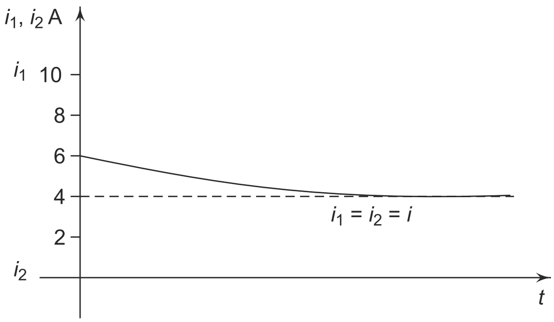

Fig. 14.35

while the current in L_2 changes instantaneously from 0 to 6A. From this value of 6A, the current decreases exponentially to a final value of 4A. Figure 14.35 shows these characteristics of i_1 and i_2.

Impulse Sources

Impulse functions can occur in sources as well as responses. Such sources are called impulsive sources. An impulse source driving a circuit imparts a finite amount of energy into the system instantaneously. In the circuit shown in Fig. 14.36 (a), an impulsive voltage source having a strength of V_0 volt-seconds is applied to a series connection of a resistor and an inductor. When the voltage source is applied, the initial energy in the inductor is zero, therefore the initial current is zero. There is no voltage drop across R. So the impulse voltage source appears directly across L. An impulse voltage at the terminals of an inductor establishes an instantaneous current. The current is

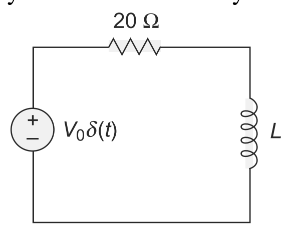

Fig. 14.36 (a)

$$i = \frac{1}{L}\int_0^t V_0\,\delta(x)\,dx \tag{14.99}$$

The integral of $\delta(t)$ over any interval that includes zero is one; thus, we have

$$i(0) = \frac{V_0}{L}\text{ A} \tag{14.100}$$

For an infinitesimal moment, the impulsive voltage source has stored in the inductor.

$$W = \frac{1}{2}L\left(\frac{V_0}{L}\right)^2 = \frac{1}{2}\frac{V_0^2}{L}\text{ J} \tag{14.101}$$

The current $\dfrac{V_0}{L}$ decays to zero in accordance with the natural response of the circuit, that is,

$$i = \frac{V_0}{L}e^{-t/\tau}u(t) \tag{14.102}$$

where $\quad \tau = \dfrac{L}{R}$.

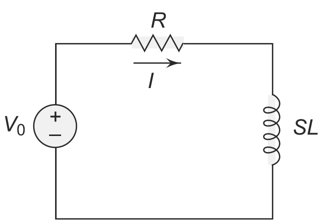

Fig. 14.36 (b)

When a circuit is driven by only an impulsive source, the total response is completely defined by the natural response. The duration of the impulse source is so infinitesimal that it does not contribute to any forced response.

We may also obtain Eq. (14.102) by direct application of the Laplace transform method. Figure 14.36 (b) shows the s-domain equivalent of the circuit in Fig. 14.36 (a).

The current I in the circuit is

$$I = \frac{V_0}{R + SL} = \frac{V_0/L}{S + R/L} \tag{14.103(a)}$$

Taking inverse Laplace transform, we get

$$i = \frac{V_0}{L}e^{-\left(\frac{R}{L}\right)t} = \frac{V_0}{L}e^{-t/\tau}u(t) \tag{14.103(b)}$$

Thus, the Laplace transform method gives the correct solution for $i \geq 0$.

Additional Solved Problems

PROBLEM 14.1

A 500 Ω resistor, a 16 mH inductor, and a 25 mF capacitor are connected in parallel. Express the admittance of this parallel combination of elements as a rational function of S.

Solution The circuit represented in *s*-domain of the above problem is shown in Fig. 14.37.

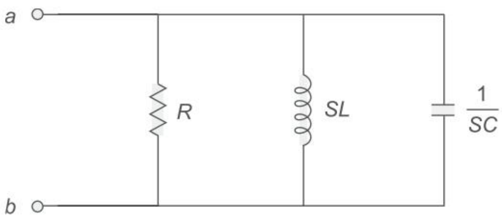

Fig. 14.37

The admittance of terminals *ab* is

$$Y(S) = \frac{1}{R} + \frac{1}{SL} + SC \tag{14.104}$$

Substituting the numerical values in the above equation,

$$Y(S) = \frac{1}{500} + \frac{1}{S \times 16 \times 10^{-3}} + S \times 25 \times 10^{-9} \tag{14.105}$$

Simplifying the above equation, we have

$$Y(S) = \frac{25 \times 10^{-9}}{S}(S^2 + 80,000\,S + 25 \times 10^8) \tag{14.106}$$

PROBLEM 14.2

The switch in the circuit shown has been in the position a for a long time. At t = 0, the switch is thrown to the position b. Find the current I as rational function of s. Find the time-domain expression for the current i.

Solution When the switch is at position for a long time, both the capacitors are charged to 100 V.

When the switch is at the position *b*, the *s*-domain circuit is shown in Fig. 14.39.

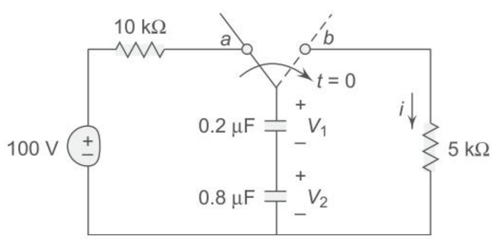

Fig. 14.38

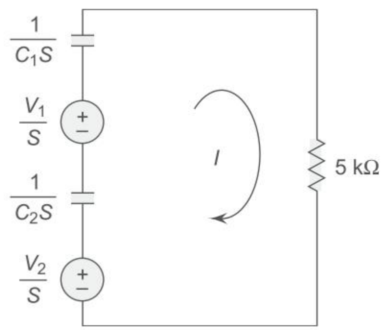

Fig. 14.39

By applying Kirchhoff's voltage law, we have

$$\frac{V_1}{S} + \frac{V_2}{S} = \frac{1}{C_1 S} I + \frac{1}{C_2 S} I + I(5K) \tag{14.107}$$

$$\frac{1}{S}(V_1+V_2)=\frac{I}{S}\left[\frac{1}{0.2\times10^{-6}}+\frac{1}{0.8\times10^{-6}}+5\times10^3\right]$$

$$\frac{I}{S}100=\frac{I}{S}\left[\frac{1}{0.16\times10^{-6}}+5\times10^3\right]$$

$$I=\frac{0.02}{S+1250} \tag{14.108}$$

By taking inverse transform, we get the time-domain expression for i

$$i = 0.02\, e^{-1250t}\ \text{A}. \tag{14.109}$$

PROBLEM 14.3

Obtain the current s-domain expression for the current I_L in the circuit shown in Fig. 14.40. Also obtain the time-domain expression for the inductor current. The switch is opened at $t = 0$. Assume initial energy stored in the circuit is zero.

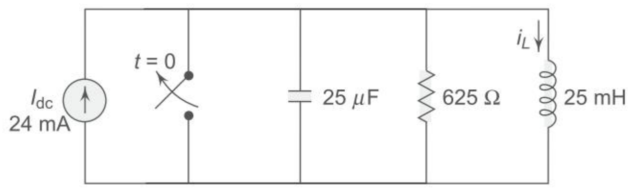

Fig. 14.40

Solution The s-domain equivalent circuit for the circuit shown in Fig. 14.40 is shown in Fig. 14.41.

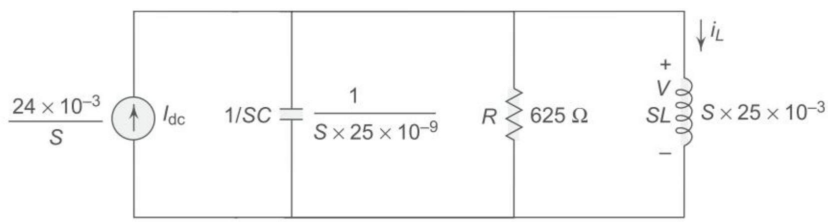

Fig. 14.41

By applying Kirchhoff's current law, we get

$$SCV+\frac{V}{R}+\frac{V}{SL}=\frac{I_{dc}}{S} \tag{14.110}$$

$$V=\frac{I_{dc}/C}{S^2+\left(\frac{1}{RC}\right)S+\frac{1}{LC}} \tag{14.111}$$

We know $$I_L=\frac{V}{SL} \tag{14.112}$$

Substituting Eq. (14.111) into Eq. (14.112), we get

$$I_L = \frac{I_{dc}/LC}{S\left[S^2 + \left(\frac{1}{RC}\right)S + \left(\frac{1}{LC}\right)\right]} \tag{14.113}$$

Substituting the numerical values yields

$$I_L = \frac{384\times10^5}{S(S^2+64000S+16\times10^8)} \tag{14.114}$$

By taking partial fractions, we get

$$I_L = \frac{384\times10^5}{S(S+32000-j24000)(S+32000+j24000)} \tag{14.115}$$

$$I_L = \frac{K_1}{S} + \frac{K_2}{S+32000-j24000} + \frac{K_2^*}{S+32000-j24000} \tag{14.116}$$

The partial fraction coefficients are

$$K_1 = \frac{384\times10^5}{16\times10^8} = 24\times10^{-3}$$

$$K_2 = \frac{384\times10^5}{(-32000+j24000)(j48000)}$$

$$= 20 \times 10^{-3} \angle 126.87° \tag{14.117}$$

Substituting the numerical values of K_1 and K_2 into Eq. (14.116) and inverse transforming the resulting expression yields

$$i_L = [24 + 40\, e^{-32,000\, t} \cos (24000\, t + 126.87°)] \text{ mA} \tag{14.118}$$

PROBLEM 14.4

Obtain the s-domain expression for the current I_L in the circuit shown in Fig. 14.40 when the dc current source is replaced by a sinusoidal current source $i_g = I_m \cos \omega t$. Where I_m = 24 mA and ω = 40,000 rad/s. Assume initial energy stored in the circuit is zero.

Solution The *s*-domain expression for the source current is

$$I_g = \frac{SI_m}{s^2+\omega^2} \tag{14.119}$$

The voltage across the parallel elements is

$$V = \frac{(I_g/C)s}{s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}} \tag{14.120}$$

Substituting Eq. (14.119) into Eq. (14.120) results in

$$V = \frac{(I_m/C)s^2}{(s^2+\omega^2)\left(s^2+\left(\frac{1}{RC}\right)s+\left(\frac{1}{LC}\right)\right)} \tag{14.121}$$

from which

$$I_L = \frac{V}{SL} = \frac{(I_m/LC)s}{(s^2+\omega^2)\left[s^2+\left(\frac{1}{RC}\right)s+\left(\frac{1}{LC}\right)\right]} \tag{14.122}$$

Substituting the numerical values of I_m, ω, R, L, and C in Eq. (14.122) gives

$$I_L = \frac{384\times10^5 s}{(s^2+16\times10^8)(s^2+64000s+16\times10^8)} \tag{14.123}$$

By factoring the denominator, we get

$$I_L = \frac{384\times10^5 s}{(s^2-j\omega)(s+j\omega)(s+\alpha-j\beta)(s+\alpha+j\beta)} \tag{14.124}$$

where $\omega = 40000$, $\alpha = 32000$ and $\beta = 24000$

By taking partial fractions, we get

$$I_L = \frac{K_1}{s-j40000}+\frac{K_1^*}{s+j40000} + \frac{K_2}{s+32000-j24000}+\frac{K_2^*}{s+32000+j24000} \tag{14.125}$$

The coefficients K_1 and K_2 are

$$K_1 = \frac{384\times10^5(j40000)}{(j80000)(32000+j16000)(32000+j64000)}$$

$$= 7.5 \times 10^{-3} \angle -90° \tag{14.126}$$

$$K_2 = \frac{384\times10^5(-32000+j24000)}{(-32000-j16000)(-32000+j64000)(j48000)}$$

$$= 12.5 \times 10^{-3} \angle 90° \tag{14.127}$$

Substituting the numerical values from (14.126) and (14.127) into (14.125) and inverse-transforming the resulting expression yields

$$i_L = [15\cos(40000\,t-90°) + 25\,e^{-32000\,t}\cos(24000\,t+90°)] \text{ mA}$$

$$i_L = [15\sin 40000\,t - 25\,e^{-32000\,t}\sin 24000\,t] \text{ mA} \tag{14.128}$$

PROBLEM 14.5

Obtain the expression for i_1 and i_2 in the circuit shown in Fig. 14.42 when dc voltage source is applied suddenly. Assume that the initial energy stored in the circuit is zero.

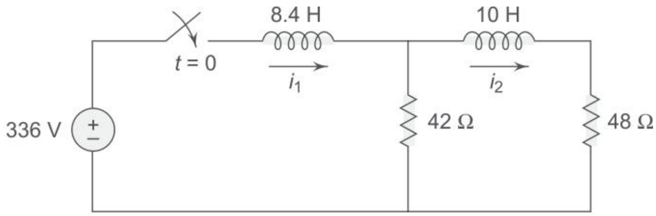

Fig. 14.42

Solution Figure 14.43 shows the *s*-domain equivalent circuit for the circuit shown in Fig. 14.42.

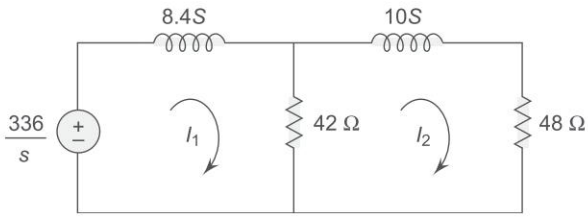

Fig. 14.43

The two mesh current equations are

$$\frac{336}{s} = (42+8.4s)I_1 - 42I_2 \tag{14.129}$$

$$0 = -42\,I_1 + (90 + 10\,s)\,I_2 \tag{14.130}$$

Using Cramer's method to solve for I_1 and I_2, we get

$$\Delta = \begin{vmatrix} 42+8.4s & -42 \\ -42 & 90+10s \end{vmatrix}$$

$$= 84\,(s^2 + 14\,s + 24)$$

$$= 84\,(s + 2)\,(s + 12) \tag{14.131}$$

$$\Delta_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90+10s \end{vmatrix} = \frac{3360(s+9)}{s} \tag{14.132}$$

$$\Delta_2 = \begin{vmatrix} 42+8.4s & 336/s \\ -42 & 0 \end{vmatrix} = \frac{14112}{s} \tag{14.133}$$

Based on Eqs (14.131) to (14.133),

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{40(s+9)}{s(s+2)(s+12)} \tag{14.134}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{168}{s(s+2)(s+12)} \tag{14.135}$$

Expanding I_1 and I_2 into a sum of partial fractions gives

$$I_1 = \frac{15}{s} - \frac{14}{s+2} - \frac{1}{s+12} \tag{14.136}$$

$$I_2 = \frac{7}{s} - \frac{8.4}{s+2} + \frac{1.4}{s+12} \tag{14.137}$$

We obtain the expressions for i_1 and i_2 by inverse transforming Eq. (14.136) and (14.137) respectively

$$i_1 = (15 - 14\, e^{-2t} - e^{-12t})\ \text{A} \tag{14.138}$$

$$i_2 = (7 - 8.4\, e^{-2t} + 1.4\, e^{-12t})\ \text{A} \tag{14.139}$$

PROBLEM 14.6

Transform the circuit shown in Fig. 14.44 to the s-domain and determine the Laplace impedance.

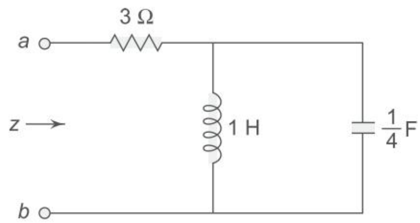

Fig. 14.44

Solution The transformed circuit for the above circuit is shown in Fig. 14.45.

The parallel combination of inductor and capacitor is in series with the resistor.

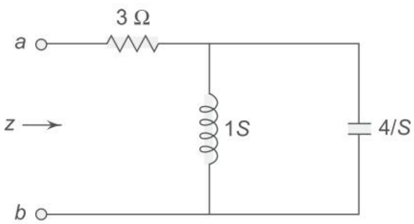

Fig. 14.45

$$z = 3 + \left\{ s \,\|\, \left(\frac{4}{s} \right) \right\} = 3 + \frac{s\left(\dfrac{4}{5}\right)}{s + \dfrac{4}{s}}$$

$$z = \frac{3s^2 + 4s + 12}{s^2 + 4} \tag{14.140}$$

PROBLEM 14.7

Determine the current i if the circuit is driven by a voltage source as shown in Fig. 14.46. The initial value of the voltage across the capacitor and the initial current through the inductor are both zero.

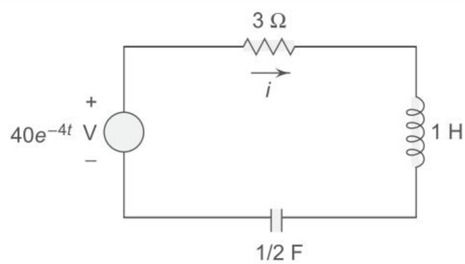

Fig. 14.46

Solution The transformed circuit is as shown in Fig. 14.47.

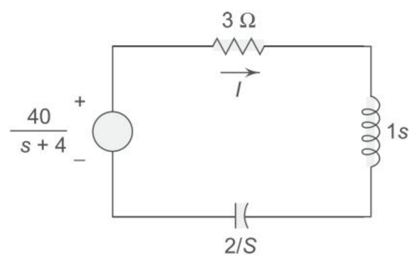

Fig. 14.47

Total Laplace impedance across the voltage source is

$$Z = 3 + s + \frac{2}{s}$$

$$Z = \frac{s^2 + 3s + 2}{s} \tag{14.141}$$

Thus, the current is

$$I = \frac{V}{Z} = \frac{40/(s+4)}{(s^2 + 3s + 2)/s} \tag{14.142}$$

$$I = \frac{40s}{(s+4)(s^2 + 3s + 2)} \tag{14.143}$$

By taking partial fractions,

$$I = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+4} \tag{14.144}$$

The coefficients K_1, K_2, and K_3 are

$$K_1 = I(s+1)\big|_{s=-1} = \frac{-40}{3}$$

$$K_2 = I(s+2)\big|_{s=-2} = 40$$

$$K_3 = I \times (s+4)\big|_{s=-4} = \frac{-80}{3}$$

Substituting the coefficients and taking inverse Laplace transform, we get

$$i = 40e^{-2t} - \frac{40}{3}e^{-t} - \frac{80}{3}e^{-4t} \tag{14.145}$$

PROBLEM 14.8

Determine the current i for $t \geq 0$ if initial current $i(0) = 1$ for the circuit shown in Fig. 14.48.

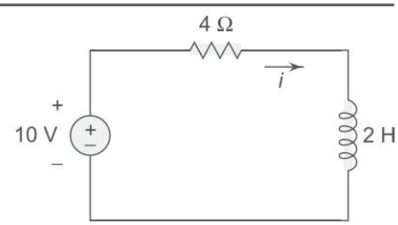

Fig. 14.48

Solution The s-domain circuit with series initial current in the inductor is shown in Fig. 14.49.

Applying Kirchhoff's voltage law results in

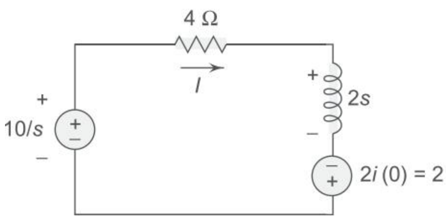

Fig. 14.49

$$\frac{10}{s} - 4I - 2sI + 2 = 0 \tag{14.146}$$

$$10 - 4sI - 2s^2I + 2s = 0$$

$$I = \frac{10+2s}{2s(s+2)}$$

$$I = \frac{5+s}{s(s+2)} \tag{14.147}$$

Taking partial fractions

$$I = \frac{K_1}{s} + \frac{K_2}{s+2} \tag{14.148}$$

The coefficients K_1 and K_2 are

$$K_1 = \frac{5}{2}; \quad K_2 = \frac{-3}{2}$$

Substituting coefficients and taking inverse Laplace transform of Eq. (14.148) gives

$$i = \frac{5}{2} - \frac{3}{2}e^{-2t} \tag{14.149}$$

Alternatively, the inductor initial condition can be represented parallelly as shown in Fig. 14.50.

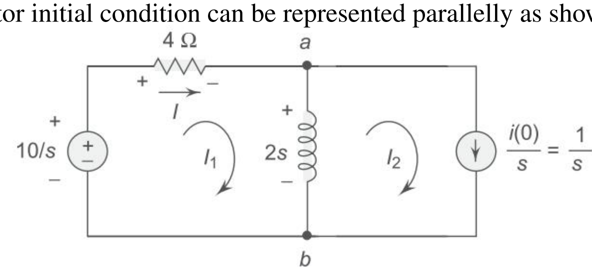

Fig. 14.50

By inspection,

$$I_2 = \frac{1}{s}$$

By applying Kirchhoff's voltage law to the mesh, we get

$$\frac{10}{s} - 4I_1 - 2sI_1 + 2sI_2 = 0 \tag{14.150}$$

From the figure, $I_1 = I$ and $I_2 = \frac{1}{s}$

$$\frac{10}{s} - 4I - 2sI + 2s\left(\frac{1}{s}\right) = 0$$

$$I = \frac{s+5}{s(s+2)} \tag{14.151}$$

Taking partial fractions and inverse Laplace transform, we get

$$i = \frac{5}{2} - \frac{3}{2}e^{-2t} \tag{14.152}$$

PROBLEM 14.9

Determine the current i for $t \geq 0$ if V_c (0) = 4 V for the circuit shown in Fig. 14.51.

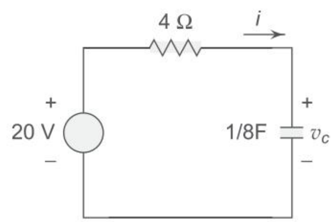

Fig. 14.51

Solution The transformed s-domain circuit is shown in Fig. 14.52.

Application of Kirchhoff's voltage law gives

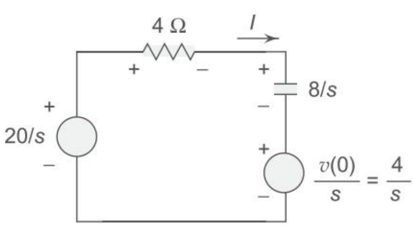

Fig. 14.52

$$\frac{20}{s} - 4I - \left(\frac{8}{5}\right)I - \frac{4}{s} = 0$$

$$I = \frac{4}{s+2} \tag{14.153}$$

Taking inverse Laplace transform gives

$$i = 4\,e^{-2t} \tag{14.154}$$

Alternatively, the initial condition can be represented as shown in Fig. 14.53.

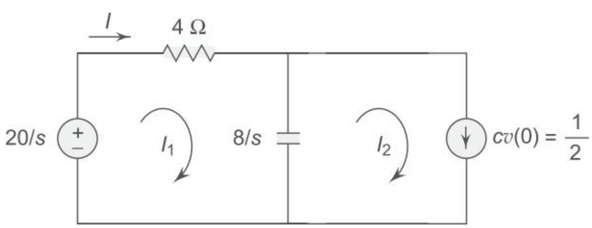

Fig. 14.53

By inspection, we have

$$I_2 = \frac{-1}{2}$$

Applying Kirchhoff's voltage law to the mesh 1 results in

$$\frac{20}{s} - 4I_1 - \frac{8}{S}I_1 + \left(\frac{8}{s}\right)\left(\frac{-1}{2}\right) = 0 \tag{14.155}$$

Because $I_1 = I$ and $I_2 = \dfrac{-1}{2}$

$$\frac{20}{s} - 4I - \frac{8}{s}I + \left(\frac{8}{s}\right)\left(\frac{-1}{2}\right) = 0$$

$$I = \frac{4}{s+2} \tag{14.156}$$

Taking inverse transform, we get

$$i = 4\,e^{-2t} \tag{14.157}$$

PROBLEM 14.10

Convert the current source in Fig. 14.54 to a voltage source in the s-domain.

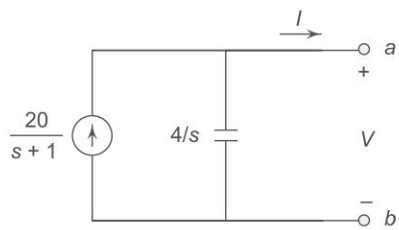

Fig. 14.54

Solution Converting the circuit in Fig. 14.54 into the voltage source results in the circuit shown in Fig. 14.55.

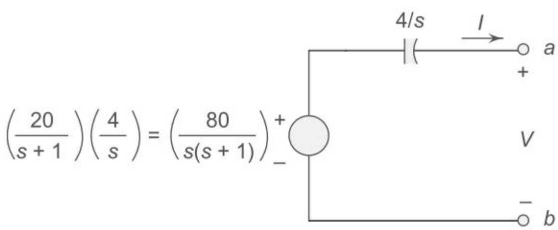

Fig. 14.55

PROBLEM 14.11

Convert the voltage source in Fig. 14.56 to a current source in the s-domain.

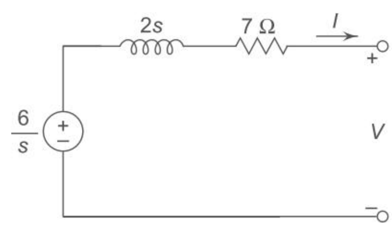

Fig. 14.56

Solution Converting Fig. 14.56 into a current source in the s-domain results in the circuit shown in Fig. 14.57.

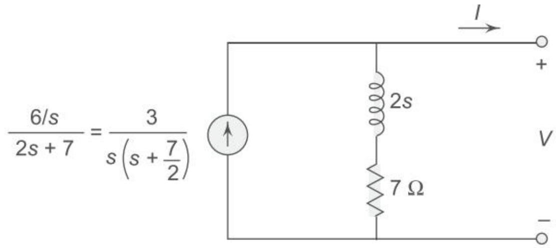

Fig. 14.57

PROBLEM 14.12

Determine v_0 for the circuit shown in Fig. 14.58.

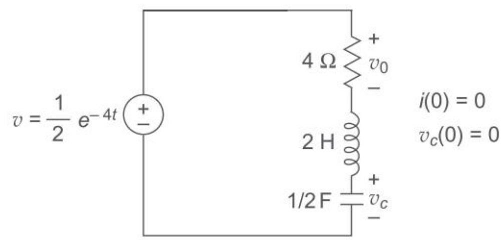

Fig. 14.58

Solution The circuit in Fig. 14.58 in the s-domain is as shown in Fig. 14.59.

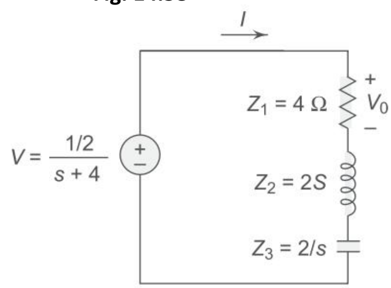

Fig. 14.59

Total impedance in the circuit

$$Z_{eq} = Z_1 + Z_2 + Z_3 \tag{14.158}$$

$$V_0 = \left(\frac{Z_1}{Z_1 + Z_2 + Z_3}\right)V$$

$$= \left\{\frac{4}{2s + 4 + \frac{2}{s}}\right\}\left\{\frac{\frac{1}{2}}{s+4}\right\} \tag{14.159}$$

$$= \frac{s}{(s+4)(s^2 + 2s + 1)}$$

$$= \frac{s}{(s+4)(s+1)^2} \tag{14.160}$$

Taking partial fractions,

$$V_0 = \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1} + \frac{K_3}{s+4} \tag{14.161}$$

The coefficients K_1, K_2, and K_3 are

$$K_1 = \frac{-1}{3}, K_2 = \frac{4}{9}, K_3 = \frac{-4}{9}$$

Thus,

$$V_0 = \frac{(-1/3)}{(s+1)^2} + \frac{4/9}{s+1} + \frac{(-4/9)}{s+4}$$

Taking inverse transform on both sides,

$$v_0 = \frac{1}{3} t\, e^{-t} + \frac{4}{9} e^{-t} - \frac{4}{9} e^{-4t} \tag{14.162}$$

PROBLEM 14.13

Determine i_1, i_2, V and V_1, for the circuit in Fig. 14.60.

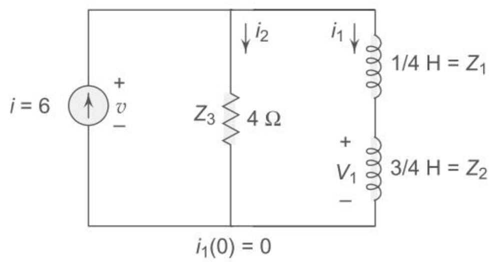

Fig. 14.60

Solution The transformed circuit in the s-domain is shown in Fig. 14.61.

Applying current division to the circuit in the s-domain, we get

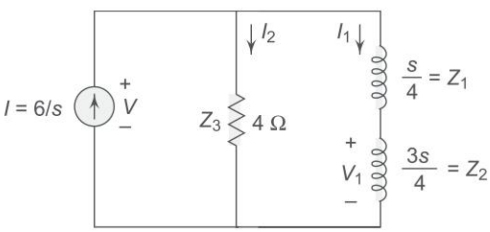

Fig. 14.61

$$Z_{eq} = 4 \,\|\, \left(\frac{s}{4} + \frac{3s}{4}\right) \tag{14.163}$$

$$I_1 = \left\{\frac{Z_{eq}}{Z_1 + Z_2}\right\}\left(\frac{6}{s}\right)$$

$$= \frac{24}{s(s+4)} \tag{14.164}$$

By taking partial fraction expansion,

$$I_1 = \frac{K_1}{s} + \frac{K_2}{s+4}$$

$$K_1 = 6; \quad K_2 = -6$$

$$I_1 = \frac{6}{s} + \frac{(-6)}{s+4} \tag{14.165}$$

Taking inverse transform, we get

$$i_1 = 6 - 6\, e^{-4t} \tag{14.166}$$

From Kirchhoff's current law,

$$\frac{6}{s} = I_2 + I_1 \tag{14.167}$$

$$= I_2 + \frac{24}{s(s+4)}$$

$$I_2 = \frac{6}{s} - \frac{24}{s(s+4)}$$

$$I_2 = \frac{6}{s+4} \tag{14.168}$$

We know

$$V = 4\,I_2 = \frac{24}{s+4} \tag{14.169}$$

$$V_1 = \left\{\frac{3s}{4}\right\} I_1$$

$$= \left\{\frac{3s}{4}\right\}\left\{\frac{24}{s(s+4)}\right\} \tag{14.170}$$

$$V_1 = \frac{18}{s+4}$$

Taking inverse transforms,

$$i_2 = \mathcal{L}^{-1}\{I_2\} = \mathcal{L}^{-1}\left\{\frac{6}{s+4}\right\} = 6\,e^{-4t} \tag{14.171}$$

$$v = \mathcal{L}^{-1}\{V\} = \mathcal{L}^{-1}\left\{\frac{24}{s+4}\right\} = 24\,e^{-4t} \tag{14.172}$$

$$v_1 = \mathcal{L}^{-1}\{V_1\} = \mathcal{L}^{-1}\left\{\frac{18}{s+4}\right\} = 18\,e^{-4} \tag{14.173}$$

PROBLEM 14.14

Determine the voltage v for the circuit shown in Fig. 14.62.

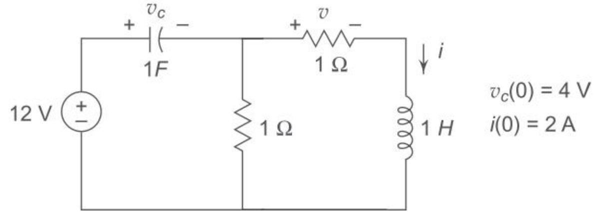

Fig. 14.62

Solution The circuit in Fig. 14.62 is transformed into s-domain as shown in Fig. 14.63.

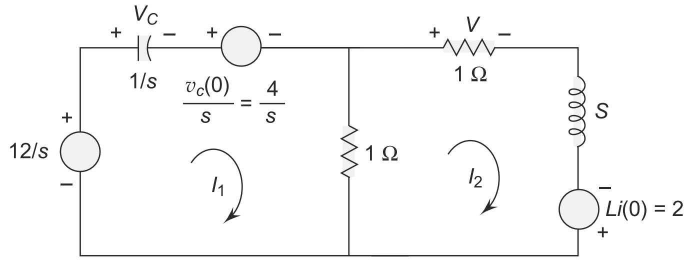

Fig. 14.63

By using mesh analysis, the current I_2 in the circuit is

$$I_2 = \frac{\begin{vmatrix} 1+\frac{1}{s} & \frac{12}{s}-\frac{4}{s} \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1+\frac{1}{s} & -1 \\ -1 & 1+1+s \end{vmatrix}} \tag{14.174}$$

$$I_2 = \frac{2\left(1+\frac{1}{s}\right)+\left(\frac{8}{s}\right)}{\left(1+\frac{1}{s}\right)(2+s)-1} = \frac{2s+10}{s^2+2s+2} \tag{14.175}$$

The voltage across the 1 Ω resistor is

$$V = RI_2 = \frac{2s+10}{s^2+2s+2} \tag{14.176}$$

The above equation can be written as

$$V = \frac{2s+2+8}{s^2+2s+2}$$

$$= 2\left\{\frac{s+1}{s^2+2s+2}\right\} + 8\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= 2\left\{\frac{s+1}{(s+1)^2+1}\right\} + 8\left\{\frac{1}{(s+1)^2+1}\right\} \tag{14.177}$$

Taking inverse Laplace transform on both sides,

$$v = 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$v = 2\,e^{-t}\cos t + 8\,e^{-t}\sin t \tag{14.178}$$

PROBLEM 14.15

Determine the voltage v for the circuit in Fig. 14.64. Assume $v_c(0) = 0$.

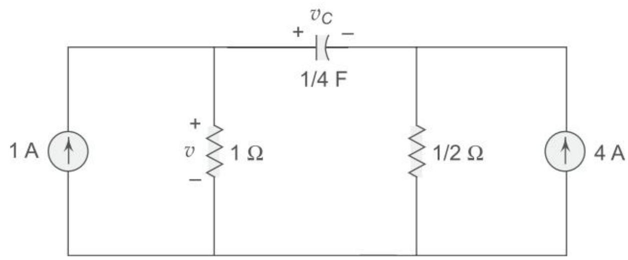

Fig. 14.64

Solution The circuit in Fig. 14.64 is transformed into the *s*-domain resulting in the circuit in Fig. 14.65.

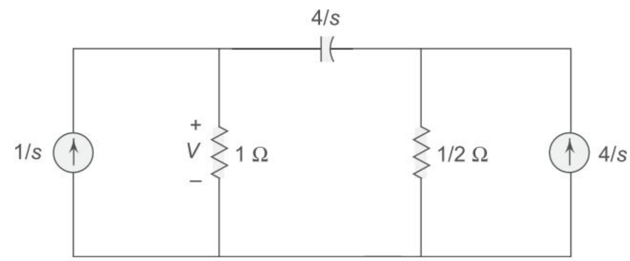

Fig. 14.65

Replacing the Laplace impedance for *R* and *C* with Laplace admittance, we get Fig. 14.66

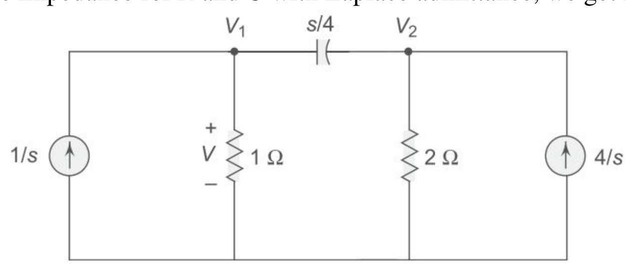

Fig. 14.66

By using nodal analysis, we get

$$V_1 = \frac{\begin{vmatrix} \frac{1}{s} & \frac{-s}{4} \\ \frac{4}{s} & 2+\frac{s}{4} \end{vmatrix}}{\begin{vmatrix} 1+\frac{s}{4} & \frac{-s}{4} \\ \frac{-s}{4} & 2+\frac{s}{4} \end{vmatrix}} \tag{14.179}$$

$$V_1 = \frac{\left(\frac{1}{s}\right)\left\{2+\left(\frac{s}{4}\right)\right\}-\left(\frac{4}{s}\right)\left(\frac{-s}{4}\right)}{\left\{1+\left(\frac{s}{4}\right)\right\}\left\{2+\left(\frac{s}{4}\right)\right\}-\left(\frac{-s}{4}\right)\left(\frac{-s}{4}\right)} \tag{14.180}$$

$$= \frac{\left(\frac{5}{3}\right)s+\left(\frac{8}{3}\right)}{s\left(s+\frac{8}{3}\right)} \tag{14.181}$$

Taking partial fraction expansion,

$$V_1 = \frac{K_1}{s}+\frac{K_2}{s+\frac{8}{3}}$$

The coefficients K_1 and K_2 are

$$K_1 = 1; \quad K_2 = \frac{2}{3}$$

$$V_1 = \frac{1}{s}+\frac{2/3}{s+\frac{8}{3}} \tag{14.182}$$

Taking the inverse Laplace transform of each side of the equation results in

$$\mathcal{L}^{-1}\{V_1\} = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{(2/3)}{\left(s+\frac{8}{3}\right)}\right\} \tag{14.183}$$

$$v_1 = \mathcal{L}^{-1}\left(\frac{1}{s}\right)+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+8/3}\right\}$$

$$v_1 = 1+\frac{2}{3}e^{-\frac{8}{3}t}$$

and because $v = v_1$

$$v = 1+\frac{2}{3}e^{\left(-\frac{8}{3}\right)t} \tag{14.184}$$

PROBLEM 14.16

Determine the voltage v for the circuit shown in Fig. 14.67 using Thevenin's theorem.

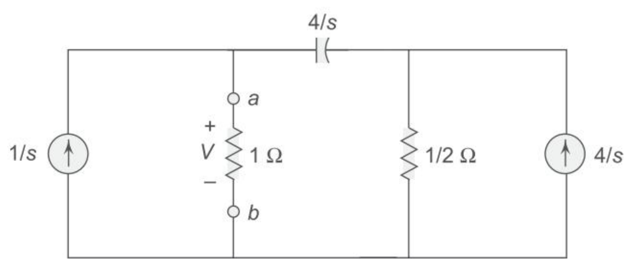

Fig. 14.67

Solution We have to find out the open-circuit voltage as shown in Fig. 14.68.

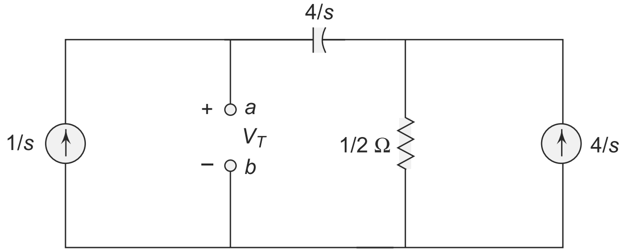

Fig. 14.68

Applying the superposition method results in the circuits in Figs 14.69 and 14.70, where V_T' and V_T'' are the contributions to V_T from the Laplace transformed sources $\left(\frac{1}{s}\right)$ and $\left(\frac{4}{s}\right)$ respectively.

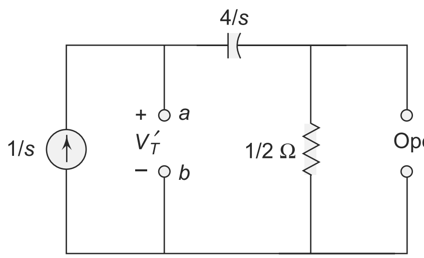

Fig. 14.69

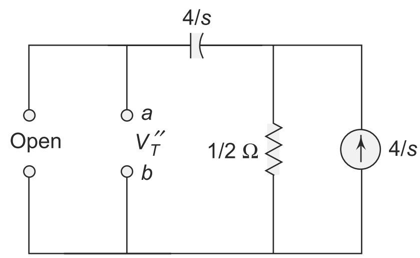

Fig. 14.70

From Fig. 14.69, the open- circuit voltage is

$$V_T' = IZ = \left(\frac{1}{s}\right)\left(\frac{4}{s}+\frac{1}{2}\right) = \frac{8+s}{2s^2} \tag{14.185}$$

From Fig. 14.70, the open-circuit voltage is

$$V_T'' = IZ = \left(\frac{4}{s}\right)\left(\frac{1}{2}\right) = \frac{2}{s} \tag{14.186}$$

because no current flows through the capacitor.

From the superposition method,

$$V_T = V_T' + V_T''$$

$$= \frac{8+s}{2s^2} + \frac{2}{s} \tag{14.187}$$

$$= \frac{8+s+4s}{2s^2}$$

$$V_T = \frac{5s+8}{2s^2} \tag{14.188}$$

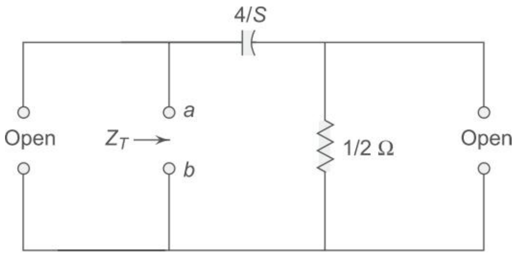

Fig. 14.71

Replacing both current sources by opens, as required, Thevenin's theorem to determine the Thevenin impedance results in Fig. 14.71.

The impedance seen into the terminals *ab*

$$Z_T = \frac{4}{s} + \frac{1}{2}$$

$$= \frac{s+8}{2s} \tag{14.189}$$

and the Thevenin equivalent circuit for terminals *a–b* is shown in Fig. 14.72.

If the 1 Ω resistor is reconnected across terminals *ab* then *V* can be determined in Fig. 14.73.

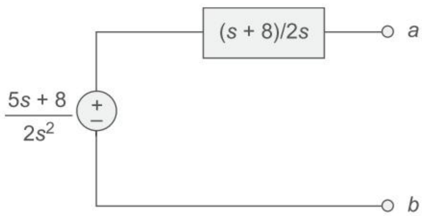

Fig. 14.72

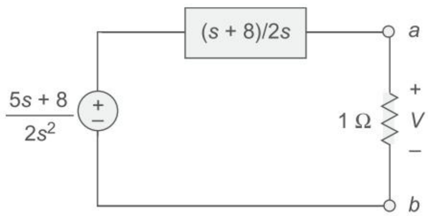

Fig. 14.73

$$V = \left\{\frac{1}{1+\dfrac{s+8}{2s}}\right\}\left\{\frac{5s+8}{2s^2}\right\}$$

$$= \left\{\frac{2s}{3s+8}\right\}\left\{\frac{5s+8}{2s^2}\right\} \tag{14.190}$$

$$V = \frac{5s+8}{s(3s+8)} \tag{14.191}$$

The inverse Laplace transform of *V* is

$$v = 1 + \left(\frac{2}{3}\right)e^{-\left(\frac{8}{3}\right)t} \tag{14.192}$$

PROBLEM 14.17

Determine the voltage V for the circuit shown in Fig. 14.74, using Norton's theorem.

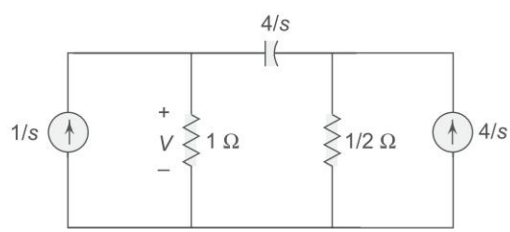

Fig. 14.74

Solution The application of Norton's theorem in the *s*-domain requires the removal of the 1 Ω resistor as shown in Fig. 14.75 and the determination of resulting short-circuited current.

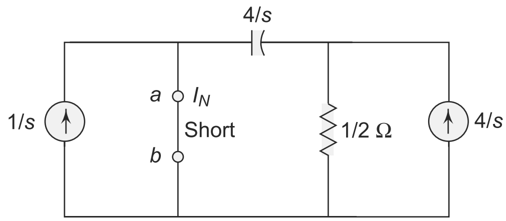

Fig. 14.75

Applying the superposition method results in the circuits in Figs 14.76 and 14.77, where I_N' and I_N'' are the contributions to I_N from the Laplace transformed current sources $\left(\frac{1}{s}\right)$ and $\left(\frac{4}{s}\right)$ respectively

By inspection of the circuit in Fig. 14.76,

$$I_N' = \frac{1}{s} \tag{14.193}$$

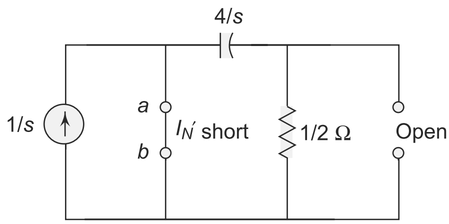

Fig. 14.76

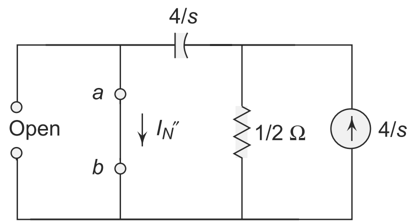

Fig. 14.77

Applying current division to the circuit in Fig. 14.77 results in

$$I_N'' = \left\{\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)+\left(\frac{4}{s}\right)}\right\}\left\{\frac{4}{s}\right\} = \frac{4}{s+8} \tag{14.194}$$

From the superposition method,

$$I_N = I'_N + I_N''$$

$$= \frac{1}{s} + \frac{4}{s+8}$$

$$I_N = \frac{5s+8}{s(s+8)} \tag{14.195}$$

Because Thevenin and Norton impedances are equal,

$$Z_N = \frac{s+8}{2s} \tag{14.196}$$

The Norton equivalent circuit for terminals *a–b* is as shown in Fig. 14.78.

If the 1 Ω resistor is reconnected across terminals *ab*, the voltage *V* can be determined in the circuit shown in Fig. 14.79.

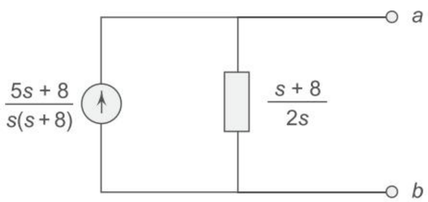

Fig. 14.78

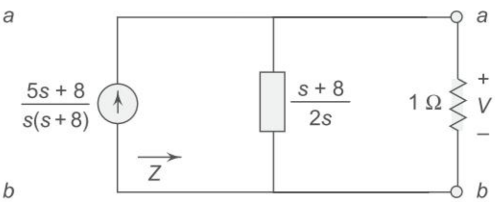

Fig. 14.79

$$Z = \left(\frac{s+8}{2s}\right) \| (1) = \frac{\left(\frac{s+8}{2s}\right)\{1\}}{1+\frac{s+8}{2s}} \tag{14.197}$$

$$Z = \frac{s+8}{3s+8} \tag{14.198}$$

We know $V = ZI$

$$= \left\{\frac{s+8}{3s+8}\right\}\left\{\frac{5s+8}{s(s+8)}\right\}$$

$$V = \frac{5s+8}{s(3s+8)} \tag{14.199}$$

Taking inverse Laplace transform of V, we get

$$v = 1 + \left(\frac{2}{3}\right)e^{-\left(\frac{8}{3}\right)t} \tag{14.200}$$

PROBLEM 14.18

The initial charge on the capacitor in the circuit shown in Fig. 14.80 is zero.
(a) Find the s-domain Thevenin equivalent circuit with respect to terminals a and b.
(b) Find the s-domain expression for the current that the circuit delivers to a load consisting of a 1 H inductor in series with a 2 Ω resistor.

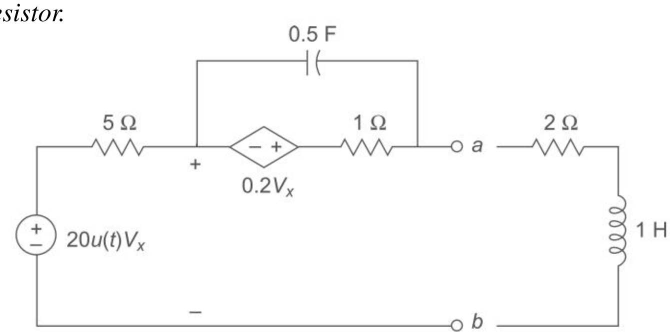

Fig. 14.80

Solution First, we have to find out the Thevenin's equivalent circuit from the *s*-domain circuit shown in Fig. 14.81.

Thevenin's voltage across terminals *ab* is

$$V_{ab} = V_x + 0.2\,V_x - I(1) \tag{14.201}$$

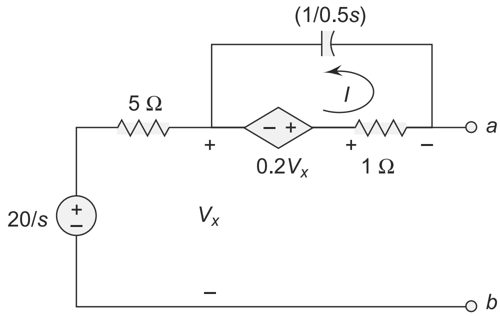

Fig. 14.81

By applying Kirchhoff's voltage law, we can determine the current I

$$0.2V_x = I\left[1+\frac{1}{0.5s}\right]$$

$$I = \frac{0.1sV_x}{(1+0.5s)} \tag{14.202}$$

Since no current is passing through the 5 Ω resistor,

$$\text{The voltage } V_x = \frac{20}{s} \tag{14.203}$$

Substituting V_x and I in Eq. (14.201), we get

$$V_{ab} = 1.2\left(\frac{20}{s}\right) - \frac{0.1s}{1+0.5s}\left(\frac{20}{s}\right) \tag{14.204}$$

$$V_{ab} = \frac{20}{s}\left[\frac{s+2.4}{s+2}\right] \tag{14.205}$$

The Thevenin's impedance after short-circuiting the voltage sources is shown in Fig. 14.82.

$$Z_{ab} = \left\{\left(\frac{1}{0.5s}\right) \| \ (1)\right\} + 5$$

$$Z_{ab} = \frac{5(s+2.4)}{s+2} \tag{14.206}$$

The Thevenin's equivalent circuit is shown in Fig. 14.83.

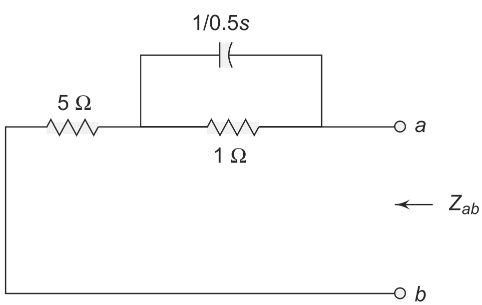

Fig. 14.82

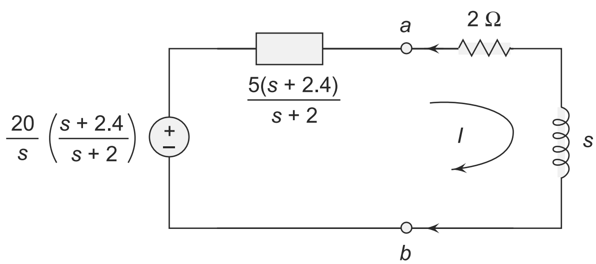

Fig. 14.83

The current I in the circuit of Fig. 14.83 is

$$I = \frac{\frac{20}{s}\left(\frac{s+2.4}{s+2}\right)}{\frac{5(s+2.4)}{s+2}+2+s} \tag{14.207}$$

$$= \frac{20}{s}\left[\frac{(s+2.4)}{5s+12+s^2+4s+4}\right] \tag{14.208}$$

$$I = \frac{20}{s}\left[\frac{s+2.4}{s^2+9s+16}\right] \tag{14.209}$$

PROBLEM 14.19

The voltage source v_g drives the circuit shown in Fig. 14.84. The response signal is the voltage across the capacitor v_o. Calculate the numerical expression for the transfer function.

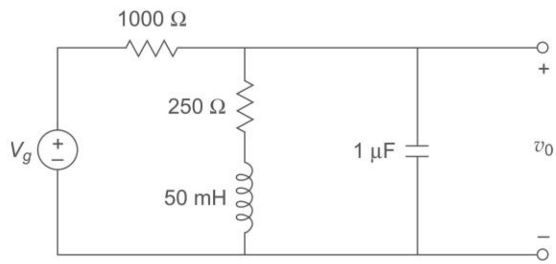

Fig. 14.84

Solution The s-domain equivalent circuit is shown in Fig. 14.85.

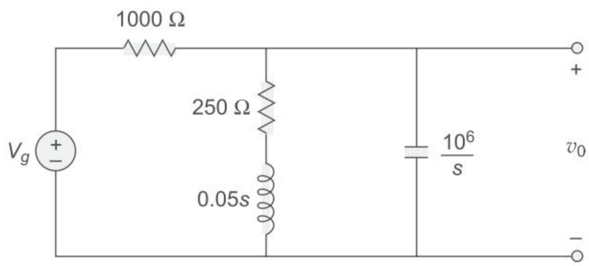

Fig. 14.85

By definition, transfer function is the ratio v_o/v_g.
By applying Kirchhoff's current law, we get

$$\frac{V_0 - V_g}{1000} + \frac{V_0}{250+0.05s} + \frac{V_0 s}{10^6} = 0 \tag{14.210}$$

Solving for V_0 yields

$$V_0 = \frac{1000(s+5000)V_g}{s^2+6000s+25\times10^6} \tag{14.211}$$

Hence, the transfer function is

$$H(s) = \frac{V_0}{V_g} = \frac{1000(s+5000)}{s^2+6000s+25\times10^6} \tag{14.212}$$

PROBLEM 14.20

The circuit shown in Fig. 14.86 is driven by a voltage source whose voltage increases linearly with time, namely, $v_0 = 50\,t\,u(t)$.

(a) *Use the transfer function to find v_0.*

(b) *Identify the transient component of the response.*

(c) *Identify the steady-state component of the response.*

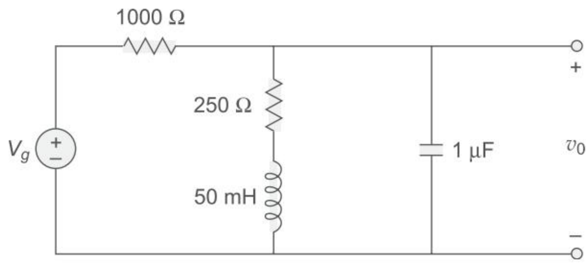

Fig. 14.86

Solution From the previous example

$$H(S) = \frac{1000(s+5000)}{s^2+6000s+25\times10^6} \tag{14.213}$$

The transform of the driving voltage is $50/s^2$, therefore, the s-domain expression for the output voltage is

$$V_0 = \frac{1000(s+5000)}{(s^2+6000s+25\times10^6)}\frac{50}{s^2} \tag{14.214}$$

The partial fraction expansion of V_0 is

$$V_0 = \frac{K_1}{s+3000-j4000}+\frac{K_1^*}{s+3000+j4000}+\frac{K_2}{s^2}+\frac{K_3}{s} \tag{14.215}$$

The involves of coefficients are

$$K_1 = 5\sqrt{5}\times10^{-4}\angle 79.70°$$

$$K_1^* = 5\sqrt{5}\times10^{-4}\angle -79.70°$$

$$K_2 = 10$$

$$K_3 = -4\times10^{-4}$$

(a) The time domain expression for v_0 is

$$v_0 = [10\sqrt{5}\times10^{-4}e^{-3000t}\cos(4000\,t+79.70°) + 10\,t - 4\times10^{-4}]\text{ V} \tag{14.216}$$

(b) The transient component of v_0 is

$$10\sqrt{5}\times10^{-4}e^{-3000t}\cos(4000\,t+79.70°)\text{V} \tag{14.217}$$

(c) The steady-state component of the response is

$$(10\,t-4\times10^{-4})\text{ V} \tag{14.218}$$

PROBLEM 14.21

The excitation voltage v_i for the circuit shown in Fig. 14.87 is shown in Fig. 14.88.

(a) *Use convolution integral to find v_0.*

(b) *Plot v_0 over the range of $0 \le t \le 15$ s.*

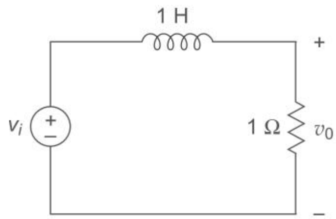

Fig. 14.87

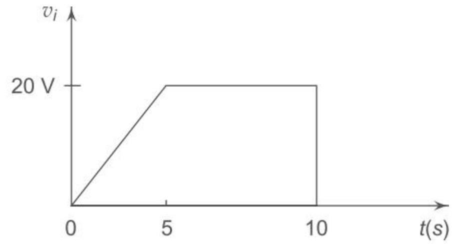

Fig. 14.88

Solution The first step in using the convolution integral is to find the unit impulse response of the circuit. Obtain the expression for V_0 from the s-domain equivalent of the circuit in Fig. 14.87.

$$V_0 = \frac{V_i}{s+1}(1) \tag{14.219}$$

When v_i is a unit impulse function $\delta(t)$,

$$\begin{aligned} v_o &= h(t) \\ &= e^{-t}\, u(t) \end{aligned} \tag{14.220}$$

from which

$$h(\tau) = e^{-\lambda}\, u(\tau) \tag{14.221}$$

The impulse response and the folded excitation function is shown in Fig. 14.89.

Sliding the folded excitation function to the right requires breaking the integration into intervals: $0 \leq t \leq 5$; $5 \leq t \leq 10$; and $10 \leq t \leq \infty$. The breaks in the excitation function at 0.5, and 10s dictate these break points. Figure 14.90 shows the positioning of the folded excitation for each of these intervals. The analytical expression for v_i in the time interval $0 \leq t \leq 5$ is

$$v_i = 4t;\ 0 \leq t \leq 5s \tag{14.222}$$

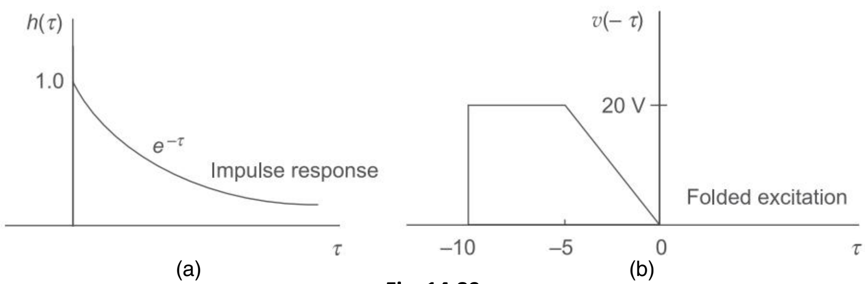

Fig. 14.89

Hence, the analytical expression for the folded excitation function in the interval $t - 5 \leq \tau \leq t$ is

$$v_i\,(t - \tau) = 4\,(t - \tau),\ t - 5 \leq \tau \leq t \tag{14.223}$$

We can now set up the three integral expression for v_0.

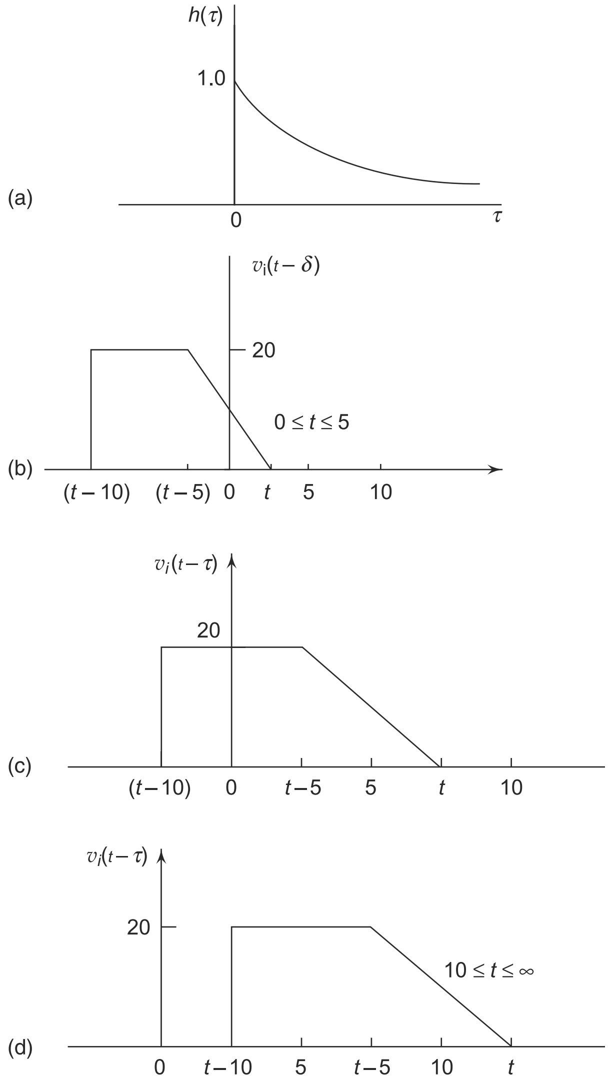

Fig. 14.90

For $0 \le t \le 5s$,

$$v_0 = \int_0^t 4(t-\tau)e^{-\tau}d\tau$$

$$= 4(e^{-t} + t - 1)\text{ V} \tag{14.224}$$

For $5 \le t \le 10s$,

$$v_0 = \int_0^{t-5} 20e^{-\tau}d\tau + \int_{t-5}^{t} 4(t-\tau)e^{-\tau}d\tau$$

$$= 4\,(5 + e^{-t} - e^{-(t-5)})\text{ V} \tag{14.225}$$

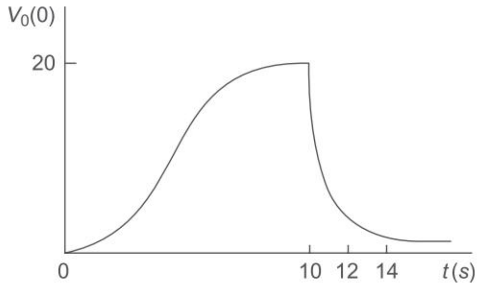

Fig. 14.91

For $0 \leq t \leq \infty$ s,

$$v_0 = \int_{t-10}^{t-5} 20e^{-\tau} d\tau + \int_{t-5}^{t} 4(t-\tau)e^{-\tau} d\tau$$

$$= 4\,(e^{-t} - e^{-(t-5)} + 5e^{-(t-10)})\text{ V} \qquad (14.226)$$

The results are computed for v_0 and tabulated in Table 14.1. The voltage response is shown graphically in Fig. 14.91.

Table 14.1 *Numerical Values of $v_0(t)$*

t	v_0	t	v_0	t	v_0
1	1.47	6	18.54	11	7.35
2	4.54	7	19.56	12	2.70
3	8.20	8	19.8	13	0.99
4	12.07	9	19.93	14	0.37
5	16.03	10	19.97	15	0.13

PROBLEM 14.22

For the circuit shown in Fig. 14.92, the sinusiodal source voltage is v_g = 120 cos (5000 t + 30°) V. Find the steady-state expression for V_0.

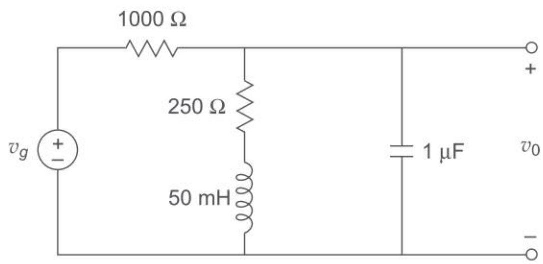

Fig. 14.92

Solution From Problem 14.19,

$$H(S) = \frac{1000(s+5000)}{s^2 + 6000s + 25\times10^6} \qquad (14.227)$$

The frequency of the voltage source is 5000 rad/s; Hence, we evaluate $H(S)$ at $H(j5000)$.

$$H(j5000) = \frac{1000(5000 + j5000)}{-25\times10^{-6} + j5000(6000) + 25\times10^6}$$

$$= \frac{1+j1}{j6} = \frac{1-j1}{6} = \frac{\sqrt{2}}{6}\angle -45°$$

Then the steady-state voltage is

$$v_{0ss} = \frac{120\sqrt{2}}{6}\cos(5000t + 30° - 45°)$$

$$= 20\sqrt{2}\cos(5000t - 15°)\text{ V}.$$

PSpice Problems

PROBLEM 14.1

The circuit shown in Fig. 14.93 consists of series RL elements. The sine wave is applied to the circuit when the switch 'S' is closed at t = 0. Find i (t) using PSpice.

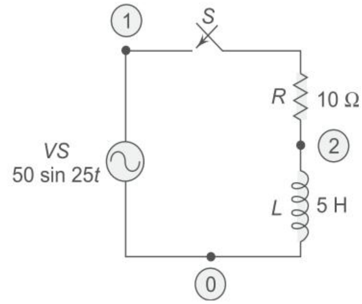

Fig. 14.93

$$f = \frac{25}{2\pi} = 3.979$$

```
RL TRANSIENT WITH SINE WAVE INPUT
VS   1     0     SIN(0 5 3.979 0)
R    1     2     10
L    2     0     5
.TRAN 0.1 1
.PROBE
.PLOT TRAN I(L)
.END
**** TRANSIENT ANALYSIS TEMPERATURE = 27.000 DEG C
**************************************************************
   TIME          I(L)
0.000E + 00   0.000E + 00
1.000E – 01   6.417E – 02
2.000E – 01   1.264E – 02
3.000E – 01   1.102E – 02
4.000E – 01   4.784E – 02
5.000E – 01   –2.371E – 02
6.000E – 01   4.252E – 02
7.000E – 01   –1.548E – 03
8.000E – 01   –4.937E – 03
9.000E – 01   3.820E – 02
1.000E + 00   –3.371E – 02
```

PROBLEM 14.2

Using PSpice, for the circuit shown in Fig. 14.94, find the voltage across the 0.5 Ω resistor when switch s is opened at t = 0. Assume there is no charge on the capacitor and no current in the inductor before switching.

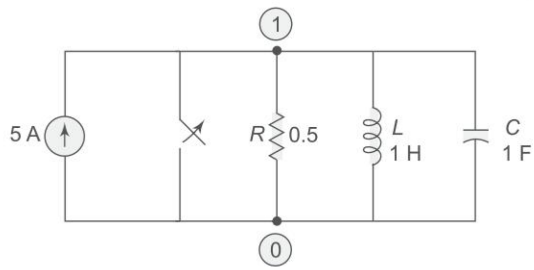

Fig. 14.94

```
I1   0   1   PWL( 0 5 0.1N 5 10 5)
R    1   0   0.5
L1   0   1   1
C1   0   1   1
.TRAN 0.05 0.5
.PROBE
```

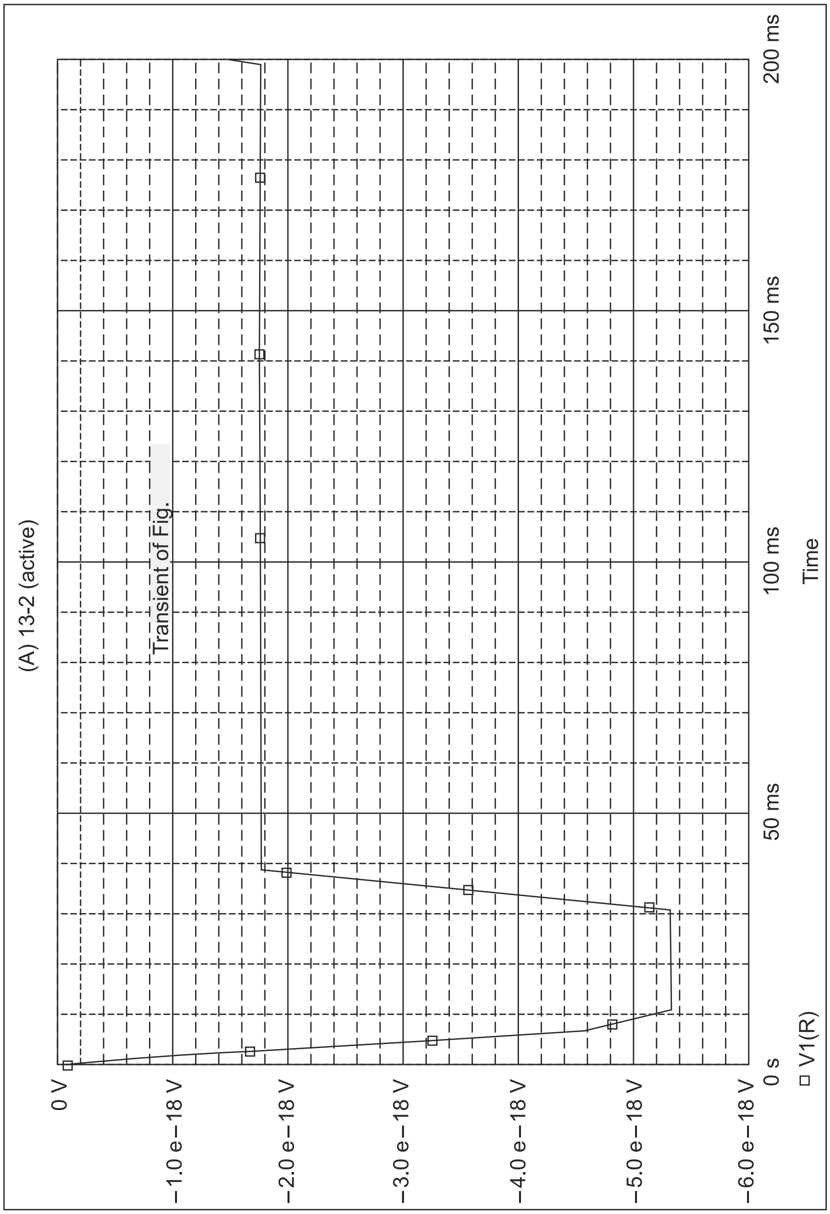

Fig. 14.95

```
.PRINT TRAN V(R)
.END
**** TRANSIENT ANALYSIS TEMPERATURE = 27.000 DEG C
******************************************************************
TIME           V(R)
0.000E + 00    0.000E + 00
5.000E – 02   –8.794E – 18
1.000E – 01   –8.794E – 18
1.500E – 01   –8.794E – 18
2.000E – 01   –8.794E – 18
2.500E – 01   –8.794E – 18
3.000E – 01   –8.794E – 18
3.500E – 01   –8.794E – 18
4.000E – 01   –8.794E – 18
4.500E – 01   –8.794E – 18
5.000E – 01   –8.281E – 18
```

Answers to Practice Problems

14-2.1 $2000\,(S + 50000)^2/(S^2 + 80000S + 25 \times 10^8)$

14-2.3 (a) $I = 40/(S^2 + 12S + 1)$

(b) $i = 50e^{-0.6t} \sin 0.8t$ A

(c) $V = 16S/(S^2 + 1.2S + 1)$

(d) $v = 200\, e^{-0.6t} \cos(0.8t + 36.87°)$ V

14-2.5 (a) $(100/3)e^{-2t} - \left(\frac{100}{3}\right)e^{-8t}$ V

(b) $\frac{50}{3}e^{-2t} - \frac{50}{3}e^{-8t}$ V

(c) $50\, e^{-2t} - 50\, e^{-8t}$ V

14-2.8 (a) $2 + \frac{10}{3}e^{-t}\cos(3t - 126.87°)$ V (b) $10.54e^{-t} \cos(3t - 18.43°)$ V

14-2.10 (a) 80 V (b) 20 V (c) 0 V (d) $32\,\delta(t)$ μA

(e) 16 V (f) 4 V (g) 20 V

14-3.1 $H(S) = 10(S + 2)/S^2 + 2S + 10$

14-4.1 $v = (1 - e^{-t})$ V $\quad 0 \le t \le 1$

$V = (e - 1)e^{-t}$ V $\quad 1 \le t \le \infty$

14-4.3 $44.7 \cos(4t - 63.43°)$ V

Objective-Type Questions

☆☆★ **14.1** An inductor in the s-domain consists of

(a) current source in series with an inductor
(b) voltage source in parallel with an inductor
(c) voltage source of LI_0 in series with an inductor
(d) current source I_0/s in series with an inductor

☆☆★ **14.2** A capacitor in the s-domain consists of

(a) current source CV_0 in parallel with capcitor
(b) current source in series with capacitor
(c) voltage source $\dfrac{V_0}{s}$ in parallel with capacitor
(d) voltage source CV_0 in parallel with capacitor

☆☆★ **14.3** The current in the circuit when the switch is closed at $t = 0$.

(a) $10\, e^{-100\, t}$ (b) $0.01\, e^{-1000\, t}$
(c) $0.1\, e^{-1000\, t}$ (d) $10\, e^{-0.1\, t}$

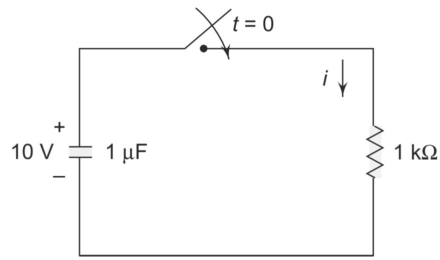

Fig. 14.96

☆★★ **14.4** The initial voltage across the capacitor when the switch s is opened at $t = 0$.

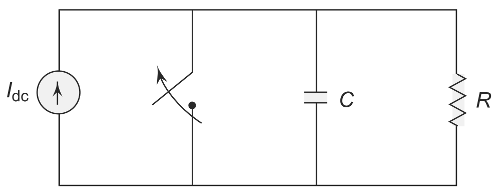

Fig. 14.97

(a) zero (b) $C.\dfrac{I_{dc}}{s}$ (c) $\dfrac{1}{CS} I_{dc}$ (d) $CS\,(I_{dc})$

☆☆★ **14.5** Thevenins equivalent circuit across terminals ab.

(a) The voltage source 10 V with $(20 + 2s)$ impedance in series
(b) The voltage source 10 V in parallel with $(20 + 2s)\ \Omega$ impedance
(c) The voltage source $\dfrac{10}{s}$ in series with an impedance of $(20 + 2s)\ \Omega$
(d) The voltage source $\dfrac{10}{s}$ in series with an impedance of $22\ \Omega$

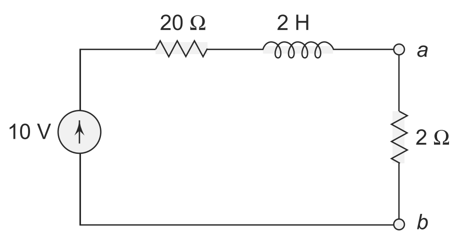

Fig. 14.98

☆☆★ **14.6** The transfer function of multiple independent sources can easily be obtained by

(a) superposition theorem
(b) Thevenin's theorem
(c) Norton's theorem
(d) reciprocity

☆★★ **14.7** In the circuit shown in Fig. 14.99, the current is defined as the response signal then the transfer function

(a) $\dfrac{10^{-6}s}{10^{-12}s^2+s+1}$

(b) $\dfrac{s}{s^2+s+1}$

(c) $\dfrac{s}{s^2+1}$

(d) $\dfrac{s}{s+1}$

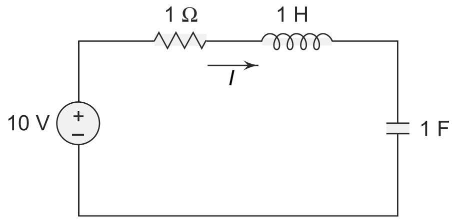

Fig. 14.99

☆☆★ **14.8** The circuit is driven by an unit impulse source then the response equals to

(a) transfer function (c) zero
(b) one (d) inverse of transfer function

☆☆★ **14.9** If the input of a circuit is represented by series of impulse functions, the response consists of

(a) sum of the series of uniformly delayed impulse responses
(b) sum of the series of responses
(c) one
(d) zero

☆☆★ **14.10** For physically realizable circuit, impulse response is

(a) zero for $t < 0$ (c) one for $t < 0$
(b) zero for $t > 0$ (d) infinite for $t > 0$

☆☆★ **14.11** The instantaneous current in an inductor when an impulse voltage V_0 applied to the terminals of an inductor

(a) zero (b) unity (c) $\dfrac{V_0}{L}$ (d) $\dfrac{V_0}{L}\delta(t)$

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/272

CHAPTER 15

S-Domain Analysis

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Explain complex frequency and its physical interpretation
- LO 2 Determine the transform impedance and admittance representation for each element
- LO 3 Analyse series and parallel connections of elements
- LO 4 Describe network functions for one-port and two-port networks
- LO 5 Describe poles and zeros of network functions and their significance
- LO 6 Explain the properties and necessary conditions for driving-point functions and transfer functions
- LO 7 Analyse the time domain, amplitude, and phase response from pole-zero plots
- LO 8 Explain the stability and Routh criteria for active networks

15.1 CONCEPT OF COMPLEX FREQUENCY

LO 1 Explain complex frequency and its physical interpretation

The solution of differential equations for networks is of the form

$$i(t) = K_n\ e^{S_n t} \tag{15.1}$$

where S_n is a complex number which is a root of the characteristic equation and may, therefore, be expressed as

$$S_n = \sigma_n + j\omega_n \tag{15.2}$$

The complex number consists of two parts, the real part σ_n and the imaginary part ω_n. The real part of the complex frequency σ_n is *neper frequency*, while the imaginary part ω_n is the *radian frequency*. The neper frequency has dimensions of neper per second. In the time-domain equations, ω_n is in the form of sin $\omega_n t$ or cos $\omega_n t$. The radian frequency ω_n is expressed in radians/second and is related to the frequency f_n in cycles/sec or the periodic time T (in seconds) by the relation.

$$\omega_n = 2\pi f_n = \frac{2\pi}{T} \tag{15.3}$$

From Eq. (15.2), we see that the real part σ_n and the imaginary part ω_n must have identical dimensions. Radian frequency ω_n is $\dfrac{2\pi}{T}$ has dimensions (time)$^{-1}$. Therefore, the dimensions of σ_n must also be (time)$^{-1}$ or the unit of σ_n must be "something per unit time". Since σ_n appears as an exponential factor,

$$I = I_0\, e^{\sigma_n t} \tag{15.4}$$

Such that

$$\sigma_n = \frac{1}{t}\ln\left(\frac{I}{I_0}\right) \tag{15.5}$$

It is fact that "something per unit time" should be nondimensional logarithmic unit. The usual unit for the natural logarithm is the neper, making the dimensions for σ the neper per second. The complex quantity

$$S_n = \sigma_n + j\omega_n \tag{15.6}$$

is defined as the complex frequency. Thus, complex frequency consists of a real part σ_n called the neper frequency and an imaginary part ω_n is called radian frequency (or *real frequency*).

15.2 PHYSICAL INTERPRETATION OF COMPLEX FREQUENCY

LO 1

The complex frequency appears in the exponential form $e^{S_n t}$. Let us consider the physical significance of complex frequency and a number of special cases for the values of S_n.

❑ ***Case (i)*** Let $S_n = \sigma_n + jo$ having zero radian frequency. The exponential function becomes

$$K_n\, e^{S_n t} = K_n e^{\sigma_n t} \tag{15.7}$$

The above exponential function increases exponentially for $\sigma_n > 0$ and decreases exponentially for $\sigma_n < 0$. For $\sigma_n = 0$, the exponential function reduces to K_n and it is a time-invariant resulting current $i(t)$ which is a dc current. Figure 15.1 shows the variation of exponential term $K_n\, e^{\sigma_n t}$ with time t for the cases of $\sigma_n > 0$, $\sigma_n < 0$ and $\sigma_n = 0$.

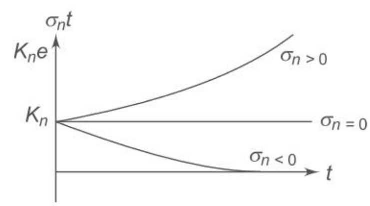

Fig. 15.1

❑ ***Case (ii)*** Let $S_n = 0 \pm j\omega_n$ having radian frequency and zero neper frequency. The exponential becomes

$$\begin{aligned} i(t) &= K_n\, e^{\pm j\omega_n t} \\ &= K_n\,(\cos \omega_n t \pm j \sin \omega_n t) \end{aligned} \tag{15.8}$$

The exponential $e^{\pm j\omega_n t}$ may be interpreted in terms of the physical model of a rotating phasor of unit length. A positive sign of exponential $e^{\pm j\omega_n t}$ implies counter- clockwise or positive rotation, while a negative sign $e^{-j\omega_n t}$ implies clockwise or negative rotation.

For positive or counter-clockwise rotation, the real part of $e^{+j\omega_n t}$ or the projection on the real axis equals $\cos \omega_n t$, while the imaginary part of $e^{+j\omega_n t}$ or the projection on the imaginary axis equals $\sin \omega_n t$ (Fig. 15.2).

The variation of exponential function $e^{+j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state.

❑ ***Case (iii)*** Let $S_n = \sigma_n + j\omega_n$ is the general case and the frequency is complex and the exponential is given by

$$i(t) = K_n\, e^{S_n t} = K_n\, e^{(\sigma_n + j\omega_n)t}$$

$$i(t) = K_n\, e^{\sigma_n t} \cdot e^{j\omega_n t} \tag{15.9}$$

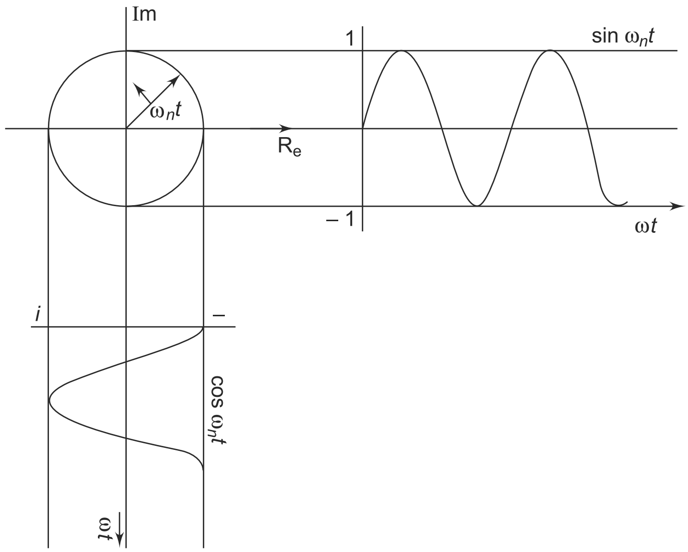

Fig. 15.2

Equation (15.9) shows that with complex frequency, the time variation of response $i(t)$ is the product of the response for $S_n = \sigma_n + jo$ and the response for $S_n = 0 + j\omega_n$. The response $e^{\sigma_n t}$ for the case of neper frequency alone, $S_n = \sigma_n + jo$ is an exponentially increasing or decaying function. The response $e^{j\omega_n t}$ for the case of radian frequency alone $S_n = 0 + j\omega_n$ may be represented by a rotating phasor. The product $e^{\sigma_n t} \cdot e^{j\omega_n t}$ may then be visualized as a rotating phasor whose magnitude is not constant at unity but changes continuously with time. Such phasors are illustrated in Fig. 15.3.

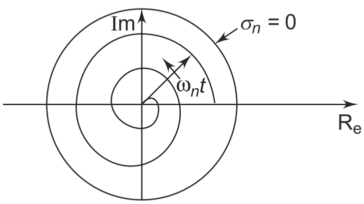

Fig. 15.3 (a)

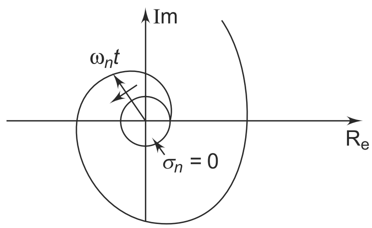

Fig. 15.3 (b)

The real and imaginary projections of this phasor are

$$\text{Re}(e^{S_n t}) = e^{\sigma_n t} \cos \omega_n t \tag{15.10}$$

and $\text{Im}(e^{S_n t}) = e^{\sigma_n t} \sin \omega_n t$

Consider the projections of this rotating phasor on the real and imaginary axes for the two cases. These projections for the case $\sigma_n < 0$ are known as a *damped sinusoid* and for $\sigma_n > 0$, the increasing oscillations are shown in Figs 15.4 (a) and (b) respectively.

From the above discussion, it is clear that the imaginary part of complex frequency governs sinusoidal oscillations and the real part of complex frequency governs the exponential decay or rise.

The roles of two types of frequency are similar even though the variations caused by them are different. This is the justification of unifying the two concepts under the name of complex frequency.

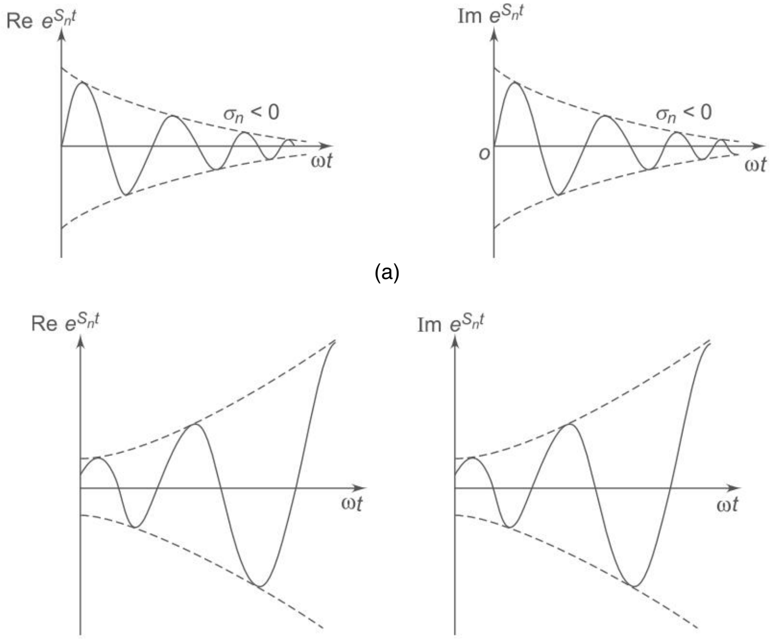

Fig. 15.4

15.3 TRANSFORM IMPEDANCE AND TRANSFORM CIRCUITS

In this section, we determine the transform impedance and admittance representations for each of the elements and initial condition sources.

LO 2 Determine the transform impedance and admittance representation for each element

15.3.1 Resistance

For a resistance, the voltage and current are related in the time domain by Ohm's law.

$$V_R(t) = Ri_R(t) \quad \text{or} \quad i_R(t) = GV_R(t); G = \frac{1}{R} \tag{15.11}$$

The corresponding transform equations are

$$V_R(S) = RI_R(S)$$

$$I_R(S) = GV_R(S) \tag{15.12}$$

The ratio of transform voltage $V_R(S)$ to the transform current $I_R(S)$ is defined as the transform impedance of the resistor, expressed as

$$Z_R(S) = \frac{V_R(S)}{I_R(S)} = R \tag{15.13}$$

Similarly, the reciprocal of this ratio is the transform admittance for the resistor, expressed as

$$Y_R(S) = \frac{I_R(S)}{V_R(S)} = G \tag{15.14}$$

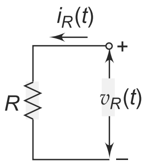

Fig. 15.5 (a)

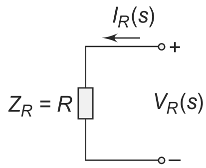

Fig. 15.5 (b)

From the above results, we can say that the resistor is frequency insensitive to the complex frequency.

Figure 15.5 (a) shows a network representing resistor R current $i_R(t)$ and voltage $V_R(t)$ in time domain. Figure 15.5 (b) gives the network representation of the same resistor and also transform current $I_R(S)$ and voltage $V_R(S)$.

15.3.2 Inductance

For inductance, the time-domain relation between the current in inductance $i_L(t)$ and the voltage $V_L(t)$ across it is expressed as

$$v_L(t) = L\frac{di_L(t)}{dt}$$

and $$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt \tag{15.15}$$

The equivalent transform equation for the voltage expression is

$$V_L(S) = L\,[SI_L(S) - i_L(0+)] \tag{15.16}$$

which, on rearranging, results

$$LS\,I_L(S) = V_L(S) + Li_L(0+) \tag{15.17}$$

In Eqs (15.15) and (15.16), $V_L(S)$ is the transform of the applied voltage $v_L(t)$ and $Li_L(0+)$ is the transform voltage caused by the initial current $i_L(0+)$ present in the inductor at time $t = 0+$.

Considering the sum of the transform voltage and the initial current voltage as $V_1(S)$, we have the transform impedance for the inductor.

$$Z_L(S) = \frac{V_1(S)}{I_L(S)} = SL \tag{15.18}$$

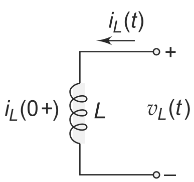

Fig. 15.6 (a)

Figure 15.6 (a) shows the time-domain network representation of the inductor L, the current $i_L(t)$ through it, and the applied voltage $v_L(t)$. Figure 15.6 (b) gives the transform representation of same inductor in terms of impedance using Eq. (15.16).

The transform equation for the current expression of Eq. (15.17) is

$$I_L(S) = \left[\frac{V_L(S)}{S} + \frac{\int_{-\infty}^{0+} v_L(t)dt}{S}\right]\frac{1}{L} \tag{15.19}$$

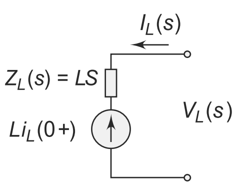

Fig. 15.6 (b)

But $$\int_{-\infty}^{0+} v_L(t)\,dt = Li_L(0+) \tag{15.20}$$

Hence, Eq. (15.19) becomes

$$I_L(S) = \frac{1}{L} \cdot \frac{V_L(S)}{S} + \frac{i_L(0+)}{S} \tag{15.21}$$

or $$\frac{1}{LS} V_L(S) = I_L(S) - \frac{i_L(0+)}{S} \tag{15.22}$$

In the above equation, $i_L(0+)/S$ is the transform caused by the initial current $i_L(0+)$ in the inductor.

Let $$I_1(S) = I_L(S) - \frac{i_L(0+)}{S} \tag{15.23}$$

Then Eq. (15.22) becomes

$$\frac{1}{LS} V_L(S) = I_1(S) \tag{15.24}$$

where $I_1(S)$ is the total transform current through the inductor L. The transform admittance becomes

$$Y_L(S) = \frac{I_1(S)}{V_L(S)} = \frac{1}{LS} \tag{15.25}$$

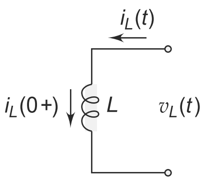

Fig. 15.7 (a)

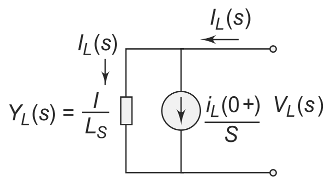

Fig. 15.7 (b)

Figure 15.7 (a) shows the time-domain representation of inductor L with initial current $i_L(0+)$. Figure 15.7 (b) shows equivalent transform circuit thus contains an admittance of value $\frac{1}{LS}$ and equivalent transform current source.

15.3.3 Capacitance

For capacitance, the time-domain relation between voltage and current is expressed as

$$i_c(t) = c\frac{dv_c(t)}{dt}$$

and $$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(t)\, dt \tag{15.26}$$

The equivalent transform equation for the voltage expression is

$$V_C(S) = \frac{1}{C}\left[\frac{I_C(S)}{S} + \frac{q(0+)}{S}\right] \tag{15.27}$$

where $\frac{q(0+)}{C} = v_C(0+)$ is the initial voltage across the capacitor.

The above equation becomes

$$\frac{1}{CS} I_C(S) = V_C(S) - \frac{v_C(0+)}{S} \tag{15.28}$$

Considering the total transform voltage across the capacitor as $V_1(S)$.

$$V_1(S) = V_C(S) - \frac{v_C(0+)}{S} \tag{15.29}$$

Then, Eq. (15.28) becomes

$$\frac{1}{CS} I_C(S) = V_1(S) \tag{15.30}$$

The transform impedance of the capacitor is the ratio of transform voltage $V_1(S)$ to the transform current $I_C(S)$ and is

$$Z_C(S) = \frac{V_1(S)}{I_C(S)} = \frac{1}{CS} \tag{15.31}$$

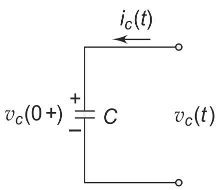

Fig. 15.8 (a)

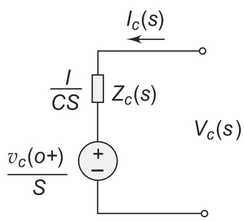

Fig. 15.8 (b)

Figure 15.8 (a) shows the time-domain representation of capacitor C with initial voltage $V_C(0+)$ across it. Voltage $V_1(S)$ includes the initial voltage $V_C(0+)$. Figure 15.8 (b) gives the transform representation of the same capacitor in terms of transform impedance.

Considering the current expression, the transform equation corresponding to Eq. (15.26) is

$$I_C(S) = C\,[SV_C(S) - v_C(0+)] \tag{15.32}$$

On rearranging,

$$CSV_C(S) = I_C(S) + CV_C(0+) \tag{15.33}$$

Considering the transform current through $Y_C(S)$ as $I_1(S)$, Eq. (15.32) may be put as

$$CSV_C(S) = I_1(S) \tag{15.34}$$

Then the transform admittance of the capacitor C is the ratio of transform current $I_1(S)$ to transform voltage $V_C(S)$, expressed as

$$Y_C(S) = \frac{I_1(S)}{V_C(S)} = CS \tag{15.35}$$

Figure 15.9 (a) shows the time-domain representation of the capacitor C with initial voltage $V_C(0+)$ across it. Figure 15.9 (b) gives the transform representation of the same capacitor in terms of admittance.

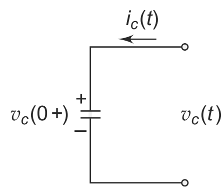

Fig. 15.9 (a)

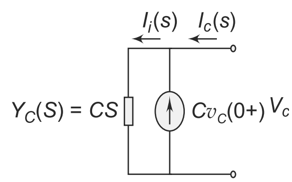

Fig. 15.9 (b)

Frequently Asked Questions linked to LO 2*

☆☆★**15-2.1** Obtain the transfer impedance for the circuit shown in Fig. Q.1. **[BPUT 2008]**

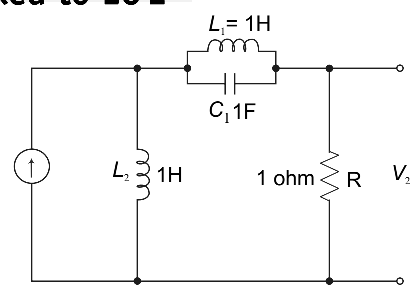

Fig. Q.1

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

15.4 SERIES AND PARALLEL COMBINATIONS OF ELEMENTS

LO 3 Analyse series and parallel connections of elements

In general, any network can be connected in series, parallel, and series-parallel combinations. Consider the series combination of elements shown in Fig. 15.10.

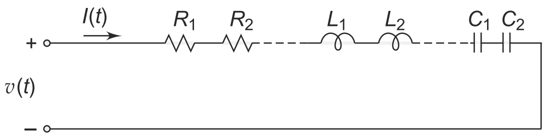

Fig. 15.10

Assume that all initial conditions are zero, i.e. the current in all inductors is zero and that the initial voltage of all capacitors is also zero. By application of Kirchhoff's voltage law, we get

$$v(t) = v_{R_1} + v_{R_2} + \dots + v_{L_1} + v_{L_2} + \dots + v_{C_1} + v_{C_2} + \dots \tag{15.36}$$

Taking transform for the above equation, we get

$$V(S) = V_{R_1}(S) + \dots + V_{L_1}(S) + \dots + V_{C_1}(S) + \dots \tag{15.37}$$

Dividing the equation by $I(S)$, the transform current through the series circuit, we get

$$Z(S) = Z_{R_1}(S) + \dots + Z_{L_1}(S) + \dots + Z_{C_1}(S) \tag{15.38}$$

or

$$Z(S) = \sum_{k=1}^{n} Z_K(S) \tag{15.39}$$

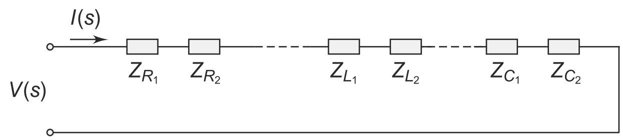

Fig. 15.11

where n is the total number of elements of all kinds in series.

Figure 15.11 shows the transform representation of the series circuit and represents Eq. (15.38).

Consider a parallel combination of resistors, inductors, and capacitors as shown in Fig. 15.12. Here, we assume that the inductors have zero initial currents and capacitors have zero initial voltages. Let $v(t)$ be the common voltage applied to all the elements in the circuit.

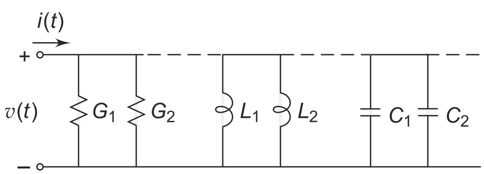

Fig. 15.12

Applying Kirchhoff's current law to the above circuit yields

$$i(t) = i_{G_1}(t) + i_{G_2}(t) + \dots + i_{L_1}(t) + i_{L_2}(t) + \dots + i_{C_1}(t) + i_{C_2}(t) + \dots \tag{15.40}$$

and the corresponding transform equation is

$$I(S) = I_{G_1}(S) + I_{G_2}(S) + \dots + I_{L_1}(S) + I_{L_2}(S) + \dots + I_{C1}(S) + I_{C_2}(S) + \dots \tag{15.41}$$

If this equation is divided by $V(S)$, we get the transform admittance which is the ratio of the current transform to the voltage transform and is

$$Y(S) = Y_{G_1}(S) + Y_{G_2}(S) + \dots + Y_{L_1}(S) + Y_{L_2}(S) + \dots + Y_{C_1}(S) + Y_{C_2}(S) + \dots \tag{15.42}$$

or

$$Y(S) = \sum_{k=1}^{n} Y_K(S) \tag{15.43}$$

where n is the total number of all kinds of elements in parallel.

Figure 15.13 gives the transform representation of the parallel network and represents Eq. (15.42).

For a series-parallel combination, rules for the combination of impedance and of admittance can be used to reduce a network to a single equivalent impedance or admittance.

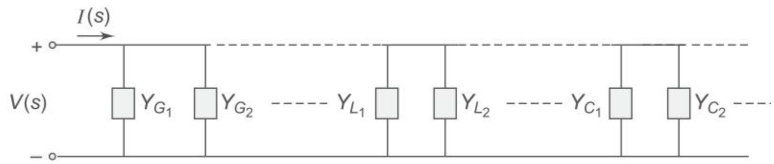

Fig. 15.13

EXAMPLE 15.1

In the circuit shown, the switch K is moved from the position 1 to the position 2 at time $t = 0$. At time $t = 0^-$, the current through inductor L is I_0 and the voltage across capacitor is V_0. Find the transform current I(S).

Solution The inductor has an initial current of I_0. It is represented by a transform impedance LS in series with a voltage source LI_0 as shown in Fig. 15.15. The capacitor has an initial voltage V_0 across it. It is represented by a transform impedance of $\frac{1}{CS}$ with an initial voltage $\frac{V_0}{S}$. The transform circuit derived from the circuit of Fig. 15.14 is shown in Fig. 15.15.

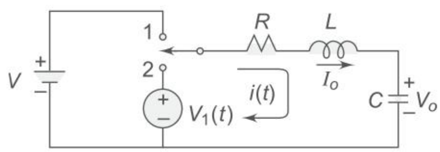

Fig. 15.14

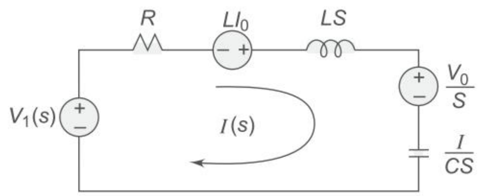

Fig. 15.15

The current $I(S)$ is given as the total transform voltage in the circuit divided by the total transform impedance. Then

$$I(S) = \frac{V(S)}{Z(S)} = \frac{V_1(S) + LI_0 - \dfrac{V_0}{S}}{R + LS + \dfrac{1}{CS}} = \frac{SV_1(S) + SLI_0 - V_0}{LS^2 + RS + \dfrac{1}{C}} \tag{15.44}$$

EXAMPLE 15.2

The network shown in Fig. 15.16 is a parallel combination of L, R and C connected across a current source I. At time $t = 0^-$, the current through inductor L is I_0 and the voltage across capacitor C is V_0. At time $t = 0^+$, the current source $I_1(t)$ is connected to the parallel RLC circuit. Find the transform voltage V(S).

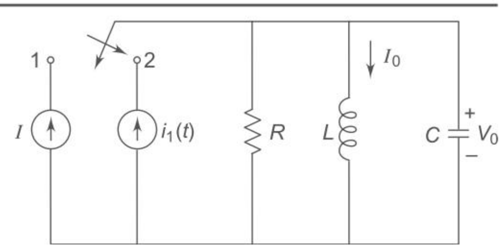

Fig. 15.16

Solution Figure 15.17 gives the transform network corresponding to the given network with the switch K moved to the position 2.

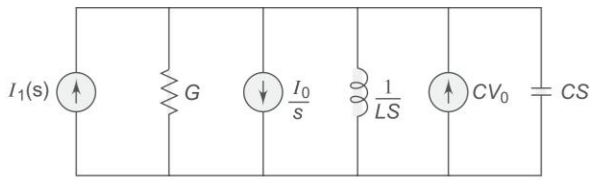

Fig. 15.17

From the above transform circuit, the transform voltage $V(S)$ may be obtained by taking the ratio of the total transform current to the total transform admittance.

The total transform current in the network is given by

$$I(S) = I_1(S) - \frac{I_0}{S} + CV_0$$

Total transform admittance is given by

$$Y(S) = G + \frac{1}{LS} + CS$$

Hence, the transform voltage is given by

$$V(S) = \frac{I(S)}{Y(S)} = \frac{I_1(S) + CV_0 - \dfrac{I_0}{S}}{G + \dfrac{1}{LS} + CS}$$

EXAMPLE 15.3

Obtain the transform impedance of the network shown in Fig. 15.18.

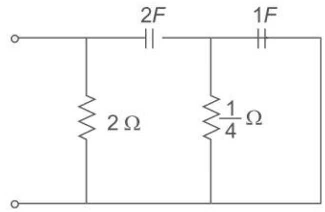

Fig. 15.18

Solution The transform network of Fig. 15.18 is shown in Fig. 15.19.

The admittance of the last two elements is the parallel combination.

$$Y_1(S) = 4 + S$$

Therefore, impedance is $Z_1(S) = \dfrac{1}{S+4}$

Series combination of last elements

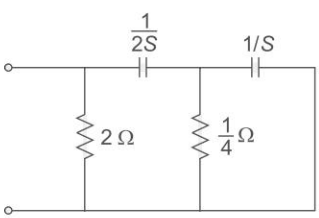

Fig. 15.19

$$Z_2(S) = \frac{1}{2S} + \frac{1}{S+4} = \frac{S+4+2S}{2S(S+4)} = \frac{3S+4}{2S(S+4)}$$

Parallel combination of elements

$$Y_2(S) = \frac{1}{2} + \frac{2S(S+4)}{3S+4} = \frac{(3S+4)+4s(S+4)}{6S+8} = \frac{4S^2+19S+4}{6S+8}$$

$$\text{Hence, the impedence } Z(S) = \frac{1}{Y_2(S)} = \frac{6s+8}{4S^2+19S+4}$$

EXAMPLE 15.4

In the given network in Fig. 15.20, the switch S is opened at t = 0, the steady state having established previously. With the switch S open, draw the transform network for analysis on the loop basis representing all elements and all initial conditions. Write transform equation for current in the loop.

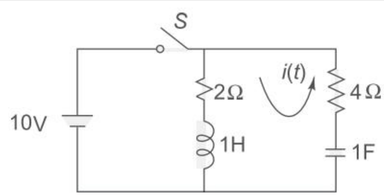

Fig. 15.20

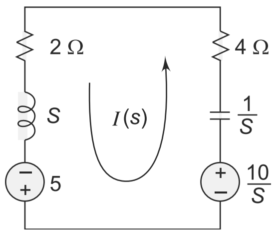

Fig. 15.21

Solution Under steady-state conditions, the capacitor is open-circuited and the inductor is short-circuited. The current through the inductor is $i_0 = \frac{10}{2} = 5\,\text{A}$. The voltage across the capacitor is $V_0 = 10\,\text{V}$. Hence, the corresponding transform network is shown in Fig. 15.21.

Hence, $I(S) = \dfrac{V(S)}{Z(S)} = \dfrac{5+\dfrac{10}{S}}{2+S+4+\dfrac{1}{S}} = \dfrac{5(S+2)}{S^2+6S+1}$

15.5 TERMINAL PAIRS OR PORTS

LO 3

Consider an arbitrary network made up of passive elements. It can be represented by a rectangular box shown in Fig. 15.22.

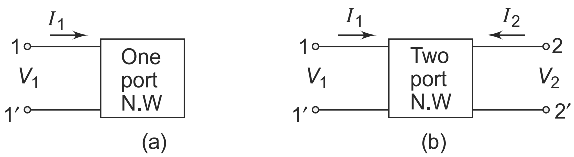

Fig. 15.22

For the network shown in Fig. 15.22 (a), only one voltage and one current exist and only one network function is defined. It constitutes one pair of terminals called a *port*. Generally, a driving source is connected to the pair of terminals. For the two-terminal pair network shown in Fig. 15.22 (b), two currents and two voltages must exist. Normally, in Fig. 15.22 (b), 1-1′ and 2-2′ are called ports. Hence, it is a called *two-port network*. If the driving source is connected across 1-1′, the load is connected across 2-2′. Otherwise, if the source is connected across 2-2′, the output is taken across 1-1′.

15.6 NETWORK FUNCTIONS FOR ONE-PORT AND TWO-PORT NETWORKS

LO 4 Describe network functions for one-port and two-port networks

For a one-port network, the driving-point impedance or impedance of the network is defined as

$$Z(s) = \frac{V(s)}{I(s)} \tag{15.45}$$

The reciprocal of the impedance function is the driving-point admittance function, and is denoted by $Y(s)$.

For the two-port network without internal sources, the driving-point impedance function at the port 1-1′ is the ratio of the transform voltage at port 1-1′ to the transform current at the same port.

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} \tag{15.46}$$

Similarly, the driving-point impedance at the port 2-2′ is the ratio of transform voltage at port 2-2′ to the transform current at the same port.

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)} \tag{15.47}$$

For the two-port network, the driving-point admittance is defined as the ratio of the transform current at any port to the transform voltage at the same port.

Therefore, $$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} \tag{15.48}$$

or $Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$, which is the driving-point admittance.

The four other network functions are called *transfer functions*. These functions give the relation between voltage or current at one port to the voltage or current at the other port as shown hereunder.

❑ ***Voltage-Transfer Ratio*** *This is the ratio of voltage transform at one port to the voltage transform at the other port, and is denoted by G(s).*

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)}$$

$$\text{and} \quad G_{12}(s) = \frac{V_1(s)}{V_2(s)} \tag{15.49}$$

❑ ***Current-Transfer Ratio*** *This is the ratio of current transform at one port to current transform at other port, and is denoted by* α*(s).*

$$\alpha_{12}(s) = \frac{I_1(s)}{I_2(s)}$$

$$\text{and} \quad \alpha_{21}(s) = \frac{I_2(s)}{I_1(s)} \tag{15.50}$$

❑ ***Transfer Impedance*** *It is defined as the ratio of voltage transform at one port to the current transform at the other port, and is denoted by Z(s).*

$$\therefore \qquad Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$$

$$\text{and} \quad Z_{12}(s) = \frac{V_1(s)}{I_2(s)} \tag{15.51}$$

❑ ***Transfer Admittance*** *It is defined as the ratio of current transform at one port to the voltage transform at the other port, and is denoted by Y(s).*

$$Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$$

$$\text{and} \quad Y_{12}(s) = \frac{I_1(s)}{V_2(s)} \tag{15.52}$$

The above network functions are found by forming the system of equations using node or mesh analysis, and taking the transforms of equations by setting the initial conditions to zero and solving for ratio of the response to excitation.

EXAMPLE 15.5

For the network shown in Fig. 15.23, obtain the driving-point impedance.

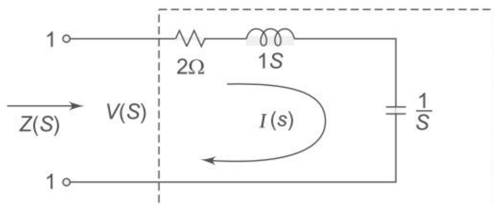

Fig. 15.23

Solution Applying Kirchhoff's law at the port 1-1′,

$$Z(S) = \frac{V(S)}{I(S)}$$

where $V(S)$ is applied at the port 1-1′ and $I(S)$ is the current flowing through the network. Then

$$Z(S)=\frac{V(S)}{I(S)}=2+S+\frac{1}{S}$$

$$Z(S)=\frac{S^2+2S+1}{S}$$

EXAMPLE 15.6

For the network shown in Fig. 15.24, obtain the transfer functions $G_{21}(S)$ and $Z_{21}(S)$ and the driving-point impedance $Z_{11}(S)$.

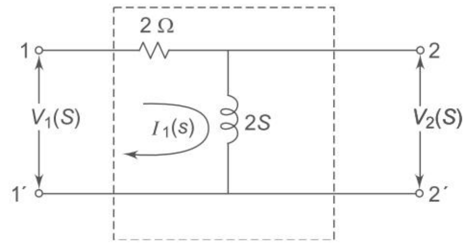

Fig. 15.24

Solution Applying Kirchhoff's law,

$$V_1(S) = 2I_1(S) + 2SI_1(S)$$

$$V_2(S) = I_1(S) \times 2S$$

Hence,

$$G_{21}(S)=\frac{V_2(S)}{V_1(S)}=\frac{2S}{2+2S}=\frac{S}{S+1}$$

$$Z_{21}(S)=\frac{V_2(S)}{I_1(S)}=2S$$

$$Z_{11}(S)=\frac{V_1(S)}{I_1(S)}=2(S+1)$$

EXAMPLE 15.7

For the network shown in Fig. 15.25, obtain the transfer functions $G_{21}(S)$, $Z_{21}(S)$, and driving-point impedance $Z_{11}(S)$.

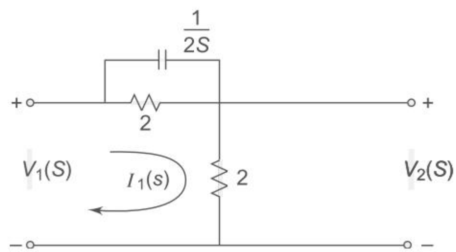

Fig. 15.25

Solution From the circuit, the parallel combination of resistance and capacitance can be combined into equivalent impedance.

$$Z_{eq}(S)=\frac{1}{2S+\frac{1}{2}}=\frac{2}{4S+1}$$

Applying Kirchhoff's laws, we have

$$V_2(S) = 2I_1(S)$$

and $$V_1(S)=I_1(S)\left[\frac{2}{4S+1}+2\right]$$

$$=I_1(S)\left[\frac{8S+4}{4S+1}\right]$$

The transfer functions

$$G_{21}(S)=\frac{V_2(S)}{V_1(S)}=\frac{2I_1(S)}{\left(\frac{8S+4}{4S+1}\right)I_1(S)}=\frac{8S+2}{8S+4}$$

$$Z_{21}(S)=\frac{V_2(S)}{I_1(S)}=2$$

The driving-point function is

$$Z_{11}(S)=\frac{V_1(S)}{I_1(S)}=\frac{8S+4}{4S+1}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★ **15-4.1** Calculate the driving-point functions for the following circuit shown in Fig.Q.1.

☆☆★ **15-4.2** Find the transfer function $G_{21}(s)$ for the two-port network shown in Fig. Q.2.

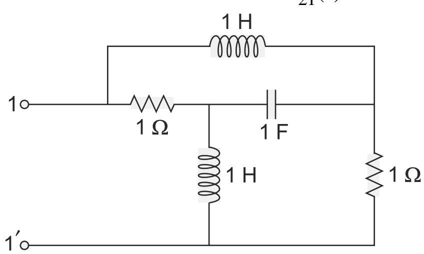

Fig. Q.1

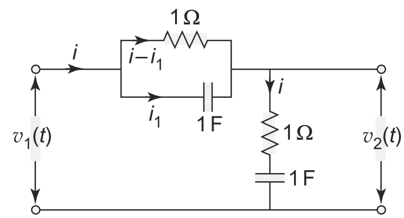

Fig. Q.2

☆★★ **15-4.3** Determine the driving-point impedance Z_d of the network shown in Fig. Q.3.

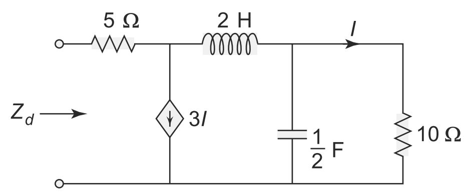

Fig. Q.3

Frequently Asked Questions linked to LO 4*

☆☆★ **15-4.1** Obtain current $I_1(s)$, $I_2(s)$, and $V_0(s)$ for circuit shown in Fig. Q.1 **[BPUT 2008]**

☆☆★ **15-4.2** State and explain the initial and final-value theorems. **[GTU Dec. 2012]**

☆☆★ **15-4.3** Compute the driving-point impedance for the network shown in Fig. Q.3. **[PTU 2011-12]**

☆★★ **15-4.4** For the network shown in Fig. Q.4, find the driving-point function $Z(s)$. Plot and zeros of $Z(s)$ on the s-plane. **[PU 2010]**

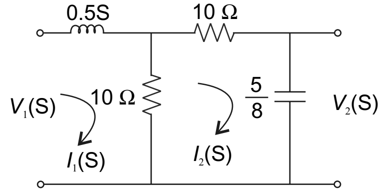

Fig. Q.1

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

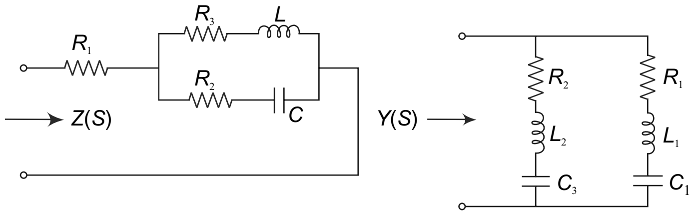

Fig. Q.3

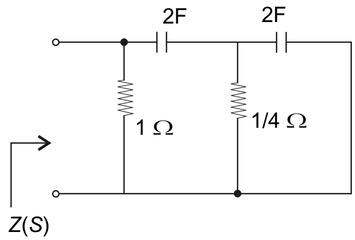

Fig. Q.4

☆☆★**15-4.5** Find the driving-point function of the network shown in Fig. Q.5. **[PU 2010]**

☆☆★**15-4.6** For the network shown in the Fig. Q.6, determine the transfer impedance. **[RGTU June 2014]**

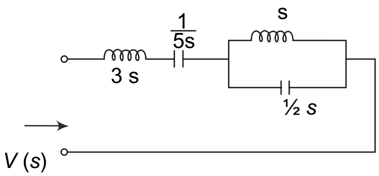

Fig. Q.5

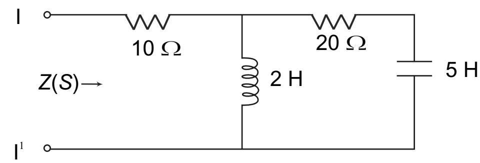

Fig. Q. 6

☆☆★**15-4.7** Calculate the transfer impedance of given circuit shown in Fig. Q.7. **[RTU Feb. 2011]**

☆☆★**15-4.8** Determine the transfer function $\dfrac{V(s)}{I(s)}$ for the network shown in Fig. Q.8. **[RTU Feb. 2011]**

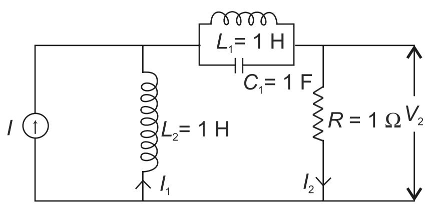

Fig. Q.7

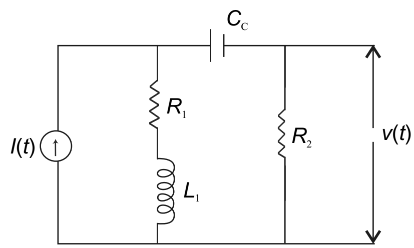

Fig. Q.8

15.7 POLES AND ZEROS OF NETWORK FUNCTIONS

> **LO 5** Describe poles and zeros of network functions and their significance

In general, the network function $N(s)$ may be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m} \tag{15.53}$$

where a_0, a_1,..., a_n and b_0, b_1, ..., b_m are the coefficients of the polynomials $P(s)$ and $Q(s)$; they are real and positive for a passive network. If the numerator and denominator of polynomial $N(s)$ are factorised, the network function can be written as

$$N(s) = \frac{P(s)}{Q(s)} = \frac{a_0(s - z_1)(s - z_2)\ldots(s - z_n)}{b_0(s - p_1)(s - p_2)\ldots(s - p_m)} \tag{15.54}$$

where z_1, z_2, ..., z_n are the n roots for $P(s) = 0$
and p_1, p_2, ..., p_m are the m roots for $Q(s) = 0$

and $a_0/b_0 = H$ is a constant called the *scale factor*.

$z_1, z_2, \ldots, z_n$ in the transfer function are called zeros, and are denoted by 0. Similarly, $p_1, p_2, \ldots, p_m$ are called poles, and are denoted by $\times$. The network function $N(s)$ becomes zero when s is equal to any one of the zeros. $N(s)$ becomes infinite when s is equal to any one of the poles. The network function is completely defined by its poles and zeros. If the poles or zeros are not repeated, then the function is said to have simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to have multiple poles, multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

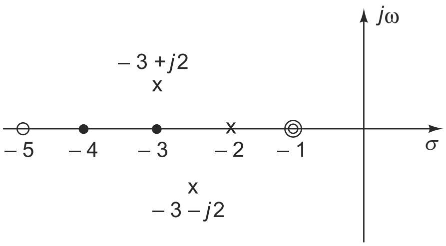

Fig. 15.26

Consider the network function

$$N(s) = \frac{(s+1)^2(s+5)}{(s+2)(s+3+j2)(s+3-j2)} \tag{15.55}$$

that has double zeros at $s = -1$ and a zero at $s = -5$; and three finite poles at $s = -2$, $s = -3 + j2$, and $s = -3 - j2$ as shown in Fig. 15.26.

The network function is said to be stable when the real parts of the poles and zeros are negative. Otherwise, the poles and zeros must lie within the left half of the s-plane.

15.8 SIGNIFICANCE OF POLES AND ZEROS

LO 5

Poles and zeros are critical frequencies. At poles, the network function become infinite, while at zeros, the network function becomes zero. At other complex frequencies, the network function has a finite non-zero value.

Poles and zeros provide useful information in network functions. Consider the following cases.

❑ *Driving Point Impedance*

$$Z(S) = \frac{V(S)}{I(S)} \tag{15.56}$$

A pole of $Z(S)$ implies zero current for a finite voltage which means an open circuit. A zero of $Z(S)$ implies no voltage for a finite current or a short circuit.

$$\text{Consider } Z(S) = \frac{1}{CS} \tag{15.57}$$

The above function has a pole at $S = 0$ and zero at $S = \infty$.

Therefore, the above function represented by capacitor acts an open circuit at pole frequency and acts as short circuit at zero frequency.

❑ *Driving Point Admittance*

$$Y(S) = \frac{I(S)}{V(S)} \tag{15.58}$$

A pole of $Y(S)$ implies zero voltage for a finite value of current which gives a short circuit. A zero of $Y(S)$ implies zero current for a finite value of voltage which gives an open circuit.

❑ *Voltage Transform Ratio*

$$G_{21}(S) = \frac{V_2(S)}{V_1(S)} \tag{15.59}$$

$$V_2(S) = G_{21}(S)\, V_1(S)$$

To obtain output voltage, we require the product of input and transfer function. The expression for $G_{21}(S)$. $V_1(S)$ is obtained in the form of a ratio of polynomials in S.

By making use of partial fractions, we can obtain a pole of either G_{21} (S) or $V_1(S)$ and no repeated roots.

$$G_{21}(S)V_1(S) = \sum_{i=1}^{n} \frac{A}{S - a_i} + \sum_{j=1}^{m} \frac{A}{S - a_j} \tag{15.60}$$

where n and m are the number of poles of $G_{21}(S)$ and $V_1(S)$ respectively.

The frequencies a_i from the natural complex frequencies corresponding to free oscillations and depend on the network function $G_{21}(S)$. While frequencies a_j constitute the complex frequencies corresponding to the forced oscillations and depend on the driving force $V_1(S)$. From the above discussion, we can say that the poles determine the time variation of the response whereas the zeros determine the magnitude response.

❑ ***Other Network Functions*** Significance of poles and zeros in other transfer functions is the same as discussed above. In each of the cases, poles determine the time-domain behaviour and zeros determine the magnitude of each of the terms of the response.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆★★ **15-5.1** For the given network function, draw the pole zero diagram and, hence, obtain the time-domain response. Verify the result analytically.

$$V(s) = \frac{5(s+5)}{(s+2)\,(s+7)}$$

☆★★ **15-5.2** Obtain the pole zero configuration of the impedance function of the network shown in Fig.Q.2.

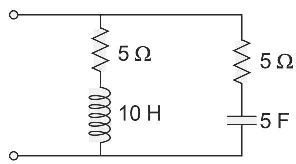

Fig. Q.2

☆☆★ **15-5.3** Draw the pole zero diagram for the given network function and hence, obtain $v(t)$.

$$V(s) = \frac{4s}{(s+2)\,(s+3)}$$

☆☆★ **15-5.4** For the network shown in Fig. Q.4, determine the following transfer functions (a) $G_{21}(s)$, (b) $Y_{21}(s)$, and (c) $\alpha_{21}(s)$.

☆☆★ **15-5.5** For the network shown in Fig. Q.5, determine the following functions (a) $Z_{11}(s)$, (b) $Y_{11}(s)$, (c) $G_{21}(s)$, and (d) $\alpha_{21}(s)$.

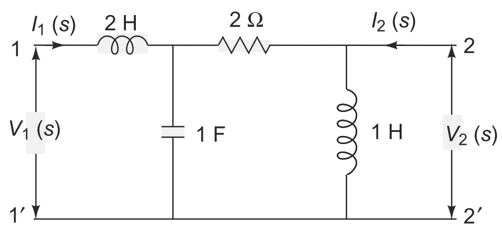

Fig. Q.4

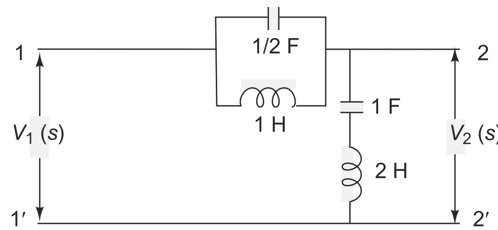

Fig. Q.5

Frequently Asked Questions linked to LO 5

☆☆★**15-5.1** Draw the pole plot of sine functions.

☆☆★**15-5.2** Explain the significance of poles and zeros.

15.9 PROPERTIES OF DRIVING POINT FUNCTIONS

LO 6 Explain the properties and necessary conditions for driving-point functions and transfer functions

1. The driving-point function is a ratio of polynomials in S. Polynomials are obtained from the transform impedances of the elements and their combinations.

Let $P(S) = a_0 S^n + a_1 S^{n-1} + \ldots + a_{n-1} S + a_n$
and $Q(S) = b_0 S^m + b_1 S^{m-1} + \ldots + b_{m-1} S + b_m$ (15.61)

be the numerator and the denominator polynomials respectively. The above equations can be factorised and therefore written as

$$\left.\begin{aligned} P(S) &= (S - Z_1)(S - Z_2)\ldots(S - Z_n) \\ Q(S) &= (S - P_1)(S - P_2)\ldots(S - P_m) \end{aligned}\right\} \quad (15.62)$$

The driving-point function $N(S)$ may be written as

$$N(S) = \frac{P(S)}{Q(S)} = \frac{(S - Z_1)(S - Z_2)\ldots(S - Z_n)}{(S - P_1)(S - P_2)\ldots(S - P_m)} \quad (15.63)$$

The quantities $Z_1, Z_2 \ldots Z_n$ are called zeros of $N(S)$ as $N(Z_1) = N(Z_2) = \ldots N(Z_n) = 0$.
The quantities P_1, P_2 ... P_m are called poles of $N(S)$ as $N(P_1) = N(P_2) = \ldots N(P_m) = \infty$.
That is, pole is that finite value of S for which $N(S)$ becomes infinity.

If the zeros and poles are not repeated then the poles or zeros are said to be *distinct* or *simple*.

A zero or a pole is said to be of *multiplicity* 'r' if $(S - Z)^r$ or $(S - P)^r$ is a factor of $P(S)$ or $Q(S)$.

A function $N(S)$ is said to have a pole (or zero) at infinity, if the function $N\left(\frac{1}{S}\right)$ has a pole (or zero) at $S = 0$.

Consider the function

$$N(S) = \frac{S+1}{(S+2)(S+4)} \quad (15.64)$$

$$N\left(\frac{1}{S}\right) = \frac{\frac{1}{S}+1}{\left(\frac{1}{S}+2\right)\left(\frac{1}{S}+4\right)} = \frac{S(S+1)}{(1+2S)(1+4S)} \quad (15.65)$$

i.e. $N\left(\frac{1}{S}\right)$ has a zero at $S = 0$

$N(S)$ has a zero at $S = \infty$

From the above example, we say that the number of zeros including zeros at ∞ equals the number of poles including poles at ∞.

2. (a) $N(S)$ be a driving-point impedance, i.e. $Z(S)$

$$Z(S) = \frac{V(S)}{I(S)} \tag{15.66}$$

A zero of $N(S)$ is a zero of $V(S)$, it signifies a short circuit. Similarly, a pole of $Z(S)$ is a zero of $I(S)$. The poles of $Z(S)$ are those frequencies corresponding to open circuit conditions.

(b) Consider a driving-point admittance function

$$Y(S) = \frac{I(S)}{V(S)} \tag{15.67}$$

A zero of $Y(S)$ means a zero of $I(S)$, i.e. the open-circuit condition and a pole of $Y(S)$ means a zero of $V(S)$ signifies a short circuit.

3. Since all the elements in the circuit are real positive quantities the coefficients $a_0, a_1, a_2, \ldots a_n$ and $b_0, b_1, b_2 \ldots b_m$ are real and positive. Therefore, any zeros or poles, if complex, must occur in conjugate pairs.

4. The real parts of all zeros and poles must be negative or zero. Consider a pole 'P' of $N(S)$, i.e. $(S - P)$ is a factor of the denominator of $N(S)$. Using partial fractions, we know that this gives rise to a term of the form $\frac{A}{S-P}$ whose inverse Laplace transform contains the term $e^{pt.}$ The real part of e^{pt} tends to zero as t tends to infinity if the real part P is negative. Therefore, for a finite input the response is finite as t tends to infinity if the real part of P is negative. A network function whose response is finite for all t, for a given finite input is said to be stable. Thus, a driving-point impedance $Z(S)$ is stable if all the poles lie in the negative half of the S-plane.

Since $Y(S) = \frac{1}{Z(S)}$, poles of $Y(S)$ are zeros of $Z(S)$. Therefore $Y(S)$ is stable if all the zeros of $Z(S)$ also lie in the negative half of the S-plane. Thus, the real parts of all zeros and poles of a driving point function must be negative or zero.

5. Poles or zeros lying on the $j\omega$-axis must be simple. Consider a pole 'P' lying on the $j\omega$-axis. If it is not simple then in the time response of the function of which it is a pole contains the term $t^k\, e^{j\omega t}$ which tends to infinity as t tends to infinite. Therefore, the function becomes unstable. Since zeros of one function will be poles of the other, therefore, the zeros of driving point function should also satisfy this condition.

6. The degree of $P(S)$ and $Q(S)$ may differ by zero or one only.

At very high frequencies, the term $a_0\, S^n$ dominates over the other terms in the numerator and the term $b_0\, S^m$ dominates over other terms in the denominator.

$$\underset{S\to\infty}{\text{Lt}}\ N(S) = \underset{S\to\infty}{\text{Lt}}\ \frac{a_0}{b_0} S^{n-m} \tag{15.68}$$

Consider the network elements R, L, C, and M. R is independent of frequency.

$\therefore$ if $n = m$, then the function behaves as a resistance $R = \frac{a_0}{b_0}$ at high frequency. The impedance LS of an inductor increases linearly with the complex frequency S and, therefore, is an open circuit at $S = \infty$. Thus if $n = m + 1$, the function $N(S)$ behaves as an inductance as S approaches infinity. A capacitor is a short circuit at infinite frequencies. Thus, $N(S)$ behaves as a capacitance if $m = n + 16$.

Now, consider the driving-point impedance $Z(S)$. $Z(S)$ will behave as an inductor as LS increases with increasing S while $\frac{1}{CS}$ decreases with increasing S and, therefore, the impedance of an inductance dominates over the capacitive impedance. If inductors are not present in the circuit, then R dominates over $\frac{1}{CS}$ as S tends to infinity. Thus, $Z(S) = LS$ or R as S tends infinity.

$\therefore n - m = 0$ or 1

If $N(S)$ is a driving-point impedance.

On the other hand, the admittance of an inductance $\frac{1}{LS}$ tends to zero as S tends to infinity. Similarly, the admittance of a capacitance CS tends to infinity as S tends to infinity. Therefore, CS dominates over $\frac{1}{LS}$ as S tends to infinity. If the network does not contain capacitors then the resistance R dominates over $\frac{1}{LS}$ at higher frequencies.

$\therefore$ If $N(S)$ is a driving-point admittance function, $m - n = 0$ or 1.

Therefore, $|n - m| = 0$ or 1

7. The lowest degree terms in $P(S)$ and $Q(S)$ may differ in degree by zero or one only.

   As S approaches zero, the higher power of S tends to zero faster than S, therefore $N(S)$ can be approximated by

$$N(S) = \frac{a_{n-1}S + a_n}{b_{m-n}S + b_m} \tag{15.69}$$

The impedance of an inductance 'LS' approaches zero as S tends to zero while that of a capacitor $\frac{1}{CS}$ approaches infinity as S tends to zero. $\frac{1}{CS}$ dominates over LS as S tends to zero. Therefore, for $Z(S)$, the capacitance dominates over an inductance as S tends to zero. If the network does not contain capacitors then R dominates over LS and S tends to zero. Thus, the network can be replaced by R or $\frac{1}{CS}$ if $Z(S)$ is of interest. Similarly, for $Y(S); \frac{1}{R}, \frac{1}{LS}$ dominate over CS as S tends to zero. Therefore, for purposes of $Y(S)$, the network is just a conductance or an inductance. Thus, the network is just one inductor or one capacitance or one resistance as S tends to zero.

$\therefore N(S)$ is of the form K_1 or $K_2\,S$ or $\frac{K_3}{S}$ where K_1, K_2, K_3 are constants.

Hence, the lowest degree of $P(S)$ and $Q(S)$ can differ at most in one degree.

8. $P(S)$ and $Q(S)$ cannot have missing terms unless all even or all odd degree terms are absent.

   We know that

$$P(S) = a_0 S^n + a_1 S^{n-1} + \ldots + a_{i+1} S^{n-i-1} + a_i S^{n-i}. + \ldots + a_{n-1} S + a_n$$

and
$$Q(S) = b_0 S^m + b_1 S^{m-1} + \ldots + b_{i+1} S^{m-i-1} + b_i S^{m-i} + \ldots + b_{m-1} S + b_m \tag{15.70}$$

The above requirement means that for any i, a_i or b_i cannot be zero unless $a_j = 0$ or $b_j = 0$ for all $i \geq j$. The only exception to this rule is when all even or all odd powers of S are missing. To understand this, consider the network under study contains only elements like R, L, C, M whose transfer impedances are R, LS, $\frac{1}{CS}$, MS respectively.

Also, R, L, C, M are positive quantities. A combination of RL or RC will give rise to a term of the form $(aS + b)$ or $a + \frac{b}{S}$. Since $\left(a + \frac{b}{S}\right) = \frac{aS + b}{S}$ means $(aS + b)$ in the numerator and S in the denominator. Similarly, R, L, C give rise terms of the form $(aS^2 + bS + C)$ and a combination of only L

and C gives rise to a term of the form $(aS^2 + b)$. Therefore, when two such factors are multiplied, since all the coefficients in each term are positive, in the expansion of the product no term can be zero. If all the terms are of the form $(aS^2 + b)$, then their product contains only even powers of S. If this is multiplied by S, the resulting function contains only odd powers of S. Given a ratio of polynomials $N(S)$, these properties can therefore be used to find out if $N(S)$ represents a driving point function of a network.

15.10 PROPERTIES OF TRANSFER FUNCTIONS

LO 6

1. The transfer function is a ratio of polynomials in S.
2. The coefficients of $P(S)$, the numerators polynomial and of $Q(S)$, the denominator polynomial must be real. Therefore, all poles and zeros, if complex, must occur in conjugate pairs.
3. The real parts of all poles must be negative and any pole on the $j\omega$-axis must be simple. As in the case of driving-point functions, this follows from the stability considerations.
4. Since poles of the transfer function are zeros of $Q(S)$, it follows that the zeroes of $Q(S)$ must lie in the negative half plane and any zero lying on the $j\omega$-axis must be simple.

   Let $P_1, P_2, \ldots P_m$ be the zeros of $Q(S)$

   Then $Q(S) = K \cdot (S - P_1)(S - P_2)(S - P_3) \ldots (S - P_m)$

   Since all poles have negative real parts and complex poles occur in conjugate pairs, the product of these factors contains all powers of S whose coefficients are positive. Therefore, $Q(S)$ does not have missing terms unless all even or all odd powers are missing. Since there are no restrictions on the zeroes of the transfer function, $P(S)$ can have missing terms. Also coefficients of powers of S in $P(S)$ can be negative.
5. For $G(S)$ and $\alpha(S)$, the degree of the numerator polynomial $P(S)$ is less than or equal to the degree of $Q(S)$.
   To prove this, we use the fact that a two-port network can be represented by an equivalent T(star) or Π (delta) shown in Fig. 15.27.
   Let a source of known voltage $V_1(S)$ be applied to a T-network of the port $11'$. Let a source of known current $I_1(S)$ be applied to the π-network at 11^1 and 22^1 be short-circuited, then assuming $I_2(S) = 0$.

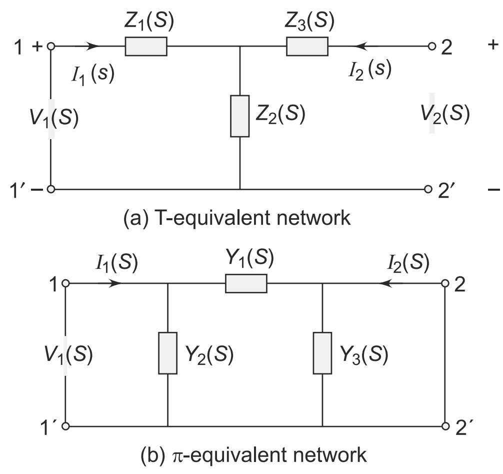

Fig. 15.27

$$G(S) = \frac{V_2(S)}{V_1(S)} = \frac{Z_2(S)}{Z_1(S) + Z_2(S)} \tag{15.71}$$

$$\text{and} \quad \alpha(S) = \frac{I_2(S)}{I_1(S)} = \frac{Y_2(S)}{Y_1(S) + Y_2(S)} \tag{15.72}$$

Since $Z_1(S), Z_2(S), Z_3(S); Y_1(S), Y_2(S)$, and $Y_3(S)$ can be thought off as the driving point functions of some one ports, they have to satisfy the properties of driving point immittance functions.

Since $Z_1(S)$ and $Z_2(S)$ are ratio of polynomials,

$$\text{Let} \quad Z_1(S) = K\frac{(S+\alpha_1)(S+\alpha_2)\dots(S+\alpha_{n1})}{(S+\beta_1)(S+\beta_2)\dots(S+\beta_{m1})} \tag{15.73}$$

$$Z_2(S) = \frac{K_2(S+r_1)(S+r_2)\dots(S+r_{n2})}{(S+\delta_1)(S+\delta_2)\dots(S+\delta_{m2})} \tag{15.74}$$

Substituting these expression in $G(S)$,

$$G(S) = \frac{Z_2(S)}{Z_1(S)+Z_2(S)} \tag{15.75}$$

$$= \frac{K_1(S+\alpha_1)(S+\alpha_2)\dots(S+\alpha_{n1})(S+\delta_1)\dots(S+\delta_{m2})}{(S+\alpha_1)(S+\alpha_2)\dots(S+\alpha_{n1})(S+\delta_1)\dots(S+\delta_{m2})(S+\gamma_1)}$$

$$(S+\gamma_2)(S+\gamma_3)\dots(S+\gamma_{n2})(S+\beta_1)\dots(S+\beta_{m1})$$

Let $P(S)$ denote the numerator polynomial of $G(S)$ and $Q(S)$, the denominator polynomial of $G(S)$.

Then degree of $P(S) = n_1 + m_2$ and degree of $Q(S) = n_1 + m_2$ or $n_2 + m_1$, whichever is greater.

Thus, if $n_1 + m_2 > n_2 + m_1$, the degree of $P(S)$ equals the degree of $Q(S)$.

If $n_1 + m_2 < n_2 + m_1$, degree of $Q(S) = n_2 + m_1$ and the degree of $P(S)$ is less than the degree of $Q(S)$.

Similarly, assuming $Y_1(S)$ and $Y_2(S)$ as ratios of polynomials and substituting those expressions in $\alpha(S)$, it can be shown that the degree of the numerator of $\alpha(S)$ is less than or equal to the degree of the denominator.

(f) The degree of the numerator polynomial of $Z_{21}(S)$ or $Y_{21}(S)$ is less than or equal to the degree of the denominator polynomial plus one.

Referring to the T and π equivalent networks of two-port network shown in Fig. 15.27,

$$Z_{21}(S) = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2=0} = \frac{Z_2(S)I_1(S)}{I_1(S)} = Z_2(S) \tag{15.76}$$

$$\text{and} \quad Y_{21}(S) = \left.\frac{I_2(S)}{V_1(S)}\right|_{V_2=0} = \frac{-V_2(S)I_1(S)}{I_1(S)} = -Y_2(S) \tag{15.77}$$

Thus, the highest degree of the numerator of $Z_{21}(S)$ equals the highest degree of the numerator of $Z_2(S)$. But as $Z_2(S)$ is a driving-point impedance, the highest degree of the numerator of $Z_2(S)$ is the degree of denominator plus one. Therefore, the highest degree of the numerator of $Z_{12}(S)$ is the degree of its denominator plus one. Similarly, since $Y_2(S)$ is a driving-point admittance, the highest degree of the numerator or $Y_{21}(S)$, which is also the numerator of $Y_2(S)$ is equal to the degree of the denominator plus one.

15.11 NECESSARY CONDITIONS FOR A DRIVING POINT FUNCTION

LO 6

The restrictions on pole and zero locations in the driving-point function with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. The coefficients in the polynomials $P(s)$ and $Q(s)$ of network function $N(s) = P(s)/Q(s)$ must be real and positive.

2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. (a) The real parts of all poles and zeros must be zero, or negative.
   (b) If the real part is zero, then the pole and zero must be simple.
4. The polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and the lowest degrees, unless all even or all odd terms are missing.
5. The degree of $P(s)$ and $Q(s)$ may differ by zero, or one only.
6. The lowest degree in $P(s)$ and $Q(s)$ may differ in degree by at the most one.

15.12 NECESSARY CONDITIONS FOR TRANSFER FUNCTIONS

LO 6

The restrictions on pole and zero location in transfer functions with common factors in $P(s)$ and $Q(s)$ cancelled are listed below.

1. (a) The coefficients in the polynomials $P(s)$ and $Q(s)$ of $N(s) = P(s)/Q(s)$ must be real.
   (b) The coefficients in $Q(s)$ must be positive, but some of the coefficients in $P(s)$ may be negative.
2. Complex or imaginary poles and zeros must occur in conjugate pairs.
3. The real part of poles must be negative, or zero. If the real part is zero, then the pole must be simple.
4. The polynomial $Q(s)$ may not have any missing terms between the highest and the lowest degree, unless all even or all odd terms are missing.
5. The polynomial $P(s)$ may have missing terms between the lowest and the highest degree.
6. The degree of $P(s)$ may be as small as zero, independent of the degree of $Q(s)$.
7. (a) For the voltage transfer ratio and the current transfer ratio, the maximum degree of $P(s)$ must equal the degree of $Q(s)$.
   (b) For transfer impedance and transfer admittance, the maximum degree of $P(s)$ must equal the degree of $Q(s)$ plus one.

Frequently Asked Questions linked to LO 6

☆★★ **14-6.1** Explain the complete procedure for making a bode plot for different types of transfer functions.

☆☆★ **14-6.2** What do you mean by impedance, admittance, and immittance functions?

☆☆★ **14-6.3** Find the driving-point impedance of the network shown in Fig. Q.3 as a function of the complex frequency s and also give the poles and zeros.

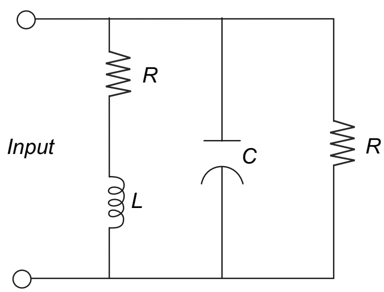

Fig. Q.3

☆☆★ **14-6.4** Discuss the restrictions on pole and zero locations in the s-plane for driving-point locations.

☆☆★ **14-6.5** Write the necessary conditions for driving-point function and transfer function.

15.13 TIME-DOMAIN RESPONSE FROM POLE ZERO PLOT

LO 7 Analyse the time domain, amplitude, and phase response from pole-zero plots

For the given network function, a pole zero plot can be drawn which gives useful information regarding the critical frequencies. The time-domain response can also be obtained from pole zero plot of a network function. Consider an array of poles shown in Fig. 15.28.

In Fig. 15.28, s_1 and s_3 are complex conjugate poles, whereas s_2 and s_4 are real poles. If the poles are real, the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta > 1$$

where δ is the damping ratio and ω_n is the undamped natural frequency.

The roots of the equation are

$$s_2, s_4 = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}; \delta > 1$$

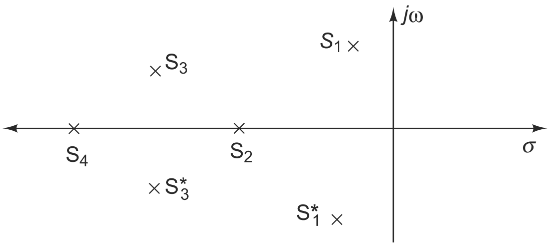

Fig. 15.28

For these poles, the time domain response is given by

$$i(t) = k_2 e^{s_2 t} + k_4 e^{s_4 t}$$

The response due to the pole s_4 dies faster compared to that of s_2 as shown in Fig. 15.29.

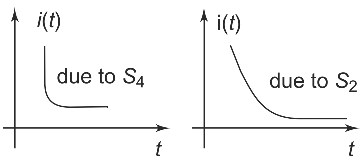

Fig. 15.29

s_1 and s_3 constitute complex conjugate poles. If the poles are complex conjugate, then the quadratic function is

$$s^2 + 2\delta\omega_n s + \omega_n^2 \text{ for } \delta < 1$$

The roots are $s_1, s_1^* = -\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$ for $\delta < 1$

For these poles, the time domain response is given by

$$i(t) = k_1 e^{-\delta\omega_n t + j\left(\omega_n\sqrt{1-\delta^2}\right)t} + k_1^* e^{-\delta\omega_n t - j\left(\omega_n\sqrt{1-\delta^2}\right)t}$$
$$= ke^{-\delta\omega_n t}\sin\left(\omega_n\sqrt{1-\delta^2}\right)t$$

From the above equation, we can conclude that the response for the conjugate poles is damped sinusoid. Similarly, s_3, s_3^* are also a complex conjugate pair. Here, the response due to s_3 dies down faster than that due to s_1 as shown in Fig. 15.30.

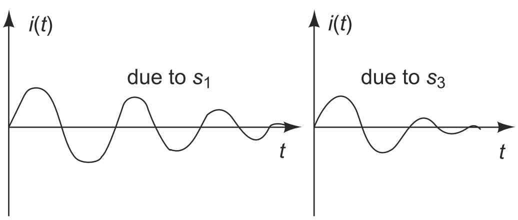

Fig. 15.30

Consider a network having transfer admittance $Y(s)$. If the input voltage $V(s)$ is applied to the network, the corresponding current is given by

$$I(s) = V(s)\,Y(s) = \frac{P(s)}{Q(s)}$$

This may be taken as

$$I(s) = H\frac{(s-s_a)(s-s_b)\ldots(s-s_n)}{(s-s_1)(s-s_2)\ldots(s-s_m)}$$

where H is the scale factor.

By taking the partial fractions, we get

$$I(s) = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \ldots + \frac{k_m}{s-s_m}$$

The time-domain response can be obtained by taking the inverse transform

$$i(t) = \mathcal{L}^{-1}\left[\frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \ldots + \frac{k_m}{s-s_m}\right]$$

Any of the above coefficients can be obtained by using the Heavisides method.
To find the coefficient k_l,

$$k_1 = H\left[\frac{(s-s_a)(s-s_b)\ldots(s-s_n)}{(s-s_1)(s-s_2)\ldots(s-s_m)}\right](s-s_l)\bigg|_{s=s_l}$$

Here, s_l, s_m, s_n are all complex numbers, the difference of $(s_l - s_n)$ is also a complex number.

$$(s_l - s_n) = M_{ln}\, e^{j\phi_{ln}}$$

Hence $$k_1 = H\frac{M_{la}M_{lb}\ldots M_{ln}}{M_{l1}M_{l2}\ldots M_{lm}} \times e^{j(\phi_{la}+\phi_{lb}+\ldots+\phi_{ln})-(\phi_{l1}+\phi_{l2}+\ldots+\phi_{lm})}$$

Similarly, all coefficients k_1, k_2, ..., k_m may be obtained, which constitute the magnitude and phase angle.

The residues may also be obtained by pole zero plot in the following way.

1. Obtain the pole zero plot for the given network function.
2. Measure the distances M_{la}, M_{lb}, ..., M_{ln} of a given pole from each of the other zeros.
3. Measure the distances M_{l1}, M_{l2}, ..., M_{lm} of a given pole from each of the other poles.
4. Measure the angle ϕ_{la}, ϕ_{lb}, ..., ϕ_{ln} of the line joining that pole to each of the other zeros.
5. Measure the angle ϕ_{l1}, ϕ_{l2}, ..., ϕ_{lm} of the line joining that pole to each of the other poles.
6. Substitute these values in required residue equation.

15.14 AMPLITUDE AND PHASE RESPONSE FROM POLE ZERO PLOT

LO 7

The steady-state response can be obtained from the pole zero plot, and it is given by

$$N(j\omega) = M(\omega)e^{j\phi(\omega)}$$

where $M(\omega)$ is the amplitude

$\phi(\omega)$ is the phase

These amplitude and phase responses are useful in the design and analysis of network functions. For different values of ω, corresponding values of $M(\omega)$ and $\phi(\omega)$ can be obtained and these are plotted to get amplitude and phase response of the given network.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆★★ **15-7.1** For the given network function, draw the pole zero diagram and hence, obtain the time domain response. Verify the result analytically.

$$I(s) = \frac{5s}{(s+3)(s^2+2s+2)}$$

Frequently Asked Questions linked to LO 7

☆☆★**15-7.1** A transform voltage is given by $V(s) = \dfrac{3s}{(s+1)(s+4)}$

Plot the pole-zero in the *S*-plane and obtain the time-domain response.

☆★★**15-7.2** The transform current $I(s)$ in a given network is given by $I(s) = \dfrac{S}{(s+2)(s^2+2s+2)}$ Obtain the pole zero plot and hence, the time-domain response.

☆★★**15-7.3** Determine the poles of a series *RLC* circuit, if *R* 120 ohms, $L = 10$ mH, and $C = 1$ micro *F*. Sketch the poles plot and comment on the nature of the response.

☆★★**15-7.4** Transform current $I(s)$ of a network is given by $I(s) = \dfrac{2s}{(s+1)(s+2)}$. Plot the poles and zeros in the *s*-plane and hence, obtain the time-domain response of it.

☆★★**15-7.5** Find the response of a network if:

$H(s) = s^2 + 3s + 5/(s+1)\,(s+2)$

and excitation $x(t) = e^{-3t}$

☆☆★**15-7.6** For the network shown in Fig. Q.6, find the admittance $Y(S)$ as seen by the source $i(t)$. Also, plot the and zeros of $Y(S)$.

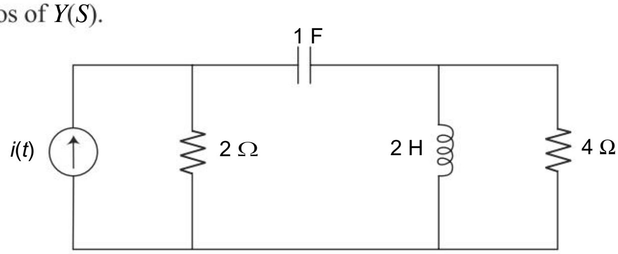

Fig. Q.6

15.15 STABILITY CRITERION FOR AN ACTIVE NETWORK

> **LO 8** Explain the stability and Routh criteria for active networks

Passive networks are said to be stable only when all the poles lie in the left half of the *s*-plane. Active networks (containing controlled sources) are not always stable. Consider transformed active network shown in Fig. 15.31.

By applying the Millman theorem, we get

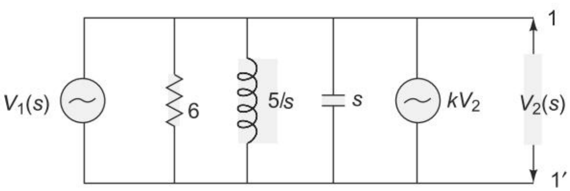

Fig. 15.31

$$V_2(s) = \frac{V_1(s) + kV_2(s)}{6 + 5/s + s}$$

$$= \frac{s[V_1(s) + kV_2(s)]}{s^2 + 6s + 5}$$

$$V_2(s)\,[s^2 + 6s + 5] - ksV_2(s) = sV_1(s)$$

$$V_2(s)\,[s^2 + (6-k)s + 5] = sV_1(s)$$

$$\therefore \quad \frac{V_2(s)}{V_1(s)} = \frac{s}{s^2 + (6-k)s + 5}$$

From the above transformed equation, the poles are dependent upon the value of *k*.

The roots of the equation are

$$s = \frac{-(6-k) \pm \sqrt{(6-k)^2 - 4 \times 5}}{2}$$

For $k = 0$, the poles are at –1, –5, which lie on the left half of the s-plane. As k increases, the poles move towards each other and meet at a point $\sqrt{(6-k)^2 - 20} = 0$, when $k = 1.53$ or 10.47. The root locus plot is shown in Fig. 15.32.

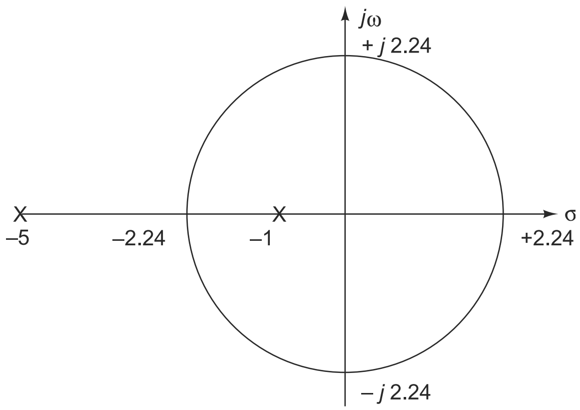

Fig. 15.32

The root locus is obtained from the characteristic equation $s^2 + (6 - k)s + 5 = 0$. As the value of k increases beyond 1.53, the locus of the root is a circle. The poles are located on the imaginary axis at $\pm j2.24$ for $k = 6$. At –2.24, the poles are coincident for $k = 1.53$ while at +2.24, the poles are coincident for $k = 10.47$. When $k > 10.47$, the poles again lie on the real axis but remain on the right half of the s-plane, one pole moving towards the origin and the other moving towards infinity. From this, we can conclude, as long as k is less than 6, the poles lie on the left half of the s-plane and the system is said to be stable. For $k = 6$, the poles lie on the imaginary axis and the system is oscillatory in nature. For values of k greater than 6, the poles lie on the right half of the s-plane. Then the system is said to be unstable.

15.16 ROUTH CRITERIA

LO 8

The locations of the poles gives us an idea about stability of the active network. Consider the denominator polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + \dots + b_m \tag{15.78}$$

To get a stable system, all the roots must have negative real parts. There should not be any positive or zero real parts. This condition is not sufficient.

Let us consider the polynomial

$$s^3 + 4s^2 + 15s + 100 = (s + 5)(s^2 - s + 20)$$

In the above polynomial, though the coefficients are positive and real, the two roots have positive real parts. From this, we conclude that the coefficients of $Q(s)$ being positive and real is not a sufficient condition to get a stable system. Therefore, we have to seek the condition for stability which is necessary and sufficient.

Consider the polynomial $Q(s) = 0$. After factorisation, we get

$$b_0 (s - s_1)(s - s_2) \dots (s - s_m) = 0 \tag{15.79}$$

On multiplication of these factors, we get

$$Q(s) = b_0 s^m - b_0(s_1 + s_2 + \dots + s_m)s^{m-1} + b_0(s_1 s_2 + s_2 s_3 + \dots)\, s^{m-2} + b_0\,(-1)^m\,(s_1 s_2 \dots s_m) = 0 \tag{15.80}$$

Equating the coefficients of Eqs (15.78) and (15.80), we have

$$\frac{b_1}{b_0} = -(s_1 + s_2 + \ldots + s_m) \tag{15.81}$$

$= -$ sum of the roots

$$\frac{b_2}{b_0} = 1(s_1 s_2 + s_2 s_3 + \ldots) \tag{15.82}$$

$=$ sum of the products of the roots taken two at a time

$$\frac{b_3}{b_0} = -(s_1 s_2 s_3 + s_2 s_3 s_4 + \ldots) \tag{15.83}$$

$= -$ sum of the products of the roots taken three at a time.

$$(-1)^m \frac{b_m}{b_0} = (s_1 s_2 s_3 \ldots s_m) = \text{product of the roots} \tag{15.84}$$

If all the roots have negative real parts, then from the above equations, it is clear that all the coefficients must have the same sign. This condition is not sufficient due to the fact that the zero value of a coefficient involves cancellation, which requires some root to have positive real parts.

The Routh criterion for stability is discussed below. Consider a polynomial

$$Q(s) = b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ \dots + b_m$$

Taking first-row coefficients and second-row coefficients separately, we have

$$\begin{matrix} b_0 & b_2 & b_4 & \dots \\ b_1 & b_3 & b_5 & \dots \end{matrix}$$

Now, we complete the Routh array as follows.

For $m = 5$,

$$\begin{array}{c|ccc} s^5 & b_0 & b_2 & b_4 \\ s^4 & b_1 & b_3 & b_5 \\ s^3 & c_1 & c_2 & \\ s^2 & d_1 & d_2 & \\ s^1 & e_1 & & \\ s^0 & f_1 & & \end{array}$$

where c_1, c_2, d_1, d_2, e_1, f_1 are determined by the algorithm given below.

$$c_1 = \frac{\begin{matrix} b_0 & b_2 \\ b_1 & b_3 \end{matrix}}{b_1} = \frac{b_1 b_2 - b_0 b_3}{b_1}$$

$$c_2 = \frac{\begin{matrix} b_0 & b_4 \\ b_1 & b_5 \end{matrix}}{b_1} = \frac{b_1 b_4 - b_0 b_5}{b_1}$$

$$d_1 = \frac{\begin{matrix} b_1 & b_3 \\ c_1 & c_2 \end{matrix}}{c_1} = \frac{c_1 b_3 - b_1 c_2}{c_1}$$

$$d_2 = \frac{\begin{matrix} b_1 & b_5 \\ c_1 & 0 \end{matrix}}{c_1} = \frac{b_5 c_1 - 0}{c_1}$$

$$e_1 = \frac{\begin{matrix} c_1 & c_2 \\ d_1 & d_2 \end{matrix}}{d_1} = \frac{c_2 d_1 - c_1 d_2}{d_1}$$

$$f_1 = \frac{\begin{matrix} d_1 & d_2 \\ e_1 & 0 \end{matrix}}{e_1} = \frac{d_2 e_1 - 0}{e_1}$$

In order to find out the element in the kth row and jth column, it is required to know the four elements. These elements in the row $(k - 1)$ and the row $(k - 2)$ just above the elements are in the column 1 of the array and the $(J + 1)$ column of the array. The product of the elements joined by a line with positive slope has positive sign while the product of elements joined with a line with negative slope has a negative sign. The difference of these products is divided by the element of the column 1 and row $(k - 1)$. The above process is repeated till $m + 1$ rows are found in the Routh array.

According to the Routh–Hurwitz theorem, the number of changes in the sign of the first column to the right of the vertical line in an array moving from top to bottom is equal to the number of roots of $Q(s) = 0$ with positive real parts. To get a stable system, the roots must have negative real parts.

According to the Routh-Hurwitz criterion, the system is stable, if and only if, there are no changes in signs of the first column of the array. This requirement is, both the necessary and sufficient condition for stability.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 8

☆☆★ **15-8.1** For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$Q(s) = s^4 + s^3 + 2s^2 + 2s + 12$

☆★★ **15-8.2** Apply the Routh criterion to the following equations and determine the number of roots (i) with positive real parts, (ii) with zero real parts, and (iii) with negative real parts:

(a) $6s^3 + 2s^2 + 5s + 2 = 0$

(b) $s^6 + 5s^5 + 13s^4 + 21s^3 + 20s^2 + 16s + 8 = 0$

(c) $s^6 - s^5 - 2s^4 + 4s^3 - 5s^2 + 21s + 30 = 0$

Frequently Asked Questions linked to LO 8

☆☆★ **15-8.1** Explain how the location of poles a effect the system performance.

☆★★ **15-8.2** Denominator polynomial of a transfer function is $P(s) = s^4 + 11s^3 + 41s^2 + 61s + 30$. Find the Routh array and verify the stability of the network.

☆☆★ **15-8.3** For the equation
$P(s) = s^6 + 11s^5 + 42s^4 + 72s^3 + 71s^2 + 61s + 30 = 0$, determine the number of roots.
(a) With positive real roots
(b) With zero real parts
(c) With negative real parts

Additional Solved Problems

PROBLEM 15.1

For the two-port network shown in Fig. 15.33, determine the driving-point impedance $z_{11}(s)$, the transfer impedance $z_{21}(s)$ and the voltage transfer ratio $G_{21}(s)$.

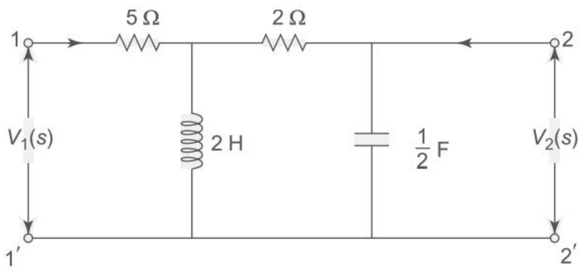

Fig. 15.33

Solution The transformed circuit is shown in Fig. 15.34.

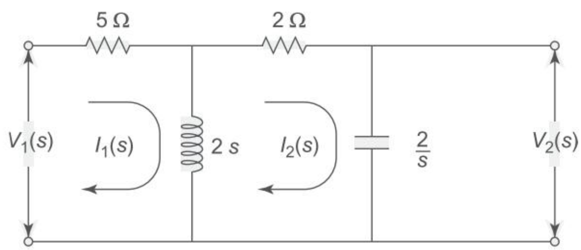

Fig. 15.34

From the above figure, by application of Kirchhoff's laws, we get

$$V_1(s) = (5 + 2s)I_1(s) - 2sI_2(s) \tag{15.85}$$

$$0 = I_2(s)\left(2 + 2s + \frac{2}{s}\right) - I_1(s)2s \tag{15.86}$$

$$V_2(s) = I_2(s)\frac{2}{s} \tag{15.87}$$

From Eq. (15.86),

$$I_2(s) = \frac{2s^2}{2s^2+2s+2} I_1(s) \tag{15.88}$$

Substituting Eq. (15.88) in Eq. (15.85),

$$V_1(s) = (5+2s)I_1(s) - \frac{4s^3}{2s^2+2s+2} I_1(s)$$

The driving-point impedance,

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = \frac{7s^2+7s+5}{s^2+s+1}$$

From Eqs (15.87) and (15.88),

$$V_2(s) = \frac{4s}{2s^2+2s+2} I_1(s)$$

The transfer impedance at the port 2 is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{2s}{s^2+s+1}$$

The voltage transfer ratio is

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{2s}{7s^2+7s+5}$$

PROBLEM 15.2

For the network shown in Fig. 15.35, determine the following transfer functions: (a) $G_{21}(s)$ (b) $Z_{21}(s)$.

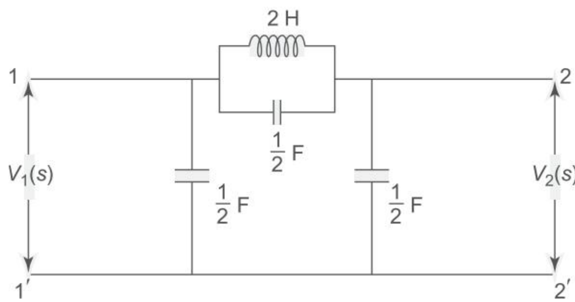

Fig. 15.35

Solution The transformed circuit is shown in Fig. 15.36.

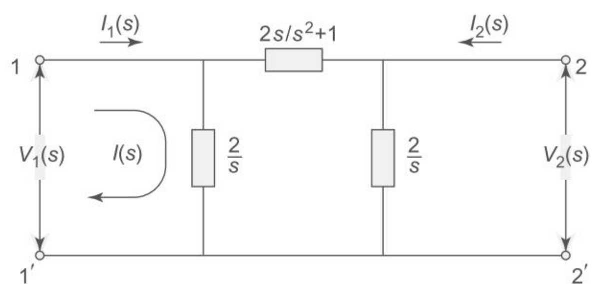

Fig. 15.36

The voltage across the port 2 is

$$V_2(s) = V_1(s) \frac{2/s}{\dfrac{2}{s} + \dfrac{2s}{s^2+1}}$$

The voltage transfer ratio at the port 1 is

$$G_{12}(s) = \frac{V_1(s)}{V_2(s)} = \frac{s^2+1}{2s^2+1}$$

The voltage at the port 1 is

$$V_1(s) = I(s)\frac{2}{s}$$

where $$V_1(s) = \frac{I_2(s)\left(\frac{2}{s}\right)\times\left(\frac{2}{s}\right)}{\frac{2}{s}+\frac{2s}{s^2+1}+\frac{2}{s}} = I_2(s)\frac{4(s^2+1)}{s(6s^2+4)}$$

The transfer impedance at the port 1 is

$$Z_{12}(s) = \frac{V_1(s)}{I_2(s)} = \frac{s^2+1}{s\left(\frac{3}{2}s^2+1\right)}$$

PROBLEM 15.3

For the network shown in Fig. 15.37, determine transfer impedance $Z_{21}(s)$ and $Y_{21}(s)$. Also find the transfer voltage ratio $G_{21}(s)$ and the transfer current ratio $\alpha_{21}(s)$.

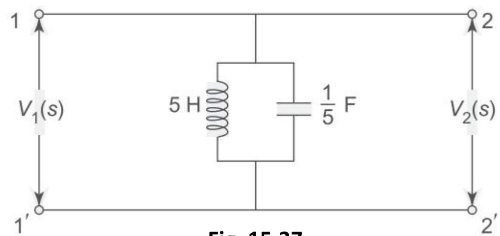

Fig. 15.37

Solution The transformed circuit is shown in Fig. 15.38.

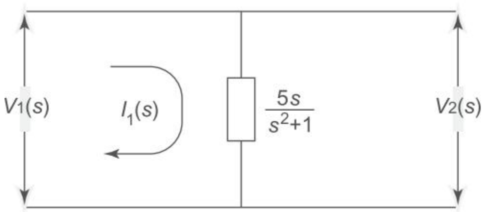

Fig. 15.38

The voltage at the port 1 is

$$V_1(s) = I_1(s)\frac{5s}{s^2+1}$$

The voltage at the port 2 is

$$V_2(s) = I_1(s)\frac{5s}{s^2+1}$$

The voltage transfer ratio at the port 2 is

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = 1$$

The transfer impedance at the port 2 is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{5s}{s^2+1}$$

The transfer admittance at the port 1 is

$$Y_{12}(s) = \frac{I_1(s)}{V_2(s)} = \frac{s^2+1}{5s}$$

PROBLEM 15.4

For the given network function, draw the pole zero diagram and hence obtain the time-domain response. Verify this result analytically.

$$I(s) = \frac{3s}{(s+1)(s+3)}$$

Solution In the network function,

$$P(s) = 3s$$

$$\text{and} \quad Q(s) = (s+1)(s+3)$$

By taking partial fractions, $I(s)$ can be written as

$$I(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

Therefore, the time-domain response is

$$i(t) = Ae^{-t} + Be^{-3t}$$

Here, the coefficients A and B are determined by using the pole zero plot as shown in Fig.15.39.
Consider the pole at –1
The distance between zero to pole at –1 is

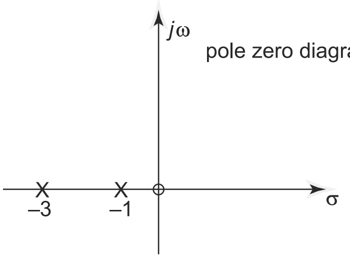

Fig. 15.39

$$M_{01} = 1$$

The angle between the line joining the pole at –1 to the zero is

$$\phi_{01} = 180°$$

Similarly, the distance between the pole at –3 to the pole to –1 is

$$M_{31} = 2$$

The angle between the line joining the pole at –1 to the pole at –3 is

$$\phi_{31} = 0°$$

$$H = 3$$

$$\text{Hence,} \quad A = H\frac{M_{01}\, e^{j\phi_{01}}}{M_{31}\, e^{j\phi_{31}}} = \frac{-3}{2}$$

$$\text{Similarly,} \quad B = H\frac{M_{03}\, e^{j\phi_{03}}}{M_{13}\, e^{j\phi_{13}}} = \frac{9}{2}$$

$$\text{where} \quad M_{03} = 3\,;\, \phi_{03} = 180°$$

$$M_{13} = 2\,;\, \phi_{13} = 180°$$

Substituting these values, we get

$$i(t) = \left(\frac{-3}{2}e^{-t} + \frac{9}{2}e^{-3t}\right)\text{A}$$

Analytically,

$$I(s) = \frac{3s}{(s+1)(s+3)}$$

By taking partial fractions,

$$I(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = I(s)(s+1)\Big|_{s=-1} = \frac{-3}{2}$$

$$B = I(s)(s+3)\Big|_{s=-3} = \frac{9}{2}$$

By substituting these values in the above equation ,

$$I(s) = \frac{-3}{2(s+1)} + \frac{9}{2(s+3)}$$

Taking inverse transform, we get

$$i(t) = \frac{-3}{2}e^{-t} + \frac{9}{2}e^{-3t}$$

PROBLEM 15.5

For the given denominator polynomial of a network function, verify the stability of the network using Routh criteria.

$$Q(s) = s^5 + 3s^4 + 4s^4 + 5s^2 + 6s + 1$$

Solution The Routh array for this polynomial is given below.

s^5	1	4	6
s^4	3	5	1
s^3	2.33	5.67	
s^2	−2.3	1	
s^1	6.68		

Since there is –ve sign in the first column, the system is unstable.

PROBLEM 15.6

Draw the pole zero diagram for the given network function and hence, obtain v(t).

$$V(s) = \frac{4(s+2)s}{(s+1)(s+3)}$$

Solution In the network function,

$$p(s) = 4s(s + 2)$$

and $Q(s) = (s + 1)(s + 3) = 0$

By taking partial fractions, we have

$$V(s) = \frac{k_1}{s+1} + \frac{k_2}{s+3}$$

The time-domain response can be obtained by taking the inverse transform

$v(t) = k_1 e^{-t} + k_2 e^{-3t}$

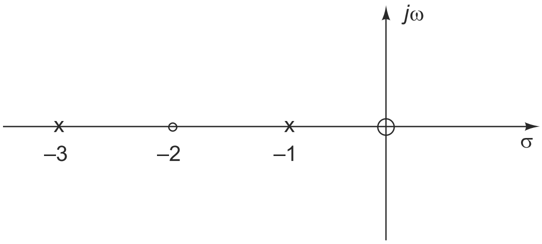

Fig. 15.40

Here, the coefficients k_1 and k_2 may be determined by using the pole zero plot as shown in Fig. 15.40.

To determine k_1, we have to find out the distances and phase angles from other zeros and poles to that particular pole.

Hence, $k_1 = H \dfrac{M_{01} M_{21} e^{j(\phi_{01} + \phi_{21})}}{M_{31} e^{j(\phi_{31})}}$

where M_{01} and M_{21} are the distances between the zeros at 0 and –2 to the pole at –1, ϕ_{01}, ϕ_{21} are the phase angle between the corresponding zeros to the pole.

Similarly, M_{31} and ϕ_{31} are the distance and phase angle, respectively, from the pole at –3 to the pole at –1.

$$\therefore \quad M_{01} = 1;\ \phi_{01} = 180°$$

$$M_{21} = 1;\ \phi_{21} = 0$$

$$M_{31} = 2;\ \phi_{31} = 0°$$

$$\therefore \quad k_1 = 4 \times \frac{1 \times 1}{2} e^{j(180°)}$$

$$k_1 = -2$$

Similarly,

$$k_2 = H \frac{M_{03} M_{23}}{M_{13}} e^{+j(\phi_{03} + \phi_{23} - \phi_{13})}$$

where $M_{03} = 3,\ \phi_{03} = 180°$

$$M_{23} = 1,\ \phi_{23} = 180°$$

$$M_{13} = 2,\ \phi_{13} = 180°$$

$$\therefore \quad k_2 = \frac{4 \times 3 \times 1}{2} e^{j(180 + 180 - 180)}$$

$$k_2 = -6$$

Substituting the values, we get

$$(t) = (-2e^{-t} - 6e^{-3t})\text{V}$$

PROBLEM 15.7

For the given network function, draw the pole zero diagram and hence, obtain the time-domain response i(t).

$$I(s) = \frac{5s}{(s+1)(s^2+4s+8)}$$

Solution In the network function,

$$P(s) = 5s$$

$$Q(s) = (s+1)(s^2+4s+8) = 0$$

By taking the partial fraction expansion of $I(s)$, we get

$$I(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2+j2)} + \frac{k_3}{(s+2-j2} \tag{15.89}$$

The time-domain response can be obtained by taking the inverse transform as under,

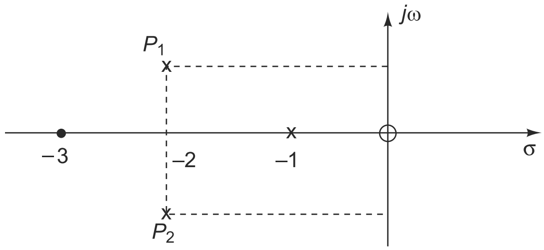

Fig. 15.41

$$i(t) = k_1 e^{-t} + k_2\, e^{-(2+j2)t} + k_2\, e^{-(2-j2)t} \tag{15.90}$$

To find the value of k_1, we have to find out the distances and phase angles from other zeros and poles to that particular pole as shown in Fig. 15.41.

Hence, $$k_1 = \frac{H\,M_{01} e^{j(\phi_{01})}}{M_{p11} M_{p21} e^{j[\phi_{p11}+\phi_{p21}]}}$$

$$M_{01} = 1;\ \phi_{01} = 180°$$

$$M_{p11} = \sqrt{5}\,;\ \phi_{p11} = -63.44°$$

$$M_{p21} = \sqrt{5}\,;\ \phi_{p21} = 63.44°$$

$$\therefore \quad k_1 = \frac{5 \times 1 e^{j180°}}{\sqrt{5} \times \sqrt{5} e^{j(-63.44°+63.44°)}}$$

$$k_1 = -1$$

Similarly, $$k_2 = \frac{HM_{0p_1} e^{j\phi_{0p_1}}}{M_{1p_1} M_{p_2p_1} e^{j(\phi_{1p_1}+\phi_{p_2p_1})}}$$

$$M_{0p_1} = \sqrt{8}\,;\ \phi_{0p_1} = 135°$$

$$M_{1p_1} = \sqrt{5}\,;\ \phi_{1p_1} = 116.56°$$

$$M_{p_1}p_2 = 4;\ \phi_{p_2}p_1 = 90°$$

Hence, $$k_2 = \frac{5 \times \sqrt{8}}{\sqrt{5} \times 4} e^{j(135°-116.56°-90°)}$$

$$= 1.58\, e^{-j(71.56°)}$$

$$k_2^* = \frac{H\, M_{0p_2} e^{j\phi_{0p_2}}}{M_{1p_2} M_{p_1p_2} e^{j(\phi_{1p_2} + \phi_{p_1p_2})}}$$

$$= \frac{5 \times \sqrt{8}\, e^{-j(135°)}}{\sqrt{5} \times 4 e^{j(-116.56° - 90°)}}$$

$$= 1.58 e^{j71.56°}$$

If we substitute the values in Eq. (15.90), we get

$$i(t) = [-1e^{-t} + 1.58\, e^{-j(71.56°)}\, e^{-(2\,+\,j2)t} + 1.58 e^{j(71.56°)}\, e^{-(2\,-\,j2)t}]\,\text{A}$$

PROBLEM 15.8

For the given denominator polynomial of a network function, verify the stability of the network by using the Routh criterion.

$$Q(s) = s^3 + 2s^2 + 8s + 10$$

Solution The Routh array for this polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 2 & 10 \\ s^1 & 3 & \\ s^0 & 10 & \end{array}$$

There is no change in sign in the first column of the array. Hence, there are no roots with positive real parts. Therefore, the network is stable.

PROBLEM 15.9

For the given denominator polynomial of a network function, verify the stability of the network using the Routh criterion.

$$Q(s) = s^3 + s^2 + 3s + 8$$

Solution The Routh array for this polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 1 & 8 \\ s^1 & -5 & \\ s^0 & +8 & \end{array}$$

There are two changes in sign of the first column, one from 1 to –5 and the other from –5 to +8. Therefore, the two roots have positive real parts. Hence, the network is not stable.

PROBLEM 15.10

For the given denominator polynomial of a network function, determine the value of k for which the network to stable.

$$Q(s) = s^3 + 2s^2 + 4s + k$$

Solution The Routh array for the given polynomial is given below.

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & k \\ s^1 & \dfrac{8-k}{2} & \\ s^0 & k & \end{array}$$

When $k < 8$, all the terms in the first column are positive. Therefore, there is no sign change in the first column. Hence, the network is stable. When $k > 8$, the $8 - k/2$ is negative. Therefore, there are two sign changes in the first column. There are two roots which have positive real parts. Hence, the network is unstable.

When $k = 8$, the Routh array becomes

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & 8 \\ s^1 & \alpha & \\ s^0 & 8 & \end{array}$$

The element in the first column and third row is zero. But we can take it as a small number. In this case, there are no changes in the sign of the first column. Hence, the network is stable.

PROBLEM 15.11

Apply the Routh criterion to the given polynomial and determine the number of roots (a) with positive real parts, (b) with zero real parts, and (c) with negative real parts.

$$Q(s) = s^4 + 4s^3 + 8s^2 + 12s + 15$$

Solution The Routh array for the polynomial is

$$\begin{array}{c|ccc} s^4 & 1 & 8 & 15 \\ s^3 & 4 & 12 & \\ s^2 & 5 & 15 & \\ s^1 & 0 & 0 & \\ s^0 & ? & ? & \end{array}$$

In this case, all the elements in the fourth row have become zero and the array cannot be completed.

The given equation is reduced by taking the new polynomial from the third row

$$5s^2 + 15 = 0$$

$$5(s^2 + 3) = 0$$

Hence, the other polynomial

$$Q_2(s) = \frac{s^4 + 4s^3 + 8s^2 + 12s + 15}{5(s^2 + 3)}$$

The equation reduces to the following polynomial.

$$(s^2 + 3)\,(s^2 + 4s + 5) = 0$$

The roots of the equation $s^2 + 3 = 0$ are $s = \pm j\sqrt{3}$
These two roots have zero real parts.
Again, forming the Routh array for the polynomial,

$$s^2 + 4s + 5 = 0$$

$$\begin{array}{c|cc} s^2 & 1 & 5 \\ s^1 & 4 & 0 \\ s^0 & 5 & \end{array}$$

There are no changes in the sign of the first column. Hence, all the roots have negative real parts. Therefore, out of the four roots, two roots have negative real parts and two roots have zero real parts.

ANSWERS TO PRACTICE PROBLEMS

15-4.1 $Z(s) = \dfrac{2s^3 + 3s^2 + 2s + 1}{s^3 + 2s^2 + 3s + 2}$

$Y(s) = \dfrac{s^3 + 2s^2 + 3s + 2}{2s^3 + 3s^2 + 2s + 1}$

15-4.2 $G_{21}(s) = \dfrac{s^2 + 2s + 1}{s^2 + 3s + 1}$

15-4.3 $z_d(s) = \dfrac{10s^2 + 27s + 30}{5s + 4}$

15-5.1 $G_{21}(S) = \dfrac{2s}{7s^2 + 7s + 5}$

15-5.2 The zeros are lying at $s = -0.5$ and $s = -0.04$

15-5.3 $v(t) = -8e^{-2t} + 12e^{-3t}$

15-5.4 $1 - \dfrac{1}{3}e^{-t/3} + \dfrac{2}{15}e^{-(t-3)/5}$

15-7.1 The poles are lying at $s = -0.98$ and $s = -0.02$

15-8.2 $i(t) = 4.5e^{-3t} - 1.5e^{-t}$

Objective-Type Questions

☆☆★ **15.1** The driving-point impedance is defined as
(a) the ratio of transform voltage to transform current at the same port
(b) the ratio of transform voltage at one port to the transform current at the other port
(c) both (a) and (b)
(d) none of the above

☆☆★ **15.2** The transfer impedance is defined as
(a) the ratio of transform voltage to transform current at the same port
(b) the ratio of transform voltage at one port to the current transform at the other port
(c) both (a) and (b)
(d) none of the above

☆☆★ **15.3** The function is said to be having simple poles and zeros only if
(a) the poles are not repeated
(b) the zeros are not repeated
(c) both poles and zeros are not repeated
(d) none of the above

☆☆★ **15.4** The necessary condition for a driving-point function is

(a) the real part of all poles and zeros must not be zero or negative
(b) the polynomials $P(s)$ and $Q(s)$ may not have any missing terms between the highest and lowest degree unless all even or all odd terms are missing
(c) the degree of $P(s)$ and $Q(s)$ may differ by more than one
(d) the lowest degree in $P(s)$ and $Q(s)$ may differ in degree by more than two

☆☆★ **15.5** The necessary condition for the transfer functions is that

(a) the coefficients in the polynomials $P(s)$ and $Q(s)$ must be real
(b) coefficients in $Q(s)$ may be negative
(c) complex or imaginary poles and zeros may not conjugate
(d) if the real part of pole is zero, then that pole must be multiple

☆☆★ **15.6** The system is said to be stable, if and only if

(a) all the poles lie on the right half of the s-plane
(b) some poles lie on the right half of the s-plane
(c) all the poles does not lie on the right half of the s-plane
(d) none of the above

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CHAPTER 16

Two-Port Networks

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Explain two-port networks
- LO 2 Derive Z- and Y-parameters of a two-port network and also draw equivalent circuits in terms of Z- and Y-parameters
- LO 3 Derive transmission and inverse transmission parameters
- LO 4 Derive hybrid and inverse-hybrid parameters
- LO 5 Determine the inter-relationship between different parameters sets of a two-port network
- LO 6 Discuss interconnected two-port networks
- LO 7 Explain T and π representation of a two-port network
- LO 8 Examine the driving-point impedance at the input and at the output of a terminated network
- LO 9 Describe lattice or bridge networks
- LO 10 Determine the image parameters of a two-port network

16.1 TWO-PORT NETWORK

LO 1 Explain two-port networks

Generally, any network may be represented schematically by a rectangular box. A network may be used for representing either source or load, or for a variety of purposes. A pair of terminals at which a signal may enter or leave a network is called a port. *A **port** is defined as any pair of terminals into which energy is supplied, or from which energy is withdrawn, or where the network variables may be measured.* One such network having only one pair of terminals (1-1′) is shown in Fig. 16.1 (a).

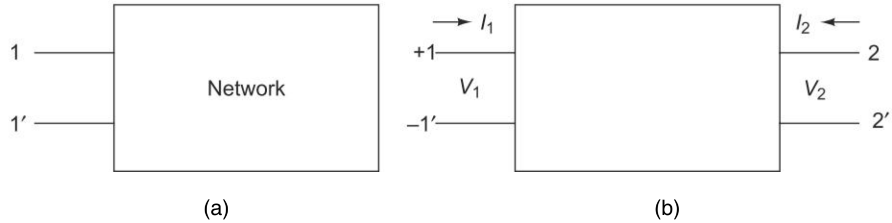


Fig. 16.1

*A **two-port network** is simply a network inside a black box, and the network has only two pairs of accessible terminals; usually one pair represents the input and the other represents the output.* Such a building block is very common in electronic systems, communication systems, transmission, and distribution systems. Figure 16.1(b) shows a two-port network, or a two terminal pair network, in which the four terminals have been paired into ports 1-1′ and 2-2′. The terminals 1-1′ together constitute a port. Similarly, the terminals 2-2′ constitute another port. Two ports containing no sources in their branches are called *passive ports*; among them are power transmission lines and transformers. Two ports containing sources in their branches are called *active ports*. A voltage and current assigned to each of the two ports. The voltage and current at the input terminals are V_1 and I_1; whereas V_2 and I_2 are specified at the output port. It is also assumed that the currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2, respectively. The variables of the two-port network are V_1, V_2, and I_1, I_2. Two of these are dependent variables, the other two are independent variables. The number of possible combinations generated by the four variables, taken two at a time, is six. Thus, there are six possible sets of equations describing a two-port network.

16.2 OPEN-CIRCUIT IMPEDANCE (*Z*) PARAMETERS

LO 2 Derive Z- and Y-parameters of a two-port network and also draw equivalent circuits in terms of Z- and Y-parameters

A general linear two-port network defined in Section 16.1 which does not contain any independent sources is shown in Fig. 16.2.

*The **Z-parameters** of a two-port for the positive directions of voltages and currents may be defined by expressing the port voltages V_1 and V_2 in terms of the currents I_1 and I_2. Here, V_1 and V_2 are dependent variables, and I_1, I_2 are independent variables. The voltage at port 1-1′ is the response produced by the two currents I_1 and I_2. Thus,*

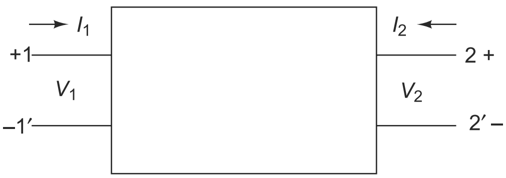


Fig. 16.2

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \qquad (16.1)$$

Similarly, $$V_2 = Z_{21} I_1 + Z_{22} I_2 \qquad (16.2)$$

Z_{11}, Z_{12}, Z_{21}, and Z_{22} are the network functions, and are called impedance (Z) parameters, and are defined by Eqs (16.1) and (16.2). These parameters can be represented by matrices.

We may write the matrix equation as $[V] = [Z]\,[I]$

where V is the column matrix $= \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Z is the square matrix $= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$

and we may write $|I|$ in the column matrix $= \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

Thus, $$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The individual Z-parameters for a given network can be defined by setting each of the port currents equal to zero. Suppose the port 2-2′ is left open-circuited, then $I_2 = 0$.

Thus, $$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

where Z_{11} is the driving-point impedance at the port 1-1′ with the port 2-2′ open circuited. It is called the *open-circuit input impedance*.

Similarly, $$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

where Z_{21} is the transfer impedance at the port 1-1′ with the port 2-2′ open-circuited. It is also called the *open-circuit forward transfer impedance*. Suppose the port 1-1′ is left open circuited, then $I_1 = 0$.

Thus, $$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

where Z_{12} is the transfer impedance at the port 2-2′, with the port 1-1′ open- circuited. It is also called the *open-circuit reverse transfer impedance.*

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

where Z_{22} is the open-circuit driving-point impedance at the port 2-2′ with the port 1-1′ open circuited. It is also called the *open-circuit output impedance*. The equivalent circuit of the two-port networks governed by Eqs (16.1) and (16.2), i.e. open-circuit impedance parameters is shown in Fig. 16.3.

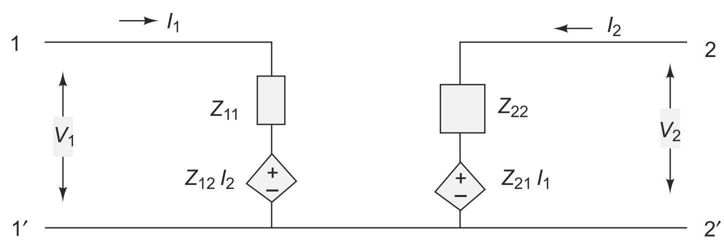


Fig. 16.3

If the network under study is reciprocal or bilateral, then in accordance with the reciprocity principle,

$$\left.\frac{V_2}{I_1}\right|_{I_2=0} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

or $$Z_{21} = Z_{12}$$

It is observed that all the parameters have the dimensions of impedance. Moreover, individual parameters are specified only when the current in one of the ports is zero. This corresponds to one of the ports being open-circuited from which the Z-parameters also derive the name *open-circuit impedance parameters*.

EXAMPLE 16.1

Find the Z-parameters for the circuit shown in Fig. 16.4.

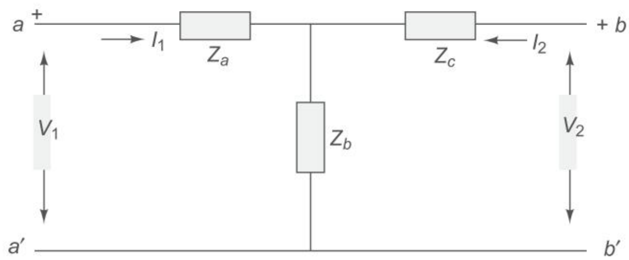


Fig. 16.4

Solution The circuit in the problem is a *T*-network. From Eqs (16.1) and (16.2) we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

When the port *b*-*b*′ is open-circuited, $Z_{11} = \dfrac{V_1}{I_1}$

where $V_1 = I_1(Z_a + Z_b)$

$$\therefore \quad Z_{11} = (Z_a + Z_b)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

where $V_2 = I_1 Z_b$

$$\therefore \quad Z_{21} = Z_b$$

When the port *a*-*a*′ is open-circuited, $I_1 = 0$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

where $V_2 = I_2(Z_b + Z_c)$

$$\therefore \quad Z_{22} = (Z_b + Z_c)$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

where $V_1 = I_2 Z_b$

$$\therefore \quad Z_{12} = Z_b$$

It can be observed that $Z_{12} = Z_{21}$, so the network is a bilateral network which satisfies the principle of reciprocity.

16.3 SHORT-CIRCUIT ADMITTANCE (Y) PARAMETERS

LO 2

A general two-port network which is considered in Section 16.2 is shown in Fig. 16.5.

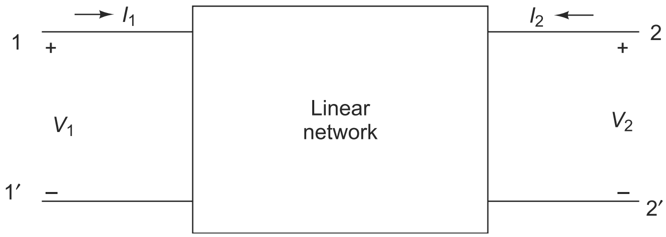


Fig. 16.5

*The **Y-parameters** of a two-port network for the positive directions of voltages and currents may be defined by expressing the port currents I_1 and I_2 in terms of the voltages V_1 and V_2. Here, I_1, I_2 are dependent variables and V_1 and V_2 are independent variables. I_1 may be considered to be the superposition of two components, one caused by V_1 and the other by V_2.*

Thus,

$$I_1 = Y_{11} V_1 + Y_{12} V_2 \tag{16.3}$$

Similarly, $$I_2 = Y_{21} V_1 + Y_{22} V_2 \tag{16.4}$$

Y_{11}, Y_{12}, Y_{21}, and Y_{22} are the network functions and are also called the admittance (Y) parameters. They are defined by Eqs (16.3) and (16.4). These parameters can be represented by matrices as follows:

$$[I] = [Y]\,[V]$$

where $$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix};\ Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

and $$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus, $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The individual Y-parameters for a given network can be defined by setting each port voltage to zero. If we let V_2 be zero by short-circuiting the port 2-2′, then

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

Y_{11} is the driving-point admittance at the port 1-1′, with the port 2-2′ short- circuited. It is also called the *short-circuit input admittance*.

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

Y_{21} is the transfer admittance at the port 1-1′ with the port 2-2′ short-circuited. It is also called *short-circuited forward transfer admittance*. If we let V_1 be zero by short- circuiting the port 1-1′, then

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

Y_{12} is the transfer admittance at the port 2-2′ with the port 1-1′ short-circuited. It is also called the *short-circuit reverse transfer admittance*.

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

Y_{22} is the short-circuit driving-point admittance at the port 2-2′ with the port 1-1′ short circuited. It is also called the *short-circuit output admittance*. The equivalent circuit of the network governed by Eqs (16.3) and (16.4) is shown in Fig . 16.6.

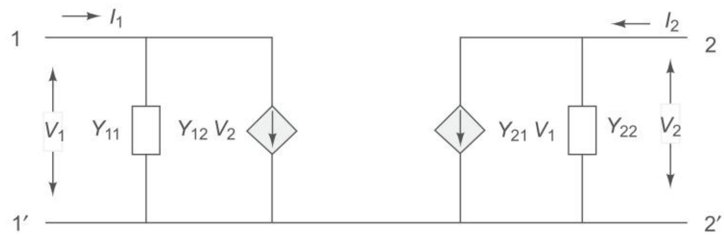


Fig. 16.6

If the network under study is reciprocal, or bilateral, then

$$\left.\frac{I_1}{V_2}\right|_{V_1=0} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

or $\qquad Y_{12} = Y_{21}$

It is observed that all the parameters have the dimensions of admittance which are obtained by short-circuiting either the output or the input port from which the parameters also derive their name, i.e. the *short-circuit admittance parameters*.

EXAMPLE 16.2

Find the Y-parameters for the network shown in Fig. 16.7.

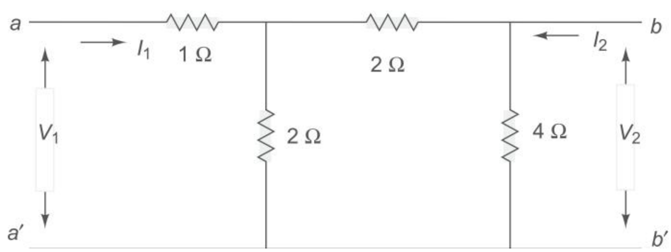


Fig. 16.7

Solution $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$

When b-b' is short-circuited, $V_2 = 0$ and the network looks as shown in Fig. 16.8 (a).

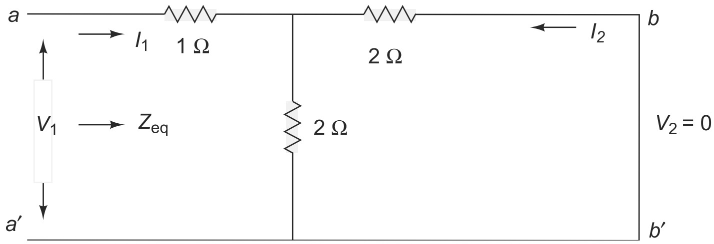


Fig. 16.8 (a)

$$V_1 = I_1 Z_{eq}$$

$$Z_{eq} = 2\ \Omega$$

$$\therefore \quad V_1 = I_1\, 2$$

$$Y_{11} = \frac{I_1}{V_1} = \frac{1}{2}\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_2}\right|_{V_2=0}$$

With the port b-b' short-circuited, $-I_2 = I_1 \times \frac{2}{4} = \frac{I_1}{2}$

$$\therefore \quad -I_2 = \frac{V_1}{4}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{1}{4}\ \mho$$

Similarly, when the port a-a' is short circuited, $V_1 = 0$ and the network looks as shown in Fig. 16.8 (b).

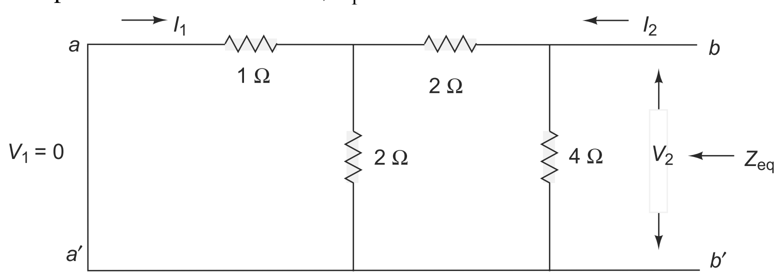


Fig. 16.8 (b)

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

$$V_2 = I_2 Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from b-b'.

$$Z_{eq} = \frac{8}{5}\ \Omega$$

$$V_2 = I_2 \times \frac{8}{5}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{5}{8}\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

With a-a' short-circuited, $-I_1 = \frac{2}{5} I_2$

Since $I_2 = \frac{5V_2}{8}$

$$-I_1 = \frac{2}{5} \times \frac{5}{8} V_2 = \frac{V_2}{4}$$

$$\therefore \quad Y_{12} = \frac{I_1}{V_2} = -\frac{1}{4}\ \mho$$

The describing equations in terms of the admittance parameters are

$$I_1 = 0.5\ V_1 - 0.25\ V_2$$

$$I_2 = -\,0.25\ V_1 + 0.625\ V_2$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2*

☆☆★ **16-2.1** Find the Z-parameters of the network shown in Fig. Q.1.

☆☆★ **16-2.2** Determine the impedance parameters for the T-network shown in Fig. Q.2 and draw the Z-parameter equivalent circuit.

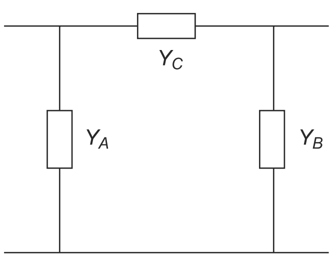


Fig. Q.1

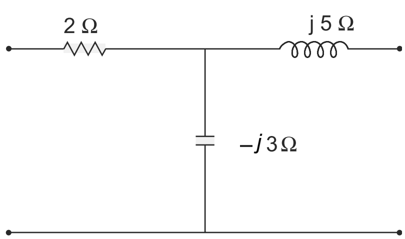


Fig. Q.2

*Note: ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★ - Level 5 and Level 6 Category

☆★★ **16-2.3** Determine the input and output impedances for the Z-parameter equivalent circuit shown in Fig. Q.3.

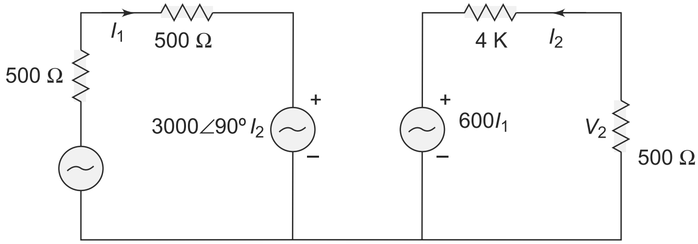


Fig. Q.3

☆☆★ **16-2.4** The Z-parameters of a two-port network shown in Fig. Q.4 are $Z_{11} = 5\ \Omega$; $Z_{12} = 4\ \Omega$; $Z_{22} = 10\ \Omega$; $Z_{21} = 5\ \Omega$. If the source voltage is 25 V, determine I_1, $V_2\,I_2$, and the driving-point impedance at the input port.

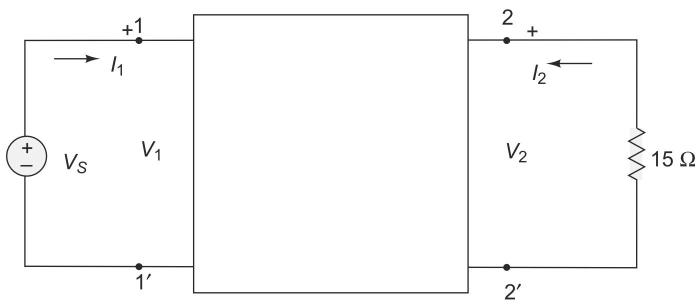


Fig. Q.4

★★★**16-2.5** For the network shown in Fig. Q.5, determine all four open-circuit impedance parameters.

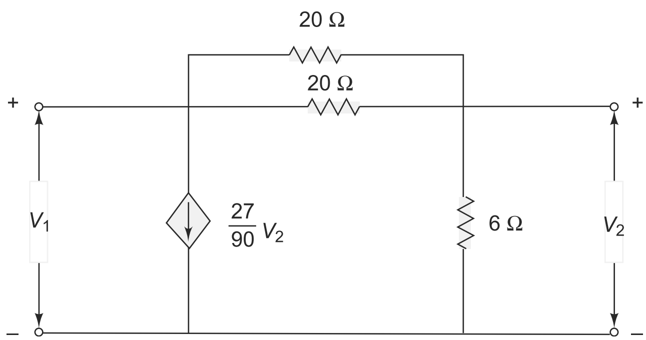


Fig. Q.5

☆☆★ **16-2.6** Find the inverse transmission parameters for the network in Fig. Q.6.

☆☆★ **16-2.7** Determine the admittance parameters for the π-network shown in Fig. Q.7 and draw the Y-parameter equivalent circuit.

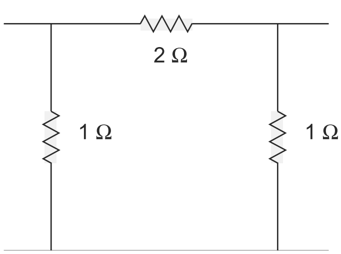


Fig. Q.6

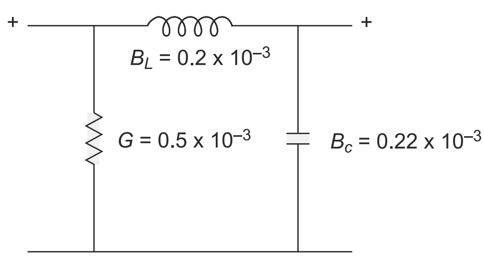


Fig. Q.7

☆★★**16-2.8** For the network shown in Fig. Q.8, determine y_{12} and y_{21}.

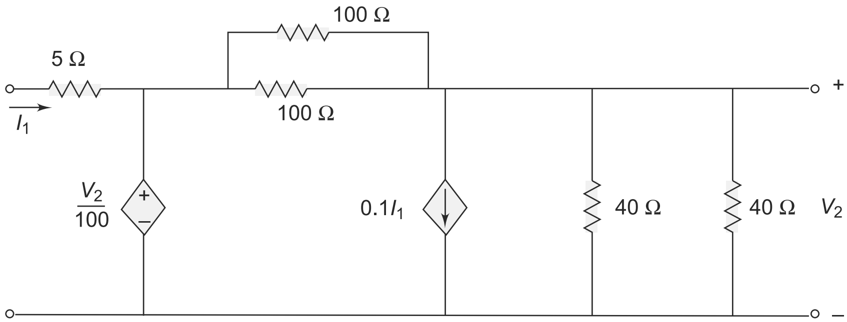


Fig. Q.8

★★★**16-2.9** For the network shown in Fig. Q.9, determine *Y*-parameters.

☆★★**16-2.10** Using PSpice, obtain γ-parameters of the two-port network shown in Fig. Q.10.

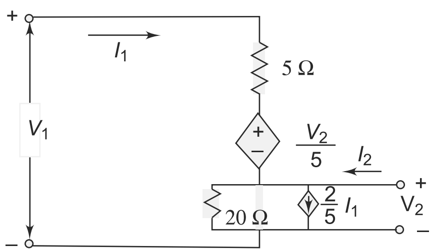


Fig. Q.9

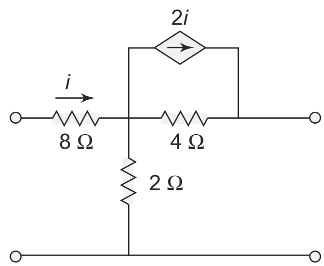


Fig. Q.10

☆★★**16-2.11** Find *Y*-parameters of the network shown in Fig. Q.11.

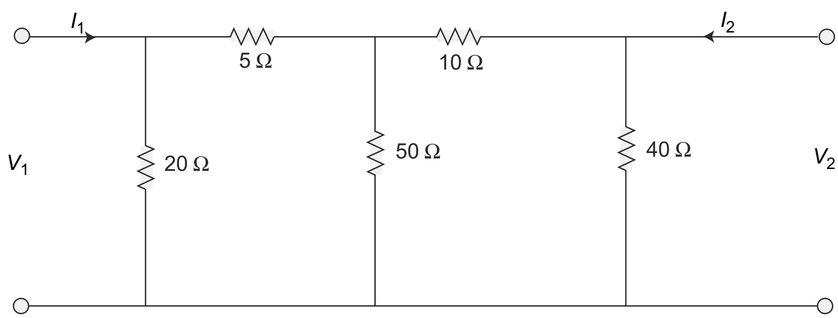


Fig. Q.11

☆☆★**16-2.12** Find *Z*- and *Y*-parameters of the given π-network. (Fig. Q. 12)

☆★★**16-2.13** Find the *Y*-parameters of the two-port network shown in Fig. Q.13.

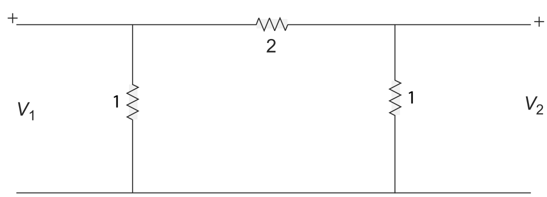


Fig. Q.12

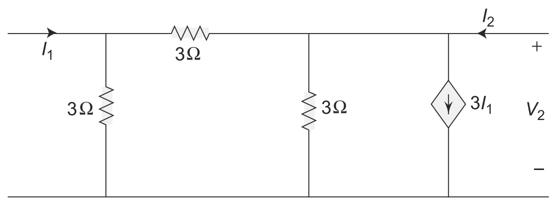


Fig. Q.13

Frequently Asked Questions linked to LO 2*

☆★★ **16-2.1** Find the network function $Z_{11}(s)$ and $Z_{21}(s)$ for the network shown in Fig. Q.1. **[PU 2010]**

☆★★ **16-2.2** Obtain the Y-parameters of the network shown in Fig. Q.2. **[PU 2010]**

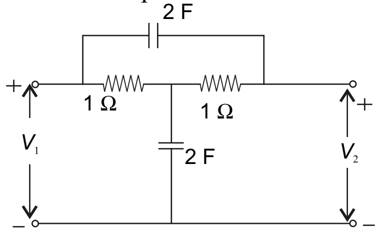


Fig. Q.1

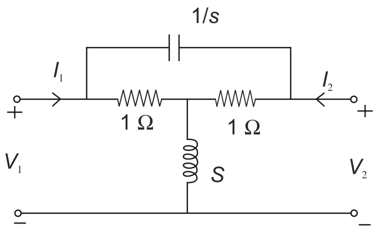


Fig. Q.2

☆☆★ **16-2.3** Find the Z-parameters for the network shown in Fig. Q.3. **[BPUT 2007]**

☆☆★ **16-2.4** Why are Z-parameters known as open-circuit parameters? **[BPTU 2008]**

☆☆★ **16-2.5** A two-port device is defined by the following pair of equations: $2V_1 + V_2$ and $i_2 = V_1 + V_2$. Write its impedance parameters $Z_{11}, Z_{12}, Z_{21}, Z_{22}$,. **[BPTU 2008]**

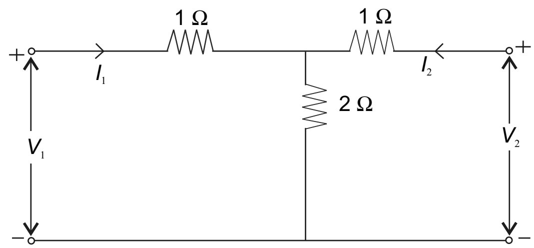


Fig. Q.3

☆☆★ **16-2.6** Explain reciprocal. **[GTU Dec. 2010]**

☆★★ **16-2.7** Find the Z-parameters for the network shown in Fig. Q.7. **[GTU Dec. 2010]**

☆★★ **16-2.8** Obtain Z-parameters and transmission parameters of the network shown in Fig. Q.8. **[JNTU Nov. 2012]**

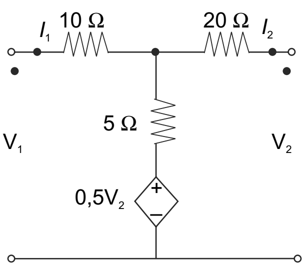


Fig. Q.7

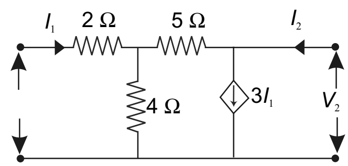


Fig. Q.8

☆☆★ **16-2.9** Obtain transmission parameters in terms of Z-parameters. **[MU 2014]**

☆★★ **16-2.10** Determine the expression for Z-parameters of lattice networks. **[PTU 2011-12]**

☆☆★ **16-2.11** Find the Z-parameters for the network shown in Fig. Q.11. **[PU 2010]**

☆★★ **16-2.12** Find the Z-parameters for the network shown in Fig. Q.12. **[RTU Feb. 2011]**

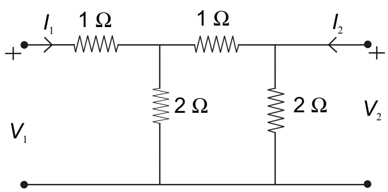


Fig. Q.11

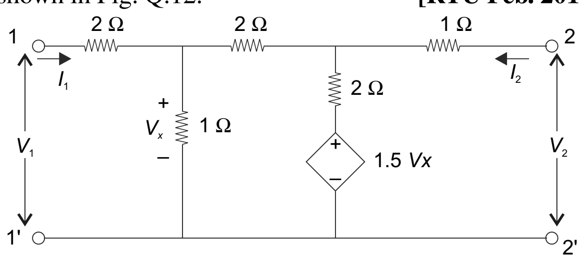


Fig. Q.12

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

☆★★ **16-2.13** What is a reciprocal network? Derive the condition for reciprocity in terms of *Z*-parameters. **[RGTU Dec. 2013]**

☆☆★ **16-2.14** Find the *Z*-parameters for the circuit shown in the Fig. Q.14. **[RGTU Dec. 2013]**

☆☆★ **16-2.15** Find the *Z*-parameters for the network given in Fig. Q.15. **[RTU Feb. 2011]**

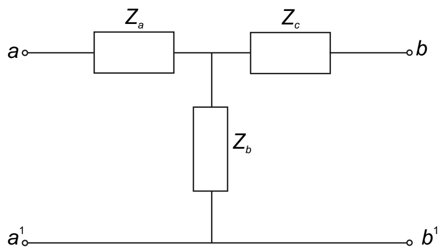


Fig. Q.14

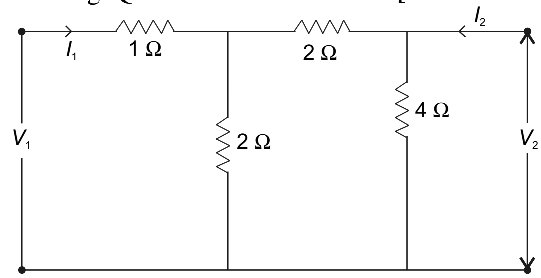


Fig. Q.15

☆☆★ **16-2.16** Find $Y_{11}(s)$ of the circuit shown in Fig. Q.16 **[BPUT 2007]**

☆★★ **16-2.17** Determine the *Y*-parameters of the given network shown in Fig. Q.17 **[BPUT 2008]**

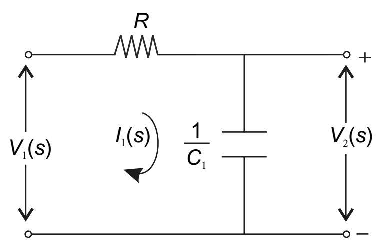


Fig. Q.16

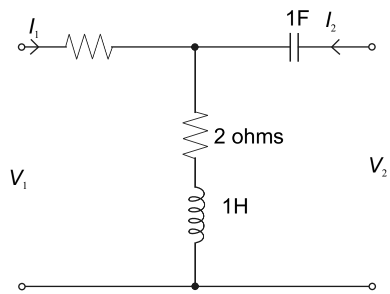


Fig. Q.17

☆☆★ **16-2.18** Determine the voltage across the capacitor in the *RLC* circuit as shown in Fig. Q.18, if $R = 400$ ohms using Laplace transform. **[GTU May 2011]**

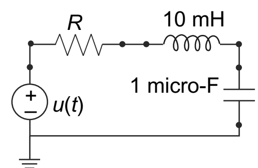


Fig. Q.18

☆☆★ **16-2.19** Explain the short-circuit admittance and the open-circuit impedance parameters for a two-port network. **[GTU May 2011]**

☆★★ **16-2.20** Find the open-circuit impedance parameters of the circuit shown in Fig. Q.20. Also find the *Y*-parameters. **[JNTU Nov. 2012]**

☆★★ **16-2.21** Determine *Y*-parameters of the network shown in Fig. Q.21. **[JNTU Nov. 2012]**

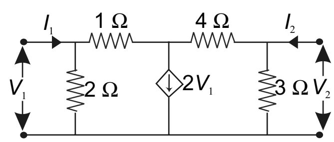


Fig. Q.20

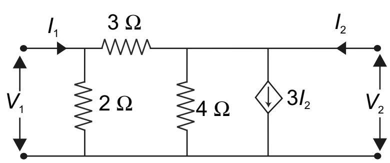


Fig. Q.21

☆★★ **16-2.22** Find the *Y* and *Z* parameters of the network in Fig. Q.22 **[PTU 2009-10]**

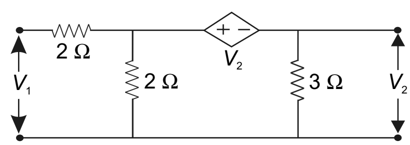


Fig. Q.22

☆☆★ **16-2.23** Derive the condition of reciprocity and symmetry for *y*-parameters. **[PU 2010]**

☆☆★ **16-2.24** Obtain the reciprocity and symmetry conditions for *Z*-and *Y*-parameters. **[PU 2012]**

☆★★ **16-2.25** The network shown in Fig. Q.25 contains a current-controlled current source. For this network find the Y-parameters. **[RGTU Dec. 2012]**

☆☆★ **16-2.26** Find the Y-parameters for the network of Fig. Q.26. **[RTU Feb. 2011]**

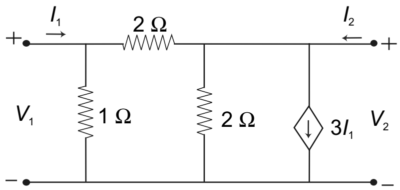


Fig. Q.25

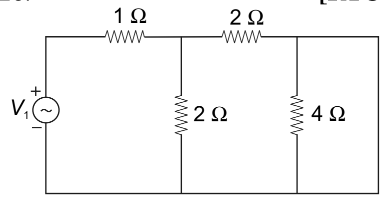


Fig. Q.26

16.4 TRANSMISSION (*ABCD*) PARAMETERS

LO 3 Derive transmission and inverse transmission parameters

Transmission parameters, or *ABCD* parameters, are widely used in transmission-line theory and cascade networks. In describing the transmission parameters, the input variables V_1 and I_1 at the port 1-1′, usually called the *sending end*, are expressed in terms of the output variables V_2 and I_2 at the port 2-2′, called the *receiving end*. The transmission parameters provide a direct relationship between input and output. Transmission parameters are also called *general circuit parameters*, or *chain parameters*. They are defined by

$$V_1 = AV_2 - BI_2 \tag{16.5}$$

$$I_1 = CV_2 - DI_2 \tag{16.6}$$

The negative sign is used with I_2, and not for the parameters B and D. Both the port currents I_1 and $-I_2$ are directed to the right, i.e. with a negative sign in Eqs (16.5) and (16.6), the current at the port 2-2′ which leaves the port is designated as positive. The parameters A, B, C and D are called the *transmission parameters*. In the matrix form, Eqs (16.5) and (16.6) are expressed as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called the *transmission matrix*.

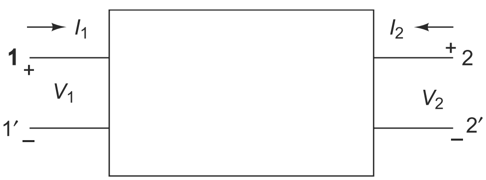


Fig. 16.9

For a given network, these parameters can be determined as follows. With the port 2-2′ open, i.e. $I_2 = 0$; applying a voltage V_1 at the port 1-1′, using Eq. (16.5), we have

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} \quad \text{and} \quad C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

$$\frac{1}{A} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = g_{21}\big|_{I_2=0}$$

$1/A$ is called the open-circuit voltage gain, a dimensionless parameter. And $\frac{1}{C} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_{21}$, which is the open-circuit transfer impedance. With the port 2-2′ short circuited, i.e. with $V_2 = 0$, applying the voltage V_1 at the port 1-1′, from Eq. (16.6), we have

$$-B = \left.\frac{V_1}{I_2}\right|_{V_2=0} \quad \text{and} \quad -D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

$$-\frac{1}{B} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = Y_{21}, \quad \text{which is the short-circuit transfer admittance}$$

$$-\frac{1}{D} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \left.\alpha_{21}\right|_{V_2=0}, \quad \text{which is the short-circuit current gain, a dimensionless parameter.}$$

16.4.1 Cascade Connection

The main use of the transmission matrix is in dealing with a cascade connection of two-port networks as shown in Fig. 16.10.

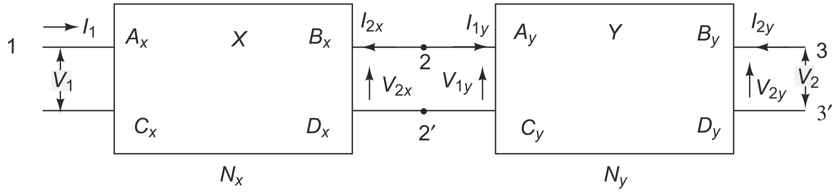


Fig. 16.10

Let us consider two two-port networks N_x and N_y connected in cascade with port voltages and currents as indicated in Fig. 16.10. The matrix representation of *ABCD* parameters for the network *X* is as under.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} V_{2x} \\ -I_{2x} \end{bmatrix}$$

And for the network *Y*, the matrix representation is

$$\begin{bmatrix} V_{1y} \\ I_{1y} \end{bmatrix} = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_{2y} \\ -I_{2y} \end{bmatrix}$$

It can also be observed that at 2-2′,

$$V_{2x} = V_{1y} \text{ and } I_{2x} = -I_{1y}.$$

Combining the results, we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_2 \\ -I_1 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the transmission-parameters matrix for the overall network.

Thus, the transmission matrix of a cascade of a two-port networks is the product of transmission matrices of the individual two-port networks. This property is used in the design of telephone systems, microwave networks, radars, etc.

EXAMPLE 16.3

Find the transmission or general circuit parameters for the circuit shown in Fig. 16.11.

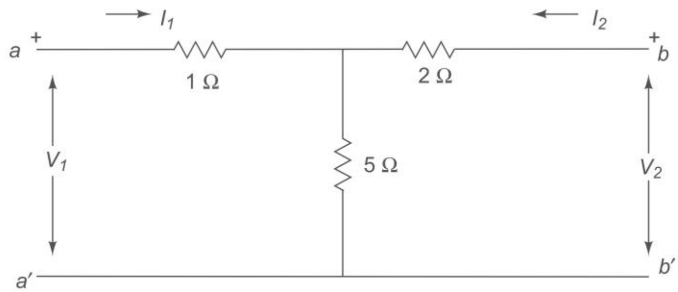


Fig. 16.11

Solution From Eqs (16.5) and (16.6) in Section 16.4, we have

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

When b-b' is open, $I_2 = 0; A = \left.\dfrac{V_1}{V_2}\right|_{I_2=0}$

where $V_1 = 6I_1$ and $V_2 = 5I_1$

$$\therefore \qquad A = \frac{6}{5}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{5}\ \mho$$

When b-b' is short-circuited; $V_2 = 0$ (see Fig. 16.12)

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0}; D = \left.\frac{-I_1}{I_2}\right|_{V_2=0}$$

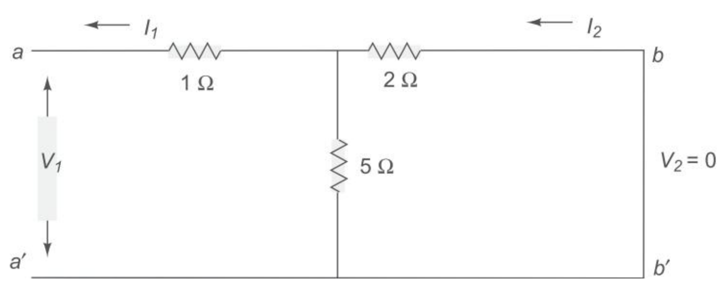


Fig. 16.12

In the circuit, $-I_2 = \dfrac{5}{17} V_1$

$$\therefore \quad B = \frac{17}{5}\ \Omega$$

Similarly, $I_1 = \dfrac{7}{17} V_1$ and $-I_2 = \dfrac{5}{17} V_1$

$$\therefore \quad D = \frac{7}{5}$$

16.5 INVERSE TRANSMISSION (A' B' C' D') PARAMETERS

LO 3

In the preceding section, the input port voltage and current are expressed in terms of output port voltage and current to describe the transmission parameters. While defining the transmission parameters, it is customary to designate the input port as the sending end and the output port as the receiving end. The voltage and current at the receiving end can also be expressed in terms of the sending end voltage and current. If the voltage and current at the port 2-2′ are expressed in terms of voltage and current at the port 1-1′, we may write the following equations.

$$V_2 = A'V_1 - B'I_1 \tag{16.7}$$

$$I_2 = C'V_1 - D'I_1 \tag{16.8}$$

The coefficients A', B', C', and D' in the above equations are called inverse transmission parameters. Because of the similarities of Eqs (16.7) and (16.8) with Eqs (16.5) and (16.6) in Section 16.4, the A', B', C', D' parameters have properties similar to $ABCD$ parameters. Thus, when the port 1-1′ is open, $I_1 = 0$.

Fig. 16.13

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0}; \quad C' = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

If the port 1-1′ is short-circuited, $V_1 = 0$

$$B' = \left.\frac{-V_2}{I_1}\right|_{V_1=0}; \quad D = \left.\frac{-I_2}{I_1}\right|_{V_1=0}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3

☆☆★ **16-3.1** Find the transmission parameters for the R–C network shown in Fig. Q.1.

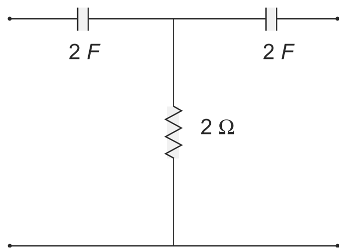


Fig. Q.1

☆★★**16-3.2** Calculate the overall transmission parameters for the cascaded network shown in Fig. Q.2.

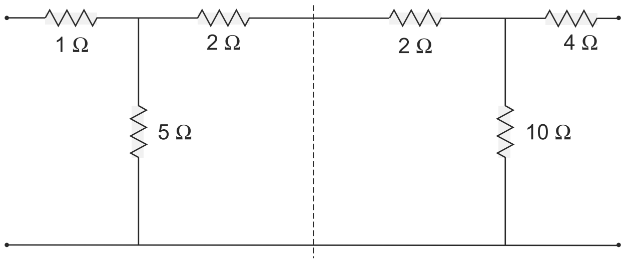


Fig. Q.2

☆★★**16-3.3** Determine the impedance parameters and the transmission parameters for the network in Fig. Q.3.

☆☆★**16-3.4** Using PSpice, find transmission parameters of the network shown in Fig. Q.4.

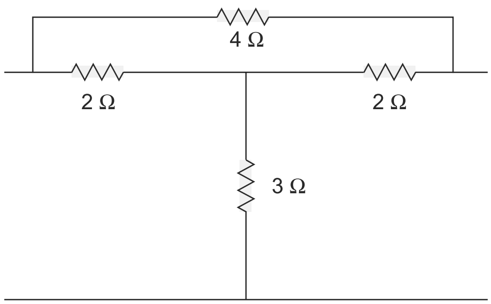


Fig. Q.3

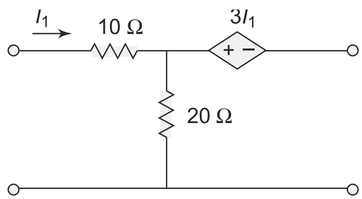


Fig. Q.4

Frequently Asked Questions linked to LO 3

☆☆★**16-3.1** *ABCD*-parameters are also known as transmission parameters and they are derived from the basic two-port network parameters. Show that, for reciprocal linear time invariant two-port network, *AD-BC* = 1. **[GTU Dec. 2010]**

☆★★**16-3.2** Find *ABCD*-parameters for the two-port network shown in Fig. Q.2. Also derive *Y*-parameters from the *ABCD*-parameters. **[GTU Dec. 2012]**

☆☆★**16-3.3** For the network shown in Fig. Q.3, find *ABCD*-parameters. **[JNTU Nov. 2012]**

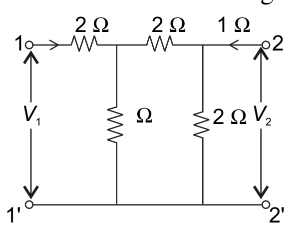


Fig. Q.2

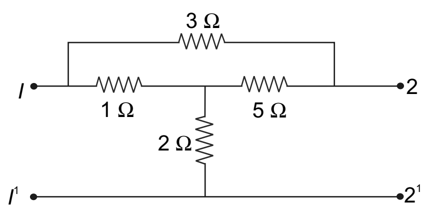


Fig. Q.3

☆☆★**16-3.4** Find transmission parameters for the network shown in Fig. Q.4. **[PU 2010]**

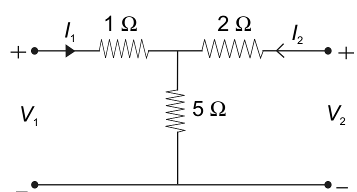


Fig. Q.4

☆☆★**16-3.5** Define *ABCD* parameters for a two-port network. **[RGTU June 2014]**

☆☆★**16-3-6** Find the *ABCD*-parameters of the network shown in Fig. Q.6. Also find the image parameters for the network.

[RTU Feb. 2011]

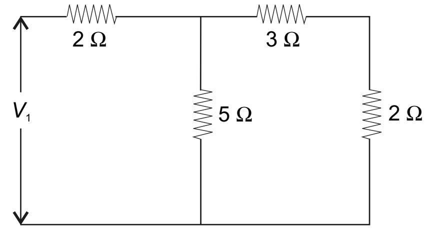


Fig. Q.6

16.6 HYBRID (H) PARAMETERS

LO 4 Derive hybrid and inverse-hybrid parameters

Hybrid parameters, or *h*-parameters find extensive use in transistor circuits. They are well suited to transistor circuits as these parameters can be most conveniently measured. The hybrid matrices describe a two-port network, when the voltage of one port and the current of other port are taken as the independent variables. Consider the network in Fig. 16.14.

If the voltage at the port 1-1′ and current at the port 2-2′ are taken as dependent variables, we can express them in terms of I_1 and V_2.

$$V_1 = h_{11} I_1 + h_{12} V_2 \tag{16.9}$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \tag{16.10}$$

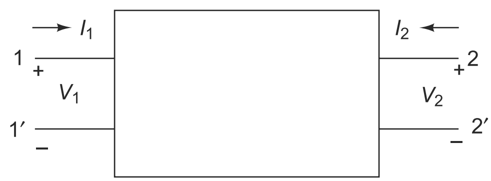


Fig. 16.14

The coefficients in the above equations are called hybrid parameters. In matrix notation,

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eqs (16.9) and (16.10), the individual *h*-parameters may be defined by letting $I_1 = 0$ and $V_2 = 0$.

When $V_2 = 0$, the port 2-2′ is short-circuited.

Then $h_{11} = \left.\dfrac{V_1}{I_1}\right|_{V_2=0}$ = Short-circuit input impedance = $\left(\dfrac{1}{Y_{11}}\right)$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \text{Short-circuit forward current gain} = \left(\frac{Y_{21}}{Y_{11}}\right)$$

Similarly, by letting port 1-1′ open, $I_1 = 0$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \quad \text{Open-circuit reverse voltage gain} = \left(\frac{Z_{12}}{Z_{22}}\right)$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \quad \text{Open-circuit output admittance} = \left(\frac{1}{Z_{22}}\right)$$

Since the *h*-parameters represent dimensionally an impedance, an admittance, a voltage gain, and a current gain, these are called hybrid parameters. An equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 16.15.

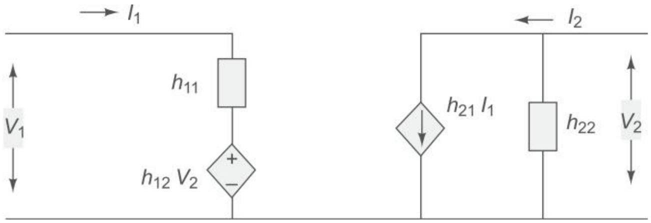


Fig. 16.15

EXAMPLE 16.4

Find the h-parameters of the network shown in Fig. 16.16.

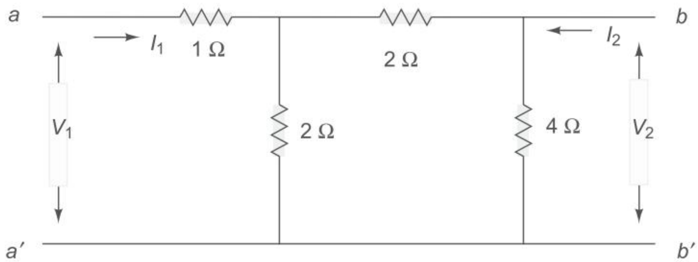


Fig. 16.16

Solution From Eqs (16.9) and (16.10), we have

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}; \; h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}; \; h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

If the port *b-b′* is short-circuited, $V_2 = 0$. The circuit is shown in Fig. 16.17 (a).

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; V_1 = I_1 Z_{eq}$$

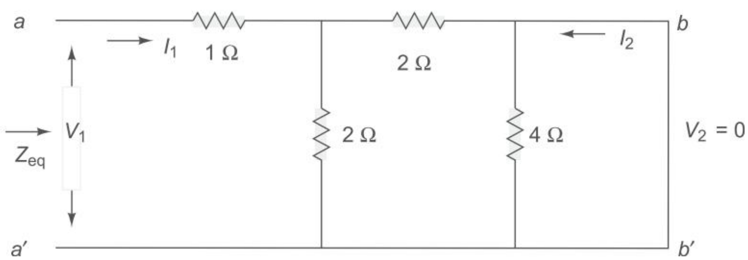


Fig. 16.17 (a)

The equivalent impedance as viewed from the port *a-a′* is 2 Ω.

$\therefore \quad V_1 = I_1\, 2\text{V}$

$$h_{11} = \frac{V_1}{I_1} = 2\ \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \quad \text{when} \quad V_2 = 0; -I_2 = \frac{I_1}{2}$$

$$\therefore \quad h_{21} = -\frac{1}{2}$$

If the port a-a' is let open, $I_1 = 0$. The circuit is shown in Fig. 16.17 (b).

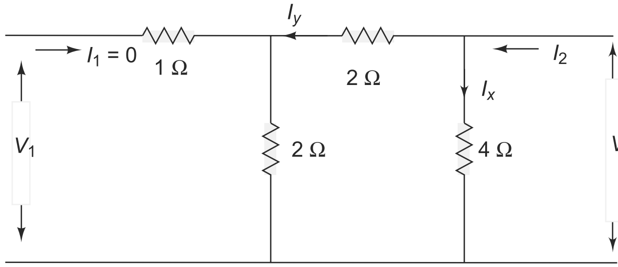


Fig. 16.17 (b)

Then,

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$V_1 = I_y 2; I_y = \frac{I_2}{2}$$

$$V_2 = I_X 4; I_X = \frac{I_2}{2}$$

$$\therefore \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{2} \text{ ℧}$$

16.7 INVERSE HYBRID (g) PARAMETERS

LO 4

Another set of hybrid matrix parameters can be defined in a similar way as was done in Section 16.6. This time the current at the input port I_1 and the voltage at the output port V_2 can be expressed in terms of I_2 and V_1. The equations are as follows.

$$I_1 = g_{11} V_1 + g_{12} I_2 \tag{16.11}$$

$$V_2 = g_{21} V_1 + g_{22} I_2 \tag{16.12}$$

The coefficients in the above equations are called the inverse hybrid parameters. In matrix notation,

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

It can be verified that $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$

The individual g-parameters may be defined by letting $I_2 = 0$ and $V_1 = 0$ in Eqs (16.11) and (16.12). Thus, when $I_2 = 0$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \text{Open-circuit input admittance} = \left(\frac{1}{Z_{11}}\right)$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \text{Open-circuit voltage gain}$$

When $V_1 = 0$,

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} = \text{Short-circuit reverse current gain}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \text{Short-circuit output impedance} = \left(\frac{1}{Y_{22}}\right)$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★ **16-4.1** For the two-port network shown in Fig. Q.1, find the h-parameters and the inverse h-parameters.

☆★★ **16-4.2** For the hybrid equivalent circuit shown in Fig. Q.2, determine (a) the input impedance, and (b) the output impedance.

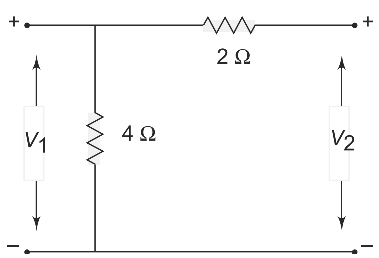


Fig. Q.1

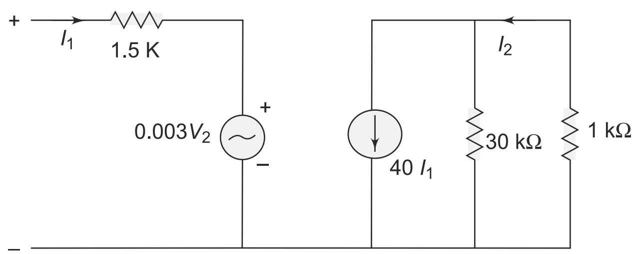


Fig. Q.2

★★★ **16-4.3** For the network shown in Fig. Q.3, determine h-parameters at $\omega = 10^8$ rad/s.

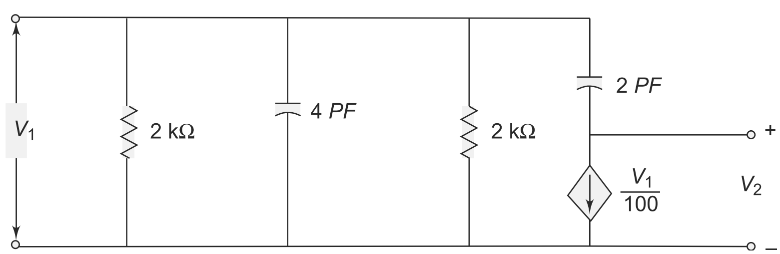


Fig. Q.3

☆★★ **16-4.4** Using PSpice, find hybrid parameters of the network shown in Fig. Q.4.

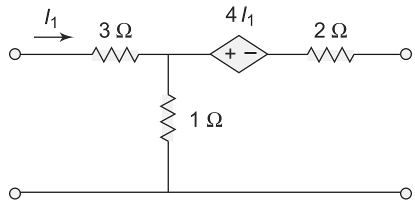


Fig. Q.4

Frequently Asked Questions linked to LO 4

★★★16-4.1 Determine the *Z-ABCD*-and *h*-parameters of the given network as shown in Fig. Q.1 **[BPUT 2008]**

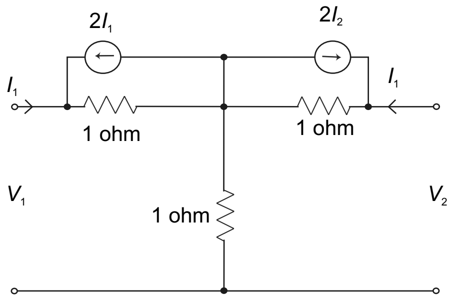


Fig. Q.1

☆☆★16-4.2 Explain hybrid parameters for two-port networks and state where one makes use of these parameters. **[GTU Dec. 2010]**

☆★★16-4.3 Obtain hybrid parameters of the interconnected 'two' 2-port networks. **[MU 2014]**

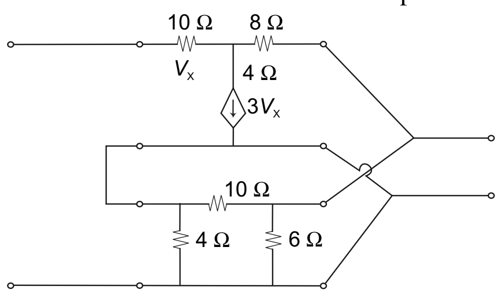


Fig. Q.3

☆☆★16-4.4 Explain the concepts of reciprocity and symmetry. Derive the above conditions for *h* and *ABCD* parameters. **[PTU 2009-10]**

☆★★16-4.5 Find the *h*-parameters of the network shown in Fig. Q.5. **[PTU 2009-10]**

☆☆★16-4.6 Determine hybrid-parameters for the network shown in Fig. Q.6. **[PU 2010]**

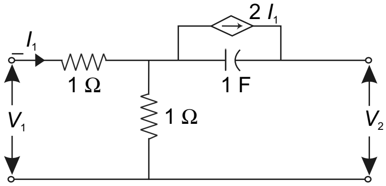


Fig. Q.5

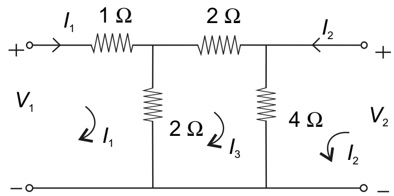


Fig. Q.6

★★★**16-4.7** Find the current transfer ratio i_2/i_1 for the network shown in Fig. Q. 7. **[PU 2012]**

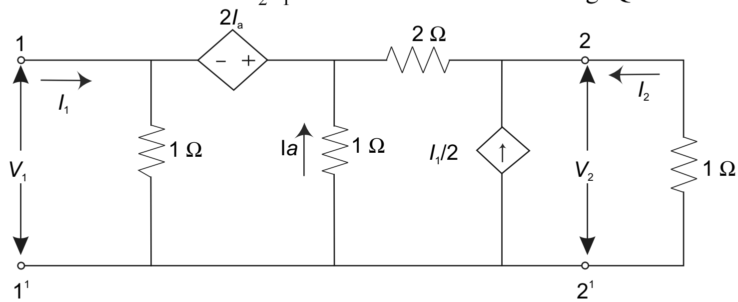


Fig. Q.7

☆☆★**16-4.8** Define hybrid parameters. **[RGTU June 2014]**

16.8 INTER-RELATIONSHIPS OF DIFFERENT PARAMETERS

16.8.1 Expression of Z-Parameters in Terms of Y-Parameters and Vice Versa

LO 5 Determine the inter-relationship between different parameters sets of a two-port network

From Eqs (16.1), (16.2), (16.3) and (16.4), it is easy to derive the relation between the open circuit impedance parameters and the short-circuit admittance parameters by means of two matrix equations of the respective parameters. By solving Eqs (16.1) and (16.2) for I_1 and I_2, we get

$$I_1 = \begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix} \bigg/ \Delta_z; \text{ and } I_2 = \begin{vmatrix} Z_{11} & V_1 \\ V_{21} & V_2 \end{vmatrix} \bigg/ \Delta_z$$

where Δ_z is the determinant of the Z-matrix

$$\Delta_z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$I_1 = \frac{Z_{22}}{\Delta_z} V_1 - \frac{Z_{12}}{\Delta_z} V_2 \tag{16.13}$$

$$I_2 = \frac{-Z_{21}}{\Delta_z} V_1 + \frac{Z_{11}}{\Delta_z} V_2 \tag{16.14}$$

Comparing Eqs (16.13) and (16.14) with Eqs (16.3) and (16.4) we have

$$Y_{11} = \frac{Z_{22}}{\Delta_z}; Y_{12} = \frac{-Z_{12}}{\Delta_z}$$

$$Y_{21} = \frac{Z_{21}}{\Delta_z}; Y_{22} = \frac{Z_{11}}{\Delta_z}$$

In a similar manner, the Z-parameters may be expressed in terms of the admittance parameters by solving Eqs (16.3) and (16.4) for V_1 and V_2.

$$V_1 = \begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix} \bigg/ \Delta_y; \text{ and } V_2 = \begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix} \bigg/ \Delta_y$$

where Δ_y is the determinant of the Y-matrix

$$\Delta_y = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}$$

$$V_1 = \frac{Y_{22}}{\Delta_y} I_1 - \frac{Y_{12}}{\Delta_y} I_2 \tag{16.15}$$

$$V_2 = \frac{-Y_{21}}{\Delta_y} I_1 + \frac{Y_{11}}{\Delta_y} I_2 \tag{16.16}$$

Comparing Eqs (16.15) and (16.16) with Eqs (16.1) and (16.2), we obtain

$$Z_{11} = \frac{Y_{22}}{\Delta_y};\ Z_{12} = \frac{-Y_{12}}{\Delta_y}$$

$$Z_{21} = \frac{-Y_{21}}{\Delta_y};\ Z_{22} = \frac{Y_{11}}{\Delta_y}$$

EXAMPLE 16.5

For a given, $Z_{11} = 3\ \Omega$; $Z_{12} = 1\ \Omega$; $Z_{21} = 2\ \Omega$, and $Z_{22} = 1\ \Omega$, find the admittance matrix, and the product of Δ_y and Δ_z.

Solution The admittance matrix $= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{22}}{\Delta_z} & \dfrac{-Z_{12}}{\Delta_z} \\ \dfrac{-Z_{21}}{\Delta_z} & \dfrac{Z_{11}}{\Delta_z} \end{bmatrix}$

$$\text{given } Z = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore\quad \Delta_z = 3 - 2 = 1$$

$$\therefore\quad \Delta_y = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = 1$$

$$(\Delta_y)\,(\Delta_z) = 1$$

16.8.2 General Circuit Parameters or *ABCD*-Parameters in Terms of Z-Parameters and Y-Parameters

We know that

$$V_1 = AV_2 - BI_2;\ V_1 = Z_{11} I_1 + Z_{12} I_2;\ I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_1 = CV_2 - DI_2;\ V_2 = Z_{21} I_1 + Z_{22} I_2;\ I_2 = Y_{21} V_1 + Y_{22} V_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0};\ C = \left.\frac{I_1}{V_2}\right|_{I_2=0};\ B = \left.\frac{-V_1}{I_2}\right|_{V_2=0};\ D = \left.\frac{-I_1}{I_2}\right|_{V_2=0}$$

Substituting the condition $I_2 = 0$ in Eqs (16.1) and (16.2), we get

$$\left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{Z_{11}}{Z_{21}} = A$$

Substituting the condition $I_2 = 0$ in Eq. (16.4), we get

$$\left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{-Y_{22}}{Y_{21}} = A$$

Substituting the condition $I_2 = 0$ in (Eq. 16.2), we get

$$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{Z_{21}} = C$$

Substituting the condition $I_2 = 0$ in Eqs (16.3) and (16.4), and solving for V_2 gives $\dfrac{-I_1 Y_{21}}{\Delta_y}$ where Δy is the determinant of the admittance matrix.

$$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{-\Delta_y}{Y_{21}} = C$$

Substituting the condition $V_2 = 0$ in Eq. (16.4), we get

$$\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{1}{Y_{21}} = B$$

Substituting the condition $V_2 = 0$ in Eqs (16.1) and (16.2) and solving for $I_2 = \dfrac{-V_1 Z_{21}}{\Delta_z}$, we get

$$-\left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{\Delta_z}{Z_{21}} = B$$

where Δ_z is the determinant of the impedance matrix.
Substituting $V_2 = 0$ in Eq. (16.2), we get

$$\left.\frac{-I_1}{I_2}\right|_{V_2=0} = \frac{Z_{22}}{Z_{21}} = D$$

Substituting $V_2 = 0$ in Eqs (16.3) and (16.4), we get

$$\left.\frac{-I_1}{I_2}\right|_{V_2=0} = \frac{-Y_{11}}{Y_{21}} = D$$

The determinant of the transmission matrix is given by

$$-AD + BC$$

Substituting the impedance parameters in A, B, C, and D, we have

$$BC - AD = \frac{\Delta_z}{Z_{21}}\frac{1}{Z_{21}} - \frac{Z_{11}}{Z_{21}}\frac{Z_{22}}{Z_{21}}$$

$$= \frac{\Delta_z}{(Z_{21})^2} - \frac{Z_{11}Z_{22}}{(Z_{21})^2}$$

$$BC - AD = \frac{-Z_{12}}{Z_{21}}$$

For a bilateral network, $Z_{12} = Z_{21}$

$\therefore \quad BC - AD = -1$

or $\quad AD - BC = 1$

Therefore, in a two-port bilateral network, if three transmission parameters are known, the fourth may be found from the equation $AD - BC = 1$.

In a similar manner, the h-parameters may be expressed in terms of the admittance parameters, impedance parameters, or transmission parameters. Transformations of this nature are possible between any of the various parameters. Each parameter has its own utility. However, we often find that it is necessary to convert from one set of parameters to another. Transformations between different parameters, and the condition under which the two-port network is reciprocal are given in Table 16.1.

Table 16.1 *Reciprocality condition for a two-port network*

	Z	**Y**	**ABCD**	**A'B'C'D'**	**h**	**g**
Z	$Z_{11}\ Z_{12}$	$\frac{Y_{22}}{\Delta_y}\ \frac{-Y_{12}}{\Delta_y}$	$\frac{A}{C}\ \frac{\Delta_T}{C}$	$\frac{D'}{C'}\ \frac{1}{C'}$	$\frac{\Delta_h}{h_{22}}\ \frac{h_{22}}{h_{22}}$	$\frac{1}{g_{11}}\ \frac{-g_{12}}{g_{11}}$
	$Z_{21}\ Z_{22}$	$\frac{-Y_{21}}{\Delta_y}\ \frac{Y_{11}}{\Delta_y}$	$\frac{1}{C}\ \frac{D}{C}$	$\frac{\Delta_{T'}}{C'}\ \frac{A'}{C'}$	$\frac{-h_{21}}{h_{22}}\ \frac{1}{h_{22}}$	$\frac{g_{21}}{g_{11}}\ \frac{\Delta_g}{g_{11}}$
Y	$\frac{Z_{22}}{\Delta_z}\ \frac{-Z_{12}}{\Delta_z}$	$Y_{11}\ Y_{12}$	$\frac{D}{B}\ \frac{-\Delta_T}{B}$	$\frac{A'}{B'}\ \frac{-1}{B'}$	$\frac{1}{h_{11}}\ \frac{-h_{12}}{h_{11}}$	$\frac{\Delta_g}{g_{22}}\ \frac{g_{12}}{g_{22}}$
	$\frac{-Z_{21}}{\Delta_z}\ \frac{-Z_{11}}{\Delta_z}$	$Y_{21}\ Y_{22}$	$\frac{-1}{B}\ \frac{A}{B}$	$\frac{-\Delta_{T'}}{B'}\ \frac{D'}{B'}$	$\frac{h_{21}}{h_{11}}\ \frac{\Delta_h}{h_{11}}$	$\frac{-g_{21}}{g_{22}}\ \frac{1}{g_{22}}$
AB	$\frac{Z_{11}}{Z_{21}}\ \frac{\Delta_z}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}\ \frac{-1}{Y_{21}}$	$A\ B$	$\frac{D'}{\Delta_{T'}}\ \frac{B'}{\Delta_{T'}}$	$\frac{\Delta_h}{h_{21}}\ \frac{h_{11}}{h_{21}}$	$\frac{1}{g_{21}}\ \frac{g_{22}}{g_{21}}$
CD	$\frac{1}{Z_{21}}\ \frac{Z_{22}}{Z_{21}}$	$\frac{\Delta Y}{Y_{21}}\ \frac{-Y_{11}}{Y_{21}}$	$C\ D$	$\frac{C'}{\Delta_{T'}}\ \frac{A'}{\Delta_{T'}}$	$\frac{-h_{22}}{h_{21}}\ \frac{-1}{h_{21}}$	$\frac{g_{11}}{g_{21}}\ \frac{\Delta_g}{g_{21}}$
$A'\ B'$	$\frac{Z_{22}}{Z_{12}}\ \frac{\Delta_z}{Z_{12}}$	$\frac{-Y_{11}}{Y_{12}}\ \frac{-1}{Y_{12}}$	$\frac{D}{\Delta_T}\ \frac{B}{\Delta_T}$	$A'\ B'$	$\frac{1}{h_{12}}\ \frac{h_{11}}{h_{12}}$	$\frac{-\Delta_g}{g_{12}}\ \frac{-g_{22}}{g_{12}}$
$C'\ D'$	$\frac{1}{Z_{12}}\ \frac{Z_{11}}{Z_{12}}$	$\frac{-\Delta_Y}{Y_{12}}\ \frac{-Y_{22}}{Y_{12}}$	$\frac{C}{\Delta_T}\ \frac{A}{\Delta_T}$	$C'\ D'$	$\frac{h_{22}}{h_{12}}\ \frac{\Delta_h}{h_{12}}$	$\frac{-g_{11}}{g_{12}}\ \frac{-1}{g_{12}}$
h	$\frac{\Delta_z}{Z_{22}}\ \frac{Z_{12}}{Z_{22}}$	$\frac{1}{Y_{11}}\ \frac{-Y_{12}}{Y_{11}}$	$\frac{B}{D}\ \frac{\Delta_T}{D}$	$\frac{B'}{A'}\ \frac{1}{A'}$	$h_{11}\ h_{12}$	$\frac{g_{22}}{\Delta_g}\ \frac{-g_{12}}{\Delta_g}$
	$\frac{-Z_{21}}{Z_{22}}\ \frac{1}{Z_{22}}$	$\frac{Y_{21}}{Y_{11}}\ \frac{\Delta_Y}{Y_{11}}$	$\frac{-1}{D}\ \frac{C}{D}$	$\frac{\Delta_{T'}}{A'}\ \frac{C'}{A'}$	$h_{21}\ h_{22}$	$\frac{-g_{21}}{\Delta_g}\ \frac{g_{11}}{\Delta_g}$
g	$\frac{1}{Z_{11}}\ \frac{-Z_{12}}{Z_{11}}$	$\frac{\Delta_Y}{Y_{22}}\ \frac{Y_{12}}{Y_{22}}$	$\frac{C}{A}\ \frac{-\Delta_T}{A}$	$\frac{C'}{D'}\ \frac{-1}{D'}$	$\frac{h_{22}}{\Delta_h}\ \frac{-h_{12}}{\Delta_h}$	$g_{11}\ g_{12}$
	$\frac{Z_{21}}{Z_{11}}\ \frac{\Delta_z}{Z_{11}}$	$\frac{-Y_{21}}{Y_{22}}\ \frac{1}{Y_{22}}$	$\frac{1}{A}\ \frac{B}{A}$	$\frac{\Delta_{T'}}{D'}\ \frac{B'}{D'}$	$\frac{-h_{21}}{\Delta_h}\ \frac{h_{11}}{\Delta_h}$	$g_{21}\ g_{22}$
The two-port is reciprocal if	$Z_{12} = Z_{21}$	$Y_{12} = Y_{21}$	The determinant of the transmission matrix = 1 ($\Delta_T = 1$)	The determinant of the inverse transmission matrix = 1	$h_{12} = -h_{21}$	$g_{12} = -g_{21}$

EXAMPLE 16.6

The impedance parameters of a two-port network are $Z_{11} = 6\ \Omega$; $Z_{22} = 4\ \Omega$; $Z_{12} = Z_{21} = 3\ \Omega$. Compute the Y-parameters and ABCD- parameters and write the describing equations.

Solution *ABCD*-parameters are given by

$$A = \frac{Z_{11}}{Z_{21}} = \frac{6}{3} = 2;\ B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} = 5\ \Omega$$

$$C = \frac{1}{Z_{21}} = \frac{1}{3}\ ℧;\quad D = \frac{Z_{22}}{Z_{21}} = \frac{4}{3}$$

Y-parameters are given by

$$Y_{11} = \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{4}{15}\ ℧;\quad Y_{12} = \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{-1}{5}\ ℧$$

$$Y_{21} = Y_{12} = \frac{-Z_{12}}{\Delta_Z} = \frac{-1}{5}\ ℧;\qquad Y_{22} = \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{2}{5}\ ℧$$

The equations, using *Z*, parameters, are

$$V_1 = 6I_1 + 3I_2$$
$$V_2 = 3I_1 + 4I_2$$

Using *Y*-parameters,

$$I_1 = \frac{4}{15}V_1 - \frac{1}{5}V_2$$

$$I_2 = \frac{-1}{5}V_1 + \frac{2}{5}V_2$$

Using *ABCD*-parameters,

$$V_1 = 2V_2 - 5I_2$$

$$I_1 = \frac{1}{3}V_2 - \frac{4}{3}I_2$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆★★**16-5.1** The hybrid parameters of a two-port network shown in Fig. Q.1 are $h_{11} = 1.5$ K; $h_{12} = 2 \times 10^{-3}$; $h_{21} = 250$; $h_{22} = 150 \times 10^{-6}$ ℧ (a) Find V_2 (b). Draw the *Z*- parameter equivalent circuit.

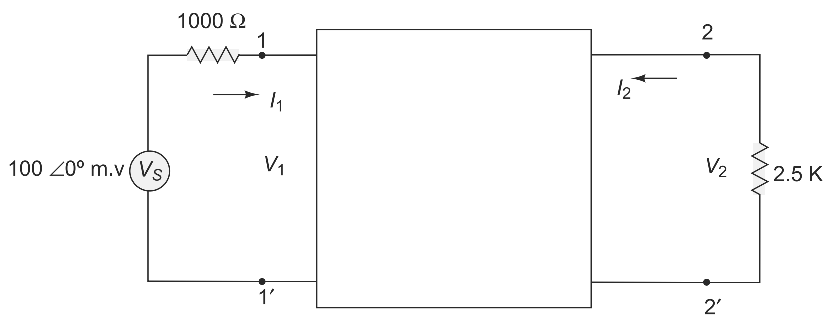


Fig. Q.1

Frequently Asked Questions linked to LO 5

☆☆★**16-5-1** Explain *ABCD*-parameters in terms of *y*-parameters. **[PTU 2011-12]**

☆☆★**16-5-2** Derive transmission parameters in terms of hybrid parameters. **[PTU 2011-12]**

☆☆★**16-5-3** Obtain hybrid-parameters in terms of admittance parameters. **[PU 2010]**

☆★★**16-5-4** The *Z*-parameters of a two-port network are $Z_{11} = 10\ \Omega$, $Z_{22} = 20\ \Omega$, $Z_{12} = Z_{21} = 5\ \Omega$.
(a) Find the *ABCD*-parameters of this two-port network.
(b) Also, find its equivalent T-network. **[RTU Feb. 2011]**

16.9 INTERCONNECTION OF TWO-PORT NETWORKS

16.9.1 Series Connection of a Two-port Network

LO 6 Discuss interconnected two-port networks

It has already been shown in Section 16.4.1 that when two-port networks are connected in cascade, the parameters of the interconnected network can be conveniently expressed with the help of *ABCD*-parameters. In a similar way, the *Z*-parameters can be used to describe the parameters of series-connected two-port networks; and *Y*-parameters can be used to describe parameters of parallel connected two-port networks. A series connection of two-port networks is shown in Fig. 16.18.

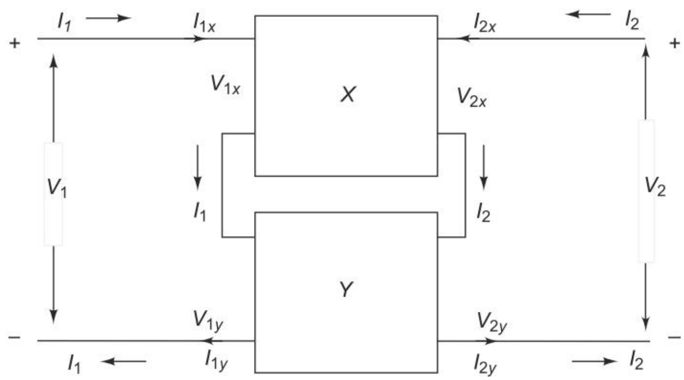


Fig. 16.18

Let us consider two two-port networks, connected in series as shown. If each port has a common reference node for its input and output, and if these references are connected together then the equations of the networks *X* and *Y* in terms of *Z*-parameters are

$$V_{1X} = Z_{11X} I_{1X} + Z_{12X} I_{2X}$$

$$V_{2X} = Z_{21X} I_{1X} + Z_{22X} I_{2X}$$

$$V_{1Y} = Z_{11Y} I_{1Y} + Z_{12Y} I_{2Y}$$

$$V_{2Y} = Z_{21Y} I_{1Y} + Z_{22Y} I_{2Y}$$

From the interconnection of the networks, it is clear that

$$I_1 = I_{1X} = I_{1Y};\ I_2 = I_{2X} = I_{2Y}$$

and $$V_1 = V_{1X} + V_{1Y};\ V_2 = V_{2X} + V_{2Y}$$

$$\therefore\quad V_1 = Z_{11X} I_1 + Z_{12X} I_2 + Z_{11Y} I_1 + Z_{12Y} I_2$$

$$= (Z_{11X} + Z_{11Y})I_1 + (Z_{12X} + Z_{12Y})\, I_2$$

$$V_2 = Z_{21X} I_1 + Z_{22X} I_2 + Z_{21Y} I_1 + Z_{22Y} I_2$$

$$= (Z_{21X} + X_{21Y})I_1 + (Z_{22X} + Z_{22Y})I_2$$

The describing equations for the series-connected two-port network are

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

where $Z_{11} = Z_{11X} + Z_{11Y}$; $Z_{12} = Z_{12X} + Z_{12Y}$

$Z_{21} = Z_{21X} + Z_{21Y}$; $Z_{22} = Z_{22X} + Z_{22Y}$

Thus, we see that each Z-parameter of the series network is given as the sum of the corresponding parameters of the individual networks.

16.9.2 Parallel Connection of Two Two-port Networks

Let us consider two two-port networks connected in parallel as shown in Fig. 16.19. If each two-port has a reference node that is common to its input and output port, and if the two ports are connected so that they have a common reference node, then the equations of the networks X and Y in terms of Y-parameters are given by

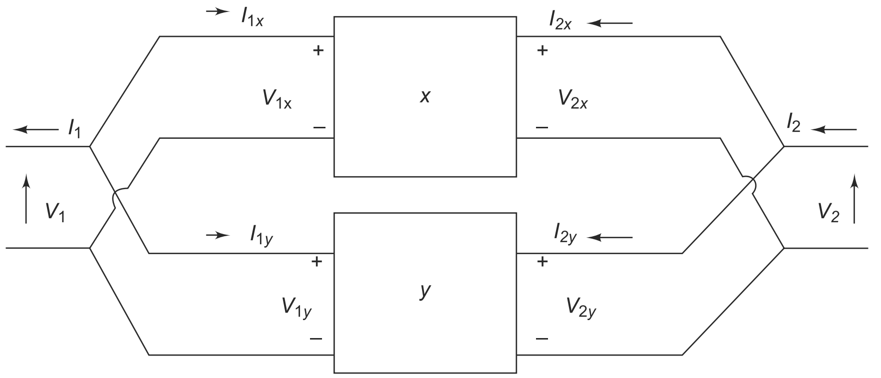


Fig. 16.19

$$I_{1X} = Y_{11X} V_{1X} + Y_{12X} V_{2X}$$
$$I_{2X} = Y_{21X} V_{1X} + Y_{22X} V_{2X}$$
$$I_{1Y} = Y_{11Y} V_{1Y} + Y_{12Y} V_{2Y}$$
$$I_{2Y} = Y_{21Y} V_{1Y} + Y_{22Y} V_{2Y}$$

From the interconnection of the networks, it is clear that

$$V_1 = V_{1X} = V_{1Y}; V_2 = V_{2X} = V_{2Y}$$

and $I_1 = I_{1X} + I_{1Y}$; $I_2 = I_{2X} + I_{2Y}$

$$\therefore \quad I_1 = Y_{11X} V_1 + Y_{12X} V_2 + Y_{11Y} V_1 + Y_{12Y} V_2$$
$$= (Y_{11X} + Y_{11Y}) V_1 + (Y_{12X} + Y_{12Y}) V_2$$
$$I_2 = Y_{21X} V_1 + Y_{22X} V_2 + Y_{21Y} V_1 + Y_{22Y} V_2$$
$$= (Y_{21X} + Y_{21Y}) V_1 + (Y_{22X} + Y_{22Y}) V_2$$

The describing equations for the parallel connected two-port networks are

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

where $Y_{11} = Y_{11X} + Y_{11Y}; Y_{12} = Y_{12X} + Y_{12Y}$

$Y_{21} = Y_{21X} + Y_{21Y}; Y_{22} = Y_{22X} + Y_{22Y}$

Thus, we see that each Y-parameter of the parallel network is given as the sum of the corresponding parameters of the individual networks.

EXAMPLE 16.7

Two networks shown in Figs 16.20 (a) and (b) are connected in series. Obtain the Z-parameters of the combination. Also verify by direct calculation.

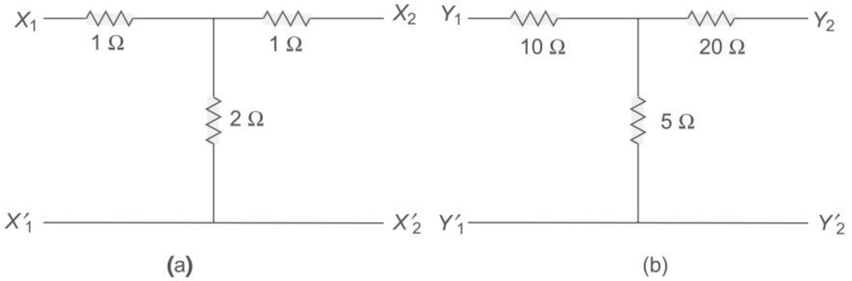


Fig. 16.20

Solution The Z-parameters of the network in Fig. 16.20 (a) are

$$Z_{11X} = 3\ \Omega\ Z_{12X} = Z_{21X} = 2\ \Omega\ Z_{22X} = 3\ \Omega$$

The Z-parameters of the network in Fig. 16.20 (b) are

$$Z_{11Y} = 15\ \Omega\ Z_{21Y} = 5\ \Omega\ Z_{22Y} = 25\ \Omega\ Z_{12Y} = 5\ \Omega$$

The Z-parameters of the combined network are

$$Z_{11} = Z_{11X} + Z_{11Y} = 18\ \Omega$$

$$Z_{12} = Z_{12X} + Z_{12Y} = 7\ \Omega$$

$$Z_{21} = Z_{21X} + Z_{21Y} = 7\ \Omega$$

$$Z_{22} = Z_{22X} + Z_{22Y} = 28\ \Omega$$

Check If the two networks are connected in series as shown in Fig. 16.20 (c), the Z-parameters are

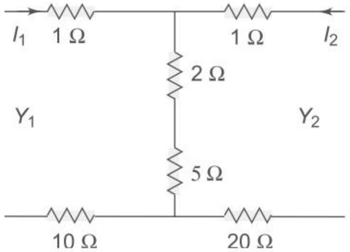


Fig. 16.20 (c)

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 18\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 7\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 28\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 7\ \Omega$$

EXAMPLE 16.8

Two identical sections of the network shown in Fig. 16.21 are connected in parallel. Obtain the Y-parameters of the combination.

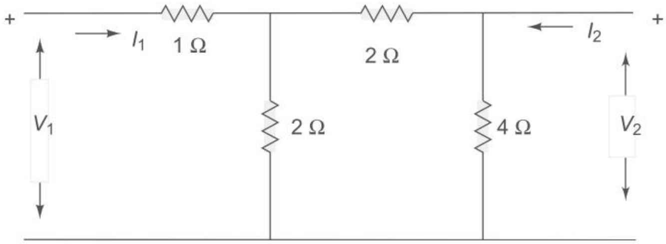


Fig. 16.21

Solution The Y-parameters of the network in Fig. 16.21 are (see Example 16.2)

$$Y_{11} = \frac{1}{2}\,℧ \quad Y_{21} = \frac{-1}{4}\,℧ \quad Y_{22} = \frac{5}{8}\,℧ \quad Y_{12} = \frac{-1}{4}\,℧$$

If two such networks are connected in parallel then the Y-parameters of the combined network are

$$Y_{11} = \frac{1}{2} + \frac{1}{2} = 1\,℧ \quad Y_{21} = \frac{-1}{4} \times 2 = \frac{-1}{2}\,℧$$

$$Y_{22} = \frac{5}{8} \times 2 = \frac{5}{4}\,℧ \quad Y_{12} = \frac{-1}{4} \times 2 = \frac{-1}{2}\,℧$$

16.10 T- AND π-REPRESENTATIONS

LO 7 Explain T and π representation of a two-port network

A two-port network with any number of elements may be converted into a two-port three-element network. Thus, a two-port network may be represented by an equivalent T-network, i.e. three impedances are connected together in the form of a T as shown in Fig. 16.22.

It is possible to express the elements of the T-network in terms of Z-parameters, or $ABCD$-parameters as explained below.

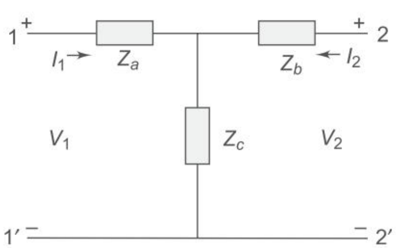


Fig. 16.22

Z-parameters of the network

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_a + Z_c$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_c$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_b + Z_c$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_c$$

From the above relations, it is clear that

$$Z_a = Z_{11} - Z_{21}$$
$$Z_b = Z_{22} - Z_{12}$$
$$Z_c = Z_{12} = Z_{21}$$

ABCD-parameters of the network

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{Z_a + Z_c}{Z_c}$$

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0}$$

When 2-2′ is short-circuited,

$$-I_2 = \frac{V_1 Z_c}{Z_b Z_c + Z_a(Z_b + Z_c)}$$

$$B = (Z_a + Z_b) + \frac{Z_a Z_b}{Z_c}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{Z_c}$$

$$D = \left.\frac{-I_1}{I_2}\right|_{V_2=0}$$

When 2-2′ is short-circuited,

$$-I_2 = I_1 \frac{Z_c}{Z_b + Z_c}$$

$$D = \frac{Z_b + Z_c}{Z_c}$$

From the above relations, we can obtain

$$Z_a = \frac{A-1}{C}; \quad Z_b = \frac{D-1}{C}; \quad Z_c = \frac{1}{C}$$

EXAMPLE 16.9

The Z-parameters of a two-port network are $Z_{11} = 10\ \Omega$; $Z_{22} = 15\ \Omega$; $Z_{12} = Z_{21} = 5\ \Omega$. Find the equivalent T-network and ABCD-parameters.

Solution The equivalent *T*-network is shown in Fig. 16.23,

where $Z_a = Z_{11} - Z_{21} = 5\ \Omega$

$Z_b = Z_{22} - Z_{12} = 10\ \Omega$

and $Z_c = 5\ \Omega$

The *ABCD*-parameters of the network are

$$A = \frac{Z_a}{Z_c} + 1 = 2; \quad B = \left(Z_a + Z_b\right) + \frac{Z_a Z_b}{Z_c} = 25\,\Omega$$

$$C = \frac{1}{Z_c} = 0.2\,\mho \quad D = 1 + \frac{Z_b}{Z_c} = 3$$

In a similar way, a two-port network may be represented by an equivalent π-network, i.e. three impedances or admittances are connected together in the form of π as shown in Fig. 16.24.

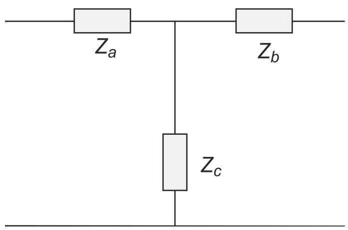


Fig. 16.23

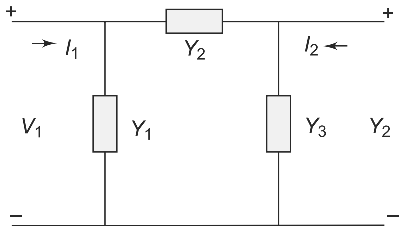


Fig. 16.24

It is possible to express the elements of the π-network in terms of *Y*-parameters or *ABCD*-parameters as explained below.

Y-parameters of the network are

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_1 + Y_2$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -Y_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_3 + Y_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_2$$

From the above relations, it is clear that

$$Y_1 = Y_{11} + Y_{21}$$

$$Y_2 = -Y_{12}$$

$$Y_3 = Y_{22} + Y_{21}$$

Writing *ABCD*-parameters in terms of *Y*-parameters yields the following results.

$$A = \frac{-Y_{22}}{Y_{21}} = \frac{Y_3 + Y_2}{Y_2}$$

$$B = \frac{-1}{Y_{21}} = \frac{1}{Y_2}$$

$$C = \frac{-\Delta y}{Y_{21}} = Y_1 + Y_3 + \frac{Y_1 Y_3}{Y_2}$$

$$D = \frac{-Y_{11}}{Y_{21}} = \frac{Y_1 + Y_2}{Y_2}$$

From the above results, we can obtain

$$Y_1 = \frac{D-1}{B}$$

$$Y_2 = \frac{1}{B}$$

$$Y_3 = \frac{A-1}{B}$$

EXAMPLE 16.10

The port currents of a two-port network are given by

$$I_1 = 2.5V_1 - V_2$$

$$I_2 = -V_1 + 5V_2$$

Find the equivalent π-network.

Solution Let us first find the Y-parameters of the network.

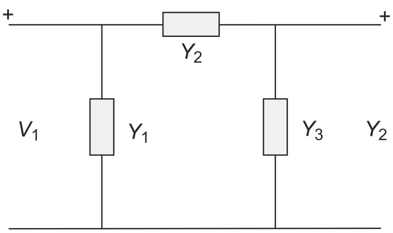


Fig. 16.25

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 2.5\ \mho; \quad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -1\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -1\ \mho; \quad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 5\ \mho$$

The equivalent π-network is shown in Fig. 16.25.

where $Y_1 = Y_{11} + Y_{21} = 1.5\ \mho$;

$Y_2 = -Y_{12} = -1\mho$

and $Y_3 = Y_{22} + Y_{12} = 4\ \mho$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆★★ **16-7.1** Obtain a π-equivalent circuit for Fig. Q.1.

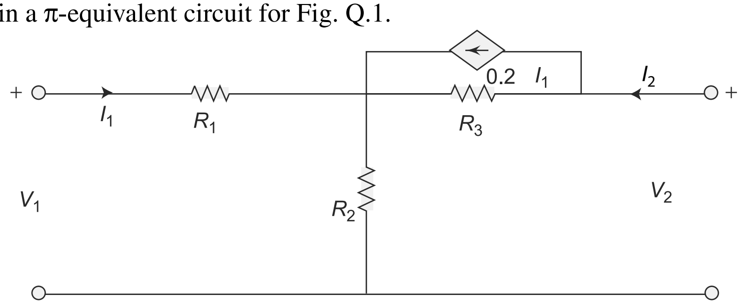


Fig. Q.1

Frequently Asked Questions linked to LO 7

☆★★**16-7.1** Obtain T and π-equivalent circuit for the network of Fig. Q.1 by using impedance and admittance parameters respectively. **[BPUT 2007]**

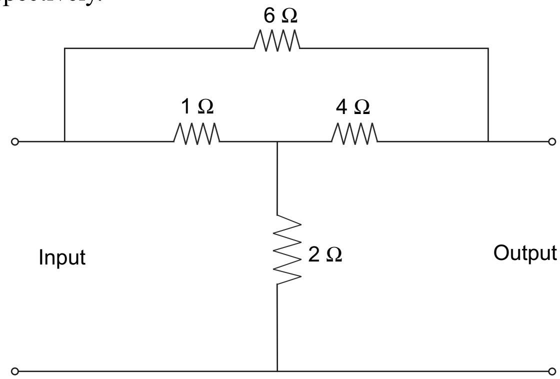


Fig. Q.1

16.11 TERMINATED TWO-PORT NETWORK

16.11.1 Driving-Point Impedance at the Input Port of a Load-Terminated Network

LO 8 Examine the driving-point impedance at the input and at the output of a terminated network

Figure 16.26 shows a two-port network connected to an ideal generator at the input port and to a load impedance at the output port. The input impedance of this network can be expressed in terms of parameters of the two-port network.

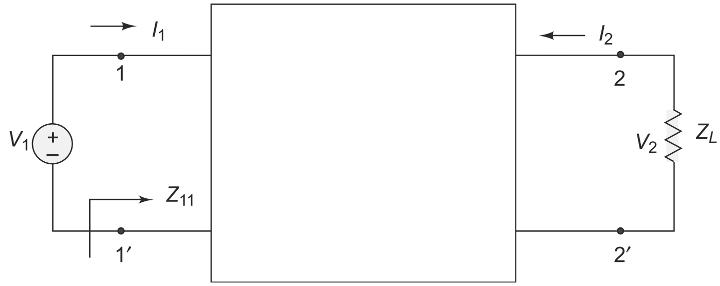


Fig. 16.26

❑ *In Terms of Z-Parameters*

The load at the output port 2-2′ imposes the following constraints on the port voltage and current,

i.e., $V_2 = -Z_L I_2$

Recalling Eqs (16.1) and (16.2), we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Substituting the value of V_2 in Eq. (16.2), we have

$$-Z_L I_2 = Z_{21} I_1 + Z_{22} I_2$$

from which $$I_2 = \frac{-I_1 Z_{21}}{Z_L + Z_{22}}$$

Substituting the value of I_2 in Eq. (16.1) gives

$$V_1 = Z_{11} I_1 - \frac{Z_{12} Z_{21} I_1}{Z_L + Z_{22}}$$

$$V_1 = I_1 \left(Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}} \right)$$

Hence, the driving-point impedance at 1-1′ is

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_L + Z_{22}}$$

If the output port is open, i.e. $Z_L \to \infty$, the input impedance is given by $V_1/I_1 = Z_{11}$
If the output port is short-circuited, i.e. $Z_L \to 0$,
The short-circuit driving-point impedance is given by

$$\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{22}} = \frac{1}{Y_{11}}$$

❑ *In Terms of Y-Parameters*

If a load admittance Y_L is connected across the output port, the constraint imposed on the output port voltage and current is

$$-I_2 = V_2 Y_L, \text{ where } Y_L = \frac{1}{Z_L}$$

Recalling Eqs (16.3) and (16.4), we have

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Substituting the value of I_2 in Eq. (16.4), we have

$$-V_2 Y_L = Y_{21} V_1 + Y_{22} V_2$$

$$V_2 = -\left(\frac{Y_{21}}{Y_L + Y_{22}} \right) V_1$$

Substituting the value of V_2 in Eq. (16.3), we have

$$I_1 = Y_{11} V_1 - \frac{Y_{12} Y_{21} V_1}{Y_L + Y_{22}}$$

From which $$\frac{I_1}{V_1} = Y_{11} - \frac{Y_{12} Y_{21}}{Y_L + Y_{22}}$$

Hence, the driving-point impedance is given by

$$\boxed{\frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}\left(Y_1 + Y_{22}\right) - Y_{12}Y_{21}}}$$

If the output port is open, i.e., $Y_L \to 0$

$$\frac{V_1}{I_1} = \frac{Y_{22}}{\Delta_y} = Z_{11}$$

If the output port is short-circuited, i.e. $Y_L \to \infty$

Then $\quad Y_{\text{in}} = Y_{11}$

In a similar way, the input impedance of the load-terminated two-port network may be expressed in terms of other parameters by simple mathematical manipulations. The results are given in Table 16.2.

Table 16.2 *Output impedance*

			In terms of			
Driving- point impedance at the input port, or input impedance $\left(\frac{V_1}{I_1}\right)$	Z parameters $\frac{\Delta_z + Z_{11}Z_L}{Z_{22} + Z_L}$	Y parameters $\frac{Y_{22} + Y_L}{\Delta_y + Y_{11}Y_L}$	$ABCD$ $\frac{AZ_L + B}{CZ_L + D}$	$A'B'C'D'$ $\frac{B' - D'Z_L}{C'Z_L - A'}$	h parameter $\frac{\Delta_h Z_L + h_{11}}{1 + h_{22}Z_L}$	g parameter $\frac{1 + g_{22}Y_L}{\Delta_{gYL} + g_{11}}$
Driving- point impedance at the output port, or output impedance $\left(\frac{V_2}{I_2}\right)$	$\frac{\Delta_z + Z_{22}Z_s}{Z_1 + Z_{11}}$	$\frac{Y_{11} + Y_s}{\Delta_y + Y_sY_{22}}$	$\frac{DZ_s + B}{CZ_s + A}$	$\frac{A'Z_s + B'}{C'Z_s + D'}$	$\frac{h_{11} + Z_s}{\Delta_h + h_{22}Z_s}$	$\frac{g_{22} + \Delta_s}{1 + g_{11}Z_s}$

Note The above relations are obtained when $V_s = 0$ and $I_s = 0$ at the input port.

16.11.2 Driving-Point Impedance at the Output Port with Source Impedance at the Input Port

Let us consider a two-port network connected to a generator at the input port with a source impedance Z_s as shown in Fig. 16.27. The output impedance, or the driving- point impedance, at the output port can be evaluated in terms of the parameters of the two-port network.

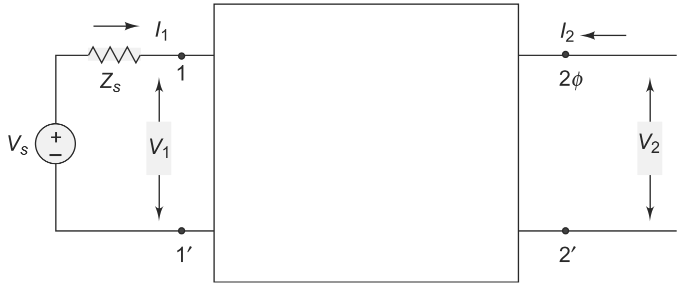


Fig. 16.27

❑ ***In terms of Z-parameters*** If I_1 is the current due to V_s at the port 1-1′, from Eqs (16.1) and (16.2), we have

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$V_1 = V_s - I_1Z_s$$

$$= Z_{11}\, I_1 + Z_{12}I_2 - (I_1)\,(Z_s + Z_{11}) = Z_{12}I_2 - V_s$$

$$-I_1 = \frac{Z_{12}I_2 - V_s}{Z_s + Z_{11}}$$

Substituting I_1 in Eq. (16.2), we get

$$V_2 = -Z_{21}\frac{(Z_{12}I_2 - V_s)}{Z_s + Z_{11}} + Z_{22}I_2$$

With no source voltage at the port 1-1′, i.e. if the source V_s is short-circuited,

$$V_2 = \frac{-Z_{21}Z_{12}}{Z_s + Z_{11}}I_2 + Z_{22}I_2$$

Hence, the driving-point impedance at the port 2-2′ $= \dfrac{V_2}{I_2}$

$$\frac{V_2}{I_2} = \frac{Z_{22}Z_s + Z_{22}Z_{11} - Z_{21}Z_{12}}{Z_s + Z_{11}} \quad \text{or} \quad \frac{\Delta_z + Z_{22}Z_s}{Z_s + Z_{11}}$$

If the input port is open, i.e. $Z_s \to \infty$

Then $$\frac{V_2}{I_2} = \left[\frac{\dfrac{\Delta_z}{Z_s} + Z_{22}}{1 + \dfrac{Z_{11}}{Z_s}}\right]_{Z_s = \infty} = Z_{22}$$

If the source impedance is zero with a short-circuited input port, the driving-point impedance at the output port is given by

$$\frac{V_2}{I_2} = \frac{\Delta_z}{Z_{11}} = \frac{1}{Y_{22}}$$

❑ ***In terms of Y-parameters*** Let us consider a two-port network connected to a current source at the input port with a source admittance Y_s as shown in Fig. 16.28.

At the port 1-1′, $\quad I_1 = I_s - V_1\, Y_s$

Recalling Eqs (16.3) and (16.4), we have

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$

$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$

Substituting I_1 in Eq. (16.3), we get

$$I_s - V_1Y_s = Y_{11}V_1 + Y_{12}V_2$$

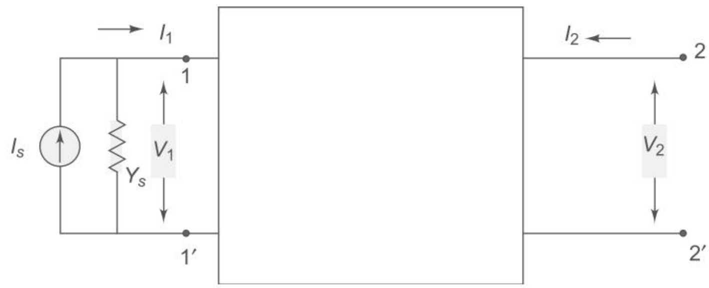


Fig. 16.28

$$-V_1(Y_s + Y_{11}) = Y_{12}\, V_2 - I_s$$

$$-V_1 = \frac{Y_{12}V_2 - I_s}{Y_s + Y_{11}}$$

Substituting V_1 in Eq. (16.4), we get

$$I_2 = -Y_{21}\left(\frac{Y_{12}V_2 - I_s}{Y_s + Y_{11}}\right) + Y_{22}V_2$$

With no source current at 1-1′, i.e. if the current source is open-circuited,

$$I_2 = \frac{-Y_{21}Y_{12}V_2}{Y_s + Y_{11}} + Y_{22}V_2$$

Hence, the driving-point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{Y_{22}Y_s + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_s + Y_{11}} \quad \text{or} \quad \frac{\Delta_y + Y_{22}Y_s}{Y_s + Y_{11}}$$

If the source admittance is zero, with an open-circuited input port, the driving-point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{\Delta_y}{Y_{11}} = \frac{1}{Z_{22}} = Y_{22}$$

In a similar way, the output impedance may be expressed in terms of the other two-port parameters by simple mathematical manipulations. The results are given in Table 16.2.

EXAMPLE 16.11

Calculate the input impedance of the network shown in Fig. 16.29.

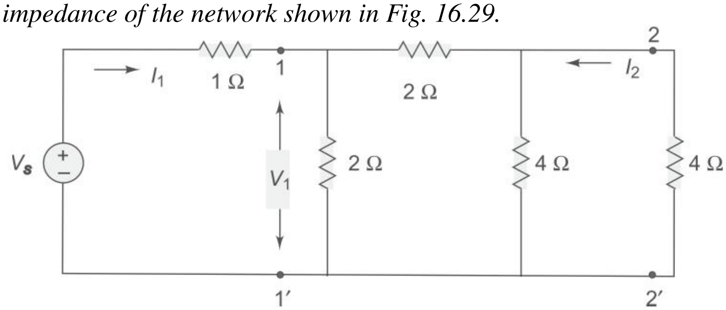


Fig. 16.29

Solution Let us calculate the input impedance in terms of Z-parameters. The Z-parameters of the given network (see Solved Problem 16.1) are $Z_{11} = 2.5\ \Omega$; $Z_{21} = 1\ \Omega$; $Z_{22} = 2\ \Omega$; $Z_{12} = 1\ \Omega$

From Section 16.11.1, we have the relation

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}$$

where Z_L is the load impedance $= 2\ \Omega$

$$\frac{V_1}{I_1} = 2.5 - \frac{1}{2+2} = 2.25\ \Omega$$

The source resistance is $1\ \Omega$.

$$\therefore \quad Z_{in} = 1 + 2.25 = 3.25\ \Omega$$

EXAMPLE 16.12

Calculate the output impedance of the network shown in Fig. 16.30 with a source admittance of 1 ℧ at the input port.

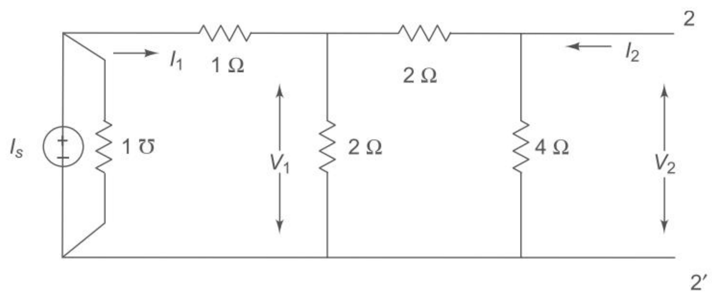


Fig. 16.30

Solution Let us calculate the output impedance in terms of Y-parameters. The Y-parameters of the given network (see Example 16.2) are

$$Y_{11} = \frac{1}{2}\ ℧;\ \ Y_{22} = \frac{5}{8}\ ℧;\ \ Y_{21} = Y_{12} = \frac{-1}{4}\ ℧$$

From Section 16.11.2, we have the relation

$$\frac{I_2}{V_2} = \frac{Y_{22}Y_s + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_s + Y_{11}}$$

where Y_s is the source admittance = 1 mho

$$Y_{22} = \frac{I_2}{V_2} = \frac{\frac{5}{8}\times 1 + \frac{5}{8}\times\frac{1}{2} - \frac{1}{16}}{1+\frac{1}{2}} = \frac{7}{12}\ ℧$$

$$\text{or} \quad Z_{22} = \frac{12}{7}\ ℧$$

16.12 LATTICE NETWORKS

LO 9 Describe lattice or bridge networks

One of the common four-terminal two-port networks is the lattice, or bridge network shown in Fig. 16.31 (a). Lattice networks are used in filter sections and are also used as attenuaters. Lattice structures are sometimes used in preference to ladder structures in some special applications. Z_a and Z_d are called *series arms*, and Z_b and Z_c are called the *diagonal arms*. It can be observed that, if Z_d is zero, the lattice structure becomes a π-section. The lattice network is redrawn as a bridge network as shown in Fig. 16.31 (b).

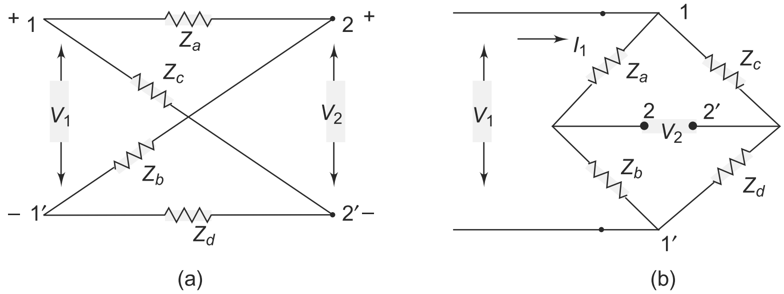


Fig. 16.31

16.12.1 Z-Parameters

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

When $I_2 = 0$; $V_1 = I_1 \dfrac{(Z_a+Z_b)(Z_d+Z_c)}{Z_a+Z_b+Z_c+Z_d}$ (16.17)

$$\therefore \quad Z_{11} = \frac{(Z_a+Z_b)(Z_d+Z_c)}{Z_a+Z_b+Z_c+Z_d}$$

If the network is symmetric, then $Z_a = Z_d$ and $Z_b = Z_c$

$$\therefore \quad Z_{11} = \frac{Z_a+Z_b}{2}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

When $I_2 = 0$, V_2 is the voltage across $2-2'$.

$$V_2 = V_1\left[\frac{Z_b}{Z_a+Z_b} - \frac{Z_d}{Z_c+Z_d}\right]$$

Substituting the value of V_1 from Eq. (16.17), we have

$$V_2 = \left[\frac{I_1(Z_a+Z_b)(Z_d+Z_c)}{Z_a+Z_b+Z_c+Z_d}\right]\left[\frac{Z_b(Z_c+Z_d)-Z_d(Z_a+Z_b)}{(Z_a+Z_b)(Z_c+Z_d)}\right]$$

$$\frac{V_2}{I_1} = \frac{Z_b(Z_c+Z_d)-Z_d(Z_a+Z_b)}{Z_a+Z_b+Z_c+Z_d} = \frac{Z_bZ_c - Z_aZ_d}{Z_a+Z_b+Z_c+Z_d}$$

$$\therefore \quad Z_{21} = \frac{Z_bZ_c - Z_aZ_d}{Z_a+Z_b+Z_c+Z_d}$$

If the network is symmetric, $Z_a = Z_d, Z_b = Z_c$

$$Z_{21} = \frac{Z_b - Z_a}{2}$$

When the input port is open, $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

The network can be redrawn as shown in Fig. 16.31 (c).

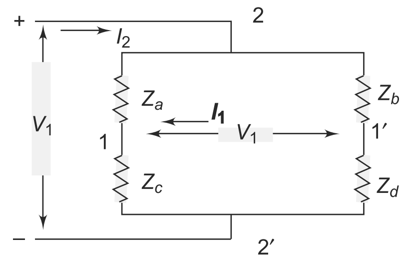


Fig. 16.31 (c)

$$V_1 = V_2\left[\frac{Z_c}{Z_a + Z_c} - \frac{Z_d}{Z_b + Z_d}\right] \tag{16.18}$$

$$V_2 = I_2\left[\frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}\right] \tag{16.19}$$

Substituting the value of V_2 in Eq. (16.18), we get

$$V_1 = I_2\left[\frac{Z_c(Z_b + Z_d) - Z_d(Z_a + Z_c)}{Z_a + Z_b + Z_c + Z_d}\right]$$

$$\frac{V_1}{I_2} = \frac{Z_c Z_b - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, $Z_a = Z_d; Z_b = Z_c$

$$\frac{V_1}{I_2} = \frac{Z_b^2 - Z_a^2}{2(Z_a + Z_b)}$$

$$\therefore \qquad Z_{12} = \frac{Z_b - Z_a}{2}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_2=0}$$

From Eq. (16.19), we have

$$\frac{V_2}{I_2} = \frac{(Z_a + Z_c) - (Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric,

$$Z_a = Z_d; Z_b = Z_c$$

$$Z_{22} = \frac{Z_a + Z_b}{2} = Z_{11}$$

From the above equations, $Z_{11} = Z_{22} = \dfrac{Z_a + Z_b}{2}$

and $Z_{12} = Z_{21} = \dfrac{Z_b - Z_a}{2}$

$\therefore$ $Z_b = Z_{11} + Z_{12}$

$Z_a = Z_{11} - Z_{12}$.

EXAMPLE 16.13

Obtain the lattice equivalent of a symmetrical T-network shown in Fig. 16.32.

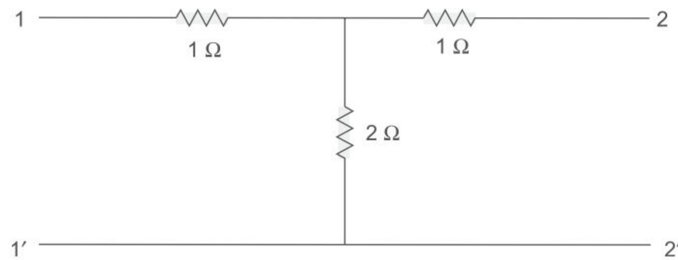


Fig. 16.32

Solution A two-port network can be realised as a symmetric lattice if it is reciprocal and symmetric. The Z-parameters of the network are (see Example 16.1). $Z_{11} = 3\ \Omega$; $Z_{12} = Z_{21} = 2\ \Omega$; $Z_{22} = 3\ \Omega$.

Since $Z_{11} = Z_{22}$; $Z_{12} = Z_{21}$, the given network is symmetrical and reciprocal

$\therefore$ the parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 1\ \Omega$$

$$Z_b = Z_{11} + Z_{12} = 5\ \Omega$$

The lattice network is shown in Fig. 16.33.

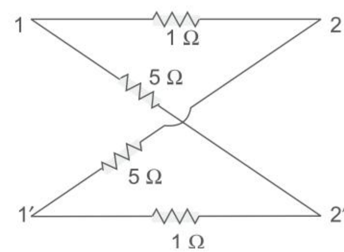


Fig. 16.33

EXAMPLE 16.14

Obtain the lattice equivalent of a symmetric π-network shown in Fig. 16.34.

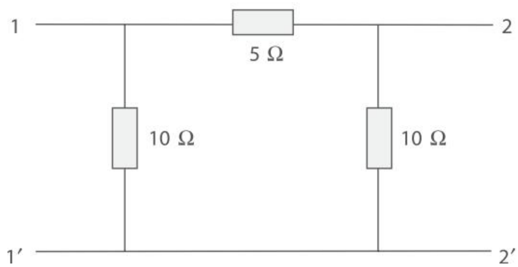


Fig. 16.34

Solution The Z-parameters of the given network are

$$Z_{11} = 6\ \Omega = Z_{22};\ Z_{12} = Z_{21} = 4\ \Omega$$

Hence, the parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 2\ \Omega$$

$$Z_b = Z_{11} + Z_{12} = 10\ \Omega$$

The lattice network is shown in Fig.16.35

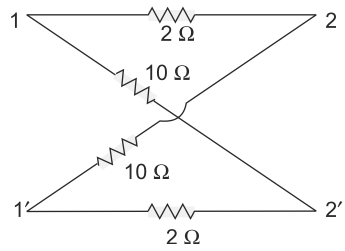


Fig. 16.35

16.13 IMAGE PARAMETERS

LO 10 Determine the image parameters of a two-port network

The image impedance Z_{I1} and Z_{I2} of a two-port network shown in Fig. 16.36 are two values of impedance such that, if the port 1-1′ of the network is terminated in Z_{I1}, the input impedance of the port 2-2′ is Z_{I2}; and if the port 2-2′ is terminated in Z_{I2}, the input impedance at the port 1-1′ is Z_{I1}.

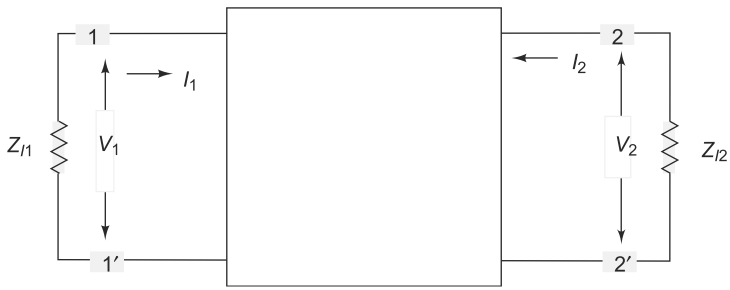


Fig. 16.36

Then, Z_{I1} and Z_{I2} are called image impedances of the two-port network shown in Fig. 16.36. These parameters can be obtained in terms of two-port parameters. Recalling Eqs (16.5) and (16.6) in Section 16.4, we have

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

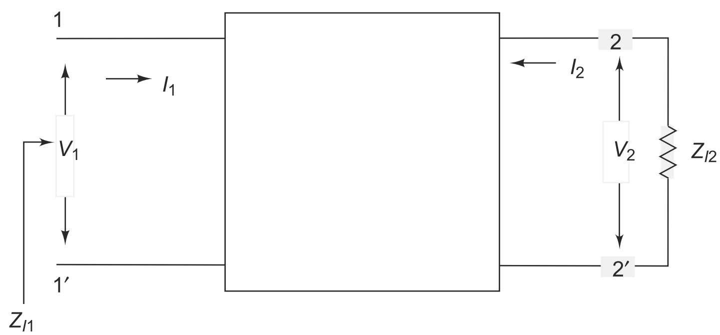


Fig. 16.37

If the network is terminated in Z_{I2} at 2-2′ as shown in Fig. 16.37,

$$V_2 = -I_2 Z_{I2}$$

$$\frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2} = Z_{I1}$$

$$Z_{I1} = \frac{-AI_2 Z_{I2} - BI_2}{-CI_2 Z_{I2} - DI_2}$$

$$Z_{I1} = \frac{-AZ_{I2} - B}{-CZ_{I2} - D}$$

or $$Z_{I1} = \frac{AZ_{I2} + B}{CZ_{I2} + D}$$

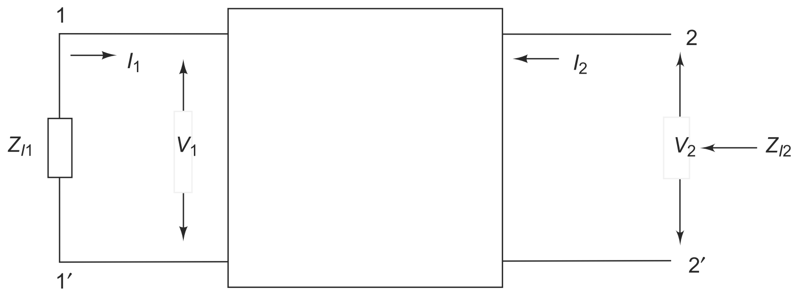


Fig. 16.38

Similarly, if the network is terminated in Z_{I1} at the port 1-1′ as shown in Fig. 16.38, then

$$V_1 = -I_1 Z_{I1}$$

$$\frac{V_2}{I_2} = Z_{I2}$$

$$\therefore \quad -Z_{I1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

$$-Z_{I1} = \frac{AI_2 Z_{I2} - BI_2}{CI_2 Z_{I2} - DI_2}$$

$$-Z_{I1} = \frac{AZ_{I2} - B}{CZ_{I2} - D}$$

from which, $$Z_{I2} = \frac{DZ_{I1} + B}{CZ_{I1} + A}$$

Substituting the value of Z_{I1} in the above equation,

$$Z_{I2}\left[C\frac{(-AZ_{I2} + B)}{(CZ_{I2} - D)} + A\right] = D\left[\frac{-AZ_{I2} + B}{CZ_{I2} - D}\right] + B$$

from which, $Z_{I2} = \sqrt{\dfrac{BD}{AC}}$

Similarly, we can find $Z_{I1} = \sqrt{\dfrac{AB}{CD}}$

If the network is symmetrical, then $A = D$

$$\therefore \quad Z_{I1} = Z_{I2} = \sqrt{\frac{B}{C}}$$

If the network is symmetrical, the image impedances Z_{I1} and Z_{I2} are equal to each other; the image impedance is then called the *characteristic* impedance, or the *iterative* impedance, i.e. if a symmetrical network is terminated in Z_L, its input impedance will also be Z_L, or its impedance transformation ratio is unity. Since a reciprocal symmetric network can be described by two independent parameters, the image parameters Z_{I1} and Z_{I2} are sufficient to characterise reciprocal symmetric networks. Z_{I1} and Z_{I2}, the two image parameters, do not completely define a network. A third parameter called *image transfer constant* ϕ is also used to describe reciprocal networks. This parameter may be obtained from the voltage and current ratios.

If the image impedance Z_{I2} is connected across the port 2-2′, then

$$V_1 = AV_2 - BI_2 \tag{16.20}$$

$$V_2 = -I_2 Z_{I2} \tag{16.21}$$

$$\therefore \quad V_1 = \left[A + \frac{B}{Z_{I2}}\right] V_2 \tag{16.22}$$

$$I_1 = CV_2 - DI_2 \tag{16.23}$$

$$I_1 = -[CZ_{I2} + D]I_2 \tag{16.24}$$

From Eq. (16.22),

$$\frac{V_1}{V_2} = \left[A + \frac{B}{Z_{I2}}\right] = A + B\sqrt{\frac{AC}{BD}}$$

$$\frac{V_1}{V_2} = A + \sqrt{\frac{ABCD}{D}} \tag{16.25}$$

From Eq. (16.24),

$$\frac{-I_1}{I_2} = [CZ_{I2} + D] = D + C\sqrt{\frac{BD}{AC}}$$

$$\frac{-I_1}{I_2} = D + \sqrt{\frac{ABCD}{A}} \tag{16.26}$$

Multiplying Eqs (16.25) and (16.26), we have

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\frac{AD + \sqrt{ABCD}}{D}\right)\left(\frac{AD + \sqrt{ABCD}}{A}\right)$$

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\sqrt{AD} + \sqrt{BC}\right)^2$$

or $$\sqrt{AD} + \sqrt{BC} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}}$$

$$\sqrt{AD} + \sqrt{AD-1} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}} \quad (\because AD - BC = 1)$$

Let $\cos h\phi = \sqrt{AD}$; $\sin h\phi = \sqrt{AD-1}$

$$\tan h\phi = \frac{\sqrt{AD-1}}{\sqrt{AD}} = \sqrt{\frac{BC}{AD}}$$

$$\therefore \quad \phi = \tan h^{-1}\sqrt{\frac{BC}{AD}}$$

Also, $$e^{\phi} = \cos h\phi + \sin h\phi = \sqrt{-\frac{V_1 I_1}{V_2 I_2}}$$

$$\phi = \log_e \sqrt{\left(-\frac{V_1 I_1}{V_2 I_2}\right)} = \frac{1}{2}\log_e\left(\frac{V_1}{V_2}\frac{I_1}{I_2}\right)$$

Since $V_1 = Z_{I1} I_1$; $V_2 = -I_2 Z_{I2}$

$$\phi = \frac{1}{2}\log_e\left[\frac{Z_{I1}}{Z_{I2}}\right] + \log\left[\frac{I_1}{I_2}\right]$$

For symmetrical reciprocal networks, $Z_{I1} = Z_{I2}$

$$\phi = \log_e\left[\frac{I_1}{I_2}\right] = \gamma$$

where γ is called the *propagation constant*.

EXAMPLE 16.15

Determine the image parameters of the T-network shown in Fig.16.39.

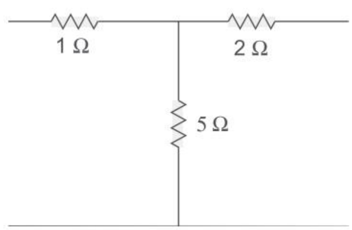


Fig. 16.39

Solution The *ABCD*-parameters of the network are

$$A = \frac{6}{5};\ B = \frac{17}{5};\ C = \frac{1}{5};\ D = \frac{7}{5} \quad \text{(See Ex. 16.3)}$$

Since the network is not symmetrical, ϕ, Z_{I1}, and Z_{I2} are to be evaluated to describe the network.

$$Z_{I1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{\frac{6}{5}\times\frac{17}{5}}{\frac{1}{5}\times\frac{7}{5}}} = 3.817\ \Omega$$

$$Z_{I2} = \sqrt{\frac{BC}{AC}} = \sqrt{\frac{\frac{17}{5}\times\frac{7}{5}}{\frac{6}{5}\times\frac{1}{5}}} = 4.453\,\Omega$$

$$\phi = \tan h^{-1}\sqrt{\frac{BC}{AD}} = \tan h^{-1}\sqrt{\frac{17}{42}}$$

or $\quad \phi = \ln\left[\sqrt{AD} + \sqrt{AD-1}\right]$

$$\phi = 0.75$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 10

★★★**16-10.1** Obtain the image parameters of the symmetric lattice network given in Fig. Q.1.

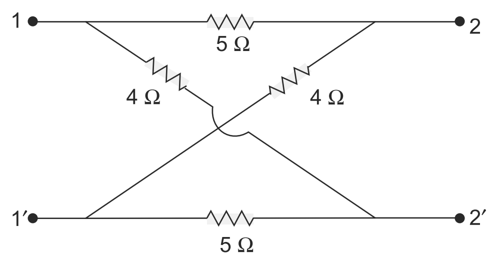


Fig. Q.1

☆★★**16-10.2** Determine the Z-parameters and image parameters of a symmetric lattice network whose series arm impedance is 10 Ω and diagonal arm impedance is 20 Ω.

Frequently Asked Questions linked to LO 10

☆☆★**16-10.1** What are image and iterative impedances? **[BPUT 2007]**

☆☆★**16-10.2** For a π-network having series impedances as Z_1 and shunt impedance is 2 Z_2, what is the image impedance? **[BPUT 2008]**

☆☆★**16-10.3** Explain reciprocal and symmetrical networks. **[PTU 2011-12]**

☆☆★**16-10.4** Determine the image parameters of the T-network shown in Fig. Q.4. **[PTU 2011-12]**

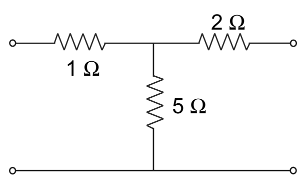


Fig. Q.4

Additional Solved Problems

PROBLEM 16.1

Find the Z-parameters for the circuit shown in Fig. 16.40.

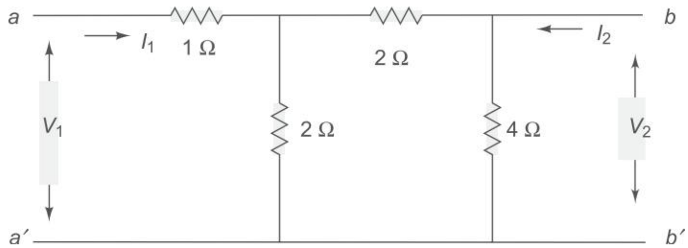


Fig. 16.40

Solution $Z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2=0}$

When $I_2 = 0$; V_1 can be expressed in terms of I_1 and the equivalent impedance of the circuit looking from the terminal a-a' is as shown in Fig. 16.41 (a).

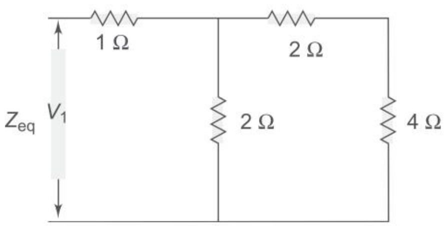


Fig. 16.41 (a)

$$Z_{eq} = 1 + \frac{6\times 2}{6+2} = 2.5\ \Omega$$

$$V_1 = I_1\, Z_{eq} = I_1\, 2.5$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 2.5\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

V_2 is the voltage across the 4 Ω impedance as shown in Fig. 16.41 (b).

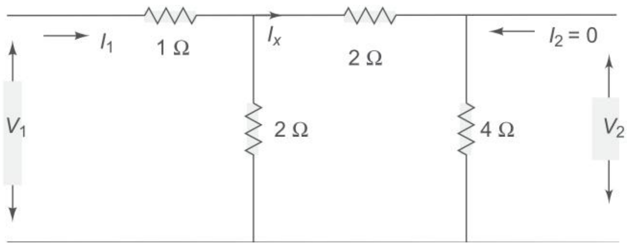


Fig. 16.41 (b)

Let the current in the 4 Ω impedance be I_x.

$$I_x = I_1 \times \frac{2}{8} = \frac{I_1}{4}$$

$$V_2 = I_x 4 = \frac{I_1}{4}\times 4 = I_1$$

$$Z_{21} = \left.\frac{V_2}{I_2}\right|_{I_2=0} = 1\Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

When the port a-a' is open-circuited, the voltage at the port b-b' can be expressed in terms of I_2, and the equivalent impedance of the circuit viewed from b-b' is as shown in Fig. 16.41 (c).

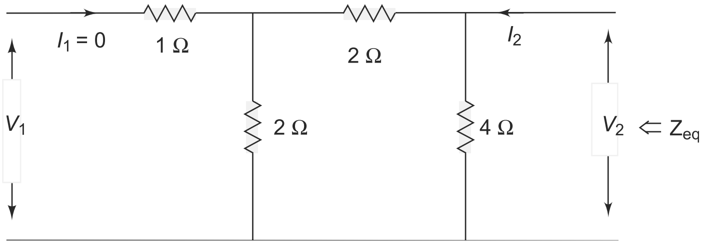


Fig. 16.41 (c)

$$V_2 = I_2 \times 2$$

$$\therefore \quad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 2\,\Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

V_1 is the voltage across the 2 Ω (parallel) impedance and let the current in the 2 Ω (parallel) impedance is I_Y as shown in Fig. 16.41 (d).

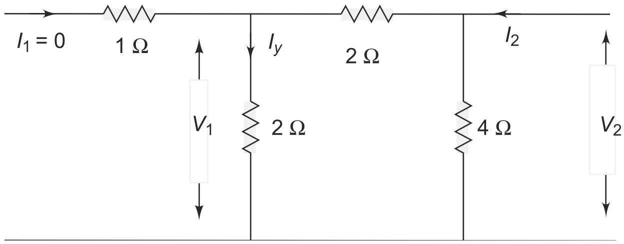


Fig. 16.41 (d)

$$I_Y = \frac{I_2}{2}$$

$$V_1 = 2I_Y$$

$$V_1 = 2\frac{I_2}{2}$$

$$\therefore \quad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 1\,\Omega$$

Here, $Z_{12} = Z_{21}$, which indicates the bilateral property of the network. The describing equations for this two-port network in terms of impedance parameters are

$$V_1 = 2.5I_1 + I_2$$

$$V_2 = I_1 + 2I_2$$

PROBLEM 16.2

Find the short-circuit admittance parameters for the circuit shown in Fig. 16.42.

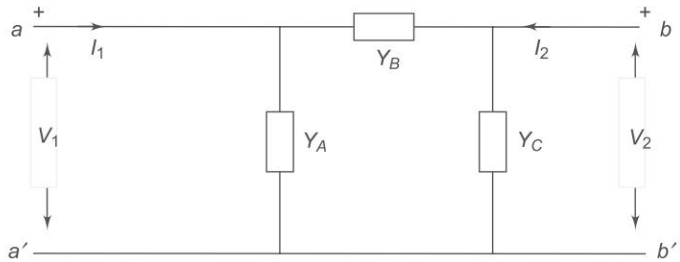


Fig. 16.42

Solution The elements in the branches of the given two-port network are admittances. The admittance parameters can be determined by short-circuiting the two-ports.

When the port *b-b′* is short-circuited, $V_2 = 0$. This circuit is shown in Fig. 16.43 (a).

$$V_1 = I_1 Z_{eq}$$

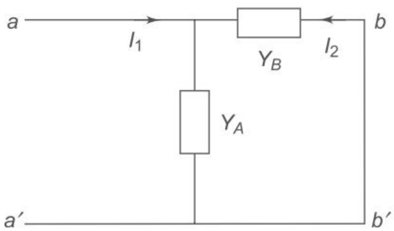


Fig. 16.43 (a)

where Z_{eq} is the equivalent impedance as viewed from *a-a′*.

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_A + Y_B$$

$$V_1 = \frac{I_1}{Y_A + Y_B}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = (Y_A + Y_B)$$

With the port *b-b′* short-circuited, the nodal equation at the node 1 gives

$$-I_2 = V_1 Y_B$$

$$\therefore \quad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -Y_B$$

When the port *a-a′* is short-circuited; $V_1 = 0$. This circuit is shown in Fig. 16.43 (b).

$$V_2 = I_2 Z_{eq}$$

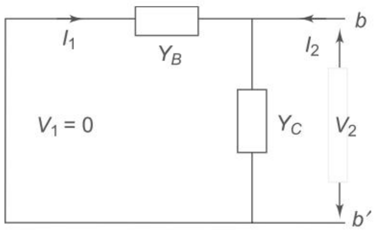


Fig. 16.43 (b)

where Z_{eq} is the equivalent impedance as viewed from *b-b′*.

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_b + Y_c$$

$$\therefore \quad V_2 = \frac{I_2}{Y_B + Y_C}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = (Y_B + Y_C)$$

With the port *a-a′* short-circuited, the nodal equation at the node 2 gives

$$-I_1 = V_2\, Y_B$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_B$$

The describing equations in terms of the admittance parameters are

$$I_1 = (Y_A + Y_B)V_1 - Y_B V_2$$

$$I_2 = -\,Y_B V_1 + (Y_C + Y_B)V_2$$

PROBLEM 16.3

Find the Z-parameters of the RC ladder network shown in Fig. 16.44.

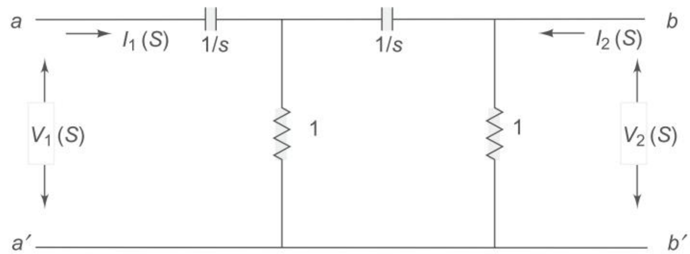


Fig. 16.44

Solution With the port *b-b′* open-circuited and assuming mesh currents with $V_1(S)$ as the voltage at *a-a′*, the corresponding network is shown in Fig. 16.45 (a).

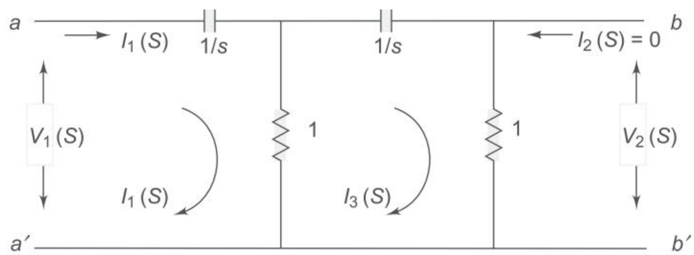


Fig. 16.45 (a)

The KVL equations are as follows:

$$V_2(S) = I_3(S) \tag{16.27}$$

$$I_3(S) \times \left(2 + \frac{1}{S}\right) = I_1(S) \tag{16.28}$$

$$\left(1 + \frac{1}{S}\right) I_1(S) - I_3(S) = V_1(S) \tag{16.29}$$

From Eq. (16.28), $I_3(S) = I_1(S)\left(\dfrac{S}{1+2S}\right)$

From Eq. (16.29), $\left(\dfrac{S+1}{S}\right)I_1(S) - I_1(S)\dfrac{S}{1+2S} = V_1(S)$

$$I_1(S)\left(\frac{1+S}{S} - \frac{S}{1+2S}\right) = V_1(S)$$

$$I_1(S)\left(\frac{S^2+3S+1}{S(1+2S)}\right) = V_1(S)$$

$$Z_{11} = \left.\frac{V_1(S)}{I_1(S)}\right|_{I_2=0} = \frac{(S^2+3S+1)}{S(1+2S)}$$

Also, $V_2(S) = I_3(S) = I_1(S)\dfrac{S}{1+2S}$

$$Z_{21} = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2=0} = \frac{S}{1+2S}$$

With the port *a*-*a*′ open-circuited and assuming mesh currents with $V_2(S)$ as the voltage as *b*-*b*′, the corresponding network is shown in Fig. 16.45 (b).

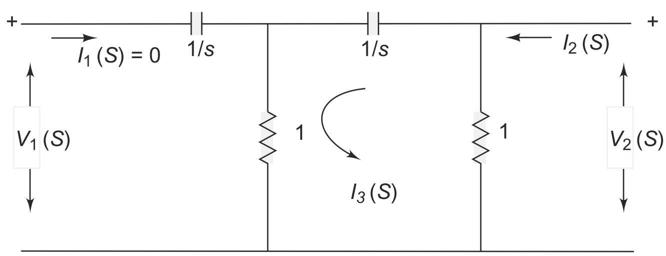


Fig. 16.45 (b)

The KVL equations are as follows:

$$V_1(S) = I_3(S) \tag{16.30}$$

$$\left(2+\frac{1}{S}\right)I_3(S) = I_2(S) \tag{16.31}$$

$$V_2(S) = I_2(S) - I_3(S) \tag{16.32}$$

From Eq. (16.31), $I_3(S) = I_2(S)\left(\dfrac{S}{2S+1}\right)$

From Eq. (16.32), $V_2(S) = I_2(S) - I_2(S)\left(\dfrac{S}{2S+1}\right)$

$$V_2(S) = I_2(S)\left(1 - \frac{S}{2S+1}\right)$$

$$Z_{22} = \left.\frac{V_2(S)}{I_2(S)}\right|_{I_1(S)=0} = \frac{S+1}{2S+1}$$

Also, $$V_1(S) = I_3(S) = I_2(S)\left(\frac{S}{2S+1}\right)$$

$$Z_{12} = \left.\frac{V_1(S)}{I_2(S)}\right|_{I_1(S)=0} = \left(\frac{S}{2S+1}\right)$$

The describing equations are

$$V_1(S) = \left[\frac{S^2+3S+1}{3(2S+1)}\right]I_1 + \left[\frac{S}{2S+1}\right]I_2$$

$$V_2(S) = \left[\frac{S}{2S+1}\right]I_1 + \left[\frac{S+1}{2S+1}\right]I_2$$

PROBLEM 16.4

Find the transmission parameters for the circuit shown in Fig. 16.46.

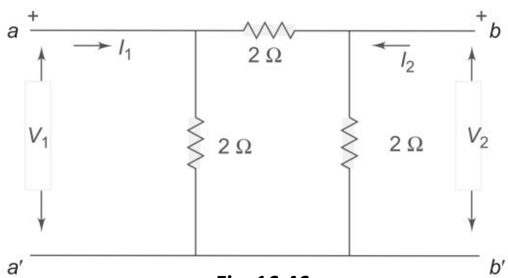


Fig. 16.46

Solution Recalling Eqs (16.5) and (16.6), we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

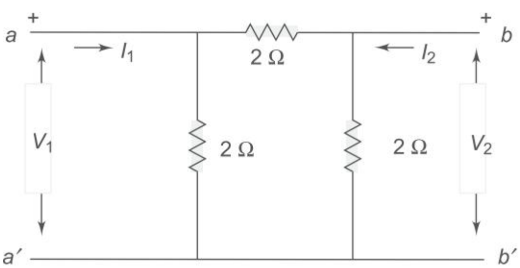


Fig. 16.47 (a)

When the port b-b' is short- circuited with V_1 across a-a', $V_2 = 0$, $B = \dfrac{-V_1}{I_2}$ and the circuit is as shown in Fig. 16.47 (a).

$$-I_2 = \frac{V_1}{2}I_1 = V_1$$

$\therefore$ $B = 2\ \Omega$

$$D = \frac{-I_1}{I_2} = 2$$

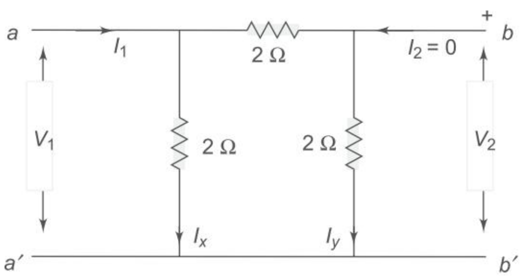


Fig. 16.47 (b)

When the port b-b' is open-circuited with V_1 across a-a', $I_2 = 0$, $A = V_1/V_2$ and the circuit is as shown in Fig. 16.47 (b), where V_1 is the voltage across the 2 Ω resistor across the port a-a' and V_2 is the voltage across the 2 Ω resistor across the port b-b' when $I_2 = 0$.

From Fig. 16.47 (b), $\quad I_Y = \dfrac{V_1}{4}$

$$V_2 = 2 \times I_Y = \frac{V_1}{2}$$

$$A = 2$$

From Fig. 16.47 (b),

$$I_x = \frac{V_1}{2}$$

$$C = \frac{I_1}{V_2}$$

where

$$I_1 = \frac{3V_1}{4}$$

Therefore,

$$C = \frac{3}{2}\ ℧$$

PROBLEM 16.5

Find the h- parameters for the network in Fig. 16.48.

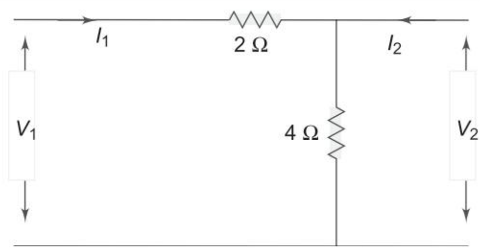


Fig. 16.48

Solution When $V_2 = 0$, the network is as shown in Fig. 16.49.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 2\,\Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}; I_2 = -I_1$$

$$\therefore \quad h_{21} = -1$$

When $\quad I_1 = 0; h_{12} = \left.\dfrac{V_1}{V_2}\right|_{I_1=0}; h_{22} = \left.\dfrac{I_2}{V_2}\right|_{I_1=0}$

$$V_1 = I_2\,4,\ V_2 = I_2\,4$$

$$\therefore \quad h_{12} = 1\ h_{22} = \frac{1}{4}\ ℧$$

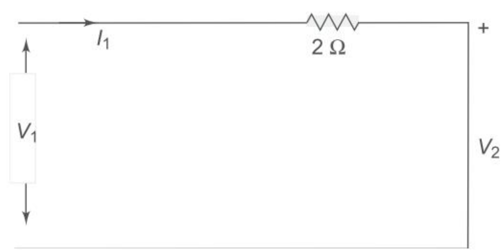


Fig. 16.49

PROBLEM 16.6

For the hybrid equivalent circuit shown in Fig. 16.50, (a) determine the current gain, and (b) determine the voltage gain.

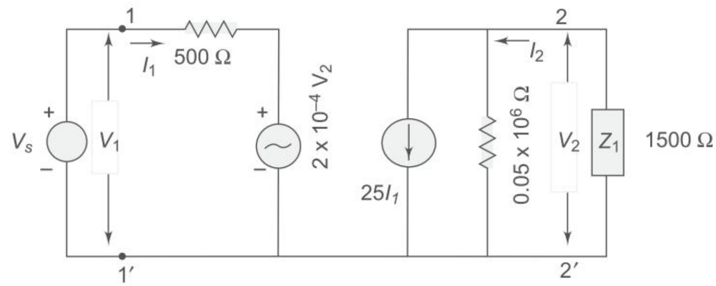


Fig. 16.50

Solution From the port 2-2′ we can find

$$I_2 = \frac{(25I_1)(0.05 \times 10^6)}{(1500 + 0.05 \times 10^6)}$$

(a) Current gain $\dfrac{I_2}{I_1} = \dfrac{1.25 \times 10^6}{0.0515 \times 10^6} = 24.3$

(b) Applying KVL at the port 1-1′,

$$V_1 = 500\, I_1 + 2 \times 10^{-4}\, V_2$$

$$I_1 = \frac{V_1 - 2 \times 10^{-4} V_2}{500} \tag{16.33}$$

Applying KCL at the port 2-2′,

$$I_2 = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

also $$I_2 = \frac{-V_2}{1500}$$

$$\frac{-V_2}{1500} = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

Substituting the value of I_1 from Eq. (16.33), in the above equation, we get

$$\frac{-V_2}{1500} = 25\left(\frac{V_1 - 2 \times 10^{-4} V_2}{500}\right) + \frac{V_2}{0.05} \times 10^{-6}$$

$$-6.6 \times 10^{-4}\, V_2 = 0.05V_1 - 0.1 \times 10^{-4}\, V_2 + 0.2 \times 10^{-4}\, V_2$$

$$\therefore \quad \frac{V_2}{V_1} = -73.89$$

The negative sign indicates that there is a 180° phase shift between input and output voltages.

PROBLEM 16.7

The hybrid parameters of a two-port network shown in Fig. 16.51 are h_{11} = 1 K; h_{12} = 0.003; h_{21} = 100; h_{22} = 50 μ℧. Find V_2 and Z-parameters of the network.

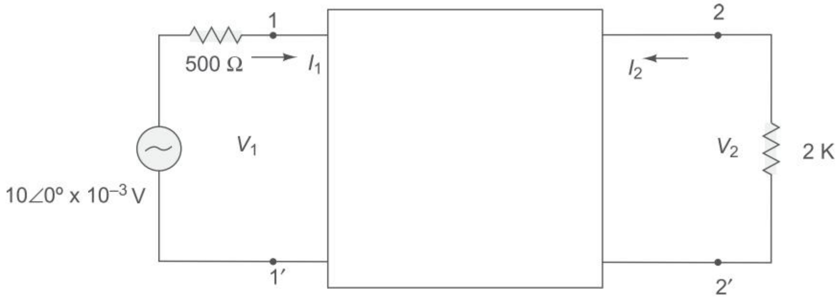


Fig. 16.51

Solution

$$V_1 = h_{11} I_1 + h_{12} V_2 \tag{16.34}$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \tag{16.35}$$

At the port 2-2′, $V_2 = -I_2\, 2000$

Substituting in Eq. (16.35), we have

$$I_2 = h_{21}I_1 - h_{22}I_2\, 2000$$

$$I_2\,(1 + h_{22}\, 2000) = h_{21}\, I_1$$

$$I_2(1 + 50 \times 10^{-6} \times 2000) = 100\, I_1$$

$$I_2 = \frac{100 I_1}{1.1}$$

Substituting the value of V_2 in Eq. (16.34), we have

$$V_1 = h_{11}\, I_1 - h_{12}\, I_2\, 2000$$

Also, at the port 1-1′, $V_1 = V_S - I_1\, 500$

$$\therefore \quad V_s - I_1 500 = h_{11}I_1 - h_{12}\frac{100 I_1}{1.1}\times 2000$$

$$\left(10\times 10^{-3}\right) - 500 I_1 = 1000 I_1 - 0.003\times\frac{100}{1.1} I_1 \times 2000$$

$$954.54 I_1 = 10 \times 10^{-3}$$

$$I_1 = 10.05 \times 10^{-6}\text{ A}$$

$$V_1 = V_S - I_1 \times 500$$

$$= 10 \times 10^{-3} - 10.5 \times 10^{-6} \times 500 = 4.75 \times 10^{-3}\text{ V}$$

$$V_2 = \frac{V_1 - h_{11}I_1}{h_{12}}$$

$$V_2 = \frac{4.75\times10^{-3} - 1000\times10.5\times10^{-6}}{0.003} = -1.916 \text{ V}$$

(b) Z-parameters of the network can be found from Table 16.1.

$$Z_{11} = \frac{\Delta_h}{h_{22}} = \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{22}} = \frac{1\times10^3\times50\times10^{-6} - 100\times0.003}{50\times10^{-6}}$$

$$= -5000\ \Omega$$

$$Z_{12} = \frac{h_{12}}{h_{22}} = \frac{0.003}{50\times10^{-6}} = 60\ \Omega$$

$$Z_{21} = \frac{-h_{21}}{h_{22}} = \frac{-100}{50\times10^{-6}} = -2\times10^6\ \Omega$$

$$Z_{22} = \frac{1}{h_{22}} = 20\times10^3\ \Omega$$

PROBLEM 16.8

The Z-parameters of a two-port network shown in Fig. 16.52 are $Z_{11} = Z_{22} = 10\ \Omega$; $Z_{21} = Z_{12} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, V_2, I_2, and input impedance.

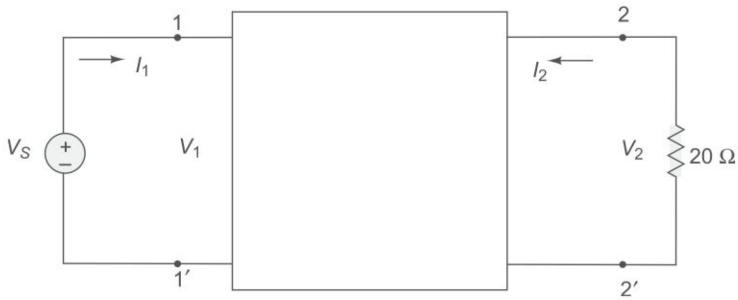


Fig. 16.52

Solution Given $V_1 = V_S = 20$ V

From Section 16.11.1, $V_1 = I_1\left(Z_{11} - \dfrac{Z_{12}Z_{21}}{Z_L + Z_{22}}\right)$

where $Z_L = 20\ \Omega$

$$\therefore \quad 20 = I_1\left(10 - \frac{4\times4}{20+10}\right)$$

$$I_1 = 2.11 \text{ A}$$

$$I_2 = -I_1\frac{Z_{21}}{Z_L + Z_{22}} = -2.11\times\frac{4}{20+10} = -0.281 \text{ A}$$

At the port 2-2′,

$$V_2 = -I_2 \times 20 = 0.281 \times 20 = 5.626 \text{ V}$$

$$\text{Input impedance} = \frac{V_1}{I_1} = \frac{20}{2.11} = 9.478\ \Omega$$

PROBLEM 16.9

The Y-parameters of the two-port network shown in Fig. 16.53 are $Y_{11} = Y_{22} = 6$ ℧ ; $Y_{12} = Y_{21} = 4$ ℧ Determine the driving-point admittance at the port 2-2′ if the source voltage is 100 V and has an impedance of 1 ohm.

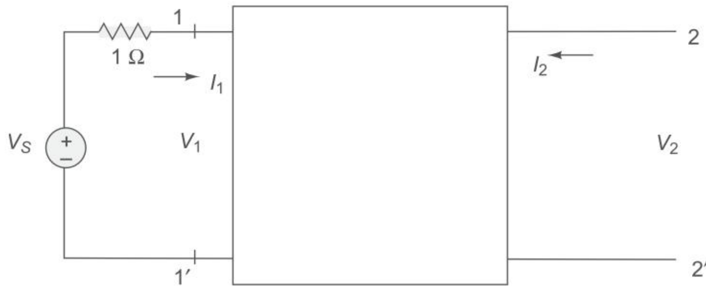


Fig. 16.53

Solution From Section 16.11.2,

$$\frac{I_2}{V_2} = \frac{Y_{22}Y_s + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_s + Y_{11}}$$

where Y_S is the source admittance = 1 ℧

$$\therefore \quad \text{the driving-point admittance} = \frac{6\times1 + 6\times6 - 4\times4}{1+6} = 3.714\ ℧$$

Or the driving-point impedance at the port 2-2′ $= \dfrac{1}{3.714}\ \Omega$

PROBLEM 16.10

Obtain the Z-parameters for the two-port unsymmetrical lattice network shown in Fig. 16.54.

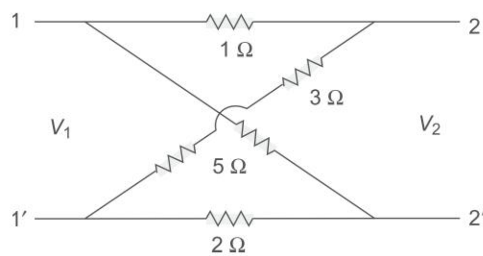


Fig. 16.54

Solution From Section 16.12, we have

$$Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+3)(2+5)}{1+3+5+2} = 2.545\ \Omega$$

$$Z_{21} = \frac{Z_b Z_c - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d} = \frac{3\times5 - 1\times2}{11} = 1.181\ \Omega$$

$$Z_{21} = Z_{12}$$

$$Z_{22} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+5)(2+3)}{11} = 2.727\ \Omega$$

PROBLEM 16.11

For the ladder two-port network shown in Fig. 16.55, find the open-circuit driving-point impedance at the port 1-2.

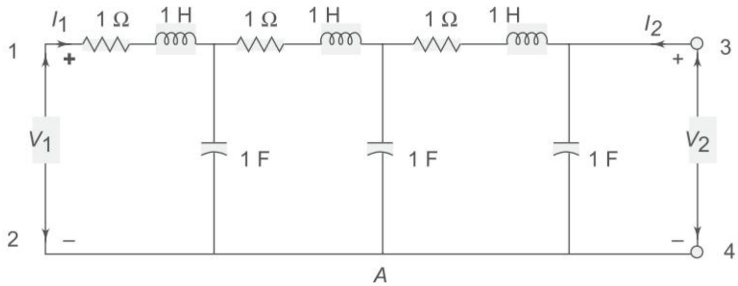


Fig. 16.55

Solution The Laplace transform of the given network is shown in Fig. 16.56.

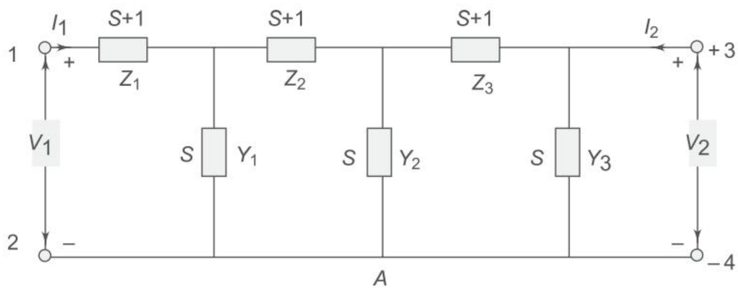


Fig. 16.56

Then the open-circuit driving-point impedance at the port 1-2 is given by

$$Z_{11} = (s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s}}}}}$$

$$= \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

PROBLEM 16.12

For the bridged T-network shown in Fig. 16.57, find the driving- point admittance y_{11} and transfer admittance y_{21} with a 2 Ω load resistor connected across the port 2.

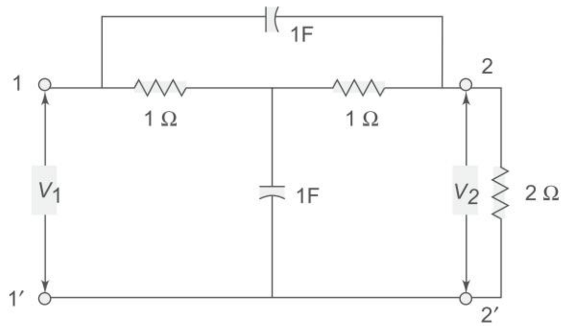


Fig. 16.57

Solution The corresponding Laplace transform network is shown in Fig. 16.58.

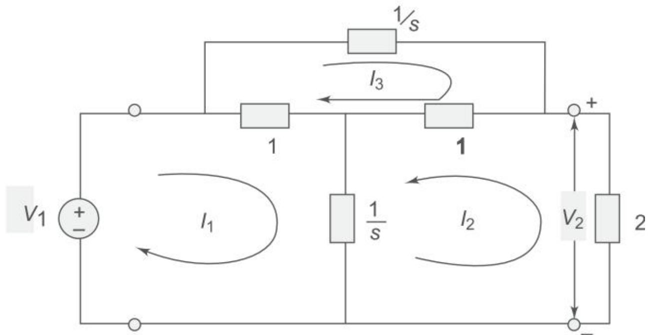


Fig. 16.58

The loop equations are

$$I_1\left(1+\frac{1}{s}\right)+I_2\left(\frac{1}{s}\right)-I_3=V_1$$

$$I_1\left(\frac{1}{s}\right)+I_2\left(1+\frac{1}{s}\right)+I_3=0$$

$$I_1(-1)+I_2+I_3\left(2+\frac{1}{s}\right)=0$$

Therefore,

$$\Delta=\begin{vmatrix}\left(1+\frac{1}{s}\right) & \frac{1}{s} & -1\\ \frac{1}{s} & 1+\frac{1}{s} & 1\\ -1 & 1 & 2+\frac{1}{s}\end{vmatrix}=\frac{s+2}{s^2}$$

Similarly, $$\Delta_{11}=\begin{vmatrix}\left(1+\frac{1}{s}\right) & \frac{1}{s}\\ 1 & \left(2+\frac{1}{s}\right)\end{vmatrix}=\frac{s^2+3s+1}{s^2}$$

$$\text{and} \quad \Delta_{12} = \begin{vmatrix} \frac{1}{s} & +1 \\ +1 & \left(2+\frac{1}{s}\right) \end{vmatrix} = \frac{s^2+2s+1}{s^2}$$

$$\text{Hence,} \quad Y_{11} = \frac{\Delta_{11}}{\Delta} = \frac{s^2+3s+1}{s+2}$$

$$\text{and} \quad Y_{21} = \frac{\Delta_{12}}{\Delta} = \frac{-(s^2+2s+1)}{s+2}$$

PROBLEM 16.13

For the two-port network shown in Fig. 16.59, determine the h-parameters. Using these parameters, calculate the output (Port 2) voltage, V_2, when the output port is terminated in a 3 Ω resistance and a 1V(dc) is applied at the input port ($V_1 = 1$ V).

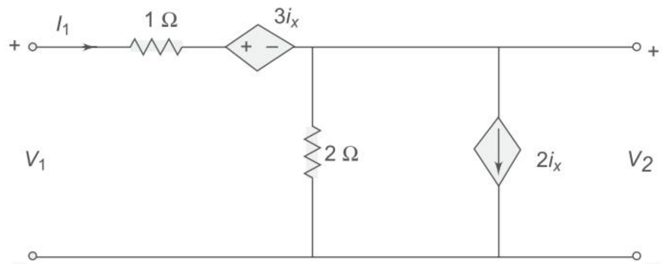


Fig. 16.59

Solution The h-parameters are defined as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

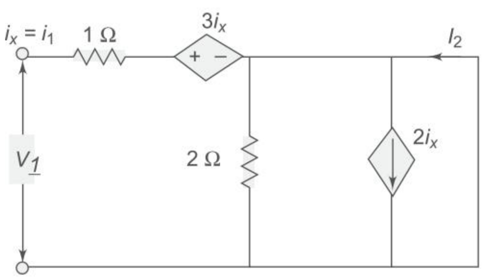


Fig. 16.60 (a)

For $V_2 = 0$, the circuit is redrawn as shown in Fig. 16.60 (a).

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{i_1 \times 1 + 3i_1}{i_1} = 4$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{i_2}{i_1} = \frac{2i_1 - i_1}{i_1} = 1$$

For $I_1 = 0$, the circuit is redrawn as shown in Fig. 16.60 (b).

$$h_{12} = \frac{V_1}{V_2} = 1; \; h_{22} = \frac{I_2}{V_2} = \frac{1}{2} = 0.5$$

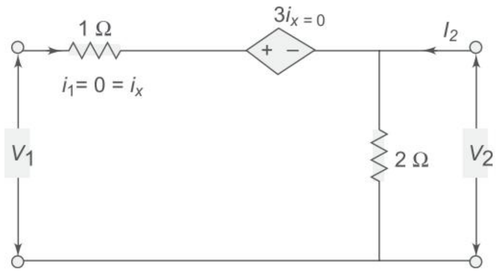


Fig. 16.60 (b)

Hence, $h = \begin{bmatrix} 4 & 1 \\ 1 & 0.5 \end{bmatrix}$

$V_1 = 1$ V

$V_1 = 4I_1 + V_2$

$I_2 = I_1 + 0.5\, V_2$

Eliminating I_1 from the above equations and putting

$V_1 = 1$ and $I_2 = \frac{-V_2}{3}$, we get $V_2 = \frac{-3}{7}$ V

PROBLEM 16.14

Find the current transfer ratio $\frac{I_2}{I_1}$ *for the network shown in Fig. 16.61.*

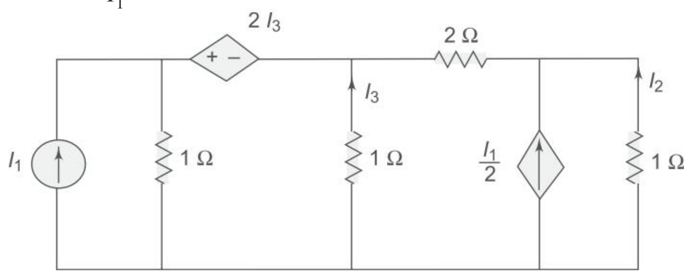


Fig. 16.61

Solution By transforming the current source into voltage source, the given circuit can be redrawn as shown in Fig. 16.62.

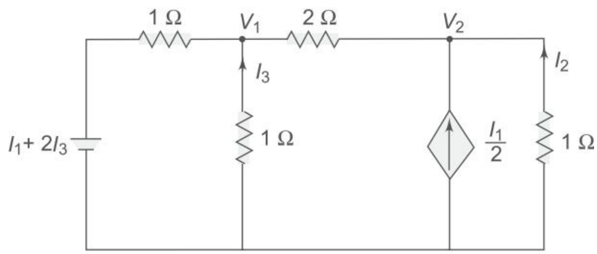


Fig. 16.62

Applying Kirchhoff's nodal analysis,

$$\frac{V_1 - (I_1 + 2I_3)}{1} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0$$

$$\text{and} \quad \frac{V_2 - V_1}{2} - \frac{I_1}{2} - I_2 = 0$$

Putting $V_1 = -I_3$ and $V_2 = -I_2$

The above equations becomes

$$-I_3 - I_1 - 2I_3 - I_3 + \frac{I_2 - I_3}{2} = 0$$

$$\text{and} \quad \frac{I_2 - I_3}{2} - \frac{I_1}{2} - I_2 = 0$$

$$\text{or} \quad I_1\ 0.5I_2 - 4.5\, I_3 = 0$$

$$\text{and} \quad -0.5\, I_1 - 1.5I_2 + 0.5I_3 = 0$$

By eliminating I_3, we get

$$\frac{I_2}{I_1} = \frac{-5.5}{13} = -0.42$$

PROBLEM 16.15

Obtain Y-parameters of the two-port network shown in Fig. 16.63.

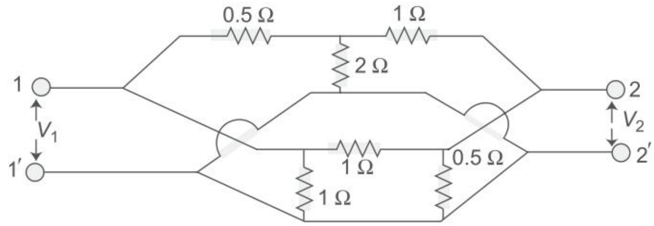


Fig. 16.63

Solution The above network is the parallel connection of two-port networks. *Y*-parameters of such networks can be found by finding individual *Y*-parameters of the respective networks.

T-and π-networks of the above figure are shown separately.

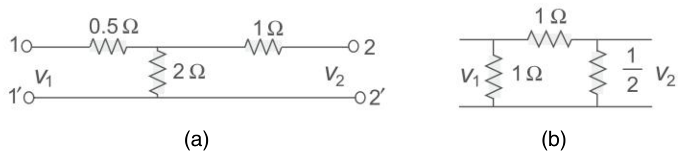


Fig. 16.64

Y-parameters of the *T*-network are given by

$$Y_{11} = \frac{6}{7};\ Y_{22} = \frac{5}{7};\ Y_{21} = Y_{12} = \frac{-4}{7}$$

The *Y*-parameters of the π-network are given by

$$Y_{11} = 2;\ Y_{12} = 1;\ Y_{22} = 3;\ Y_{21} = -1$$

Y-parameters of the combination are given by

$$Y_{11} = \frac{6}{7} + 2 = \frac{20}{7},\ Y_{22} = \frac{5}{7} + 3 = \frac{26}{7}$$

$$Y_{12} = Y_{21} = \frac{-4}{7} - 1 = \frac{-5}{7}$$

PROBLEM 16.16

Find the transmission parameters for the network shown in Fig. 16.65.

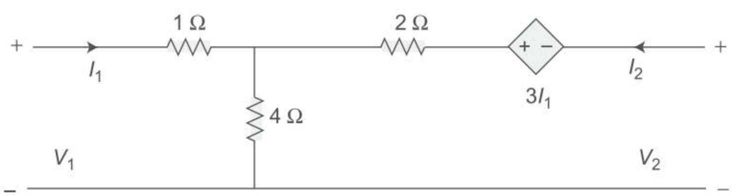


Fig. 16.65

Solution Equations of transmission parameters are

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} ; \; -B = \left.\frac{V_1}{I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} ; \; -D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

When $I_2 = 0$; $V_1 = I_1 + 4I_1 = 5I_1$

and $-V_2 - 3I_1 + 4I_1 = 0$

$$V_2 = I_1 \Rightarrow \frac{I_1}{I_2} = 1$$

$$\therefore \quad V_1 = 5V_2 \Rightarrow \frac{V_1}{V_2} = 5$$

When $V_2 = 0$; the network is shown in Fig. 16.66.

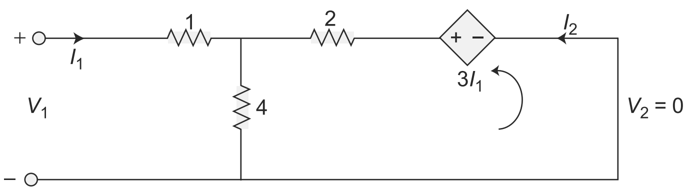


Fig. 16.66

Loop equations $V_1 = 5I_1 + 4I_2$

$$-3I_1 + 2I_2 + 4I_2 + 4I_1 = 0$$

from which, $I_1 = -6I_2 \Rightarrow \dfrac{I_1}{I_2} = -6$

$$\therefore \quad V_1 = -30I_2 + 4I_2$$

$$= -26I_2$$

$$\frac{V_1}{I_2} = -26$$

$$A = \frac{V_1}{V_2} = 5; \; B = \frac{-V_1}{I_2} = 26$$

$$C = \frac{I_1}{V_2} = 1; \; D = \frac{-I_1}{I_2} = 6$$

PROBLEM 16.17

Determine Z- and Y-parameters for the circuit shown in Fig. 16.67.

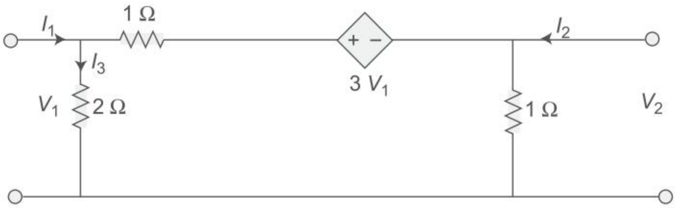


Fig. 16.67

Solution *Z-parameters*

Let I_3 be the current in $2\,\Omega$ resistor

$$V_1 = 2I_3 \Rightarrow I_3 = V_1/2$$

Applying KVL to the outer loop,

$$2I_3 - (I_1 - I_3) - 3V_1 - (I_1 + I_2 - I_3) = 0$$

$$3V_1 = -2I_1 - I_2 + 4I_3$$

$$V_2 = I_1 - I_3 + I_2$$

$$V_2 = I_1 + I_2 - \frac{V_1}{2}$$

$$3V_1 = -2I_1 - I_2 + 2V_1$$

$$V_1 = -2I_1 - I_2$$

$$V_2 = I_1 + I_2 - \frac{1}{2}(-2I_1 - I_2)$$

$$V_2 = 2I_1 + \frac{3}{2}I_2$$

$$\therefore \quad Z_{11} = -2;\ Z_{12} = -1;\ Z_{21} = 2;\ Z_{22} = \frac{3}{2}$$

Y-parameters

From the above equations,

$$V_2 + V_1 = I_1 + I_2 - \frac{V_1}{2} - 2I_1 - I_2$$

$$I_1 = \frac{-3}{2}V_1 - V_2$$

Multiply equations V_2 with 2 and add the equation V_1

$$2V_2 + V_1 = 2I_1 + 2I_2 - V_1$$

$$2V_2 + 2V_1 = 2I_1 + 2I_2$$

$$\text{Also } V_1 = -2I_1 - I_2$$

$$2V_2 + 3V_1 = I_2$$

$$\therefore \quad Y_{11} = \frac{-3}{2}\,℧$$

$$Y_{12} = -1\,℧$$

$$Y_{21} = 3\,℧$$

$$Y_{22} = 2\,℧$$

PROBLEM 16.18

Find Z_{21} and Z_{22} for the network shown in Fig. 16.68.

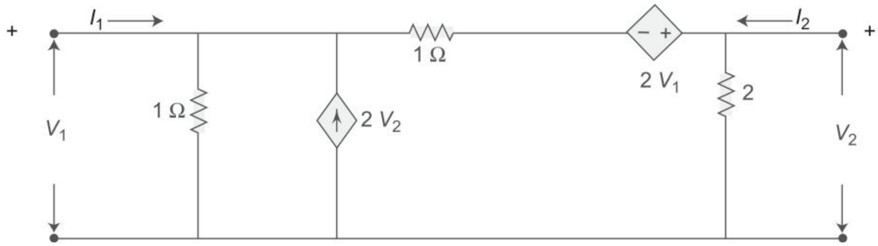


Fig. 16.68

Solution Transforming the dependent current source into voltage source, the network is shown as follows:

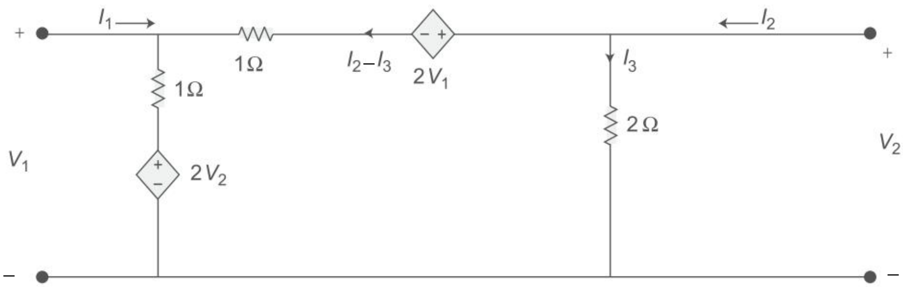


Fig. 16.69

Let I_3 be the current through $2\,\Omega$. KVL to the outer loop,

$$-V_2 + 2V_1 + I_2 - I_3 + V_1 = 0$$

$$-V_2 + 3V_1 + I_2 - I_3 = 0$$

Also, $\quad -V_1 + (I_1 + I_2 - I_3) + 2V_2 = 0$

$$V_1 = I_1 + I_2 - I_3 + 2V_2$$

from which

$$-7V_2 - 3I_1 - 2I_2 + 2I_3 = 0$$

where $\quad I_3 = \dfrac{V_2}{2}$

$$\therefore \quad V_2 = \frac{-I_1}{2} - \frac{I_2}{3}$$

Hence, $\quad Z_{21} = \dfrac{-1}{2};\; Z_{22} = \dfrac{-1}{3}$

PROBLEM 16.19

Obtain the transmission parameters for the following T-network and verify the reciprocity theorem.

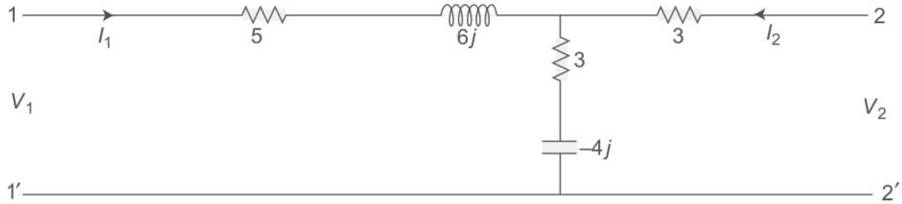


Fig. 16.70

Solution

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

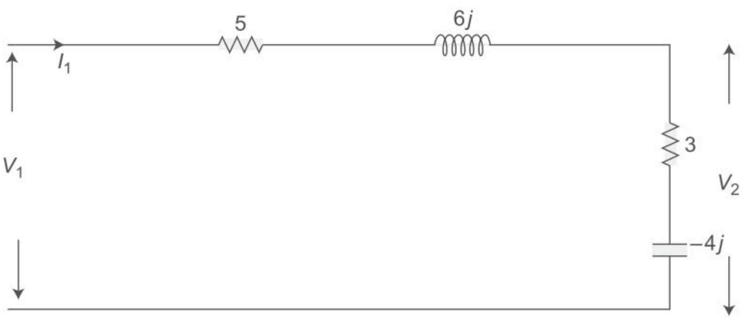


Fig. 16.71

When $I_2 = 0$,

$$V_1 = I_1\,(8 + 2j)$$

$$V_2 = I_1\,(3-4j)$$

$$A = \frac{V_1}{V_2} = \frac{I_1(8+2j)}{I_1(3-4j)} = \frac{8+2j}{3-4j} = 0.64+1.52j$$

$$C = \frac{I_1}{V_2} = \frac{I_1}{I_1(3-4j)} = \frac{1}{3-4j} = 0.12+0.16j\ \mho$$

When $V_2 = 0$,

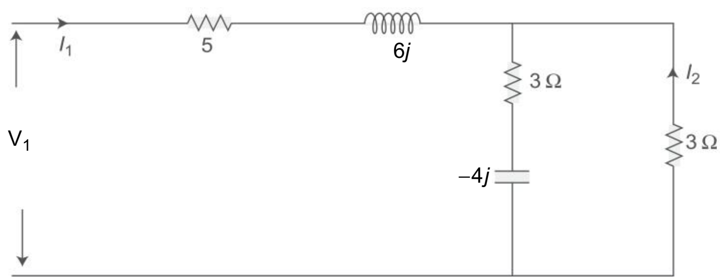


Fig. 16.72

$$B = \frac{-V_1}{I_2}$$

$$-I_2 = \frac{I_1(3-4j)}{6-4j}$$

$$-I_2 = I_1\,(0.65 - 0.23j)$$

$$V_1 = I_1\{(5 + 6j) + [(3 - 4j)\|3]\}$$

$$= I_1[6.96 + 5.3j]$$

$$B = \frac{-V_1}{I_2} = \frac{I_1(6.96+5.3j)}{I_1(0.65-0.23j)}$$

$$\therefore \quad B = 6.95 + 10.61j$$

$$D = -I_1 / I_2 = \frac{I_1}{I_1(0.65-0.23j)} = 1.367 + 0.48j$$

Reciprocity condition is satisfied when $AD - BC = 1$.

$$\left[(0.64+1.52j)\ \ (1.367+0.48j)\right] - \left[(6.95+10.61j)(0.12+0.16j)\right]$$

$$= 1.00 - 1.6\times10^{-4} = 1.008\underline{|-0.009} \simeq 1$$

Hence, the condition of reciprocity is verified.

PSpice Problems

PROBLEM 16.1

Using PSpice, find the Z-parameters for the circuit in Fig. 16.73.

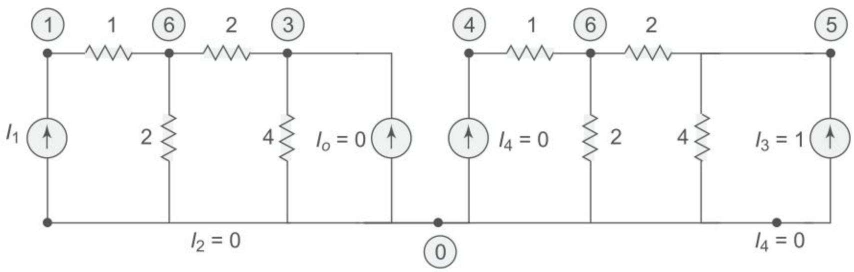


Fig. 16.73

$$V_1 = Z_{11}\,I_1 + Z_{12}\,I_2$$

$$V_2 = Z_{21}\,I_1 + Z_{22}\,I_2$$

```
* DETERMINATION OF Z PARAMETERS
   .SUBCKT AMP 1 3
   R1       1       6       1
```

```
R2      6      3      2
R3      0      6      2
R4      0      3      4
.ENDS
I1      0      1      1
I2      0      3      0
I3      0      5      1
I4      0      4      0
X1      1      3 AMP
X2      4      5 AMP
.OP
.END
**** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
*************************************************************
NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
(1)   2.5000   (3)   1.0000      (4)   1.0000      (5)   2.0000
(X1.6)   1.5000         (X2.6)   1.0000
```

Result

Subcircuit 1 is with $I_2 = 0$ ($I_2 = 0$)

$\Rightarrow$ output port is open-circuited.

$$Z_{11} = \frac{V_1}{I_1} = 2.5\ \Omega$$

$$Z_{21} = \frac{V_3}{I_1} = 1\ \Omega$$

Subcircuit 2 is with $I_4 = 0$

$\Rightarrow$ input port open-circuited.

$$Z_{12} = \frac{V_4}{I_3} = \frac{V_4}{1} = 1\ \Omega$$

$$Z_{22} = \frac{V_5}{I_3} = 2\ \Omega$$

PROBLEM 16.2

Using PSpice, find the transmission parameters for the circuit shown in Fig. 16.74.

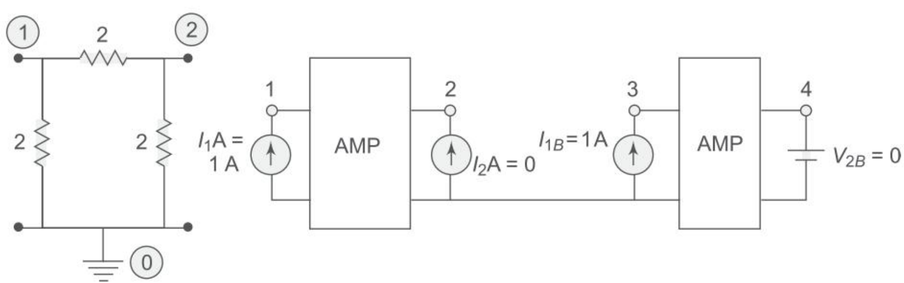


Fig. 16.74

$$V_1 = AV_2 - BI_2$$
$$V_1 = CV_2 - DI_2$$

```
* TRANSMISSION PARAMETERS
   .SUBCKT AMP 1 2
   R1     1     2     2
   R2     1     0     2
   R3     2     0     2
   .ENDS
   I1A    0     1     1
   I2A    0     2     0
   I1B    0     3     1
   V2B    4     0     0
   X1     1     2 AMP
   X2     3     4 AMP
   .OP
   .END
   **** SMALL SIGNAL BIAS SOLUTION TEMPERATURE = 27.000 DEG C
   ***************************************************************
   NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE NODE VOLTAGE
   (1)  1.3333    (2)  .6667    (3)  1.0000    (4)  0.0000
   VOLTAGE SOURCE CURRENTS
   NAME           CURRENT
   V2B            5.000E – 01
```

Result

Subcircuit 1 is with output node open-circuited.

$$A = \frac{V_1}{V_2} = \frac{1.333}{0.667} = 2$$

$$C = \frac{I_1}{V_2} = \frac{1}{0.667} = 1.5\ \mho$$

Subcircuit 2 is with output node short-circuited

$$B = \frac{-V_3}{I_2} = \frac{-1}{0.5} = 2\ \Omega$$

$$D = \frac{-I_1}{I_2} = \frac{-1}{0.5} = 2$$

Answers to Practice Problems

16-2.1 $Z_{11}=\dfrac{Y_B+Y_C}{\Delta Y};Z_{12}=Z_{21}=\dfrac{Y_C}{\Delta Y};Z_{22}=\dfrac{Y_A+Y_C}{\Delta Y}$

$\Delta_Y = Y_A\,Y_B + Y_B\,Y_C + Y_C Y_A$

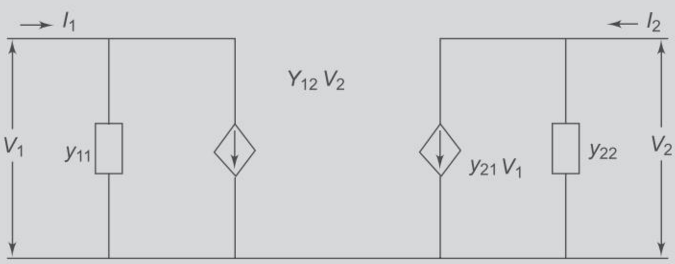
16-2.5 $\begin{bmatrix}5.71 & -4.29\\ 2.14 & 2.14\end{bmatrix}$

16-2.6 $A' = 3; B' = 2; C' = 4; D' = 3$

16-2.7 $Y_{11} = (0.5 - j0.2)10^{-3};$

$Y_{12} = Y_{21} = (j0.2 \times 10^{-3})$

$Y_{22} = j(0.02 \times 10^{-3})$



16-2.11 $Y_{11}= -0.5$ $Y_{12}=0.25$ $Y_{22}= 0.2$

16-2.12 $Z=\begin{bmatrix}0.75 & 0.25\\ 0.25 & 0.75\end{bmatrix};\quad Y=\begin{bmatrix}1.5 & -0.5\\ -0.5 & 1.5\end{bmatrix}$

16-2.13 $Y_{11}=\dfrac{2}{3};\quad Y_{12}=\dfrac{-1}{3};\quad Y_{21}=\dfrac{5}{3};\quad Y_{22}=\dfrac{-1}{3}$

16-4.1 $h_{11}=\dfrac{4}{3};h_{21}=\dfrac{-2}{3};h_{22}=\dfrac{1}{6};h_{12}=\dfrac{2}{3}$

$g_{11}=\dfrac{1}{4};g_{12}=-1;g_{21}=1;g_{22}=2$

16-4.2 $Z_i = 1.5\ \text{k}\Omega; Z_0 = 0.033 \times 10^{-3}\ \Omega$

16-4.3 $\begin{bmatrix}0.857\angle-31^\circ\text{k}\Omega & 0.17\angle 59^\circ\\ 8.58\angle-32.1^\circ & 1.89\angle 61.1^\circ\text{m}\end{bmatrix}$

16-7.1 $Y_1=\dfrac{2R_2-0.8R_3}{\Delta Z};\quad Y_2=\dfrac{-R_2}{\Delta Z}$

$Y_3=\dfrac{R_1}{\Delta Z};\quad \Delta Z=\begin{vmatrix}R_1+R_2 & R_2\\ R_2-0.2R_3 & R_2+R_3\end{vmatrix}$

Objective-Type Questions

☆☆★ **16.1** A two-port network is simply a network inside a black box, and the network has only

(a) two terminals (b) two pairs of accessible terminals (c) two pairs of ports

☆☆★ **16.2** The number of possible combinations generated by four variables taken two at a time in a two-port network is

(a) four (b) two (c) six

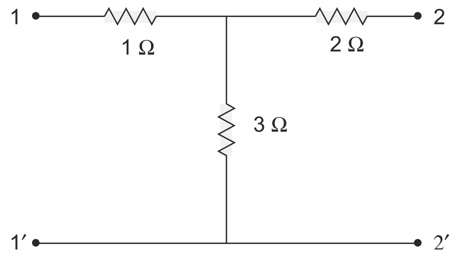


Fig. 16.75

☆☆★ **16.3** What is the driving-point impedance at port one with port two open- circuited for the network in Fig. 16.75?

(a) 4 Ω (c) 3 Ω
(b) 5 Ω

☆☆★ **16.4** What is the transfer impedance of the two-port network shown in Practice Problem 1 of LO 10?

(a) 1 Ω (b) 2 Ω
(c) 3 Ω

☆☆★ **16.5** If the two-port network in Practice Problem 1 of LO 10 is reciprocal or bilateral then

(a) $Z_{11} = Z_{22}$ (b) $Z_{12} = Z_{21}$ (c) $Z_{11} = Z_{12}$

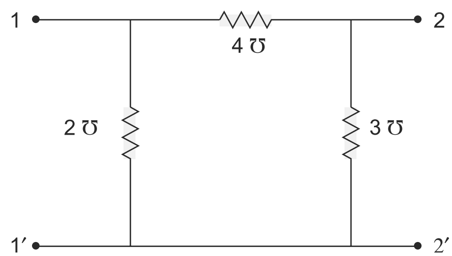


Fig. 16.76

☆☆★ **16.6** What is the transfer admittance of the network shown in Fig. 16.76.

(a) –2 ℧ (c) –4 ℧
(b) –3 ℧

☆☆★ **16.7** If the two-port network in Practice Problem 5 of LO 2 is reciprocal then

(a) $Y_{11} = Y_{22}$ (b) $Y_{12} = Y_{22}$
(c) $Y_{12} = Y_{11}$

☆☆★ **16.8** In describing the transmission parameters,

(a) the input voltage and current are expressed in terms of output voltage and current
(b) the input voltage and output voltage are expressed in terms of output current and input current
(c) the input voltage and output current are expressed in terms of input current and output voltage

☆☆★ **16.9** If $Z_{11} = 2\ \Omega$; $Z_{12} = 1\ \Omega$; $Z_{21} = 1\ \Omega$; and $Z_{22} = 3\ \Omega$, what is the determinant of admittance matrix?

(a) 5 (b) 1/5 (c) 1

☆☆★ **16.10** For a two-port bilateral network, the three transmission parameters are given by $A = \frac{6}{5}$; $B = \frac{17}{5}$ and $C = \frac{1}{5}$, what is the value of D?

(a) 1 (b) $\frac{1}{5}$ (c) $\frac{7}{5}$

☆☆★ **16.11** The impedance matrices of two two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 5 & 25 \end{bmatrix}$. If the two networks are connected in series, what is the impedance matrix of the combination?

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$

☆☆★ **16.12** The admittance matrices of two two-port networks are given by $\begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 5/8 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$. If the two networks are connected in parallel, what is the admittance matrix of the combination?

(a) $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -1 & 5/2 \end{bmatrix}$ (c) $\begin{bmatrix} 3/2 & -3/4 \\ -3/4 & 15/8 \end{bmatrix}$

☆☆★ **16.13** If the *Z*-parameters of a two-port network are $Z_{11} = 5\ \Omega$ $Z_{22} = 7\ \Omega$; $Z_{12} = Z_{21} = 3\ \Omega$ then the *A*, *B*, *C*, *D* parameters are respectively given by

(a) $\frac{5}{3}; \frac{26}{3}; \frac{1}{3}; \frac{7}{3}$ (b) $\frac{10}{3}; \frac{52}{3}; \frac{2}{3}; \frac{14}{3}$ (c) $\frac{15}{3}; \frac{78}{3}; \frac{3}{3}; \frac{21}{3}$

☆☆★ **16.14** For a symmetric lattice network, the value of the series impedance is 3 Ω and that of the diagonal impedance is 5 Ω, then the *Z*-parameters of the network are given by

(a) $Z_{11} = Z_{22} = 2\ \Omega$
$Z_{12} = Z_{21} = 1/2\ \Omega$

(b) $Z_{11} = Z_{22} = 4\ \Omega$
$Z_{12} = Z_{21} = 1\ \Omega$

(c) $Z_{11} = Z_{22} = 8\ \Omega$
$Z_{12} = Z_{21} = 2\ \Omega$

☆☆★ **16.15** For a two-port network to be reciprocal,

(a) $Z_{11} = Z_{22}$ (b) $y_{21} = y_{22}$ (c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

☆☆★ **16.16** Two-port networks are connected in cascade. The combination is to be represented as a single two-port network. The parameters of the network are obtained by adding the individual

(a) *Z*-parameter matrix
(b) *h*-parameter matrix
(c) $A^1\ B^1\ C^1\ D^1$ matrix
(d) *ABCD*-parameter matrix

☆☆★ **16.17** The *h*-parameters h_{11} and h_{12} are obtained

(a) by shorting output terminals
(b) by opening input terminals
(c) by shorting input terminals
(d) by opening output terminals

☆☆★ **16.18** Which parameters are widely used in transmission-line theory?

(a) *Z*-parameters (b) *Y*-parameters (c) *ABCD*-parameters (d) *h*-parameters

For interactive quiz with answers,
scan the QR code given here
OR
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http://qrcode.flipick.com/index.php/274

CHAPTER 17

Filters and Attenuators

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- LO 1 Understand different types of filters
- LO 2 Define filter networks and write filter equations
- LO 3 Classify pass-band and stop-band filters and explain characteristic impedance in them
- LO 4 Analyse constant-K low-pass filters and constant-*K* high-pass filter
- LO 5 Explain m-derived T-section filter
- LO 6 Explain band-pass and band-elimination filter
- LO 7 Explain functionality of an attenuator and its various types
- LO 8 Explain equalisers and their types

17.1 CLASSIFICATION OF FILTERS

LO 1 Understand different types of filters

Wave filters were first invented by G A Campbell and O I Lobel of the Bell Telephone Laboratories. A filter is a reactive network that freely passes the desired bands of frequencies while almost totally suppressing all other bands. A filter is constructed from purely reactive elements, for otherwise the attenuation would never become zero in the passband of the filter network. Filters differ from simple resonant circuits in providing a substantially constant transmission over the band which they accept; this band may lie between any limits depending on the design. Ideally, filters should produce no attenuation in the desired band, called the *transmission* band or *pass* band, and should provide total or infinite attenuation at all other frequencies, called *attenuation* band or *stop* band. The frequency which separates the transmission band and the attenuation band is defined as the cut-off frequency of the wave filters, and is designated by f_c.

Filter networks are widely used in communication systems to separate various voice channels in carrier frequency telephone circuits. Filters also find applications in instrumentation, telemetering equipment, etc. where it is necessary to transmit or attenuate a limited range of frequencies.

A filter may, in principle, have any number of pass bands separated by attenuation bands. However, they are classified into four common types, viz. low-pass, high-pass, band-pass and band-elimination.

17.1.1 Decibel and Neper

The attenuation of a wave filter can be expressed in decibels or nepers. Neper is defined as the natural logarithm of the ratio of input voltage (or current) to the output voltage (or current), provided that the network is properly terminated in its characteristic impedance Z_0.

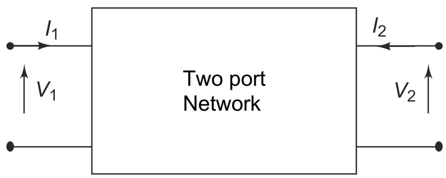


Fig. 17.1 (a)

From Fig. 17.1 (a), the number of nepers, $N = \log_e \left|\frac{V_1}{V_2}\right|$ or $\log_e \left|\frac{I_1}{I_2}\right|$.

A neper can also be expressed in terms of input power, P_1 and the output power P_2 as $N = 1/2 \log_e P_1/P_2$.

A decibel is defined as ten times the common logarithms of the ratio of the input power to the output power.

$$\therefore \quad \text{Decibel } D = 10 \log_{10} \frac{P_1}{P_2}$$

The decibel can be expressed in terms of the ratio of input voltage (or current) and the output voltage (or current.)

$$D = 20 \log_{10} \left|\frac{V_1}{V_2}\right| = 20 \log_{10} \left|\frac{I_1}{I_2}\right|$$

$\therefore$ One decibel is equal to 0.115 N.

17.1.2 Low-pass Filter

By definition, a low-pass (LP) filter is one which passes without attenuation all frequencies up to the cut-off frequency f_c, and attenuates all other frequencies greater than f_c. The attenuation characteristic of an ideal LP filter is shown in Fig. 17.1 (b). This transmits currents of all frequencies from zero up to the cut-off frequency. The band is called pass band or transmission band. Thus, the pass band for the LP filter is the frequency range 0 to f_c. The frequency range over which transmission does not take place is called the stop band or attenuation band. The stop band for an LP filter is the frequency range above f_c.

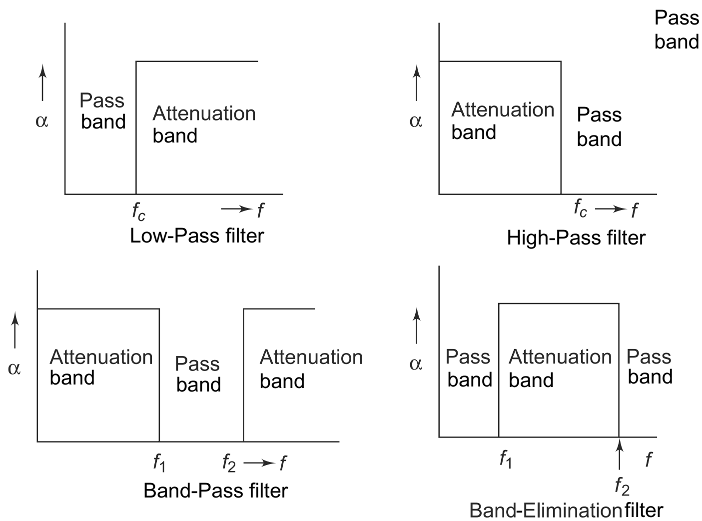


Fig. 17.1 (b)

17.1.3 High-Pass Filter

A high-pass (HP) filter attenuates all frequencies below a designated cut-off frequency, f_c, and passes all frequencies above f_c. Thus, the pass band of this filter is the frequency range above f_c, and the stop band is the frequency range below f_c. The attenuation characteristic of an HP filter is shown in Fig. 17.1 (b).

17.1.4 Band-Pass Filter

A band-pass filter passes frequencies between two designated cut-off frequencies and attenuates all other frequencies. It is abbreviated as *BP filter*. As shown in Fig.17.1 (b), a BP filter has two cut-off frequencies and will have the pass band $f_2 - f_1$; f_1 is called the lower cut-off frequency, while f_2 is called the upper cut-off frequency.

17.1.5 Band-Elimination Filter

A band-elimination filter passes all frequencies lying outside a certain range, while it attenuates all frequencies between the two designated frequencies. It is also referred as band stop filter. The characteristic of an ideal band elimination filter is shown in Fig. 17.1 (b).

All frequencies between f_1 and f_2 will be attenuated while frequencies below f_1 and above f_2 will be passed.

Frequently Asked Questions linked to LO 1*

☆☆★**17-1.1** What are the properties of filters? **[JNTU Nov. 2012]**

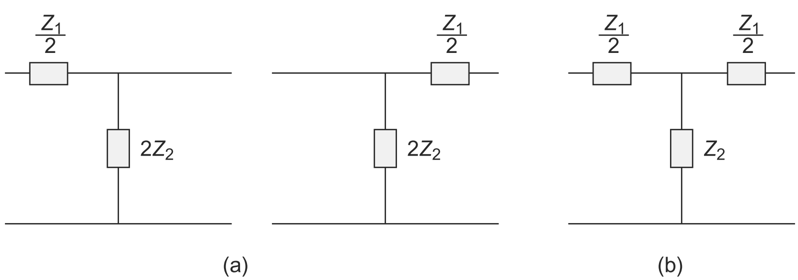
☆☆★**17-1.2** What are the classifications of filter? Discuss them briefly. **[JNTU Nov. 2012]**

☆☆★**17-1.3** Explain the concept of insertion loss. **[PU 2012]**

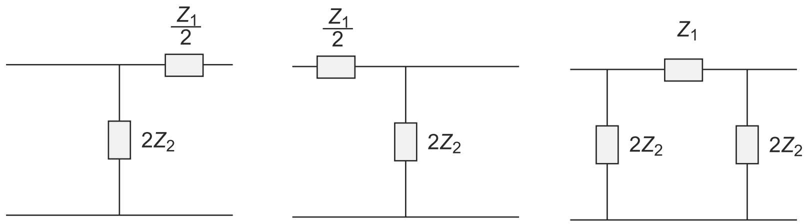
17.2 FILTER NETWORKS

LO 2 Define filter networks and write filter equations

Ideally a filter should have zero attenuation in the pass band. This condition can only be satisfied if the elements of the filter are dissipationless, which cannot be realised in practice. Filters are designed with an assumption that the elements of the filters are purely reactive. Filters are made of symmetrical T, or π sections. T and π sections can be considered as combinations of unsymmetrical L sections as shown in Fig. 17.2.



*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600



(c) (d)

Fig. 17.2

The ladder structure is one of the commonest forms of filter network. A cascade connection of several T and π sections constitutes a ladder network. A common form of the ladder network is shown in Fig. 17.3.

Figure 17.3 (a) represents a T-section ladder network, whereas Fig. 17.3 (b) represents the π-section ladder network. It can be observed that both networks are identical except at the ends.

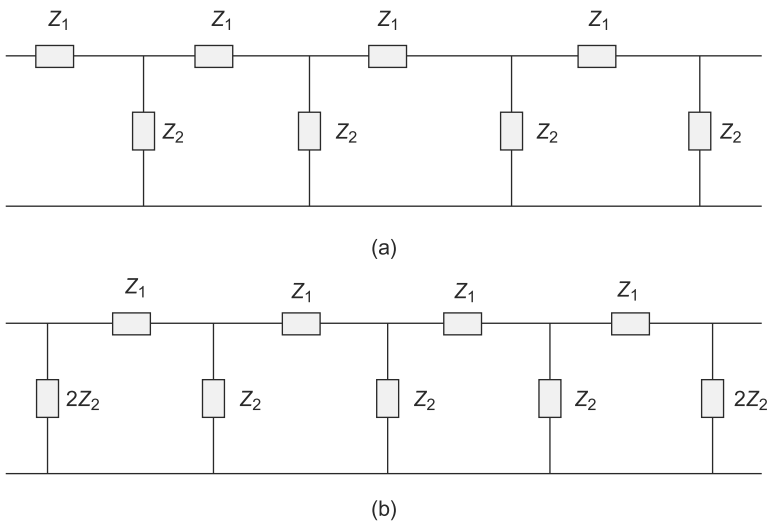


Fig. 17.3

17.3 EQUATIONS OF FILTER NETWORKS

LO 2

The study of the behaviour of any filter requires the calculation of its propagation constant γ, attenuation α, phase shift β, and its characteristic impedance Z_0.

17.3.1 T-Network

Consider a symmetrical T-network as shown in Fig. 17.4.

As has already been mentioned in Section 16.13, if the image impedances at the port 1-1′ and port 2-2′ are equal to each other, the image impedance is then called the characteristic, or the iterative impedance, Z_0. Thus, if the network in Fig. 17.4 is terminated in Z_0, its input impedance will also be Z_0. The value of input impedance for the T-network when it is terminated in Z_0

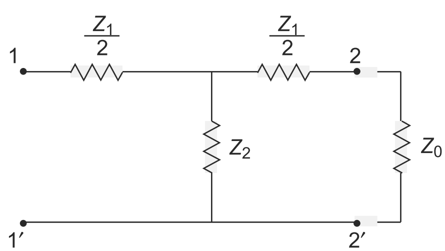


Fig. 17.4

is given by

$$Z_{\text{in}} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{\frac{Z_1}{2} + Z_2 + Z_0}$$

also $\qquad \mathrm{Z}_{\text{in}} = \mathrm{Z}_0$

$$\therefore \qquad Z_0 = \frac{Z_1}{2} + \frac{2Z_2\left(\frac{Z_1}{2} + Z_0\right)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1}{2} + \frac{(Z_1Z_2 + 2Z_2Z_0)}{Z_1 + 2Z_2 + 2Z_0}$$

$$Z_0 = \frac{Z_1^2 + 2Z_1Z_2 + 2Z_1Z_0 + 2Z_1Z_2 + 4Z_0Z_2}{2(Z_1 + 2Z_2 + 2Z_0)}$$

$$4Z_0^2 = Z_1^2 + 4Z_1Z_2$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

The characteristic impedance of a symmetrical T-section is

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2} \tag{17.1}$$

Z_{0T} can also be expressed in terms of open-circuit impedance Z_{0c} and short circuit impedance Z_{sc} of the T-network. From Fig. 17.4, the open-circuit impedance $Z_{0c} = \frac{Z_1}{2} + Z_2$ and

$$Z_{sc} = \frac{Z_1}{2} + \frac{\frac{Z_1}{2} \times Z_2}{\frac{Z_1}{2} + Z_2}$$

$$Z_{sc} = \frac{Z_1^2 + 4Z_1Z_2}{2Z_1 + 4Z_2}$$

$$Z_{0c} \times Z_{sc} = Z_1Z_2 + \frac{Z_1^2}{4}$$

$$= Z_{0T}^2 \quad \text{or} \quad Z_{0T} = \sqrt{Z_{0c}Z_{sc}} \tag{17.2}$$

❑ ***Propagation Constant of T-Network*** By definition, the propagation constant γ of the network in Fig. 17.5 is given by $\gamma = \log_e I_1/I_2$

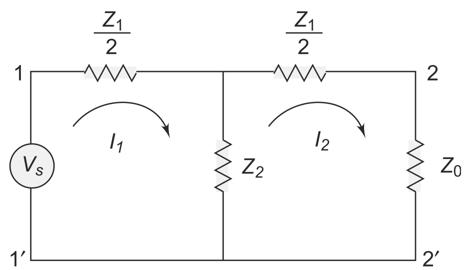


Fig. 17.5

Writing the mesh equation for the second mesh, we get

$$I_1Z_2 = I_2\left(\frac{Z_1}{2} + Z_2 + Z_0\right)$$

$$\frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_2 + Z_0}{Z_2} = e^{\gamma}$$

$$\therefore \qquad \frac{Z_1}{2} + Z_2 + Z_0 = Z_2e^{\gamma}$$

$$Z_0 = Z_2(e^{\gamma} - 1) - \frac{Z_1}{2} \tag{17.3}$$

The characteristic impedance of a *T*-network is given by

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2} \tag{17.4}$$

Squaring Eqs (17.3) and (17.4) and subtracting Eq. (17.4) from Eq. (17.3), we get

$$Z_2^2(e^{\gamma} - 1)^2 + \frac{Z_1^2}{4} - Z_1Z_2(e^{\gamma} - 1) - \frac{Z_1^2}{4} - Z_1Z_2 = 0$$

$$Z_2^2(e^{\gamma} - 1)^2 - Z_1Z_2(1 + e^{\gamma} - 1) = 0$$

$$Z_2^2(e^{\gamma} - 1)^2 - Z_1Z_2e^{\gamma} = 0$$

$$Z_2(e^{\gamma} - 1)^2 - Z_1e^{\gamma} = 0$$

$$(e^{\gamma} - 1)^2 = \frac{Z_1e^{\gamma}}{Z_2}$$

$$e^{2\gamma} + 1 - 2e^{\gamma} = \frac{Z_1}{Z_2e^{-\gamma}}$$

Rearranging the above equation, we have

$$e^{-\gamma}(e^{2\gamma} + 1 - 2e^{\gamma}) = \frac{Z_1}{Z_2}$$

$$(e^{\gamma} + e^{-\gamma} - 2) = \frac{Z_1}{Z_2}$$

Dividing both sides by 2, we have

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cosh\gamma = 1 + \frac{Z_1}{2Z_2} \tag{17.5}$$

Still another expression may be obtained for the complex propagation constant in terms of the hyperbolic tangent rather than hyperbolic cosine.

$$\sinh \gamma = \sqrt{\cos h^2\gamma - 1}$$

$$= \sqrt{\left(1+\frac{Z_1}{2Z_2}\right)^2 - 1} = \sqrt{\frac{Z_1}{Z_1} + \left(\frac{Z_1}{2Z_2}\right)^2}$$

$$\sinh \gamma = \frac{1}{Z_2}\sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \frac{Z_{0T}}{Z_2} \tag{17.6}$$

Dividing Eq. (17.6) by Eq. (17.5), we get

$$\tanh \gamma = \frac{Z_{0T}}{Z_2 + \dfrac{Z_1}{2}}$$

But $$Z_2 + \frac{Z_1}{2} = Z_{0c}$$

Also, from Eq. (17.2), $Z_{0T} = \sqrt{Z_{0c} Z_{sc}}$

$$\tanh \gamma = \sqrt{\frac{Z_{sc}}{Z_{0c}}}$$

Also, $$\sinh \frac{\gamma}{2} = \sqrt{\frac{1}{2}(\cosh \gamma - 1)}$$

where $$\cosh \gamma = 1 + (Z_1/2Z_2)$$

$$= \sqrt{\frac{Z_1}{4Z_2}} \tag{17.7}$$

17.3.2 π-Network

Consider an asymmetrical π-section shown in Fig. 17.6. When the network is terminated in Z_0 at the port 2-2′, its input impedance is given by

$$Z_{in} = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2\, Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \dfrac{2Z_2\, Z_0}{2Z_2 + Z_0} + 2Z_2}$$

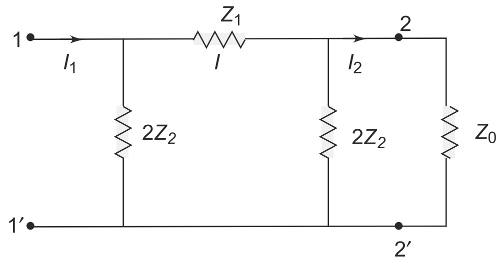


Fig. 17.6

By definition of characteristic impedance, $Z_{in} = Z_0$

$$Z_0 = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2\, Z_0}{2Z_2 + Z_0}\right]}{Z_1 + \dfrac{2Z_2\, Z_0}{2Z_2 + Z_0} + 2Z_2}$$

$$Z_0Z_1 + \frac{2Z_2Z_0^2}{2Z_2 + Z_0} + 2Z_0Z_2 = \frac{2Z_2(2Z_1Z_2 + Z_0Z_1 + 2Z_0Z_2)}{(2Z_2 + Z_0)}$$

$$2Z_0Z_1Z_2 + Z_1Z_0^2 + 2Z_0^2Z_2 + 4Z_2^2Z_0 + 2Z_2Z_0^2$$

$$= 4Z_1Z_2^2 + 2Z_0Z_1Z_2 + 4Z_0Z_2^2$$

$$Z_1Z_0^2 + 4Z_2Z_0^2 = 4Z_1Z_2^2$$

$$Z_0^2(Z_1 + 4Z_2) = 4Z_1Z_2^2$$

$$Z_0^2 = \frac{4Z_1Z_2^2}{Z_1 + 4Z_2}$$

Rearranging the above equation leads to

$$Z_0 = \sqrt{\frac{Z_1Z_2}{1 + Z_1 / 4Z_2}} \tag{17.8}$$

which is the characteristic impedance of a symmetrical π-network,

$$Z_{0\pi} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2 + Z_1^2 / 4}}$$

From Eq. (17.1), $Z_{0T} = \sqrt{\dfrac{Z_1^2}{4} + Z_1Z_2}$

$$\therefore \quad Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}} \tag{17.9}$$

$Z_{0\pi}$ can be expressed in terms of the open-circuit impedance Z_{0c} and short-circuit impedance Z_{sc} of the π network shown in Fig. 17.6 exclusive of the load Z_0.

From Fig. 17.6, the input impedance at the port 1-1′ when the port 2-2′ is open is given by $Z_{0C} = \dfrac{2Z_2(Z_1 + 2Z_2)}{Z_1 + 4Z_2}$

Similarly, the input impedance at the port 1-1′ when the port 2-2′ is short-circuited is given by $Z_{sc} = \dfrac{2Z_1Z_2}{2Z_2 + Z_1}$

Hence, $Z_{0c} \times Z_{sc} = \dfrac{4Z_1Z_2^2}{Z_1 + 4Z_2} = \dfrac{Z_1Z_2}{1 + Z_1 / 4Z_2}$

Thus, from Eq. (17.8),

$$Z_{0\pi} = \sqrt{Z_{0c}\ Z_{sc}} \tag{17.10}$$

❑ ***Propagation Constant of π-Network*** The propagation constant of a symmetrical π-section is the same as that for a symmetrical *T*-section,

i.e., $$\cosh \gamma = 1 + \frac{Z_1}{2Z_2}$$

Frequently Asked Questions linked to LO 2

☆☆★ **17-2.1** Compare a first-order low-pass filter to a second-order low-pass filter in terms of (a) voltage gain and (b) cut-off frequency, etc. **[BPUT 2008]**

☆☆★ **17-2.2** Find the characteristic impedance of a T-section as shown in Fig. Q.2. Verify the value of impedance with the help of open-and short-circuit impedances. **[PU 2010]**

☆☆★ **17-2.3** A symmetrical T-network consisting of pure resistances has open-and short-circuit impedances $Z_{oc} = 800 \angle 0° \ \Omega$, and $Z_{sc} = 600 \angle 0° \ \Omega$. Design a symmetrical T-network. **[PU 2010]**

☆☆★ **17-2.4** For a symmetrical T-network, prove that: **[PU 2010]**

$$R_{1/2} = R_0 \left[\frac{N-1}{N+1} \right]$$

$$R_2 = R_0 \left[\frac{2N}{N^2 - 1} \right]$$

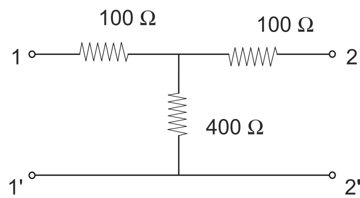


Fig. Q.2

☆★★ **17-2.5** Design a suitable matching half-section to match a symmetrical T-network with Z_{OT} = 500 Ω to a generator having internal resistance equal to 200 Ω. **[PU 2012]**

☆★★ **17-2.6** Design a π-type attenuator to give 20 dB attenuation and to have a characteristic impedance of 100 Ω. **[PTU 2011-12]**

☆☆★ **17-2.7** Calculate image impedance and iterative impedance of the T-network shown in Fig. Q.7 **[PU 2010]**

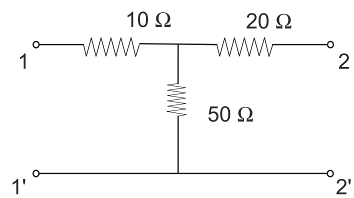


Fig. Q.7

17.4 CLASSIFICATION OF PASS BAND AND STOP BAND

LO 3 *Classify pass-band and stop-band filters and explain characteristic impedance in them*

It is possible to verify the characteristics of filters from the propagation constant of the network. The propagation constant γ, being a function of frequency, the pass band, stop band, and the cut-off point, i.e. the point of separation between the two bands, can be identified. For symmetrical *T* or π-sections, the expression for propagation constant γ in terms of the hyperbolic functions is given by Eqs (17.5) and (17.7) in Section 17.3. From Eq. (17.7), $\sinh \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}}$.

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

If Z_1 and Z_2 are both pure imaginary values, their ratio, and hence $Z_1/4Z_2$, will be a pure real number. Since Z_1 and Z_2 may be anywhere in the range from $-j_\propto$ t$_0$ + $j_\propto$, $Z_1/4Z_2$ may also have any real value between the infinite limits. Then $\sinh\frac{\gamma}{2} = \sqrt{Z_1}/\sqrt{4Z_2}$ will also have infinite limits, but may be either real or imaginary depending upon whether $Z_1/4Z_2$ is positive or negative.

We know that the propagation constant is a complex function $\gamma = \alpha + j\beta$, the real part of the complex propagation constant α, is a measure of the change in magnitude of the current or voltage in the network, known as the attenuation constant. β is a measure of the difference in phase between the input and output currents or voltages, known as phase shift constant. Therefore, α and β take on different values depending upon the range of $Z_1/4Z_2$. From Eq. (17.7), we have

$$\sinh\frac{\gamma}{2} = \sinh\left(\frac{\alpha}{2} + \frac{j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2}$$

$$= \sqrt{\frac{Z_1}{4Z_2}} \tag{17.11}$$

❑ ***Case A*** *If Z_1 and Z_2 are the same type of reactances, then* $\left|\frac{Z_1}{4Z_2}\right|$ *is real and equal to say $\alpha + x$.*

The imaginary part of the Eq. (17.11) must be zero.

$$\therefore \quad \cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = 0 \tag{17.12}$$

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = x \tag{17.13}$$

α and β must satisfy both the above equations.

Equation (17.12) can be satisfied if $\beta/2 = 0$ or $n\pi$, where $n = 0, 1, 2, \ldots$, then $\cos\beta/2 = 1$ and $\sinh\alpha/2 = x = \sqrt{\frac{Z_1}{4Z_2}}$

That x should be always positive implies that

$$\left|\frac{Z_1}{4Z_2}\right| > 0 \text{ and } \alpha = 2\sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.14}$$

Since $\alpha \neq 0$, it indicates that the attenuation exists.

❑ ***Case B*** *Consider the case of Z_1 and Z_2 being opposite type of reactances, i.e. $Z_1/4Z_2$ is negative, making $\sqrt{Z_1/4Z_2}$ imaginary and equal to say Jx*

$\therefore$ the real part of Eq. (17.11) must be zero.

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0 \tag{17.15}$$

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = x \tag{17.16}$$

Both the above equations must be satisfied simultaneously by α and β. Equation (17.15) may be satisfied when $\alpha = 0$, or when $\beta = \pi$. These conditions are considered separately hereunder.

1. When $\alpha = 0$; from Eq. (17.15), $\sinh \alpha/2 = 0$. And from Eq. (17.16) $\sin\beta/2 = x = \sqrt{Z_1/4Z_2}$. But the sine can have a maximum value of 1. Therefore, the above solution is valid only for negative $Z_1/4Z_2$, and having maximum value of unity. It indicates the condition of pass band with zero attenuation and follows the condition as

$$-1 \le \frac{Z_1}{4Z_2} \le 0$$

$$\beta = 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.17}$$

2. When $\beta = \pi$, from Eq. (17.15), $\cos \beta/2 = 0$. And from Eq. (17.16), $\sin \beta/2 = \pm 1$; $\cosh \alpha/2 = x = \sqrt{Z_1/4Z_2}$.

Since $\cosh \alpha/2 \ge 1$, this solution is valid for negative $Z_1/4Z_2$, and having magnitude greater than, or equal to unity. It indicates the condition of stop band since $\alpha \ne 0$.

$$-\alpha \le \frac{Z_1}{4Z_2} \le -1$$

$$\alpha = 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{17.18a}$$

It can be observed that there are three limits for case *A* and *B*. Knowing the values of Z_1 and Z_2, it is possible to determine the case to be applied to the filter. Z_1 and Z_2 are made of different types of reactances, or combinations of reactances, so that, as the frequency changes, a filter may pass from one case to another. Case A and (ii) in case B are attenuation bands, whereas (i) in Case B is the transmission band.

The frequency which separates the attenuation band from pass band or vice versa is called cut-off frequency. The cut-off frequency is denoted by f_c, and is also termed *nominal frequency*. Since Z_0 is real in the pass band and imaginary in an attenuation band, f_c is the frequency at which Z_0 changes from being real to being imaginary. These frequencies occur at

$$\left.\begin{aligned} \frac{Z_1}{4Z_2} &= 0 \text{ or } Z_1 = 0 \qquad (17.18b) \\ \frac{Z_1}{4Z_2} &= -1 \text{ or } Z_1 + 4Z_2 = 0 \qquad (17.18c) \end{aligned}\right\}$$

The above conditions can be represented graphically, as in Fig. 17.7.

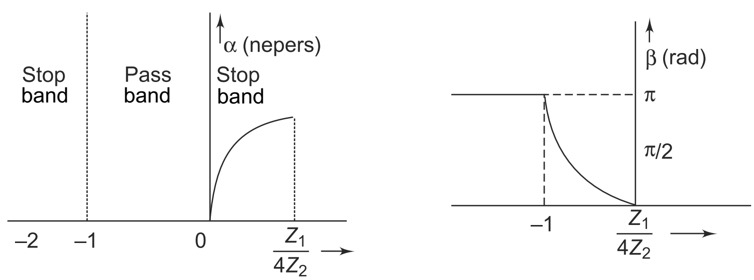


Fig. 17.7

7.5 CHARACTERISTIC IMPEDANCE IN THE PASS AND STOP BANDS

LO 3

Referring to the characteristic impedance of a symmetrical T-network, from Eq. (17.1), we have

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)}$$

If Z_1 and Z_2 are purely reactive, let $Z_1 = jx_1$ and $Z_2 = jx_2$, then

$$Z_{0T} = \sqrt{-x_1 x_2 \left(1 + \frac{x_1}{4x_2}\right)} \tag{17.19}$$

A pass band exists when x_1 and x_2 are of opposite reactances and

$$-1 < \frac{x_1}{4x_2} < 0$$

Substituting these conditions in Eq. (17.19), we find that Z_{0T} is positive and real. Now consider the stop band. A stop band exists when x_1 and x_2 are of the same type of reactances; then $x_1/4x_2 > 0$. Substituting these conditions in Eq. (17.19), we find that Z_{0T} is purely imaginary in this attenuation region. Another stop band exists when x_1 and x_2 are of the same type of reactances, but with $x_1/4x_2 < -1$. Then from Eq. (17.19), Z_{0T} is again purely imaginary in the attenuation region.

Thus, in a pass band if a network is terminated in a pure resistance $R_0 (Z_{0T} = R_0)$, the input impedance is R_0 and the network transmits the power received from the source to R_0 without any attenuation. In a stop band, Z_{0T} is reactive. Therefore, if the network is terminated in a pure reactance (Z_0 = pure reactance), the input impedance is reactive, and cannot receive or transmit power. However, the network transmits voltage and current with 90° phase difference and with attenuation. It has already been shown that the characteristic impedance of a symmetrical π-section can be expressed in terms of T. Thus, from Eq. (17.9), $Z_{0\pi} = Z_1 Z_2 / Z_{0T}$.

Since Z_1 and Z_2 are purely reactive, $Z_{0\pi}$ is real if Z_{0T} is real, and Z_{0x} is imaginary if Z_{0T} is imaginary. Thus, the conditions developed for T-sections are valid for π-sections.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 3*

☆☆★**17-3.1.** For a π-section filter network shown in Fig. Q.1. calculate the cut-off frequency and the value of nominal impedance in the pass band.

☆☆★**17-3.2** A π-section filter network is shown in Fig. Q.2. Calculate the cut-off frequency and phase shift at 10 kHz. What is the value of nominal impedance in the pass band?

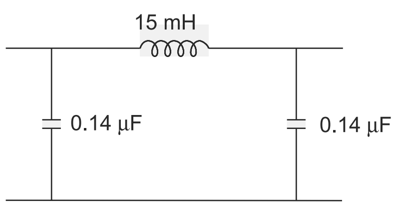


Fig. Q.1

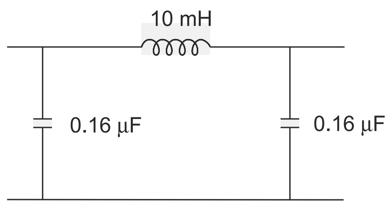


Fig. Q.2

17.6 CONSTANT-*K* LOW-PASS FILTER

LO 4 Analyse constant-K low-pass filters and constant-K high-pass filter

A network, either T or π, is said to be of the constant-k type if Z_1 and Z_2 of the network satisfy the relation

$$Z_1 Z_2 = k^2 \tag{17.20}$$

where Z_1 and Z_2 are impedances in the T and π-sections as shown in Fig. 17.8. Equation (17.20) states that Z_1 and Z_2 are inverse if their product is a constant, independent of frequency. k is a real constant, that is the resistance. k is often termed as design impedance or nominal impedance of the constant k-filter.

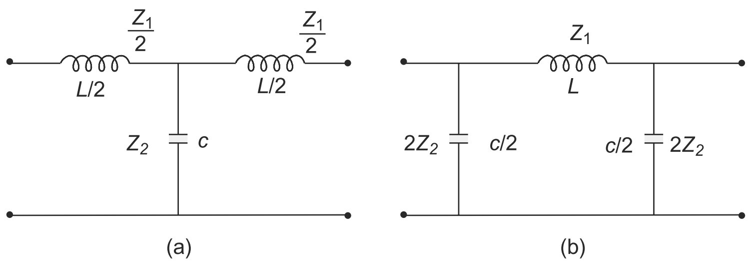


Fig. 17.8

The constant k, T or π-type filter is also known as the *prototype* because other more complex networks can be derived from it. A prototype T and π-sections are shown in Fig. 17.8 (a) and (b), where $Z_1 = j\omega_L$ and $Z_2 = 1/j\omega_C$. Hence, $Z_1 Z_2 = \dfrac{L}{C} = k^2$ which is independent of frequency.

$$Z_1 Z_2 = k^2 = \frac{L}{C} \quad \text{or} \quad k = \sqrt{\frac{L}{C}} \tag{17.21}$$

Since the product Z_1 and Z_2 is constant, the filter is a constant-k type. From Eq. (17.18 (a)), the cut-off frequencies are $Z_1/4Z_2 = 0$,

i.e., $$\frac{-\omega^2 LC}{4} = 0$$

i.e., $$f = 0 \text{ and } \frac{Z_1}{4Z_2} = -1$$

$$\frac{-\omega^2 LC}{4} = -1$$

or $$f_c = \frac{1}{\pi\sqrt{LC}} \tag{17.22}$$

The pass band can be determined graphically. The reactances of Z_1 and $4Z_2$ will vary with frequency as drawn in Fig. 17.9. The cut-off frequency at the intersection of the curves Z_1 and $-4Z_2$ is indicated as f_c. On the X-axis, as $Z_1 = -4Z_2$ at the cut-off frequency, the pass band lies between the frequencies at which $Z_1 = 0$, and $Z_1 = -4Z_2$. All the frequencies above f_c lie in a stop or attenuation band. Thus, the network is called a low-pass filter.

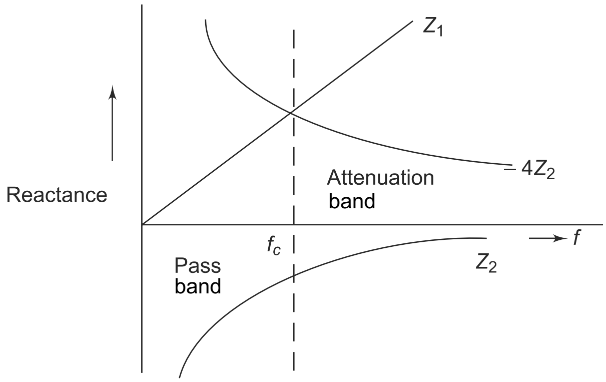


Fig. 17.9

We also have from Eq. (17.7) that

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-\omega^2 LC}{4}} = \frac{J\omega\sqrt{LC}}{2}$$

From Eq. (17.22), $\sqrt{LC} = \dfrac{1}{f_c\pi}$

$$\therefore \qquad \sinh\frac{\gamma}{2} = \frac{j2\pi f}{2\pi f_c} = j\frac{f}{f_c}$$

We also know that in the pass band

$$-1 < \frac{Z_1}{4Z_2} < 0$$

$$-1 < \frac{-\omega^2 LC}{4} < 0$$

$$-1 < -\left(\frac{f}{f_c}\right)^2 < 0$$

or
$$\frac{f}{f_c} < 1$$

and
$$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right); \alpha = 0$$

In the attenuation band,

$$\frac{Z_1}{4Z_2} < -1, \text{ i.e.} \frac{f}{f_c} < 1$$

$$\alpha = 2\cosh^{-1}\left[\frac{Z_1}{4Z_2}\right] = 2\cosh^{-1}\left(\frac{f}{f_c}\right); \beta = \pi$$

The plots of α and β for pass and stop bands are shown in Fig. 17.10.

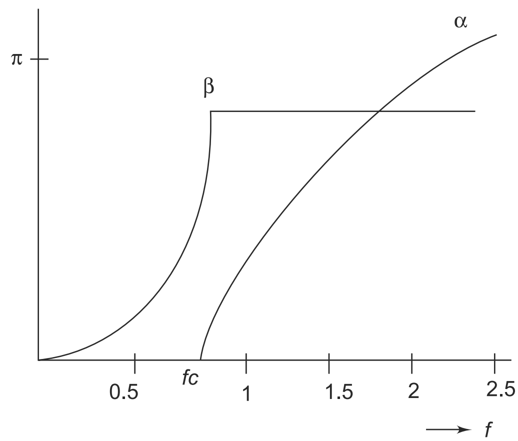


Fig. 17.10

Thus, from Fig. 17.10,

$$\alpha = 0, \beta = 2\sinh^{-1}\left(\frac{f}{f_c}\right) \text{for } f < f_c$$

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right); \beta = \pi \text{ for } f > f_c$$

The characteristic impedance can be calculated as follows:

$$Z_{0T} = \sqrt{Z_1 Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)}$$

$$Z_{0T} = k\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \qquad (17.23)$$

From Eq. (17.23), Z_{0T} is real when $f < f_c$, i.e. in the pass band at $f = f_c$, $Z_{0T} = 0$; and for $f > f_c$, Z_{0T} is imaginary in the attenuation band, rising to infinite reactance at infinite frequency. The variation of Z_{0T} with frequency is shown in Fig. 17.11.

Similarly, the characteristic impedance of a π-network is given by

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}} \qquad (17.24)$$

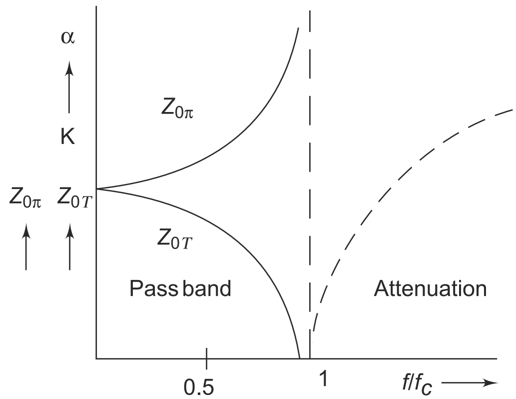


Fig. 17.11

The variation of $Z_{0\pi}$ with frequency is shown in Fig. 17.11. For $f < f_c$, $Z_{0\pi}$ is real; at $f = f_c$, $Z_{0\pi}$ is infinite, and for $f > f_c$, $Z_{0\pi}$ is imaginary. A low-pass filter can be designed from the specifications of cut-off frequency and load resistance.

At cut-off frequency, $Z_1 = -4Z_2$

$$j\omega_c L = \frac{-4}{j\omega_c C}$$

$$\pi^2 f_c^2 LC = 1$$

Also, we know that $k = \sqrt{L/C}$ is called the design impedance or the load resistance

$$\therefore \qquad k^2 = \frac{L}{C}$$

$$\pi^2 f_c^2 k^2 C^2 = 1$$

$C = \dfrac{1}{\pi f_c k}$ gives the value of the *shunt capacitance*

and $L = k^2 C = \dfrac{k}{\pi f_c}$ gives the value of the series inductance.

EXAMPLE 17.1

Design a low-pass filter (both π and T-sections) having a cut-off frequency of 2 kHz to operate with a terminated load resistance of 500 Ω.

Solution It is given that $k = \sqrt{\frac{L}{C}} = 500\ \Omega$, and $f_c = 2000$ Hz

We know that
$$L = \frac{k}{\pi f_c} = \frac{500}{3.14 \times 2000} = 79.6 \text{ mH}$$
$$C = \frac{1}{\pi f_c k} = \frac{1}{3.14 \cdot 2000 \cdot 500} = 0.318\ \mu\text{F}$$

The T and π-sections of this filter are shown in Fig. 17.12 (a) and (b) respectively.

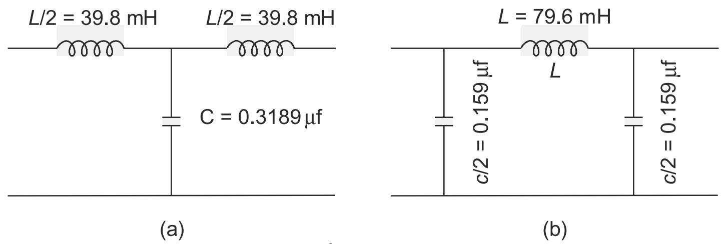


Fig. 17.12

17.7 CONSTANT-*K* HIGH PASS FILTER

LO 4

A constant-*K* high-pass filter can be obtained by changing the positions of series and shunt-arms of the networks shown in Fig. 17.8. The prototype high-pass filters are shown in Fig. 17.13, where $Z_1 = -j/\omega_C$ and $Z_2 = j\omega L$.

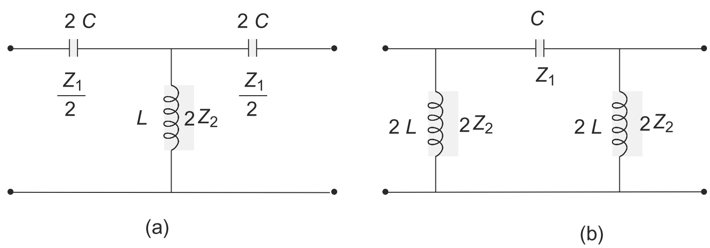


Fig. 17.13

Again, it can be observed that the product of Z_1 and Z_2 is independent of frequency, and the filter design obtained will be of the constant-k type. Thus, Z_1Z_2 are given by

$$Z_1Z_2 = \frac{-j}{\omega C} j\omega L = \frac{L}{C} = k^2$$
$$k = \sqrt{\frac{L}{C}}$$

The cut-off frequencies are given by $Z_1 = 0$ and $Z_1 = -4Z_2$.

$$Z_1 = 0 \text{ indicates } \frac{j}{\omega C} = 0 \text{, or } \omega \to \infty$$

From $Z_1 = -4Z_2$,

$$\frac{-j}{\omega C} = -4 j\omega L$$

$$\omega^2 LC = \frac{1}{4}$$

or

$$f_c = \frac{1}{4\pi\sqrt{LC}} \qquad (17.25)$$

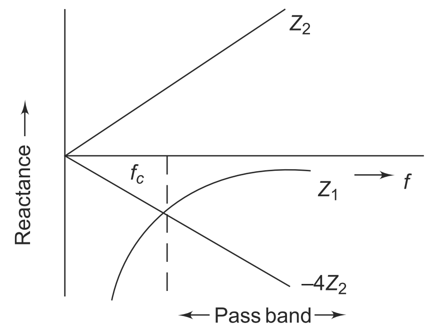


Fig. 17.14

The reactances of Z_1 and Z_2 are sketched as functions of frequency as shown in Fig. (17.14).

As seen from Fig. 17.14, the filter transmits all frequencies between $f = f_c$ and $f = \infty$. The point f_c from the graph is a point at which $Z_1 = -4Z_2$.

From Eq. (17.7),

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-1}{4\omega^2 LC}}$$

From Eq. (17.25), $f_c = \dfrac{1}{4\pi\sqrt{LC}}$

$$\therefore \quad \sqrt{LC} = \frac{1}{4\pi f_c}$$

$$\therefore \quad \sinh\frac{\gamma}{2} = \sqrt{\frac{-(4\pi)^2(f_c)^2}{4\omega^2}} = j\frac{f_c}{f}$$

In the pass band, $-1 < \dfrac{Z_1}{4Z_2} < 0$, $\alpha = 0$ or the region in which $\dfrac{f_c}{f} < 1$ is a pass band $\beta = 2\sin^{-1}\left(\dfrac{f_c}{f}\right)$

In the attenuation band, $\dfrac{Z_1}{4Z_2} < -1$, i.e. $\dfrac{f_c}{f} > 1$

$$\alpha = 2\cosh^{-1}\left[\frac{Z_1}{4Z_2}\right]$$

$$= 2\cos^{-1}\left(\frac{f_c}{f}\right); \quad \beta = -\pi$$

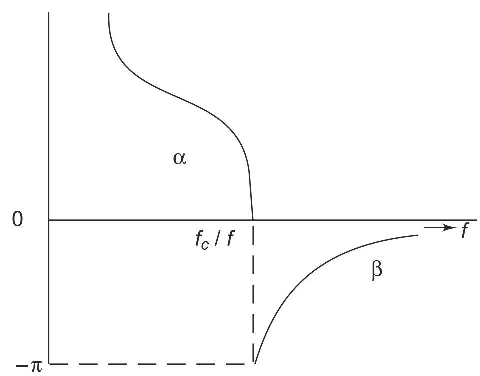


Fig. 17.15

The plots of α and β for pass and stop bands of a high-pass filter network are shown in Fig. 17.15.

A high-pass filter may be designed similar to the low-pass filter by choosing a resistive load *r* equal to the constant *k*, such that $R = k = \sqrt{L/C}$

$$f_c = \frac{1}{4\pi\sqrt{L/C}}$$

$$f_c = \frac{k}{4\pi L} = \frac{1}{4\pi C k}$$

Since $\sqrt{C} = \frac{L}{k}$,

$$L = \frac{k}{4\pi f_c} \text{ and } C = \frac{1}{4\pi f_c k}$$

The characteristic impedance can be calculated using the relation

$$Z_{0T} = \sqrt{Z_1 Z_2\left(1+\frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{L}{C}\left(1-\frac{1}{4\omega^2 LC}\right)}$$

$$Z_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Similarly, the characteristic impedance of a π-network is given by

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{k^2}{Z_{0T}}$$

$$= \frac{k}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad (17.26)$$

The plot of characteristic impedances with respect to frequency is shown in Fig. 17.16.

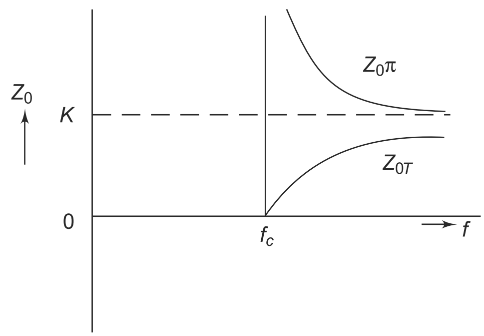


Fig. 17.16

EXAMPLE 17.2

Design a high-pass filter having a cut-off frequency of 1 kHz with a load resistance of 600 Ω.

Solution It is given that $R_L = K = 600\ \Omega$ and $f_c = 1000$ Hz

$$\therefore \qquad L = \frac{K}{4\pi f_c} = \frac{600}{4\times\pi\times 1000} = 47.74 \text{ mH}$$

$$C = \frac{1}{4\pi k f_c} = \frac{1}{4\pi\times 600\times 1000} = 0.133\ \mu\text{F}$$

The T and π-sections of the filter are shown in Fig. 17.17.

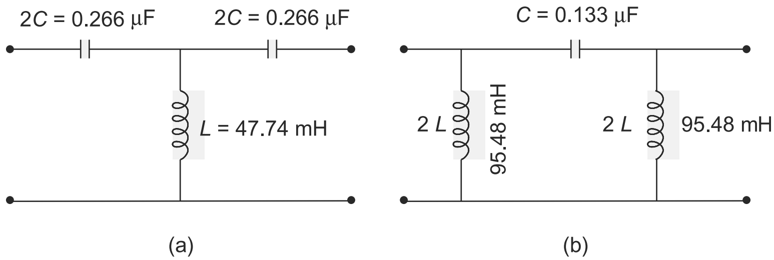


Fig. 17.17

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆☆★ **17-4.1** Design a high-pass filter with a cut-off frequency of 1 kHz with a terminated design impedance of 800 Ω.

Frequently Asked Questions linked to LO 4

☆☆★ **17-4.1** Analyse a prototype low-pass filter with derivation of all necessary equations and also discuss the different characteristics of the filter. **[BPUT 2007]**

☆★★ **17-4.2** A constant-*K* low-pass filter is designed to cut-off at a frequency of 1000 Hz and the resistance of the load circuit is 50 Ω. Calculate the values of the components required and the attenuation constant per section at a frequency of 1500 Hz. **[BPUT 2008]**

☆☆★ **17-4.3** Design a constant-*k* low-pass filter having f_c = 2 kHz and design impedance R_o = 600 Ω,. Obtain the value of attenuation at 25 kHz. **[JNTU Nov. 2012]**

☆☆★ **17-4.4** Design constant-*k* low-pass T and π-section filter to be terminated in 600 Ω having cut-off frequency of 3 kHz. **[PTU 2009-10]**

☆☆★ **17-4.5** Draw the reactance curve for a constant *k* low-pass filter and derive the expression for cut-off frequency and design impedance (R_0). **[PU 2012]**

☆☆★ **17-4.6** Design a constant-*K* high-pass filter, having f_c = 4 kHz and design impedance R_o = 600 Ω. **[BPUT 2007]**

☆☆★ **17-4.7** Can you design a filter (low and high-pass) for a cut-off frequency of 50 Hz. If you can, what is the value of parameters? **[BPUT 2008]**

☆☆★ **17-4.8** A constant-*K* high-pass filter is required for a cut-off frequency of 1500 Hz. The resistance of the load circuit is 600 Ω. Determine the values of the components required. **[BPUT 2008]**

☆☆★ **17-4.9** Design a T-section constant-*K* high-pass filter having cut-off frequency of 10 kHZ and design impedance R_o = 600 ohms. Find its characteristic impedance and constant at 25 kHz. **[JNTU Nov. 2012]**

☆☆★ **17-4.10** A prototype high-pass filter has a cut-off frequency of 10 kHz and design impedance of 600 Ω. Find the values of *L* and *C*. Also find attenuation in dB and phase shift in degrees at a frequency of 8 kHz. **[PU 2010]**

17.8 *m*-DERIVED T-SECTION

LO 5 Explain m-derived T-section filter

It is clear from Figs 17.10 and 17.15 that the attenuation is not sharp in the stop band for *k*-type filters. The characteristic impedance, Z_0 is a function of frequency and varies widely in the transmission band. Attenuation can be increased in the stop band by using ladder section, i.e. by connecting two or more identical sections. In order to join the filter sections, it would be necessary that their characteristic impedances be equal to each other at all frequencies. If their characteristic impedances match at all frequencies, they would also have the same pass band. However, cascading is not a proper solution from a practical point of view. This is because practical elements have a certain resistance, which gives rise to attenuation in the pass band also. Therefore, any attempt to increase attenuation in stop band by cascading also results in an increase of 'α' in the pass band. If the constant-k section is regarded as the prototype, it is possible to design a filter to have rapid attenuation in the stop band, and the same characteristic impedance as the prototype at all frequencies. Such a filter is called *m-derived filter*. Suppose a prototype T-network shown in Fig. 17.18 (a) has the series arm modified as shown in Fig. 17.18 (b), where m is a constant. Equating the characteristic impedance of the networks in Fig. 17.18, we have where $Z_{0T'}$ is the characteristic impedance of the modified (*m-derived*) T-network.

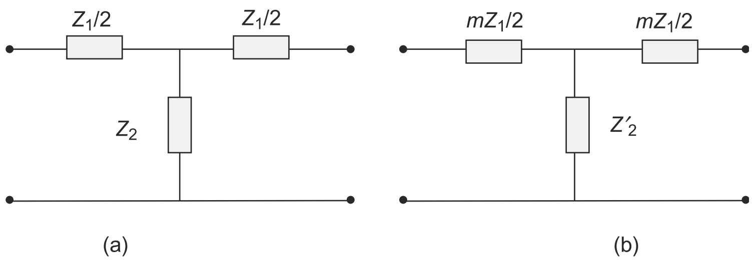


Fig. 17.18

$$Z_{0T} = Z_{0T'}$$

$$\sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{\frac{m^2 Z_1^2}{4} + m Z_1 Z_2'}$$

$$\frac{Z_1^2}{4} + Z_1 Z_2 = \frac{m^2 Z_1^2}{4} + m Z_1 Z_2'$$

$$m Z_1 Z_2' = \frac{Z_1^2}{4}(1 - m^2) + Z_1 Z_2$$

$$Z_2' = \frac{Z_1}{4m}(1 - m^2) + \frac{Z_2}{m} \tag{17.27}$$

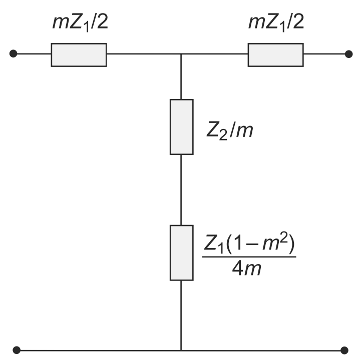


Fig. 17.19

It appears that the shunt-arm Z_2' consists of two impedances in series as shown in Fig. 17.19.

From Eq. (17.27), $\dfrac{1 - m^2}{4m}$ should be positive to realise the impedance Z_2' physically, i.e. $0 < m < 1$. Thus, the m-derived section can be obtained from the prototype by modifying its series and shunt-arms. The same technique can be applied to π-section network. Suppose a prototype π-network shown in Fig. 17.20 (a) has the shunt-arm modified as shown in Fig. 17.20 (b).

The characteristic impedances of the prototype and its modified sections have to be equal for matching.

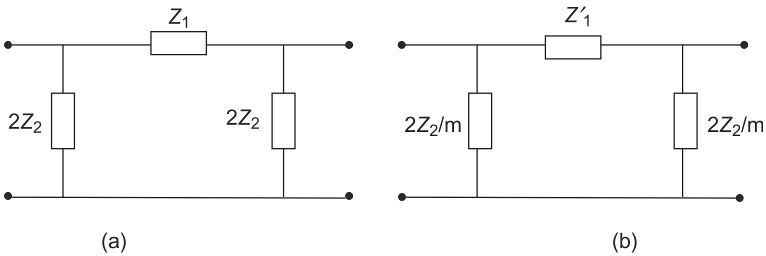


Fig. 17.20

$$Z_{0\pi} = Z'_{0\pi}$$

where $Z'_{0\pi}$ is the characteristic impedance of the modified (m-derived) π-network.

$$\therefore \qquad \sqrt{\frac{Z_1 Z_2}{1+\dfrac{Z_1}{4Z_2}}} = \sqrt{\frac{Z_1' \dfrac{Z_2}{m}}{1+\dfrac{Z_1'}{4\cdot Z_2/m}}}$$

Squaring and cross multiplying the above equation results as under.

$$(4Z_1Z_2 + mZ_1'Z_1) = \frac{4Z_1'Z_2 + Z_1Z_1'}{m}$$

$$Z_1'\left(\frac{Z_1}{m} + \frac{4Z_2}{m} - mZ_1\right) = 4Z_1Z_2$$

or

$$Z_1' = \frac{Z_1Z_2}{\dfrac{Z_1}{4m} + \dfrac{Z_2}{m} - \dfrac{mZ_1}{4}}$$

$$= \frac{Z_1Z_2}{\dfrac{Z_2}{m} + \dfrac{Z_1}{4m}(1-m^2)}$$

$$Z_1' = \frac{Z_1Z_2\dfrac{4m^2}{(1-m^2)}}{\dfrac{Z_2 4m^2}{m(1-m^2)} + Z_1 m} = \frac{mZ_1\dfrac{Z_2 4m}{(1-m^2)}}{mZ_1 + \dfrac{Z_2 4m}{(1-m^2)}} \qquad (17.28)$$

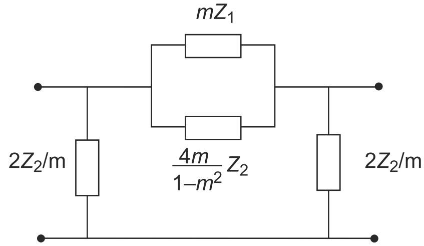


Fig. 17.21

It appears that the series arm of the m-derived π-section is a parallel combination of mZ_1 and $4mZ_2 / 1 - m^2$. The derived m-section is shown in Fig. 17.21.

17.8.1 *m*-Derived Low-Pass Filter

In Fig. 17.22, both m-derived low-pass T and π-filter sections are shown. For the T-section shown in Fig. 17.22 (a), the shunt-arm is to be chosen so that it is resonant at some frequency f_∞ above cut-off frequency f_c.

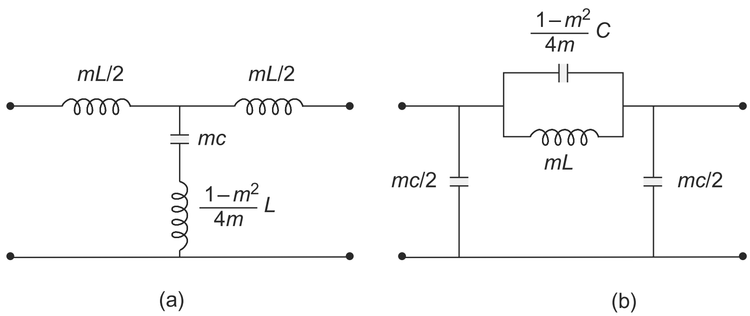


Fig. 17.22

If the shunt-arm is series resonant, its impedance will be minimum or zero. Therefore, the output is zero and will correspond to infinite attenuation at this particular frequency. Thus, at f_∞

$$\frac{1}{m\omega_r C} = \frac{1-m^2}{4m}\,\omega_r L, \text{ where } \omega_r \text{ is the resonant frequency}$$

$$\omega_r^2 = \frac{4}{(1-m^2)LC}$$

$$f_r = \frac{1}{\pi\sqrt{LC(1-m^2)}} = f_\infty$$

Since the cut-off frequency for the low-pass filter is $f_c = \dfrac{1}{\pi\sqrt{LC}}$

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}} \tag{17.29}$$

or
$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} \tag{17.30}$$

If a sharp cut-off is desired, f_∞ should be near to f_c. From Eq. (17.29), it is clear that the smaller the value of m, f_∞ comes close to f_c. Equation (17.30) shows that if f_c and f_∞ are specified, the necessary value of m may then be calculated. Similarly, for m-derived π-section, the inductance and capacitance in the series arm constitute a resonant circuit. Thus, at f_∞ a frequency corresponds to infinite attenuation, i.e., at f_∞

$$m\omega_r L = \frac{1}{\left(\dfrac{1-m^2}{4m}\right)\omega_r C}$$

$$\omega_r^2 = \frac{4}{LC(1-m^2)}$$

$$f_r = \frac{1}{\pi\sqrt{LC(1-m^2)}}$$

Since, $$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_r = \frac{f_c}{\sqrt{1-m^2}} = f_\infty \tag{17.31}$$

Thus, for both m-derived low-pass networks for a positive value of m $(0 < m < 1)$, $f_\infty > f_c$. Equations (17.30) or (17.31) can be used to choose the value of m, knowing f_c and f_r. After the value of m is evaluated, the elements of the T or π-networks can be found from Fig. 17.22. The variation of attenuation for a low-pass m-derived section can be verified from $\alpha = 2\cosh^{-1}\sqrt{Z_1/4Z_2}$ for $f_c < f < f_\infty$. For $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the prototype.

$$\therefore \qquad \alpha = 2\cosh^{-1}\frac{m\dfrac{f}{f_c}}{\sqrt{1-\left(\dfrac{f}{f_\infty}\right)^2}}$$

and $$\beta = 2\sin^{-1}\sqrt{\left|\frac{Z_1}{4Z_1}\right|} = 2\sin^{-1}\frac{m\dfrac{f}{f_c}}{\sqrt{1-\left(\dfrac{f}{f_c}\right)^2(1-m)^2}}$$

Figure 17.23 Shows the variation of α, β, and Z_0 with respect to frequency for an m-derived low-pass filter.

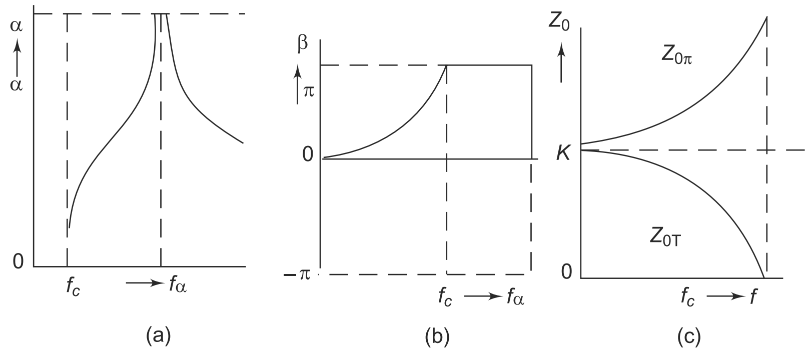


Fig. 17.23

EXAMPLE 17.3

Design a m-derived low-pass filter having cut-off frequency of 1 kHz, design impedance of 400 Ω, and the resonant frequency of 1100 Hz.

Solution $k = 400\ \Omega, f_c = 1000$ Hz; $f_\infty = 1100$ Hz

From Eq. (17.30),

$$m=\sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}=\sqrt{1-\left(\frac{1000}{1100}\right)^2}=0.416$$

Let us design the values of L and C for a low-pass, K-type filter (prototype filter). Thus,

$$L=\frac{k}{\pi f_c}=\frac{400}{\pi\times 1000}=127.32 \text{ mH}$$

$$C=\frac{1}{\pi k f_c}=\frac{1}{\pi\times 400\times 1000}=0.795\ \mu\text{F}$$

The elements of m-derived low-pass sections can be obtained with reference to Fig. 17.22. Thus, the T-section elements are

$$\frac{mL}{2}=\frac{0.416\times 127.32\times 10^{-3}}{2}=26.48 \text{ mH}$$

$$mC = 0.416 \times 0.795 \times 10^{-6} = 0.33\ \mu\text{F}$$

$$\frac{1-m^2}{4m}L=\frac{1-(0.416)^2}{4\cdot 0.416}\times 127.32\times 10^{-3}=63.27 \text{ mH}$$

The π-section elements are

$$\frac{mC}{2}=\frac{0.416\times 0.795\times 10^{-6}}{2}=0.165\ \mu\text{F}$$

$$\frac{1-m^2}{4m}\times C=\frac{1-(0.416)^2}{4\times 0.416}\times 0.795\times 10^{-6}=0.395\ \mu\text{F}$$

$$mL = 0.416 \times 127.32 \times 10^{-3} = 52.965 \text{ mH}$$

The m-derived LP filter sections are shown in Fig. 17.24.

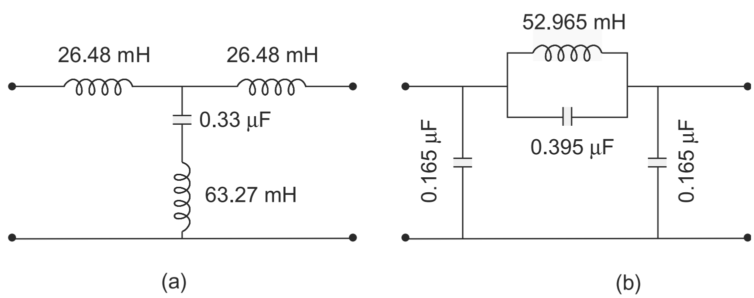


Fig. 17.24

17.8.2 *m*-derived High-Pass Filter

In Fig. 17.25, both m-derived high-pass T and π-sections are shown.

If the shunt-arm in T-section is series resonant, it offers minimum or zero impedance. Therefore, the output is zero and, thus, at resonance frequency, or the frequency corresponds to infinite attenuation.

$$\omega_r \frac{L}{m} = \frac{1}{\omega_r \frac{4m}{1-m^2} C}$$

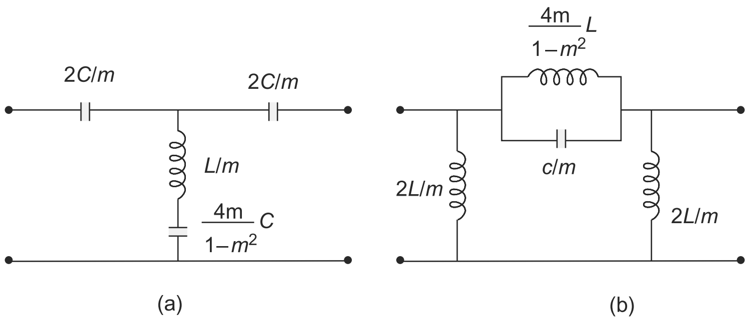


Fig. 17.25

$$\omega_r^2 = \omega_\infty^2 = \frac{1}{\frac{L}{m}\frac{4m}{1-m^2}C} = \frac{1-m^2}{4LC}$$

$$\omega_\infty = \frac{\sqrt{1-m^2}}{2\sqrt{LC}} \text{ or } f_\infty = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

From Eq. (17.25), the cut-off frequency f_c of a high-pass prototype filter is given by

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2} \tag{17.32}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} \tag{17.33}$$

Similarly, for the m-derived π-section, the resonant circuit is constituted by the series arm inductance and capacitance. Thus, at f_∞

$$\frac{4m}{1-m^2}\omega_r L = \frac{1}{\frac{\omega_r}{m}C}$$

$$\omega_r^2 = \omega_\infty^2 = \frac{1-m^2}{4LC}$$

$$\omega_\infty = \frac{\sqrt{1-m^2}}{2\sqrt{LC}} \text{ or } f_\infty = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

Thus, the frequency corresponding to infinite attenuation is the same for both sections.

Equation (17.33) may be used to determine m for a given f_∞ and f_c. The elements of the m-derived high-pass T or π-sections can be found from Fig. 17.25. The variation of α, β and Z_0 with frequency is shown in Fig. 17.26.

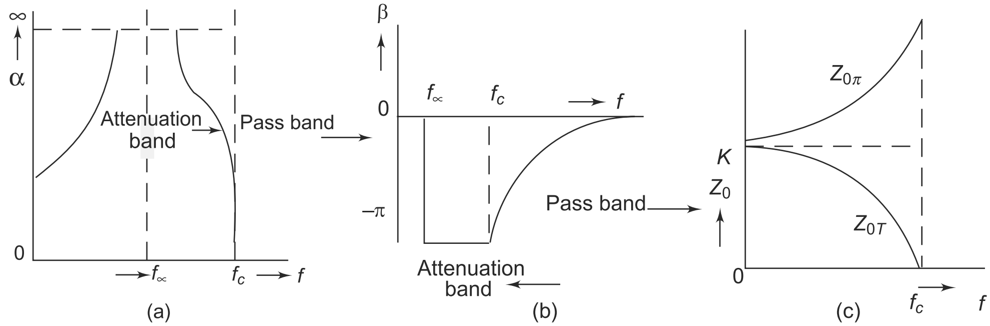


Fig. 17.26

EXAMPLE 17.4

Design an m-derived high-pass filter with a cut-off frequency of 10 kHz; design impedance of 5 Ω and m = 0.4.

Solution For the prototype high-pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{500}{4\times\pi\times 10000} = 3.978 \text{ mH}$$

$$C = \frac{1}{4\pi k f_c} = \frac{1}{4\pi\times 500\times 10000} = 0.0159 \text{ μF}$$

The elements of an m-derived high-pass sections can be obtained with reference to Fig. 17.25. Thus, the T-section elements are

$$\frac{2C}{m} = \frac{2\times 0.0159\times 10^{-6}}{0.4} = 0.0795 \text{ μF}$$

$$\frac{L}{m} = \frac{3.978\times 10^{-3}}{0.4} = 9.945 \text{ mH}$$

$$\frac{4m}{1-m^2}C = \frac{4\times 0.4}{1-(0.4)^2}\times 0.0159\times 10^{-6} = 0.0302 \text{ μF}$$

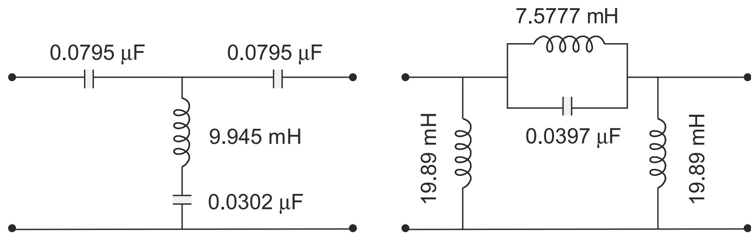
The π-section elements are

$$\frac{2L}{m} = \frac{2\times 0.0159\times 10^{-3}}{0.4} = 19.89 \text{ mH}$$

$$\frac{4m}{1-m^2}\times L=\frac{4\times 0.4}{1-(0.4)^2}\times 3.978\times 10^{-3}=7.577\text{ mH}$$

$$\frac{C}{m}=\frac{0.0159}{0.4}\times 10^{-6}=0.0397\ \mu\text{F}$$

T and π-sections of the m-derived high-pass filter are shown in Fig. 17.27.



(a) (b)

Fig. 17.27

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 5

☆☆★**17-5.1** Design a low-pass T-section filter having a cut-off frequency of 1.5 kHz to operate with a terminated load resistance of 600 Ω.

☆☆★**17-5.2** A T-section low-pass filter has series inductance of 80 mH and a shunt capacitance of 0.022 μF. Determine the cut-off frequency and nominal design impedance. Obtain the equivalent π-section.

☆☆★**17-5.3** Design an m-derived high-pass filter having a design impedance of 500 Ω and a cut-off frequency of 1 kHz. Take $m = 0.2$.

☆☆★**17-5.4** Design an m-derived high-pass filter with a cut-off frequency of 10 kHz, design impedance of 600 Ω and $m = 0.3$.

☆☆★**17-5.5** Determine the cut-off frequency and design impedance for the T-section shown in Fig. Q.5.

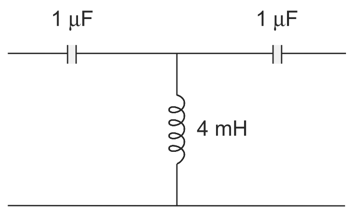


Fig. Q.5

Frequently Asked Questions linked to LO 5

☆☆★**17-5.1** Design an m-derived T-section low-pass filter having cut-off frequency, f_c = 7000 Hz, design impedance R_o = 600 Ω, and frequency of infinite attenuation. **[JNTU Nov. 2012]**

☆☆★**17-5.2** Explain the concept of m-derived filters. **[JNTU Nov. 2012]**

☆☆★**17-5.3** The T-section of an m-derived LP filter is shown in Q.3. **[PTU 2011-12]**

☆☆★**17-5.4** Draw the m-derived figh-pass filter. Plot attenuation phase shift, and characteristic impedance vs frequency for m-derived filter. **[PTU 2011-12]**

☆★★**17-5.5** Design a composite low-pass T-section filter having a design impedance of 600 Ω, a cut-off frequency of 2000 Hz, and a frequency of infinite attenuation of 2100 Hz. **[PU 2012]**

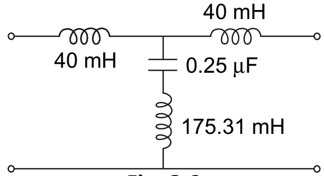


Fig. Q.3

17.9 BAND-PASS FILTER

LO 6 Explain bandpass and band-elimination filter

As already explained in Section 17.1, a band-pass filter is one which attenuates all frequencies below a lower cut-off frequency f_1 and above an upper cut-off frequency f_2. Frequencies lying between f_1 and f_2 comprise the pass band, and are transmitted with zero attenuation. A band-pass filter may be obtained by using a low-pass filter followed by a high-pass filter in which the cut-off frequency of the LP filter is above the cut-off frequency of the HP filter, the overlap thus allowing only a band of frequencies to pass. This is not economical in practice; it is more economical to combine the low and high pass functions into a single filter section.

Consider the circuit in Fig. 17.28, each arm has a resonant circuit with same resonant frequency, i.e. the resonant frequency of the series arm and the resonant frequency of the shunt-arm are made equal to obtain the band-pass characteristic.

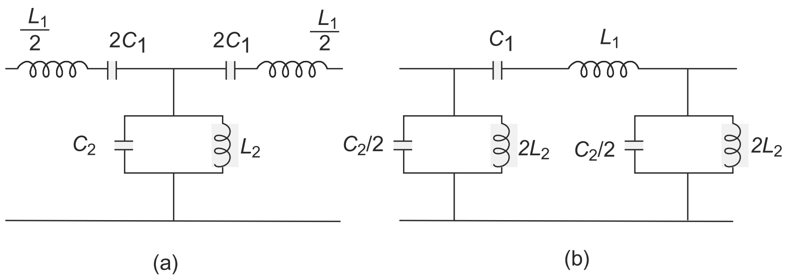


Fig. 17.28

For this condition of equal resonant frequencies.

$$\omega_0 \frac{L_1}{2} = \frac{1}{2\omega_0 C_1} \quad \text{for the series arm}$$

from which, $\quad \omega^2_0 L_1 C_1 = 1 \qquad (17.34)$

and $$\frac{1}{\omega_0 C_2} = \omega_0 L_2 \quad \text{for the shunt-arm}$$

from which, $\quad \omega_0^2 L_2 C_2 = 1 \qquad (17.35)$

$$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2$$

$$L_1 C_1 = L_2 C_2 \qquad (17.36)$$

The impedance of the series arm, Z_1 is given by

$$Z_1 = \left(j\omega L_1 - \frac{j}{\omega C_1} \right) = j \left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1} \right)$$

The impedance of the shunt-arm, Z_2 is given by

$$Z_2 = \frac{j\omega L_2 \dfrac{1}{j\omega C_2}}{j\omega L_2 + \dfrac{1}{j\omega C_2}} = \frac{j\omega L_2}{1-\omega^2 L_2 C_2}$$

$$Z_1 Z_2 = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)\left(\frac{j\omega L_2}{1-\omega^2 L_2 C_2}\right)$$

$$= \frac{-L_2}{C_1}\left(\frac{\omega^2 L_1 C_1 - 1}{1-\omega^2 L_2 C_2}\right)$$

From Eq. (17.36), $L_1 C_1 = L_2 C_2$

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

where k is constant. Thus, the filter is a constant k-type. Therefore, for a constant k-type in the pass band, $-1 < \dfrac{Z_1}{4Z_2} < 0$, and at cut-off frequency

$$Z_1 = -4Z_2$$

$$Z_1^2 = -4Z_1 Z_2 = -4k^2$$

$$\therefore \qquad Z_1 = \pm j2k$$

i.e. the value of Z_1 at lower cut-off frequency is equal to the negative of the value of Z_1 at the upper cut-off frequency.

$$\therefore \qquad \left(\frac{1}{j\omega_1 C_1} + j\omega_1 L_1\right) = -\left(\frac{1}{j\omega_2 C_1} + j\omega_2 L_1\right)$$

or

$$\left(\omega_1 L_1 - \frac{1}{\omega_1 C_1}\right) = \left(\frac{1}{\omega_2 C_1} - \omega_2 L_1\right)$$

$$(1-\omega_1^2 L_1 C_1) = \frac{\omega_1}{\omega_2}(\omega_2^2 L_1 C_1 - 1) \tag{17.37}$$

From Eq. (17.34), $L_1 C_1 = \dfrac{1}{\omega_0^2}$

Hence, Eq. (17.37) may be written as

$$\left(1-\frac{\omega_1^2}{\omega_0^2}\right) = \frac{\omega_1}{\omega_2}\left(\frac{\omega_2^2}{\omega_0^2} - 1\right)$$

$$(\omega_0^2 - \omega_1^2)\omega_2 = \omega_1(\omega_2^2 - \omega_0^2)$$

$$\omega_0^2\omega_2 - \omega_1^2\omega_2 = \omega_1\omega_2^2 - \omega_1\omega_0^2$$

$$\omega_0^2(\omega_1 + \omega_2) = \omega_1\omega_2(\omega_2 + \omega_1)$$

$$\omega_0^2 = \omega_1\omega_2$$

$$f_0 = \sqrt{f_1 f_2} \tag{17.38}$$

Thus, the resonant frequency is the geometric mean of the cut-off frequencies. The variation of the reactances with respect to frequency is shown in Fig. 17.29.

Design If the filter is terminated in a load resistance $R = K$, then at the lower cut-off frequency,

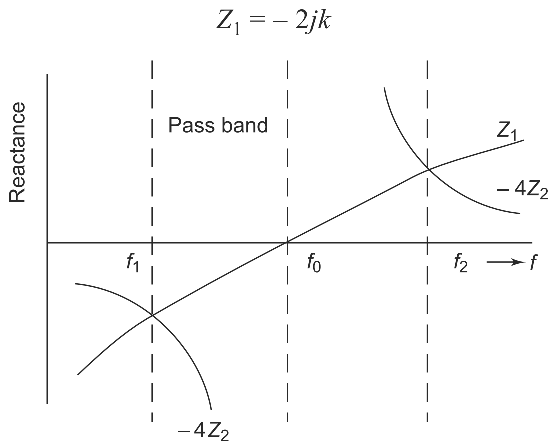


Fig. 17.29

$$\left(\frac{1}{j\omega_1 C_1} + j\omega_1 L_1\right) = -2jk$$

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2k$$

$$1 - \omega^2{}_1 C_1 L_1 = 2k\omega_1 C_1$$

Since $$L_1 C_1 = \frac{1}{\omega_0^2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = 2k\omega_1 C_1$$

or $$1 - \left(\frac{f_1}{f_0}\right)^2 = 4\pi k f_1 C_1$$

$$1 - \frac{f_1^2}{f_1 f_2} = 4\pi k f_1 C_1 \qquad (\because f_0 = \sqrt{f_1 f_2})$$

$$f_2 - f_1 = 4\pi k f_1 f_2 C_1$$

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2} \tag{17.39}$$

Since $$L_1 C_1 = \frac{1}{\omega_0^2}$$

$$L_1 = \frac{1}{\omega_0^2 C_1} = \frac{4\pi k f_1 f_2}{\omega_0^2 (f_2 - f_1)}$$

$$L_1 = \frac{k}{\pi(f_2 - f_1)} \tag{17.40}$$

To evaluate the values for the shunt-arm, consider the equation

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

$\therefore$ $$L_2 = C_1 k^2 = \frac{(f_2 - f_1)k}{4\pi f_1 f_2} \tag{17.41}$$

and $$C_2 = \frac{L_1}{k^2} = \frac{1}{\pi(f_2 - f_1)k} \tag{17.42}$$

Equations (17.39) through (17.42) are the design equations of a prototype band- pass filter. The variation of α, β with respect to frequency is shown in Fig. 17.30.

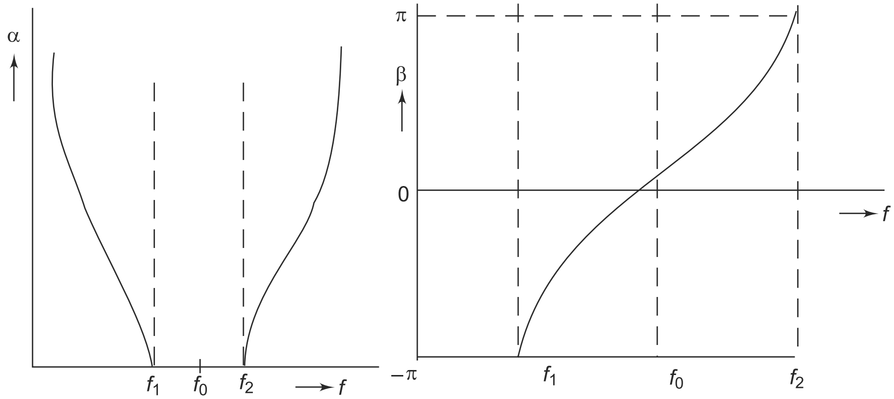


Fig. 17.30

EXAMPLE 17.5

Design a k-type band-pass filter having a design impedance of 500 Ω and cut-off frequencies 1 kHz and 10 kHz.

Solution $k = 500\ \Omega; f_1 = 1000$ Hz; $f_2 = 10000$ Hz

From Eq. (17.40),

$$L_1 = \frac{k}{\pi(f_2 - f_1)} = \frac{500}{\pi 9000} = \frac{55.55}{\pi}\text{ mH} = 16.68\text{ mH}$$

From Eq. (17.39),

$$C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2} = \frac{9000}{4 \times \pi \times 500 \times 1000 \times 10000} = 0.143\ \mu\text{F}$$

From Eq. (17.41),

$$L_2 = C_1 k^2 = 3.57\text{ mH}$$

From Eq. (17.42),

$$C_2 = \frac{L_1}{k^2} = 0.0707\ \mu\text{F}$$

Each of the two series arms of the constant k, T-section filter is given by

$$\frac{L_1}{2} = \frac{17.68}{2} = 8.84\text{ mH}$$

$$2C_1 = 2 \times 0.143 = 0.286\ \mu\text{F}$$

And the shunt-arm elements of the network are given by

$$C_2 = 0.0707\ \mu\text{F and } L_2 = 3.57\ \text{mH}$$

For the constant-k, π-section filter, the elements of the series arm are

$$C_1 = 0.143\ \mu\text{F and } L_1 = 16.68\ \text{mH}$$

The elements of the shunt-arms are

$$\frac{C_2}{2} = \frac{0.0707}{2} = 0.035\ \mu\text{F}$$

$$2L_2 = 2 \times 0.0358 = 0.0716\ \text{H}$$

17.10 BAND-ELIMINATION FILTER

LO 6

A band-elimination filter is one which passes without attenuation all frequencies less than the lower cut-off frequency f_1, and greater than the upper cut-off frequency f_2. Frequencies lying between f_1 and f_2 are attenuated. It is also known as band-stop filter. Therefore, a band-stop filter can be realised by connecting a low-pass filter in parallel with a highpass section, in which the cut-off frequency of low-pass filter is below that of a high pass filter. The configurations of T and π constant k band-stop sections are shown in Fig. 17.31. The band-elimination filter is designed in the same manner as is the band-pass filter.

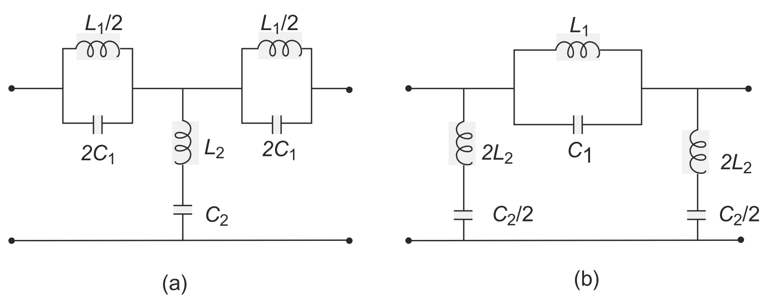


Fig. 17.31

As for the band-pass filter, the series and shunt-arms are chosen to resonate at the same frequency ω_0. Therefore, from Fig. 17.31 (a), for the condition of equal resonant frequencies,

$$\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1} \quad \text{for the series arm}$$

or
$$\omega_0^2 = \frac{1}{L_1 C_1} \tag{17.43}$$

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2} \quad \text{for the shunt-arm}$$

$$\omega_0^2 = \frac{1}{L_2 C_2} \tag{17.44}$$

$$\frac{1}{L_1C_1} = \frac{1}{L_2C_2} = k$$

Thus, $$L_1C_1 = L_2C_2 \tag{17.45}$$

It can be also verified that

$$Z_1Z_2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} = k^2 \tag{17.46}$$

and $$f_0 = \sqrt{f_1 f_2} \tag{17.47}$$

At cut-off frequencies, $Z_1 = -4Z_2$

Multiplying both sides with Z_2, we get

$$Z_1Z_2 = -4Z_2^2 = k^2$$

$$Z_2 = \pm j\frac{k}{2} \tag{17.48}$$

If the load is terminated in a load resistance, $R = k$, then at lower cut-off frequency,

$$Z_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = j\frac{k}{2}$$

$$\frac{1}{\omega_1 C_2} - \omega_1 L_2 = \frac{k}{2}$$

$$1 - \omega_1^2 C_2 L_2 = \omega_1 C_2 \frac{k}{2}$$

From Eq. (17.44), $L_2C_2 = \dfrac{1}{\omega_0^2}$

$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{k}{2}\omega_1 C_2$$

$$1 - \left(\frac{f_1}{f_0}\right)^2 = k\pi f_1 C_2$$

$$C_2 = \frac{1}{k\pi f_1}\left[1 - \left(\frac{f_1}{f_0}\right)^2\right]$$

Since $$f_0 = \sqrt{f_1 f_2}$$

$$C_2 = \frac{1}{k\pi}\left[\frac{1}{f_1} - \frac{1}{f_2}\right]$$

$$C_2 = \frac{1}{k\pi}\left[\frac{f_2 - f_1}{f_1 f_2}\right] \tag{17.49}$$

From Eq. (17.44), $\omega_0^2 = \frac{1}{L_2 C_2}$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{\pi k f_1 f_2}{\omega_0^2 (f_2 - f_1)}$$

Since $$f_0 = \sqrt{f_1 f_2}$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)} \tag{17.50}$$

Also, from Eq. (17.46),

$$k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1}$$

$$\therefore \quad L_1 = k^2 C_2 = \frac{k}{\pi}\left(\frac{f_2 - f_1}{f_1 f_2}\right) \tag{17.51}$$

$$\text{and } C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi k(f_2 - f_1)} \tag{17.52}$$

The variation of the reactances with respect to frequency is shown in Fig. 17.32.

Equation (17.49) through Eq. (17.52) are the design equations of a prototype band elimination filter. The variation of α, β with respect to frequency is shown in Fig. 17.33.

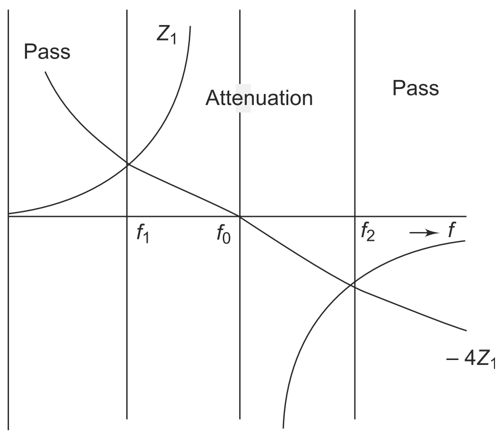


Fig. 17.32

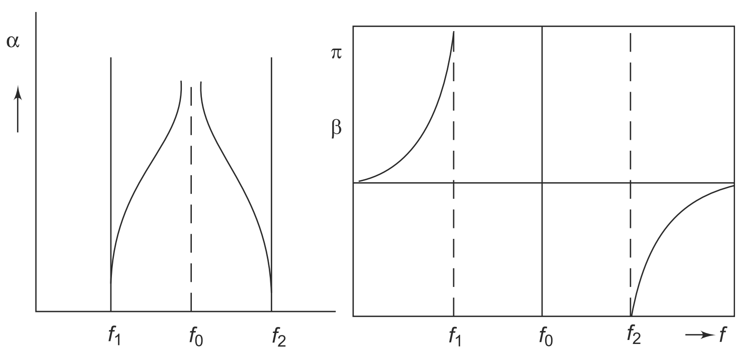


Fig. 17.33

EXAMPLE 17.6

Design a band-elimination filter having a design impedance of 600 Ω and cut-off frequencies $f_1 = 2$ kHz and $f_2 = 6$ kHz.

Solution $(f_2 - f_1) = 4$ kHz

Makinkg use of Eqs (17.49) through (17.52) in Section 17.10, we have

$$L_1 = \frac{k}{\pi}\left(\frac{f_2 - f_1}{f_2 f_1}\right) = \frac{600 \times 4000}{\pi \times 2000 \times 6000} = 63 \text{ mH}$$

$$C_1 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{1}{4 \times \pi \times 600(4000)} = 0.033\ \mu\text{F}$$

$$L_2 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{600}{4\pi(4000)} = 12 \text{ mH}$$

$$C_2 = \frac{1}{k\pi}\left[\frac{f_2 - f_1}{f_1 f_2}\right] = \frac{1}{600 \times \pi}\left[\frac{4000}{2000 \times 6000}\right] = 0.176\ \mu\text{F}$$

Each of the two series arms of the constant-k, T-section filter is given by

$$\frac{L_1}{2} = 31.5 \text{ mH}$$

$$2C_1 = 0.066\ \mu\text{F}$$

And the shunt-arm elements of the network are

$$L_2 = 12 \text{ mH and } C_2 = 0.176\ \mu\text{F}$$

For the constant k, π-section filter, the elements of the series arm are

$$L_1 = 63 \text{ mH}, C_1 = 0.033\ \mu\text{F}$$

and the elements of the shunt-arms are

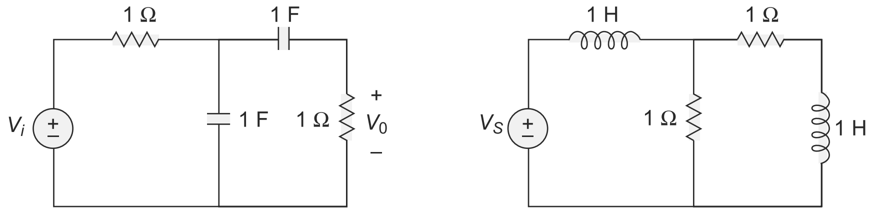
$$2L_2 = 24 \text{ mH and } \frac{C_2}{2} = 0.088\ \mu\text{F}$$

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 6

☆☆★ **17-6.1** Design a prototype band-pass filter with both T and π-sections having cut-off frequencies of 3000 Hz and 6000 Hz and nominal characteristic impedance of 600 Ω. Also, find the resonant frequency of shunt-arm or series arm.

☆☆★ **17-6.2** Using PSpice, determine center frequency and bandwidth of the band-pass filter shown in Fig. Q.2 (a) and (b).



(a) (b)

Fig. Q.2

☆☆★ **17-6.3** A low-quality factor, double-tuned band-pass filter is shown in Fig. Q.3. Use PSpice, to generate the magnitude plot of V_0 (w).

☆★★ **17-6.4** Using PSpice, find the bandwidth and center frequency of the band stop filter of Fig. Q.4.

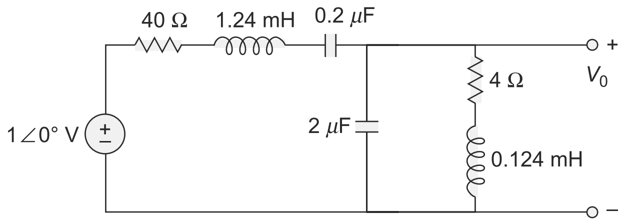


Fig. Q.3

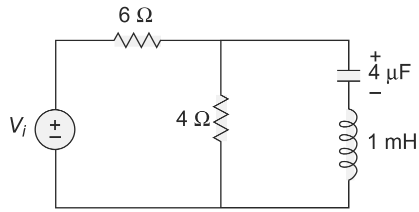


Fig. Q.4

Frequently Asked Questions linked to LO 6

☆☆★ **17-6.1** In a series resonance-type band-pass filter, $L = 60$ mH, $C = 150$ nF, and $R = 70$. **[BPUT 2007]**

☆☆★ **17-6.2** Explain the analysis of band-pass filter. **[JNTU Nov. 2012]**

☆☆★ **17-6.3** With neat diagrams, explain the resistance curves for a band-pass filter. Also obtain the design equations for a band-pass filter. **[PU 2012]**

☆☆★ **17-6.4** Explain various types of filters. **[MU 2014]**

☆☆★ **17-6.5** Draw T-section and π-section of a band-stop filter. **[PTU 2011-12]**

☆☆★ **17-6.6** How is a band-stop filter designed? **[PTU 2011-12]**

☆★★ **17-6.7** Design a prototype band-stop filter section having cut-off frequencies of 2000 Hz and 5000 Hz and design resistance of 600 Ω. **[JNTU Nov. 2012]**

17.11 ATTENUATORS

LO 7 Explain functionality of an attenuator and its various types

An attenuator is a two-port resistive network and is used to reduce the signal level by a given amount. In a number of applications, it is necessary to introduce a specified loss between source and a matched load without altering the impedance relationship. Attenuators may be used for this purpose. Attenuators may be symmetrical or asymmetrical, and can be either fixed or variable. A fixed attenuator with constant attenuation is called a *pad*. Variable attenuators are used as volume controls in radio broadcasting sections. Attenuators are also used the in laboratory to obtain small values of voltage or current for testing circuits.

The increase or decrease in power due to insertion or substitution of a new element in a network can be conveniently expressed in decibels (dB), or in nepers. In other words, attenuation is expressed either in decibels (dB) or in nepers.

Accordingly, the attenuation offered by a network in decibels is

$$\text{Attenuation in dB} = 10 \log_{10}\left(\frac{P_1}{P_2}\right) \tag{17.53}$$

where P_1 is the input power and P_2 is the output power.

For a properly matched network, both terminal pairs are matched to the characteristic resistance, R_0 of the attenuator.\

Hence,
$$\frac{P_1}{P_2} = \frac{I_1^2 R_0}{I_2^2 R_0} = \frac{I_1^2}{I_2^2} \tag{17.54}$$

where I_1 is the input current and I_2 is the output current leaving the port.

or
$$\frac{P_1}{P_2} = \frac{V_1^2}{V_2^2} \tag{17.55}$$

where V_1 is the voltage at the port 1 and V_2 is the voltage at the port 2

$$\text{Hence, attenuation in dB} = 20 \log_{10}\left(\frac{V_1}{V_2}\right) \tag{17.56}$$

$$= 20 \log_{10}\left(\frac{I_1}{I_2}\right) \tag{17.57}$$

If
$$\frac{V_1}{V_2} = \frac{I_1}{I_2} = N \tag{17.58}$$

then
$$\frac{P_1}{P_2} = N^2$$

and
$$\text{dB} = 20 \log_{10} N \tag{17.59}$$

or
$$N = \text{antilog}\left(\frac{\text{dB}}{20}\right) \tag{17.60}$$

17.12 *T*-TYPE ATTENUATOR

LO 7

Basically, there are four types of attenuators, T, π, lattice and bridged T-type. The basic design principles are discussed in the following sections. Figure 17.34 shows the symmetrical T-attenuator. An attenuator is to be designed for desired values of characteristic resistance, R_0 and attenuation.

The values of the arms of the network can be specified in terms of characteristic impedance, Z_0, and propagation constant, γ, of the network. The network in the figure is a symmetrical resistive circuit; hence, $Z_0 = R_0$ and $\gamma = \alpha$. The design equations can be obtained by applying Kirchhoff's law to the network in Fig. 17.34.

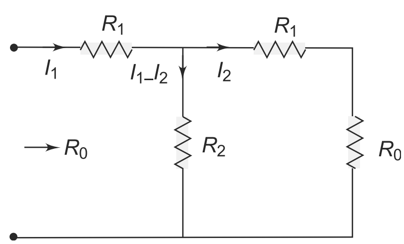


Fig. 17.34

$$R_2 (I_1 - I_2) = I_2 (R_1 + R_0)$$

$$I_2 (R_2 + R_1 + R_0) = I_1 R_2$$

$$\frac{I_1}{I_2} = \frac{R_1 + R_0 + R_2}{R_2} = N \tag{17.61}$$

The characteristic impedance of the attenuator is R_0 when it is terminated in a load of R_0

Hence, $$R_0 = R_1 + \frac{R_2(R_1 + R_0)}{R_1 + R_0 + R_2}$$

Substituting in Eq. (17.61), we have

$$R_0 = R_1 + \frac{(R_1 + R_0)}{N}$$

$$NR_0 = NR_1 + R_1 + R_0$$

$$R_0(N - 1) = R_1 (N + 1)$$

$$\boxed{R_1 = \frac{R_0(N-1)}{N+1}} \tag{17.62}$$

From Eq. (17.61), we have

$$NR_2 = R_1 + R_0 + R_2$$

$$(N - 1)R_2 = (R_1 + R_0)$$

Substituting the value of R_1 from Eq. (17.62), we have

$$(N-1)R_2 = R_0\frac{(N-1)}{N+1} + R_0$$

$$(N-1)R_2 = \frac{2NR_0}{(N+1)}$$

$$\boxed{R_2 = \frac{2NR_0}{N^2-1}} \tag{17.63}$$

Equations (17.62) and (17.63) are the design equations for the symmetrical *T*-attenuator.

EXAMPLE 17.7

Design a T-pad attenuator to give an attenuation of 60 dB and to work in a line of 500 Ω impedance.

Solution $$N = \frac{I_1}{I_2} = \text{antilog}\,\frac{D}{20}$$

$$= \text{antilog}\,\frac{60}{20} = 1000$$

Each of the series arm is given by

$$R_1 = \frac{R_0(N-1)}{N+1} = 500\frac{(1000-1)}{(1000+1)} = 499\ \Omega$$

The shunt-arm resistor R_2 is given by

$$R_2 = \frac{2N}{N^2-1} = R_0 = \frac{2\times 1000}{(1000)^2-1}\times 500 = 1\ \Omega$$

17.13 π-TYPE ATTENUATOR

LO 7

Figure 17.35 shows a symmetrical attenuator. The series and shunt-arms of the attenuator can be specified in terms of Z_0 and propagation constant γ. In this case also, the network is formed by resistors and is symmetrical, hence $Z_0 = R_0$ and $\gamma = \alpha$. From the fundamental equations, we have

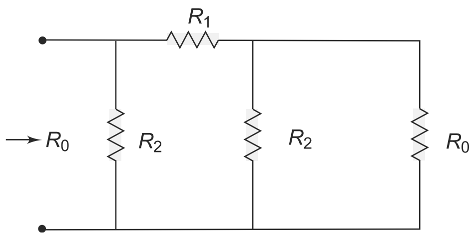


Fig. 17.35

$$R_1 = R_0 \sinh \alpha \qquad (17.64)$$

$$R_2 = R_0 \coth \alpha/2 \qquad (17.65)$$

$$\therefore \quad R_1 = R_0 \frac{e^{\alpha} - e^{-\alpha}}{2} \qquad (17.66)$$

By definition of propagation constant,

$$e^{\gamma} = \frac{I_1}{I_2} = N$$

Here, $\gamma = \alpha$ and $e^{\alpha} = N$

Therefore, Eq. (17.66) can be written as

$$\boxed{R_1 = R_0 \frac{N - \dfrac{1}{N}}{2} = R_0 \frac{N^2 - 1}{2N}} \qquad (17.67)$$

Similarly, from Eq. (17.65),

$$R_2 = R_0 \frac{\cosh \alpha / 2}{\sinh \alpha / 2} = R_0 \frac{e^{\alpha/2} + e^{-\alpha/2}}{e^{\alpha/2} - e^{-\alpha/2}}$$

$$\boxed{R_2 = R_0 \frac{e^{\alpha} + 1}{e^{\alpha} - 1} = R_0 \frac{(N-1)}{(N+1)}} \qquad (17.68)$$

Equations (17.67) and (17.68) are the design equations for the symmetrical π-attenuator.

EXAMPLE 17.8

Design a π-type attenuator to give 20 dB attenuation and to have a characteristic impedance of 100 Ω.

Solution Given $R_0 = 100\ \Omega$, $D = 20$ dB.

$$N = \text{Antilog}\ \frac{D}{20} = 10$$

$$R_1 = R_0 \frac{(N^2 - 1)}{2N} = 100 \frac{(10^2 - 1)}{2 + 10} = 495\ \Omega$$

$$R_2 = R_0 \frac{(N+1)}{(N-1)} = 100 \left(\frac{10+1}{10-1} \right) = 122.22\ \Omega$$

17.14 LATTICE ATTENUATOR

LO 7

A symmetrical resistance lattice is shown in Fig. 17.36. The series and the diagonal arm of the network can be specified in terms of the characteristic impedance Z_0 and propagation constant γ.

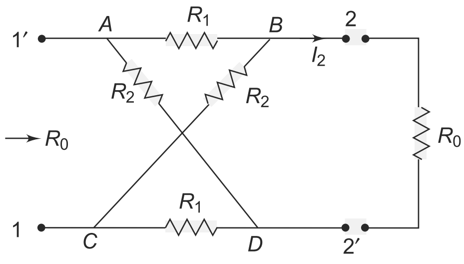


Fig. 17.36

It has already been stated and proved that characteristic impedance of symmetrical network is the geometric mean of the open and short- circuit impedance. The circuit in Fig. 17.36 is redrawn as in Fig. 17.37 to calculate the open-and short-circuit impedances.

Thus, $$Z_{sc} = \frac{2R_1R_2}{R_1+R_2}$$

$$Z_{0c} = \frac{R_1+R_2}{2}$$

Hence, $$Z_0 = R_0 = \sqrt{Z_{0c}Z_{sc}}$$

$$R_0 = \sqrt{R_1R_2}$$

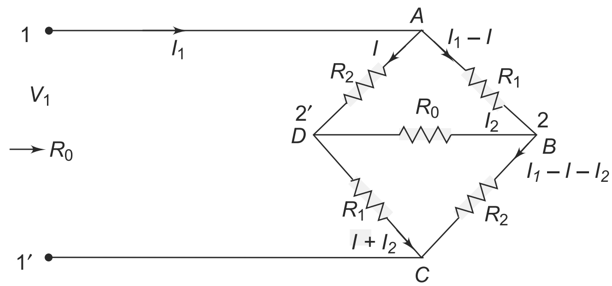


Fig. 17.37

In Fig. 17.37, the input impedance at 1-1′ is R_0 when the network is terminated in R_0 at 2-2′. By applying Kirchhoff's voltage law, we get

$$V_1 = I_1R_0 = (I_1 - I)R_1 + I_2R_0 + (1 + I_2)R_1$$

$$I_1R_0 = R_1(I_1 + I_2) + I_2R_0$$

$$I_1(R_0 - R_1) = I_2(R_1 + R_0)$$

$$\frac{I_1}{I_2} = \frac{R_1+R_0}{R_0-R_1} = \frac{1+\dfrac{R_1}{R_0}}{1-\dfrac{R_1}{R_0}} \tag{17.69}$$

$$N = e^{\alpha} = \frac{I_1}{I_2} = \frac{1+\dfrac{R_1}{R_0}}{1-\dfrac{R_1}{R_0}} \tag{17.70}$$

$$e^{\alpha} = \frac{1+\sqrt{R_1/R_2}}{1-\sqrt{R_1/R_2}}$$

The propagation constant $$\alpha = \log\left[\frac{1+\sqrt{\dfrac{R_1}{R_2}}}{1-\sqrt{\dfrac{R_1}{R_2}}}\right] \tag{17.71}$$

From Eq. (17.70),

$$N\left(1-\frac{R_1}{R_0}\right) = \left(1+\frac{R_1}{R_0}\right)$$

$$\boxed{R_1 = R_0\frac{(N-1)}{(N+1)}} \tag{17.72}$$

Similarly, we can express $$\boxed{R_2 = R_0\frac{(N+1)}{(N-1)}} \tag{17.73}$$

Equations (17.72) and (17.73) are the design equations for lattice attenuator.

EXAMPLE 17.9

Design a symmetrical lattice attenuator to have characteristic impedance of 800 Ω and attenuation of 20 dB.

Solution Given $R_0 = 800\ \Omega$ and $D = 20$ dB

$$N = \text{antilog}\frac{D}{20} = \text{antilog}\frac{20}{20} = 10$$

From the design equations of lattice attenuator,

Series-arm resistance $$R_1 = R_0\frac{(N-1)}{(N+1)}$$

$$= 800\frac{(10-1)}{(10+1)} = 654.545\ \Omega$$

Diagonal-arm resistance $$R_2 = R_0\frac{(N+1)}{(N-1)}$$

$$= 800\frac{(10+1)}{(10-1)} = 977.777\ \Omega$$

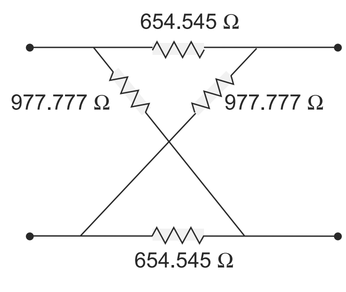


Fig. 17.38

The resulting lattice attenuator is shown in Fig. 17.38.

17.15 BRIDGED-*T* ATTENUATOR

LO 7

A bridged-*T* attenuator is shown in Fig. 17.39. In this case also, since the attenuator is formed by resistors only, $Z_0 = R_0$ and $\gamma = \alpha$. The bridged-*T* network may be designed to have any characteristic resistance R_0 and desired attenuation by making $R_A\ R_B = R^2_0$. Here, R_A and R_B are variable resistances and all other resistances are equal to the characteristic resistance R_0 of the network.

A symmetrical resistance lattice network can be converted into an equivalent T, π or bridged-T resistance network using the bisection theorem. We can obtain the design equations of the bridged-T attenuator by bisection theorem. A bisected half-section is shown in Fig. 17.40. According to the bisection theorem, a network having mirror image symmetry can be reduced to an equivalent lattice structure. The series arm of the equivalent lattice is found by bisecting the given network into two parts, short-circuiting all the cut wires and equating the series impedance of the lattice to the input impedance of the bisected network; the diagonal arm is equal to the input impedance of the bisected network when cut wires are open circuited.

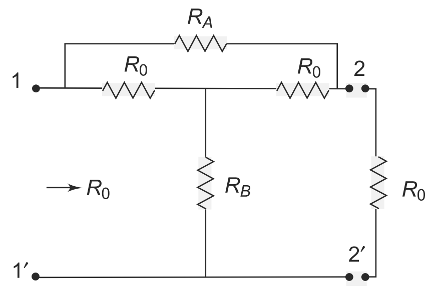


Fig. 17.39

From Fig. 17.40, when the cut wires A, B, C are shorted, the input resistance of the network is given by

$$R_{sc} = \frac{R_0 \times R_{A/2}}{R_0 + R_{A/2}} = \frac{R_0 R_A}{2R_0 + R_A} \quad (17.74)$$

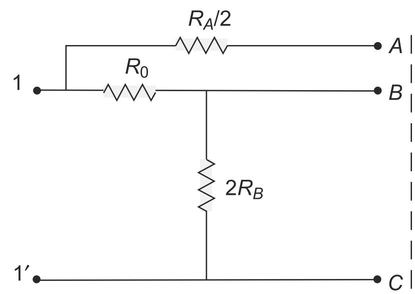


Fig. 17.40

This resistance is equal to the series-arm resistance of the lattice network shown in Fig. 17.36.

$$\therefore \quad \frac{R_0 R_A}{2R_0 + R_A} = R_1 \quad (17.75)$$

From Eq. (17.72), we have

$$R_1 = R_0 \frac{(N-1)}{(N+1)}$$

Hence, $$\frac{R_0 R_A}{(2R_0 + R_A)} = R_0 \frac{(N-1)}{(N+1)}$$

from which, $R_A = R_0\,(N - 1)$ (17.76)

From Fig. 17.40, when the cut wires A, B, C are open, the input resistance of the network is given by

$$R_{0c} = (R_0 + 2R_B) \quad (17.77)$$

This resistance is equal to the diagonal arm resistance of the lattice network shown in Fig. 17.36.

$$\therefore \quad R_0 + 2R_B = R_2 \quad (17.78)$$

From Eq. (17.73), we have

$$R_2 = R_0 \frac{(N+1)}{(N-1)}$$

Hence, $$(R_0 + 2R_B) = R_0 \frac{(N+1)}{(N-1)}$$

from which, $$R_B = \frac{R_0}{N-1} \quad (17.79)$$

Equations (17.76) and (17.79) are the design equations for bridged-T attenuator.

EXAMPLE 17.10

Design a symmetrical bridged-T attenuator with an attenuation of 20 dB and terminated into a load of 500 Ω.

Solution $D = 20\ dB;\ R_0 = 500\,\Omega$

$$N = \text{antilog}\frac{D}{20} = \text{antilog}\frac{20}{20} = 10$$

$$R_A = R_0(N-1) = 500(10-1) = 4500\,\Omega$$

$$R_B = \frac{R_0}{(N-1)} = \frac{500}{(10-1)} = 55.555\,\Omega$$

The desired configuration of the attenuator is shown in Fig.17.41.

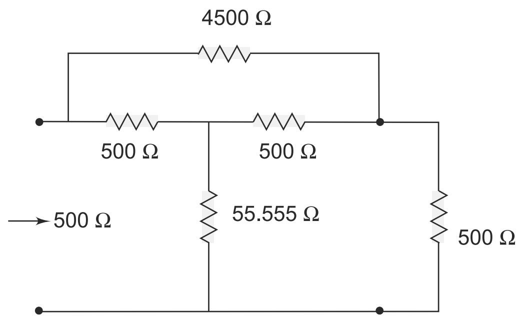


Fig. 17.41

17.16 L-TYPE ATTENUATOR

LO 7

An L-type asymmetrical attenuator is shown in Fig. 17.42. The attenuator is connected between a source with source resistance $R_s = R_0$ and load resistance $R_L = R_0$.

The design equations can be obtained by applying simple laws.

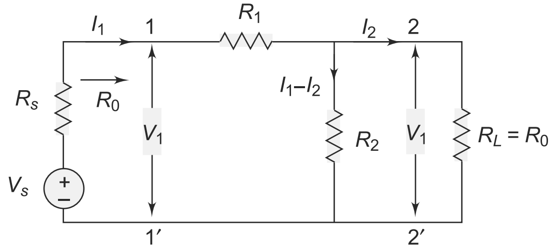


Fig. 17.42

$$V_2 = (I_1 - I_2)R_2 = I_2R_L$$

or $\quad I_1R_2 = I_2(R_2 + R_L)$

$$\frac{I_1}{I_2} = \frac{R_2 + R_L}{R_2} = N \tag{17.80}$$

$$1 + \frac{R_L}{R_2} = N$$

$$R_2 = \frac{R_L}{N-1} \tag{17.81}$$

As $R_L = R_0$, Eq. (17.81) can be written as

$$R_2 = \frac{R_0}{N-1} \tag{17.82}$$

The resistance of the network as viewed from 1-1′ into the network is

$$R_0 = R_1 + \frac{R_2R_0}{R_2 + R_0}$$

$$R_1 = \frac{R_0^2}{R_2 + R_0} \tag{17.83}$$

Substituting the value of R_2 from Eq. (17.82), we have

$$R_1 = \frac{R_0^2}{\dfrac{R_0}{N-1} + R_0} = \frac{R_0^2(N-1)}{R_0 + R_0(N-1)}$$

$$\boxed{R_1 = R_0 \frac{(N-1)}{N}} \tag{17.84}$$

Equations (17.82) and (17.84) are the design equations. Attenuation N of the network can be varied by varying the values of R_1 and R_2.

EXAMPLE 17.11

Design a L-type attenuator to operate into a load resistance of 600 Ω with an attenuation of 20 dB.

Solution $N = \text{antilog}\,\dfrac{\text{dB}}{20} = \text{antilog}\,\dfrac{20}{20} = 10$

The series arm of the attenuator is given by

$$R_1 = R_0\left(\frac{N-1}{N}\right) = 600\left(\frac{10-1}{10}\right) = 540\,\Omega$$

The shunt-arm of the attenuator is given by

$$R_2 = \frac{R_0}{N-1} - \frac{600}{9} = 66.66\,\Omega$$

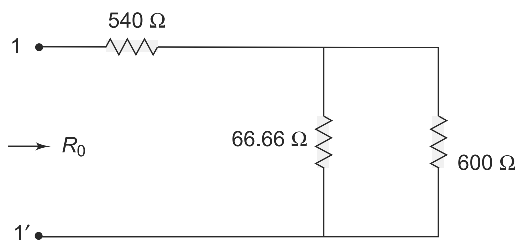


Fig. 17.43

The desired configuration of the network is shown in Fig. 17.43.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 7

☆☆★ **17-7.1** An attenuator is composed of symmetrical π-section having a series arm of 275 Ω and shunt-arm each of 450 Ω. Find
(a) The characteristic impedance of the network
(b) Attenuation per section

Frequently Asked Questions linked to LO 7

☆☆★ **17-7.1** Derive the relationship between neper and decibel. **[PU 2012]**

☆☆★ **17-7.2** Design a π-type attenuator with the following specifications: Attenuation = 20 dB, characteristic resistance = 20 Ω. **[PU 2012]**

☆★★ **17-7.3** Design an asymmetrical T-attenuator so that it works between a source and load impedance of 260 Ω and 490 Ω respectively and provides an attenuation of 40 dB. **[PTU 2011-12]**

17.17 EQUALISERS

LO 8 Explain equalizers and their types

Equalisers *are networks designed to provide compensation against distortions that occur in a signal while passing through an electrical network.* In general, any electrical network has attenuation distortion and phase distortion.

Attenuation distortion occurs due to non-uniform attenuation against frequency characteristics. Phase distortion occurs due to phase delay against frequency characteristics. An attenuation equaliser is used to compensate attenuation distortion in any network. These equalisers are used in medium to high frequency-carrier telephone systems, amplifiers, transmission lines, and speech reproduction, etc. A phase equaliser is used to compensate phase distortion in any network. These equalisers are used in TV signal transmission lines and in facsimile.

17.18 INVERSE NETWORK

LO 8

The geometrical mean of two impedances Z_1 and Z_2 is a real number and they are said to be inverse if

$$Z_1 Z_2 = R_0^2$$

where R_0 is a resistance.

Consider $Z_1 = R_1$ and $Z_2 = R_2$

The product $Z_1 Z_2$ is a real number.

Therefore, the two impedances are said to be inverse if they satisfy the relation $Z_1 Z_2 = R_1 R_2 = R_0^2$.

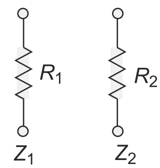


Fig. 17.44

In another case, consider $Z_1 = j\omega L$ and $Z_2 = \dfrac{1}{j\omega C}$

$$Z_1 Z_2 = \frac{j\omega L}{j\omega C} = \frac{L}{C}$$

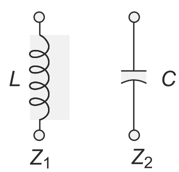


Fig. 17.45

The product $Z_1 Z_2$ is a real number.

Therefore, the two impedances are inverse.

Similarly,

$$\text{Let} \quad Z_1 = R_1 + j\omega L \tag{17.85}$$

$$\text{and} \quad Z_2 = \frac{R_2 \dfrac{1}{j\omega C}}{R_2 + \dfrac{1}{j\omega C}} = \frac{-jR_2}{\omega C R_2 - j} \cdot \frac{\omega C R_2 + j}{\omega C R_2 + j} \tag{17.86}$$

$$= \frac{R_2 - j\omega C R_2^2}{1 + \omega^2 C^2 R_2^2}$$

$$Z_1 Z_2 = (R_1 + j\omega L)\left(\frac{R_2 - j\omega C R_2^2}{1 + \omega^2 C^2 R_2^2}\right)$$

$$= \frac{R_1 R_2 + \omega^2 R_2^2 LC + j(\omega L R_2 - \omega C R_1 R_2^2)}{1 + \omega^2 C^2 R_2^2} \tag{17.87}$$

The imaginary part of the above equation must be zero.

Therefore, we get $\omega L R_2 = \omega C R_1 R_2^2$

$$\frac{L}{C} = R_1 R_2 = R_0^2 \tag{17.88}$$

The two impedances Z_1 and Z_2 are inverse, when the above condition is satisfied.
An inverse network may be obtained by

1. Converting each series branch into parallel branch and vice versa.
2. Converting each resistance element R into a corresponding resistive element $\frac{R_0^2}{R}$.
3. Converting each inductance L into capacitance $C^1 = \frac{L}{R_0^2}$.
4. Converting each capacitance C into inductance $L^1 = CR_0^2$.

EXAMPLE 17.12

Obtain the inverse network of the network shown in Fig. 17.46.

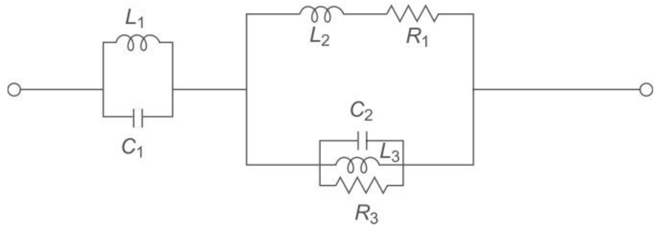


Fig. 17.46

Solution The parallel branch is converted into a series branch and vice versa. The capacitance is replaced by inductance and vice versa. The resistance is replaced by another resistance as shown in Fig. 17.47.

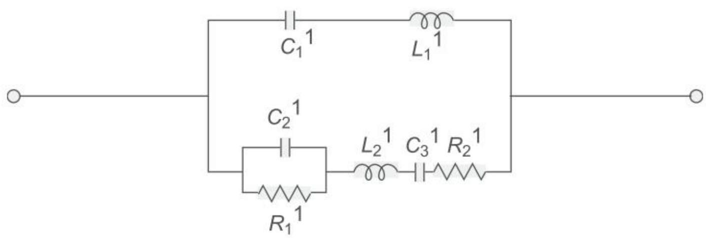


Fig. 17.47

Here, $C_1^1 = \frac{L_1}{R_0^2}, iL_1^1 = C_1 R_0^2, R_1^1 = \frac{R_0^2}{R_1}$

$$C_2^1 = \frac{L_2}{R_0^2}, iL_2^1 = C_2 R_0^2, R_2^1 = \frac{R_0^2}{R_2}$$

$$C_3^1 = \frac{L_3}{R_0^2} \text{ and } R_0 = \text{design impedance.}$$

17.19 SERIES EQUALISER

LO 8

The series equaliser is a two-terminal network connected in series with a network to be corrected. (see Fig. 17.48)

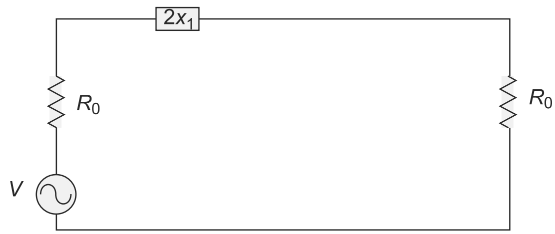


Fig. 17.48

Let N = Input to output power ratio of the load
D = Attenuation in decibels
R_0 = resistance of the load as well as source
P_1 = input power
P_1 = load power
$2X_1$ = reactance of the equaliser
V_{max} = Voltage applied to the network

Attenuation $D = \log_{10} N$

$$\text{or} \quad N = \text{antilog}\left(\frac{D}{10}\right) \tag{17.89}$$

$$N = \frac{\text{Maximum power delivered to the load when equlizer is not present}}{\text{Power delivered to the load when equalizer is present}}$$

$$N = \frac{P_i}{P_l}$$

$$P_i = \left(\frac{V_{max}}{2R_0}\right)^2 R_0 = \frac{V_{max}^2}{4R_0}$$

When the equaliser is connected,

$$l_1 = \frac{V_{max}}{\sqrt{(2R_0)^2 + (2X_1)^2}}$$

$$P_l = \left[\frac{V_{max}}{\sqrt{(2R_0)^2 + (2X_1)^2}}\right]^2 R_0$$

$$= \left[\frac{V_{max}^2}{4(R_0^2 + X_1^2)}\right] R_0 \tag{17.90}$$

$$\text{Therefore,} \quad N = \frac{P_i}{P_l} = \frac{V_{max}^2 / 4R_0}{\dfrac{V_{max}^2 R_0}{4(R_0^2 + X_1^2)}} = 1 + \frac{X_1^2}{R_0^2} \tag{17.91}$$

By knowing the values of R_0 and N, X_1 can be determined.

17.20 FULL-SERIES EQUALISER

LO 8

Figure 17.49 shows the configuration of a full-series equaliser.

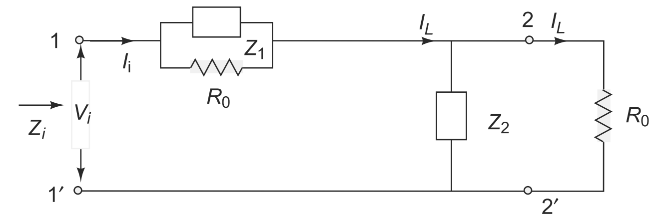


Fig. 17.49

The circuit is a constant resistance equaliser satisfying the relation $Z_1Z_2 = R_0^2$. The input impedance is given by

$$Z_i = \frac{R_0Z_1}{R_0+Z_1} + \frac{R_0Z_2}{R_0+Z_2}$$

$$= \frac{R_0[2Z_1Z_2 + R_0(Z_1+Z_2)]}{R_0^2 + R_0(Z_1+Z_2) + Z_1Z_2} \tag{17.92}$$

If we substitute $Z_1Z_2 = R_0^2$ in the above equation,

$$Z_i = R_0$$

$$|V_i| = I_iZ_i = I_i R_0$$

$$|V_l| = I_iZ_i = I_i \frac{R_0Z_2}{R_0+Z_2} \tag{17.93}$$

$$N = \left|\frac{V_i}{V_l}\right|^2 = \left|\frac{R_0+Z_2}{Z_2}\right|^2 = 1 + \frac{R_0^2}{X_2^2} \tag{17.94}$$

$$= 1 + \frac{X_1^2}{R_0^2}$$

Since Z_1 and Z_2 are pure reactances and $X_1X_2 = R_0^2$

1. When $X_1 = \omega L$,

$$X_2 = \frac{1}{\omega C_1} \text{ since both are inverse.}$$

The full-series equaliser is shown in Fig. 17.50.

where $\dfrac{L_1}{C_1} = R_0^2$

From the equation, $N = 1 + \dfrac{X_1^2}{R_0^2}$

$$= 1 + \frac{\omega^2 L_1^2}{R_0^2}$$

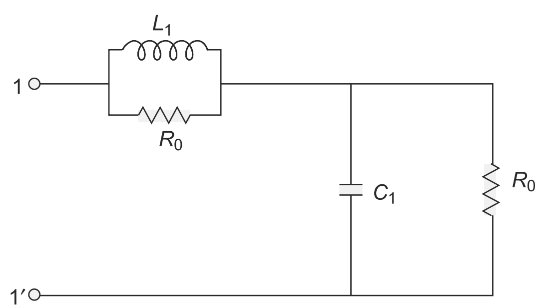


Fig. 17.50

By knowing the values of N and R_0, the elemental values of L_1, C_1 may be obtained.

2. When $X_1 = \dfrac{1}{\omega C_1}$,

$$X_2 = \omega L_1$$

The full, series equaliser is shown in Fig. 17.51.

Here, $\dfrac{L_1}{C_1} = R_0^2$

From the equation, $N = 1 + \dfrac{R_0^2}{X_2^2} = 1 + \dfrac{R_0^2}{\omega^2 L_1^2}$

By knowing the values of N and R_0, the values of L_1, C_1 may be obtained.

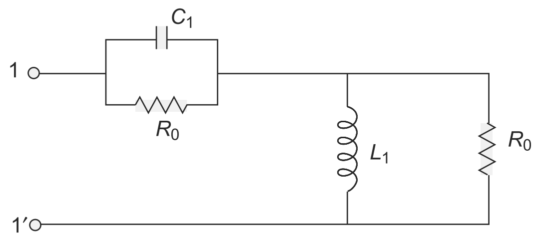


Fig. 17.51

17.21 SHUNT EQUALISER

LO 8

The shunt equaliser is a two-terminal network connected in shunt with a network to be corrected.

Let N = input to output power ratio
D = attenuation in decibels
R_0 = source resistance/load resistance
I_s = source current
I_l = load current
P_i = input power
P_l = load power
$\dfrac{X_1}{2}$ = reactance of shunt equaliser

The shunt equaliser connected to the network is shown in Fig. 17.52.

$$\text{Source current } I_s = \frac{V_{\max}}{R_0 + \left(R_0 // \dfrac{jX_1}{2}\right)} \tag{17.95}$$

$$= \frac{V_{\max}}{R_0 + \left[\dfrac{jX_1 R_0}{2R_0 + jX_1}\right]}$$

$$= \frac{V_{\max}[2R_0 + jX_1]}{2R_0(R_0 + jX_1)}$$

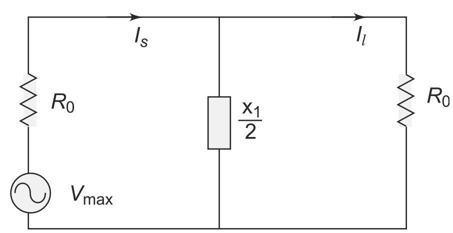


Fig. 17.52

$$\text{Load current } I_l = I_s \frac{jX_1/2}{R_0 + \dfrac{jX_1}{2}} = I_s \frac{jX_1}{2R_0 + jX_1} \tag{17.96}$$

Substituting I_s in the above equation,

$$I_l = \frac{V_{\max} jX_1}{2R_0(R_0 + jX_1)} \tag{17.97}$$

Power delivered to the load

$$P_l = |I_l|^2 R_0 = \frac{V_{\max}^2 X_1^2}{4R_0(R_0^2 + X_1^2)} \tag{17.98}$$

and $\quad P_i = V_{\max}^2 / 4R_0$

Therefore, $\quad N = \dfrac{P_i}{P_l} = \dfrac{\dfrac{V_{\max}^2}{4R_0}}{\dfrac{V_{\max}^2 X_1^2}{4R_0(R_0^2 + X_1^2)}}$

$$\therefore \quad N = 1 + \left(\frac{R_0}{X_1}\right)^2 \tag{17.99}$$

By knowing the values of R_0 and N, X_1 can be determined.

17.22 FULL-SHUNT EQUALISER

LO 8

Figure 17.53 shows the full-shunt equaliser. It is also a constant-resistance equaliser which satisfies the equation $Z_1 Z_2 = R_0^2$.

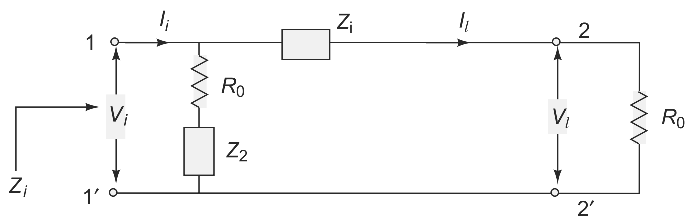


Fig. 17.53

The input impedance is given by

$$Z_i = \frac{(R_0 + Z_2)(R_0 + Z_1)}{2R_0 + Z_1 + Z_2} \tag{17.100}$$

$$= \frac{Z_1 Z_2 + R_0^2 + R_0(Z_1 + Z_2)}{2R_0 + Z_1 + Z_2}$$

$$Z_l = R_0$$

Since $\quad Z_1 Z_2 = R_0^2$,

$$V_i = I_i Z_i = I_i R_0$$

$$V_l = I_l R_0$$

$$\frac{V_i}{V_l} = \frac{I_i}{I_l}$$

But $\quad I_l = I_i \dfrac{(R_0 + Z_2)}{2R_0 + Z_1 + Z_2}$ (17.101)

$$\frac{I_i}{I_l} = \frac{Z_1 + Z_2 + 2R_0}{R_0 + Z_2} \tag{17.102}$$

Multiplying both numerator and denominator by Z_1, we get

$$\frac{I_i}{I_l} = \frac{Z_1^2 + Z_1 Z_2 + 2R_0 Z_1}{Z_1 R_0 + Z_1 Z_2}$$

$$\frac{I_i}{I_l} = \frac{(Z_1 + R_0)^2}{R_0(Z_1 + R_0)} = \frac{Z_1 + R_0}{R_0}$$

Therefore, $N = \left|\dfrac{V_i}{V_l}\right|^2 = \left|\dfrac{I_i}{I_l}\right|^2 = \left|\dfrac{R_0 + Z_1}{R_0}\right|^2$

$$N = 1 + \frac{X_1^2}{R_0^2} = 1 + \frac{R_0^2}{X_2^2} \tag{17.103}$$

since Z_1 and Z_2 are pure reactances and are equal to X_1 and X_2 respectively.

By knowing the values or R_0 and N_1 the elemental values X_1 and X_2 can be obtained.

1. When $X_1 = \omega L_1$

   X_2 becomes $\dfrac{1}{\omega C_1}$

   The circuit is shown in Fig. 17.54.

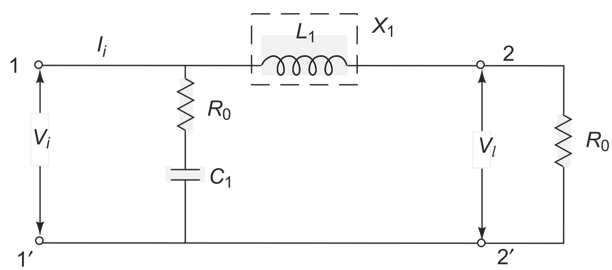


Fig. 17.54

2. When $X_1 = \dfrac{1}{\omega C_1}$,

   X_2 becomes ωL_1

The circuit is shown in Fig. 17.55.

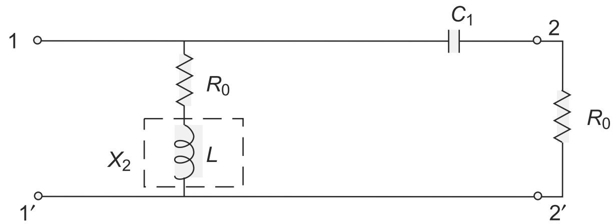


Fig. 17.55

17.23 CONSTANT-RESISTANCE EQUALISER

LO 8

The disadvantage of a reactance equaliser either in a shunt equaliser or a series equaliser is that, the variation of impedance with frequency causes impedance mismatch which results in reflection losses. A four-terminal equaliser which offers a constant resistance at all frequencies avoids reflection loss when terminated in its design impedance. Constant-resistance equaliser is a four terminal network which can be T, π, lattice and bridged-T network type. All these types have characteristic impedance satisfying the relation $Z_1Z_2 = R_0^2$.

17.24 BRIDGED-*T* ATTENUATION EQUALISER

LO 8

The network shown in Fig. 17.56 is a bridged-T attenuation equaliser. Let Z_1 be a parallel combination of resistor R_1 and inductance L_1. To provide a constant resistance, the impedance Z_2 must be an inverse of Z_1 which is a series combination of R_2 and a capacitor C_1. Let R_0 be the design resistance.

Then, $\quad Z_1Z_2 = R_0^2$

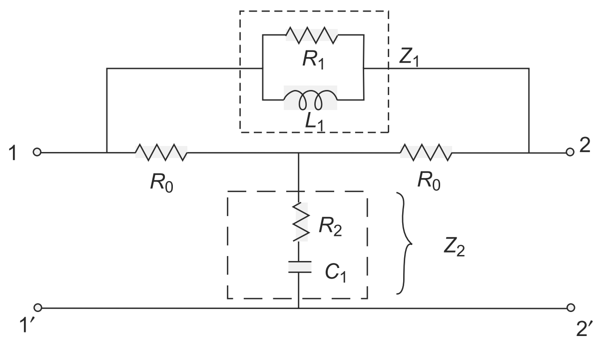


Fig. 17.56 *Bridged-T attenuation equaliser*

The propagation constant for a bridged-T network is given by

$$\gamma = \ln\left[1+\frac{Z_1}{Z_0}\right] = \ln\left[1+\frac{Z_0}{Z_2}\right] \tag{17.104}$$

But $Z_0 = R_0$

And $$Z_1 = \frac{jR_1\,\omega L_1}{R_1 + \omega L_1} \tag{17.105}$$

Therefore, the propagation constant

$$\gamma = \ln\left[1 + \frac{jR_1\omega L_1}{R_0(R_1 + j\omega L_1)}\right] \tag{17.106}$$

$$\alpha + j\beta = \ln\left[\frac{R_0R_1 + j\omega L_1(R_0 + R_1)}{R_0R_1 + j\omega L_1R_0}\right] \tag{17.107}$$

Equating real parts on both sides

$$\alpha = \ln\left[\frac{(R_0R_1)^2 + \omega^2L_1^2R_0^2 + \omega^2L_1^2R_1^2 + 2\omega^2L_1^2R_0R_1}{R_0^2R_1^2 + \omega^2L_1^2R_0^2}\right]^{1/2}$$

$$= \frac{1}{2}\ln\left[1 + \frac{\omega^2L_1^2R_1(2R_0 + R_1)}{R_0^2(R_1^2 + \omega^2L_1^2)}\right] \tag{17.108}$$

and $$R_1R_2 = R_0^2 = \frac{L_1}{C_1} \tag{17.109}$$

The elements may be calculated from the above design from Eqs (17.108) and (17.109).

17.25 BRIDGED-*T* PHASE EQUALISER

LO 8

A bridged-*T* phase equaliser is shown in Fig. 17.57.

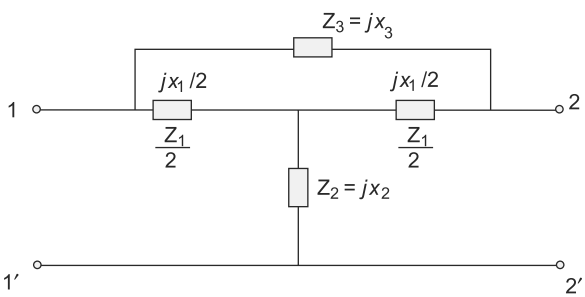


Fig. 17.57

It consists of only pure reactances.
The characteristics impedance is given by

$$Z_0 = \left[\frac{Z_1Z_3(Z_1 + 4Z_2)}{4(Z_1 + Z_3)}\right]^{1/2} \tag{17.110}$$

From Fig. 17.57, $Z_3 = jX_3, \dfrac{Z_1}{2} = jX_1, Z_2 = jX_2$ and $Z_0 = R_0$.

$$R_0^2 = \frac{2jX_1 \cdot jX_3(2jX_1 + 4jX_2)}{4(2jX_1 + 4jX_3)} \tag{17.111}$$

$$= \frac{-X_1 X_3 (X_1 + 2X_2)}{2X_1 + X_3}$$

Let X_1 and X_3 be made inverse.

$$jX_1 . jX_3 = R_0^2$$

$$-X_1 X_3 = R_0^2 \tag{17.112}$$

Substituting this in the above equation, we get

$$X_2 = \frac{X_1 + X_3}{2} \tag{17.113}$$

The propagation constant is given by

$$e^{\gamma} = \frac{Z_0(Z_1 + Z_3) + (Z_1 Z_3 / 2)}{Z_0(Z_1 + Z_3) - (Z_1 Z_3 / 2)} \tag{17.114}$$

$$e^{\gamma} - 1 = \frac{Z_1 Z_3}{Z_0(Z_1 + Z_3) - (Z_1 Z_3 / 2)}$$

and similarly,

$$e^{\gamma} + 1 = \frac{2Z_0(Z_1 + Z_3)}{Z_0(Z_1 + Z_3) - \left(\dfrac{Z_1 Z_3}{2}\right)}$$

From the above equations,

$$\frac{e^{\gamma} - 1}{e^{\gamma} + 1} = \tanh\frac{\gamma}{2} = \frac{Z_1 Z_3}{2Z_0(Z_1 + Z_3)} = \frac{2jX_1 \cdot jX_3}{2R_0(2jX_1 + jX_3)}$$

$$= \frac{2R_0^2}{2R_0 j(2X_1 + X_3)} \tag{17.115}$$

$$\tanh\frac{\gamma}{2} = \frac{2R_0^2}{2R_0 j\left(2X_1 - \dfrac{R_0^2}{X_1}\right)}$$

$$\frac{\gamma}{2} = \tanh^{-1}\frac{jR_0 X_1}{R_0^2 - 2X_1^2}$$

$$\therefore \quad \alpha + j\beta = 2j\tan^{-1}\frac{R_0X_1}{R_0^2 - 2X_1^2}$$

Equating the real and imaginary parts, we get

$$\alpha = 0$$

$$\beta = 2\tan^{-1}\left(\frac{R_0X_1}{R_0^2 - 2X_1^2}\right) \tag{17.116}$$

Equations (17.112), (17.113), and (17.116) are the design equations of a bridged-*T* phase equaliser.

17.26 LATTICE-ATTENUATION EQUALISER

LO 8

The constant-resistance lattice attenuation equaliser is shown in Fig. 17.58. The element Z_1 represents series arm and Z_2 represents diagonal arm as shown in Fig. 17.58. The equaliser is a constant-resistance equaliser such that Z_1 must be inverse of Z_2 to the design resistance R_0.

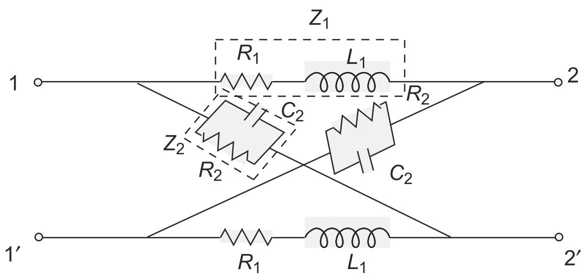


Fig. 17.58

$$Z_1Z_2 = R_0^2$$

$$R_1R_2 = \frac{L_1}{C_1}R_0^2 \tag{17.117}$$

The propagation constant of a lattice network is given by

$$\gamma = \ln\left(\frac{1+\dfrac{Z_1}{R_0}}{1-\dfrac{Z_1}{R_0}}\right) = \ln\left(\frac{1+\dfrac{Z_2}{R_0}}{\dfrac{Z_2}{R_0}-1}\right) \tag{17.118}$$

$$\alpha + j\beta = \ln\left(\frac{1+\dfrac{R_1 + j\omega L_1}{R_0}}{1-\dfrac{R_1 + j\omega L_1}{R_0}}\right) \tag{17.119}$$

$$\alpha + j\beta = \ln\left[\frac{(R_0 + R_1) + j\omega L_1}{(R_0 - R_1) - j\omega L_1}\right]$$

Equating real parts on both sides,

$$\alpha = \ln\left[\frac{(R_0+R_1)^2+\omega^2 L_1^2}{(R_0-R_1)^2+\omega^2 L_1^2}\right]^{1/2}$$

$$N = e^{\alpha} = \left[\frac{(R_0+R_1)^2+\omega^2 L_1^2}{(R_0-R_1)^2+\omega^2 L_1^2}\right]^{1/2} \tag{17.120}$$

On the other hand, if $X_1 = \dfrac{1}{\omega C_1}$

$$N = e^{\alpha} = \left[\frac{(R_0+R_1)^2+\dfrac{1}{\omega^2 C_1^2}}{(R_0-R_1)^2+\dfrac{1}{\omega^2 C_1^2}}\right]^{1/2} \tag{17.121}$$

Equations (17.117) and (17.121) are called design equations for the lattice- attenuator network.

17.27 LATTICE-PHASE EQUALISER

LO 8

The lattice-phase equaliser is shown in Fig. 17.59. It consists of only reactive components. This is also a constant-resistance equaliser which satisfies the equation $Z_1 Z_2 = R_0^2$.

Z_1 is the series-arm impedance and Z_2 is the shunt-arm impedance as shown in Fig. 17.59.

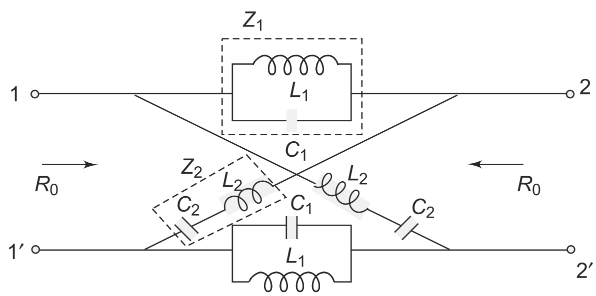


Fig. 17.59

The propagation constant is given by

$$\tanh\left(\frac{\gamma}{2}\right) = \left(\frac{Z_1}{R_0}\right) = \sqrt{\frac{Z_1}{Z_2}}$$

$$\therefore \qquad \tanh\left(\frac{\gamma}{2}\right) = \frac{j\omega L_1 / j\omega C_1}{R_0\left(j\omega L_1 + \dfrac{1}{j\omega C_1}\right)}$$

$$\tanh\left(\frac{\gamma}{2}\right) = \frac{j\omega L_1}{R_0(1-\omega^2 L_1 C_1)}$$

$$\gamma = 2\tanh^{-1}\left[\frac{j\omega L_1}{R_0(1-\omega^2 L_1 C_1)}\right]$$

$$= 2j\tan^{-1}\left[\frac{\omega L_1}{R_0(1-\omega^2 L_1 C_1)}\right]$$

$$\alpha + j\beta = 2j\tan^{-1}\left[\frac{\omega L_1}{R_0(1-\omega^2 L_1 C_1)}\right]$$

Here, $\alpha = 0$

$$\beta = 2\tan^{-1}\left[\frac{\omega L_1}{R_0(1-\omega^2 L_1 C_1)}\right]$$

The above expression gives the phase delay in a lattice phase equaliser.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 8

☆☆★ **17-8.1** Design a full series equaliser for a design resistance R_0 = 600 Ω and attenuation of 20 dB at 400 Hz. Calculate the attenuation *M* at 1000 MHz.

☆☆★ **17-8.2** Design the full-shunt equaliser, for design resistance R_0 = 600 Ω and attenuation at frequencies of 600 Hz and 1200 Hz.

☆☆★ **17-8.3** Design a constant-resistance lattice attenuation equaliser to produce an attenuation of 20 dB at 50 Hz and 3 dB at 3000 Hz. Calculate its loss at 500 Hz. The equaliser is working between two impedances of 500 Ω each.

Additional Solved Problems

PROBLEM 17.1

Design a low-pass π-section filter with a cut-off frequency of 2 kHz to operate with a load resistance of 400 Ω.

Solution The π-section low-pass filter is shown in Fig. 17.60.

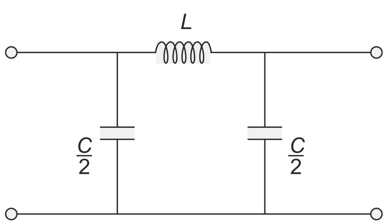


Fig. 17.60

Cut-off frequency f_c = 2 kHz
Load resistance $K = 400\,\Omega = R_L$

Inductance $L = \dfrac{K}{\pi f_c} = \dfrac{400}{\pi\times 2\times 10^3} = 63.66$ mH

Capacitance $C = \dfrac{1}{K\pi f_c} = \dfrac{1}{400\times\pi\times 2\times 10^3} = 0.3978\ \mu$F

PROBLEM 17.2

Design an m-derived low-pass filter having cut-off frequency of 1.5 kHz with a nominal impedance of 500 Ω, and resonant frequency is 1600 Hz.

Solution We have $f_c = 1.5$ kHz ; $k = 500\ \Omega$, and $f_\alpha = 1600$ Hz

For an m-derived section, the value of $m = \sqrt{1-\left(\dfrac{f_c}{f_\alpha}\right)^2}$

$$= \sqrt{1-\left(\frac{1.5\times10^3}{1600}\right)^2} = 0.3479$$

For the prototype low-pass section, $L = \dfrac{k}{\pi f_c}$

$$= \frac{500}{\pi\times1.5\times10^3} = 0.1061\text{H} = \text{z}_1$$

$$C = \frac{1}{\pi k f_c} = \frac{1}{\pi\times500\times1.5\times10^3} = 0.4244\ \mu\text{F} = \text{z}$$

The T-section elements are

$$\frac{mz_1}{2} = \frac{mL}{2} = \frac{0.3479\times0.1061}{2} = 18.45\text{ mH}$$

$$mz = mc = 0.3479 \times 0.4244 \times 10^{-6} = 0.147\ \mu\text{F}$$

and $\left(\dfrac{1-m^2}{4m}\right)z_1 = \left(\dfrac{1-m^2}{4m}\right)L = 6$ mH

The π-section elements are

$$\frac{mc}{2} = \frac{0.3479\times0.4244\times10^{-6}}{2} = 0.0738\ \mu\text{F}$$

$$mL = 0.3479 \times 0.1061 = 36.91\text{ mH}$$

and $\left(\dfrac{1-m^2}{4m}\right)c = 0.268\ \mu$F

The filter sections are shown in Fig. 17.61.

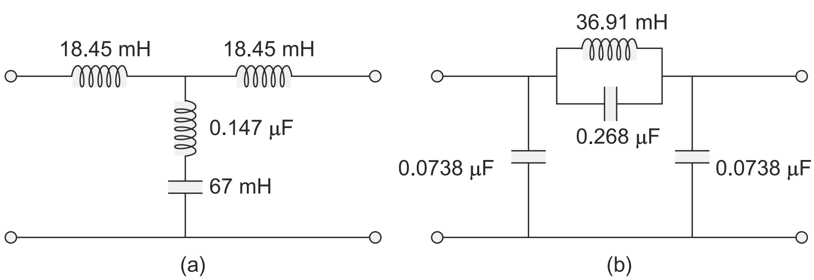


Fig. 17.61

PROBLEM 17.3

Design a band-elimination filter having a design impedance of 500 Ω and cut-off frequencies f_1 = 1 kHz and f_2 = 5 kHz.

Solution We have $f_1 = 1$ kHz ; $f_2 = 5$ kHz ; $k = 500\ \Omega$

$$\text{and} \quad f_0 = \sqrt{f_1 f_2} = 2.236\,\text{kHz}$$

$$\text{B}\omega = f_2 - f_1 = 4\ \text{kHz}$$

$$L_1 = \frac{k(f_2 - f_1)}{\pi f_1 f_2} = \frac{500 \times 4 \times 10^3}{\pi \times 1 \times 10^3 \times 5 \times 10^3} = 0.127\,\text{H}$$

$$C_1 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{1}{4\pi \times 500 \times (4 \times 10^3)} = 3.971 \times 10^{-8}\ \text{F}$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)} = \frac{500}{4 \times \pi \times 4 \times 10^3} = 9.94\ \text{mH}$$

$$C_2 = \frac{1(f_2 - f_1)}{\pi k(f_2\ f_1)} = \frac{4 \times 10^3}{\pi \times 500 \times 10^3 \times 5 \times 10^3} = 0.5\ \mu\text{F}$$

Each of the two series arms of the constant K, T-section filter is given by $\frac{L_1}{2} = 63.5\,\text{mH};\ 2C_1 = 0.08\,\mu\text{F}$ and the shunt-arm elements of the network are

$$L_2 = 9.9\ \text{mH};\ C_2 = 0.5\ \mu\text{F}$$

For constant-K, π-section filter, the elements of the series arm are L_1 = 127 mH; C_1 = 0.04 μF and elements of the shunt arm are

$$2L_2 = 19.8\ \text{mH};\ \frac{C_2}{2} = 0.25\ \mu\text{F}$$

PROBLEM 17.4

A T-section filter is shown is Fig. 17.62. Calculate the value of cut-off frequency and determine the iterative impedance and the phase shift of the network at 1.5 kHz.

Solution We have $\frac{L}{2} = 20\ \text{mH} \Rightarrow L = 40\ \text{mH}$

$$C = 0.12\ \mu\text{F}$$

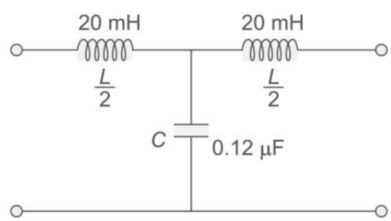


Fig. 17.62

The cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{Lc}} = \frac{`1}{\pi\sqrt{40\times10^{-3}\times0.12\times10^{-6}}}$$

$$f_c = 4.6\,\text{kHz}$$

The iterative impedance is given by

$$z_{oT} = \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$

$$= 577\sqrt{1-\left(\frac{1.5}{4.6}\right)^2} = 545.5\ \Omega$$

Phase shift $\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right) = 2\sin^{-1}\left(\frac{1.5}{4.6}\right) = 38°$

PROBLEM 17.5

Find the frequency at which a prototype π-section low-pass filter having a cut-off frequency f_c has an attenuation of 20 dB.

Solution We have $\alpha = 20\,\text{dB} = \frac{20}{8.696}$ nepers

$= 2.23$ nepers.

If f is the desired frequency for 20 dB, then

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

$$2.23 = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

$$\cosh(1.115) = \frac{f}{f_c}$$

$$f = f_c \cosh 1.115 = 1.689 f_c$$

The frequency at which low-pass π-section filter has an attenuation of 20 dB will be 1.689 times the cut-off frequency.

PROBLEM 17.6

Design an m-derived LPF (T-section) having a cut-off frequency of 6 kHz and a design impedance of 500 Ω. The frequency of infinite attenuation should be 1.75 times the cut-off frequency.

Solution We have $f_c = 6000$ Hz; $k = 500\ \Omega$, and $f_\propto = 1.75 f_c$.

For the prototype low-pass section, $L = \frac{k}{\pi f_c}$

$$= \frac{500}{\pi\cdot 6000} = 26.525\text{ mH}$$

and $C = \dfrac{1}{\pi k f_c} = \dfrac{1}{\pi \times 500 \times 6000} = 0.106\ \mu\text{F}$

For an m-derived section, the value of $m = \sqrt{1 - \left(\dfrac{f_c}{f_\infty}\right)^2}$

$$= \sqrt{1 - \left(\frac{6000}{1.75 \times 6000}\right)^2} = 0.820$$

Now each of the series element of low-pass T-section is given by

$$m\frac{L}{2} = \frac{0.820 \times 26.525 \times 10^{-3}}{2} = 10.68\ \text{mH}$$

The shunt-arm elements are $mC = 0.82 \times 0.106 \times 10^{-6} = 0.087\ \mu\text{F}$

and $\dfrac{1-m^2}{4m} \times L = \dfrac{1-(0.82)^2}{4 \cdot 0.82} \times (26.525 \times 10^{-3})$

$$= 2.65\ \text{mH}$$

The required m-derived network is shown in Fig. 17.63.

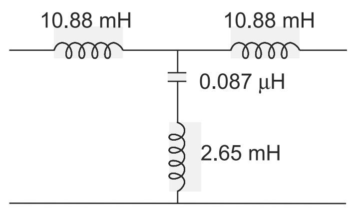


Fig. 17.63

PROBLEM 17.7

A π-section filter network consists of a series-arm inductance of 10 mH and two shunt-arm capacitances of 0.16 μF each. Calculate the cut-off frequency, attenuation and phase shift at 12 kHz. What is the value of nominal impedance in the pass band?

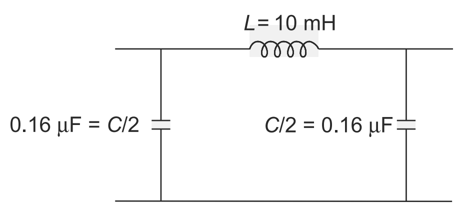
Solution The given filter is shown in Fig. 17.64, it is a low-pass filter; given $L = 10$ mH; $C/2 = 0.16\ \mu$F; $C = 0.32\ \mu$F.

For π-section low-pass filter,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$= \frac{1}{\pi\sqrt{10 \times 10^{-3} \times 0.32 \times 10^{-6}}}$$

$$= 5.627\ \text{kHz}$$



(a)

Fig. 17.64

Nominal terminating impedance is given by

$$k = \sqrt{\frac{L}{C}}$$

$$= \sqrt{\frac{10 \times 10^{-3}}{0.32 \times 10^{-6}}} = 176.77\ \Omega$$

The attenuation constant $= 2\cosh^{-1}\left(\dfrac{\omega}{\omega_c}\right)$ nepers

$$= 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{12\times10^3}{5.627\times10^3}\right) = 2.78 \text{ nepers}$$

The phase shift introduced by the LPF will be π rad in the attenuation band.

PROBLEM 17.8

Each of the two series elements of a T-type low-pass filter consists of an inductance of 30 mH having negligible resistance and a shunt element having capacitance of 0.16 μF. Calculate the value of cut-off frequency and determine the iterative impedance and the phase shift of the network at 2 kHz.

Solution We have $L/2 = 30$ mH $\Rightarrow L = 60$ mH, $C = 0.16$ μF

The cut-off frequency $f_c = \dfrac{1}{\pi\sqrt{LC}}$

$$= \frac{1}{\pi\sqrt{60\times10^{-3}\times0.16\times10^{-6}}}$$

$$f_c = 3.24 \text{ kHz}$$

The characteristic impedance is given by

$$Z_{0T} = \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}$$

$$= \sqrt{\frac{L}{C}}\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$

$$= \sqrt{\frac{60\times10^{-3}}{0.16\times10^{-6}}}\sqrt{1-\frac{2\times10^3}{3.248\times10^3}} = (612)\,(0.619) = 379.05\ \Omega$$

Since $f < f_c$ the attenuation $\alpha = 0$ and the phase shift in the pass band is given by

$$\beta = 2\sin^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\sin^{-1}\left(\frac{2}{3.248}\right) = 76°$$

PROBLEM 17.9

Design the full- series equaliser shown in Fig. 17.65. The design resistance R_0 = 600 Ω and attenuation of 12 dB at 800 Hz. Compute the elemental values.

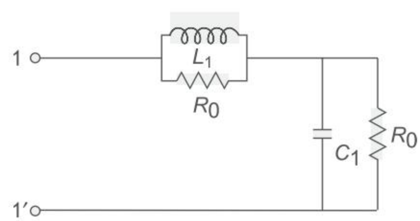


Fig. 17.65

Solution $D = 10 \log N$

$$12 = 10 \log N$$

$$N = \text{Antilog}\left(\frac{12}{10}\right)$$

$$= 15.85$$

We know that $N = 1 + \dfrac{\omega^2 L_1^2}{R_0^2}$

$$L_1 = \frac{R_0\sqrt{N-1}}{\omega}$$

$$L_1 = \frac{600 \times \sqrt{15.58 - 1}}{2\pi \times 800} = 0.46 \text{ henry}$$

$$\frac{L_1}{C_1} = R_0^2$$

$$C_1 = \frac{L_1}{R_0^2} = \frac{0.46}{600 \times 600} = 1.28\ \mu\text{F}$$

PROBLEM 17.10

Design the full- shunt equaliser shown in Fig. 17.66 for a design resistance $R_0 = 600\ \Omega$ and attenuation of 10 dB at 600 Hz. Calculate the elemental values.

Solution $D = 10 \log N$

$$D = 10 \text{ dB}$$

$$N = \text{Antilog } 1 = 10$$

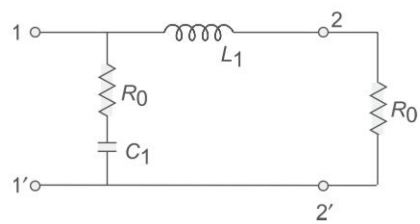


Fig. 17.66

$$N = 1 + \frac{X_1^2}{R_0^2} = 1 + \frac{R_0^2}{X_1^2}$$

$$X_1 = R_0\sqrt{N-1}$$

$$\omega L_1 = R_0\sqrt{N-1}$$

$$L_1 = \frac{R_0\sqrt{N-1}}{2\pi f} = \frac{600\sqrt{10-1}}{2\pi \times 600}$$

$$L_1 = 0.48 \text{ H}$$

$$X_2 = \frac{R_0}{\sqrt{N-1}} = \frac{600}{3}$$

$$\frac{1}{\omega C_1} = \frac{600}{3}$$

$$C_1 = \frac{3}{2\pi \times 600 \times 600} = 1.33\ \mu\text{F}$$

PROBLEM 17.11

Design a constant-resistance lattice attenuation equaliser shown in Fig. 17.67. The series arm consists of $R_1 = 2\ k\Omega$ in series with $L_1 = 30$ mH. If $R_2 = 300\ \Omega$, calculate the values of R_0 and capacitance C_1 of the shunt arm.

Solution $R_1 = 2000\ \Omega \quad L_1 = 30$ mH

$R_2 = 300\ \Omega$

$$R_1 R_2 = R_0^2$$

$$R = \sqrt{R_1 R_2} = 774.6\ \Omega$$

$$C_1 = \frac{L_1}{R_0^2} = \frac{0.03}{(774.6)^2} = 0.049\ \mu\text{F}$$

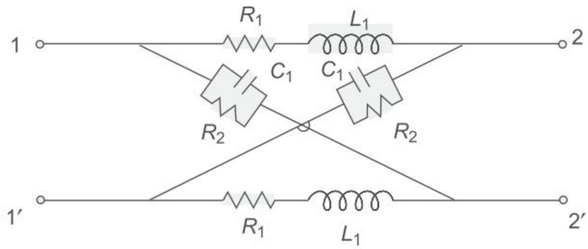


Fig. 17.67

PROBLEM 17.12

Determine the series arm of a constant-resistance lattice- attenuation equaliser shown in Fig. 17.68 having design impedance of 2 Ω, the shunt arm consists of $R_2 = 2\ \Omega$ in series with a capacitor $C_2 = 0.1$ F.

Solution The shunt-arm values are given as follows:

$$R_2 = 2\ \Omega$$

$$C_2 = 0.1\text{ F}$$

$$R_0 = 2\ \Omega$$

$$R_1 R_2 = \frac{L_1}{C_2} = R_0^2$$

$$R_1 = \frac{4}{2} = 2\,\Omega$$

$$L_1 = C_2\, R_0^2$$

$$= (0.1)\,(2)^2 = 0.4\text{ H}$$

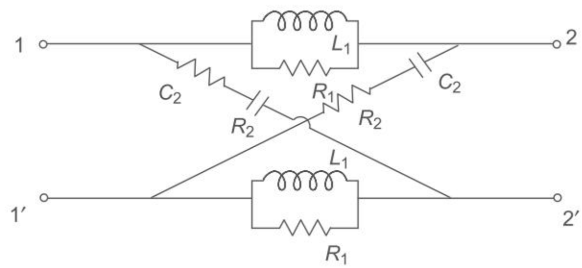


Fig. 17.68

PROBLEM 17.13

Obtain the inverse network for the network shown in Fig. 17.69.

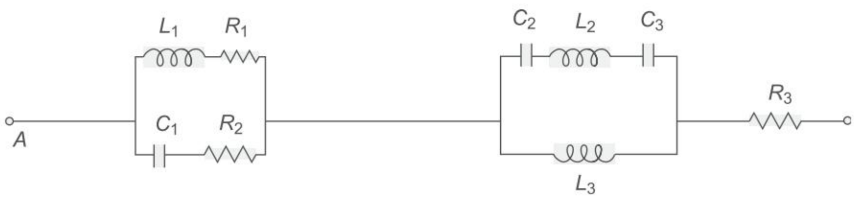


Fig. 17.69

Solution The elements of the inverse network are given by

$$C_1^1 = \frac{L_1}{R_0^2} \quad C_3^1 = \frac{L_3}{R_0^2} \quad L_2^1 = C_2 R_0^2 \quad R_1^1 = \frac{R_0^2}{R_1}$$

$$C_2^1 = \frac{L_2}{R_0^2} \quad L_1^1 = C_1 R_0^2 \quad L_3^1 = C_3 R_0^2 \quad R_2^1 = \frac{R_0^2}{R_2} \quad R_3^1 = \frac{R_0^2}{R_3}$$

The inverse network is shown in Fig. 17.70.

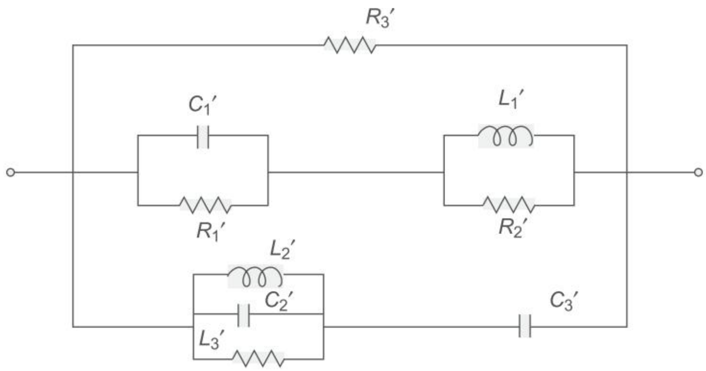


Fig. 17.70

PSpice Problems

PROBLEM 17.1

Determine the response of the twin t-band stop filter shown in Fig. 17.71(a) using PSpice.

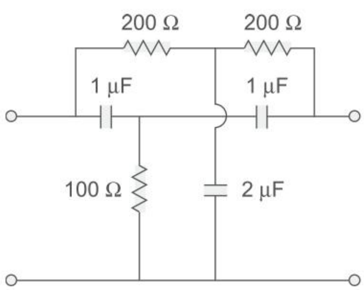


Fig. 17.71 (a)

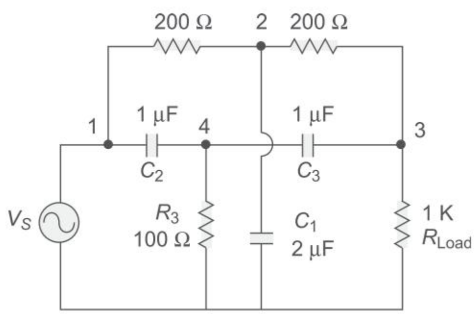


Fig. 17.71 (b)

```
TWIN-T BANDSTOP FILTER
V1   1   0   AC 1 0
R1   1   2   200
C1   2   0   2 U
R2   2   3   200
C2   1   4   1 U
R3   4   0   100
C3   4   3   1 U
RLOAD 3 0 1K
.AC DEC 20 1 1000K
.PLOT AC V(3)
```

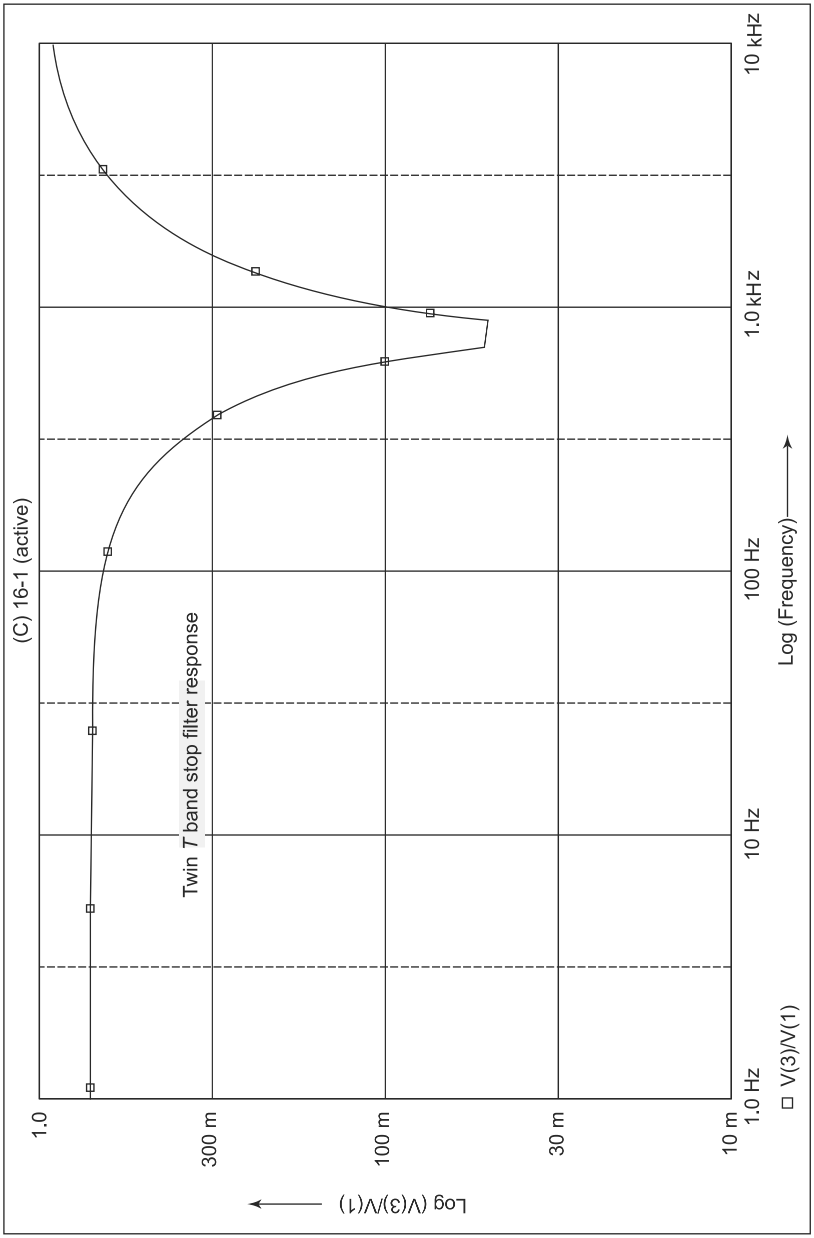


Fig. 17.72

.PROBE
.END

Result

FREQ V(3)

(*)--------- 1.0000E – 04 1.0000E – 03 1.0000E – 02 1.0000E – 01
1.0000E + 00

```
                 _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _
1.000E + 00    7.143E – 01.        .        .        .          *.
1.122E + 00    7.143E – 01.        .        .        .          *.
1.259E + 00    7.143E – 01.        .        .        .          *.
1.413E + 00    7.143E – 01.        .        .        .          *.
1.585E + 00    7.143E – 01.        .        .        .          *.
1.778E + 00    7.143E – 01.        .        .        .          *.
1.995E + 00    7.143E – 01.        .        .        .          *.
2.239E + 00    7.143E – 01.        .        .        .          *.
2.512E + 00    7.142E – 01.        .        .        .          *.
2.818E + 00    7.142E – 01.        .        .        .          *.
3.162E + 00    7.142E – 01.        .        .        .          *.
3.548E + 00    7.142E – 01.        .        .        .          *.
3.981E + 00    7.142E – 01.        .        .        .          *.
4.467E + 00    7.142E – 01.        .        .        .          *.
3.162E + 02    3.926E – 01.        .        .        .       *   .
3.548E + 02    3.483E – 01.        .        .        .      *    .
3.981E + 02    3.019E – 01.        .        .        .     *     .
4.467E + 02    2.539E – 01.        .        .        .    *      .
5.012E + 02    2.047E – 01.        .        .        .  *        .
5.623E + 02    1.547E – 01.        .        .        . *         .
6.310E + 02    1.039E – 01.        .        .        .*          .
7.079E + 02    5.265E – 02.        .        .        *.          .
7.943E + 02    8.235E – 04.        *.       .          .         .
8.913E + 02    5.159E – 02.        .        .        *.          .
1.000E + 03    1.047E – 01.        .        .          .*        .
1.122E + 03    1.585E – 01.        .        .          . *       .
1.259E + 03    2.132E – 01.        .        .          .  *      .
1.413E + 03    2.687E – 01.        .        .          .   *     .
1.585E + 03    3.249E – 01.        .        .          .    *    .
1.778E + 03    3.815E – 01.        .        .          .     *   .
1.995E + 03    4.381E – 01.        .        .          .      *  .
2.239E + 03    4.943E – 01.        .        .          .       * .
                 - - - - - - - - - - - - - - - - - - - - - - - -
```

PROBLEM 17.2

Using PSpice, determine the response of the LC low-pass filter shown in Fig. 17.73 (a).

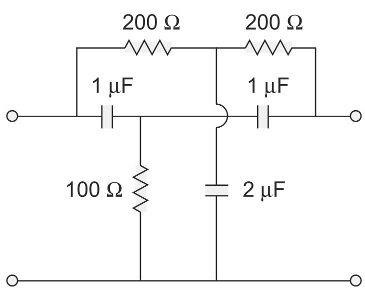


Fig. 17.73 (a)

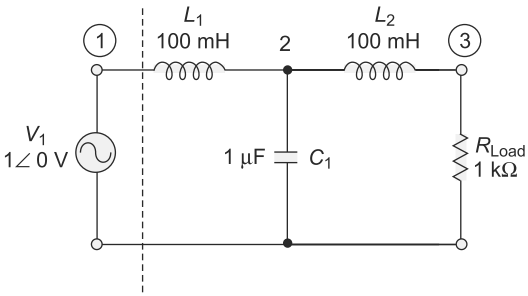


Fig. 17.73 (b)

```
LC LOWPASS FILTER
V1    1      0    AC 1 0
L1           1    2    100 M
C1           2    0    1 U
L2           2    3    100 M
RLOAD 3 0 1K
.AC LIN 20 100 1.5 K
.PLOT AC V(3)
.PROBE
.END
```

Result

```
FREQ          V(3)
(*)--------   1.0000E – 04    1.0000E – 02    1.0000E + 001.0000E + 02         1.0000E + 04
```

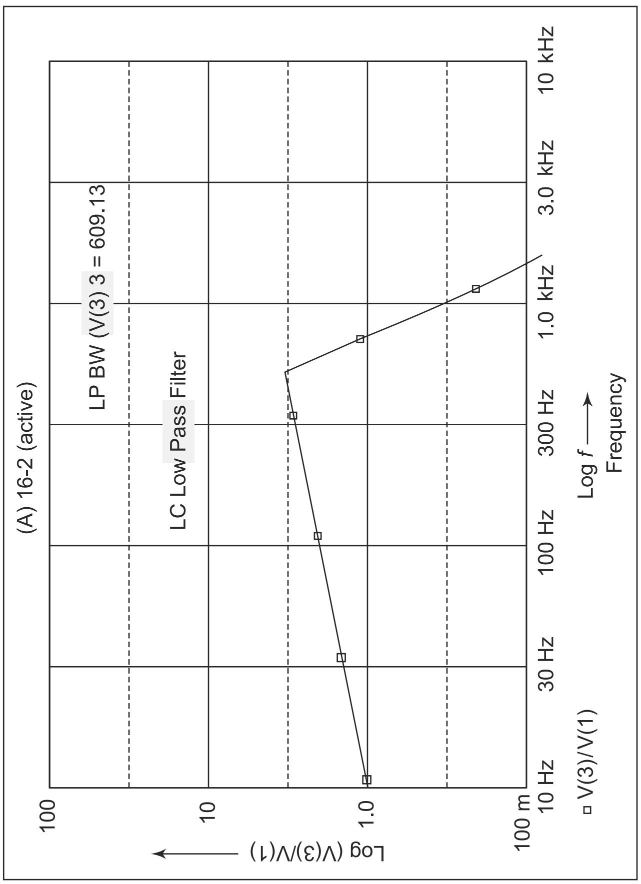


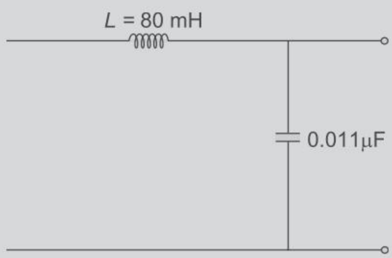
Fig. 17.74

```
           ––––––––––––––––––––––––––––––
1.000E + 01    1.000E + 00 .       .     .*     .         .
5.358E + 02    3.117E + 00 .       .     . *    .         .
1.062E + 03    2.620E – 01 .       .   *  .     .         .
```

```
1.587E + 03    8.366E – 02 .          .  *   .     .      .
2.113E + 03    3.761E – 02 .          .  *   .     .      .
2.639E + 03    2.005E – 02 .          . *    .     .      .
3.165E + 03    1.190E – 02 .          .*     .     .      .
3.691E + 03    7.626E – 03 .          *      .     .      .
4.216E + 03    5.170E – 03 .        *.       .     .      .
4.742E + 03    3.662E – 03 .      *  .       .     .      .
5.268E + 03    2.687E – 03 .     *   .       .     .      .
5.794E + 03    2.028E – 03 .   *     .       .     .      .
6.319E + 03    1.568E – 03 .   *     .       .     .      .
6.845E + 03    1.237E – 03 .   *     .       .     .      .
7.371E + 03    9.930E – 04 .   *     .       .     .      .
7.897E + 03    8.090E – 04 .  *      .       .     .      .
8.423E + 03    6.677E – 04 .  *      .       .     .      .
8.948E + 03    5.574E – 04 .  *      .       .     .      .
9.474E + 03    4.701E – 04 . *       .       .     .      .
1.000E + 04    4.001E – 04 . *       .       .     .      .
 - - - - - - - - - - - - - - - - - - - - - - - - - - -
```

Answers to Practice Problems

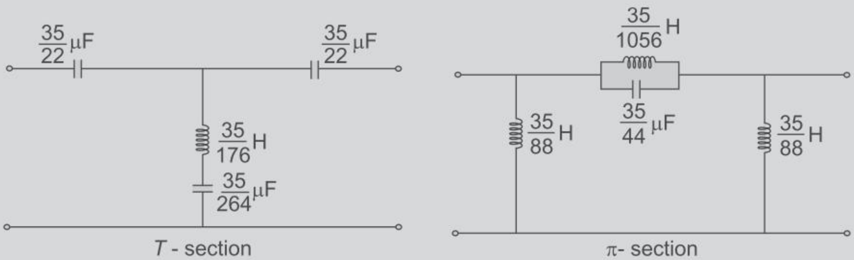
17-4.1 0.09 μF; 0.06 H



17.5.1 $L = 0.127$ H;
$C = 0.35$ μF

17-5.2 $f_c = 7.587$ kHz;
$R_0 = 1.907$ kΩ

17-5.3

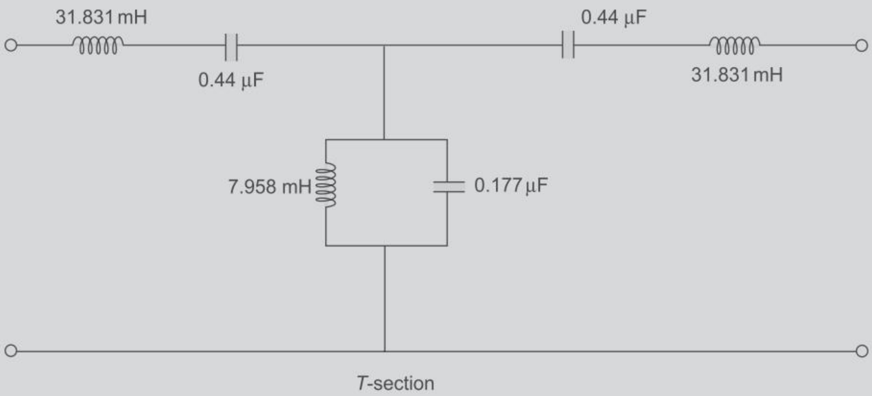


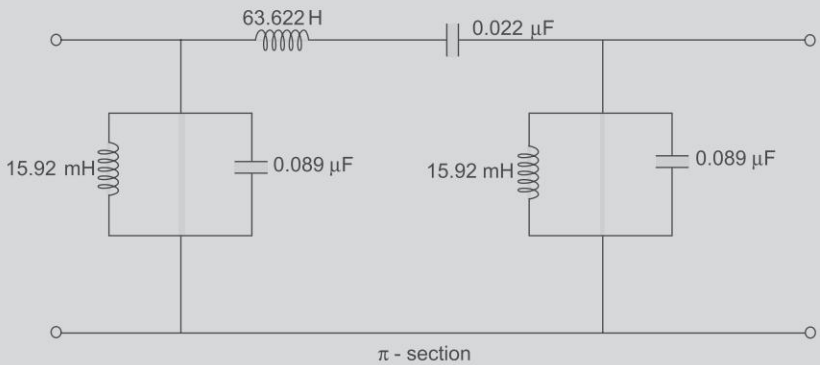
17-5.4 For *T*-section; series arm component 6.66×10^{-9} F; shunt arm 0.015 mH, 1.3×10^{-9} F

For π-section series arm 6.19 mH: 3.33×10^{-9} F; shunt arm 0.031 H

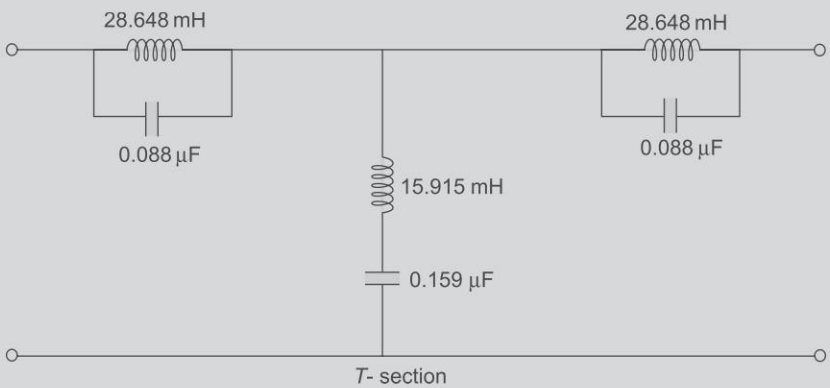
17-5.5 1.779 kHz; 89.44 Ω

17-6.1 $f_0 = 4242.7$ Hz





17-6.5



17-7.1 (a) $R_0 = 217.731\ \Omega$

(b) 9.17 dB

Objective-Type Questions

☆☆★ **17.1** A low-pass filter is one which
(a) passes all low frequencies
(b) attenuates all high frequencies
(c) passes all frequencies up to cut-off frequency, and attenuates all other frequencies

☆☆★ **17.2** A high-pass filter is one which
(a) passes all high frequencies
(b) attenuates all low frequencies
(c) attenuates all frequencies below a designated cut-off frequency, and passes all frequencies above cut-off

☆☆★ **17.3** A band-pass filter is one which
(a) attenuates frequencies between two designated cut-off frequencies and passes all other frequencies
(b) passes frequencies between two designated cut-off frequencies, and attenuates all other frequencies
(c) passes all frequencies

☆☆★ **17.4** An ideal filter should have
(a) zero attenuation in the pass band
(b) infinite attenuation in the pass band
(c) zero attenuation in the attenuation band

☆☆★ **17.5** The propagation constant of a symmetrical T-section and π-section are the same.
(a) true (b) false

☆★★ **17.6** The values of L and C for a low-pass filter with cut-off frequency of 2.5 kHz to operate with a terminated load resistance of 450 ohms are given by
(a) 57.32 mH; 0.283 μF (b) 28.66 mH; 0.14 μF (c) 114.64 μH; 0.566 μF

☆☆★ **17.7** The attenuation is sharp in the stop band for K-type filter.
(a) True (b) False

☆☆★ **17.8** The attenuation is not sharp in the stop band for an m-derived filter.
(a) True (b) False

☆☆★ **17.9** In m-derived low-pass filters, the resonant frequency is to be chosen so that it is
(a) above the cut-off frequency (b) below the cut-off frequency
(c) none of the above

☆☆★ **17.10** In m-derived high-pass filters, the resonant frequency is to be chosen so that it is
(a) above the cut-off frequency (b) below the cut-off frequency
(c) none of the above

☆☆★ **17.11** A band-pass filter may be obtained by using a high-pass filter followed by a low-pass filter
(a) true (b) false

☆☆★ **17.12** A band-elimination filter is one
(a) which attenuates all frequencies less than lower cut-off frequency f_1
(b) which attenuates all frequencies greater than upper cut-off frequency f_2
(c) frequencies lying between f_1 and f_2 are attenuated and all other frequencies are passed

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/275

CHAPTER 18

Elements of Realizability and Synthesis of One-port Networks

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

- **LO 1** Describe Hurwitz polynomials
- **LO 2** Explain the properties of positive real functions
- **LO 3** Analyse the frequency response of reactive one-ports
- **LO 4** Synthesis of reactive one-ports by Foster and Cauer methods
- **LO 5** Synthesis of *R-L* network by Foster and Cauer methods
- **LO 6** Synthesis of *R-C* network by Foster and Cauer methods

18.1 HURWITZ POLYNOMIALS

LO 1 Describe Hurwitz polynomials

As stated in Chapter 15, the poles of the stable system must lie on the left half of the *s*-plane. Any network function can be written as the ratio of two polynomials, and is given by

$$Z(s)=\frac{P(s)}{Q(s)}$$

A polynomial must satisfy the following conditions.

(a) $Z(s)$ must be a real function of s

$$Z(s)=\frac{P(s)}{Q(s)}=\frac{a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}{b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m}$$

where all the quotients a_i, b_j are real, and hence, $Z(s)$ is real if s is real.

(b) All the roots of $P(s)$ must have zero real parts, or negative real parts.

Hurwitz polynomials have the following properties.

1. All the quotients in the polynomial
   $P(s) = a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n$
   are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing. For example, the polynomial $P(s) = s^5 + 3s^3 + 5s^2 + 2s + 1$ is not Hurwitz as the term s^4 is missing. At the same time, the polynomial $P(s) = s^3 + 3s$ is Hurwitz because all quotient terms are positive and all even terms are missing.

2. The roots of the odd and even parts of a Hurwitz polynomial $P(s)$ lie on the $j\omega$ axis. Consider the polynomial $P(s)$ having odd and even parts $o(s)$ and $e(s)$, respectively; then

   $$P(s) = o(s) + e(s)$$

   Both have roots on the $j\omega$ axis.
3. If the polynomial $P(s)$ is either even or odd, the roots of $P(s)$ lie on the $j\omega$ axis.
4. All the quotient terms are positive in the continued fraction expansion of the ratio of the odd to even, or even to odd parts of the polynomial $P(s)$. Consider a polynomial

   $$P(s) = s^4 + s^3 + 6s^2 + 3s + 4$$

   The even parts of the polynomial, $e(s) = s^4 + 6s^2 + 4$

   The odd parts of the polynomial $o(s) = s^3 + 3s$

   The continued fraction expansion is given by

   $$
   \begin{array}{l}
   s^3 + 3s)\,s^4 + 6s^2 + 4\,(s \\
   \qquad \underline{s^4 + 3s^2 \qquad\qquad} \\
   \qquad\quad 3s^2 + 4)\,s^3 + 3s\,(\frac{s}{3} \\
   \qquad\qquad\quad \underline{s^3 + \frac{4s}{3} \qquad\quad} \\
   \qquad\qquad\qquad \frac{5s}{3})\,3s^2 + 4\,(\frac{9s}{5} \\
   \qquad\qquad\qquad\quad \underline{3s^2 \qquad\qquad} \\
   \qquad\qquad\qquad\qquad 4)\,\frac{5s}{3}\,(\frac{5s}{12} \\
   \qquad\qquad\qquad\qquad\quad \underline{\frac{5s}{3}} \\
   \qquad\qquad\qquad\qquad\quad\ 0
   \end{array}
   $$

   The continued fraction expansion can be written as

   $$c(s) = \frac{e(s)}{o(s)} = s + \cfrac{1}{\frac{s}{3} + \cfrac{1}{\frac{9s}{5} + \cfrac{1}{\frac{5s}{12}}}}$$

   Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.
5. If the polynomial satisfies the condition of Hurwitz, then the polynomial must be Hurwitz to within an even multiplicative factor $\omega(s)$, that is, if

   $$P_1(s) = \omega(s)\, P(s), \text{ then } P(s) \text{ is Hurwitz}$$

   If $\omega(s)$ is Hurwitz, $P_1(s)$ must be Hurwitz.

   Consider the polynomial $P_1(s) = s^3 + 3s^2 + 6s + 18$

   The continued fraction expansion is obtained from the division

   $$
   \begin{array}{l}
   3s^2 + 18)\,s^3 + 6s\,(s/3 \\
   \qquad\quad \underline{s^3 + 6s} \\
   \qquad\qquad 0
   \end{array}
   $$

The continued fraction expansion has been terminated abruptly. So, the polynomial can be written as

$$P_1(s) = (s^3 + 6s)\left(1 + \frac{3}{s}\right)$$

Here, $(1 + 3/s)$ term is Hurwitz. Since the terms $(s^3 + 6s)$ is Hurwitz, then $P_1(s)$ also is Hurwitz.

6. If the ratio of the polynomial $P(s)$ and its derivative $P(s)$ gives a continued fraction expansion with all positive coefficients, then the polynomial $P(s)$ is Hurwitz.
Consider the polynomial

$P(s) = s^4 + 3s^2 + 2$

The derivative is $P'(s) = 4s^3 + 6s$

By taking continued fraction expansion, we get

$$4s^3 + 6s)\, s^4 + 3s^2 + 2\, (s/4$$
$$\underline{s^4 + \frac{6}{4}s^2}$$
$$\frac{3}{2}s^2 + 2\Big)\, 4s^3 + 6s\, \Big(\frac{8}{3}s$$
$$\underline{4s^3 + \frac{16}{3}s}$$
$$\frac{2}{3}s\Big)\, \frac{3}{2}s^2 + 2\, \Big(\frac{9}{4}s$$
$$\underline{\frac{3}{2}s^2}$$
$$2)\, \frac{2}{3}s\, \Big(\frac{1}{3}s$$
$$\underline{\frac{2}{3}s}$$
$$0$$

Since all the quotients in the continued fraction expansion are positive, the polynomial $P(s)$ is Hurwitz.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 1*

☆☆★ **18-1.1** Test whether the following polynomials are Hurwitz.

(a) $P(s) = s^3 + 2s^2 + 4s + 2$

(b) $P(s) = s^4 + s^3 + 4s^2 + 2s + 3$

(c) $P(s) = s^4 + 2s^3 + 2s^2 + 6s + 10$

***Note:** ☆☆★ - Level 1 and Level 2 Category
☆★★ - Level 3 and Level 4 Category
★★★- Level 5 and Level 6 Category

Frequently Asked Questions linked to LO 1*

☆☆★ **18-1.1** Check the following for Hurwitz polynominal: **[MU 2014]**

$Q(s) = s^5 + s^3 + s^1$

$Q(s) = s^4 + 6s^3 + 8s^2 + 10$

☆☆★ **18-1.2** What are the conditions to be satisfied for a polynomial $P(s)$ to be Hurwitz? **[PTU 2011-12]**

18.2 POSITIVE REAL FUNCTIONS

LO 2 Explain the properties of positive real functions

As discussed in Chapter 15, the driving-point impedance function $Z(s)$ and driving-point admittance function $Y(s)$ of a one-port network can be expressed as the ratio of two polynomials,

$$Z(s) = Y(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}$$

Functions possessing the following properties are called positive real functions, and are abbreviated as *prf*.

1. When s is real, $Z(s)$ and $Y(s)$ are real functions because the quotients of the polynomials $P(s)$ and $Q(s)$, that is, a_k and b_k are real. When $Z(s)$ is determined from the impedances of the individual branches, the quotients a_k and b_k are obtained by adding together, multiplying or dividing the branch parameters which are real.
2. The poles are zeros of $Z(s)$ and $Y(s)$ all lie in the left half of the s-plane, or on the imaginary axis of the s-plane. In the latter case, the poles and zeros are simple.

   From the above property, it should be noted that if the roots of the characteristic equation were lying on the imaginary axis, and the roots $s = \pm j\omega$, were multiples, the solution of the characteristic equation would be of the form

   $$x_t = (c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1}\, t^{m-1}) \sin \omega_1 t$$

   This would cause the transients to build up, which cannot happen in a passive one-port. Under these conditions, all quotients a_n and b_n of the polynomials $P(s)$ and $Q(s)$ must be positive. This can be proved by writing the polynomial $P(s)$ as

   $$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$$
   $$= a_0 (s - s_1)(s - s_2) \cdots (s - s_n)$$

   For each pair of complex and conjugate roots, $s_k = +j\omega_k$ and $s_{k+1} = -j\omega_k$, we have

   $$(s - s_k)(s - s_{k+1}) = (s - j\omega_k)(s + j\omega_k)$$
   $$= s^2 + \omega_k^2$$

   For real roots of s_k, all the quotients of s in $s^2 + \omega_k^2$ of the polynomial $P(s)$ are non-negative. So by multiplying all factors in $P(s)$, we find that all quotients $a_0, a_1, \cdots, a_n$ are positive.
3. The real parts of the driving-point functions $Z(s)$ and $Y(s)$ are positive, or zero, that is, Re $Z(s) > 0$ or Re $Y(s) > 0$ provided for all Re$(s) > 0$.

   Let $\quad Z(s) = \dfrac{P(s)}{Q(s)}$

   where $P(s)$ and $Q(s)$ are polynomials in s and have real coefficients. Hence, $Z(s)$ is real, when s is real. Further, $P(s)$ and $Q(s)$, are real when s is real. Since the poles and zeros of a network function $Z(s)$ are real, complex zeros must appear in conjugate pairs.

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 2

☆★★ **18-2.1** Check the positive realness of the following functions.

(a) $\dfrac{(2s+4)}{s+5}$ (b) $\dfrac{s^2+2s+4}{(s+3)(s+1)}$ (c) $(s^2+2s)/(s^2+1)$

☆★★ **18-2.2** Investigate if the following partially factored driving-point impedance function is a minimum positive real function.

$$Z(s)=\frac{2s^4+3s^3+5s^2+5s+1}{(s^2+1)(2s^2+2s+1)}$$

Frequently Asked Questions linked to LO 2

☆☆★ **18-2.1** Check whether $F(s)=\dfrac{s+2}{s+1}$ is a positive real function. **[BPUT 2007]**

☆☆★ **18-2.2** Check whether the following function is positive real. **[BPUT 2008]**

$$F(s)=\frac{s^2-s-8}{s^2+2s-2}$$

☆☆★ **18-2.3** Define positive real function and mention its properties. Also write the properties of *RL*, *RC*, and *LC* driving-point functions. **[PTU 2009-10]**

18.3 FREQUENCY RESPONSE OF REACTIVE ONE-PORTS

LO 3 Analyse the frequency response of reactive one-ports

Based on the locations of zeros and poles, a reactive one-port can have the following four types of frequency response.

1. A frequency response with two external poles is shown in Fig. 18.1 (a). In this case, the driving-point impedance with poles at $\omega=0$ and $\omega=\infty$ must have an s in the denominator polynomial and one excess term $(s^2+\omega_n^2)$ in the numerator than in the denominator.

$$\therefore \qquad Z(s)=\frac{H(s^2+\omega_1^2)\cdots(s^2+\omega_n^2)}{s(s^2+\omega_2^2)\cdots(s^2+\omega_{n-1}^2)}$$

The driving-point impedance of the one-port is infinite, and it will not pass either direct current $(\omega=0)$ or alternating current of an infinitely high frequency.

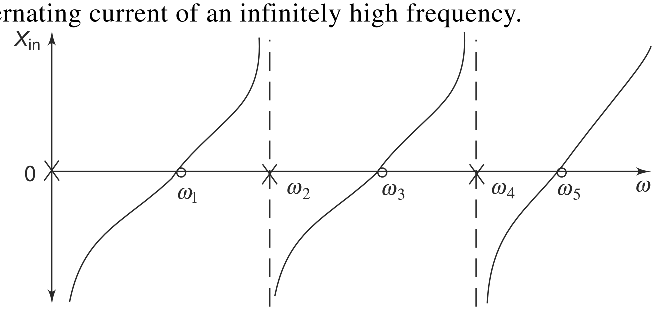


Fig. 18.1 (a)

2. A frequency response with two external zeros is shown in Fig. 18.1 (b). In this case, the driving- point impedance with zeros at $\omega = 0$ and $\omega = \infty$ must have an s term in the numerator and an excess $(s^2 + \omega_n^2)$ term in the denominator polynomial.

$$\therefore \quad Z(s) = \frac{Hs(s^2+\omega_2^2)\cdots(s^2+\omega_{n-1}^2)}{(s^2+\omega_1^2)\cdots(s^2+\omega_n^2)}$$

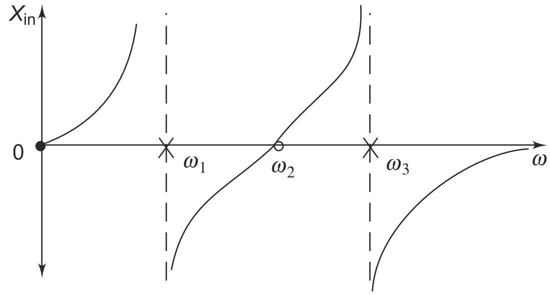


Fig. 18.1 (b)

The driving-point impedance of the one-port is zero, and it will pass both direct current and an alternating current of an infinitely high frequency.

3. A frequency response with an external zero at $\omega = 0$ and an external pole at $\omega = \infty$ is shown in Fig. 18.1 (c). In this case, the driving- point impedance with zero at $\omega = 0$ and pole at $\omega = \infty$ must have a term s in the numerator and equal number of $(s^2 + \omega_n^2)$ type terms in the numerator and the denominator.

$$\therefore \quad Z(s) = \frac{Hs(s^2+\omega_2^2)\cdots(s^2+\omega_n^2)}{(s^2+\omega_1^2)\cdots(s^2+\omega_{n-1}^2)}$$

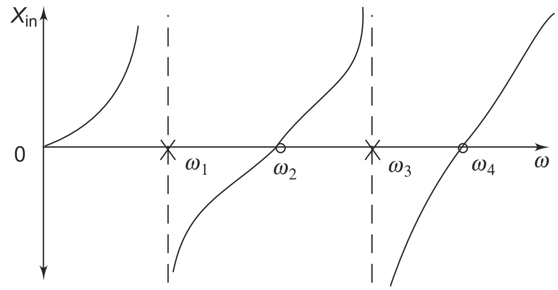


Fig. 18.1 (c)

In this case, the one-port will pass direct current and block an alternating current of an infinitely high frequency.

4. A frequency response with an external pole at $\omega = 0$ and an external zero at $\omega = \infty$ is shown in Fig. 18.1 (d). In this case, the driving-point impedance with pole at $\omega = 0$ and zero at $\omega = \infty$ must have a term s in the denominator and equal number of $(s^2 + \omega_n^2)$ terms in the numerator and the denominator.

$$\therefore \quad Z(s) = \frac{H(s^2+\omega_1^2)\cdots(s^2+\omega_{n-1}^2)}{s(s^2+\omega_2^2)\cdots(s^2+\omega_n^2)}$$

Here, the one-port will block direct current and pass an alternating current of an infinitely high frequency.

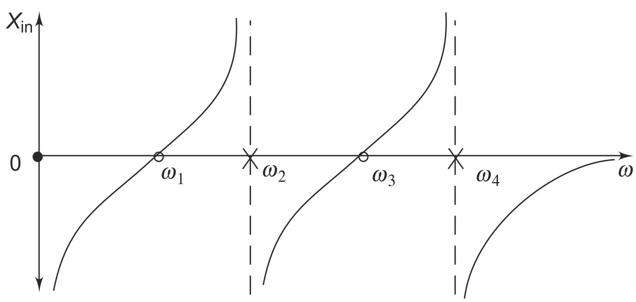


Fig. 18.1 (d)

The function of factor H is to fix the scale of the reactance, and hence it is referred to as the *multiplying factor*, or the *scale factor*. It is to be noted that as the number of zeros and poles in $X_{in}(\omega)$ increases, there will be an increasing number of reactive one-ports having the same form of frequency response.

18.4 SYNTHESIS OF REACTIVE ONE-PORTS BY FOSTER'S METHOD

LO 4 Synthesis of reactive one-ports by Foster and Cauer methods

The driving-point function of a reactive one-port $Z(s)$ is given by

$$Z(s)=\frac{H(s^2+\omega_1^2)(s^2+\omega_3^2)(s^2+\omega_5^2)\cdots}{s(s^2+\omega_2^2)(s^2+\omega_4^2)(s^2+\omega_6^2)\cdots} \tag{18.1}$$

Let us determine the circuit and parameters that implement its frequency response $Z_{in}(j\omega) = jX_{in}(\omega)$. There are two forms of Foster networks for reactive one ports. One is a series combination of parallel LC circuits with capacitance C_0 and inductance L_∞ as shown in Fig. 18.2 known as first *Foster form* or *impedance form*.

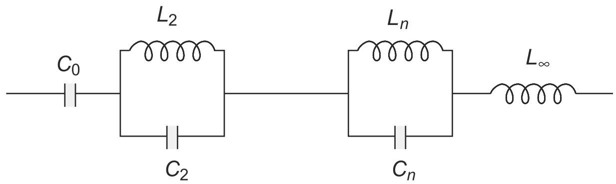


Fig. 18.2

The other form (known as second Foster form or admittance form) is a parallel combination of series LC circuits with inductance L_0 and capacitance C_∞ as shown in Fig. 18.3.

To synthesize the impedance form or first Foster form, we shall write the expression for LC parallel combination in the network of Fig. (18.2)

$$Z(s)=\frac{1}{Cs+\dfrac{1}{Ls}}=\frac{\left(\dfrac{1}{C}\right)s}{s^2+\dfrac{1}{LC}} \tag{18.2}$$

To synthesise the first Foster network, the first step is to express $Z(s)$ as the sum of rational fractions of the form of Eq. (18.2), to which is added the term $1/C_0s$ and $L_\infty s$

Equation (18.1) can be written as

$$Z(s)=\frac{P_0}{s}+\frac{2P_2\,s}{s^2+\omega_2^2}+\frac{2P_4s}{s^2+\omega_4^2}+\cdots+Hs \tag{18.3}$$

If we divide the total impedance into a series combination of impedances $Z_1(s), Z_2(s), \cdots, Z_n(s)$.

$$Z(s) = Z_1(s) + Z_2(s) + Z_3(s) + \cdots + Z_n(s) \tag{18.4}$$

By comparing Eqs (18.3) and (18.4), we have impedance $Z_1(s) = P_0/s$ that represents a capacitor C_0 of value $1/P_0$, and the impedance $Z_n(s) = Hs$ that represents an inductor L_∞ of value H henrys. The remaining intermediate terms represents parallel combination of an inductor and a capacitor. By comparing Eq. 18.2 and the middle terms of Eq. (18.3), we get

$$C_n=\frac{1}{2P_n} \text{ and } L_n=\frac{2P_n}{\omega_n^2}$$

where n refers to the term $2P_ns/s^2+\omega_n^2$ in Eq. (18.3).

The presence of first element capacitor C_0 and the last element inductor L_∞ depends on the pole-zero configuration. If there is pole at $\omega = 0$, the first element C_0 is present in the network. Similarly, if there is pole at $\omega = \infty$, the last element L_∞ is present in the network.

The second canonical form, known as the second Foster network, is a parallel combination of series LC circuits. Because all branches in the network of Fig. 18.3 are connected in parallel, the network can be

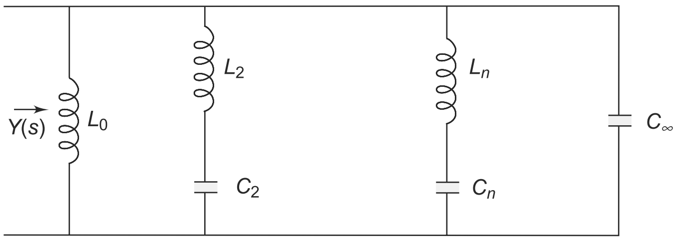


Fig. 18.3

simplified by taking the driving-point admittance $Y(s)$. Therefore, we have

$$Y(s) = \frac{H}{s}\frac{(s^2+\omega_1^2)(s^2+\omega_3^2)}{(s^2+\omega_2^2)(s^2+\omega_4^2)} \tag{18.5}$$

To synthesise the parallel Foster network, we shall write the expression for LC series combination in the network of Fig. 18.3.

$$Y(s) = \frac{1}{Ls+\dfrac{1}{Cs}} = \frac{(s/L)}{s^2+\dfrac{1}{LC}} \tag{18.6}$$

Now, to synthesise the second Foster network, the first step is to express $Y(s)$ as the sum of rational fractions of Eq. (18.6), to which is added the term $C_\infty s$ and $1/L_0 s$.

Equation (18.5) can be written as

$$Y(s) = \frac{P_0}{s} + \frac{2P_2 s}{s^2+\omega_2^2} + \frac{2P_4 s}{s^2+\omega_4^2} + \cdots + Hs \tag{18.7}$$

If we divide the total admittance into a parallel combination of admittance $Y_1(s)$, $Y_2(s)$, $\cdots$, $Y_n(s)$

$$\therefore \qquad Y(s) = Y_1(s) + Y_2(s) + \cdots + Y_n(s) \tag{18.8}$$

By comparing Eqs (18.7) and (18.8), we have the admittance $Y_1(s) = P_0/s$ which represents an inductor L_0 of value $1/P_0$, and the admittance $Y_n(s) = Hs$ which represents a capacitor C_∞ of value H. The remaining intermediate terms represents series combination of an inductor and a capacitor. By comparing Eq. (18.6) and middle terms of Eq. (18.7), we get

$$L_n = \frac{1}{2P_n} \text{ and } C_n = \frac{2P_n}{\omega_n^2}$$

where n refers to the terms $2P_n s/(s^2 + \omega_n^2)$ in Eq. (18.7).

The presence of first element inductor L_0 and the last element capacitor C_∞ depends on the pole-zero configuration. If there is pole at $\omega = 0$, the first element L_0 is present in the network. Similarly, if there is pole at $\omega = \infty$, the last element C_∞ is present in the network.

EXAMPLE 18.1

The driving-point impedance of a one-port reactive network is given by

$$Z(s) = 5\,\frac{(s^2+4)(s^2+25)}{s(s^2+16)}$$

Obtain the first and second Foster networks.

Solution Since there is an extra term in the numerator compared to the denominator, and also an s term in the denominator, the two poles exists at $\omega = 0$ and at $\omega = \infty$. Therefore, the network consists of first element and last element.

By taking the partial fraction expansion of $Z(s)$, we have

$$Z(s) = \frac{P_0}{s} + \frac{P_2}{s+j4} + \frac{P_2^*}{s-j4} + Hs$$

By applying the Heaviside method, from the above equation, we have

$$P_0 = \left.\frac{5(s^2+4)(s^2+25)}{s^2+16}\right|_{s=0}$$

$$= \frac{5\times4\times25}{16} = \frac{125}{4}$$

$$P_2 = \left.\frac{5(s^2+4)(s^2+25)}{s(s-j4)}\right|_{s=-j4} = \frac{135}{8}$$

By inspection, $H = 5$

Therefore, $C_0 = \frac{1}{P_0} = \frac{4}{125}$ farad

$$L_\infty = H = 5\text{ H}$$

$$C_2 = \frac{1}{2P_2} = \frac{8}{2\times135} = \frac{8}{270}\text{F}$$

$$L_2 = \frac{2P_2}{\omega_n^2} = \frac{2\times135}{16\times8} = \frac{135}{64}\text{H}$$

The element values in the first Foster form are shown in Fig. 18.4.

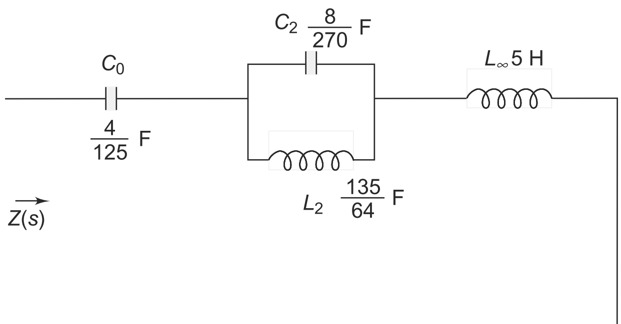


Fig. 18.4

To find the second Foster form, first we have to take the function into admittance form.

$$Y(s) = \frac{s(s^2+16)}{5(s^2+4)(s^2+25)}$$

Since, there is an s term in the numerator and an excess term in the denominator, the two zeros exists at $\omega = 0$ and at $\omega = \infty$. Therefore, the network consists of a series LC combination of parallel elements.

By taking the partial fraction expansion of $Y(s)$, we get

$$Y(s) = \frac{2P_1 s}{s^2+4} + \frac{2P_2 s}{s^2+25}$$

By applying the Heaviside method, we get

$$P_1 = \frac{1}{5}\frac{s(s^2+16)}{(s-j2)(s^2+25)}\bigg|_{s=-j2} = \frac{2}{35}$$

$$P_2 = \frac{1}{5}\frac{s(s^2+16)}{(s^2-j2)(s^2+25)}\bigg|_{s=-j2} = \frac{2}{35}$$

Therefore, the elemental values are

$$L_1 = \frac{1}{2P_1} = \frac{35}{4}\text{H}$$

$$C_1 = \frac{2P_1}{\omega_1^2} = \frac{1}{35}\text{F}$$

$$L_2 = \frac{1}{2P_2} = \frac{35}{3}\text{H}$$

$$C_2 = \frac{2P_2}{\omega_2^2} = \frac{2\times 3}{70\times 25} = \frac{3}{875}\text{F}$$

The circuit of the second Foster form is shown in Fig. 18.5.

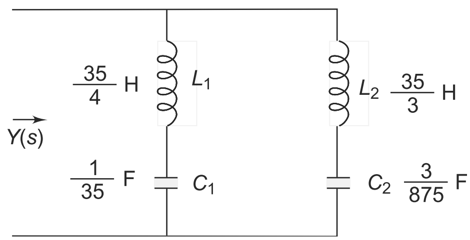


Fig. 18.5

18.5 SYNTHESIS OF REACTIVE ONE-PORTS BY THE CAUER METHOD

LO 4

In the Cauer method, there are two types of ladder networks to realise the one-port network. In one type of network, the series arms are inductors and the shunt arms are capacitors as shown in Fig. 18.6 (a).

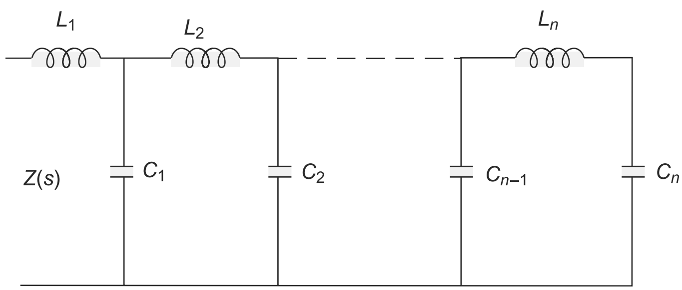


Fig. 18.6 (a)

In the other network, the series arms are capacitors and the shunt arms are inductors as shown in Fig. 18.6 (b).

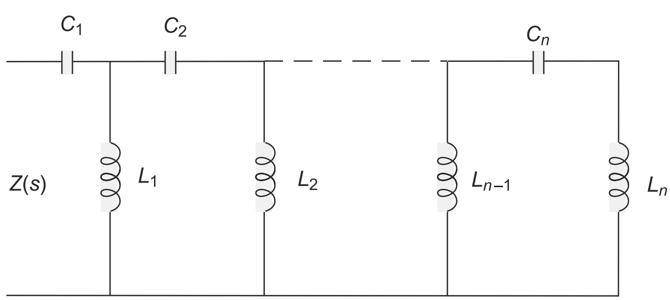


Fig. 18.6 (b)

From the driving-point function $Z(s)$ or $Y(s)$, there is always a zero or a pole at $s = \infty$. We can remove this pole or zero by remaining an impedance $Z_1(s)$ or admittance $Y_1(s)$. Then from each remainder left, an inductor or a capacitor is removed, depending upon the driving-point function. It may be an impedance or an admittance function. This process continues until the remainder is zero. From the above, the impedance $Z(s)$ may be written as a continued fraction as under.

$$Z(s)=Z_1(s)+\cfrac{1}{Y_2(s)+\cfrac{1}{Z_3(s)+\cfrac{1}{Y_4(s)+\cfrac{1}{Z_5(s)+\cdots}}}}$$

Let us realise the first Cauer form. Consider a driving-point function having a pole at infinity. This implies that the degree of the numerator is greater than that of the denominator. We always remove pole at infinity by inverting the remainder, and dividing. That means an LC driving-point function can be synthesised by the continued fraction expansion.

If $Z(s)$ is the function to be synthesised, then the continued fraction expansion is as follows.

$$Z(s)=L_1s+\cfrac{1}{C_1s+\cfrac{1}{L_2s+\cfrac{1}{C_2s+\cdots}}}$$

Therefore, in the first Cauer network shown in Fig. 18.6 (a), the inductors are connected in series and the capacitors are connected in shunt.

If the driving-point function, $Z(s)$ has zero at infinite, that is, if the degree of its numerator is less than that of its denominator, the driving-point function is inverted. In this case, the continued fraction will give a capacitive admittance as first element, and a series inductance.

Now let us realise the second Cauer network. In this case, the removal of the pole at zero gives the network shown in Fig. 18.6 (b), where the capacitors are connected in series and the inductors are connected in shunt. If $Z(s)$ is the function to be synthesised, then the continued fraction expansion is

$$Z(s)=\frac{1}{C_1s}+\cfrac{1}{\cfrac{1}{L_1s}+\cfrac{1}{\cfrac{1}{C_2s}+\cfrac{1}{\cfrac{1}{L_2s}+\cdots}}}$$

If the driving-point function, $Z(s)$ has a zero at zero, the continued fraction expansion will give an inductive admittance as first element and a series capacitance.

From the above discussion, we can conclude that in the first Cauer network, the first element in a series inductor when the driving-point function consists of a pole at infinity, and it is a shunt capacitor when the driving-point function consists of zero at infinity. Similarly, the last element is an inductor when the function consists of zero at $\omega = 0$, and it is a capacitor when the function consists of pole at $\omega = 0$.

In case of second Cauer network, the first element is a series capacitor when the driving-point function consists of a pole at zero and it is shunt inductance when the function consists of a zero at zero. Similarly, the last element is an inductor when the driving-point function consists of a pole at infinity; and it is a capacitor when impedance function consists of zero at infinity.

EXAMPLE 18.2

The driving-point impedance of an LC network is given by $Z(s) = \dfrac{2s^5 + 12s^3 + 16s}{s^4 + 4s^2 + 3}$ *Determine the first Cauer form of the network.*

Solution By taking continued fraction expansion, we get

$$s^4 + 4s^2 + 3)\,2s^5 + 12s^3 + 16s\,(2s - L_1$$
$$\underline{2s^5 + 8s^3 + 6s}$$
$$4s^3 + 10s)\,s^4 + 4s^2 + 3\,(\frac{s}{4} - C_2$$
$$\underline{s^4 + \frac{5}{2}s^2}$$
$$\frac{3s^2}{2} + 3)\,4s^3 + 10s\,(\frac{8}{3}s - L_3$$
$$\underline{4s^3 + 8s}$$
$$2s)\,\frac{3s^2}{2} + 3\,(\frac{3}{4}s - C_3$$
$$\underline{\frac{3s^2}{2}}$$
$$3)\,2s\,(\frac{2}{3}s - L_5$$
$$\underline{2s}$$
$$0$$

Hence,

$$Z(s) = 2s + \cfrac{1}{\cfrac{s}{4} + \cfrac{1}{\cfrac{8}{3}s + \cfrac{1}{\cfrac{3}{4}s + \cfrac{1}{\cfrac{2}{3}s}}}}$$

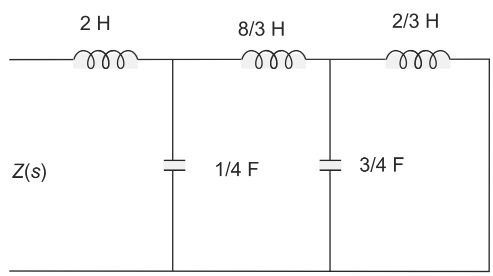


Fig. 18.7

The resulting network shown in Fig. 18.7 is called the first Cauer form.

EXAMPLE 18.3

The driving-point impedance of an LC network is given by

$$Z(s) = s^4 + 4s^2 + 3/(s^3 + 2s)$$

Determine the second Cauer form of the network.

Solution To obtain the second Cauer form, we have to arrange the numerator and the denominator of given $Z(s)$ in assending powers of s before starting the continued fraction expansion.

By taking continued fraction expansion, we get

$$2s + s^3)\,3 + 4s^2 + s^4\,\left(\frac{3}{2s} - C_1\right.$$

$$\underline{3 + \frac{3}{2}s^2}$$

$$\frac{5s^2}{2} + s^4)\,2s + s^3\,\left(\frac{4}{5s} - L_2\right.$$

$$\underline{2s + \frac{4}{5}s^3}$$

$$\frac{s^3}{5})\,\frac{5s^2}{2} + s^4\,\left(\frac{25}{2s} - C_3\right.$$

$$\underline{\frac{5s^2}{2}}$$

$$s^4)\,\frac{s^3}{5}\,\left(\frac{1}{5s} - L_4\right.$$

$$\underline{\frac{s^3}{5}}$$

$$0$$

Hence, $$Z(s) = \frac{3}{2s} + \cfrac{1}{\cfrac{4}{5s} + \cfrac{1}{\cfrac{25}{2s} + \cfrac{1}{\cfrac{1}{5s}}}}$$

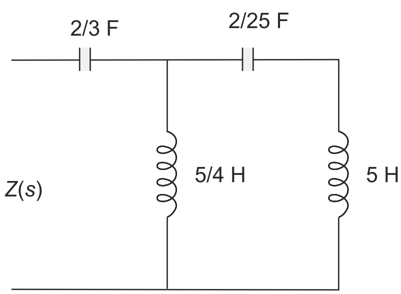


Fig. 18.8

The resulting network shown in Fig. 18.8 is called the second Cauer form.

NOTE: YOU ARE NOW READY TO ATTEMPT THE FOLLOWING PROBLEMS

Practice Problems linked to LO 4

☆★★ **18-4.1** Find the two canonical Foster networks with elements for the impedance function $Z(s)$ given by

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

☆☆★ **18-4.2** Find the first and second Foster forms of the function

$$Z(s) = \frac{10^9 s^3 + 16 \times 10^{21} s}{s^4 + 37 \times 10^{12} s^2 + 36 \times 10^2 s}$$

☆★★ **18-4.3** Synthesise the first and second Foster form of *LC* network for the impedance

$$Z(s) = \frac{(s^2 + 1^2)(s^2 + 3^2)}{(s^2)(s^2 + 2^2)}$$

☆★★ **18-4.4** Find the first and second Cauer networks of the given functions.

$$Z_1(s) = \frac{2s^3 + 8s}{s^2 + 1}$$

$$Z_2(s) = \frac{s^3 + 4s}{2s^4 + 20s^2 + 18}$$

☆☆★**18-4.5** An impedance function has the pole-zero diagram as shown in Fig. Q.2 Find the impedance function to $z(-4) = \dfrac{3}{8}$ and realise it in Cauer form.

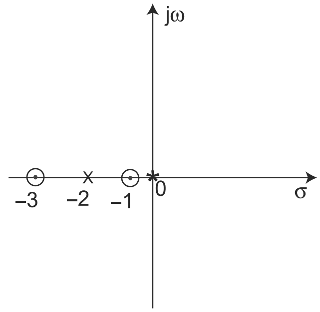


Fig. Q.2

☆★★**18-4.6** Find the first Foster form and the second Cauer form of the function

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

☆★★**18-4.7** Find the second Cauer form of the function

$$Z(s) = \frac{s^2 + 4s + 3}{s^2 + 8s + 12}$$

☆★★**18-4.8** Find the first Foster form and the second Cauer form after synthesising the impedance function given by

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

☆★★**18-4.9** For the given function

$$Z(s) = \frac{(s+1)(s+3)(s+5)}{s(s+2)(s+4)(s+6)}$$

determine the first and second Foster forms of realisation, and the Cauer, first and second forms of realisation.

Frequently Asked Questions linked to LO 4

☆★★**18-4.1** The driving point impedance of an *LC* network is given by

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s(s^2 + 1)}$$

Obtain the first form of Cauer *LC* network. **[BPUT 2007]**

☆★★**18-4.2** A function is given by **[BPUT 2008]**

$$Z(s) = \frac{(s^2 + 1)(s^2 + 16)}{s(s^2 + 4)}.$$

Realise it in the first and second form of foster *LC* forms

☆★★**18-4.3** Compare and obtain Forter form I and form II using an example of *RC* circuit

$$Z_o(s) = \frac{(s+1)(S+6)}{s(s+4)(s+8)}$$

Also give a example of *LC* and *RL* circuits. **[MU 2014]**

☆★★**18-4.4** Synthesise the Foster I and II forms of realization of the following driving-point function:

$$Z_o(s) = \frac{2s^2 + 12s + 16}{s^2 + 4s + 3}$$ **[PTU 2009-10]**

☆☆★**18-4.5** Synthesise in Foster II form, $Z(s) = \dfrac{(s+5)(s+7)}{(s+1)(s+6)(s+8)}$ **[PTU 2011-12]**

☆☆★**18-4.6** Synthesise

$Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$ in Cauer I form. **[BPUT 2008]**

☆★★**18-4.7** Compare Cauer form I and Cauer form II of the network. **[MU 2014]**

$$Z(s) = \frac{s(s^2 + 4)}{(s^2 + 1)(s^2 + 9)}$$

☆☆★**18-4.8** Differentiate between Foster form and Cauer form. **[PTU 2011-12]**

☆★★**18-4.9** Given the driving-point impedance function of an *LC* network, determine the Cauer first and second forms of realization for the *LC* network given as $Z(s) = \dfrac{s(s^2+4)}{(s^2+1)(s^2+9)}$ **[PTU 2011-12]**

☆★★**18-4.10** An impedance function is given by **[PTU 2011-12]**

$Z(s) = 8(s^2 + 4)(s^2 + 25)/s(s^2 + 16)$

Find the Foster I, II forms and Cauer I and II forms.

☆☆★**18-4.11** (a) Find the Cauer II form of the *RC* function. **[RTU Feb. 2011]**

$$Z(s) = \frac{s^2+4s+3}{(s)(s^2+2)}$$

(b) Find the Cauer II form of the *RC* network.

$$Z(s) = \frac{2s^2+12s+16}{s^2+4s+3}$$

☆★★**18-4.12** Find the Cauer I and II forms of the *RL* function. **[RTU Feb. 2011]**

$$Z_{RL} = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

☆☆★**18-4.13** Synthesise the network $F(s)$ in Cauer form if **[RTU Feb. 2011]**

(a) $F(s)$ is an impedance function

(b) $F(s)$ is an admittance function

$$F(s) = \frac{3(s+2)(s+4)}{s(s+3)}$$

18.6 SYNTHESIS OF *R-L* NETWORK BY THE FOSTER METHOD

LO 5 Synthesis of R-L network by Foster and Cauer methods

The driving-point impedance function of an *RL* network $Z(s)$ is given by

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)\cdots}{(s+\sigma_2)(s+\sigma_4)\cdots} \tag{18.9}$$

The first form of the Foster network is shown in Fig. 18.9.

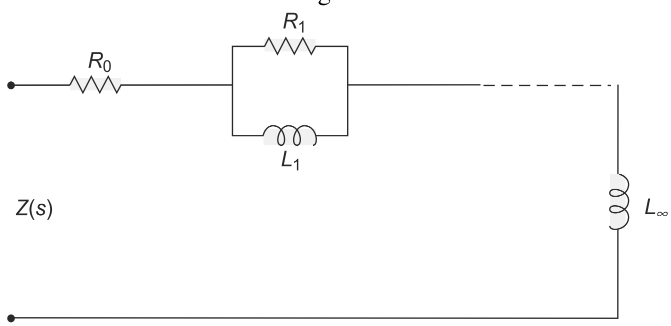


Fig. 18.9

The above impedance function possess the following properties.

1. The poles and zeros of the RL driving-point impedance function are located on the negative real axis of the s-plane.
2. Poles and zeros alternate along the negative real axis.

3. The singularity at the origin, or s 5 0 is a zero.
4. The singularity at s 5 ` is a pole.
5. The slope of the impedance curve is positive.
6. The impedance at s 5 ` is always greater than the impedance at v 5 0.
7. The residues at the poles of Z(s) are real and negative. The residues of Z(s)/s are real and positive.

To synthesise the first Foster network, we shall write the expression for the *RL* parallel combination in the network of Fig. 18.9.

$$Z_1(s) = \frac{R_1 s}{s + \sigma_1}$$

where
$$\sigma_1 = \frac{R_1}{L_1} \tag{18.10}$$

or
$$\frac{Z_1(s)}{s} = \frac{R_1}{s + \sigma_1}$$

We have another form of the equation as discussed in Chapter 15.

$$Z(s) = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}{b_0 s^m + b_1 s^{m-1} + \cdots + b_m} \tag{18.11}$$

where $n > m$

The degree of the numerator is greater than that of the denominator by one.

At $s = 0$,

$$Z(s) = \frac{a_n}{b_n} \text{ (when } a_n \neq 0)$$

$$= 0 \text{ (when } a_n = 0)$$

And at $s = \infty$, $\quad Z(s) = s\left(\frac{a_0}{b_0}\right)$ (when $a_0 \neq 0$)

$$= \frac{a_1}{b_1} \text{ (when } a_0 = 0)$$

By separating the constant term and linear term in Eq. (18.11), the *RL* impedance function can be written as

$$Z(s) = P_0 + \frac{P_i s}{s + \sigma_i} + \cdots + Hs \tag{18.12}$$

If we divide the total impedance into a series combination of impedance $Z_1(s)$, $Z_2(s)$, $\cdots Z_n(s)$

$$Z(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s) \tag{18.13}$$

By comparing Eqs (18.12) and (18.13), we have the impedance $Z_1(s) = P_0$, which is constant. The term P_0 represents a resistor R_0, and the impedance $Z_n(s) = Hs$ represents L_∞ of value H henries. The remaining terms represent parallel combination of an inductor and a resistor. By comparing Eq. (18.10) and middle terms of Eq. (18.12), we have

$$P_n = R_n \text{ and } \sigma_n = \frac{R_n}{L_n}$$

where n refers to the term $P_n s/(s + \sigma_n)$ in Eq. (18.12).

Consider a function $Z(s) = 5\dfrac{(s+1)(s+4)}{(s+3)(s+5)}$

$Z(s)$ represents *RL* impedance, because it satisfies all the properties, but the signs of $Z(s)$ at its poles are negative as shown.

$$Z(s) = \frac{5(s+1)(s+4)}{(s+3)(s+5)} = 5 - \frac{5}{s+3} - \frac{10}{s+5}$$

Therefore, we have to expand $\dfrac{Z(s)}{s}$

$$\frac{Z(s)}{s} = \frac{5(s+1)(s+4)}{s(s+3)(s+5)} = \frac{4}{3s} + \frac{5}{3(s+3)} + \frac{2}{s+5}$$

If we multiply both sides by s, we get

$$Z(s) = \frac{4}{3} + \frac{5}{3}\frac{s}{s+3} + \frac{2s}{s+5}$$

Hence, the impedance $Z(s)$ can be realised as a series Foster form of *RL* network shown in Fig. 18.10.

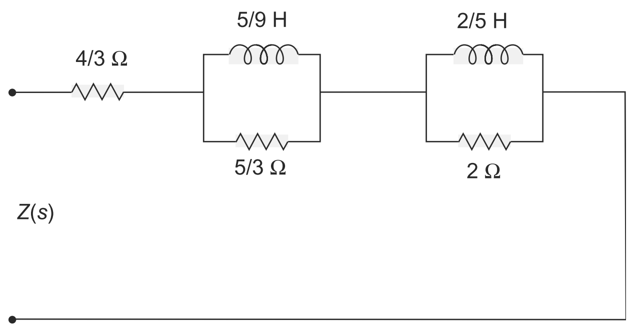


Fig. 18.10

Similarly, the driving-point admittance function of the *RL* network, $Y(s)$ is given by

$$Y(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \tag{18.14}$$

The second form of the Foster network is shown in Fig. 18.11.

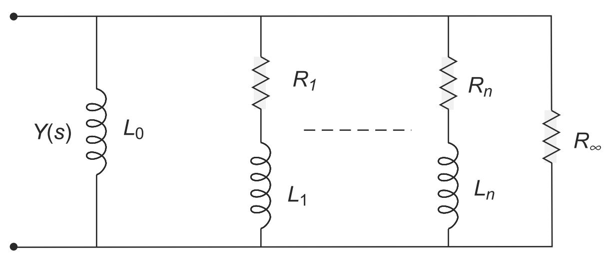


Fig. 18.11

The above admittance function must possess the following properties.

1. The poles and zeros of the *RL* driving-point admittance function are located on the negative real axis of the *s*-plane.
2. Poles and zeros alternate along the negative real axis.
3. The singularity at the origin, or s 5 0, is a pole.
4. The singularity at s 5 ` is a zero.
5. The slope of the admittance curve is negative.
6. The admittance at s 5 0 is always greater than the admittance at s 5 `.
7. The residues at the poles of $Y(s)$ are real and positive.

The *RL* admittance function can be written as

$$Y(s) = \frac{P_0}{s} + \frac{P_i}{s+\sigma_i} + \cdots + H \tag{18.15}$$

If we observe Eq. (18.15), we have the first term P_0/s representing inductance $L_0 = 1/P_0$, and the last term representing a resistance $R_\infty = H$. The intermediate terms represent admittance function of the series *RL* network. We, therefore, have

$$Y_n = \frac{1}{R_n + sL_n} \tag{18.16}$$

Comparing Eq. (18.16) with the middle terms of Eq. (18.15), we have $R_n = \sigma_n/P_n$ and $L_n = 1/P_n$ where n refers to the nth term of Eq. (18.15), i.e. $P_n/(s + \sigma_n)$,

Consider an admittance function

$$Y(s) = \frac{2s^2 + 16s + 30}{s^2 + 6s + 8}$$

The poles and zeros are positive, real and simple. The poles are at −2, −4, and the zeros are at −3, and −5. For the second Foster form of realisation by partial fraction expansion,

$$Y(s) = 2 + \frac{4s + 14}{s^2 + 6s + 8}$$

$$= 2 + \frac{A}{s+2} + \frac{B}{s+4}$$

where $$A = \left.\frac{4s+14}{(s+4)}\right|_{s=-2} = 3$$

$$B = \left.\frac{4s+14}{s+2}\right|_{s=-4} = 1$$

The residues are positive. Hence,

$$Y(s) = 2 + \frac{3}{s+2} + \frac{1}{s+4}$$

Comparing with Eq. (18.15), we have $R_\infty = 2$, $R_1 = 2/3\ \Omega$, $L_1 = 1/3$ H and $R_2 = 4\,\Omega$, $L_2 = 1\,$H.

The second Foster form of the *RL* admittance function with various values is shown in Fig. 18.12.

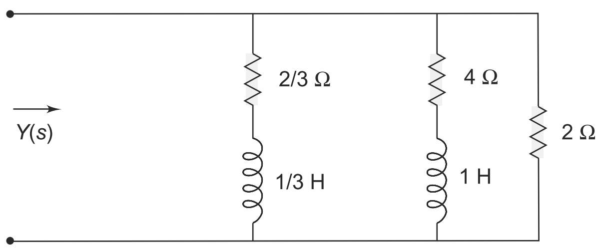


Fig. 18.12

18.7 SYNTHESIS OF *R-L* NETWORK BY THE CAUER METHOD

LO 5

To synthesise the *RL* network, the basic step to know is that the impedance function at infinity is always greater than the impedance function at zero. Similarly, the admittance function at zero is always greater than the admittance function at infinity. In case of *RL* network synthesis, we remove the minimum real part from the function $Z(s)$. If the minimum real part is Re $[Z(j\omega)] = Z(0)$, by removing $Z(0)$ from $Z(s)$, the remainder will have a zero at $s = 0$. After inverting the remaining function, we can remove the pole at $s = 0$. By carrying on this process, we obtain a continued fraction expansion.

The first form of continued fraction expansion is called the first Cauer form, which is

$$Z(s) = sL_1 + \cfrac{1}{\cfrac{1}{R_1} + \cfrac{1}{sL_2 + \cfrac{1}{\cfrac{1}{R_2} + \cdots}}}$$

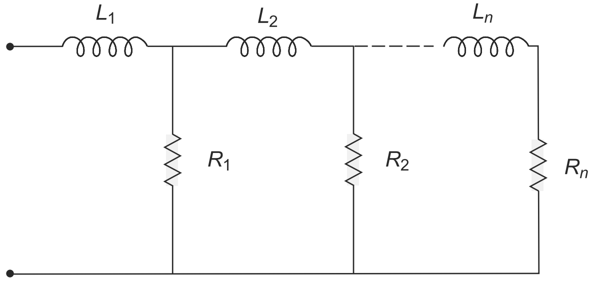


Fig. 18.13

The Cauer network for realising the above function is shown in Fig. 18.13.

In the network shown above, if $Z(s)$ has a pole at $s = \infty$, the first element is L_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a zero at $s = 0$, the last element is L_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

The second form of the continued fraction expansion is

$$Z(s) = R_1 + \cfrac{1}{\cfrac{1}{sL_1} + \cfrac{1}{R_2 + \cfrac{1}{\cfrac{1}{sL_2} + \cfrac{1}{R_3 + \cdots}}}}$$

The second Cauer form of the network for the above function $Z(s)$ is shown in Fig. 18.14.

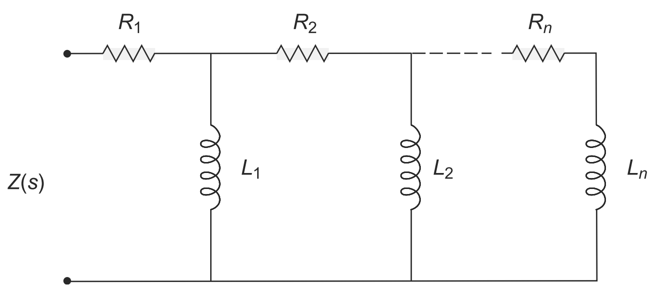


Fig. 18.14

Here also, the presence of the first and the last element depends on the characteristics of impedance function, $Z(s)$. If $Z(s)$ has a zero at $s = 0$, the first element is L_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_1. If $Z(s)$ has a pole at $s = \infty$, the last element is L_n. If $Z(s)$ is a constant at $s = \infty$, the last element is R_n.

The first form of the Cauer network can be obtained by continued fraction expansion and arranging the numerator and denominator polynomials of $Z(s)$ in descending powers of s. The second form of the Cauer network can be obtained by continued fraction expansion and arranging the numerator and denominator polynomials of $Z(s)$ in ascending powers of s. Consider a function

$$Z(s) = \frac{(s+4)(s+8)}{(s+2)(s+6)}$$

To find out the first Cauer form, let us take the continued fraction expansion of $Z(s)$.

$$\begin{array}{l}
s^2+8s+12)\,s^2+12s+32\,(1 \\
\qquad\qquad\quad \underline{s^2+8s+12} \\
\qquad 4s+20)\,s^2+8s+12\,(\frac{s}{4} \\
\qquad\qquad\quad \underline{s^2+5s} \\
\qquad 3s+12)\,4s+20\,(\frac{4}{3} \\
\qquad\qquad\quad \underline{4s+16} \\
\qquad\qquad 4)\,3s+12\,(\frac{3}{4}s \\
\qquad\qquad\quad \underline{3s} \\
\qquad\qquad\quad 12)\,4\,(\frac{1}{3} \\
\qquad\qquad\qquad \underline{4} \\
\qquad\qquad\qquad 0
\end{array}$$

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{4} + \cfrac{1}{\cfrac{4}{3} + \cfrac{1}{\cfrac{3}{4}s + \cfrac{1}{\cfrac{1}{3}}}}}$$

Therefore, the impedance function $Z(s)$ can be realised as an *RL* network as shown in Fig. 18.15.

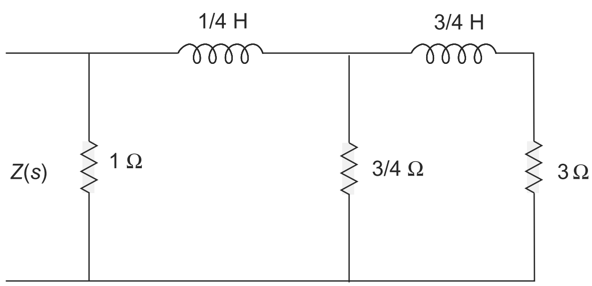


Fig. 18.15

Similarly, consider another function

$$Z(s) = \frac{2s^2+8s+6}{s^2+8s+12}$$

To find out the second Cauer network, we have to write the impedance function in ascending powers.

By taking the continued fraction expansion of $Z(s)$, we have

$$
\begin{array}{l}
12 + 8s + s^2)\,6 + 8s + 2s^2\,(\frac{1}{2} \\
\qquad\quad \underline{6 + 4s + \frac{1}{2}s^2} \\
\qquad\qquad 4s + \frac{3}{2}s^2)\,12 + 8s + s^2\,(\frac{3}{s} \\
\qquad\qquad\qquad\qquad \underline{12 + \frac{9}{2}s} \\
\qquad\qquad\qquad\qquad\quad \frac{7}{2}s + s^2)\,4s + \frac{3}{2}s^2\,(\frac{8}{7} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{4s + \frac{8}{7}s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{5}{14}s^2)\,\frac{7}{2}s + s^2\,(\frac{49}{5s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\frac{7}{2}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad s^2)\,\frac{5}{14}s^2\,(\frac{5}{14} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\frac{5}{14}s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$Z(s) = \frac{1}{2} + \cfrac{1}{\frac{3}{s} + \cfrac{1}{\frac{8}{7} + \cfrac{1}{\frac{49}{5s} + \cfrac{1}{\frac{5}{14}}}}}$$

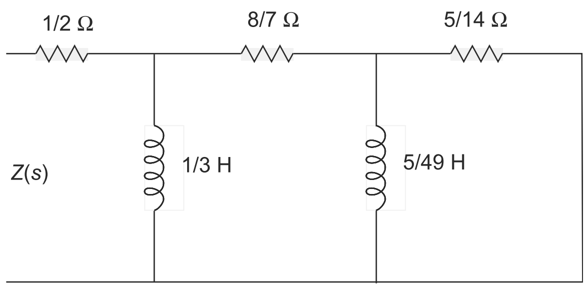


Fig. 18.16

Therefore, the impedance function $Z(s)$ can be realised as an *RL* network shown in Fig. 18.16.

18.8 SYNTHESIS OF *R-C* NETWORK BY THE FOSTER METHOD

LO 6 Synthesis of R-C network by Foster and Cauer methods

The driving-point impedance *RC* network, $Z(s)$ is given by

$$Z(s) = \frac{\mathrm{H}(s+\sigma_1)(s+\sigma_3)\cdots}{s(s+\sigma_2)(s+\sigma_4)\cdots} \qquad (18.17)$$

The first form of the *RC* Foster network is shown in Fig. 18.17.

Here, the *RC* impedance possesses the same properties as the *RL* admittance function. To synthesise the first Foster form of the *RC* network, we shall write the expression for the *RC* parallel combination in the network of Fig. 18.17.

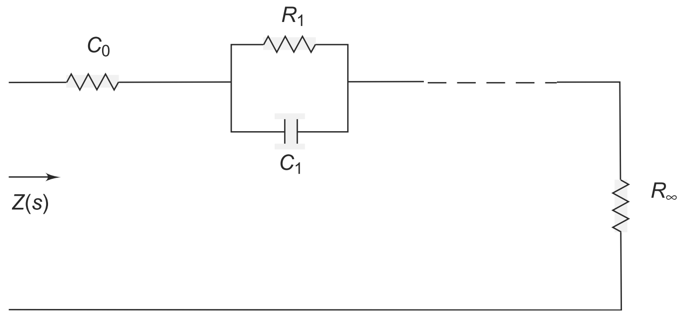


Fig. 18.17

$$Z_1(s) = \frac{\frac{1}{C_1}}{s + \frac{1}{R_1 C_1}} \tag{18.18}$$

where $\sigma_1 = \frac{1}{R_1 C_1} \quad P_1 = \frac{1}{C_1}$

We have the other form of the impedance function

$$Z(s) = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}{b_0 s^m + b_1 s^{m-1} + \cdots + b_n s} \tag{18.19}$$

Obviously, the degree in s of the numerator polynomial is greater than that of the denominator polynomial by one. The roots of the polynomials are real and negative.

At $s = \infty$, $\quad Z(s) = \frac{a_0}{b_0} = R_\infty, \;$ when $a_0 \neq 0$

$\qquad = 0, \;$ when $a_0 = 0$

The total impedance can be written as the combination of impedances $Z_1(s), Z_2(s), \cdots, Z_n(s)$

$$Z(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s) \tag{18.20}$$

From Fig. 18.18, we have the impedance

$$Z(s) = \frac{P_0}{s} + \frac{P_i}{s + \sigma_i} + \cdots + \mathrm{H} \tag{18.21}$$

By comparing Eqs (18.20) and (18.21), we have the impedance $Z_1(s) = P_0/s$ representing a capacitance term $1/P_0$, and the impedance $Z_n(s) = \mathrm{H}$, a constant term representing resistor R_∞. The remaining terms represent a parallel combination of a capacitor and resistor. By comparing Eq. (18.18) with the middle terms of Eq. (18.21), we have

$$P_n = \frac{1}{C_n}$$

and $\quad \sigma_n = \frac{1}{R_n C_n}$

where n refers to the term $P_n/(s + \sigma_n)$ in Eq. (18.21).

Similarly, the driving-point function of an *RC* network *Y*(*s*) is given by

$$Y(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m s}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \tag{18.22}$$

The second form of the Foster network is shown in Fig. 18.18.

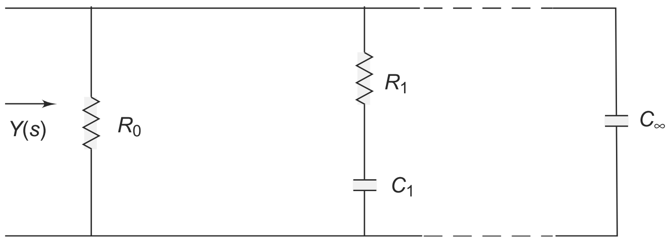


Fig. 18.18

The *RC* admittance function possesses the same properties as the *RL* impedance function. By taking the partial fraction expansion of Eq. (18.22) we can write the *RC* admittance function as

$$Y(s) = P_0 + \frac{P_i s}{s + \sigma_i} + \cdots + \text{H}s \tag{18.23}$$

If we observe Eq. (18.23), we have the first term P_0 representing resistance $R_0 = 1/\,P_0$, and the last term represents capacitance $C_\infty = \text{H}$ and the intermediate terms representing admittance function of series *RC* network.

$$Y_n = \frac{1}{R_n + \dfrac{1}{sC_n}} \tag{18.24}$$

Comparing Eq. (18.24) and the middle terms of Eq. (18.23), we have

$$R_n = \frac{1}{P_n} \text{ and } C_n = \frac{1}{\sigma_n R_n}$$

Consider a function $Z(s) = \dfrac{3(s+2)(s+4)}{(s+1)(s+3)}$

The first Foster form can be realised by taking the partial fraction of *Z*(*s*)

$$Z(s) = 3 + \frac{6s+15}{s^2+4s+3}$$

$$= 3 + \frac{6s+15}{(s+1)(s+3)} = 3 + \frac{A}{s+1} + \frac{B}{s+3}$$

where $A = \left.\dfrac{6s+15}{s+3}\right|_{s=-1} = \dfrac{9}{2}$

$$B = \left.\frac{6s+15}{s+1}\right|_{s=-3} = \frac{3}{2}$$

The residues are positive, and hence

$$Z(s) = 3 + \frac{9/2}{s+1} + \frac{3/2}{s+3}$$

Comparing with Eq. (18.21), we have

$$R_\infty = 3, R_1 = \frac{9}{2}, \quad C_1 = \frac{2}{9}\,\text{F}$$

and $$R_2 = \frac{1}{2}, \; C_2 = \frac{2}{3}\,\text{F}$$

The network with elemental values is shown in Fig. 18.19.

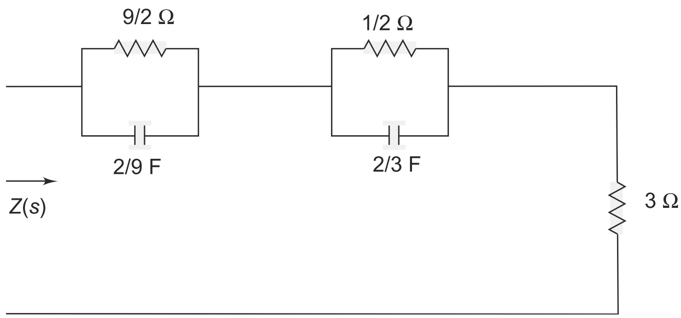


Fig. 18.19

The second Foster form can be realised by taking the reciprocal of the impedance function and partial fraction expansion as

$$Y(s) = \frac{s^2 + 4s + 3}{3s^2 + 18s + 24} = \left(\frac{s^2 + 4s + 3}{3s^2 + 18s + 24} - \frac{1}{8}\right) + \frac{1}{8}$$

$$Y(s) = \frac{s(5s+14)}{8(3)(s+2)(s+4)} + \frac{1}{8}$$

$$\frac{Y(s)}{s} = \frac{5s+14}{24(s+2)(s+4)} + \frac{1}{8s} = \frac{1}{8s} + \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = \left.\frac{5s+14}{24(s+4)}\right|_{s=-2} = \frac{1}{12}$$

$$B = \left.\frac{5s+14}{24(s+2)}\right|_{s=-4} = \frac{1}{8}$$

$$\therefore \qquad Y(s) = \frac{1}{8} + \frac{(1/12)s}{s+2} + \frac{(1/8)s}{s+4}$$

The network with elemental values is shown in Fig. 18.20.

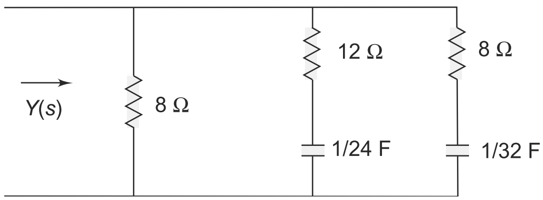


Fig. 18.20

18.9 SYNTHESIS OF *R-C* NETWORK BY THE CAUER METHOD

LO 6

To synthesise the *RC* network function, the basic step to know is that the impedance function at zero is always greater than the impedance function at infinity. Similarly, the admittance function at infinite is always greater than the admittance function at zero.

To synthesise an *RC* network, we remove the minimum real part from the function, $Z(s)$. If the minimum real part is Re $[Z(j\omega)] = Z(\infty)$, by removing $Z(\infty)$ from $Z(s)$, the remainder will have a zero at $s = \infty$. After inverting the remaining function, we can remove a pole at $s = \infty$. By carrying on this process, we obtain a continued fraction expansion. The first form of continued fraction expansion is called the first Cauer form, and is given by

$$Z(s) = R_1 + \cfrac{1}{C_1 s + \cfrac{1}{R_2 + \cfrac{1}{C_2 s + \cdots}}}$$

The Cauer network for realising the above function is shown in Fig. 18.21.

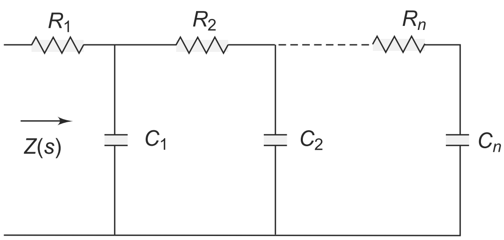


Fig. 18.21

In the network shown, if $Z(s)$ has a zero at $s = \infty$, the first element is C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is constant at $s = 0$, the last element is R_n.

The second form of continued fraction expansion is

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_1} + \cfrac{1}{\cfrac{1}{C_2 s} + \cfrac{1}{\cfrac{1}{R_2} + \cdots}}}$$

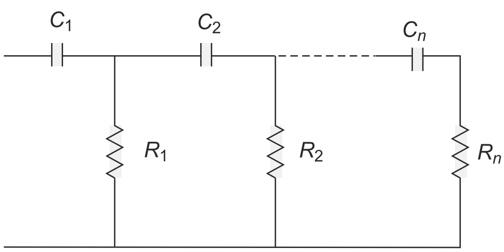


Fig. 18.22

The second Cauer form of network for the above function $Z(s)$ is shown in Fig. 18.22.

In the network shown in Fig. 18.22, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Consider a function $Z(s) = (s + 2)(s + 4)/s(s + 3)$. To find the first Cauer form, we take the continued fraction expansion by the divide, invert, divide procedure as follows.

$$s^2 + 3s)\, s^2 + 6s + 8 \,(1 - R_1$$

$$\underline{s^2 + 3s}$$

$$3s + 8)\, s^2 + 3s \,(\frac{s}{3} - C_1 s$$

$$\underline{s^2 + \frac{8s}{3}}$$

$$\frac{s}{3})\, 3s + 8 \,(9 - R_2$$

$$\underline{3s}$$

$$8)\, \frac{s}{3} \,(\frac{s}{24} - C_2 s$$

$$\underline{\frac{s}{3}}$$

$$0$$

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{3} + \cfrac{1}{9 + \cfrac{1}{\cfrac{s}{24}}}}$$

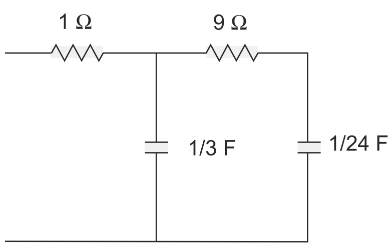


Fig. 18.23

Therefore, the impedance function $Z(s)$ can be realised as an RC network shown in Fig. 18.23.

Similarly, the second Cauer network can be obtained by arranging the numerator and denominator polynomials of $Z(s)$ in ascending powers of s. The continued fraction expansion is

$$3s + s^2)\, 8 + 6s + s^2 \,(\frac{8}{3s}$$

$$\underline{8 + \frac{8s}{3}}$$

$$\frac{10s}{3} + s^2)\, 3s + s^2 \,(\frac{9}{10}$$

$$\underline{3s + \frac{9s^2}{10}}$$

$$\frac{s^2}{10})\, \frac{10s}{3} + s^2 \,(\frac{100}{3s}$$

$$\underline{\frac{10s}{3}}$$

$$s^2)\, \frac{s^2}{10} \,(\frac{1}{10}$$

$$\underline{\frac{s^2}{10}}$$

$$0$$

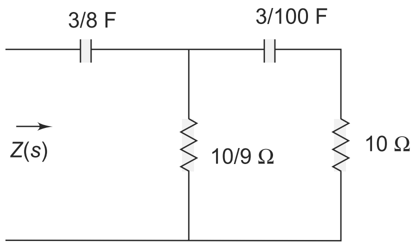


Fig. 18.24

$$Z(s) = \frac{8}{3s} + \cfrac{1}{\cfrac{9}{10} + \cfrac{1}{\cfrac{100}{3s} + \cfrac{1}{\cfrac{1}{10}}}}$$

Therefore, the impedance function $Z(s)$ can be realised as an RC network shown in Fig. 18.24.

Additional Solved Problems

PROBLEM 18.1

Find the two Foster realisations of the given function.

$$Z(s) = \frac{2s^3 + 8s}{s^2 + 1}$$

Solution For the first Foster network, we expand $Z(s)$ into partial fractions.

$$Z(s) = 2s + \frac{6s}{s^2+1}$$
$$= 2s + \frac{A}{s+j} + \frac{A^*}{s-j}$$

By applying Heaviside method, we get

$$A = \frac{6s}{(s+j)(s-j)}(s+j)\bigg|_{s=-j} = 3$$
$$A^* = \frac{6s}{(s+j)(s-j)}(s-j)\bigg|_{s=j} = 3$$

By inspection, $H = 2$

$$\therefore \quad Z(s) = 2s + \frac{3}{s+j} + \frac{3}{s-j}$$

$$\therefore \quad L_2 = \frac{2A}{\omega_n^2} = 6;$$

$$C_2 = \frac{1}{2A} = \frac{1}{6}$$

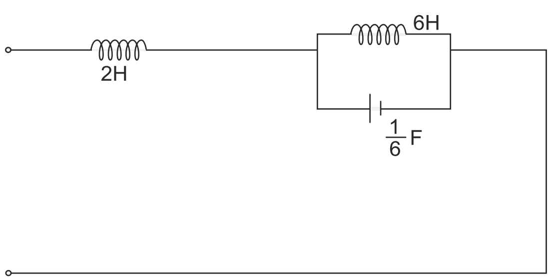


Fig. 18.25

The first Foster network with elemental values is shown in Fig. 18.25.

The second Foster network can be obtained by taking the admittance function

$$Y(s) = \frac{s^2+1}{2s(s+4)}$$

By taking partial fractions, we have

$$Y(s) = \frac{A}{s} + \frac{B}{s+j2} + \frac{B^*}{s-j2}$$

By applying the Heaviside method, we get

$$A = \left.\frac{s^2+1}{2s(s+4)}s\right|_{s=0} = \frac{1}{8}$$

$$B = \left.\frac{s^2+1}{2s(s+4)}(s+j2)\right|_{s=-j2} = \frac{3}{16}$$

$$B^* = \left.\frac{s^2+1}{2s(s+4)}(s-j2)\right|_{s=j2} = \frac{3}{16}$$

Therefore, the elemental values are

$$L_0 = \frac{1}{A} = 8 \text{ H}$$

$$L_1 = \frac{1}{2B} = \frac{8}{3} \text{ H}$$

$$C_1 = \frac{2B}{\omega_n^2} = \frac{3}{32} \text{ F}$$

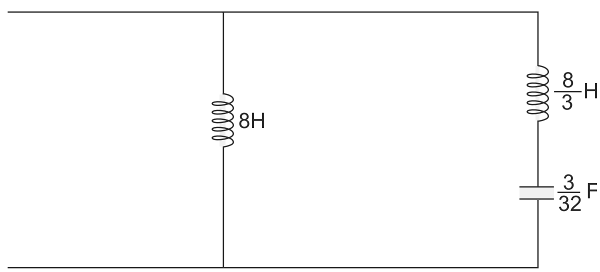


Fig. 18.26

The second Foster network with elemental values is shown in Fig. 18.26.

PROBLEM 18.2

Find the two Foster realisations of the given function.

$$z(s) = \frac{3(s^2+1)(s^2+16)}{s(s^2+9)}$$

Solution For the first Foster network, we expand $Z(s)$ into partial fractions

$$Z(s) = 3s + \frac{24(s^2+2)}{s(s^2+9)}$$

$$= 3s + \frac{A}{s} + \frac{B}{s+j3} + \frac{B^*}{s-j3}$$

By applying the Heaviside method,

$$A = \left.\frac{24(s^2+2)}{s(s^2+9)}s\right|_{s=0} = \frac{16}{3}$$

$$B = \frac{24(s^2+2)}{s(s^2+9)}(s+j3)\bigg|_{s=-j3} = \frac{28}{3} = B^*$$

$$\therefore \qquad Z(s) = 3s + \frac{16}{3s} + \frac{28}{3(s+j3)} + \frac{28}{3(s-j3)}$$

By inspection $H = L_\infty = 3$

$$A = \frac{16}{3} \text{ and } C_0 = \frac{1}{A} = \frac{3}{16} \text{ F}$$

$$C_2 = \frac{1}{2B} = \frac{3}{56} \text{ F} ; L_2 = \frac{2B}{\omega_n^2} = \frac{56}{27}$$

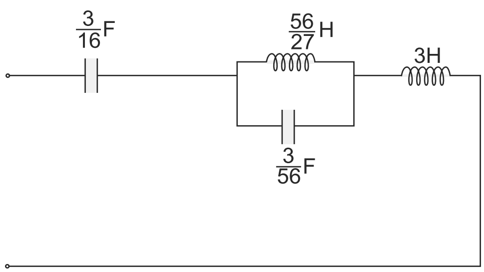


Fig. 18.27

The first Foster network with elemental values is shown in Fig. 18.27.

The second Foster network can be obtained by taking admittance function

$$Y(s) = \frac{s(s^2+9)}{s(s^2+1)(s^2+16)}$$

By taking partial fraction expansion, we have

$$Y(s) = \frac{2AS}{s^2+1} + \frac{2BS}{s^2+16}$$

By applying the Heaviside method, we get

$$A = \frac{8}{90} ; B = \frac{7}{90}$$

Therefore, the elemental values are

$$L_1 = \frac{1}{2\text{A}} = \frac{90}{16} \text{ H} ; C_1 = \frac{2A}{\omega_1^2} = \frac{8}{45} \text{ F}$$

$$L_2 = \frac{1}{2\text{B}} = \frac{90}{14} \text{ H} ; C_2 = \frac{2B}{\omega_1^2} = \frac{7}{720} \text{ F}$$

The second Foster network with elemental values is shown in Fig. 18.28.

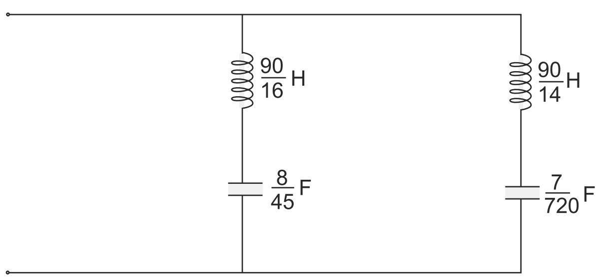


Fig. 18.28

PROBLEM 18.3

Find the second Cauer network of the given function.

$$z(s) = \frac{s^4 + 6s^2 + 4}{s^3 + 2s}$$

Solution The second Cauer network can be realised by arranging the numerator and denominator polynomials of $Z(s)$ in ascending power of s and taking continued fraction expansion, we get

$$
\begin{array}{l}
2s + s^3)\ 4 + 6s^2 + s^4\ (\frac{2}{s} \\
\qquad\ \underline{4 + 2s^2} \\
\qquad\qquad 4s^2 + s^4)\ 2s + s^3\ (\frac{1}{2s} \\
\qquad\qquad\qquad\qquad \underline{2s + \frac{s^3}{2}} \\
\qquad\qquad\qquad\qquad\ \frac{s^3}{2})\ 4s^2 + s^4\ (\frac{8}{s} \\
\qquad\qquad\qquad\qquad\qquad \underline{4s^2\qquad} \\
\qquad\qquad\qquad\qquad\qquad s^4)\ \frac{s^3}{2}\ (\frac{1}{2s} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{s^3}{2}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\ 0
\end{array}
$$

$$Z(s) = \frac{2}{s} + \cfrac{1}{\cfrac{1}{2s} + \cfrac{1}{\cfrac{8}{s} + \cfrac{1}{1/2s}}}$$

Therefore, the impedance function, $Z(s)$, can be realised as the *RC* network shown in Fig. 18.29.

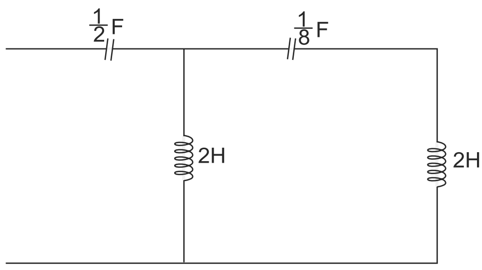


Fig. 18.29

PROBLEM 18.4

Find the first and second Cauer forms of the function.

$$z(s) = \frac{2s^2 + 8s + 6}{s^2 + 2s}$$

Solution The first Cauer network can be realised by taking continued fraction expansion

$$
\begin{array}{l}
s^2+2s)\,2s^2+8s+6\,(2 \\
\qquad \underline{2s^2+4s} \\
\qquad\qquad 4s+6)\,s^2+2s\,(\dfrac{s}{4} \\
\qquad\qquad\qquad \underline{s^2+\dfrac{6s}{4}} \\
\qquad\qquad\qquad\quad \dfrac{s}{2})\,4s+6\,(8 \\
\qquad\qquad\qquad\qquad \underline{4s} \\
\qquad\qquad\qquad\qquad 6)\,s/2\,(s/12 \\
\qquad\qquad\qquad\qquad\quad \underline{s/2} \\
\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$Z(s)=2+\cfrac{1}{\dfrac{s}{4}+\cfrac{1}{8+\cfrac{1}{\dfrac{s}{12}}}}$$

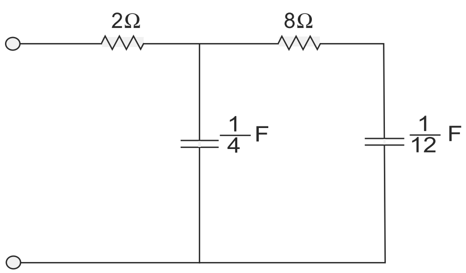


Fig. 18.30

Therefore, the impedance function $Z(s)$ can be realised as RC network shown in Fig. 18.30.

The second Cauer network can be realised by arranging the numerator and denominator polynomials of $Z(s)$ in ascending power of s and taking continued fraction expansion, we get

$$
\begin{array}{l}
2s+s^2)\,6+8s+2s^2\,(\dfrac{3}{s} \\
\qquad \underline{6+3s} \\
\qquad\quad 5s+2s^2)\,2s+s^2\,(\dfrac{2}{5} \\
\qquad\qquad\qquad \underline{2s+\dfrac{4}{5}s^2} \\
\qquad\qquad\qquad\quad \dfrac{s^2}{5})\,5s+2s^2\,(\dfrac{25}{s} \\
\qquad\qquad\qquad\qquad \underline{5s} \\
\qquad\qquad\qquad\qquad 2s^2)\,\dfrac{s^2}{5}\,(\dfrac{1}{10} \\
\qquad\qquad\qquad\qquad\quad \underline{\dfrac{s^2}{5}} \\
\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$Z(s)=\frac{3}{s}+\cfrac{1}{\dfrac{2}{5}+\cfrac{1}{\dfrac{25}{s}+\dfrac{1}{10}}}$$

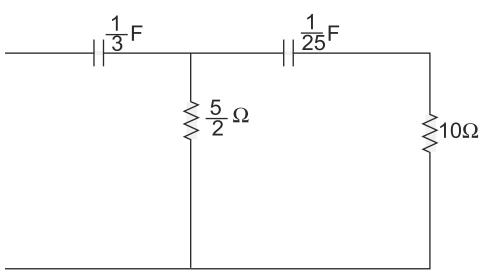


Fig. 18.31

Therefore, the impedance function, $Z(s)$, can be realised as the RC network shown in Fig. 18.31.

PROBLEM 18.5

Find the second Foster form and the first Cauer form of the network whose driving-point admittance is

$$Y(s)=\frac{3(s+2)(s+5)}{s(s+3)}$$

Solution By taking partial fraction expansion, we get

$$Y(s)=3+\frac{9s+24}{s^2+3s}=3+\frac{A}{s}+\frac{B}{s+3}$$

By applying the Heaviside method, we get

$$A=\frac{9s+24}{s(s+3)}s\bigg|_{s=0}=8$$

$$B=\frac{9s+24}{s^2+3s}(s+3)\bigg|_{s=-3}=1$$

$$\therefore\quad Y(s)=3+\frac{8}{s}+\frac{1}{s+3}$$

Therefore, the elemental values are

$$R=\frac{1}{3}\,\Omega,\quad L=\frac{1}{8}\,\text{H}\,;\,R_1=3\,\Omega\,;\,L_1=1\,\text{H}$$

Therefore, the second Foster network is shown in Fig. 18.32.

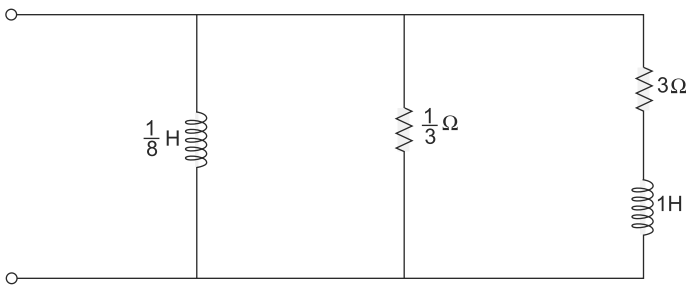


Fig. 18.32

To get the first Cauer realisation, we take continued fraction expansion from the expression.

$$Y(s)=\frac{3(s+2)(s+4)}{s(s+3)}$$

$$s^2+3s)\,3s^2+18s+24\,(3$$
$$\underline{3s^2+9s}$$
$$9s+24)\,s^2+3s\,\Big(\frac{s}{9}$$
$$\underline{s^2+\frac{8}{3}s}$$
$$\frac{s}{3}\Big)\,9s+24\,\Big(27$$
$$\underline{9s}$$
$$24\Big)\,\frac{s}{3}\,\Big(\frac{s}{72}$$
$$\underline{\frac{s}{3}}$$
$$0$$

$$\therefore \qquad Y(s) = \cfrac{1}{3 + \cfrac{1}{\cfrac{s}{9} + \cfrac{1}{27 + \cfrac{1}{\cfrac{s}{72}}}}}$$

Therefore, the admittance, $Y(s)$, can be realised as RL network shown in Fig. 18.33.

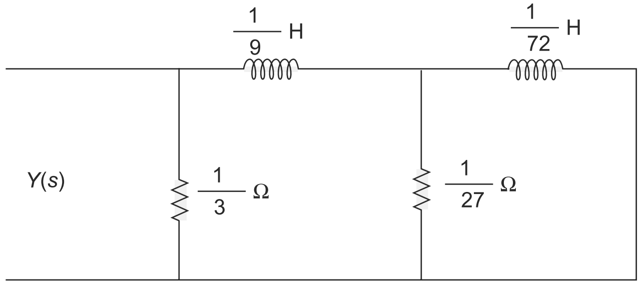


Fig. 18.33

PROBLEM 18.6

Find the two Foster realisations of

$$Z(s) = \frac{4(s^2+1)(s^2+16)}{s(s^2+4)}$$

Solution For the first Foster network, we expand $Z(s)$ into partial fractions.

$$Z(s) = \frac{P_0}{s} + \frac{P_2}{s+j2} + \frac{P_2^*}{s-j2} + \mathrm{H}s$$

By applying the Heaviside method, from the above equation we have

$$P_0 = \left.\frac{4(s^2+1)(s^2+16)}{s^2+4}\right|_{s=0} = 16$$

$$P_2 = \left.\frac{4(s^2+1)(s^2+16)}{s(s-j2)}\right|_{s=-j2} = -\frac{4\times 3\times 12}{-j4\times -j2} = 18$$

By inspection, $H = 4$

$$\therefore \qquad C_0 = \frac{1}{P_0} = \frac{1}{16}\,\mathrm{F};\, L_\infty = H = 4\ \mathrm{H}$$

$$C_2 = \frac{1}{2P_2} = \frac{1}{36}\,\mathrm{F}$$

$$L_2 = \frac{2P_2}{\omega_n^2} = \frac{2\times 18}{4} = 9\,\mathrm{H}$$

The first Foster network with elemental values is shown in Fig. 18.34 (a).

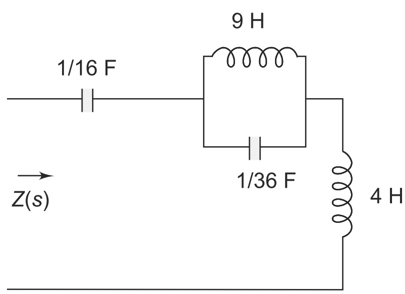


Fig. 18.34 (a)

The second Foster network can be obtained by taking admittance function

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+16)}$$

Let us take the partial fraction expansion, we have

$$Y(s) = \frac{2P_1 s}{s^2+1} + \frac{2P_2 s}{s^2+16}$$

By applying the Heaviside method, we get

$$P_1 = \frac{1}{4}\frac{s(s^2+4)}{(s^2+16)(s+j1)}\bigg|_{s=-j1} = \frac{1}{4}\frac{(-j1)(3)}{(15)(-j2)} = \frac{1}{40}$$

$$P_2 = \frac{1}{4}\frac{s(s^2+4)}{(s-j4)(s^2+1)}\bigg|_{s=-j4} = \frac{1}{4}\frac{(-j4)(-12)}{(-j8)(-15)} = \frac{1}{10}$$

Therefore, the element values are

$$L_1 = \frac{1}{2P_1} = 20\,\text{H}$$

$$C_1 = \frac{2P_1}{\omega_1^2} = \frac{2\times 1}{40\times 1} = \frac{1}{20}\,\text{F}$$

$$L_2 = \frac{1}{2P_2} = \frac{10}{2} = 5\,\text{H}$$

$$C_2 = \frac{2P_2}{\omega_1^2} = \frac{2}{10\times 16} = \frac{1}{80}\,\text{F}$$

The second Foster network with elemental values is shown in Fig. 18.34 (b).

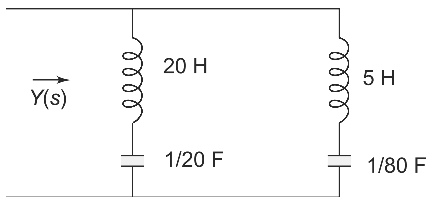


Fig. 18.34 (b)

PROBLEM 18.7

Find the two Cauer realisations of driving-point function given by

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

Solution By taking the continued fraction expansion, we get

$$2s^3 + 2s)\,10s^4 + 12s^2 + 1\,(5s$$
$$\underline{10s^4 + 10s^2}$$
$$2s^2 + 1)\,2s^3 + 2s\,(s$$
$$\underline{2s^3 + s}$$
$$s)\,2s^2 + 1\,(2s$$
$$\underline{2s^2}$$
$$1)\,s\,(s$$
$$\underline{s}$$
$$0$$

Hence $$Z(s) = 5s + \cfrac{1}{s + \cfrac{1}{2s + \cfrac{1}{s}}}$$

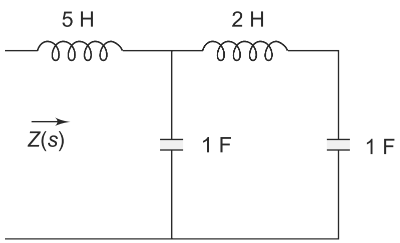


Fig. 18.35 (a)

The resulting network is called the first Cauer form with elemental values shown in Fig. 18.35 (a).

To realise the second Cauer network, we have to take ascending powers of the impedance function.

Continued fraction expansion gives

$$2s + 2s^3)\,1 + 12s^2 + 10s^4\,(\frac{1}{2s}$$
$$\underline{1 + s^2}$$
$$11s^2 + 10s^4)\,2s + 2s^3\,(\frac{2}{11s}$$
$$\underline{2s + \frac{20s^3}{11}}$$
$$\frac{2s^3}{11})\,11s^2 + 10s^4\,(\frac{121}{2s}$$
$$\underline{11s^2}$$
$$10s^4)\,\frac{2}{11}s^3\,(\frac{2}{110s}$$
$$\underline{\frac{2}{11}s^3}$$
$$0$$

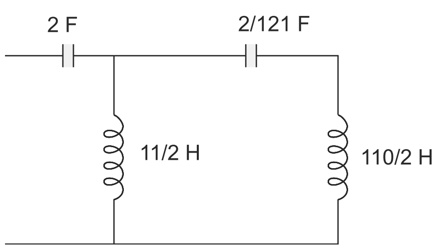


Fig. 18.35 (b)

Hence, $$Z(s) = \frac{1}{2s} + \cfrac{1}{\cfrac{2}{11s} + \cfrac{1}{\cfrac{121}{2s} + \cfrac{1}{\cfrac{2}{110s}}}}$$

The resulting network shown in Fig. 18.35 (b) is called the second Cauer form.

PROBLEM 18.8

Find the first Foster form of the driving-point function of

$$Z(s) = \frac{2(s+2)\,(s+5)}{(s+4)\,(s+6)}$$

Solution If we take the partial fraction of $Z(s)$, the signs of the function and its poles are negative as shown.

$$Z(s) = 2 - \frac{2}{s+4} - \frac{4}{s+6}$$

Therefore, we have to expand $Z(s)/s$

$$\frac{Z(s)}{s} = \frac{2(s+2)(s+5)}{s(s+4)(s+6)}$$

By taking partial fractions, we get

$$\frac{2(s+2)(s+5)}{s(s+4)(s+6)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+6}$$

$$= \frac{5}{6s} + \frac{1}{2(s+4)} + \frac{2}{3(s+6)}$$

If we multiply both sides by s, we get

$$Z(s) = \frac{5}{6} + \frac{s}{2(s+4)} + \frac{2s}{3(s+6)}$$

Hence, impedance $Z(s)$ can be realised as a series Foster form of RL network shown in Fig. 18.36.

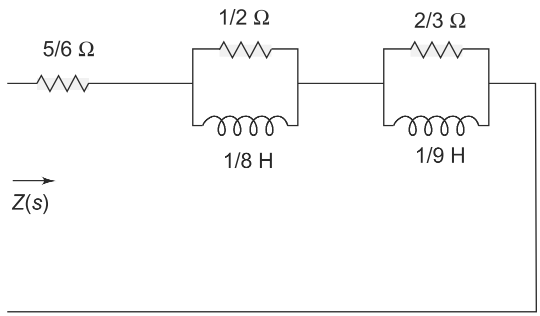


Fig. 18.36

PROBLEM 18.9

Find the second Foster form of RL network for the function.

$$Y(s) = \frac{s^2 + 8s + 15}{s^2 + 5s + 4}$$

Solution By taking partial fraction expansion, we get

$$Y(s) = 1 + \frac{3s+11}{s^2+5s+4} = 1 + \frac{A}{(s+1)} + \frac{B}{s+4}$$

where $A = \left.\dfrac{3s+11}{s+4}\right|_{s=-1} = \dfrac{8}{3}$

$$B = \left.\frac{3s+11}{s+1}\right|_{s=-4} = \frac{1}{3}$$

The residues are positive. Hence,

$$Y(s) = 1 + \frac{8}{3(s+1)} + \frac{1}{3(s+4)}$$

Therefore $R = 1\,\Omega, R_1 = \dfrac{3}{8}\,\Omega, L_1 = \dfrac{3}{8}$ H

The second Foster form of RL admittance function with various elemental values is shown in Fig. 18.37.

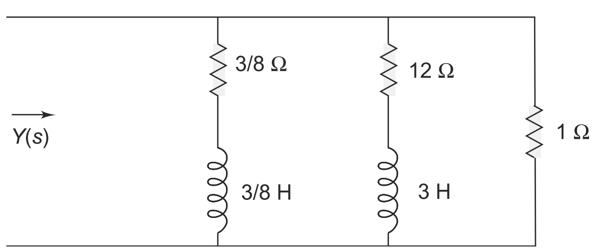


Fig. 18.37

PROBLEM 18.10

Find the first Cauer form of the function.

$$Z(s) = \frac{(s+3)(s+7)}{(s+2)(s+4)}$$

Solution By taking continued fraction expansion, we get

$$
\begin{array}{l}
s^2+6s+8)\, s^2+10s+21\,(1 \\
\qquad\qquad\quad \underline{s^2+6s+8} \\
\qquad\qquad\qquad 4s+13)\, s^2+6s+8\,(\dfrac{s}{4} \\
\qquad\qquad\qquad\qquad\qquad \underline{s^2+\dfrac{13s}{4}} \\
\qquad\qquad\qquad\qquad\qquad \dfrac{11s}{4}+8)\, 4s+13\,(\dfrac{16}{11} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4s+\dfrac{128}{11}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \dfrac{15}{11})\, \dfrac{11s}{4}+8\,(\dfrac{121}{60}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\dfrac{11s}{4}\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 8)\, \dfrac{15}{11}\,(\dfrac{15}{88} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{\dfrac{15}{11}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{4} + \cfrac{1}{\cfrac{1}{16/11} + \cfrac{1}{\cfrac{121}{60}s + \cfrac{1}{\cfrac{15}{88}}}}}$$

Therefore, the impedance function can be realised as the *RL* network shown in Fig. 18.38.

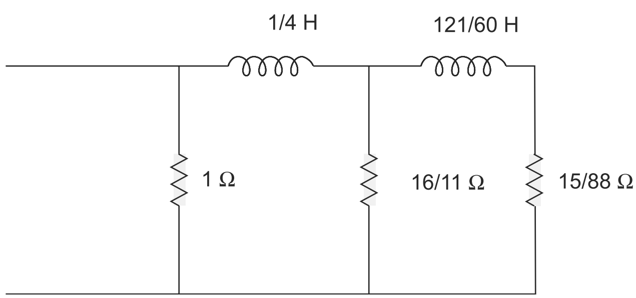


Fig. 18.38

PROBLEM 18.11

Find the first and second Foster forms of the function.

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution By taking the partial fraction expansion, we get

$$Z(s) = 1 + \frac{2s+3}{s(s+2)} = 1 + \frac{A}{s} + \frac{B}{s+2}$$

$$Z(s) = 1 + \frac{3}{2s} + \frac{1}{2(s+2)}$$

Hence, the impedance function Z (s) can be realised as series Foster form of RC network shown in Fig. 18.39 (a).

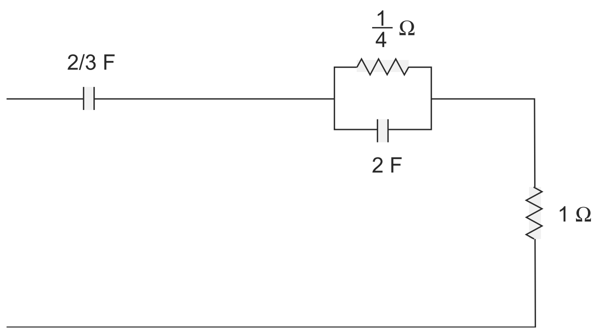


Fig. 18.39 (a)

The second Foster form can be realised by taking Y (s) as under.

$$Y(s) = \frac{s(s+2)}{(s+1)(s+3)}$$

$$= 1 - \frac{2s+3}{(s+1)(s+3)} = 1 - \frac{1}{2(s+1)} - \frac{3}{2(s+3)}$$

Since negative quotients appear, we have to expand $Y(s)/s$ as follows.

$$\frac{Y(s)}{s} = \frac{(s+2)}{(s+1)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+3} = \frac{1/2}{s+2} + \frac{1/2}{2(s+3)}$$

Multiplying both sides by s, we get

$$Y(s) = \frac{s/2}{s+1} + \frac{s/2}{s+3}$$

The network with elemental values are shown in Fig. 18.39 (b).

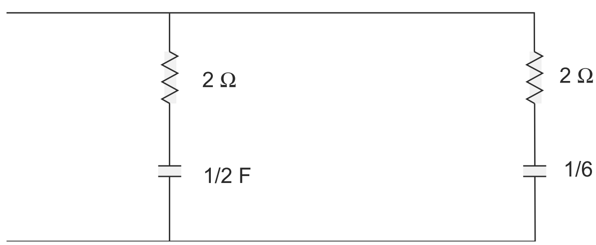


Fig. 18.39 (b)

PROBLEM 18.12

Find the first and second Cauer forms of the given function.

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution The first Cauer network can be realised by taking continued fraction expansion.

$$s^2 + 2s)\, s^2 + 4s + 3\, (1$$
$$\underline{s^2 + 2s}$$
$$2s + 3)\, s^2 + 2s\, (\frac{s}{2}$$
$$\underline{s^2 + \frac{3s}{2}}$$
$$\frac{s}{2})\, 2s + 3\, (4$$
$$\underline{2s}$$
$$3)\, \frac{s}{2}\, (\frac{1}{6}s$$
$$\underline{\frac{s}{2}}$$
$$0$$

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4 + \cfrac{1}{\cfrac{s}{6}}}}$$

Therefore, the impedance function, $Z(s)$, can be realised as the *RC* network shown in Fig. 18.40 (a).

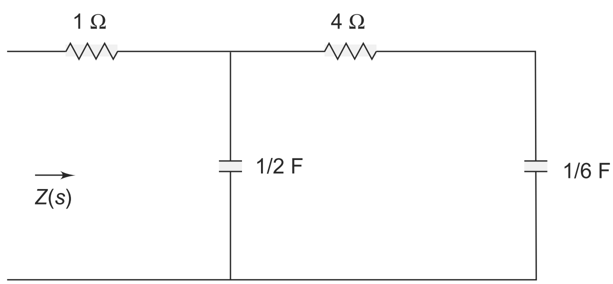


Fig. 18.40 (a)

The second Cauer network can be realised by arranging the numerator and denominator polynomials of $Z(s)$ in ascending power of s and taking continued fraction expansion; we get

$$2s+s^2\,)\,3+4s+s^2\,\left(\frac{3}{2s}\right.$$

$$\frac{3+\frac{3}{2}s}{\quad}$$

$$\frac{5s}{2}+s^2\,)\,2s+s^2\,\left(\frac{4}{5}\right.$$

$$\frac{2s+\frac{4}{5}s^2}{\quad}$$

$$\frac{s^2}{5}\,)\,\frac{5s}{2}+s^2\,\left(\frac{25}{2s}\right.$$

$$\frac{\frac{5s}{2}}{\quad}$$

$$s^2\,)\,\frac{s^2}{5}\,\left(\frac{1}{5}\right.$$

$$\frac{\frac{s^2}{5}}{0}$$

$$Z(s)=\frac{3}{2s}+\cfrac{1}{\cfrac{4}{5}+\cfrac{1}{\cfrac{25}{2s}+\cfrac{1}{\cfrac{1}{5}}}}$$

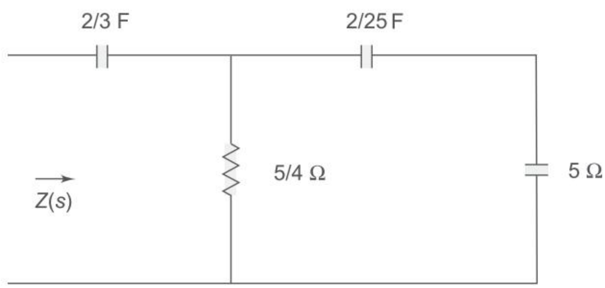


Fig. 18.40 (b)

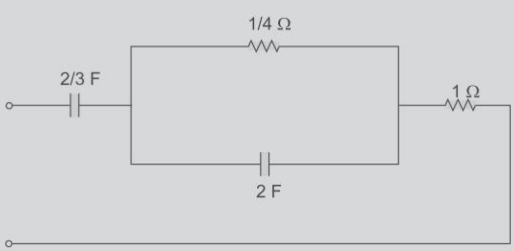
Therefore, the impedance function, $Z(s)$, can be realised as the *RC* network shown in Fig. 18.40 (b).

Answers to Practice Problems

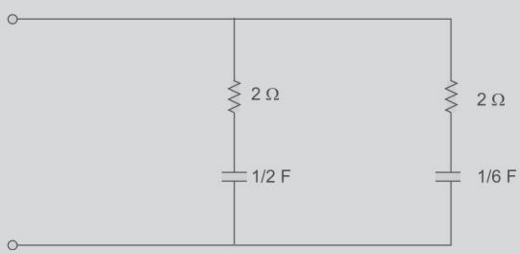
18-1.1 (a) Hurwitz (b) Hurwitz (c) Not Hurwitz

18-2.2 The function is a minimum positive real function

18-4.1

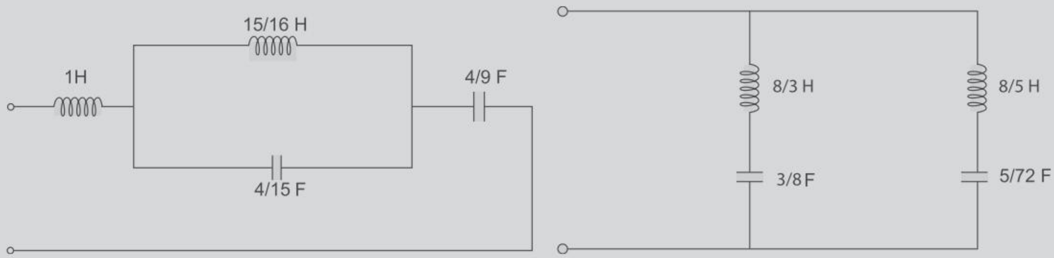


(a) First Foster form



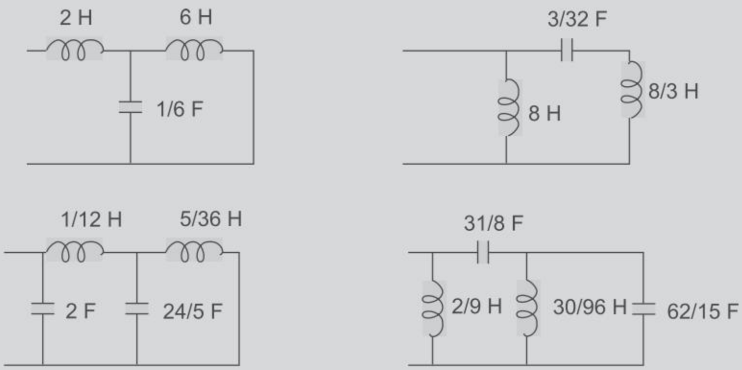
(b) Second Foster form

18.4.3

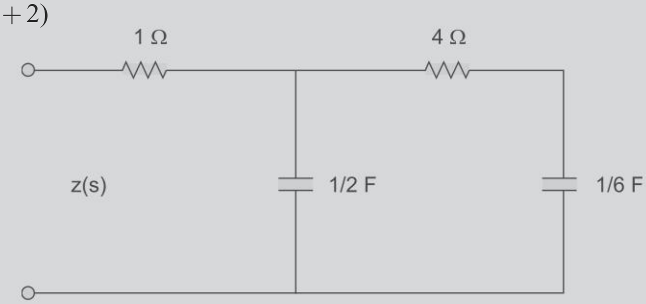


(a) First Foster form (b) Second Foster form

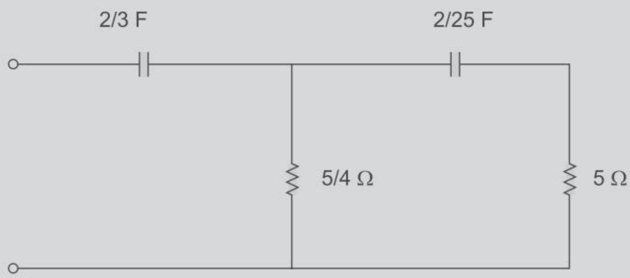
18-4.4



18-4.5 $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$



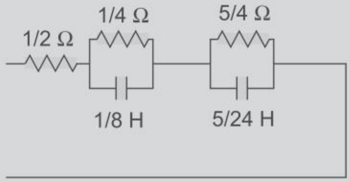
(a) First Cauer form



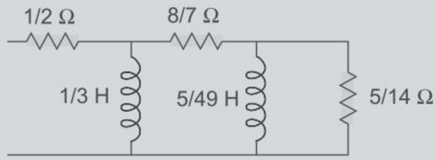
(b) Second Cauer form

Fig. 18.44

18-4.6

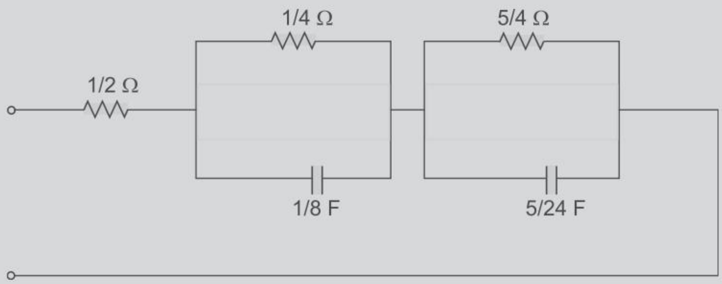


I Foster form

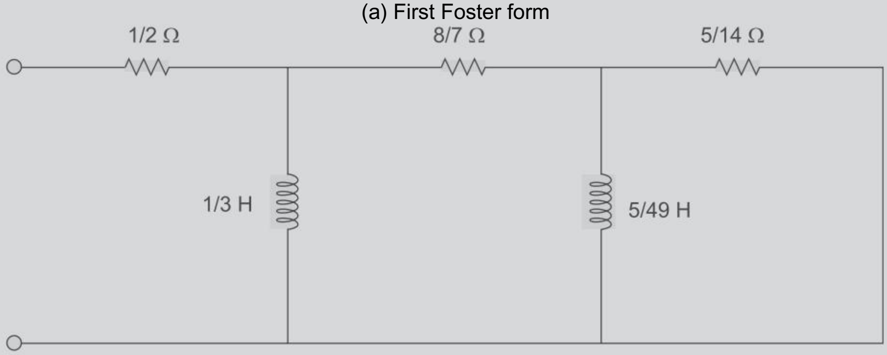


II Cauer form

18.4.8



(a) First Foster form



(b) Second Cauer form

Objective-Type Questions

☆☆★**18.1** A polynomial must satisfy the condition that

(a) $Z(s)$ is a real function
(b) all the roots of $P(s)$ have zero real parts, or negative real parts
(c) both (a) and (b)
(d) none of the above

☆☆★**18.2** Hurwitz polynomial possesses one of the conditions that

(a) all the quotients in the polynomial $P(s)$ must be positive
(b) the roots of $P(s)$ must lie on the right half of the S-plane
(c) the ratio of $P(s)$ and $P'(s)$ gives negative quotients
(d) $P(s)$ may have missing terms

☆☆★ **18.3** The function is said to be positive real, when
(a) the poles and zeros lie on the right half of the S-plane
(b) the poles and zeros lie on the left half of the S-plane
(c) the poles and zeros are simple and lie on the imaginary axis
(d) both (b) and (c)

☆☆★ **18.4** The driving-point impedance with poles at $\omega = 0$ and $\omega = \infty$ must have the
(a) s terms in the denominator and an excess term in the numerator
(b) s term in the numerator and an excess term in the denominator
(c) s term in the numerator and equal number of terms in the numerator and the denominator
(d) s term in the denominator and equal number of terms in the numerator and the denominator

☆☆★ **18.5** In the first Foster form, the presence of the first element capacitor C_0 indicates
(a) pole at $\omega = 0$ (b) pole at $\omega = \infty$ (c) zero at $\omega = 0$ (d) zero at $\omega = \infty$

☆☆★ **18.6** In the first Foster form, the presence of last element inductor L_∞ indicates
(a) pole at $\omega = 0$ (b) pole at $\omega = \infty$ (c) zero at $\omega = 0$ (d) zero at $\omega = \infty$

☆☆★ **18.7** Pole at infinity indicates that the
(a) degree of numerator is greater than that of denominator
(b) degree of denominator is greater than that of numerator
(c) degree of numerator is equal to the degree of denominator
(d) none of the above

☆☆★ **18.8** In the first Cauer LC network, the first element is a series inductor when the driving-point function consists of
(a) pole at $\omega = \infty$ (b) zero at $\omega = \infty$ (c) pole at $\omega = 0$ (d) zero at $\omega = 0$

☆☆★ **18.9** In the second Cauer LC network, the last element is an inductor, when the driving-point function consists of
(a) pole at $\omega = 0$ (b) pole at $\omega = \infty$ (c) zero at $\omega = \infty$ (d) zero at $\omega = 0$

For interactive quiz with answers,
scan the QR code given here
OR
visit
http://qrcode.flipick.com/index.php/276

CHAPTER 19

An Introduction to PSpice

LEARNING OBJECTIVES

After reading this chapter, the reader should be able to

LO 1 Understand PSpice
LO 2 How to get started with PSpice; describe the simulation steps and the component values
LO 3 Analyse dc circuits using control statements
LO 4 Analyse dependent sources using control statements
LO 5 Explain the concept of dc sweep
LO 6 Analyse ac circuits using control statements
LO 7 Analyse the time-domain response using PSpice

19.1 INTRODUCTION

SPICE is a universal standard simulator used to simulate the operation of various electric circuits and devices. PSpice is one of the many commercial derivatives of SPICE. PSpice helps to simulate electrical circuit design before they are set up. This allows the designer to decide if changes are needed, without touching any hardware. PSpice also helps check the design and response of the network. In short, PSpice is a simulated lab bench on which the test circuit can be created and measurement can be made.

SPICE stands for Simulation Program with Integrated Circuit Emphasis. PSpice is a member of the spice family of circuit simulators, developed at the University of California, Berkeley. PSpice is a commercial product developed by Microsim Corporation.

19.2 WHAT IS PSPICE?

LO 1 Understand PSpice

In 1968, a junior faculty member at the University of California, Berkeley, started a course on circuit simulation, hoping to develop a new circuit simulator for his work in circuit optimisation. He, along with a few students, assembled a non-linear circuit simulator which was to become the foundation for SPICE. The first simulator was named CANCER (Computer Analysis of Non-linear Circuits Excluding Radiation). But its capability was limited as it could not handle more components and/or circuit nodes.

During the 1970s, improvements in CANCER continued. In 1971, an improved version of CANCER named SPICE 1 (Simulation Program with Integrated Circuit Emphasis 1) was released. The next major breakthrough was in 1975 with the introduction of SPICE 2. From 1975 through 1983, Berkeley continued improving and upgrading the SPICE 2 program. In 1983, SPICE 2G.6 version was released. All these versions were written in FORTRAN source code. Later it was rewritten in C. The new C version of the program was known as SPICE 3. SPICE 3 offers several technical advantages as compared to SPICE 2. Several vendor-offered versions of SPICE are there in the market. Some of the better-known simulators include Meta-Software's HSPICE, Intusoft's IS-SPICE, Spectrum Software's MICRO-CAP, and Microsim's PSpice. All these were developed from the original SPICE 2. Although many other SPICE-based programs exist, these four represent the best known simulators. Majority of the SPICE-like simulators are still based on SPICE 2G.6, that is, a SPICE 2 version. PSpice, which uses the same algorithms as SPICE 2 (and confirms to its output syntax), shares this emphasis on micro circuit technology. However, the electrical concepts are general and are useful for all sizes of circuits and a wide range of applications.

19.3 GETTING STARTED WITH PSPICE

LO 2 How to get started with PSpice; describe the simulation steps and the component values

SPICE is widely used in the academic and industrial worlds to simulate the operation of various electric circuits and devices. In order to use the educational version of PSpice from Microsim or elsewhere, the minimum requirement for any PC are PC/XT/AT with atleast 512 KB of RAM, a fixed disk, MS-DOS version 3.0 or later and a monochrome or colour graphic monitor with a 20 MB hard disc. PSpice was developed by Microsim Corporation in California and made available in 1984, and later by ORCAD. PSpice has been made available in different operating systems such as DOS; WINDOWS or UNIX, etc. Though the Windows version of PSpice is becoming more and more popular, a general description is presented in this chapter. PSpice can analyse upto roughly 125 elements and over 100 nodes. It is capable of performing dc analysis transient analysis and ac analysis. In addition it can also perform transfer function analysis, Fourier analysis and operating point analysis. The circuit may contain resistors, inductors, capacitors, independent and dependent sources, OP amps, transformers, transmission lines, and semiconductor devices.

Make sure that the operating system and PSpice is already installed in your P.C. with the necessary configuration. The best way to learn a circuit simulator is to do simulations. Running this simulation involves the following main steps.

1. Create the input file or circuit file. It is also called a program for the simulator.
2. Run the simulator.
3. Find where the output is available.
4. Check the output. A text editor is required to create the input file, then the PSpice program can be run specifying the input file. If everything works, PSpice will read the input file executes and place the results in an output file. This output file may also be directed to a printer to get a print out.

Though PSpice is a powerful program that can carry out many different procedures, a brief introduction for the elementary types of dc, ac and transient analysis is presented in this chapter. The procedure described in this chapter is general, many advanced versions of spice packages are now available, students are advised to consult the user's guide supplied by the vendor for a specific PSpice simulation and design.

19.4 SIMULATION STEPS

LO 2

As a first step in simulation, an input file must be created for the given circuit which is also called the circuit file. Always begin with a complete sketch of the circuit. Label the nodes using distinct markings. There must be always a zero (0) node, which will be the reference node. The other nodes can have either numerical or alphabetical designations.

Title or Comment Line

The input file must be given a name (title or description of the file). Any line beginning with an asterisk (*) will be printed or displayed with the program, but will otherwise be ignored by the computer. Any line may be a title line, by starting it with a "*" in the first column. It is always better to include a statement for every element in the circuit. PSpice allows the user to insert comments or statements on any line by starting the comment with a ";" (semicolon). Everything on the line after the ";" is ignored. PSpice always expects the first line of the circuit file to be a title line. If it describes an element, it will be ignored. Any statement that begins with a "." (period) is called a control statement. The last statement must be the .END statement which completes the description of the entire circuit. After the .END statement, PSpice will let you start another completely different circuit simulation. Upper and lowercase alphabetic characters may be used in PSpice; RSHUNT, Rshunt, refer to the same device.

19.5 COMPONENT VALUES

LO 2

While representing either large or small component values, the following letters with corresponding scale factors are to be used in PSpice.

Table 19.1 Letters used in PSpice

Symbol	*Meaning*	*Value*	*Exponential form*
F	Femto	10^{-15}	IE-15
P	Pico	10^{-12}	IE-12
N	Nano	10^{-9}	IE-9
U	Micro	10^{-6}	IE-6
M	Milli	10^{-3}	IE-3
K	Kilo	10^{3}	IE 3
MEG	Mega	10^{6}	IE 6
G	Giga	10^{9}	IE 9
T	Tera	10^{12}	IE 12

The symbolic form may be written either using upper or lower case letters. For example M or m indicates milli or 10^{-3}; mega or 10^{6} is written by MEG or meg. All the quantities, or values, in PSpice may be expressed as decimal or floating point values as used by all computer programs. The symbols in Table 19.1, when used as suffixes multiply the number they follow by a power of ten as an example 25N indicates the value of $25 \times 10^{-9} = 0.025\text{E-6}$.

19.6 DC ANALYSIS AND CONTROL STATEMENTS

LO 3 Analyse dc circuits using control statements

In dc circuit for SPICE, only seven circuit elements are used. These are the resistors—two independent sources and four dependent sources. Let us consider the voltage divider circuit shown in Fig. 19.1 (a) to investigate using PSpice.

Figure 19.1 (a) shows a series circuit with a dc voltage source and 3 resistors R1, R2, and R3. To specify the device or element in the circuit file we have to include the name of the

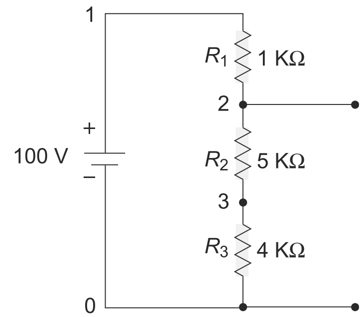


Fig. 19.1 (a)

element; the location of the element i.e. the nodes between which the element is connected and its value. PSpice uses the basic electrical units for voltage (volts); current (amps) and also uses, ohms, Farads and Henrys. We can specify the elements merely by using appropriate letter as the first letter of the device name as R for resistor, L for inductor C for capacitor, V for independent voltage source and I for independent current source. Now let us write the input file or circuit file for the circuit shown in Fig. 19.1 (a).

```
*voltage divider circuit
VIN   1   0   100V
R1    1   2   1K
R2    2   3   5K
R3    3   0   4K
.OP
.END
```

An editor such as the MS-DOS editor notepad or MS word from MS Office is to be used to enter the circuit file. The file might be suitably named. After running the PSpice program, the result would appear in the output file as follows:

❑ *Simulation Result of Fig. 19.1 (a).*

```
NODE  VOLTAGE   NODE   VOLTAGE   NODE   VOLTAGE
(1)    100.0     (2)    90.000    (3)    40.000
VOLTAGE SOURCE CURRENTS
NAME        CURRENT
VIN         –1.00E − 02
TOTAL POWER DISSIPATION 1.00E + 00W
Total job time 1.05.
```

Let us examine the statements in the input file. There is a statement for each element of the circuit. Each line of the input file is a statement. The first line in the program indicates the title of the file. Four lines are used to describe four elements in the circuit. The second line describes the independent voltage source. It is identified by using the first letter of the source (It can be followed by any combination of seven additional letters or numbers). The name (VIN) is followed by a blank, the node (1) to which the positive reference to the source is connected, another blank, and then the node (0) at which the negative terminal is located. Another blank precedes the numerical value of the voltage in volts. 3rd, 4th and 5th lines describes the three resistances in the circuit. A resistor is identified by its first letter (It can be followed by another seven additional letters or integers), the name R1 is followed by one or more blanks, followed by the first node (1), followed by one or more blanks and then the second node (2) and one more blank precedes the value (1 kΩ) of R1. The last two lines in the input file are called control statements. After incorporating all the circuit data in the program, it is necessary to specify the operations that are to be performed. This is done by control statements.

19.6.1 .OP Statement

The .OP Statement is a control statement which instructs the computer to calculate the dc voltage between each node and the reference node.

19.6.2 .END Statement

The .END statement is the another most important control statement which must be used as the last line in every input file program. The .END statement marks the end of the circuit. All the data and commands must come before it. When the .END statement is reached, PSpice does all the specified analysis on the circuit.

There may be more than one circuits in an input file. Each circuit and its commands are marked by a .END statement. PSpice processes all the analysis for each circuit before going on to the next one. Everything is reset at the beginning of each circuit. Having several circuits in one file gives the same results as having them in separate files and running each one separately. Having finished with the file, exit the editor and run the PSpice program. If there are no errors the output analysis of the circuit will be available in the output file.

The control statement .OP gives maximum amount of information. It produces detailed bias point information, that is the voltage of all nodes, the currents and power dissipation of all the voltage sources. If the number of nodes are more, the computer generates a lot of output data that we may not really need.

19.6.3 .PRINT Statement

Instead of the .OP statement, we can use another control statement the .PRINT, for specific outputs. The print control statement consists of .PRINT followed by a space and DC, another space, and the desired node voltage or node voltages separated by at least one space. For example the following statement indicates the voltage at the node 2 and the node 3 with reference to zero node.

PRINT DC V(2) V(3)

In addition, the voltage between two nodes current values may be specified by .PRINT statement as .PRINT DC V(1, 3) I(R1). The above statement indicates in the output file, the voltage between nodes 1 and 3, and the current through resistor R1. One important point is that the .PRINT command does not result in printing of any value on paper. It is merely made available in the output file in computer memory. If the printer is connected to the system, then the appropriate command will produce a printed output. The currents through the branches can be measured in PSpice. If an independent voltage source exists in that branch. Thus, if, we want to calculate the value of current in some branch of a circuit that does not contain an independent voltage source, then we have to insert a voltage source with a value of 0 volts in the branch. Let us consider Fig. 19.1 (b). It is required to write the input file to calculate the current through 3 Ω resistor with the indicated direction where no voltage source exists. The four nodes and the reference node have been numbered, the current through 3 Ω is desired, we shall therefore insert (V_3) a 0 volt voltage source in the branch as shown in Fig. 19.1 (c). In SPICE a voltage source current is positive if it were directed from plus to minus through V. Hence, the assumed polarities for V_3 are correct.

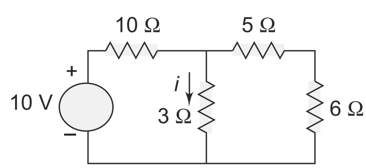


Fig. 19.1 (b)

The data for the circuit file is given by

*current measurement

VIN	1	0	10 V
R_1	1	2	10
R_2	2	3	5
R_3	2	4	3
R_4	3	0	6
V_3	4	0	

.PRINT DC I (R3)
.END

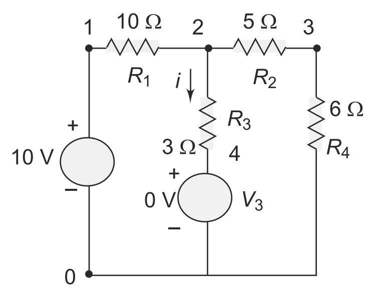


Fig. 19.1 (c)

The result of this program is as follows

❑ *Simulation Result of Fig. 19.1 (c)*

```
NODE   VOLTAGE     N      V     N     V     N     V
(1)    10.0000    (2)  1.9075 (3)  1.040  (4)  0.0000
VOLTAGE SOURCE CURRENTS
VIN - 8.092 E - 01
V3   6.358 E - 01
```

Total power dissipation 8.09E + 00 WATTS

As mentioned earlier, the .PRINT statement can be used for several outputs in one table, and mix voltages and currents. The output values you can print are basically node voltages and device (also source) currents. Node voltages can be printed relative to ground (0 node) or relative to another node. Examine the following statements for the circuit shown in Fig. 19.1 (c).

```
.PRINT DC V(1) to print voltage at node 1 (i.e. source voltage 10V)
.PRINT DC V(1,2) to print voltage between node 1 & 2.
.PRINT DC V(R4) to print voltage across R4
.PRINT DC V(2) V(3) I(R1) to print voltage at node 2, node 3 and current
through R1.
```

19.6.4 Current Source

If a current source is present in the circuit, the first node listed in the SPICE statement is the one at the tail of the current arrow and the rest of the listing is similar to the voltage source representation. As an example consider the circuit in Fig. 19.1 (d).

Now the statement for the independent current source in the input file is I IN 0 1 100M.

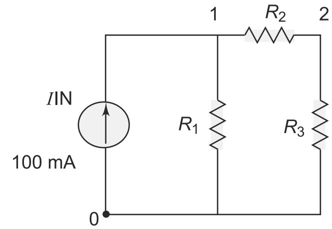


Fig. 19.1 (d)

19.7 DEPENDENT SOURCES

LO 4 *Analyse dependent sources using control statements*

In the seven circuit elements/devices mentioned in dc analysis, we have discussed only three; they are resistor, independent current, and voltage sources. The other four elements are dependent sources. They are (VCVS) voltage-controlled voltage sources, (CCCS) current-controlled current source, (VCCS) voltage-controlled current source, and (CCVS) current-controlled voltage source. These sources are described in the input file in a way that is similar to the passive devices. The names of VCVS; and VCCS are identified in the circuit file with a name beginning with letters E and G respectively followed by the connecting nodes, control nodes, and gain factor in the order mentioned, of course with blanks in between. The following examples illustrate the description of the dependent sources in the file. Let us consider the current in Fig. 19.2 (a) where we have one.

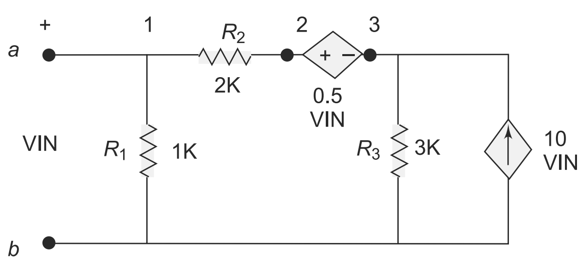


Fig. 19.2 (a)

The VCVS in the circuit of Fig. 19.2 (a) is written in the input file as

E 2 3 1 0 0.5

where, E—Device name

2—Positive node of the device

3—Negative node of the device

1—Controlled voltage positive node

0—Controlled voltage negative node

0.5—Gain factor of the voltage

Similarly, VCCS in the circuit of Fig. 19.2 (a) is written in the input file as

G 0 3 1 0 10

G—Device name

0—node at the tail of the arrow in current source

3—node at the head of the arrow in current source

1—Controlled voltage positive node

0—Controlled voltage 2nd node

10—Gain factor.

Let us consider the current in Fig. 19.2 (b) where we have a CCCS and a CCVS.

The statement for current controlled sources has a name beginning with F and H for CCCS and CCVS respectively, followed by the two nodes defining the direction of the current flow through the dependent source and the name of the V-device that has the controlling current, as PSpice measures the currents through voltage sources only. In the circuit shown, the controlling current I_2 is in the branch R_2 which does not have an independent voltage source, that is, no V-type element. Hence, a slight modification is required in the above circuit. Insert a zero volt independent voltage source in the branch R_2 and name this as VO, and change the above circuit to the circuit shown in Fig. 19.2 (c).

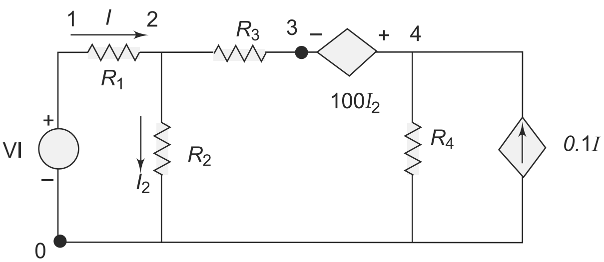


Fig. 19.2 (b)

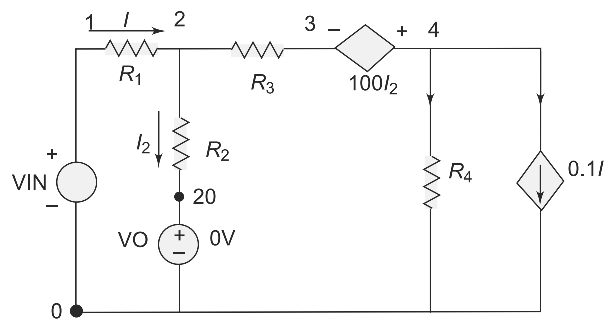


Fig. 19.2 (c)

Now, the CCVS in the circuit of Fig. 19.2 (c) is written in the circuit file as

H 4 3 VO 100

where, H—Device name

4—Positive node of the device

2—Second node (negative node) of the device

VO—The name of the zero volt source in the control branch

100—Gain factor of the controlling current.

Similarly, CCCS of Fig. 19.2 (c) in the circuit file is listed as

F 4 0 VIN –0.1

where, F—Device name

4—Node at the tail of the arrow of the CCCS

0—Node at the head of the arrow of the CCCS
VIN—Name of the independent voltage source through which the controlling current is carried.
– 0.1—Gain factor of the controlling current.

The reason for using minus sign is that the controlling current *I* is directed from minus to plus through VIN in the circuit.

EXAMPLE 19.1

Write a SPICE program for the circuit shown in Fig. 19.2 (a) to determine the voltages at node 2 and 3, if VIN = 10 volts dc.

Solution *Voltage Dependent sources

```
VIN   1   0   10
R1    1   0   1K
R2    1   2   2K
R3    3   0   3K
E     2   3   1   0   0.5
G     0   3   1   0   10
.OP
.END
```

Simulation Result of Example 19.1

NODE	VOLTAGE	N	V	N	V
(1)	10	(2)	12.01EW + 03	(3)	12.00E + 03

VOLTAGE SOURCE CURRENTS
VIN 5.989E + 00
POWER DISSIPATION − 5.99E + 01W; Time 2.43.

EXAMPLE 19.2

Write a SPICE program for the circuit shown in Fig. 19.2 (c) to determine the voltages of all nodes, and the power dissipation of all sources. Assume VIN = 100 volts; $R_1 = R_2 = 2$ kΩ; $R_3 = R_4 = 0.5$ kΩ.

Solution * Current Dependent source

```
VIN   1    0    100
R1    1    2    2K
R2    2    20   2K
R3    2    3    0.5K
R4    4    0    0.5K
VO    20   0
H     4    3    VO     100
F     4    0    VIN    −0.1
.OP
.END
```

Simulation Result of Example 19.2

NODE	VOLTAGE	N	V	N	V	N	V	N	V
(1)	100	(2)	18.4810	(3)	4.2208	(4)	5.1948	(20)	0.000

Voltage source currents

NAME CURRENT

VIN −4.026E − 0.2
VO 9.740E − 03
Total power dissipation 4.03 W.
Current–controlled sources
NAME F
I SOURCE 2.013E − 02
Current–controlled voltage sources
NAME
V-SOURCE 9.740E − 01
I-SOURCE −3.052E − 02
Total job time 2.32.

19.8 DC SWEEP

LO 5 Explain the concept of dc sweep

In the calculations so far, the values for sources maintained fixed values. But when the PSpice analysis is used with a range of input voltages which is called dc sweep, where the sources vary, though the analysis will still calculate quiescent operation. Using this analysis allows to look at the results from many .OP analysis in a single simulation run. That is, when you sweep a source the simulator starts with one value for a source (voltage or current), calculate the DC bias point as it does for the .OP statement, then increments the value and does another dc bias point calculation and so on until the last source value has been analysed.

19.8.1 .DC Statement

The dc sweep analysis is controlled with a .DC statement. The .DC statement gives a range to voltages/currents. This is called a sweep of voltage/current. This statement specifies the values used during the dc sweep. The statement says which source value is to be swept, the starting value, the end value and the amount of increment in each step. Let us insert the .DC statement in the circuit file of Fig. 19.1 (a) and rewrite the file.

```
*voltage divider circuit
VIN    1    0    100V
R1     1    2    1K
R2     2    3    5K
R3     3    0    4K
.OP
.DC   VIN  0  100  10
.END
```

While adding a .DC statement to the circuit file, the other lines used to describe the circuit need not be changed. Adding the .DC statement will override the fixed value indicated by the independent source VIN during DC sweep analysis. After running the PSpice program, the output file contains the following simulation result.

❑ *Small-Signal Bias Solution*

```
N   V       N      V      N      V
(1) 100    (2)    90     (3)    40
```

```
VOLTAGE SOURCE CURRENTS
NAME    CURRENT
VIN     21.00E − 02
Total power dissipation 1.00 + 0.0watt
Time 1.66.
```

The .DC statement followed by name of the source VIN whose voltage is to be swept, the next two values 0 and 100, are for the start and stop voltage values of the sweep, and the last value 10 is the increment.

19.8.2 .PROBE Statement

In PSpice, we have a facility called PROBE which provides us the powerful graphic capability of PSpice. To use the above statement, we must instruct PSPICE to create a data file for probe which is done by including the .PROBE statement in the input file. This statement is similar to the .PRINT with the .PROBE you may select node voltages and device currents to be output from the simulation. The .PROBE statement writes the results from DC, AC and transient analysis to a data file named PROBE.DAT for graphics analysis by post-processor. The general forms of the .PROBE statement are

.PROBE
.PROBE V (1) V (4 3) I (R4)

The first form without any output variable writes all the node voltages and all the device currents to the data file. The second form writes the following output variables to the data file. The voltage of node 1, voltage between node, 4 and 3 and the current through R_4. Another important difference between .PRINT and .PROBE statement is that the analysis name (DC, AC or transient) is absent before the output variable in .PROBE statement.

19.9 AC ANALYSIS AND CONTROL STATEMENTS

LO 6 *Analyse ac circuits using control statements*

Another important application of the PSpice simulator is to verify the frequency response of various devices and circuits. The response calculates all the ac node voltages and branch currents over a specified range of frequencies. PSpice calculates the dc node voltage without any special requirements, but in ac analysis, we must specifically ask for it.

Let us consider the circuit shown in Fig. 19.3 (a) which is a series *RLC* circuit with a voltage source of 100V at an angle of 15°. Each independent voltage and current source in ac analysis is characterised by its amplitude and phase with the source statement in the file. The source frequency is specified in a control statement. (.AC statement). Let us write a suitable input file for the circuit shown in Fig. 19.3 (a).

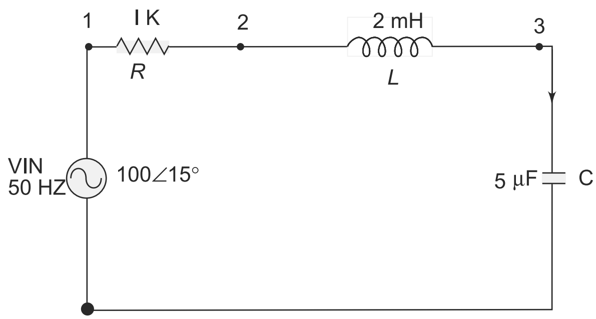


Fig. 19.3

```
*RLC SERIES CIRCUIT
 VIN    1     0     AC     100    15
 R      1     2     1K
 L      2     3     2mH
 C      3     0     5mF
 .AC    LIN   1     50     50
```

```
.PRINT AC IM (R) IR(R) II(R) IP(R)
.END
```

The statement VIN 1 0 AC 100 15 indicates the ac source which is connected between nodes 1 and zero with an amplitude of 100 V and a phase angle of 15°. If the phase angle is 0°, the last term in the statement can be omitted.

Simulation Result

FREQ	IM(R)	IR(R)	II(R)	IP(R)
5.00E + 0.1	8.438×10^{-2}	5.705×10^{-2}	6.217×10^{-2}	4.746×10^{1}

Time 2.28.

19.9.1 .AC Statement

This statement begins with the specification .AC and continues with four additional terms. The first term indicates the type of frequency sweep (linear, octave and decade). The second term indictes the number of points in the sweep and the third and fourth terms indicate the beginning and ending frequencies. Hence, .AC LIN 1 50 50 statement gives a linear sweep for one frequency only with beginning and ending value of 50 Hz that is it selects only one frequency. If it is required to seep 10 frequencies linearly from 5 Hz to 50 Hz then the ac statement would be

(.AC LIN 10 5 50)

This statement would provide results at the starting and stopping frequencies and eight intermediate frequencies that are uniformly spaced (5, 10, 15, 20, 25, 30, 35, 40, 45, 50).

19.9.2 .PRINT AC Statement

Output from ac analysis may be generated by .PRINT statement, just as in DC analysis. The phase AC replaces DC. The output values that can be printed are node voltages and device currents (source currents) with some special considerations for AC analysis. The voltages and currents may be specified as magnitude, phase, real part, imaginary part or magnitude in dB by adding M, P, R, I and DB respectively as a suffix to "V" (Voltage magnitude) or "I" (Current magnitude). Thus the statement.

```
(.PRINT AC IM(R) IR(R) II(R) IP(R))
```

would yield the magnitude, the real component, the imaginary component and the phase angle in degree of the current through R. This is shown in the input file and its simulation result of Fig. 19.3 (a). As mentioned earlier, the frequency sweep can be done in octave and decade also. Their syntax is similar, only thing required is, .AC DEC is used for decade sweep and .AC OCT is used for octave sweep.

19.10 TRANSIENT ANALYSIS

LO 7 Analyse the time-domain response using PSpice

PSpice can be effectively used for transient or time domain analysis. It is used very often for circuit simulation, because this analysis is the tedious and difficult analysis as it involves lengthy integro-differential equations with boundary conditions.

InPSpice,wecaninvestigatethecircuittransientresponseforvarioustypesofinputwaveforms,likeexponential (EXP), pulse (PULSE); piecewise-linear (PWL); frequency modulated wave (SFFM) and for sinusoidal wave (SIN) forms. Hence, the independent voltage and current sources may be specified in any of the above time-varying waveforms by giving a proper format. The following are the General formats of the statements used to describe the applied voltages (waveforms) in transient analysis.

PWL (T1, V1 T2, V2 … TNVN) describes a piecewise linear waveform. The arguments in parenthesis represent time voltage pairs at the corners of the waveform.

EXP (V1 V2 TD1 TR1 TD2 TR2) describes the exponential waveform initial voltage V1 upto a delay time of TD1 seconds. V2 is the peak voltage with a fall delay time of TD2, TR1 and TR2 are the rise time constant and fall time constants respectively.

PULSE (V1 V2 TD TR TF PW PER) describes the pulse form of voltage with initial voltage (V1); peak value of pulse (V2); delay time (TD); rise time (TR); fall time (TF); width of the pulse (PW); and period of the pulse (PER).

SFFM (VO VA FC MD FS) describes the single frequencies modulated wave, with offset voltage VO peak amplitude (VA): Carrier Frequency (FC), Modulation Index (MD), and Single Frequency (FS).

SIN (VO VA FREQ TD DF PHASE) describes the sinusoidal waveform with an offset voltage of VO, peak value of VA, frequency FREQ, delay time td, damping factor DF, and a phase angle. The SIN waveform format is only for transient analysis only.

19.10.1 .TRAN Statement

.TRAN statement specifies the time interval over which the transient analysis takes place. This statement is followed by two values. The first value indicates the print-step (interval) value and the second value indicates the final value of time (length of the time for the analysis). Observe the following .TRAN statement.

.TRAN 2 ms 20 ms

wherein the time interval (time step) is 2 ms and the maximum value of time limit is 20 ms. Apart from the time step and final time some more options like starting time (default value is zero), max time for analysis and initial conditions can also be used along with .TRAN statement. Output from transient analysis may be generated by .PRINT statement just as in dc and ac analysis. Hence, transient analysis requires a .PRINT command similar to dc or ac analysis except that the term dc/ac is replaced by TRAN. The statement form is .PRINT TRAN (Any of the eight output variables).

As an example of transient analysis, let us calculate the voltage at node 2 in the circuit shown in Fig. 19.4 (a).

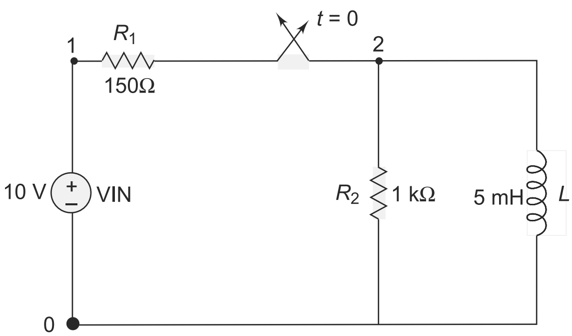


Fig. 19.4 (a)

Let us apply piecewise linear transition

```
*RL TRANSIENT
VIN    1     0     PWL   (0, 0   10 µs,   1 V   10 ms,   10 V)
R1     1     2     150
R2     2     0     1K
L      2     0     5M    IC = 0
TRAN   1 ms  10 ms
.PRINT TRAN  V(2)
.END
```

IC = 0 in the above file indicates zero initial current in the inductor. After running the PSpice analysis, we obtain the following simulation result.

❑ *Simulation Result*

N	V	N	V
(1)	0.40	(2)	0.40

Voltage source currents

V	0.000E + 00
Total power dissipation	0.00E + 00W

❑ *Transient Analysis* Temperature 27 DEG C

Time	V(2)
0.000E + 00	0.000E + 00
1.000E − 03	3.002E − 02
2.000E − 03	3.003E − 02
3.000E − 03	3.003E − 02
4.000E − 03	3.003E − 02
5.000E − 03	3.003E − 02
6.000E − 03	3.003E − 02
7.000E − 03	3.003E − 02
8.000E − 03	3.003E − 02
9.000E − 03	3.003E − 02
1.000E − 03	3.003E − 02

Total job time 0.08 second

19.10.2 .PROBE Statement

Using probe with transient analysis is identical to what we have done with dc and ac analysis. Include a .PROBE statement to the circuit file. Try the above example with .PROBE statement, and verify on the graph the voltage variation.

Consider another example with non-zero initial current as shown in Fig. 19.4 (b). The switch is closed at $t = 0$.

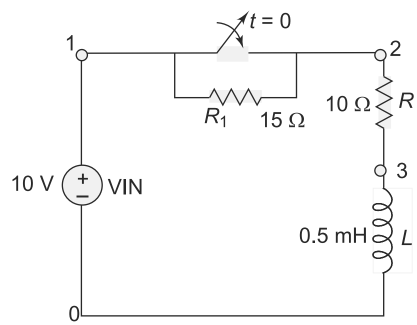


Fig. 19.4 (b)

The switch is opened before $t = 0$, the initial current before the closure of the switch is $i(o) = 10/25 = 0.4$A (through inductor). After the switch is closed, the current rises exponentially. The following is the input file for the transient analysis of the circuit in Fig. 19.4 (b).

*Transient analysis with I.C.

```
VIN      1       0     PWL    (0, 4V   1 µs,   10V   1 ms,   10V)
R        2       3     10
L        3       0     0.5M    IC = 0.5A
.TRAN    10 µs   1 ms
.PROBE
.END
```

Notice that, the first time voltage pair in PWL parenthesis is written as 0, 4. This is because when the switch is closed at $t\,(o^+)$ the voltage 4 V will appear across R.(0.5 × 10 = 4 V). Run the PSpice program and verify the result.

Sometimes, capacitors have initial voltages, if a 0.5 mF capacitor connected between nodes 3 and 4, carrying an initial voltage of 50 V may be specified in input with the following description.

C 3 4 0.5 μF IC = 50

We can also use sinusoidal excitation in transient analysis to verify the frequency response of the circuit. If the input source is a simple sinusoidal voltage source without any offset values and time delays with a maximum value of 215 V and frequency of 50 Hz. It can be represented in the input file as

VIN 0 1 sin (0 215 50 Hz).

Additional Solved Problems

PROBLEM 19.1

For the circuit shown in Fig. 19.5 write the input file, run the PSpice program and obtain the current through R_1; R_5, voltage at nodes 2 and voltage between node 2 and 3.

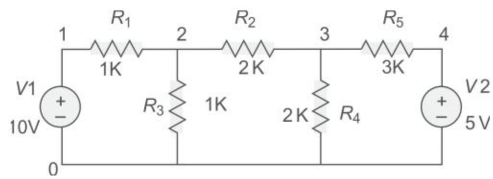


Fig. 19.5

Solution

```
*TWO VOLTAGE SOURCES
V1      1    0     10
R1      1    2     1K
R2      2    3     2K
R3      2    0     1K
R4      3    0     2K
R5      3    4     3K
V2      4    0     5
.OP
.DC     VI   50    50     5
.PRINT  DC   1(R1) 1(R5)  V(2)  V(2,3)
.ENDs
```

(.DC is a sweep statement it allows to sweep through a set of voltage of source V1. Though we are not interested in sweep in this problem, it is required for the next (.PRINT) statement. Without .DC statement, .PRINT is not valid).

The order in which the elements in the input file are listed makes no difference in the PSpice analysis.

Simulation Result

```
                         DC transfer curves

VI          I(R1)         I(R5)        V(2)       V(2,3)
5.0 × 109 2.811 × 10^-2 1.486 × 10^-3 2.189 × 109 1.243 × 109

                    Small signal bias solution i
N   V     N      V      N      V      N      V
(1) 10   (2)   4.5946  (3)  2.9730  (4)    5.000

                         Voltage source currents
```

V1 $- 5.405 \times 10^{-3}$
V2 $- 6.757 \times 10^{-4}$

Total power dissipation 5.74×10^{-2} watts
Total job time 2.24.

PROBLEM 19.2

For the circuit shown in Fig. 19.6, write the input file to obtain voltage across R_L and current through R_1 when the input voltage varies from 0 to 100 V.

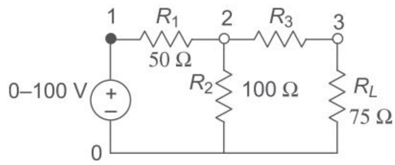


Fig. 19.6

Solution *DC SWEEP

```
VIN      1     0      100
R1       1     2      50
R2       2     0      100
R3       2     3      25
RL       3     0      75
.OP
.DC VIN  0     100    10
.PRINT   DC    V(RL)  I(RI)
.END
```

The variation of the source voltage from 0 to 100 is set with 10 V increment.

After running the PSpice program, the output file consists a table; showing the relation between VIN V(RL) and I(RI).

Following is the simulation result.

DC Transfer Curves

VIN	V(RL)	I(RI)
0	0	0
10	3.750	$1\ 3\ 10^{-1}$
20	7.5	$2\ 3\ 10^{-1}$
30	$1.125\ 3\ 10^{1}$	$3\ 3\ 10^{-1}$
40	$1.5\ 3\ 10^{1}$	$4\ 3\ 10^{-1}$
50	$1.875\ 3\ 10^{1}$	$5\ 3\ 10^{-1}$
60	$2.25\ 3\ 10^{1}$	$6\ 3\ 10^{-1}$
70	$2.625\ 3\ 10^{1}$	$7\ 3\ 10^{-1}$
80	$3\ 3\ 10^{1}$	$8\ 3\ 10^{-1}$
90	$3.375\ 3\ 10^{1}$	$9\ 3\ 10^{-1}$
100	$3.75\ 3\ 10^{1}$	1.00

Small signal bias solution

N	V	N	V	N	V
(1)	100	(2)	50	(3)	37.5

VOLTAGE SOURCE CURRENTS
VIN -1.00
Total power dissipation $1\ 3\ 10^{2}$ W
Time 1.47.

PROBLEM 19.3

Obtain the SPICE solution for the voltages at all nodes for the circuit shown in Fig. 19.7.
Assume $V_{IN} = 100$ *V;* $R_1 = 1$ *K;* $R_2 = 500\ \Omega$*;* $R_3 = 100\ \Omega$*;* $R_4 = 2$ *K;* $I_1 = 20$ *mA and* $I_2 = 25$ *mA.*

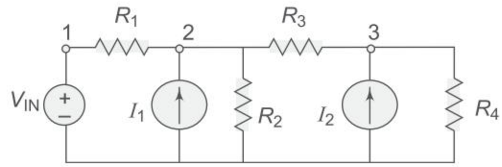


Fig. 19.7

Solution

```
*CURRENT SOURCES
VIN   1   0   100
R1    1   2   1K
R2    2   0   500
R3    2   3   100
R4    3   0   2K
I1    0   2   20M
I2    0   3   25M
.OP
.END
```

Run the PSpice analysis and the output of the result is as follows.

```
Small signal bias solution
N       V      N      V      N      V
(1)    100    (2)   4.37   (3)  41.7810
Voltage source currents
VIN   -5.863 × 10^-2
```

Total power dissipation = 5.86 W
Total time = 1.31.

PROBLEM 19.4

For the circuit shown in Fig. 19.8 (a). Find the current, I, and voltage at node 3.

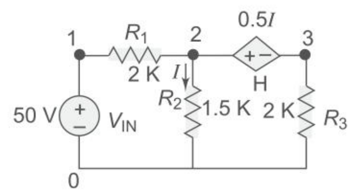


Fig. 19.8 (a)

Solution Since the branch in which the current is to be found, does not have an independent voltage source, assume = 0-volt voltage source with proper polarities as shown in Fig. 19.8 (b).

```
*CURRENT DEPENDENT SOURCE
 VIN    1     0     50
 R1     1     2     2K
 R2     2     20   1.5K
 R3     3     0     2K
 VO     20    0
 H      2     3     VO 0.5
 .OP
 .END
```

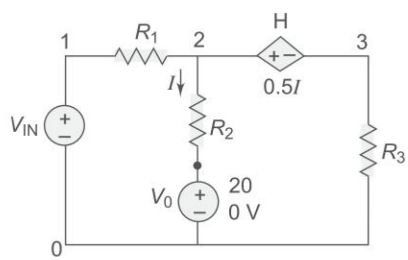


Fig. 19.8 (b)

Simulation Result

```
Small signal bias solution
N    V      N      V      N      V      N      V
(1) 50     (2)    15     (3)    14.997       20     0.0
Voltage source currents
VI  21.75 3 10^-2
VO    1 3 10^-2
Total power dissipation 8.75 3 10^-1
Current-controlled voltage sources
Name          H
V-SOURCE      5 3 10^-3
I-SOURCE      7.498 3 10^-3
Time 5.12.
```

PROBLEM 19.5

Find the magnitude of the current, its real, imaginary components, and its phase with respect to source in the series R-L circuit shown in Fig. 19.9.

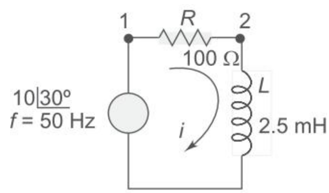


Fig. 19.9

Solution

```
*AC RL series circuit
V_IN      1    0     AC      10     30
R         1    2     100
L         2    0     2.5M
.AC       LIN  1     50      50
.PRINT    AC   IM(R) IR(R)   II(R)  IP(R)
.END
```

Simulation Result

```
Small signal bias solution
In dc bias calculations, all node voltages; source currents and powers
are zero.
AC Analysis
FREQ       IM(R)      IR(R)        II(R)       IP(R)
5.00E 1 011.000E 2 01 8.699E 2 02  4.932E 2 02  2.955E 1 01
Total job time 0.96.
```

PROBLEM 19.6

For the given series RLC circuit shown in Fig. 19.10, find the resonant condition and plot the graphs using PSpice program.
Assume R = 25 Ω; L = 10 mH and C = 100 μF; V = 100 V.

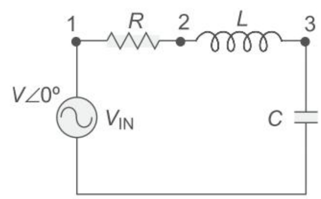


Fig. 19.10

Solution The resonance frequencies

$$f_r = \frac{1}{2\pi\sqrt{10\times10^{-3}\times100\times10^{-6}}}$$

$$= 159\,\text{Hz}$$

It is observed that the resonance frequency is 159 Hz. Hence, to plot a wide range of frequencies in ac sweep, the .AC statement calls for a linear sweep. Let us fix the starting frequency as 5 Hz and stop frequency as 1000 Hz in 100 steps. Now the input file is given by

```
*RLC series resonance
 VIN     1    0       AC      100
 R       1    2       25
 L       2    3       10M
 C       3    0       100u
 .AC     LIN          100     5    1000
 .PROBE
 .PRINT               AC      I(R)
 .END
```

Run the PSpice analysis and see the PROBE screen display. There are many variables that can be displayed. These are all menu-driven and can be easily learned on screen. You can simultaneously display many quantities on the same graph by incorporating .PROBE statement.

In the AC analysis you will find a linear sweep of 100 frequencies starting from 5 Hz to 1000 Hz. The maximum current 3.999 A is observed at 1.558×10^2 Hz.

Total job time is 1.31.

PROBLEM 19.7

For the coupled circuit shown in Fig. 19.11, the coefficient of coupling is 0.5. Use the SPICE program to find currents in L_1, L_2. Take $R_1 = R_2 = 10\ \Omega$; $L_1 = L_2 = 20\,mH$; $C = 5\ \mu F$ and $R_L = 50\ \Omega$.

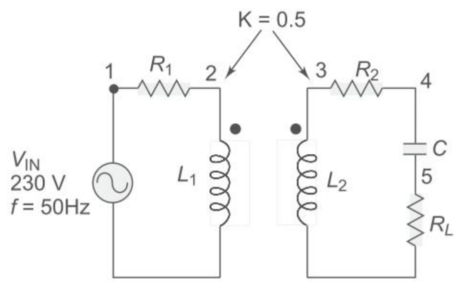


Fig. 19.11

Solution Coupled coils may also be specified in a SPICE input file. The coefficient of coupling is always greater than zero and maximum value is one. If two coils with self-inductances of L_1 and L_2 are mutually coupled with a mutual inductance of M then the coefficient of coupling is given by $K = \dfrac{M}{\sqrt{L_1 L_2}}$

```
*Coupled coils
 VIN   1    0     AC     230
 R1    1    2     10
 R2    3    4     10
 L1    2    0     20M
 L2    3    0     20M
 RL    5    0     50
 C     4    5     5u
 K     L1   L2    0.5
 .AC   LIN  1     50Hz   50Hz
 .PRINT     AC    I(R1)  I(R2)
 .END
```

Simulation Output

```
Small signal DC biasing values will be zero.
```

```
AC analysis
FREQ          I(R1)       I(R2)
5.000E + 01  1.946E + 01  9.655E − 02
Time 1.66.
```

PROBLEM 19.8

For the circuit shown in Fig. 19.12, determine the voltage at nodes 1 and 2, when the switch is closed at time $t = 0$. Use piecewise linear function. Use $R_1 = R_2 = 50\ \Omega$ and $L = 10$ mH.

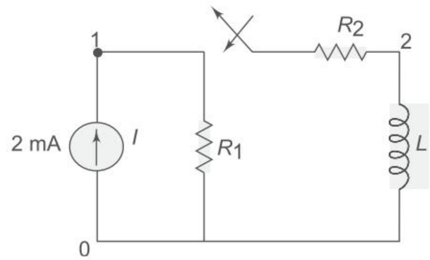


Fig. 19.12

Solution As already mentioned in transient analysis, the .TRAN statement should specify time interval (TSTEP) and length of time. Generally, PSpice employs a variable time interval which is larger when the output is relatively constant, and smaller when the output changes more rapidly. Generally, for a circuit having a time constant of τ, we can use TSTEP as 0.1 τ and a maximum time of 10τ or use the step time as one tenth of the length of the time for analysis.

$$\tau = \frac{L}{R_{eq}} = \frac{10 \times 10^{-3}}{100} = 100\ \mu\ \text{sec}$$

Therefore, let us use TSTEP as 10 μs and max time as 100 μs. Now the input file for the transient analysis is given by

```
*RL TRANSIENT
 I        0    1     PWL (0,0   10μs,   2M   100μs,   2M)
 R1       1    0     50
 R2       1    2     50
 L        2    0     10M
 .TRAN               10μs      100μs
 .PRINT              TRAN      v(1)      v(2)
 .END
```

TRANSIENT ANALYSIS

TIME	V(1)	V(2)
0	0	0
1×10^{-5}	9.758×10^{-2}	9.516×10^{-2}
2×10^{-5}	9.306×10^{-2}	8.611×10^{-2}
3×10^{-5}	8.896×10^{-2}	7.792×10^{-2}
4×10^{-5}	8.525×10^{-2}	6.379×10^{-2}
5×10^{-5}	8.19×1^{-2}	5.772×10^{-2}
6×10^{-5}	7.886×10^{-2}	5.772×10^{-2}
7×10^{-5}	7.611×10^{-2}	5.223×10^{-2}
8×10^{-5}	7.363×10^{-2}	4.726×10^{-2}
9×10^{-5}	7.138×10^{-2}	4.276×10^{-2}
10×10^{-5}	6.935×10^{-2}	3.869×10^{-2}

Time 2.53 seconds

PROBLEM 19.9

For the source- free circuit shown in Fig. 19.13 (a) the initial current in the inductor is 50 mA and the switch is closed at time $t = 0$. Obtain the growth of the current through L. Assume $R = 50$; $L = 5\,H$.

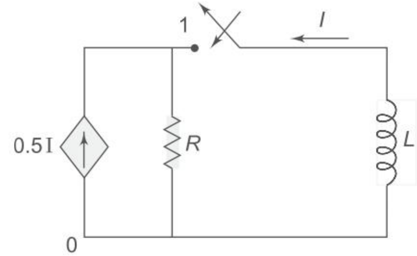


Fig. 19.13 (a)

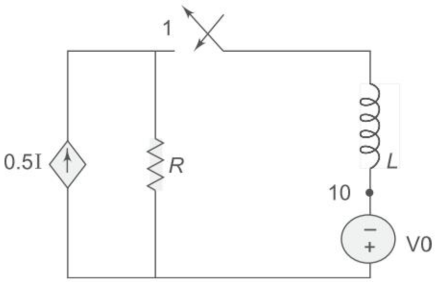


Fig. 19.13 (b)

Solution To find I in L where the source is absent, we assume a zero volt voltage source with proper polarities as shown in Fig. 19.13 (b).

```
*SOURCE    FREE   RL   CIRCUIT
VO         O     10
L          10     1     5   IC = 50M
R           1     0    50
F           0     1    VO  0.5
.TRAN            IM  300M UIC
.PRINT          TRAN  I(L)
.END
```

UIC indicates the use initial condition. The transient analysis results are given up to 10 milliseconds. The current through the inductor will be zero at ≈ 275 ms. By incorporating the .PROBE statement, the variation of *I*(*L*) can be observed in the graphic display.

Time	*I*(*L*)
0.0	5×10^{-2}
1×10^{-3}	4.926×10^{-2}
2×10^{-3}	4.825×10^{-2}
3×10^{-3}	4.78×10^{-2}
4×10^{-3}	4.709×10^{-2}
5×10^{-3}	4.639×10^{-2}
6×10^{-3}	4.570×10^{-2}
7×10^{-3}	4.502×10^{-2}
8×10^{-3}	4.435×10^{-2}
9×10^{-3}	4.369×10^{-2}
10×10^{-3}	4.30×10^{-2}

PROBLEM 19.10

For the circuit shown in Fig. 19.14, observe the voltage at the node 2 and current through R_1 up to 1 s from the instant when the switch is closed at $t = 0$; before the closure of the switch the voltage across the capacitor is 5 mV.

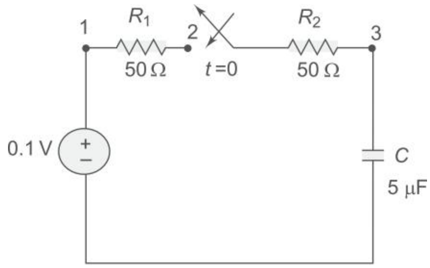


Fig. 19.14

```
*RC TRANSIENT
 VIN    1   0    PWL (0,0   10μ   10M,   1,10M)
 R1     1   2    50
 R2     2   3    50
 C      3   0    5μ     IC = 5M
 .TRAN      10M  I
 .PRINT
 .PRINT     TRAN V(2)   I(R1)
 .END
```

Simulation Result

Initial Transient Solution

N	V	N	V	N	V
(1)	0	(2)	0.0025	(3)	0.0050

VOLTAGE SOURCE CURRENTS

VIN 5×10^{-5}

TRANSIENT ANALYSIS

TIME	V(2)	I(R1)
0	2.5×10^{-3}	-5×10^{-5}
1×10^{-2}	1×10^{-2}	4.476×10^{-7}
2×10^{-2}	1.178×10^{-2}	4.514×10^{-7}
…	…	…
…	…	…
…	…	…
1 sec	9.998×10^{-2}	4.5×10^{-7}

Total job time 2.82. The inclusion of .PROBE statement provides the variation of V2 and *I* with respect to time up to 1 second in the circuit.

Test Your Skill in PSpice

19.1 For the circuit shown in Fig. 19.15, find the voltage across 40 Ω.

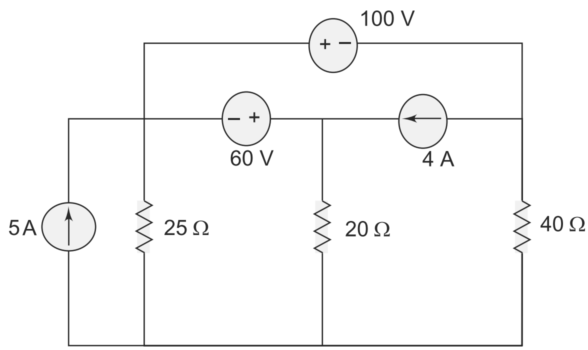


Fig. 19.15

19.2 Write an input file to print the current through 4 V source for the circuit in Fig. 19.16.

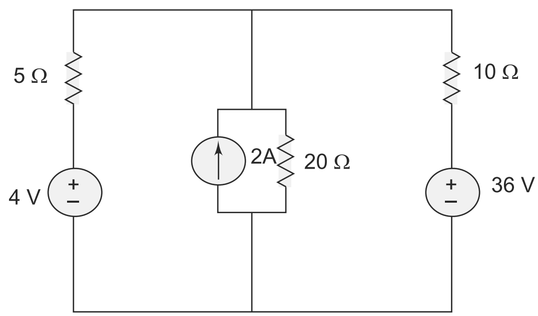


Fig. 19.16

19.3 Find the current in R_2 and total power in the circuit of Fig. 19.17.

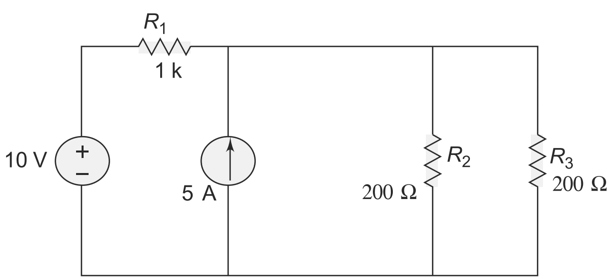


Fig. 19.17

19.4 Find current I and voltage at the node 2 for the circuit in Fig. 19.18.

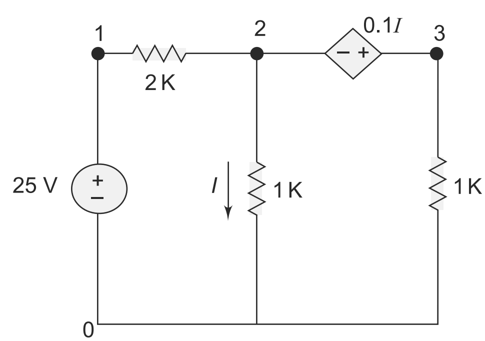


Fig. 19.18

19.5 Find the magnitude of current, its real, imaginary components and its phase angle with respect to the source in the series *RL* circuit shown in Fig. 19.19.

19.6 For the given *RLC* circuit shown in Fig. 19.20, find the resonant condition using PSpice, $R = 20$ ohms; $L = 5$ mH; $C = 80\ \mu$F; $V = 120$ V.

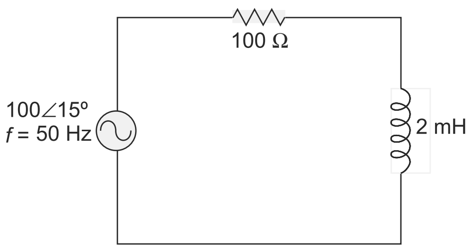


Fig. 19.19

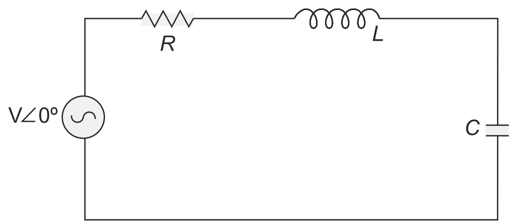


Fig. 19.20

19.7 For the coupled circuit shown in Fig. 19.21, find the primary and secondary currents with the following data $M = 10$ mH; $L_1 = L_2 = 15$ mH; $R_1 = R_2 = 25$ ohms. $C = 10\ \mu$F, $R_L = 40$ ohms.

19.8 For the circuit shown in Fig. 19.22, calculate the growth of voltages at Node 1 and Node 2 when the switch is closed at time $t = 0$; $R_1 = R_2 = 40$ ohms, $L = 5$ mH.

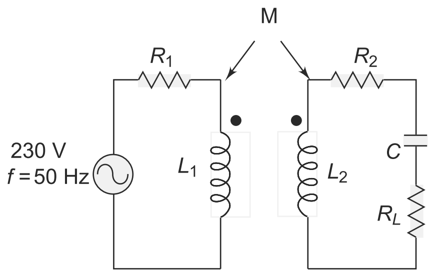


Fig. 19.21

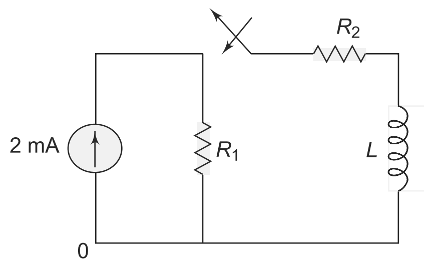


Fig. 19.22

19.9 For the source-free circuit shown in Fig. 19.23, calculate the variation of current through the inductor, when the switch is closed at $t = 0$. Before the closure of the switch, the current through L is 10 mA. Take $R = 60\ \Omega$; $L = 5$ mH.

19.10 For the circuit shown in Fig. 19.24, write an input file in SPICE and run to display the variation of current through the inductor up to five time constants when the switch is closed at $t = 0$, before the closure of the switch the current through the inductor is 40 mA. ($L = 2$ H; $R = 25\ \Omega$).

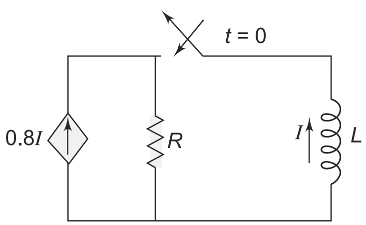


Fig. 19.23

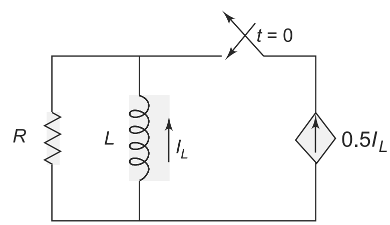


Fig. 19.24

APPENDIX A

Active Filters

A.1 INTRODUCTION

A filter is generally a frequency-selective circuit. Signals having certain frequencies are passed, whereas signals having other frequencies are blocked or attenuated. The bands of frequencies that are passed through the filter are called passbands and those that are blocked in bands are called stopbands. The frequency band between the passband and stopband is the transition band. Filters circuits are classified into two groups as per the type of elements used in the circuit. Filter circuits having passive elements such as resistors, capacitors, and inductors are passive filters. On the other hand, filter circuits employing active elements like transistors or op-amps in addition to resistors and capacitors are active filters. Active filters eliminate use of inductors that are problematic at audio frequencies as the inductors are bulky and expensive. Also, op-amps use in active filters offer high input impedance and low output impedance. This will reduce the loading effect. The active filters have few disadvantages. High frequency response is limited by the gain-bandwidth product and slow rate of the op-amp. The most commonly used active filter circuits are

1. **Low-pass filter** – Passes all frequencies less than the cut-off frequency f_C and attenuates or stops beyond the cut-off frequency f_C.
2. **High-pass filter** – Passes higher frequency signals above the cut-off frequency f_C.
3. **Band-pass filter** – Passes a band of frequencies and blocks other frequencies.
4. **Band-reject filter** – Blocks a band of frequencies and passes other frequencies.

The frequency response of these filter circuits are shown in Fig. A.1 where the dashed curve indicates the ideal response and the solid curve represents the practical filter response.

Active filters are specified by voltage transfer function

$$H(s) = \frac{V_0(s)}{V_i(s)} \tag{1}$$

Under-steady state condition, (i.e., $s = j\omega$),

$$H(j\omega) = |H(j\omega)|\,|\phi(\omega) \tag{2}$$

where $|H(j\omega)|$ is the magnitude or the amplitude function and $\phi(\omega)$ is the phase function.

Filters are classified based on their frequency response and depend on the order of S in the transfer function. These circuits are further classified according to their characteristics.

1. Butterworth filters
2. Chebyshev filters

Butterworth, Chebyshev filters are some of the most commonly used practical filters that approximate the ideal response.

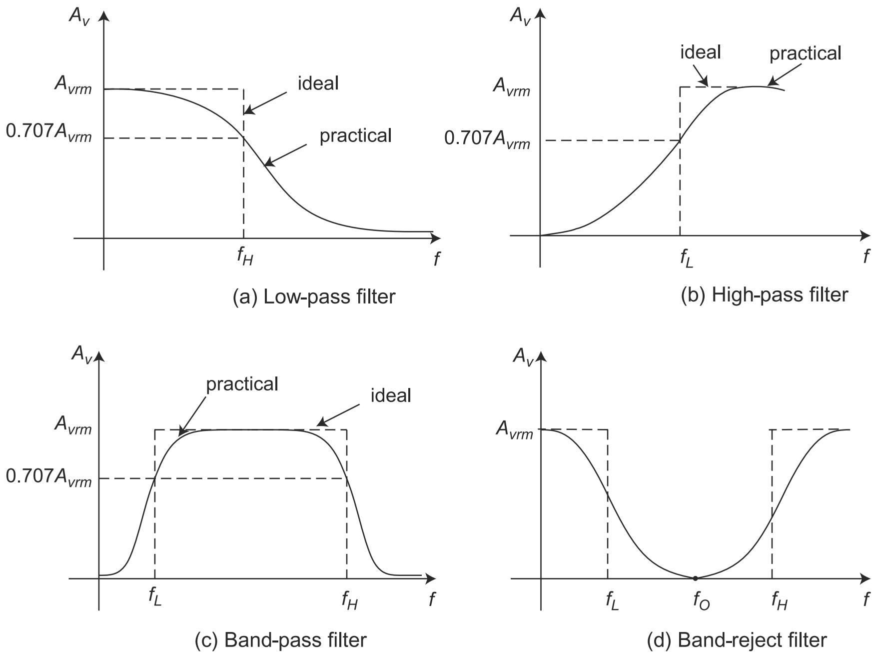


Fig. A.1 *Frequency response of the major active filters*

A.2 FIRST-ORDER LOW-PASS BUTTERWORTH FILTER

A first-order low-pass Butterworth filter is shown in Fig. A.2. It consists of a single *RC* network connected to the non-inverting input terminal of an op-amp.

The voltage V_1 at the non-inventing terminal across the capacitor C is

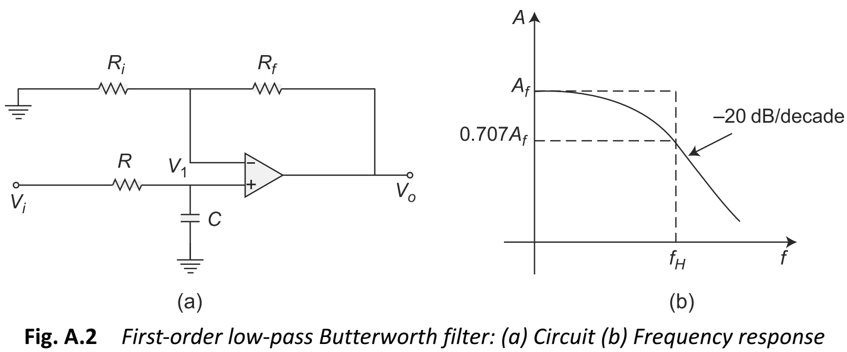


Fig. A.2 *First-order low-pass Butterworth filter: (a) Circuit (b) Frequency response*

$$V_1 = \frac{\dfrac{1}{SC}}{R + \dfrac{1}{SC}} V_i \tag{3}$$

Simplifying Eq. (3), we get

$$\frac{V_1}{V_i} = \frac{1}{RCs + 1} \tag{4}$$

and the output voltage

$$V_0 = \left(1 + \frac{R_f}{R_i}\right) V_1$$

Therefore,

$$V_0 = \left(1 + \frac{R_f}{R_i}\right) \frac{V_i}{1 + RCs}$$

or

$$\frac{V_0}{V_i} = \frac{\left(1 + \frac{R_f}{R_i}\right)}{1 + RCs}$$

$$\frac{V_0}{V_i} = \frac{A_f}{1 + j\omega RC} = \frac{A_f}{1 + j\left(\frac{f}{f_H}\right)} \tag{5}$$

where $\frac{V_0}{V_i} =$ gain of the filter as a function of frequency

$A_f = 1 + \frac{R_f}{R_i} =$ pass-band gain of the filter

$f =$ frequency of the input singal

$f_H = \frac{1}{2\pi RC} =$ high cut-off frequency of the filter

The polar form of Eq. (5)

$$\left|\frac{V_0}{V_i}\right| = \frac{A_f}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}} \tag{6}$$

$$\phi = -\tan^{-1}\left(\frac{f}{f_H}\right)$$

From Eq. (6), at very low frequencies, that is, $f < f_H$, the filter has a constant gain A_f. At $f = f_H$, the gain is $0.707H$, and at $f > f_H$ the gain decreases at a constant rate with an increase in frequency.

The following steps are to be implemented to design a low-pass filter.

1. Choose values of high cut-off frequency f_H and capacitor C. (normally selecting value of C less than $1\mu F$).

2. Calculate the value of R by using the formula

$$R = \frac{1}{2\pi f_H C}$$

3. Desired passband gain A_f can be obtained by selecting values of R_i and R_f.

A.3 SECOND-ORDER LOW-PASS BUTTERWORTH FILTER

A second-order low-pass Butterworth filter is shown in Fig. A.3.

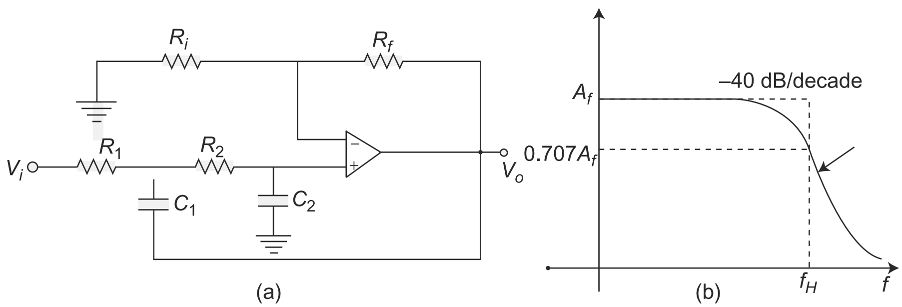


Fig. A.3 *Second-order low-pass Butterworth filter: (a) Circuit (b) Frequency response*

The high cut-off frequency f_H is determined by values R_1, C_1, R_2, and C_2 and is given by

$$f_H = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} \tag{7}$$

The pass-band gain of the filter is

$$A_f = 1 + \frac{R_f}{R_i}$$

The voltage gain magnitude equation is

$$\left|\frac{V_0}{V_i}\right| = \frac{A_f}{\sqrt{1 + \left(\frac{f}{f_H}\right)^4}} \tag{8}$$

A stopband response having a 40 dB/decade roll-off is obtained with the second- order filter. The following steps are implemented to design second order low-pass Butterworth filter.

1. Set $R_1 = R_2 = R$ and $C_1 = C_2 = C$ and choose a value of $C \leq 1\mu F$.
2. Choose a value of higher cut-off frequency f_H.
3. Calculate the value of R by using the equation

$$R = \frac{1}{2\pi f_H C}$$

4. Choose values of R_f and R_i to obtain passband voltage gain A_f.

A.4 FIRST-ORDER HIGH-PASS BUTTERWORTH FILTER

A first-order high-pass Butterworth filter is shown in Fig. A.4 (a) and its frequency response is shown in Fig. A.4 (b). The frequency f_L is the frequency at which the magnitude of the gain is 0.707 times the maximum value in its passband.

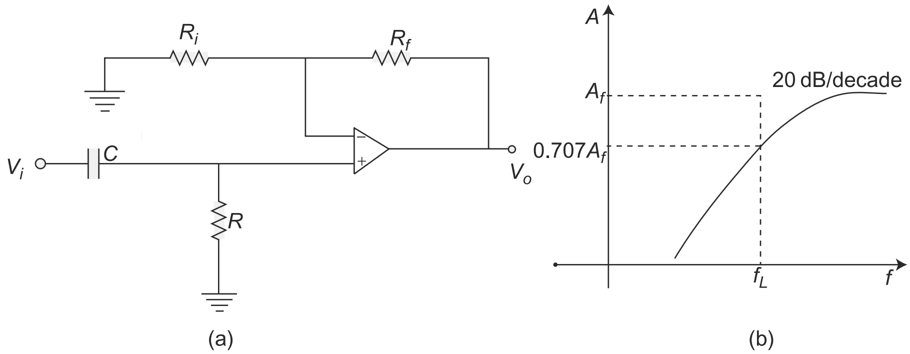


Fig. A.4 *(a) First-order high-pass Butterworth filter (b) Frequency response*

The output voltage $V_0 = \left(1+\frac{R_f}{R_i}\right)\frac{sRC}{1+sRC}V_i$

or

$$\frac{V_0}{V_i} = A_f\left[\frac{j\left(\frac{f}{f_L}\right)}{1+j\left(\frac{f}{f_L}\right)}\right] \tag{9}$$

The magnitude of the voltage gain

$$\left|\frac{V_0}{V_i}\right| = \frac{A_f\left(\frac{f}{f_L}\right)}{\sqrt{1+\left(\frac{f}{f_L}\right)^2}} \tag{10}$$

where $A_f = 1+\frac{R_f}{R_i} =$ passband gain of the filter

$f =$ frequency of the input signal

$f_L = \frac{1}{2\pi RC} =$ lower cut-off frequency

A.5 SECOND-ORDER HIGH-PASS BUTTERWORTH FILTER

A second-order high-pass Butterworth filter is shown in Fig. A.5(a) and its frequency response is shown in Fig. A.5(b). The stopband gain of the second-order filter changes at the rate of 40 dB/decade.

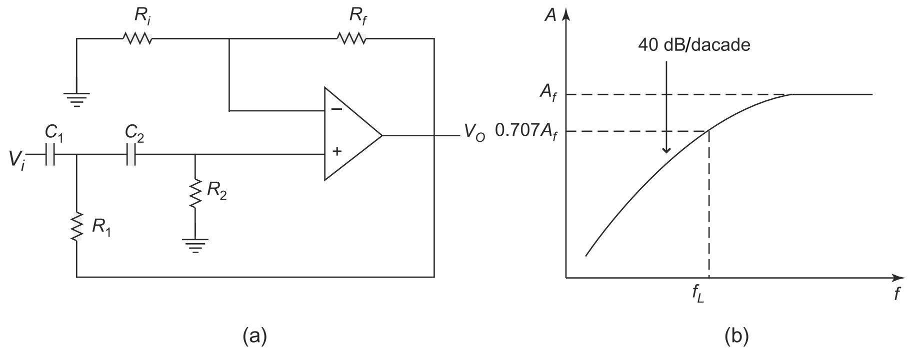


Fig. A.5 *(a) Second-order high-pass Butterworth filter: (b) Frequency response*

The magnitude of the voltage gain of the second-order high-pass filter is

$$\left|\frac{V_0}{V_i}\right| = \frac{A_f}{\sqrt{1+\left(\frac{f_L}{f}\right)^4}} \tag{11}$$

where A_f = passband gain

f = frequency of the input signal

f_L = lower cutoff frequency.

A.6 HIGHER ORDER FILTERS

Higher order filters can be obtained by connecting in series or cascading a proper number of first and second-order filters. For n-th order filter, the roll-off rate will be n × 20 dB/decade.

The transfer function will be given by

$$H(s) = \frac{A_1}{s+1} + \frac{A_2}{s^2+\beta_1 s+1} + \frac{A_3}{s^2+\beta_2 s+1} \tag{12}$$

The transfer function of a low-pass Butterworth filter is

$$H_n(s) = \frac{1}{B_n(s)}$$

where $B_n(s)$ is a Butterworth polynomial.

Table A.1 *The first five Butterworth polynomials*

order m	Butterworth polynomials $B_m(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$

A.7 BANDPASS FILTER

A bandpass filter and its frequency response is shown in Fig. A.6. A bandpass filter has a passband between two cut-off frequencies f_L and f_H such that $f_H > f_L$. The width of passband in a filter depends on its figure of merit or quality factor Q. If $Q < 10$, the filter is called wide bandpass filter on the otherhand, if $Q > 10$, the filter is a narrow band-pass filter.

The quality factor is given by

$$Q = \frac{f_C}{\text{BW}} = \frac{f_C}{f_H - f_L} \tag{13}$$

where $f_C = \sqrt{f_H f_L}$

and f_H = higher cut-off frequency

f_L = lower cut-off frequency

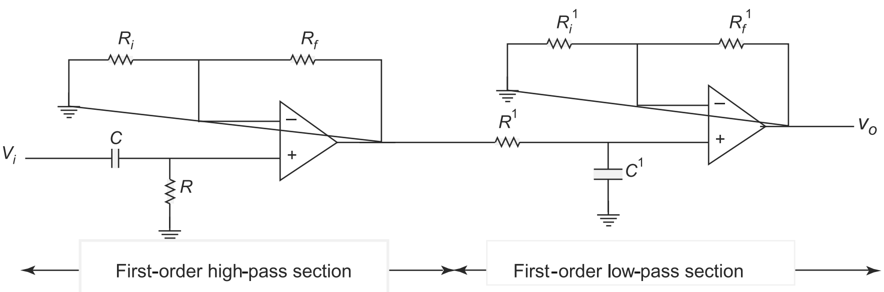


Fig. A.6 *(a) Band-pass filter*

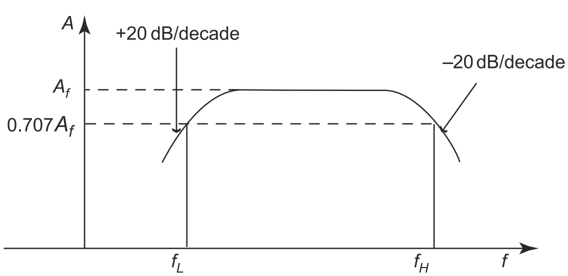


Fig. A.6 *(b) Frequency response*

A bandpass filter can be obtained by cascading high-pass and low-pass sections. The magnitude of the voltage gain for the bandpass filter is equal to the product of the magnitudes of voltage gain of the high-pass and low-pass filters.

Therefore, the voltage gain of the bandpass filter is

$$\left|\frac{V_0}{V_i}\right| = \frac{A_f\left(\dfrac{f}{f_L}\right)}{\sqrt{\left(1+\left(\dfrac{f}{f_L}\right)^2\right)\left(1+\left(\dfrac{f}{f_H}\right)^2\right)}} \tag{14}$$

where A_f = total passband gain
f = frequency of the input signal
f_L = lower cut-off frequency
f_H = higher cut-off frequency

A.8 BAND-REJECT FILTERS

A band-reject filter and its frequency response is shown in Fig. A.7. The band reject filter is also called a bandstop or band-elimination filter. A band-reject filter can be made using a low-pass filter, high-pass filter, and a summer. The lower cut-off frequency f_L of the high-pass filter should be much greater than the higher cut-off frequency f_H of the low pass filter and the passband gain of both the filters should be same.

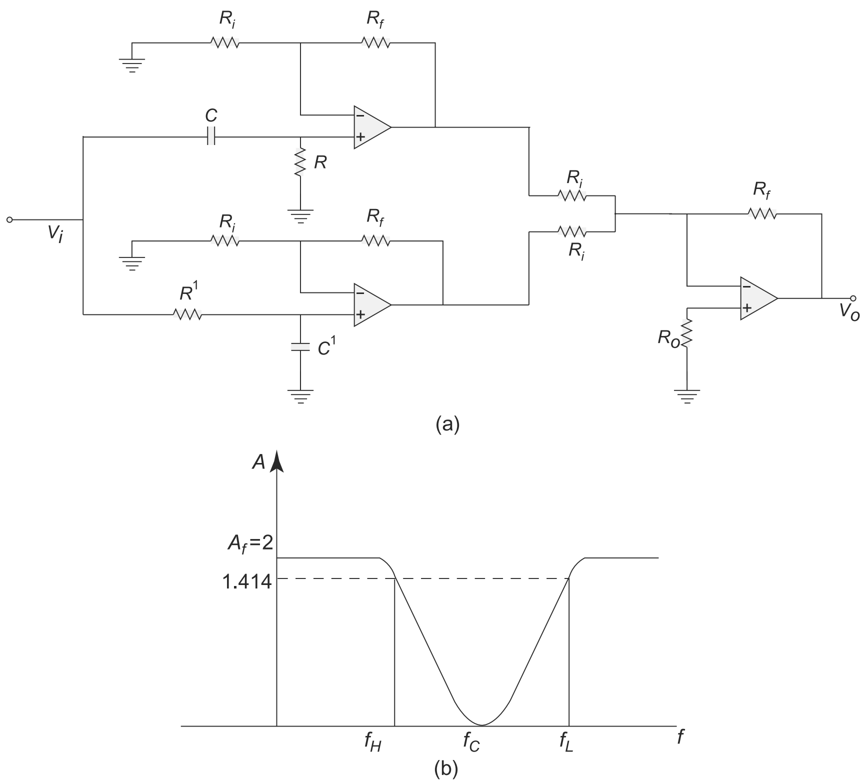


Fig. A.7 *(a) Band-reject filter circuit (b) Frequency response*

The voltage gain changes at the rate of 20 dB/decade above f_H and below f_L, with a maximum attenuation occuring at f_C. $f_C = \sqrt{f_H f_L}$

A.9 CHEBYSHEV FILTERS

The magnitude function of a normalised low-pass Chebyshev filter is characterised by

$$\left|H_m(j\Omega)\right|^2 = \frac{1}{1+\in^2 C_m{}^2(\Omega)} \tag{15}$$

where $C_m(\Omega)$ is a Chebyshev function of order m, which can be written in its trigonometric form as

$$C_m(\Omega) = \begin{cases} \cos(m\cos^{-1}\Omega), 0 \le \Omega \le 1 \\ \cosh(m\cos^{-1}\Omega), \Omega > 1 \end{cases} \tag{16}$$

The Chebyshev polynominals can also be generated and defined from the following recursive formula

$$C_m(\Omega) = 2\Omega C_{m\text{-}1}(\Omega) - C_{m\text{-}2}(\Omega),\ m > 2 \tag{17}$$

with $T_0(\Omega) = 1$ and $T_1(\Omega) = \Omega$

A list of the first five Chebyshev polynominals is given in,Table A. 2 for reference

Table A.2 *The first five Chebyshev polynominals*

m	$T_m(\Omega)$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$

Taking logarithm for the magnitude function, we get

$$20\log|H(j\Omega)| = 10\log 1 - 10\log[1+\in^2 \mathrm{C}_\mathrm{m}^2(\Omega)] \tag{18}$$

Since $C_m(\Omega) = 1$ at $\Omega = 1$, we have attenuation at the passband frequency

$$A_p = A_{dB}(1) = 10\log_{10}[1 + \in^2]$$

and then

$$\in = \sqrt{10^{0.1Ap} - 1}$$

From equations (15) and (16), when $\Omega = \Omega_r$, we find attenuation at the stopband frequency

$$A_\mathrm{r} = A_{\mathrm{dB}}(\Omega_r) = 10\log_{10}[1 + \in^2 \cosh^2(m\cosh^{-1}\Omega_r)]$$

and thus the order of the normalized low-pass Chebyshev filter that satisfies the required stopband attenuation is the smallest integer number that satisfies

$$m \geq \frac{\cos h^{-1}\sqrt{\dfrac{10^{0.1A_r}-1}{\in^2}}}{\cos h^{-1}(\Omega_r)}$$

The magnitude responses for the normalised Chebyshev filter of odd and even n is shown in Fig. A.8.

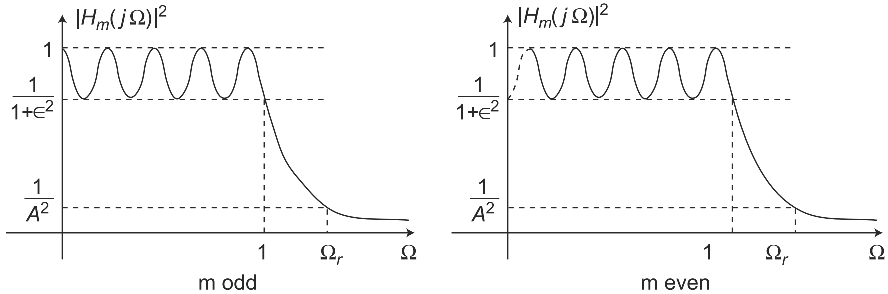


Fig. A.8 *Magnitude response for normalised Chebyshev filter of odd and even m*

It is interesting to note that in a Butterworth filter, the frequency response is monotonic in both passband and stop band and is maximally flat at $\Omega = 0$, where the magnitude response of the Chebyshev filter exhibits ripples in the passband or stopband. Chebyshev filters usually require lower-order transfer functions than Butterworth filters.

Frequently Asked Questions

☆☆★**A-1.1** Explain the advantages of active filters in comparison to passive filters. **[PTU 2011-12]**
The specifications of a LPF are
Pass band ripple = 1 db
Pass band = 0 to 1.75 MHz
Stop band loss = 20 db at 2.5 MHz
Find η and ε.

*For answers to **Frequently Asked Questions**, please visit the link http://highered.mheducation.com/sites/9339219600

APPENDIX B

The j Factor

B.1 DEFINITION OF j FACTOR

j is used in all electrical circuits to denote imaginary numbers. Alternate symbol for j is $\sqrt{-1}$, and is known as j factor or j operator.

Thus

$$\sqrt{-1} = \sqrt{(-1)(1)} = j(1)$$
$$\sqrt{-2} = \sqrt{(-1)2} = j\sqrt{2}$$
$$\sqrt{-4} = \sqrt{(-1)4} = j2$$
$$\sqrt{-5} = \sqrt{(-1)5} = j\sqrt{5}$$

Since j is defined as $\sqrt{-1}$, it follows that $(j)(j) = j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$

$\therefore \quad (j3)(j3) = j^2 3^2$

Since $\quad j^2 = -1$

$$(j3)(j3) = -9$$

(i.e.) the square root of −9 is $j3$.

Therefore, $j3$ is a square root of −9.

The use of j factor provides a solution to an equation of the form $x^2 = -4$

Thus, $\quad x = \sqrt{-4} = \sqrt{(-1)4}$

$$x = (\sqrt{-1})2$$

With $\quad j = \sqrt{-1}, x = j2$

The real number 9 when multiplied three times by j becomes $-j9$.

$$(j)\,(j)\,(j) = (j)^2\, j = (-1)j = -j$$

Finally, when the real number 10 is multiplied four times by j, it becomes 10.

$$j = +j$$
$$j^2 = (j)\,(j) = -1$$
$$j^3 = (j^2)\,(j) = (-1)j = -j$$
$$j^4 = (j^2)\,(j)^2 = (-1)\,(-1) = +1$$

EXAMPLE B.1

Express the following imaginary numbers using the j factor:

(a) $\sqrt{-13}$ (b) $\sqrt{-9}$ (c) $\sqrt{-29}$ (d) $\sqrt{-49}$

Solution

(a) $\sqrt{-13} = \sqrt{(-1)(13)} = j\sqrt{13}$
(b) $\sqrt{-9} = \sqrt{(-1)9} = j3$
(c) $\sqrt{-29} = \sqrt{(-1)29} = j\sqrt{29}$
(d) $\sqrt{-49} = \sqrt{(-1)(49)} = j7$

B.2 RECTANGULAR AND POLAR FORMS

A complex number $(a + jb)$ can be represented by a point whose coordinates are (a, b). Thus, the complex number $3 + j4$ is located on the complex plane at a point having rectangular coordinates (3, 4).

This method of representing complex numbers is known as the rectangular form. In ac analysis, impedances, currents, and voltages are commonly represented by complex numbers that may be either in the rectangular form or in the polar form. In Fig. B.1, the complex number in the polar form is represented. Here, R is the magnitude of the complex number and ϕ is the angle of the complex number. Thus, the polar form of the complex number is $R \angle\phi$. If the rectangular coordinates (a, b) are known, they can be converted into polar form. Similarly, if the polar coordinates (R, ϕ) are known, they can be converted into rectangular form.

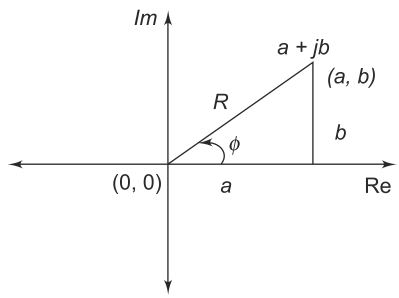


Fig. B.1

In Fig. B.1, a and b are the horizontal and vertical components of the vector R, respectively. From Fig. B.1, R can be found as $R = \sqrt{a^2 + b^2}$.

Also from Fig. B.1,

$$\sin\phi = \frac{b}{R}$$
$$\cos\phi = \frac{a}{R}$$
$$\tan\phi = \frac{b}{a}$$
$$\phi = \tan^{-1}\frac{b}{a}$$
$$R = \sqrt{a^2 + b^2}$$

EXAMPLE B.2

Express 10 ∠53.1° in rectangular form.

Solution $a + jb = R(\cos\phi + j\sin\phi)$

$$R = 10;\ \angle\phi = \angle 53.1^\circ$$
$$a + jb = R\cos\phi + jR\sin\phi$$
$$R\cos\phi = 10\cos 53.1^\circ = 6$$
$$R\sin\phi = 10\sin 53.1^\circ = 8$$
$$a + jb = 6 + j8$$

EXAMPLE B.3

Express 3 + j4 in polar form.

Solution $R\cos\phi = 3$ (1)

$R\sin\phi = 4$ (2)

Squaring and adding the above equations, we get

$$R^2 = 3^2 + 4^2$$
$$R = \sqrt{3^2 + 4^2} = 5$$

From (1) and (2), $\tan\phi = 4/3$

$$\phi = \tan^{-1}\frac{4}{3} = 53.13^\circ$$

Hence, the polar form is $5\angle 53.13^\circ$

B.3 OPERATIONS WITH COMPLEX NUMBERS

The basic operations such as addition, subtraction, multiplication, and division can be performed using complex numbers.

❑ ***Addition*** It is very easy to add two complex numbers in the rectangular form. The real parts of the two complex numbers are added and the imaginary parts of the two complex numbers are added. For example,

$$(3 + j4) + (4 + j5) = (3 + 4) + j(4 + 5)$$
$$= 7 + j9$$

❑ ***Subtraction*** Subtraction can also be performed by using the rectangular form. To subtract, the sign of the subtrehand is changed and the components are added. For example, subtract $5 + j3$ from $10 + j6$:

$$10 + j6 - 5 - j3 = 5 + j3$$

❑ ***Multiplication*** To multiply two complex numbers, it is easy to operate in polar form. Here, we multiply the magnitudes of the two numbers and add the angles algebraically. For example, when we multiply $3\angle 30^\circ$ with $4\angle 20^\circ$, it becomes $(3)(4)\angle 30^\circ + 20^\circ = 12\angle 50^\circ$.

❑ ***Division*** To divide two complex numbers, it is easy to operate in polar form. Here, we divide the magnitudes of the two numbers and subtract the angles. For example, the division of

$$9\angle 50^\circ \text{ by } 3\angle 15^\circ = \frac{9\angle 50^\circ}{3\angle 15^\circ} = 3\underline{|50^\circ - 15^\circ} = 3\underline{|35^\circ}.$$

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