Circuit Theory and Transmission Lines

Second Edition

About the Author

Ravish R Singh is presently Academic Advisor at Thakur Educational Trust, Mumbai. He obtained a BE degree from University of Mumbai in 1991, an MTech degree from IIT Bombay in 2001, and a PhD degree from Faculty of Technology, University of Mumbai, in 2013. He has published several books with McGraw Hill Education (India) on varied subjects like *Engineering Mathematics (I and II), Applied Mathematics, Electrical Engineering, Electrical and Electronics Engineering*, etc., for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Dr A P J Abdul Kalam Technical University. Dr Singh is a member of IEEE, ISTE, and IETE, and has published research papers in national and international journals. His fields of interest include Circuits, Signals and Systems, and Engineering Mathematics.

Circuit Theory and Transmission Lines

Second Edition

Ravish R Singh

Academic Advisor
Thakur Educational Trust
Mumbai, Maharashtra

McGraw Hill Education (India) Private Limited

NEW DELHI

McGraw Hill Education Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal San Juan
Santiago Singapore Sydney Tokyo Toronto

McGraw Hill Education (India) Private Limited

Published by the McGraw Hill Education (India) Private Limited
P-24, Green Park Extension, New Delhi 110 016.

Circuit Theory and Transmission Lines, 2e

Copyright© 2016, 2014, by McGraw Hill Education (India) Private Limited.
No part of this publication may be reproduced or distributed in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise or stored in a database or retrieval system without the prior written permission of the publishers. The program listings (if any) may be entered, stored and executed in a computer system, but they may not be reproduced for publication. This edition can be exported from India only by the publishers, McGraw Hill Education (India) Private Limited.

ISBN (13): 978-93-5260-270-4
ISBN (10): 93-5260-270-6

Managing Director: *Kaushik Bellani*

Director—Products (Higher Education and Professional): *Vibha Mahajan*

Manager—Product Development: *Koyel Ghosh*
Specialist—Product Development: *Piyali Chatterjee*

Head—Production (Higher Education and Professional): *Satinder S Baveja*
Senior Production Executive: *Jagriti Kundu*

AGM—Product Management (Higher Education and Professional): *Shalini Jha*
Manager—Product Management: *Ritwick Dutta*

General Manager—Production: *Rajender P Ghansela*
Manager—Production: *Reji Kumar*

Information contained in this work has been obtained by McGraw Hill Education (India), from sources believed to be reliable. However, neither McGraw Hill Education (India) nor its authors guarantee the accuracy or completeness of any information published herein, and neither McGraw Hill Education (India) nor its authors shall be responsible for any errors, omissions, or damages arising out of use of this information. This work is published with the understanding that McGraw Hill Education (India) and its authors are supplying information but are not attempting to render engineering or other professional services. If such services are required, the assistance of an appropriate professional should be sought.

Typeset at Text-o-Graphics, B-1/56, Aravali Apartment, Sector-34, Noida 201 301, and printed at

Cover Printer:

Visit us at: www.mheducation.co.in

Dedicated to

My Father

Late Shri Ramsagar Singh

and

My Mother

Late Shrimati Premsheela Singh

Contents

Preface *xiii*

1. BASIC NETWORK CONCEPTS 1.1

1.1 Introduction *1.1*
1.2 Resistance *1.1*
1.3 Inductance *1.2*
1.4 Capacitance *1.3*
1.5 Sources *1.4*
1.6 Some Definitions *1.6*
1.7 Series and Parallel Combination of Resistors *1.7*
1.8 Series and Parallel Combination of Inductors *1.9*
1.9 Series and Parallel Combination of Capacitors *1.10*
1.10 Star-Delta Transformation *1.10*
1.11 Source Transformation *1.13*
1.12 Source Shifting *1.19*
Exercises *1.21*
Objective-Type Questions *1.22*
Answers to Objective-Type Questions *1.23*

2. ELEMENTARY NETWORK THEOREMS 2.1

2.1 Introduction *2.1*
2.2 Kirchhoff's Laws *2.1*
2.3 Mesh Analysis *2.2*
2.4 Supermesh Analysis *2.17*
2.5 Node Analysis *2.26*
2.6 Supernode Analysis *2.46*
Exercises *2.52*
Objective-Type Questions *2.56*
Answers to Objective-Type Questions *2.57*

3. NETWORK THEOREMS (APPLICATION TO DC NETWORKS) 3.1

3.1 Introduction *3.1*
3.2 Superposition Theorem *3.1*
3.3 Thevenin's Theorem *3.30*
3.4 Norton's Theorem *3.64*
3.5 Maximum Power Transfer Theorem *3.91*
3.6 Millman's Theorem *3.112*
Exercises *3.117*
Objective-Type Questions *3.122*
Answers to Objective-Type Questions *3.123*

4. COUPLED CIRCUITS 4.1

4.1 Introduction *4.1*
4.2 Self-Inductance *4.1*
4.3 Mutual Inductance *4.2*
4.4 Coefficient of Coupling (k) *4.2*
4.5 Inductances in Series *4.3*
4.6 Inductances in Parallel *4.4*
4.7 Dot Convention *4.9*
4.8 Coupled Circuits *4.15*
4.9 Conductively Coupled Equivalent Circuits *4.37*
4.10 Tuned Circuits *4.40*
Exercises 4.47
Objective-Type Questions 4.49
Answers to Objective-Type Questions 4.50

5. RESONANCE 5.1

5.1 Introduction *5.1*
5.2 Series Resonance *5.1*
5.3 Parallel Resonance *5.18*
5.4 Comparison of Series and Parallel Resonant Circuits *5.21*
Exercises 5.38
Objective-Type Questions 5.39
Answers to Objective-Type Questions 5.40

6. TRANSIENT ANALYSIS 6.1

6.1 Introduction *6.1*
6.2 Initial Conditions *6.1*
6.3 Resistor–Inductor Circuit *6.27*
6.4 Resistor–Capacitor Circuit *6.49*
6.5 Resistor–Inductor–Capacitor Circuit *6.66*
Exercises 6.79
Objective-Type Questions 6.82
Answers to Objective-Type Questions 6.85

7. LAPLACE TRANSFORM AND ITS APPLICATION 7.1

7.1 Introduction *7.1*
7.2 Laplace Transformation *7.1*
7.3 Laplace Transforms of Some Important Functions *7.2*
7.4 Properties of Laplace Transform *7.4*
7.5 Inverse Laplace Transform *7.7*
7.6 The Transformed Circuit *7.12*
7.7 Resistor–Inductor Circuit *7.13*
7.8 Resistor–Capacitor Circuit *7.19*

7.9 Resistor–Inductor–Capacitor Circuit *7.25*
7.10 Response of *RL* Circuit to Various Functions *7.31*
7.11 Response of *RC* Circuit to Various Functions *7.39*
Exercises 7.49
Objective-Type Questions 7.52
Answers to Objective-Type Questions 7.53

8. NETWORK FUNCTIONS 8.1

8.1 Introduction *8.1*
8.2 Driving-Point Functions *8.1*
8.3 Transfer Functions *8.2*
8.4 Analysis of Ladder Networks *8.5*
8.5 Analysis of Non-Ladder Networks *8.15*
8.6 Poles and Zeros of Network Functions *8.20*
8.7 Restrictions on Pole and Zero Locations for Driving-Point Functions [Common Factors in N(*s*) and D(*s*) Cancelled] *8.21*
8.8 Restrictions on Pole and Zero Locations for Transfer Functions [Common Factors in N(*s*) and D(*s*) Cancelled] *8.21*
8.9 Time-Domain Behaviour from the Pole-Zero Plot *8.39*
8.10 Graphical Method for Determination of Residue *8.42*
Exercises 8.50
Objective-Type Questions 8.53
Answers to Objective-Type Questions 8.55

9. NETWORK SYNTHESIS 9.1

9.1 Introduction *9.1*
9.2 Hurwitz Polynomials *9.1*
9.3 Positive Real Functions *9.16*
9.4 Elementary Synthesis Concepts *9.24*
9.5 Realisation of *LC* Functions *9.30*
9.6 Realisation of *RC* Functions *9.47*
9.7 Realisation of *RL* Functions *9.63*
Exercises 9.72
Objective-Type Questions 9.74
Answers to Objective-Type Questions 9.76

10. TWO-PORT NETWORKS 10.1

10.1 Introduction *10.1*
10.2 Open-Circuit Impedance Parameters (Z Parameters) *10.2*
10.3 Short-Circuit Admittance Parameters (Y Parameters) *10.8*
10.4 Transmission Parameters (*ABCD* Parameters) *10.18*
10.5 Inverse Transmission Parameters ($A'B'C'D'$ Parameters) *10.24*
10.6 Hybrid Parameters (*h* Parameters) *10.28*
10.7 Inverse Hybrid Parameters (*g* Parameters) *10.33*

10.8 Inter-relationships between the Parameters *10.37*
10.9 Interconnection of Two-Port Networks *10.63*
10.10 *T*-Network *10.79*
10.11 *Pi* (π)-Network *10.79*
10.12 Lattice Networks *10.84*
10.13 Terminated Two-Port Networks *10.87*
Exercises 10.97
Objective-Type Questions 10.100
Answers to Objective-Type Questions 10.103

11. FILTERS AND ATTENUATORS 11.1

11.1 Introduction *11.1*
11.2 Classification of Filters *11.1*
11.3 T-Network *11.1*
11.4 π Network *11.4*
11.5 Characteristic of Filters *11.6*
11.6 Constant-*k* Low Pass Filter *11.7*
11.7 Constant-*k* High-pass Filter *11.14*
11.8 Band-pass Filter *11.18*
11.9 Band-stop Filter *11.22*
11.10 *m*-Derived Filters *11.25*
11.11 *m*-Derived Low-Pass Filter *11.28*
11.12 *m*-Derived High-Pass Filter *11.31*
11.13 Terminating Half Sections *11.34*
11.14 Composite Filter *11.37*
11.15 Attenuator *11.40*
11.16 Lattice Attenuator *11.41*
11.17 T-Type Attenuator *11.42*
11.18 π-Type Attenuator *11.45*
11.19 Ladder-Type Attenuator *11.47*
Exercises 11.48
Objective-Type Questions 11.49
Answers to Objective-Type Questions 11.50

12. TRANSMISSION LINES 12.1

12.1 Introduction *12.1*
12.2 Types of Transmission Lines *12.1*
12.3 Primary Constants of a Transmission Line *12.2*
12.4 Equivalent Circuit Representation of Transmission Line *12.2*
12.5 Secondary Constants of a Transmission Line *12.3*
12.6 Infinite Line *12.10*
12.7 Terminated Transmission Lines *12.11*
12.8 General Solution of a Transmission Line *12.11*
12.9 Reflection *12.15*

12.10 Reflection Coefficient *12.15*
12.11 Standing Waves *12.16*
12.12 Standing Wave Ratio *12.16*
12.13 Scattering parameters or *S*-parameters *12.18*
12.14 Impedance Smith Chart *12.19*
12.15 Impedance Matching Using Smith Chart *12.36*
Exercises 12.42
Objective-Type Questions 12.42
Answers to Objective-Type Questions 12.43

Solved Question Paper December 2015 (EXC 304) ***SQP.1***
Solved Question Paper December 2015 (ETC 304) ***SQP.1***
Solved Question Paper May 2016 (EXC 304) ***SQP.1***
Solved Question Paper May 2016 (ETC 304) ***SQP.1***

Index ***I.1***

Preface

Overview

Circuit Theory and Transmission Lines is an important subject for third-semester students of Electronics Engineering and Electronics & Telecommunication Engineering. With lucid and brief theory, this textbook provides thorough understanding of the topics of this subject. Following a problem-solving approach and discussing both analysis and synthesis of networks, it offers good coverage of dc circuits, network theorems, two-port networks, network synthesis, and transmission lines.

Generally, numerical problems are expected in university examinations in this subject. The weightage given to problems in examinations is more than 70–80%. Questions from important topics of this subject are part of competitive examinations such as IAS, IES, etc. Hence, numerous solved examples and exercise problems are included in each chapter of this book to help students develop and master problem-solving skills required to ace any examination with confidence. Objective-type questions from various competitive examinations are also included at the end of each chapter for easy revision of core concepts.

Salient Features

- Up-to-date and full coverage of the latest revised syllabus of University of Mumbai
- Covers both analysis and synthesis of networks
- Uses problem-solving approach to explain topics
- Lucid coverage of network theorems, transient analysis, two-port networks, network synthesis
- Separate chapter on transmission lines
- Extensively supported by illustrations
- Solution of 2015 and 2016 University of Mumbai question papers is provided at the end of the book.
- Examination-oriented excellent pedagogy:
  - ✔ *Illustrations: 1500+*
  - ✔ *Solved Examples within chapters 570*
  - ✔ *Unsolved Problems: 185*
  - ✔ *Objective Type Questions: 135*

Chapter Organisation

This text is organised into 12 chapters. Chapter 1 covers basic circuit elements and basic laws comprising of networks. Further, dc network theorems are elucidated in Chapters 2 and 3. Chapters 4 and 5 discuss coupled circuits and resonance, respectively. Chapters 6 and 7 discuss transient analysis in time domain and frequency domain, respectively. Chapters 8 and 9 cover network functions and network synthesis. Chapter 10 elucidates two-port networks. Chapter 11 describes filters and attenuators. Lastly, transmission lines and radio frequency is covered in Chapter 12.

Acknowledgements

My acknowledgements would be incomplete without a mention of the contribution of my family members. I feel indebted to my father and mother for their lifelong inspiration. I also send a heartfelt thanks to my wife Nitu; son Aman; and daughter Aditri, for always motivating and supporting me during the preparation of the

project. I appreciate the support extended by the team at McGraw Hill Education (India), especially Koyel Ghosh, Piyali Chatterjee, Satinder Singh Baveja, Anuj Shrivastava and Jagriti Kundu during the editorial, copyediting and production stages of this book.

I am grateful to the reviewers mentioned below for taking out time to review certain chapters of the book and sharing their valuable suggestions:

Amit Bagade	*Ramrao Adik Institute of Technology, Nerul, Navi Mumbai, Maharashtra*
Gajraj Singh	*Ramrao Adik Institute of Technology, Nerul, Navi Mumbai, Maharashtra*
Reena Sonkusare	*Sardar Patel Institute of Technology, Mumbai, Maharashtra*

Suggestions for improvements will always be welcome.

Ravish R Singh

Publisher's Note

Remember to write to us. We look forward to receiving your feedback, comments, and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Please mention the title and authors' name as the subject. In case you spot piracy of this book, please do let us know.

Roadmap to the Syllabus

(As per latest revised syllabus of University of Mumbai)
This text is useful for Electronics Engineering
Circuit Theory—EXC304

Module 1: Analysis of Electrical Circuits

1.1 Analysis of DC Circuits: Analysis of circuits with and without controlled sources using generalized loop, node matrix, superposition, Thevenin, Norton, Millman theorems

1.2 Analysis of Coupled Circuits: Self and mutual inductances, coefficient of coupling, dot convention, equivalent circuit, solution using loop analysis

1.3 Series and Parallel Resonance Circuits: Selectivity, bandwidth, quality factor

GO TO:
CHAPTER 1. BASIC NETWORK CONCEPTS
CHAPTER 2. ELEMENTARY NETWORK THEOREMS
CHAPTER 3. NETWORK THEOREMS (APPLICATION TO dc NETWORKS)
CHAPTER 4. COUPLED CIRCUITS
CHAPTER 5. RESONANCE

Module 2: Time and Frequency Domain Analysis

2.1 Time Domain Analysis of R-L and R-C Circuits: Forced and natural response, time constant, initial and final values
Solution Using First Order Equation for Standard Input Signals: Transient and steady state time response, solution using universal formula

2.2 Time Domain Analysis of R-L-C Circuits: Forced and natural response, effect of damping
Solution Using Second Order Equation for Standard Input Signals: Transient and steady state time response

2.3 Frequency Domain Analysis of RLC Circuits: S-domain representation, applications of Laplace transform in solving electrical networks, driving point and transfer function, poles and zeros, calculation of residues by analytical and graphical method, frequency response

GO TO:
CHAPTER 6. TRANSIENT ANALYSIS
CHAPTER 7. LAPLACE TRANSFORM AND ITS APPLICATION
CHAPTER 8. NETWORK FUNCTIONS

Module 3: Synthesis of RLC Circuits

3.1 Positive Real Functions: Concept of positive real function, testing for Hurwitz polynomials, testing for necessary and sufficient conditions for positive real functions

3.2 Synthesis of RC, RL, LC Circuits: Concepts of synthesis of RC, RL, LC driving point functions

GO TO:

CHAPTER 9. NETWORK SYNTHESIS

Module 4: Two-Port Networks

4.1 Parameters: Open circuit, short circuit, transmission and hybrid parameters, relationships among parameters, reciprocity and symmetry conditions

4.2 Series/Parallel Connection: T and Pi representations, interconnection of two-port networks

GO TO:

CHAPTER 10. TWO-PORT NETWORKS

Module 5: Filters and Attenuators

5.1 Basic Filter Circuits: Low pass, high pass, band pass and band stop filters, transfer function, frequency response, cutoff frequency, bandwidth, quality factor, attenuation constant, phase shift, characteristic impedance

5.2 Concept of Design and Analysis of Filters: Constant K, M derived and composite filters

5.3 Attenuators: Basic concepts, classification, attenuation in dB, K factor (impedance factor) and design concepts

GO TO:

CHAPTER 11. FILTERS AND ATTENUATORS

Module 6: Transmission Lines

6.1 Power Frequency Lines: Representation, losses and efficiency in power lines, effect of length, calculation of inductance and capacitance

6.2 Radio Frequency Lines: Representation, propagation constant, attenuation constant, phase constant, group velocity, input impedance, characteristic impedance, reflection coefficient, standing wave ratio, VSWR, ISWR, S-parameters

6.3 Smith Chart: Impedance locus diagram, impedance matching

GO TO:

CHAPTER 12. TRANSMISSION LINES

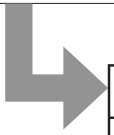
This text is useful for Electronics and Telecommunication Engineering Circuits and Transmission Lines—ETC304

Module 1: Electrical Circuit Analysis

1.1 Analysis of DC Circuits: Analysis of circuits with and without controlled sources using generalized loop and node matrix methods and source transformation, superposition, Thevenin, Norton, Millman theorems

1.2 Magnetic Circuits: Self and mutual inductances, coefficient of coupling, dot convention, equivalent circuit, solution using loop analysis

1.3 Tuned Coupled Circuits: Analysis of tuned coupled circuits



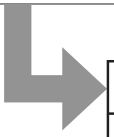
GO TO:
CHAPTER 1. BASIC NETWORK CONCEPTS
CHAPTER 2. ELEMENTARY NETWORK THEOREMS
CHAPTER 3. NETWORK THEOREMS (APPLICATION TO dc NETWORKS)
CHAPTER 4. COUPLED CIRCUITS

Module 2: Time and Frequency Domain Analysis

2.1 Time Domain Analysis of R-L and R-C Circuits: Forced and natural response, time constant, initial and final values
Solution Using First Order Equation for Standard Input Signals: Transient and steady state time response, solution using universal formula

2.2 Time Domain Analysis of R-L-C Circuits: Forced and natural response, effect of damping
Solution Using Second Order Equation for Standard Input Signals: Transient and steady state time response

2.3 Frequency Domain Analysis of RLC Circuits: S-domain representation, applications of Laplace transform in solving electrical networks, driving point and transfer function, poles and zeros, calculation of residues by analytical and graphical method, analysis of ladder and lattice network
Response to Standard Signals: Transient and steady state time response of R-L-C circuits



GO TO:
CHAPTER 6. TRANSIENT ANALYSIS
CHAPTER 7. LAPLACE TRANSFORM AND ITS APPLICATION
CHAPTER 8. NETWORK FUNCTIONS

Module 3: Synthesis of RLC circuits

3.1 Positive Real Functions: Concept of positive real function, testing for Hurwitz polynomials, testing for necessary and sufficient conditions for positive real functions

3.2 Synthesis of RC, RL, LC and RLC Circuits: Properties and synthesis of RC, RL, LC driving point functions

GO TO:

CHAPTER 9. NETWORK SYNTHESIS

Module 4: Two-Port Circuits

4.1 Parameters: Open circuits, short circuit, transmission and hybrid parameters, relationship among parameters, reciprocity and symmetry conditions

4.2 Interconnections of two-port circuits, T & π representation

4.3 Terminated two-port circuits

GO TO:

CHAPTER 10. TWO-PORT NETWORKS

Module 5: Radio Frequency Transmission Lines

5.1 Transmission Line Representation: T and π representations, terminated transmission line, infinite line

5.2 Parameters of Radio Frequency Lines: Propagation constant, attenuation constant, phase constant, group velocity, input impedance, characteristic impedance, reflection coefficient, standing wave ratio, VSWR, ISWR, S-parameters

5.3 Smith Chart: Impedance locus diagram, impedance matching

GO TO:

CHAPTER 12. TRANSMISSION LINES

1

Basic Network Concepts

1.1 INTRODUCTION

We know that like charges repel each other whereas unlike charges attract each other. To overcome this force of attraction, a certain amount of work or energy is required. When the charges are separated, it is said that a potential difference exists and the work or energy per unit charge utilised in this process is known as voltage or potential difference.

The phenomenon of transfer of charge from one point to another is termed current. Current (I) is defined as the rate of flow of electrons in a conductor. It is measured by the number of electrons that flow in unit time.

Energy is the total work done in the electric circuit. The rate at which the work is done in an electric circuit is called electric power. Energy is measured in joules (J) and power in watts (W).

1.2 RESISTANCE

Resistance is the property of a material due to which it opposes the flow of electric current through it.

Certain materials offer very little opposition to the flow of electric current and are called conductors, e.g., metals, acids and salt solutions. Certain materials offer very high resistance to the flow of electric current and are called insulators, e.g., mica, glass, rubber, Bakelite, etc.

The practical unit of resistance is **ohm** and is represented by the symbol Ω. A conductor is said to have resistance of one ohm if a potential difference of one volt across its terminals causes a current of one ampere to flow through it.

The resistance of a conductor depends on the following factors.

(i) It is directly proportional to its length.
(ii) It is inversely proportional to the area of cross section of the conductor.
(iii) It depends on the nature of the material.
(iv) It also depends on the temperature of the conductor.

Hence,

$$R \propto \frac{l}{A}$$

$$R = \rho \frac{l}{A}$$

where l is length of the conductor, A is the cross-sectional area and ρ is a constant known as specific resistance or resistivity of the material.

1. Power Dissipated in a Resistor We know that $v = R\,i$

When current flows through any resistor, power is absorbed by the resistor which is given by

$$p = v\,i$$

The power dissipated in the resistor is converted to heat which is given by

$$E = \int_0^t v\,i\,dt = \int_0^t R\,i\,i\,dt = i^2\,Rt$$

1.3 INDUCTANCE

Inductance is the property of a coil that opposes any change in the amount of current flowing through it. If the current in the coil is increasing, the self-induced emf is set up in such a direction so as to oppose the rise of current. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage.

Inductance is defined as the ratio of flux linkage to the current flowing through the coil. The practical unit of inductance is **henry** and is represented by the symbol H. A coil is said to have an inductance of one henry if a current of one ampere when flowing through it produces flux linkages of one weber-turn in it.

The inductance of an inductor depends on the following factors.

(i) It is directly proportional to the square of the number of turns.
(ii) It is directly proportional to the area of cross section.
(iii) It is inversely proportional to the length.
(iv) It depends on the absolute permeability of the magnetic material.

Hence,

$$L \propto \frac{N^2 A}{l}$$

$$L = \mu \frac{N^2 A}{l}$$

where l is the mean length, A is the cross-sectional area and μ is the absolute permeability of the magnetic material.

1. Current–Voltage Relationships in an Inductor We know that

$$v = L\frac{di}{dt}$$

Expressing inductor current as a function of voltage,

$$di = \frac{1}{L} v\,dt$$

Integrating both the sides,

$$\int_{i(0)}^{i(t)} di = \frac{l}{L}\int_0^t v\,dt$$

$$i(t) = \frac{1}{L}\int_0^t v\,dt + i(0)$$

The quantity $i(0)$ denotes the initial current through the inductor. When there is no initial current through the inductor,

$$i(t) = \frac{1}{L}\int_0^t v\, dt$$

2. *Energy Stored in an Inductor* Consider a coil of inductance L carrying a changing current I. When the current is changed from zero to a maximum value I, every change is opposed by the self-induced emf produced. To overcome this opposition, some energy is needed and this energy is stored in the magnetic field. The voltage v is given by

$$v = L\frac{di}{dt}$$

Energy supplied to the inductor during interval dt is given by

$$dE = v\, i\, dt = L\frac{di}{dt} i\, dt = L\, i\, dt$$

Hence, total energy supplied to the inductor when current is increased from 0 to I amperes is

$$E = \int_0^I dE = \int_0^I L\, i\, di = \frac{1}{2} L I^2$$

1.4 CAPACITANCE

Capacitance is the property of a capacitor to store an electric charge when its plates are at different potentials. If Q coulombs of charge is given to one of the plates of a capacitor and if a potential difference of V volts is applied between the two plates then its capacitance is given by

$$C = \frac{Q}{V}$$

The practical unit of capacitance is **farad** and is represented by the symbol F. A capacitor is said to have capacitance of one farad if a charge of one coulomb is required to establish a potential difference of one volt between its plates.

The capacitance of a capacitor depends on the following factors.

(i) It is directly proportional to the area of the plates.
(ii) It is inversely proportional to the distance between two plates.
(iii) It depends on the absolute permittivity of the medium between the plates.

Hence,

$$C \propto \frac{A}{d}$$

$$C = \varepsilon\frac{A}{d}$$

where d is the distance between two plates, A is the cross-sectional area of the plates and ε is absolute permittivity of the medium between the plates.

1. Current–Voltage Relationships in a Capacitor The charge on a capacitor is given by

$$q = Cv$$

where q denotes the charge and v is the potential difference across the plates at any instant.
We know that

$$i = \frac{dq}{dt} = \frac{d}{dt}Cv = C\frac{dv}{dt}$$

Expressing capacitor voltage as a function of current,

$$dv = \frac{1}{C}i\,dt$$

Integrating both the sides,

$$\int_{v(0)}^{v(t)} dv = \frac{1}{C}\int_0^t i\,dt$$

$$v(t) = \frac{1}{C}\int_0^t i\,dt + v(0)$$

The quantity v (0) denotes the initial voltage across the capacitor. When there is no initial voltage on the capacitor,

$$v(t) = \frac{1}{C}\int_0^t i\,dt$$

2. Energy Stored in a Capacitor Let a capacitor of capacitance C farads be charged from a source of V volts. Then current i is given by

$$i = C\frac{dv}{dt}$$

Energy supplied to the capacitor during interval dt is given by

$$dE = v\,i\,dt = v\,C\frac{dv}{dt}dt$$

Hence, total energy supplied to the capacitor when potential difference is increased from 0 to V volts is

$$E = \int_0^V dE = \int_0^V C\,v\,dv = \frac{1}{2}CV^2$$

1.5 SOURCES

Source is a basic network element which supplies energy to the networks. There are two classes of sources, namely,

1. Independent sources
2. Dependent sources

1.5.1 Independent Sources

Output characteristics of an independent source are not dependent on any network variable such as a current or voltage. Its characteristics, however, may be time-varying. There are two types of independent sources:

1. Independent voltage source
2. Independent current source

1. ***Independent Voltage Source*** An independent voltage source is a two-terminal network element that establishes a specified voltage across its terminals. The value of this voltage at any instant is independent of the value or direction of the current that flows through it. The symbols for such voltage sources are shown in Fig. 1.1.

   The terminal voltage may be a constant, or it may be some specified function of time.

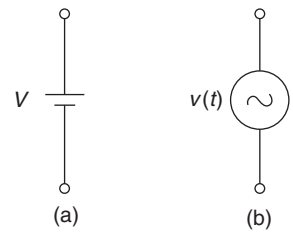


Fig. 1.1 *Symbols for independent voltage source*

2. ***Independent Current Source*** An independent current source is a two-terminal network element which produces a specified current. The value and direction of this current at any instant is independent of the value or direction of the voltage that appears across the terminals of the source. The symbols for such current sources are shown in Fig. 1.2.

   The output current may be a constant or it may be a function of time.

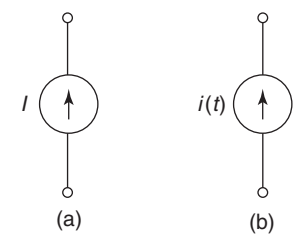


Fig. 1.2 *Symbols for independent current source*

1.5.2 Dependent Sources

If the voltage or current of a source depends in turn upon some other voltage or current, it is called as dependent or controlled source. The dependent sources are of four kinds, depending on whether the control variable is voltage or current and the controlled source is a voltage source or current source.

1. ***Voltage-Controlled Voltage Source (VCVS)*** A voltage-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to a voltage v_{ab} between two points a and b.

   The symbol for such a source is shown in Fig. 1.3.

   The (+) and (−) sign inside the diamond of the component symbol identifies the component as a voltage source.

$$v_{cd} = \mu\, v_{ab}$$

   The voltage v_{cd} depends upon the control voltage v_{ab} and the constant μ, a dimensionless constant called voltage gain.

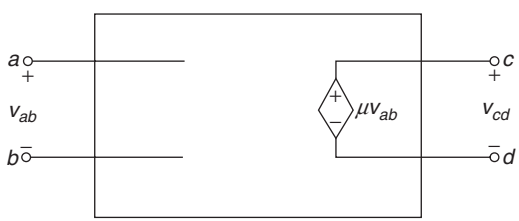


Fig. 1.3 *Symbol for VCVS*

2. ***Voltage-Controlled Current Source (VCCS)*** A voltage-controlled current source is a four-terminal network component that establishes a current i_{cd} in a branch of the circuit that is proportional to the voltage v_{ab} between two points a and b.

   The symbol for such a source is shown in Fig. 1.4.

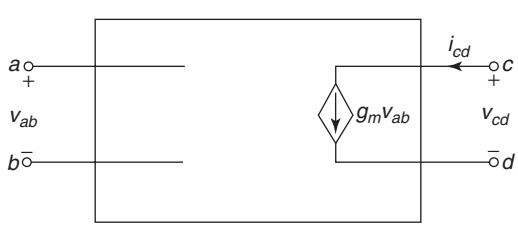


Fig. 1.4 *Symbol for VCCS*

The arrow inside the diamond of the component symbol identifies the component as a current source.

$$i_{cd} = g_m v_{ab}$$

The current i_{cd} depends only on the control voltage v_{ab} and the constant g_m, called the transconductance or mutual conductance. The constant g_m has dimension of ampere per volt or siemens (S).

3. ***Current-Controlled Voltage Source (CCVS)*** A current-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to the current i_{ab} in some branch of the circuit.

The symbol for such a source is shown in Fig. 1.5.

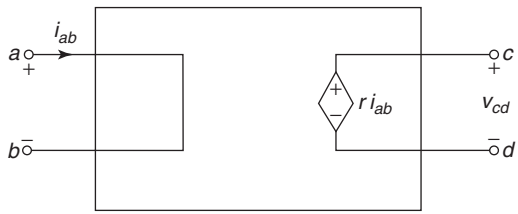


Fig. 1.5 *Symbol for CCVS*

$$v_{cd} = r\, i_{ab}$$

The voltage v_{cd} depends only on the control current i_{ab} and the constant r called the transresistance or mutual resistance. The constant r has dimension of volt per ampere or ohm (Ω).

4. ***Current-Controlled Current Source (CCCS)*** A current-controlled current source is a four-terminal network component that establishes a current i_{cd} in one branch of a circuit that is proportional to the current i_{ab} in some branch of the network.

The symbol for such a source is shown in Fig. 1.6.

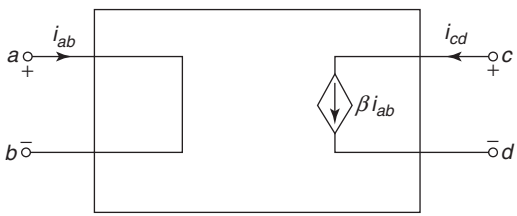


Fig. 1.6 *Symbol for CCCS*

$$i_{cd} = \beta\, i_{ab}$$

The current i_{cd} depends only on the control current i_{ab} and the dimensionless constant β, called the current gain.

1.6 SOME DEFINITIONS

1. ***Network and Circuit*** The interconnection of two or more circuit elements (viz., voltage sources, resistors, inductors and capacitors) is called an *electric network*. If the network contains at least one closed path, it is called an electric circuit. Every circuit is a network, but all networks are not circuits. Figure 1.7(a) shows a network which is not a circuit and Fig. 1.7(b) shows a network which is a circuit.

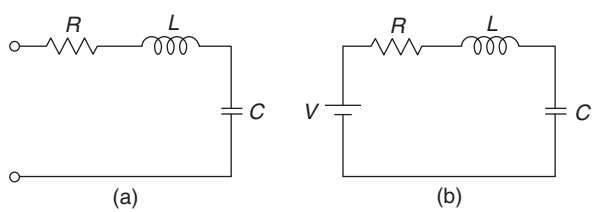


Fig. 1.7 *(a) Network which is not a circuit*
(b) Network which is a circuit

2. ***Linear and Non-linear Elements*** If the resistance, inductance or capacitance offered by an element does not change linearly with the change in applied voltage or circuit current, the element is termed as *linear element*. Such an element shows a linear relation between voltage and current as shown in Fig. 1.8. Ordinary resistors, capacitors and inductors are examples of linear elements.

A non-linear circuit element is one in which the current does not change linearly with the change in applied voltage. A semiconductor diode operating in the curved region of characteristics as shown in Fig. 1.8 is common example of non-linear element.

Other examples of non-linear elements are voltage-dependent resistor (VDR), voltage-dependent capacitor (varactor), temperature-dependent resistor (thermistor), light-dependent resistor (LDR), etc. Linear elements obey Ohm's law whereas non-linear elements do not obey Ohm's law.

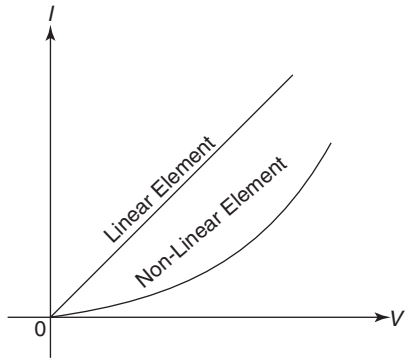


Fig. 1.8 *V-I characteristics of linear and non-linear elements*

3. ***Active and Passive Elements*** An element which is a source of electrical signal or which is capable of increasing the level of signal energy is termed as *active element*. Batteries, BJTs, FETs or OP-AMPs are treated as active elements because these can be used for the amplification or generation of signals. All other circuit elements, such as resistors, capacitors, inductors, VDR, LDR, thermistors, etc., are termed *passive elements*. The behaviour of active elements cannot be described by Ohm's law.

4. ***Unilateral and Bilateral Elements*** If the magnitude of current flowing through a circuit element is affected when the polarity of the applied voltage is changed, the element is termed *unilateral element*. Consider the example of a semiconductor diode. Current flows through the diode only in one direction. Hence, it is called an unilateral element. Next, consider the example of a resistor. When the voltage is applied, current starts to flow. If we change the polarity of the applied voltage, the direction of the current is changed but its magnitude is not affected. Such an element is called a *bilateral element*.

5. ***Lumped and Distributed Elements*** A lumped element is the element which is separated physically, like resistors, inductors and capacitors. Distributed elements are those which are not separable for analysis purposes. Examples of distributed elements are transmission lines in which the resistance, inductance and capacitance are distributed along its length.

6. ***Active and Passive Networks*** A network which contains at least one active element such as an independent voltage or current source is an active network. A network which does not contain any active element is a passive network.

7. ***Time-invariant and Time-variant Networks*** A network is said to be time-invariant or fixed if its input–output relationship does not change with time. In other words, a network is said to time-invariant, if for any time shift in input, an identical time-shift occurs for output. In time-variant networks, the input–output relationship changes with time.

1.7 SERIES AND PARALLEL COMBINATIONS OF RESISTORS

Let R_1, R_2 and R_3 be the resistances of three resistors connected in series across a dc voltage source V as shown in Fig. 1.9. Let V_1, V_2 and V_3 be the voltages across resistances R_1, R_2 and R_3 respectively.

In series combination, the same current flows through each resistor but voltage across each resistor is different.

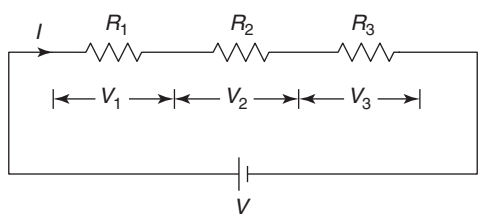


Fig. 1.9 *Series combination of resistors*

$$V = V_1 + V_2 + V_3$$
$$R_T I = R_1 I + R_2 I + R_3 I$$
$$R_T = R_1 + R_2 + R_3$$

Hence, when a number of resistors are connected in series, the equivalent resistance is the sum of all the individual resistance.

1. Voltage Division and Power in a Series Circuit

$$I = \frac{V}{R_1 + R_2 + R_3}$$
$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2 + R_3} V$$
$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2 + R_3} V$$
$$V_3 = R_3 I = \frac{R_3}{R_1 + R_2 + R_3} V$$

Total power
$$P_T = P_1 + P_2 + P_3$$
$$= I^2 R_1 + I^2 R_2 + I^2 R_3$$
$$= \frac{V_1^2}{R_1} + \frac{V_2^2}{R_2} + \frac{V_3^2}{R_3}$$

Figure 1.10 shows three resistors connected in parallel across a dc voltage source V. Let I_1, I_2 and I_3 be the current flowing through resistors R_1, R_2 and R_3 respectively.

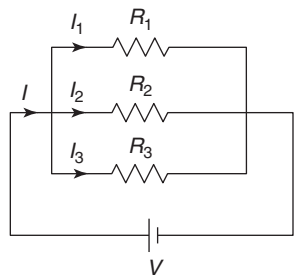


Fig. 1.10 *Parallel combination of resistors*

In parallel combination, the voltage across each resistor is same but current through each resistor is different.

$$I = I_1 + I_2 + I_3$$
$$\frac{V}{R_T} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
$$R_T = \frac{R_1 R_2 R_3}{R_2 R_3 + R_3 R_1 + R_1 R_2}$$

Hence, when a number of resistors are connected in parallel, the reciprocal of the equivalent resistance is equal to the sum of reciprocals of individual resistances.

2. Current Division and Power in a Parallel Circuit

$$V = R_T I = R_1 I_1 = R_2 I_2 = R_3 I_3$$
$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$
$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$
$$I_3 = \frac{V}{R_3} = \frac{R_T I}{R_3} = \frac{R_1 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$

Total power

$$P_T = P_1 + P_2 + P_3$$
$$= I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$$
$$= \frac{V^2}{R_1} + \frac{V^2}{R_2} + \frac{V^3}{R_3}$$

Note: For two branch circuits,

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$
$$V = R_T I = R_1 I_1 = R_2 I_2$$
$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2}{R_1 + R_2} I$$
$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

1.8 SERIES AND PARALLEL COMBINATION OF INDUCTORS

Let L_1, L_2 and L_3 be the inductances of three inductors connected in series across an ac voltage source v as shown in Fig. 1.11. Let v_1, v_2 and v_3 be the voltages across inductances L_1, L_2 and L_3 respectively.

In series combination, the same current flows through each inductor but the voltage across each inductor is different.

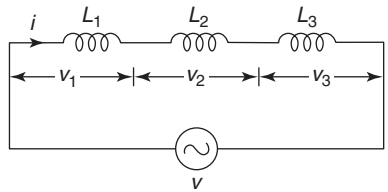


Fig. 1.11 *Series connection of inductors*

$$v = v_1 + v_1 + v_3$$
$$L_T \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$
$$L_T = L_1 + L_2 + L_3$$

Hence, when a number of inductors are connected in series, the equivalent inductance is the sum of all the individual inductances.

Figure 1.12 shows three inductors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each inductance L_1, L_2 and L_3 respectively.

In parallel combination, the voltage across each inductor is same but the current through each inductor is different.

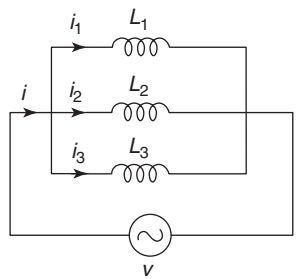


Fig. 1.12 *Parallel connection of inductors*

$$i = i_1 + i_2 + i_3$$

$$\frac{1}{L_T} \int v\,dt = \frac{1}{L_1} \int v\,dt + \frac{1}{L_2} \int v\,dt + \frac{1}{L_3} \int v\,dt$$

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

$$L_T = \frac{L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}$$

Hence, when a number of inductors are connected in parallel, the reciprocal of the equivalent inductance is equal to the sum of reciprocals of individual inductances.

1.9 SERIES AND PARALLEL COMBINATION OF CAPACITORS

Let C_1, C_2 and C_3 be the capacitances of three capacitors connected in series across an ac voltage source v as shown in Fig 1.13. Let v_1, v_2 and v_3 be the voltages across capacitances C_1, C_2 and C_3 respectively.

In series combination, the charge on each capacitor is same but voltage across each capacitor is different.

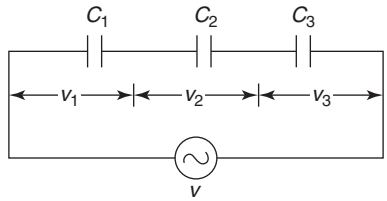


Fig. 1.13 *Series combination of capacitors*

$$v = v_1 + v_2 + v_3$$

$$\frac{1}{C_T}\int i\,dt = \frac{1}{C_1}\int i\,dt + \frac{1}{C_2}\int i\,dt + \frac{1}{C_3}\int i\,dt$$

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_T = \frac{C_1 C_2 C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1}$$

Hence, when a number of capacitors are connected in series, the reciprocal of the equivalent capacitance is equal to the sum of reciprocals of individual capacitances.

1. Voltage Division in a Series Circuit

$$Q = C_T V = C_1 V_1 = C_2 V_2 = C_3 V_3$$

$$V_1 = \frac{Q}{C_1} = \frac{C_T V}{C_1} = \frac{C_2 C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

$$V_2 = \frac{Q}{C_2} = \frac{C_T V}{C_2} = \frac{C_1 C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

$$V_3 = \frac{Q}{C_3} = \frac{C_T V}{C_3} = \frac{C_1 C_2}{C_1 C_2 + C_2 C_3 + C_3 C_1} V$$

Figure 1.14 shows three capacitors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each capacitance C_1, C_2 and C_3 respectively.

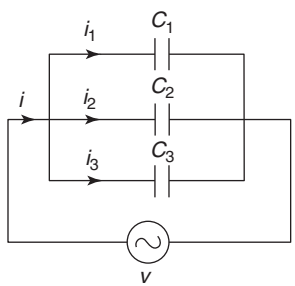


Fig. 1.14 *Parallel combination of capacitors*

In parallel combination, the voltage across each capacitor is same but current through each capacitor is different.

$$i = i_1 + i_2 + i_3$$

$$C_T \frac{dv}{dt} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

$$C_T = C_1 + C_2 + C_3$$

Hence, when a number of capacitors are connected in parallel, the equivalent capacitance is the sum of all the individual capacitance.

1.10 STAR-DELTA TRANSFORMATION

When a circuit cannot be simplified by normal series–parallel reduction technique, the star-delta transformation can be used.

Figure 1.15 (a) shows three resistors R_A, R_B and R_C connected in delta.

Figure 1.15 (b) shows three resistors R_1, R_2 and R_3 connected in star.

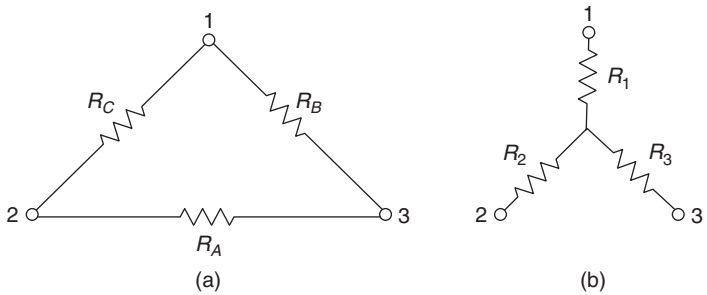


Fig. 1.15 *(a) Delta network (b) Star network*

These two networks will be electrically equivalent if the resistance as measured between any pair of terminals is the same in both the arrangements.

1.10.1 Delta to Star Transformation

Referring to delta network shown in Fig. 1.15 (a), the resistance between terminals 1 and 2 $= R_C \parallel (R_A + R_B) = \dfrac{R_C(R_A + R_B)}{R_A + R_B + R_C}$

Referring to the star network shown in Fig. 1.15 (b), the resistance between terminals 1 and 2 = $R_1 + R_2$. Since the two networks are electrically equivalent,

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \quad ...(1.1)$$

Similarly,
$$R_2 + R_3 = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} \quad ...(1.2)$$

and
$$R_3 + R_1 = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} \quad ...(1.3)$$

Subtracting Eq. (1.2) from Eq. (1.1),

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \quad ...(1.4)$$

Adding Eq. (1.4) and Eq. (1.3),

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

Similarly,
$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

Thus, star resistor connected to a terminal is equal to the product of the two delta resistors connected to the same terminal divided by the sum of the delta resistors.

1.10.2 Star to Delta Transformation

Multiplying the above equations,

$$R_1R_2 = \frac{R_AR_BR_C^2}{(R_A + R_B + R_C)^2} \quad ...(1.5)$$

$$R_2R_3 = \frac{R_A^2R_BR_C}{(R_A + R_B + R_C)^2} \quad ...(1.6)$$

$$R_3R_1 = \frac{R_AR_B^2R_C}{(R_A + R_B + R_C)^2} \quad ...(1.7)$$

Adding Eqs (1.5), (1.6) and (1.7),

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_AR_BR_C(R_A + R_B + R_C)}{(R_A + R_B + R_C)^2} = \frac{R_AR_BR_C}{R_A + R_B + R_C}$$

$$= R_AR_1 = R_BR_2 = R_CR_3$$

Hence,

$$R_A = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = R_2 + R_3 + \frac{R_2R_3}{R_1}$$

$$R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} = R_1 + R_3 + \frac{R_3R_1}{R_2}$$

$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = R_1 + R_2 + \frac{R_1R_2}{R_3}$$

Thus, delta resistor connected between the two terminals is the sum of two star resistors connected to the same terminals plus the product of the two resistors divided by the remaining third star resistor.

Note: (1) When three equal resistors are connected in delta (Fig. 1.16), the equivalent star resistance is given by

$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta} = \frac{R_\Delta}{3}$$

$$R_\Delta = 3R_Y$$

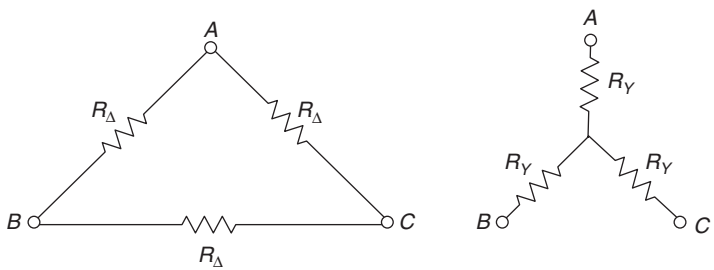


Fig. 1.16 *Equivalent star resistance for three equal delta resistors*

(2) Star-delta transformation can also be applied to network containing inductors and capacitors.

1.11 SOURCE TRANSFORMATION

A voltage source with a series resistor can be converted into a equivalent current source with a parallel resistor. Conversely, a current source with a parallel resistor can be converted into a voltage source with a series resistor as shown in Fig. 1.17.

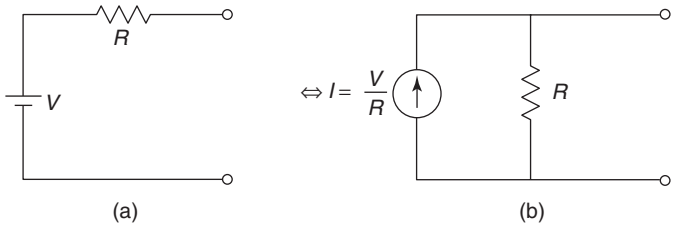


Fig. 1.17 *Source transformation*

Source transformation can be applied to dependent sources as well. The controlling variable, however must not be tampered with any way since the operation of the controlled sources depends on it.

Example 1.1 *Replace the given network of Fig. 1.18 with a single current source and a resistor.*

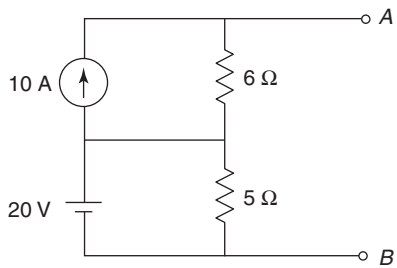


Fig. 1.18

Solution Since the resistor of 5 Ω is connected in parallel with the voltage source of 20 V it becomes redundant. Converting parallel combination of current source and resistor into equivalent voltage source and resistor (Fig. 1.19),

By source transformation (Fig. 1.20),

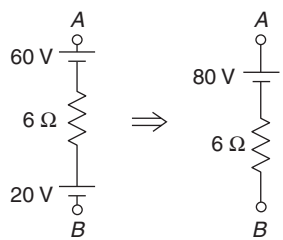


Fig. 1.19

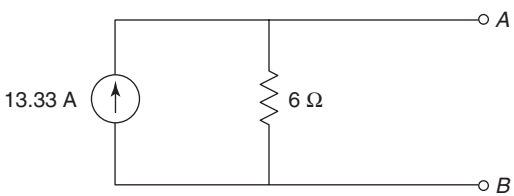


Fig. 1.20

Example 1.2 *Reduce the network shown in Fig. 1.21 into a single source and a single resistor between terminals A and B.*

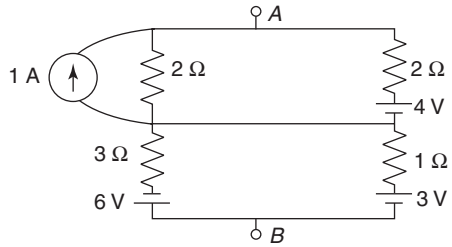


Fig. 1.21

Solution Converting all voltage sources into equivalent current sources (Fig. 1.22),

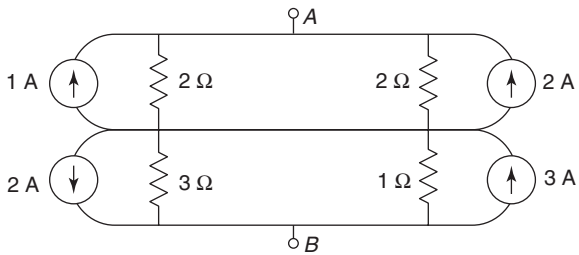


Fig. 1.22

Adding the current sources and simplifying the network (Fig. 1.23),

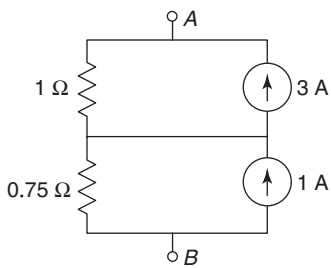


Fig. 1.23

Converting the current sources into equivalent voltage sources (Fig. 1.24),

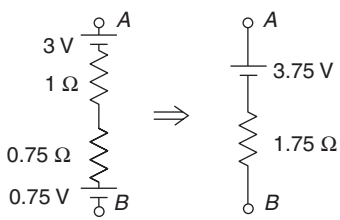


Fig. 1.24

Example 1.3 *Replace the circuit between A and B in Fig. 1.25 with a voltage source in series with a single resistor.*

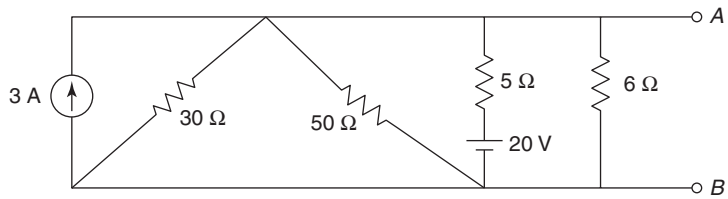


Fig. 1.25

Solution Converting the series combination of voltage source of 20 V and a resistor of 5 Ω into equivalent parallel combination of current source and resistor (Fig. 1.26),

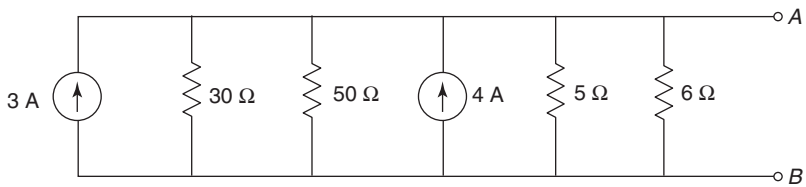


Fig. 1.26

Adding the two current sources and simplifying the circuit (Fig. 1.27),

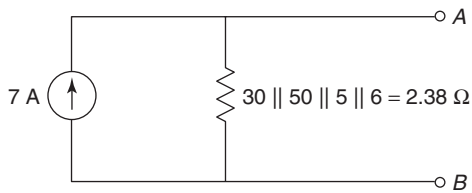


Fig. 1.27

By source transformation (Fig. 1.28),

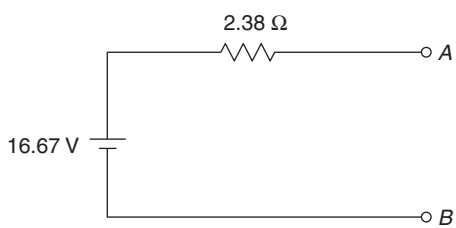


Fig. 1.28

Example 1.4 *Find the power delivered by the 50 V source in the network of Fig. 1.29.*

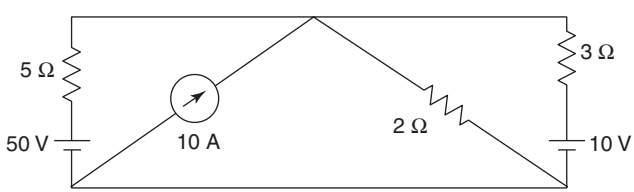


Fig. 1.29

Solution Converting the series combination of voltage source of 10 V and resistor of 3 Ω into equivalent current source and resistor (Fig. 1.30),

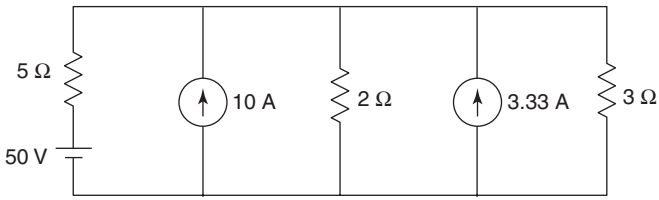


Fig. 1.30

Adding the two current sources and simplifying the network (Fig. 1.31),

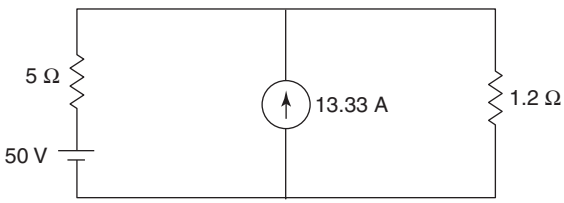


Fig. 1.31

By source transformation (Fig. 1.32),

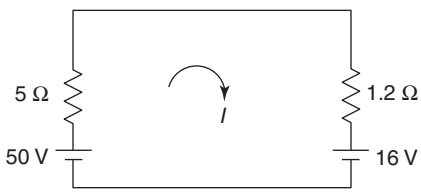


Fig. 1.32

$$I = \frac{50-16}{5+1.2} = 5.48 \text{ A}$$

Power delivered by the 50 V source = $50 \times 5.48 = 274$ W

Example 1.5 *Find the current in the 4 Ω resistor shown in network of Fig. 1.33.*

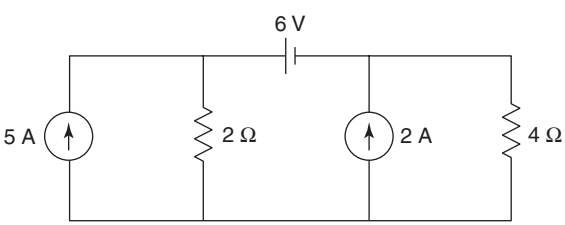


Fig. 1.33

Solution Converting the parallel combination of the current source of 5 A and the resistor of 2 Ω into an equivalent series combination of voltage source and resistor (Fig. 1.34),

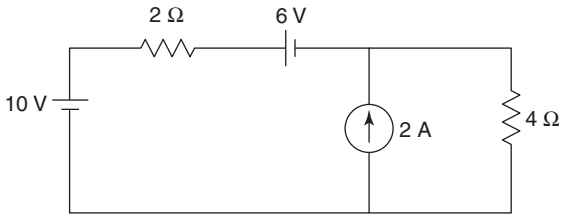


Fig. 1.34

Adding two voltage sources (Fig. 1.35),

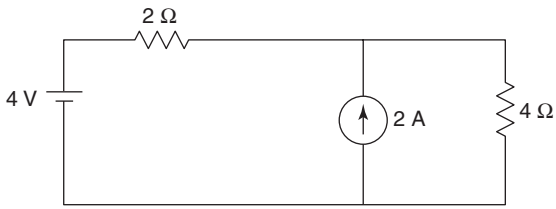


Fig. 1.35

Again by source transformation (Fig. 1.36),

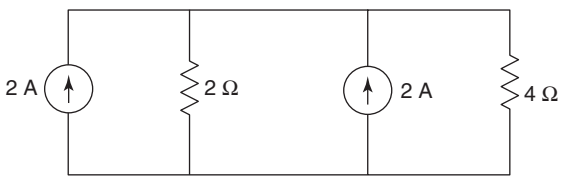


Fig. 1.36

Adding two current sources (Fig. 1.37),

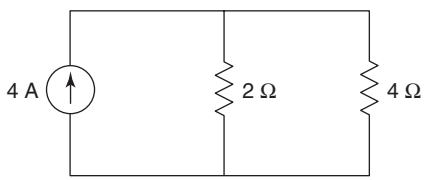


Fig. 1.37

By current-division rule,

$$I_{4\,\Omega} = 4 \times \frac{2}{2+4} = 1.33 \text{ A}$$

Example 1.6 *Find the voltage across the 4 Ω resistor shown in network of Fig. 1.38.*

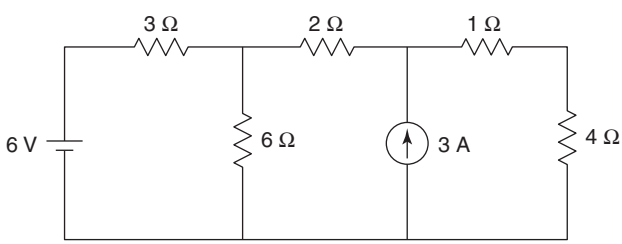


Fig. 1.38

Solution Converting the series combination of the voltage source of 6 V and the resistor of 3 Ω into equivalent current source and resistor (Fig. 1.39),

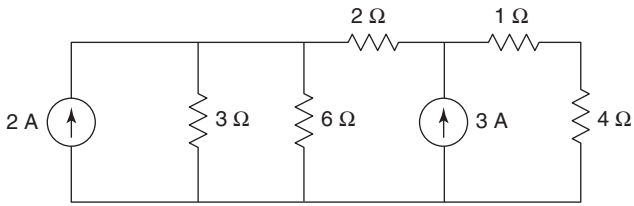


Fig. 1.39

By series–parallel reduction technique (Fig. 1.40),

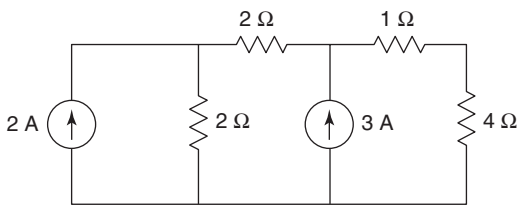


Fig. 1.40

By source transformation (Fig. 1.41),

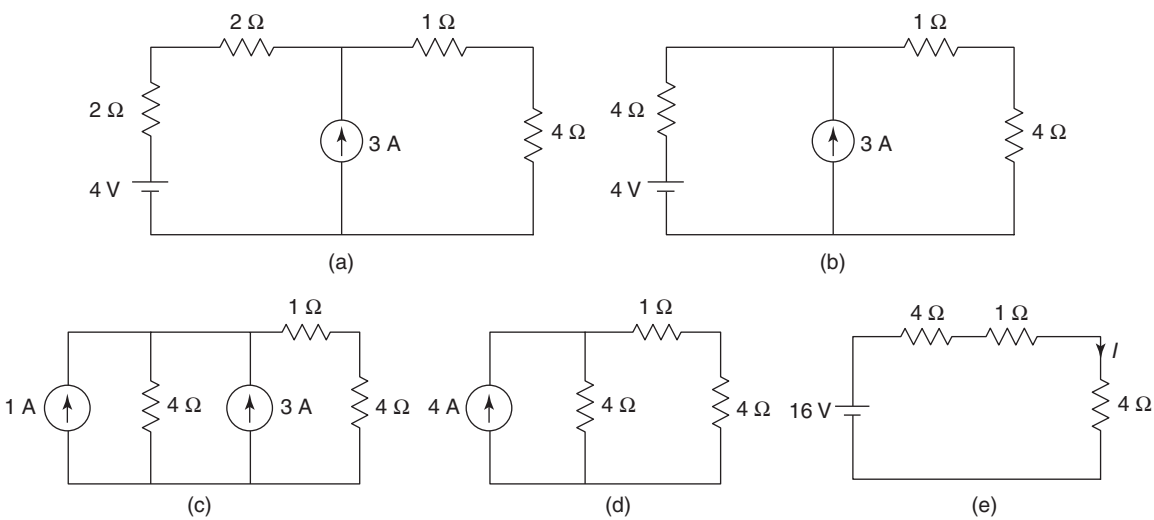


Fig. 1.41

$$I = \frac{16}{4+1+4} = 1.78 \text{ A}$$

Voltage across the 4 Ω resistor $= 4I = 4 \times 1.78 = 7.12$ V

Example 1.7 *Find the voltage at Node 2 of the network shown in Fig. 1.42.*

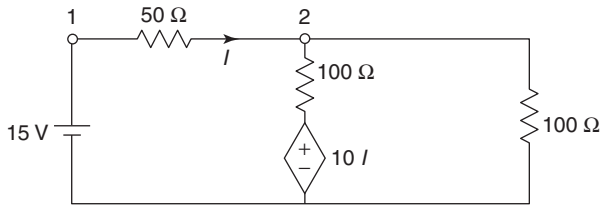


Fig. 1.42

Solution We cannot change the network between nodes 1 and 2 since the controlling current I, for the controlled source, is in the resistor between these nodes. Applying source transformation to series combination of controlled source and the 100 Ω resistor (Fig. 1.43),

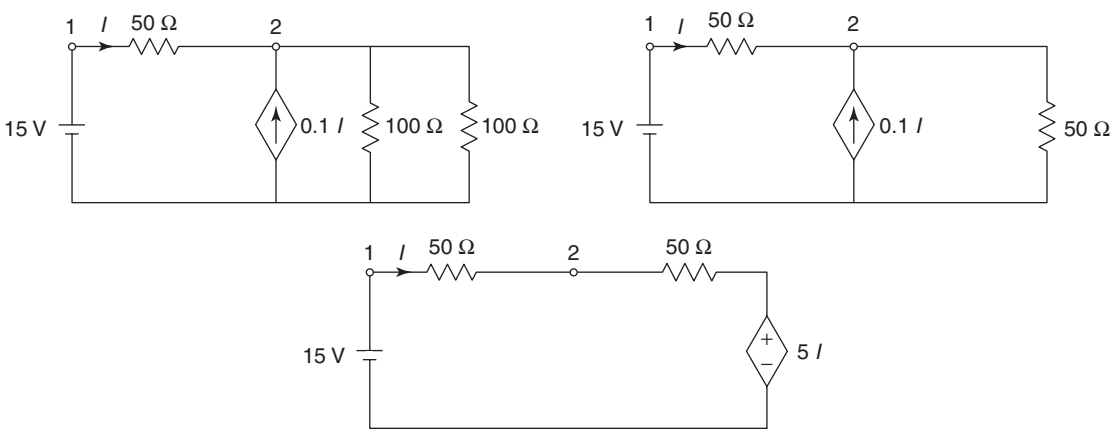


Fig. 1.43

Applying KVL to the mesh,

$$15 - 50I - 50I - 5I = 0$$

$$I = \frac{15}{105} = 0.143 \text{ A}$$

Voltage at Node 2 $= 15 - 50I = 15 - 50 \times 0.143 = 7.86$ V

1.12 SOURCE SHIFTING

Source shifting is the simplification technique used when there is no resistor in series with a voltage source or a resistor in parallel with a current source.

Example 1.8 *Calculate the voltage across the 6 Ω resistor in the network of Fig. 1.44 using source-shifting technique.*

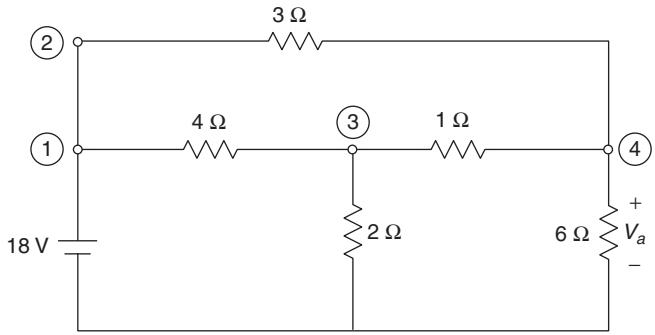


Fig. 1.44

Solution Adding a voltage source of 18 V to the network and connecting to Node 2 (Fig. 1.45), we have

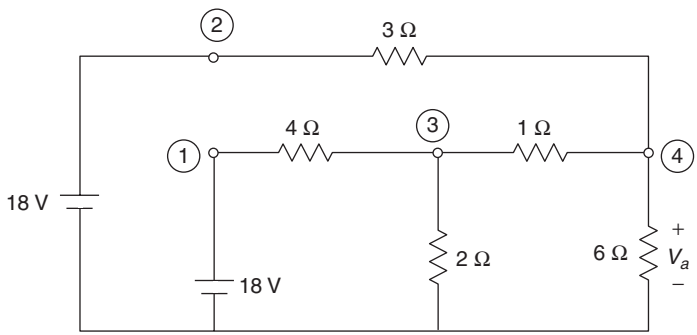


Fig. 1.45

Since nodes 1 and 2 are maintained at the same voltage by the sources, the connection between nodes 1 and 2 is removed. Now the two voltage sources have resistors in series and source transformation can be applied (Fig. 1.46).

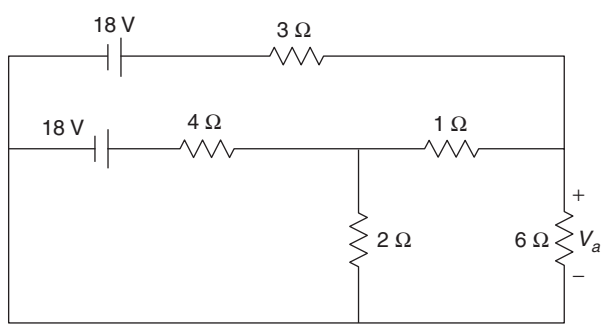


Fig. 1.46

Simplifying the network (Fig. 1.47),

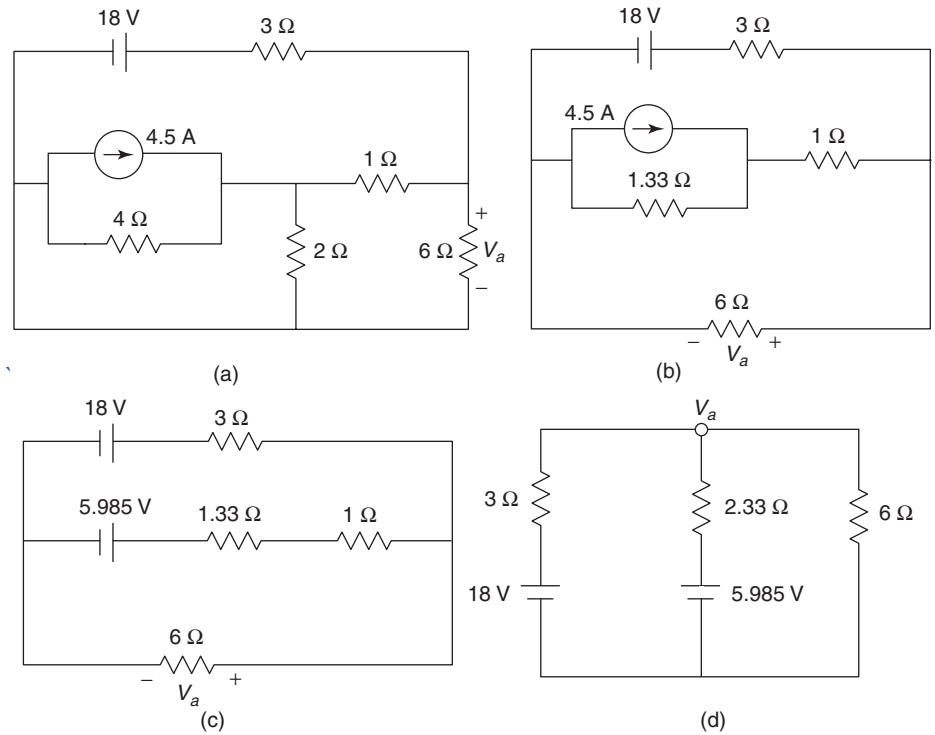


Fig. 1.47

Applying KCL at the node,

$$\frac{V_a - 18}{3} + \frac{V_a - 5.985}{2.33} + \frac{V_a}{6} = 0$$

$$V_a = 9.23 \text{ V}$$

Exercises

1.1 Use source transformation to simplify the network until two elements remain to the left of terminals *A* and *B*.

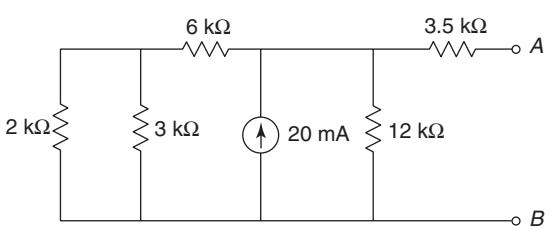


Fig. 1.48

[88.42 V, 7.92 k Ω]

1.2 Determine the voltage V_x in the network of Fig. 1.49 by source-shifting technique.

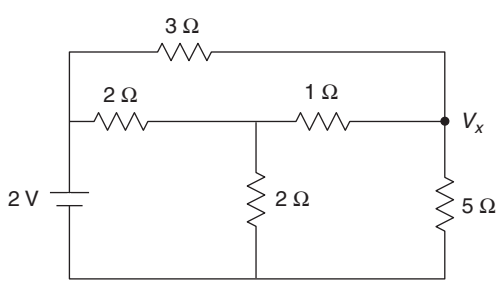


Fig. 1.49

[1.129 V]

Objective-Type Questions

1.1 A network contains linear resistors and ideal voltage sources. If values of all the resistors are doubled then the voltage across each resistor is

(a) halved
(b) doubled
(c) increased by four times
(d) not changed

1.2 Four resistances 80 Ω, 50 Ω, 25 Ω, and R are connected in parallel. Current through 25 Ω resistor is 4 A. Total current of the supply is 10 A. The value of R will be

(a) 66.66 Ω (b) 40.25 Ω
(c) 36.36 Ω (d) 76.56 Ω

1.3 Viewed from the terminal AB, the network of Fig. 1.50 can be reduced to an equivalent network of a single voltage source in series with a single resistor with the following parameters

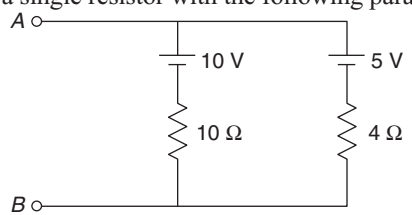


Fig. 1.50

(a) 5 V source in series with a 10 Ω resistor
(b) 1 V source in series with a 2.4 Ω resistor
(c) 15 V source in series with a 2.4 Ω resistor
(d) 1 V source in series with a 10 Ω resistor

1.4 A 10 V battery with an internal resistance of 1 Ω is connected across a nonlinear load whose *V-I* characteristic is given by $7I = V^2 + 2V$. The current delivered by the battery is

(a) 0 (b) 10 A
(c) 5 A (d) 8 A

1.5 If the length of a wire of resistance *R* is uniformly stretched to *n* times its original value, its new resistance is

(a) nR (b) $\frac{R}{n}$
(c) n^2R (d) $\frac{R}{n^2}$

1.6 All the resistances in Fig. 1.51 are 1 Ω each. The value of *I* will be

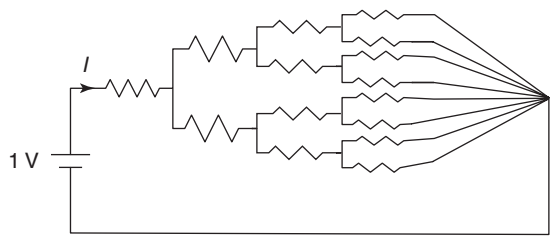


Fig. 1.51

(a) $\frac{1}{15}$ A (b) $\frac{2}{15}$ A
(c) $\frac{4}{15}$ A (d) $\frac{8}{15}$ A

1.7 The current waveform in a pure resistor at 10 Ω is shown in Fig. 1.52. Power dissipated in the resistor is

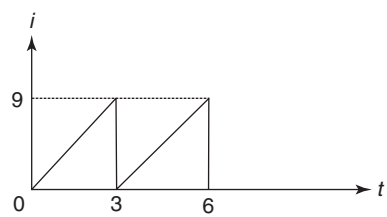


Fig. 1.52

(a) 7.29 W (b) 52.4 W
(c) 135 W (d) 270 W

1.8 Two wires *A* and *B* of the same material and length *L* and 2*L* have radius *r* and 2*r* respectively. The ratio of their specific resistance will be

(a) 1 : 1 (b) 1 : 2
(c) 1 : 4 (d) 1 : 8

Answers to Objective-Type Questions

1.1. (d) 1.2. (c) 1.3. (b) 1.4. (c) 1.5. (c) 1.6. (d) 1.7. (d)
1.8. (b)

2

Elementary Network Theorems

2.1 INTRODUCTION

In Chapter 1, we have studied basic network concepts. In network analysis, we have to find currents and voltages in various parts of networks. In this chapter, we will study elementary network theorems like Kirchhoff's laws, mesh analysis and node analysis. These methods are applicable to all types of networks. The first step in analyzing networks is to apply Ohm's law and Kirchhoff's laws. The second step is the solving of these equations by mathematical tools.

2.2 KIRCHHOFF'S LAWS

The entire study of electric network analysis is based mainly on Kirchhoff's laws. But before discussing this, it is essential to familiarise ourselves with the following terms:

Node A node is a junction where two or more network elements are connected together.

Branch An element or number of elements connected between two nodes constitute a branch.

Loop A loop is any closed part of the circuit.

Mesh A mesh is the most elementary form of a loop and cannot be further divided into other loops. All meshes are loops but all loops are not meshes.

1. ***Kirchhoff's Current Law (KCL)*** The algebraic sum of currents meeting at a junction or node in an electric circuit is zero.

   Consider five conductors, carrying currents I_1, I_2, I_3, I_4 and I_5 meeting at a point O as shown in Fig. 2.1. Assuming the incoming currents to be positive and outgoing currents negative, we have

$$I_1 + (-I_2) + I_3 + (-I_4) + I_5 = 0$$
$$I_1 - I_2 + I_3 - I_4 + I_5 = 0$$
$$I_1 + I_3 + I_5 = I_2 + I_4$$

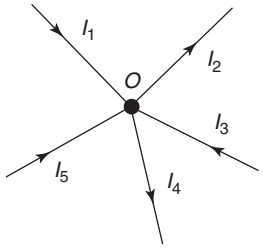


Fig. 2.1 *Kirchhoff's current law*

   Thus, the above law can also be stated as the sum of currents flowing towards any junction in an electric circuit is equal to the sum of the currents flowing away from that junction.

2. ***Kirchhoff's Voltage Law (KVL)*** The algebraic sum of all the voltages in any closed circuit or mesh or loop is zero.

   If we start from any point in a closed circuit and go back to that point, after going round the circuit, there is no increase or decrease in potential at that point. This means that the sum of emfs and the sum of voltage drops or rises meeting on the way is zero.

3. Determination of Sign A rise in potential can be assumed to be positive while a fall in potential can be considered negative. The reverse is also possible and both conventions will give the same result.

(i) If we go from the positive terminal of the battery or source to the negative terminal, there is a fall in potential and so the emf should be assigned a negative sign (Fig. 2.2a). If we go from the negative terminal of the battery or source to the positive terminal, there is a rise in potential and so the emf should be given a positive sign (Fig. 2.2b).

(a) Fall in potential (b) Rise in potential

Fig. 2.2 *Sign convention*

(ii) When current flows through a resistor, there is a voltage drop across it. If we go through the resistor in the same direction as the current, there is a fall in the potential and so the sign of this voltage drop is negative (Fig. 2.3a). If we go opposite to the direction of the current flow, there is a rise in potential and hence, this voltage drop should be given a positive sign (Fig. 2.3b).

(a) Fall in potential (b) Rise in potential

Fig. 2.3 *Sign convention*

2.3 MESH ANALYSIS

A mesh is defined as a loop which does not contain any other loops within it. Mesh analysis is applicable only for planar networks. A network is said to be planar if it can be drawn on a plane surface without crossovers. In this method, the currents in different meshes are assigned continuous paths so that they do not split at a junction into branch currents. If a network has a large number of voltage sources, it is useful to use mesh analysis. Basically, this analysis consists of writing mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents.

Steps to be Followed in Mesh Analysis

1. Identify the mesh, assign a direction to it and assign an unknown current in each mesh.
2. Assign the polarities for voltage across the branches.
3. Apply KVL around the mesh and use Ohm's law to express the branch voltages in terms of unknown mesh currents and the resistance.
4. Solve the simultaneous equations for unknown mesh currents.

Consider the network shown in Fig. 2.4 which has three meshes. Let the mesh currents for the three meshes be I_1, I_2, and I_3 and all the three mesh currents may be assumed to flow in the clockwise direction. The choice of direction for any mesh current is arbitrary.

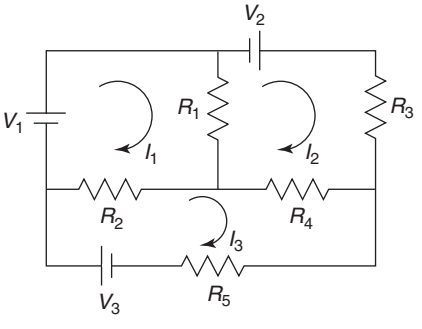


Fig. 2.4 *Circuit for mesh analysis*

Applying KVL to Mesh 1,

$$V_1 - R_1(I_1 - I_2) - R_2(I_1 - I_3) = 0$$

$$(R_1 + R_2)I_1 - R_1 I_2 - R_2 I_3 = V_1 \quad \ldots(2.1)$$

Applying KVL to Mesh 2,

$$V_2 - R_3 I_2 - R_4(I_2 - I_3) - R_1(I_2 - I_1) = 0$$
$$-R_1 I_1 + (R_1 + R_3 + R_4) I_2 - R_4 I_3 = V_2 \quad \ldots(2.2)$$

Applying KVL to Mesh 3,

$$-R_2(I_3 - I_1) - R_4(I_3 - I_2) - R_5 I_3 + V_3 = 0$$
$$-R_2 I_1 - R_4 I_2 + (R_2 + R_4 + R_5) I_3 = V_3 \quad \ldots(2.3)$$

Writing Eqs (2.1), (2.2), and (2.3) in matrix form,

$$\begin{bmatrix} R_1 + R_2 & -R_1 & -R_2 \\ -R_1 & R_1 + R_3 + R_4 & -R_4 \\ -R_2 & -R_4 & R_2 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

In general,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where, R_{11} = Self-resistance or sum of all the resistance of mesh 1
$R_{12} = R_{21}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 2
$R_{13} = R_{31}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 3
R_{22} = Self-resistance or sum of all the resistance of mesh 2
$R_{23} = R_{32}$ = Mutual resistance or sum of all the resistances common to meshes 2 and 3
R_{33} = Self-resistance or sum of all the resistance of mesh 3

If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign, and if the direction of the currents passing through common resistance are opposite then the mutual resistance will have a negative sign. If each mesh current is assumed to flow in the clockwise direction then all self-resistances will always be positive and all mutual resistances will always be negative.

The voltages V_1, V_2 and V_3 represent the algebraic sum of all the voltages in meshes 1, 2 and 3 respectively. While going along the current, if we go from negative terminal of the battery to the positive terminal then its emf is taken as positive. Otherwise, it is taken as negative.

Example 2.1 *Find the current through the 5 Ω resistor is shown in Fig. 2.5.*

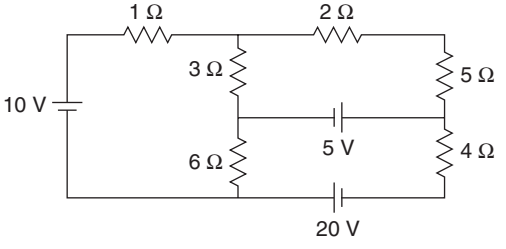


Fig. 2.5

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.6.
Applying KVL to Mesh 1,

$$10 - 1I_1 - 3(I_1 - I_2) - 6(I_1 - I_3) = 0$$
$$10 I_1 - 3 I_2 - 6 I_3 = 10 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 5I_2 - 5 = 0$$

$$-3I_1 + 10I_2 = -5 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) + 5 - 4I_3 + 20 = 0$$

$$-6I_1 + 10I_3 = 25 \qquad \ldots(iii)$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 10 & -3 & -6 \\ -3 & 10 & 0 \\ -6 & 0 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 25 \end{bmatrix}$$

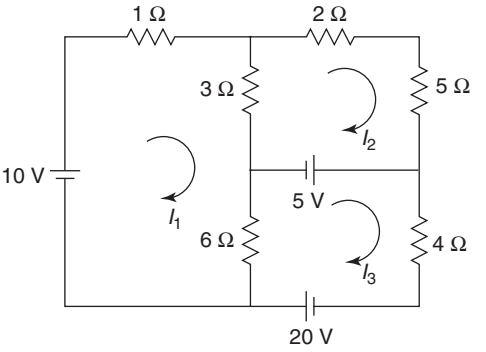


Fig. 2.6

We can write matrix equation directly from Fig. 2.6,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where R_{11} = Self-resistance of Mesh 1 = 1 + 3 + 6 = 10 Ω

R_{12} = Mutual resistance common to meshes 1 and 2 = –3 Ω

Here, negative sign indicates that the current through common resistance are in opposite direction.

R_{13} = Mutual resistance common to meshes 1 and 3 = –6 Ω

Similarly,

$R_{21} = -3\ \Omega$

$R_{22} = 3 + 2 + 5 = 10\ \Omega$

$R_{23} = 0$

$R_{31} = -6\ \Omega$

$R_{32} = 0$

$R_{33} = 6 + 4 = 10\ \Omega$

For voltage matrix,

$V_1 = 10$ V

$V_2 = -5$ V

V_3 = algebraic sum of all the voltages in mesh 3 = 5 + 20 = 25 V

Solving Eqs (i), (ii) and (iii),

$I_1 = 4.27$ A

$I_2 = 0.78$ A

$I_3 = 5.06$ A

$I_{5\Omega} = I_2 = 0.78$ A

Example 2.2 *Find the current through the 2 Ω resistor of the network shown in Fig. 2.7.*

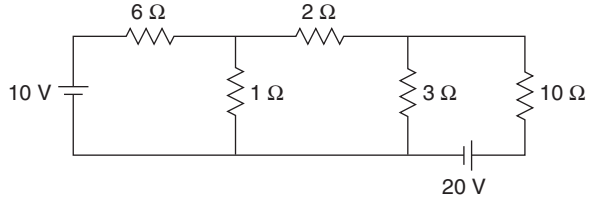


Fig. 2.7

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.8,
Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \quad \ldots(i)$$

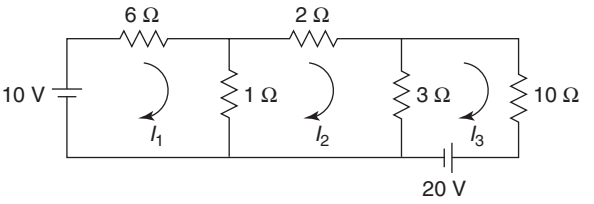


Fig. 2.8

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$-I_1 + 6I_2 - 3I_3 = 0 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-3(I_3 - I_2) - 10I_3 - 20 = 0$$
$$-3I_2 + 13I_3 = -20 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.34 \text{ A}$$
$$I_2 = -0.62 \text{ A}$$
$$I_3 = -1.68 \text{ A}$$
$$I_{2\Omega} = I_2 = -0.62 \text{ A}$$

Example 2.3

Determine the current through the 5 Ω resistor of the network shown in Fig. 2.9.

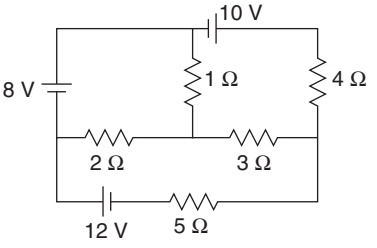


Fig. 2.9

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.10.
Applying KVL to Mesh 1,

$$8 - 1(I_1 - I_2) - 2(I_1 - I_3) = 0$$
$$3I_1 - I_2 - 2I_3 = 8 \quad \ldots(i)$$

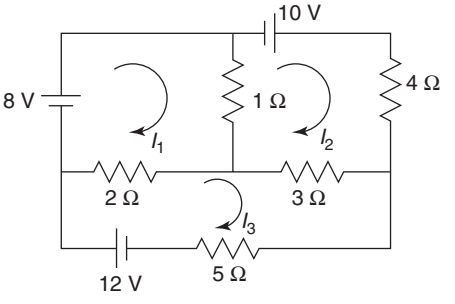


Fig. 2.10

Applying KVL to Mesh 2,

$$10 - 4I_2 - 3(I_2 - I_3) - 1(I_2 - I_1) = 0$$
$$-I_1 + 8I_2 - 3I_3 = 10 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 3(I_3 - I_2) - 5I_3 + 12 = 0$$
$$-2I_1 - 3I_2 + 10I_3 = 12 \quad \ldots(iii)$$

Solving Eqs (i), (ii), and (iii),

$$I_1 = 6.01 \text{ A}$$
$$I_2 = 3.27 \text{ A}$$

$$I_3 = 3.38 \text{ A}$$
$$I_{5\Omega} = I_3 = 3.38 \text{ A}$$

Example 2.4 *Find the current supplied by the battery of the network shown in Fig. 2.11.*

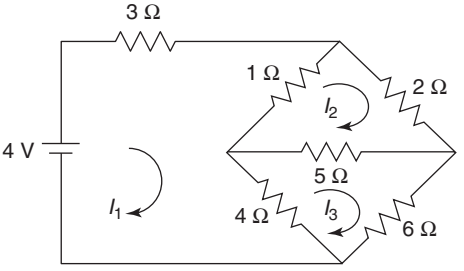


Fig. 2.11

Solution

Applying KVL to Mesh 1,

$$4 - 3I_1 - 1(I_1 - I_2) - 4(I_1 - I_3) = 0$$
$$8I_1 - I_2 - 4I_3 = 4 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-2I_2 - 5(I_2 - I_3) - 1(I_2 - I_1) = 0$$
$$-I_1 + 8I_2 - 5I_3 = 0 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-6I_3 - 4(I_3 - I_1) - 5(I_3 - I_2) = 0$$
$$-4I_1 - 5I_2 + 15I_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.66 \text{ A}$$
$$I_2 = 0.24 \text{ A}$$
$$I_3 = 0.26 \text{ A}$$

Current supplied by the battery $= I_1 = 0.66$ A.

Example 2.5 *Find the current through the 4 Ω resistor in the network of Fig. 2.12.*

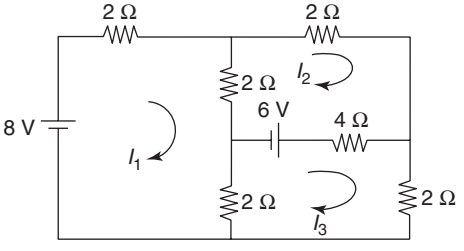


Fig. 2.12

Solution Applying KVL to Mesh 1,

$$8 - 2I_1 - 2(I_1 - I_2) - 2(I_1 - I_3) = 0$$
$$6I_1 - 2I_2 - 2I_3 = 8 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 2I_2 - 4(I_2 - I_3) - 6 = 0$$
$$-2I_1 + 8I_2 - 4I_3 = -6 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) + 6 - 4(I_3 - I_2) - 2I_3 = 0$$
$$-2I_1 - 4I_2 + 8I_3 = 6 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 0.5 \text{ A}$$
$$I_3 = 1.5 \text{ A}$$
$$I_{4\Omega} = I_3 - I_2 = 1.5 - 0.5 = 1 \text{ A}$$

Example 2.6 *Determine the voltage V which causes the current I_1 to be zero in the network of Fig. 2.13.*

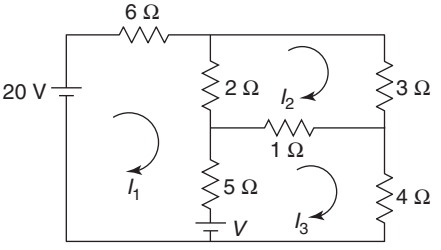


Fig. 2.13

Solution Applying KVL to Mesh 1,

$$20 - 6I_1 - 2(I_1 - I_2) - 5(I_1 - I_3) - V = 0$$
$$V + 13I_1 - 2I_2 - 5I_3 = 20 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 3I_2 - 1(I_2 - I_3) = 0$$
$$2I_1 - 6I_2 + I_3 = 0 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-1(I_3 - I_2) - 4I_3 + V - 5(I_3 - I_1) = 0$$
$$V + 5I_1 + I_2 - 10I_3 = 0 \quad \ldots(iii)$$

Putting $I_1 = 0$ in Eqs (i), (ii) and (iii),

$$V - 2I_2 - 5I_3 = 20$$
$$-6I_2 + I_3 = 0$$
$$V + I_2 - 10I_3 = 0 \quad \ldots(iv)$$

Solving Eqs (i), (ii) and (iii),

$$V = 43.7 \text{ V}$$

Example 2.7 *Find the current through the 2 Ω resistor in the network of Fig. 2.14.*

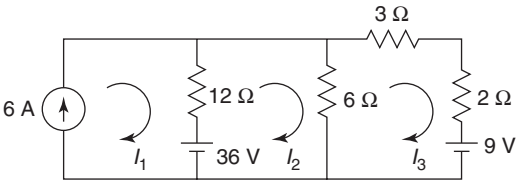


Fig. 2.14

Solution Mesh 1 contains a current source of 6 A. Hence, we can write current equation for Mesh 1. Since direction of current source and mesh current I_1 are same,

$$I_1 = 6 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$36 - 12(I_2 - I_1) - 6(I_2 - I_3) = 0$$

$$36 - 12(I_2 - 6) - 6I_2 + 6I_3 = 0$$

$$18I_2 - 6I_3 = 108 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 3I_3 - 2I_3 - 9 = 0$$

$$6I_2 - 11I_3 = 9 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_3 = 3\,\text{A}$$

$$I_{2\Omega} = I_3 = 3\,\text{A}$$

Example 2.8 *Determine the mesh currents I_1, I_2 and I_3 in the network of Fig. 2.15.*

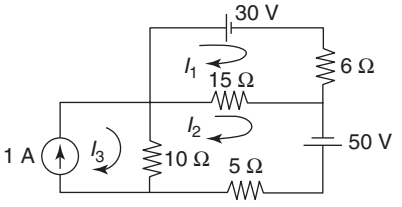


Fig. 2.15

Solution Applying KVL to Mesh 1,

$$-30 - 6I_1 - 15(I_1 - I_2) = 0$$

$$21I_1 - 15I_2 = -30 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_3) - 15(I_2 - I_1) + 50 - 5I_2 = 0$$

$$-15I_1 + 30I_2 - 10I_3 = 50 \qquad \ldots(ii)$$

For Mesh 3,

$$I_3 = 1 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0$$
$$I_2 = 2 \text{ A}$$
$$I_3 = 1 \text{ A}$$

Example 2.9 *Find the current through the 5 Ω resistor in the network of Fig. 2.16.*

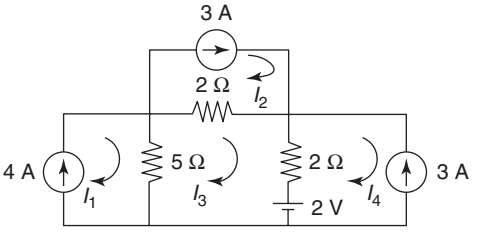


Fig. 2.16

Solution Writing current equations for Meshes 1, 2 and 4,

$$I_1 = 4 \quad \ldots(i)$$
$$I_2 = 3 \quad \ldots(ii)$$
$$I_4 = -3 \quad \ldots(iii)$$

Applying KVL to Mesh 3,

$$-5(I_3 - I_1) - 2(I_3 - I_2) - 2(I_3 - I_4) - 2 = 0 \quad \ldots(iv)$$

Substituting Eqs (i), (ii) and (iii) in Eq. (iv),

$$-5(I_3 - 4) - 2(I_3 - 3) - 2(I_3 + 3) - 2 = 0$$
$$I_3 = 2 \text{ A}$$
$$I_{5\Omega} = I_1 - I_3 = 4 - 2 = 2 \text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.10 *Obtain the branch currents in the network shown in Fig. 2.17.*

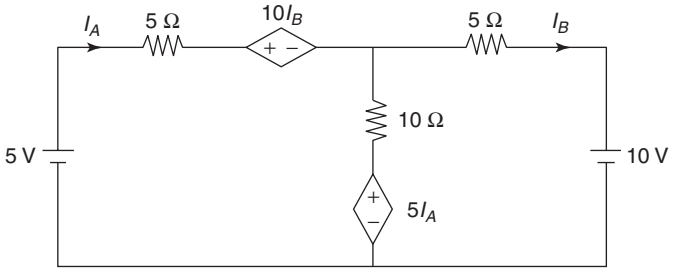


Fig. 2.17

Solution We assign clockwise currents in two meshes as shown in Fig. 2.18.

From Fig. 2.18,

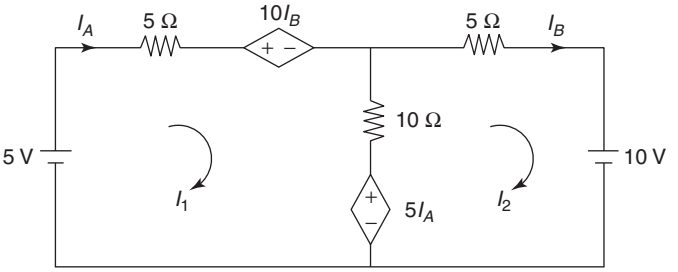


Fig. 2.18

$$I_A = I_1 \quad \ldots(i)$$

$$I_B = I_2 \quad \ldots(ii)$$

Applying KVL to Mesh 1,

$$5 - 5I_1 - 10I_B - 10(I_1 - I_2) - 5I_A = 0$$

$$5 - 5I_1 - 10I_2 - 10I_1 + 10I_2 - 5I_1 = 0$$

$$-20I_1 = -5$$

$$I_1 = \frac{1}{4} = 0.25\,\text{A} \quad \ldots(iii)$$

Applying KVL to Mesh 2,

$$5I_A - 10(I_2 - I_1) - 5I_2 - 10 = 0$$

$$5I_1 - 10I_2 + 10I_1 - 5I_2 = 10$$

$$15I_1 - 15I_2 = 10 \quad \ldots(iv)$$

Putting $I_1 = 0.25$ A in Eq. (iv),

$$15(0.25) - 15I_2 = 10$$

$$I_2 = -0.416\ \text{A}$$

Example 2.11

Find the mesh currents in the network shown in Fig. 2.19.

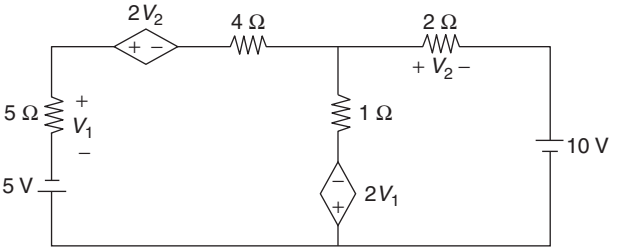


Fig. 2.19

Solution Assigning clockwise currents in the two meshes as shown in Fig. 2.20,

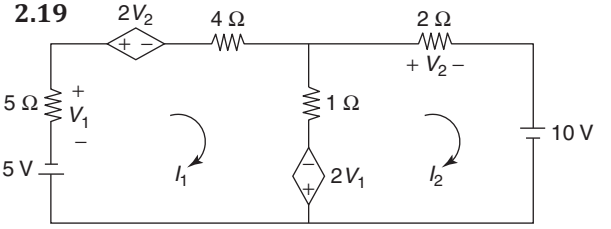


Fig. 2.20

From Fig. 2.20,

$$V_1 = -5I_1 \quad \ldots(i)$$

$$V_2 = 2I_2 \quad \ldots(ii)$$

Applying KVL to Mesh 1,

$$-5-5I_1-2V_2-4I_1-1(I_1-I_2)+2V_1=0$$
$$-5-5I_1-2(2I_2)-4I_1-I_1+I_2+2(-5I_1)=0$$
$$20I_1+3I_2=-5 \quad \ldots(iii)$$

Applying KVL to Mesh 2,

$$-2V_1-1(I_2-I_1)-2I_2-10=0$$
$$-2(-5I_1)-I_2+I_1-2I_2=10$$
$$11I_1-3I_2=10 \quad \ldots(iv)$$

Solving Eqs (iii) and (iv),

$$I_1=0.161 \text{ A}$$
$$I_2=-2.742 \text{ A}$$

Example 2.12 *Find currents I_x and I_y of the network shown in Fig. 2.21.*

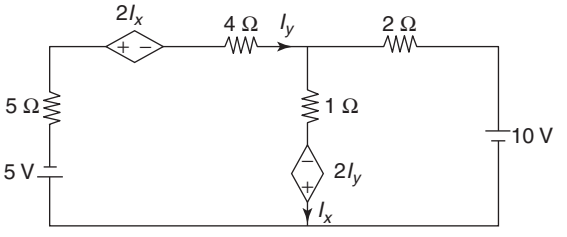


Fig. 2.21

Solution Assigning clockwise currents in the two Meshes as shown in Fig. 2.22.
From Fig. 2.22,

$$I_y=I_1 \quad \ldots(i)$$
$$I_x=I_1-I_2 \quad \ldots(ii)$$

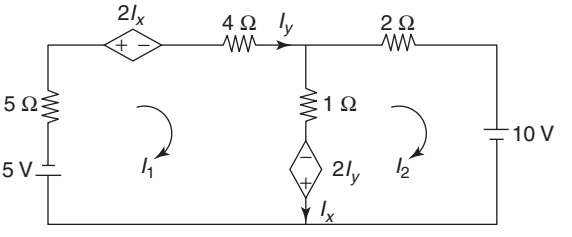


Fig. 2.22

Applying KVL to Mesh 1,

$$-5-5I_1-2I_x-4I_1-1(I_1-I_2)+2I_y=0$$
$$-5-5I_1-2(I_1-I_2)-4I_1-I_1+I_2+2I_1=0$$
$$-5-5I_1-2I_1+2I_2-4I_1-I_1+I_2+2I_1=0 \quad \ldots(iii)$$
$$-10I_1+3I_2=5$$

Applying KVL to Mesh 2,

$$-2I_y-1(I_2-I_1)-2I_2-10=0$$
$$-2I_1-I_2+I_1-2I_2=10$$
$$-I_1-3I_2=10 \quad \ldots(iv)$$

Solving Eqs (iii) and (iv),

$$I_1=-\frac{15}{11}=-1.364 \text{ A}$$
$$I_2=-2.878 \text{ A}$$
$$I_y=-1.364 \text{ A}$$
$$I_x=I_1-I_2=-1.364+2.878=1.514 \text{ A}$$

Example 2.13 *Find the currents in the three meshes of the network shown in Fig. 2.23.*

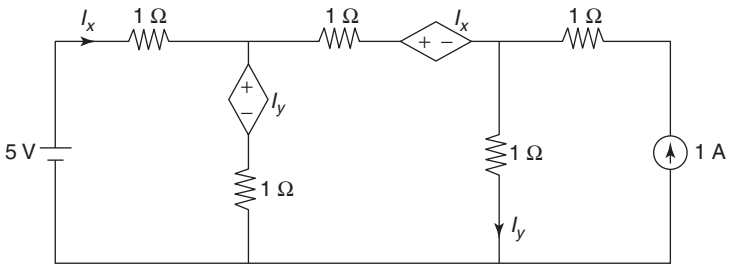


Fig. 2.23

Solution Assigning clockwise currents in the three meshes is shown in Fig. 2.24.

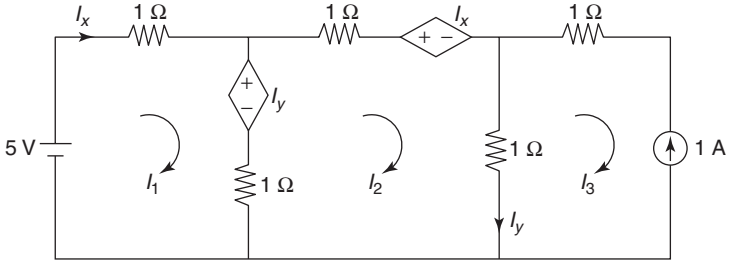


Fig. 2.24

From Fig. 2.24,

$$I_x = I_1 \qquad \text{...(i)}$$

$$I_x = I_1$$

$$I_y = I_2 - I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$5 - 1I_1 - I_y - 1(I_1 - I_2) = 0$$

$$5 - I_1 - (I_2 - I_3) - (I_1 - I_2) = 0$$

$$-2I_1 + I_3 = -5 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + I_y - 1I_2 - I_x - 1(I_2 - I_3) = 0$$

$$-(I_2 - I_1) + (I_2 - I_3) - I_2 - I_1 - (I_2 - I_3) = 0$$

$$-2I_2 = 0 \qquad \text{...(iv)}$$

For Mesh 3,

$$I_3 = -1 \qquad \text{...(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 2 \text{ A}$$

$$I_2 = 0$$

$$I_3 = -1 \text{ A}$$

Example 2.14 *For the network shown in Fig. 2.25, find the power supplied by the dependent voltage source.*

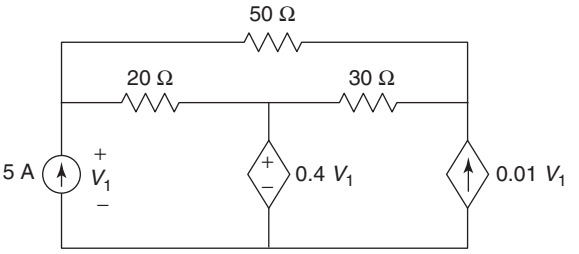


Fig. 2.25

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.26.

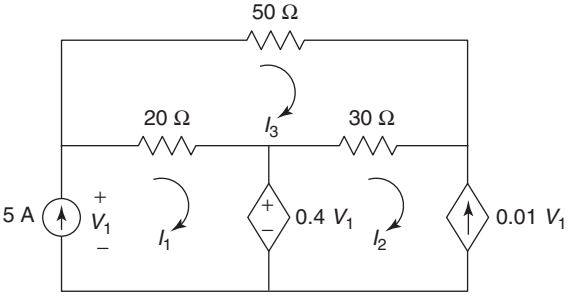


Fig. 2.26

From Fig. 2.26,

$$V_1 - 20(I_1 - I_3) - 0.4\,V_1 = 0$$

$$0.6\,V_1 = 20\,I_1 - 20\,I_3$$

$$V_1 = 33.33\,I_1 - 33.33\,I_3 \quad \ldots(\text{i})$$

For Mesh 1,

$$I_1 = 5 \quad \ldots(\text{ii})$$

For Mesh 2,

$$I_2 = -0.01\,V_1 = -0.01(33.33\,I_1 - 33.33\,I_3)$$

$$0.33\,I_1 + I_2 - 0.33\,I_3 = 0 \quad \ldots(\text{iii})$$

Applying KVL to Mesh 3,

$$-50\,I_3 - 30(I_3 - I_2) - 20(I_3 - I_1) = 0$$

$$-20\,I_1 - 30\,I_2 + 100\,I_3 = 0 \quad \ldots(\text{iv})$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5 \text{ A}$$

$$I_2 = -1.47 \text{ A}$$

$$I_3 = 0.56 \text{ A}$$

$V_1 = 33.33\,I_1 - 33.33\,I_3 = 33.33(5) - 33.33(0.56) = 148$ V

Power supplied by the dependent voltage source $= 0.4\,V_1(I_1 - I_2) = 0.4\,(148)(5 + 1.47) = 383.02$ W

Example 2.15 *Find the voltage V_x in the network shown in Fig. 2.27.*

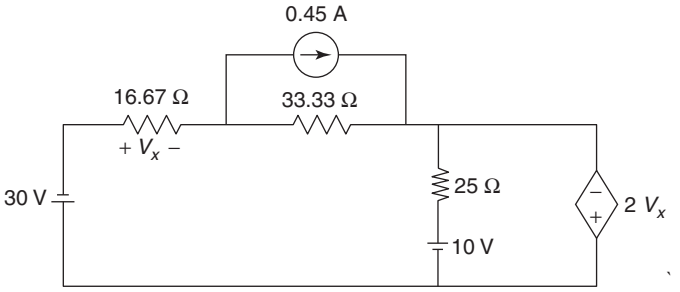


Fig. 2.27

Solution Assigning clockwise currents in the three meshes as shown in Fig. 2.28.

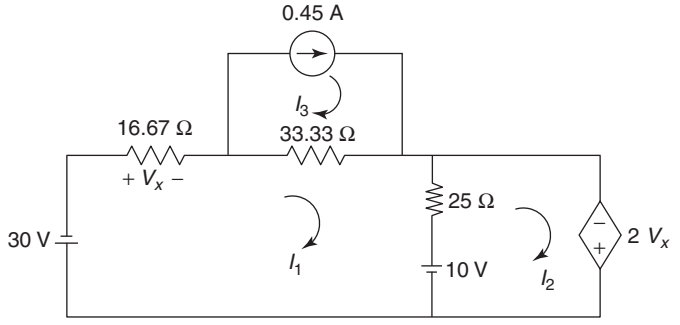


Fig. 2.28

From Fig. 2.28,

$$V_x = 16.67\, I_1 \qquad \ldots(i)$$

Applying KVL to Mesh 1,

$$-30 - 16.67\, I_1 - 33.33(I_1 - I_3) - 25(I_1 - I_2) - 10 = 0$$

$$-30 - 16.67\, I_1 - 33.33\, I_1 + 33.33\, I_3 - 25\, I_1 + 25\, I_2 - 10 = 0$$

$$-75\, I_1 + 25\, I_2 + 33.33\, I_3 = 40 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$10 - 25(I_2 - I_1) + 2\, V_x = 0$$

$$10 - 25(I_2 - I_1) + 2(16.67\, I_1) = 0$$

$$10 - 25\, I_2 + 25\, I_1 + 33.34\, I_1 = 0$$

$$58.34\, I_1 - 25\, I_2 = -10 \qquad \ldots(iii)$$

For Mesh 3,

$$I_3 = 0.45 \qquad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.9 \text{ A}$$

$$I_2 = -1.7 \text{ A}$$

$$I_3 = 0.45 \text{ A}$$

$$V_x = 16.67\, I_1 = 16.67\,(-0.9) = -15 \text{ V} \qquad \ldots(v)$$

Example 2.16 *For the network shown in Fig. 2.29, find the mesh currents I_1, I_2 and I_3.*

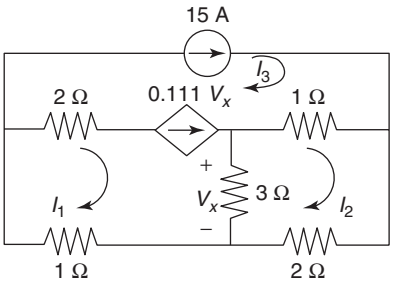


Fig. 2.29

Solution From Fig. 2.29,

$$V_x = 3(I_1 - I_2) \quad \ldots(i)$$

Writing current equation for the two current sources,

$$I_3 = 15 \quad \ldots(ii)$$

and

$$0.111\,V_x = I_1 - I_3$$

$$0.111\,[3\,(I_1 - I_2) = I_1 - I_3$$

$$0.333\,I_1 - 0.333\,I_2 - I_1 + I_3 = 0$$

$$-0.667\,I_1 - 0.333\,I_2 + I_3 = 0 \quad \ldots(iii)$$

Applying KVL to Mesh 2,

$$-3\,(I_2 - I_1) - 1(I_2 - I_3) - 2\,I_2 = 0$$

$$-3\,I_1 + 6\,I_2 - I_3 = 0 \quad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 17 \text{ A}$$

$$I_2 = 11 \text{ A}$$

$$I_3 = 15 \text{ A}$$

Example 2.17 *For the network shown in Fig. 2.30, find the magnitude of V_0 and the current supplied by it, given that power loss in R_L = 2 Ω resistor is 18 W.*

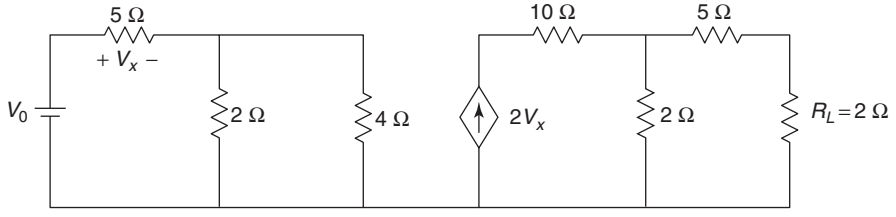


Fig. 2.30

Solution Assigning clockwise currents in meshes is shown in Fig. 2.31.

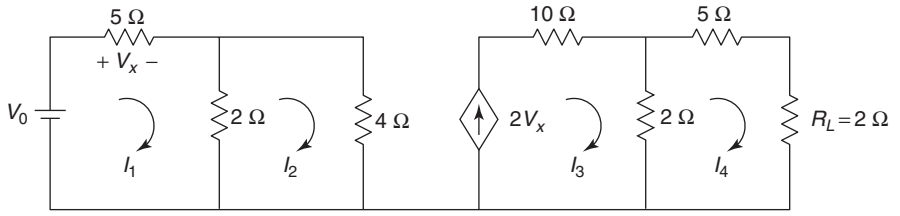


Fig. 2.31

From Fig. 2.31,

$$V_x = 5I_1 \quad \ldots(i)$$

Also,

$$I_4^2 R_L = 18$$

$$I_4^2(2) = 18$$

$$I_4 = 3 \text{ A} \quad \ldots(ii)$$

Applying KVL to Mesh 1,

$$V_0 - 5I_1 - 2(I_1 - I_2) = 0$$

$$7I_1 - 2I_2 = V_0 \quad \ldots(iii)$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 4I_2 = 0$$

$$-2I_1 + 6I_2 = 0 \quad \ldots(iv)$$

For Mesh 3,

$$I_3 = 2V_x = 2(5I_1) = 10I_1$$

$$10I_1 - I_3 = 0 \quad \ldots(v)$$

Applying KVL to Mesh 4,

$$-2(I_4 - I_3) - 5I_4 - 2I_4 = 0$$

$$-2I_3 + 9I_4 = 0$$

$$-2I_3 + 9(3) = 0$$

$$I_3 = 13.5 \text{ A}$$

From Eq. (v),

$$I_1 = \frac{I_3}{10} = \frac{13.5}{10} = 1.35 \text{ A}$$

From Eq. (iv),

$$-2(1.35) + 6I_2 = 0$$

$$I_2 = 0.45 \text{ A}$$

From Eq. (iii),

$$7(1.35) - 2(0.45) = V_0$$

$$V_0 = 8.55 \text{ V}$$

Current supplied by voltage source $V_0 = I_1 = 1.35$ A

Example 2.18 *In the network shown in Fig. 2.32, find voltage V_2 such that $V_x = 0$.*

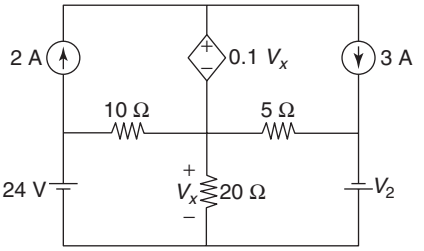


Fig. 2.32

Solution Assigning clockwise currents in four meshes as shown in Fig. 2.33.

From Fig. 2.33,

$$V_x = 20\,(I_3 - I_4) \qquad \ldots(i)$$

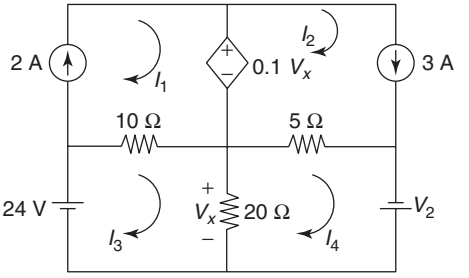


Fig. 2.33

Writing current equations for Meshes 1 and 2,

$$I_1 = 2 \qquad \ldots(ii)$$

$$I_2 = 3 \qquad \ldots(iii)$$

Applying KVL to Mesh 3,

$$24 - 10(I_3 - I_1) - 20(I_3 - I_4) = 0$$

$$24 - 10(I_3 - 2) - 20(I_3 - I_4) = 0$$

$$-30\,I_3 + 20\,I_4 = -44 \qquad \ldots(iv)$$

Applying KVL to Mesh 4,

$$-20(I_4 - I_3) - 5(I_4 - I_2) + V_2 = 0$$

$$-20(I_4 - I_3) - 5(I_4 - 3) + V_2 = 0$$

$$20\,I_3 - 25\,I_4 = -V_2 - 15 \qquad \ldots(v)$$

But

$$V_x = 0$$

$$20\,(I_3 - I_4) = 0$$

$$I_3 = I_4$$

From Eq. (iv),

$$-30\,I_3 + 20\,I_3 = -44$$

$$I_3 = 4.4 \text{ A}$$

$$I_4 = 4.4 \text{ A}$$

From Eq. (v),

$$20\,(4.4) - 25\,(4.4) = -V_2 - 15$$

$$V_2 = 7 \text{ V}$$

2.4 SUPERMESH ANALYSIS

Meshes that share a current source with other meshes, none of which contains a current source in the outer loop, form a supermesh. A path around a supermesh doesn't pass through a current source. A path around each mesh contained within a supermesh passes through a current source. The total number of equations required for a supermesh is equal to the number of meshes contained in the supermesh. A supermesh requires one mesh current equation, that is, a KVL equation. The remaining mesh current equations are KCL equations.

Example 2.19 *Find the current through the 10 Ω resistor of the network shown in Fig. 2.34.*

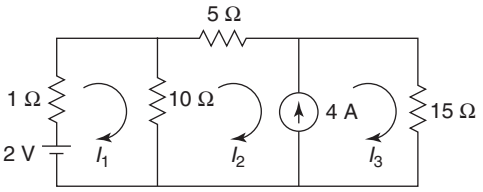


Fig. 2.34

Solution Applying KVL to Mesh 1,

$$2 - 1I_1 - 10(I_1 - I_2) = 0$$
$$11I_1 - 10I_2 = 2 \quad \ldots(i)$$

Since meshes 2 and 3 contain a current source of 4 A, these two meshes will form a supermesh. A supermesh is formed by two adjacent meshes that have a common current source. The direction of the current source of 4 A and current $(I_3 - I_2)$ are same, i.e., in the upward direction.
Writing current equation to the supermesh,1

$$I_3 - I_2 = 4 \quad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-10(I_2 - I_1) - 5I_2 - 15I_3 = 0$$
$$10I_1 - 15I_2 - 15I_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = -2.35 \text{ A}$$
$$I_2 = -2.78 \text{ A}$$
$$I_3 = 1.22 \text{ A} \quad \ldots(iv)$$

Current through the 10 Ω resistor $= I_1 - I_2 = -(2.35) - (-2.78) = 0.43$ A

Example 2.20 *Find the current in the 3 Ω resistor of the network shown in Fig. 2.35.*

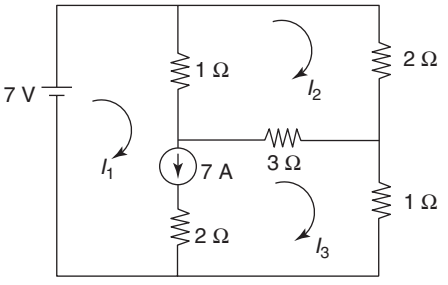


Fig. 2.35

Solution Meshes 1 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_3 = 7 \quad \ldots(i)$$

Applying KVL to the outer path of the supermesh,

$$7 - 1(I_1 - I_2) - 3(I_3 - I_2) - 1I_3 = 0$$
$$-I_1 + 4I_2 - 4I_3 = -7 \quad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$I_1 - 6I_2 + 3I_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 9 \text{ A}$$
$$I_2 = 2.5 \text{ A}$$
$$I_3 = 2 \text{ A}$$

Current through the 3 Ω resistor = $I_2 - I_3 = 2.5 - 2 = 0.5$ A

Example 2.21 *Find the current in the 5 Ω resistor of the network shown in Fig. 2.36.*

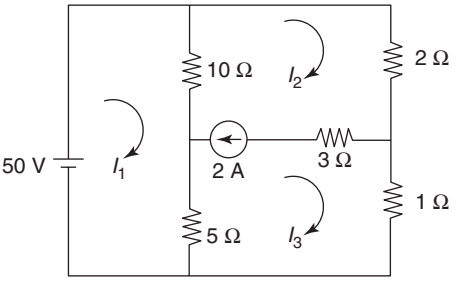


Fig. 2.36

Solution Applying KVL to Mesh 1,

$$50 - 10(I_1 - I_2) - 5(I_1 - I_3) = 0$$
$$15I_1 - 10I_2 - 5I_3 = 50 \quad \ldots(i)$$

Meshes 2 and 3 will form a supermesh as these two meshes share a common current source of 2 A. Writing current equation for the supermesh,

$$I_2 - I_3 = 2 \quad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-10(I_2 - I_1) - 2I_2 - 1I_3 - 5(I_3 - I_1) = 0$$
$$-15I_1 + 12I_2 + 6I_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 20 \text{ A}$$
$$I_2 = 17.33 \text{ A}$$
$$I_3 = 15.33 \text{ A}$$

Current through the 5 Ω resistor = $I_1 - I_3 = 20 - 15.33 = 4.67$ A

Example 2.22 *Determine the power delivered by the voltage source and the current in the 10 Ω resistor of the network shown in Fig. 2.37.*

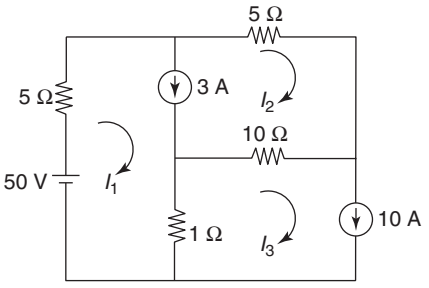


Fig. 2.37

Solution Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 3 \qquad \ldots(i)$$

Applying KVL to the outer path of the supermesh,

$$50 - 5I_1 - 5I_2 - 10(I_2 - I_3) - 1(I_1 - I_3) = 0$$

$$-6I_1 - 15I_2 + 11I_3 = -50 \qquad \ldots(ii)$$

For Mesh 3,

$$I_3 = 10 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 9.76 \text{ A}$$

$$I_2 = 6.76 \text{ A}$$

$$I_3 = 10 \text{ A}$$

Power delivered by the voltage source = $50\, I_1 = 50 \times 9.76 = 488$ W

$$I_{10\,\Omega} = I_3 - I_2 = 10 - 6.76 = 3.24 \text{ A}$$

Example 2.23 *For the network shown in Fig. 2.38, find current through the 8 Ω resistor.*

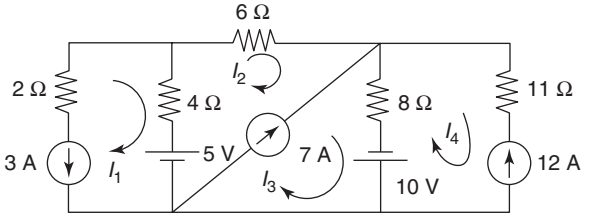


Fig. 2.38

Writing current equations for meshes 1 and 4,

$$I_1 = -3 \qquad \ldots(i)$$

$$I_4 = -12 \qquad \ldots(ii)$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 7 \qquad \ldots(iii)$$

Applying KVL to the outer path of the supermesh,

$$5-4(I_2-I_1)-6I_2-8(I_3-I_4)+10=0$$
$$5-4(I_2+3)-6I_2-8(I_3+12)+10=0$$
$$-10I_2-8I_3=93 \quad \ldots(iv)$$

Solving Eqs (iii) and (iv),

$$I_2=-8.28 \text{ A}$$
$$I_3=-1.28 \text{ A}$$
$$I_{8\Omega}=I_3-I_4=-1.28+12=10.72 \text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.24 *In the network of Fig. 2.39, find currents I_1 and I_2.*

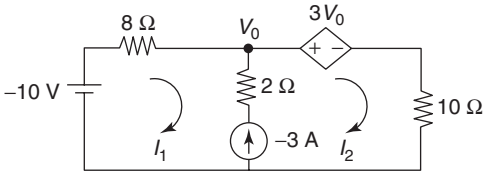


Fig. 2.39

Solution From Fig. 2.39,

$$-10-8I_1-V_0=0$$
$$V_0=-10-8I_1 \quad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equations for the supermesh,

$$I_2-I_1=-3 \quad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-10-8I_1-3V_0-10I_2=0$$
$$-10-8I_1-3(-10-8I_1)-10I_2=0$$
$$16I_1-10I_2=-20 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1=-8.33 \text{ A}$$
$$I_2=-11.33 \text{ A}$$

Example 2.25 *In the network of Fig. 2.40, find the current through the 3 Ω resistor.*

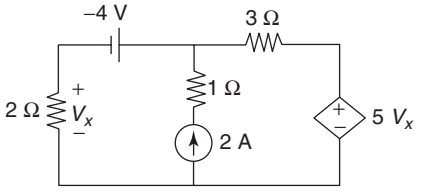


Fig. 2.40

Solution Assigning clockwise currents in two meshes as shown in Fig. 2.41.
From Fig. 2.41,

$$V_x = -2\,I_1 \qquad \ldots(i)$$

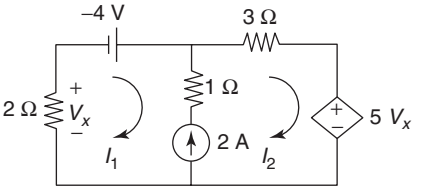


Fig. 2.41

Meshes 1 and 2 will form a supermesh.
Writing current equations for the supermesh,

$$I_2 - I_1 = 2 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-2\,I_1 - 4 - 3\,I_2 - 5V_x = 0$$
$$-2\,I_1 - 4 - 3\,I_2 - 5(-2\,I_1) = 0$$
$$8\,I_1 - 3\,I_2 = 4 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 4 \text{ A}$$
$$I_{3\,\Omega} = I_2 = 4 \text{ A}$$

Example 2.26

Find the currents I_1 and I_2 at the network shown in Fig. 2.42.

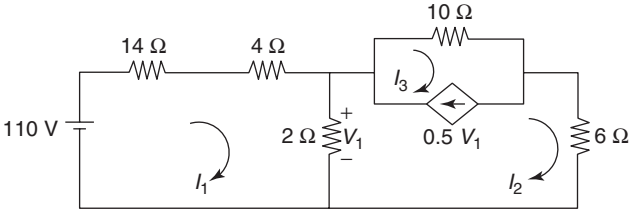


Fig. 2.42

Solution From Fig. 2.42,

$$V_1 = 2\,(I_1 - I_2)$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 0.5 \text{ V}_1 = 0.5 \times 2\,(I_1 - I_2) = I_1 - I_2$$
$$I_3 = I_1$$

Applying KVL to outer path of the supermesh,

$$-2\,(I_2 - I_1) - 10\,I_3 - 6\,I_2 = 0$$
$$-2\,I_2 + 2\,I_1 - 10\,I_1 - 6\,I_2 = 0$$
$$I_1 = -I_2$$

Applying KVL to Mesh 1,

$$110 - 14\,I_1 - 4\,I_1 - 2\,(I_1 - I_2) = 0$$
$$110 - 20\,I_1 + 2\,I_2 = 0$$
$$110 + 20\,I_1 + 2\,I_2 = 0$$
$$I_2 = -5 \text{ A}$$
$$I_1 = -I_2 = 5 \text{ A}$$

Example 2.27 *For the network of Fig. 2.43, find current through the 8 Ω resistor.*

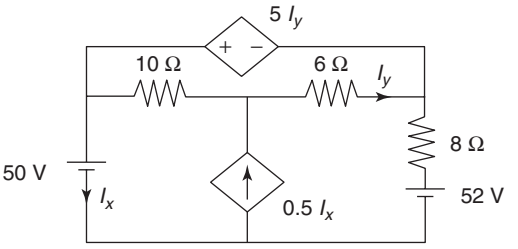


Fig. 2.43

Solution Assigning clockwise currents to the three meshes as shown in Fig. 2.44.
From Fig. 2.44,

$$I_x = -I_1 \qquad \ldots(i)$$

$$I_y = I_2 - I_3 \qquad \ldots(ii)$$

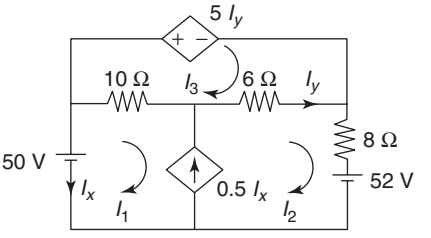


Fig. 2.44

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 0.5\, I_x = 0.5\,(-I_1)$$

$$-0.5\, I_1 + I_2 = 0 \qquad \ldots(iii)$$

Applying KVL to the outer path of the supermesh,

$$50 - 10(I_1 - I_3) - 6(I_2 - I_3) - 8I_2 - 52 = 0$$

$$-10\, I_1 - 14\, I_2 + 16\, I_3 = 2 \qquad \ldots(iv)$$

Applying KVL to Mesh 3,

$$-5\, I_y - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$

$$-5\,(I_2 - I_3) - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$

$$10\, I_1 + I_2 - 11\, I_3 = 0 \qquad \ldots(v)$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = -1.56 \text{ A}$$

$$I_2 = -0.58 \text{ A}$$

$$I_3 = -1.11 \text{ A}$$

$$I_{8\,\Omega} = I_2 = -0.58 \text{ A}$$

Example 2.28 *For the network shown in Fig. 2.45, find the current through the 10 Ω resistor.*

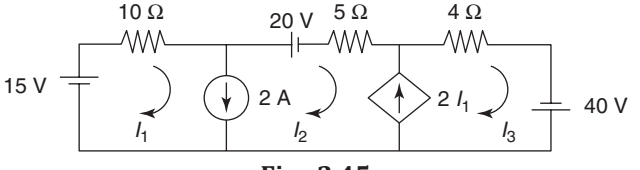


Fig. 2.45

Solution Meshes 1, 2 and 3 will form a supermesh.
Writing current equations for the supermesh,

$$I_1 - I_2 = 2 \qquad \text{...(i)}$$

$$I_3 - I_2 = 2\,I_1$$

and

$$2\,I_1 + I_2 - I_3 = 0 \qquad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$15 - 10\,I_1 - 20 - 5\,I_2 - 4I_3 + 40 = 0$$

$$10\,I_1 + 5\,I_2 + 4\,I_3 = 35 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.96 \text{ A}$$

$$I_2 = -0.04 \text{ A}$$

$$I_3 = 3.89 \text{ A}$$

$$I_{10\,\Omega} = I_1 = 1.96 \text{ A}$$

Example 2.29 *In the network shown in Fig. 2.46, find the power delivered by the 4 V source and voltage across the 2 Ω resistor.*

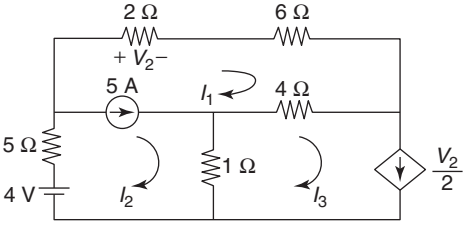


Fig. 2.46

Solution From Fig. 2.46,

$$V_2 = 2\,I_1 \qquad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \qquad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$4 - 5\,I_2 - 2\,I_1 - 6\,I_1 - 4(I_1 - I_3) - 1(I_2 - I_3) = 0$$

$$-12\,I_1 - 6\,I_2 + 5\,I_3 = -4 \qquad \text{...(iii)}$$

For Mesh 3,

$$I_3 = \frac{V_2}{2} = \frac{2\,I_1}{2} = I_1$$

$$I_1 - I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -2 \text{ A}$$

$$I_2 = 3 \text{ A}$$

$$I_3 = -2 \text{ A}$$

Power delivered by the 4 V source $= 4I_2 = 4(3) = 12$ W

$$V_{2\Omega} = 2I_1 = 2(-2) = -4 \text{ V}$$

Example 2.30 *Find currents I_1, I_2, I_3, I_4 of the network shown in Fig. 2.47.*

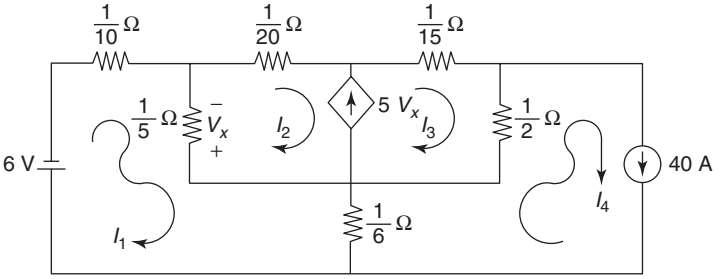


Fig. 2.47

From Fig. 2.47,

$$V_x = \frac{1}{5}(I_2 - I_1) \quad \ldots(i)$$

For Mesh 4,

$$I_4 = 40 \quad \ldots(ii)$$

Applying KVL to Mesh 1,

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}(I_1 - I_2) - \frac{1}{6}(I_1 - I_4) = 0$$

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}I_1 + \frac{1}{5}I_2 - \frac{1}{6}I_1 + \frac{1}{6}(40) = 0$$

$$-\frac{7}{15}I_1 + \frac{1}{5}I_2 = -\frac{2}{3} \quad \ldots(iii)$$

Mesh 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 5V_x = 5\left[\frac{1}{5}(I_2 - I_1)\right] = I_2 - I_1$$

$$I_1 - 2I_2 + I_3 = 0 \quad \ldots(iv)$$

Applying KVL to the outer path of the supermesh,

$$-\frac{1}{5}(I_2 - I_1) - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}(I_3 - I_4) = 0$$

$$-\frac{1}{5}I_2 + \frac{1}{5}I_1 - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}I_3 + \frac{1}{2}(40) = 0$$

$$\frac{1}{5}I_1 - \frac{1}{4}I_2 - \frac{17}{30}I_3 = -20 \quad \ldots(v)$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 10 \text{ A}$$
$$I_2 = 20 \text{ A}$$
$$I_3 = 30 \text{ A}$$
$$I_4 = 40 \text{ A}$$

2.5 NODE ANALYSIS

Node analysis is based on Kirchhoff's current law which states that the algebraic sum of currents meeting at a point is zero. Every junction where two or more branches meet is regarded as a node. One of the nodes in the network is taken as *reference node* or *datum node*. If there are n nodes in any network, the number of simultaneous equations to be solved will be $(n - 1)$.

Steps to be followed in Node Analysis

1. Assuming that a network has n nodes, assign a reference node and the reference directions, and assign a current and a voltage name for each branch and node respectively.
2. Apply KCL at each node except for the reference node and apply Ohm's law to the branch currents.
3. Solve the simultaneous equations for the unknown node voltages.
4. Using these voltages, find any branch currents required.

Example 2.31 *Calculate the current through 2 Ω resistor for the network shown in Fig. 2.48.*

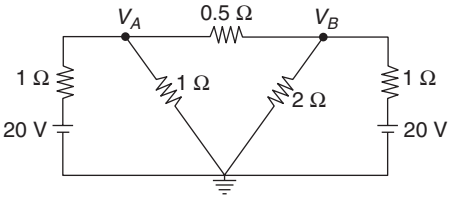


Fig. 2.48

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node A,

$$\frac{V_A - 20}{1} + \frac{V_A}{1} + \frac{V_A - V_B}{0.5} = 0$$

$$\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{0.5}\right) V_A - \frac{1}{0.5} V_B = \frac{20}{1}$$

$$4 V_A - 2 V_B = 20 \qquad \ldots\text{(i)}$$

Applying KCL at Node B,

$$\frac{V_B - V_A}{0.5} + \frac{V_B}{2} + \frac{V_B - 20}{1} = 0$$

$$-\frac{1}{0.5} V_A \left(\frac{1}{0.5} + \frac{1}{2} + \frac{1}{1}\right) V_B = \frac{20}{1}$$

$$-2 V_A + 3.5 V_B = 20 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_A = 11 \text{ V}$$
$$V_B = 12 \text{ V}$$

Current through the 2 Ω resistor $= \frac{V_B}{2} = \frac{12}{2} = 6$ A

Example 2.32 *Find the voltage at nodes 1 and 2 for the network shown in Fig. 2.49.*

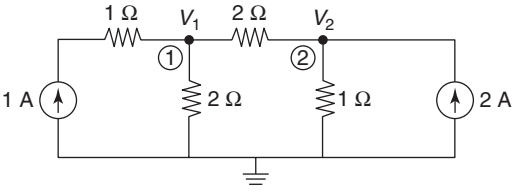


Fig. 2.49

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$1 = \frac{V_1}{2} + \frac{V_1 - V_2}{2}$$
$$\left(\frac{1}{2} + \frac{1}{2}\right) V_1 - \frac{1}{2} V_2 = 1$$
$$V_1 - 0.5\, V_2 = 1 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$2 = \frac{V_2}{1} + \frac{V_2 - V_1}{2}$$
$$-\frac{1}{2} V_1 + \left(1 + \frac{1}{2}\right) V_2 = 2$$
$$-0.5 V_1 + 1.5\, V_2 = 4 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 2 \text{ V}$$
$$V_2 = 2 \text{ V}$$

Example 2.33 *Find the current in the 100 Ω resistor for the network shown in Fig. 2.50.*

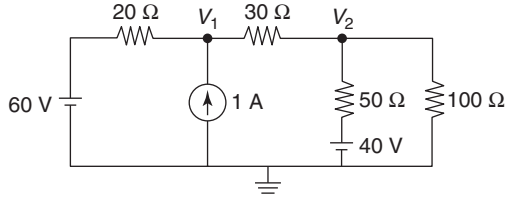


Fig. 2.50

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 60}{20} + \frac{V_1 - V_2}{30} = 1$$

$$\left(\frac{1}{20} + \frac{1}{30}\right) V_1 - \frac{1}{30} V_2 = \frac{60}{20} + 1$$

$$0.083\, V_1 - 0.033\, V_2 = 4 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{30} + \frac{V_2 - 40}{50} + \frac{V_2}{100} = 0$$

$$-\frac{1}{30} V_1 + \left(\frac{1}{30} + \frac{1}{50} + \frac{1}{100}\right) V_2 = \frac{40}{50}$$

$$-0.033\, V_1 + 0.063\, V_2 = 0.8 \qquad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$V_1 = 67.25 \text{ V}$$

$$V_2 = 48 \text{ V}$$

$$\text{Current through the 100 } \Omega \text{ resistor} = \frac{V_2}{100} = \frac{48}{100} = 0.48 \text{ A}$$

Example 2.34 *Find V_A and V_B for the network shown in Fig. 2.51.*

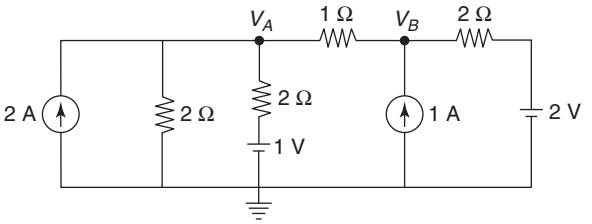


Fig. 2.51

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node *A*,

$$2 = \frac{V_A}{2} + \frac{V_A - 1}{2} + \frac{V_A - V_B}{1}$$

$$\left(\frac{1}{2} + \frac{1}{2} + 1\right) V_A - V_B = 2 + \frac{1}{2}$$

$$2V_A - V_B = 2.5 \qquad \ldots(i)$$

Applying KCL at Node *B*,

$$\frac{V_B - V_A}{1} + \frac{V_B - 2}{2} = 1 \qquad \ldots(ii)$$

$$-V_A + \left(1 + \frac{1}{2}\right) V_B = 1 + \frac{2}{2}$$

$$-V_A + 1.5\, V_B = 2$$

Solving Eqs (i) and (ii),

$$V_A = 2.875 \text{ V}$$

$$V_B = 3.25 \text{ V}$$

Example 2.35 *Find currents I_1, I_2 and I_3 for the network shown in Fig. 2.52.*

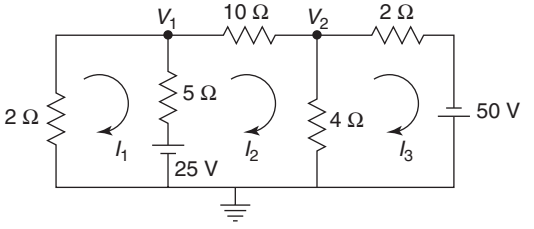


Fig. 2.52

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1}{2} + \frac{V_1 - 25}{5} + \frac{V_1 - V_2}{10} = 0$$

$$\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right) V_1 - \frac{1}{10} V_2 = \frac{25}{5}$$

$$0.8\, V_1 - 0.1\, V_2 = 5 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{10} + \frac{V_2}{4} + \frac{V_2 - (-50)}{2} = 0$$

$$-\frac{1}{10} V_1 + \left(\frac{1}{10} + \frac{1}{4} + \frac{1}{2}\right) V_2 = -\frac{50}{2}$$

$$-0.1\, V_1 + 0.85\, V_2 = -25 \qquad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$V_1 = 2.61 \text{V}$$

$$V_2 = -29.1 \text{ V}$$

$$I_1 = -\frac{V_1}{2} = -\frac{2.61}{2} = -1.31 \text{ A}$$

$$I_2 = \frac{V_1 - V_2}{10} = \frac{2.61 - (-2.91)}{10} = 3.17 \text{A}$$

$$I_3 = \frac{V_2 + 50}{2} = \frac{-29.1 + 50}{2} = 10.45 \text{ A}$$

Example 2.36 *Find currents I_1, I_2 and I_3 and voltages V_a and V_b for the network shown in Fig. 2.53.*

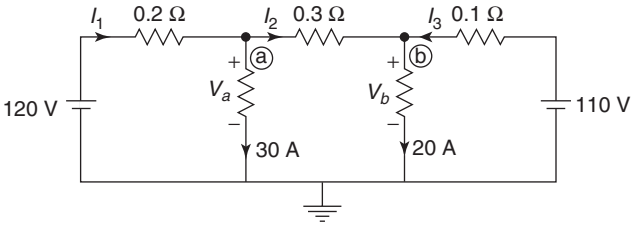


Fig. 2.53

Solution Applying KCL at Node *a*,

$$I_1 = 30 + I_2$$

$$\frac{120 - V_a}{0.2} = 30 + \frac{V_a - V_b}{0.3}$$

$$36 - 0.3\,V_a = 1.8 + 0.2\,V_a - 0.2\,V_b$$

$$0.5\,V_a - 0.2\,V_b = 34.2 \qquad \ldots(i)$$

Applying KCL at Node *b*,

$$I_2 + I_3 = 20$$

$$\frac{V_a - V_b}{0.3} + \frac{110 - V_b}{0.1} = 20$$

$$\frac{0.1\,V_a - 0.1\,V_b + 33 - 0.3\,V_b}{0.03} = 20$$

$$0.1\,V_a - 0.4\,V_b = -32.4 \qquad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$V_a = 112 \text{ V}$$

$$V_b = 109 \text{ V}$$

$$I_1 = \frac{120 - V_a}{0.2} = \frac{120 - 112}{0.2} = 40 \text{ A}$$

$$I_2 = \frac{V_a - V_b}{0.3} = \frac{112 - 109}{0.3} = 10 \text{ A}$$

$$I_3 = \frac{110 - V_b}{0.1} = \frac{110 - 109}{0.1} = 10 \text{ A}$$

Example 2.37 *Calculate the current through the 5 Ω resistor for the network shown in Fig. 2.54.*

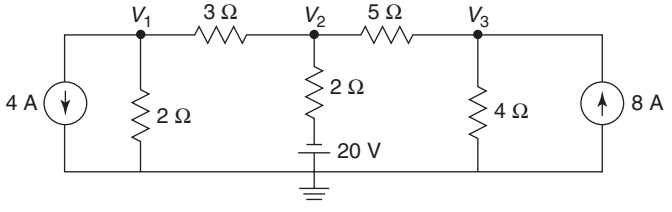


Fig. 2.54

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$4+\frac{V_1}{2}+\frac{V_1-V_2}{3}=0$$

$$\left(\frac{1}{2}+\frac{1}{3}\right)V_1-\frac{1}{3}V_2=4$$

$$0.83\,V_1-0.33\,V_2=-4 \quad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{3}+\frac{V_2-(-20)}{2}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{3}V_1+\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=-\frac{20}{2}$$

$$-0.33\,V_1+1.03\,V_2-0.2\,V_3=-10 \quad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{4}=8$$

$$-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{4}\right)V_3=8$$

$$-0.2\,V_2+0.45\,V_3=8 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1=-8.76\text{ V}$$

$$V_2=-9.92\text{ V}$$

$$V_3=13.37\text{ V}$$

$$\text{Current through the 5 }\Omega\text{ resistor}=\frac{V_3-V_2}{5}=\frac{13.37-(-9.92)}{5}=4.66\text{ A}$$

Example 2.38 *Find the voltage across the 5 Ω resistor for the network shown in Fig. 2.55.*

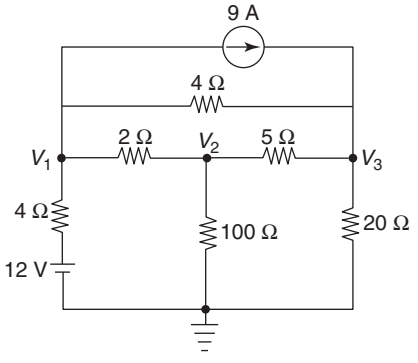


Fig. 2.55

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 12}{4} + \frac{V_1 - V_2}{2} + \frac{V_1 - V_3}{4} + 9 = 0$$

$$\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right) V_1 - \frac{1}{2} V_2 - \frac{1}{4} V_3 = -9 + \frac{12}{4}$$

$$V_1 - 0.5\, V_2 - 0.25\, V_3 = -6 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{2} + \frac{V_2}{100} + \frac{V_2 - V_3}{5} = 0$$

$$-\frac{1}{2} V_1 + \left(\frac{1}{2} + \frac{1}{100} + \frac{1}{5}\right) V_2 - \frac{1}{5} V_3 = 0$$

$$-0.5\, V_1 + 0.71\, V_2 - 0.2\, V_3 = 0 \qquad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3 - V_2}{5} + \frac{V_3}{20} + \frac{V_3 - V_1}{4} = 9$$

$$-\frac{1}{4} V_1 - \frac{1}{5} V_2 + \left(\frac{1}{5} + \frac{1}{20} + \frac{1}{4}\right) V_3 = 9$$

$$-0.25\, V_1 - 0.2\, V_2 + 0.5\, V_3 = 9 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 6.35 \text{ V}$$

$$V_2 = 11.76 \text{ V}$$

$$V_3 = 25.88 \text{ V}$$

Voltages across the 5 Ω resistor $= V_3 - V_2 = 25.88 - 11.76 = 14.12$ V

Example 2.39 *Determine the current through the 5 Ω resistor for the network shown in Fig. 2.56.*

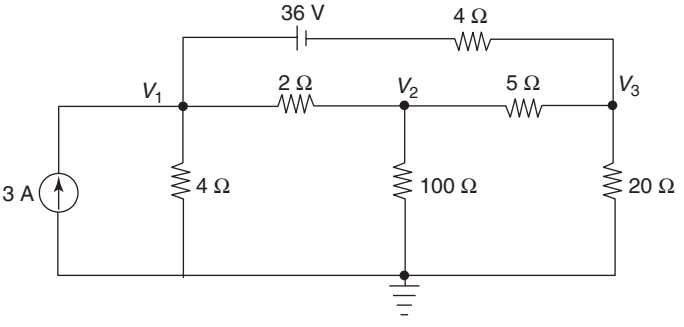


Fig. 2.56

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1}{4}+\frac{V_1-V_2}{2}+\frac{V_1-36-V_3}{4}=3$$

$$\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right)V_1-\frac{1}{2}V_2-\frac{1}{4}V_3=3+\frac{36}{4}$$

$$V_1-0.5\,V_2-0.25\,V_3=12 \qquad \ldots(\text{i})$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{2}+\frac{V_2}{100}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{2}V_1+\left(\frac{1}{2}+\frac{1}{100}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=0$$

$$-0.5\,V_1+0.71\,V_2-0.2\,V_3=0 \qquad \ldots(\text{ii})$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{20}+\frac{V_3-(-36)-V_1}{4}=0$$

$$-\frac{1}{4}V_1-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{20}+\frac{1}{4}\right)V_3=-9$$

$$-0.25\,V_1-0.2\,V_2+0.5\,V_3=-9 \qquad \ldots(\text{iii})$$

Solving Eqs (i), (ii) and (iii),

$$V_1=13.41\text{ V}$$
$$V_2=7.06\text{ V}$$
$$V_3=-8.47\text{ V}$$

$$\text{Current through the 5 }\Omega\text{ resistor}=\frac{V_2-V_3}{5}=\frac{7.06-(-8.47)}{5}=3.11\text{ A}$$

Example 2.40 *Find the voltage drop across the 5 Ω resistor in the network shown in Fig. 2.57.*

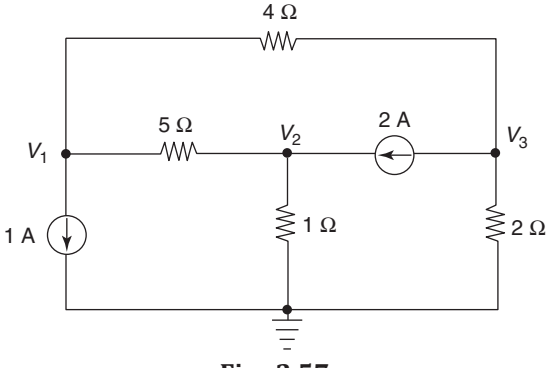


Fig 2.57

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$1+\frac{V_1-V_2}{5}+\frac{V_1-V_3}{4}=0$$

$$\left(\frac{1}{5}+\frac{1}{4}\right)V_1-\frac{1}{5}V_2-\frac{1}{4}V_3=-1$$

$$0.45\,V_1-0.2\,V_2-0.25\,V_3=-1 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{5}+\frac{V_2}{1}=2$$

$$-\frac{1}{5}V_1+\left(\frac{1}{5}+1\right)V_2=2$$

$$-0.2\,V_1+1.2\,V_2=2 \qquad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3}{2}+\frac{V_3-V_1}{4}+2=0$$

$$-\frac{1}{4}V_1+\left(\frac{1}{2}+\frac{1}{4}\right)V_3=-2$$

$$-0.25\,V_1+0.75\,V_3=-2 \qquad \ldots(iii)$$

.

Solving Eqs (i), (ii) and (iii),

$$V_1=-4\text{ V}$$

$$V_2=1\text{ V}$$

$$V_3=-4\text{ V}$$

$$V_{5\,\Omega}=V_2-V_1=1-(-4)=5\text{ V}$$

Example 2.41 *Find the power dissipated in the 6 Ω resistor for the network shown in Fig. 2.58.*

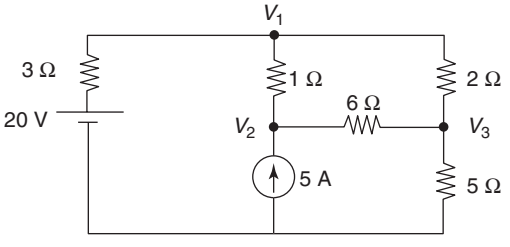


Fig. 2.58

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 20}{3} + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{2} = 0$$

$$\left(\frac{1}{3} + 1 + \frac{1}{2}\right) V_1 - V_2 - \frac{1}{2} V_3 = \frac{20}{3}$$

$$1.83\, V_1 - V_2 - 0.5\, V_3 = 6.67 \quad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{6} = 5$$

$$-V_1 + \left(1 + \frac{1}{6}\right) V_2 - \frac{1}{6} V_3 = 5$$

$$-V_1 + 1.17\, V_2 - 0.17\, V_3 = 5 \quad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{2} + \frac{V_3}{5} + \frac{V_3 - V_2}{6} = 0$$

$$-\frac{1}{2} V_1 - \frac{1}{6} V_2 + \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{6}\right) V_3 = 0$$

$$-0.5\, V_1 - 0.17\, V_2 + 0.87\, V_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 23.82 \text{ V}$$

$$V_2 = 27.4 \text{ V}$$

$$V_3 = 19.04 \text{ V}$$

$$I_{6\,\Omega} = \frac{V_2 - V_3}{6} = \frac{27.4 - 19.04}{6} = 1.39 \text{ A}$$

Power dissipated in the 6 Ω resistor $= (1.39)^2 \times 6 = 11.59$ W

Example 2.42 *Find the voltage V in the network shown in Fig. 2.59 which makes the current in the 10 Ω resistor zero.*

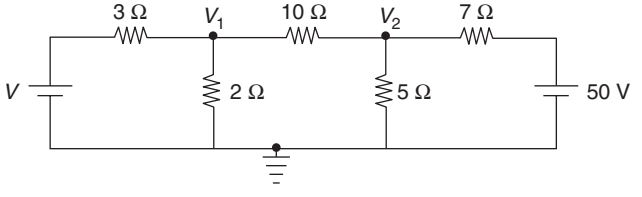


Fig. 2.59

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - V}{3} + \frac{V_1}{2} + \frac{V_1 - V_2}{10} = 0$$

$$\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{10}\right) V_1 - \frac{1}{10} V_2 - \frac{1}{3} V = 0$$

$$0.93\, V_1 - 0.1\, V_2 - 0.33\, V = 0 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - 50}{7} = 0$$

$$-\frac{1}{10} V_1 + \left(\frac{1}{10} + \frac{1}{5} + \frac{1}{7}\right) V_2 = \frac{50}{7}$$

$$-0.1\, V_1 + 0.44\, V_2 = 7.14 \qquad \ldots(ii)$$

$$I_{10\,\Omega} = \frac{V_1 - V_2}{10} = 0$$

$$V_1 - V_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V = 52.82 \text{ V}$$

Example 2.43 *Find V_1 and V_2 for the network shown in Fig. 2.60.*

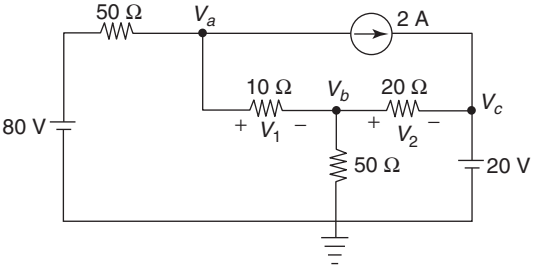


Fig. 2.60

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node *a*,

$$\frac{V_a - 80}{50} + \frac{V_a - V_b}{10} + 2 = 0$$

$$\left(\frac{1}{50} + \frac{1}{10}\right) V_a - \frac{1}{10} V_b = \frac{80}{50} - 2$$

$$0.12\, V_a - 0.1\, V_b = -0.4 \qquad \ldots(i)$$

Applying KCL at Node *b*,

$$\frac{V_b - V_a}{10} + \frac{V_b}{50} + \frac{V_b - V_c}{20} = 0$$

$$-\frac{1}{10} V_a + \left(\frac{1}{10} + \frac{1}{50} + \frac{1}{20}\right) V_b - \frac{1}{20} V_c = 0$$

$$-0.1\, V_a + 0.17\, V_b - 0.05\, V_c = 0 \qquad \ldots(ii)$$

Node *c* is directly connected to a voltage source of 20 V. Hence, we can write voltage equation at Node *c*.

$$V_c = 20 \quad \ldots(iii)$$

Solving Eqs (i), (ii), and (iii),

$$V_a = 3.08 \text{ V}$$
$$V_b = 7.69 \text{ V}$$
$$V_1 = V_a - V_b = 3.08 - 7.69 = -4.61 \text{ V}$$
$$V_2 = V_b - V_c = 7.69 - 20 = -12.31 \text{ V}$$

Example 2.44 *Find the voltage across the 100 Ω resistor for the network shown in Fig. 2.61.*

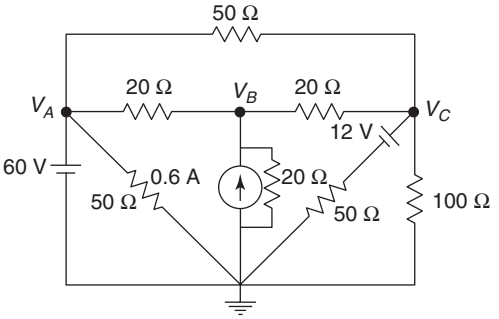


Fig. 2.61

Solution

Node *A* is directly connected to a voltage source of 20 V. Hence, we can write voltage equation at Node *A*.

$$V_A = 60 \quad \ldots(i)$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node *B*,

$$\frac{V_B - V_A}{20} + \frac{V_B - V_C}{20} + \frac{V_B}{20} = 0.6$$
$$-\frac{1}{20}V_A + \left(\frac{1}{20} + \frac{1}{20} + \frac{1}{20}\right)V_B - \frac{1}{20}V_C = 0.6$$
$$-0.05\,V_A + 0.15\,V_B - 0.05\,V_C = 0.6 \quad \ldots(ii)$$

Applying KCL at Node *C*,

$$\frac{V_C - V_A}{50} + \frac{V_C - V_B}{20} + \frac{V_C - 12}{50} + \frac{V_C}{100} = 0$$
$$-\frac{1}{50}V_A - \frac{1}{20}V_B + \left(\frac{1}{50} + \frac{1}{20} + \frac{1}{50} + \frac{1}{100}\right)V_C = \frac{12}{50}$$
$$-0.02\,V_A - 0.05\,V_B + 0.1\,V_C = 0.24 \quad \ldots(iii)$$

Solving Eqs (i), (ii), and (iii),

$$V_C = 31.68 \text{ V}$$

Voltages across the 100 Ω resistor = 31.68 V

EXAMPLES WITH DEPENDENT SOURCES

Example 2.45 *Find the voltage across the 5 Ω resistor in the network shown in Fig. 2.62.*

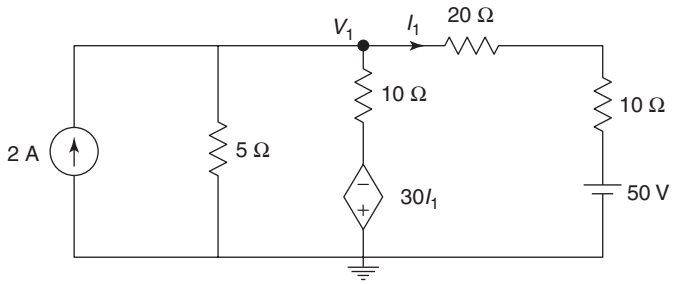


Fig. 2.62

Solution From Fig. 2.62,

$$I_1 = \frac{V_1 - 50}{20 + 10} = \frac{V_1 - 50}{30} \qquad \ldots(i)$$

Assume that the currents are moving away from the node.
Applying KCL at Node 1,

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\, I_1}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\left(\frac{V_1 - 50}{30}\right)}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{2V_1 - 50}{10} + \frac{V_1 - 50}{30} \qquad \ldots(ii)$$

Solving Eq. (ii),

$$V_1 = 20 \text{ V}$$

Voltage across the 5 Ω resistor = 20 V

Example 2.46 *For the network shown in Fig. 2.63, find the voltage V_x.*

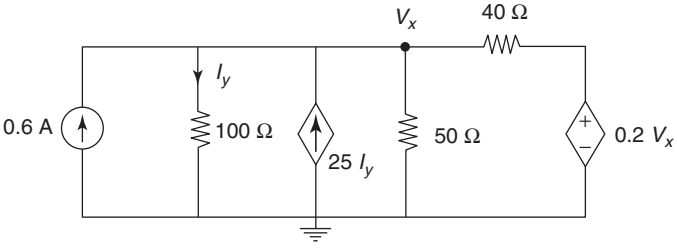


Fig. 2.63

Solution From Fig. 2.63,

$$I_y = \frac{V_x}{100} \qquad \ldots(i)$$

Assume that the currents are moving away from the node.
Applying KCL at Node x,

$$25 I_y + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{V_x - 0.2 V_x}{40}$$

$$25\left(\frac{V_x}{100}\right) + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{0.8 V_x}{40}$$

$$\left(\frac{1}{4} - \frac{1}{100} - \frac{1}{50} - \frac{0.8}{40}\right) V_x = -0.6$$

$$0.2 V_x = -0.6$$

$$V_x = -3 \text{ V}$$

Example 2.47

For the network shown in Fig. 2.64, find voltages V_1 and V_2.

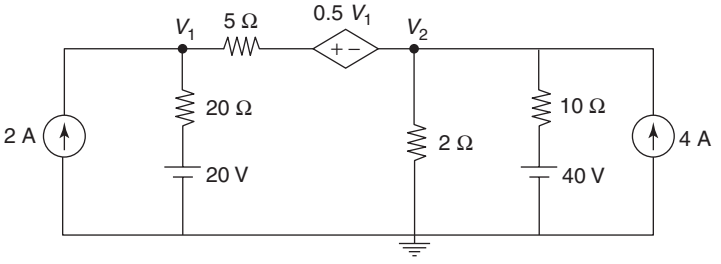


Fig. 2.64

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1 - 20}{20} + \frac{V_1 - 0.5 V_1 - V_2}{5}$$

$$\left(\frac{1}{20} + \frac{1}{5} - \frac{0.5}{5}\right) V_1 - \frac{1}{5} V_2 = 3$$

$$0.15 V_1 - 0.2 V_2 = 3 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 + 0.5 V_1 - V_1}{5} + \frac{V_2}{2} + \frac{V_2 - 40}{10} = 4$$

$$\left(\frac{0.5}{5} - \frac{1}{5}\right) V_1 + \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{10}\right) V_2 = 4 + \frac{40}{10}$$

$$-0.1 V_1 + 0.8 V_2 = 8 \qquad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$V_1 = 40 \text{ V}$$

$$V_2 = 15 \text{ V}$$

Example 2.48 *Determine the voltages V_1 and V_2 in the network of Fig. 2.65.*

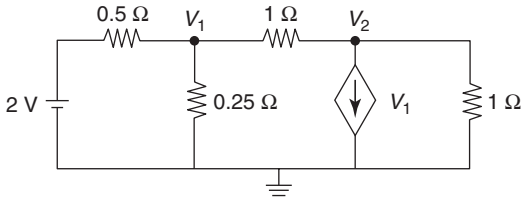


Fig. 2.65

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 2}{0.5} + \frac{V_1}{0.25} + \frac{V_1 - V_2}{1} = 0$$

$$\left(\frac{1}{0.5} + \frac{1}{0.25} + 1\right) V_1 - V_2 = \frac{2}{0.5}$$

$$7V_7 - V_2 = 4 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} + V_1 = 0$$

$$2V_2 = 0$$

$$V_2 = 0 \qquad \ldots(ii)$$

From Eq. (i),

$$V_1 = \frac{4}{7}\text{ V}$$

Example 2.49 *In the network of Fig. 2.66, find the node voltages V_1, V_2 and V_3.*

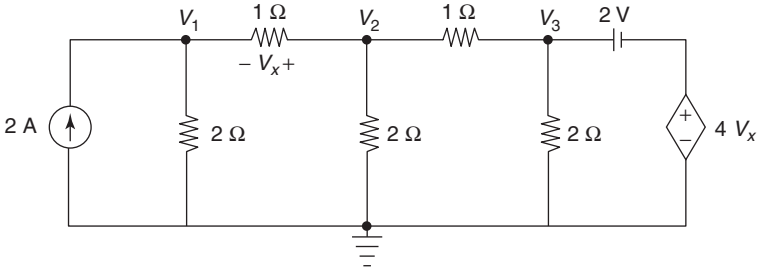


Fig. 2.66

Solution From Fig. 2.66,

$$V_x = V_2 - V_1 \qquad \ldots(i)$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1}{2} + \frac{V_1 - V_2}{1}$$

$$\left(\frac{1}{2} + 1\right) V_1 - V_2 = 2$$

$$1.5\, V_1 - V_2 = 2 \qquad \ldots(ii)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{2} + \frac{V_2 - V_3}{1} = 0$$

$$-V_1 + \left(1 + \frac{1}{2} + 1\right) V_2 - V_3 = 0$$

$$-V_1 + 2.5\, V_2 - V_3 = 0 \qquad \ldots(iii)$$

At Node 3,

$$V_3 - 4\, V_x = 2$$

$$V_3 - 4(V_2 - V_1) = 2$$

$$4\, V_1 - 4\, V_2 + V_3 = 2 \qquad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$V_1 = -1.33 \text{ V}$$

$$V_2 = -4 \text{ V}$$

$$V_3 = -8.67 \text{ V}$$

Example 2.50 *For the network shown in Fig. 2.67, find the node voltages V_1 and V_2.*

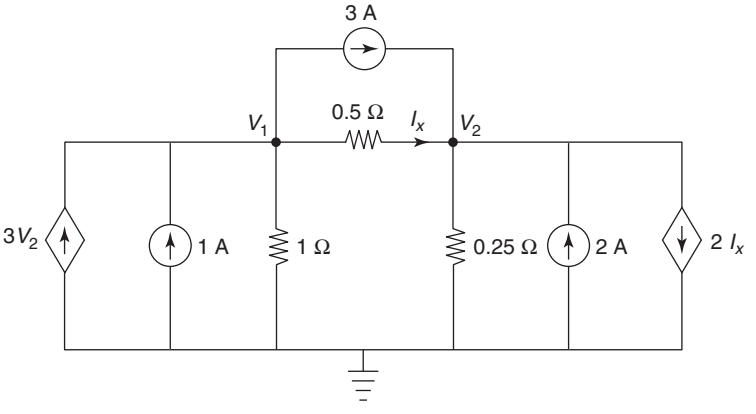


Fig. 2.67

Solution From Fig. 2.67,

$$I_x = \frac{V_1 - V_2}{0.5} \qquad \ldots(i)$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$3V_2 + 1 = \frac{V_1}{1} + \frac{V_1 - V_2}{0.5} + 3$$

$$\left(1 + \frac{1}{0.5}\right)V_1 - \left(3 + \frac{1}{0.5}\right)V_2 = -2$$

$$3V_1 - 5V_2 = -2 \qquad \ldots(ii)$$

Applying KCL at Node 2,

$$3 + 2 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2I_x$$

$$5 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2\left(\frac{V_1 - V_2}{0.5}\right)$$

$$\left(-\frac{1}{0.5} + \frac{2}{0.5}\right)V_1 + \left(\frac{1}{0.5} + \frac{1}{0.25} - \frac{2}{0.5}\right)V_2 = 5$$

$$2V_1 + 2V_2 = 5 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$V_1 = 1.31 \text{ V}$$
$$V_2 = 1.19 \text{ V}$$

Example 2.51 *Find voltages V_1 and V_2 in the network shown in Fig. 2.68.*

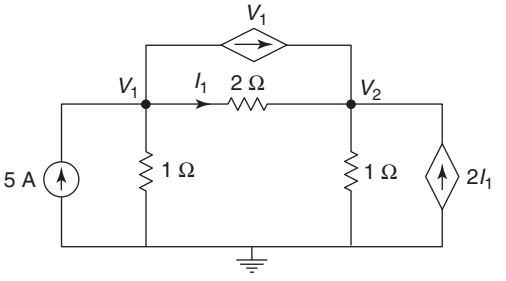


Fig. 2.68

Solution From Fig. 2.68,

$$I_1 = \frac{V_1 - V_2}{2} \qquad \ldots(i)$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$5 = \frac{V_1}{1} + \frac{V_1 - V_2}{2} + V_1$$

$$\left(1 + \frac{1}{2} + 1\right) V_1 - \frac{1}{2} V_2 = 5$$

$$2.5\,V_1 - 0.5\,V_2 = 5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} = 2I_1 + V_1$$

$$V_2 - V_1 + V_2 = 2\left(\frac{V_1 - V_2}{2}\right) + V_1$$

$$3\,V_1 = 3\,V_2$$

$$V_1 = V_2 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = 2.5 \text{ V}$$

$$V_2 = 2.5 \text{ V}$$

Example 2.52 *Find the power supplied by the 10 V source in the network shown in Fig. 2.69.*

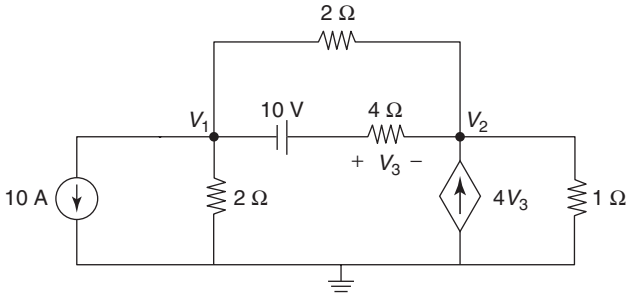


Fig. 2.69

Solution Assume that the currents are moving away from the nodes.
From Fig. 2.69,

$$V_3 = V_1 + 10 - V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 1,

$$10 + \frac{V_1}{2} + \frac{V_1 + 10 - V_2}{4} + \frac{V_1 - V_2}{2} = 0$$

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right) V_1 - \left(\frac{1}{4} + \frac{1}{2}\right) V_2 = -10 - \frac{10}{4}$$

$$1.25\,V_1 - 0.75\,V_2 = -12.5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2-10-V_1}{4}+\frac{V_2-V_1}{2}+\frac{V_2}{1}=4V_3$$

$$\frac{V_2-10-V_1}{4}+\frac{V_2-V_1}{2}+\frac{V_2}{1}=4(V_1+10-V_2)$$

$$\left(-\frac{1}{4}-\frac{1}{2}-4\right)V_1+\left(\frac{1}{4}+\frac{1}{2}+1+4\right)V_2=\frac{10}{4}+40$$

$$-4.75V_1+5.75V_2=42.5 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$V_1=-11.03\text{ V}$$

$$V_2=-1.72\text{ V}$$

$$I_{10\text{ V}}=\frac{V_1+10-V_2}{4}=\frac{-11.03+10-(-1.72)}{4}=0.173\text{ A}$$

Power supplied by the 10 V source = $10\times0.173=1.73$ W

Example 2.53 *For the network shown in Fig. 2.70, find voltages V_1 and V_2.*

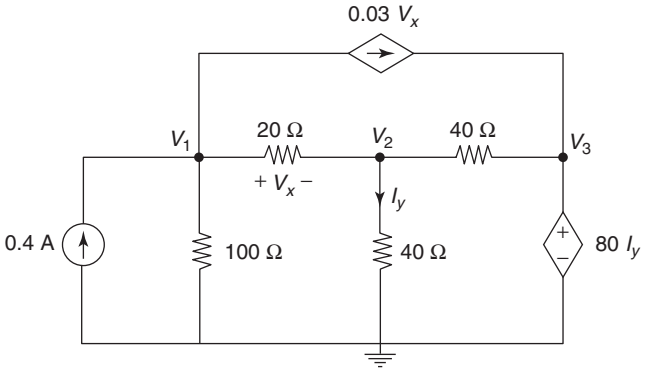


Fig. 2.70

Solution From Fig. 2.70,

$$V_x=V_1-V_2 \qquad \ldots(i)$$

$$I_y=\frac{V_2}{40} \qquad \ldots(ii)$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$0.4=\frac{V_1}{100}+\frac{V_1-V_2}{20}+0.03V_x$$

$$0.4=\frac{V_1}{100}+\frac{V_1-V_2}{20}+0.03(V_1-V_2)$$

$$\left(\frac{1}{100}+\frac{1}{20}+0.03\right)V_1-\left(\frac{1}{20}+0.03\right)V_2=0.4$$

$$0.09V_1-0.08V_2=0.4 \qquad \ldots(iii)$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{20} + \frac{V_2}{40} + \frac{V_2 - V_3}{40} = 0$$

$$-\frac{1}{20}V_1 + \left(\frac{1}{20} + \frac{1}{40} + \frac{1}{40}\right)V_2 - \frac{1}{40}V_3 = 0$$

$$-0.05\,V_1 + 0.1\,V_2 - 0.025\,V_3 = 0 \quad \ldots(iv)$$

For Node 3,

$$V_3 = 80\,I_y = 80\left(\frac{V_2}{40}\right) = 2\,V_2$$

$$2\,V_2 - V_3 = 0 \quad \ldots(v)$$

Solving Eqs (iii), (iv) and (v),

$$V_1 = 40 \text{ V}$$
$$V_2 = 40 \text{ V}$$
$$V_3 = 80 \text{ V}$$

Example 2.54 *Find voltages V_a, V_b and V_c in the network shown in Fig. 2.71.*

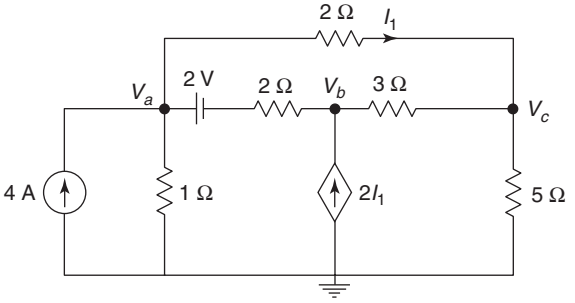


Fig. 2.71

Solution From Fig. 2.71,

$$I_1 = \frac{V_a - V_c}{2}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node *a*,

$$4 = \frac{V_a}{1} + \frac{V_a - V_c}{2} + \frac{V_a - 2 - V_b}{2}$$

$$\left(1 + \frac{1}{2} + \frac{1}{2}\right)V_a - \frac{1}{2}V_b - \frac{1}{2}V_c = 5$$

$$2\,V_a - 0.5\,V_b - 0.5\,V_c = 5 \quad \ldots(i)$$

Applying KCL at Node *b*,

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2I_1$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2\left(\frac{V_a - V_c}{2}\right)$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = V_a - V_c$$

$$\left(-\frac{1}{2} - 1\right)V_a + \left(\frac{1}{2} + \frac{1}{3}\right)V_b + \left(1 - \frac{1}{3}\right)V_c = -1$$

$$-1.5\,V_a + 0.83\,V_b + 0.67\,V_c = -1 \qquad \ldots(ii)$$

Applying KCL at Node *c*,

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = I_1$$

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = \frac{V_a - V_c}{2}$$

$$-\frac{1}{2}V_a - \frac{1}{3}V_b + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{2}\right)V_c = 0$$

$$-0.5\,V_a - 0.33\,V_b + 1.033\,V_c = 0 \qquad \ldots(iii)$$

Solving Eqs (i), (ii), and (iii),

$$V_a = 4.303 \text{ V}$$

$$V_b = 3.88 \text{ V}$$

$$V_c = 3.33 \text{ V}$$

2.6 SUPERNODE ANALYSIS

Nodes that are connected to each other by voltage sources, but not to the reference node by a path of voltage sources, form a *supernode*. A supernode requires one node voltage equation, that is, a KCL equation. The remaining node voltage equations are KVL equations.

Example 2.55 *Determine the current in the 5 Ω resistor for the network shown in Fig. 2.72.*

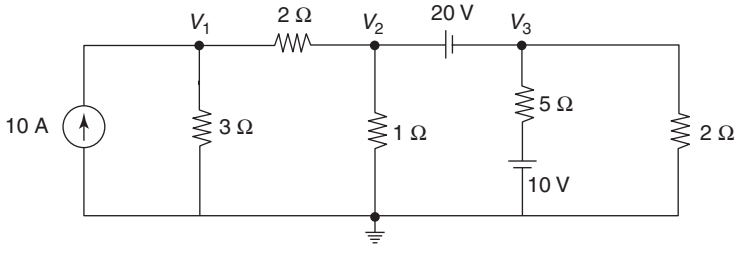


Fig. 2.72

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

$$\left(\frac{1}{3} + \frac{1}{2}\right) V_1 - \frac{1}{2} V_2 = 10$$

$$0.83 V_1 - 0.5 V_2 = 10 \quad \ldots(i)$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_2 - V_3 = 20 \quad \ldots(ii)$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

$$-\frac{1}{2} V_1 + \left(\frac{1}{2} + 1\right) V_2 + \left(\frac{1}{5} + \frac{1}{2}\right) V_3 = 2$$

$$-0.5 V_1 + 1.5 V_2 + 0.7 V_3 = 2 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 19.04 \text{ V}$$

$$V_2 = 11.6 \text{ V}$$

$$V_3 = -8.4 \text{ V}$$

$$I_{5\Omega} = \frac{V_3 - 10}{5} = \frac{-8.4 - 10}{5} = -3.68 \text{ A}$$

Example 2.56 *Find the power delivered by the 5 A current source in the network shown in Fig. 2.73.*

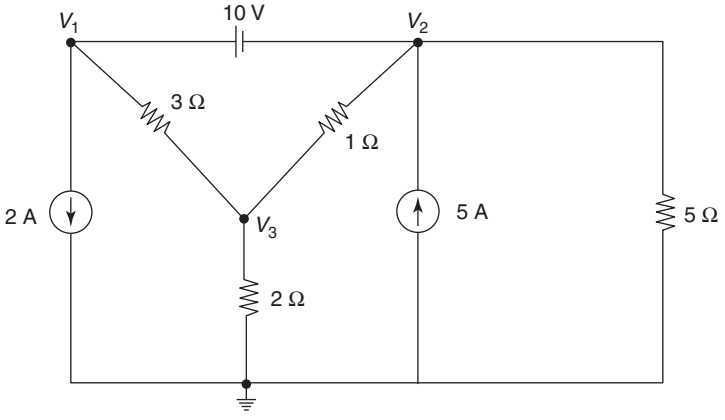


Fig. 2.73

Solution Assume that the currents are moving away from the nodes.
Nodes 1 and 2 will form a supernode.
Writing voltage equation for the supernode,

$$V_1 - V_2 = 10 \qquad \ldots(i)$$

Applying KCL at the supernode,

$$2 + \frac{V_1 - V_3}{3} + \frac{V_2}{5} + \frac{V_2 - V_3}{1} = 5$$

$$\frac{1}{3}V_1 + \left(\frac{1}{5} + 1\right)V_2 - \left(\frac{1}{3} + 1\right)V_3 = 3$$

$$0.33\,V_1 + 1.2\,V_2 - 1.33\,V_3 = 3 \qquad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{3} + \frac{V_3 - V_2}{1} + \frac{V_3}{2} = 0$$

$$-\frac{1}{3}V_1 - V_2 + \left(\frac{1}{3} + 1 + \frac{1}{2}\right)V_3 = 0$$

$$-0.33\,V_1 - V_2 + 1.83\,V_3 = 0 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 13.72 \text{ V}$$

$$V_2 = 3.72 \text{ V}$$

$$V_3 = 4.51 \text{ V}$$

Power delivered by the 5 A source = $5\,V_2 = 5 \times 3.72 = 18.6$ W

Example 2.57 *In the network of Fig. 2.74, find the node voltages V_1, V_2 and V_3.*

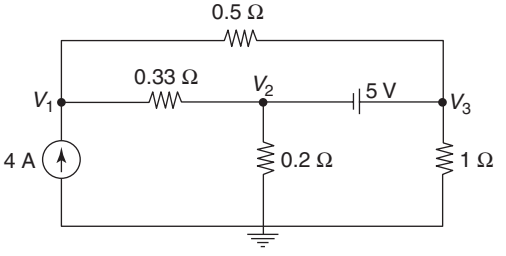


Fig. 2.74

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$4 = \frac{V_1 - V_2}{0.33} + \frac{V_1 - V_3}{0.5}$$

$$\left(\frac{1}{0.33} + \frac{1}{0.5}\right)V_1 - \frac{1}{0.33}V_2 - \frac{1}{0.5}V_3 = 4$$

$$5.03\,V_1 - 3.03\,V_2 - 2\,V_3 = 4 \qquad \ldots(i)$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_3 - V_2 = 5 \quad \ldots(ii)$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{0.33} + \frac{V_2}{0.2} + \frac{V_3}{1} + \frac{V_3 - V_1}{0.5} = 0$$

$$\left(-\frac{1}{0.33} - \frac{1}{0.5}\right)V_1 + \left(\frac{1}{0.33} + \frac{1}{0.2}\right)V_2 + \left(1 + \frac{1}{0.5}\right)V_3 = 0$$

$$-5.03\,V_1 + 8.03\,V_2 + 3\,V_3 = 0 \quad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 2.62 \text{ V}$$
$$V_2 = -0.17 \text{ V}$$
$$V_3 = 4.83 \text{ V}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.58 *For the network shown in Fig. 2.75, determine the voltage V_x.*

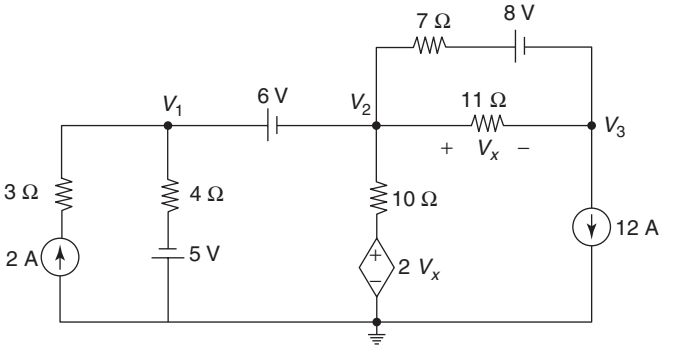


Fig. 2.75

Solution From Fig. 2.75,

$$V_x = V_2 - V_3$$

Assume that the currents are moving away from the nodes.
Node 1 and 2 will form a supernode.
Writing voltage equations for the supernode,

$$V_1 - V_2 = 6 \quad \ldots(i)$$

Applying KCL at the supernode,

$$2 = \frac{V_1 + 5}{4} + \frac{V_2 - 2V_x}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$2 = \frac{V_1 + 5}{4} + \frac{V_2 - 2(V_2 - V_3)}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$\frac{1}{4}V_1 + \left(\frac{1}{10} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11}\right)V_2 + \left(\frac{1}{5} - \frac{1}{7} - \frac{1}{11}\right)V_3 = 2 - \frac{5}{4} + \frac{8}{7}$$

$$0.25\,V_1 + 0.133\,V_2 - 0.033\,V_3 = 1.89 \quad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3 - V_2}{11} + \frac{V_3 + 8 - V_2}{7} + 12 = 0$$

$$\left(-\frac{1}{11} - \frac{1}{7}\right)V_2 + \left(\frac{1}{11} + \frac{1}{7}\right)V_3 = -12 - \frac{8}{7}$$

$$-0.233\,V_2 + 0.233\,V_3 = -13.14 \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 1.8 \text{ V}$$
$$V_2 = -4.2 \text{ V}$$
$$V_3 = -60.6 \text{ V}$$

Example 2.59 *Find the node voltages in the network shown in Fig. 2.76.*

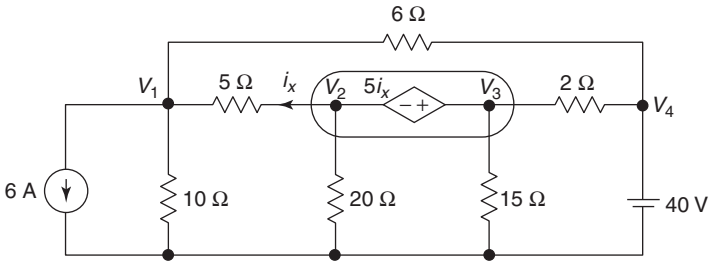


Fig. 2.76

Solution From Fig. 2.76,

$$I_x = \frac{V_2 - V_1}{5} \qquad \ldots(i)$$

For Node 4,

$$V_4 = 40 \qquad \ldots(ii)$$

Applying KCL at Node 1,

$$6 + \frac{V_1}{10} + \frac{V_1 - V_2}{5} + \frac{V_1 - V_4}{6} = 0$$

$$6 + \frac{V_1}{10} + \frac{V_1 - V_2}{5} + \frac{V_1 - 40}{6} = 0$$

$$\left(\frac{1}{10} + \frac{1}{5} + \frac{1}{6}\right)V_1 - \frac{1}{5}V_2 = \frac{40}{6} - 6$$

$$\frac{7}{15}V_1 - \frac{1}{5}V_2 = \frac{2}{3} \qquad \ldots(iii)$$

Nodes 2 and 3 will form a supernode,
Writing voltage equation for the supernode,

$$V_3 - V_2 = 5\, I_x = 5\left(\frac{V_2 - V_1}{5}\right) = V_2 - V_1$$

$$V_1 - 2V_2 + V_3 = 0 \qquad \ldots(\text{iv})$$

Applying KCL to the supernode,

$$\frac{V_2 - V_1}{5} + \frac{V_2}{20} + \frac{V_3}{15} + \frac{V_3 - V_4}{2} = 0$$

$$\frac{V_2 - V_1}{5} + \frac{V_2}{20} + \frac{V_3}{15} + \frac{V_3 - 40}{2} = 0$$

$$-\frac{1}{5}V_1 + \left(\frac{1}{5} + \frac{1}{20}\right)V_2 + \left(\frac{1}{15} + \frac{1}{2}\right)V_3 = 20$$

$$-\frac{1}{5}V_1 + \frac{1}{4}V_2 + \frac{17}{30}V_3 = 20 \qquad \ldots(\text{v})$$

Solving Eqs (iii), (iv) and (v),

$$V_1 = 10\,\text{V}$$
$$V_2 = 20\,\text{V}$$
$$V_3 = 30\,\text{V}$$
$$V_4 = 40\,\text{V}$$

Example 2.60 *Find the node voltages in the network shown in Fig. 2.77.*

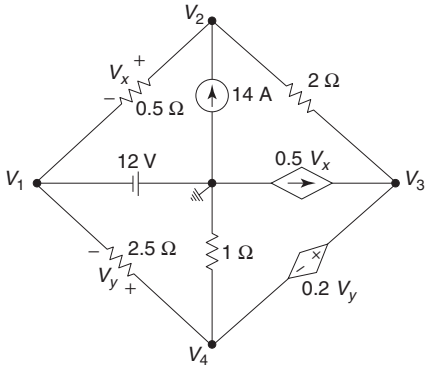


Fig. 2.77

Solution Selecting the central node as reference node,

$$V_1 = -12\,\text{V} \qquad \ldots(\text{i})$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{0.5} + \frac{V_2 - V_3}{2} = 14$$

$$-\frac{1}{0.5}V_1 + \left(\frac{1}{0.5} + \frac{1}{2}\right)V_2 - \frac{1}{2}V_3 = 14$$

$$-2V_1 + 2.5V_2 - 0.5V_3 = 14 \quad \ldots(ii)$$

Nodes 3 and 4 will form a supernode,
Writing voltage equation for the supernode,

$$V_3 - V_4 = 0.2V_y = 0.2(V_4 - V_1)$$

$$0.2V_1 + V_3 - 1.2V_4 = 0 \quad \ldots(iii)$$

Applying KCL to the supernode,

$$\frac{V_3 - V_2}{2} - 0.5V_x + \frac{V_4}{1} + \frac{V_4 - V_1}{2.5} = 0$$

$$\frac{V_3 - V_2}{2} - 0.5(V_2 - V_1) + V_4 + \frac{V_4 - V_1}{2.5} = 0$$

$$\left(0.5 - \frac{1}{2.5}\right)V_1 - \left(\frac{1}{2} + 0.5\right)V_2 + \frac{1}{2}V_3 + \left(1 + \frac{1}{2.5}\right)V_4 = 0$$

$$0.1V_1 - V_2 + 0.5V_3 + 1.4V_3 = 0 \quad \ldots(iv)$$

Solving Eqs (i), (ii), (iii) and (iv),

$$V_1 = -12 \text{ V}$$
$$V_2 = -4 \text{ V}$$
$$V_3 = 0$$
$$V_4 = -2 \text{ V}$$

Exercises

MESH ANALYSIS

2.1 Find the current through the 10 Ω resistor in the network shown in Fig. 2.78.

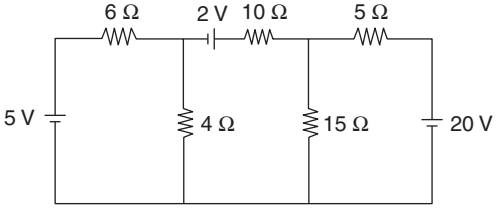


Fig. 2.78

[0.68 A]

2.2 Find the current through the 20 Ω resistor in the network shown in Fig. 2.79.

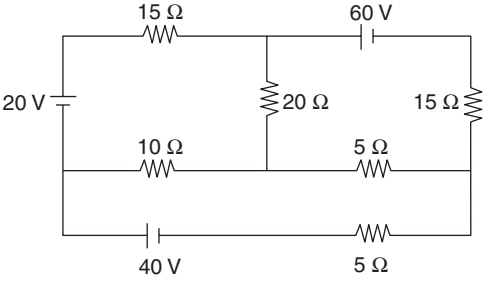


Fig. 2.80

[1.46 A]

2.3 Find the current through the 10 Ω resistor in the network shown in Fig. 2.80.

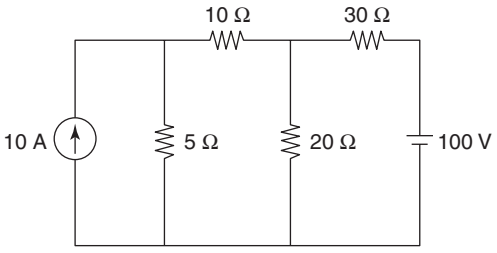


Fig. 2.80

[0.37 A]

2.4 Find the current through the 1 Ω resistor in the network shown in Fig. 2.81.

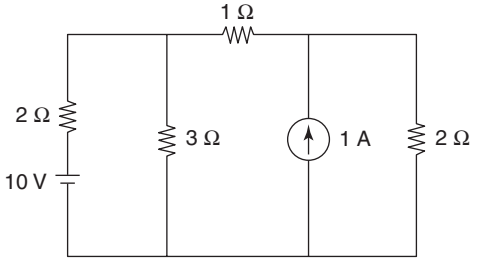


Fig. 2.81

[0.95 A]

2.5 Find the current through the 4 Ω resistor in the network shown in Fig. 2.82.

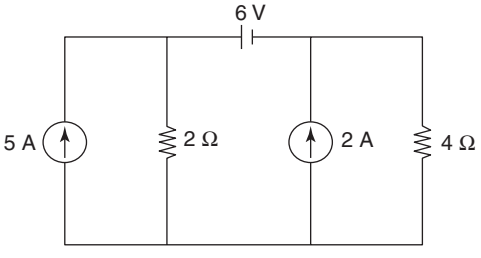


Fig. 2.82

[1.33 A]

2.6 Find currents I_x and I_y in the network shown in Fig. 2.83.

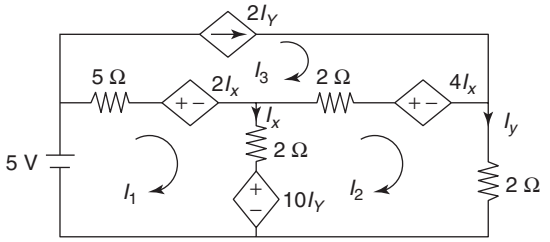


Fig. 2.83

[0.5 A, 0.1 A]

2.7 In the network shown in Fig. 2.84, find V_3 if element A is a

(i) short circuit

(ii) 5 Ω resistor

(iii) 20 V independent voltage source, positive reference on the right

(iv) dependent voltage source of 1.5 i_1, with positive reference on the right

(v) dependent current source 5 i_1, arrow directed to the right

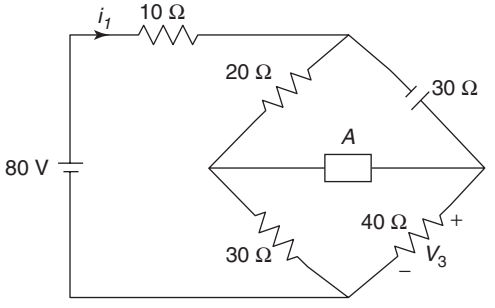


Fig. 2.84

[69.4 V, 72.38 V, 73.68 V, 70.71 V, 97.39 V]

2.8 Find currents I_1, I_2, and I_3 in the network shown in Fig. 2.85.

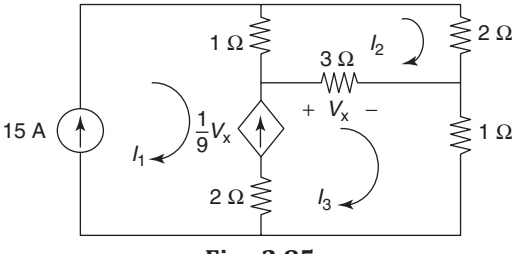


Fig. 2.85

[15 A, 11 A, 17 A]

2.9 Find currents I_x in the network shown in Fig. 2.86.

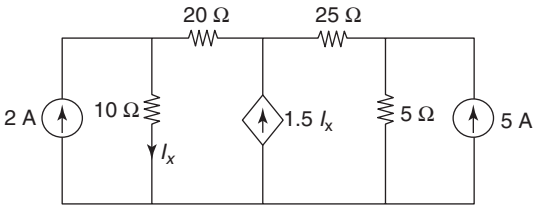


Fig. 2.86

[8.33 A]

2.10 Find currents I_1 in the network shown in Fig. 2.87.

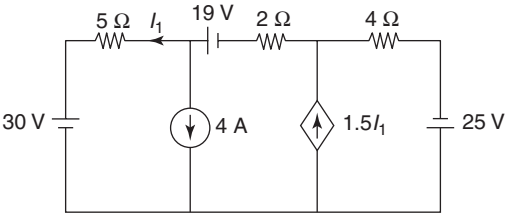


Fig. 2.87

[–12 A]

NODE ANALYSIS

2.11 Find the current I_x in the network shown in Fig. 2.88.

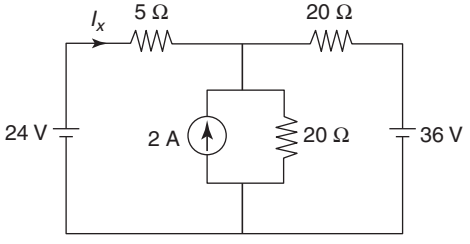


Fig. 2.88

[–0.93 A]

2.12 Find V_A and V_B in the network shown in Fig. 2.89.

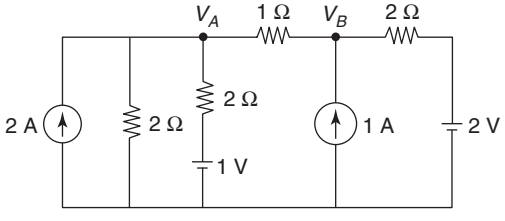


Fig. 2.89

[2.88 V, 3.25 V]

2.13 Find the current through the 6 Ω resistor in the network shown in Fig. 2.90.

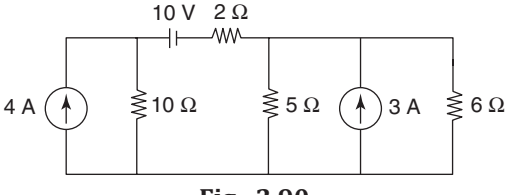


Fig. 2.90

[2.04 A]

2.14 Calculate the current through the 10 Ω resistor in the network shown in Fig. 2.91.

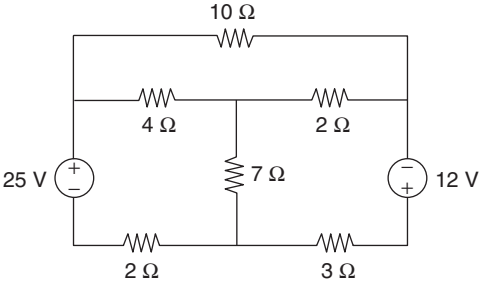


Fig. 2.91

[1.62 A]

2.15 Find the current through the branch *ab* in the network shown in Fig. 2.92.

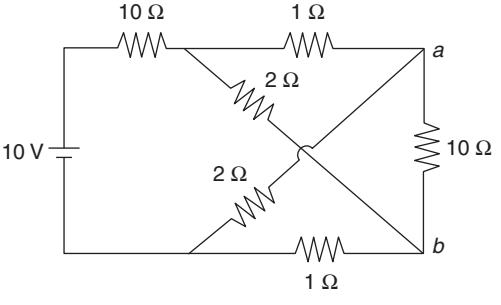


Fig. 2.92

[0.038 A]

2.16 Find the current through the 4 Ω resistor in the network shown in Fig. 2.93.

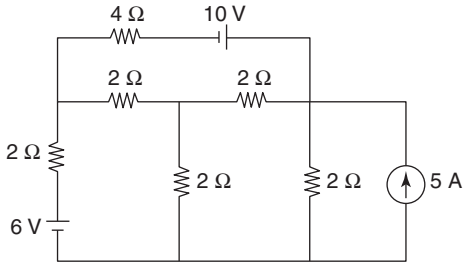


Fig. 2.93

[1.34 A]

2.17 Find the current through the 4 Ω resistor in the network shown in Fig. 2.94.

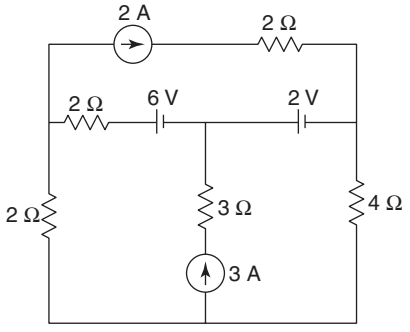


Fig. 2.94

[1 A]

2.18 Find the voltage V_x in the network shown in Fig. 2.95.

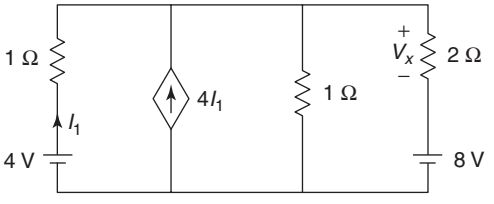


Fig. 2.95

[–4.31 V]

2.19 Find the currents V_x in the network shown in Fig. 2.96.

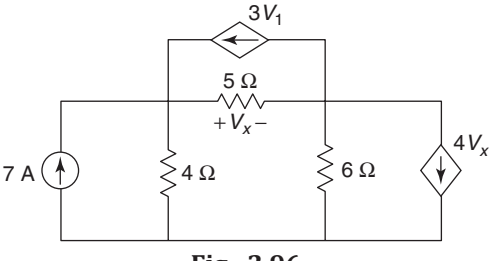


Fig. 2.96

[2.09 V]

2.20 Find the voltage V_x in the network shown in Fig. 2.97.

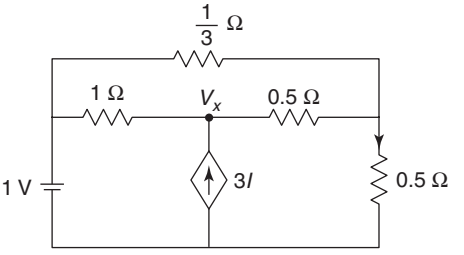


Fig. 2.97

[6.2 V]

2.21 Determine V_1 in the network shown in Fig. 2.98.

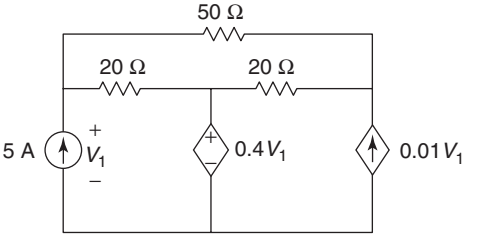


Fig. 2.98

[140 V]

2.22 Find the voltage V_y in the network shown in Fig. 2.99.

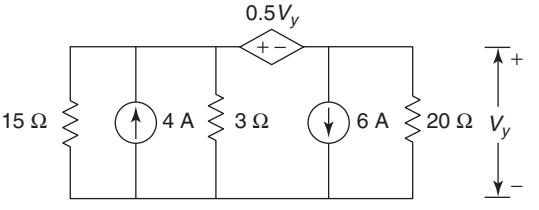


Fig. 2.99

[–10 V]

2.23 Find the voltage V_2 in the network shown in Fig. 2.100.

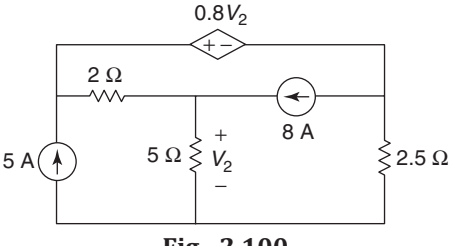


Fig. 2.100

[25.9 V]

Objective–Type Questions

2.1 Two electrical sub-networks N_1 and N_2 are connected through three resistors as shown in Fig. 2.101. The voltages across the 5 Ω resistor and 1 Ω resistor are given to be 10 V and 5 V respectively. Then the voltage across the 15 Ω resistor is

(a) −105 V (b) 105 V
(c) −15 V (d) 15 V

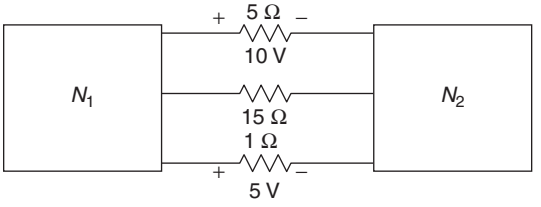


Fig. 2.101

2.2 The nodal method of circuit analysis is based on

(a) KVL and Ohm's law
(b) KCL and Ohm's law
(c) KCL and KVL
(d) KCL, KVL and Ohm's law

2.3 The voltage across terminals *a* and *b* in Fig. 2.102 is

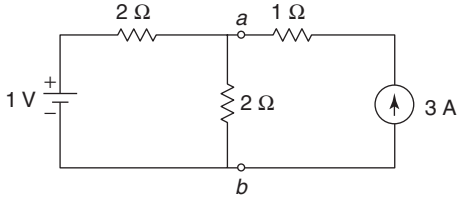


Fig. 2.102

(a) 0.5 V (b) 3 V
(c) 3.5 V (d) 4 V

2.4 The voltage V_0 in Fig. 2.103 is

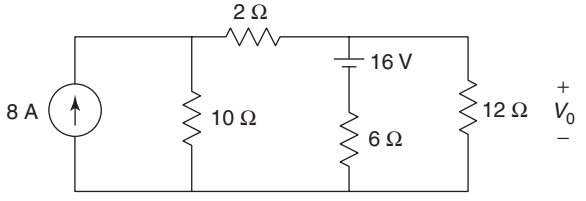


Fig. 2.103

(a) 48 V (b) 24 V
(c) 36 V (d) 28 V

2.5 The dependent current source shown in Fig. 2.104.

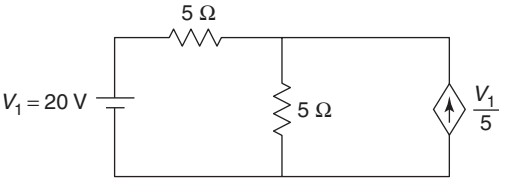


Fig. 2.104

(a) delivers 80 W
(b) absorbs 80 W
(c) delivers 40 W
(d) absorbs 40 W

2.6 If $V = 4$ in Fig. 2.105, the value of I_s is given by

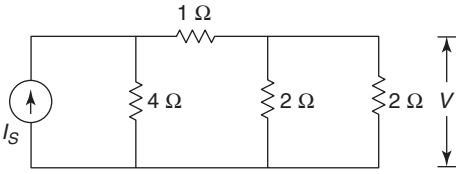


Fig. 2.105

(a) 6 A (b) 2.5 A
(c) 12 A (d) none of these

2.7 The value of V_x, V_y and V_z in Fig. 2.106 shown are

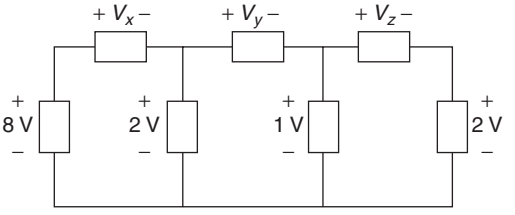


Fig. 2.106

(a) −6, 3, −3 (b) −6, −3, 1
(c) 6, 3, 3 (d) 6, 1, 3

2.8 The circuit shown in Fig. 2.107 is equivalent to a load of

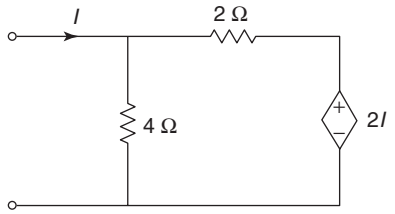


Fig. 2.107

(a) $\frac{4}{3}\,\Omega$ (b) $\frac{8}{3}\,\Omega$

(b) $4\,\Omega$ (d) $2\,\Omega$

2.9 In the network shown in Fig. 2.108, the effective resistance faced by the voltage source is

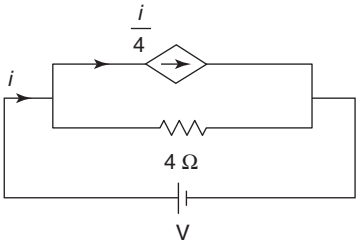


Fig. 2.108

(a) $4\,\Omega$ (b) $3\,\Omega$

(c) $2\,\Omega$ (d) $1\,\Omega$

2.10 A network contains only an independent current source and resistors. If the values of all resistors are doubled, the value of the node voltages will

(a) become half
(b) remain unchanged
(c) become double
(d) none of these

Answers to Objective-Type Questions

2.1. (a)	2.2. (b)	2.3. (c)	2.4. (d)	2.5. (a)	2.6. (d)	2.7. (a)
2.8. (a)	2.9. (d)	2.10. (b)	2.11. (b)	2.12. (b)	2.13. (c)	2.14. (b)

3

Network Theorems (Application to dc Networks)

3.1 INTRODUCTION

In Chapter 2, we have studied elementary network theorems like Kirchhoff's laws, mesh analysis and node analysis. There are some other methods also to analyse circuits. In this chapter, we will study superposition theorem, Thevenin's theorem, Norton's theorem, maximum power transfer theorem and Millman's theorem. We can find currents and voltages in various parts of the circuits with these methods.

3.2 SUPERPOSITION THEOREM

It states that '*in a linear network containing more than one independent source and dependent source, the resultant current in any element is the algebraic sum of the currents that would be produced by each independent source acting alone, all the other independent sources being represented meanwhile by their respective internal resistances.*'

The independent voltage sources are represented by their internal resistances if given or simply with zero resistances, i.e., short circuits if internal resistances are not mentioned. The independent current sources are represented by infinite resistances, i.e., open circuits.

The dependent sources are not sources but dissipative components—hence they are active at all times. A dependent source has zero value only when its control voltage or current is zero.

A linear network is one whose parameters are constant, i.e., they do not change with voltage and current.

Explanation Consider the network shown in Fig. 3.1. Suppose we have to find current I_4 through resistor R_4.

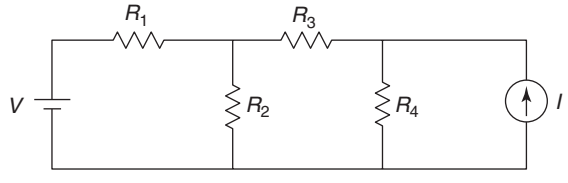


Fig. 3.1 *Network to illustrate superposition theorem*

The current flowing through resistor R_4 due to constant voltage source V is found to be say I_4' (with proper direction), representing constant current source with infinite resistance, i.e., open circuit.

The current flowing through resistor R_4 due to constant current source I is found to be say I_4'' (with proper direction), representing the constant voltage source with zero resistance or short circuit.

The resultant current I_4 through resistor R_4 is found by superposition theorem.

$$I_4 = I_4' + I_4''$$

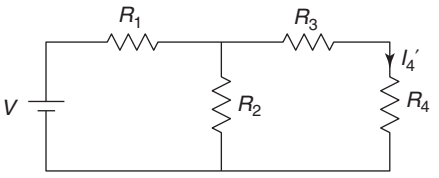


Fig. 3.2 *When voltage source V is acting alone*

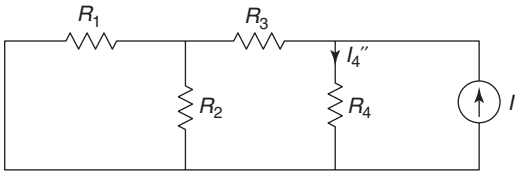


Fig. 3.3 *When current source I is acting alone*

Steps to be followed in Superposition Theorem

1. Find the current through the resistance when only one independent source is acting, replacing all other independent sources by respective internal resistances.
2. Find the current through the resistance for each of the independent sources.
3. Find the resultant current through the resistance by the superposition theorem considering magnitude and direction of each current.

Example 3.1 *Find the current through the 2 Ω resistor in Fig. 3.4.*

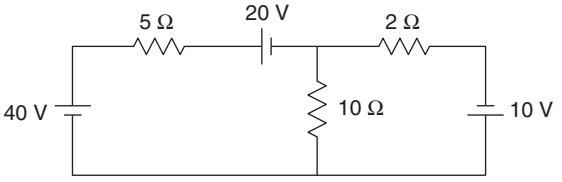


Fig. 3.4

Solution

Step I When the 40 V source is acting alone (Fig. 3.5)

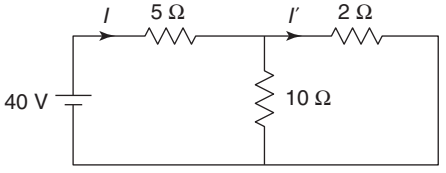


Fig. 3.5

By series parallel reduction technique (Fig. 3.6),

$$I = \frac{40}{5 + 1.67} = 6 \text{ A}$$

From Fig. 3.5, by current-division rule,

$$I' = 6 \times \frac{10}{10 + 2} = 5 \text{ A } (\rightarrow)$$

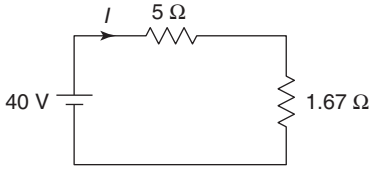


Fig. 3.6

Step II When the 20 V source is acting alone (Fig. 3.7)

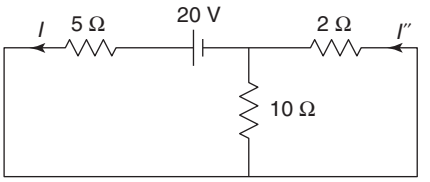


Fig. 3.7

By series–parallel reduction technique (Fig. 3.8),

$$I = \frac{20}{5+1.67} = 3 \text{ A}$$

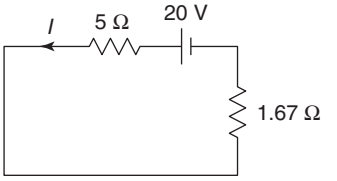


Fig. 3.8

From Fig. 3.7, by current-division rule,

$$I'' = 3 \times \frac{10}{10+2} = 2.5 \text{ A}(\leftarrow) = -2.5 \text{ A} \quad (\rightarrow)$$

Step III When the 10 V source is acting alone (Fig. 3.9)

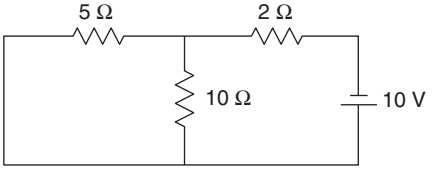


Fig. 3.9

By series–parallel reduction technique (Fig. 3.10),

$$I''' = \frac{10}{3.33+2} = 1.88 \text{ A}(\rightarrow)$$

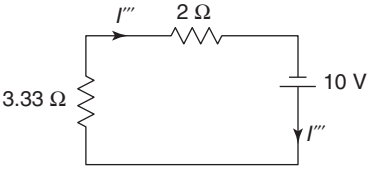


Fig. 3.10

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 5 - 2.5 + 1.88 = 4.38 \text{ A} \quad (\rightarrow)$$

Example 3.2 *Find the current through the 1 Ω resistor in Fig. 3.11.*

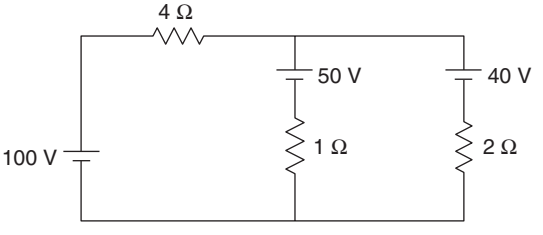


Fig. 3.11

Solution

Step I When the 100 V source is acting alone (Fig. 3.12)

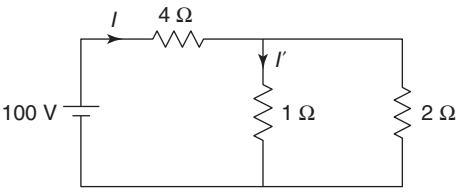


Fig. 3.12

By series–parallel reduction technique (Fig. 3.13),

$$I = \frac{100}{4 + 0.67} = 21.41 \text{ A}$$

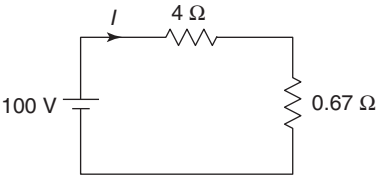


Fig. 3.13

From Fig. 3.12, by current-division rule,

$$I' = 21.41 \times \frac{2}{2+1} = 14.27 \text{ A}(\downarrow)$$

Step II When the 50 V source is acting alone (Fig. 3.14)

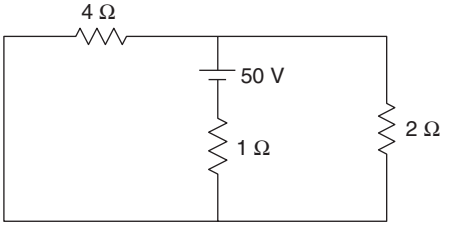


Fig. 3.14

By series–parallel reduction technique (Fig. 3.15),

$$I'' = \frac{50}{1+1.33} = 21.46 \text{ A}(\uparrow) = -21.46 \text{ A}(\downarrow)$$

Fig. 3.15

Step III When the 40 V source is acting alone (Fig. 3.16)

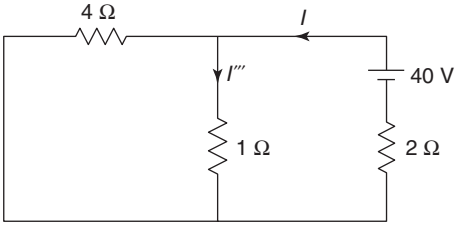


Fig. 3.16

By series–parallel reduction technique (Fig. 3.17),

$$I = \frac{40}{0.8+2} = 14.29 \text{ A}$$

From Fig. 3.16, by current-division rule,

$$I''' = 14.29 \times \frac{4}{4+1} = 11.43 \text{ A}(\downarrow)$$

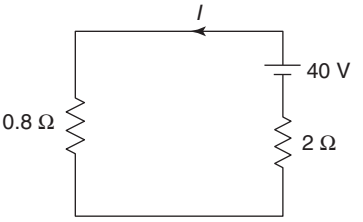


Fig. 3.17

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 14.27 - 21.46 + 11.43 = 4.24 \text{ A}(\downarrow)$$

Example 3.3

Find the current through the 8 Ω resistor in Fig. 3.18.

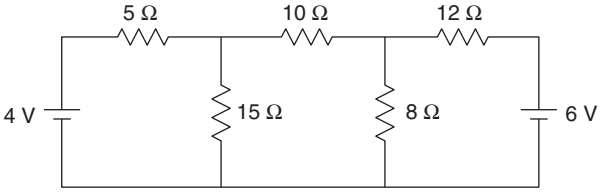


Fig. 3.18

Solution

Step I When the 4 V source is acting alone (Fig. 3.19)

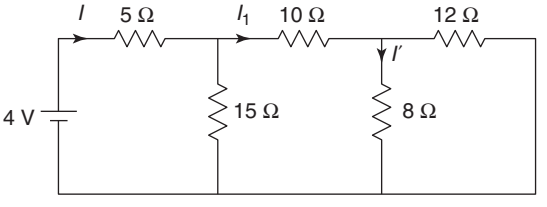
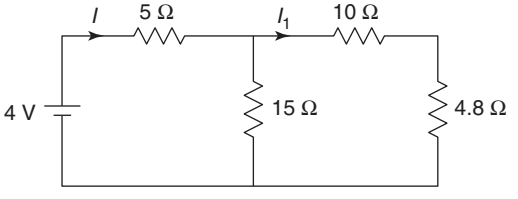
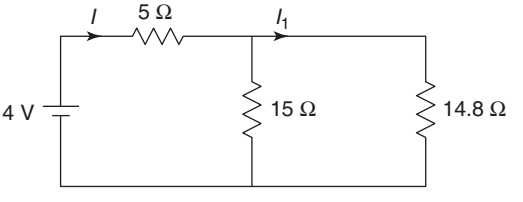


Fig. 3.19

By series–parallel reduction technique (Fig. 3.20),

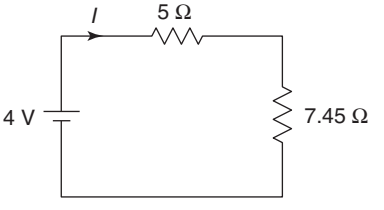


(a)



(b)

$$I = \frac{4}{5+7.45} = 0.32 \text{ A}$$



(c)

From Fig. 3.20(b), by current-division rule,

$$I_1 = 0.32 \times \frac{15}{15+14.8} = 0.16 \text{ A}$$

From Fig. 3.19, by current-division rule,

$$I' = 0.16 \times \frac{12}{12+8} = 0.096 \text{ A}(\downarrow)$$

Fig. 3.20

Step II When the 6 V source is acting alone (Fig. 3.21)

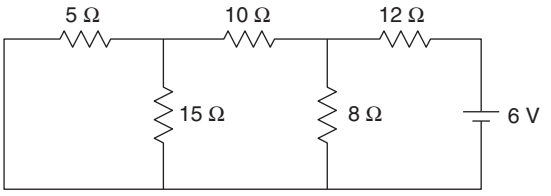
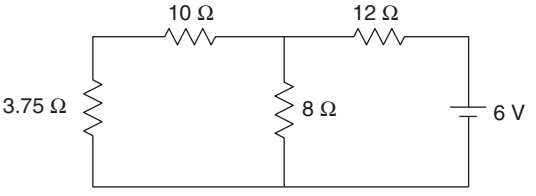
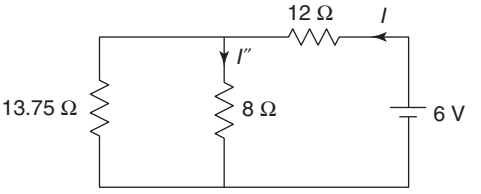


Fig. 3.21

By series–parallel reduction technique (Fig. 3.22),



(a)

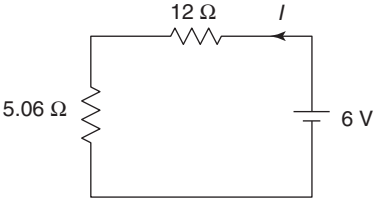


(b)

$$I = \frac{6}{12 + 5.06} = 0.35 \text{ A}$$

From Fig. 3.22(b), by current division rule,

$$I'' = 0.35 \times \frac{13.75}{13.75 + 8} = 0.22 \text{ A}(\downarrow)$$



(c)

Fig. 3.22

Step III By superposition theorem,

$$I = I' + I'' = 0.096 + 0.22 = 0.316 \text{ A}(\downarrow)$$

Example 3.4

Find the current through the 4 Ω resistor in Fig. 3.23.

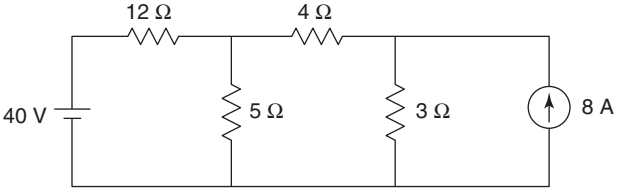


Fig. 3.23

Solution

Step I When the 40 V source is acting alone (Fig. 3.24)

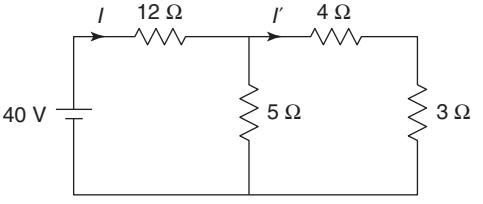


Fig. 3.24

By series–parallel reduction technique (Fig. 3.25),

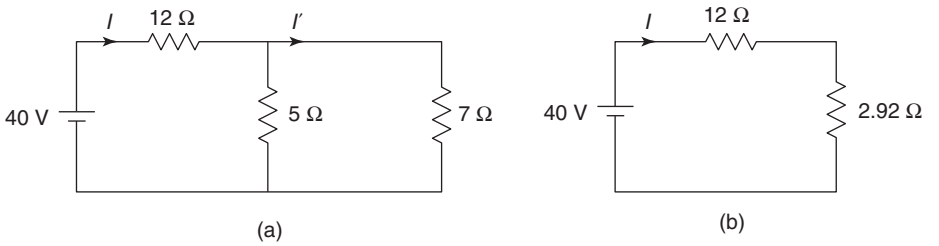


Fig. 3.25

$$I = \frac{40}{12 + 2.92} = 2.68 \text{ A}$$

From Fig. 3.25(a), by current-division rule,

$$I' = 2.68 \times \frac{5}{5+7} = 1.12 \text{ A}(\rightarrow) = -1.12 \text{ A}\ (\leftarrow)$$

Step II When the 8 A source is acting alone (Fig. 3.26)

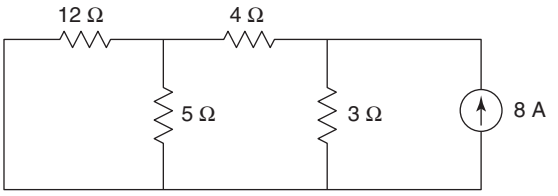


Fig. 3.26

By series–parallel reduction technique (Fig. 3.27),

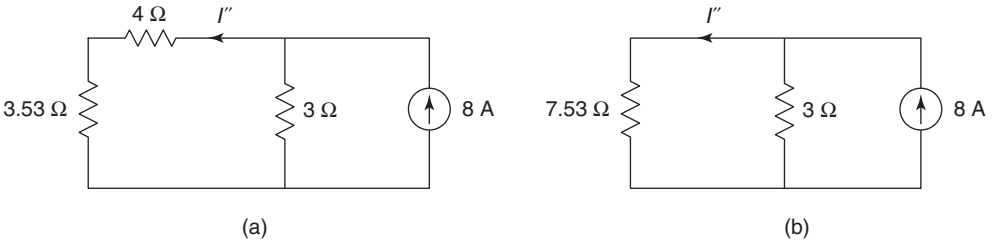


Fig. 3.27

$$I'' = 8 \times \frac{3}{7.53 + 3} = 2.28 \text{ A}(\leftarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = -1.12 + 2.28 = 1.16 \text{ A}\ (\leftarrow)$$

Example 3.5 *Find the current in the 10 Ω resistor in Fig. 3.28.*

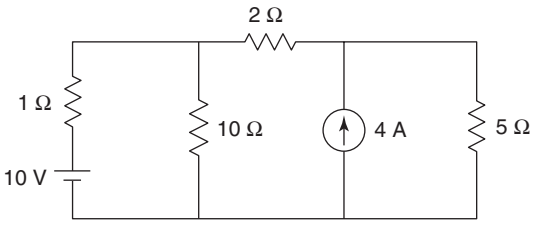


Fig. 3.28

Solution

Step I When the 10 V source is acting alone (Fig. 3.29)

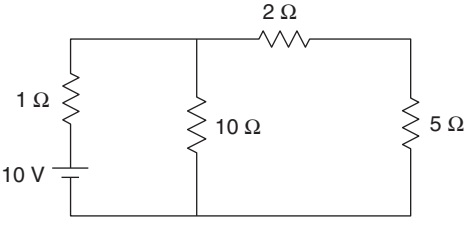


Fig. 3.29

By series–parallel reduction technique (Fig. 3.30),

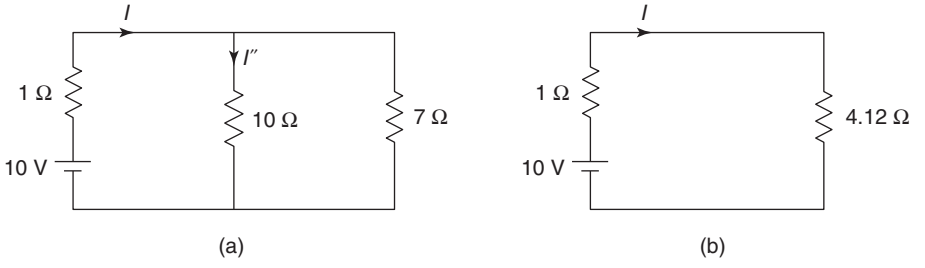


Fig. 3.30

$$I = \frac{10}{1+4.12} = 1.95 \text{ A}$$

From Fig. 3.30 (a), by current-division rule,

$$I' = 1.95 \times \frac{7}{7+10} = 0.8 \text{ A}(\downarrow)$$

Step II When the 4 A source is acting alone (Fig. 3.31)

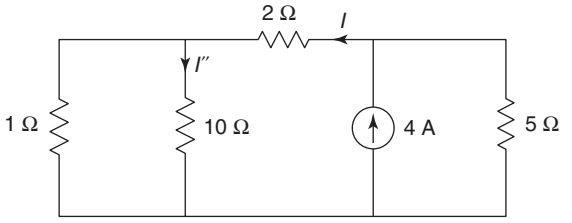


Fig. 3.31

By series–parallel reduction technique (Fig. 3.32),

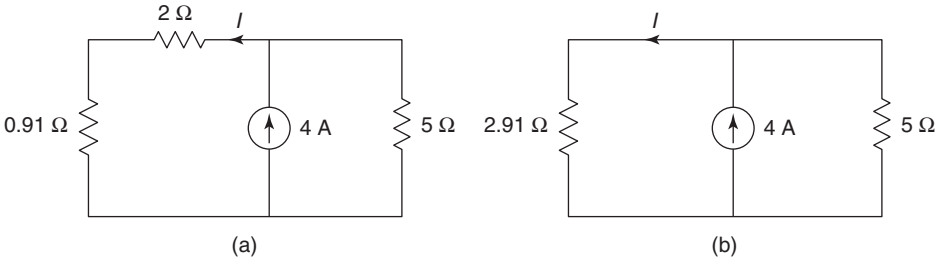


Fig. 3.32

$$I = 4 \times \frac{5}{2.91 + 5} = 2.53 \text{ A}$$

From Fig. 3.31, by current-division rule,

$$I'' = 2.53 \times \frac{1}{1+10} - 0.23 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.8 + 0.23 = 1.03 \text{ A}(\downarrow)$$

Example 3.6 *Find the current through the 8 Ω resistor in Fig. 3.33.*

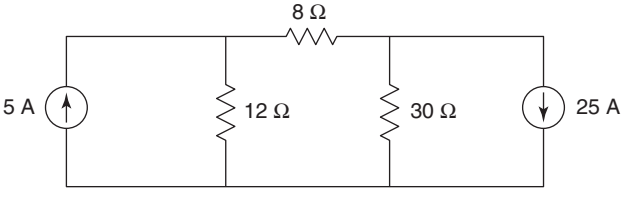


Fig. 3.33

Solution

Step I When the 5 A source is acting alone (Fig. 3.34)

$$I' = 5 \times \frac{12}{12+8+30} = 1.2 \text{ A } (\rightarrow)$$

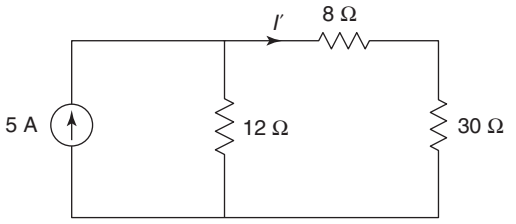


Fig. 3.34

Step II When the 25 A source is acting alone (Fig. 3.35)

$$I'' = 25 \times \frac{30}{30+12+8} = 15 \text{ A } (\rightarrow)$$

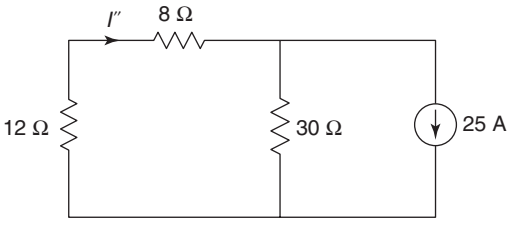


Fig. 3.35

Step III By superposition theorem,

$$I = I' + I'' = 1.2 + 15 = 16.2 \text{ A}(\rightarrow)$$

Example 3.7 *Find the current through the 4 Ω resistor in Fig. 3.36.*

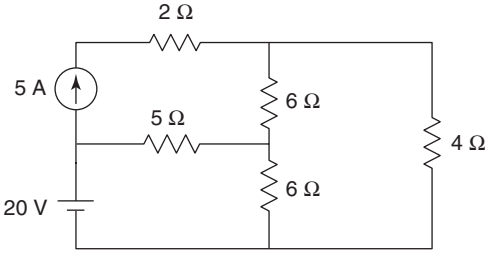


Fig. 3.36

Solution

Step I When the 5 A source is acting alone (Fig. 3.37)

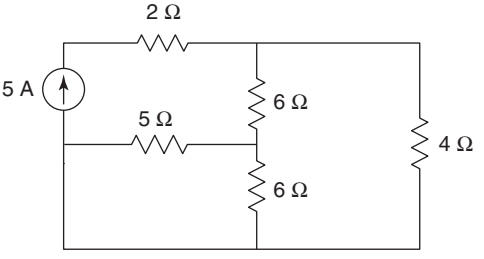


Fig. 3.37

By series–parallel reduction technique (Fig. 3.38),

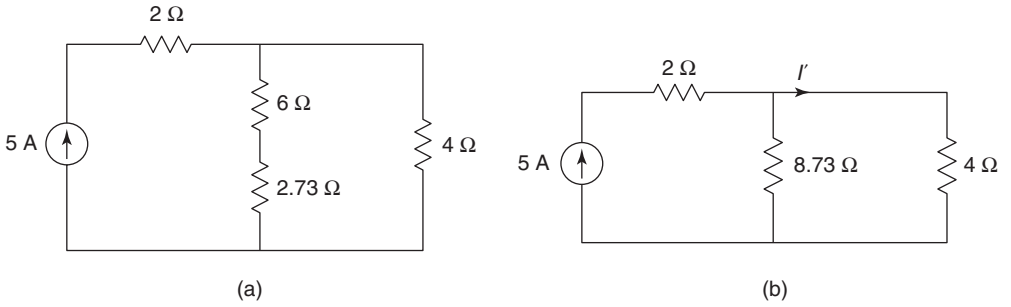


Fig. 3.38

$$I' = 5 \times \frac{8.73}{8.73 + 4} = 3.43 \text{ A}(\downarrow)$$

Step II When the 20 V source is acting alone (Fig. 3.39)

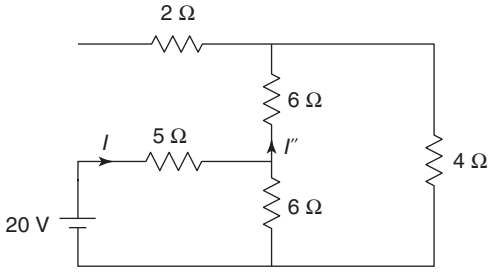


Fig. 3.39

By series–parallel reduction technique (Fig. 3.40),

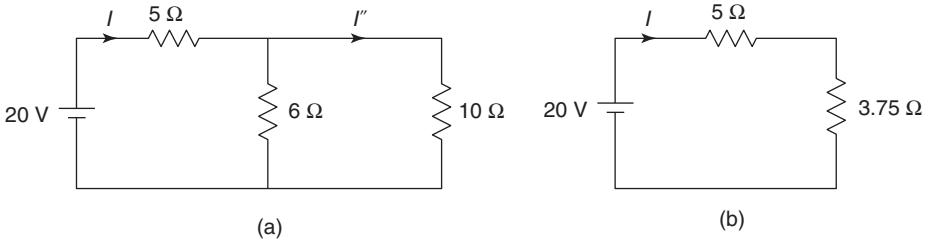


Fig. 3.40

$$I = \frac{20}{5 + 3.75} = 2.29 \text{ A}$$

From Fig. 3.40 (a), by current-division rule,

$$I'' = 2.29 \times \frac{6}{6 + 10} = 0.86 \text{ A}(\downarrow)$$

Step III By superposition theorem

$$I = I' + I'' = 3.43 + 0.86 = 4.29 \text{ A } (\downarrow)$$

Example 3.8 *Find the current through the 3 Ω resistor in Fig. 3.41.*

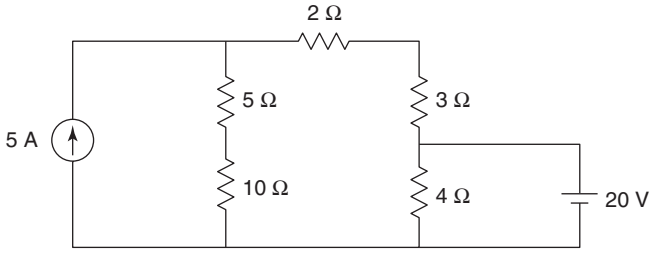


Fig. 3.41

Solution

Step I When the 5 A source is acting alone (Fig. 3.42)

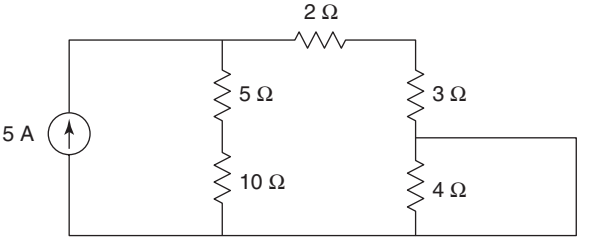


Fig. 3.42

By series–parallel reduction technique (Fig. 3.43),

$$I' = 5 \times \frac{15}{15+2+3} = 3.75 \text{ A}(\downarrow)$$

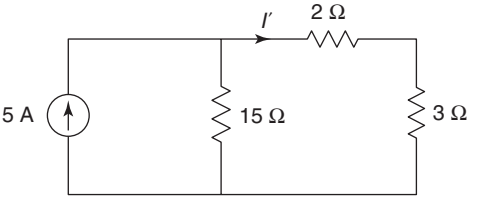


Fig. 3.43

Step II When the 20 V source is acting alone (Fig. 3.44)

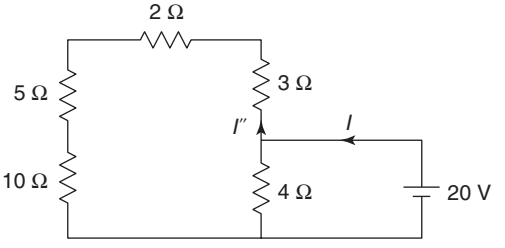


Fig. 3.44

By series–parallel reduction technique (Fig. 3.45),

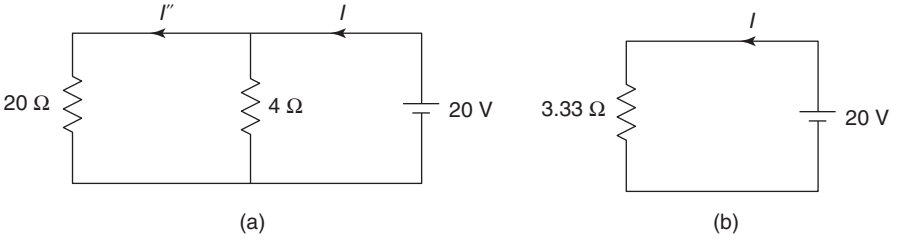


Fig. 3.45

$$I = \frac{20}{3.33} = 6 \text{ A}$$

From Fig. 3.45(a), by current-division rule,

$$I'' = 6 \times \frac{4}{20+4} = 1 \text{ A}(\uparrow) = -1 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.75 - 1 = 2.75 \text{ A } (\downarrow)$$

Example 3.9 *Find the current in the 1 Ω resistors in Fig. 3.46.*

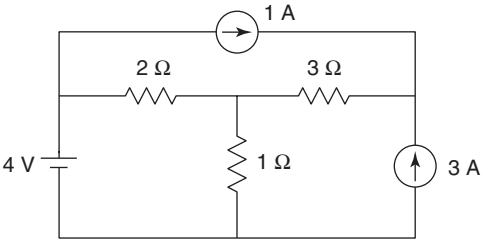


Fig. 3.46

Solution

Step I When the 4 V source is acting alone (Fig. 3.47)

$$I' = \frac{4}{2+1} = 1.33 \text{ A}(\downarrow)$$

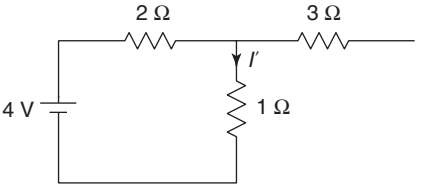


Fig. 3.47

Step II When the 3 A source is acting alone (Fig. 3.48)
By current-division rule,

$$I'' = 3 \times \frac{2}{1+2} = 2 \text{ A } (\downarrow)$$

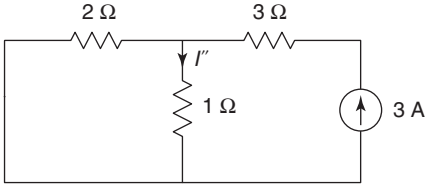


Fig. 3.48

Step III When the 1 A source is acting alone (Fig. 3.49)

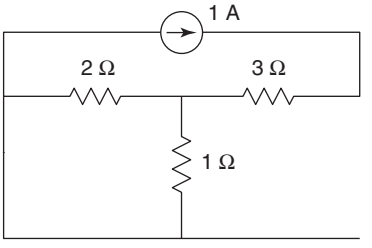


Fig. 3.49

Redrawing the network (Fig. 3.50),
By current-division rule,

$$I''' = 1 \times \frac{2}{2+1} = 0.66 \text{ A}(\downarrow)$$

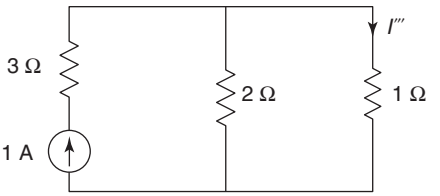


Fig. 3.50

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 1.33 + 2 + 0.66 = 4 \text{ A } (\downarrow)$$

Example 3.10

Find the voltage V_{AB} in Fig. 3.51.

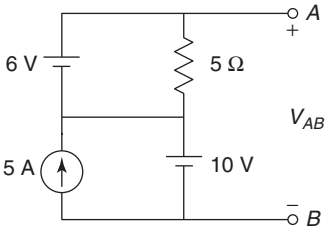


Fig. 3.51

Solution

Step I When the 6 V source is acting alone (Fig. 3.52)

$$V'_{AB} = 6 \text{ V}$$

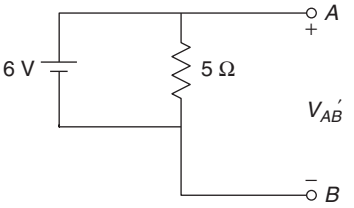


Fig. 3.52

Step II When the 10 V source is acting alone (Fig. 3.53)
Since the resistor of 5 Ω is shorted, the voltage across it is zero.

$$V''_{AB} = 10 \text{ V}$$

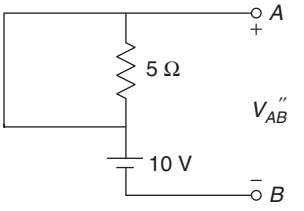


Fig. 3.53

Step III When the 5 A source is acting alone (Fig. 3.54)
Due to short circuit in both the parts,

$$V'''_{AB} = 0$$

Step IV By superposition theorem,

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB} = 6 + 10 + 0 = 16 \text{ V}$$

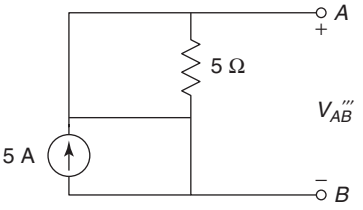


Fig. 3.54

Example 3.11 *Find the current through the 5 Ω resistor in Fig. 3.55.*

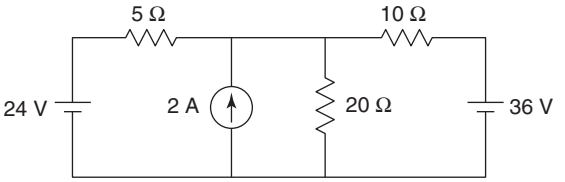


Fig. 3.55

Solution

Step I When the 24 V source is acting alone (Fig. 3.56)

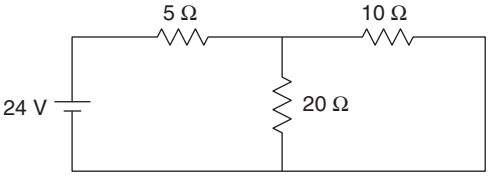


Fig. 3.56

By series–parallel reduction technique (Fig. 3.57),

$$I' = \frac{24}{5+6.67} = 2.06\ \text{A}(\rightarrow) = -2.06\ \text{A}(\leftarrow)$$

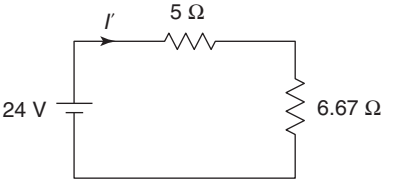


Fig. 3.57

Step II When the 2 A source is acting alone
By series–parallel reduction technique (Fig. 3.58),

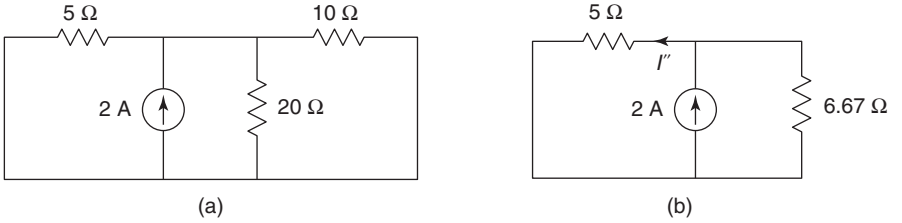


Fig. 3.58

By current-division rule,

$$I'' = 2 \times \frac{6.67}{5+6.67} = 1.14\ \text{A}(\leftarrow)$$

Step III When the 36 V source is acting alone (Fig. 3.59)
By series–parallel reduction technique,

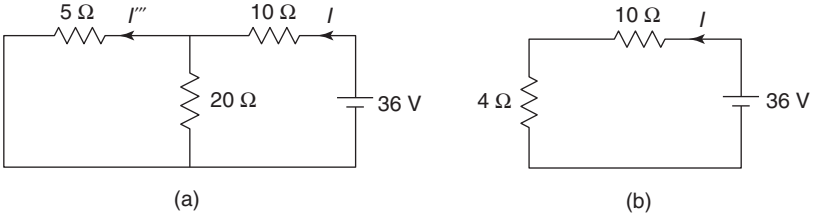


Fig. 3.59

$$I = \frac{36}{10+4} = 2.57 \text{ A}$$

By current-division rule,

$$I''' = 2.57 \times \frac{20}{20+5} = 2.06 \text{ A}(\leftarrow)$$

Step IV By superposition theorem,

$$I = I' + I'' + I''' = -2.06 + 1.14 + 2.06 = 1.14 \text{ A } (\leftarrow)$$

Example 13.12

Find the current through the 4 Ω resistor in Fig. 3.60.

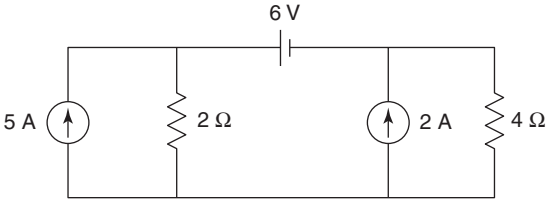


Fig. 3.60

Solution

Step I When the 5 A source is acting alone (Fig. 3.61)
By current-division rule,

$$I' = 5 \times \frac{2}{2+4} = 1.67 \text{ A}(\downarrow)$$

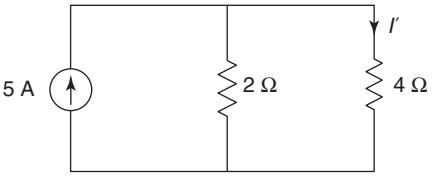


Fig. 3.61

Step II When the 2 A source is acting alone (Fig. 3.62)
By current-division rule,

$$I'' = 2 \times \frac{2}{2+4} = 0.67 \text{ A}(\downarrow)$$

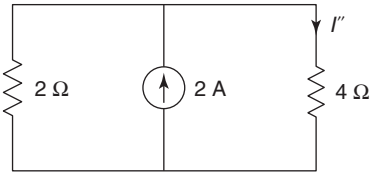


Fig. 3.62

Step III When the 6 V source is acting alone (Fig. 3.63)
Applying KVL to the mesh,

$$-2I''' - 6 - 4I''' = 0$$

$$I''' = -1 \text{ A } (\downarrow)$$

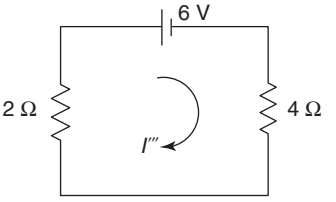


Fig. 3.63

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 1.67 + 0.67 - 1 = 1.34 \text{ A } (\downarrow)$$

EXAMPLES WITH DEPENDENT SOURCES

Example 3.13 *Find the current through the 6 Ω resistor in Fig. 3.64.*

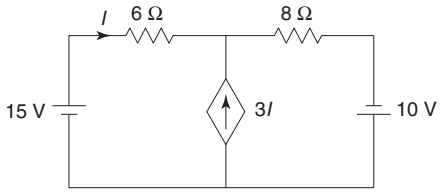


Fig. 3.64

Solution

Step I When the 15 V source is acting alone (Fig 3.65)
From Fig. 3.65,

$$I' = I_1 \qquad \ldots(i)$$

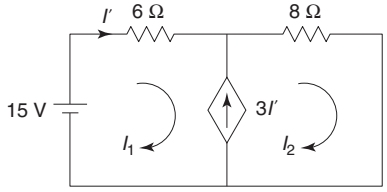


Fig. 3.65

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3I' = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$15 - 6I_1 - 8I_2 = 0$$

$$6I_1 + 8I_2 = 15 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.39\text{A}$$
$$I_2 = 1.58\text{A}$$
$$I' = I_1 = 0.39 \text{ A } (\rightarrow)$$

Step II When the 10 V source is acting alone (Fig 3.66)
From Fig. 3.66,

$$I'' = I_1 \qquad \ldots(i)$$

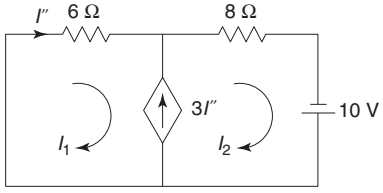


Fig. 3.66

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3I'' = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-6I_1 - 8I_2 + 10 = 0$$

$$6I_1 + 8I_2 = 10 \qquad \ldots..(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.26 \text{ A}$$
$$I_2 = 1.05 \text{ A}$$
$$I'' = I_1 = 0.26 \text{ A} \quad (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.39 + 0.26 = 0.65 \text{ A} \ (\rightarrow)$$

Example 3.14

Find the current I_x in Fig. 3.67.

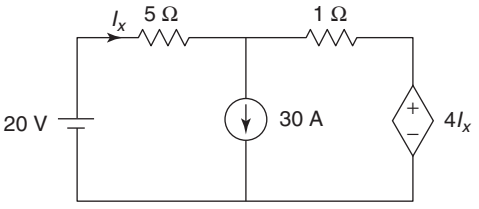


Fig. 3.67

Solution

Step I When the 30 A source is acting alone (Fig. 3.68)
From Fig. 3.68,

$$I_x' = I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 30 \qquad \ldots(ii)$$

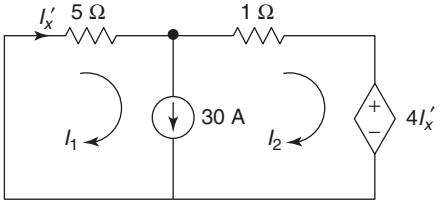


Fig. 3.68

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 1I_2 - 4I_x' = 0$$
$$-5I_1 - I_2 - 4I_1 = 0$$
$$9I_1 + I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 3 \text{ A}$$
$$I_2 = -27 \text{ A}$$
$$I_x' = I_1 = 3 \text{ A} \ (\rightarrow)$$

Step II When the 20 V source is acting alone (Fig. 3.69)
Applying KVL to the mesh,

$$20 - 5I_x'' - 1I_x'' - 4I_x'' = 0$$
$$I_x'' = 2 \text{ A} (\rightarrow)$$

Step III By superposition theorem,

$$I_x = I_x' + I_x'' = 3 + 2 = 5 \text{ A} (\rightarrow)$$

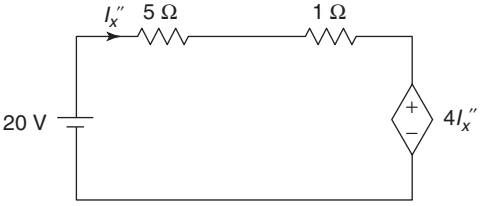


Fig. 3.69

Example 3.15 *Find the current I_1. in Fig. 3.70.*

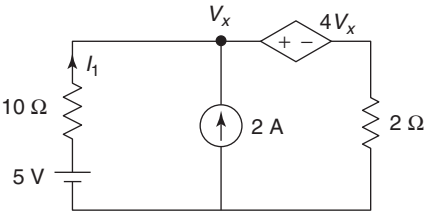


Fig. 3.70

Solution

Step I When the 5 V source is acting alone (Fig 3.71)
From the figure,

$$V_x = 5 - 10I_1'$$

Applying KVL to the mesh,

$$5 - 10I_1' - 4V_x - 2I_1' = 0$$
$$5 - 10I_1' - 4\,(5 - 10I_1') - 2I_1' = 0$$
$$5 - 10I_1' - 20 + 40I_1' - 2I_1' = 0$$
$$I_1' = \frac{15}{28} = 0.54 \text{ A}(\uparrow)$$

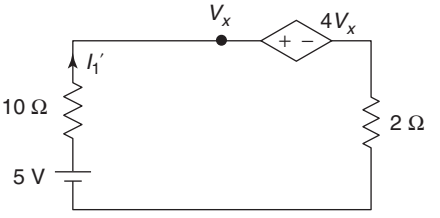


Fig. 3.71

Step II When the 2 A source is acting alone (Fig 3.72)
From Fig . 3.72,

$$V_x = -10I_1' \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1' = 2 \qquad \ldots(ii)$$

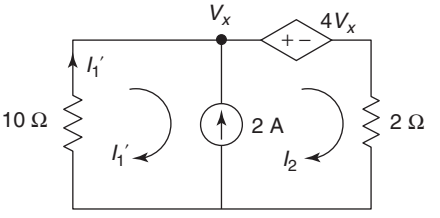


Fig. 3.72

Applying KVL to the outer path of the supermesh,

$$-10I_1' - 4V_x - 2I_2 = 0$$
$$-10I_1' - 4(-10I_1') - 2I_2 = 0$$
$$30I_1' - 2I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.14 \text{ A } (\uparrow)$$
$$I_2 = 2.14 \text{ A}$$

Step III By superposition theorem,

$$I_1 = I_1' + I_1'' = 0.54 + 0.14 = 0.68 \text{ A } (\uparrow)$$

Example 3.16 *Determine the current through the 10 Ω resistor in Fig. 3.73.*

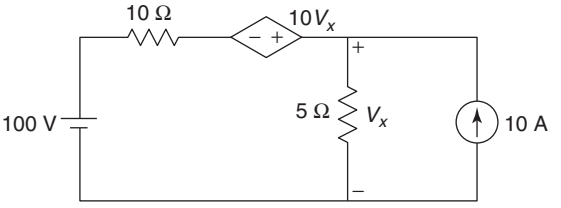


Fig. 3.73

Solution

Step I When the 100 V source is acting alone (Fig. 3.74)
From the figure,

$$V_x = 5I'$$

Applying KVL to the mesh,

$$100 - 10I' + 10V_x - 5I' = 0$$
$$100 - 10I' + 10(5I') - 5I' = 0$$
$$I' = -2.86 \text{ A } (\rightarrow)$$

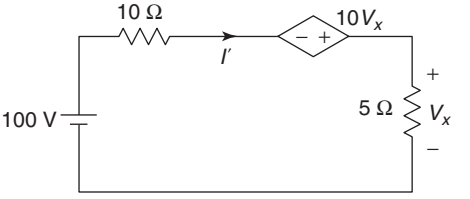


Fig. 3.74

Step II When the 10 A source is acting alone (Fig. 3.75)
From Fig. 3.75,

$$V_x = 5(I_1 - I_2) \quad \ldots(i)$$

Applying KVL to Mesh 1,

$$-10I_1 + 10V_x - 5(I_1 - I_2) = 0$$
$$-10I_1 + 10\{5(I_1 - I_2)\} - 5(I_1 - I_2) = 0$$
$$35I_1 - 45I_2 = 0 \quad \ldots(ii)$$

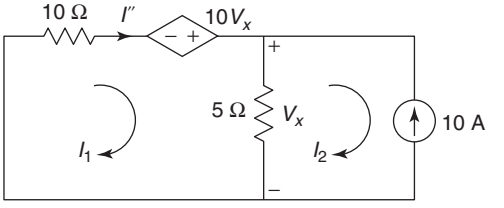


Fig. 3.75

For Mesh 2,

$$I_2 = -10 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = -12.86 \text{ A}$$
$$I_2 = -10 \text{ A}$$
$$I'' = I_1 = -12.86 \text{ A } (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = -2.86 - 12.86 = -15.72 \text{ A } (\rightarrow)$$

Example 3.17 *Find the current I in the network of Fig. 3.76.*

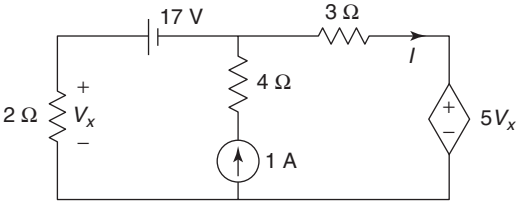


Fig. 3.76

Solution

Step I When the 17 V source is acting alone (Fig. 3.77)
From the figure,

$$V_x = -2I'$$

Applying KVL to the mesh,

$$-2I' - 17 - 3I' - 5V_x = 0$$
$$-2I' - 17 - 3I' - 5(-2I') = 0$$
$$I' = 3.4 \text{ A } (\rightarrow)$$

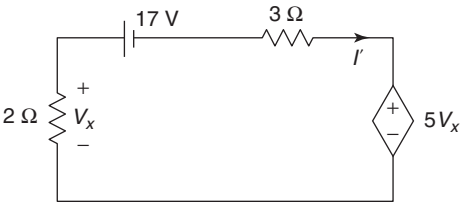


Fig. 3.77

Step II When the 1 A source is acting alone (Fig. 3.78)
From Fig. 3.78,

$$V_x = -2I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$-2I_1 - 3I_2 - 5V_x = 0$$
$$-2I_1 - 3I_2 - 5(-2I_1) = 0$$
$$8I_1 - 3I_2 = 0 \qquad \ldots(iii)$$

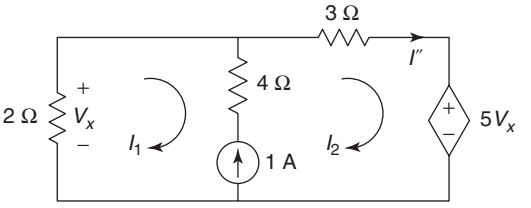


Fig. 3.78

Solving Eqs (ii) and (iii),

$$I_1 = 0.6 \text{ A}$$
$$I_2 = 1.6 \text{ A}$$
$$I'' = I_2 = 1.6 \text{ A } (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.4 + 1.6 = 5 \text{ A } (\rightarrow)$$

Example 3.18

Find the voltage V_1 in Fig. 3.79.

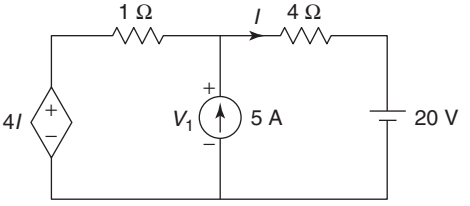


Fig. 3.79

Solution

Step I When the 5 A source is acting alone (Fig. 3.80)
From Fig. 3.80,

$$I = \frac{V_1'}{4}$$

Applying KCL at Node 1,

$$\frac{V_1' - 4I}{1} + \frac{V_1'}{4} = 5$$

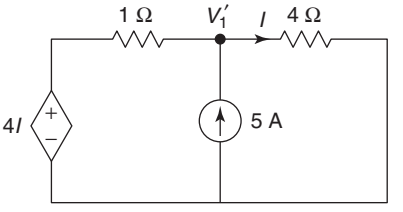


Fig. 3.80

$$V_1' - 4\left(\frac{V_1'}{4}\right) + \frac{V_1'}{4} = 5$$

$$V_1' = 20 \text{ V}$$

Step II When the 20 V source is acting alone (Fig. 3.81)
Applying KVL to the mesh,

$$4I - I - 4I - 20 = 0$$

$$I = -20 \text{ A}$$

$$V_1'' = 4I - 1(I) = 3I = 3(-20) = -60 \text{ V}$$

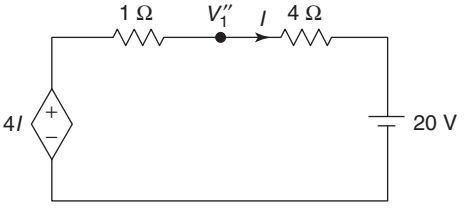


Fig. 3.81

Step III By superposition theorem,

$$V_1 = V_1' + V_1'' = 20 - 60 = -40 \text{ V}$$

Example 3.19

Find the current in the 6 Ω resistor in Fig. 3.82.

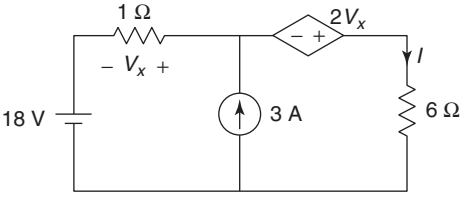


Fig. 3.82

Solution

Step I When the 18 V source is acting alone (Fig. 3.83)
From Fig. 3.83,

$$V_x = -I'$$

Applying KVL to the mesh,

$$18 - I' + 2V_x - 6I' = 0$$

$$18 - I' - 2I' - 6I' = 0$$

$$I' = 2 \text{ A } (\downarrow)$$

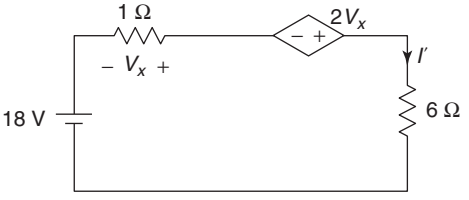


Fig. 3.83

Step II When the 3 A source is acting alone (Fig. 3.84)
From Fig. 3.84,

$$V_x = -1\, I_1 = -I_1 \qquad \ldots(\text{i})$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3 \qquad \ldots(\text{ii})$$

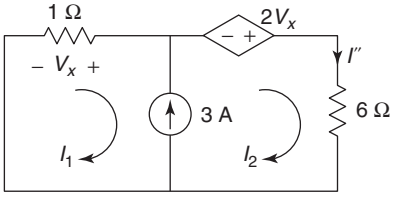


Fig. 3.84

Applying KVL to the outerpath of the supermesh,

$$-1I_1 + 2V_x - 6I_2 = 0$$

$$-I_1 + 2(-I_1) - 6I_2 = 0$$

$$3I_1 + 6I_2 = 0 \qquad \ldots(\text{iii})$$

Solving Eqs (ii) and (iii),

$$I_1 = -2 \text{ A}$$
$$I_2 = 1 \text{ A}$$
$$I'' = I_2 = 1 \text{ A } (\downarrow)$$

Step III By superposition theorem,

$$I_{6\,\Omega} = I' + I'' = 2 + 1 = 3 \text{ A } (\downarrow)$$

Example 3.20 *Find the current I_y in Fig. 3.85.*

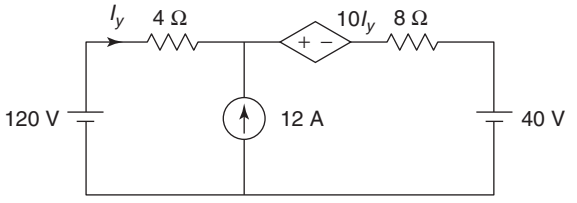


Fig. 3.85

Solution

Step I When the 120 V source is acting alone (Fig. 3.86)
Applying KVL to the mesh,

$$120 - 4I_y' - 10I_y' - 8I_y' = 0$$

$$I_y' = 5.45 \text{ A } (\rightarrow)$$

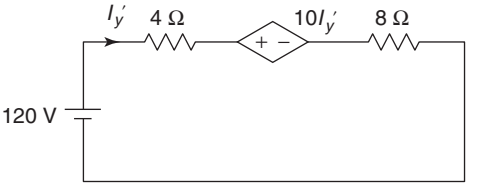


Fig. 3.86

Step II When the 12 A source is acting alone (Fig. 3.87)
From Fig. 3.87,

$$I_y'' = I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 12 \qquad \ldots(ii)$$

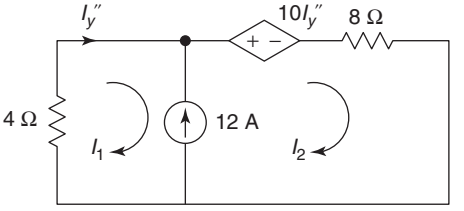


Fig. 3.87

Applying KVL to the outer path of the supermesh,

$$-4I_1 - 10I_y'' - 8I_2 = 0$$
$$-4I_1 - 10I_1 - 8I_2 = 0$$
$$14I_1 + 8I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = -4.36 \text{ A}$$
$$I_2 = 7.64 \text{ A}$$
$$Iy'' = I_1 = -4.36 \text{ A } (\rightarrow)$$

Step III When the 40 V source is acting alone (Fig. 3.88)
Applying KVL to the mesh,

$$-4\,I_y''' - 10I_y''' - 8I_y''' - 40 = 0$$

$$I_y''' = -\frac{40}{22} = -1.82 \text{ A } (\rightarrow)$$

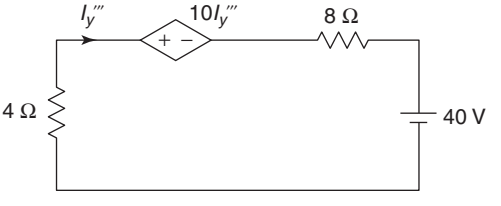


Fig. 3.88

Step IV By superposition theorem,

$$I_y = I_y' + I_y'' + I_y''' = 5.45 - 4.36 - 1.82 = -0.73 \text{ A } (\rightarrow)$$

Example 3.21

Find the voltage V_x in Fig. 3.89.

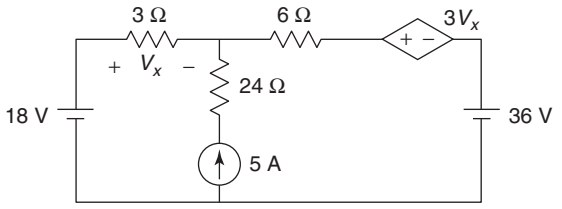


Fig. 3.89

Solution

Step I When the 18 V source is acting alone (Fig. 3.90)
From Fig. 3.90,

$$V_x' = 3I$$

Applying KVL to the mesh,

$$18 - 3I - 6I - 3V_x' = 0$$
$$18 - 3I - 6I - 3(3I) = 0$$
$$I = 1 \text{ A}$$
$$V_x' = 3 \text{ V}$$

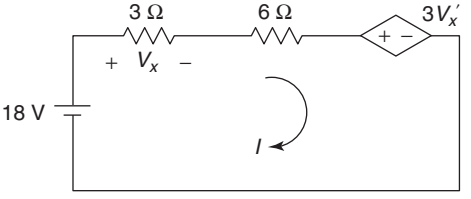


Fig. 3.90

Step II When the 5 A source is acting alone (Fig. 3.91)
From Fig. 3.91,

$$V_x'' = -3I_1 \quad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \quad \ldots(ii)$$

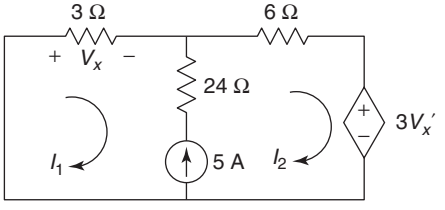


Fig. 3.91

Applying KVL to the outer path of the supermesh,

$$-3I_1 - 6I_2 - 3V_x'' = 0$$
$$-3I_1 - 6I_2 - 3(3I_1) = 0$$
$$12I_1 + 6I_2 = 0 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = -1.67\text{A}$$
$$I_2 = 3.33 \text{ A}$$
$$V_x'' = 3I_1 = 3(-1.67) = -5 \text{ V}$$

Step III When the 36 V source is acting alone (Fig. 3.92)
From the figure,

$$V_x''' = -3I$$

Applying KVL to the mesh,

$$36 + 3V_x''' - 6I - 3I = 0$$

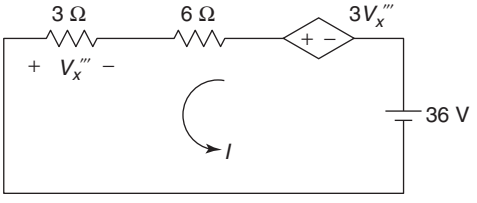


Fig. 3.92

$$36 + 3V_x''' - 6\left(\frac{-V_x'''}{3}\right) - 3\left(\frac{-V_x'''}{3}\right) = 0$$

$$36 + 3V_x''' + 2V_x''' + V_x''' = 0$$

$$V_x''' = -6 \text{ V}$$

Step IV By superposition theorem,

$$V_x = V_x' + V_x'' + V_x''' = 3 - 5 - 6 = -8 \text{ V}$$

Example 3.22 *Find the voltage V in the network of Fig. 3.93.*

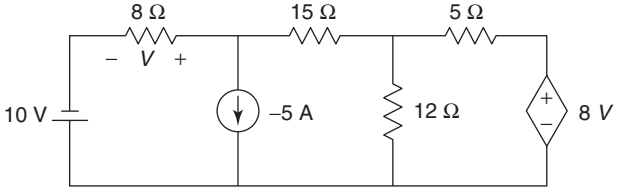


Fig. 3.93

Solution

Step I When the 10 V source is acting alone (Fig. 3.94)
From Fig. 3.94,

$$V' = -8I_1 \quad ...(i)$$

Applying KVL to Mesh 1,

$$-10 - 8I_1 - 15\ I_1 - 12\ (I_1 - I_2) = 0$$

$$35I_1 - 12I_2 = -10 \quad ...(ii)$$

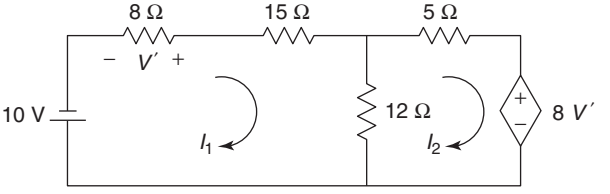


Fig. 3.94

Applying KVL to Mesh 2,

$$-12(I_2 - I_1) - 5I_2 - 8V' = 0$$

$$-12I_2 + 12I_1 - 5I_2 - 8(-8I_1) = 0$$

$$76I_1 - 17I_2 = 0 \quad ...(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.54 \text{ A}$$

$$I_2 = 2.4 \text{ A}$$

$$V' = -8I_1 = -8(0.54) = -4.32 \text{ V}$$

Step II When the –5 A source is acting alone (Fig. 3.95)
From Fig. 3.95,

$$V'' = -8I_1 \quad ...(i)$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = -5 \quad ...(ii)$$

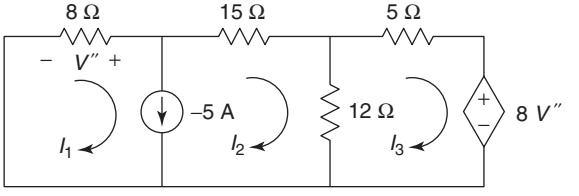


Fig. 3.95

Applying KVL to the outer path of the supermesh,

$$-8I_1 - 15I_2 - 12(I_2 - I_3) = 0$$

$$-8I_1 - 27I_2 + 12I_3 = 0 \quad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-12(I_3 - I_2) - 5I_3 - 8V'' = 0$$

$$-12I_3 + 12I_2 - 5I_3 - 8(-8I_1) = 0$$

$$64I_1 + 12I_2 - 17I_3 = 0 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.97 \text{ A}$$

$$I_2 = 9.97 \text{ A}$$

$$I_3 = 25.74 \text{ A}$$

$$V'' = -8I_1 = -8(-4.97) = -39.76 \text{ V}$$

Step III By superposition theorem,

$$V = V' + V'' = -4.32 - 39.76 = -44.08 \text{ V}$$

Example 3.23

For the network shown in Fig. 3.96, find the voltage V_0.

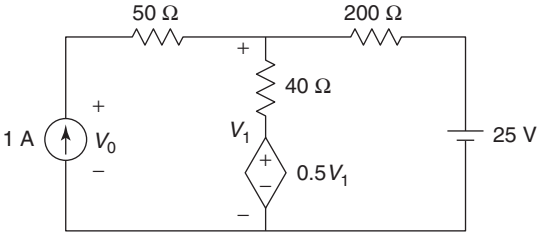


Fig. 3.96

Solution

Step I When the 1 A source is acting alone (Fig. 3.97)

From Fig. 3.97,

$$V_1 = 200\, I_2 \quad \text{...(i)}$$

For Mesh 1,

$$I_1 = 1 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$0.5V_1 - 40(I_2 - I_1) - 200\, I_2 = 0$$

$$0.5(200I_2) - 40I_2 + 40I_1 - 200I_2 = 0$$

$$40I_1 - 140\, I_2 = 0 \quad \text{...(iii)}$$

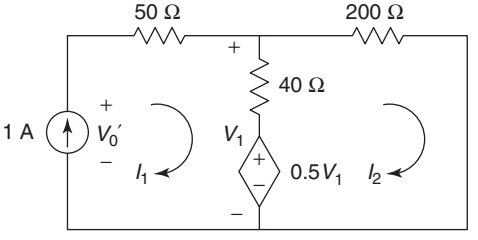


Fig. 3.97

Solving Eqs (ii) and (iii),

$$I_1 = 1 \text{ A}$$

$$I_2 = 0.29 \text{ A}$$

$$V_0' - 50\, I_1 - 200I_2 = 0$$

$$V_0' - 50(1) - 200(0.29) = 0$$

$$V_0' = 108 \text{ V}$$

Step II When the 25 V source is acting alone (Fig. 3.98)
From Fig. 3.98,

$$V_1 - 200\,I - 25 = 0$$

$$V_1 = 200I + 25 \quad ...(i)$$

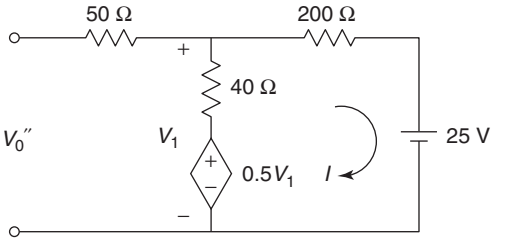


Fig. 3.98

Applying KVL to Mesh 1,

$$0.5V_1 - 40\,I - 200\,I - 25 = 0$$

$$0.5(200\,I + 25) - 40\,I - 200\,I - 25 = 0$$

$$I = -0.09 \text{ A}$$

$$V_0'' = V_1 = 200\,I + 25 = 200(-0.09) + 25 = 7 \text{ V}$$

Step III By superposition theorem,

$$V_0 = V_0' + V_0'' = 108 + 7 = 115 \text{ V}$$

Example 3.24 *For the network shown in Fig. 3.99, find the voltage V_x.*

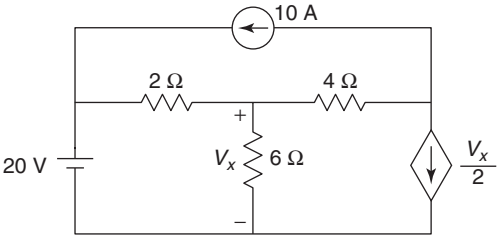


Fig. 3.99

Solution

Step I When the 20 V source is acting alone (Fig. 3.100)
From Fig. 3.100,

$$V_x' = 6(I_1 - I_2) \quad ...(i)$$

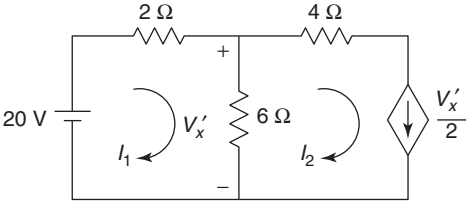


Fig. 3.100

Applying KVL to Mesh 1,

$$20 - 2I_1 - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 = 20 \quad ...(ii)$$

For Mesh 2,

$$I_2 = \frac{V_x'}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \quad ...(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 5.71 \text{ A}$$

$$I_2 = 4.29 \text{ A}$$

$$V_x' = 6(I_1 - I_2) = 6(5.71 - 4.29) = 8.52 \text{ V}$$

Step II When the 10 A source is acting alone (Fig. 3.101)
From Fig. 3.101,

$$V_x'' = 6(I_1 - I_2) \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2(I_1 - I_3) - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 - 2I_3 = 0 \quad \text{...(ii)}$$

For Mesh 2,

$$I_2 = \frac{V_x''}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \quad \text{...(iii)}$$

For Mesh 3,

$$I_3 = -10 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -5.71 \text{ A}$$

$$I_2 = -4.29 \text{ A}$$

$$I_3 = -10 \text{ A}$$

$$V_x'' = 6(I_1 - I_2) = 6(-5.71 + 4.29) = -8.52 \text{ V}$$

Step III By superposition theorem,

$$V_x = V_x' + V_x'' = 8.52 - 8.52 = 0$$

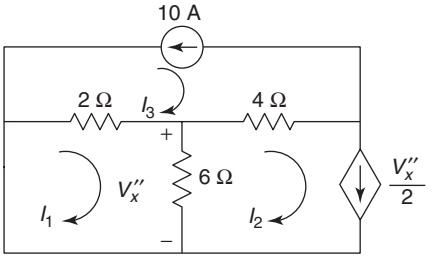


Fig. 3.101

Example 3.25

Calculate the current I in the network shown in Fig. 3.102.

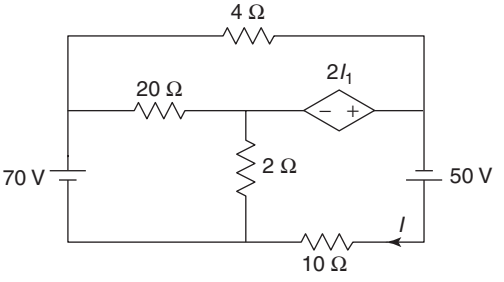


Fig. 3.102

Solution

Step I When the 70 V source is acting alone (Fig. 3.103)
From Fig. 3.103,

$$I' = I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$70 - 20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 70 \quad \text{...(iii)}$$

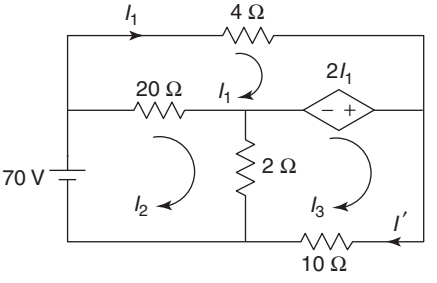


Fig. 3.103

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = 0 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 8.94\,\text{A}$$
$$I_2 = 11.62\,\text{A}$$
$$I_3 = 3.43\,\text{A}$$
$$I' = I_3 = 3.43\,\text{A}\,(\leftarrow)$$

Step II When the 50 V source is acting alone (Fig. 3.104)

From Fig. 3.104,

$$I'' = I_3 \quad \ldots(\text{i})$$

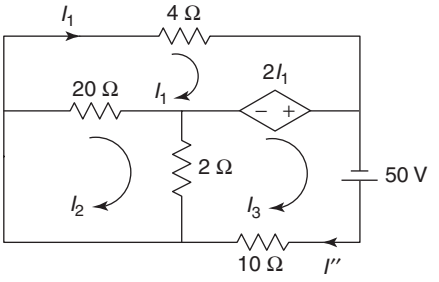


Fig. 3.104

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \quad \ldots(\text{ii})$$

Applying KVL to Mesh 2,

$$-20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 0 \quad \ldots(\text{iii})$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 + 50 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = -50 \quad \ldots(\text{iv})$$

Solving Eqs (ii), (iii), and (iv),

$$I_1 = 1.06\ \text{A}$$
$$I_2 = 1.38\ \text{A}$$
$$I_3 = 4.57\ \text{A}$$
$$I'' = I_3 = 4.57\ \text{A}\ (\leftarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.43 + 4.57 = 8\ \text{A}\ (\leftarrow)$$

Example 3.26 *Find the voltage V_0 in the network of Fig. 3.105.*

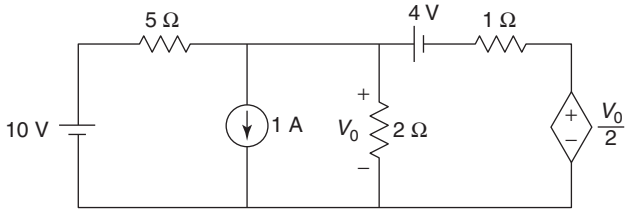


Fig. 3.105

Solution

Step I When the 10 V source is acting alone (Fig. 3.106)
Applying KCL at the node,

$$\frac{V_0' - 10}{5} + \frac{V_0'}{2} + \frac{V_0' - \frac{V_0'}{2}}{1} = 0$$

$$\left(\frac{1}{5} + \frac{1}{2} + \frac{1}{2}\right) V_0' = 2$$

$$V_0' = 1.67 \text{ V}$$

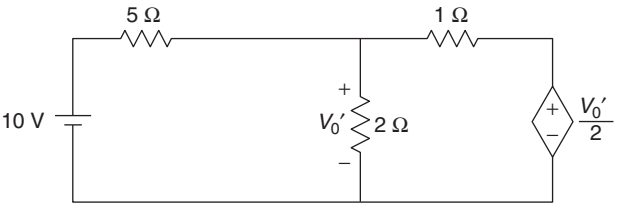


Fig. 3.106

Step II When the 1A current source is acting alone (Fig. 3.107)
Applying KCL at the node,

$$\frac{V_0''}{5} + 1 + \frac{V_0''}{2} + \frac{V_0'' - \frac{V_0''}{2}}{1}$$

$$\left(\frac{1}{5} + \frac{1}{2} + \frac{1}{2}\right) V_0'' = -1$$

$$V_0'' = -0.83 \text{ V}$$

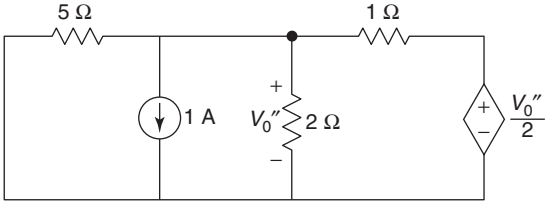


Fig. 3.107

Step III When the 4 V source is acting alone (Fig. 3.108)
Applying KCL at the node,

$$\frac{V_0'''}{5} + \frac{V_0'''}{2} + \frac{V_0''' - 4 - \frac{V_0'''}{2}}{1} = 0$$

$$\left(\frac{1}{5} + \frac{1}{2} + \frac{1}{2}\right) V_0''' = 4$$

$$V_0''' = 3.33 \text{ V}$$

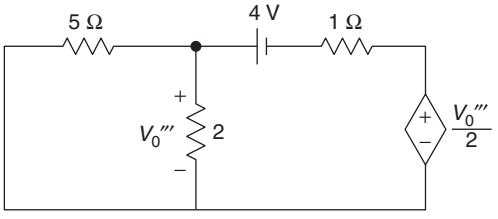


Fig. 3.108

Step IV By superposition theorem,

$$V_0 = V_0' + V_0'' + V_0''' = 1.67 - 0.83 + 3.33 = 4.17 \text{ V}$$

3.3 THEVENIN'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent voltage source and an equivalent series resistance. The voltage source is the voltage across the two terminals with load, if any, removed. The series resistance is the resistance of the network measured between two terminals with load removed and constant voltage source being replaced by its internal resistance (or if it is not given with zero resistance, i.e., short circuit) and constant current source replaced by infinite resistance, i.e., open circuit*.'

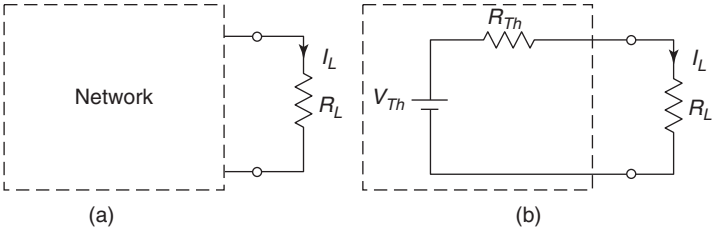


Fig. 3.109 *Network illustrating Thevenin's theorem*

Explanation Consider a simple network as shown in Fig. 3.110.

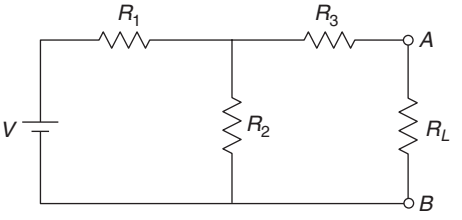


Fig. 3.110 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate open circuit voltage V_{Th} across points A and B as shown in Fig. 3.111.

$$V_{\text{Th}} = \frac{R_2}{R_1 + R_2} \text{ V}$$

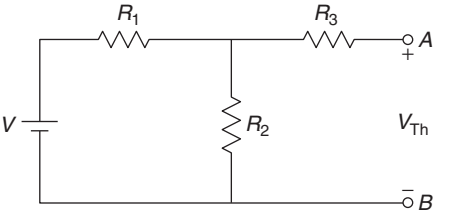


Fig. 3.111 *Calculation of V_{Th}*

For finding series resistance R_{Th}, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 3.112.

$$R_{\text{Th}} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

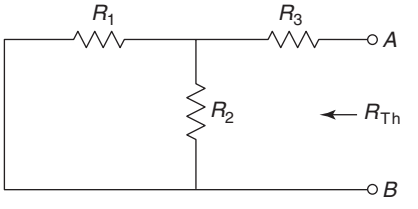


Fig. 3.112 *Calculation of R_{Th}*

Thevenin's equivalent network is shown in Fig. 3.113.

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L}$$

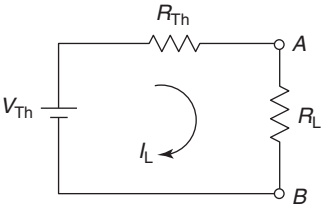


Fig 3.113 *Thevenin's equivalent network*

If the network contains both independent and dependent sources, Thevenin's resistance R_{Th} is calculated as,

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N}$$

where I_N is the short-circuit current which would flow in a short circuit placed across the terminals A and B. Dependent sources are active at all times. They have zero values only when the control voltage or current is zero. R_{Th} may be negative in

some cases which indicates negative resistance region of the device, i.e., as voltage increases, current decreases in the region and vice-versa.

If the network contains only dependent sources then

$$V_{\text{Th}} = 0$$
$$I_N = 0$$

For finding R_{Th} in such a network, a known voltage V is applied across the terminals A and B and current is calculated through the path AB.

$$R_{\text{Th}} = \frac{V}{I}$$

or a known current source I is connected across the terminals A and B and voltage is calculated across the terminals A and B.

$$R_{\text{Th}} = \frac{V}{I}$$

Thevenin's equivalent network for such a network is shown in Fig. 3.114.

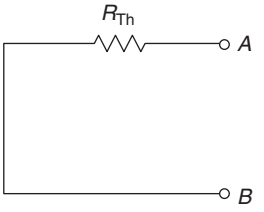


Fig. 3.114 *Thevenin's equivalent network*

Steps to be Followed in Thevenin's Theorem

1. Remove the load resistance R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Replace the network by a voltage source V_{Th} in series with resistance R_{Th}.
5. Find the current through R_L using Ohm's law.

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L}$$

Example 3.27

Find the current through the 2 Ω resistor in Fig. 3.115.

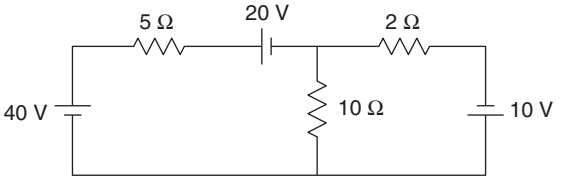


Fig. 3.115

Solution

Step I Calculation of V_{Th} (Fig. 3.116)
Applying KVL to the mesh,

$$40 - 5I - 20 - 10I = 0$$
$$15I = 20$$
$$I = 1.33 \text{ A}$$

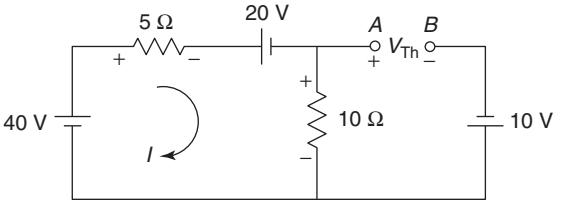


Fig. 3.116

Writing the V_{Th} equation,

$$10I - V_{Th} + 10 = 0$$

$$V_{Th} = 10I + 10 = 10(1.33) + 10 = 23.33 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.117)

$$R_{Th} = 5 \,||\, 10 = 3.33\ \Omega$$

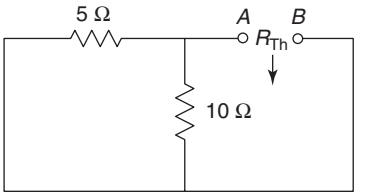


Fig. 3.117

Step III Calculation of I_L (Fig. 3.118)

$$I_L = \frac{23.33}{3.33 + 2} = 4.38 \text{ A}$$

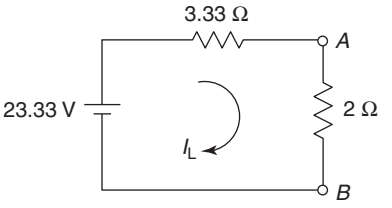


Fig. 3.118

Example 3.28 *Find the current through the 8 Ω resistor in Fig. 3.119.*

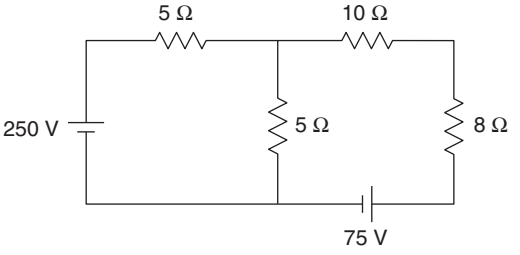


Fig. 3.119

Solution

Step I Calculation of V_{Th} (Fig. 3.120)

$$I = \frac{250}{5 + 5} = 25 \text{ A}$$

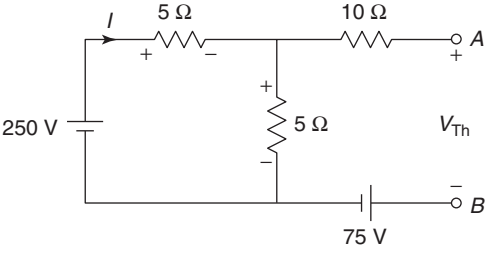


Fig. 3.120

Writing the V_{Th} equation,

$$250 - 5I - V_{Th} - 75 = 0$$

$$V_{Th} = 175 - 5I = 175 - 5(25) = 50 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.121)

$$R_{Th} = (5 \,||\, 5) + 10 = 12.5\ \Omega$$

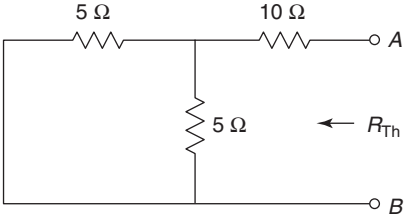


Fig. 3.121

Step III Calculation of I_L (Fig. 3.122)

$$I_L = \frac{50}{12.5+8} = 2.44 \text{ A}$$

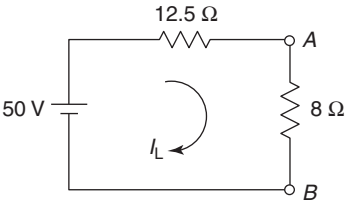


Fig. 3.122

Example 3.29 *Find the current through the 2 Ω resistor connected between terminals A and B in the Fig. 3.123.*

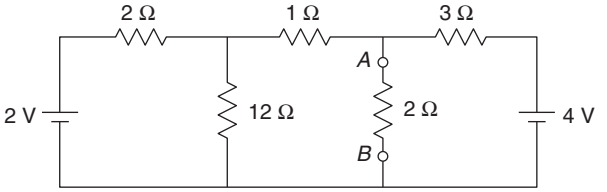


Fig. 3.123

Solution

Step I Calculation of V_{Th} (Fig. 3.124)

Applying KVL to Mesh 1,

$$2 - 2I_1 - 12(I_1 - I_2) = 0$$
$$14I_1 - 12I_2 = 2 \quad ...(i)$$

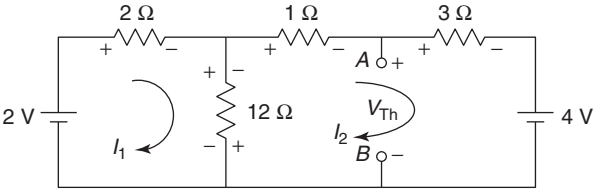


Fig. 3.124

Applying KVL to Mesh 2,

$$-12(I_2 - I_1) - 1I_2 - 3I_2 - 4 = 0$$
$$-12I_1 + 16I_2 = -4 \quad ...(ii)$$

Solving Eqs (i) and (ii),

$$I_2 = -0.4 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 3I_2 - 4 = 0$$
$$V_{Th} = 4 + 3I_2 = 4 + 3(-0.4) = 2.8 \text{ V}$$

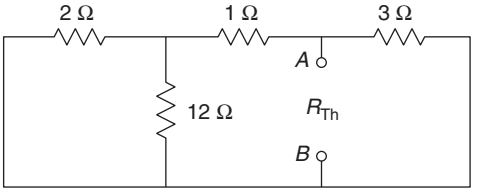


Fig. 3.125

Step II Calculation of R_{Th} (Fig. 3.125)

$$R_{Th} = [(2 \,||\, 12) + 1] \,||\, 3 = 1.43 \,\Omega$$

Step III Calculation of I_L (Fig. 3.126)

$$I_L = \frac{2.8}{1.43+2} = 0.82 \text{ A}$$

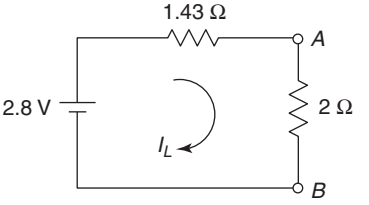


Fig. 3.126

Example 3.30 *Find the current through the 10 Ω resistor in Fig. 3.127.*

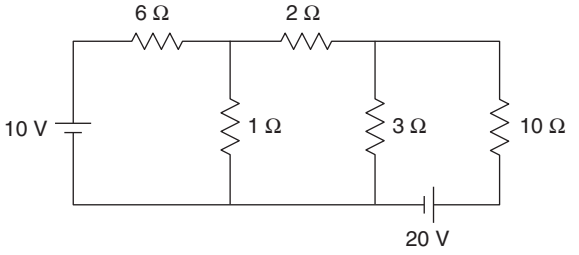


Fig. 3.127

Solution

Step I Calculation of V_{Th} (Fig. 3.128)

Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \quad \text{.....(i)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3I_2 = 0$$
$$-I_1 + 6I_2 = 0 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = 0.24 \text{ A} \quad \text{...(ii)}$$

Writing the V_{Th} equation,

$$3I_2 - V_{Th} - 20 = 0$$
$$V_{Th} = 3I_2 - 20 = 3(0.24) - 20 = -19.28 \text{ V}$$
$$= 19.28 \text{ V (terminal } B \text{ is positive w.r.t } A)$$

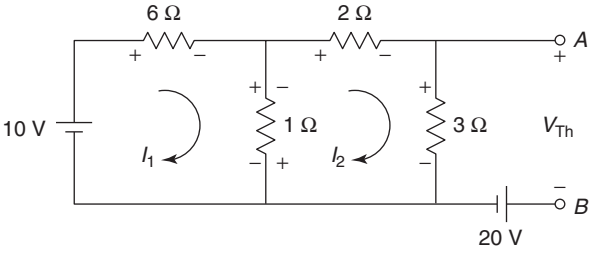


Fig. 3.128

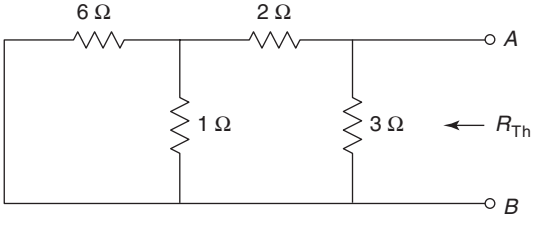


Fig. 3.129

Step II Calculation of R_{Th} (Fig. 3.129)

$$R_{Th} = [(6 \,||\, 1) + 2] \,||\, 3 = 1.47 \,\Omega$$

Step III Calculation of I_L (Fig. 3.130)

$$I_L = \frac{19.28}{1.47 + 10} = 1.68 \text{ A}(\uparrow)$$

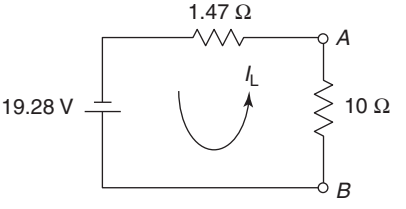


Fig. 3.130

Example 3.31 *Find the current through the 10 Ω resistor in Fig. 3.131.*

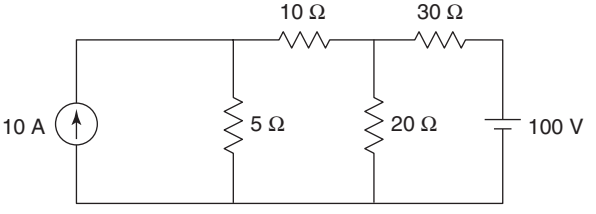


Fig. 3.131

Solution

Step I Calculation of V_{Th} (Fig. 3.132)
For Mesh 1,

$$I_1 = 10$$

Applying KVL to Mesh 2,

$$100 - 30I_2 - 20I_2 = 0$$

$$I_2 = 2 \text{ A}$$

Writing the V_{Th} equation,

$$5I_1 - V_{Th} - 20I_2 = 0$$

$$V_{Th} = 5I_1 - 20I_2 = 5(10) - 20(2) = 10 \text{ V}$$

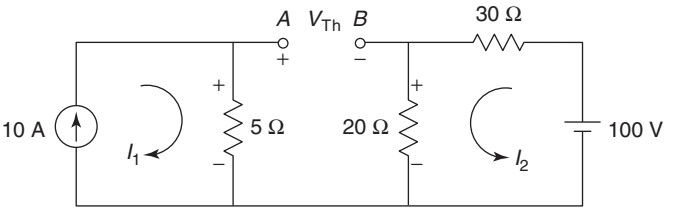


Fig. 3.132

Step II Calculation of R_{Th} (Fig. 3.133)

$$R_{Th} = 5 + (20 \parallel 30) = 17 \ \Omega$$

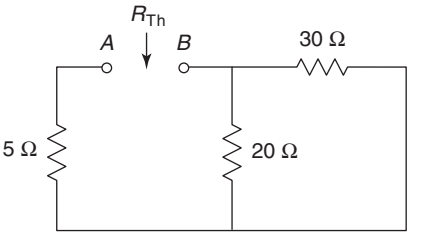


Fig. 3.133

Step III Calculation of I_L (Fig. 3.134)

$$I_L = \frac{10}{17+10} = 0.37 \text{ A}$$

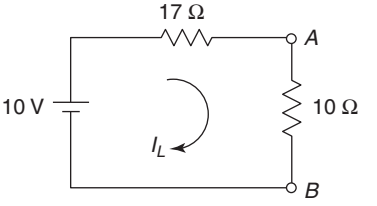


Fig. 3.134

Example 3.32

Find the current through the 40 Ω resistor in Fig. 3.135.

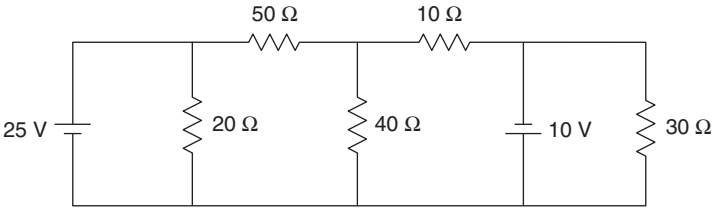


Fig. 3.135

Solution

Step I Calculation of V_{Th} (Fig. 3.136)

Since the 20 Ω resistor is connected across the 25 V source, the resistor becomes redundant.

$$V_{20\,\Omega} = 25 \text{ V}$$

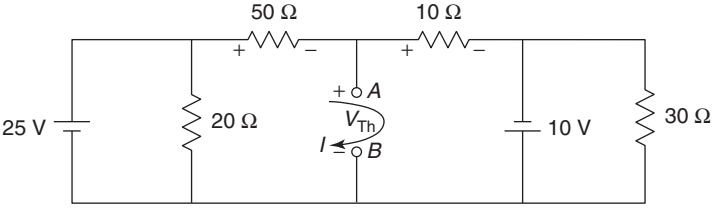


Fig. 3.136

Applying KVL to the mesh,

$$25 - 50I - 10I + 10 = 0$$

$$I = 0.58 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 10I + 10 = 0$$

$$V_{Th} = 10(I) - 10 = 10(0.58) - 10 = -4.2 \text{ V}$$

$$= 4.2 \text{ V(terminal } B \text{ is positive w.r.t } A)$$

Step II Calculation of R_{Th} (Fig. 3.137)

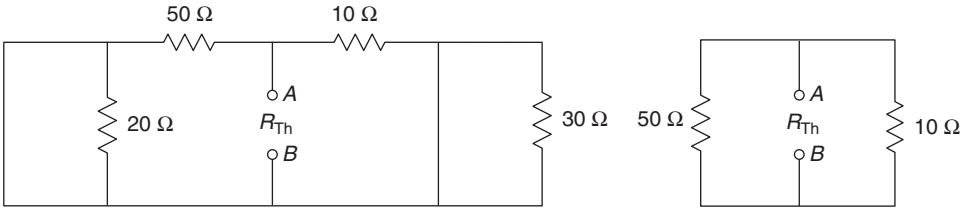


Fig. 3.137

$$R_{Th} = 50 \,||\, 10 = 8.33 \,\Omega$$

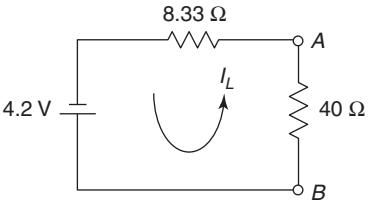


Fig. 3.138

Step III Calculation of I_L (Fig. 3.138)

$$I_L = \frac{4.2}{8.33 + 40} = 0.09 \text{ A}(\uparrow)$$

Example 3.33 *Find the current through the 10 Ω resistor in Fig. 3.139.*

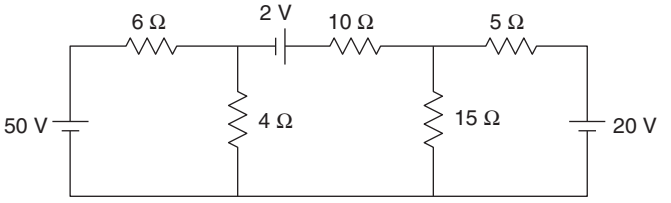


Fig. 3.139

Solution

Step I Calculation of V_{Th} (Fig. 3.140)

$$I_1 = \frac{50}{6 + 4} = 5 \text{ A}$$

$$I_2 = \frac{20}{5 + 15} = 1 \text{ A}$$

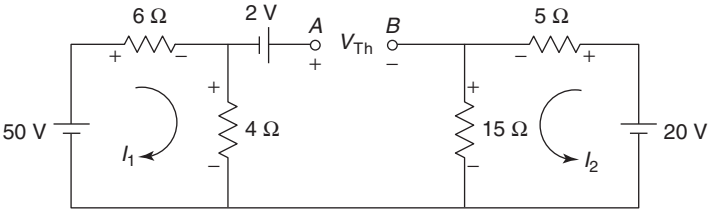


Fig. 3.140

Writing the V_{Th} equation,

$$4I_1 + 2 - V_{Th} - 15I_2 = 0$$

$$V_{Th} = 4I_1 + 2 - 15I_2 = 4(5) + 2 - 15(1) = 7 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.141)

$$R_{Th} = (6 \,||\, 4) + (5 \,||\, 15) = 6.15\ \Omega$$

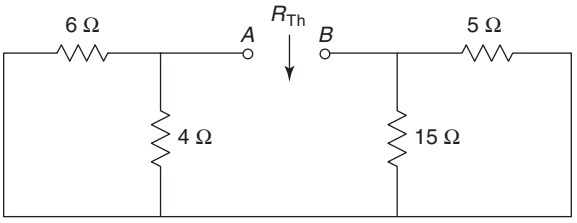


Fig. 3.141

Step III Calculation of I_L (Fig. 3.142)

$$I_L = \frac{7}{6.15 + 10} = 0.43 \text{ A}$$

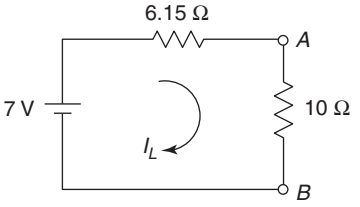


Fig. 3.142

Example 3.34 *Determine the current through the 24 Ω resistor in Fig. 3.143.*

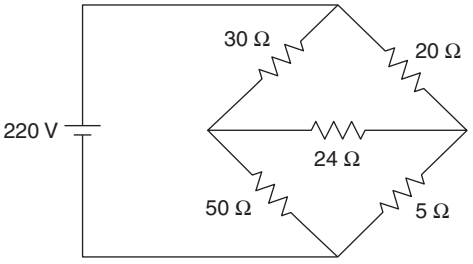


Fig. 3.143

Solution

Step I Calculation of V_{Th} (Fig. 3.144)

$$I_1 = \frac{220}{30 + 50} = 2.75 \text{ A}$$

$$I_2 = \frac{220}{20 + 5} = 8.8 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} + 30I_1 - 20I_2 = 0$$

$$V_{Th} = 20I_2 - 30I_1 = 20(8.8) - 30(2.75) = 93.5 \text{ V}$$

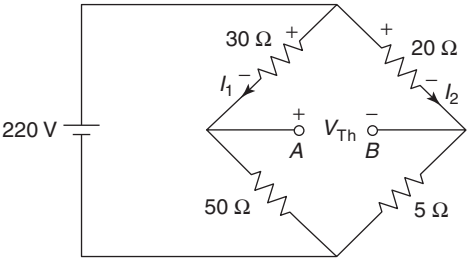


Fig. 3.144

Step II Calculation of R_{Th} (Fig. 3.145)

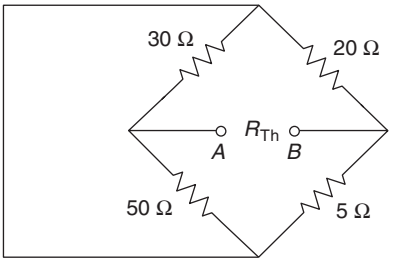


Fig. 3.145

Redrawing the circuit (Fig. 3.146),

$$R_{Th} = (30 \,||\, 50) + (20 \,||\, 5) = 22.75\ \Omega$$

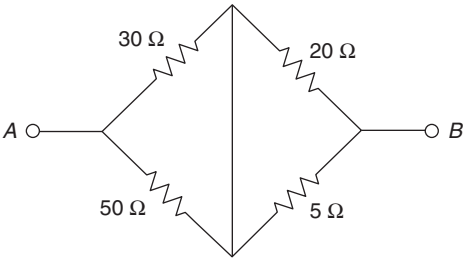


Fig. 3.146

Step III Calculation of I_L (Fig. 3.147)

$$I_L = \frac{93.5}{22.75 + 24} = 2 \text{ A}$$

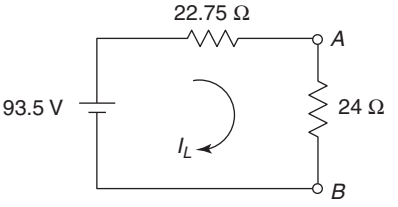


Fig. 3.147

Example 3.35 *Find the current through the 3 Ω resistor in Fig. 3.148.*

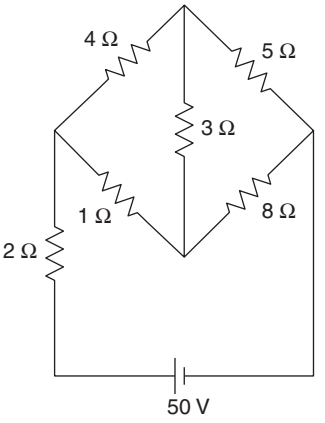


Fig. 3.148

Solution

Step I Calculation of V_{Th} (Fig. 3.149)

Applying KVL to Mesh 1,

$$50 - 2I_1 - 1(I_1 - I_2) - 8(I_1 - I_2) = 0$$
$$11I_1 - 9I_2 = 50 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-4I_2 - 5I_2 - 8(I_2 - I_1) - 1(I_2 - I_1) = 0$$
$$-9I_1 + 18I_2 = 0 \quad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$I_1 = 7.69 \text{ A}$$
$$I_2 = 3.85 \text{ A}$$

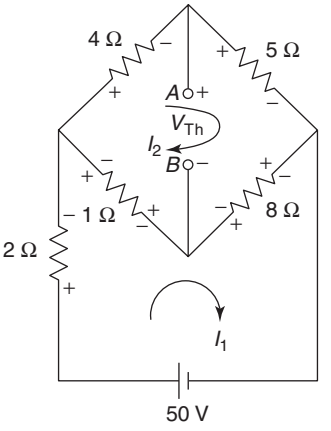


Fig. 3.149

Writing the V_{Th} equation,

$$V_{Th} - 5I_2 - 8(I_2 - I_1) = 0$$
$$V_{Th} = 5I_2 + 8(I_2 - I_1) = 5(3.85) + 8(3.85 - 7.69) = -11.47 \text{ V}$$
$$= 11.47 \text{ V (the terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.150)

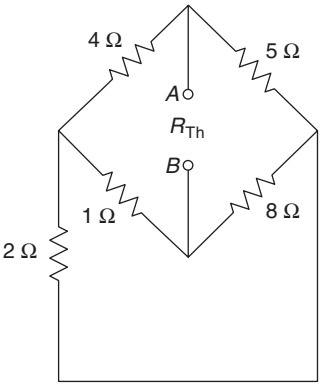


Fig. 3.150

Redrawing the network (Fig. 3.151),

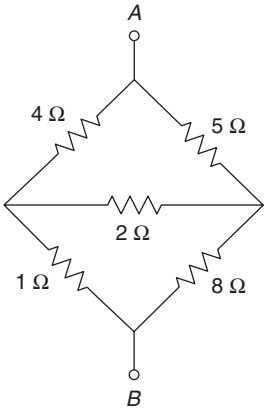


Fig. 3.151

Converting the upper delta into equivalent star network (Fig. 3.152),

$$R_1 = \frac{4 \times 2}{4+2+5} = 0.73\ \Omega$$

$$R_2 = \frac{4 \times 5}{4+2+5} = 1.82\ \Omega$$

$$R_3 = \frac{5 \times 2}{4+2+5} = 0.91\ \Omega$$

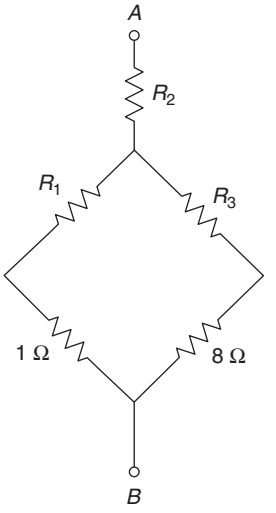


Fig. 3.152

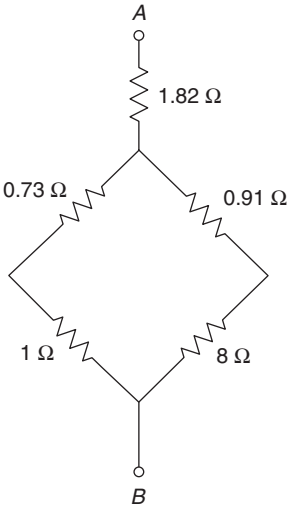


Fig. 3.153

Simplifying the network (Fig. 3.153),

$$R_{\text{Th}} = 1.82 + (1.73 \,||\, 8.91) = 3.27\ \Omega$$

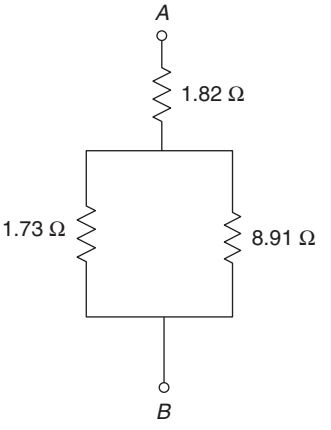


Fig. 3.154

Step III Calculation of I_L (Fig. 3.155)

$$I_L = \frac{11.47}{3.27+3} = 1.83\,\text{A}(\uparrow)$$

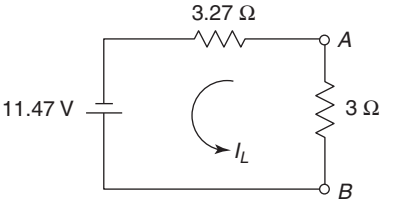


Fig. 3.155

Example 3.36 *Find the current through the 20 Ω resistor in Fig. 3.156.*

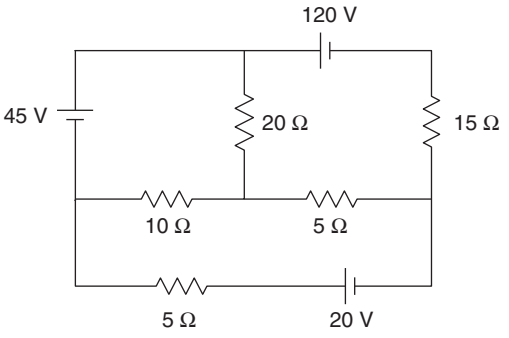


Fig. 3.156

Solution

Step I Calculation of V_{Th} (Fig. 3.157)

Applying KVL to Mesh 1,

$$45 - 120 - 15I_1 - 5(I_1 - I_2) - 10(I_1 - I_2) = 0$$

$$30I_1 - 15I_2 = -75 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$20 - 5I_2 - 10(I_2 - I_1) - 5(I_2 - I_1) = 0$$

$$-15I_1 + 20I_2 = 20 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = -3.2 \text{ A}$$

$$I_2 = -1.4 \text{ A}$$

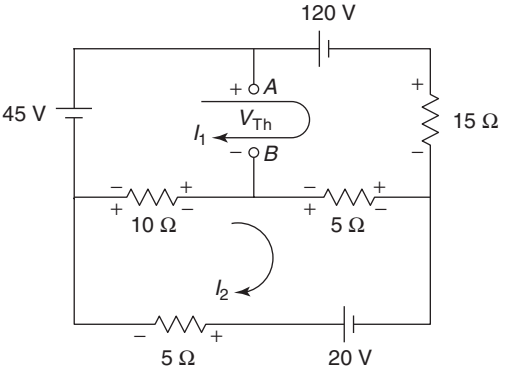


Fig. 3.157

Writing the V_{Th} equation,

$$45 - V_{Th} - 10(I_1 - I_2) = 0$$

$$V_{Th} = 45 - 10(I_1 - I_2) = 45 - 10[-3.2 - (-1.4)] = 63 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.158)

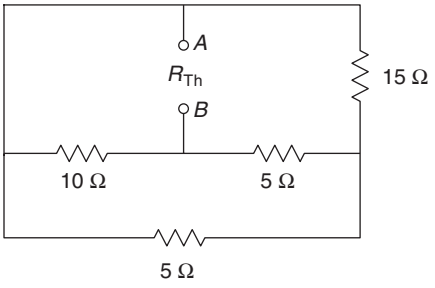


Fig. 3.158

Converting the delta formed by resistors of 10 Ω, 5 Ω and 5 Ω into an equivalent star network (Fig. 3.159),

$$R_1 = \frac{10 \times 5}{20} = 2.5\ \Omega$$

$$R_2 = \frac{10 \times 5}{20} = 2.5\ \Omega$$

$$R_3 = \frac{5 \times 5}{20} = 1.25\ \Omega$$

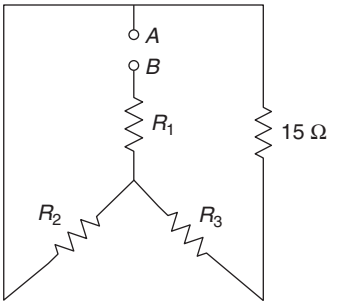


Fig. 3.159

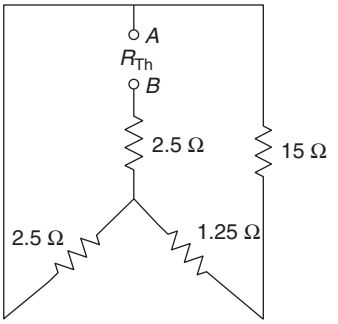


Fig. 3.160

Simplifying the network (Fig. 3.160),

$$R_{Th} = (16.25 \,||\, 2.5) + 2.5 = 4.67\ \Omega$$

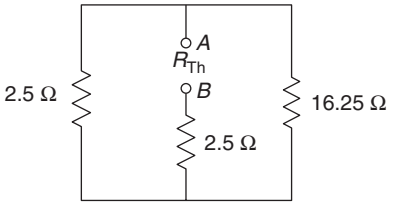


Fig. 3.161

Step III Calculation of I_L

$$I_L = \frac{63}{4.67 + 20} = 2.55\,\text{A}$$

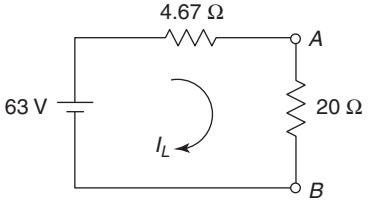


Fig. 3.162

Example 3.37

Find the current through the 3 Ω resistor in Fig. 3.163.

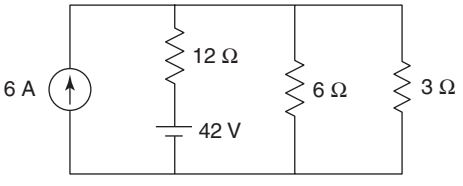


Fig. 3.163

Solution

Step I Calculation of V_{Th} (Fig. 3.164)

For Mesh 1,

$$I_1 = 6 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$42 - 12(I_2 - I_1) - 6I_2 = 0$$

$$-12I_1 + 18I_2 = 42 \qquad \ldots(ii)$$

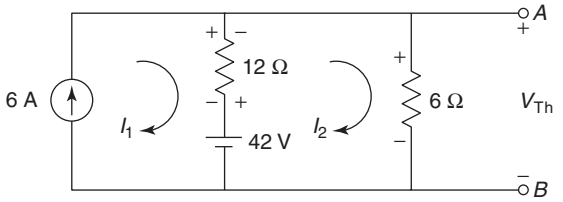


Fig. 3.164

Solving Eqs (i) and (ii),

$$I_2 = 6.33 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} = 6I_2 = 38 \text{ V}$$

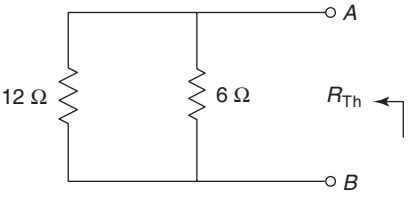


Fig. 3.165

Step II Calculation of R_{Th} (Fig. 3.165)

$$R_{Th} = 6 \,||\, 12 = 4\,\Omega$$

Step III Calculation of I_L (Fig. 3.166)

$$I_L = \frac{38}{4+3} = 5.43\,\text{A}$$

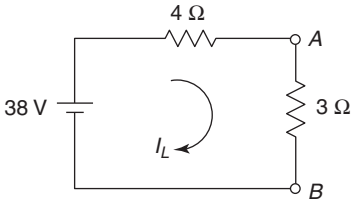


Fig. 3.166

Example 3.38

Find the current through the 30 Ω resistor in Fig. 3.167.

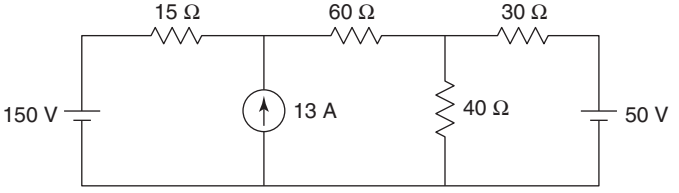


Fig. 3.167

Solution

Step I Calculation of V_{Th} (Fig. 3.168)

Meshes 1 and 2 form a supermesh.

Writing current equation for supermesh,

$$I_2 - I_1 = 13 \qquad ...(i)$$

Writing voltage equation for supermesh,

$$150 - 15I_1 - 60I_2 - 40I_2 = 0$$

$$15I_1 + 100I_2 = 150 \quad ...(ii)$$

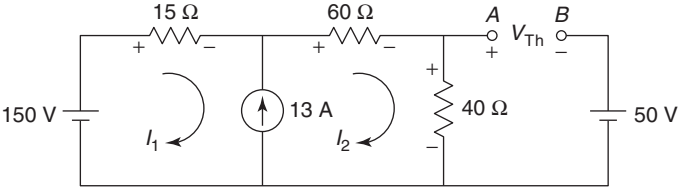


Fig. 3.168

Solving Eqs (i) and (ii),

$$I_1 = -10 \text{ A}$$

$$I_2 = 3 \text{ A}$$

Writing the V_{Th} equation,

$$40I_2 - V_{Th} - 50 = 0$$

$$V_{Th} = 40I_2 - 50 = 40(3) - 50 = 70 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.169)

$$R_{Th} = 75 \,||\, 40 = 26.09\,\Omega$$

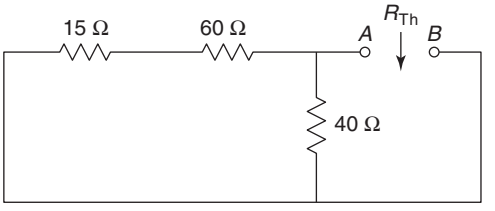


Fig. 3.169

Step III Calculation of I_L (Fig. 3.170)

$$I_L = \frac{70}{26.09 + 30} = 1.25\,\text{A}$$

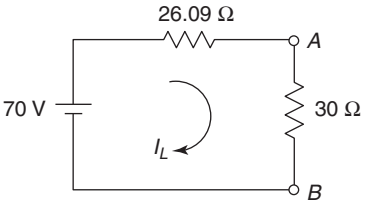


Fig. 3.170

Example 3.39 *Find the current through the 20 Ω resistor in Fig. 3.171.*

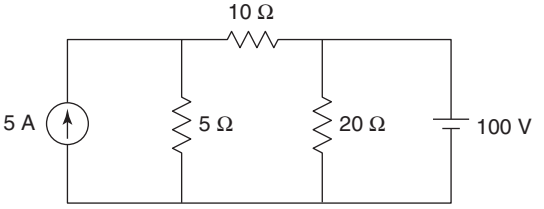


Fig. 3.171

Solution

Step I Calculation of V_{Th} (Fig. 3.172)

$$V_{\text{Th}} = 100\,\text{V}$$

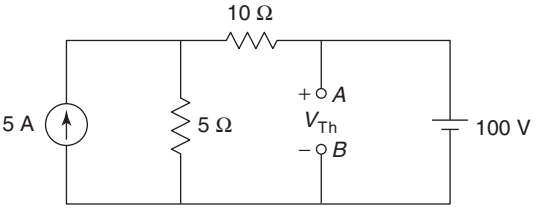


Fig. 3.172

Step II Calculation of R_{Th} (Fig.3.173)

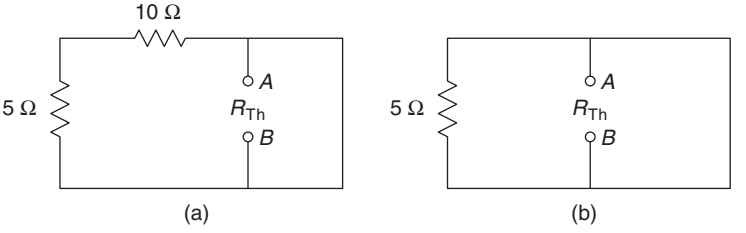


Fig. 3.173

$$R_{\text{Th}} = 0$$

Step III Calculation of I_L (Fig. 3.174)

$$I_L = \frac{100}{20} = 5\,\text{A}$$

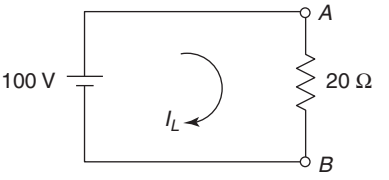


Fig. 3.174

Example 3.40 *Find the current through the 20 Ω resistor in Fig. 3.175.*

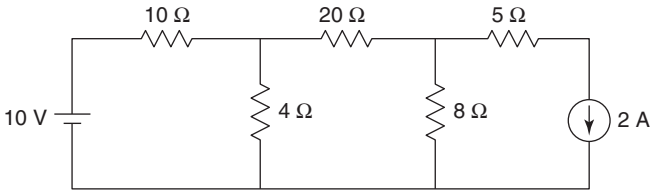


Fig. 3.175

Solution

Step I Calculation of V_{Th} (Fig. 3.176)

$$I_1 = \frac{10}{10+4} = 0.71\text{ A}$$

$$I_2 = 2\text{ A}$$

Writing the V_{Th} equation,

$$4I_1 - V_{Th} + 8I_2 = 0$$

$$V_{Th} = 4(0.71) + 8(2) = 18.84\text{ V}$$

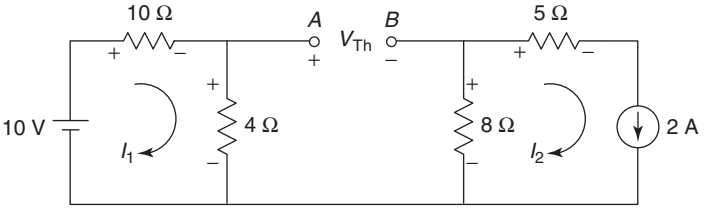


Fig. 3.176

Step II Calculation of R_{Th} (Fig. 3.177).

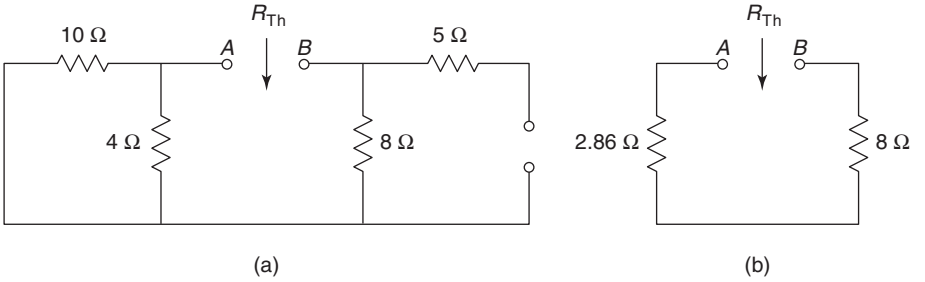


Fig. 3.177

$$R_{Th} = 10.86\ \Omega$$

Step III Calculation of I_L (Fig. 3.178)

$$I_L = \frac{18.84}{10.86+20} = 0.61\text{A}$$

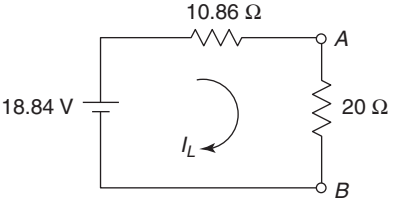


Fig. 3.178

Example 3.41 *Find the current through the 5 Ω resistor in Fig. 3.179.*

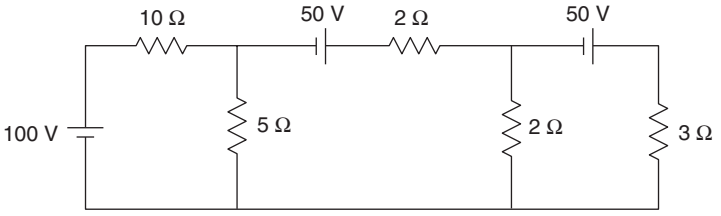


Fig. 3.179

Solution

Step I Calculation of V_{Th} (Fig. 3.180)

Applying KVL to Mesh 1,

$$100 - 10I_1 + 50 - 2I_1 - 2(I_1 - I_2) = 0$$

$$14I_2 - 2I_2 = 150 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) + 50 - 3I_2 = 0$$

$$-2I_1 + 5I_2 = 50 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 12.88 \text{ A}$$

$$I_2 = 15.15 \text{ A}$$

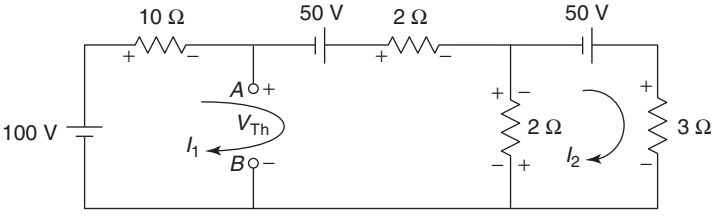


Fig. 3.180

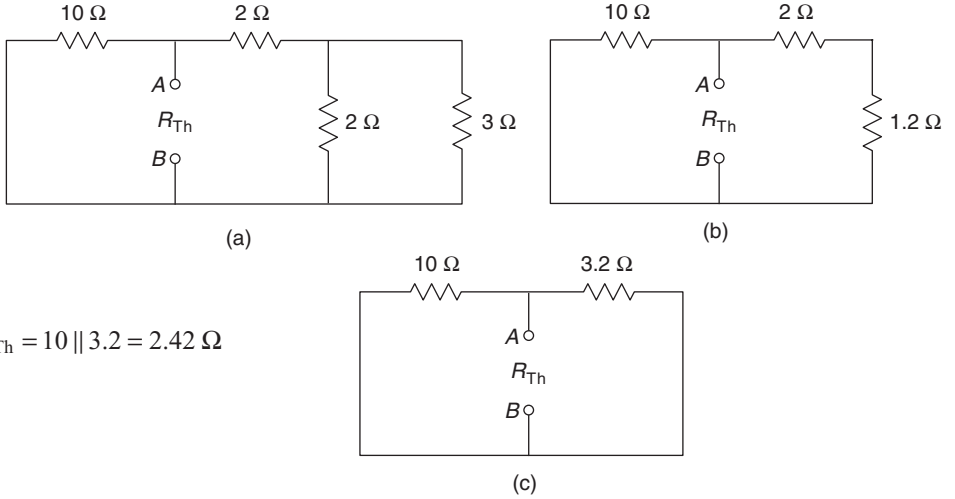
Writing the V_{Th} equation,

$$100 - 10I_1 - V_{Th} = 0$$

$$V_{Th} = 100 - 10(12.88) = -28.8 \text{ V}$$

$$= 28.8 \text{ V (terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.181)



$$R_{Th} = 10 \,||\, 3.2 = 2.42 \ \Omega$$

Fig. 3.181

Step III Calculation of I_L (Fig. 3.182)

$$I_L = \frac{28.8}{2.42+5} = 3.88 \text{ A } (\uparrow)$$

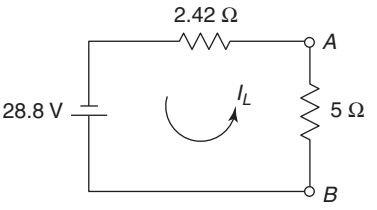


Fig. 3.182

Example 3.42

Find the current through the 10 Ω resister in Fig. 3.183.

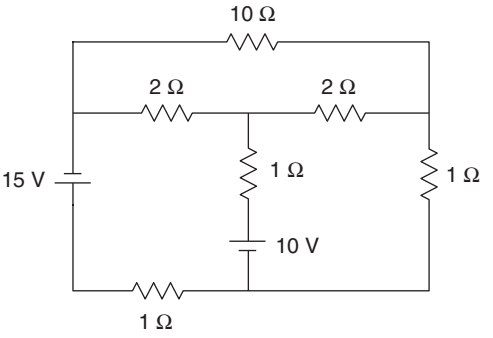


Fig. 3.183

Solution

Step I Calculation of V_{Th} (Fig. 3.183)

Applying KVL to Mesh 1,

$$-15 - 2I_1 - 1(I_1 - I_2) - 10 - 1I_1 = 0$$

$$4I_1 - I_2 = -25 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$10 - 1(I_2 - I_1) - 2I_2 - 1I_2 = 0$$

$$-I_1 + 4I_2 = 10 \quad \text{...(ii)}$$

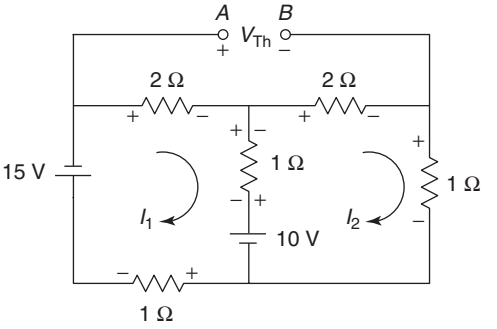


Fig. 3.184

Solving Eqs (i) and (ii),

$$I_1 = -6 \text{ A}$$

$$I_2 = 1 \text{ A}$$

Writing the V_{Th} equation,

$$-V_{Th} + 2I_2 + 2I_1 = 0$$

$$V_{Th} = 2I_1 + 2I_2 = 2(-6) + 2(1) = -10 \text{ V}$$

$$= 10 \text{ V} (\text{the terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.185)

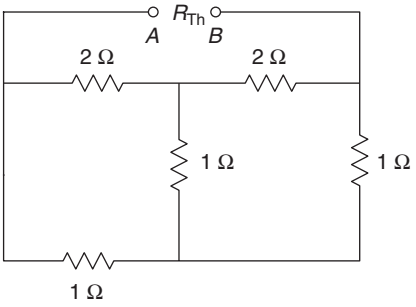


Fig. 3.185

Converting the star network formed by resistors of $2\,\Omega$, $2\,\Omega$ and $1\,\Omega$ into an equivalent delta network (Fig. 3.186),

$$R_1 = 2 + 2 + \frac{2 \times 2}{1} = 8\,\Omega$$

$$R_2 = 2 + 1 + \frac{2 \times 1}{2} = 4\,\Omega$$

$$R_3 = 2 + 1 + \frac{2 \times 1}{2} = 4\,\Omega$$

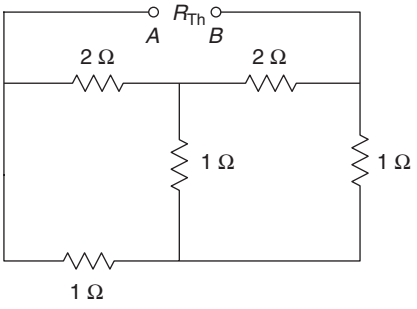


Fig. 3.186

Simplifying the network (Fig. 3.187),

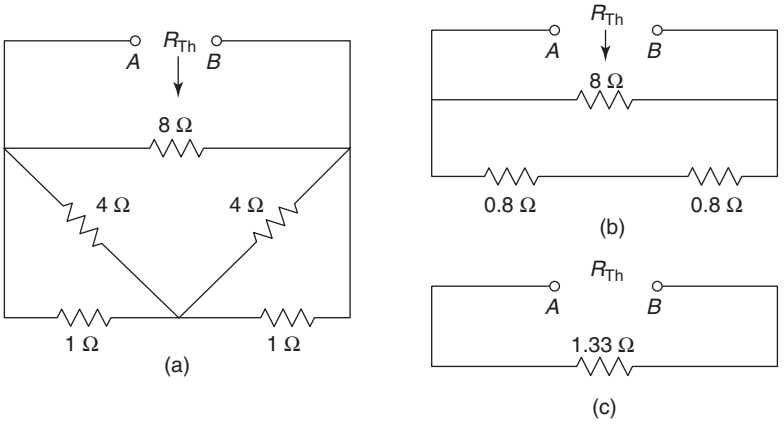


Fig. 3.187

$$R_{Th} = 1.33\,\Omega$$

Step III Calculation of I_L (Fig. 3.188)

$$I_L = \frac{10}{1.33+10} = 0.88 \text{ A } (\uparrow)$$

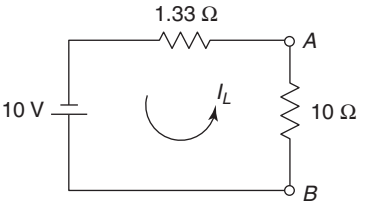


Fig. 3.188

Example 3.43 *Find the current through the 1 Ω resistor in Fig. 3.189.*

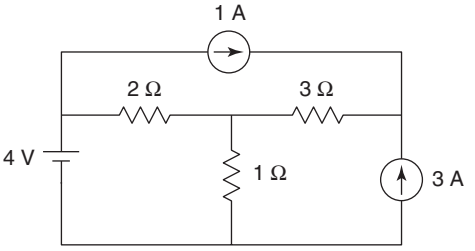


Fig. 3.189

Solution

Step I Calculation of V_{Th} (Fig. 3.190)

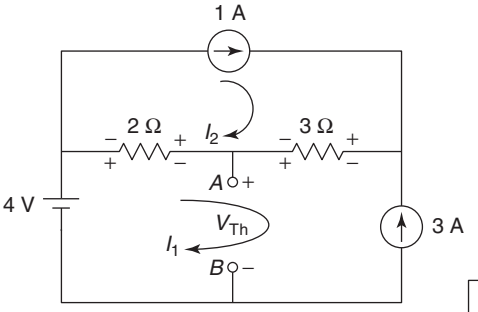


Fig. 3.190

Writing the current equations for Meshes 1 and 2,

$$I_1 = -3$$

$$I_2 = 1$$

Writing the V_{Th} equation,

$$4 - 2(I_1 - I_2) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 4 - 2(I_1 - I_2) = 4 - 2(-4) = 12 \text{ V}$$

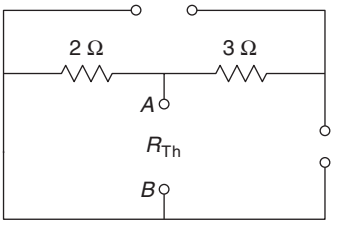


Fig. 3.191

Step II Calculation of R_{Th} (Fig. 3.191)

$$R_{\text{Th}} = 2 \text{ Ω}$$

Step III Calculation of I_L (Fig. 3.192)

$$I_L = \frac{12}{2+1} = 4 \text{ A}$$

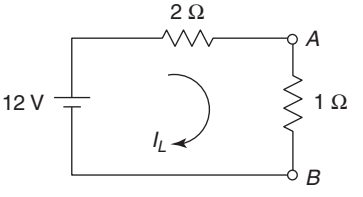


Fig. 3.192

Example 3.44 *Find the current through the 3 Ω resistor in Fig. 3.193.*

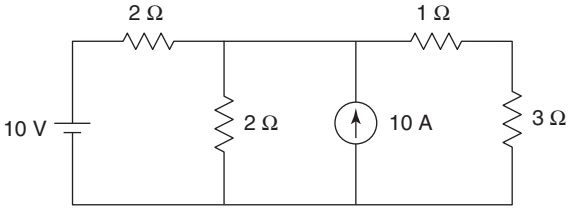


Fig. 3.193

Solution

Step I Calculation of V_{Th} (Fig. 3.194)

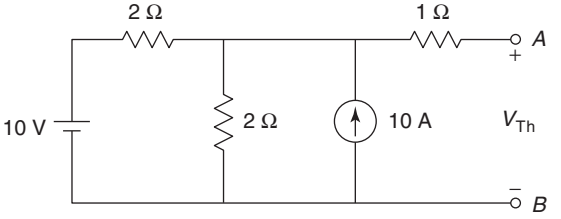


Fig. 3.194

By source transformation (Fig. 3.195),

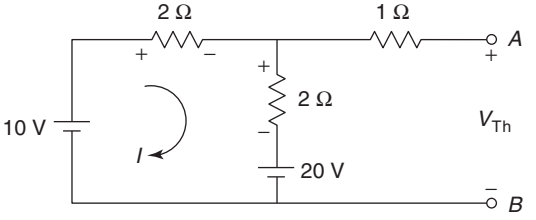


Fig. 3.195

Applying KVL to the mesh,

$$10 - 2I - 2I - 20 = 0$$
$$4I = -10$$
$$I = -2.5\,\text{A}$$

Writing the V_{Th} equation,

$$10 - 2I - V_{Th} = 0$$
$$V_{Th} = 10 - 2I = 10 - 2(-2.5) = 15\,\text{V}$$

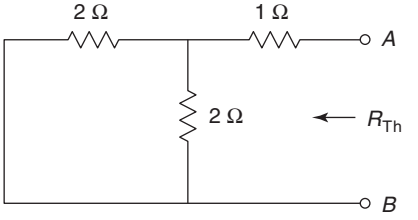


Fig. 3.196

Step II Calculation of R_{Th} (Fig. 3.196)

$$R_{Th} = (2 \parallel 2) + 1 = 1 + 1 = 2\,\Omega$$

Step III Calculation of I_L (Fig. 3.197)

$$I_L = \frac{15}{2+3} = 3\,\text{A}$$

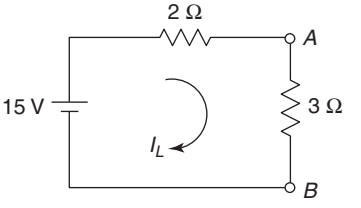


Fig. 3.197

EXAMPLES WITH DEPENDENT SOURCES

Example 3.45 *Obtain the Thevenin equivalent network for the given network of Fig. 3.198 at terminals A and B.*

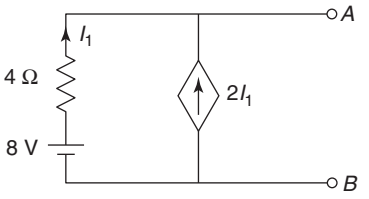


Fig. 3.198

Solution

Step I Calculation of V_{Th} (Fig. 3.199)
From Fig. 3.199,

$$I_1 = -2I_1$$
$$3I_1 = 0$$
$$I_1 = 0$$

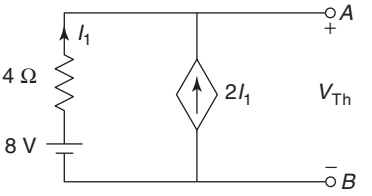


Fig. 3.199

Writing the V_{Th} equation,

$$8 - 0 - V_{Th} = 0$$
$$V_{Th} = 8 \text{ V}$$

Step II Calculation of I_N (Fig. 3.200),
Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 2I_1$$
$$3I_1 - I_2 = 0 \qquad \text{...(i)}$$

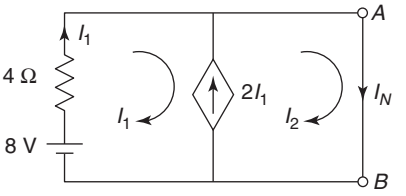


Fig. 3.200

Applying KVL to the outer path of the supermesh,

$$8 - 4I_1 = 0$$
$$I_1 = 2 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = 6 \text{ A}$$
$$I_N = I_2 = 6 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{8}{6} = 1.33\ \Omega$$

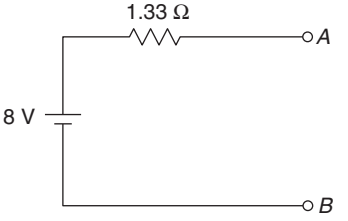


Fig. 3.201

Step IV Thevenin's Equivalent Network (Fig. 3.201)

Example 3.46 *Find Thevenin's equivalent network of Fig. 3.202.*

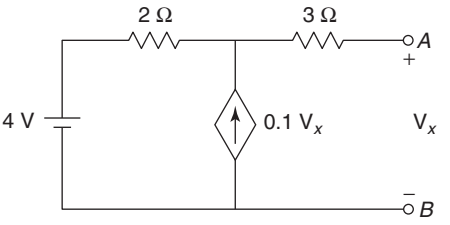


Fig. 3.202

Solution

Step I Calculation of V_{Th} (Fig. 3.203)

$$V_x = V_{Th}$$

$$I_1 = -0.1\,V_x$$

Writing the V_{Th} equation,

$$4 - 2I_1 - V_x = 0$$

$$4 - 2(-0.1V_x) - V_x = 0$$

$$4 - 0.8V_x = 0$$

$$V_x = 5 \text{ V}$$

$$V_x = V_{Th} = 5 \text{ V}$$

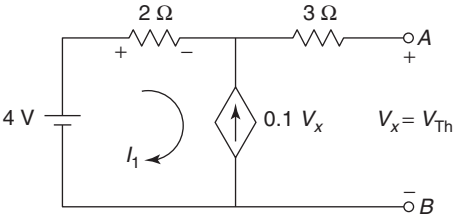


Fig. 3.203

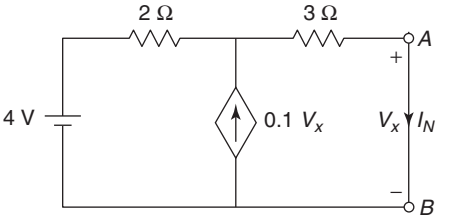


Fig. 3.204

Step II Calculation of I_N (Fig. 3.204)
From Fig. 3.204,

$$V_x = 0$$

The dependent source 0.1 V_x depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e., 0.1 $V_x = 0$ as shown in Fig. 3.205.

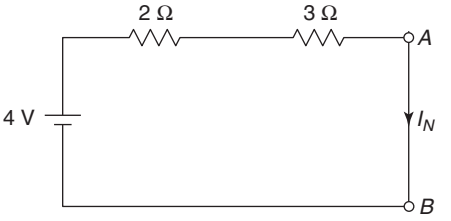


Fig. 3.205

$$I_N = \frac{4}{2+3} = 0.8 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{5}{0.8} = 6.25\ \Omega$$

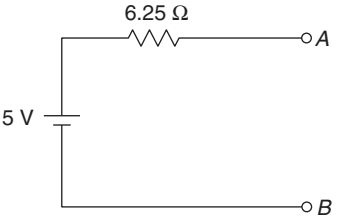


Fig. 3.206

Step IV Thevenin's Equivalent Network (Fig. 3.206)

Example 3.47 *Obtain the Thevenin equivalent network of Fig. 3.207 for the terminals A and B.*

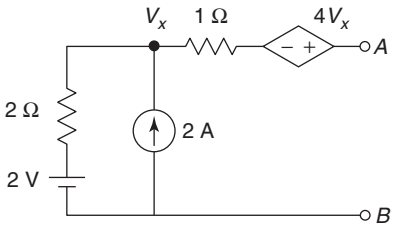


Fig. 3.207

Solution

Step I Calculation of V_{Th} (Fig. 3.208)
From Fig. 3.208,

$$2 - 2I_1 - V_x = 0$$
$$V_x = 2 - 2I_1$$

For Mesh 1,

$$I_1 = -2 \text{ A}$$
$$V_x = 2 - 2(-2) = 6 \text{ V}$$

Writing the V_{Th} equation,

$$2 - 2I_1 - 0 + 4V_x - V_{Th} = 0$$
$$2 - 2(-2) - 0 + 4(6) - V_{Th} = 0$$
$$V_{Th} = 30 \text{ V}$$

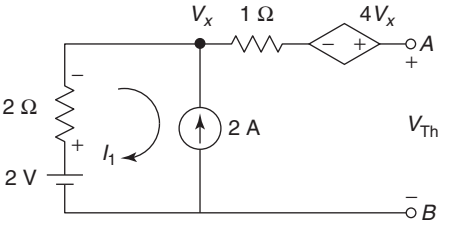


Fig. 3.208

Step II Calculation of I_N (Fig. 3.209)
From Fig. 3.209,

$$V_x = 2 - 2I_1 \quad \ldots(i)$$

Meshes 1 and 2 will form a supermesh,
Writing current equation for the supermesh

$$I_2 - I_1 = 2 \quad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$2 - 2I_1 - 1I_2 + 4V_x = 0$$
$$2 - 2I_1 - I_2 + 4(2 - 2I_1) = 0$$
$$10I_1 + I_2 = 10 \quad \ldots(iii)$$

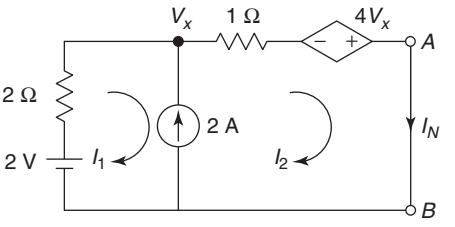


Fig. 3.209

Solving Eqs (ii) and (iii),

$$I_1 = 0.73 \text{ A}$$
$$I_2 = 2.73 \text{ A}$$
$$I_N = I_2 = 2.73 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{2.73} = 10.98 \ \Omega$$

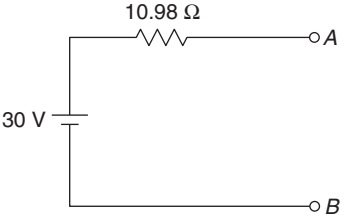


Fig. 3.210

Step IV Thevenin's Equivalent Network (Fig. 3.210)

Example 3.48 *Find the Thevenin equivalent network of Fig. 3.211 for the terminals A and B.*

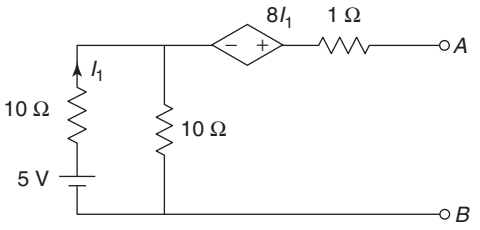


Fig. 3.211

Solution

Step I Calculation of V_{Th} (Fig. 3.212)
Applying KVL to the mesh,

$$5 - 10I_1 - 10I_1 = 0$$

$$I_1 = \frac{5}{20} = 0.25 \text{ A}$$

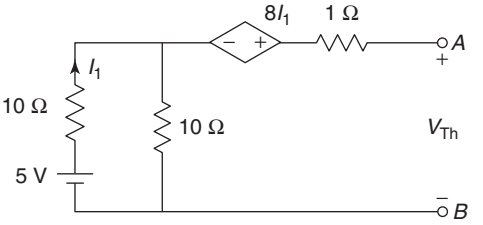


Fig. 3.212

Writing the V_{Th} equation,

$$5 - 10I_1 + 8I_1 - 0 - V_{Th} = 0$$

$$V_{Th} = 5 - 2I_1 = 5 - 2(0.25) = 4.5 \text{ V}$$

Step II Calculation of I_N (Fig. 3.213)
Applying KVL to Mesh 1,

$$5 - 10I_1 - 10(I_1 - I_2) = 0$$

$$20I_1 - 10I_2 = 5 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) + 8I_1 - 1I_2 = 0$$

$$18I_1 - 11I_2 = 0 \quad \text{...(ii)}$$

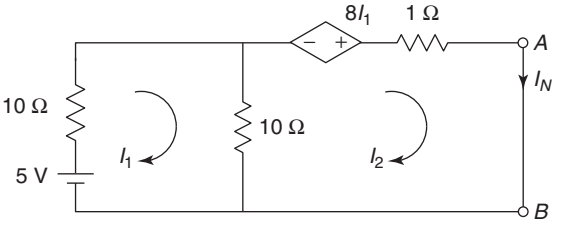


Fig. 3.213

Solving Eqs (i) and (ii),

$$I_1 = 1.375 \text{ A}$$

$$I_2 = 2.25 \text{ A}$$

$$I_N = I_2 = 2.25 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{4.5}{2.25} = 2\ \Omega$$

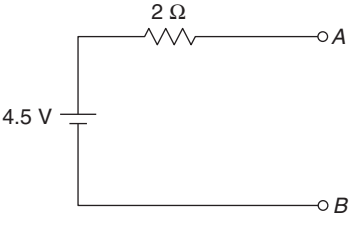


Fig. 3.214

Step IV Thevenin's Equivalent Network (Fig. 3.214)

Example 3.49 *Find V_{Th} and R_{Th} between terminals A and B of the network shown in Fig. 3.215 .*

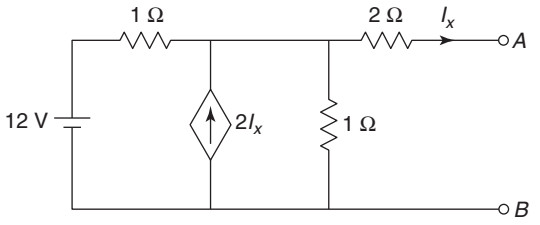


Fig. 3.215

Solution

Step I Calculation of V_{Th} (Fig. 3.216)

$$I_x = 0$$

The dependent source $2I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $2I_x = 0$ as shown in Fig. 3.216.

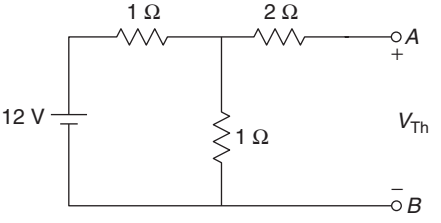


Fig. 3.216

Writing the V_{Th} equation,

$$V_{\text{Th}} = 12 \times \frac{1}{1+1} = 6 \text{ V}$$

Step II Calculation of I_N (Fig. 3.217)
From Fig. 3.217,

$$I_x = \frac{V_1}{2}$$

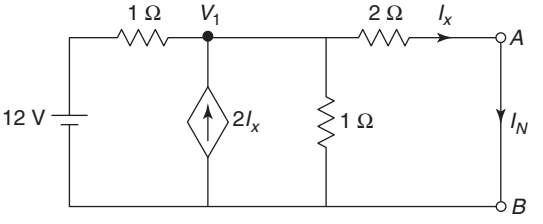


Fig. 3.217

Applying KCL at Node 1,

$$\frac{V_1 - 12}{1} + \frac{V_1}{1} + \frac{V_1}{2} = 2I_x$$

$$V_1 + V_1 + \frac{V_1}{2} - 12 = 2\left(\frac{V_1}{2}\right)$$

$$V_1 = 8 \text{ V}$$

$$I_N = \frac{V_1}{2} = \frac{8}{2} = 4 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{6}{4} = 1.5 \ \Omega$$

Example 3.50 *Obtain the Thevenin equivalent network of Fig. 3.218 for the given network at terminals a and b .*

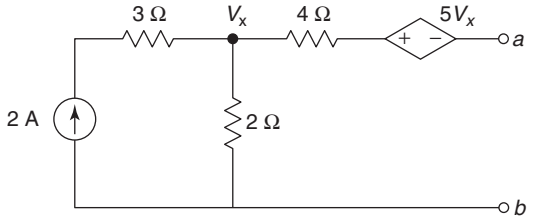


Fig. 3.218

Solution

Step I Calculation of V_{Th} (Fig. 3.219)
Applying KCL at Node x,

$$2 = \frac{V_x}{2}$$

$$V_x = 4 \text{ V}$$

Writing the V_{Th} equation,

$$V_{Th} = V_x - 5V_x = -4V_x$$

$$= -16 \text{ V} \quad (\text{the terminal } a \text{ is negative w.r.t. } b)$$

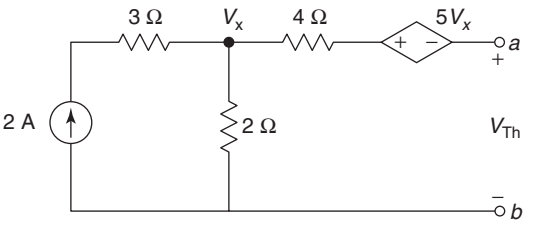


Fig. 3.219

Step II Calculation of I_N (Fig. 3.220)
Applying KCL at Node x,

$$2 = \frac{V_x}{2} + \frac{V_x - 5V_x}{4}$$

$$2 = \frac{V_x}{2} - V_x = -\frac{V_x}{2}$$

$$V_x = -4 \text{ V}$$

$$I_N = \frac{V_x - 5V_x}{4} = -V_x = 4 \text{ A}$$

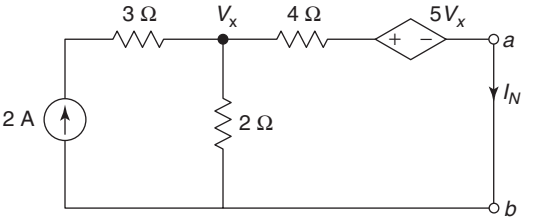


Fig. 3.220

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{-16}{4} = -4 \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.221)

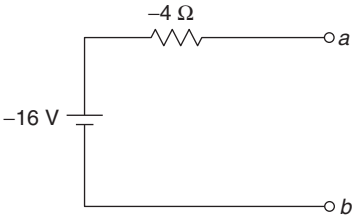


Fig. 3.221

Example 3.51 *Obtain the Thevenin equivalent network of Fig. 3.222 for the given network.*

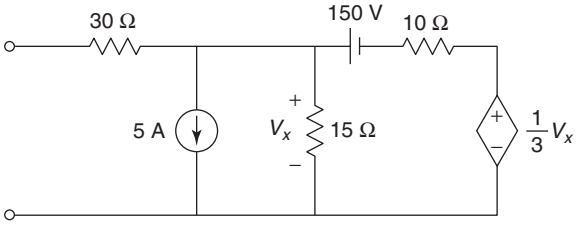


Fig. 3.222

Solution

Step I Calculation of V_{Th} (Fig. 3.223)

From Fig. 3.223

$$V_x = V_{Th}$$

Applying KCL at the node,

$$\frac{V_x - 150 - \frac{1}{3}V_x}{10} + \frac{V_x}{15} + 5 = 0$$

$$V_x = 75 \text{ V}$$

$$V_{Th} = 75 \text{ V}$$

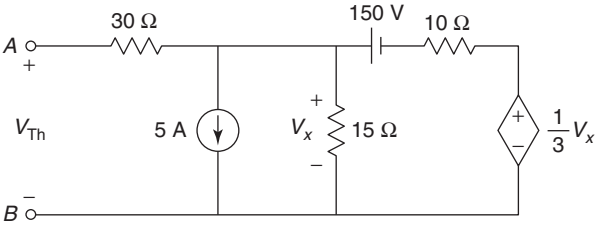


Fig. 3.223

Step II Calculation of I_N (Fig. 3.224)

Applying KCL at Node x,

$$\frac{V_x}{30} + 5 + \frac{V_x}{15} + \frac{V_x - 150 - \frac{1}{3}V_x}{10} = 0$$

$$\frac{V_x}{30} + \frac{V_x}{15} + \frac{V_x}{10} - \frac{V_x}{30} = 15 - 5$$

$$V_x = 60 \text{ V}$$

$$I_N = \frac{V_x}{30} = \frac{60}{30} = 2 \text{ A}$$

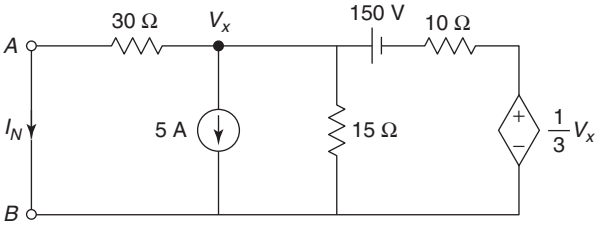


Fig. 3.224

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{75}{2} = 37.5 \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.225)

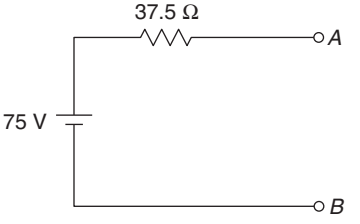


Fig. 3.225

Example 3.52 *Find the Thevenin's equivalent network of the network to the left of A-B in the Fig. 3.226.*

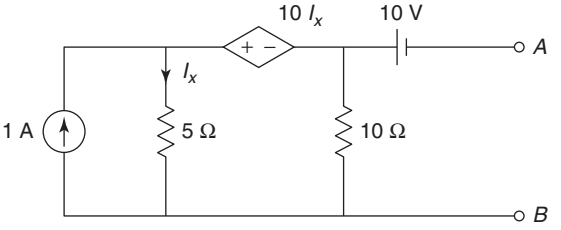


Fig. 3.226

Solution

Step I Calculation of V_{Th} (Fig. 3.227)

From Fig. 3.227,

$$I_x = I_1 - I_2 \quad \ldots(i)$$

For Mesh 1,

$$I_1 = 1 \quad \ldots(ii)$$

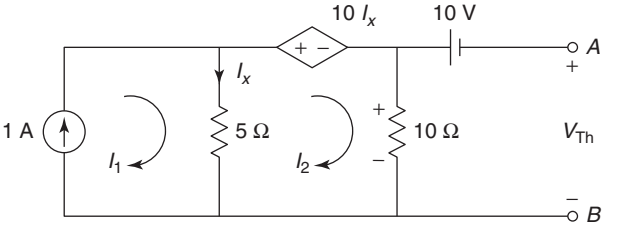


Fig. 3.227

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10I_2 = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10I_2 = 0$$

$$5I_1 + 5I_2 = 0 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 1 \text{ A}$$

$$I_2 = -1 \text{ A}$$

$$I_x = I_1 - I_2 = 1 - (-1) = 2 \text{ A}$$

Writing the V_{Th} equation,

$$10I_2 - 10 - V_{Th} = 0$$

$$10(-1) - 10 - V_{Th} = 0$$

$$V_{Th} = -20 \text{ V}$$

Step II Calculation of I_N (Fig. 3.228)

From Fig. 3.228,

$$I_x = I_1 - I_2 \quad \ldots(i)$$

For Mesh 1,

$$I_1 = 1 \quad \ldots(ii)$$

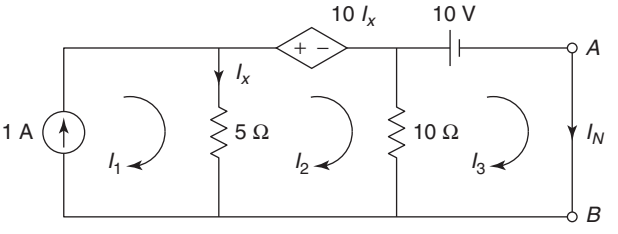


Fig. 3.228

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10(I_2 - I_3) = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10(I_2 - I_3) = 0$$

$$-5I_1 - 5I_2 + 10I_3 = 0 \quad \ldots(iii)$$

Applying KVL to Mesh 3,

$$-10(I_3 - I_2) - 10 = 0$$

$$10I_2 - 10I_3 = 10 \quad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 1 \text{ A}$$

$$I_2 = 3 \text{ A}$$

$$I_3 = 2 \text{ A}$$

$$I_N = I_3 = 2 \text{ A}$$

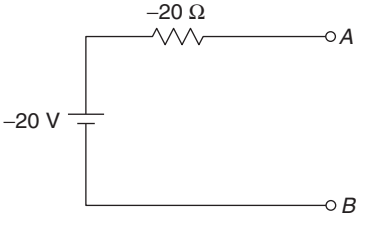


Fig. 3.229

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{-20}{2} = -10 \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.229)

Example 3.53 *Find Thevenin's equivalent network at terminals A and B in the network of Fig. 3.230.*

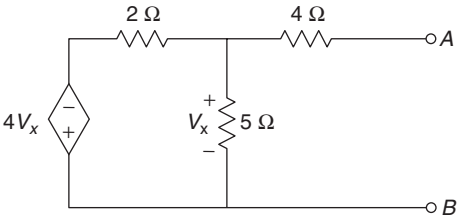


Fig. 3.230

Solution

Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$

$$I_N = 0$$

But the R_{Th} can be calculated by applying a known voltage source of 1 V at the terminals A and B as shown in Fig. 3.231.

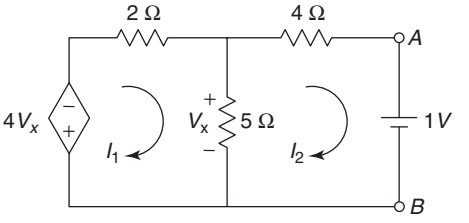


Fig. 3.231

$$R_{\text{Th}} = \frac{V}{I} = \frac{1}{I}$$

From Fig. 3.231,

$$V_x = 5(I_1 - I_2) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4V_x - 2I_1 - 5(I_1 - I_2) = 0$$

$$-4\left[5(I_1 - I_2)\right] - 2I_1 - 5I_1 + 5I_2 = 0$$

$$-27I_1 + 25I_2 = 0 \qquad \text{…(ii)}$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 4I_2 - 1 = 0$$

$$5I_1 - 9I_2 = 1 \qquad \text{…(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -0.21 \text{ A}$$

$$I_2 = -0.23 \text{ A}$$

Hence, current supplied by voltage source of 1 V is 0.23 A.

$$R_{\text{Th}} = \frac{1}{0.23} = 4.35\ \Omega$$

Hence, Thevenin's equivalent network is shown in Fig. 3.232.

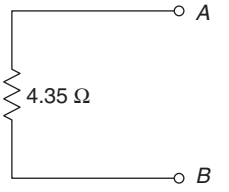


Fig. 3.232

Example 3.54 *Find the current in the 9 Ω resistor in Fig. 3.233.*

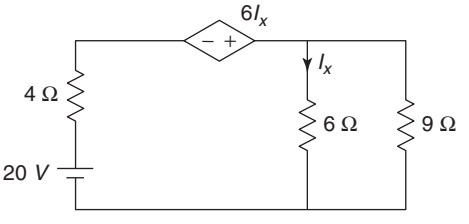


Fig. 3.233

Solution

Step I Calculation of V_{Th} (Fig. 3.234)

Applying KVL to the mesh,

$$20 - 4I_x + 6I_x - 6I_x = 0$$

$$I_x = 5 \text{ A}$$

Writing the V_{Th} equation,

$$6I_x - V_{Th} = 0$$

$$6(5) - V_{Th} = 0$$

$$V_{Th} = 30 \text{ V}$$

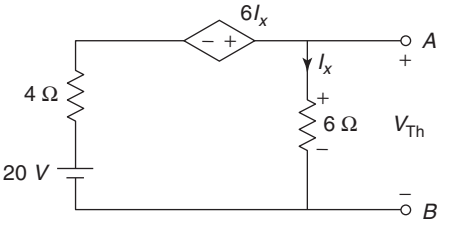


Fig. 3.234

Step II Calculation of I_N (Fig. 3.235).

From Fig. 3.235,

$$I_x = 0$$

The dependent source $6I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $6I_x = 0$ as shown in Fig. 3.236.

$$I_N = \frac{20}{4} = 5 \text{ A}$$

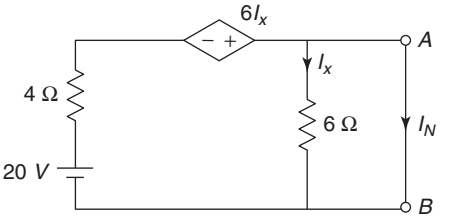


Fig. 3.235

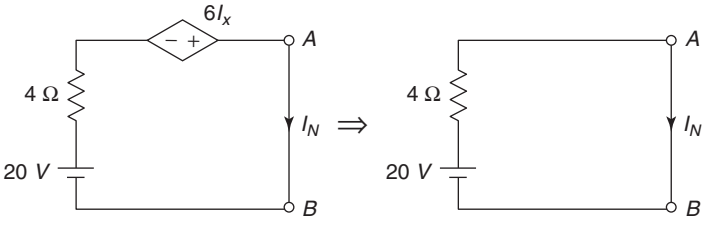


Fig. 3.236

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6 \ \Omega$$

Step IV Calculation of I_L (Fig. 3.237)

$$I_L = \frac{30}{6+9} = 2 \text{ A}$$

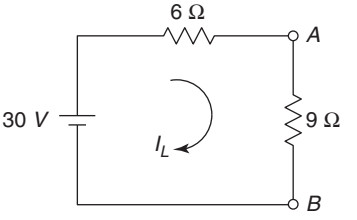


Fig. 3.237

Example 3.55 *Determine the current in the 16 Ω resistor in Fig. 3.238.*

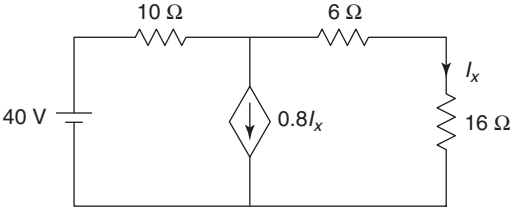


Fig 3.238

Solution

Step I Calculation of V_{Th} (Fig. 3.239)
From Fig. 3.239,

$$I_x = 0$$

The dependent source $0.8I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, as shown in Fig. 3.240.

i.e., $$0.8I_x = 0$$

$$V_{Th} = 40 \text{ V}$$

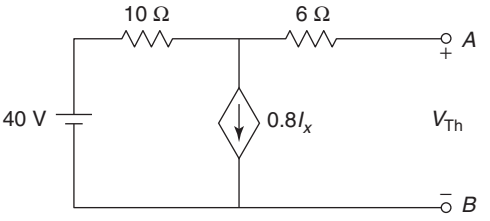


Fig. 3.239

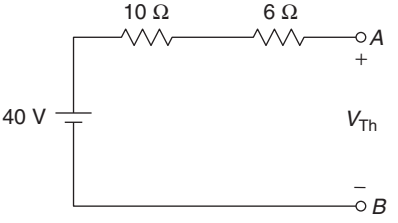


Fig. 3.240

Step II Calculation of I_N (Fig. 3.241)
From Fig. 3.241,

$$I_x = I_2 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh,
Writing current equation for the supermesh,

$$I_1 - I_2 = 0.8\, I_x = 0.8\, I_2$$

$$I_1 - 1.8\, I_2 = 0 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$40 - 10\, I_1 - 6\, I_2 = 0$$

$$10\, I_1 + 6\, I_2 = 40 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 3 \text{ A}$$

$$I_2 = \frac{5}{3} \text{ A}$$

$$I_N = I_2 = \frac{5}{3} \text{ A}$$

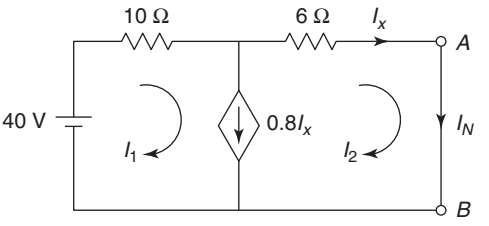


Fig. 3.241

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{40}{\frac{5}{3}} = 24\ \Omega$$

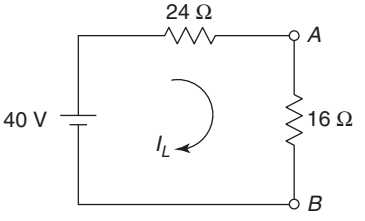


Fig. 3.242

Step IV Calculation of I_L (Fig. 3.242)

$$I_L = \frac{40}{24+16} = 1 \text{ A}$$

Example 3.56 *Find the current in the 6 Ω resistor in Fig. 3.243.*

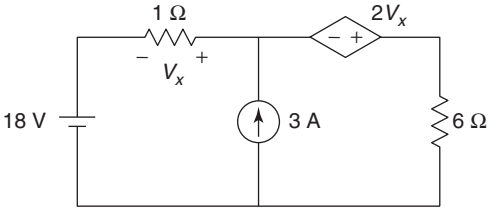


Fig. 3.243

Solution

Step I Calculation of V_{Th} (Fig. 3.244)

From Fig. 3.244,

$$V_x = -1I_1 = -I_1 \qquad \ldots(i)$$

For Mesh 1,

$$I_1 = -3 \text{ A} \qquad \ldots(ii)$$

$$V_x = 3 \text{ V}$$

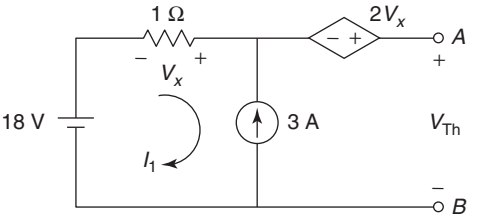


Fig. 3.244

Writing the V_{Th} equation,

$$18 - 1\,I_1 + 2\,V_x - V_{Th} = 0$$

$$18 + 3 + 2(3) - V_{Th} = 0$$

$$V_{Th} = 27 \text{ V}$$

Step II Calculation of I_N (Fig. 3.245)

From Fig. 3.245,

$$V_x = -I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for supermesh,

$$I_2 - I_1 = 3 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$18 - 1I_1 + 2\,V_x = 0$$

$$18 - I_1 + 2(-I_1) = 0 \qquad \ldots(iii)$$

$$I_1 = 6 \text{ A}$$

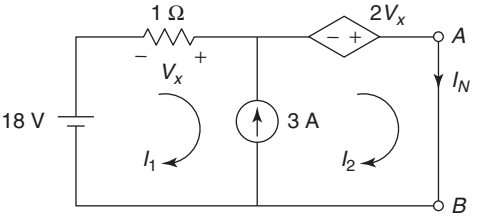


Fig. 3.245

Solving Eqs (ii) and (iii),

$$I_2 = 9 \text{ A}$$

$$I_N = I_2 = 9 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{27}{9} = 3\ \Omega$$

Step IV Calculation of I_L (Fig. 3.246)

$$I_L = \frac{27}{3+6} = 3 \text{ A}$$

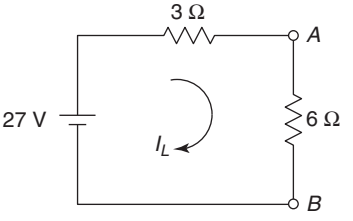


Fig. 3.246

Example 3.57

Find the current in the 10 Ω resistor.

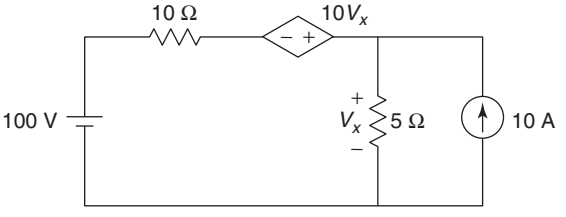


Fig. 3.247

Solution

Step I Calculation of V_{Th} (Fig. 3.248)

From the figure,

$$V_x = 10 \times 5 = 50 \text{ V}$$

Writing the V_{Th} equation,

$$100 - V_{Th} + 10V_x - V_x = 0$$
$$100 - V_{Th} + 9V_x = 0$$
$$100 - V_{Th} + 9(50) = 0$$
$$V_{Th} = 550 \text{ V}$$

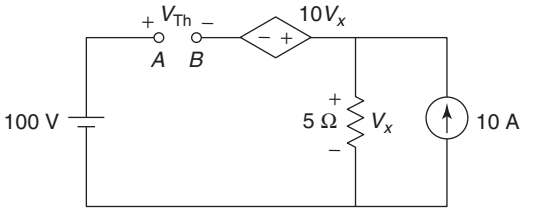


Fig. 3.248

Step II Calculation of I_N (Fig. 3.249)

From Fig. 3.249,

$$V_x = 5(I_N + 10)$$

Applying KVL to Mesh 1,

$$100 + 10V_x - V_x = 0$$
$$V_x = -\frac{100}{9}$$
$$-\frac{100}{9} = 5I_N + 50$$
$$I_N = -\frac{550}{45} \text{ A}$$

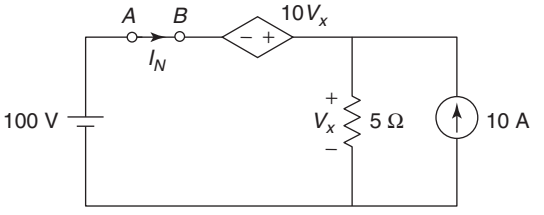


Fig. 3.249

Step III Calculation of R_{Th}

$$R_{Th} = \frac{550}{-\frac{550}{45}} = -45 \ \Omega$$

Step IV Calculation of I_L (Fig. 3.250)

$$I_L = \frac{550}{-45+10} = -\frac{110}{7} \text{ A}$$

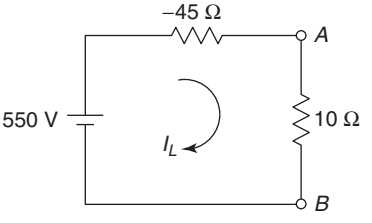


Fig. 3.250

3.4 NORTON'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent current source and an equivalent parallel resistance.*' The constant current is equal to the current which would flow in a short circuit placed across the terminals. The parallel resistance is the resistance of the network when viewed from these open-circuited terminals after all voltage and current sources have been removed and replaced by internal resistances.

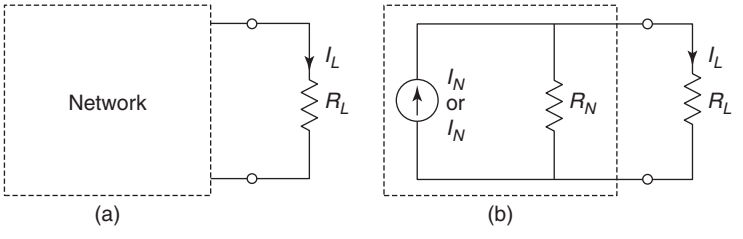


Fig. 3.251 *Network illustrating Norton's theorem*

Explanation Consider a simple network as shown in Fig.3.252

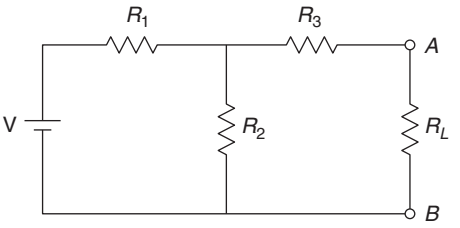


Fig. 3.252 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate short circuit current I_{SC} or I_N which would flow in a short circuit placed across terminals A and B as shown in Fig. 3.253.

For finding parallel resistance R_N, replace the voltage source by a short circuit and calculate resistance between points *A* and *B* as shown in Fig. 3.254.

$$R_N = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

Norton's equivalent network is shown in Fig.3.255.

$$I_L = I_N \frac{R_N}{R_N + R_L}$$

If the network contains both independent and dependent sources, Norton's resistances R_N is calculated as

$$R_N = \frac{V_{\text{Th}}}{I_N}$$

where V_{Th} is the open-circuit voltage across terminals *A* and *B*. If the network contains only dependent sources, then

$$V_{\text{Th}} = 0$$

$$I_N = 0$$

To find R_{Th} in such network, a known voltage *V* or current *I* is applied across the terminals *A* and *B*, and the current *I* or the voltage *V* is calculated respectively.

$$R_N = \frac{V}{I}$$

Norton's equivalent network for such a network is shown in Fig. 3.256.

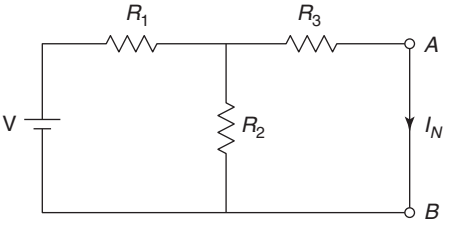


Fig. 3.253 *Calculation of I_N*

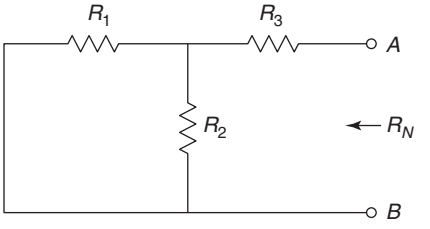


Fig. 3.254 *Calculation of R_N*

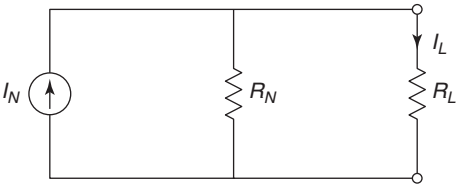


Fig. 3.255 *Norton's equivalent network*

Fig. 3.256 *Norton's equivalent network*

Steps to be followed in Norton's Theorem

1. Remove the load resistance R_L and put a short circuit across the terminals.
2. Find the short-circuit current I_{SC} or I_N.
3. Find the resistance R_N as seen from points A and B.
4. Replace the network by a current source I_N in parallel with resistance R_N.
5. Find current through R_L by current–division rule.

$$I_L = \frac{I_N R_N}{R_N + R_L}$$

Example 3.58 *Find the current through the 10 Ω resistor in Fig. 3.257.*

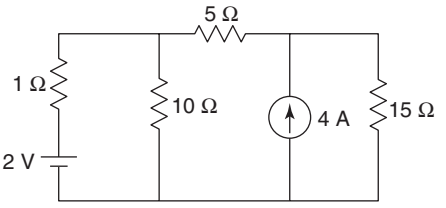


Fig. 3.257

Solution

Step I Calculation of I_N (Fig. 3.258)

Applying KVL to Mesh 1,

$$2 - 1I_1 = 0$$

$$I_1 = 2 \qquad ...(i)$$

Meshes 2 and 3 will form a supermesh.

Writing the current equation for the supermesh,

$$I_3 - I_2 = 4 \qquad ...(ii)$$

Applying KVL to the supermesh,

$$-5I_2 - 15I_3 = 0 \qquad ...(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 2\,\text{A}$$

$$I_2 = -3\,\text{A}$$

$$I_3 = 1\,\text{A}$$

$$I_N = I_1 - I_2 = 2 - (-3) = 5\,\text{A}$$

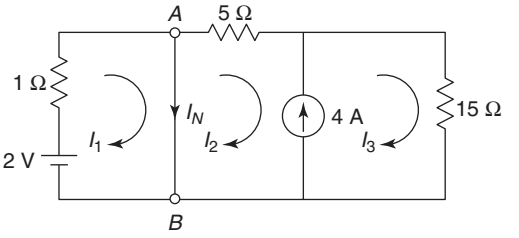


Fig. 3.258

Step II Calculation of R_N (Fig. 3.259)

$$R_N = 1 \,||\, (5+15) = 0.95\ \Omega$$

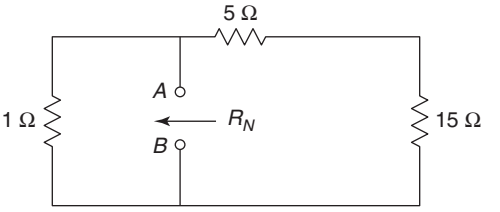


Fig. 3.259

Step III Calculation of I_L (Fig.3.260)

$$I_L = 5 \times \frac{0.95}{0.95 + 10} = 0.43\,\text{A}$$

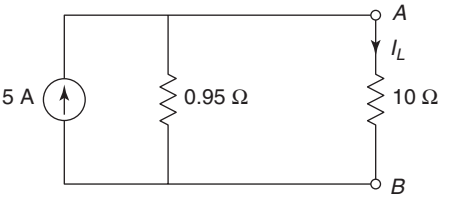


Fig. 3.260

Example 3.59 *Find the current through the 10 Ω resistor in Fig. 3.258.*

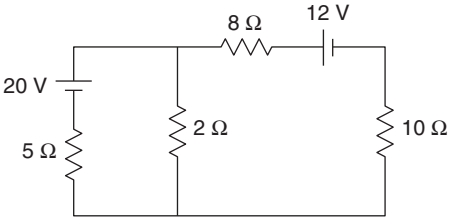


Fig. 3.261

Solution

Step I Calculation of I_N (Fig. 3.262)

Applying KVL to Mesh 1,

$$-5I_1 + 20 - 2(I_1 - I_2) = 0$$
$$7I_1 - 2I_2 = 20 \quad \text{...(i)}$$

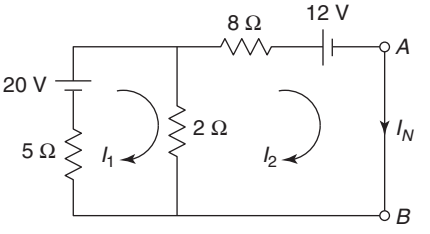


Fig. 3.262

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 8I_2 - 12 = 0$$
$$-2I_1 + 10I_2 = -12 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = -0.67\ \text{A}$$
$$I_N = I_2 = -0.67\ \text{A}$$

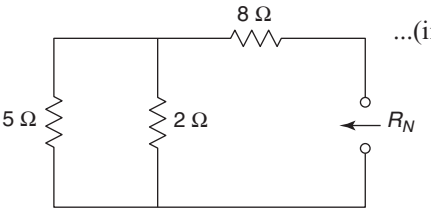


Fig. 3.263

Step II Calculation of R_N (Fig. 3.263)

$$R_N = (5 \parallel 2) + 8 = 9.43\ \Omega$$

Step III Calculation of I_L (Fig. 3.264)

$$I_L = 0.67 \times \frac{9.43}{9.43 + 10} = 0.33\ \text{A}\ (\uparrow)$$

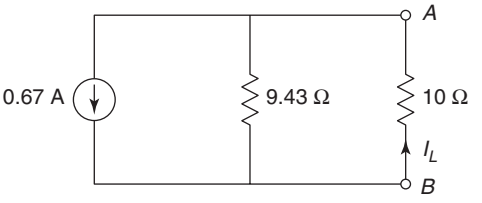


Fig. 3.264

Example 3.60 *Find the current in the 10 Ω resistor in Fig. 3.265.*

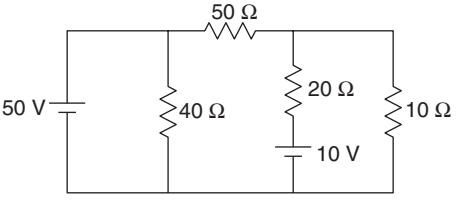


Fig. 3.265

Solution

Step I Calculation of I_N (Fig. 3.266)

The resistance of $40\ \Omega$ becomes redundant as it is connected across the 50 V source (Fig. 3.267).

Applying KVL to Mesh 1,

$$50 - 50I_1 - 20(I_1 - I_2) - 10 = 0$$
$$70I_1 - 20I_2 = 40 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$10 - 20(I_2 - I_1) = 0$$
$$-20I_1 + 20I_2 = 0 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 1\,\text{A}$$
$$I_2 = 1.5\,\text{A}$$
$$I_N = I_2 = 1.5\,\text{A}$$

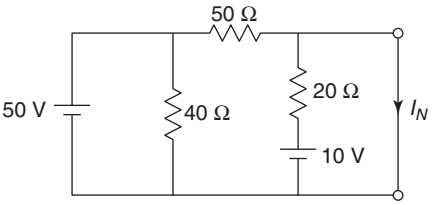


Fig. 3.266

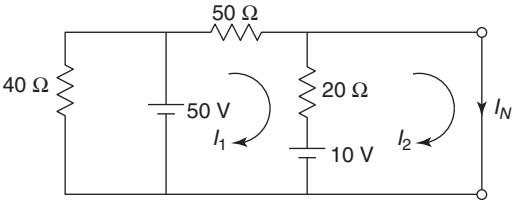


Fig. 3.267

Step II Calculation of R_N (Fig. 3.268)

$$R_N = 50 \,||\, 20 = 14.28\ \Omega$$

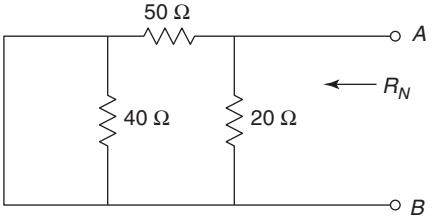


Fig. 3.268

Step III Calculation of I_L (Fig. 3.269)

$$I_L = 1.5 \times \frac{14.28}{14.28 + 10} = 0.88\,\text{A}$$

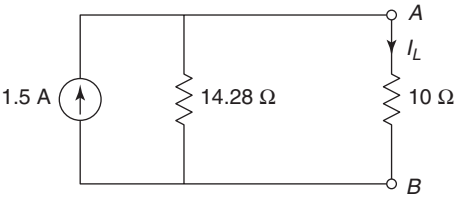


Fig. 3.269

Example 3.61 *Find the current through the 10 Ω resistor in Fig. 3.270.*

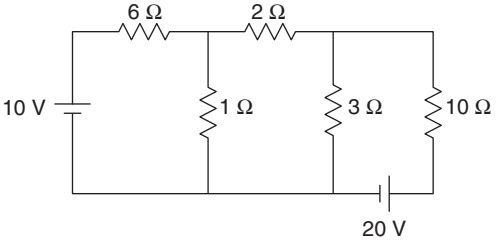


Fig. 3.270

Solution

Step I Calculation of I_N (Fig. 3.271)

Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$-I_1 + 6I_2 - 3I_3 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-3(I_3 - I_2) - 20 = 0$$
$$3I_2 - 3I_3 = 20 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_3 = -13.17\,\text{A}$$
$$I_N = I_3 = -13.17\,\text{A}$$

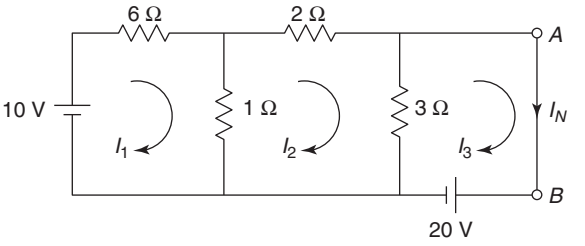


Fig. 3.271

Step II Calculation of R_N (Fig. 3.272)

$$R_N = [(6 \,\|\, 1) + 2] \,\|\, 3 = 1.46\,\Omega$$

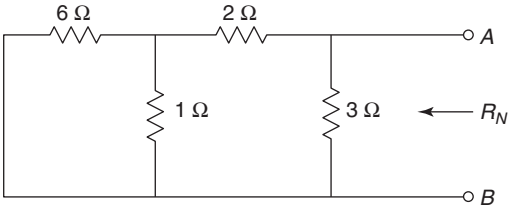


Fig. 3.272

Step III Calculation of I_L (Fig. 3.273)

$$I_L = 13.17 \times \frac{1.46}{1.46 + 10} = 1.68\,\text{A}\,(\uparrow)$$

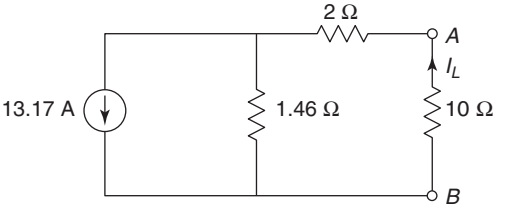


Fig. 3.273

Example 3.62 *Find the current through the 10 Ω resistor in Fig. 3.274.*

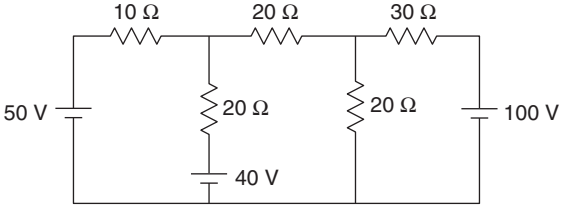


Fig. 3.274

Solution

Step I Calculation of I_N (Fig. 3.274)

Applying KVL to Mesh 1,

$$50 - 20(I_1 - I_2) - 40 = 0$$
$$20I_1 - 20I_2 = 10 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$40 - 20(I_2 - I_1) - 20I_2 - 20(I_2 - I_3) = 0$$
$$-20I_1 + 60I_2 - 20I_3 = 40 \quad \text{...(ii)}$$

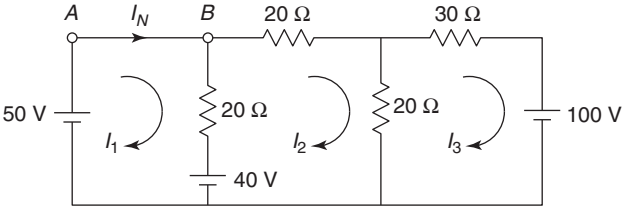


Fig. 3.275

Applying KVL to Mesh 3,

$$-20(I_3 - I_2) - 30I_3 - 100 = 0$$

$$-20I_2 + 50I_3 = -100 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.81\,\text{A}$$

$$I_N = I_1 = 0.81\,\text{A}$$

Step II Calculation of R_N (Fig. 3.276)

$$R_N = [(20 \,||\, 30) + 20] \,||\, 20 = 12.3\,\Omega$$

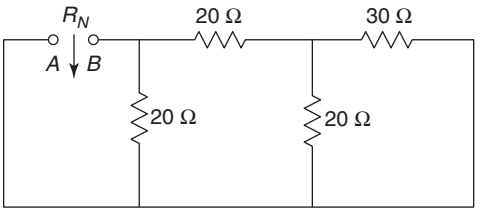


Fig. 3.276

Step III Calculation of I_L (Fig. 3.277)

$$I_L = 0.81 \times \frac{12.3}{12.3 + 10} = 0.45\,\text{A}$$

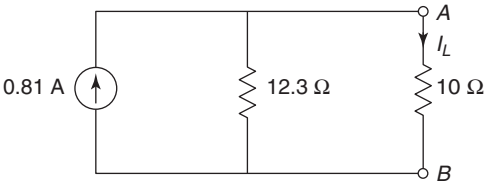


Fig. 3.277

Example 3.63 *Obtain Norton's equivalent network as seen by R_L in Fig. 3.278.*

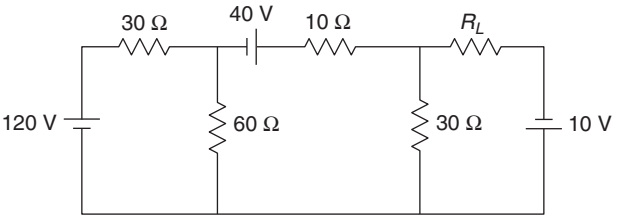


Fig. 3.278

Solution

Step I Calculation of I_N (Fig. 3.279)

Applying KVL to Mesh 1,

$$120 - 30I_1 - 60(I_1 - I_2) = 0$$

$$90I_1 - 60I_2 = 120 \quad \text{...(i)}$$

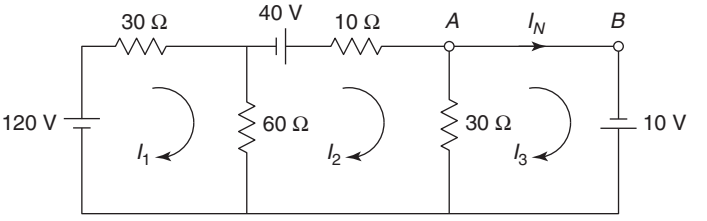


Fig. 3.279

Applying KVL to Mesh 2,

$$-60(I_2 - I_1) + 40 - 10I_2 - 30(I_2 - I_3) = 0$$

$$-60I_1 + 100I_2 - 30I_3 = 40 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-30(I_3 - I_2) + 10 = 0$$

$$30I_2 - 30I_3 = -10 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_3 = 4.67\,\text{A}$$

$$I_N = I_3 = 4.67\,\text{A}$$

Step II Calculation of R_N (Fig. 3.280)

$$R_N = [(30 \,||\, 60) + 10] \,||\, 30 = 15\,\Omega$$

Step III Norton's equivalent network (Fig. 3.281)

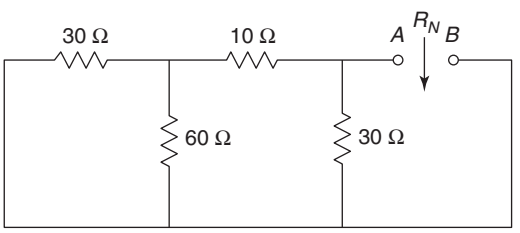


Fig. 3.280

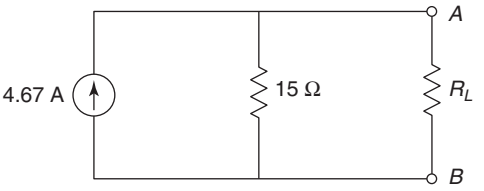


Fig. 3.281

Example 3.64 *Find the current through the 8 Ω resistor in Fig. 3.282.*

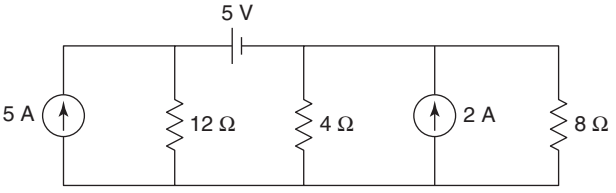


Fig. 3.282

Solution

Step I Calculation of I_N (Fig. 3.283)

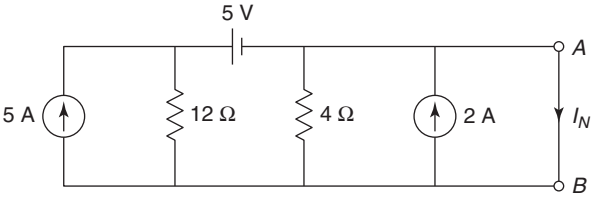


Fig. 3.283

The resistor of the 4 Ω gets shorted as it is in parallel with the short circuit. Simplifying the network by source transformation (Fig. 3.284),

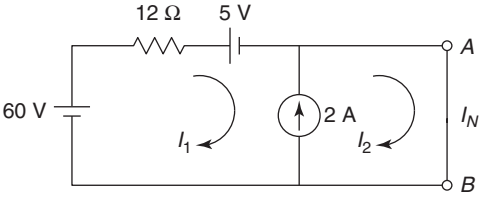


Fig. 3.284

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \text{...(i)}$$

Applying KVL to the supermesh,

$$60 - 12I_1 - 5 = 0$$
$$12I_1 = 55 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 4.58\,\text{A}$$
$$I_2 = 6.58\,\text{A}$$
$$I_N = I_2 = 6.58\,\text{A}$$

Step II Calculation of R_N (Fig. 3.285)

$$R_N = 12 \,||\, 4 = 3\,\Omega$$

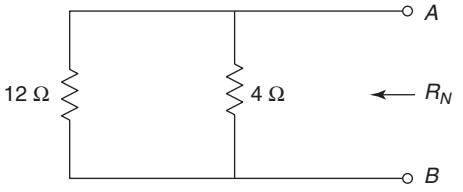


Fig. 3.285

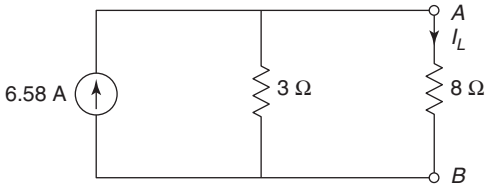


Fig. 3.286

Step III Calculation of I_L (Fig. 3.286)

$$I_L = 6.58 \times \frac{3}{3+8} = 1.79\,\text{A}$$

Example 3.65 *Find the current through the 1 Ω resistor in Fig. 3.287.*

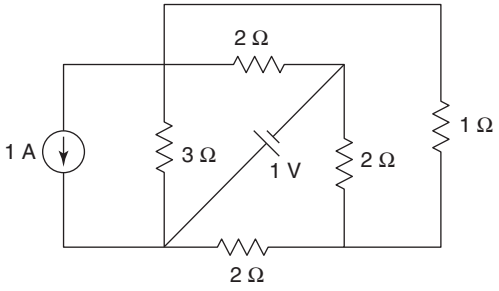


Fig. 3.287

Solution

Step I Calculation of I_N (Fig. 3.288)

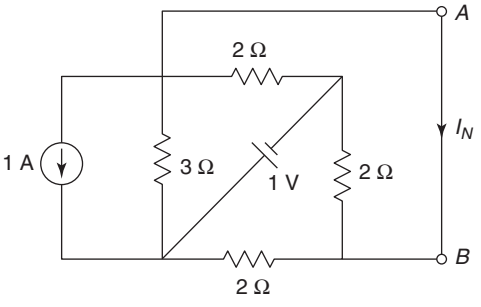


Fig. 3.288

By source transformation (Fig. 3.289),

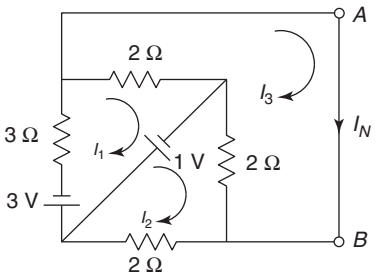


Fig. 3.289

Applying KVL to Mesh 1,

$$-3-3I_1-2(I_1-I_3)+1=0$$

$$5I_1-2I_3=-2 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-1-2(I_2-I_3)-2I_2=0$$

$$4I_2-2I_3=-1 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3-I_1)-2(I_3-I_2)=0$$

$$-2I_1-2I_2+4I_3=0 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=-0.64\,\text{A}$$

$$I_2=-0.55\,\text{A}$$

$$I_3=-0.59\,\text{A}$$

$$I_N=I_3=-0.59\,\text{A}$$

Step II Calculation of R_N (Fig. 3.290)

$$R_N=2.2\,\Omega$$

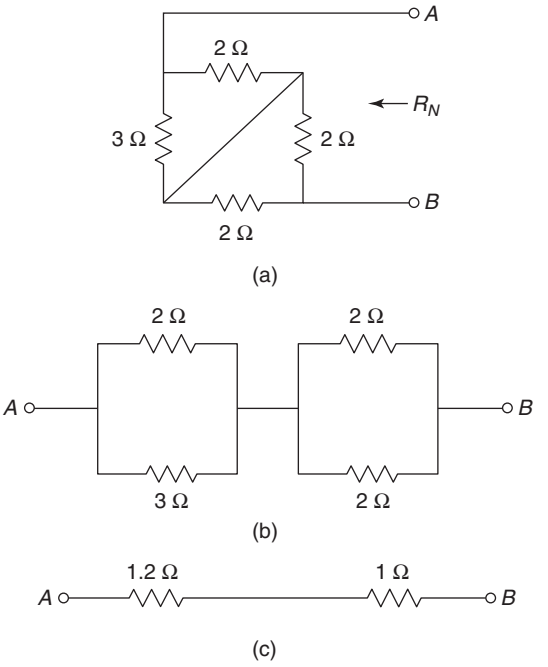


Fig. 3.290

Step III Calculation of I_L (Fig. 3.291)

$$I_L=0.59\times\frac{2.2}{2.2+1}=0.41\,\text{A}$$

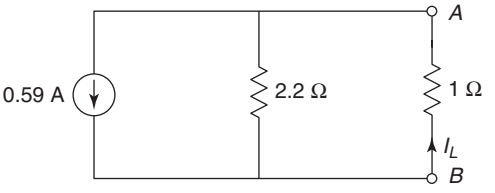


Fig. 3.291

EXAMPLES WITH DEPENDENT SOURCES

Example 3.66 *Find Norton's equivalent network across terminals A and B of Fig. 3.292.*

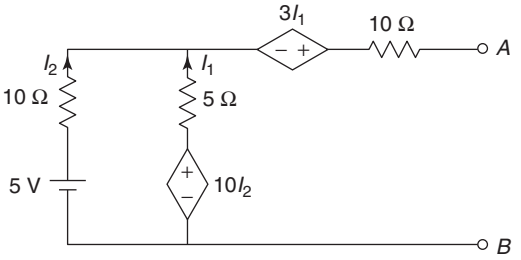


Fig. 3.292

Solution

Step I Calculation of V_{Th} (Fig. 3.293)

From Fig. 3.293,

$$I_2 = I_x$$
$$I_1 = -I_x$$

Applying KVL to the mesh,

$$5 - 10I_x - 5I_x - 10I_2 = 0$$
$$5 - 10I_x - 5I_x - 10I_x = 0$$
$$I_x = 0.2 \text{ A}$$
$$I_1 = -0.2 \text{ A}$$

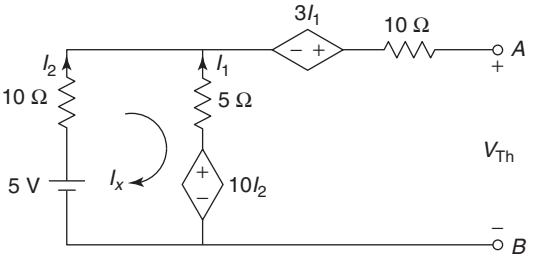


Fig. 3.293

Writing the V_{Th} equation,

$$5 - 10I_x + 3I_1 - V_{Th} = 0$$
$$5 - 10(0.2) + 3(-0.2) - V_{Th} = 0$$
$$V_{Th} = 2.4 \text{ V}$$

Step II Calculation of I_N (Fig. 3.294)

From Fig. 3.294,

$$I_2 = I_x \quad \ldots(i)$$
$$I_1 = I_y - I_x \quad \ldots(ii)$$

Applying KVL to Mesh 1,

$$5 - 10I_x - 5(I_x - I_y) - 10I_2 = 0$$
$$5 - 10I_x - 5I_x + 5I_y - 10I_x = 0$$
$$25I_x - 5I_y = 5 \quad \ldots(iii)$$

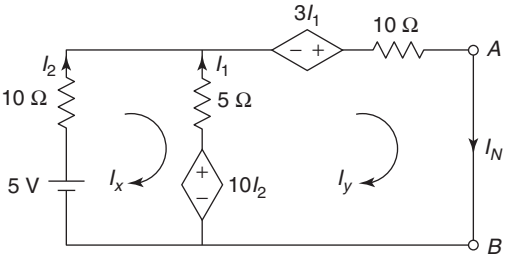


Fig. 3.294

Applying KVL to Mesh 2,

$$10I_2 - 5(I_y - I_x) + 3I_1 - 10I_y = 0$$
$$10I_x - 5I_y + 5I_x + 3(I_y - I_x) - 10I_y = 0$$
$$12I_x - 12I_y = 0 \quad \ldots(iv)$$

Solving Eqs (iii) and (iv),

$$I_x = 0.25 \text{ A}$$
$$I_y = 0.25 \text{ A}$$
$$I_N = I_y = 0.25 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{2.4}{0.25} = 9.6\ \Omega$$

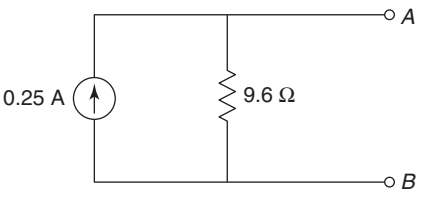


Fig. 3.295

Step IV Norton's Equivalent Network (Fig. 3.295)

Example 3.67 *For the network shown in Fig. 3.296, find Norton's equivalent network.*

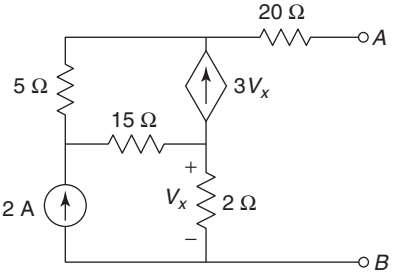


Fig. 3.296

Solution

Step I Calculation of V_{Th} (Fig. 3.297)

From Fig. 3.297,

$$V_x = 2I_2 \qquad \ldots(\text{i})$$

For Mesh 1,

$$I_1 = -3V_x = -3(2I_2) = -6I_2 \qquad \ldots(\text{ii})$$

For Mesh 2,

$$I_2 = 2 \qquad \ldots(\text{iii})$$

$$I_1 = -6I_2 = -6(2) = -12 \text{ A}$$

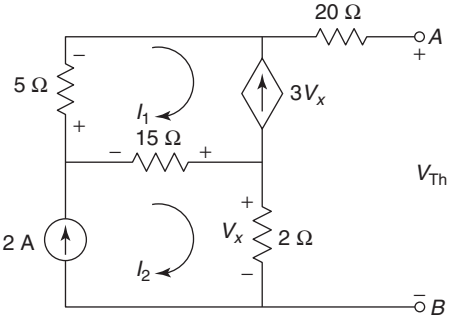


Fig. 3.297

Writing the V_{Th} equation,

$$V_{\text{Th}} - 0 + 5I_1 + 15(I_1 - I_2) - 2I_2 = 0$$
$$V_{\text{Th}} + 5(-12) + 15(-12 - 2) - 2(2) = 0$$
$$V_{\text{Th}} = 274 \text{ V}$$

Step II Calculation of I_N (Fig. 3.298)

From Fig. 3.298,

$$V_x = 2(I_2 - I_3) \qquad \ldots(\text{i})$$

For Mesh 2,

$$I_2 = 2 \qquad \ldots(\text{ii})$$

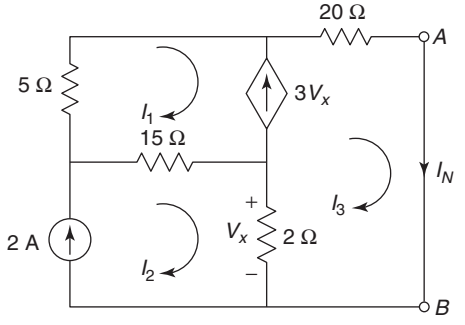


Fig. 3.298

Meshes 1 and 3 will form a supermesh.
Writing the current equation for the supermesh,

$$I_3 - I_1 = 3V_x = 3\left[2(I_2 - I_3)\right] = 6I_2 - 6I_3$$

$$I_1 + 6I_2 - 7I_3 = 0 \quad \ldots(\text{iii})$$

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 20I_3 - 2(I_3 - I_2) - 15(I_1 - I_2) = 0$$

$$-20I_1 + 17I_2 - 22I_3 = 0 \quad \ldots(\text{iv})$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.16 \text{ A}$$
$$I_2 = 2 \text{ A}$$
$$I_3 = 1.69 \text{ A}$$
$$I_N = I_3 = 1.69 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{274}{1.69} = 162.13\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.299)

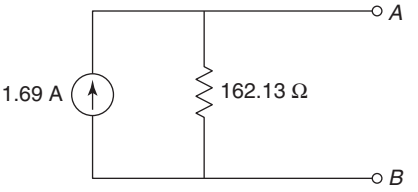


Fig. 3.299

Example 3.68

Obtain Norton's equivalent network across A-B in the network of Fig. 3.300.

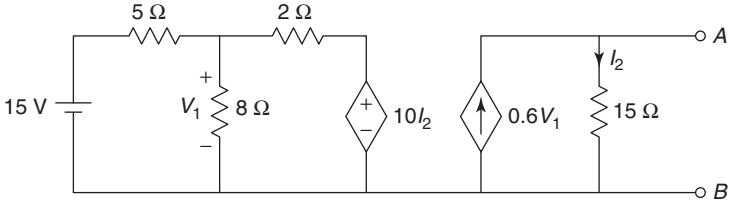


Fig. 3.300

Solution

Step I Calculation of V_{Th} (Fig. 3.301)
From Fig. 3.301,

$$V_1 = 8(I_x - I_y) \quad \ldots(\text{i})$$

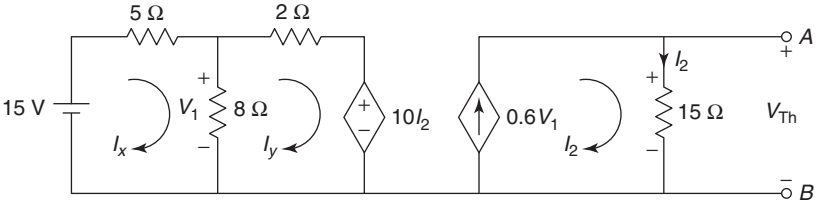


Fig. 3.301

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$

$$13I_x - 8I_y = 15 \quad \ldots(\text{ii})$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y - 10I_2 = 0$$

$$8I_x - 10I_y - 10I_2 = 0 \quad \ldots(\text{iii})$$

For Mesh 3,

$$I_2 = 0.6V_1 = 0.6\left[8(I_x - I_y)\right]$$

$$4.8I_x - 4.8I_y - I_2 = 0 \quad \ldots(\text{iv})$$

Solving Eqs (ii), (iii) and (iv),

$$I_x = 3.28 \text{ A}$$
$$I_y = 3.45 \text{ A}$$
$$I_2 = -0.83 \text{ A}$$

Writing the V_{Th} equation,

$$15I_2 - V_{\text{Th}} = 0$$
$$15(-0.83) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -12.45 \text{ V}$$

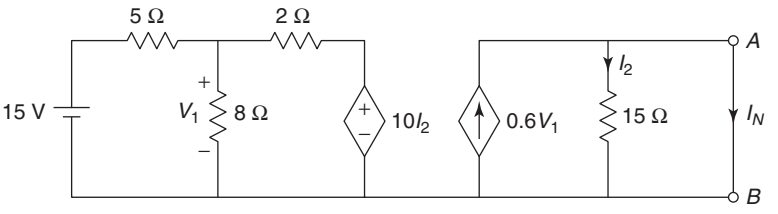


Fig. 3.302

Step II Calculation of I_N (Fig. 3.302)
From Fig. 3.302,

$$I_2 = 0$$

The dependent source of $10\,I_2$ depends on the controlling variable I_2. When $I_2 = 0$, the dependent source vanishes, i.e. $10I_2 = 0$ as shown in Fig. 3.303.
From Fig. 3.303,

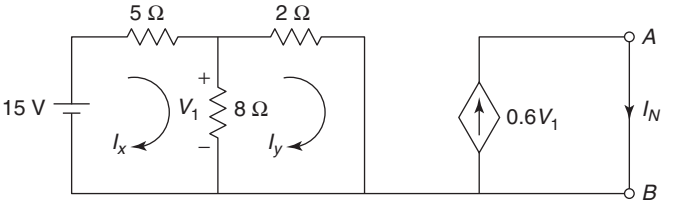


Fig. 3.303

$$V_1 = 8(I_x - I_y) \quad \ldots(\text{i})$$

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$
$$13I_x - 8I_y = 15 \quad \ldots(\text{ii})$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y = 0$$
$$-8I_x + 10I_y = 0 \quad \ldots(\text{iii})$$

Solving Eqs (ii) and (iii),

$$I_x = 2.27 \text{ A}$$
$$I_y = 1.82 \text{ A}$$
$$V_1 = 8(I_x - I_y) = 8(2.27 - 1.82) = 3.6 \text{ V}$$

For Mesh 3,

$$I_N = 0.6V_1 = 0.6(3.6) = 2.16 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-12.45}{2.16} = -5.76 \ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.304)

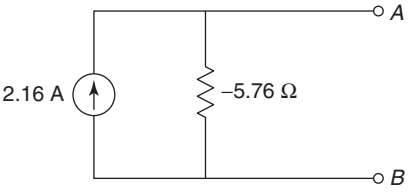


Fig. 3.304

Example 3.69 *Find Norton's equivalent network of Fig. 3.305.*

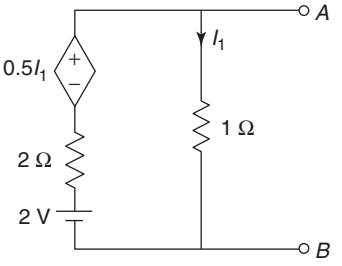


Fig. 3.305

Solution

Step I Calculation of V_{Th} (Fig. 3.306)

Applying KVL to the mesh,

$$2 - 2I_1 + 0.5I_1 - 1I_1 = 0$$
$$2 - 2.5I_1 = 0$$
$$I_1 = 0.8 \text{ A}$$

Writing the V_{Th} equation,

$$1I_1 - V_{Th} = 0$$
$$1(0.8) - V_{Th} = 0$$
$$V_{Th} = 0.8 \text{ V}$$

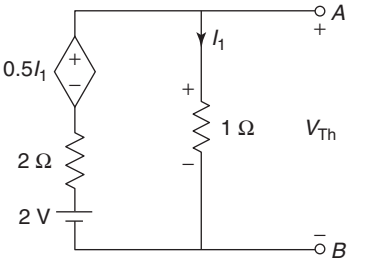


Fig. 3.306

Step II Calculation of I_N (Fig. 3.307)

When a short circuit is placed across the 1 Ω resistor, it gets shorted.

$$I_1 = 0$$

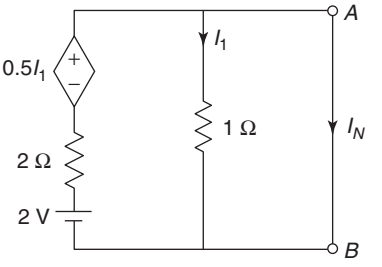


Fig. 3.307

The dependent source of $0.5I_1$ depends on the controlling variable I_1. When $I_1 = 0$, the dependent source vanishes, i.e. $0.5\,I_1 = 0$ as shown in Fig. 3.308.

$$I_N = \frac{2}{2} = 1 \text{ A}$$

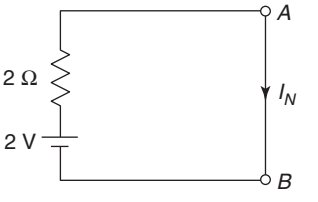


Fig. 3.308

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{0.8}{1} = 0.8 \text{ Ω}$$

Step IV Norton's Equivalent Network (Fig. 3.309)

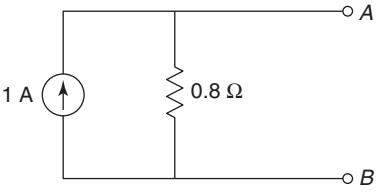


Fig. 3.309

Example 3.70 *Find Norton's equivalent network at the terminals A and B of Fig. 3.310.*

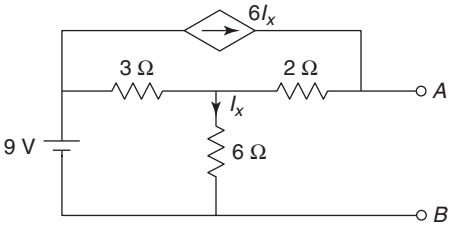


Fig. 3.310

Solution

Step I Calculation of V_{Th} (Fig. 3.311)

From Fig. 3.311,

$$I_x = I_1 \qquad \ldots(i)$$

Applying KVL to Mesh 1,

$$9 - 3(I_1 - I_2) - 6I_1 = 0$$

$$9I_1 - 3I_2 = 9 \qquad \ldots(ii)$$

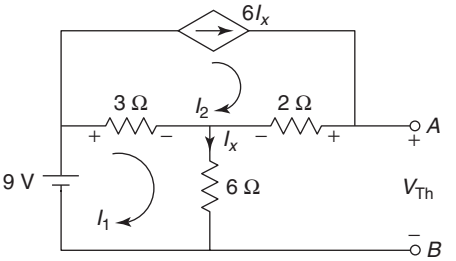


Fig. 3.311

For Mesh 2,

$$I_2 = 6I_x = 6I_1$$

$$6I_1 - I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = -1 \text{ A}$$

$$I_2 = -6 \text{ A}$$

Writing the V_{Th} equation,

$$9 - 3(I_1 - I_2) + 2I_2 - V_{Th} = 0$$

$$9 - 3(-1 + 6) + 2(-6) - V_{Th} = 0$$

$$V_{Th} = -18 \text{ V}$$

Step II Calculation of I_N (Fig. 3.312)

From Fig. 3.312,

$$I_x = I_1 - I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 1,

$$9 - 3(I_1 - I_2) - 6(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 6I_3 = 9 \qquad \ldots(ii)$$

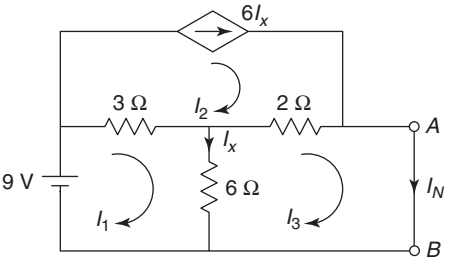


Fig. 3.312

For Mesh 2,

$$I_2 = 6I_x = 6(I_1 - I_3)$$

$$6I_1 - I_2 - 6I_3 = 0 \qquad \ldots(iii)$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) - 2(I_3 - I_2) = 0$$

$$-6I_1 - 2I_2 + 8I_3 = 0 \qquad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5 \text{ A}$$
$$I_2 = 3 \text{ A}$$
$$I_3 = 4.5 \text{ A}$$
$$I_N = I_3 = 4.5 \text{ A}$$

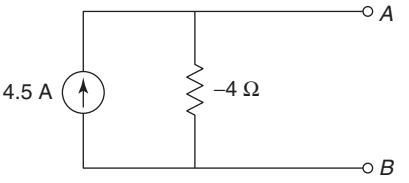


Fig. 3.313

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4 \ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.313)

Example 3.71 *Find Norton's equivalent network to the left of terminal A-B in Fig. 3.314.*

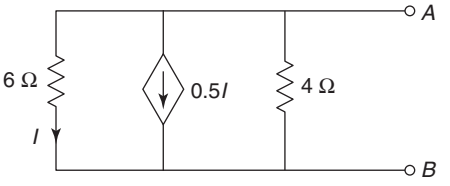


Fig. 3.314

Solution Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$
$$I_N = 0$$

But R_N can be calculated by applying a known current source of 1 A at the terminals A and B as shown in Fig. 3.315.

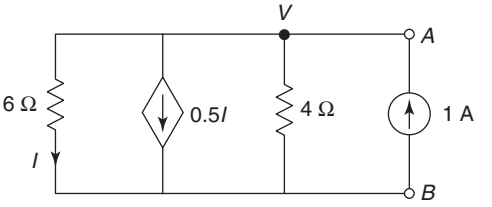


Fig. 3.315

From Fig. 3.315,

$$I = \frac{V}{6}$$

Applying KCL at the node,

$$\frac{V}{6} + 0.5I + \frac{V}{4} = 1$$

$$\frac{V}{6} + 0.5\left(\frac{V}{6}\right) + \frac{V}{4} = 1$$

$$\left(\frac{1}{6} + \frac{0.5}{6} + \frac{1}{4}\right)V = 1$$

$$V = 2$$

$$R_N = \frac{V}{1} = \frac{2}{1} = 2 \ \Omega$$

Hence, Norton's equivalent network is shown in Fig. 3.316.

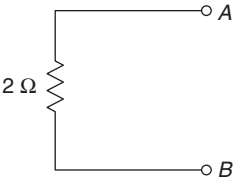


Fig. 3.316

Example 3.72 *Find the current through the 2 Ω resistor in the network shown in Fig. 3.317.*

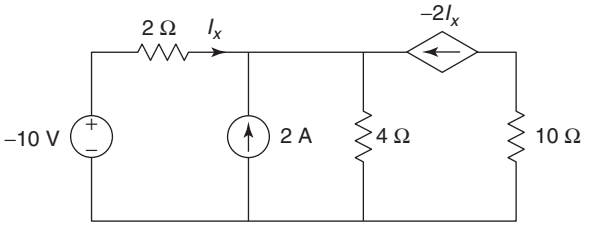


Fig. 3.317

Solution

Step I Calculation of V_{Th} (Fig. 3.318)
From Fig. 3.318,

$$I_x = 0$$

The dependent source of $-2\ I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e. $-2I_x = 0$ as shown in Fig. 3.319.

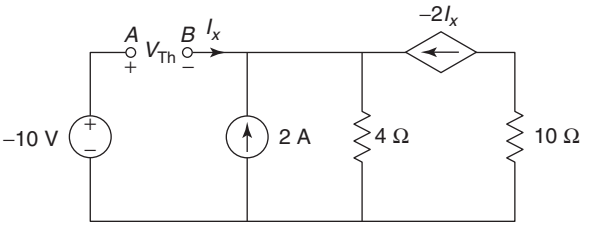


Fig. 3.318

$$I_1 = 2$$

Writing the V_{Th} equation,

$$-10 - V_{Th} - 4I_1 = 0$$

$$-10 - V_{Th} - 4(2) = 0$$

$$V_{Th} = -18 \text{ V}$$

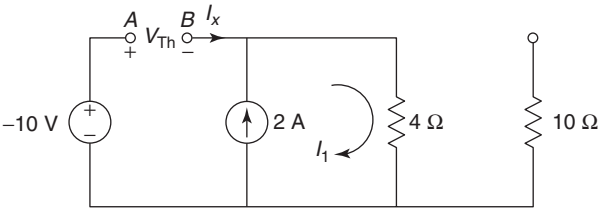


Fig. 3.319

Step II Calculation of I_N (Fig. 3.320)
From Fig. 3.320,

$$I_x = I_1 \qquad \ldots(i)$$

Mesh 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

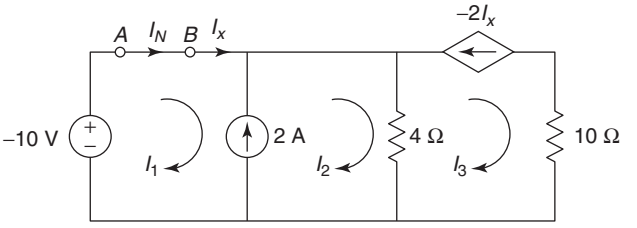


Fig. 3.320

$$-10 - 4(I_2 - I_3) = 0$$

$$-4I_2 + 4I_3 = 10 \qquad \ldots(iii)$$

For Mesh 3,

$$I_3 = -(-2I_x) = 2I_x = 2I_1$$

$$2I_1 - I_3 = 0 \qquad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.5 \text{ A}$$
$$I_2 = 6.5 \text{ A}$$
$$I_3 = 9 \text{ A}$$
$$I_N = I_1 = 4.5 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4\ \Omega$$

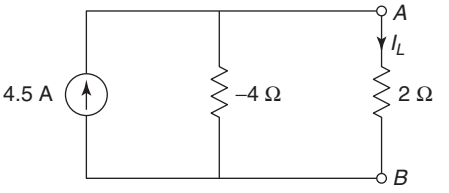


Fig. 3.321

Step IV Calculation of I_L (Fig. 3.321)

$$I_L = 4.5 \times \frac{-4}{-4+2} = 9 \text{ A}$$

Example 3.73 *Find the current through the 2Ω resistor in the network of Fig. 3.322.*

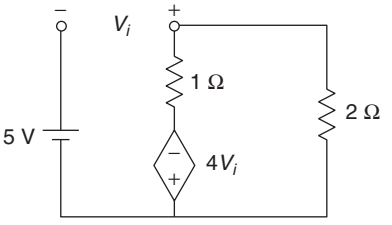


Fig. 3.322

Solution

Step I Calculation of V_{Th} (Fig. 3.322)
From Fig. 3.323,

$$5 + V_i + 4V_i = 0$$
$$V_i = -1 \text{ V}$$

Writing the V_{Th} equation,

$$-4V_i - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -4V_i = -4(-1) = 4 \text{ V}$$

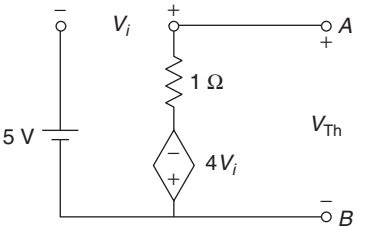


Fig. 3.323

Step II Calculation of I_N (Fig. 3.324)
From Fig. 3.324,

$$5 + V_i = 0$$
$$V_i = -5 \text{ V}$$

Applying KVL to the mesh,

$$-4V_i - 1I_N = 0$$
$$I_N = -4V_i = -4(-5) = 20 \text{ A}$$

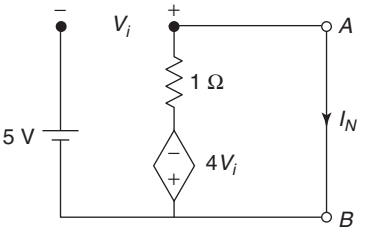


Fig. 3.324

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{4}{20} = 0.2\ \Omega$$

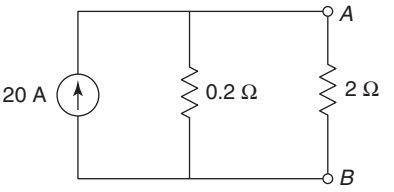


Fig. 3.325

Step IV Calculation of I_L (Fig. 3.325)

$$I_L = 20 \times \frac{0.2}{0.2+2} = 1.82\ \text{A}$$

Example 3.74

Find the current in the 2 Ω resistor in the network of Fig. 3.326.

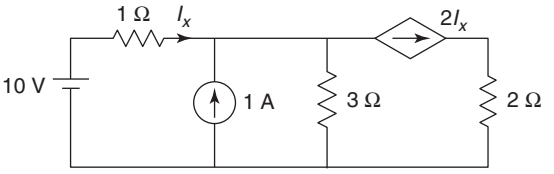


Fig. 3.326

Solution

Step I Calculation of V_{Th} (Fig. 3.327)

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots(\text{i})$$

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3I_2 = 0$$

$$I_1 + 3I_2 = 10 \qquad \ldots(\text{ii})$$

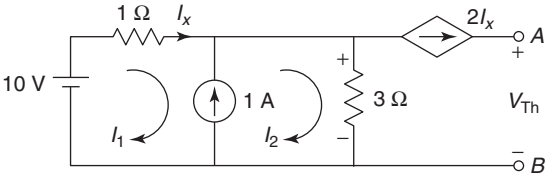


Fig. 3.327

Solving Eqs (i) and (ii),

$$I_1 = 1.75\ \text{A}$$

$$I_2 = 2.75\ \text{A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{\text{Th}} = 0$$

$$3(2.75) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 8.25\ \text{V}$$

Step II Calculation of I_N (Fig. 3.328)

From Fig. 3.328,

$$I_x = I_1 \qquad \ldots(\text{i})$$

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots(\text{ii})$$

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3(I_2 - I_3) = 0$$

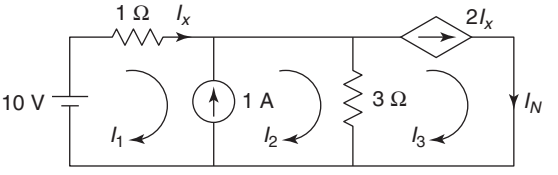


Fig. 3.328

$$I_1 + 3I_2 - 3I_3 = 10 \quad \ldots(iii)$$

For Mesh 3,

$$I_3 = 2I_x = 2I_1$$
$$2I_1 - I_3 = 0 \quad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -3.5 \text{ A}$$
$$I_2 = -2.5 \text{ A}$$
$$I_3 = -7 \text{ A}$$
$$I_N = I_3 = -7 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8.25}{-7} = -1.18\ \Omega$$

Step IV Calculation of I_L (Fig. 3.329)

$$I_L = -7 \times \frac{-1.18}{-1.18 + 2} = 10.07 \text{ A}$$

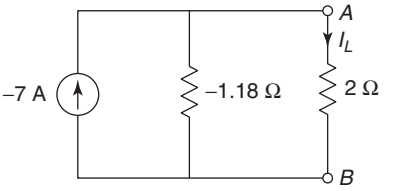


Fig. 3.329

Example 3.75 *Find the current through the 10 Ω resistor for the network of Fig. 3.330.*

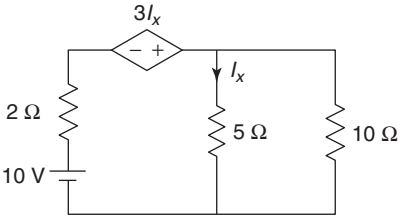


Fig. 3.330

Solution

Step I Calculation of V_{Th} (Fig. 3.331)
Applying KVL to the mesh,

$$10 - 2I_x + 3I_x - 5I_x = 0$$
$$I_x = 2.5 \text{ A}$$

Writing the V_{Th} equation,

$$5I_x - V_{\text{Th}} = 0$$
$$5(2.5) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 12.5 \text{ V}$$

Step II Calculation of I_N (Fig. 3.332)
From Fig. 3.332,

$$I_x = 0$$

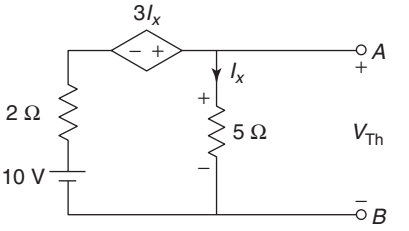


Fig. 3.331

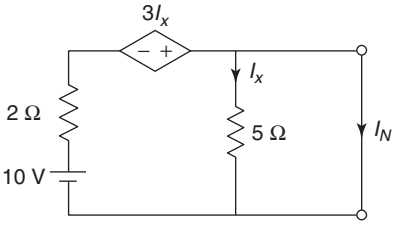


Fig. 3.332

The dependent source of 3 I_x depends on the controlling variable I_x. When $I_x = 0$, the dependent source 3 I_x vanishes, i.e. 3 $I_x = 0$ as shown in Fig. 3.333.

$$I_N = \frac{10}{2} = 5 \text{ A}$$

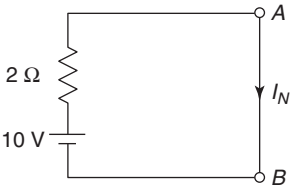


Fig. 3.333

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{12.5}{5} = 2.5\ \Omega$$

Step IV Calculation of I_L (Fig. 3.334)

$$I_L = 5 \times \frac{2.5}{2.5+10} = 1 \text{ A}$$

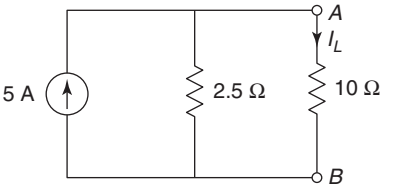


Fig. 3.334

Example 3.76 *Find the current through the 5 Ω resistor in the network of Fig. 3.335.*

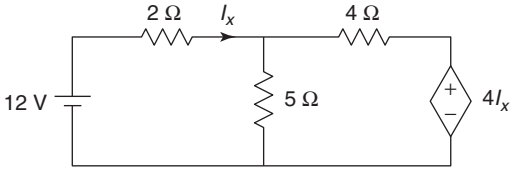


Fig. 3.335

Solution

Step I Calculation of V_{Th} (Fig. 3.336)

Applying KVL to the mesh,

$$12 - 2I_x - 4I_x - 4I_x = 0$$
$$12 - 10I_x = 0$$
$$I_x = 1.2 \text{ A}$$

Writing the V_{Th} equation,

$$12 - 2I_x - V_{\text{Th}} = 0$$
$$12 - 2(1.2) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 9.6 \text{ V}$$

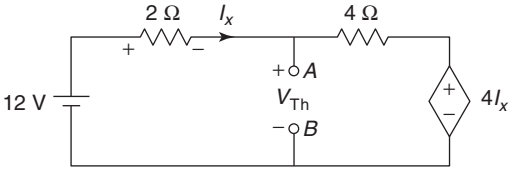


Fig. 3.336

Step II Calculation of I_N (Fig. 3.337)

From Fig. 3.337,

$$I_x = I_1 \quad \ldots(\text{i})$$

Applying KVL to Mesh 1,

$$12 - 2I_1 = 0$$
$$I_1 = 6 \text{ A} \quad \ldots(\text{ii})$$

Applying KVL to Mesh 2,

$$-4I_2 - 4I_x = 0$$
$$-4I_2 - 4I_1 = 0 \quad \ldots(\text{iii})$$

Solving Eqs (ii) and (iii),

$$I_2 = -6 \text{ A}$$
$$I_N = I_1 - I_2 = 6 - (-6) = 12 \text{ A}$$

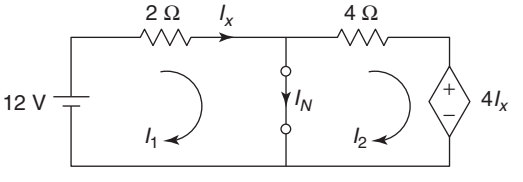


Fig. 3.337

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{9.6}{12} = 0.8\ \Omega$$

Step IV Calculation of I_L (Fig. 3.338)

$$I_L = 12 \times \frac{0.8}{0.8+5} = 1.66\ \text{A}$$

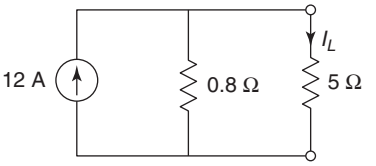


Fig. 3.338

Example 3.77 *Find the current through the 10 Ω resistor for the network of Fig. 3.339.*

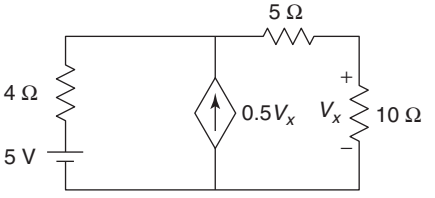


Fig. 3.339

Solution

Step I Calculation of V_{Th} (Fig. 3.340)
For the mesh,

$$I = -0.5V_x = -0.5V_{\text{Th}}$$

Writing the V_{Th} equation,

$$5 - 4I - 0 - V_{\text{Th}} = 0$$

$$5 - 4(-0.5V_{\text{Th}}) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = -5\ \text{V}$$

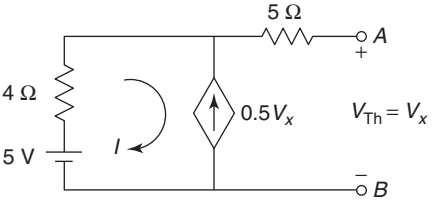


Fig. 3.340

Step II Calculation of I_N (Fig. 3.341)
From Fig. 3.341,

$$V_x = 0$$

The dependent source of $0.5V_x$ depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. 0.5 $V_x = 0$ as shown in Fig. 3.342.

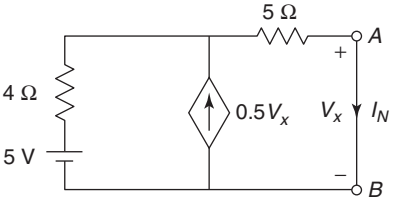


Fig. 3.341

$$I_N = \frac{5}{4+5} = \frac{5}{9}\ \text{A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-5}{\frac{5}{9}} = -9\ \Omega$$

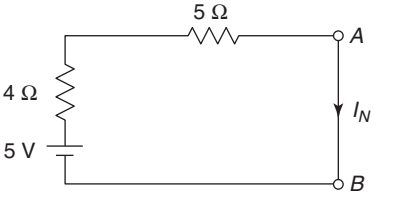


Fig. 3.342

Step IV Calculation of I_L (Fig. 3.343)

$$I_L = \frac{5}{9} \times \frac{-9}{-9+10} = -5\ \text{A}$$

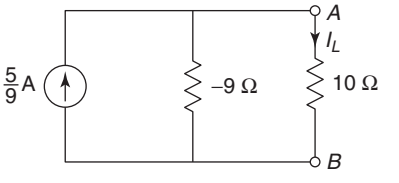


Fig. 3.343

Example 3.78 *Find the current through the 10 Ω resistor in the network shown in Fig. 3.344.*

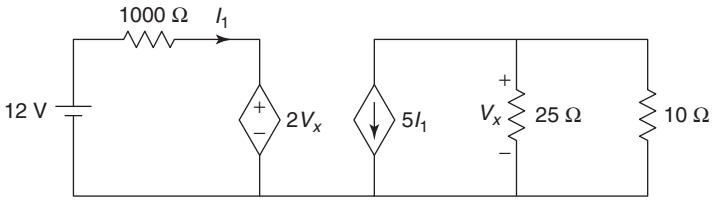


Fig. 3.344

Solution

Step I Calculation of V_{Th} (Fig. 3.345)
From Fig. 3.345,

$$V_x = -25(5I_1) = -125\,I_1 \quad \ldots(i)$$

Applying KVL to Mesh 1,

$$12 - 1000I_1 - 2V_x = 0$$

$$12 - 1000I_1 - 2(-125I_1) = 0 \quad \ldots(ii)$$

$$I_1 = 0.016 \text{ A}$$

$$V_x = -125I_1 = -125(0.016) = -2 \text{ V}$$

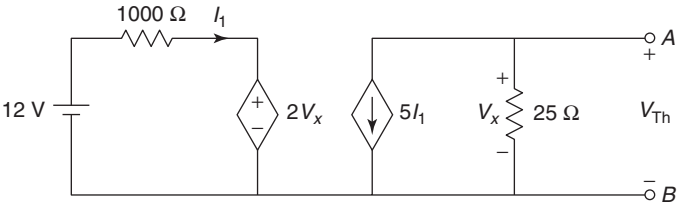


Fig. 3.345

Writing the V_{Th} equation,

$$V_{Th} = V_x = -2 \text{ V}$$

Step II Calculation of I_N (Fig. 3.346)
From Fig. 3.346,

$$V_x = 0$$

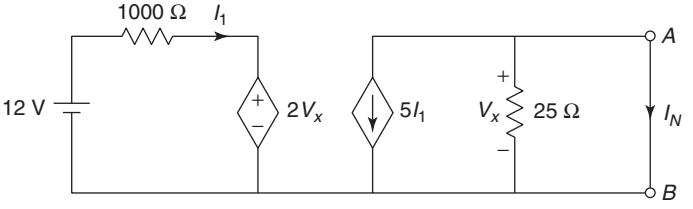


Fig. 3.346

The dependent source of $2V_x$ depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. $2V_x = 0$ as shown in Fig. 3.347.

$$I_1 = \frac{12}{1000} = 0.012 \text{ A}$$

$$I_N = -5I_1 = -5(0.012) = -0.06 \text{ A}$$

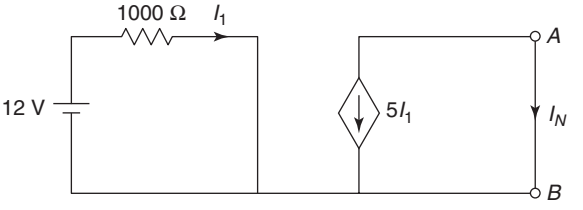


Fig. 3.347

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{-2}{-0.06} = 33.33\ \Omega$$

Step IV Calculation of I_L (Fig. 3.348)

$$I_L = -0.06 \times \frac{33.33}{33.33 + 10} = -0.046 \text{ A}$$

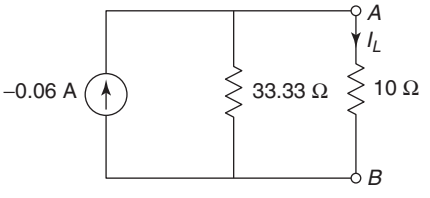


Fig. 3.348

Example 3.79 *Find the current through the 5 Ω resistor for the network of Fig. 3.349.*

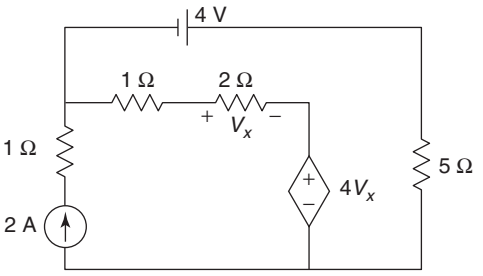


Fig. 3.349

Solution

Step I Calculation of V_{Th} (Fig. 3.350)

From Fig. 3.350,

$$V_x = 2I \quad \ldots(i)$$

For the mesh,

$$I = 2 \quad \ldots(ii)$$

$$V_x = 2(2) = 4 \text{ V}$$

Writing the V_{Th} equation,

$$4V_x + 2I + 1I + 4 - V_{Th} = 0$$

$$4(4) + 2(2) + 2 + 4 - V_{Th} = 0$$

$$V_{Th} = 26 \text{ V}$$

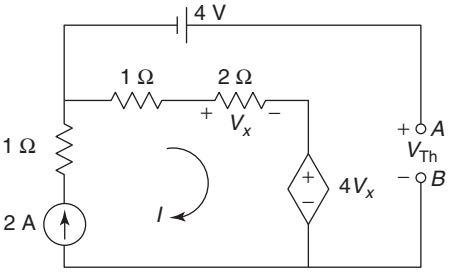


Fig. 3.350

Step II Calculation of I_N (Fig. 3.351)

From Fig. 3.351,

$$V_x = 2(I_1 - I_2) \quad \ldots(i)$$

For Mesh 1,

$$I_1 = 2 \quad \ldots(ii)$$

Applying KVL to Mesh 2,

$$4V_x - 2(I_2 - I_1) - 1(I_2 - I_1) + 4 = 0$$

$$4[2(I_1 - I_2)] - 2I_2 + 2I_1 - I_2 + I_1 + 4 = 0$$

$$11I_1 - 11I_2 = -4 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$

$$I_2 = 2.36 \text{ A}$$

$$I_N = I_2 = 2.36 \text{ A}$$

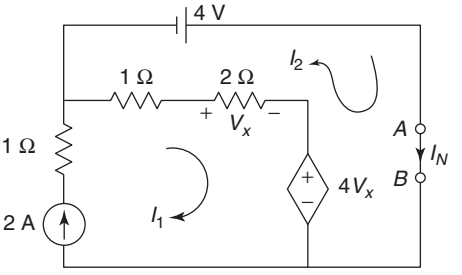


Fig. 3.351

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{26}{2.36} = 11.02 \ \Omega$$

Step IV Calculation of I_L (Fig. 3.352)

$$I_L = 2.36 \times \frac{11.02}{11.02 + 5} = 1.62 \text{ A}$$

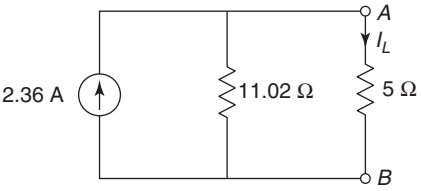


Fig. 3.352

Example 3.80 *Find the current through the 1 Ω resistor in the network of Fig. 3.353.*

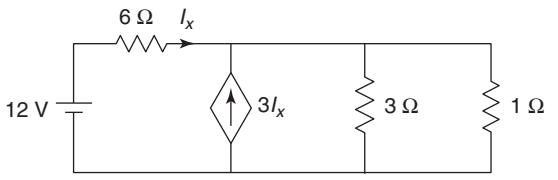


Fig. 3.353

Solution

Step I Calculation of V_{Th} (Fig. 3.354)
From Fig. 3.354,

$$I_x = I_1 \quad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \quad \ldots(ii)$$

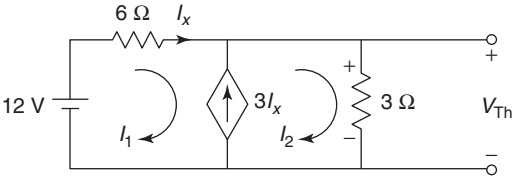


Fig. 3.354

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 - 3I_2 = 0$$

$$6I_1 + 3I_2 = 12 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.67 \text{ A}$$

$$I_2 = 2.67 \text{ A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{Th} = 0$$

$$3(2.67) - V_{Th} = 0$$

$$V_{Th} = 8 \text{ V}$$

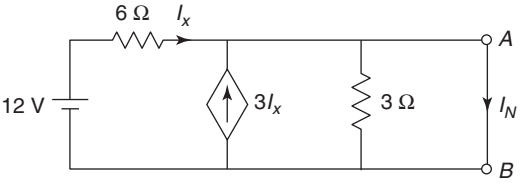


Fig. 3.355

Step II Calculation of I_N (Fig. 3.355)
When a short circuit is placed across a 3 Ω resistor, it gets shorted as shown in Fig. 3.356.
From Fig. 3.356,

$$I_x = I_1 \quad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \quad \ldots(ii)$$

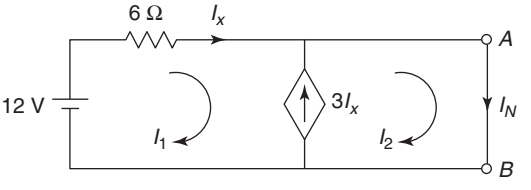


Fig. 3.356

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 = 0$$

$$I_1 = 2 \quad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 8 \text{ A}$$
$$I_N = I_2 = 8 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8}{8} = 1\ \Omega$$

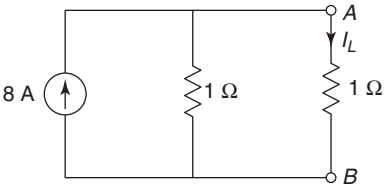


Fig. 3.357

Step IV Calculation of I_L (Fig. 3.357)

$$I_L = 8 \times \frac{1}{1+1} = 4 \text{ A}$$

Example 3.81 *Find the current through the 1.6 Ω resistor in the network of Fig. 3.358.*

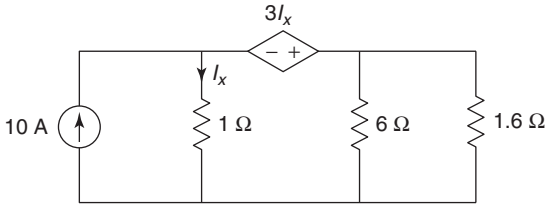


Fig. 3.358

Solution

Step I Calculation of V_{Th} (Fig. 3.359)

From Fig. 3.359,

$$I_x = I_1 - I_2 \quad \ldots(\text{i})$$

For Mesh 1,

$$I_1 = 10 \quad \ldots(\text{ii})$$

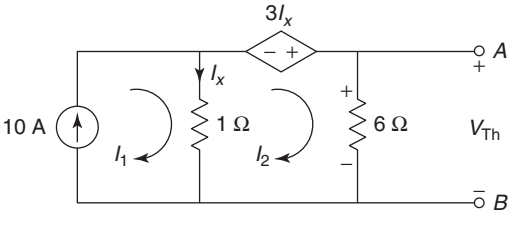


Fig. 3.359

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x - 6I_2 = 0$$
$$-I_2 + I_1 + 3(I_1 - I_2) - 6I_2 = 0$$
$$4I_1 - 10I_2 = 0 \quad \ldots(\text{iii})$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 4 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - V_{\text{Th}} = 0$$
$$6(4) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 24 \text{ V}$$

Step II Calculation of I_N (Fig. 3.360)

When a short circuit is placed across the 3 Ω resistor, it gets shorted as shown in Fig. 3.361.

From Fig. 3.361,

$$I_x = I_1 - I_2 \qquad \ldots(i)$$

For Mesh 1,

$$I_1 = 10 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x = 0$$
$$-I_2 + I_1 + 3(I_1 - I_2) = 0$$
$$4I_1 - 4I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 10 \text{ A}$$
$$I_N = I_2 = 10 \text{ A}$$

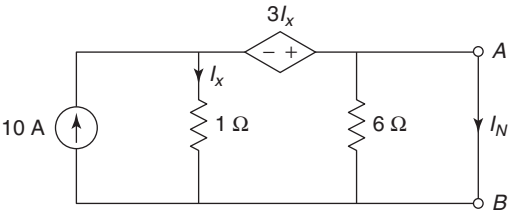


Fig. 3.360

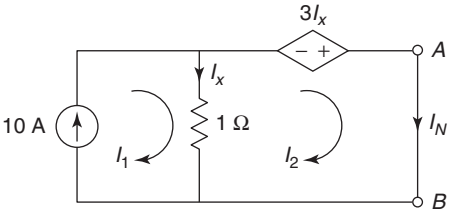


Fig. 3.361

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{24}{10} = 2.4 \ \Omega$$

Step IV Calculation of I_L (Fig. 3.362)

$$I_L = 10 \times \frac{2.4}{2.4 + 1.6} = 6 \text{ A}$$

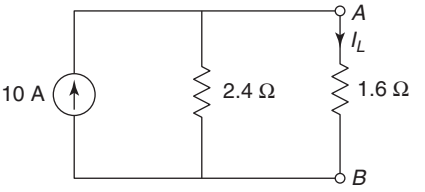


Fig. 3.362

3.5 MAXIMUM POWER TRANSFER THEOREM

It states that '*the maximum power is delivered from a source to a load when the load resistance is equal to the source resistance.*'

Proof From Fig. 3.363,

$$I = \frac{V}{R_s + R_L}$$

Power delivered to the load $R_L = P = I^2 \ R_L = \dfrac{V^2 R_L}{(R_s + R_L)^2}$

To determine the value of R_L for maximum power to be transferred to the load,

$$\frac{dP}{dR_L} = 0$$

$$\frac{dP}{dR_L} = \frac{d}{dR_L} \frac{V^2}{(R_s + R_L)^2} R_L$$

$$= \frac{V^2[(R_s + R_L)^2 - (2R_L)(R_s + R_L)]}{(R_s + R_L)^4}$$

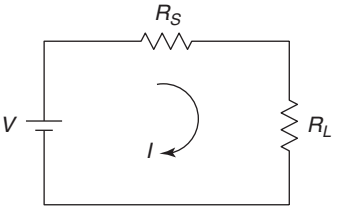


Fig. 3.363 *Network illustrating maximum power transfer theorem*

$$(R_s + R_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s{}^2 + R_L{}^2 + 2R_sR_L - 2R_LR_s - 2R_L^2 = 0$$

$$R_s = R_L$$

Hence, the maximum power will be transferred to the load when load resistance is equal to the source resistance.

Steps to be followed in Maximum Power Transfer Theorem

1. Remove the variable load resistor R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Find the resistance R_L for maximum power transfer.

$$R_L = R_{Th}$$

5. Find the maximum power (Fig. 3.364).

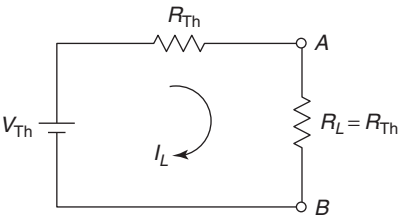


Fig. 3.364 *Thevenin's equivalent network*

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{V_{Th}}{2R_{Th}}$$

$$P_{max} = I_L^2\,R_L = \frac{V_{Th}^2}{4R_{Th}^2} \times R_{Th} = \frac{V_{Th}^2}{4R_{Th}}$$

Example 3.82 *Find the value of resistance R_L in Fig. 3.365 for maximum power transfer and calculate maximum power.*

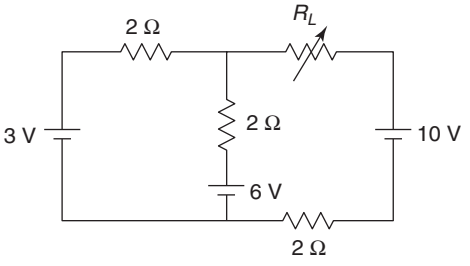


Fig. 3.365

Solution

Step I Calculation of V_{Th} (Fig. 3.366)

Applying KVL to the mesh,

$$3 - 2I - 2I - 6 = 0$$

$$I = -0.75\,\text{A}$$

Writing the V_{Th} equation,

$$6 + 2I - V_{Th} - 10 = 0$$

$$V_{Th} = 6 + 2I - 10 = 6 + 2(-0.75) - 10 = -5.5\,\text{V}$$

$$= 5.5\ \text{V (terminal } B \text{ is positive w.r.t } A)$$

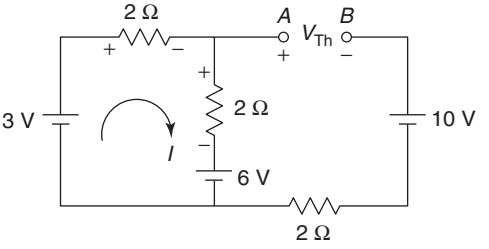


Fig. 3.366

Step II Calculation of R_{Th} (Fig. 3.367)

$$R_{Th} = (2 \,||\, 2) + 2 = 3\,\Omega$$

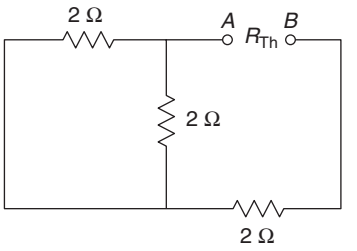


Fig. 3.367

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3\,\Omega$$

Step IV Calculation of P_{max} (Fig. 3.368)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(5.5)^2}{4 \times 3} = 2.52\,\text{W}$$

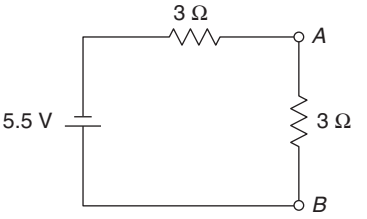


Fig. 3.368

Example 3.83 *Find the value of resistance R_L in Fig. 3.369 for maximum power transfer and calculate maximum power.*

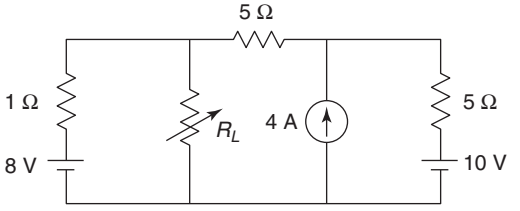


Fig. 3.369

Solution

Step I Calculation of V_{Th} (Fig. 3.370)

$$I_2 - I_1 = 4 \qquad \text{...(i)}$$

Applying KVL to the outer path,

$$8 - 1I_1 - 5I_1 - 5I_2 - 10 = 0$$

$$-6I_1 - 5I_2 = 2 \qquad \text{...(ii)}$$

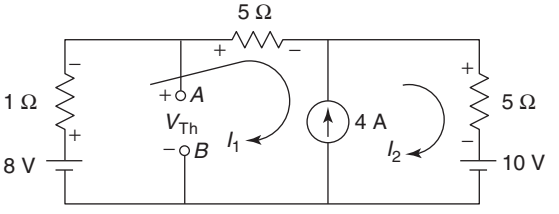


Fig. 3.370

Solving Eqs (i) and (ii),

$$I_1 = -2\,\text{A}$$

$$I_2 = 2\,\text{A} \qquad \text{...(ii)}$$

Writing the V_{Th} equation,

$$8 - 1I_1 - V_{Th} = 0$$

$$V_{Th} = 8 - I_1 = 8 - (-2) = 10\,\text{V}$$

Step II Calculation of R_{Th} (Fig. 3.371)

$$R_{Th} = 10 \,||\, 1 = 0.91\,\Omega$$

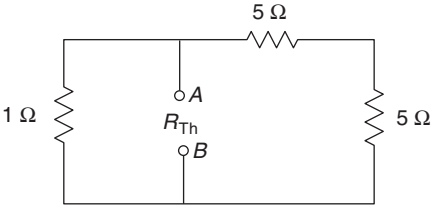


Fig. 3.371

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 0.91\,\Omega$$

Step IV Calculation of $P_{\max}$

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(10)^2}{4 \times 0.91} = 27.47 \text{ W}$$

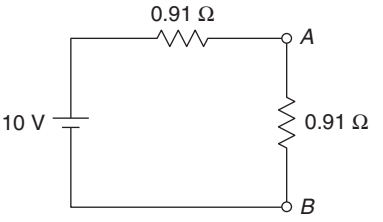


Fig. 3.372

Example 3.84 *Find the value of the resistance R_L in Fig. 3.373 for maximum power transfer and calculate the maximum power.*

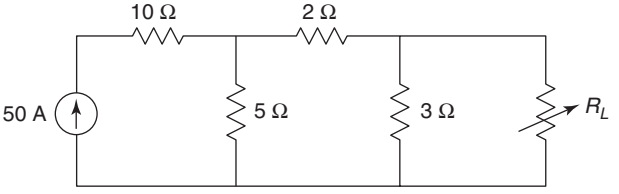


Fig. 3.373

Solution

Step I Calculation of V_{Th} (Fig. 3.374)
For Mesh 1,

$$I_1 = 50$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 2I_2 - 3I_2 = 0$$
$$5I_1 - 10I_2 = 0$$
$$I_1 = 2I_2$$
$$I_2 = 25\,\text{A}$$
$$V_{\text{Th}} = 3I_2 = 3(25) = 75\,\text{V}$$

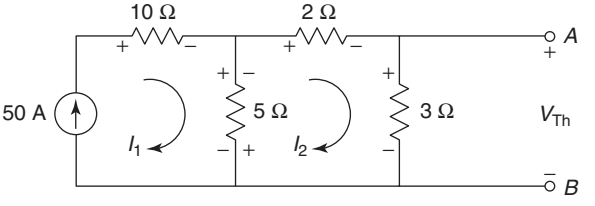


Fig. 3.374

Step II Calculation of R_{Th} (Fig. 3.375)

$$R_{\text{Th}} = (5+2) \,||\, 3 = 2.1\,\Omega$$

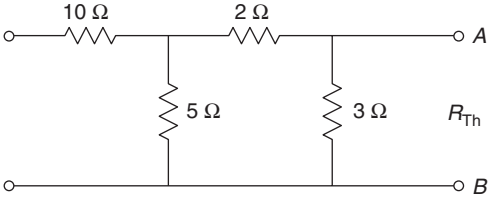


Fig. 3.375

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 2.1\,\Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.376)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(75)^2}{4 \times 2.1} = 669.64 \text{ W}$$

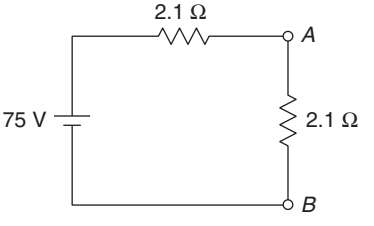


Fig. 3.376

Example 3.85 *Find the value of resistance R_L in Fig. 3.377 for maximum power transfer and calculate maximum power.*

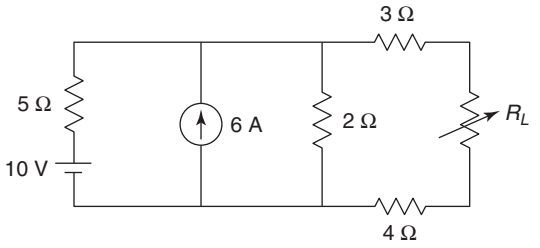


Fig. 3.377

Solution

Step I Calculation of V_{Th} (Fig. 3.378)
Writing the current equation for the supermesh,

$$I_2 - I_1 = 6 \quad \text{...(i)}$$

Applying KVL to the supermesh,

$$10 - 5I_1 - 2I_2 = 0$$
$$5I_1 + 2I_2 = 10 \quad \text{...(ii)}$$

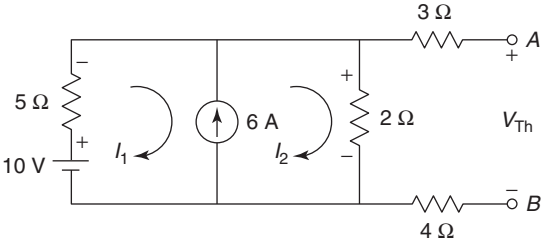


Fig. 3.378

Solving Eqs (i) and (ii),

$$I_1 = -0.29\,\text{A}$$
$$I_2 = 5.71\,\text{A}$$

Writing the V_{Th} equation,

$$V_{Th} = 2I_2 = 11.42\,\text{V}$$

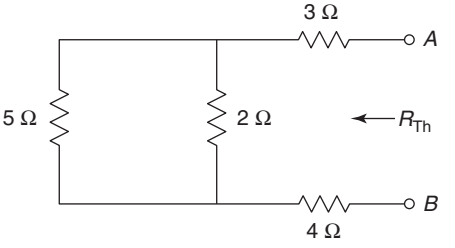


Fig. 3.379

Step II Calculation of R_{Th} (Fig. 3.379)

$$R_{Th} = (5 \,||\, 2) + 3 + 4 = 8.43\,\Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{Th} = 8.43\,\Omega$$

Step IV Calculation of P_{max} (Fig. 3.380)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(11.42)^2}{4 \times 8.43} = 3.87\ \text{W}$$

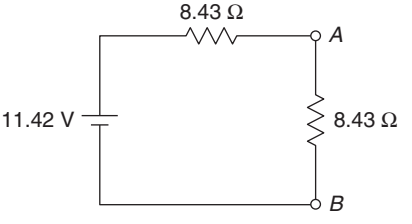


Fig. 3.380

Example 3.86 *Find the value of resistance R_L in Fig. 3.381 for maximum power transfer and calculate the maximum power.*

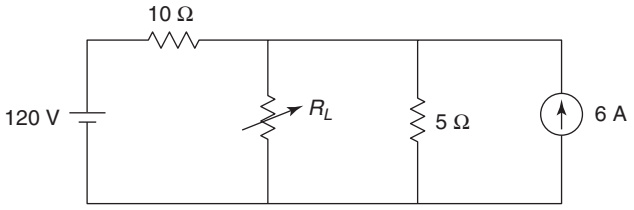


Fig. 3.381

Solution

Step I Calculation of V_{Th} (Fig. 3.382)

Applying KVL to Mesh 1,

$$120 - 10I_1 - 5(I_1 - I_2) = 0$$

$$15I_1 - 5I_2 = 120 \qquad ...(i)$$

Writing current equation for Mesh 2,

$$I_2 = -6 \qquad ...(ii)$$

Solving Eqs (i) and (ii),

$$I_1 = 6\,\text{A}$$

Writing the V_{Th} equation,

$$120 - 10I_1 - V_{Th} = 0$$

$$V_{Th} = 120 - 10(6) = 60\,\text{V}$$

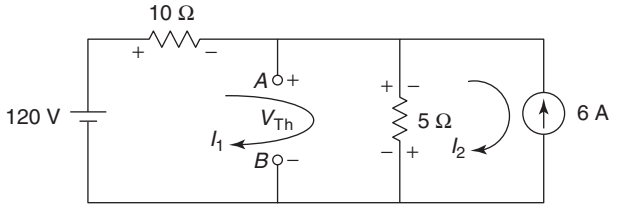


Fig. 3.382

Step II Calculation of R_{Th} (Fig. 3.383)

$$R_{Th} = 10 \,||\, 5 = 3.33\ \Omega$$

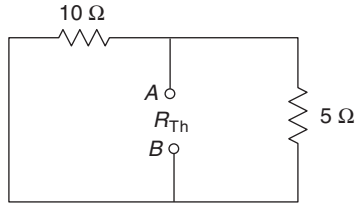


Fig. 3.383

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3.33\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.384)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(60)^2}{4 \times 3.33} = 270.27\ \text{W}$$

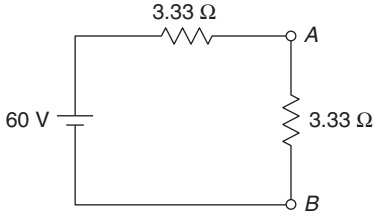


Fig. 3.384

Example 3.87 *Find the value of resistance R_L in Fig. 3.385 for maximum power transfer and calculate the maximum power.*

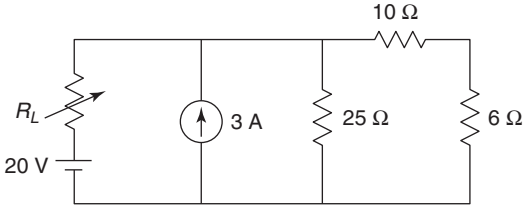


Fig. 3.385

Solution

Step I Calculation of V_{Th} (Fig. 3.386)

$$I_1 = 3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-25(I_2 - I_1) - 10I_2 - 6I_2 = 0$$

$$-25I_1 + 41I_2 = 0 \qquad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$I_2 = 1.83 \text{ A}$$

Writing the V_{Th} equation,

$$20 + V_{Th} - 10I_2 - 6I_2 = 0$$

$$V_{Th} = -20 + 10\,(1.83) + 6\,(1.83) = 9.28 \text{ V}$$

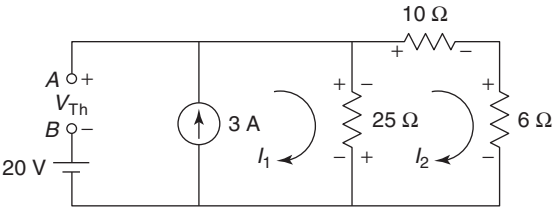


Fig. 3.386

Step II Calculation of R_{Th} (Fig. 3.387)

$$R_{Th} = 25 \,||\, (10 + 6) = 9.76\ \Omega$$

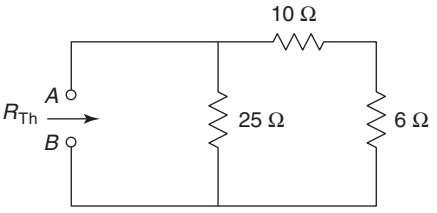


Fig. 3.387

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 9.76\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.388)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(9.28)^2}{4 \times 9.76} = 2.21 \text{ W}$$

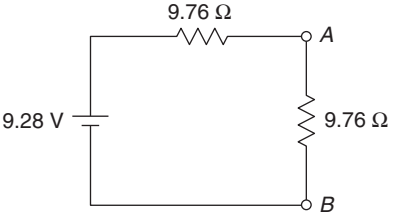


Fig. 3.388

Example 3.88 *Find the value of resistance R_L in Fig. 3.389 for maximum power transfer and calculate maximum power.*

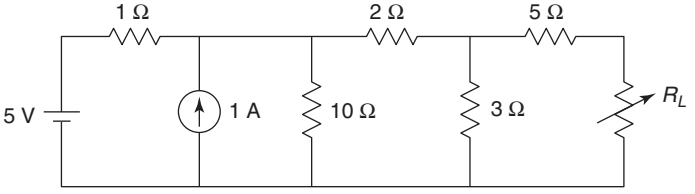


Fig. 3.389

Solution

Step I Calculation of V_{Th} (Fig. 3.390)

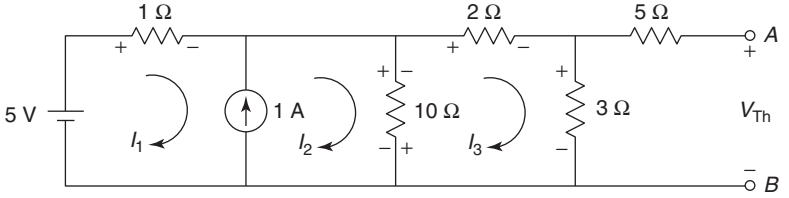


Fig. 3.390

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots(i)$$

Writing the voltage equation for the supermesh,

$$5 - 1I_1 - 10(I_2 - I_3) = 0$$
$$I_1 + 10I_2 - 10I_3 = 5 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-10(I_3 - I_2) - 2I_3 - 3I_3 = 0$$
$$-10I_2 + 15I_3 = 0 \qquad \ldots(iii)$$

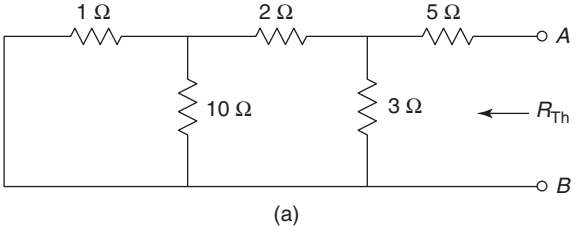
Solving Eqs (i), (ii) and (iii),

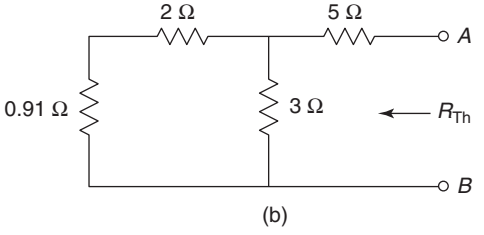
$$I_1 = 0.38 \text{ A}$$
$$I_2 = 1.38 \text{ A}$$
$$I_3 = 0.92 \text{ A}$$

Writing the V_{Th} equation,

$$V_{\text{Th}} = 3I_3 = 2.76 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.391)





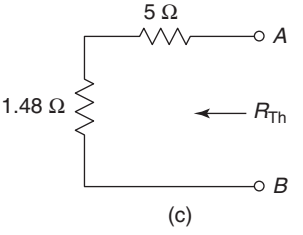


Fig. 3.391

$$R_{\text{Th}} = 6.48 \ \Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 6.48 \ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.392)

$$P_{\text{max}} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(2.76)^2}{4 \times 6.48} = 0.29 \text{ W}$$

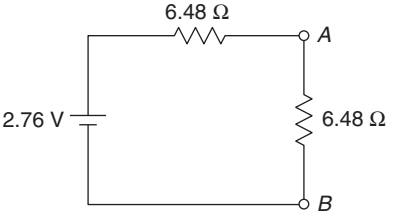


Fig. 3.392

Example 3.89 *For the network shown in Fig. 3.393, find the value of the resistance R_L for maximum power transfer and calculate the maximum power.*

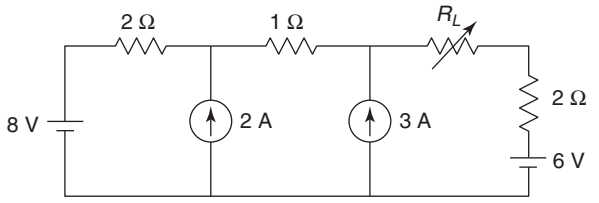


Fig. 3.393

Solution

Step I Calculation of V_{Th} (Fig. 3.394)

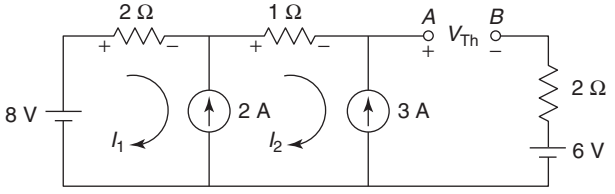


Fig. 3.394

$$I_2 - I_1 = 2 \quad \ldots(i)$$

$$I_2 = -3 \text{ A} \quad \ldots(ii)$$

Solving Eqs (i), and (ii),

$$I_1 = -5 \text{ A}$$

Writing the V_{Th} equation,

$$8 - 2I_1 - 1I_2 - V_{Th} - 6 = 0$$

$$V_{Th} = 8 - 2(-5) - (-3) - 6 = 15 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.395)

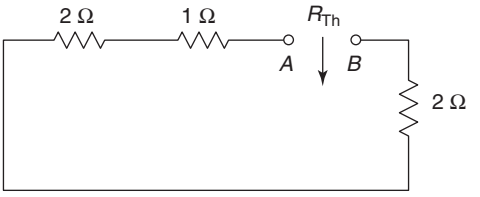


Fig. 3.395

$$R_{Th} = 5\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 5\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.396)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(15)^2}{4 \times 5} = 11.25 \text{ W}$$

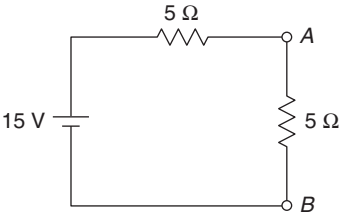


Fig. 3.396

Example 3.90 *For the value of resistance R_L in Fig. 3.397 for maximum power transfer and calculate the maximum power.*

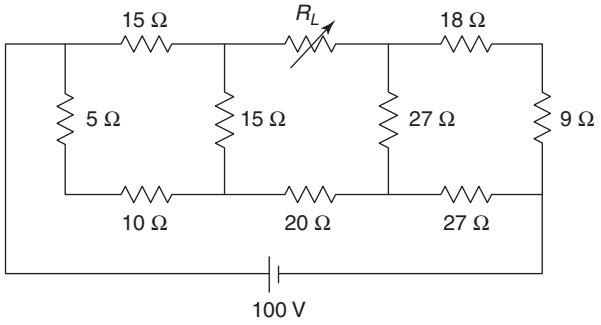


Fig. 3.397

Solution

Step I Calculation of V_{Th} (Fig. 3.398)

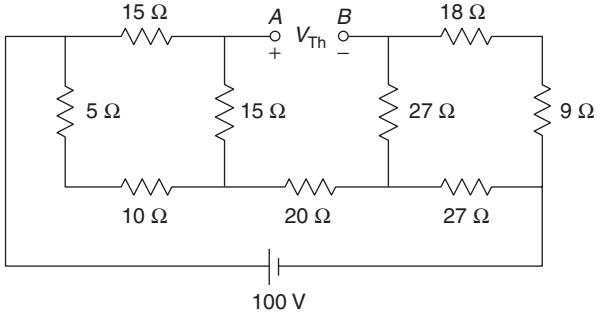


Fig. 3.398

By star-delta transformation (Fig. 3.399),

$$I = \frac{100}{5+5+20+9+9} = 2.08 \text{ A}$$

Writing the V_{Th} equation,

$$100 - 5I - V_{Th} - 9I = 0$$

$$V_{Th} = 100 - 14I$$

$$= 100 - 14(2.08)$$

$$= 70.88 \text{ V}$$

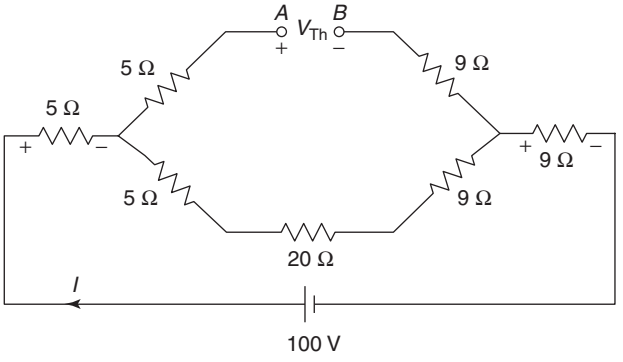


Fig. 3.399

Step II Calculation of R_{Th} (Fig. 3.400)

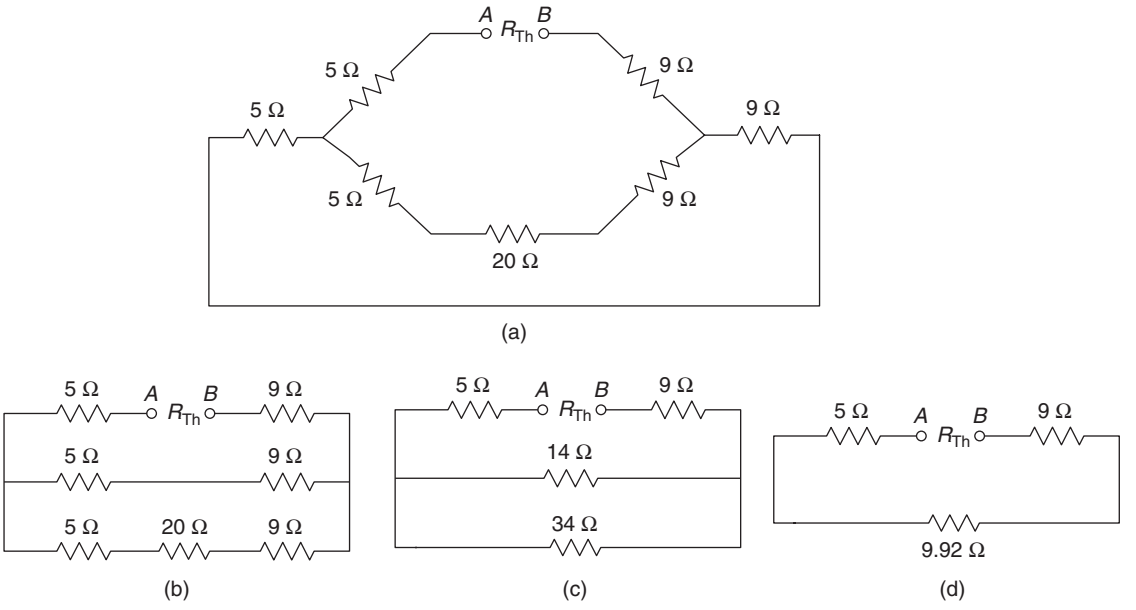


Fig. 3.400

$$R_{Th} = 23.92\ \Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{Th} = 23.92\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.401)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(70.88)^2}{4 \times 23.92} = 52.51\ \text{W}$$

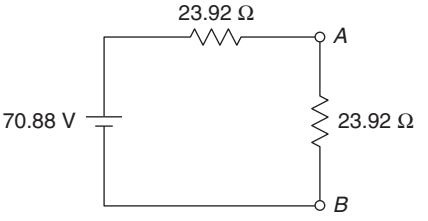


Fig. 3.401

Example 3.91 *For the value of resistance R_L in Fig. 3.402 for maximum power transfer and calculate the maximum power.*

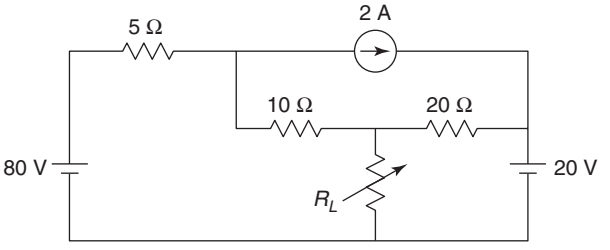


Fig. 3.402

Solution

Step I Calculation of V_{Th} (Fig. 3.403)

Applying KVL to Mesh 1,

$$80 - 5I_1 - 10(I_1 - I_2) - 20(I_1 - I_2) - 20 = 0$$

$$35I_1 - 30I_2 = 60 \quad \ldots(i)$$

Writing the current equation for Mesh 2,

$$I_2 = 2 \quad \ldots(ii)$$

Solving Eqs (i) and (ii),

$$I_1 = 3.43 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 20\,(I_1 - I_2) - 20 = 0$$

$$V_{Th} = 20(3.43 - 2) + 20 = 48.6 \text{ V}$$

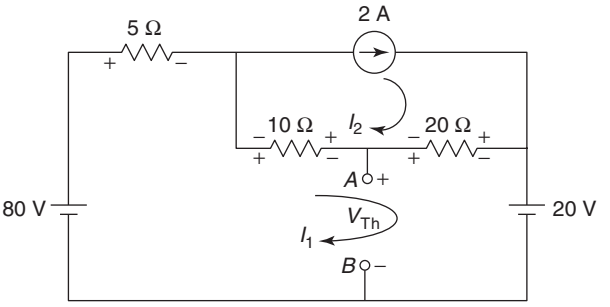


Fig. 3.403

Step II Calculation of R_{Th} (Fig. 3.404)

$$R_{Th} = 15 \,||\, 20 = 8.57\ \Omega$$

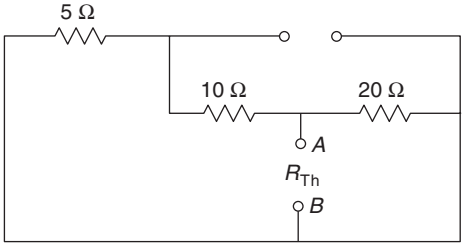


Fig. 3.404

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 8.57\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.405)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(48.6)^2}{4 \times 8.57} = 68.9 \text{ W}$$

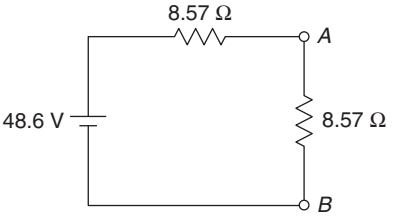


Fig. 3.405

Example 3.92 *For the value of resistance R_L in Fig. 3.406 for maximum power transfer and calculate the maximum power.*

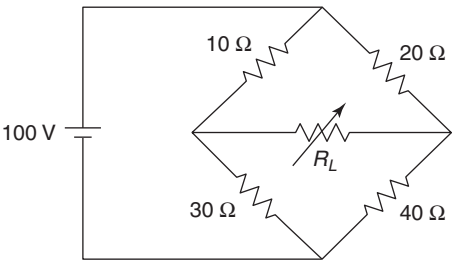


Fig. 3.406

Solution

Step I Calculation of V_{Th} (Fig. 3.407)

$$I_1 = \frac{100}{10+30} = 2.5 \text{ A}$$

$$I_2 = \frac{100}{20+40} = 1.66 \text{ A}$$

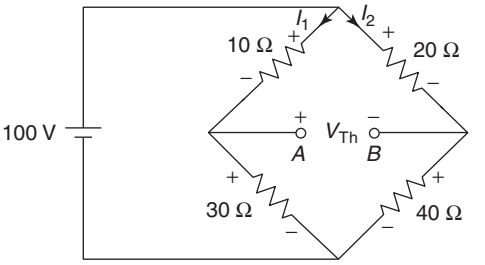


Fig. 3.407

Writing the V_{Th} equation,

$$V_{Th} + 10\, I_1 - 20 I_2 = 0$$

$$V_{Th} = 20 I_2 - 10 I_1 = 20(1.66) - 10(2.5) = 8.2 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.408)

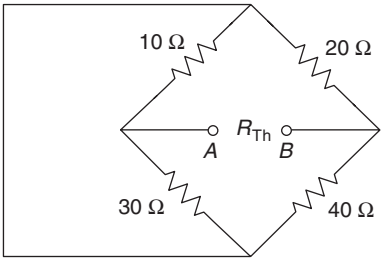


Fig. 3.408

Redrawing the network (Fig. 3.409),

$$R_{Th} = (10 \,||\, 30) + (20 \,||\, 40) = 20.83\ \Omega$$

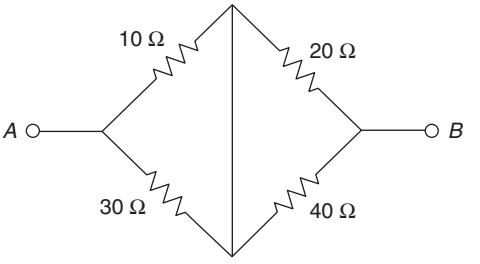


Fig. 3.409

Step III Value of R_L

For maximum power transfer,

$$R_L = R_{Th} = 20.83\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.410)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(8.2)^2}{4 \times 20.83} = 0.81 \text{ W}$$

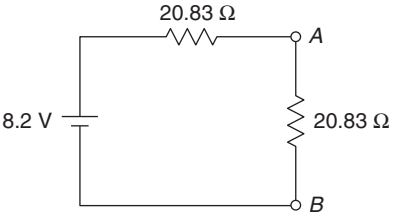


Fig. 3.410

Example 3.93 *For the value of resistance R_L in Fig. 3.411 for maximum power transfer and calculate the maximum power.*

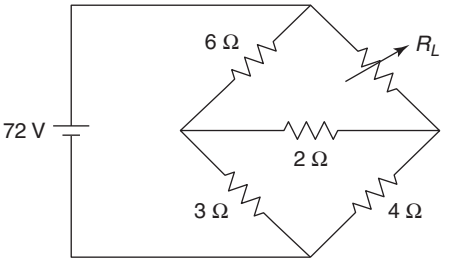


Fig. 3.411

Solution

Step I Calculation of V_{Th} (Fig. 3.412)

Applying KVL to Mesh 1,

$$72 - 6I_1 - 3(I_1 - I_2) = 0$$
$$9I_1 - 3I_2 = 72 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 4I_2 = 0$$
$$-3I_1 + 9I_2 = 0 \quad \ldots(ii)$$

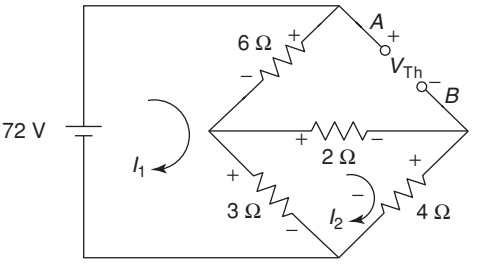


Fig. 3.412

Solving Eqs (i) and (ii),

$$I_1 = 9 \text{ A}$$
$$I_2 = 3 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 6I_1 - 2I_2 = 0$$
$$V_{Th} = 6I_1 + 2I_2 = 6(9) + 2(3) = 60 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.413)

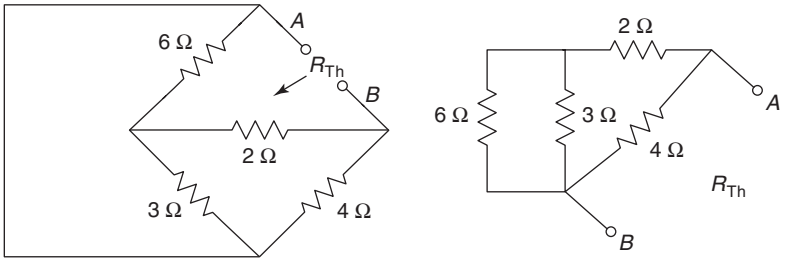


Fig. 3.413

$$R_{Th} = [(6 \parallel 3) + 2] \parallel 4 = 2 \ \Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 2\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.414)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(60)^2}{4 \times 2} = 450\ \text{W}$$

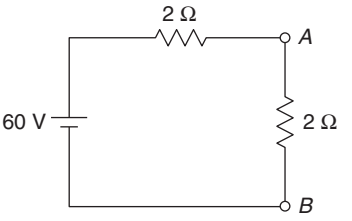


Fig. 3.414

Example 3.94 *For the network shown in Fig. 3.415 find the value of the resistance R_L for maximum power transfer and calculate maximum power.*

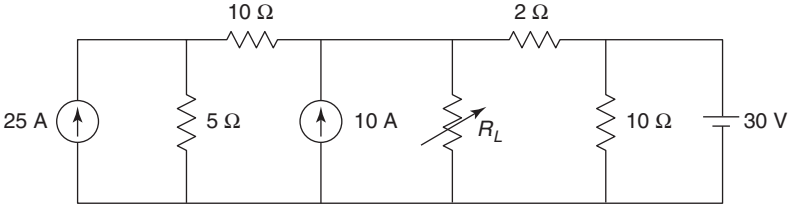


Fig. 3.415

Solution

Step I Calculation of V_{Th} (Fig. 3.416)

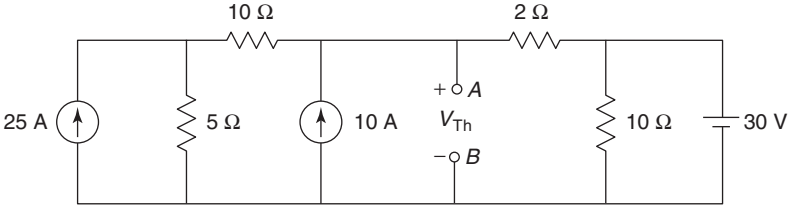


Fig. 3.416

By source transformation, the current source of 25 A and the 5 Ω resistor is converted into an equivalent voltage source of 125 V and a series resistor of 5 Ω. Also the voltage source of 30 V is connected across the 10 Ω resistor. Hence, the 10 Ω resistor becomes redundant (Fig. 3.417).

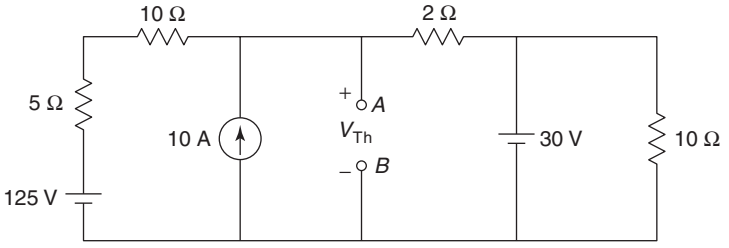


Fig. 3.417

Applying KCL at the node,

$$\frac{V_{Th}-125}{15}-10+\frac{V_{Th}-30}{2}=0$$

$$V_{Th}=58.81\text{ V}$$

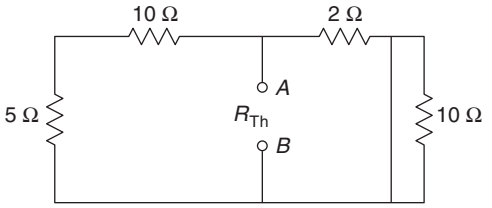


Fig. 3.418

Step II Calculation of R_{Th} (Fig. 3.418)

$$R_{Th}=15\,\|\,2=1.76\ \Omega$$

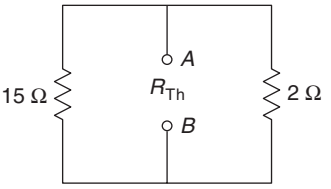


Fig. 3.419

Step III Value of R_L
For maximum power transfer

$$R_L=R_{Th}=1.76\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.420)

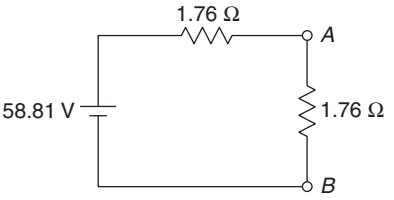


Fig. 3.420

$$P_{max}=\frac{V_{Th}^2}{4R_{Th}}=\frac{(58.81)^2}{4\times1.76}=491.28\text{ W}$$

Example 3.95 *For the network shown in Fig. 3.421, find the value of the resistance R_L for maximum power transfer and calculate maximum power.*

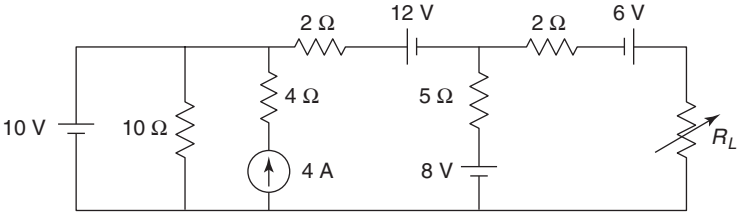


Fig. 3.421

Solution

Step I Calculation of V_{Th} (Fig. 3.422)

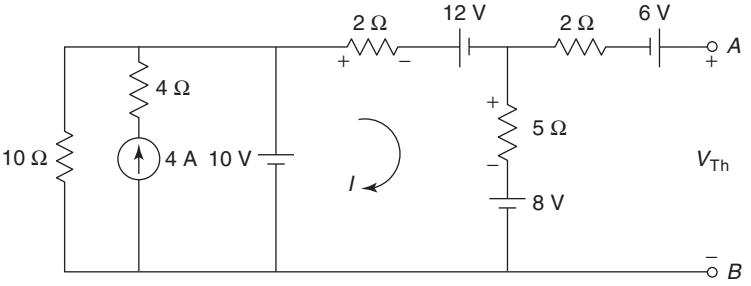


Fig. 3.422

Applying KVL to the outer path,

$$10 - 2I - 12 - 5I - 8 = 0$$

$$I = -\frac{10}{7} = -1.43 \text{ A}$$

Writing the V_{Th} equation,

$$8 + 5I + 6 - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 8 + 6 + 5I = 8 + 6 + 5(-1.43) = 6.85 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.423)

$$R_{\text{Th}} = (2 \,||\, 5) + 2$$
$$= 3.43\ \Omega$$

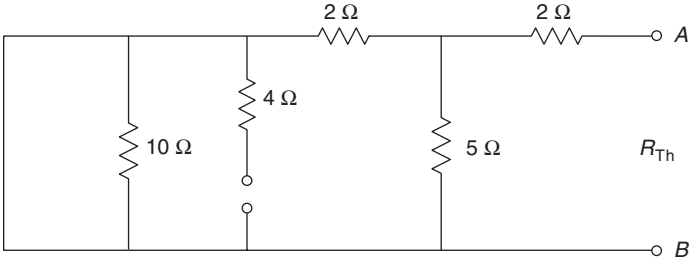


Fig. 3.423

Step III Value of R_L

For maximum power transfer,

$$R_L = R_{\text{Th}} = 3.43\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.424)

$$P_{\text{max}} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(6.85)^2}{4 \times 3.43} = 3.42 \text{ W}$$

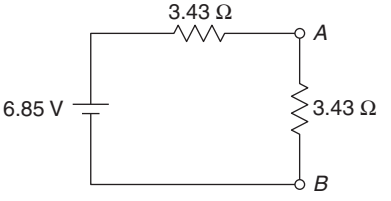


Fig. 3.424

EXAMPLES WITH DEPENDENT SOURCES

Example 3.96 *For the network shown in Fig. 3.425, find the value of R_L for maximum power transfer. Also, calculate maximum power.*

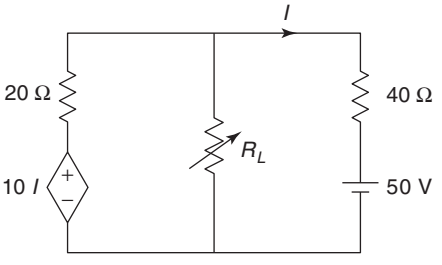


Fig. 3.425

Solution

Step I Calculation of V_{Th} (Fig. 3.426)

Applying KVL to the mesh,

$$10I - 20I - 40I - 50 = 0$$

$$I = -1 \text{ A}$$

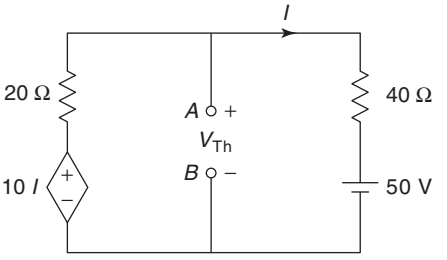


Fig. 3.426

Writing the V_{Th} equation,

$$V_{Th} - 40I - 50 = 0$$

$$V_{Th} - 40(-1) - 50 = 0$$

$$V_{Th} = 10 \text{ V}$$

Step II Calculation of I_N (Fig. 3.427)

From Fig. 3.427,

$$I = I_2 \qquad \ldots(i)$$

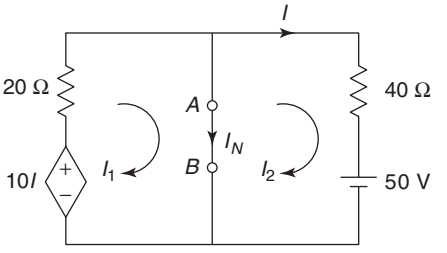


Fig. 3.427

Applying KVL to Mesh 1,

$$10I - 20I_1 = 0$$

$$10I_2 - 20I_1 = 0 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-40I_2 - 50 = 0$$

$$I_2 = -1.25 \text{ A} \qquad \ldots(iii)$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = -0.625 \text{ A}$$

$$I_N = I_1 - I_2 = -0.625 + 1.25 = 0.625 \text{ A}$$

Step III Calculation of R_N

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{10}{0.625} = 16 \ \Omega$$

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 16 \ \Omega$$

Step V Calculation of P_{max} (Fig. 3.428)

$$P_{max} = \frac{V_{Th}}{4R_{Th}} = \frac{(10)^2}{4 \times 16} = 1.56 \text{ W}$$

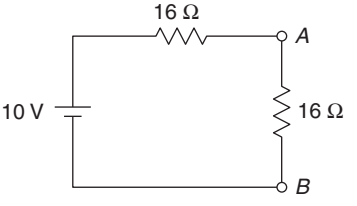


Fig. 3.428

Example 3.97 *For the network shown in Fig. 3.429, calculate the maximum power that may be dissipated in the load resistor R_L.*

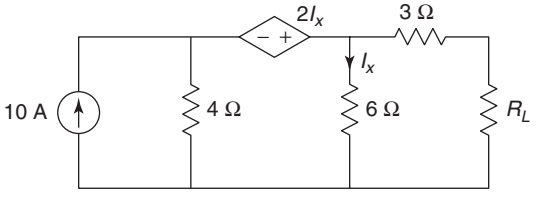


Fig. 3.429

Solution

Step I Calculation of V_{Th} (Fig. 3.430)

From Fig. 3.430,

$$I_x = I_2 \qquad \ldots(i)$$

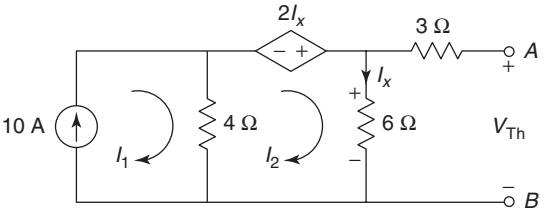


Fig. 3.430

For Mesh 1,

$$I_1 = 10 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6I_2 = 0$$
$$-4I_2 + 4I_1 + 2I_2 - 6I_2 = 0$$
$$4I_1 - 8I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 5 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - 0 - V_{Th} = 0$$
$$V_{Th} = 6I_2 = 6(5) = 30 \text{ V}$$

Step II Calculation of I_N (Fig. 3.431)

From Fig. 3.431,

$$I_x = I_2 - I_3 \qquad \ldots(i)$$

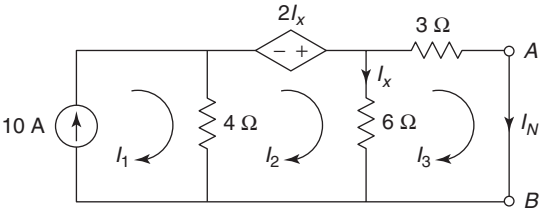


Fig. 3.431

For Mesh 1,

$$I_1 = 10 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6(I_2 - I_3) = 0$$
$$-4I_2 + 4I_1 + 2(I_2 - I_3) - 6I_2 + 6I_3 = 0$$
$$4I_1 - 8I_2 + 4I_3 = 0 \qquad \ldots(iii)$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 3I_3 = 0$$
$$6I_2 - 9I_3 = 0 \qquad \ldots(iv)$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 10 \text{ A}$$
$$I_2 = 7.5 \text{ A}$$
$$I_3 = 5 \text{ A}$$
$$I_N = I_3 = 5 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6\ \Omega$$

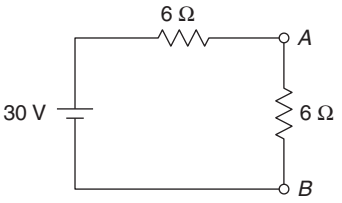


Fig. 3.432

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 6\ \Omega$$

Step V Calculation of P_{max}

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(30)^2}{4 \times 16} = 37.5\ \text{W}$$

Example 3.98 *For the network shown in Fig. 3.433, find the value of R_L for maximum power transfer. Also, find maximum power.*

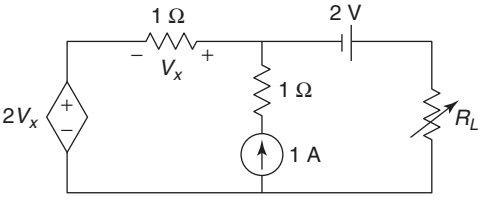


Fig. 3.433

Solution

Step I Calculation of V_{Th} (Fig. 3.434)

From Fig. 3.434,

$$V_x = -1I = -I \qquad \ldots(i)$$

For Mesh 1,

$$I = -1 \qquad \ldots(ii)$$

$$V_x = 1\ \text{V}$$

Writing the V_{Th} equation,

$$2V_x - 1I + 2 - V_{Th} = 0$$

$$2(1) - (-1) + 2 - V_{Th} = 0$$

$$V_{Th} = 5\ \text{V}$$

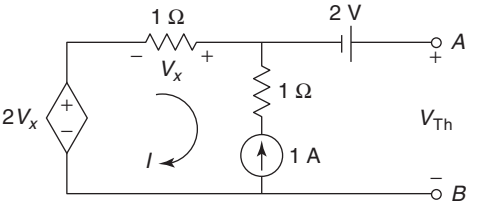


Fig. 3.434

Step II Calculation of I_N (Fig. 3.435)

From Fig. 3.435,

$$V_x = -1I_1 = -I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots(ii)$$

Applying KVL to the outer path of the supermesh,

$$2V_x - 1I_1 + 2 = 0$$

$$2(-I_1) - I_1 + 2 = 0$$

$$3I_1 = 0 \qquad \ldots(iii)$$

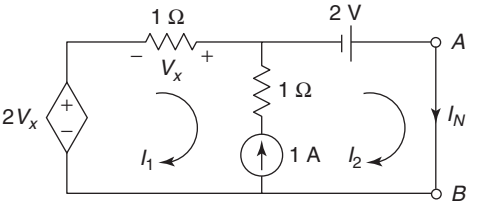


Fig. 3.435

Solving Eqs (ii) and (iii),

$$I_1 = 0.67 \text{ A}$$
$$I_2 = 1.67 \text{ A}$$
$$I_N = I_2 = 1.67 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{5}{1.67} = 3\ \Omega$$

Step IV Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 3\ \Omega$$

Step V Calculation of P_{max} (Fig. 3.436)

$$P_{\text{max}} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(5)^2}{4 \times 3} = 2.08 \text{ W}$$

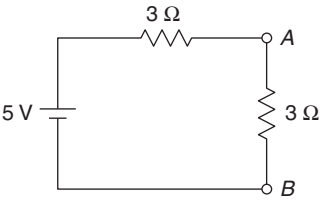


Fig. 3.436

Example 3.99 *What will be the value of R_L in Fig. 3.437 to get maximum power delivered to it? What is the value of this power?*

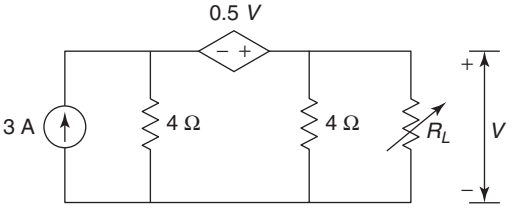


Fig. 3.437

Solution

Step I Calculation of V_{Th} (Fig. 3.438)
By source transformation,
From Fig. 3.438,

$$V_{\text{Th}} = 4I$$

Applying KVL to the mesh,

$$12 - 4I + 0.5\,V_{\text{Th}} - 4I = 0$$
$$12 - V_{\text{Th}} + 0.5\,V_{\text{Th}} - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 8 \text{ V}$$

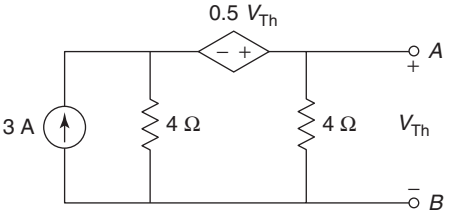


Fig. 3.438

Step II Calculation of I_N (Fig. 3.439)
If two terminals A and B are shorted, the 4 Ω resistor gets shorted.

$$V = 0$$

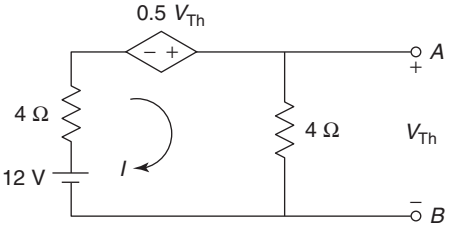


Fig. 3.439

Dependent source 0.5 V depends on the controlling variable V. When $V = 0$, the dependent source vanishes, ie. 0.5 $V = 0$ as shown in Fig. 3.441.

$$I_N = \frac{12}{4} = 3 \text{ A}$$

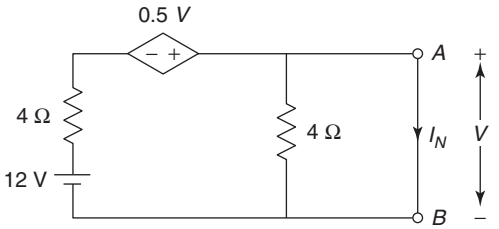


Fig. 3.440

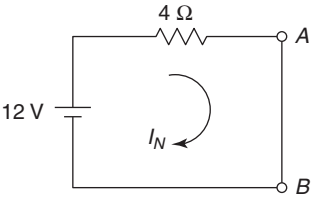


Fig. 3.441

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{8}{3} = 2.67\ \Omega$$

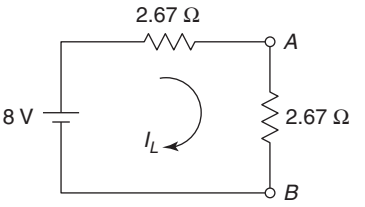


Fig. 3.442

Step IV *Calculation of* R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 2.67\ \Omega$$

Step V Calculation of $P_{\max}$ (Fig. 3.442)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(8)^2}{4 \times 2.67} = 6 \text{ W}$$

3.6 MILLMAN'S THEOREM

It states that '*if there are n voltage sources* $V_1, V_2, \ldots, V_n$ *with internal resistances* $R_1, R_2, \ldots, R_n$ *respectively connected in parallel then these voltage sources can be replaced by a single voltage source* V_m *and a single series resistance* R_m, '(Fig. 3.443).

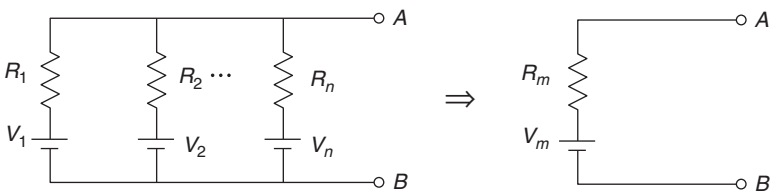


Fig. 3.443 *Millman's network*

where

$$V_m = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

and

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

Explanation By source transformation, each voltage source in series with a resistance can be converted to a current source in parallel with a resistance as shown in Fig. 3.444.

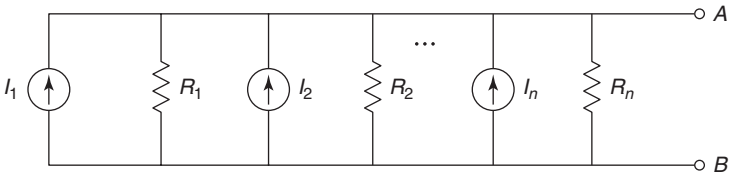


Fig. 3.444 *Equivalent network*

Let I_m be the resultant current of the parallel current sources and R_m be the equivalent resistance as shown in Fig. 3.445.

$$I_m = I_1 + I_2 + \ldots + I_n = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \ldots + \frac{V_n}{R_n} = V_1 G_1 + V_2 G_2 + \ldots V_n G_n$$

$$\frac{1}{R_m} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

$$G_m = G_1 + G_2 + \ldots + G_n$$

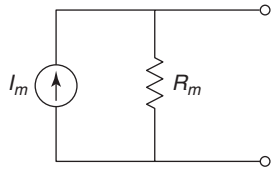


Fig. 3.445 *Equivalent network*

By source transformation, the parallel circuit can be converted into a series circuit as shown in Fig. 3.446.

$$V_m = I_m R_m = \frac{I_m}{G_m} = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

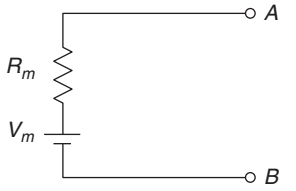


Fig. 3.446 *Millman's equivalent network*

Dual of Millman's Theorem

It states that '*if there are n current sources* $I_1, I_2, \ldots, I_n$ *with internal resistances* $R_1, R_2, \ldots, R_n$ *respectively, connected in series then these current sources can be replaced by a single current source* I_m *and a single parallel resistance* R_m ' (Fig. 3.447).

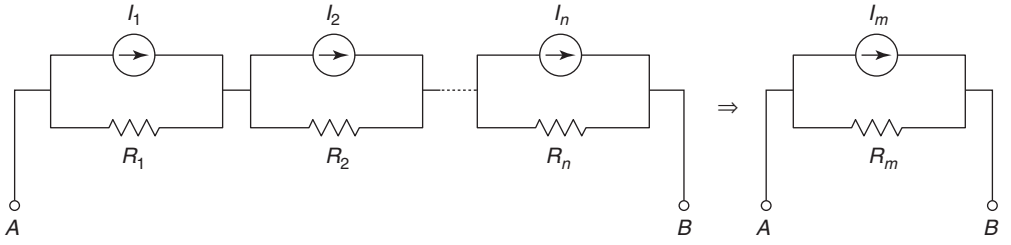


Fig. 3.447 *Millman's network*

where

$$I_m = \frac{I_1 R_1 + I_2 R_2 + \ldots + I_n R_n}{R_1 + R_2 + \ldots + R_n}$$

$$R_m = R_1 + R_2 + \ldots + R_n$$

Steps to be followed in Millman's Theorem

1. Remove the load resistance R_L.
2. Find Millman's voltage across points A and B.

$$V_m = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

3. Find the resistance R_m between points A and B.

$$R_m = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

4. Replace the network by a voltage source V_m in series with the resistance R_m.
5. Find the current through R_L using ohm's law.

$$I_L = \frac{V_m}{R_m + R_L}$$

Example 3.100 *Find Millman's equivalent for the left of the terminals A-B in Fig. 3.448.*

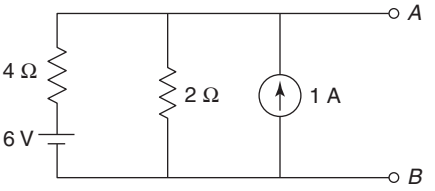


Fig. 3.448

Solution By source transformation, the network is redrawn as shown in Fig. 3.449.

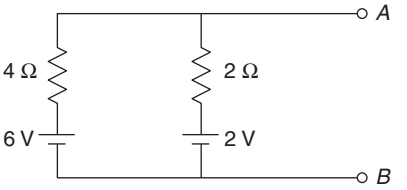


Fig. 3.449

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{6\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right)}{\frac{1}{4} + \frac{1}{2}} = 3.33 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{4} + \frac{1}{2}} = 1.33\ \Omega$$

Step III Millman's Equivalent Network (Fig. 3.450)

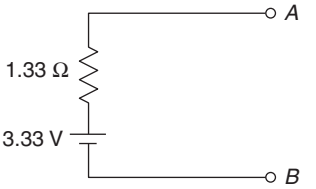


Fig. 3.450

Example 3.101

Find the current through the 10 Ω resistor in the network of Fig. 3.451.

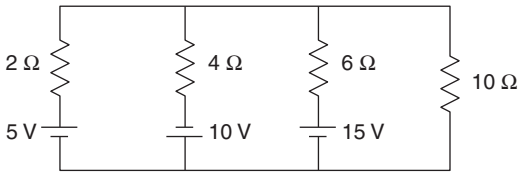


Fig. 3.451

Solution

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{5\left(\frac{1}{2}\right) - 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{6}\right)}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 2.73 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 1.09\ \Omega$$

Step III Calculation of I_L (Fig. 3.452)

$$I_L = \frac{2.73}{1.09 + 10} = 0.25 \text{ A}$$

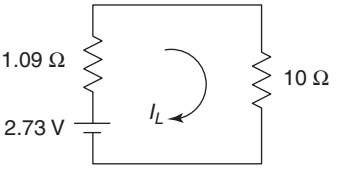


Fig. 3.452

Example 3.102

Find the current through the 10 Ω resistor in the network of Fig. 3.453.

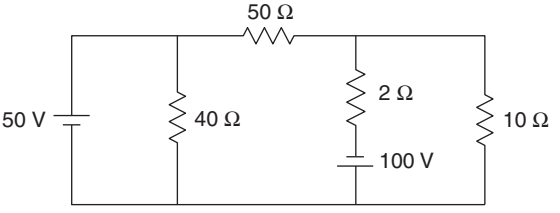


Fig. 3.453

Solution

Since the 40 Ω resistor is connected in parallel with the 50 V source, it becomes redundant. The network can be redrawn as shown in Fig. 3.454.

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{50\left(\frac{1}{50}\right) - 100\left(\frac{1}{20}\right)}{\frac{1}{50} + \frac{1}{20}} = -57.15 \text{ V}$$

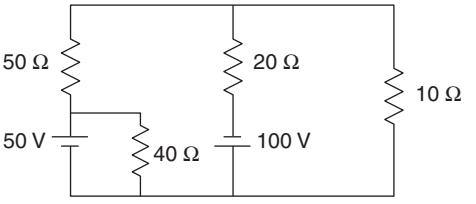


Fig. 3.454

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{50} + \frac{1}{20}} = 14.29\ \Omega$$

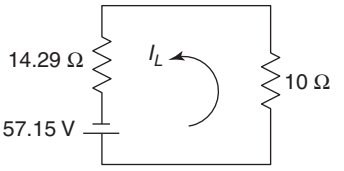


Fig. 3.455

Step III Calculation of I_L (Fig. 3.455)

$$I_L = \frac{57.15}{14.29 + 10} = 2.35\ \text{A}$$

Example 3.103 *Draw Millman's equivalent network across terminals AB in the network of Fig. 3.456.*

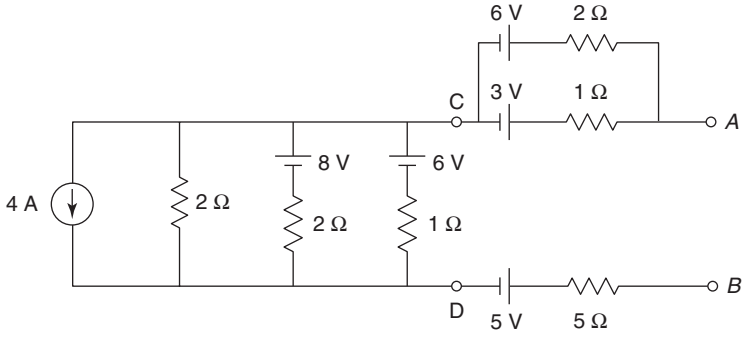


Fig. 3.456

Step I By source transformation, the network is redrawn as shown in Fig. 3.457.

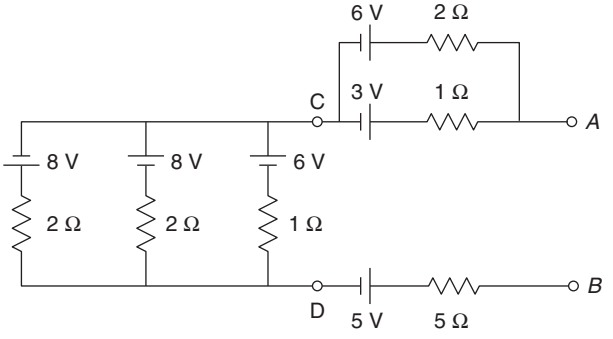


Fig. 3.457

Step II Applying Millman's theorem at terminals *CD,*

$$V_{m_1} = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{-8\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right) + 6(1)}{\frac{1}{2} + \frac{1}{2} + 1} = 3\ \text{V}$$

$$R_{m_1} = \frac{1}{G_{m_1}} = \frac{1}{G_1 + G_2 + G_3} = \frac{1}{\frac{1}{2} + \frac{1}{2} + 1} = 0.5\ \Omega$$

Step III Applying Millman's theorem at terminals *CA*,

$$V_{m_2} = \frac{V_4 G_4 + V_5 G_5}{G_4 + G_5} = \frac{6\left(\frac{1}{2}\right) + 3(1)}{\frac{1}{2} + 1} = 4\ \text{V}$$

$$R_{m_2} = \frac{1}{G_{m_2}} = \frac{1}{G_4 + G_4} = \frac{1}{\frac{1}{2} + 1} = 0.67\ \Omega$$

Step IV Millman's Equivalent Network (Fig. 3.458) Simplifying Fig. 3.458 further, the Millman's equivalent network is shown in Fig. 3.459.

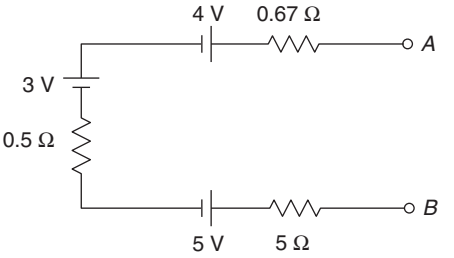


Fig. 3.458

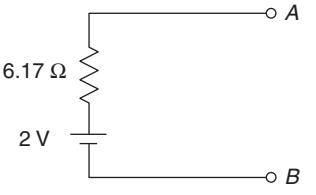


Fig. 3.459

Exercises

Superposition Theorem

3.1 Find the current through the 10 Ω resistor in Fig. 3.460.

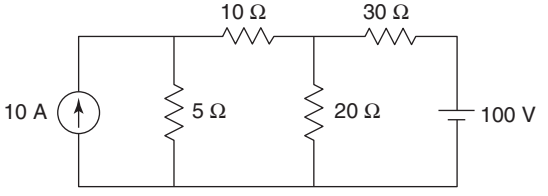


Fig. 3.460

[0.37 A]

3.2 Find the current through the 8 Ω resistor in Fig. 3.461.

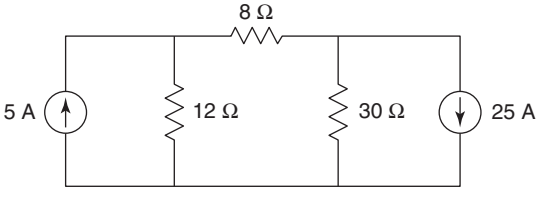


Fig. 3.461

[16.2 A]

3.3 Find the potential across the 3 Ω resistor in Fig. 3.462.

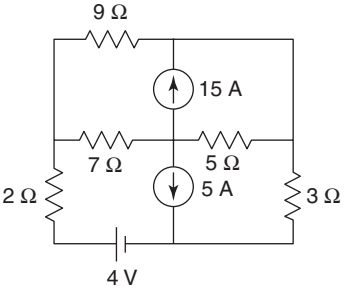


Fig. 3.462

[3.3 V]

3.4 Calculate the current through the 10 Ω resistor in Fig. 3.463.

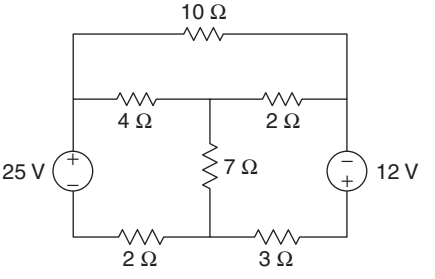


Fig. 3.463

[1.62 A]

3.5 Find the current through the 1Ω resistor in Fig. 3.464.

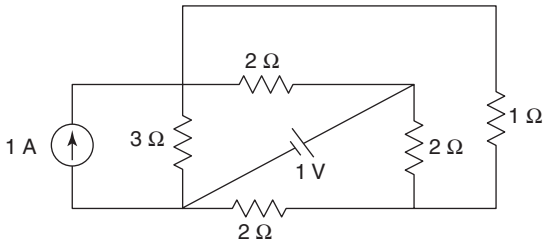


Fig. 3.464

[0.41 A]

3.6 Find the current through the 4 Ω resistor in Fig. 3.465.

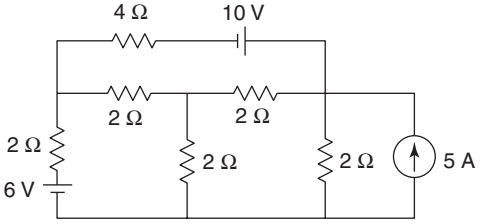


Fig. 3.465

[1.33 A]

3.7 Find the current I_x in Fig. 3.466.

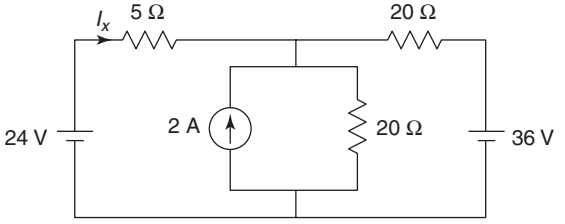


Fig. 3.466

[−1.143 A]

3.8 Find the voltage V_x in Fig. 3.467.

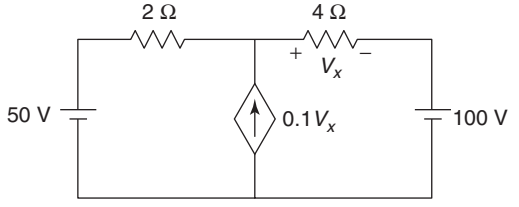


Fig. 3.467

[−38.5 V]

3.9 Determine the voltages V_1 and V_2 in Fig. 3.468.

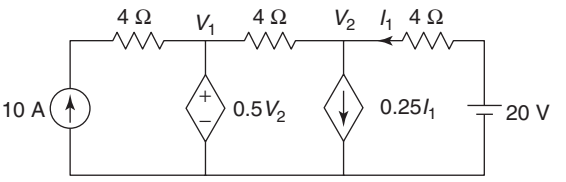


Fig. 3.468

[6 V,12 V]

3.10 Find the voltage V_x in Fig. 3.469.

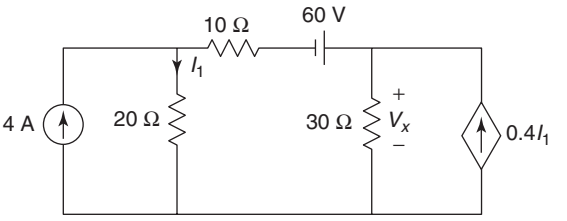


Fig. 3.469

[7.5 V]

Thevenin's Theorem

3.11 Find the current through the 5 Ω resistor in Fig. 3.470.

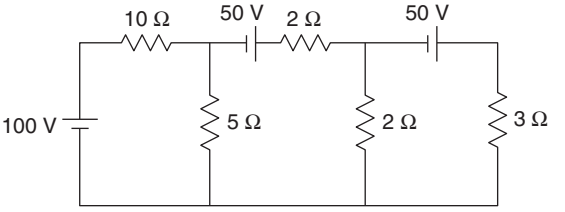


Fig. 3.470

[3.87 A]

3.12 Find the current through the 6 Ω resistor in Fig. 3.471.

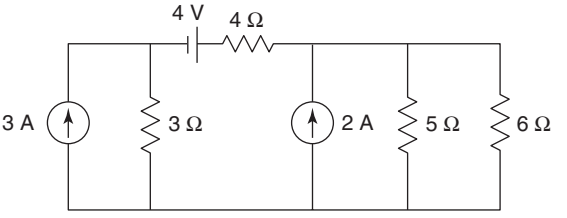


Fig. 3.471

[1.26 A]

3.13 Find the current through the 6 Ω resistor in Fig. 3.472.

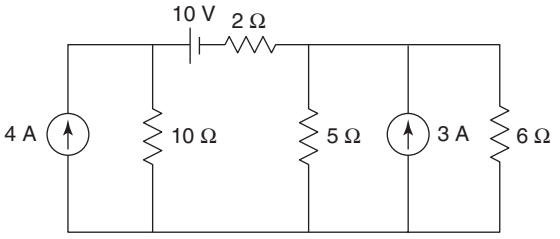


Fig. 3.472

[2.04 A]

3.14 Find the current through the 2 Ω resistor connected between terminals *A* and *B* in Fig. 3.473.

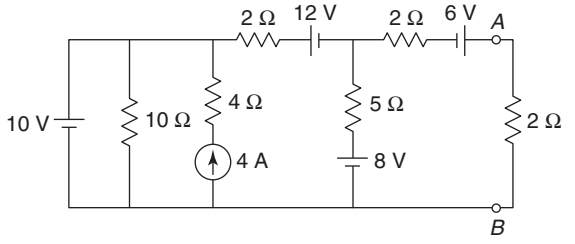


Fig. 3.473

[1.26 A]

3.15 Find the current through the 5 Ω resistor in Fig. 3.474.

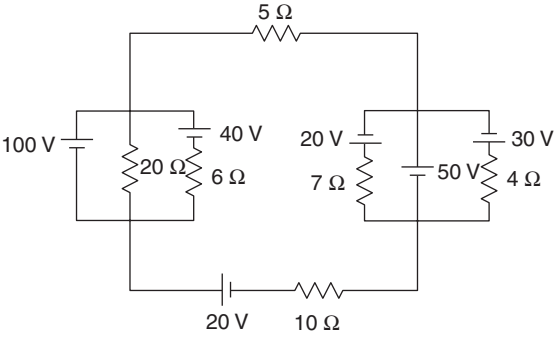


Fig. 3.474

[4.67 A]

3.16 Find the current through the 20 Ω resistor in Fig. 3.475.

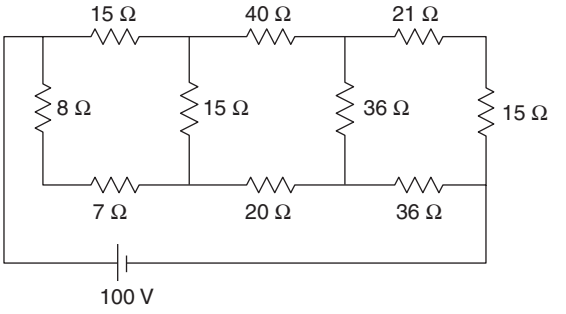


Fig. 3.475

[1.54 A]

3.17 Calculate the current through the 10 Ω resistor in Fig. 3.476.

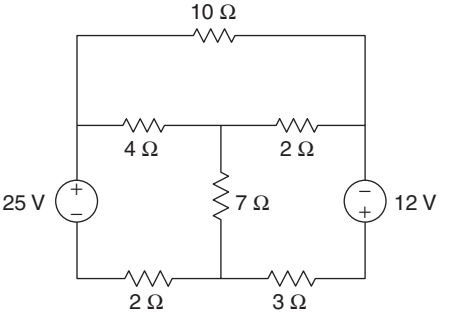


Fig. 3.476

[1.62 A]

3.18 Determine Thevenin's equivalent network for figures 3.477 to 3.480 shown below.

(i)

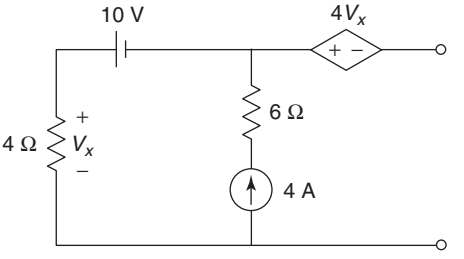


Fig. 3.477

[–58 V,12 Ω]

(ii)

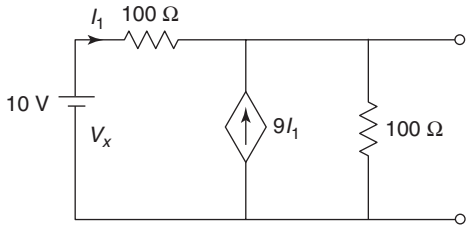


Fig. 3.478

[9.09 V, 9.09 Ω]

(iii)

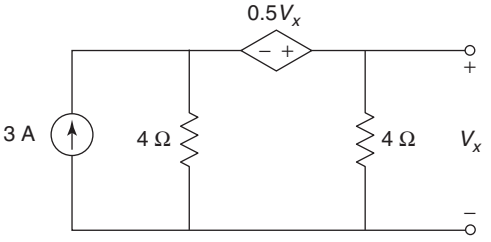


Fig. 3.479

[8 V, 2.66 Ω]

(iv)

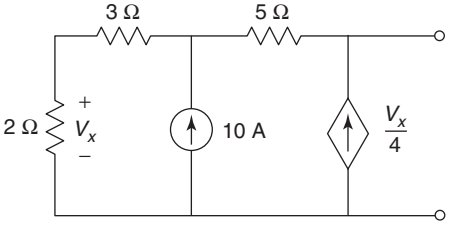


Fig. 3.480

[150 V, 20 Ω]

3.19 Find the current I_x in Fig. 3.481.

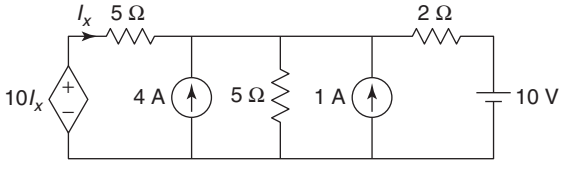


Fig. 3.481

[4 A]

3.20 Find the current in the 24 Ω resistor in Fig. 3.482.

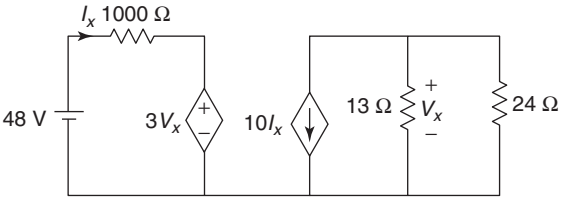


Fig. 3.482

[0.225 A]

Norton's Theorem

3.21 Find the current through the 10 Ω resistor in Fig. 3.483.

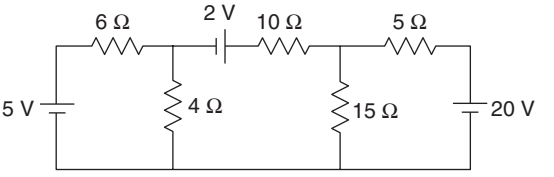


Fig. 3.483

[0.68 A]

3.22 Find the current through the 20 Ω resistor in Fig. 3.484.

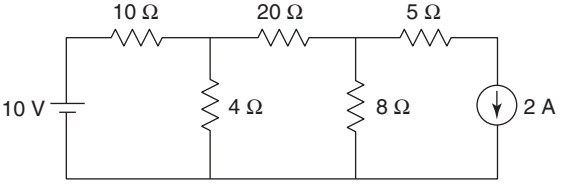


Fig. 3.484

[0.61 A]

3.23 Find the current through the 2 Ω resistor in Fig. 3.485.

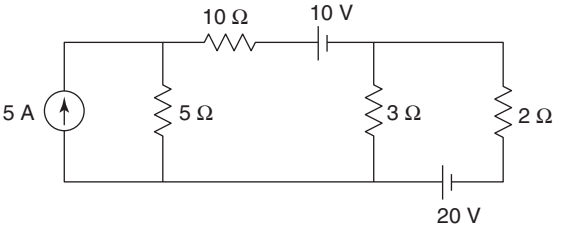


Fig. 3.485

[5 A]

3.24 Find the current through the 5 Ω resistor in Fig. 3.486.

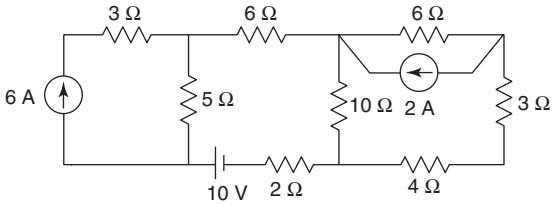


Fig. 3.486

[4.13 A]

3.25 Find Norton's equivalent circuit for the portion of network shown in Fig. 3.487 to the left of *ab*. Hence obtain the current in the 10 Ω resistor.

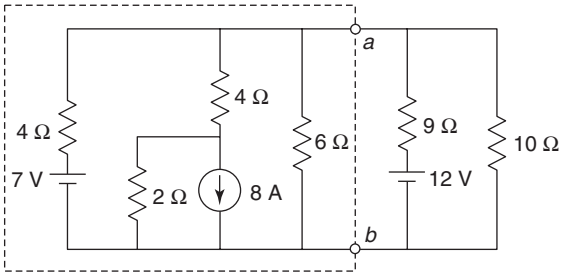


Fig. 3.487

[0.053 A]

3.26 Find Norton's equivalent network and hence find the current in the 10 Ω resistor in Fig. 3.488.

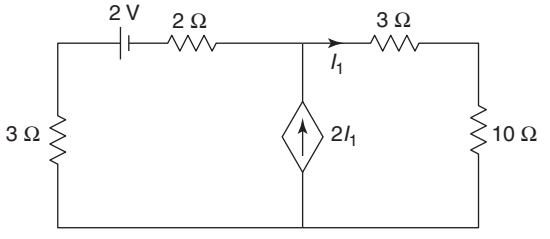


Fig. 3.488

[0.25 A]

3.27 Find Norton's equivalent network In Fig. 3.489.

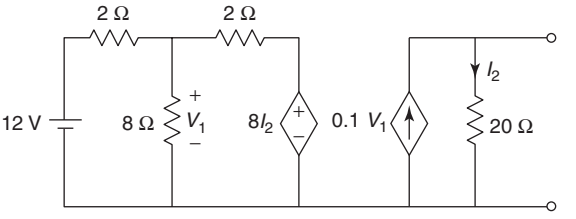


Fig. 3.489

[0.533 A, 31 Ω]

Maximum Power Transfer Theorem

3.28 Find the value of the resistance R_L in Fig. 3.490 for maximum power transfer and calculate the maximum power.

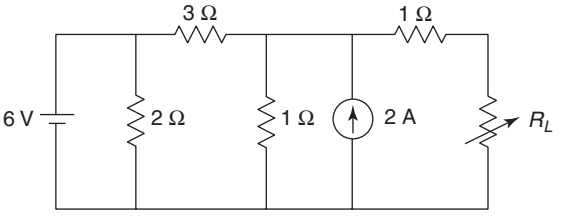


Fig. 3.490

[1.75 Ω, 1.29 W]

3.29 Find the value of the resistance R_L in Fig. 3.491 for maximum power transfer and calculate the maximum power.

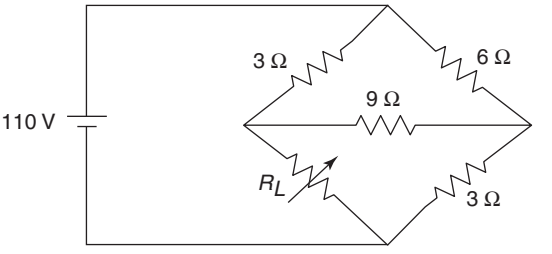


Fig. 3.491

[2.36 Ω, 940 W]

3.30 Find the value of the resistance R_L in Fig. 3.492 for maximum power transfer and calculate the maximum power.

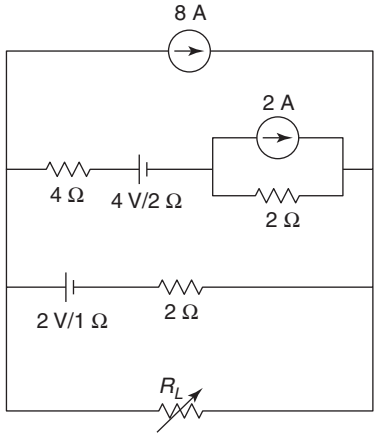


Fig. 3.492

$[2.18\ \Omega, 29.35\ \text{W}]$

3.31 Find the value of the resistance R_L in Fig. 3.493 for maximum power transfer and calculate the maximum power.

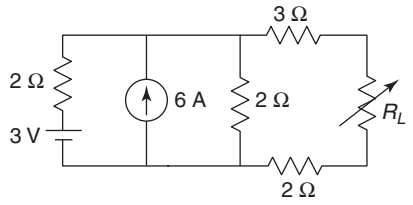


Fig. 3.493

$[3\ \Omega, 2.52\ \text{W}]$

3.32 Find the value of the resistance R_L in Fig. 3.494 for maximum power transfer and calculate the maximum power.

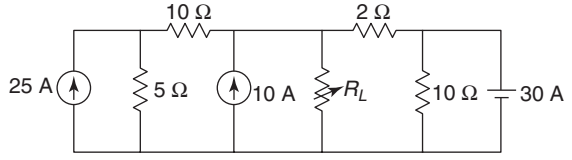


Fig. 3.494

$[1.76\ \Omega, 490.187\ \text{W}]$

Objective-Type Questions

3.1 The value of the resistance R connected across the terminals A and B in Fig. 3.495, which will absorb the maximum power is

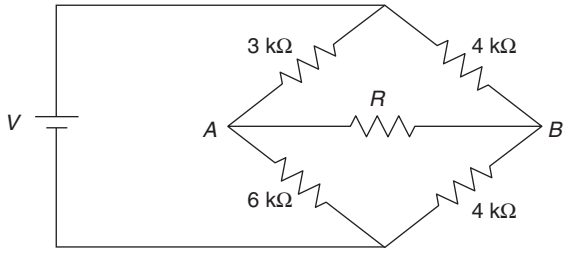


Fig. 3.495

(a) 4 kΩ (b) 4.11 kΩ
(c) 8 kΩ (d) 9 k Ω

3.2 Superposition theorem is not applicable to networks containing

(a) nonlinear elements
(b) dependent voltage source
(c) dependent current source
(d) transformers

3.3 The value of R required for maximum power transfer in the network shown in Fig. 3.496 is

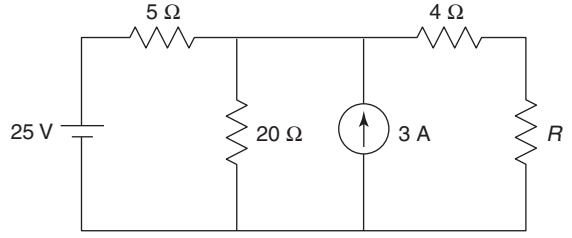


Fig. 3.496

(a) 2 Ω (b) 4 Ω (c) 8 Ω (d) 16 Ω

3.4 In the network of Fig. 3.497, the maximum power is delivered to R_L if its value is

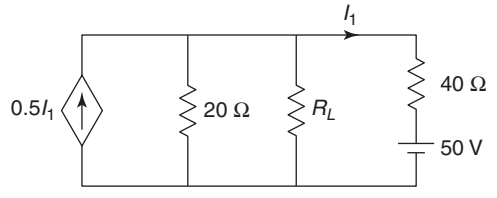


Fig. 3.497

(a) 16 Ω (b) $\frac{40}{3}$ Ω

(c) 60 Ω (d) 20 Ω

3.5 The maximum power that can be transferred to the load R_L from the voltage source in Fig. 3.498 is

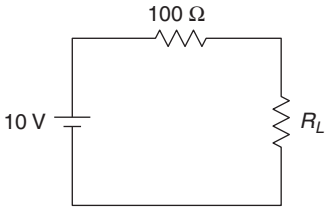


Fig. 3.498

(a) 1 W (b) 10 W

(c) 0.25 W (d) 0.5 W

3.6 For the circuit shown in Fig. 3.499, Thevenin's voltage and Thevenin's equivalent resistance at terminals *a*-*b* is

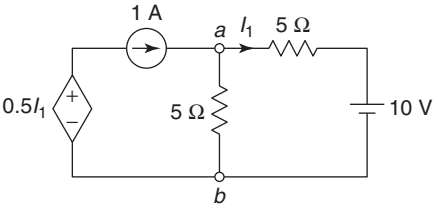


Fig. 3.499

(a) 5 V and 2 Ω (b) 7.5 V and 2.5 Ω

(c) 4 V and 2 Ω (d) 3 V and 2.5 Ω

3.7 The value of R_L in Fig. 3.500 for maximum power transfer is

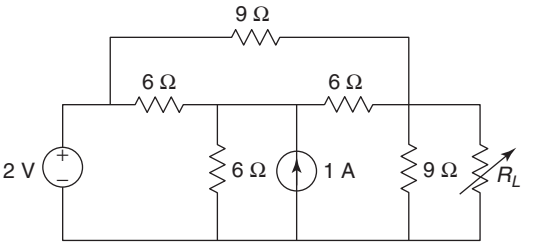


Fig. 3.500

(a) 3 Ω (b) 1.125 Ω

(c) 4.1785 Ω (d) none of these

Answers to Objective-Type Questions

3.1 (a) 3.2 (a) 3.3 (c) 3.4 (a) 3.5 (c) 3.6 (b) 3.7 (a)

4

Coupled Circuits

4.1 INTRODUCTION

Two circuits are said to be coupled circuits when energy transfer takes place from one circuit to the other without having any electrical connection between them. Such coupled circuits are frequently used in network analysis and synthesis. Common examples of coupled circuits are transformer, gyrator, etc. In this chapter, we will discuss self and mutual inductance, magnetically coupled circuits, dot conventions and tuned circuits.

4.2 SELF-INDUCTANCE

Consider a coil of N turns carrying a current i as shown in Fig. 4.1.

When current flows through the coil, a flux ϕ is produced in the coil. The flux produced by the coil links with the coil itself. If the current flowing through the coil changes, the flux linking the coil also changes. Hence, an emf is induced in the coil. This is known as self-induced emf. The direction of this emf is given by *Lenz's law*.

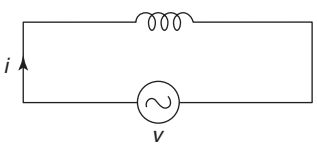


Fig. 4.1 *Coil carrying current*

We know that

$$\phi \propto i$$

$$\frac{\phi}{i} = k, \text{ a constant}$$

$$\phi = k\, i$$

Hence, rate of change of flux = $k \times$ rate of change of current

$$\frac{d\phi}{dt} = k\frac{di}{dt}$$

According to Faraday's laws of electromagnetic induction, a self-induced emf can be expressed as

$$v = -N\frac{d\phi}{dt} = -Nk\frac{di}{dt} = -N\frac{\phi}{i}\frac{di}{dt} = -L\frac{di}{dt}$$

where $L = \dfrac{N\phi}{i}$ and is called coefficient of self-inductance.

The property of a coil that opposes any change in the current flowing through it is called self-inductance or inductance of the coil. If the current in the coil is increasing, the self-induced emf is set up in such a direction so

as to oppose the rise in current, i.e., the direction of self-induced emf is opposite to that of the applied voltage. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage. Self-inductance does not prevent the current from changing, it serves only to delay the change.

4.3 MUTUAL INDUCTANCE

If the flux produced by one coil links with the other coil, placed closed to the first coil, an emf is induced in the second coil due to change in the flux produced by the first coil. This is known as mutually induced emf.

Consider two coils 1 and 2 placed adjacent to each other as shown in Fig. 4.2. Let Coil 1 has N_1 turns while Coil 2 has N_2 turns.

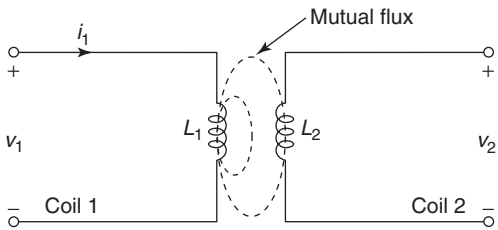


Fig. 4.2 *Two adjacent coils*

If a current i_1 flows in Coil 1, flux is produced and a part of this flux links Coil 2. The emf induced in Coil 2 is called mutually induced emf.

We know that

$$\phi_2 \propto i_1$$

$$\frac{\phi_2}{i_1} = k, \text{ a constant}$$

$$\phi_2 = k\, i_1$$

Hence, rate of change of flux = $k \times$ rate of change of current i_1

$$\frac{d\phi_2}{dt} = k\frac{di_1}{dt}$$

According to Faraday's law of electromagnetic induction, the induced emf is expressed as

$$v_2 = -N_2\frac{d\phi_2}{dt} = -N_2\, k\frac{di_1}{dt} = -N_2\frac{\phi_2}{i_1}\frac{di_1}{dt} = -M\frac{di_1}{dt}$$

where $M = \dfrac{N_2\,\phi_2}{i_1}$ and is called *coefficient of mutual inductance*.

4.4 COEFFICIENT OF COUPLING (k)

The coefficient of coupling (k) between coils is defined as fraction of magnetic flux produced by the current in one coil that links the other.

Consider two coils having number of turns N_1 and N_2 respectively. When a current i_1 is flowing in Coil 1 and is changing, an emf is induced in Coil 2.

$$M = \frac{N_2\,\phi_2}{i_1}$$

Let
$$k_1 = \frac{\phi_2}{\phi_1}$$
$$\phi_2 = k_1\ \phi_1$$
$$M = \frac{N_2\ k_1\ \phi_1}{i_1} \quad \ldots(4.1)$$

If the current i_2 is flowing in Coil 2 and is changing, an emf is induced in Coil 1,
$$M = \frac{N_1\ \phi_1}{i_2}$$

Let
$$k_2 = \frac{\phi_1}{\phi_2}$$
$$\phi_1 = k_2\ \phi_2$$
$$M = \frac{N_1\ k_2\ \phi_2}{i_2} \quad \ldots(4.2)$$

Multiplying Eqs (4.1) and (4.2),
$$M^2 = k_1 k_2 \times \frac{N_1\phi_1}{i_1} \times \frac{N_2\phi_2}{i_2} = k^2 L_1 L_2$$
$$M = k\sqrt{L_1 L_2}$$
where
$$k = \sqrt{k_1 k_2}$$

4.5 INDUCTANCES IN SERIES

1. Cumulative Coupling Figure 4.3 shows two coils 1 and 2 connected in series, so that currents through the two coils are in the same direction in order to produce flux in the same direction. Such a connection of two coils is known as *cumulative coupling*.

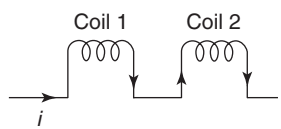


Fig. 4.3 *Cumulative coupling*

Let
L_1 = coefficient of self-inductance of Coil 1
L_2 = coefficient of self-inductance of Coil 2
M = coefficient of mutual inductance

If the current in the coil increases by *di* amperes in *dt* seconds then

Self-induced emf in Coil 1 $= -L_1 \frac{di}{dt}$

Self-induced emf in Coil 2 $= -L_2 \frac{di}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M \frac{di}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M \frac{di}{dt}$

Total induced emf
$$v = -(L_1 + L_2 + 2M)\frac{di}{dt} \quad \ldots(4.3)$$

If L is the equivalent inductance then total induced emf in that single coil would have been

$$v = -L\frac{di}{dt} \qquad \ldots(4.4)$$

Equating Eqs (4.3) and (4.4),

$$L = L_1 + L_2 + 2M$$

2. *Differential Coupling* Figure 4.4 shows the coils connected in series but the direction of current in Coil 2 is now opposite to that in 1. Such a connection of two coils is known as *differential coupling*.

Hence, total induced emf in coils 1 and 2.

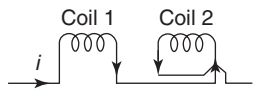


Fig. 4.4 *Differential coupling*

$$v = -L_1\frac{di}{dt} - L_2\frac{di}{dt} + 2M\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

Coils 1 and 2 connected in series can be considered as a single coil with equivalent inductance L. The induced emf in the equivalent single coil with same rate of change of current is given by,

$$v = -L\frac{di}{dt}$$

$$-L\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

$$L = L_1 + L_2 - 2M$$

4.6 INDUCTANCES IN PARALLEL

1. *Cumulative Coupling* Figure 4.5 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the same direction. Such a connection of two coils is known as cumulative coupling.

Let L_1 = coefficient of self-inductance of Coil 1

L_2 = coefficient of self-inductance of Coil 2

M = coefficient of mutual inductance

If the current in the coils changes by di amperes in dt seconds then

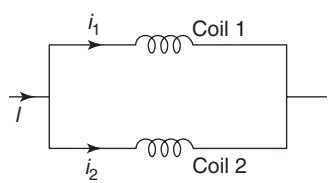


Fig. 4.5 *Cumulative coupling*

Self-induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2\dfrac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M\dfrac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M\dfrac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt} - M\dfrac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in both the coils must be equal.

$$-L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = -L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} - M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} - M \frac{di_2}{dt}$$

$$(L_1 - M) \frac{di_1}{dt} = (L_2 - M) \frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 - M}{L_1 - M} \right) \frac{di_2}{dt} \qquad ...(4.5)$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M} \right) \frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M} + 1 \right) \frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 - 2M}{L_1 - M} \right) \frac{di_2}{dt} \qquad ...(4.6)$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L \frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L \frac{di}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L} \left(L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right)$$

$$= \frac{1}{L} \left[L_1 \left(\frac{L_2 - M}{L_1 - M} \right) \frac{di_2}{dt} + M \frac{di_2}{dt} \right]$$

$$= \frac{1}{L} \left[L_1 \left(\frac{L_2 - M}{L_1 - M} \right) + M \right] \frac{di_2}{dt} \qquad ...(4.7)$$

Substituting Eq. (4.6) in Eq. (4.7),

$$\left(\frac{L_1 + L_2 - 2M}{L_1 - M} \right) \frac{di_2}{dt} = \frac{1}{L} \left[L_1 \left(\frac{L_2 - M}{L_1 - M} \right) + M \right] \frac{di_2}{dt}$$

$$L = \frac{L_1 \left(\frac{L_2 - M}{L_1 - M} \right) + M}{\frac{L_1 + L_2 - 2M}{L_1 - M}}$$

$$= \frac{L_1L_2 - L_1M + L_1M - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

2. *Differential Coupling* Figure 4.6 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the opposite direction. Such a connection of two coils is known as differential coupling.

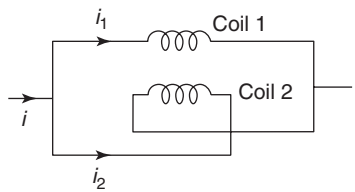


Fig. 4.6 *Differential coupling*

Self-induced emf in Coil 1 $= -L_1 \frac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2 \frac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= M \frac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= M \frac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in the coils must be equal.

$$-L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} + M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_2}{dt}$$

$$(L_1 + M)\frac{di_1}{dt} = (L_2 + M)\frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 + M}{L_1 + M}\right)\frac{di_2}{dt} \qquad ...(4.8)$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M}\right)\frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M} + 1\right)\frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 + 2M}{L_1 + M}\right)\frac{di_2}{dt} \qquad ...(4.9)$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L\frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L\frac{di}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}\left(L_1\frac{di_1}{dt} - M\frac{di_2}{dt}\right)$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right)\frac{di_2}{dt} - M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt} \qquad ...(4.10)$$

Substituting Eq. (4.9) in Eq. (4.10),

$$\left(\frac{L_1+L_2+2M}{L_1+M}\right)\frac{di_2}{dt} = \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt}$$

$$L = \frac{L_1\left(\frac{L_2+M}{L_1+M}\right) - M}{\frac{L_1+L_2+2M}{L_1+M}}$$

$$= \frac{L_1L_2 + L_1M - L_1M - M^2}{L_1+L_2+2M}$$

$$= \frac{L_1L_2 - M^2}{L_1+L_2+2M}$$

Example 4.1 *The combined inductance of two coils connected in series is 0.6 H or 0.1 H depending on relative directions of currents in the two coils. If one of the coils has a self-inductance of 0.2 H, find (a) mutual inductance, and (b) coefficient of coupling.*

Solution $L_1 = 0.2\,\text{H}, \quad L_{\text{diff}} = 0.1\,\text{H}, \quad L_{\text{cum}} = 0.6\,\text{H}$

(a) Mutual inductance

$$L_{\text{cum}} = L_1 + L_2 + 2M = 0.6 \qquad ...(i)$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.1 \qquad ...(ii)$$

Adding Eqs (i) and (ii),

$$2(L_1 + L_2) = 0.7$$

$$L_1 + L_2 = 0.35$$

$$L_2 = 0.35 - 0.2 = 0.15\,\text{H}$$

Subtracting Eqs (ii) from Eqs (i),

$$4M = 0.5$$
$$M = 0.125 \text{ H}$$

(b) Coefficient of coupling

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.125}{\sqrt{0.2 \times 0.15}} = 0.72$$

Example 4.2 *Two coils with a coefficient of coupling of 0.6 between them are connected in series so as to magnetise in (a) same direction, and (b) opposite direction. The total inductance in the same direction is 1.5 H and in the opposite direction is 0.5 H. Find the self-inductance of the coils.*

Solution

$$k = 0.6, \quad L_{\text{diff}} = 0.5 \text{ H}, \quad L_{\text{cum}} = 1.5 \text{ H}$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.5 \quad \text{...(i)}$$
$$L_{\text{cum}} = L_1 + L_2 + 2M = 1.5 \quad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$4M = 1$$
$$M = 0.25 \text{ H}$$

Adding Eq. (i) and (ii),

$$2(L_1 + L_2) = 2$$
$$L_1 + L_2 = 1 \quad \text{...(iii)}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}$$
$$0.6 = \frac{0.25}{\sqrt{L_1 L_2}}$$
$$L_1 L_2 = 0.1736 \quad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$L_1 = 0.22 \text{ H}$$
$$L_2 = 0.78 \text{ H}$$

Example 4.3 *Two coils having self-inductances of 4 mH and 7 mH respectively are connected in parallel. If the mutual inductance between them is 5 mH, find the equivalent inductance.*

Solution

$$L_1 = 4 \text{ mH}, \quad L_2 = 7 \text{ mH}, \quad M = 5 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 - 2(5)} = 3 \text{ mH}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 + 2(5)} = 0.143 \text{ mH}$$

Example 4.4 *Two inductors are connected in parallel. Their equivalent inductance when the mutual inductance aids the self-inductance is 6 mH and it is 2 mH when the mutual inductance opposes the self-inductance. If the ratio of the self- inductances is 1:3 and the mutual inductance between the coils is 4 mH, find the self-inductances.*

Solution

$$L_{\text{cum}} = 6 \text{ mH}, \quad L_{\text{diff}} = 2 \text{ mH}, \quad \frac{L_1}{L_2} = 1.3, \quad M = 4 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$6 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 - 2(4)}$$

$$6 = \frac{L_1 L_2 - 16}{L_1 + L_2 - 8} \qquad \text{...(i)}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$2 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 + 8}$$

$$2 = \frac{L_1 L_2 - 16}{L_1 + L_2 + 8} \qquad \text{...(ii)}$$

From Eqs (i) and (ii),

$$2(L_1 + L_2 + 8) = 6(L_1 + L_2 - 8)$$

$$L_1 + L_2 + 8 = 3L_1 + 3L_2 - 24$$

$$L_1 + L_2 = 16$$

But

$$\frac{L_1}{L_2} = 1.3$$

$$1.3\, L_2 + L_2 = 16$$

$$2.3\, L_2 = 16$$

$$L_2 = 6.95 \text{ mH}$$

$$L_1 = 1.3\, L_2 = 9.035 \text{ mH}$$

4.7 DOT CONVENTION

Consider two coils of inductances L_1 and L_2 respectively connected in series as shown in Fig. 4.7. Each coil will contribute the same mutual flux (since it is in a series connection, the same current flows through L_1 and L_2) and hence, same mutual inductance (M). If the mutual fluxes of the two coils aid each other as

shown in Fig 4.7 (a), the inductances of each coil will be increased by M, i.e., the inductance of coils will become $(L_1 + M)$ and $(L_2 + M)$. If the mutual fluxes oppose each other as shown in Fig. 4.7 (b), inductance of the coils will become $(L_1 - M)$ and $(L_2 - M)$. Whether the two mutual fluxes aid to each other or oppose will depend upon the manner in which coils are wound. The method described above is very inconvenient because we have to include the pictures of the coils in the circuit. There is another simple method of defining the directions of currents in the coils. This is known as dot convention.

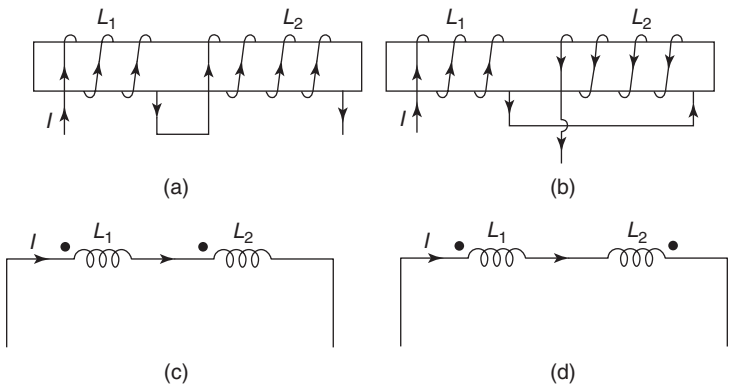


Fig. 4.7 *Dot convention*

Figure 4.7 shows the schematic connection of the two coils. It is not possible to state from Fig. 4.7(a) and Fig. 4.7(b) whether the mutual fluxes are additive or in opposition. However dot convention removes this confusion.

If the current enters from both the dotted ends of Coil 1 and Coil 2, the mutual fluxes of the two coils aid each other as shown in Fig. 4.7(c). If the current enters from the dotted end of Coil 1 and leaves from the dotted end of Coil 2, the mutual fluxes of the two coils oppose each other as shown in Fig. 4.7(d).

When two mutual fluxes aid each other, the mutual inductance is positive and polarity of the mutually induced emf is same as that of the self-induced emf. When two mutual fluxes oppose each other, the mutual inductance is negative and polarity of the mutually induced emf is opposite to that of the self-induced emf.

Example 4.5 *Obtain the dotted equivalent circuit for Fig. 4.8 shown below.*

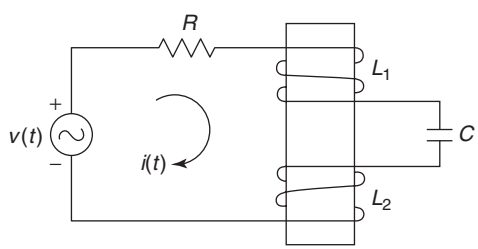


Fig. 4.8

Solution The current in the two coils is shown in Fig. 4.9. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

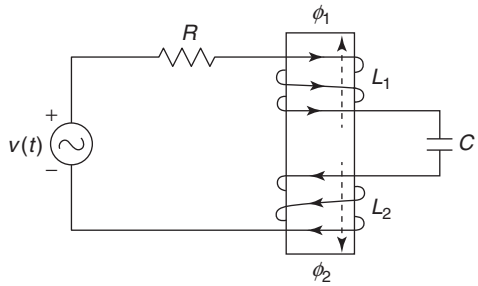


Fig. 4.9

From Fig. 4.9, it is seen that, the flux ϕ_1 is in upward direction in Coil 1, and flux ϕ_2 is in downward direction in Coil 2. Hence, fluxes are opposing each other. The mutual inductances are negative and mutually induced emfs have opposite polarities as that of self-induced emf. The dots are placed in two coils to illustrate these conditions. Hence, current $i(t)$ enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2.

The dotted equivalent circuit is shown in Fig. 4.10.

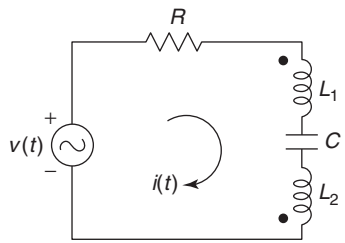


Fig. 4.10

Example 4.6 *Obtain the dotted equivalent circuit for the circuit of Fig. 4.11.*

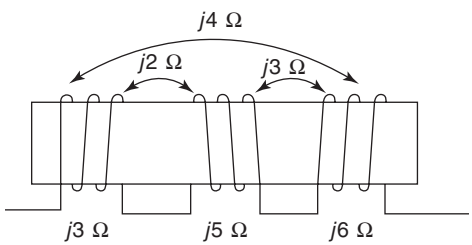


Fig. 4.11

Solution The current in the three coils is shown in Fig. 4.12. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

From Fig. 4.12, it is seen that the flux is towards the left in Coil 1, towards the right in Coil 2 and towards the left in Coil 3. Hence, fluxes ϕ_1 and ϕ_2 oppose each other in coils 1 and 2, fluxes ϕ_2 and ϕ_3 oppose each other in coils 2 and 3, and fluxes ϕ_1 and ϕ_3 aid each other in coils 1 and 3. The dots are placed in three coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1, leaves from the dotted end in Coil 2 and enters from the dotted end in Coil 3.

The dotted equivalent circuit is shown in Fig. 4.13.

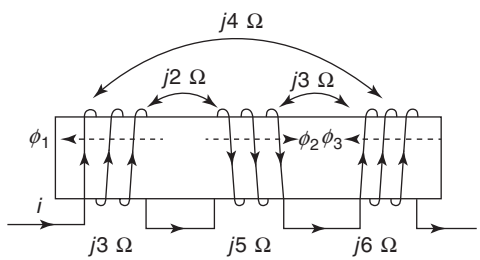


Fig. 4.12

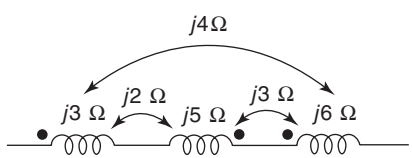


Fig. 4.13

Example 4.7 *Obtain the dotted equivalent circuit for the circuit shown in Fig. 4.14.*

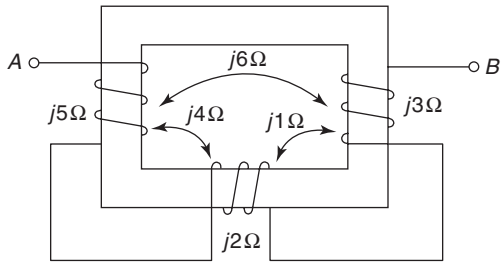


Fig. 4.14

Solution The current in the three coils is shown in Fig. 4.15. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

From Fig. 4.15, it is seen that all the three fluxes ϕ_1, ϕ_2, ϕ_3 aid each other. Hence, all the mutual reactances are positive and mutually induced emfs have same polarities as that of self-induced emfs. The dots are placed in three coils to illustrate these conditions. Hence, currents enter from the dotted end in each of the three coils. The dotted equivalent circuit is shown in Fig. 4.16.

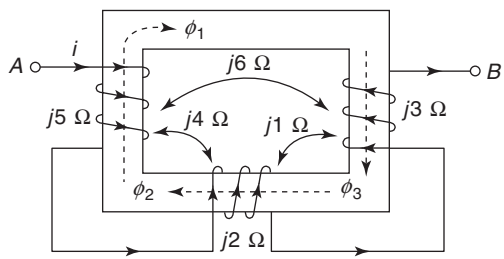


Fig. 4.15

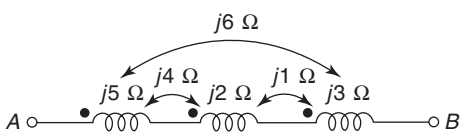


Fig. 4.16

Example 4.8 *Obtain the dotted equivalent circuit for the coupled circuit of Fig. 4.17.*

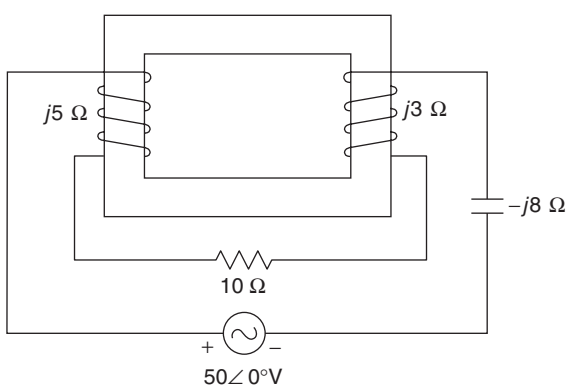


Fig. 4.17

Solution The current in the two coils is shown in Fig. 4.18 . The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

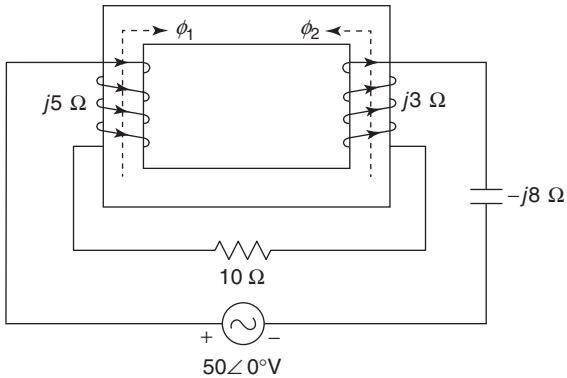


Fig. 4.18

From Fig. 4.18, it is seen that the flux ϕ_1 is in clockwise direction in Coil 1 and in anti-clockwise direction in Coil 2. Hence, fluxes are opposing each other. The dots are placed in two coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2. The dotted equivalent circuit is shown in Fig. 4.19.

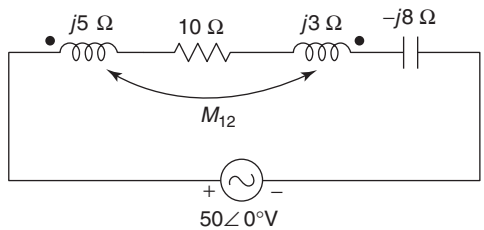


Fig. 4.19

Example 4.9 *Find the equivalent inductance of the network shown in Fig. 4.20.*

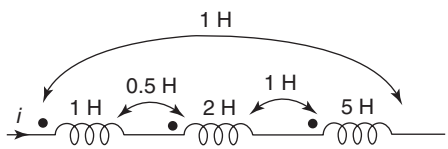


Fig. 4.20

Solution

$$L = (L_1 + M_{12} + M_{13}) + (L_2 + M_{23} + M_{21}) + (L_3 + M_{31} + M_{32})$$
$$= (1 + 0.5 + 1) + (2 + 1 + 0.5) + (5 + 1 + 1)$$
$$= 13\ \text{H}$$

Example 4.10 *Find the equivalent inductance of the network shown in Fig. 4.21.*

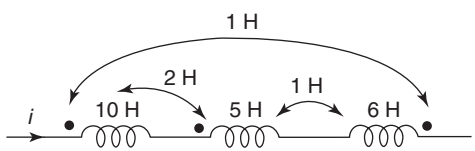


Fig. 4.21

Solution

$$L = (L_1 + M_{12} - M_{13}) + (L_2 - M_{23} + M_{21}) + (L_3 - M_{31} - M_{23})$$
$$= (10 + 2 - 1) + (5 - 1 + 2) + (6 - 1 - 1) = 21\ \text{H}$$

Example 4.11 *Find the equivalent inductance of the network shown in Fig. 4.22.*

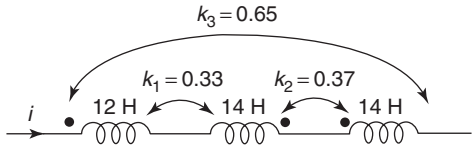


Fig. 4.22

Solution

$$M_{12} = M_{21} = k_1\sqrt{L_1L_2} = 0.33\sqrt{(12)(14)} = 4.28 \text{ H}$$

$$M_{23} = M_{32} = k_2\sqrt{L_2L_3} = 0.37\sqrt{(14)(14)} = 5.18 \text{ H}$$

$$M_{31} = M_{13} = k_3\sqrt{L_3L_1} = 0.65\sqrt{(12)(14)} = 8.42 \text{ H}$$

$$L = (L_1 - M_{12} + M_{13}) + (L_2 - M_{23} - M_{21}) + (L_3 + M_{31} - M_{32})$$

$$= (12 - 4.28 + 8.42) + (14 - 5.18 - 4.28) + (14 + 8.42 - 5.18)$$

$$= 37.92 \text{ H}$$

Example 4.12 *Find the equivalent inductance of the network shown in Fig. 4.23.*

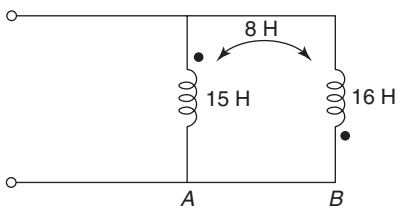


Fig. 4.23

Solution For Coil *A*,

$$L_A = L_1 - M_{12} = 15 - 8 = 7 \text{ H}$$

For Coil *B*,

$$L_B = L_2 - M_{12} = 16 - 8 = 8 \text{ H}$$

$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} = \frac{1}{7} + \frac{1}{8} = \frac{15}{56}$$

$$L = \frac{56}{15} = 3.73 \text{ H}$$

Example 4.13 *Find the equivalent inductance of the network shown in Fig. 4.24.*

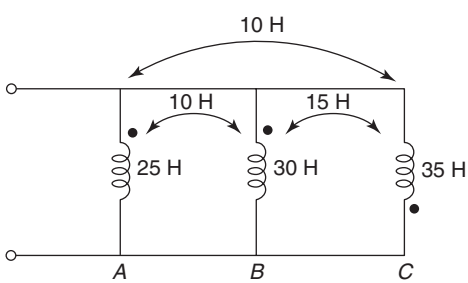


Fig. 4.24

Solution For Coil A,

$$L_A = L_1 + M_{12} - M_{13} = 25 + 10 - 10 = 25 \text{ H}$$

For Coil B,

$$L_B = L_2 - M_{23} + M_{21} = 35 - 15 + 10 = 25 \text{ H}$$

For Coil C,

$$L_C = L_3 - M_{32} - M_{31} = 35 - 15 - 10 = 10 \text{ H}$$

$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} + \frac{1}{L_C} = \frac{1}{25} + \frac{1}{25} + \frac{1}{10} = \frac{9}{50}$$

$$L = \frac{50}{9} = 5.55 \text{ H}$$

Example 4.14 *Find the equivalent impedance across the terminals A and B in Fig. 4.25.*

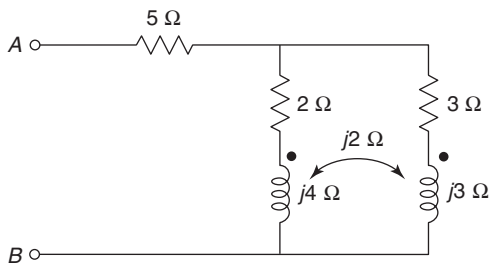


Fig. 4.25

Solution $\mathbf{Z}_1 = 5\,\Omega, \quad \mathbf{Z}_2 = (2 + j4)\,\Omega, \quad \mathbf{Z}_3 = (3 + j3)\,\Omega, \quad \mathbf{Z}_M = j2\,\Omega$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3 - \mathbf{Z}_M^2}{\mathbf{Z}_2 + \mathbf{Z}_3 - 2\mathbf{Z}_M} = 5 + \frac{(2 + j4)(3 + j3) - (j2)^2}{2 + j4 + 3 + j3 - 2(j2)} = 6.9 \angle 24.16° \,\Omega$$

4.8 COUPLED CIRCUITS

Consider two coils located physically close to one another as shown in Fig. 4.26.

When current i_1 flows in the first coil and $i_2 = 0$ in the second coil, flux ϕ_1 is produced in the coil. A fraction of this flux also links the second coil and induces a voltage in this coil. The voltage v_1 induced in the first coil is

$$v_1 = L_1 \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

The voltage v_2 induced in the second coil is

$$v_2 = M \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

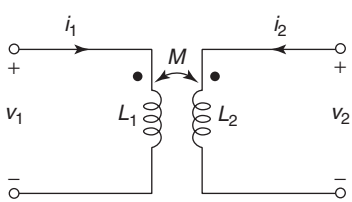


Fig. 4.26 *Coupled circuit*

The polarity of the voltage induced in the second coil depends on the way the coils are wound and it is usually indicated by dots. The dots signify that the induced voltages in the two coils (due to single current) have the same polarities at the dotted ends of the coils. Thus, due to i_1, the induced voltage v_1 must be positive at the dotted end of Coil 1. The voltage v_2 is also positive at the dotted end in Coil 2.

The same reasoning applies if a current i_2 flows in Coil 2 and $i_1 = 0$ in Coil 1. The induced voltages v_2 and v_1 are

$$v_2 = L_2 \frac{di_2}{dt}\bigg|_{i_1 = 0}$$

and

$$v_1 = M \frac{di_2}{dt}\bigg|_{i_1 = 0}$$

The polarities of v_1 and v_2 follow the dot convention. The voltage polarity is positive at the doted end of inductor L_2 when the current direction for i_2 is as shown in Fig. 4.26. Therefore, the voltage induced in Coil 1 must be positive at the dotted end also.

Now if both currents i_1 and i_2 are present, by using superposition principle, we can write

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This can be represented in terms of dependent sources, as shown in Fig. 4.27.

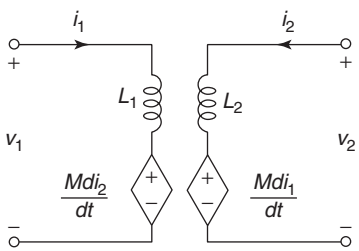


Fig. 4.27 *Equivalent circuit*

Now consider the case when the dots are placed at the opposite ends in the two coils, as shown in Fig. 4.28.

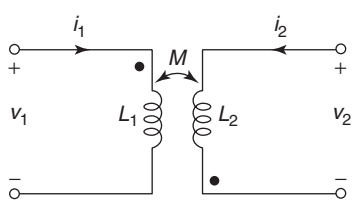


Fig. 4.28 *Coupled circuit*

Due to i_1, with $i_2 = 0$, the dotted end in Coil 1 is positive, so the induced voltage in Coil 2 is positive at the dot, which is the reverse of the designated polarity for v_2. Similarly, due to i_2, with $i_1 = 0$, the dotted ends have negative polarities for the induced voltages. The mutually induced voltages in both cases have polarities that are the reverse of terminal voltages and the equations are

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This can be repressed in terms of dependent sources as shown in Fig. 4.29.

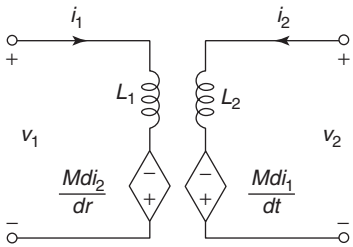


Fig. 4.29 *Equivalent circuit*

The various cases are summarised in the table shown in Fig. 4.30.

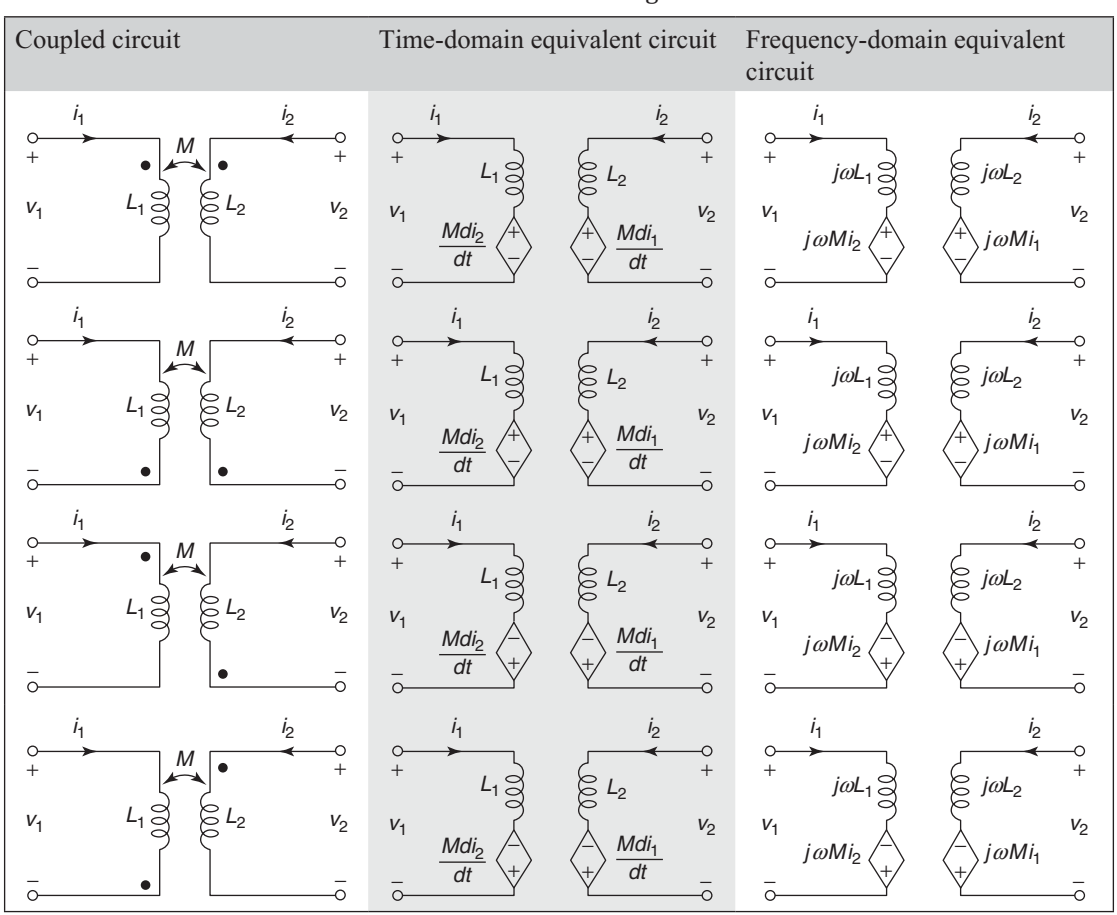


Fig. 4.30 *Coupled circuits for various cases*

Example 4.15 *Write mesh equations for the network shown in Fig. 4.31.*

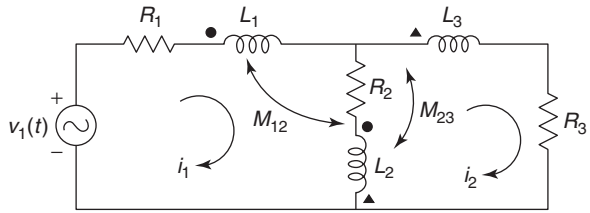


Fig. 4.31

Solution Coil 1 is magnetically coupled to Coil 2. Similarly, Coil 2 is magnetically coupled with Coil 1 and Coil 3. By applying dot convention, the equivalent circuit is drawn with the dependent sources.

The equivalent circuit in terms of dependent sources is shown in Fig. 4.32.

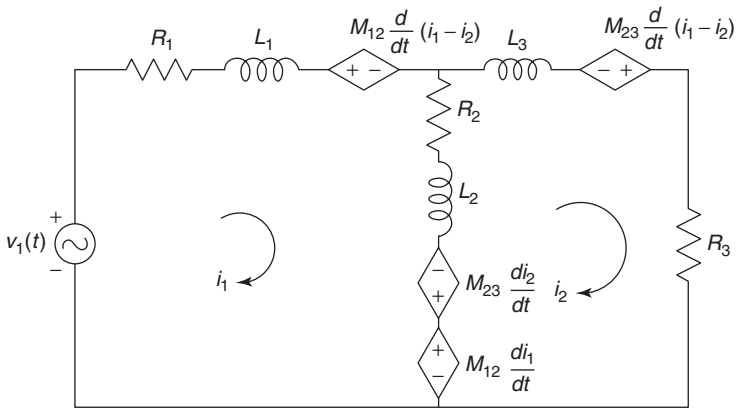


Fig. 4.32

(a) In Coil 1, there is a mutually induced emf due to current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is same as that of self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from the dotted ends.

(b) In Coil 2, there are two mutually induced emfs, one due to current i_1 in Coil 1 and the other due to current i_2 in Coil 3. The polarity of the mutually induced emf in Coil 2 due to the current i_1 is same as that of the self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from dotted ends. The polarity of the mutually induced emf in Coil 2 due to the current i_2 is opposite to that of the self-induced emf because current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

(c) In Coil 3, there is a mutually induced emf due to the current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is opposite to that of self-induced emf because the current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

Applying KVL to Mesh 1,

$$v_1(t) - R_1 i_1 - L_1 \frac{di_1}{dt} - M_{12} \frac{d}{dt}(i_1 - i_2) - R_2(i_1 - i_2) - L_2 \frac{d}{dt}(i_1 - i_2) + M_{23} \frac{di_2}{dt} - M_{12} \frac{di_1}{dt} = 0$$

$$(R_1 + R_2)\, i_1 + (L_1 + L_2 + 2M_{12})\frac{di_1}{dt} - R_2 i_2 - (L_2 + M_{12} + M_{23})\frac{di_2}{dt} = v_1(t) \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$M_{12}\frac{di_1}{dt} - M_{23}\frac{di_2}{dt} - L_2\frac{d}{dt}(i_2 - i_1) - R_2(i_2 - i_1) - L_3\frac{di_2}{dt} + M_{23}\frac{d}{dt}(i_1 - i_2) - R_3 i_2 = 0$$

$$-R_2\, i_1 - (L_2 + M_{12} + M_{23})\frac{di_1}{dt} + (R_2 + R_3)\, i_2 + (L_2 + L_3 + 2M_{23})\frac{di_2}{dt} = 0 \qquad \ldots(\text{ii})$$

Example 4.16

Write KVL equations for the circuit shown in Fig. 4.33.

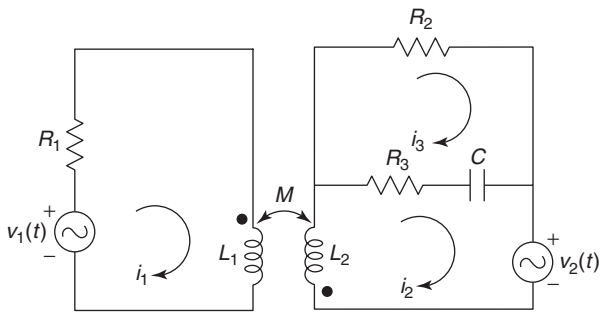


Fig. 4.33

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.34.

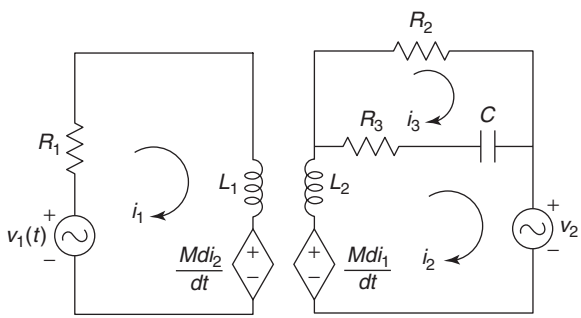


Fig. 4.34

Applying KVL to Mesh 1,

$$v_1(t) - R_1\, i_1 - L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

$$R_1\, i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = v_1(t) \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$M\frac{di_1}{dt} - L_2\frac{di_2}{dt} - R_3(i_2 - i_3) - \frac{1}{C}\int_0^t (i_2 - i_3)\, dt - v_2(t) = 0$$

$$M\frac{di_1}{dt} - L_2\frac{di_2}{dt} - R_3(i_2 - i_3) - \frac{1}{C}\int_0^t (i_2 - i_3)\,dt = v_2(t) \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-R_2 i_3 - \frac{1}{C}\int_0^t (i_3 - i_2)dt - R_3(i_3 - i_2) = 0 \qquad \ldots(iii)$$

Example 4.17 *Write down the mesh equations for the network shown in Fig. 4.35.*

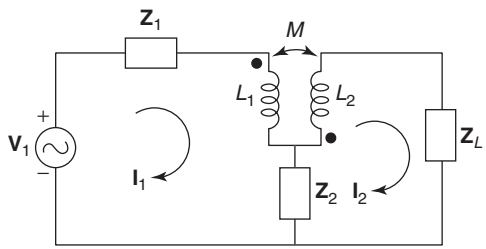


Fig. 4.35

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.36.

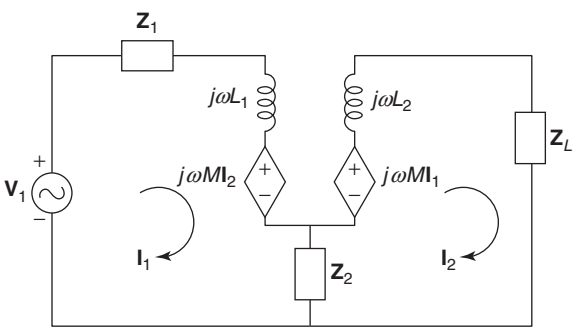


Fig. 4.36

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{Z}_1\mathbf{I}_1 - j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$(\mathbf{Z}_1 + j\omega L_1 + \mathbf{Z}_2)\,\mathbf{I}_1 - (\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) + j\omega M\mathbf{I}_1 - j\omega L_2\mathbf{I}_2 - \mathbf{Z}_L\mathbf{I}_2 = 0$$
$$-(\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_1 + (\mathbf{Z}_2 + j\omega L_2 + \mathbf{Z}_L)\,\mathbf{I}_2 = 0 \qquad \ldots(ii)$$

Example 4.18 *Write mesh equations for the network shown in Fig. 4.37.*

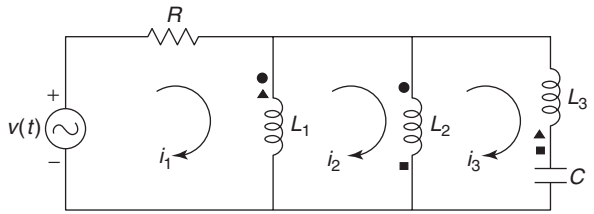


Fig. 4.37

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.38.

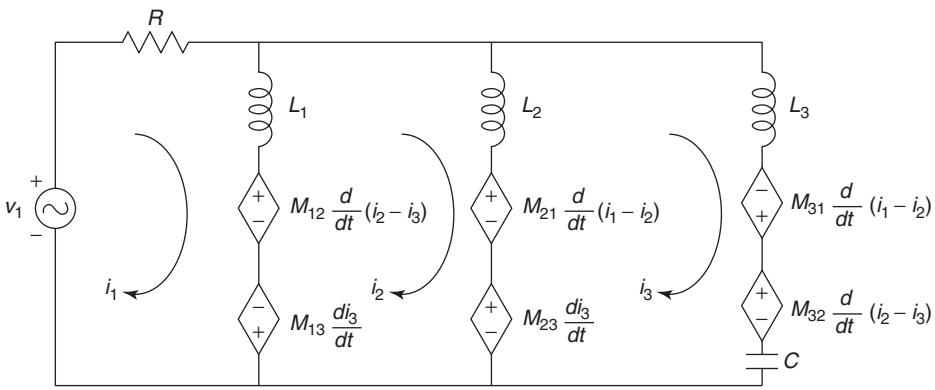


Fig. 4.38

Applying KVL to Mesh 1,

$$v(t) - Ri_1 - L_1\frac{d}{dt}(i_1 - i_2) - M_{12}\frac{d}{dt}(i_2 - i_3) + M_{13}\frac{di_3}{dt} = 0$$

$$Ri_1 + L_1\frac{d}{dt}(i_1 - i_2) + M_{12}\frac{d}{dt}(i_2 - i_3) - M_{13}\frac{di_3}{dt} = v(t) \quad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-M_{13}\frac{di_3}{dt} + M_{12}\frac{d}{dt}(i_2 - i_3) - L_1\frac{d}{dt}(i_2 - i_1) - L_2\frac{d}{dt}(i_2 - i_3) - M_{21}\frac{d}{dt}(i_1 - i_2) - M_{23}\frac{di_3}{dt} = 0$$

$$M_{13}\frac{di_3}{dt} - M_{12}\frac{d}{dt}(i_2 - i_3) + L_1\frac{d}{dt}(i_2 - i_1) + L_2\frac{d}{dt}(i_2 - i_3) + M_{21}\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{di_3}{dt} = 0 \quad \ldots(\text{ii})$$

Applying KVL to Mesh 3,

$$M_{23}\frac{di_3}{dt} + M_{21}\frac{d}{dt}(i_1 - i_2) - L_2\frac{d}{dt}(i_3 - i_2) - L_3\frac{di_3}{dt} + M_{31}\frac{d}{dt}(i_1 - i_2) - M_{32}\frac{d}{dt}(i_2 - i_3) - \frac{1}{C}\int i_3\, dt = 0$$

$$-M_{23}\frac{di_3}{dt} - M_{21}\frac{d}{dt}(i_1 - i_2) + L_2\frac{d}{dt}(i_3 - i_2) + L_3\frac{di_3}{dt} - M_{31}\frac{d}{dt}(i_1 - i_2) + M_{32}\frac{d}{dt}(i_2 - i_3) + \frac{1}{C}\int i_3\, dt = 0 \quad \ldots(\text{iii})$$

Example 4.19 *Write KVL equations for the network shown in Fig. 4.39.*

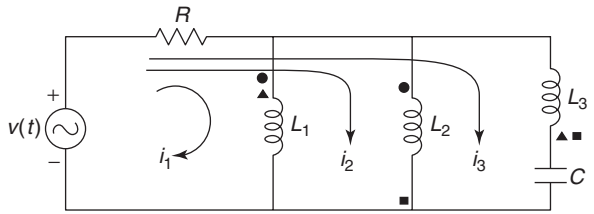


Fig. 4.39

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.40.

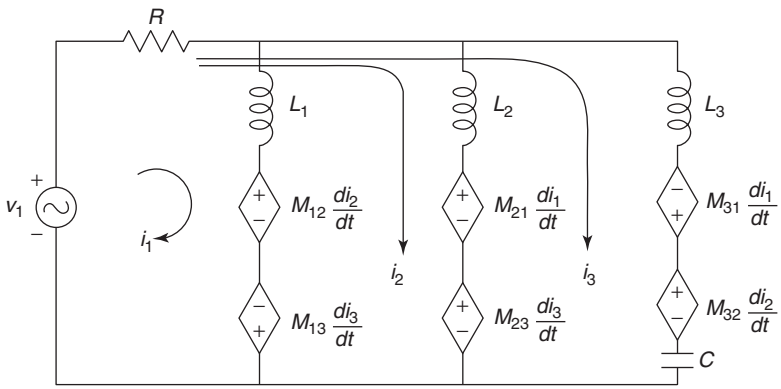


Fig. 4.40

Applying KVL to Loop 1,

$$v(t) - R(i_1 + i_2 + i_3) - L_1 \frac{di_1}{dt} - M_{12} \frac{di_2}{dt} + M_{13} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} - M_{13} \frac{di_3}{dt} = v(t) \quad \ldots(i)$$

Applying KVL to Loop 2,

$$v(t) - R(i_1 + i_2 + i_3) - L_2 \frac{di_2}{dt} - M_{21} \frac{di_1}{dt} - M_{23} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt} + M_{23} \frac{di_3}{dt} = v(t) \quad \ldots(ii)$$

Applying KVL to Loop 3,

$$v(t) - R(i_1 + i_2 + i_3) - L_3 \frac{di_3}{dt} + M_{31} \frac{di_1}{dt} - M_{32} \frac{di_2}{dt} - \frac{1}{C} \int i_3 \, dt = 0$$

$$R(i_1 + i_2 + i_3) + L_3 \frac{di_3}{dt} - M_{31} \frac{di_1}{dt} + M_{32} \frac{di_2}{dt} + \frac{1}{C} \int i_3 \, dt = v(t) \quad \ldots(iii)$$

Example 4.20 *In the network shown in Fig. 4.41, find the voltages V_1 and V_2.*

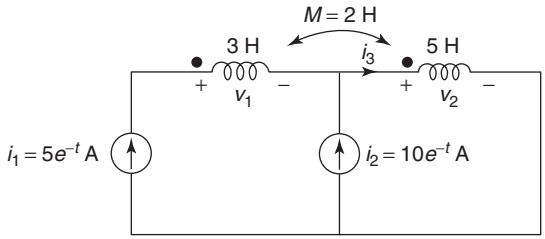


Fig. 4.41

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.42.

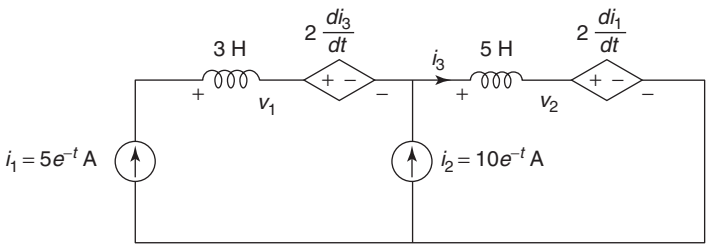


Fig. 4.42

From Fig. 4.42,

$$i_3 = i_1 + i_2 = 5\,e^{-t} + 10\,e^{-t} = 15\,e^{-t}\ \text{A}$$

$$v_1 = 3\frac{di_1}{dt} + 2\frac{di_3}{dt} = 3\frac{d}{dt}(5\,e^{-t}) + 2\frac{d}{dt}(15\,e^{-t}) = -15\,e^{-t} - 30\,e^{-t} = -45\,e^{-t}\text{V}$$

$$v_2 = 5\frac{di_3}{dt} + 2\frac{di_1}{dt} = 5\frac{d}{dt}(15\,e^{-t}) + 2\frac{d}{dt}(5\,e^{-t}) = -75\,e^{-t} - 10\,e^{-t} = -85\,e^{-t}\text{V}$$

Example 4.21 *In the network shown in Fig. 4.43, find the voltages V_1 and V_2.*

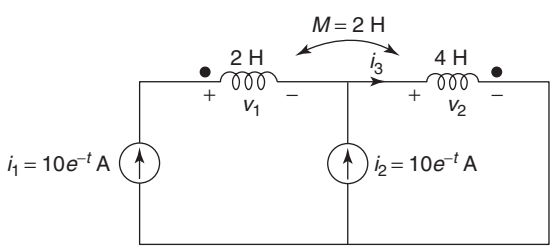


Fig. 4.43

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.44.

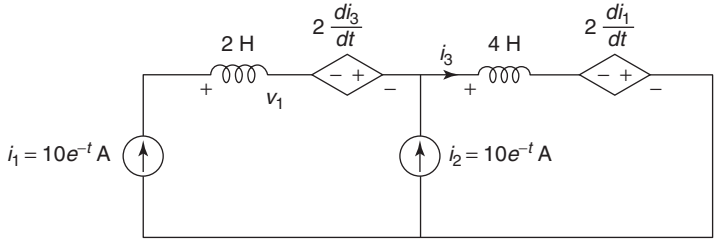


Fig. 4.44

From Fig. 4.44,

$$i_3 = i_1 + i_2 = 10\,e^{-t} + 10\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_1 = 2\frac{di_1}{dt} - 2\frac{di_3}{dt} = 2\frac{d}{dt}(10\,e^{-t}) - 2\frac{d}{dt}(20\,e^{-t}) = -20\,e^{-t} + 40\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_2 = 4\frac{di_3}{dt} - 2\frac{di_1}{dt} = 4\frac{d}{dt}(20\,e^{-t}) - 2\frac{d}{dt}(10\,e^{-t}) = -80\,e^{-t} + 20\,e^{-t} = -60\,e^{-t}\text{A}$$

Example 4.22 *Calculate the current $i_2(t)$ in the coupled circuit of Fig. 4.45.*

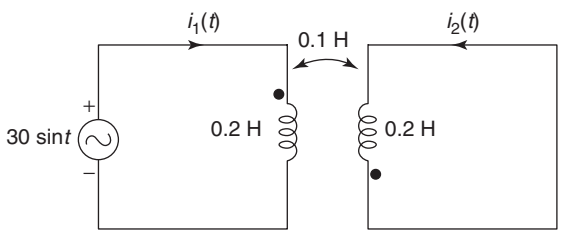


Fig. 4.45

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.46.

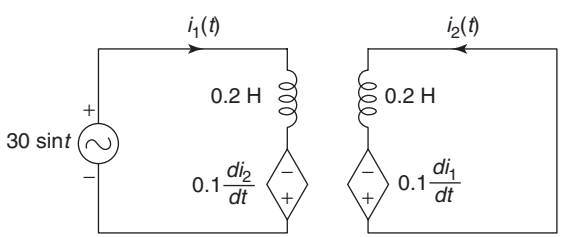


Fig. 4.46

Applying KVL to Mesh 1,

$$30 \sin t - 0.2\frac{di_1}{dt} + 0.1\frac{di_2}{dt} = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-0.2\frac{di_2}{dt}+0.1\frac{di_1}{dt}=0$$

$$\frac{di_1}{dt}=2\frac{di_2}{dt} \qquad \ldots(ii)$$

Substituting Eq. (ii) in Eq. (i),

$$30\sin t-0.2\left(2\frac{di_2}{dt}\right)+0.1\frac{di_2}{dt}=0$$

$$0.3\frac{di_2}{dt}=30\sin t$$

$$\frac{di_2}{dt}=100\sin t$$

$$di_2=100\sin t\,dt$$

Integrating both the sides,

$$i_2(t)=100\int_0^t \sin t\,dt$$

$$=100\left[-\cos t\right]_0^t$$

$$=100\,(1-\cos t)$$

Example 4.23 *Find the voltage V_2 in the circuit shown in Fig. 4.47 such that the current in the left-hand loop (Loop 1) is zero.*

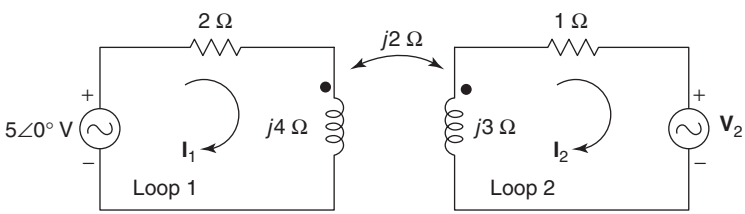


Fig. 4.47

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.48.

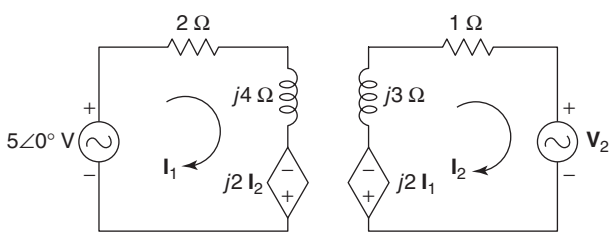


Fig. 4.48

Applying KVL to Loop 1,

$$5 \angle 0° - 2\,\mathbf{I}_1 - j4\mathbf{I}_1 + j2\mathbf{I}_2 = 0$$

$$(2 + j4)\,\mathbf{I}_1 - j2\mathbf{I}_2 = 5 \angle 0° \quad \ldots(i)$$

Applying KVL to Loop 2,

$$-j2\mathbf{I}_1 - j3\mathbf{I}_2 - 1\mathbf{I}_2 - \mathbf{V}_2 = 0$$

$$-j2\mathbf{I}_1 - (1 + j3)\,\mathbf{I}_2 = \mathbf{V}_2 \quad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 2 + j4 & -j2 \\ -j2 & -(1 + j3) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 5 \angle 0° \\ \mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 5 \angle 0° & -j2 \\ \mathbf{V}_2 & -(1 + j3) \end{vmatrix}}{\begin{vmatrix} 2 + j4 & -j2 \\ -j2 & -(1 + j3) \end{vmatrix}}$$

But $\mathbf{I}_1 = 0$.

$$\therefore -(5 \angle 0°)(1 + j3) + j2\,\mathbf{V}_2 = 0$$

$$\mathbf{V}_2 = \frac{(5 \angle 0°)(1 + j3)}{j2} = 7.91 \angle -18.43° \text{ V}$$

Example 4.24 *Determine the ratio* $\frac{V_2}{V_1}$ *in the circuit of Fig. 4.49, if* $\boldsymbol{I}_1 = 0$.

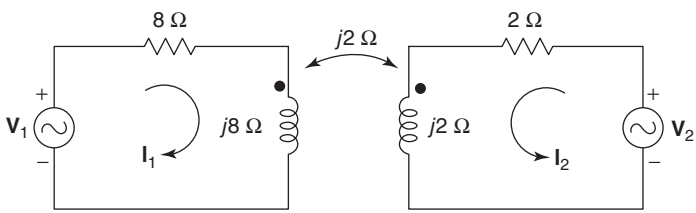


Fig. 4.49

Solution The equivalent circuit in terms of dependent sources is as shown in Fig. 4.50.

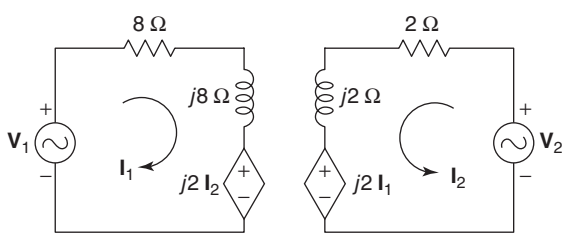


Fig. 4.50

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 8\mathbf{I}_1 - j8\mathbf{I}_1 - j2\mathbf{I}_2 = 0$$

$$(8 + j8)\,\mathbf{I}_1 + j2\mathbf{I}_2 = \mathbf{V}_1 \quad \ldots(i)$$

Putting $\mathbf{I}_1 = 0$ in Eq (i),

$$j2\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(ii)$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - 2\mathbf{I}_2 - j2\mathbf{I}_2 - j2\mathbf{I}_1 = 0$$

$$j2\mathbf{I}_1 + (2 + j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots(iii)$$

Putting $\mathbf{I}_1 = 0$ in Eq (iii),

$$(2 + j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots(iv)$$

From Eqs (ii) and (iv),

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{(2 + j2)\mathbf{I}_2}{j2\mathbf{I}_2} = \frac{2 + j2}{j2} = 1.41 \angle -45° \text{ V}$$

Example 4.25 *For the coupled circuit shown in Fig. 4.51, find input impedance at terminals A and B.*

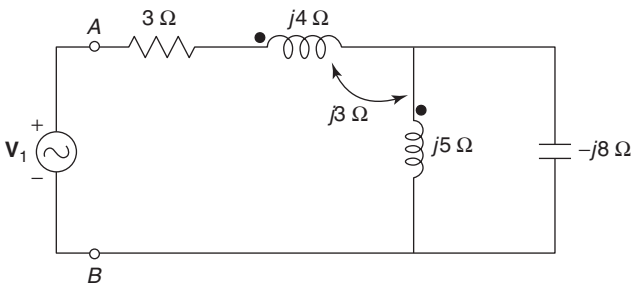


Fig. 4.51

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.52.

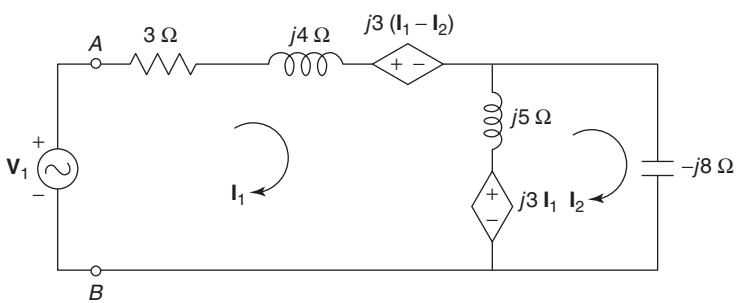


Fig. 4.52

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 3\mathbf{I}_1 - j4\mathbf{I}_1 - j3\,(\mathbf{I}_1 - \mathbf{I}_2) - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - j3\mathbf{I}_1 = 0$$

$$(3 + j15)\,\mathbf{I}_1 - j8\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$

$$j8\,\mathbf{I}_1 + j3\,\mathbf{I}_2 = 0$$

$$\mathbf{I}_2 = -\frac{j8}{j3}\mathbf{I}_1 = -2.67\,\mathbf{I}_1 \qquad \ldots(ii)$$

Substituting Eq (ii) in Eq (i),

$$(3 + j15)\mathbf{I}_1 - j8\,(-2.67\,\mathbf{I}_1) = \mathbf{V}_1$$

$$(3 + j\,36.36)\,\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = (3 + j\,36.36)\,\Omega = 36.48 \angle 85.28°\,\Omega$$

Example 4.26 *Find equivalent impedance of the network shown in Fig. 4.53.*

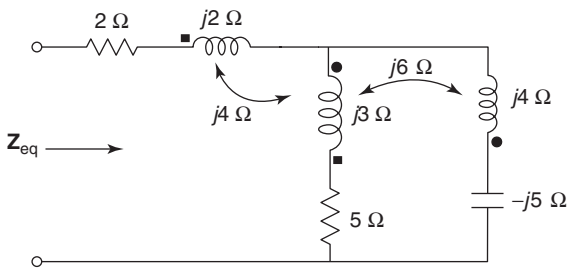


Fig. 4.53

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.54.

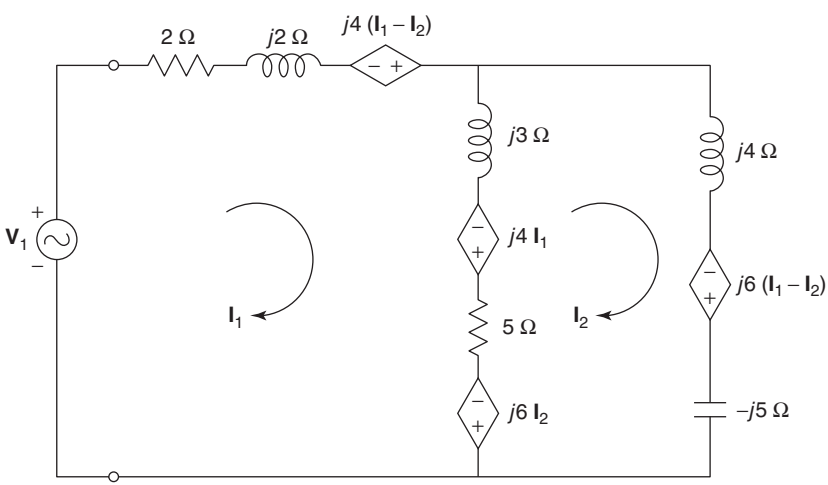


Fig. 4.54

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 2\mathbf{I}_1 - j2\mathbf{I}_1 + j4\,(\mathbf{I}_1 - \mathbf{I}_2) - j3\,(\mathbf{I}_1 - \mathbf{I}_2) + j4\,\mathbf{I}_1 - 5(\mathbf{I}_1 - \mathbf{I}_2) + j6\mathbf{I}_2 = 0$$

$$(7 - j3)\,\mathbf{I}_1 - (5 + j5)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-j6\mathbf{I}_2 - 5(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_1 - j3(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_2 + j6(\mathbf{I}_1 - \mathbf{I}_2) + j5\mathbf{I}_2 = 0$$

$$(5 + j5)\mathbf{I}_1 = (5 + j4)\,\mathbf{I}_2$$

$$\mathbf{I}_2 = \left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 \qquad \ldots(ii)$$

Substituting Eq. (ii) in Eq. (i),

$$(7 - j3)\mathbf{I}_1 - (5 + j5)\left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 7 - j3 - \frac{(5 + j5)\,(5 + j5)}{5 + j4} = 5.63 \angle -47.15° \ \Omega$$

Example 4.27 *Find the voltage across the 5 Ω resistor in Fig. 4.55 using mesh analysis.*

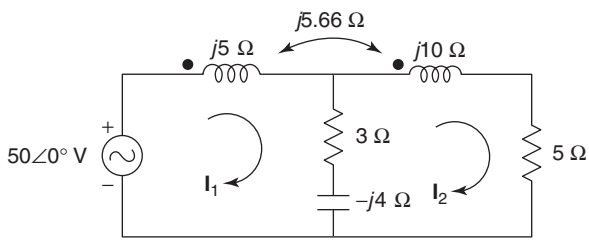


Fig. 4.55

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.56.

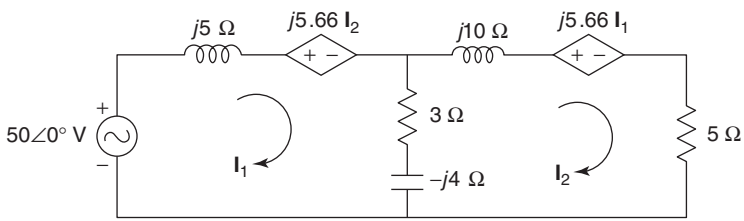


Fig. 4.56

Applying KVL to Mesh 1,

$$50 \angle 0° - j5\,\mathbf{I}_1 - j\,5.66\,\mathbf{I}_2 - (3 - j4)\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j\,1)\,\mathbf{I}_1 - (3 - j\,9.66)\,\mathbf{I}_2 = 50 \angle 0° \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-(3 - j4)\,(\mathbf{I}_2 - \mathbf{I}_1) - j\,10\mathbf{I}_2 - j\,5.66\,\mathbf{I}_1 - 5\mathbf{I}_2 = 0$$

$$-(3 - j\,9.66)\,\mathbf{I}_1 + (8 + j6)\,\mathbf{I}_2 = 0 \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}=\begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2=\frac{\begin{vmatrix} 3+j1 & 50\angle 0^\circ \\ -(3-j9.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{vmatrix}}=3.82\angle -112.14^\circ \text{ A}$$

$$\mathbf{V}_{5\,\Omega}=5\,\mathbf{I}_2=5\,(3.82\angle -112.14^\circ)=19.1\angle -112.14^\circ \text{ V}$$

Example 4.28 *Find the voltage across the 5 Ω resistor in Fig. 4.57 using mesh analysis.*

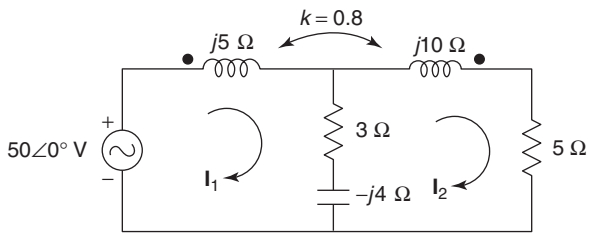


Fig. 4.57

Solution For a magnetically coupled circuit,

$$X_M=k\sqrt{X_{L_1}X_{L_2}}$$
$$=0.8\sqrt{(5)\,(10)}$$
$$=5.66\ \Omega$$

The equivalent circuit in terms of dependent sources is shown in Fig. 4.58.

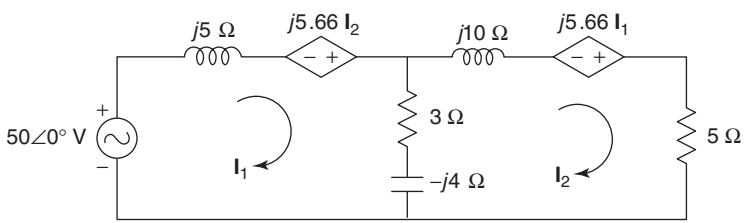


Fig. 4.58

Applying KVL to Mesh 1,

$$50\angle 0^\circ - j5\,\mathbf{I}_1 + j\,5.66\,\mathbf{I}_2-(3-j4)\,(\mathbf{I}_1-\mathbf{I}_2)=0$$
$$(3+j\,1)\,\mathbf{I}_1-(3+j\,1.66)\,\mathbf{I}_2=50\angle 0^\circ \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-(3-j4)(\mathbf{I}_2-\mathbf{I}_1)-j10\mathbf{I}_2+j5.66\,\mathbf{I}_1-5\mathbf{I}_2=0$$
$$-(3+j1.66)\,\mathbf{I}_1+(8+j6)\,\mathbf{I}_2=0 \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j1 & -(3+j1.66) \\ -(3+j1.66) & 8+j6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}=\begin{bmatrix} 50\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2=\frac{\begin{vmatrix} 3+j1 & 50\angle 0° \\ -(3+j1.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -(3+j1.66) \\ -(3+j1.66) & 8+j6 \end{vmatrix}}=8.62\angle -24.79°\ \text{A}$$

$$\mathbf{V}_{5\,\Omega}=5\,\mathbf{I}_2=5\,(8.62\angle -24.79°)=43.1\angle -24.79°\ \text{A}$$

Example 4.29 *Find the current through the capacitor in Fig. 4.59 using mesh analysis.*

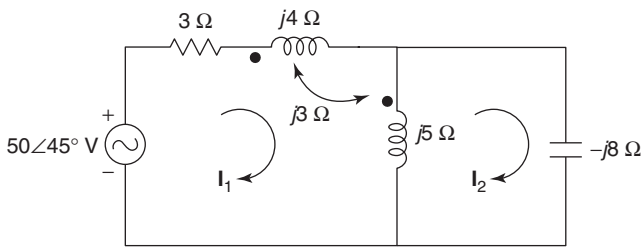


Fig. 4.59

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.60.

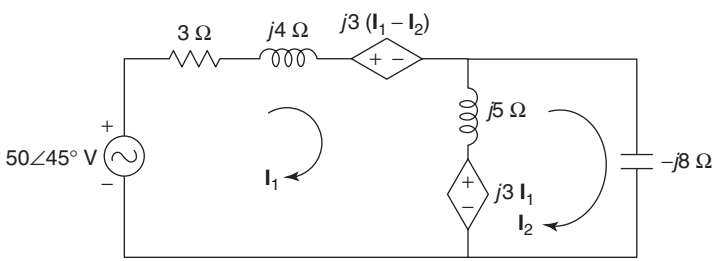


Fig. 4.60

Applying KVL to Mesh 1,

$$50\angle 45°-(3+j4)\,\mathbf{I}_1-j3(\mathbf{I}_1-\mathbf{I}_2)-j5\,(\mathbf{I}_1-\mathbf{I}_2)-j3\,\mathbf{I}_1=0$$
$$(3+j15)\,\mathbf{I}_1-j8\,\mathbf{I}_2=50\angle 45° \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$
$$-j8\,\mathbf{I}_1 - j3\,\mathbf{I}_2 = 0 \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3+j15 & 50\angle 45^\circ \\ -j8 & 0 \end{vmatrix}}{\begin{vmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{vmatrix}} = 3.66\angle -310.33^\circ \text{ A}$$

$$\mathbf{I}_C = \mathbf{I}_2 = 3.66\angle -310.33^\circ \text{A}$$

Example 4.30 *Find the voltage across the 15 Ω resistor in Fig. 4.61 using mesh analysis.*

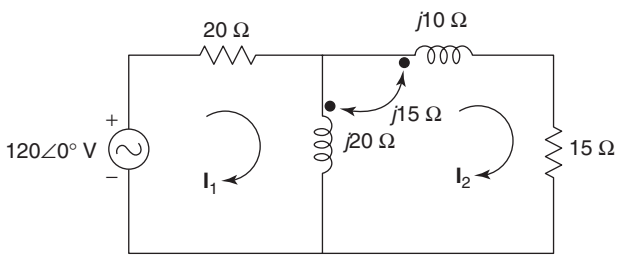


Fig. 4.61

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.62.

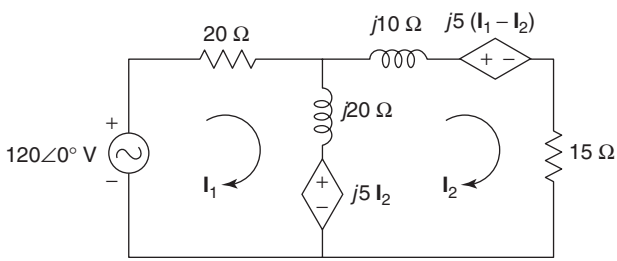


Fig. 4.62

Applying KVL to Mesh 1,

$$120\angle 0^\circ - 20\,\mathbf{I}_1 - j20(\mathbf{I}_1 - \mathbf{I}_2) - j5\,\mathbf{I}_2 = 0$$
$$(20 + j\,20)\,\mathbf{I}_1 - j\,15\,\mathbf{I}_2 = 120\angle 0^\circ \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$j5\,\mathbf{I}_2 - j\,20\,(\mathbf{I}_2 - \mathbf{I}_1) - j10\,\mathbf{I}_2 - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - 15\,\mathbf{I}_2 = 0$$

$$-j15\,\mathbf{I}_1 + (15 + j20)\,\mathbf{I}_2 = 0 \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120 \angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 20 + j\,20 & 120 \angle 0° \\ -j15 & 0 \end{vmatrix}}{\begin{vmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{vmatrix}} = 2.53 \angle 10.12° \text{A}$$

$$\mathbf{V}_{15\,\Omega} = 15\,\mathbf{I}_2 = 15\,(2.53 \angle 10.12°) = 37.95 \angle 10.12° \text{V}$$

Example 4.31 *Find the current through the 6 Ω resistor in Fig. 4.63 using mesh analysis.*

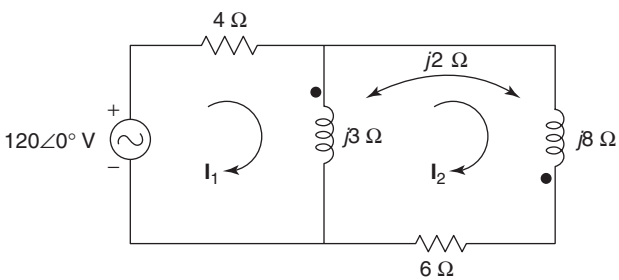


Fig. 4.63

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.64.

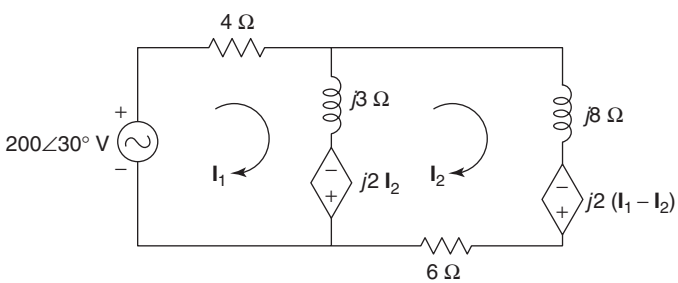


Fig. 4.64

Applying KVL to Mesh 1,

$$120 \angle 0° - 4\,\mathbf{I}_1 - j3(\mathbf{I}_1 - \mathbf{I}_2) + j2\,\mathbf{I}_2 = 0$$

$$(4 + j\,3)\,\mathbf{I}_1 - j\,5\,\mathbf{I}_2 = 120 \angle 0° \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-j\,2\,\mathbf{I}_2 - j\,3\,(\mathbf{I}_2 - \mathbf{I}_1) - j8\,\mathbf{I}_2 + j2\,(\mathbf{I}_1 - \mathbf{I}_2) - 6\,\mathbf{I}_2 = 0$$

$$-j5\,\mathbf{I}_1 + (6 + j15)\,\mathbf{I}_2 = 0 \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 4 + j\,3 & -j5 \\ -j5 & 6 + j\,15 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120 \angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4 + j\,3 & 120 \angle 0° \\ -j5 & 0 \end{vmatrix}}{\begin{vmatrix} 4 + j\,3 & -j5 \\ -j5 & 6 + j\,15 \end{vmatrix}} = 7.68 \angle 2.94° \text{A}$$

Example 4.32 *Determine the mesh current $\boldsymbol{I}_3$ in the network of Fig. 4.65.*

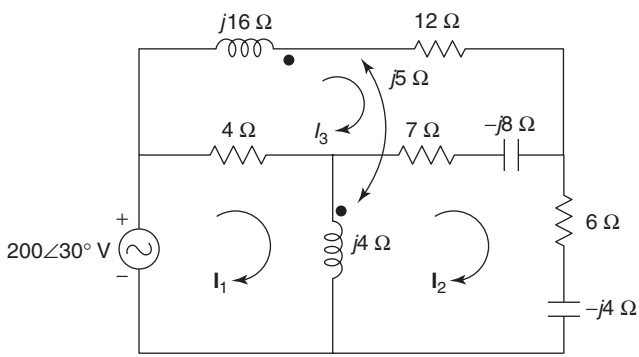


Fig. 4.65

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 4.66.

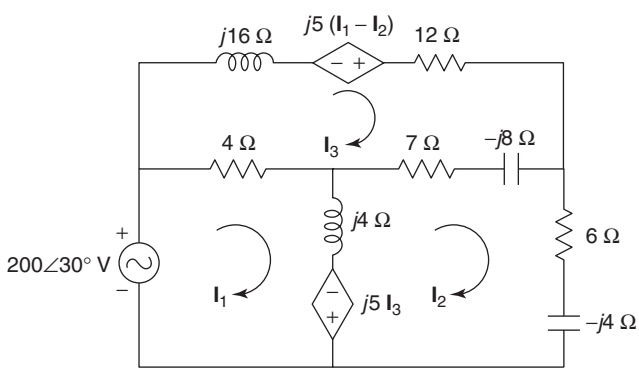


Fig. 4.66

Applying KVL to Mesh 1,

$$200 \angle 30° - 4(\mathbf{I}_1 - \mathbf{I}_3) - j4(\mathbf{I}_1 - \mathbf{I}_2) + j5\,\mathbf{I}_3 = 0$$
$$(4 + j\,4)\,\mathbf{I}_1 - j\,4\,\mathbf{I}_2 - (4 + j5)\,\mathbf{I}_3 = 200 \angle 30° \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$-j\,5\,\mathbf{I}_3 - j\,4(\mathbf{I}_2 - \mathbf{I}_1) - (7 - j8)(\mathbf{I}_2 - \mathbf{I}_3) - (6 - j\,4)\,\mathbf{I}_2 = 0$$
$$-j4\,\mathbf{I}_1 + (13 - j8)\,\mathbf{I}_2 - (7 - j13)\,\mathbf{I}_3 = 0 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-j\,16\,\mathbf{I}_3 + j\,5(\mathbf{I}_1 - \mathbf{I}_2) - 12\,\mathbf{I}_3 - (7 - j\,8)(\mathbf{I}_3 - \mathbf{I}_2) - 4(\mathbf{I}_3 - \mathbf{I}_1) = 0$$
$$-(4 + j5)\,\mathbf{I}_1 - (7 - j13)\,\mathbf{I}_2 + (23 + j8)\,\mathbf{I}_3 = 0 \quad \ldots(iii)$$

Writing Eqs. (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 200 \angle 30° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4 + j4 & -j4 & 200 \angle 30° \\ -j4 & 13 - j8 & 0 \\ -(4 + j5) & -(7 - j13) & 0 \end{vmatrix}}{\begin{vmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{vmatrix}} = 16.28 \angle 16.87° \text{A}$$

Example 4.33 *Obtain the dotted equivalent circuit for the coupled circuit shown in Fig. 4.67 and find mesh currents. Also find the voltage across the capacitor.*

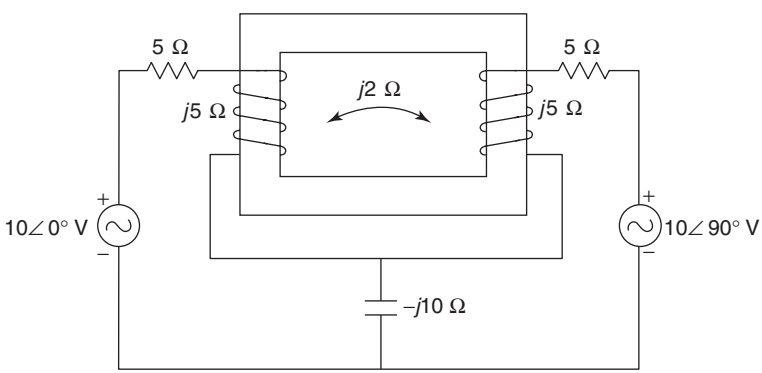


Fig. 4.67

Solution The currents in the coils are as shown in Fig. 4.68. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

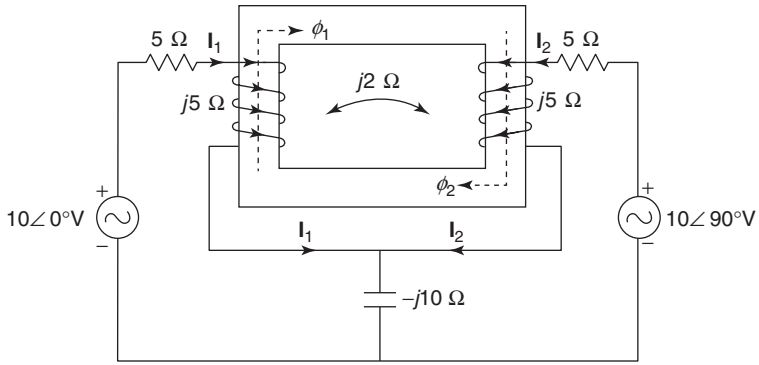


Fig. 4.68

From Fig. 4.68, it is seen that two fluxes ϕ_1 and ϕ_2 aid each other. Hence, dots are placed at the two coils as shown in Fig. 4.69.

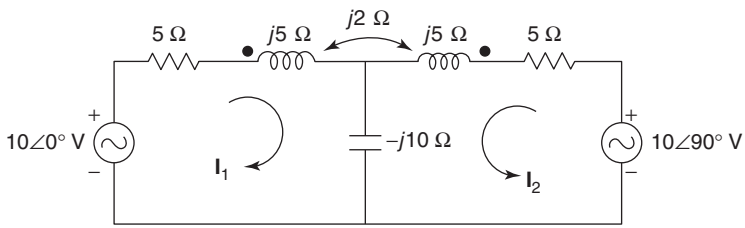


Fig. 4.69

The equivalent circuit in terms of dependent sources is shown in Fig. 4.70.

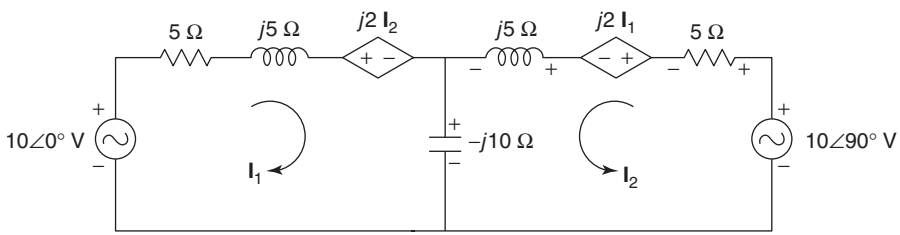


Fig. 4.70

Applying KVL to Mesh 1,

$$10 \angle 0° - (5 + j5)\,\mathbf{I}_1 - j2\,\mathbf{I}_2 + j10\,(\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$(5 - j\,5)\,\mathbf{I}_1 - j\,8\,\mathbf{I}_2 = 10 \angle 0° \quad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-j10\,(\mathbf{I}_2 + \mathbf{I}_1) + j5\,\mathbf{I}_2 - j2\,\mathbf{I}_1 + 5\,\mathbf{I}_2 - 10 \angle 90° = 0$$

$$-j\,8\,\mathbf{I}_1 + (5 - j5)\,\mathbf{I}_2 = 10 \angle 90° \quad \ldots(\text{ii})$$

Writing Eqs. (i) and (ii) in matrix form,

$$\begin{bmatrix} 5 - j\,5 & -j8 \\ -j8 & 5 - j\,5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10 \angle 0° \\ 10 \angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0° & -j8 \\ 10\angle 90° & 5-j5 \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 0.72\angle -82.97°\text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5-j5 & 10\angle 0° \\ -j8 & 10\angle 90° \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 1.71\angle 106.96°\text{A}$$

$$\mathbf{V}_C = -j\,10\,(\mathbf{I}_1 + \mathbf{I}_2) = (-j\,10)\,(0.72\angle -82.97° + 1.71\angle 106.96°\text{ A})$$
$$= 10.08\angle 24.03°\text{V}$$

4.9 CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS

For simplifying circuit analysis, it is desirable to replace a magnetically coupled circuit with an equivalent circuit called conductively coupled circuit. In this circuit, no magnetic coupling is involved. The dot convention is also not needed in the conductively coupled circuit.

Consider a coupled circuit as shown in Fig. 4.71.

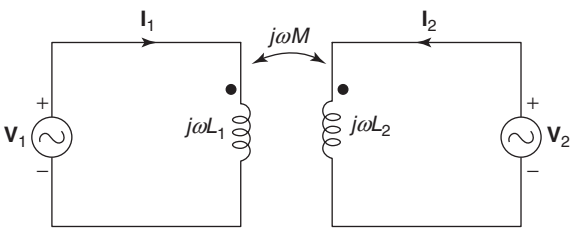


Fig. 4.71 *Coupled circuit*

The equivalent circuit in terms of dependent sources is shown in Fig. 4.72.

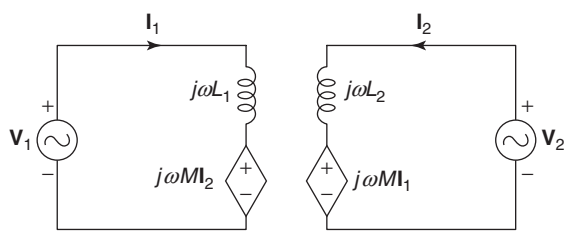


Fig. 4.72 *Equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - j\omega L_1\,\mathbf{I}_1 - j\omega M\,\mathbf{I}_2 =$$
$$j\omega L_1\,\mathbf{I}_1 + j\omega M\,\mathbf{I}_2 = \quad \ldots(4.11)$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - j\omega L_2\,\mathbf{I}_2 - j\omega M\,\mathbf{I}_1 = 0$$
$$j\omega M\,\mathbf{I}_1 + j\omega L_2\,\mathbf{I}_2 = \mathbf{V}_2 \quad \ldots(4.12)$$

Writing Eqs (4.11) and (4.12) in matrix form,

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \quad \ldots(4.13)$$

Consider a T-network as shown in Fig. 4.73.

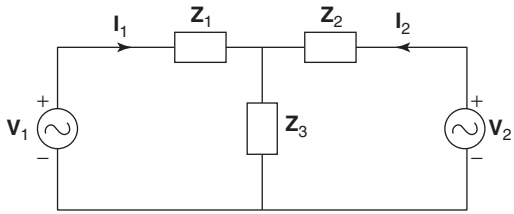


Fig. 4.73 *T-network*

Applying KVL to Mesh 1,

$$\begin{aligned} \mathbf{V}_1 - \mathbf{Z}_1\,\mathbf{I}_1 - \mathbf{Z}_3(\mathbf{I}_1 + \mathbf{I}_2) &= 0 \\ (\mathbf{Z}_1 + \mathbf{Z}_3)\,\mathbf{I}_1 + \mathbf{Z}_3\,\mathbf{I}_2 &= \mathbf{V}_1 \end{aligned} \quad \ldots (4.14)$$

Applying KVL to Mesh 2,

$$R_{\text{Th}} = [(2 \,||\, 12) + 1] \,||\, 3 = 1.43\ \Omega \quad \ldots (4.15)$$

Writing Eqs (4.14) and (4.15) in matrix form,

$$\begin{bmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & \mathbf{Z}_3 \\ \mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Comparing matrix equations,

$$\begin{aligned} \mathbf{Z}_1 + \mathbf{Z}_3 &= j\,\omega\,L_1 \\ \mathbf{Z}_3 &= j\,\omega\,M \\ \mathbf{Z}_2 + \mathbf{Z}_3 &= j\,\omega\,L_2 \end{aligned}$$

Solving these equations,

$$\begin{aligned} \mathbf{Z}_1 &= j\,\omega\,L_1 - j\,\omega\,M = j\,\omega(L_1 - M) \\ \mathbf{Z}_2 &= j\,\omega\,L_2 - j\,\omega\,M = j\,\omega(L_2 - M) \\ \mathbf{Z}_3 &= j\,\omega\,M \end{aligned}$$

Hence, the conductively coupled circuit of a magnetically coupled circuit is shown in Fig. 4.74.

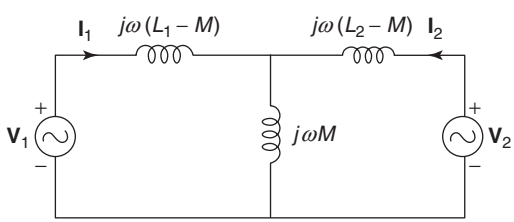


Fig. 4.74 *Conductively coupled equivalent circuit*

Example 4.34 *Find the conductively coupled equivalent circuit for the network shown in Fig. 4.75.*

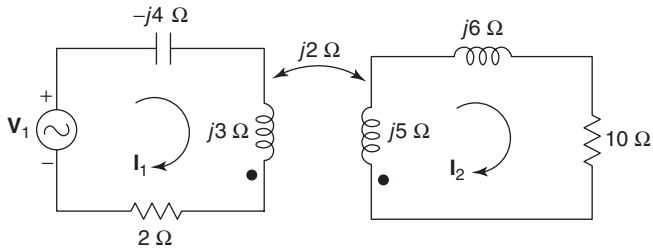


Fig. 4.75

Solution The current $\mathbf{I}_1$ leaves from the dotted end and $\mathbf{I}_2$ enters from the dotted end. Hence, mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j3 - j2 = j1\ \Omega$$

$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j5 - j2 = j3\ \Omega$$

$$\mathbf{Z}_3 = j\omega M = j2\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.76.

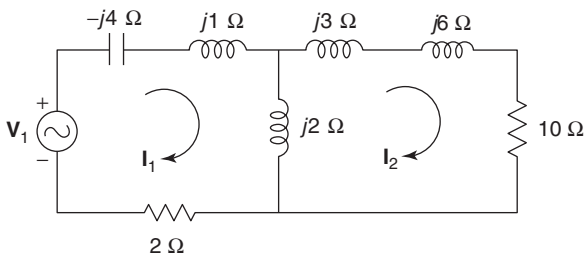


Fig. 4.76

Example 4.35 *Draw the conductively coupled equivalent circuit of Fig. 4.77.*

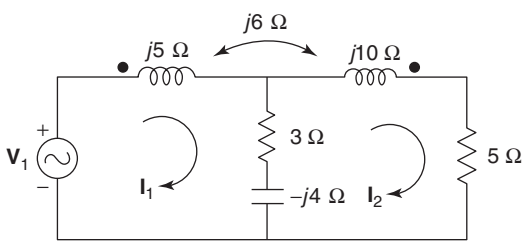


Fig. 4.77

Solution The current $\mathbf{I}_1$ enters from the dotted end and $\mathbf{I}_2$ leaves from the dotted end. Hence, the mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j5 - j6 = -j1\ \Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j10 - j6 = j4\ \Omega$$
$$\mathbf{Z}_3 = j\omega M = j6\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.78.

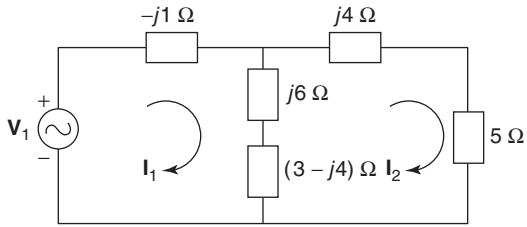


Fig. 4.78

Example 4.36 *Find the conductively coupled equivalent circuit of the network in Fig. 4.79.*

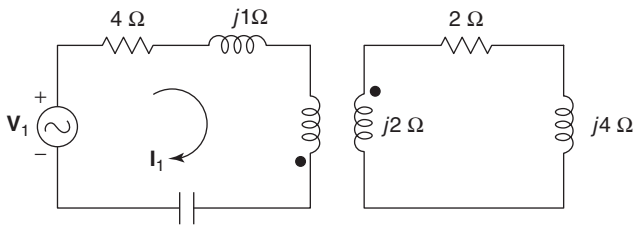


Fig. 4.79

Solution The currents $\mathbf{I}_1$ and $\mathbf{I}_2$ leave from the dotted terminals. Hence, mutual inductance is positive. In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 + M) = j\omega L_1 + j\omega M = j4 + j2 = j6\ \Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 + M) = j\omega L_2 + j\omega M = j2 + j2 = j4\ \Omega$$
$$\mathbf{Z}_3 = -j\omega M = -j2\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 4.80.

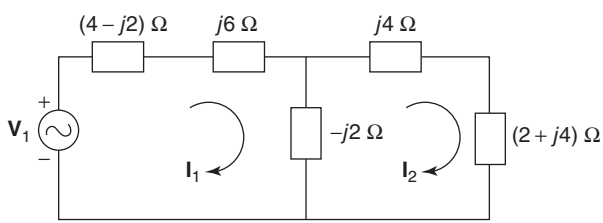


Fig. 4.80

4.10 TUNED CIRCUITS

At radio frequencies, the amplification of signals is done by tuned circuits. There are two types of tuned circuits—single-tuned circuit and double-tuned circuit. In the single-tuned circuit, only the secondary is tuned. In the double-tuned circuits, both primary and secondary of the coupled circuits are tuned. The tuned circuits are used in TV receivers to couple the antenna to the RF stage. In RF circuit design, the tuned circuits are generally employed for obtaining maximum power transfer to the load connected to secondary or for obtaining maximum possible value of the secondary voltage.

4.10.1 Single-Tuned Circuit

Consider a single-tuned circuit as shown in Fig. 4.81.

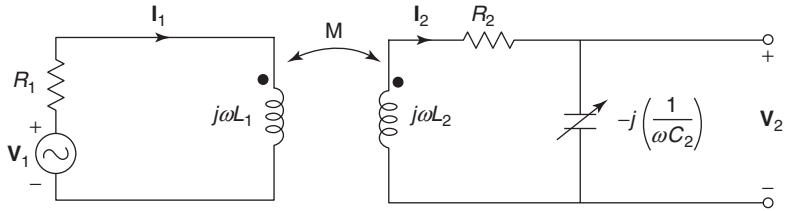


Fig. 4.81 *Single-tuned circuit*

The conductively coupled equivalent circuit is shown in Fig. 4.82.

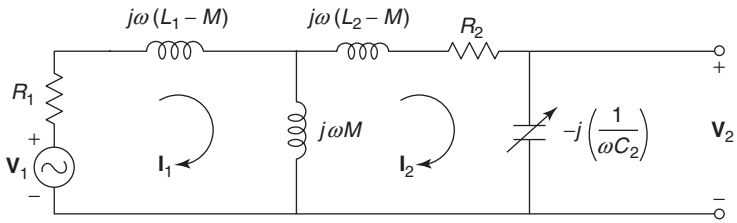


Fig. 4.82 *Conductively coupled equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - R_1\mathbf{I}_1 - j\omega(L_1 - M)\mathbf{I}_1 - j\omega M(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(4.16)$$

Applying KVL to Mesh 2,

$$-j\omega M(\mathbf{I}_2 - \mathbf{I}_1) - j\omega(L_2 - M)\mathbf{I}_2 - R_2\mathbf{I}_2 + j\frac{1}{\omega C_2}\mathbf{I}_2 = 0$$

$$-j\omega M\mathbf{I}_1 + \left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right]\mathbf{I}_2 = 0 \qquad \ldots(4.17)$$

From Eq. (4.17),

$$\mathbf{I}_1 = \frac{R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{j\omega M}\mathbf{I}_2 \qquad \ldots(4.18)$$

Substituting Eq. (4.18) in Eq. (4.16),

$$\left(R_1 + j\omega L_1\right)\left[\frac{R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{j\omega M}\right]\mathbf{I}_2 - j\omega M\mathbf{I}_2 = \mathbf{V}_1$$

$$\mathbf{I}_2\left\{\frac{\left(R_1 + j\omega L_1\right)\left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right] + \omega^2 M^2}{j\omega M}\right\} = \mathbf{V}_1$$

$$\mathbf{I}_2 = \frac{j\omega M\,\mathbf{V}_1}{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}$$

1. Output Voltage

$$\mathbf{V}_2 = -j\left(\frac{1}{\omega C_2}\right)\mathbf{I}_2 = \frac{1}{j\omega C_2}\mathbf{I}_2$$

$$= \frac{MV_1}{C_2\left\{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

2. Voltage Amplification

$$A = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{M}{C_2\left\{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

If $R_1 >> j\omega L_1$ and secondary circuit resonates at

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}}$$

then, at the resonant frequency ω_0,

$$I_2 = \frac{\omega_0 M V_1}{R_1 R_2 + \omega_0^2 M^2}$$

$$V_2 = \frac{MV_1}{C_2(R_1 R_2 + \omega_0^2 M^2)}$$

$$A = \frac{M}{C_2(R_1 R_2 + \omega_0^2 M^2)}$$

Output current, output voltage and voltage amplification depends on the mutual inductance M at resonant frequency. The maximum amplification or maximum output voltage can be obtained by

$$\frac{dA}{dM} = 0$$

$$\frac{C_2(R_1 R_2 + \omega_0^2 M^2) - 2C_2\omega_0^2 M^2}{\left[C_2(R_1 R_2 + \omega_0^2 M^2)\right]^2} = 0$$

$$R_1 R_2 + \omega_0^2 M^2 - 2\omega_0^2 M^2 = 0$$

$$\omega_0^2 M^2 = R_1 R_2$$

$$M = \frac{\sqrt{R_1 R_2}}{\omega_0}$$

where $M = \frac{\sqrt{R_1 R_2}}{\omega_0}$ is the critical value of mutual inductance.

Hence, maximum value of $I_{2,}\ V_2$ and A is

$$I_{2_{max}} = \frac{V_1}{2\sqrt{R_1 R_2}}$$

$$V_{2_{max}} = \frac{V_1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

$$A_{max} = \frac{1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

The variation of output voltage V_0 with angular frequency ω for different values of coefficient of coupling k is shown in Fig. 4.83.

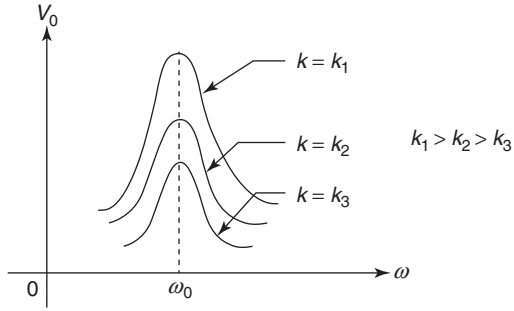


Fig. 4.83 *Variation of V_0 with ω for different values of k*

4.10.2 Double-Tuned Circuit

Consider a double-tuned circuit with tuning capacitors placed both in primary and secondary circuits as shown in Fig. 4.84.

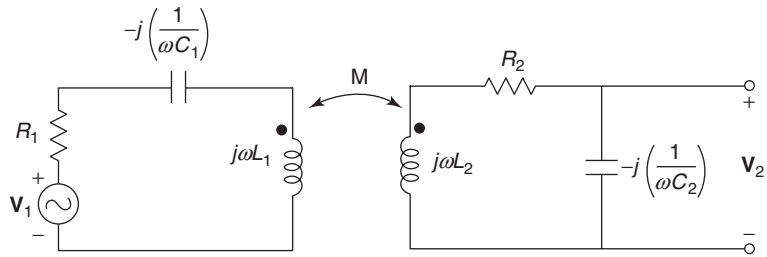


Fig. 4.84 *Double-tuned circuit*

The conductively coupled equivalent circuit is shown in Fig. 4.85.

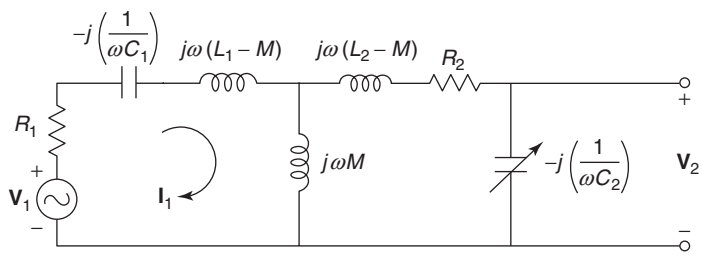


Fig. 4.85 *Conductively coupled equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - R_1\mathbf{I}_1 + j\frac{1}{\omega C_2}\mathbf{I}_1 - j\omega(L_1 - M)\mathbf{I}_1 - j\omega M(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$\left(R_1 + j\omega L_1 - j\frac{1}{\omega C_1}\right)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V}_1 \qquad ...(4.19)$$

Applying KVL to Mesh 2,

$$-j\omega M(\mathbf{I}_2 - \mathbf{I}_1) - j\omega(L_2 - M)\mathbf{I}_2 - R_2\mathbf{I}_2 + j\frac{1}{\omega C_2}\mathbf{I}_2 = 0$$

$$-j\omega M\mathbf{I}_1 + \left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right]\mathbf{I}_2 = 0 \qquad ...(4.20)$$

From Eq. (4.20),

$$\mathbf{I}_1 = \frac{R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)}{j\omega M}\mathbf{I}_2 \qquad ...(4.21)$$

Substituting Eq. (4.21) in Eq. (4.19),

$$\left(R_1 + j\omega L_1 - j\frac{1}{\omega C_2}\right)\left[\frac{R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)}{j\omega M}\right]\mathbf{I}_2 - j\omega M\mathbf{I}_2 = \mathbf{V}_1$$

$$\mathbf{I}_2\left\{\frac{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}{j\omega M}\right\} = \mathbf{V}_1$$

$$\mathbf{I}_2 = \frac{j\omega M\mathbf{V}_1}{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right) + \omega^2 M^2\right]}$$

1. Output Voltage

$$\mathbf{V}_2 = -j\left(\frac{1}{\omega C_2}\right)\mathbf{I}_2 = \frac{1}{j\omega C_2}\mathbf{I}_2$$

$$= \frac{M\mathbf{V}_1}{C_2\left\{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

2. Voltage Amplification

$$A = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{M}{C_2\left\{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

If the primary and secondary circuits resonate at the same frequency ω_0,

$$\omega_0 = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$$

Then, at the resonant frequency $\omega_{0,}$

$$I_2 = \frac{\omega_0 M V_1}{R_1 R_2 + \omega_0^2 M^2}$$

$$V_2 = \frac{M V_1}{C_2 (R_1 R_2 + \omega_0^2 M^2)}$$

$$A = \frac{M}{C_2 (R_1 R_2 + \omega_0^2 M^2)}$$

The maximum amplification or maximum output voltage can be obtained by,

$$\frac{dA}{dM} = 0$$

$$\frac{C_2 (R_1 R_2 + \omega_0^2 M^2) - 2C_2 \omega_0^2 M^2}{\left[C_2 (R_1 R_2 + \omega_0^2 M^2)\right]^2} = 0$$

$$\left(R_1 R_2 + \omega_0^2 M^2\right) - 2\omega_0^2 M^2 = 0$$

$$\omega_0^2 M^2 = R_1 R_2$$

$$M = \frac{\sqrt{R_1 R_2}}{\omega_0}$$

where $M = \dfrac{\sqrt{R_1 R_2}}{\omega_0}$ is the critical value of mutual inductance.

Hence, maximum value of $I_{2,}$ V_2 and A is

$$I_{2_{\max}} = \frac{V_1}{2\sqrt{R_1 R_2}}$$

$$V_{2_{\max}} = \frac{V_1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

$$A_{\max} = \frac{1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

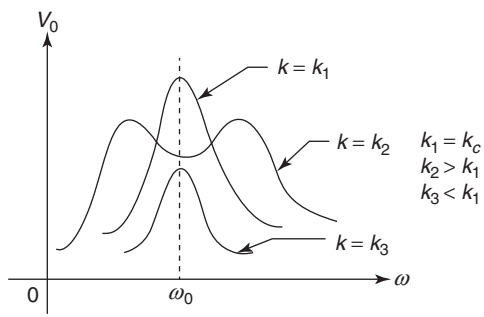


Fig. 4.86 *Variation of V_0 with ω for different values of k*

The variation of the output voltage V_0 with the angular frequency ω for different values of coefficient of coupling k is shown in Fig. 4.86.

Example 4.37 *For the single-tuned circuit shown in Fig. 4.87, determine (a) the resonant frequency, (b) Output voltage at resonance, and (c) maximum output voltage. Assume $R_1 >> \omega_0 L_1$ and $k = 0.8$.*

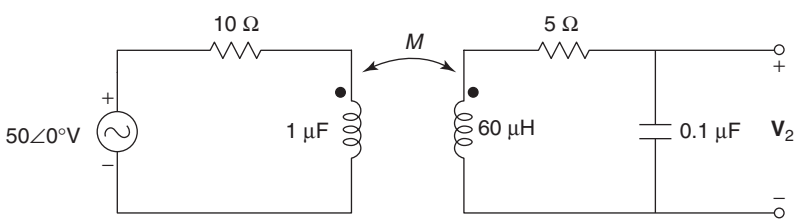


Fig. 4.87

Solution

$$M = k\sqrt{L_1 L_2} = 0.8\sqrt{(1\times10^{-6})(60\times10^{-6})} = 6.2\ \mu\text{H}$$

(a) Resonant frequency

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\sqrt{(60\times10^{-6})(0.1\times10^{-6})}} = 0.41\times10^{6}\ \text{rad/s}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{0.41\times10^{6}}{2\pi} = 0.0653\ \text{MHz} = 65.3\ \text{kHz}$$

(b) Output voltage at resonance

$$V_0 = \frac{MV_1}{C_2\left(R_1R_2 + \omega_0^2 M^2\right)} = \frac{6.2\times10^{-6}\times50}{0.1\times10^{-6}\left[(10\times5)+\left(0.42\times10^{6}\right)^2\left(6.2\times10^{-6}\right)^2\right]} = 54.6\ \text{V}$$

(c) Maximum output voltage

$$V_{2_{\max}} = \frac{V_1}{2\omega_0 C_2\sqrt{R_1R_2}} = \frac{50}{2(0.42\times10^{6})(0.1\times10^{-6})\sqrt{10\times5}} = 84.18\ \text{V}$$

Example 4.38 *The resonant frequency of the tuned circuit shown in Fig. 4.88 is 1000 rad/s. Calculate the self-inductances of two coils and critical value of mutual inductance.*

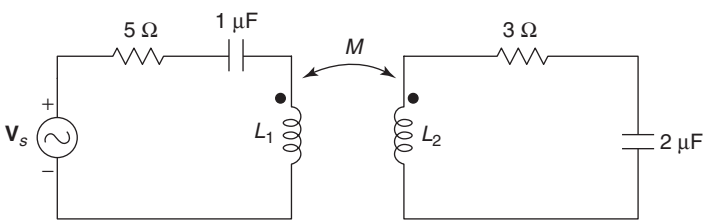


Fig. 4.88

Solution

$$\omega_0 = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$$

$$L_1 = \frac{1}{\omega_0^2 C_1} = \frac{1}{(1000)^2\left(1\times10^{-6}\right)} = 1\ \text{H}$$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{1}{(1000)^2\left(2\times10^{-6}\right)} = 0.5\ \text{H}$$

$$M_c = \frac{\sqrt{R_1R_2}}{\omega_0} = \frac{\sqrt{5\times3}}{1000} = 3.87\ \text{mH}$$

Example 4.39 *The tuned frequency of a double-tuned circuit shown in Fig. 4.89 is 10^4 rad/s. If the source voltage is 2 V and has a resistance of 0.1 Ω, calculate the maximum output voltage.*

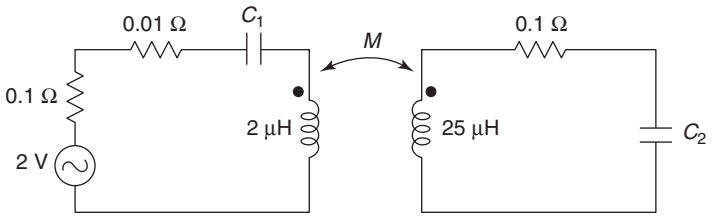


Fig. 4.89

Solution

$$M_C = \frac{\sqrt{(R_1 + R_S)R_2}}{\omega_0} = \frac{\sqrt{(0.01+0.1)(0.1)}}{10^4} = 10.49\ \mu\text{H}$$

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}}$$

$$10^4 = \frac{1}{\sqrt{(25\times10^{-6})C_2}}$$

$$C_2 = 0.4\ \text{mF}$$

$$V_{0_{\max}} = \frac{V_1}{2\omega_0 C_2\sqrt{(R_S + R_1)R_2}} = \frac{2}{2\times10^4\times0.4\times10^{-3}\times\sqrt{(0.1+0.01)(0.1)}} = 2.38\ \text{V}$$

Exercises

4.1 Two coupled coils have inductances of 0.8 H and 0.2 H. The coefficient of coupling is 0.90. Find the mutual inductance and the turns ratio $\frac{N_1}{N_2}$.

[0.36 H, 2]

4.2 Two coils with coefficient of coupling 0.5 are connected in such a way that they magnetise (i) in the same direction, and (ii) in opposite directions. The corresponding equivalent inductances are 1.9 H and 0.7 H. Find self-inductances of the two coils and the mutual inductance between them.

[0.4 H, 0.9 H, 0.3 H]

4.3 Two coils having 3000 and 2000 turns are wound on a magnetic ring. 60% of the flux produced in the first coil links with the second coil. A current of 3 A produce a flux of 0.5 mwb in the first coil and 0.3 mwb in the second coil. Determine the mutual inductance and coefficient of coupling.

[0.2 H, 0.63]

4.4 Find the equivalent inductance of the network shown in Fig. 4.90.

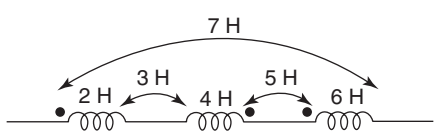


Fig. 4.90

[10 H]

4.5 Find the effective inductance of the network shown in Fig. 4.91.

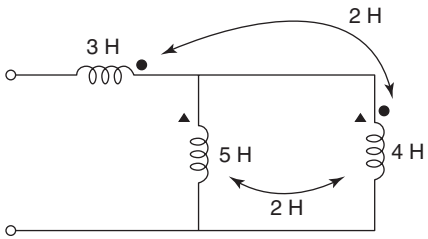


Fig. 4.91

[4.8 H]

4.6 Write mesh equations of the network shown in Fig. 4.92.

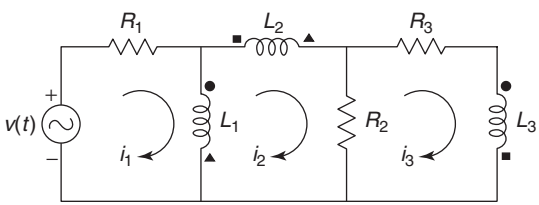


Fig. 4.92

$$\left[\begin{array}{l} v = i_1 R_1 + L_1 \dfrac{d}{dt}(i_1 - i_2) + M_{12} \dfrac{di_2}{dt} + M_{13} \dfrac{di_3}{dt} \\ R_2(i_3 - i_2) + R_3 i_3 + L_3 \dfrac{di_3}{dt} + M_{13} \dfrac{d}{dt}(i_1 - i_2) \\ -M_{23} \dfrac{di_2}{dt} = 0 \end{array}\right]$$

4.7 Find the input impedance at terminals *AB* of the coupled circuits shown in Fig. 4.93 to 4.95.

(i)

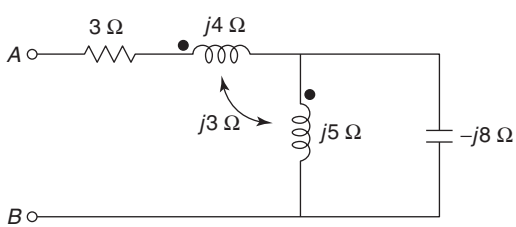


Fig. 4.93

(ii)

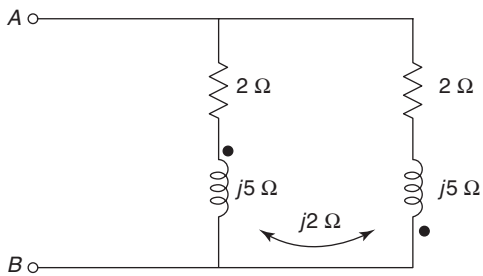


Fig. 4.94

(iii)

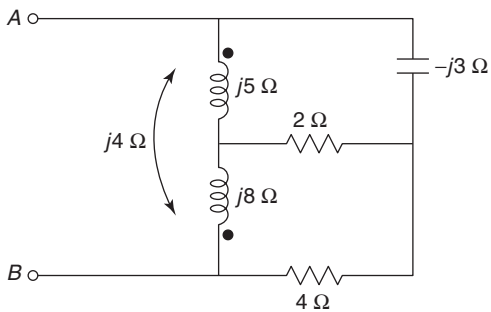


Fig. 4.95

$$\left[\begin{array}{ll} \text{(a)}\,(3 + j36.3)\,\Omega & \text{(b)}\,(1 + j1.5)\,\Omega \\ \text{(c)}\,(6.22 + j4.65)\,\Omega & \end{array}\right]$$

4.8 In the coupled circuit shown in Fig. 4.96, find $\mathbf{V}_2$ for which $\mathbf{I}_1 = 0$. What voltage appears at the 8 Ω inductive reactance under this condition?

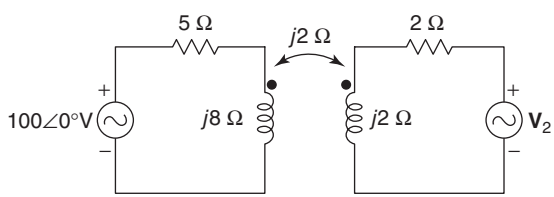


Fig. 4.96

[141.5∠ − 45°V, 100∠0°V]

4.9 For the coupled circuit shown in Fig. 4.97, find the components of the current $\mathbf{I}_2$ resulting from each source $\mathbf{V}_1$ and $\mathbf{V}_2$.

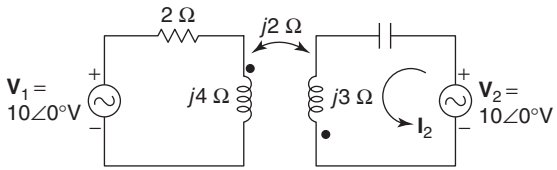


Fig. 4.97

[0.77∠112.6°A, 1.72∠86.05°A]

4.10 Find the voltage across the 5 Ω resistor in the network shown in Fig. 4.98.

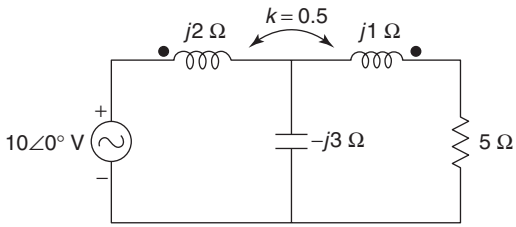


Fig. 4.98

[19.2∠ – 33.02°V]

4.11 Find the power dissipated in the 5 Ω resistor in the network of Fig. 4.99.

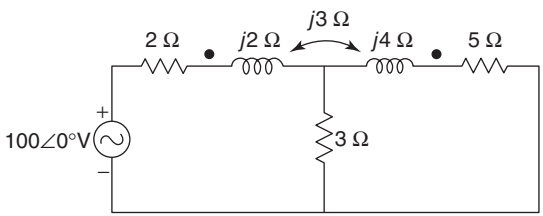


Fig. 4.99

[668.16 W]

4.12 Find the current **I** in the circuit of Fig. 4.100.

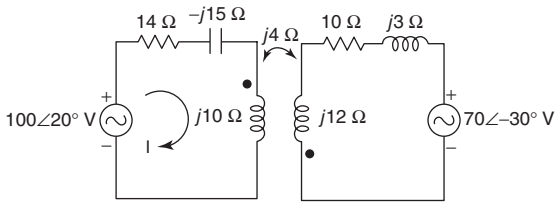


Fig. 4.100

[7.07∠45° V, 1.

4.13 Obtain a conductively coupled circuit for the circuit shown in Fig. 4.101.

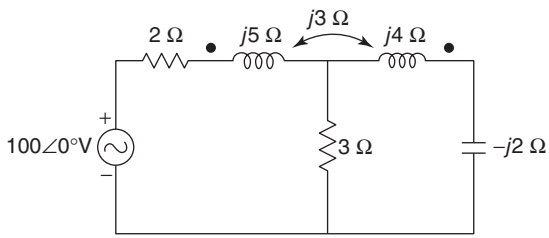


Fig. 4.101

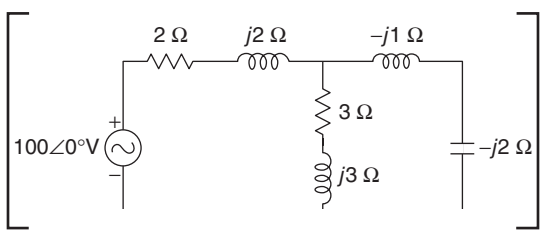


Fig. 4.102

Objective-Type Questions

4.1 Two coils are wound on a common magnetic core. The sign of mutual inductance M for finding out effective inductance of each coil is positive if the

(a) two coils are wound in the same sense.
(b) fluxes produced by the two coils are equal
(c) fluxes produced by the coils act in the same direction
(d) fluxes produced by the two coils act in opposition

4.2 When two coils having self-inductances of L_1 and L_2 are coupled through a mutual inductance M, the coefficient of coupling k is given by

(a) $k = \dfrac{M}{\sqrt{2L_1L_2}}$ (b) $k = \dfrac{M}{\sqrt{L_1L_2}}$

(c) $k = \dfrac{2M}{\sqrt{L_1L_2}}$ (d) $k = \dfrac{L_1L_2}{M}$

4.3 The overall inductance of two coils connected in series, with mutual inductance aiding self-inductance is L_1; with mutual inductance opposing self-inductance, the overall inductance is L_2. The mutual inductance M is given by

(a) $L_1 + L_2$ (b) $L_1 - L_2$

(c) $\frac{1}{4}(L_1 - L_2)$ (d) $\frac{1}{2}(L_1 + L_2)$

4.4 Consider the following statements:
The coefficient of coupling between two oils depends upon

1. Orientation of the coils
2. Core material
3. Number of turns on the two coils
4. Self-inductance of the two coils

of these statements,

(a) 1, 2 and 3 are correct
(b) 1 and 2 are correct
(c) 3 and 4 are correct
(d) 1, 2 and 4 are correct

4.5 Two coupled coils connected in series have an equivalent inductance of 16 mH or 8 mH depending on the inter connection.
Then the mutual inductance M between the coils is

(a) 12 mH (b) $8\sqrt{2}$ mH
(c) 4 mH (d) 2 mH

4.6 Two coupled coils with $L_1 = L_2 = 0.6$ H have a coupling coefficient of $k = 0.8$. The turns ratio $\frac{N_1}{N_2}$ is

(a) 4 (b) 2
(c) 1 (d) 0.5

4.7 The coupling between two magnetically coupled coils is said to be ideal if the coefficient of coupling is

(a) zero (b) 0.5
(c) 1 (d) 2

4.8 The mutual inductance between two coupled coils is 10 mH . If the turns in one coil are doubled and that in the other are halved then the mutual inductance will be

(a) 5 mH (b) 10 mH
(c) 14 mH (d) 20 mH

4.9 Two perfectly coupled coils each of 1 H self-inductance are connected in parallel so as to aid each other. The overall inductance in henrys is

(a) 2 (b) 1
(c) $\frac{1}{2}$ (d) Zero

4.10 The impedance Z as shown in Fig.4.103 is

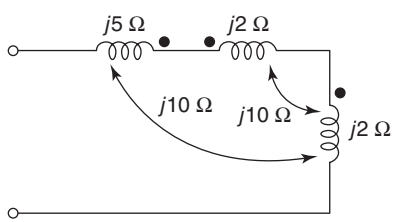


Fig. 4.103

(a) j 29 Ω (b) j9 Ω
(c) j19 Ω (d) j39 Ω

Answers to Objective-Type Questions

4.1	(c)	4.2	(b)	4.3	(c)	4.4	(d)	4.5	(d)	4.6	(c)
4.7	(c)	4.8	(b)	4.9	(b)	4.10	(b)				

5

Resonance

5.1 INTRODUCTION

Resonance is a very important phenomenon in many electrical applications. The study of resonance is very useful in the telecommunication field. A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it. At resonance, the circuit thus behaves as a pure resistor and the net reactance is zero.

5.2 SERIES RESONANCE

Consider the series *RLC* circuit as shown in Fig. 5.1. The impedance of the circuit is

$$\mathbf{Z} = R + jX_L - jX_C = R + j(X_L - X_C)$$

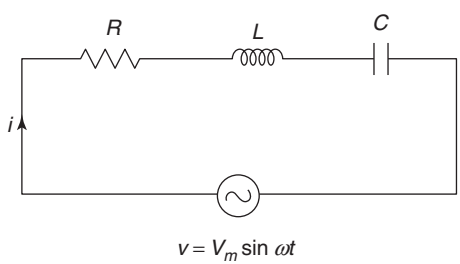


Fig. 5.1 *Series circuit*

At resonance, the circuit is purely resistive.

$$X_L - X_C = 0$$

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

where f_0 is called the resonant frequency of the circuit.

1. Power Factor

$$\text{Power factor} = \cos\phi = \frac{R}{Z}$$

At resonance $Z = R$

$$\text{Power factor} = \frac{R}{R} = 1$$

2. Current Since impedance is minimum, the current is maximum at resonance. Thus, the circuit accepts more current and as such, an *RLC* circuit under resonance is called an *accepator circuit*.

$$I_0 = \frac{V}{Z} = \frac{V}{R}$$

3. Voltage At resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 L I_0 = \frac{1}{\omega_0 C} I_0$$

$$V_{L_0} = V_{C_0}$$

Thus, potential difference across inductor equal to potential difference across capacitor being equal and opposite cancel each other. Also, since I_0 is maximum, V_{L_0} and V_{C_0} will also be maximum. Thus, voltage magnification takes place during resonance. Hence, it is also referred to as voltage magnification circuit.

4. Phasor Diagram Figure 5.2 shows the phasor diagram of a series resonant circuit.

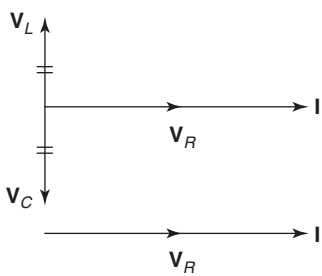


Fig. 5.2 *Phasor diagram*

5. Behaviour of R, L and C with Change in Frequency Resistance remains constant with the change in frequencies. Inductive reactance X_L is directly proportional to frequency *f*. It can be drawn as a straight line passing through the origin. Capacitive reactance X_C is inversely proportional to the frequency *f*. It can be drawn as a rectangular hyperbola in the fourth quadrant. Figure 5.3 shows the behaviour of *R*, *L* and *C* with change in frequency.

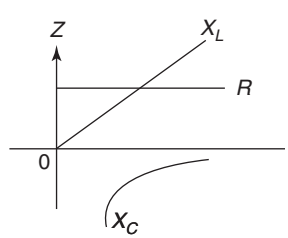


Fig. 5.3 *Behaviour of R, L and C with change in frequency*

Total impedance $\mathbf{Z} = R + j(X_L - X_C)$

(a) When $f < f_0$, impedance is capacitive and decreases up to f_0. The power factor is leading in nature.
(b) At $f = f_0$, impedance is resistive. The power factor is unity.
(c) When $f > f_0$, impedance is inductive and goes on increasing beyond f_0. The power factor is lagging in nature. Figure 5.4 shows the impedance curve of series resonant circuit.

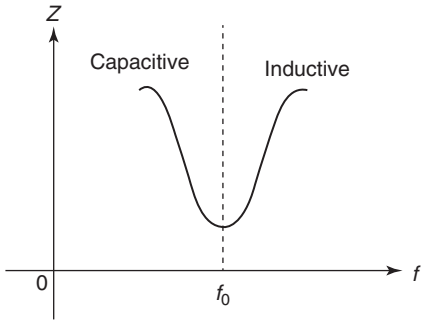


Fig. 5.4 *Impedance curve*

6. *Bandwidth* For the series *RLC* circuit, bandwidth is defined as the range of frequencies for which the power delivered to *R* is greater than or equal to $\frac{P_0}{2}$ where P_0 is the power delivered to *R* at resonance.

From the shape of the resonance curve shown in Fig. 5.5, it is clear that there are two frequencies for which the power delivered to *R* is half the power at resonance. For this reason, these frequencies are referred as those corresponding to the half-power points. The magnitude of the current at each half-power point is the same.

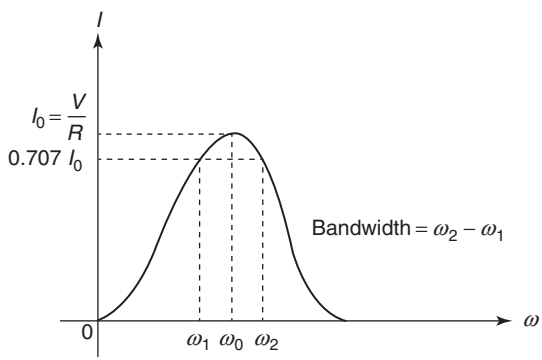


Fig. 5.5 *Resonance curve*

Hence,
$$I_1^2 R = \frac{1}{2} I_0^2 R = I_2^2 R$$

where the subscript 1 denotes the lower half point and the subscript 2, the higher half point. It follows then that

$$I_1 = I_2 = \frac{I_0}{\sqrt{2}} = 0.707 I_0$$

Accordingly, the bandwidth may be identified on the resonance curve as the range of frequencies over which the magnitude of the current is equal to or greater than 0.707 of the current at resonance. In Fig. 5.5, the bandwidth is $\omega_2 - \omega_1$.

Expression for Bandwidth Generally, at any frequency ω,

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \ldots(i)$$

At half-power points,

$$I = \frac{I_0}{\sqrt{2}}$$

But $$I_0 = \frac{V}{R}$$

$$I = \frac{V}{\sqrt{2}R} \qquad \ldots(ii)$$

From Eqs (i) and (ii),

$$\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V}{\sqrt{2}R}$$

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}R$$

Squaring both the sides,

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\omega L - \frac{1}{\omega C} \pm R = 0$$

$$\omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$$

$$\omega = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

For low values of R, the term $\left(\frac{R^2}{4L^2}\right)$ can be neglected in comparison with the term $\frac{1}{LC}$.

Then ω is given by, $$\omega = \pm\frac{R}{2L} \pm \sqrt{\frac{1}{LC}} = \pm\frac{R}{2L} \pm \frac{1}{\sqrt{LC}}$$

The resonant frequency for this circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega = \pm\frac{R}{2L} + \omega_0 \qquad \text{(considering only posititve sign of } \omega_0\text{)}$$

$$\omega_1 = \omega_0 - \frac{R}{2L}$$

and $$\omega_2 = \omega_0 + \frac{R}{2L}$$

$$f_1 = f_0 - \frac{R}{4\pi L}$$

and $$f_2 = f_0 + \frac{R}{4\pi L}$$

$$\text{Bandwidth} = \omega_2 - \omega_1 = \frac{R}{L}$$

or $$\text{Bandwidth} = f_2 - f_1 = \frac{R}{2\pi L}$$

7. Quality Factor It is measure of voltage magnification in the series resonant circuit. It is also a measure of selectivity or sharpness of the series resonant circuit.

$$Q_0 = \frac{\text{Voltage across inductor or capacitor}}{\text{Voltage at resonance}} = \frac{V_{L_0}}{V} = \frac{V_{C_0}}{V}$$

Substituting values of V_{L_0} and V,

$$Q_0 = \frac{I_0 X_{L_0}}{I_0 R} = \frac{X_{L_0}}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Substituting values of ω_0,

$$Q_0 = \frac{\left(\frac{1}{\sqrt{LC}}\right)L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Example 5.1 *A series RLC circuit has the following parameter values: R = 10 Ω, L = 0.01 H, C = 100 μF. Compute the resonant frequency, bandwidth, and lower and upper frequencies of the bandwidth.*

Solution $R = 10\ \Omega, \quad L = 0.01\ \text{H}, \quad C = 100\ \mu\text{F}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01 \times 100 \times 10^{-6}}} = 159.15\ \text{Hz}$$

(b) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{10}{2\pi \times 0.01} = 159.15\ \text{Hz}$$

(c) Lower frequency of bandwidth

$$f_1 = f_0 - \frac{BW}{2} = 159.15 - \frac{159.15}{2} = 79.58\ \text{Hz}$$

(d) Upper frequency of bandwidth

$$f_2 = f_0 + \frac{BW}{2} = 159.15 + \frac{159.15}{2} = 238.73\ \text{Hz}$$

Example 5.2 *An RLC series circuit with a resistance of 10 Ω, inductance of 0.2 H and a capacitance of 40 μF is supplied with a 100 V supply at variable frequency. Find the following w.r.t. the series resonant circuit:*

(a) frequency of which resonance takes place
(b) current
(c) power
(d) power factor
(e) voltage across RLC at that time
(f) quality factor
(g) half-power points
(h) resonance and phasor diagrams

Solution $R = 10\ \Omega, \quad L = 0.2\ \text{H}, \quad C = 40\ \mu\text{F}, \quad V = 100\ \text{V}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 40 \times 10^{-6}}} = 56.3\ \text{Hz}$$

(b) Current

$$I_0 = \frac{V}{R} = \frac{100}{10} = 10\ \text{A}$$

(c) Power

$$P_0 = I_0^2 R = (10)^2 \times 10 = 1000\ \text{W}$$

(d) Power factor

$$\text{pf} = 1$$

(e) Voltage across R, L, C

$$V_{R_0} = R\, I_0 = 10 \times 10 = 100\ \text{V}$$

$$V_{L_0} = X_{L_0}\, I_0 = 2\pi f_0 L I_0 = 2\pi \times 56.3 \times 0.2 \times 10 = 707.5\ \text{V}$$

$$V_{C_0} = X_{C_0} I_0 = \frac{1}{2\pi f_0 C} I_0 = \frac{1}{2\pi \times 56.3 \times 40 \times 10^{-6}} \times 10 = 707.5\ \text{V}$$

(f) Quality factor

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.2}{40 \times 10^{-6}}} = 7.07$$

(g) Half-power points

$$f_1 = f_0 - \frac{R}{4\pi L} = 56.3 - \frac{10}{4\pi \times 0.2} = 52.32\ \text{Hz}$$

$$f_2 = f_0 + \frac{R}{4\pi L} = 56.3 + \frac{10}{4\pi \times 0.2} = 60.3\ \text{Hz}$$

(h) Resonance and phasor diagram are shown in Fig. 5.6.

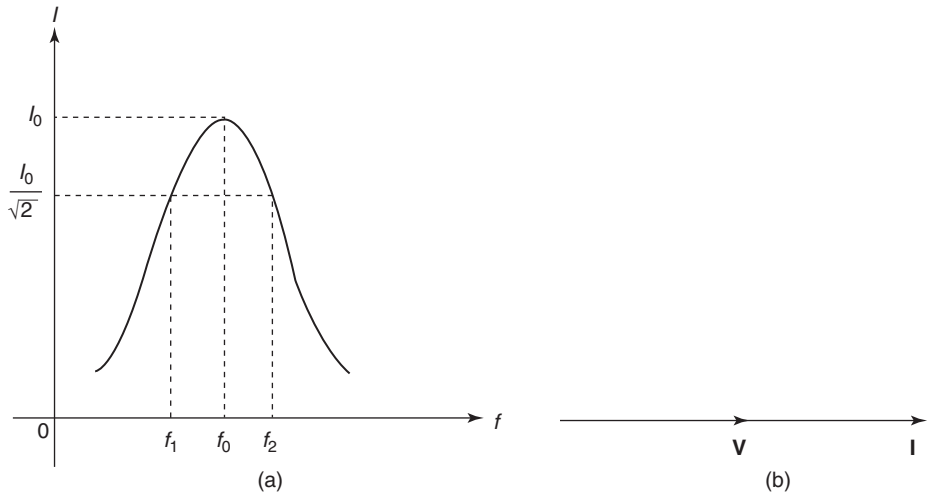


Fig. 5.6

Example 5.3 *A series RLC circuit is connected to a 200 V ac supply. The current drawn by the circuit at resonance is 20 A. The voltage drop across the capacitor is 5000 V at series resonance. Calculate resistance and inductance if capacitance is 4 μF. Also, calculate the resonant frequency.*

Solution $V = 200$ V, $I_0 = 20$ A, $V_{C_0} = 5000$ V, $C = 4$ μF

(a) Resistance

$$R = \frac{V}{I_0} = \frac{200}{20} = 10\ \Omega$$

(b) Resonant frequency

$$X_{C_0} = \frac{V_{C_0}}{I_0} = \frac{5000}{20} = 250\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$250 = \frac{1}{2\pi \times f_0 \times 4 \times 10^{-6}}$$

$$f_0 = 159.15 \text{ Hz}$$

(c) Inductance
At resonance

$$X_{C_0} = X_{L_0} = 250\ \Omega$$

$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 159.15 \times L$$

$$L = 0.25 \text{ H}$$

Example 5.4 *A resistor and a capacitor are connected in series with a variable inductor. When the circuit is connected to a 230 V, 50 Hz supply, the maximum current obtained by varying the inductance is 2 A. The voltage across the capacitor is 500 V. Calculate the resistance, inductance and capacitance of the circuit.*

Solution

$$V = 230 \text{ V}, \quad f_0 = 50 \text{ Hz}, \quad I_0 = 2 \text{ A}, \quad V_{C_0} = 500 \text{ V}$$

$$I_0 = 2 \text{ A}$$

$$V_{C_0} = 500 \text{ V}$$

(a) Resistance

$$R = \frac{V}{I_0} = \frac{230}{2} = 115\ \Omega$$

(b) Capacitance

$$X_{C_0} = \frac{V_{C_0}}{I_0} = \frac{500}{2} = 250\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$250 = \frac{1}{2\pi \times 50 \times C}$$

$$C = 12.73\ \mu\text{F}$$

(c) Inductance
At resonance

$$X_{C_0} = X_{L_0} = 250\ \Omega$$

$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 50 \times L$$

$$L = 0.795 \text{ H}$$

Example 5.5 *A coil of 2 Ω resistance and 0.01 H inductance is connected in series with a capacitor across 200 V mains. What must be the capacitance in order that maximum current occurs at a frequency of 50 Hz? Find also the current and the voltage across the capacitor.*

Solution

$$R = 2\ \Omega, \quad L = 0.01 \text{ H}, \quad V = 200 \text{ V}, \quad f_0 = 50 \text{ Hz}$$

(a) Capacitance

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$50 = \frac{1}{2\pi\sqrt{0.01 \times C}}$$

$$C = 1013.2\ \mu\text{F}$$

(b) Current

$$I_0 = \frac{V}{R} = \frac{200}{2} = 100 \text{ A}$$

(c) Voltage across capacitor

$$V_{C_0} = I_0 X_{C_0} = I_0 \frac{1}{2\pi f_0 C} = 100 \times \frac{1}{2\pi \times 50 \times 1013.2 \times 10^{-6}} = 314.16 \text{ V}$$

Example 5.6 *A series RLC circuit has a quality factor of 5 at 50 rad/s. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. Find the circuit constants.*

Solution

$$Q_0 = 5, \quad \omega_0 = 50 \text{ rad/s}, \quad I_0 = 10 \text{ A}, \quad V = 100 \text{ V}$$

$$I_0 = \frac{V}{R}$$

$$10 = \frac{100}{R}$$

$$R = 10 \ \Omega$$

$$Q_0 = \frac{\omega_0 L}{R}$$

$$5 = \frac{50 \times L}{10}$$

$$L = 1 \text{ H}$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$5 = \frac{1}{10}\sqrt{\frac{1}{C}}$$

$$C = 400 \ \mu\text{F}$$

Example 5.7 *A series RLC circuit has R = 2.5 Ω, C = 100 μF and a variable inductance. The applied voltage is 50 V at 800 rad/s. The inductance is varied till the voltage across the resistance is maximum. Find (a) value of inductance under this condition, (b) Q, (c) current, and (d) voltage across resistance, capacitance and inductance.*

Solution

$$R = 2.5 \ \Omega, \quad C = 100 \ \mu\text{F}, \quad V = 50 \text{ V}, \quad \omega = 800 \text{ rad/s}$$

(a) Value of inductance

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$800 = \frac{1}{\sqrt{L \times 100 \times 10^{-6}}}$$

$$L = 0.0156 \text{ H}$$

(b) Quality factor

$$Q_0 = \frac{\omega_0 L}{R} = \frac{800 \times 0.0156}{2.5} = 5$$

(c) Current

$$I_0 = \frac{V}{R} = \frac{50}{2.5} = 20 \text{ A}$$

(d) Voltage across resistance, capacitance and inductance

$$V_{R_0} = I_0 \, R = 20 \times 2.5 = 50 \text{ V}$$
$$V_{L_0} = Q_0 \, V = 5 \times 50 = 250 \text{ V}$$
$$V_{C_0} = Q_0 \, V = 5 \times 50 = 250 \text{ V}$$

Example 5.8 *A series RLC circuit which resonates at 500 kHz has $R = 25\ \Omega$, $L = 100\ \mu H$ and $C = 1000$ pF. Determine the quality factor, new value of C required to resonate at 500 kHz when the value of L is doubled, and the new quality factor.*

Solution $f_0 = 500$ kHz, $R = 25\ \Omega$, $L = 100\ \mu$H, $C = 1000$ pF

(a) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{\frac{100 \times 10^{-6}}{1000 \times 10^{-12}}} = 12.65$$

(b) New value of C when $L = 200\ \mu$H

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$500 \times 10^3 = \frac{1}{2\pi\sqrt{200 \times 10^{-6} \times C}}$$
$$C = 506.61 \text{ pF}$$

(c) New Q factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{\frac{200 \times 10^{-6}}{506.61 \times 10^{-12}}} = 25.13$$

Example 5.9 *Find the values of R,L and C in a series RLC circuit that resonates at 1.5 kHz and consumes 50 W from a 50 V ac source operating at the resonant frequency. The bandwidth is 0.75 kHz.*

Solution $f_0 = 1.5$ kHz, $P = 50$ W, $V = 50$ V, $BW = 0.75$ kHz

$$P = \frac{V^2}{R}$$
$$50 = \frac{(50)^2}{R}$$
$$R = 50\ \Omega$$

$$BW = \frac{R}{2\pi L}$$

$$0.75\times10^{3} = \frac{50}{2\pi\times L}$$

$$L = 0.011\ \text{H}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$1.5\times10^{3} = \frac{1}{2\pi\sqrt{0.011\times C}}$$

$$C = 1.023\ \mu\text{F}$$

Example 5.10 *A 50 Ω resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 5.7. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.*

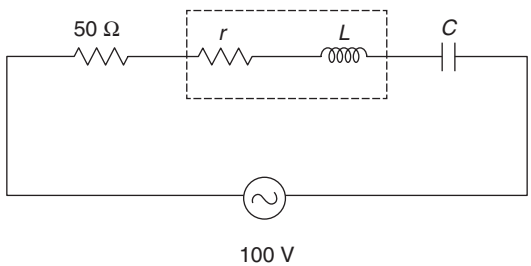


Fig. 5.7

Solution

$$V = 100\ \text{V}, \quad R = 50\ \Omega, \quad f_0 = 200\ \text{Hz}, \quad I_0 = 0.7\ \text{A}$$

$$V_{C_0} = 200\ \text{V}$$

$$V_{C_0} = I_0 X_{C_0}$$

$$200 = 0.7 X_{C_0}$$

$$X_{C_0} = 285.71\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$285.71 = \frac{1}{2\pi\times 200\times C}$$

$$C = 2.79\ \mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$200 = \frac{1}{2\pi\sqrt{L\times 2.79\times10^{-6}}}$$

$$L = 0.23\ \text{H}$$

At resonance, total resistance is

$$R_T = \frac{V}{I} = \frac{100}{0.7} = 142.86\ \Omega$$
$$R_T = R + r$$
$$142.86 = 50 + r$$
$$r = 92.86\ \Omega$$

Example 5.11 *In the circuit shown in Fig. 5.8, a maximum current of 0.1 A flows through the circuit when the capacitor is at 5 μF with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, quality factor Q and the value of resistance at resonant frequency.*

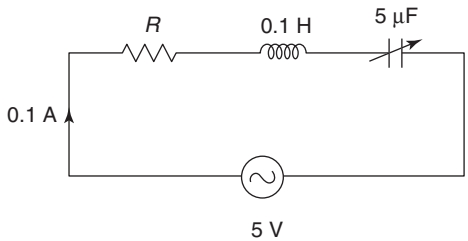


Fig. 5.8

Solution

$$I_0 = 0.1\text{ A}, \quad C = 5\ \mu\text{F}, \quad V = 5\text{ V}, \quad L = 0.1\text{ H}$$

$$R = \frac{V}{I_0} = \frac{5}{0.1} = 50\ \Omega$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 5\times 10^{-6}}} = 225.08\text{ Hz}$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{50}\sqrt{\frac{0.1}{5\times 10^{-6}}} = 2.83$$

$$BW = \frac{R}{2\pi L} = \frac{50}{2\pi\times 0.1} = 79.58\text{ Hz}$$

Example 5.12 *A series RLC circuit has R = 10 Ω and L = 60 mH. At a frequency of 25 Hz, the pf of the circuit is 45° lead. At what frequency will the circuit be resonant?*

Solution

$$R = 10\ \Omega, \quad L = 60\text{ mH}$$
$$X_L = 2\pi fL = 2\pi\times 25\times 60\times 10^{-3} = 9.42\ \Omega$$

For a series *RLC* circuit,

$$\tan\phi = \frac{X_C - X_L}{R}$$
$$\tan 45° = \frac{X_C - 9.42}{10}$$

$$1 = \frac{X_C - 9.42}{10}$$

$$X_C = 19.42\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$19.42 = \frac{1}{2\pi \times 25 \times C}$$

$$C = 327.82\ \mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{60\times10^{-3}\times327.82\times10^{-6}}} = 35.89\ \text{Hz}$$

Example 5.13 *A voltage $v(t) = 10 \sin \omega t$ is applied to a series RLC circuit. At the resonant frequency of the circuit, the voltage across the capacitor is found to be 500 V. The bandwidth of the circuit is known to be 400 rad/s and the impedance of the circuit at resonance is 100 Ω. Determine inductance and capacitance, resonant frequency, upper and lower cut-off frequencies.*

Solution $v(t) = 10 \sin \omega t, \quad V_{C_0} = 500\ \text{V}, \quad BW = 400\ \text{rad/s}, \quad R = 100\ \Omega$

(a) Inductance and capacitance

$$V = \frac{V_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.07\ \text{V}$$

$$I_0 = \frac{V}{R} = \frac{7.07}{100} = 0.0707\ \text{A}$$

$$BW = \frac{R}{L}$$

$$400 = \frac{100}{L}$$

$$L = 0.25\ \text{H}$$

$$Q_0 = \frac{V_{C_0}}{V} = \frac{500}{7.07} = 70.72$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$70.72 = \frac{1}{100}\sqrt{\frac{0.25}{C}}$$

$$C = 4.99\ \text{nF}$$

(b) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\times\sqrt{0.25\times4.99\times10^{-9}}} = 4506.09\ \text{Hz}$$

(c) Lower cut-off frequency

$$f_1 = f_0 - \frac{R}{4\pi L} = 4506.09 - \frac{100}{4\pi \times 0.25} = 4474.26 \text{ Hz}$$

(d) Upper cut-off frequency

$$f_2 = f_0 + \frac{R}{4\pi L} = 4506.09 + \frac{100}{4\pi \times 0.25} = 4537.92 \text{ Hz}$$

Example 5.14 *A series resonant circuit has an impedance of 500 Ω at resonant frequency. Cut-off frequencies are 10 kHz and 100 Hz. Determine (a) resonant frequency, (b) value of L, C, and (c) quality factor at resonant frequency.*

Solution $R = 500\ \Omega, \quad f_1 = 100 \text{ Hz}, \quad f_2 = 10 \text{ kHz}$

(a) Resonant frequency

$$BW = f_2 - f_1 = 10000 - 100 = 9900 \text{ Hz}$$

$$f_1 = f_0 - \frac{R}{4\pi L} \quad \ldots(i)$$

$$f_2 = f_0 + \frac{R}{4\pi L} \quad \ldots(ii)$$

Adding Eqs (i) and (ii),

$$f_1 + f_2 = 2f_0$$

$$f_0 = \frac{f_1 + f_2}{2} = \frac{100 + 10000}{2} = 5050 \text{ Hz}$$

(b) Values of L and C

$$BW = \frac{R}{2\pi L}$$

$$9900 = \frac{500}{2\pi L}$$

$$L = 8.038 \text{ mH}$$

$$X_{L_0} = 2\pi f_0 L = 2\pi \times 5050 \times 8.038 \times 10^{-3} = 255.05\ \Omega$$

At resonance $X_{L_0} = X_{C_0} = 255.05\ \Omega$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$255.05 = \frac{1}{2\pi \times 5050 \times C}$$

$$C = 0.12\ \mu\text{F}$$

(c) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{500}\sqrt{\frac{8.038 \times 10^{-3}}{0.12 \times 10^{-6}}} = 0.5176$$

Example 5.15 *Impedance of a circuit is observed to be capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing. Find the values of circuit elements if the power drawn by this circuit is 100 W at 100 Hz, when the current is 1 A. The power factor of the circuit at 70 Hz is 0.707.*

Solution $f_0 = 100 \text{ Hz}, \quad P_0 = 100 \text{ W}, \quad I_0 = 1\text{A}, \quad (\text{pf})_{70\text{ Hz}} = 0.707$

The impedance of the circuit is capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing.

$$f_0 = 100 \text{ Hz}$$
$$P_0 = I_0^2 R$$
$$100 = (1)^2 \times R$$
$$R = 100\ \Omega$$
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$100 = \frac{1}{2\pi\sqrt{LC}}$$
$$LC = 2.53 \times 10^{-6} \quad \ldots(\text{i})$$

Power factor at 70 Hz is 0.707.

$$\frac{R}{Z} = 0.707$$
$$\frac{100}{Z} = 0.707$$
$$Z = 141.44\ \Omega$$

$$\text{Impedance at 70 Hz} = Z_{70} = \sqrt{R^2 + (X_C - X_L)^2}$$
$$141.44 = \sqrt{(100)^2 + \left(\frac{1}{2\pi \times 70 \times C} - 2\pi \times 70 \times L\right)^2}$$
$$\frac{2.27 \times 10^{-3}}{C} - 439.82\ L = 100.02 \quad \ldots(\text{ii})$$

Solving Eqs (i) and (ii),

$$L = 0.2187 \text{ H}$$
$$C = 11.58\ \mu\text{F}$$

Example 5.16 *A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor. When the capacitor is set to 500 pF, the current has its maximum value while it is reduced to one-half when the capacitance is 600 pF. Find resistance, inductance and Q-factor of inductor.*

Solution

$$f_0 = 1\text{ MHz}, \quad C_1 = 500\text{ pF}, \quad C_2 = 600\text{ pF}$$

(a) Resistance and inductance of inductor

At resonance

$$C = 500\text{ pF}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$10^6 = \frac{1}{2\pi\sqrt{L \times 500 \times 10^{-12}}}$$

$$L = 0.05\text{ mH}$$

$$X_L = 2\pi f_0 L = 2\pi \times 10^6 \times 0.05 \times 10^{-3} = 314.16\ \Omega$$

When capacitance is 600 pF, the current reduces to one-half of the current at resonance.

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 10^6 \times 600 \times 10^{-12}} = 265.26\ \Omega$$

$$I = \frac{1}{2} I_0$$

$$\frac{V}{Z} = \frac{1}{2}\frac{V}{R}$$

$$Z = 2R$$

$$\sqrt{R^2 + (X_L - X_C)^2} = 2R$$

$$R^2 + (314.16 - 265.26)^2 = 4R^2$$

$$3R^2 = 2391.21$$

$$R = 28.23\ \Omega$$

(b) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{28.23}\sqrt{\frac{0.05 \times 10^{-3}}{500 \times 10^{-12}}} = 11.2$$

Example 5.17 *In the circuit shown in Fig. 5.9, determine the circuit constants when the circuit draws a maximum current at 10 μF with a 10 V, 100 Hz supply. When the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/s. Also find the maximum current that flows through the circuit.*

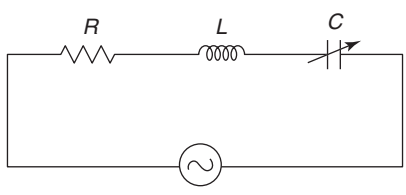


Fig. 5.9

Solution

$$C = 10\ \mu\text{F}, \quad V = 10\ \text{V}, \quad f_0 = 100\ \text{Hz}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$100 = \frac{1}{2\pi\sqrt{L \times 10 \times 10^{-6}}}$$

$$L = 0.25\ \text{H}$$

When $C = 12\ \mu\text{F}$, $\omega = 900$ rad/s

$$I = 0.707\, I_0$$

$$\frac{V}{Z} = 0.707\, \frac{V}{R}$$

$$Z = \frac{R}{0.707}$$

$$\sqrt{R^2 + (X_L - X_C)^2} = \frac{R}{0.707}$$

$$\sqrt{R^2 + \left(900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}}\right)^2} = \frac{R}{0.707}$$

$$R = 132.37\ \Omega$$

$$Q_0 = \frac{\omega L}{R} = \frac{900 \times 0.25}{132.37} = 1.7$$

$$I_0 = \frac{V}{R} = \frac{10}{132.37} = 0.076\ \text{A}$$

Example 5.18 *A series RLC circuit is excited from a constant voltage variable frequency source. The current in the circuit becomes maximum at a frequency of* $\frac{600}{2\pi}$ *Hz and falls to half the maximum value at* $\frac{400}{2\pi}$ *Hz. If the resistance in the circuit is 3Ω, find L and C.*

Solution

$$f_0 = \frac{600}{2\pi}\ \text{Hz}, \quad R = 3\ \Omega$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$\frac{600}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$LC = 2.78 \times 10^{-6} \qquad \ldots(\text{i})$$

At $f = \frac{400}{2\pi}\ \text{Hz}$,

$$I = \frac{I_0}{2}$$

$$\frac{V}{Z} = \frac{V}{2R}$$

$$Z = 2R$$

$$\sqrt{R^2 + (X_L - X_C)^2} = 2R$$

$$R^2 + (X_L - X_C)^2 = 4R^2$$

$$(X_L - X_C)^2 = 3R^2 = 3(3)^2 = 27$$

$$\left(2\pi \times \frac{400}{2\pi} \times L - \frac{1}{2\pi \times \frac{400}{2\pi} \times C}\right)^2 = 27$$

$$\left(400\,L - \frac{1}{400C}\right)^2 = 27 \qquad \ldots(ii)$$

$$\left(400 \times \frac{2.78 \times 10^{-6}}{C} - \frac{1}{400\,C}\right)^2 = 27$$

$$C = 267.12\ \mu\text{F}$$

$$L = \frac{2.78 \times 10^{-6}}{C} = \frac{2.78 \times 10^{-6}}{267.12 \times 10^{-6}} = 10.41\ \text{mH}$$

5.3 PARALLEL RESONANCE

Consider a parallel circuit consisting of a coil and a capacitor as shown in Fig. 5.10. The impedances of two branches are

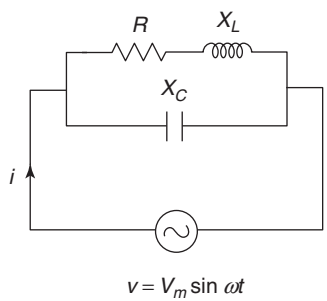


Fig. 5.10 *Parallel circuit*

$$\mathbf{Z}_1 = R + jX_L$$

$$\mathbf{Z}_2 = -jX_C$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{R + jX_L} = \frac{R - jX_L}{R^2 + X_L^2}$$

$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{-jX_C} = \frac{j}{X_C}$$

Admittance of the circuit $$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \frac{R - jX_L}{R^2 + X_L^2} + \frac{j}{X_C} = \frac{R}{R^2 + X_L^2} - j\left(\frac{X_L}{R^2 + X_L^2} - \frac{1}{X_C}\right)$$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{X_L}{R^2 + X_L^2} - \frac{1}{X_C} = 0$$

$$\frac{X_L}{R^2 + X_L^2} = \frac{1}{X_C}$$

$$X_L X_C = R^2 + X_L^2$$

$$\omega_0 L \frac{1}{\omega_0 c} = R^2 + \omega_0^2 L^2$$

$$\frac{L}{C} = R^2 + \omega_0^2 L^2$$

$$\omega_0^2 L^2 = \frac{L}{C} - R^2$$

$$\omega_0{}^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

where f_0 is called the resonant frequency of the circuit.
If R is very small as compared to L then

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

1. *Dynamic Impedance of a Parallel Circuit* At resonance, the circuit is purely resistive. The real part of admittance is $\frac{R}{R^2 + X_L^2}$. Hence, the dynamic impedance at resonance is given by

$$Z_D = \frac{R^2 + X_L^2}{R}$$

At resonance,

$$R^2 + X_L^2 = X_L X_C = \frac{L}{C}$$

$$Z_D = \frac{L}{CR}$$

2. *Current* Since impedance is maximum at resonance, the current is minimum at resonance.

$$I_0 = \frac{V}{Z_D} = \frac{V}{\frac{L}{CR}} = \frac{VCR}{L}$$

3. *Phasor Diagram* At resonance, power factor of the circuit is unity and the total current drawn by the circuit is in phase with the voltage. This will happen only when the current I_C is equal to the reactive component of the current in the inductive branch, i.e., $I_C = I_L \sin\phi$

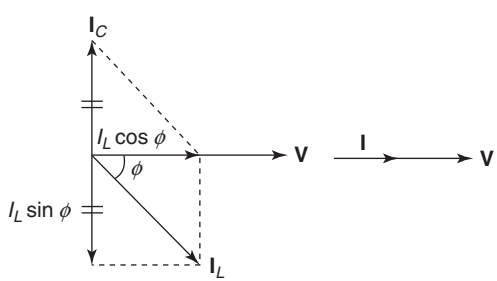


Fig. 5.11 *Phasor diagram*

Hence, at resonance

$$I_C = I_L \sin\phi$$

and

$$I = I_L \cos\phi$$

The phasor diagram of parallel resonant circuit is shown in Fig. 5.11.

4. *Behaviour of Conductance G, Inductive Susceptance B_L and Capacitive Susceptance with Change in Frequency* Conductance remains constant with the change in frequencies. Inductive susceptance B_L is

$$B_L = \frac{1}{jX_L} = -j\frac{1}{X_L} = -j\frac{1}{2\pi fL}$$

It is inversely proportional to the frequency. Thus, it decreases with the increase in the frequency. Hence, it can be drawn as a rectangular hyperbola in the fourth quadrant.
Capacitive susceptance B_C is

$$B_C = \frac{1}{-jX_C} = -j\frac{1}{X_C} = j2\pi fC$$

It is directly proportional to the frequency. It can be drawn as a straight line passing through the origin. Figure 5.12 shows the behaviour of G, B_L and B_C with change in frequency.

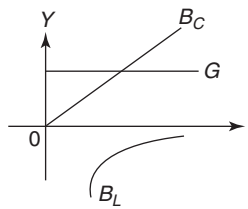


Fig. 5.12 *Behaviour of G, B_L and B_c with change in frequency*

(a) When $f < f_0$, inductive susceptance predominates. Hence, the current lags behind the voltage and the power factor is lagging in nature.

(b) When $f = f_0$, net susceptance is zero. Hence, the admittance is minimum and impedance is maximum. At f_0, the current is in phase with the voltage and the power factor is unity.

(c) When $f > f_0$, capacitive susceptance predominates. Hence, the current leads the voltage and power factor is leading in nature.

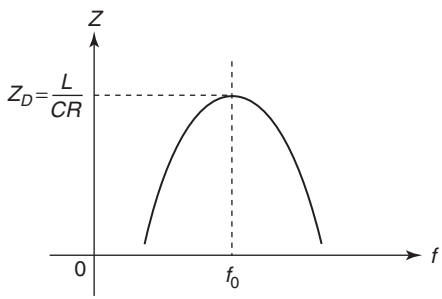


Fig. 5.13 *Impedance curve*

Figure 5.13 shows the impedance curve of a parallel resonant circuit.

5. *Bandwidth* The bandwidth of a parallel resonant circuit is defined in the same way as that for a series resonant circuit. Figure 5.14 shows the resonance curve of a parallel resonant circuit.

6. *Quality Factor* It is a measure of current magnification in a parallel resonant circuit.

$$Q_0 = \frac{\text{Current through inductor or capacitor}}{\text{Current at resonance}} = \frac{I_{C_0}}{I_0}$$

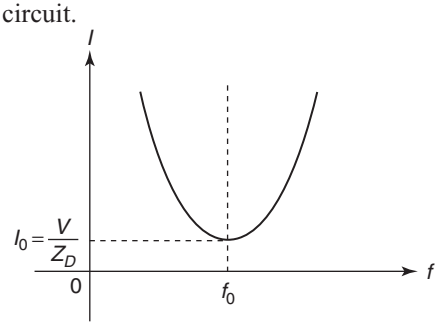


Fig. 5.14 *Resonance curve*

Substituting values of I_{C_0} and I_0,

$$Q_0 = \frac{\dfrac{V}{X_{C_0}}}{\dfrac{VCR}{L}} = \frac{\dfrac{1}{X_{C_0}}}{\dfrac{CR}{L}} = \frac{\omega_0 C}{\dfrac{CR}{L}} = \frac{\omega_0 L}{R}$$

Neglecting the resistance R, the resonant frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q_0 = \frac{\left(\dfrac{1}{\sqrt{LC}}\right)L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

5.4 COMPARISON OF SERIES AND PARALLEL RESONANT CIRCUITS

Parameter	Series Circuit	Parallel Circuit
Current at resonance	$I = \dfrac{V}{R}$ and is maximum	$I = \dfrac{VCR}{L}$ and is minimum
Impedance at resonance	$Z = R$ and is minimum	$Z = \dfrac{L}{CR}$ and is maximum
Power factor at resonance	Unity	Unity
Resonant frequency	$f_0 = \dfrac{1}{2\pi\sqrt{LC}}$	$f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$
Q-factor	$Q = \dfrac{2\pi f_0 L}{R}$	$Q = \dfrac{2\pi f_0 L}{R}$
It magnifies	Voltage across L and C	Current through L and C

Example 5.19 *A coil having an inductance of L henries and a resistance of 12 Ω is connected in parallel with a variable capacitor. At $\omega = 2.3 \times 10^6$ rad/s, resonance is achieved and at this instant, capacitance $C = 0.021\ \mu F$. Find the inductance of the coil.*

Solution

$$R = 12\ \Omega, \quad \omega_0 = 2.3 \times 10^6\ \text{rad/s}, \quad C = 0.021\ \mu\text{F}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$2.3\times10^{6} = \sqrt{\frac{1}{L\times0.021\times10^{-6}} - \frac{(12)^2}{L^2}}$$

$$L = 89.7\ \mu\text{H}$$

Example 5.20 *A coil of 20 Ω resistance has an inductance of 0.2 H and is connected in parallel with a condenser of 100 μF capacitance. Calculate the frequency at which this circuit will be have as a non-inductive resistance. Find also the value of dynamic resistance.*

Solution $R = 20\ \Omega,\quad L = 0.2\ \text{H},\quad C = 100\ \mu\text{F}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.2\times100\times10^{-6}} - \left(\frac{20}{0.2}\right)^2} = 31.83\ \text{Hz}$$

(b) Dynamic resistance

$$Z_D = \frac{L}{CR} = \frac{0.2}{100\times10^{-6}\times20} = 100\ \Omega$$

Example 5.21 *In the circuit shown in Fig. 5.15, an inductance at 0.1 H having a Q_0 of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at a resonant frequency of 500 rad/s.*

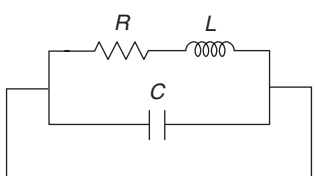


Fig. 5.15

Solution $L = 0.1\ \text{H},\quad Q_0 = 5,\quad \omega_0 = 500\ \text{rad/s}$

$$Q_0 = \frac{\omega_0 L}{R}$$

$$5 = \frac{500\times0.1}{R}$$

$$R = 10\ \Omega$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$500 = \sqrt{\frac{1}{0.1\times C} - \frac{(10)^2}{(0.1)^2}}$$

$$C = 38.46\ \mu\text{F}$$

Example 5.22 *A coil of 10 Ω resistance and 0.2 H inductance is connected in parallel with a variable capacitor across a 220 V, 50 Hz supply. Calculate (a) the capacitance of the capacitor for resonance, (b) the dynamic impedance of the circuit, and (c) supply current.*

Solution $R = 10\ \Omega, \quad L = 0.2\ \text{H}, \quad V = 220\ \text{V}, \quad f_0 = 50\ \text{Hz}$

(a) Capacitance of the capacitor

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$50 = \frac{1}{2\pi}\sqrt{\frac{1}{0.2 \times C} - \frac{(10)^2}{(0.2)^2}}$$

$$C = 49.41\ \mu\text{F}$$

(b) Dynamic impedance of the circuit

$$Z_D = \frac{L}{CR} = \frac{0.2}{49.41 \times 10^{-6} \times 10} = 404.78\ \Omega$$

(c) Supply current

$$I = \frac{V}{Z_D} = \frac{220}{404.78} = 0.543\ \text{A}$$

Example 5.23 *Find the value of the dynamic impedance of the circuit shown in Fig. 5.16 at a frequency of 500 kHz and for bandwidth of operation equal to 20 kHz. The resistance of the coil is 5 Ω.*

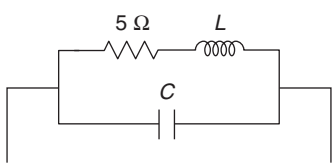


Fig. 5.16

Solution $f_0 = 500\ \text{kHz}, \quad BW = 20\ \text{kHz}, \quad R = 5\ \Omega$

$$BW = \frac{R}{2\pi L}$$

$$20 \times 10^3 = \frac{5}{2\pi L}$$

$$L = 39.79\ \mu\text{H}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$500 \times 10^3 = \frac{1}{2\pi}\sqrt{\frac{1}{39.79 \times 10^{-6} \times C} - \frac{(5)^2}{(39.79 \times 10^{-6})^2}}$$

$$C = 2.54 \text{ nF}$$

$$Z_D = \frac{L}{CR} = \frac{39.79 \times 10^{-6}}{2.54 \times 10^{-9} \times 5} = 3.13 \text{ k } \Omega$$

Example 5.24 *A coil having a resistance of 20 Ω and an inductance of 200 μH is connected in parallel with a variable capacitor. This parallel combination is connected in series with a resistance of 8000 Ω. A voltage of 230 V at a frequency of 10^6 Hz is applied across the circuit as shown in Fig. 5.17. Calculate (a) the value of capacitance at resonance, (b) Q factor of the circuit, (c) dynamic impedance of the circuit, and (d) total circuit current.*

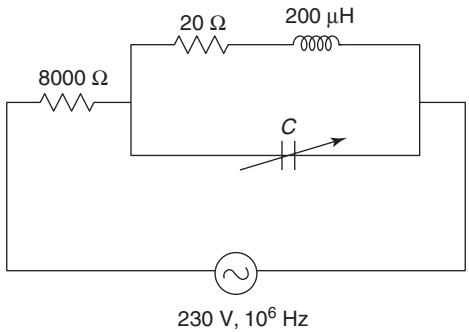


Fig. 5.17

Solution $R = 20\ \Omega, \quad L = 200\ \mu\text{H}, \quad f_0 = 10^6 \text{ Hz}, \quad V = 230 \text{ V}, \quad R_S = 8000\ \Omega$

(a) Value of capacitance

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$10^6 = \frac{1}{2\pi}\sqrt{\frac{1}{200 \times 10^{-6} \times C} - \frac{(20)^2}{(200 \times 10^{-6})^2}}$$

$$C = 126.65 \text{ pF}$$

(b) Quality factor

$$Q_0 = \frac{2\pi f_0 L}{R} = \frac{2\pi \times 10^6 \times 200 \times 10^{-6}}{20} = 62.83$$

(c) Dynamic impedance

$$Z_D = \frac{L}{CR} = \frac{200 \times 10^{-6}}{126.65 \times 10^{-12} \times 20} = 78958\ \Omega$$

(d) Current

Total impedance of the circuit at resonance $= Z_D + R_S = 78958 + 8000 = 86958\ \Omega$

$$I = \frac{230}{86958} = 2.65 \text{ mA}$$

Example 5.25 *Derive the expression for resonant frequency for the parallel circuit shown in Fig. 5.18. Also calculate the impedance and current at resonance.*

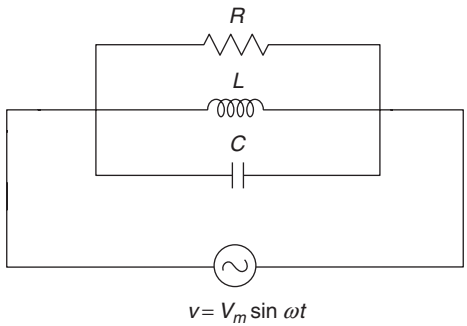


Fig. 5.18

Solution $\mathbf{Z}_1 = R, \quad \mathbf{Z}_2 = jX_L, \quad \mathbf{Z}_3 = -jX_C$

(a) Resonant frequency of the circuit
For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}$$

$$\mathbf{Y} = \frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C} = \frac{1}{R} - j\frac{1}{X_L} + j\frac{1}{X_C} = \frac{1}{R} - j\left(\frac{1}{X_L} - \frac{1}{X_C}\right)$$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{1}{X_L} - \frac{1}{X_C} = 0$$

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

where f_0 is called the resonant frequency of the circuit.

(b) Impedance at resonance.
At resonance, the circuit is purely resistive. Hence, the imaginary part of $\mathbf{Y}$ is zero.

$$Y_D = \frac{1}{R}$$

$$Z_D = R$$

(c) Current at resonance

$$I_0 = \frac{V}{Z_D} = \frac{V}{R}$$

Example 5.26 *Derive the expression for the resonant frequency of the parallel circuit as shown in Fig. 5.19.*

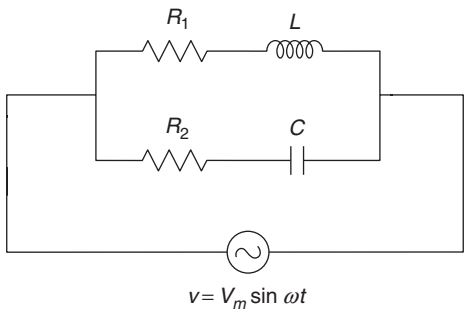


Fig. 5.19

Solution

$$\mathbf{Z}_1 = R_1 + jX_L$$
$$\mathbf{Z}_2 = R_2 - jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{R_1 + jX_L} + \frac{1}{R_2 - jX_C} = \frac{R_1 - jX_L}{R_1^2 + X_L^2} + \frac{R_2 + jX_C}{R_2^2 + X_C^2} = \frac{R_1}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} - j\left(\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2}\right)$$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2} = 0$$

$$\frac{X_L}{R_1^2 + X_L^2} = \frac{X_C}{R_2^2 + X_C^2}$$

$$\frac{\omega_0 L}{R_1^2 + \omega_0^2 L^2} = \frac{\frac{1}{\omega_0 C}}{R_2^2 + \frac{1}{\omega_0^2 C^2}}$$

$$\frac{\omega_0^2 LC}{R_1^2 + \omega_0^2 L^2} = \frac{\omega_0^2 C^2}{R_2^2 \omega_0^2 C^2 + 1}$$

$$LC(R_2^2 \omega_0^2 C^2 + 1) = C^2(R_1^2 + \omega_0^2 L^2)$$

$$\omega_0^2 R_2^2 LC^3 + LC = C^2 R_1^2 + \omega_0^2 L^2 C^2$$

$$\omega_0^2 R_2^2 LC^3 - \omega_0^2 L^2 C^2 = C^2 R_1^2 - LC$$

$$\omega_0^2 LC^2(CR_2^2 - L) = C(CR_1^2 - L)$$

$$\omega_0^2 LC(CR_2^2 - L) = CR_1^2 - L$$

$$\omega_0^2 = \frac{CR_1^2 - L}{LC(CR_2^2 - L)}$$

$$\omega_0 = \sqrt{\frac{CR_1^2 - L}{LC(CR_2^2 - L)}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{CR_1^2 - L}{CR_2^2 - L}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{CR_1^2 - L}{CR_2^2 - L}}$$

where f_0 is called the resonant frequency of the circuit.

Example 5.27 *Derive the expression for resonant frequency for the parallel circuit shown in Fig. 5.20.*

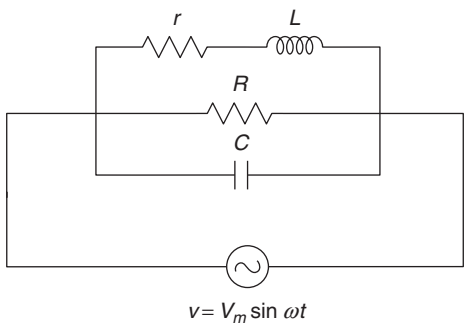


Fig. 5.20

Solution $\mathbf{Z}_1 = r + jX_L, \quad \mathbf{Z}_2 = R, \quad \mathbf{Z}_3 = -jX_C$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}$$

$$\mathbf{Y} = \frac{1}{r + jX_L} + \frac{1}{R} + \frac{1}{-jX_C} = \frac{r - jX_L}{r^2 + {X_L}^2} + \frac{1}{R} + j\frac{1}{X_C} = \left(\frac{r}{r^2 + {X_L}^2} + \frac{1}{R}\right) + j\left(\frac{1}{X_C} - \frac{X_L}{r^2 + {X_L}^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{X_L}{r^2 + {X_L}^2} = 0$$

$$\frac{1}{X_C} = \frac{X_L}{r^2 + {X_L}^2}$$

$$X_L\, X_C = r^2 + {X_L}^2$$

$$\omega_0 L \frac{1}{\omega_0 C} = r^2 + \omega_0{}^2 L^2$$

$$\frac{L}{C} = r^2 + \omega_0{}^2 L^2$$

$$\omega_0{}^2 L^2 = \frac{L}{C} - r^2$$

$$\omega_0{}^2 = \frac{1}{LC} - \frac{r^2}{L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{r^2}{L^2}}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{r^2}{L^2}}$$

where f_0 is called the resonant frequency of the circuit.

Example 5.28 *Find the value of R in the circuit shown in Fig. 5.21 to achieve resonance.*

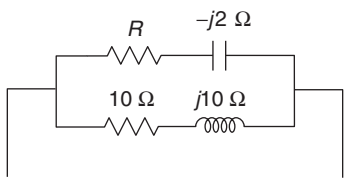


Fig. 5.21

Solution

$$\mathbf{Z}_1 = 10 + j10\ \Omega, \quad \mathbf{Z}_2 = R - j2\ \Omega$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{10 + j10} + \frac{1}{R - j2} = \frac{10 - j10}{100 + 100} + \frac{R + j2}{R^2 + 4} = \left(\frac{10}{200} + \frac{R}{R^2 + 4}\right) - j\left(\frac{10}{200} - \frac{2}{R^2 + 4}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{10}{200} - \frac{2}{R^2 + 4} = 0$$

$$\frac{2}{R^2 + 4} = \frac{1}{20}$$

$$R^2 + 4 = 40$$

$$R^2 = 36$$

$$R = 6\ \Omega$$

Example 5.29 *Find the value of R_1 such that the circuit given in Fig. 5.22 is in resonance.*

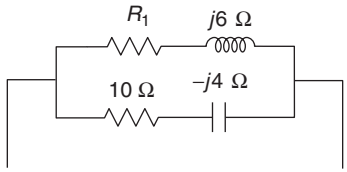


Fig. 5.22

Solution

$$\mathbf{Z}_1 = R_1 + j6\ \Omega, \quad \mathbf{Z}_2 = 10 - j4\ \Omega$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{R_1 + j6} + \frac{1}{10 - j4} = \frac{R_1 - j6}{R_1^2 + 36} + \frac{10 + j4}{100 + 16} = \left(\frac{R_1}{R_1^2 + 36} + \frac{10}{116}\right) - j\left(\frac{6}{R_1^2 + 36} - \frac{4}{116}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{6}{R_1^2 + 36} - \frac{4}{116} = 0$$

$$\frac{6}{R_1^2 + 36} = \frac{4}{116}$$

$$R_1^2 + 36 = \frac{116 \times 6}{4} = 174$$

$$R_1^2 = 138$$

$$R_1 = 11.75\ \Omega$$

Example 5.30 *Find the value of C in the circuit shown in Fig. 5.23 to give resonance.*

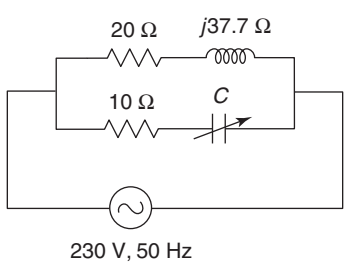


Fig. 5.23

Solution

$$\mathbf{Z}_1 = 20 + j37.7\ \Omega, \quad \mathbf{Z}_2 = 10 - jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{20 + j37.7} + \frac{1}{10 - jX_C} = \frac{20 - j37.7}{400 + 1421.29} + \frac{10 + jX_C}{100 + X_C^2}$$

$$= \left(\frac{20}{1821.29} + \frac{10}{100 + X_C^2}\right) - j\left(\frac{37.7}{1821.29} - \frac{X_C}{100 + X_C^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{37.7}{1821.29} - \frac{X_C}{100 + X_C^2} = 0$$

$$\frac{X_C}{100 + X_C^2} = 0.021$$

$$0.021\, X_C^2 - X_C + 2.1 = 0$$

Solving the quadratic equation,

$$X_C = 45.41\,\Omega \quad \text{or} \quad X_C = 2.2\,\Omega$$

$$\frac{1}{2\pi \times 50 \times C} = 45.41 \quad \text{or} \quad \frac{1}{2\pi \times 50 \times C} = 2.2$$

$$C = 70.1\,\mu\text{F} \quad \text{or} \quad C = 1.44\ \text{mF}$$

Example 5.31 *Find the value of L at which the circuit resonates at a frequency of 1000 rad/s in the circuit shown in Fig. 5.24.*

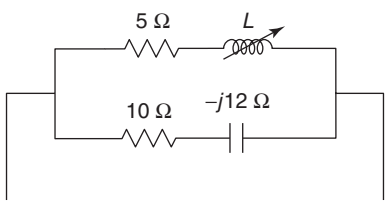


Fig. 5.24

Solution $\mathbf{Z}_1 = 5 + jX_L\ \Omega, \quad \mathbf{Z}_2 = 10 - j12\ \Omega$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{5 + jX_L} + \frac{1}{10 - j12} = \frac{5 - jX_L}{25 + X_L^2} + \frac{10 + j12}{100 + 144} = \left(\frac{5}{25 + X_L^2} + \frac{10}{244}\right) - j\left(\frac{X_L}{25 + X_L^2} - \frac{12}{244}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{X_L}{25+X_L^2}-\frac{12}{244}=0$$

$$\frac{X_L}{25+X_L^2}=\frac{12}{244}$$

$$12X_L^2-244X_L+300=0$$

Solving the quadratic equation,

$$X_L=19.01 \quad \text{or} \quad X_L=1.31\,\Omega$$

$$2\pi\times 50\times L=19.01 \quad \text{or} \quad 2\pi\times 50\times L=1.31$$

$$L=19.01\text{ mH} \quad \text{or} \quad L=1.31\text{ mH}$$

Example 5.32 *Two impedances $Z_1 = (20 + j10)\ \Omega$ and $Z_2 = (10 - j30)\ \Omega$ are connected in parallel and this combination is connected in series with $Z_3 = (30 + jX)\ \Omega$. Find the value of X which will produce resonance.*

Solution $\mathbf{Z}_1 = 20 + j10\ \Omega, \quad \mathbf{Z}_2 = 10 - j30\ \Omega, \quad \mathbf{Z}_3 = 30 + jX\ \Omega$

$$\mathbf{Z}=\frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1+\mathbf{Z}_2}+\mathbf{Z}_3=\frac{(20+j10)(10-j30)}{20+j10+10-j30}+(30+jX)=19.23-j3.85+30+jX=49.23+j(X-3.85)$$

At resonance, the circuit is purely resistive, i.e., imaginary part of $\mathbf{Z}$ is equal to zero.

$$X-3.85=0$$

$$X=3.85\ \Omega$$

Example 5.33 *Impedances Z_2 and Z_3 in parallel are in series with an impedance Z_1 across a 100 V, 50 Hz ac supply. $Z_1 = (6.25 + j1.25)\ \Omega, Z_2 = 5\ \Omega, Z_3 = (5 - jX_C)\ \Omega$. Determine the value of capacitance of X_C such that the total current of the circuit will be in phase with the total voltage.*

Solution $\mathbf{Z}_1 = 6.25 + j1.25\ \Omega, \quad \mathbf{Z}_2 = 5\ \Omega, \quad \mathbf{Z}_3 = 5 - jX_C\ \Omega$

$$\mathbf{Z}=\frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2+\mathbf{Z}_3}+\mathbf{Z}_1=\frac{5(5-jX_C)}{5+5-jX_C}+(6.25+j1.25)$$

$$=\frac{25-j5X_C}{10-jX_C}+6.25+j1.25=\frac{25-j5X_C}{10-jX_C}\times\frac{10+jX_C}{10+jX_C}+6.25+j1.25$$

$$=\frac{250+5X_C{}^2}{100+X_C{}^2}-j\frac{25X_C}{100+X_C{}^2}+6.25+j1.25=\left(6.25+\frac{250+5X_C^2}{100+X_C^2}\right)-j\left(\frac{25X_C}{100+X_C^2}-1.25\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero.

$$\frac{25X_C}{100+X_C{}^2}-1.25=0$$

$$\frac{25X_C}{100+X_C{}^2}=1.25$$

$$1.25\,X_C{}^2-25X_C+125=0$$

$$X_C{}^2 - 20X_C + 100 = 0$$

$$X_c = 10$$

$$\frac{1}{2\pi \times 50 \times C} = 10$$

$$C = 318.31\,\mu\text{F}$$

Example 5.34 *Find the frequency at which the circuit shown in Fig. 5.25 will be at resonance. If the capacitor and inductors are interchanged, what would be the value of resonant frequency?*

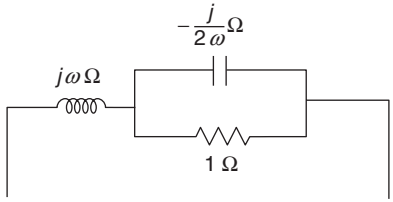


Fig. 5.25

Solution $\mathbf{Z}_1 = j\omega\ \Omega, \quad \mathbf{Z}_2 = -\frac{j}{2\omega}\,\Omega, \quad \mathbf{Z}_3 = 1\,\Omega$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = j\omega + \frac{\left(-\frac{j}{2\omega}\right)(1)}{-\frac{j}{2\omega}+1} = j\omega - \frac{j}{2\omega - j} = j\omega - \frac{j(2\omega + j)}{4\omega^2 + 1} = \frac{1}{4\omega^2+1} + j\left(\omega - \frac{2\omega}{4\omega^2+1}\right)$$

At resonance, the circuit is purely resistive, i.e., imaginary part of $\mathbf{Z}$ is equal to zero.

$$\omega - \frac{2\omega}{4\omega^2+1} = 0$$

$$\omega = \frac{2\omega}{4\omega^2+1}$$

$$\frac{1}{2} = \frac{1}{4\omega^2+1}$$

$$4\omega^2 + 1 = 2$$

$$4\omega^2 - 1 = 0$$

$$\omega = \pm 0.5 \text{ rad/s}$$

Considering positive values of ω,

$$\omega = 0.5 \text{ rad/s}$$

If the capacitor and inductor are interchanged,

$$\mathbf{Z} = -\frac{j}{2\omega} + \frac{(1)(j\omega)}{1 + j\omega} = -\frac{j}{2\omega} + \frac{j\omega(1 - j\omega)}{1+\omega^2} = -\frac{j}{2\omega} + \frac{(\omega^2 + j\omega)}{1+\omega^2} = \frac{\omega^2}{1+\omega^2} - j\left(\frac{1}{2\omega} - \frac{\omega}{1+\omega^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero.

$$\frac{1}{2\omega} - \frac{\omega}{1+\omega^2} = 0$$

$$\frac{1}{2\omega} = \frac{\omega}{1+\omega^2}$$

$$2\omega^2 - \omega^2 - 1 = 0$$

$$\omega^2 - 1 = 0$$

$$\omega = \pm 1 \text{ rad/s}$$

Considering positive values of ω,

$$\omega = 1 \text{ rad/s}$$

Example 5.35 *A 400 V, 50 Hz supply feeds energy to a parallel circuit consisting of $10\angle 30°\ \Omega$ branch and $10\angle -60°\ \Omega$ branch. Determine the impedance of a circuit element such that if connected in series with the source, the system comes at resonance.*

Solution

$$\mathbf{Z}_p = \frac{(10\angle 30°)\,(10\angle -60°)}{10\angle 30° + 10\angle -60°} = 7.07\angle -15° = 6.83 - j1.83\ \Omega$$

$$\mathbf{Z}_{\text{eq}} = \mathbf{Z} + \mathbf{Z}_p = \mathbf{Z} + 6.83 - j1.83$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero. Hence $\mathbf{Z}$ must be equal to $j1.83\ \Omega$.

$$\mathbf{Z} = j1.83\ \Omega = jX_L$$

$$X_L = 1.83$$

$$2\pi \times 50 \times L = 1.83$$

$$L = 5.82 \text{ mH}$$

Example 5.36 *A coil of 400 Ω resistance and 318 μH inductance is connected in parallel with a capacitor and the circuit resonates at 1 MHz. If a second capacitor of 23.5 pF capacitance is connected in parallel with the first capacitor, find the frequency at which the circuit resonates.*

Solution

$$R = 400\ \Omega, \quad L = 318\ \mu\text{H}, \quad f_0 = 1\ \text{MHz}, \quad C_2 = 23.5\ \text{pF}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC_1} - \frac{R^2}{L^2}}$$

$$10^6 = \frac{1}{2\pi}\sqrt{\frac{1}{318\times 10^{-6}\times C_1} - \frac{(400)^2}{(318\times 10^{-6})^2}}$$

$$C_1 = 76.59 \text{ pF}$$

When capacitor of 23.5 pF is connected in parallel with $C_1 = 76.59$ pF,

$$C_T = C_1 + C_2 = 76.59 + 23.5 = 100.09 \text{ pF}$$

The new resonant frequency $f_0' = \frac{1}{2\pi}\sqrt{\frac{1}{LC_T} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{318\times10^{-6}\times100.09\times10^{-12}} - \frac{(400)^2}{(318\times10^{-6})^2}}$

$= 869.34$ kHz

Example 5.37 *A series circuit consisting of a 12 Ω resistor, 0.3 H inductor and a variable capacitor is connected across a 100 V, 50 Hz ac supply as shown in Fig. 5.26. The capacitance value is adjusted to obtain maximum current. Find the capacitance value. Now, the supply frequency is raised to 60 Hz, the voltage remaining same at 100 V. Find the value of the capacitance C_2 to be connected across the above series circuit so that current drawn from the supply is minimum.*

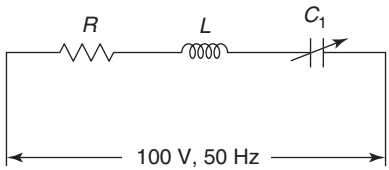


Fig. 5.26

Solution $R = 12\ \Omega, \quad L = 0.3$ H, $\quad V = 100$ V, $\quad f = 50$ Hz

(a) Value of capacitance C_1

The resonance occurs at $f = 50$ Hz

$$f_0 = \frac{1}{2\pi\sqrt{LC_1}}$$

$$50 = \frac{1}{2\pi\sqrt{0.3\times C_1}}$$

$$C_1 = 33.77\ \mu\text{F}$$

(b) Value of capacitance C_2

The resonance occurs at 60 Hz for the circuit shown in Fig. 5.27.

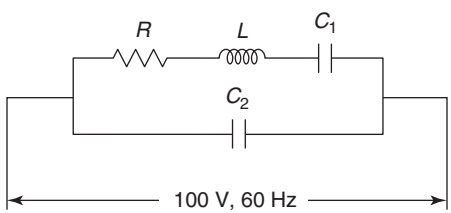


Fig. 5.27

$$X_L = 2\pi f_0 L = 2\pi\times60\times0.3 = 113.1\ \Omega$$

$$X_{C_1} = \frac{1}{2\pi f_0 C_1} = \frac{1}{2\pi\times60\times33.77\times10^{-6}} = 78.55\ \Omega$$

$$\mathbf{Z}_1 = 12 + j113.1 - j78.55 = 36.57\angle70.85°\ \Omega$$

$$\mathbf{Y}_1 = \frac{1}{Z_1} = \frac{1}{36.57\angle70.85°} = 0.027\angle70.85° = 8.86\times10^{-3} - j0.0255\ \text{℧}$$

$$\mathbf{Y}_{\text{req}} = \mathbf{Y}_1 + \mathbf{Y}_2 = 8.86\times10^{-3} - j0.0255 + \mathbf{Y}_2$$

At resonance, the imaginary part of $\mathbf{Y}_{req}$ becomes zero.

$$\mathbf{Y}_2 = j0.0255\ \text{℧}$$

$$\mathbf{Y}_2 = \frac{1}{X_{C_2}} = 0.0255\ \text{℧}$$

$$X_{C_2} = 39.22\ \Omega$$

$$X_{C_2} = \frac{1}{2\pi f_0 C_2}$$

$$39.22 = \frac{1}{2\pi \times 60 \times C_2}$$

$$C_2 = 67.63\ \mu\text{F}$$

Example 5.38 *Find active and reactive components of the current taken by a series circuit consisting of a coil of inductance 0.1 H and a resistance of 8 Ω and a capacitor of 120 μF connected to a 240 V, 50 Hz supply. Find the value of the capacitor that has to be connected in parallel with the above series circuit so that the pf of the entire circuit is unity.*

Solution

$$L = 0.1\ \text{H}, \quad R = 8\ \Omega, \quad C = 120\ \mu\text{F}, \quad V = 240\ \text{V}, \quad f = 50\ \text{Hz}$$

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 120 \times 10^{-6}} = 26.53\ \Omega$$

$$\mathbf{Z}_1 = R + jX_L - jX_C = 8 + j31.42 - j26.53 = 9.38\ \angle 31.44°\ \Omega$$

$$I = \frac{V}{Z_1} = \frac{240}{9.43} = 25.45\ \text{A}$$

$$\text{Active component of current} = I\cos\phi = 25.45\cos(31.44°) = 21.71\ \text{A}$$

$$\text{Reactive component of current} = I\sin\phi = 25.45\sin(31.44°) = 13.27\ \text{A}$$

When a capacitor is connected in parallel with the above series circuit, to make pf of the entire circuit unity,

$$\mathbf{Z}_1 = R + jX_L - jX_C = 8 + j31.42 - j26.53 = 8 + j4.89\ \Omega$$

$$\mathbf{Z}_2 = -jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{8 + j4.89} + \frac{1}{-jX_C} = \frac{8 - j4.89}{64 + 23.91} + j\frac{1}{X_C} = \frac{8}{87.91} + j\left(\frac{1}{X_C} - \frac{4.89}{87.91}\right)$$

At resonance, pf is unity and the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{4.89}{87.91} = 0$$

$$\frac{1}{X_C} = 0.056$$

$$X_C = 17.86$$

$$\frac{1}{2\pi \times 50 \times C} = 17.86$$

$$C = 178.23\ \mu\text{F}$$

Example 5.39 *A coil of 10 Ω resistance and 0.5 H inductance is connected in series with a capacitor. On applying a sinusoidal voltage the current is maximum when the frequency is 50 Hz. A second capacitor is connected in parallel with the circuit. What capacitance should it have so that the combination acts as a non-inductive resistor at 100 Hz?*

Solution

$$R = 10\ \Omega, \quad L = 0.5\ \text{H}, \quad f_0 = 50\ \text{Hz}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$50 = \frac{1}{2\pi\sqrt{0.5 \times C}}$$

$$C = 20.26\ \mu\text{F}$$

When a second capacitor is connected in parallel with the circuit so that the combination acts as a non-inductive resistor at 100 Hz,

$$X_L = 2\pi \times 100 \times 0.5 = 314.16\ \Omega$$

$$X_C = \frac{1}{2\pi \times 100 \times 20.26 \times 10^{-6}} = 78.56\ \Omega$$

$$\mathbf{Z}_1 = 10 + j314.16 - j78.56 = 10 + j235.6\ \Omega$$

$$\mathbf{Z}_2 = -jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{10 + j235.6} + \frac{1}{-jX_C} = \frac{10 - j235.6}{100 + 55507.36} + j\frac{1}{X_C} = \frac{10}{55607.36} + j\left(\frac{1}{X_C} - \frac{235.6}{55607.36}\right)$$

At resonance, the circuit acts as a non-inductive resistor, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{235.6}{55607.36} = 0$$

$$X_C = 236.02$$

$$\frac{1}{2\pi \times 100 \times C} = 236.02$$

$$C = 6.7\ \mu\text{F}$$

Example 5.40 *A coil having a resistance and inductance of 15 Ω and 8 mH respectively is connected in parallel with another coil having a resistance and inductance of 4 Ω and 18 mH as shown in Fig. 5.28. If this parallel combination is to be replaced by a single coil, calculate the value of resistance and inductance*

of that coil. What value of capacitance should be connected in parallel with this coil in order to get unity power factor? Assume operating frequency to be 50 Hz.

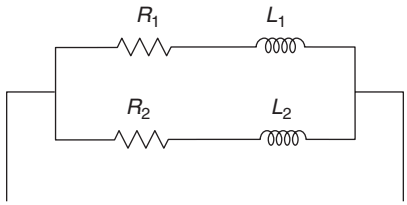


Fig. 5.28

Solution $R_1 = 15\ \Omega, \quad L_1 = 8\ \text{mH}, \quad R_2 = 4\ \Omega, \quad L_2 = 18\ \text{mH}, \quad f = 50\ \text{Hz}$

(a) Value of resistance and inductance of the coil

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 8 \times 10^{-3} = 2.51\ \Omega$$

$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 18 \times 10^{-3} = 5.65\ \Omega$$

$$\mathbf{Z}_1 = R_1 + jX_{L_1} = 15 + j2.51 = 15.21\angle 9.5°\ \Omega$$

$$\mathbf{Z}_2 = R_2 + jX_{L_2} = 4 + j5.65 = 6.92\angle 54.7°\ \Omega$$

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(15.21\angle 9.5°)(6.92\angle 54.7°)}{15.21\angle 9.5° + 6.92\angle 54.7°} = 5.09\angle 40.96°\ \Omega = 3.84 + j3.34\ \Omega$$

$$R = 3.84\ \Omega$$

$$X_L = 3.34\ \Omega$$

$$X_L = 2\pi f L$$

$$3.34 = 2\pi \times 50 \times L$$

$$L = 10.63\ \text{mH}$$

(b) Value of capacitance

When a capacitance C is connected in parallel with this coil as shown in Fig. 5.29, power factor becomes unity. Hence, the circuit behaves like a parallel resonant circuit at 50 Hz.

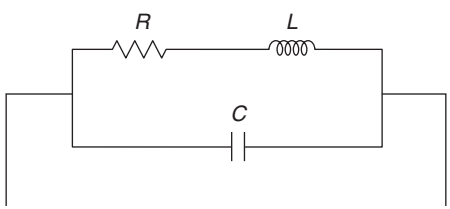


Fig. 5.29

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$50 = \frac{1}{2\pi}\sqrt{\frac{1}{10.63 \times 10^{-3} \times C} - \frac{(3.84)^2}{(10.63 \times 10^{-3})^2}}$$

$$C = 410.46\ \mu\text{F}$$

Exercises

5.1 A series *RLC* circuit has the following parameter values:
$R = 10\ \Omega$, $L = 0.014$ H, $C = 100\ \mu$F. Compute the following.
(i) resonant frequency in rad/s
(ii) quality factor of the circuit
(iii) bandwidth
(iv) lower and upper frequency points of the bandwidth
[845.2 rad/s, 1.185, 714.3 rad/s 487.9 rad/s, 1202.1 rad/s]

5.2 A series *RLC* circuit as $R = 500\ \Omega$, $L = 60$ mH and $C = 0.053\ \mu$F. Estimate
(i) circuit impedance at 50 Hz
(ii) resonance frequency in rad/s
(iii) quality factor of the circuit
(iv) bandwidth
(v) current at resonance frequency with an applied voltage of 30 V
[0.0041 Ω, 17733 rad/s 2.128, 8333.33 rad/s 60 mA]

5.3 A choking coil of 10 Ω resistance and 0.1 H inductance is connected in series with a capacitor of 200 μF capacitance across a supply voltage of 230 V, 50 Hz ac. Find the following w.r.t. the resonant circuit.
(i) Neat circuit diagram indicating all the relevant parameters.
(ii) At what frequency will the circuit resonate.
(iii) Value of reactances at resonance condition
(iv) Impedance of the coil at resonance condition
(v) Voltage across the coil and voltage across the capacitance
(vi) Half-power frequencies f_1 and f_2
(vii) Q factor of the circuit
(vii) Power and power factor
[35.59 Hz, 22.36 Ω, 10 Ω, 744.28 V, 514.28 V, 27.63 Hz, 43.55 Hz, 2.24, 5.29 kW, 1]

5.4 A coil of 40 Ω resistance and 0.75 H inductance forms part of a series circuit for which the resonant frequency is 55 Hz. If the supply is 250 V, 50 Hz, find
(i) the line current
(ii) the power factor
(iii) the voltage across the coil
[3.85 A, 0.616 (leading), 920.07 V]

5.5 An *RLC* series circuit of 10 Ω resistance should be designed to have a bandwidth of 100 Hz. Determine the values of L and C so that the circuit resonates at 250 Hz.
[0.0159 H, 25 μF]

5.6 A resistor and capacitor are connected in series with a variable inductor. When the circuit is connected to a 220 V, 50 Hz supply, the maximum current obtained by varying the inductance is 0.314 A. The voltage across the capacitor is 800 V. Calculate the resistance, inductance and capacitance of the circuit.
[700.63 Ω, 1.25 μF, 8.10 H]

5.7 A coil of 2 Ω resistance and 0.01 H inductance is connected in series with a capacitor across 230 V mains. What must be the capacitance, in order that maximum current occurs at a frequency of 50 Hz? Find also the current and the voltage across the capacitor.
[1 mF, 115 A, 361.28 V]

5.8 A choking coil of 10-ohm resistance and 0.1 H inductance is connected in series with a capacitor of 200 μF capacitance. Calculate the current, the coil voltage and the capacitor voltage. The supply voltage is 230 V at 50 Hz. At what frequency will the circuit resonate? Calculate the voltages at resonant frequency across the coil and capacitor. For this, assume that supply voltage is 230 V of variable frequency.
[12.47 A, 411.17, 198.47 V, 35.6 Hz, 563.3 V, 514.2 V]

5.9 An inductive coil having an inductance of 0.04 H and a resistance of 25 Ω has been connected in series with another inductive coil of 0.2 H

inductance and 15 Ω resistance. The whole circuit has been energized from 230 V, 50 Hz mains. Calculate power dissipation in each coil and power factor of the whole circuit. Draw the phasor diagram. Suggest a suitable capacitor for the above circuit to resonate at 50 Hz.

[181.57 W,108.95 W, 0.468(lagging), 42.22 μF]

5.10 A voltage $v(t) = 10\sqrt{2}\sin \omega t$ is applied to a series *RLC* circuit. At resonant frequency, voltage across the capacitor is found to be 500 V. The bandwidth of the circuit is known to be 400 rad/s and impedance at resonance, is 10 Ω. Determine resonant frequency, upper and lower cut-off frequencies, *L* and *C*.

[3.183 kHz, 3.151 kHz, 3.214 kHz, 0.025 H, 0.1 μF]

5.11 An inductive coil having a resistance of 20 Ω and an inductance of 0.2 H is connected in parallel with a 20 μF capacitor with variable frequency, 230 V supply. Find the resonant frequency at which the total current taken from the supply is in phase with the supply voltage. Also, find the value of this current.

[19.49 Hz, 4.6 A]

Objective-Type Questions

5.1 For a series *RLC* resonant circuit, the total reactance at the lower half power frequency is

(a) $\sqrt{2}R\angle 45°$ (b) $\sqrt{2}R\angle -45°$
(c) R (d) $-R$

5.2 In a series *RLC* high *Q* circuit, the current peaks at a frequency

(a) equal to the resonant frequency
(b) greater than the resonant frequency
(c) less than the resonant frequency
(d) none of the above.

5.3 In a series *RLC* circuit, $R = 2\text{k}\,\Omega$, $L = 1\,\text{H}$, $C = \frac{1}{400}\,\mu\text{F}$. The resonant frequency is

(a) $2\times 10^4\,\text{Hz}$ (b) $\frac{1}{\pi}\times 10^4\,\text{Hz}$
(c) $10^4\,\text{Hz}$ (d) $2\pi\times 10^4\,\text{Hz}$

5.4 For a series resonant circuit at low frequency, circuit impedance is _____ and at high frequency circuit impedance is _____.

(a) capacitive, inductive
(b) inductive, capacitive
(c) resistive, inductive
(d) capacitive, resistive

5.5 A circuit with a resistor, inductor and capacitor in series in resonant of f_0 Hz. If all the component values are now doubled, the new resonant frequency is

(a) $2f_0$ (b) f_0
(c) $\frac{f_0}{4}$ (d) $\frac{f_0}{2}$

5.6 In a series *RLC* circuit at resonance, the magnitude of the voltage developed across the capacitor

(a) is always zero
(b) can never be greater than the input voltage
(c) can be greater than the input voltage, however it is 90° out of phase with the input voltage
(d) can be greater than the input voltage and is in phase with the input voltage

5.7 The power in a series *RLC* circuit will be half of that at resonance when the magnitude of the current is equal to

(a) $\frac{V}{2R}$ (b) $\frac{V}{\sqrt{3}R}$
(c) $\frac{V}{\sqrt{2}R}$ (d) $\frac{\sqrt{2}V}{R}$

5.8 A series *RLC* circuit has a resonance frequency of 1 kHz and a quality factor *Q* of 100. If each of *R*, *L* and *C* is doubled from its original value, the new *Q* of the circuit is

(a) 25 (b) 50
(c) 100 (d) 200

5.9 A series *RLC* circuit has a Q of 100 and an impedance of $(100 + j0)\,\Omega$ at its resonant angular frequency of 10^7 rad/s. The values of R and L are

(a) $100\,\Omega, 10^3\,\text{H}$ (b) $100\,\Omega, 10^{-3}\,\text{H}$
(c) $10\,\Omega, 10\,\text{H}$ (d) $10\,\Omega, 0.1\,\text{H}$

5.10 The following circuit in Fig. 5.30 resonates at

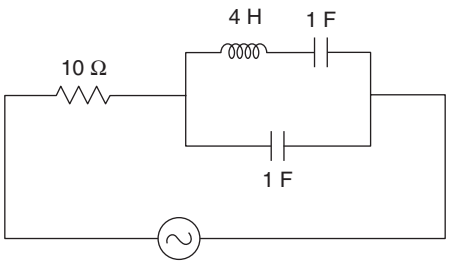


Fig. 5.30

(a) all frequencies (b) 0.5 rad/s
(c) 5 rad/s (d) 1 rad/s

5.11 At resonance, the parallel circuit shown in Fig. 5.31 behaves like

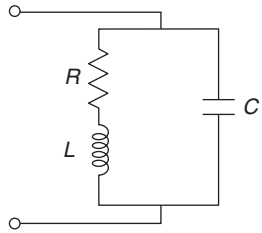


Fig. 5.31

(a) an open circuit
(b) a short circuit
(c) a pure resistor of value R
(d) a pure resistor of value much higher than R

Answers to Objective-Type Questions

5.1 (d)	5.2 (a)	5.3 (b)	5.4 (a)	5.5 (d)	5.6 (c)
5.7 (c)	5.8 (b)	5.9 (b)	5.10 (b)	5.11 (d)	

6

Transient Analysis

6.1 INTRODUCTION

Whenever a network containing energy storage elements such as inductor or capacitor is switched from one condition to another, either by change in applied source or change in network elements, the response current and voltage change from one state to the other state. The time taken to change from an initial steady state to the final steady state is known as the *transient period*. This response is known as *transient response* or *transients*. The response of the network after it attains a final steady value is independent of time and is called the steady-state response. The complete response of the network is determined with the help of a differential equation.

6.2 INITIAL CONDITIONS

In solving the differential equations in the network, we get some arbitary constant. Initial conditions are used to determine these arbitrary constants. It helps us to know the behaviour of elements at the instant of switching.

To differentiate between the time immediately before and immediately after the switching, the signs '–' and '+' are used. The conditions existing just before switching are denoted as $i(0^-)$, $v(0^-)$, etc. Conditions just after switching are denoted as $i(0^+)$, $v(0^+)$.

Sometimes conditions at $t = \infty$ are used in the evaluation of arbitrary constants. These are known as *final conditions*.

In solving the problems for initial conditions in the network, we divide the time period in the following ways:

1. Just before switching (from $t = -\infty$ to $t = 0^-$)
2. Just after switching (at $t = 0^+$)
3. After switching (for $t > 0$)

If the network remains in one condition for a long time without any switching action, it is said to be under steady-state condition.

1. Initial Conditions for the Resistor For a resistor, current and voltage are related by $v(t) = Ri(t)$. The current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, the voltage will change instantaneously if the current changes instantaneously.

2. Initial Conditions for the Inductor For an inductor, current and voltage are related by,

$$v(t) = L\frac{di}{dt}$$

Voltage across the inductor is proportional to the rate of change of current. It is impossible to change the current through an inductor by a finite amount in zero time. This requires an infinite voltage across the inductor. An inductor does not allow an abrupt change in the current through it.

The current through the inductor is given by,

$$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current through the inductor.

If there is no current flowing through the inductor at $t = 0^-$, the inductor will act as an open circuit at $t = 0^+$. If a current of value I_0 flows through the inductor at $t = 0^-$, the inductor can be regarded as a current source of I_0 ampere at $t = 0^+$.

3. Initial Conditions for the Capacitor For the capacitor, current and voltage are related by,

$$i(t) = C\frac{dv(t)}{dt}$$

Current through a capacitor is proportional to the rate of change of voltage. It is impossible to change the voltage across a capacitor by a finite amount in zero time. This requires an infinite current through the capacitor. A capacitor does not allow an abrupt change in voltage across it.

The voltage across the capacitor is given by,

$$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

where $v(0)$ is the initial voltage across the capacitor.

If there is no voltage across the capacitor at $t = 0^-$, the capacitor will act as a short circuit at $t = 0^+$. If the capacitor is charged to a voltage V_0 at $t = 0^-$, it can be regarded as a voltage source of V_0 volt at $t = 0^+$. These conditions are summarized in Fig. 6.1.

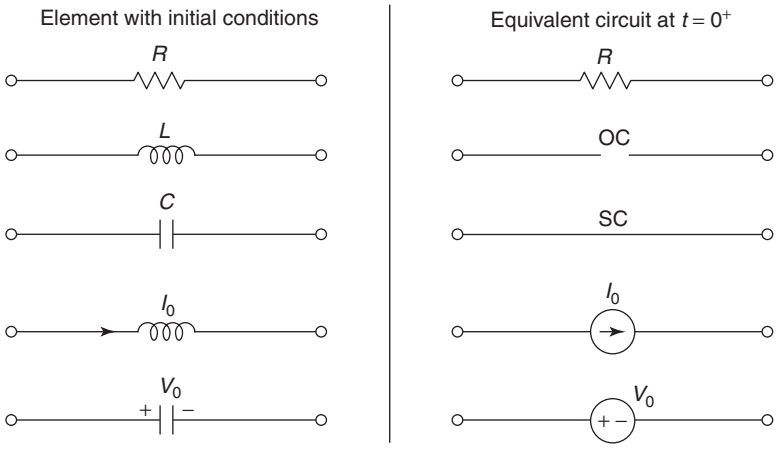


Fig. 6.1 *Initial conditions*

Similarly, we can draw the chart for final conditions as shown in Fig. 6.2

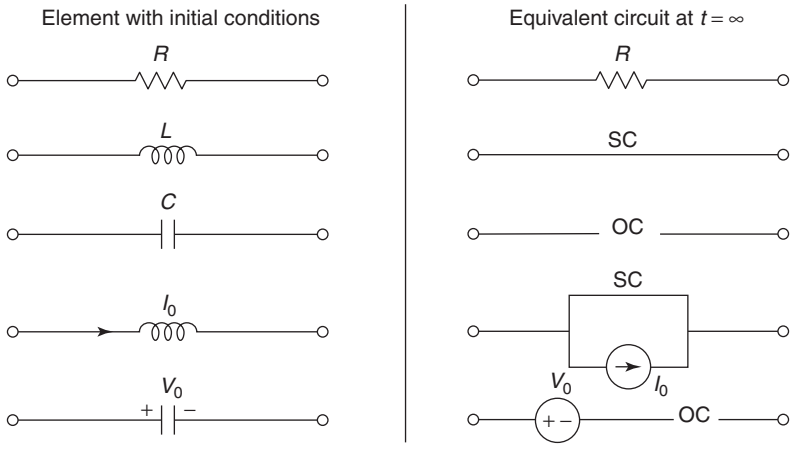


Fig. 6.2 *Final conditions*

4. Procedure for Evaluating Initial Conditions

(a) Draw the equivalent network at $t = 0^-$. Before switching action takes place, i.e., for $t = -\infty$ to $t = 0^-$, the network is under steady-state conditions. Hence, find the current flowing through the inductors $i_L(0^-)$ and voltage across the capacitor $v_C(0^-)$.

(b) Draw the equivalent network at $t = 0^+$, i.e., immediately after switching. Replace all the inductors with open circuits or with current sources $i_L(0^+)$ and replace all capacitors by short circuits or voltage sources $v_C(0^+)$. Resistors are kept as it is in the network.

(c) Initial voltages or currents in the network are determined from the equivalent network at $t = 0^+$.

(d) Initial conditions, i.e., $\frac{di}{dt}(0^+), \frac{dv}{dt}(0^+), \frac{d^2i}{dt^2}(0^+), \frac{d^2v}{dt^2}(0^+)$ are determined by writing integro-differential equations for the network for $t > 0$, i.e., after the switching action by making use of initial condition.

Example 6.1 *In the given network of Fig. 6.3, the switch is closed at t = 0. With zero current in the inductor, find i,* $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

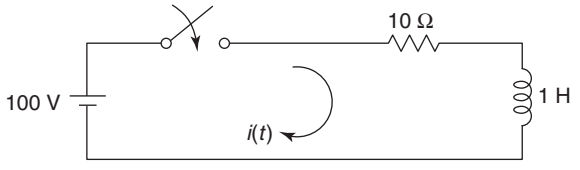


Fig. 6.3

Solution

At $t = 0^-$, no current flows through the inductor.

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.4.

At $t = 0^+$, the inductor acts as an open circuit.

$$i(0^+) = 0$$

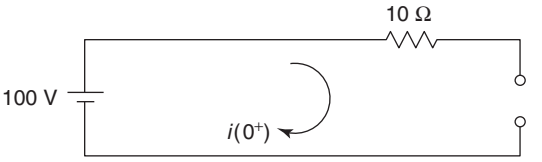


Fig. 6.4

For $t > 0$, the network is shown in Fig. 6.5.
Writing the KVL equation for $t > 0$,

$$100 - 10i - 1\frac{di}{dt} = 0 \qquad \ldots(i)$$

$$\frac{di}{dt} = 100 - 10i \qquad \ldots(ii)$$

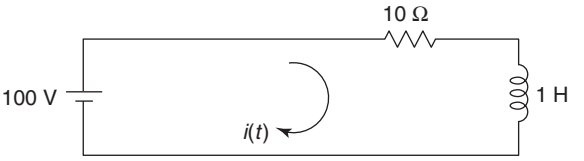


Fig. 6.5

At $t = 0^+$, $\quad \frac{di}{dt}(0^+) = 100 - 10i(0^+) = 100 - 10(0) = 100 \text{ A/s}$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -10\frac{di}{dt}$$

At $t = 0^+$, $\quad \frac{d^2i}{dt^2}(0^+) = -10\frac{di}{dt}(0^+) = -10(100) = -1000 \text{ A/s}^2$

Example 6.2 *In the network of Fig. 6.6, the switch is closed at t = 0. With the capacitor uncharged, find value for i,* $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

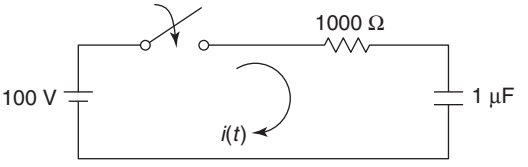


Fig. 6.6

Solution

At $t = 0^-$, the capacitor is uncharged.

$$v_C(0^-) = 0$$

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.7.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{100}{1000} = 0.1 \text{ A}$$

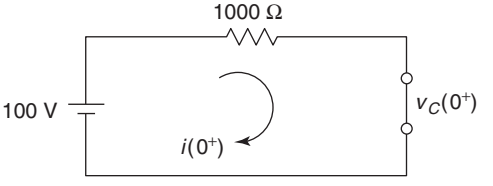


Fig. 6.7

For $t > 0$, the network is shown in Fig. 6.8.
Writing the KVL equation for $t > 0$,

$$100 - 1000i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots(\text{i})$$

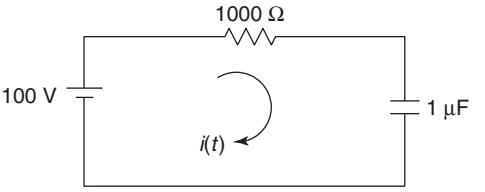


Fig. 6.8

Differentiating Eq. (i),

$$0 - 1000\frac{di}{dt} - 10^6 i = 0$$

$$\frac{di}{dt} = -\frac{10^6}{1000}i \qquad \ldots(\text{ii})$$

At $t = 0^+$, $$\frac{di}{dt}(0^+) = -\frac{10^6}{1000}i(0^+) = -\frac{10^6}{1000}(0.1) = -100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -\frac{10^6}{1000}\frac{di}{dt}$$

At $t = 0^+$, $$\frac{d^2i}{dt^2}(0^+) = -\frac{10^6}{1000}\frac{di}{dt}(0^+) = -\frac{10^6}{1000}(-100) = 10^5 \text{ A/s}^2$$

Example 6.3 *In the network shown in Fig. 6.9, the switch is closed. Assuming all initial conditions as zero, find* i, $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

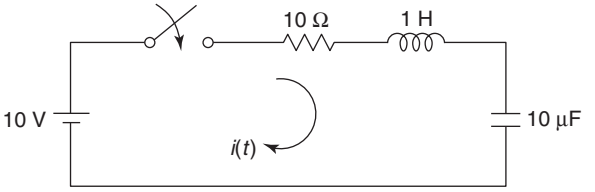


Fig. 6.9

Solution

At $t = 0^-$,

$$i(0^-) = 0$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.10.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i(0^+) = 0$$

$$v_C(0^+) = 0$$

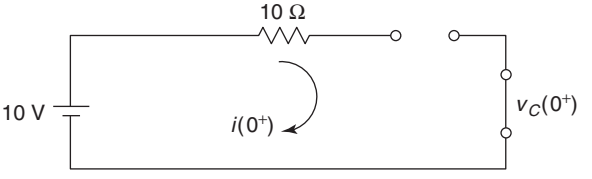


Fig. 6.10

For $t > 0$, the network is shown in Fig. 6.11.
Writing the KVL equation for $t > 0$,

$$10 - 10i - 1\frac{di}{dt} - \frac{1}{10\times10^{-6}}\int_0^t i\,dt = 0 \quad \ldots(\text{i})$$

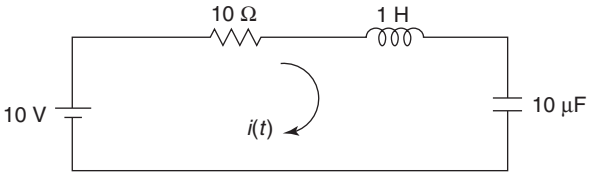


Fig. 6.11

At $t = 0^+$, $\quad 10 - 10i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$

$$\frac{di}{dt}(0^+) = 10\text{A/s}$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - \frac{d^2i}{dt^2} - \frac{1}{10\times10^{-6}}i = 0$$

At $t = 0^+$, $\quad 0 - 10\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) - \frac{1}{10^{-5}}i(0^+) = 0$

$$\frac{d^2i}{dt^2}(0^+) = -10\times10 = -100\ \text{A/s}^2$$

Example 6.4 *In the network shown in Fig. 6.12, at t = 0, the switch is opened. Calculate* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

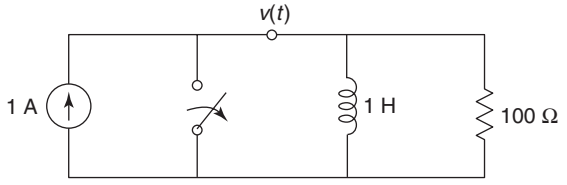


Fig. 6.12

Solution At $t = 0^-$, the switch is closed. Hence, no current flows through the inductor.

$$i_L(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.13.
At $t = 0^+$, the inductor acts as an open circuit.

$$i_L(0^+) = 0$$

$$v(0^+) = 100\times1 = 100\ \text{V}$$

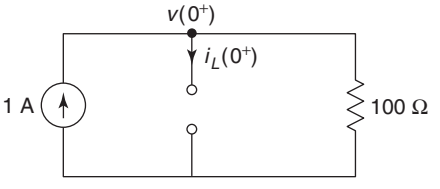


Fig. 6.13

For $t > 0$, the network is shown in Fig. 6.14.
Writing the KCL equation for $t > 0$,

$$\frac{v}{100} + \frac{1}{1}\int_0^t v\,dt = 1 \qquad \ldots(i)$$

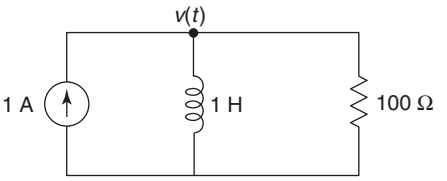


Fig. 6.14

Differentiating Eq. (i),

$$\frac{1}{100}\frac{dv}{dt} + v = 0 \qquad \ldots(ii)$$

At $t = 0^+$,

$$\frac{dv}{dt}(0^+) = -100v(0^+) = -100 \times 100 = -10000 \text{ V/s}$$

Differentiating Eq. (ii),

$$\frac{1}{100}\frac{d^2v}{dt^2} + \frac{dv}{dt} = 0$$

At $t = 0^+$,

$$\frac{d^2v}{dt^2}(0^+) = -100\frac{dv}{dt}(0^+) = -100 \times (-10^4) = 10^6 \text{ V/s}^2$$

Example 6.5 *In the given network of Fig. 6.15, the switch is opened at t = 0. Solve for* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

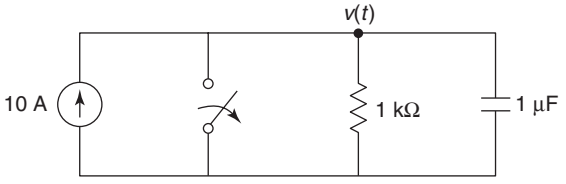


Fig. 6.15

Solution At $t = 0^-$, switch is closed. Hence, the voltage across the capacitor is zero.

$$v(0^-) = v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.16.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v(0^+) = v_C(0^+) = 0$$

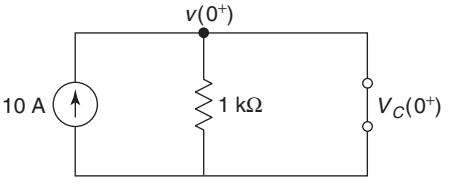


Fig. 6.16

For $t > 0$, the network is shown in Fig. 6.17.
Writing the KCL equation for $t > 0$,

$$\frac{v}{1000} + 10^{-6}\frac{dv}{dt} = 10 \qquad \ldots(i)$$

At $t = 0^+$,

$$\frac{v(0^+)}{1000} + 10^{-6}\frac{dv}{dt}(0^+) = 10$$

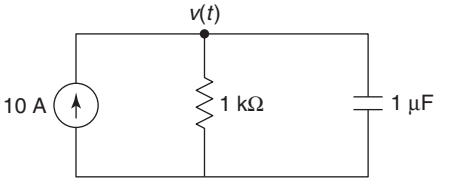


Fig. 6.17

$$\frac{dv}{dt}(0^+) = \frac{10}{10^{-6}} = 10 \times 10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{1000}\frac{dv}{dt} + 10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$, $$\frac{1}{1000}\frac{dv}{dt}(0^+) + 10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

$$\frac{d^2v}{dt^2}(0^+) = -\frac{1}{1000 \times 10^{-6}} \times 10 \times 10^6 = -10 \times 10^9 \text{ V/s}^2$$

Example 6.6 *For the network shown in Fig. 6.18, the switch is closed at t = 0, determine v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at $t = 0^+$.*

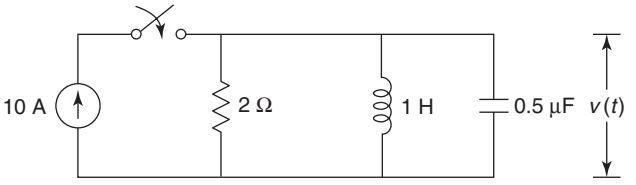


Fig. 6.18

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.19.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

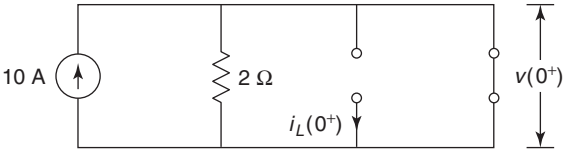


Fig. 6.19

For $t > 0$, the network is shown in Fig. 6.20.

Writing the KCL equation for $t > 0$,

$$\frac{v}{2} + \frac{1}{1}\int_1^t v\,dt + 0.5 \times 10^{-6}\frac{dv}{dt} = 10 \qquad \ldots(i)$$

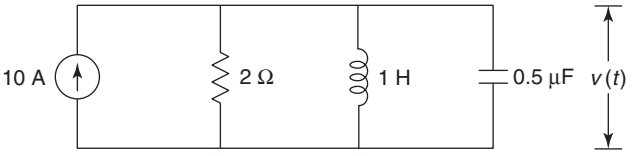


Fig. 6.20

At $t = 0^+$, $$\frac{v(0^+)}{2} + 0 + 0.5 \times 10^{-6}\frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = 20 \times 10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{2}\frac{dv}{dt} + v + 0.5\times10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{1}{2}\frac{dv}{dt}(0^+) + v(0^+) + 0.5\times10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

$$\frac{d^2v}{dt^2}(0^+) = -20\times10^{12}\ \text{V/s}^2$$

Example 6.7 *In the network shown in Fig. 6.21, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* $i, \frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

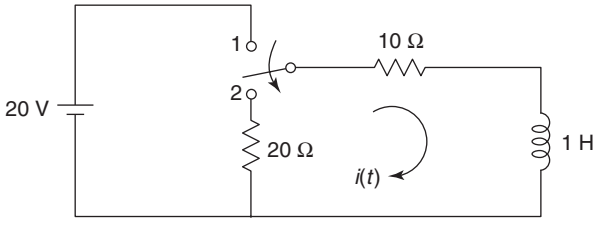


Fig. 6.21

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

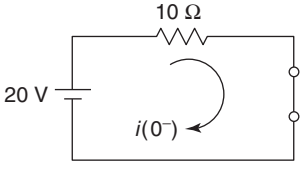


Fig. 6.22

$$i(0^-) = \frac{20}{10} = 2\ \text{A}$$

At $t = 0^+$, the network is shown in Fig. 6.23.

At $t = 0^+$, the inductor acts as a current source of 2 A.

$$i(0^+) = 2\ \text{A}$$

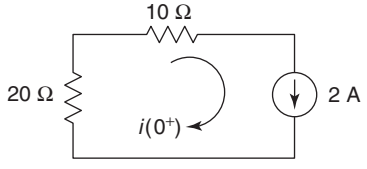


Fig. 6.23

For $t > 0$, the network is shown in Fig. 6.24.

Writing the KVL equation for $t > 0$,

$$-20i - 10i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

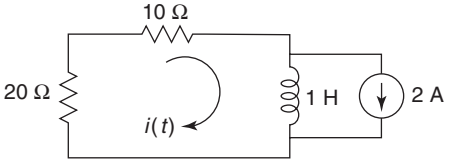


Fig. 6.24

At $t = 0^+$,

$$-30i(0^+) - \frac{di}{dt}(0^+) = 0$$

$$\frac{di}{dt}(0^+) = -30\times2 = -60\ \text{A/s}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-30\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$
$$\frac{d^2i}{dt^2}(0^+) = 1800 \text{ A/s}^2$$

Example 6.8 *In the network shown in Fig.6.25, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.*

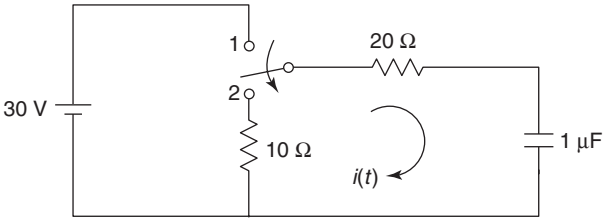


Fig. 6.25

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

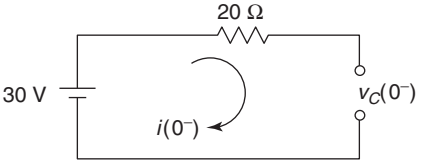


Fig. 6.26

$$v_C(0^-) = 30 \text{ V}$$
$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.27.
At $t = 0^+$, the capacitor acts as a voltage source of 30 V.

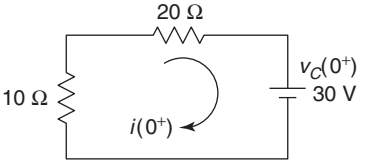


Fig. 6.27

$$v_C(0^+) = 30 \text{ V}$$
$$i(0^+) = -\frac{30}{30} = -1 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.28.
Writing the KVL equation for $t > 0$,

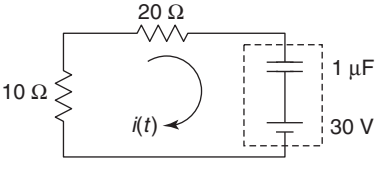


Fig. 6.28

$$-10i - 20i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt - 30 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt} - 10^6 i = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$-30\frac{di}{dt}(0^+) - 10^6 i(0^+) = 0$$
$$\frac{di}{dt}(0^+) = \frac{10^6(-1)}{30} = 0.33\times10^5 \text{ A/s}$$

Differentiating Eq. (ii),

$$-30\frac{d^2i}{dt^2}-10^6\frac{di}{dt}=0$$

At $t=0^+$, $$-30\frac{d^2i}{dt^2}(0^+)-10^6\frac{di}{dt}(0^+)=0$$

$$\frac{d^2i}{dt^3}(0^+)=-\frac{10^6\times0.33\times10^5}{30}=-1.1\times10^9\ \text{A/s}^2$$

Example 6.9 *In the network shown in Fig. 6.29, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* $i, \frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

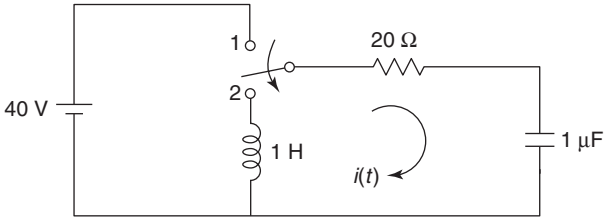


Fig. 6.29

Solution At $t = 0^-$, the network attains steady state. Hence, the capacitor acts as an open circuit.

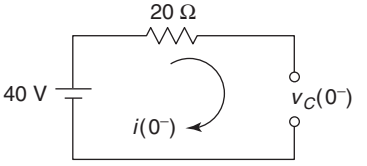


Fig. 6.30

$$v_C(0^-)=40\ \text{V}$$

$$i(0^-)=0$$

At $t = 0^+$, the network is shown in Fig. 6.31.

At $t = 0^+$, the capacitor acts as a voltage source of 40 V and the inductor acts as an open circuit.

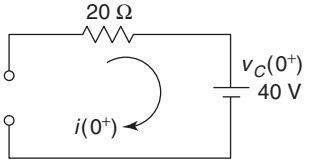


Fig. 6.31

$$v_C(0^+)=40\ \text{V}$$

$$i(0^+)=0$$

For $t > 0$, the network is shown in Fig. 6.32.

Writing the KVL equation fo $t > 0$,

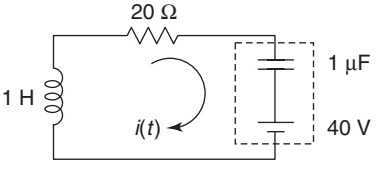


Fig. 6.32

$$-1\frac{di}{dt}-20i-\frac{1}{1\times10^{-6}}\int_0^t i\,dt-40=0 \qquad \ldots(\text{i})$$

At $t=0^+$, $$-\frac{di}{dt}(0^+)-20i(0^+)-0-40=0$$

$$\frac{di}{dt}(0^+)=-40\ \text{A/s}$$

Differentiating Eq. (i),

$$-\frac{d^2i}{dt^2}-20\frac{di}{dt}-10^6 i-0=0$$

At $t=0^+$,

$$-\frac{d^2i}{dt^2}(0^+)-20\frac{di}{dt}(0^+)-10^6 i(0^+)=0$$

$$\frac{d^2i}{dt^2}(0^+)=800\ \text{A/s}^2$$

Example 6.10 *In the network of Fig. 6.33, the switch is changed from the position 'a' to 'b' at $t=0$. Solve for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t=0^+$.*

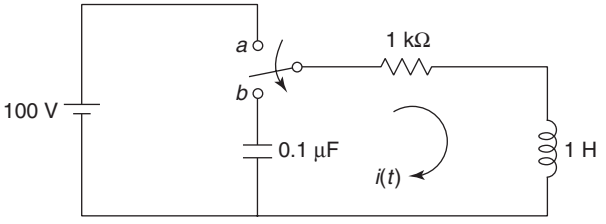


Fig. 6.33

Solution At $t=0^-$, the network attains steady condition. Hence, the inductor acts as a short circuit.

$$i(0^-)=\frac{100}{1000}=0.1\ \text{A}$$

$$v_C(0^-)=0$$

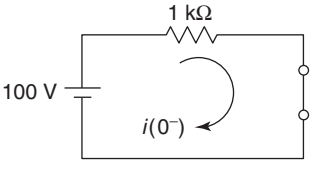


Fig. 6.34

At $t=0^+$, the network is shown in Fig. 6.35.

At $t=0^+$, the inductor acts as a current source of 0.1 A and the capacitor acts as a short circuit.

$$i(0^+)=0.1\ \text{A}$$

$$v_C(0^+)=0$$

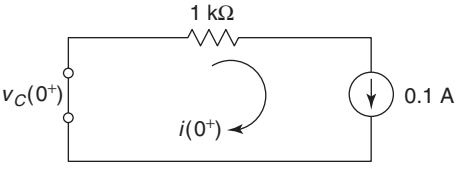


Fig. 6.35

For $t>0$, the network is shown in Fig. 6.36.

Writing the KVL equation for $t>0$,

$$-\frac{1}{0.1\times10^{-6}}\int_0^t i\,dt-1000i-1\frac{di}{dt}=0 \qquad \ldots(\text{i})$$

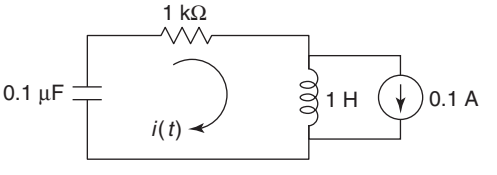


Fig. 6.36

At $t = 0^+$,
$$-0 - 1000i(0^+) - \frac{di}{dt}(0^+) = 0$$
$$\frac{di}{dt}(0^+) = -1000\, i(0^+) = -1000 \times 0.1 = -100 \text{ A/s}$$

Differentiating Eq. (i),
$$-\frac{1}{10^{-7}} i - 1000\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-10^7 i(0^+) - 1000\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$
$$\frac{d^2i}{dt^2}(0^+) = -10^7(0.1) - 1000(-100) = -9 \times 10^5 \text{ A/s}^2$$

Example 6.11 *The network of Fig. 6.37 attains steady-state with the switch closed. At $t = 0$, the switch is opened. Find the voltage across the switch v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.*

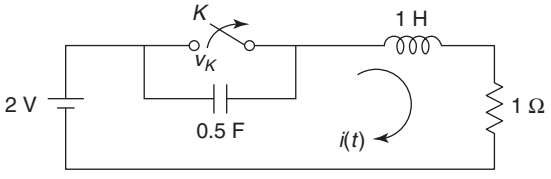


Fig. 6.37

Solution: At $t = 0^-$, the network is shown in Fig. 6.38. At $t = 0^-$, the network attains steady-state condition. The capacitor acts as an open circuit and the inductor acts as a short circuit.

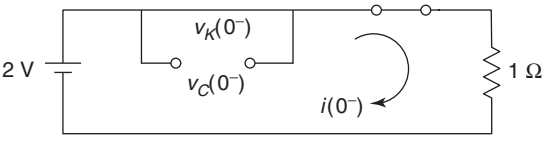


Fig. 6.38

$$i(0^-) = \frac{2}{1} = 2 \text{ A}$$
$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.39.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as a current source of 2 A.

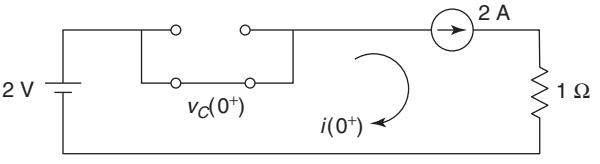


Fig. 6.39

$$i(0^+) = 2 \text{ A}$$
$$v_C(0^+) = 0$$
$$v_K(0^+) = 0$$

Also,
$$v_K = \frac{1}{C}\int i\,dt$$
$$\frac{dv_K}{dt} = \frac{i}{C}$$

At $t = 0^+$,
$$\frac{dv_K}{dt}(0^+) = \frac{i(0^+)}{C} = \frac{2}{0.5} = 4 \text{ A/s}$$

Example 6.12 *In the network shown in Fig. 6.40, assuming all initial conditions as zero, find* $i_1(0^+), i_2(0^+), \frac{di_1}{dt}(0^+), \frac{di_2}{dt}(0^+), \frac{d^2i_1}{dt^2}(0^+)$ *and* $\frac{d^2i_2}{dt^2}(0^+)$.

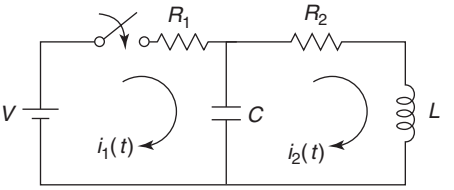


Fig. 6.40

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.41.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

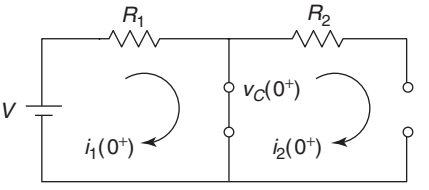


Fig. 6.41

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.42.

Writing the KVL equations for two meshes for $t > 0$,

$$V - R_1 i_1 - \frac{1}{C}\int_0^t (i_1 - i_2)dt = 0 \qquad \ldots(i)$$

and
$$-\frac{1}{C}\int (i_2 - i_1)dt - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots(ii)$$

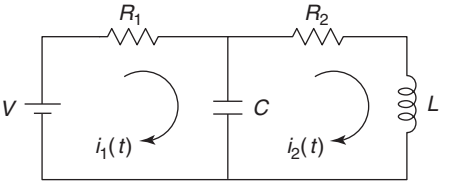


Fig. 6.42

From Eq. (ii), at $t = 0^+$,

$$-\frac{1}{C}\int_0^{0^+}(i_2 - i_1)dt - R_2 i_2(0^+) - L\frac{di_2}{dt}(0^+) = 0$$

$$\frac{di_2}{dt}(0^+) = 0$$

Differentiating Eq. (i),

$$0 - R_1\frac{di_1}{dt} - \frac{1}{C}(i_1 - i_2) = 0 \qquad \ldots(iii)$$

At $t = 0^+$, $$0 - R_1\frac{di_1}{dt}(0^+) - \frac{1}{C}i_1(0^+) + \frac{1}{C}i_2(0^+) = 0$$

$$R_1\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{V}{R_1} = 0$$

$$\frac{di_1}{dt}(0^+) = -\frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$-R_1\frac{d^2 i_1}{dt^2} - \frac{1}{C}\frac{di_1}{dt} + \frac{1}{C}\frac{di_2}{dt} = 0$$

At $t = 0^+$, $$-R_1\frac{d^2 i_1}{dt^2}(0^+) - \frac{1}{C}\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{di_2}{dt}(0^+) = 0$$

$$\frac{d^2 i_1}{dt^2}(0^+) = \frac{V}{R_1^3 C^2}$$

Differentiating Eq. (ii),

$$-\frac{1}{C}(i_2 - i_1) - R_2\frac{di_2}{dt} - L\frac{d^2 i_2}{dt^2} = 0$$

At $t = 0^+$, $$\frac{d^2 i_2}{dt^2}(0^+) = -\frac{R_2}{L}\frac{di_2}{dt}(0^+) - \frac{1}{LC}[i_2(0^+) - i_1(0^+)] = \frac{V}{R_1 LC}$$

Example 6.13 *In the network shown in Fig. 6.43, assuming all initial conditions as zero, find* $i_1, i_2 \frac{di_1}{dt}, \frac{di_2}{dt}, \frac{d^2 i_1}{dt^2}$ *and* $\frac{d^2 i_2}{dt^2}$ *at* $t = 0^+$.

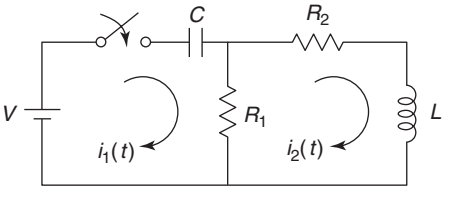


Fig. 6.43

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.44.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as an open circuit.

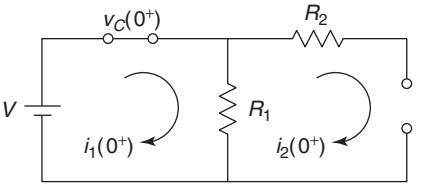


Fig. 6.44

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.45.

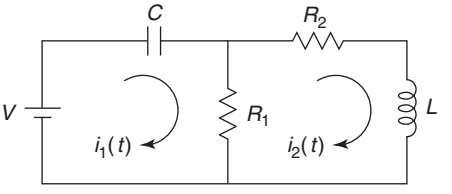


Fig. 6.45

Writing the KVL equation for $t > 0$,

$$V - \frac{1}{C}\int_0^t i_1\, dt - R_1(i_1 - i_2) = 0 \qquad \ldots(i)$$

and
$$-R_1(i_2 - i_1) - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots(ii)$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{1}{L}[R_1 i_1 - (R_1 + R_2)i_2] \qquad \ldots(iii)$$

At $t = 0^+$,
$$\frac{di_2}{dt}(0^+) = \frac{1}{L}[R_1 i_1(0^+) - (R_1 + R_2)i_2(0^+)] = \frac{1}{L}\left[R_1\frac{V}{R_1} - (R_1 + R_2)0\right] = \frac{V}{L}$$

Differentiating Eq. (i),

$$0 - \frac{i_1}{C} - R_1\frac{di_1}{dt} + R_1\frac{di_2}{dt} = 0$$
$$\frac{di_1}{dt} = \frac{di_2}{dt} - \frac{i_1}{R_1 C} \qquad \ldots(iv)$$

At $t = 0^+$,
$$\frac{di_1}{dt}(0^+) = \frac{di_2}{dt}(0^+) - \frac{i_1(0^+)}{R_1 C} = \frac{V}{L} - \frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$\frac{d^2 i_2}{dt^2} = \frac{1}{L}\left[R_1\frac{di_1}{dt} - (R_1 + R_2)\frac{di_2}{dt}\right]$$

At $t = 0^+$,
$$\frac{d^2 i_2}{dt^2}(0^+) = -V\left(\frac{1}{R_1 LC} + \frac{R_2}{L^2}\right)$$

Differentiating Eq. (iv),

$$\frac{d^2 i_1}{dt^2} = \frac{d^2 i_2}{dt^2} - \frac{1}{R_1 C}\frac{di_1}{dt}$$

At $t = 0^+$, $\frac{d^2 i_1}{dt^2}(0^+) = \frac{d^2 i_2}{dt^2}(0^+) - \frac{1}{R_1 C}\frac{di_1}{dt}(0^+) = -\frac{V}{R_1 LC} - \frac{VR_2}{L^2} - \frac{1}{R_1 C}\left(\frac{V}{L} - \frac{V}{R_1^2 C}\right) = \frac{V}{R_1^3 C^2} - \frac{2V}{R_1 LC} - \frac{VR_2}{L^2}$

Example 6.14 *In the network shown in Fig. 6.46, a steady state is reached with the switch open. At t = 0, the switch is closed. For the element values given, determine the value of $v_a(0^-)$, $v_b(0^-)$, $v_a(0^+)$ and $v_b(0^+)$.*

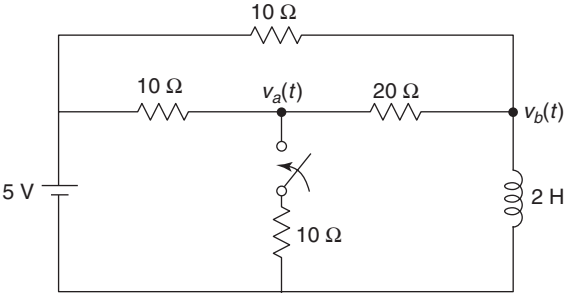


Fig. 6.46

Solution At $t = 0^-$, the network is shown in Fig. 6.47.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{5}{(10 \,||\, 30)} = \frac{5}{7.5} = \frac{2}{3}\text{ A}$$

$$v_b(0^-) = 0$$

$$v_a(0^-) = 5 \times \frac{20}{30} = 3.33\text{ V}$$

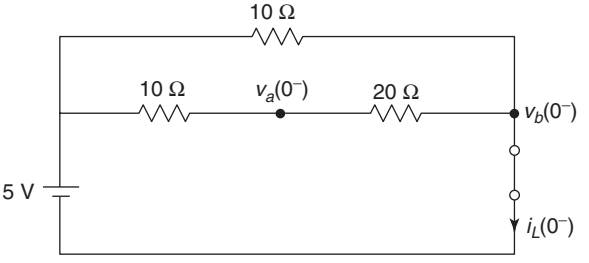


Fig. 6.47

At $t = 0^+$, the network is shown in Fig. 6.48.
At $t = 0^+$, the inductor acts as a current source of $\frac{2}{3}$ A.

$$i_L(0^+) = \frac{2}{3}\text{ A}$$

Writing the KCL equations at $t = 0^+$,

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

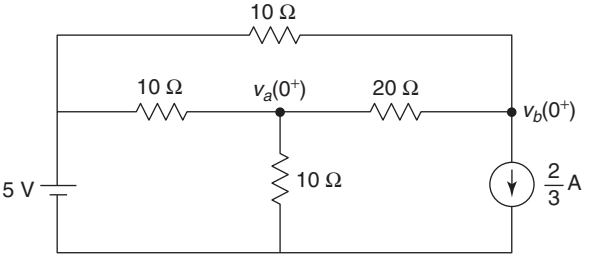


Fig. 6.48

and
$$\frac{v_b(0^+)-v_a(0^+)}{20}+\frac{v_b(0^+)-5}{10}+\frac{2}{3}=0$$

Solving these two equations,

$$v_a(0^+)=1.9\text{ V}$$
$$v_b(0^+)=-0.477\text{ V}$$

Example 6.15 *In the accompanying Fig. 6.49 is shown a network in which a steady state is reached with switch open. At t = 0, switch is closed. Determine $v_a(0^-)$, $v_a(0^+)$, $v_b(0^-)$ and $v_b(0^+)$.*

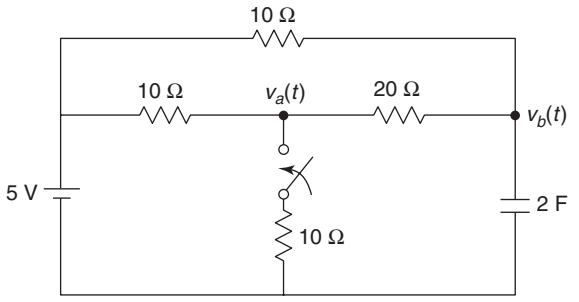


Fig. 6.49

Solution At $t=0^-$, the network is shown in Fig. 6.50.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_a(0^-)=5\text{ V}$$
$$v_b(0^-)=5\text{ V}$$

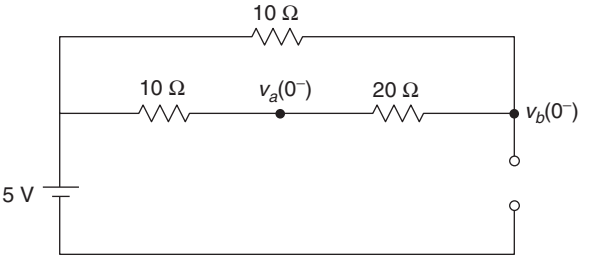


Fig. 6.50

At $t=0^+$, the network is shown in Fig. 6.51.
At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$v_b(0^+)=5\text{ V}$$

Writing the KCL equation at $t = 0^+$,

$$\frac{v_a(0^+)-5}{10}+\frac{v_a(0^+)}{10}+\frac{v_a(0^+)-5}{20}=0$$
$$0.25v_a(0^+)=0.75$$
$$v_a(0^+)=3\text{ V}$$

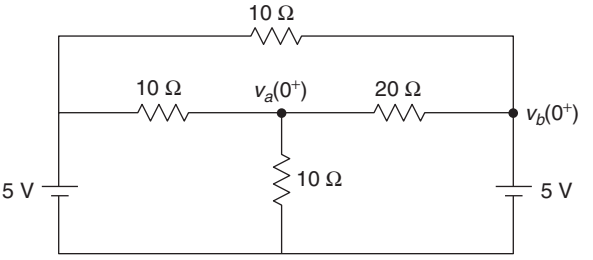


Fig. 6.51

Example 6.16 *The network shown in Fig. 6.52 has two independent node pairs. If the switch is opened at t = 0. Find v_1, v_2, $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0^+$.*

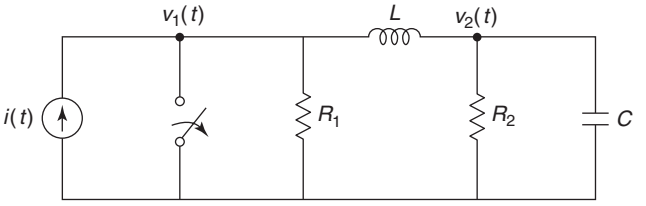


Fig. 6.52

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$
$$v_C(0^-) = v_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.53.
At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

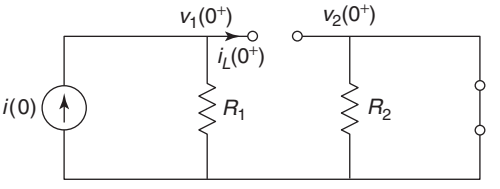


Fig. 6.53

$$i_L(0^+) = 0$$
$$v_1(0^+) = R_1\, i(0^+)$$
$$v_2(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.54.
Writing the KCL equation at Node 1 for $t > 0$,

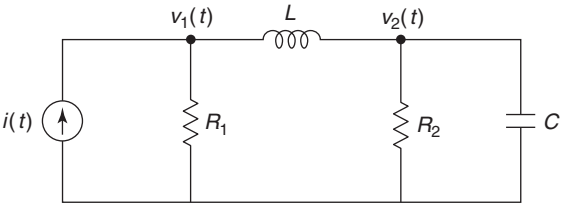


Fig. 6.54

$$\frac{v_1}{R_1} + \frac{1}{L}\int_0^t (v_1 - v_2)dt = i(t) \qquad \ldots(\text{i})$$

Differentiating Eq. (i),

$$\frac{1}{R_1}\frac{dv_1}{dt} + \frac{1}{L}(v_1 - v_2) = \frac{di}{dt}$$

At $t = 0^+$,

$$\frac{dv_1}{dt}(0^+) = R_1\left[\frac{di}{dt}(0^+) - \frac{1}{L}R_1 i(0^+)\right]$$

Writing the KCL equation at Node 2 for $t > 0$,

$$\frac{1}{L}\int_0^t (v_2 - v_1)dt + \frac{v_2}{R_2} + C\frac{dv_2}{dt} = 0 \qquad \ldots(\text{ii})$$

At $t = 0^+$,

$$0 + \frac{v_2(0^+)}{R_2} + C\frac{dv_2}{dt}(0^+) = 0$$
$$\frac{dv_2}{dt}(0^+) = 0$$

Example 6.17 *In the network shown in Fig. 6.55, the switch is closed at t = 0, with zero capacitor voltage and zero inductor current. Solve for* $v_1, v_2, \frac{dv_1}{dt}, \frac{dv_2}{dt}$ *and* $\frac{d^2v_2}{dt^2}$ *at* $t = 0^+$.

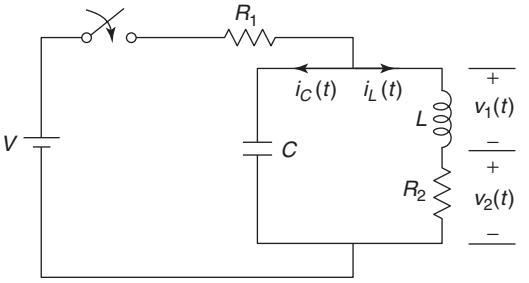


Fig. 6.55

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$v_1(0^-) = 0$$

$$v_2(0^-) = 0$$

$$i_L(0^-) = 0$$

$$i_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.56.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$

$$v_1(0^+) = 0$$

$$v_2(0^+) = 0$$

$$i_L(0^+) = 0$$

$$i_C(0^+) = \frac{V}{R_1}$$

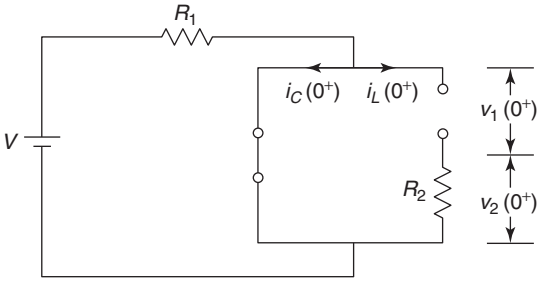


Fig. 6.56

For $t > 0$, the network is shown in Fig. 6.57.

Writing the KVL equation for $t > 0$,

$$v_C(t) = v_1(t) + v_2(t) \qquad \ldots(\text{i})$$

Differentiating Eq. (i),

$$\frac{dv_C}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt} \qquad \ldots(\text{ii})$$

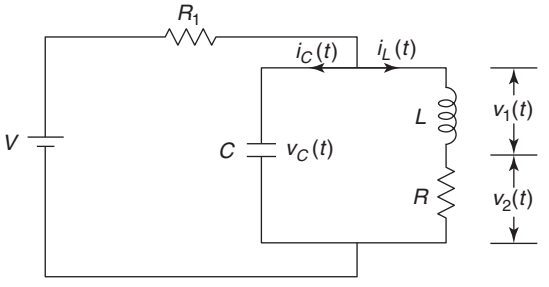


Fig. 6.57

Now, $$v_C = \frac{1}{C}\int_0^t i_C\, dt \quad \text{…(iii)}$$

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

At $t = 0^+$, $$\frac{dv_C}{dt}(0^+) = \frac{i_C(0^+)}{C} = \frac{V}{R_1 C}\ \text{V/s}$$

Also $$v_1 = L\frac{di_L}{dt} \quad \text{…(iv)}$$

$$\frac{di_L}{dt} = \frac{v_1}{L} \quad \text{…(v)}$$

At $t = 0^+$, $$\frac{di_L}{dt}(0^+) = \frac{v_1(0^+)}{L} = 0$$

Also, $$v_2 = R_2 i_L \quad \text{…(vi)}$$

$$\frac{dv_2}{dt} = R_2\frac{di_L}{dt} \quad \text{…(vii)}$$

At $t = 0^+$, $$\frac{dv_2}{dt}(0^+) = R_2\frac{di_L}{dt}(0^+) = 0$$

$$\frac{dv_C}{dt}(0^+) = \frac{dv_1}{dt}(0^+) + \frac{dv_2}{dt}(0^+)$$

$$\frac{dv_1}{dt}(0^+) = \frac{V}{R_1 C}\ \text{V/s}$$

Differentiating Eq. (vii),

$$\frac{d^2v_2}{dt^2} = R_2\frac{d^2i_L}{dt^2}$$

At $t = 0^+$, $$\frac{d^2v_2}{dt^2}(0^+) = R_2\frac{d^2i_L}{dt}(0^+)$$

Differentiating Eq. (v),

$$\frac{d^2i_L}{dt^2} = \frac{1}{L}\frac{dv_1}{dt}$$

At $t = 0^+$, $$\frac{d^2i_L}{dt^2}(0^+) = \frac{1}{L}\frac{dv_1}{dt}(0^+) = \frac{1}{L}\frac{V}{R_1 C}$$

$$\frac{d^2v_2}{dt^2}(0^+) = \frac{R_2 V}{R_1 LC}\ \text{V/s}^2$$

Example 6.18 *In the network shown in Fig. 6.58, a steady state is reached with switch open. At t = 0, switch is closed. Find the three loop currents at t = 0⁺.*

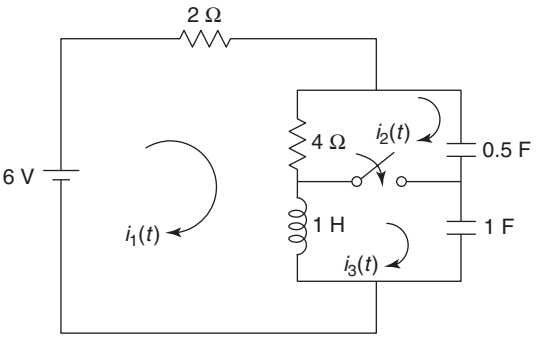


Fig. 6.58

Solution At $t = 0^-$, the network is shown in Fig. 6.59.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor act as a short circuit and the capacitors act as open circuits.

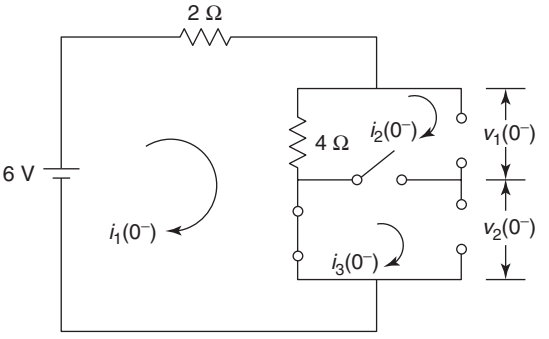


Fig. 6.59

$$i_{4\Omega}(0^-) = i_1(0^-) = \frac{6}{6} = 1 \text{ A}$$

$$i_2(0^-) = 0$$

$$i_3(0^-) = 0$$

$$v_1(0^-) + v_2(0^-) = 6 \times \frac{4}{6} = 4 \text{ V}$$

Since the charges on capacitors are equal when connected in series,

$$Q_1 = Q_2$$

$$C_1 v_1 = C_2 v_2$$

$$\frac{v_1(0^-)}{v_2(0^-)} = \frac{C_2}{C_1} = \frac{1}{0.5} = 2$$

$$v_1(0^-) = \frac{8}{3} \text{ V}$$

and

$$v_2(0^-) = \frac{4}{3} \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.60.

At $t = 0^+$, the inductor is replaced by a current source of 1 A and the capacitors are replaced by a voltage source of $\frac{8}{3}$ V and $\frac{4}{3}$ V respectively.

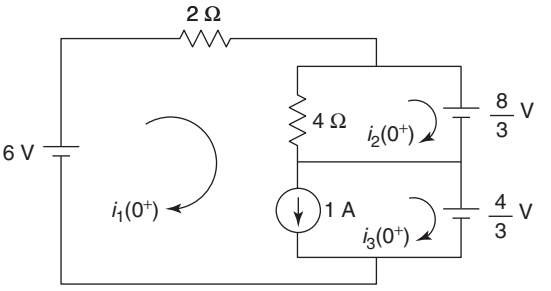


Fig. 6.60

$$v_1(0^+) = \frac{8}{3}\text{ V}$$

$$v_2(0^+) = \frac{4}{3}\text{ V}$$

At $t = 0^+$, $\quad 6 - 2i_1(0^+) - \frac{8}{3} - \frac{4}{3} = 0$

$$i_1(0^+) = 1\text{ A}$$

Now, $\quad i_1(0^+) - i_3(0^+) = 1$

$$i_3(0^+) = 0$$

Writing the KVL equation for Mesh 2,

$$-4[i_2(0^+) - i_1(0^+)] - \frac{8}{3} = 0$$

$$-4i_2(0^+) + 4 - \frac{8}{3} = 0$$

$$i_2(0^+) = \frac{1}{3}\text{ A}$$

Example 6.19 *In the network shown in Fig. 6.61, the switch K is closed at t = 0 connecting a voltage V_0 sin ωt to the parallel RL-RC circuit. Find (a) $i_1(0^+)$ and $i_2(0^+)$ (b) $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.*

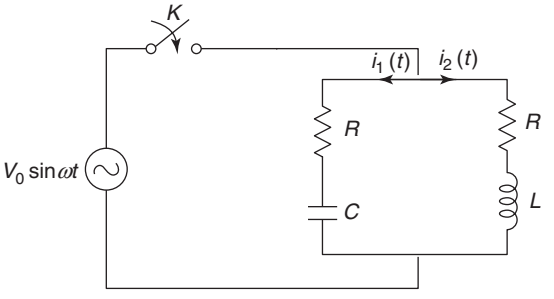


Fig. 6.61

Solution At $t = 0^-$, no current flows in the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.62.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit. The voltage source $V_0 \sin \omega t$ acts as a short circuit.

$$i_1(0^+) = 0$$

$$i_2(0^+) = 0$$

$$v_C(0^+) = 0$$

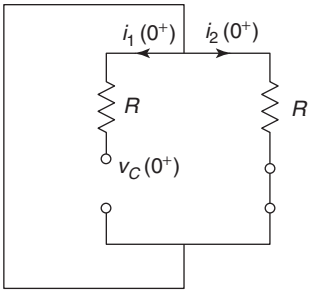


Fig. 6.62

For $t > 0$, the network is shown in Fig. 6.63.

Writing the KVL equation for $t > 0$,

$$V_0 \sin \omega t - R\, i_1 - \frac{1}{C}\int i_1\, dt = 0 \qquad \ldots(i)$$

and

$$V_0 \sin \omega t - R\, i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots(ii)$$

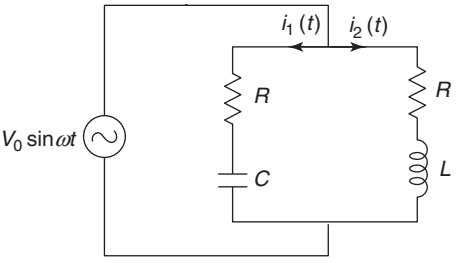


Fig. 6.63

Differentiating Eq. (i),

$$V_0\, \omega \cos \omega t - R\frac{di_1}{dt} - \frac{i_1}{C} = 0$$

$$\frac{di_1}{dt} = \frac{V_0 \omega}{R}\cos \omega t - \frac{i_1}{RC} \qquad \ldots(iii)$$

At $t = 0^+$,

$$\frac{di_1}{dt}(0^+) = \frac{V_0 \omega}{R}\cos \omega t\Big|_{t=0^+} - \frac{i_1(0^+)}{RC} = \frac{V_0 \omega}{R}$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{V_0}{L}\sin \omega t - \frac{R}{L} i_2$$

At $t = 0^+$,

$$\frac{di_2}{dt}(0^+) = \frac{V_0}{L}\sin \omega t\Big|_{t=0^+} - \frac{R}{L} i_2(0^+) = 0$$

Example 6.20 *In the network of Fig. 6.64, the switch K is changed from 'a' to 'b' at $t = 0$ (a steady state having been established at the position a). Find i_1, i_2 and i_3 at $t = 0^+$.*

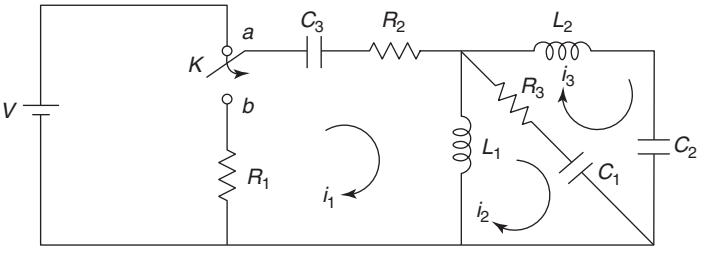


Fig. 6.64

Solution At $t = 0^-$, the network is shown in Fig. 6.65.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitors act as open circuits and inductors act as short circuits.

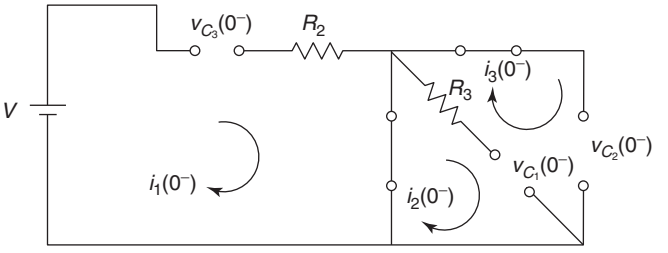


Fig. 6.65

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_3(0^-) = 0$$
$$v_{C_3}(0^-) = V$$
$$v_{C_2}(0^-) = 0$$
$$v_{C_1}(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.66.

At $t = 0^+$, the capacitor C_3 acts as a voltage source of V volts and capacitors C_1 and C_2 act as short circuits. The inductors act as open circuits.

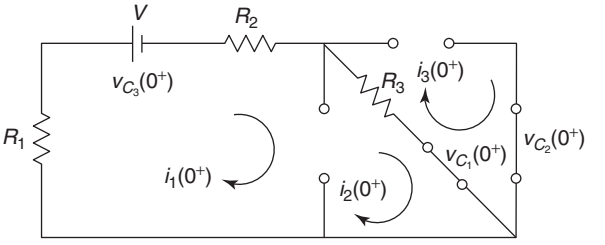


Fig. 6.66

$$i_1(0^+) = i_2(0^+) = -\frac{V}{R_1 + R_2 + R_3}$$

$$i_3(0^+) = 0$$

Example 6.21 *In the network of Fig. 6. 67, the switch K_1 has been closed for a long time prior to $t = 0$. At $t = 0$, the switch K_2 is closed. Find $v_C(0^+)$ and $i_C(0^+)$.*

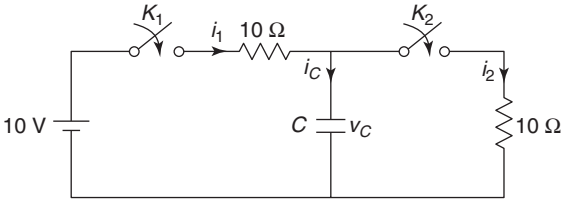


Fig. 6.67

Solution At $t = 0^-$, the network is shown in Fig. 6.68. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

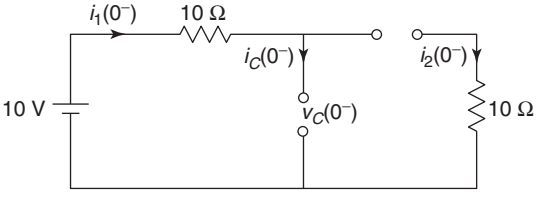


Fig. 6.68

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_C(0^-) = 0$$
$$v_C(0^-) = 10 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 6.69.

At $t = 0^+$, the capacitor acts as a voltage source of voltage V.

$$v_C(0^+) = 10 \text{ V}$$

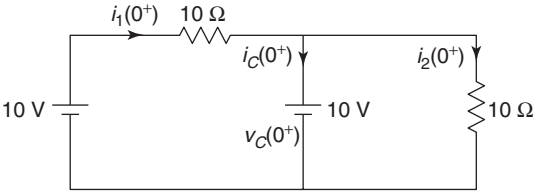


Fig. 6.69

Writing the KVL equation at $t = 0^+$,

$$10 - 10\, i_1(0^+) - 10 = 0$$

and

$$10 - 10\, i_2(0^+) = 0$$

$$i_1(0^+) = 0$$

$$i_2(0^+) = -1 \text{ A}$$

$$i_1(0^+) = i_C(0^+) + i_2(0^+)$$

$$i_C(0^+) = 1 \text{ A}$$

Example 6.22 *In the network shown in Fig. 6.70, a steady state is reached with the switch open. At $t = 0$, the switch is closed. Determine $v_C(0^-), i_1(0^+), i_2(0^+), \frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.*

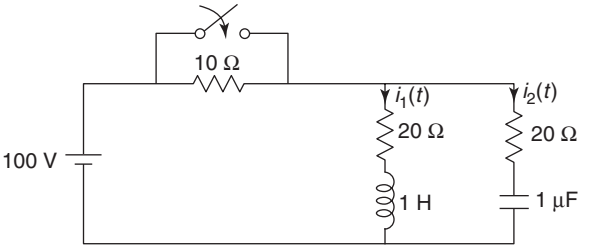


Fig. 6.70

Solution At $t = 0^-$, the network is shown in Fig. 6.71.

At $t = 0^-$, the network is in steady-state. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

$$v_C(0^-) = 100 \times \frac{20}{20+10} = 66.67 \text{ V}$$

$$i_1(0^-) = \frac{66.67}{20} = 3.33 \text{ A}$$

$$i_2(0^-) = 0$$

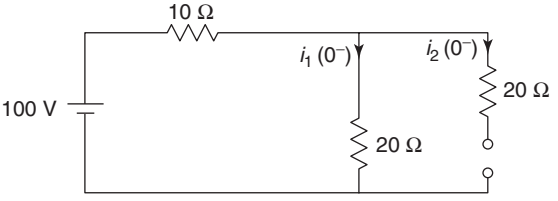


Fig. 6.71

At $t = 0^+$, the network is shown in Fig. 6.72.

At $t = 0^+$, the inductor acts as a current source of 3.33 A and the capacitor acts as a voltage source of 66.67 V.

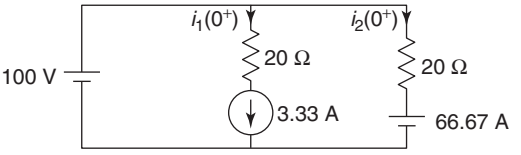


Fig. 6.72

$$v_C(0^+) = 66.67 \text{ V}$$

$$i_1(0^+) = 3.33 \text{ A}$$

$$i_2(0^+) = \frac{100 - 66.67}{20} = 1.67 \text{ A}$$

For $t > 0^-$, the network is shown in Fig. 6.73.

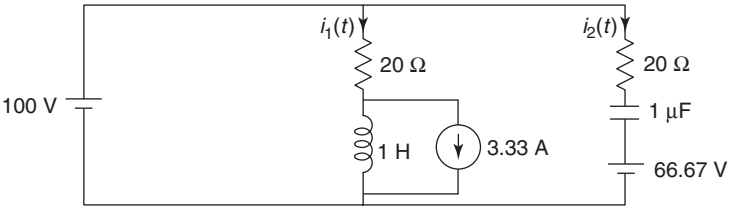


Fig. 6.73

Writing the KVL equations for $t > 0$,

$$100 - 20i_1 - 1\frac{di_1}{dt} = 0 \quad \ldots(\text{i})$$

and $$100 - 20\, i_2 - \frac{1}{10^{-6}} \int i_2\, dt - 66.67 = 0 \quad \ldots(\text{ii})$$

At $t = 0^+$, $$\frac{di_1}{dt}(0^+) = 100 - 20i_1(0^+) = 100 - 20\,(3.33) = 33.3 \text{ A/s}$$

Differentiating Eq. (ii),

$$0 - 20\frac{di_2}{dt} - 10^6 i_2 = 0$$

At $t = 0^+$, $$20\frac{di_2}{dt}(0^+) = -10^6\, i_2(0^+)$$

$$\frac{di_2}{dt}(0^+) = -\frac{10^6}{20} \times 1.67 = -83500 \text{ A/s}^2$$

6.3 RESISTOR-INDUCTOR CIRCUIT

Consider a series *RL* circuit as shown in Fig. 6.74. The switch is closed at time $t = 0$. The inductor in the circuit is initially un-energised.

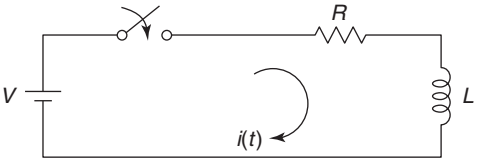


Fig. 6.74 *Series RL circuit*

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} = 0$$

This is a linear differential equation of first order. It can be solved if the variables can be separated.

$$(V - Ri)\, dt = L\, di$$

$$\frac{L\, di}{V - Ri} = dt$$

Integrating both the sides,

$$-\frac{L}{R} l_n(V - Ri) = t + k$$

where l_n denotes that the logarithm is of base e and k is an arbitrary constant. k can be evaluated from the initial condition. In the circuit, the switch is closed at $t = 0$, i.e., just before closing the switch, the current in the inductor is zero. Since the inductor does not allow sudden change in current, at $t = 0^+$, just after the switch is closed, the current remains zero.

Setting $i = 0$ at $t = 0$,

$$-\frac{L}{R} l_n V = k$$

$$-\frac{L}{R} l_n (V - Ri) = t - \frac{L}{R} l_n V$$

$$-\frac{L}{R}\left[l_n (V - Ri) - l_n V\right] = t$$

$$\frac{V - Ri}{V} = e^{-\frac{R}{L}t}$$

$$V - Ri = Ve^{-\frac{R}{L}t}$$

$$Ri = V - Ve^{-\frac{R}{t}t}$$

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

The complete response is composed of two parts, the steady-state response or forced response or zero state response $\frac{V}{R}$ and transient response or natural response or zero input response $\frac{V}{R} e^{-\frac{R}{L}t}$.

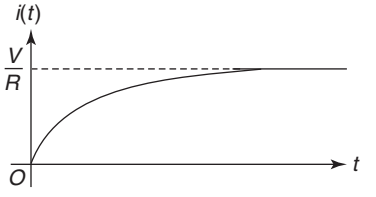


Fig. 6.75 *Current response of series RL circuit*

The natural response is a characteristic of the circuit. Its form may be found by considering the source-free circuit. The forced response has the characteristics of forcing function, i.e., applied voltage. Thus, when the switch is closed, response reaches the steady-state value after some time interval as shown in Fig. 6.75.

Here, the transient period is defined as the time taken for the current to reach its final or steady state value from its initial value.

The term $\frac{L}{R}$ is called time constant and is denoted by T.

$$T = \frac{L}{R}$$

At one time constant, the current reaches 63.2 per cent of its final value $\frac{V}{R}$.

$$i(T) = \frac{V}{R} - \frac{V}{R} e^{-\frac{1}{T}t} = \frac{V}{R} - \frac{V}{R} e^{-1} = \frac{V}{R} - 0.368\frac{V}{R} = 0.632\frac{V}{R}$$

Similarly,

$$i(2T) = \frac{V}{R} - \frac{V}{R}e^{-2} = \frac{V}{R} - 0.135\frac{V}{R} = 0.865\frac{V}{R}$$

$$i(3T) = \frac{V}{R} - \frac{V}{R}e^{-3} = \frac{V}{R} - 0.0498\frac{V}{R} = 0.950\frac{V}{R}$$

$$i(5T) = \frac{V}{R} - \frac{V}{R}e^{-5} = \frac{V}{R} - 0.0067\frac{V}{R} = 0.993\frac{V}{R}$$

After 5 time constants, the current reaches 99.33 per cent of its final value. The voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= V\left(1 - e^{-\frac{R}{L}t}\right) \qquad \text{for } t > 0$$

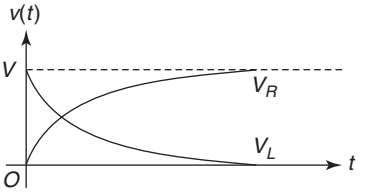


Fig. 6.76 *Voltage response of series RL circuit*

Similarly, the voltage across the inductor is

$$v_L = \frac{di}{dt} = L\frac{V}{R}\frac{d}{dt}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= Ve^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

Note:

1. Consider a homogeneous equation,

$$\frac{di}{dt} + Pi = 0 \qquad \text{where } P \text{ is a constant.}$$

The solution of this equation is given by,

$$i\,(t) = ke^{-Pt}$$

The value of k is obtained by putting $t = 0$ in the equation for $i\,(t)$.

2. Consider a non-homogeneous equation,

$$\frac{di}{dt} + Pi = Q$$

where P is a constant and Q may be a function of the independent variable t or a constant.
The solution of this equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

The value of k is obtained by putting $t = 0$ in the equation of $i(t)$.

Example 6.23 *In the network of Fig. 6.77, the switch is initially at the position 1. On the steady state having reached, the switch is changed to the position 2. Find current i(t).*

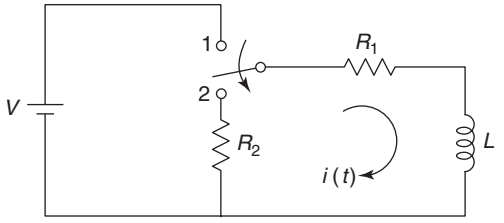


Fig. 6.77

Solution At $t = 0^-$, the network is shown in Fig. 6.78.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

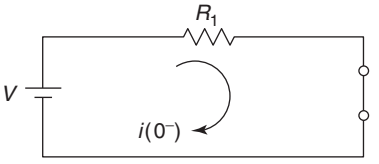


Fig. 6.78

$$i(0^-) = \frac{V}{R_1}$$

Since the inductor does not allow sudden change in current,

$$i(0^+) = \frac{V}{R_1}$$

For $t > 0$, the network is shown in Fig. 6.79.

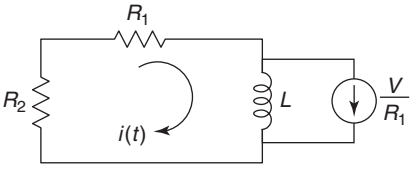


Fig. 6.79

Writing the KVL equation for $t > 0$,

$$-R_2 i - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{(R_1 + R_2)}{L} i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = \frac{R_1 + R_2}{L}$$

The solution of this differential equation is given by,

$$i(t) = k\, e^{-Pt}$$

$$i(t) = k\, e^{-\left(\frac{R_1 + R_2}{L}\right)t}$$

At $t = 0$, $i(0) = \frac{V}{R_1}$

$$\frac{V}{R_1} = k\, e^0 = k$$

$$i(t) = \frac{V}{R_1} e^{-\left(\frac{R_1 + R_2}{L}\right)t} \quad \text{for } t > 0$$

Example 6.24 *In the network shown in Fig. 6.80, the switch is closed at t = 0, a steady state having previously been attained. Find the current i (t).*

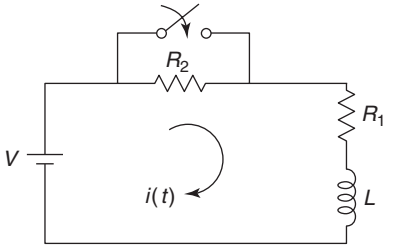


Fig. 6.80

Solution At $t = 0^-$, the network is shown in Fig. 6.81.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{V}{R_1 + R_2}$$

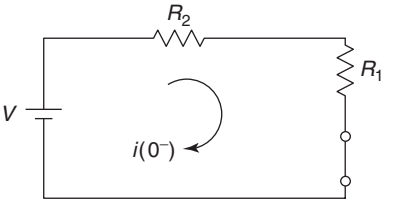


Fig. 6.81

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = \frac{V}{R_1 + R_2}$$

For $t > 0$, the network is shown in Fig. 6.82.

Writing the KVL equation for $t > 0$,

$$V - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R_1}{L} i = \frac{V}{L}$$

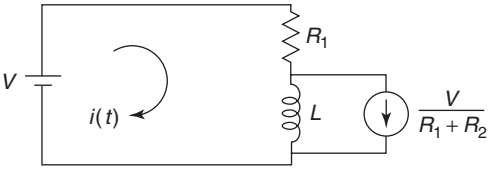


Fig. 6.82

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = \frac{R_1}{L},\ Q = \frac{V}{L}$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt} \int Q e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-\frac{R_1}{L}t} \int \frac{V}{L} e^{-\frac{R_1}{L}t} dt + k\, e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1} + k\, e^{-\frac{R_1}{L}t}$$

At $t = 0,\ i(0) = \dfrac{V}{R_1 + R_2}$

$$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + k$$

$$k = -\frac{VR_2}{R_1(R_1 + R_2)}$$

$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} e^{-\frac{R_1}{L}t}\right) \quad \text{for } t > 0$$

Example 6.25 *In the network of Fig. 6.83, a steady state is reached with the switch K open. At $t = 0$, the switch K is closed. Find the current $i(t)$ for $t > 0$.*

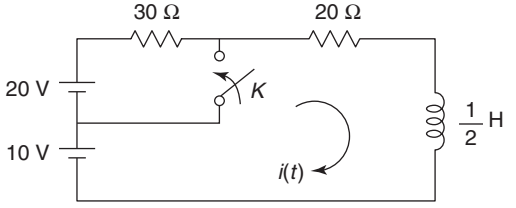


Fig. 6.83

Solution At $t = 0^-$, the network is shown in Fig. 6.84.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{20 + 10}{30 + 20} = 0.6 \text{ A}$$

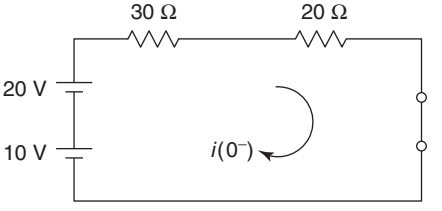


Fig. 6.84

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 0.6 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.85.

Writing the KVL equation for $t > 0$.

$$10 - 20i - \frac{1}{2}\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 40i = 20$$

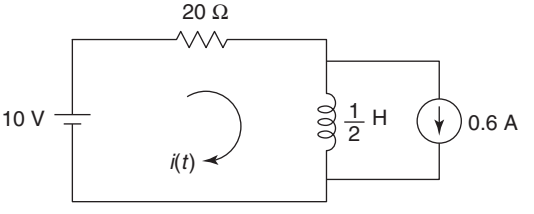


Fig. 6.85

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 40, \quad Q = 20$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-40t} \int 20\, e^{40t} dt + k\, e^{-40t}$$

$$= \frac{20}{40} + k\, e^{-40t}$$

$$= 0.5 + k\, e^{-40t}$$

At $t = 0, i(0) = 0.6$ A

$$0.6 = 0.5 + k$$

$$k = 0.1$$

$$i(t) = 0.5 + 0.1\, e^{-40t} \qquad \text{for } t > 0$$

Example 6.26 *The network of Fig. 6.86 is under steady state with switch at the position 1. At t = 0, switch is moved to position 2. Find i (t).*

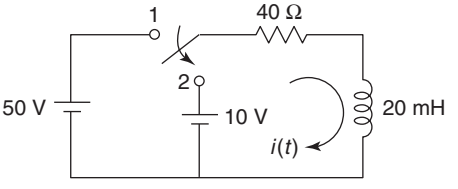


Fig. 6.86

Solution At $t = 0^-$, the network is shown in Fig. 6.87.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{50}{40} = 1.25 \text{ A}$$

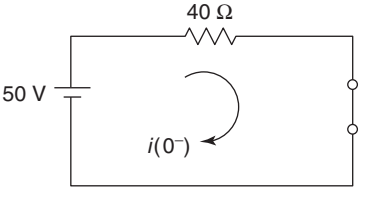


Fig. 6.87

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.25 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.88.

Writing the KVL equation for $t > 0$,

$$10 - 40i - 20 \times 10^{-3} \frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2000i = 500$$

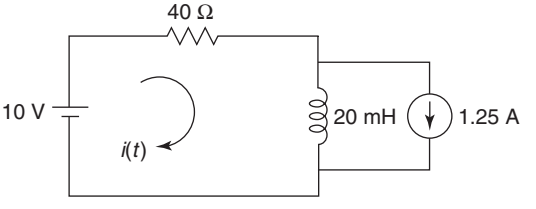


Fig. 6.88

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2000, \; Q = 500$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-2000t} \int 500\, e^{2000t} + k\, e^{-2000t} \\ &= \frac{500}{2000} + k\, e^{-2000t} \\ &= 0.25 + k\, e^{-2000t} \end{aligned}$$

At $t = 0, \; i(0) = 1.25$ A

$$\begin{aligned} 1.25 &= 0.25 + k \\ k &= 1 \\ i(t) &= 0.25 + e^{-2000t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.27 *In the network of Fig. 6.89, the switch is moved from 1 to 2 at t = 0. Determine i(t).*

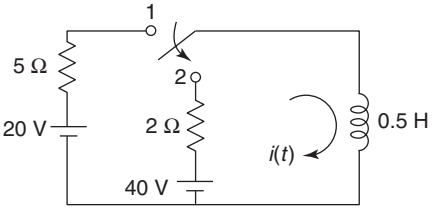


Fig. 6.89

Solution At $t = 0^-$, the network is shown in Fig. 6.90.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{20}{5} = 4 \text{ A}$$

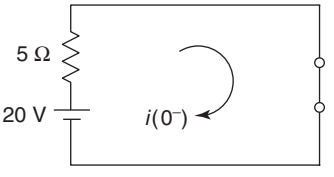


Fig. 6.90

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 4 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.91.

Writing the KVL equation for $t > 0$,

$$40 - 2i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 4i = 80$$

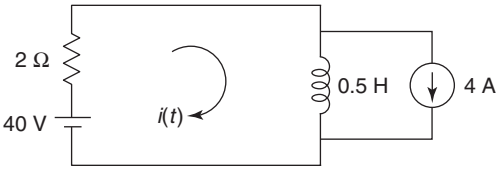


Fig. 6.91

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 4, \quad Q = 80$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-4t} \int 80\, e^{4t}\, dt + k\, e^{-4t} \\ &= \frac{80}{4} + k\, e^{-4t} \\ &= 20 + k\, e^{-4t} \end{aligned}$$

At $t = 0, i(0) = 4$ A

$$\begin{aligned} 4 &= 20 + k \\ k &= -16 \\ i(t) &= 20 - 16\, e^{-4t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.28 *For the network shown in Fig. 6.92, steady state is reached with the switch closed. The switch is opened at t = 0. Obtain expressions for i_L (t) and v_L (t).*

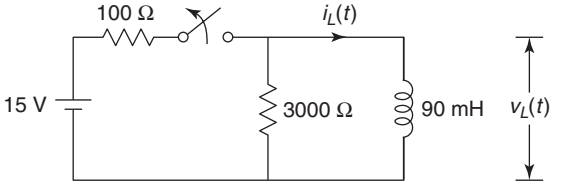


Fig. 6.92

Solution At $t = 0^-$, the network is shown in Fig. 6.93.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{15}{100} = 0.15 \text{ A}$$

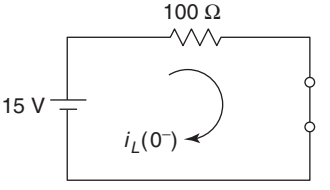


Fig. 6.93

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 0.15 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.94.

Writing the KVL equation for $t > 0$,

$$-3000 i_L - 90 \times 10^{-3} \frac{di_L}{dt} = 0$$

$$\frac{di_L}{dt} + 33.33 \times 10^3\, i_L = 0$$

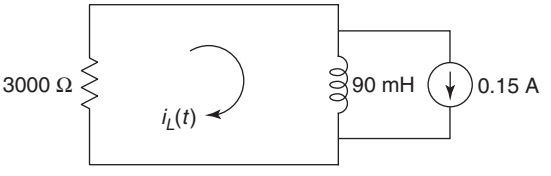


Fig. 6.94

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 33.33 \times 10^3$$

The solution of this differential equation is given by,

$$i_L(t) = k\,e^{-Pt}$$

$$i_L(t) = k\,e^{-33.33\times10^3 t}$$

At $t = 0$, $i_L(0) = 0.15$ A

$$0.15 = k$$

$$i_L(t) = 0.15\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Also,

$$v_L(t) = L\frac{di_L}{dt}$$

$$= 90\times10^{-3}\,\frac{d}{dt}(0.15\,e^{-33.33\times10^3 t})$$

$$= -90\times10^{-3}\times0.15\times33.33\times10^3\times e^{-33.33\times10^3 t}$$

$$= -450\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Example 6.29 *In the network of Fig. 6.95, the switch is open for a long time and it closes at t = 0. Find i (t).*

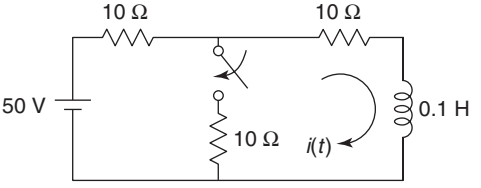


Fig. 6.95

Solution At $t = 0^-$, the network is shown in Fig. 6.96.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

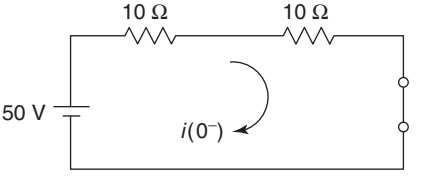


Fig. 6.96

$$i(0^-) = \frac{50}{10+10} = 2.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.97.

For $t > 0$, representing the network to the left of the inductor by Thevenin's equivalent network,

$$V_{eq} = 50\times\frac{10}{10+10} = 25 \text{ V}$$

$$R_{eq} = (10 \,||\, 10) + 10 = 15\ \Omega$$

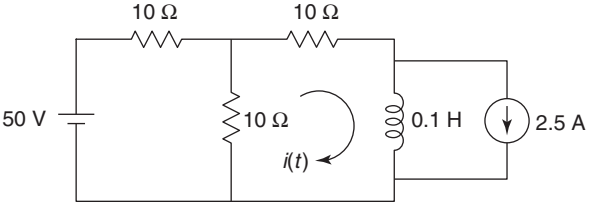


Fig. 6.97

For $t > 0$, Thevenin's equivalent network is shown in Fig. 6.98.
Writing the KVL equation for $t > 0$,

$$25 - 15i - 0.1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 150i = 250$$

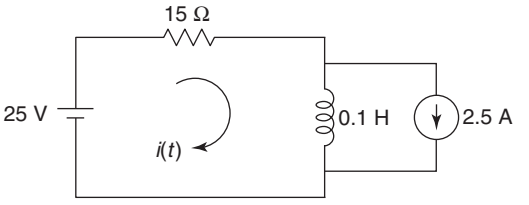


Fig. 6.98

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 150, \quad Q = 250$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-150t}\int 250\, e^{150t}\, dt + k\, e^{-Pt}$$

$$= \frac{250}{150} + k\, e^{-150t}$$

$$= 1.667 + k\, e^{-150t}$$

At $t = 0$, $i(0) = 2.5$ A

$$2.5 = 1.667 + k$$

$$k = 0.833$$

$$i(t) = 1.667 + 0.833\, e^{-150t} \qquad \text{for } t > 0$$

Example 6.30 *In Fig. 6.99, the switch is closed at t = 0. Find i (t) for t > 0.*

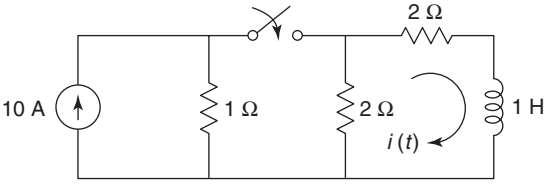


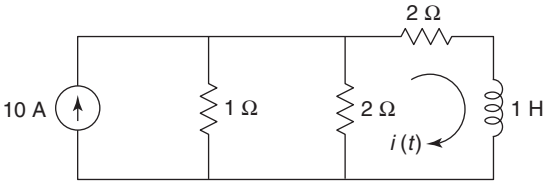
Fig. 6.99

Solution At $t = 0^-$, $i(0^-) = 0$

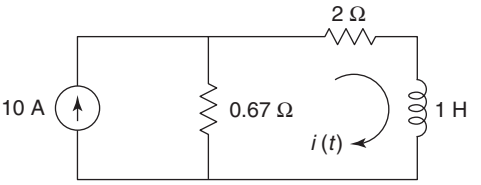
Since current through inductor cannot change instantaneously,

$$i(0^+) = 0$$

For $t > 0$, simplifying the network by source-transformation technique as shown in Fig. 6.100.



(a)

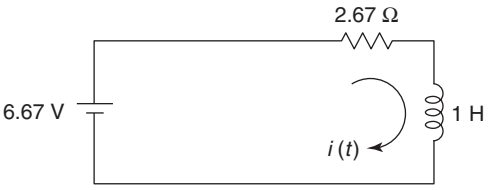


(b)

Writing the KVL equation for $t > 0$,

$$6.67 - 2.67i - 1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2.67i = 6.67$$



(c)

Fig. 6.100

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2.67,\ Q = 6.67$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-2.67t}\int 6.67\, e^{2.67t}\, dt + k\, e^{-2.67t}$$

$$= \frac{6.67}{2.67} + k\, e^{-2.67t}$$

$$= 2.5 + k\, e^{-2.67t}$$

At $t = 0$, $i\,(0) = 0$

$$0 = 2.5 + k$$

$$k = -2.5$$

$$i(t) = 2.5 - 2.5\, e^{-2.67t}$$

$$= 2.5(1 - e^{-2.67t}) \qquad \text{for } t > 0$$

Example 6.31 *Find the current i (t) for t > 0.*

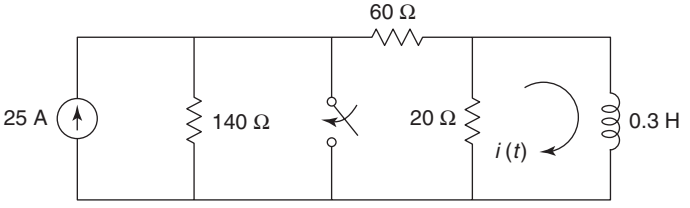


Fig. 6.101

Solution At $t = 0^-$, the inductor acts as a short circuit. Simplifying the network as shown in Fig. 6.102.

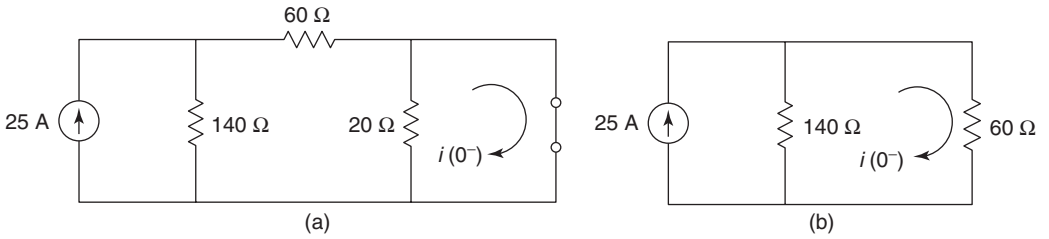


Fig. 6.102

$$i(0^-) = 25 \times \frac{140}{140 + 60} = 17.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 17.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.103.

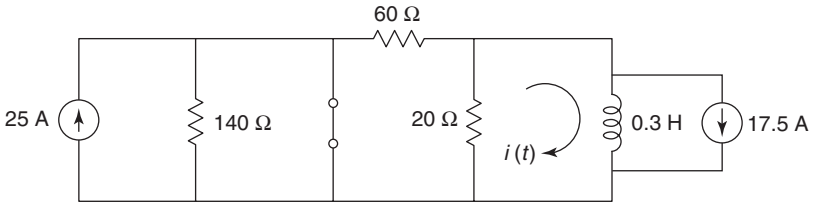


Fig. 6.103

Simplifying the network by source transformation as shown in Fig. 6.104,

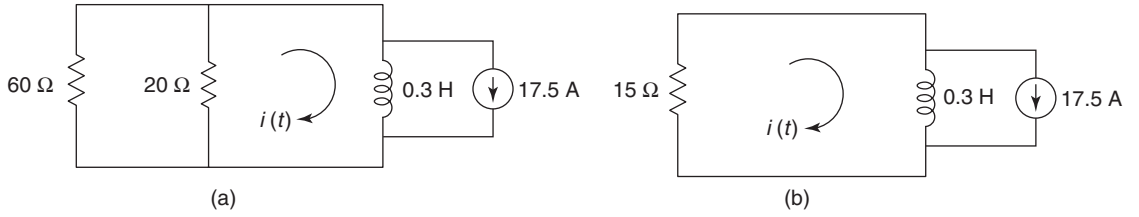


Fig. 6.104

Writing the KVL equation for $t > 0$,

$$-15i - 0.3\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 50i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 50$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50t}$$

At $t = 0$, $i\,(0) = 17.5$ A

$$k = 17.5$$

$$i(t) = 17.5\,e^{-50t} \qquad \text{for } t > 0$$

Example 6.32 *In the network of Fig. 6.105, the switch is in position 'a' for a long time. At t = 0, the switch is moved from a to b. Find v_2 (t). Assume that the initial current in the 2 H inductor is zero.*

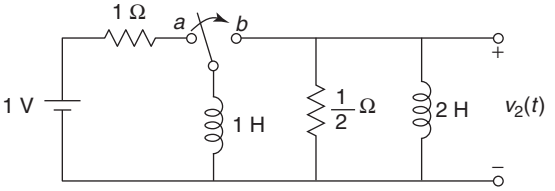


Fig. 6.105

Solution At $t = 0^-$, the switch is in the position *a*. The network has attained steady-state condition. Hence, the inductor acts as a short circuit.
Current through the 1 H inductor is given by

$$i(0^-) = \frac{1}{1} = 1\text{ A}$$

$$v_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1\text{ A}$$

$$v_2(0^+) = -1 \times \frac{1}{2} = -0.5\text{ V}$$

For $t > 0$, the network is shown in Fig. 6.106.
Writing the KCL equation for $t > 0$,

$$\frac{1}{1}\int_0^t v_2 dt + 1 + \frac{v_2}{\frac{1}{2}} + \frac{1}{2}\int_0^t v_2 dt = 0 \qquad \ldots\text{(i)}$$

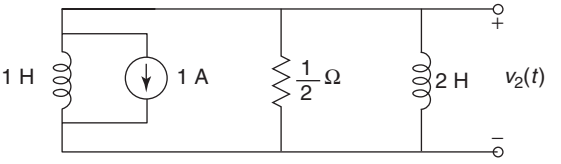


Fig. 6.106

Differentiating Eq. (i),

$$v_2 + 2\frac{dv_2}{dt} + \frac{1}{2}v_2 = 0$$

$$\frac{dv_2}{dt} + \frac{3}{4}v_2 = 0$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = 0$,

$$P = \frac{3}{4}$$

The solution of this differential equation is given by,

$$v_2(t) = K\,e^{-Pt} = k\,e^{-\frac{3}{4}t}$$

At $t = 0$, $v_2(0) = -0.5$ V

$$-0.5 = k\,e^{\circ}$$
$$k = -0.5$$
$$v_2(t) = -0.5\,e^{-\frac{3}{4}t} \qquad \text{for } t > 0$$

Example 6.33 *In the network shown in Fig. 6.107, a steady-state condition is achieved with switch open. At t = 0 switch is closed. Find v_a (t).*

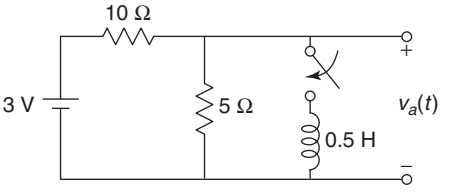


Fig. 6.107

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = 0$$
$$v_a(0^-) = 3 \times \frac{5}{10+5} = 1\text{ V}$$

Since current through inductor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v_a(0^+) = 1\text{ V}$$

For $t > 0$, the network is shown in Fig. 6.108.

Writing the KCL equation for $t > 0$,

$$\frac{1}{0.5}\int_0^t v_a\,dt + \frac{v_a}{5} + \frac{v_a - 3}{10} = 0$$

Differentiating Eq. (i),

$$2v_a + 0.2\frac{dv_a}{dt} + 0.1\frac{dv_a}{dt} = 0$$
$$\frac{dv_a}{dt} + \frac{20}{3}v_a = 0$$

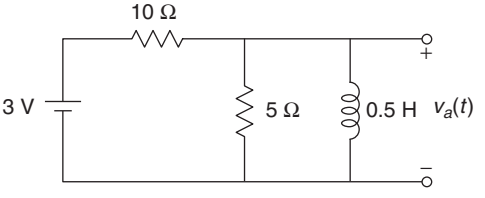


Fig. 6.108

Comparing with the differential equation $\frac{dv}{dt} + Pv = 0$,

$$P = \frac{20}{3}$$

The solution of this differential equation is given by,

$$v_a(t) = k\,e^{-Pt} = k\,e^{-\frac{20}{3}t}$$

At $t = 0$, $v_a(0) = 1$ V

$$1 = k$$

$$v_a(t) = e^{-\frac{20}{3}t} \qquad \text{for } t > 0$$

Example 6.34 *In the network of Fig. 6.109, determine currents $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$.*

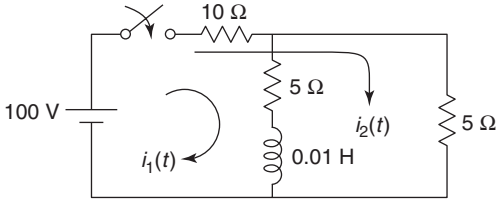


Fig. 6.109

Solution At $t = 0^-$, $i_1(0^-) = i_2(0^-) = 0$

At $t = 0^+$,

$$i_1(0^+) = 0$$

$$i_2(0^+) = \frac{100}{15} = 6.67 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.110.
Writing the KVL equations for $t > 0$,

$$100 - 10(i_1 + i_2) - 5i_1 - 0.01\frac{di_1}{dt} = 0 \qquad \ldots\text{(i)}$$

and

$$100 - 10(i_1 + i_2) - 5i_2 = 0 \qquad \ldots\text{(ii)}$$

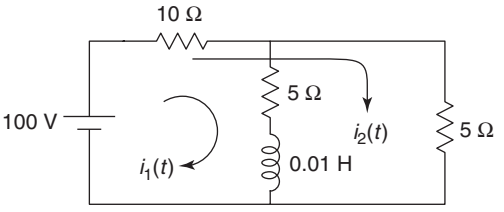


Fig. 6.110

From Eq. (ii),

$$i_2 = \frac{100 - 10i_1}{15}$$

Substituting in Eq. (i),

$$\frac{di_1}{dt} + 833i_1 = 3333$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 833, Q = 3333$$

The solution of this differential equation is given by,

$$\begin{aligned} i_1(t) &= e^{-Pt} \int Q e^{Pt} dt + k e^{-Pt} \\ &= e^{-833t} \int 3333\, e^{833t} dt + k e^{-833t} \\ &= \frac{3333}{833} + k e^{-833t} \\ &= 4 + k e^{-833t} \end{aligned}$$

At $t = 0$, $i_1(0) = 0$

$$\begin{aligned} 0 &= 4 + k \\ k &= -4 \\ i_1(t) &= 4 - 4e^{-833t} \\ &= 4(1 - e^{-833t}) \qquad \text{for } t > 0 \\ i_2(t) &= \frac{100 - 10 i_1}{15} \\ &= \frac{100 - 10(4 - 4e^{-833t})}{15} \\ &= 4 + 2.67 e^{-833t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.35 *The switch in the network shown in Fig. 6.111 is closed at $t = 0$. Find $v_2(t)$ for all $t > 0$. Assume zero initial current in the inductor.*

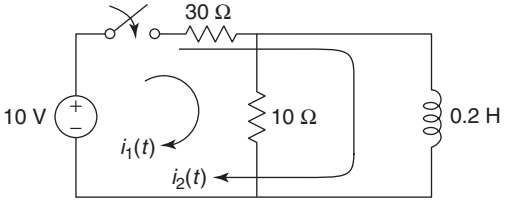


Fig. 6.111

Solution At $t = 0^-$, $i_1(0^-) = 0$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_2(0^+) = 0$$

$$i_1(0^+) = \frac{10}{30 + 10} = 0.25 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.112.
Writing the KVL equations for $t > 0$,

$$10 - 30(i_1 + i_2) - 10\, i_1 = 0 \qquad \ldots(i)$$

and $$10 - 30(i_1 + i_2) - 0.2\frac{di_2}{dt} = 0 \qquad \ldots(ii)$$

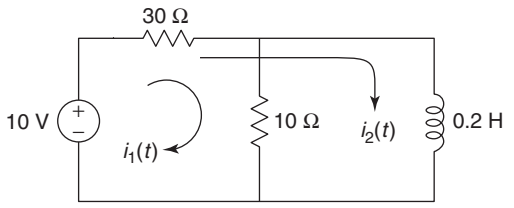


Fig. 6.112

From Eq. (i),

$$i_1 = \frac{10 - 30\, i_2}{40} = 0.25 - 0.75\, i_2 \qquad \ldots(iii)$$

Substituting Eq. (iii) into Eq. (ii),

$$\frac{di_2}{dt} + 37.5\, i_2 = 2.5$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 37.5,\ Q = 2.5$$

The solution of this differential equation is given by,

$$\begin{aligned} i_2(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-37.5t}\int 2.5\, e^{37.5t}\, dt + k\, e^{-37.5t} \\ &= \frac{2.5}{37.5} + k\, e^{-37.5t} \\ &= 0.067 + k\, e^{-37.5t} \end{aligned}$$

At $t = 0,\ i_2(0) = 0$

$$\begin{aligned} 0 &= 0.067 + k \\ k &= -0.067 \\ i_2(t) &= 0.067 - 0.067\, e^{-37.5t} \\ v_2(t) &= 0.2\frac{di_2}{dt} \\ &= 0.2\frac{d}{dt}(0.067 - 0.067\, e^{-37.5t}) \\ &= 0.5\, e^{-37.5t} \qquad \text{for } t > 0 \end{aligned}$$

Example 6.36 *For the network shown in Fig. 6.113, find the current i(t) when the switch is changed from the position 1 to 2 at t = 0.*

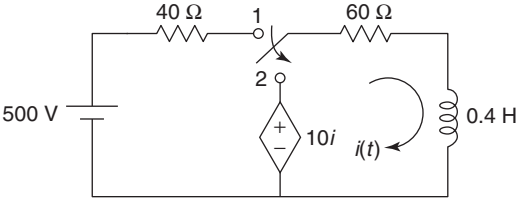


Fig. 6.113

Solution At $t = 0^-$, the network is shown in Fig. 6.114. At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

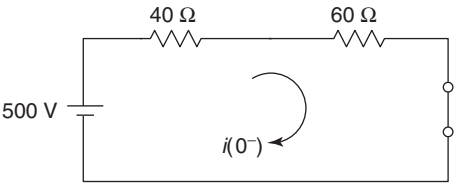


Fig. 6.114

$$i(0^-) = \frac{500}{40+60} = 5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.115. Writing the KVL equation for $t > 0$,

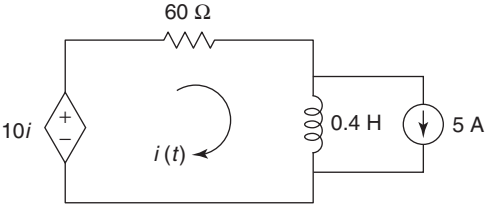


Fig. 6.115

$$10i - 60i - 0.4\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 125i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 125$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-125t}$$

At $t = 0, i(0) = 5$ A

$$5 = k$$

$$i(t) = 5e^{-125t} \qquad \text{for } t > 0$$

Example 6.37 *For the network shown in Fig. 6.116, find the current in the 20 Ω resistor when the switch is opened at $t = 0$.*

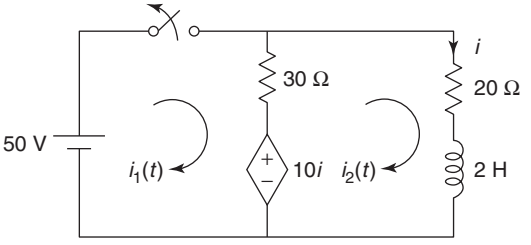


Fig. 6.116

Solution At $t = 0^-$, the network is shown in Fig. 6.117.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

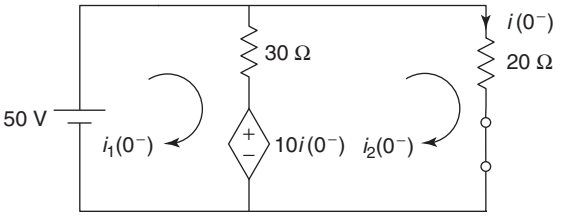


Fig. 6.117

$$i(0^-) = i_2(0^-)$$

Writing the KVL equations at $t = 0^-$,

$$50 - 30(i_1 - i_2) - 10i_2 = 0$$
$$10i_2 - 30(i_2 - i_1) - 20i_2 = 0$$

Solving these equations,

$$i_1(0^-) = 3.33 \text{ A}$$
$$i_2(0^-) = 2.5 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i_2(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 6.118.

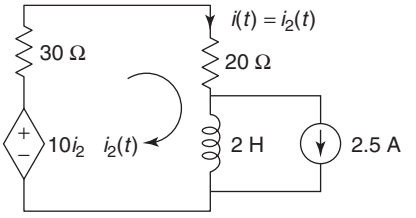


Fig. 6.118

Writing the KVL equation for $t > 0$,

$$10i_2 - 30i_2 - 20i_2 - 2\frac{di_2}{dt} = 0$$
$$\frac{di_2}{dt} + 20i_2 = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 20$$

The solution of this differential equation is given by,

$$i_2(t) = k\,e^{-Pt} = k\,e^{-20t}$$

At $t = 0$, $i_2(0) = 2.5$ A

$$2.5 = k$$
$$i_2(t) = 2.5\,e^{-20t} \qquad \text{for } t > 0$$

Example 6.38 *In the network of Fig. 6.119, an exponential voltage $v(t) = 4\,e^{-3t}$ is applied at $t = 0$. Find the expression for current i(t). Assume zero current through inductor at $t = 0$.*

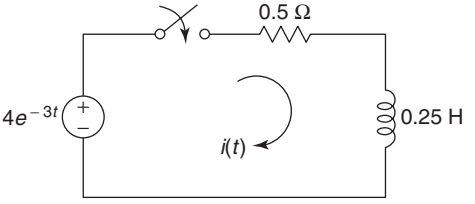


Fig. 6.119

Solution At $t = 0^-$, $i(0^-) = 0$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 0$$

Writing the KVL equation for $t > 0$,

$$4e^{-3t} - 0.5i - 0.25\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2i = 16e^{-3t}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2, \quad Q = 16\,e^{-3t}$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt} \\ &= e^{-2t}\int 16\,e^{-3t}\,e^{2t}\,dt + + k\,e^{-2t} \\ &= 16\,e^{-2t}\int e^{-t}\,dt + k\,e^{-2t} \\ &= -16\,e^{-3t} + k\,e^{-2t} \end{aligned}$$

At $t = 0, i(0) = 0$

$$0 = -16 + k$$

$$k = 16$$

$$i(t) = -16\,e^{-3t} + 16\,e^{-2t} \qquad \text{for } t > 0$$

Example 6.39 *For the network shown in Fig. 6.120, a sinusoidal voltage source $v = 150\sin(500t + \theta)$ volts is applied at a time when $\theta = 0$. Find the expression for the current $i(t)$.*

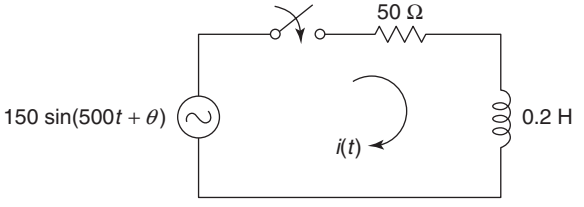


Fig. 6.120

Solution

Writing the KVL equation for $t > 0$,

$$150\sin(500t + \theta) - 50i - 0.2\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 250i = 750\sin(500t + \theta)$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 250, \quad Q = 750\sin(500t + \theta)$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

$$= e^{-250t}\int 750\sin(500t+\theta)\,e^{250t} + k\,e^{-250t}$$

$$= 750\,e^{-250t}\left[\frac{e^{250t}}{(250)^2+(500)^2}\{250\sin(500t+\theta)-500\cos(500t+\theta)\}\right] + k\,e^{-250t}$$

$$= 0.6\sin(500t+\theta) - 1.2\cos(500t+\theta) + k\,e^{-250t}$$

Let $A\cos\phi = 0.6$

and $A\sin\phi = 1.2$

$$A^2\cos^2\phi + A^2\sin^2\phi = (0.6)^2 + (1.2)^2 = 1.8$$

$$A = 1.342$$

and $$\phi = \tan^{-1}\left(\frac{1.2}{0.6}\right) = 63.43^\circ$$

$$i(t) = 1.342\cos(63.43^\circ)\sin(500t+\theta) - 1.342\sin(63.43^\circ)\cos(500t+\theta) + k\,e^{-250t}$$

$$i(t) = 1.342\sin(500t+\theta-63.43^\circ) + k\,e^{-250t}$$

At $t = 0, \theta = 0, i(0) = 0$

$$0 = 1.342\sin(-63.43^\circ) + k$$

$$k = 1.2$$

$$i(t) = 1.342\sin(500t+\theta-63.43^\circ) + 1.2\,e^{-250t} \qquad \text{for } t > 0$$

Example 6.40 *For the network shown in Fig. 6.121, find the transient current when the switch is moved from the position 1 to 2 at $t = 0$. The network is in steady state with the switch in the position 1. The voltage applied to the network is $v = 150\cos(200t + 30^\circ)\,V$.*

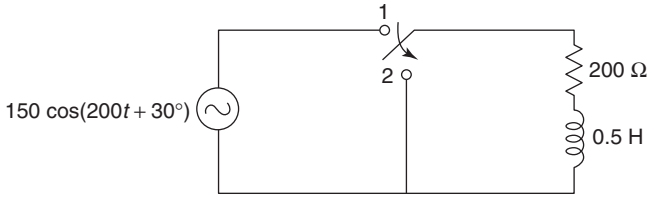


Fig. 6.121

Solution At $t = 0^-$, the network is shown in Fig 6.122.

At $t = 0^-$ the network attains steady-state condition.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{150\angle 30^\circ}{200 + j\times 200\times 0.5} = 0.67\angle 3.43^\circ \text{ A}$$

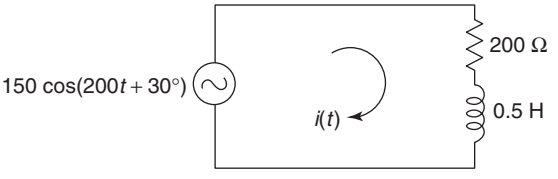


Fig. 6.122

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 0.67\cos(200t + 3.43°) \qquad \ldots(i)$$

For $t > 0$, the network is shown in Fig. 6.123.
Writing the KVL equation for $t > 0$,

$$-200\,i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 400\,i = 0$$

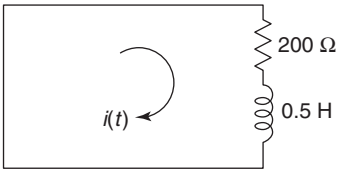


Fig. 6.123

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 400$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-pt} = k\,e^{-400t} \qquad \ldots(ii)$$

From Eqs (i) and (ii),

$$0.67\cos(200t + 3.43°) = k\,e^{-400t}$$

$$0.67\cos(3.43°) = k$$

At $t = 0$, $k = 6.67$

$$i(t) = 0.67\,e^{-400t} \qquad \text{for } t > 0$$

6.4 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 6.124. The switch is closed at time $t = 0$. The capacitor is initially uncharged. Applying KVL to the circuit for $t > 0$,

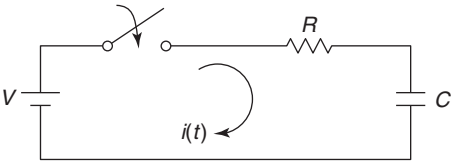


Fig. 6.124 *Series RC circuit*

$$V - Ri - \frac{1}{C}\int_0^1 i\,dt = 0$$

Differentiating the above equation,

$$0 - R\frac{di}{dt} - \frac{i}{C} = 0$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

This is a linear differential equation of first order. The variables may be separated to solve the equation.

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating both the sides,

$$l_n i = -\frac{1}{RC}t + k$$

The constant k can be evaluated from initial condition. In the circuit shown, the switch is closed at $t = 0$. Since the capacitor never allows sudden change in voltage, it will act as short circuit at $t = 0^+$. Hence, current in the circuit at $t = 0^+$ is $\frac{V}{R}$.

Setting $i = \frac{V}{R}$ at $t = 0$,

$$l_n \frac{V}{R} = k$$

$$l_n i = -\frac{1}{RC} t + l_n \frac{V}{R}$$

$$l_n i - l_n \frac{V}{R} = -\frac{1}{RC} t$$

$$l_n \frac{i}{\left(\frac{V}{R}\right)} = -\frac{1}{RC} t$$

$$\frac{i}{\frac{V}{R}} = e^{-\frac{1}{RC}t}$$

$$i = \frac{V}{R} e^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

When the switch is closed, the response decays with time as shown in Fig. 6.125(a).

The term RC is called time constant and is denoted by T.

$$T = RC$$

After 5 time constants, the current drops to 99 per cent of its initial value.

The voltage across the resistor is

$$v_R = Ri = R \frac{V}{R} e^{-\frac{1}{RC}t}$$

$$= Ve^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

Similarly, the voltage across the capacitor is

$$v_C = \frac{1}{C} \int_0^t i \, dt$$

$$= \frac{1}{C} \int_0^t \frac{V}{R} e^{-\frac{1}{RC}t}$$

$$= -Ve^{-\frac{1}{RC}t} + k$$

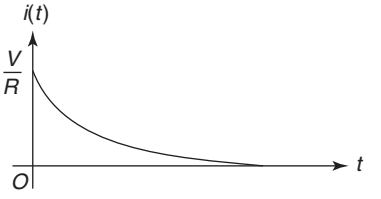


Fig. 6.125(a) *Current response of series RC circuit*

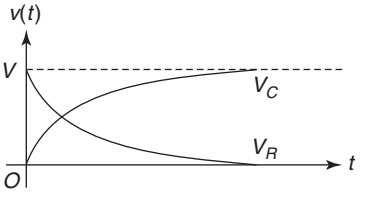


Fig. 6.125(b) *Voltage response of series RC circuit*

At $t = 0$, $v_C(0) = 0$

$$k = V$$

Hence,
$$v_C = V\left(1 - e^{-\frac{1}{RC}t}\right) \quad \text{for } t > 0$$

Example 6.41 *The switch in the circuit of Fig. 6.126 is moved from the position 1 to 2 at t = 0. Find $v_C(t)$.*

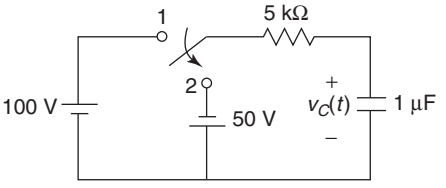


Fig. 6.126

Solution At $t = 0^-$, the network is shown in Fig. 6.127.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 100 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 100 \text{ V}$$

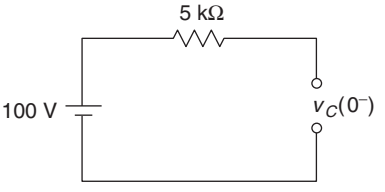


Fig. 6.127

For $t > 0$, the network is shown in Fig. 6.128.

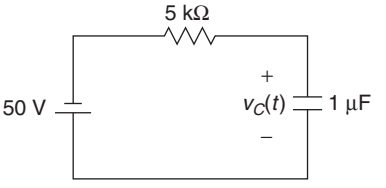


Fig. 6.128

Writing the KCL equation for $t > 0$,

$$1 \times 10^{-6}\frac{dv_C}{dt} + \frac{v_C + 50}{5000} = 0$$

$$\frac{dv_C}{dt} + 200v_C = 10^4$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 200, \quad Q = 10^4$$

Solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-200t}\int 10^4\, e^{200t}\, dt + k\, e^{-200t}$$

$$= \frac{10^4}{200} + k\, e^{-200t}$$

$$= -50 + k\, e^{-200t}$$

At $t = 0$, $v_C(0) = 100$ V

$$100 = -50 + k$$
$$k = 150$$
$$v_C(t) = -50 + 150\, e^{-200t} \qquad \text{for } t > 0$$

Example 6.42 *In the network shown in Fig. 6.129, the switch closes at t = 0. The capacitor is initially uncharged. Find v_C (t) and i_C (t).*

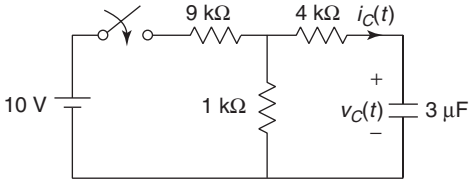


Fig. 6.129

Solution At $t = 0^-$, the capacitor is uncharged. Hence, it acts as a short circuit.

$$v_C(0^-) = 0$$
$$i_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.130.

Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

At $t = 0^+$, $\quad i_T(0^+) = \left[\dfrac{10}{9\text{ k} + (4\text{ k} \,\|\, 1\text{ k})}\right] = \dfrac{10}{9.8\text{ k}} = 1.02\text{ mA}$

$$i_C(0^+) = 1.02\text{ m} \times \frac{1\text{ k}}{1\text{ k} + 4\text{ k}} = 0.204\text{ mA}$$

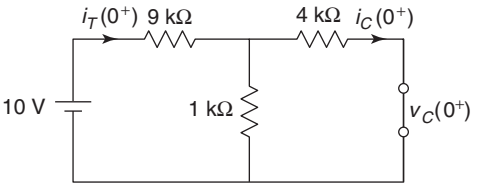


Fig. 6.130

For $t > 0$, the network is shown in Fig. 6.131.

For $t > 0$, representing the network to the left of the capacitor by Thevenin's equivalent network,

$$V_{eq} = 10 \times \frac{1\text{ k}}{9\text{ k} + 1\text{ k}} = 1\text{ V}$$
$$R_{eq} = (9\text{ k} \,\|\, 1\text{ k}) + 4\text{ k} = 4.9\text{ k}\Omega$$

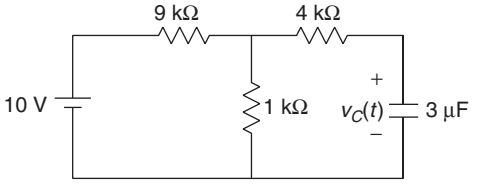


Fig. 6.131

For $t > 0$, Thevenin's equivalent network is shown in Fig. 6.132. Writing the KCL equation for $t > 0$,

$$3 \times 10^{-6} \frac{dv_C}{dt} + \frac{v_C - 1}{4.9 \times 10^3} = 0$$
$$\frac{dv_C}{dt} + 68.02\, v_C = 68.02$$

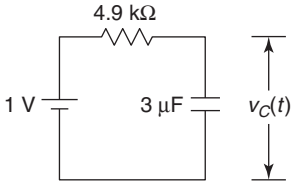


Fig. 6.132

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 68.02, \quad Q = 68.02$$

The solution of this differential equation is given by,

$$v_C(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-68.02t} \int 68.02\, e^{68.02t}\, dt + k\, e^{-68.02t}$$

$$= 1 + k\, e^{-68.02t}$$

At $t = 0, v_C(0) = 0$

$$0 = 1 + k$$

$$k = -1$$

$$v_C(t) = 1 - e^{-68.02t} \qquad \text{for } t > 0$$

$$i_C(t) = C \frac{dv_C}{dt}$$

$$= 3 \times 10^{-6} \frac{d}{dt}(1 - e^{-68.02t})$$

$$= 3 \times 10^{-6} \times 68.02 e^{-68.02t}$$

$$= 204.06 \times 10^{-6}\, e^{-68.02t} \qquad \text{for } t > 0$$

Example 6.43 *For the network shown in Fig. 6.133, the switch is open for a long time and closes at t = 0. Determine v_C (t).*

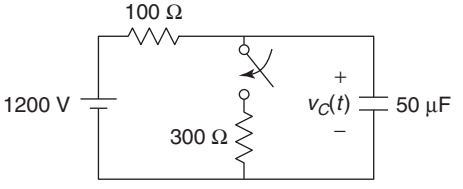


Fig. 6.133

Solution At $t = 0^-$, the network is shown in Fig. 6.134.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 1200 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 1200 \text{ V}$$

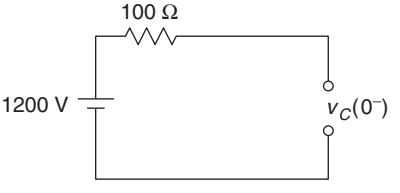


Fig. 6.134

For $t > 0$, the network is shown in Fig. 6.135.
Writing the KCL equation for $t > 0$,

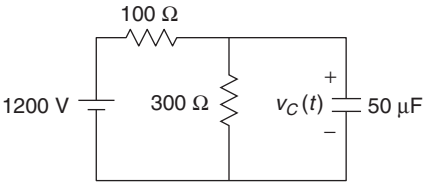


Fig. 6.135

$$50\times10^{-6}\frac{dv_C}{dt}+\frac{v_C}{300}+\frac{v_C-1200}{100}=0$$

$$\frac{dv_C}{dt}+266.67\,v_C=0.24\times10^6$$

Comparing with the differential equation $\frac{dv}{dt}+Pv=Q$,

$$P=266.67,\quad Q=0.24\times10^6$$

The solution of this differential equation is given by,

$$v_C(t)=e^{-Pt}\int Q e^{Pt}\,dt+k\,e^{-Pt}$$

$$=e^{-266.67t}\int 0.24\times10^6\,e^{266.67t}\,dt+k\,e^{-266.67t}$$

$$=\frac{0.24\times10^6}{266.67}+k\,e^{-266.67t}$$

$$=900+k\,e^{-266.67t}$$

At $t=0, v_C(0)=1200$ V

$$1200=900+k$$

$$k=300$$

$$v_C(t)=900+300\,e^{-266.67t}\qquad \text{for } t>0$$

Example 6.44 *In Fig. 6.136, the switch is closed at t = 0 Find v_C (t) for t > 0.*

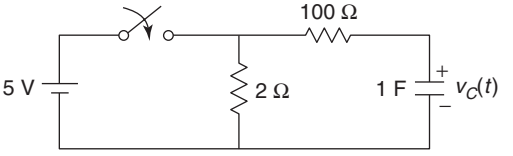


Fig. 6.136

Solution At $t=0^-$, $\quad v_C(0^-)=0$
Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+)=0$$

Since the resistor of 2 Ω is connected in parallel with the voltage source of 5 V, it becomes redundant.

For $t > 0$, the network is as shown in Fig. 6.137.
Writing KCL equation for $t > 0$,

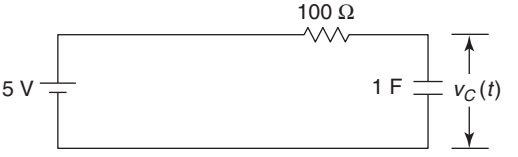


Fig. 6.137

$$\frac{v_C-5}{100}+1\frac{dv_C}{dt}=0$$

$$100\frac{dv_C}{dt}+v_C=5$$

$$\frac{dv_C}{dt} + 0.01v_C = 0.05$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 0.01, \quad Q = 0.05$$

The solution of this differential equation is given by,

$$\begin{aligned} v_C(t) &= e^{-Pt}\int Q e^{Pt}\, dt + k e^{-Pt} \\ &= e^{-0.01}\int 0.05\, e^{0.01t}\, dt + k e^{-0.01t} \\ &= \frac{0.05}{0.01} + k e^{-0.01t} \\ &= 5 + k e^{-0.01t} \end{aligned}$$

At $t = 0, v_C(0) = 0$

$$\begin{aligned} 0 &= 5 + k \\ k &= -5 \\ v_C(t) &= 5 - 5e^{-0.01t} \\ &= 5(1 - e^{-0.01t}) \qquad \text{for } t > 0 \end{aligned}$$

Example 6.45 *In the network shown, the switch is shifted to position b at t = 0. Find v (t) for t > 0.*

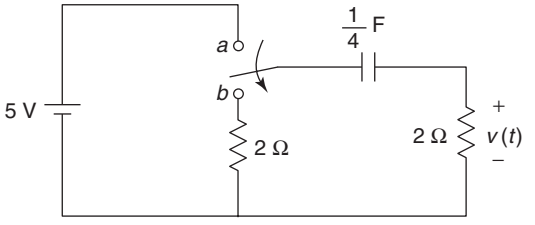


Fig. 6.138

Solution At $t = 0^-$, the network is shown in Fig. 6.139.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 5 \text{ V}$$

$$v(0^-) = 0$$

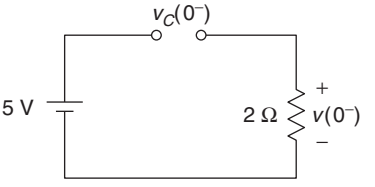


Fig. 6.139

At $t = 0^+$, the network is shown in Fig. 6.140.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$i(0^+) = -\frac{5}{4} = -1.25 \text{ A}$$

$$v(0^+) = -1.25 \times 2 = -2.5 \text{ V}$$

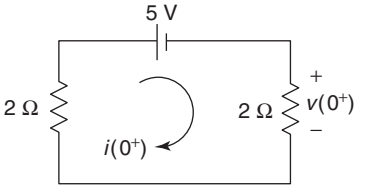


Fig. 6.140

For $t > 0$, the network is shown in Fig. 6.141.
Writing the KVL equation for $t > 0$,

$$-2i - 5 - \frac{1}{\frac{1}{4}}\int_0^t i\,dt - 2i = 0 \qquad \ldots(i)$$

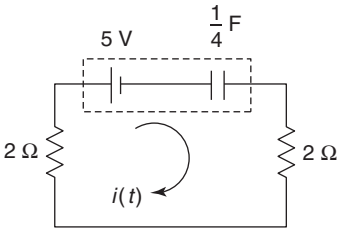


Fig. 6.141

Differentiating Eq. (i),

$$-4\frac{di}{dt} - 4i = 0$$

$$\frac{di}{dt} + i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-t}$$

At $t = 0$, $i(0) = -1.25$ A

$$k = -1.25$$

$$i(t) = -1.25\,e^{-t} \quad \text{for } t > 0$$

$$v(t) = 2i(t)$$

$$= -2.5e^{-t} \quad \text{for } t > 0$$

Example 6.46 *In the network of Fig. 6.142, the switch is open for a long time and at t = 0, it is closed. Determine v_2 (t).*

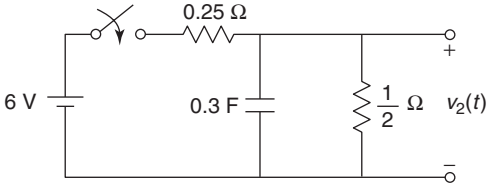


Fig. 6.142

Solution At $t = 0^-$, the switch is open.

$$v_2(0^-) = 0$$

Since voltage across capacitor cannot change instantaneously,

$$v_2(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.143.

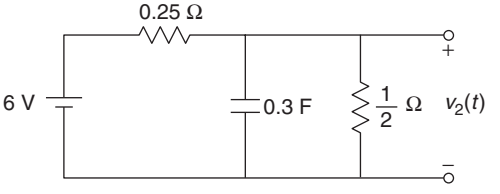


Fig 6.143

Writing KCL equation for $t > 0$,

$$\frac{v_2}{\frac{1}{2}} + 0.3\frac{dv_2}{dt} + \frac{v_2 - 6}{0.25} = 0$$

$$\frac{dv_2}{dt} + 20v_2 = 80$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 20, \quad Q = 80$$

The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-20t} \int 80\, e^{20t}\, dt + k\, e^{-20t} \\ &= \frac{80}{20} + k\, e^{-20t} \\ v_2 t &= 4 + k\, e^{-20t} \end{aligned}$$

At $t = 0, v_2(0) = 0$

$$\begin{aligned} 0 &= 4 + k \\ k &= -4 \\ v_2(t) &= 4 - 4e^{-20t} \\ &= 4(1 - e^{-20t}) \qquad \text{for } t > 0. \end{aligned}$$

Example 6.47 *The switch is moved from the position a to b at t = 0, having been in the position a for a long time before t = 0. The capacitor C_2 is uncharged at t = 0. Find i (t) and v_2 (t) for t > 0.*

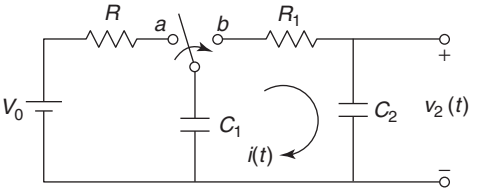


Fig. 6.144

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor C_1 acts as an open circuit and it will charge to V_0 volt.

$$v_{C_1}(0^-) = V_0$$

$$v_{C_2}(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_{C_1}(0^+) = V_0$$

$$v_{C_2}(0^+) = 0$$

$$i(0^+) = \frac{V_0}{R_1}$$

For $t > 0$, the network is shown in Fig. 6.145.
Writing the KVL equation for $t > 0$,

$$V_0 - \frac{1}{C_1}\int_0^t i\,dt - R_1 i - \frac{1}{C_2}\int_0^t i\,dt = 0 \qquad \text{...(i)}$$

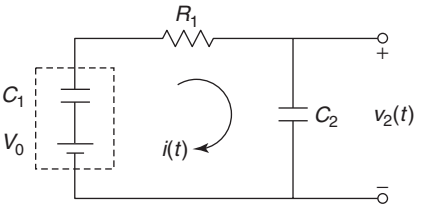


Fig. 6.145

Differentiating Eq. (i),

$$-\frac{i}{C_1} - R_1\frac{di}{dt} - \frac{i}{C_2} = 0$$

$$\frac{di}{dt} + \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right) i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right)$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

At $t = 0$, $i(0) = \frac{V_0}{R_1}$

$$k = \frac{V_0}{R_1}$$

$$i(t) = \frac{V_0}{R_1} e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

$$= \frac{V_0}{R_1} e^{-\frac{1}{R_1 C}t} \qquad \text{where, } C = \frac{C_1 C_2}{C_1 + C_2}$$

$$v_2(t) = \frac{1}{C_2}\int_0^t i\,dt$$

$$= \frac{1}{C_2}\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{R_1 C}}\,dt$$

$$= \frac{V_0}{R_1 C_2} R_1 C\left(1 - e^{-\frac{1}{R_1 C}t}\right)$$

$$= \frac{V_0 C_1}{C_1 + C_2}\left(1 - e^{-\frac{t}{R_1 C}t}\right) \qquad \text{for } t > 0$$

Example 6.48 *For the network shown in Fig.6.146, the switch is opened at $t = 0$. Find v (t) for $t > 0$.*

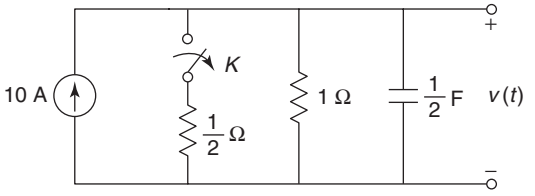


Fig. 6.146

Solution At $t = 0^-$, the network is shown in Fig. 6.147.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 0$$

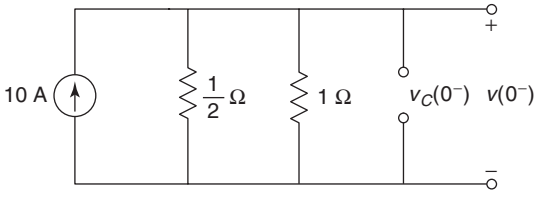


Fig. 6.147

Writing the KCL equation at $t = 0^-$,

$$\frac{v(0^-)}{1} + \frac{v(0^-)}{\frac{1}{2}} = 10$$

$$3v(0^-) = 10$$

$$v(0^-) = 3.33 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = v(0^+) = 3.33 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.148.

Writing the KCL equation for $t > 0$,

$$\frac{1}{2}\frac{dv}{dt} + \frac{v}{1} = 10$$

$$\frac{dv}{dt} + 2v = 20$$

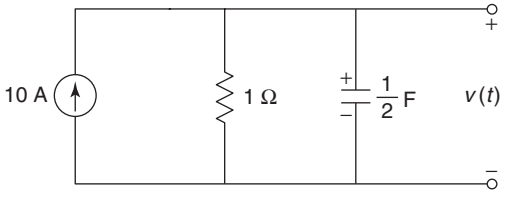


Fig. 6.148

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 2, \quad Q = 20$$

The solution of this differential equation is given by,

$$v(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-2t} \int 20\, e^{-2t}\, dt + k\, e^{-2t}$$

$$= \frac{20}{2} + k\, e^{-2t}$$

$$= 10 + k\, e^{-2t}$$

At $t = 0$, $v(0) = 3.33$ V

$$3.33 = 10 + k$$
$$k = 6.67$$
$$v(t) = 10 + 6.67\, e^{-2t}$$

Example 6.49 *For the network shown in Fig. 6.149, find the current i(t) when the switch is opened at $t = 0$.*

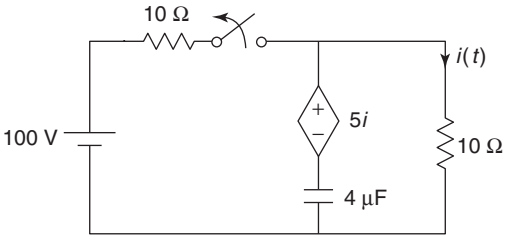


Fig. 6.149

Solution At $t = 0^-$, the network is shown in Fig. 6.150.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as open circuit.

$$i(0^-) = \frac{100}{10+10} = 5 \text{ A}$$

$$v_C(0^-) = 100 - 10i(0^-) - 5i(0^-) = 100 - 10(5) - 5(5) = 25 \text{ V}$$

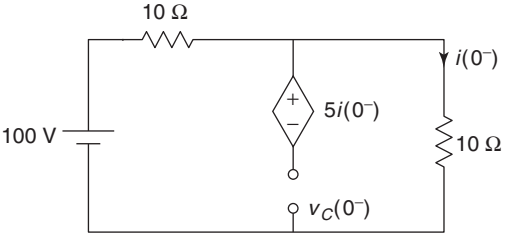


Fig. 6.150

At $t = 0^+$, the network is shown in Fig. 6.151.

$$25 + 5i(0^-) - 10i(0^+) = 0$$
$$i(0^+) = 5 \text{ A}$$

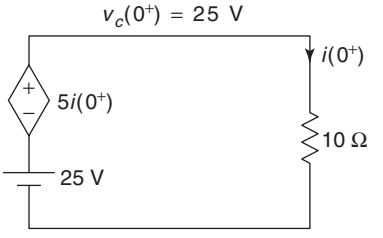


Fig. 6.151

For $t > 0$, the network is shown in Fig. 6.152.

$$25 - \frac{1}{4\times10^{-6}} \int_0^t i\, dt + 5i - 10i = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$0 - 0.25\times10^6\, i + 5\frac{di}{dt} - 10\frac{di}{dt} = 0$$
$$\frac{di}{dt} + 50000\, i = 0$$

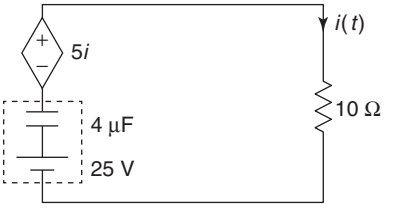


Fig. 6.152

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 50000$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50000\,t}$$

At $t = 0$, $i(0) = 5$ A

$$5 = k$$

$$i(t) = 5\,e^{-50000\,t} \qquad \text{for } t > 0$$

Example 6.50 *For the network shown in Fig. 6.153, find the current i(t) when the switch is opened at t = 0.*

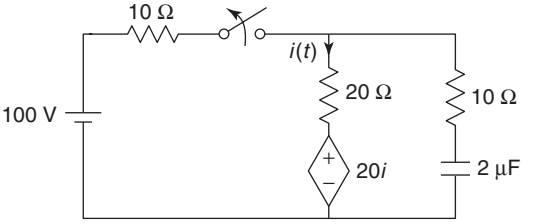


Fig. 6.153

Solution At $t = 0^-$, the network is shown in Fig. 6.154. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

Writing the KVL equation at $t = 0^-$,

$$100 - 10\,i(0^-) - 20\,i(0^-) - 20\,i(0^-) = 0$$

$$i(0^-) = 2 \text{ A}$$

Also, $$20\,i(0^-) + 20\,i(0^-) - 0 - v_C(0^-) = 0$$

$$v_C(0^-) = 40\,i(0^-) = 40(2) = 80 \text{ V}$$

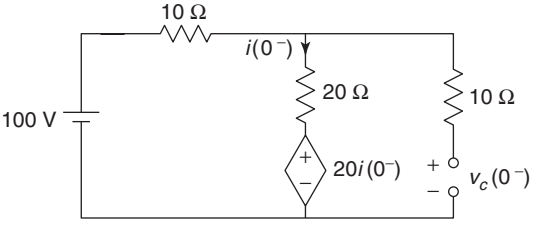


Fig. 6.154

At $t = 0^+$, the network is shown in Fig. 6.155.

From Fig. 6.155, $i(0^+) = -i_2(0^+)$

$$20\,i(0^+) - 20\,i_2(0^+) - 10\,i_2(0^+) - 80 = 0$$

$$20\,i(0^+) + 20\,i(0^+) + 10\,i(0^+) - 80 = 0$$

$$i(0^+) = 1.6 \text{ A}$$

$$v_C(0^+) = 80 \text{ V}$$

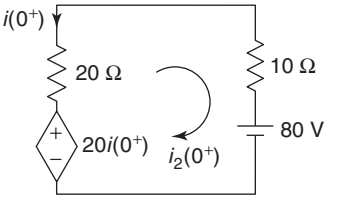


Fig. 6.155

For $t > 0$, the network is shown in Fig. 6.156.

From Fig. 6.156, $i(t) = -i_2(t)$

Writing the KVL equation for $t > 0$,

$$20i - 20i_2 - 10i_2 - \frac{1}{2\times10^{-6}}\int_0^t i_2\,dt - 80 = 0$$

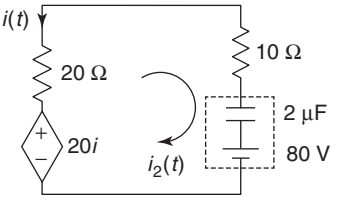


Fig. 6.156

$$20i + 20i_2 + 10i_2 + \frac{1}{2\times10^{-6}}\int_0^t i\,dt - 80 = 0$$

$$50i + \frac{1}{2\times10^{-6}}\int i\,dt - 80 = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$50\frac{di}{dt} + 5\times10^5 i = 0$$

$$\frac{di}{dt} + 1\times10^4 i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1\times10^4$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-1\times10^4 t}$$

At $t = 0$, $i(0) = 1.6$ A

$$1.6 = k$$

$$i(t) = 1.6\,e^{-1\times10^4 t} \qquad \text{for } t > 0$$

Example 6.51 *In the network of Fig. 6.157, an exponential voltage $4\,e^{-5t}$ is applied at time $t = 0$. Find the expression for current i(t). Assume zero voltage across the capacitor at $t = 0$.*

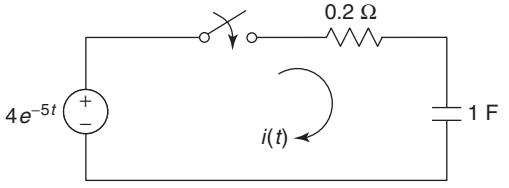


Fig. 6.157

Solution At $t = 0^-$,

$$v_C(0^-) = 0$$

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.158.
Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{4}{0.2} = 20 \text{ A}$$

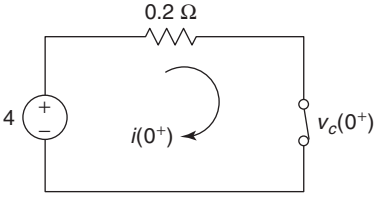


Fig. 6.158

Writing the KVL equation for $t > 0$,

$$4\,e^{-5t} - 0.2i - \frac{1}{1}\int_0^t i\,dt = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$-20e^{-5t} - 0.2\frac{di}{dt} - i = 0$$

$$\frac{di}{dt} + 5i = -100e^{-5t}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 5, \quad Q = -100e^{-5t}$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-5t}\int -100e^{-5t}\, e^{5t}\, dt + k\, e^{-5t}$$

$$= -100t\, e^{-5t} + k\, e^{-5t}$$

At $t = 0$, $i(0) = 20$ A

$$20 = k$$

$$i(t) = -100t\, e^{-5t} + 20e^{-5t} \qquad \text{for } t > 0$$

Example 6.52 *In the network shown in Fig. 6.159, the switch is closed at $t = 0$ connecting a source e^{-t} to the network. At $t = 0$, $v_C(0) = 0.5$ V. Determine $v(t)$.*

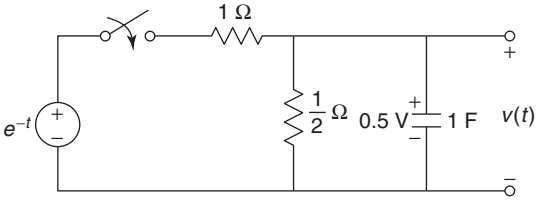


Fig. 6.159

Solution At $t = 0^-$, $v(0) = v_C(0^-) = 0.5$ V

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = v_C(0^+) = 0.5 \text{ V}$$

Writing the KCL equation for $t > 0$,

$$\frac{v - e^{-t}}{1} + \frac{v}{\frac{1}{2}} + 1\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + 3v = e^{-t}$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 3, \quad Q = e^{-t}$$

The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-3t}\int e^{-t}\, e^{3t}\, dt + k\, e^{-3t} \\ &= e^{-3t}\int e^{2t}\, dt + k\, e^{-3t} \\ &= \frac{1}{2}e^{-t} + k\, e^{-3t} \end{aligned}$$

At $t = 0$, $\quad v(0) = 0.5$ V

$$0.5 = \frac{1}{2} + k$$
$$k = 0$$
$$v(t) = 0.5\, e^{-t}$$

Example 6.53 *In the network shown in Fig. 6.160, a sinusoidal voltage $v = 100\ \sin(500t + \theta)$ volts is applied to the circuit at a time corresponding to $\theta = 45°$. Obtain the expression for the current i(t).*

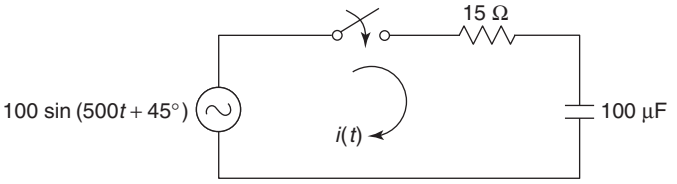


Fig. 6.160

Solution

Writing the KVL equation for $t > 0$,

$$100\sin(500t + 45°) - 15i - \frac{1}{100\times10^{-6}}\int_0^t i\, dt = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$(100)(500)\cos(500t + 45°) - 15\frac{di}{dt} - 10^4 i = 0$$
$$\frac{di}{dt} + 666.67i = 3333.33\cos(500t + 45°)$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 666.67,\ Q = 3333.33\cos(500t + 45°)$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-666.67t}\int 3333.33\cos(500t + 45°)\, e^{666.67t} + k\, e^{-666.67t} \end{aligned}$$

$$= 3333.33\, e^{-666.67t}\left[\frac{e^{666.67t}}{(666.67)^2+(500)^2}\{666.67\cos(500t+45°)+500\sin(500t+45°)\}\right]+k\,e^{-666.67t}$$

$$= 3.2\cos(500t+45°)+2.4\sin(500t+45°)+k\,e^{-666.67t}$$

Let $A\sin\phi = 3.2$

and $A\cos\phi = 2.4$

$$A^2\sin^2\phi + A^2\cos^2\phi = (3.2)^2+(2.4)^2 = 16$$

$$A = 4$$

and $$\phi = \tan^{-1}\left(\frac{3.2}{2.4}\right) = 53.13°$$

$$i(t) = 4\sin(53.13°)\cos(500t+45°)+4\cos(53.13°)\sin(500t+45°)+k\,e^{-666.67t}$$

$$= 4\sin(500t+98.13°)+k\,e^{-666.67t}$$

Putting $t = 0$, in Eq. (i),

$$100\sin(45°)-15\,i(0)-0 = 0$$

$$i(0) = 4.71$$

$$4.71 = 4\sin(98.13°)+k$$

$$k = 0.75$$

$$i(t) = 4\sin(500t+98.13°)+0.75\,e^{-666.67t} \qquad \text{for } t>0$$

Example 6.54 *In the network of Fig. 6.161, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression i(t), when $\omega = 1000$ rad/s.*

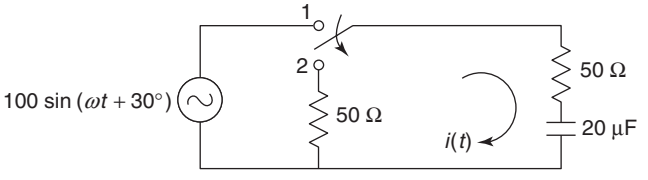


Fig. 6.161

Solution At $t = 0^-$, the network is shown in Fig. 6.162. At $t = 0^-$, the network attains steady-state condition.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100\angle 30°}{50 - j\dfrac{1}{1000\times20\times10^{-6}}} = 1.41\angle 75°\text{ A}$$

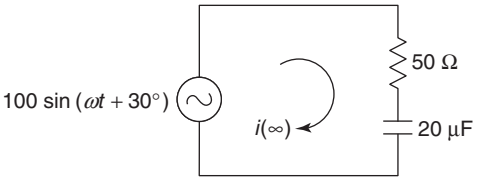


Fig. 6.162

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 1.41\sin(1000t+75°) \qquad \ldots(i)$$

For $t > 0$, the network is as shown in Fig. 6.163.
Writing the KVL equation for $t > 0$,

$$-50i - 50i - \frac{1}{20\times10^{-6}}\int_0^t i\,dt - v_C(0) = 0 \qquad \ldots(ii)$$

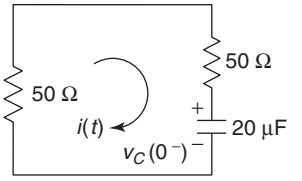


Fig. 6.163

Differentiating Eq. (ii),

$$-50\frac{di}{dt} - 50\frac{di}{dt} - \frac{1}{20\times10^{-6}}i = 0$$

$$\frac{di}{dt} + 500i = 0$$

Comparing with differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 500$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-500t} \qquad \ldots(iii)$$

From Eqs (i) and (ii),

$$1.41\sin(1000t + 75°) = k\,e^{-500t}$$

At $t = 0$,

$$1.41\sin(75°) = k$$

$$k = 1.36$$

$$i(t) = 1.36\,e^{-500t} \quad \text{for } t > 0$$

6.5 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit as shown in Fig. 6.164. The switch is closed at time $t = 0$. The capacitor and inductor are initially uncharged.

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} - \frac{1}{C}\int_0^t i\,dt = 0$$

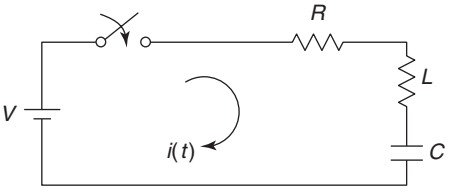


Fig. 6.164 *Series RLC circuit*

Differentiating the above equation,

$$0 - R\frac{di}{dt} - L\frac{d^2i}{dt^2} - \frac{1}{C}i = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a second-order differential equation. The auxiliary equation or characteristic equation will be given by,

$$s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$$

Let s_1 and s_2 be the roots of the equation.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $\alpha = \frac{R}{2L}$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $\beta = \sqrt{\alpha^2 - \omega_0^2}$

The solution of the above second-order differential equation will be given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation.
Now depending upon the values of α and ω_0, we have 3 cases of the response.

Case I *When* $\alpha > \omega_0$

i.e., $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$

The roots are real and unequal and it gives an overdamped response.
In this case, the solution is given by,

$$i = e^{-\alpha t}\,(k_1\, e^{\beta t} + k_2\, e^{-\beta t})$$

or $i = k_1\, e^{s_1 t} + k_2\, e^{s_2 t}$ for $t > 0$

The current curve for an overdamped case is shown in Fig. 6.165.

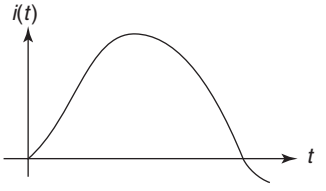


Fig. 6.165 *Overdamped response*

Case II *When* $\alpha = \omega_0$

i.e., $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$

The roots are real and equal and it gives a critically damped response.
In this case the solution is given by,

$$i = e^{-\alpha t}(k_1 + k_2 t) \quad \text{for } t > 0$$

The current curve for critically damped case is shown in Fig. 6.166.

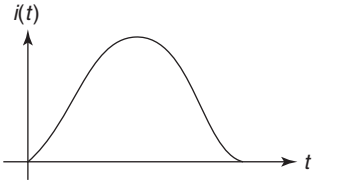


Fig. 6.166 *Critically damped response*

Case III *When* $\alpha < \omega_0$

i.e., $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$

The roots are complex conjugate and it gives an underdamped response.

In this case, the solution is given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 + \omega_0^2}$

Let $\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$

where $j = \sqrt{-1}$

and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Hence,
$$i(t) = e^{-\alpha t}(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t})$$
$$= e^{-\alpha t}\left\{(k_1 + k_2)\left[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right] + j(k_1 - k_2)\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{j2}\right]\right\}$$
$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \quad \text{for } t > 0$$

The current curve for an underdamped case is shown in Fig. 6.167.

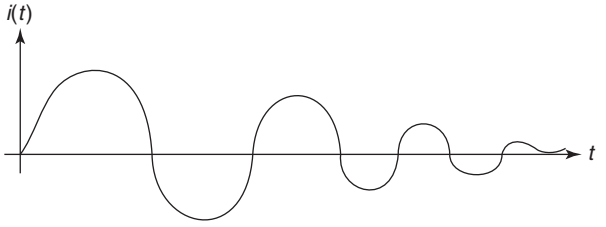


Fig. 6.167 *Underdamped response*

Example 6.55 *In the network of Fig. 6.168, the switch is closed at $t = 0$. Obtain the expression for current $i(t)$ for $t > 0$.*

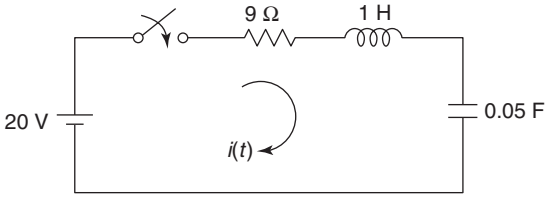


Fig. 6.168

Solution At $t = 0^-$, the switch is open.

$$i(0^-) = 0$$
$$v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.169.

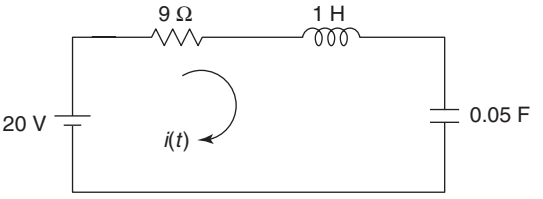


Fig. 6.169

Writing the KVL equation for $t > 0$,

$$20 - 9i - 1\frac{di}{dt} - \frac{1}{0.05}\int_0^t i\,dt = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$0 - 9\frac{di}{dt} - \frac{d^2i}{dt^2} - 20i = 0$$

$$\frac{d^2i}{dt^2} + 9\frac{di}{dt} + 20i = 0$$

$$(D^2 + 9D + 20)i = 0$$

$$D_1 = -4,\ D_2 = -5$$

The solution of this differential equation is given by,

$$i(t) = k_1 e^{-4t} + k_2 e^{-5t} \qquad \ldots(ii)$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -4k_1 e^{-4t} - 5k_2 e^{-5t} \qquad \ldots(iii)$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \ldots(iv)$$

$$\frac{di}{dt}(0) = -4k_1 - 5k_2 \qquad \ldots(v)$$

Putting $t = 0$ in Eq. (i),

$$20 - 9i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 20 - 9i(0^+) = 20 \text{ A/s}$$

From Eq. (v),

$$20 = -4k_1 - 5k_2 \qquad \ldots(vi)$$

Solving Eqs (iv) and (vi),

$$k_1 = 20$$

$$k_2 = 20$$

$$i(t) = 20e^{-4t} - 20e^{-5t} \qquad \text{for } t > 0$$

Example 6.56 *In the network shown in Fig. 6.170, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Determine the expression for the current $i(t)$.*

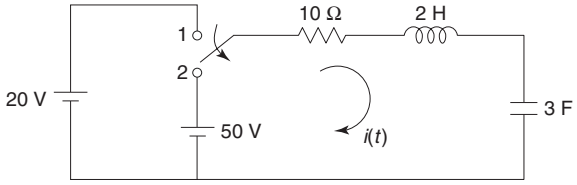


Fig. 6.170

Solution At $t = 0^-$, the network is shown in Fig. 6.171.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

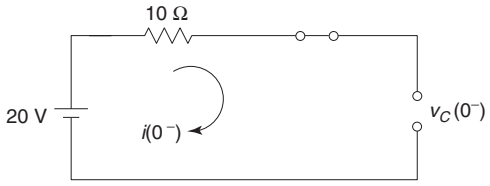


Fig. 6.171

$$v_C(0^-) = 20 \text{ V}$$

$$i(0^-) = 0$$

Since the current through the inductor and the voltage across the capacitor cannot change instaneously,

$$v_C(0^+) = 20 \text{ V}$$

$$i(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.172. Writing the KVL equation for $t > 0$,

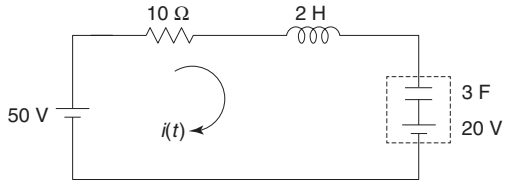


Fig. 6.172

$$50 - 10i - 2\frac{di}{dt} - \frac{1}{3}\int_0^t i\,dt - 20 = 0 \qquad \ldots(\text{i})$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - 2\frac{d^2i}{dt^2} - \frac{1}{3}i - 0 = 0$$

$$\frac{d^2i}{dt^2} + 5\frac{di}{dt} + \frac{1}{6}i = 0$$

$$\left(D^2 + 5D + \frac{1}{6}\right)i = 0$$

$$D_1 = -0.03,\ D_2 = -4.97$$

The solution of this differential equation is given by,

$$i(t) = k_1\, e^{-0.03t} + k_2\, e^{-4.97t} \qquad \ldots(\text{ii})$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -0.03k_1\, e^{-0.03t} - 4.97k_2\, e^{-4.97t} \qquad \ldots(\text{iii})$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \ldots(\text{iv})$$

$$\frac{di}{dt}(0) = -0.03\, k_1 - 4.97\, k_2 \qquad \ldots(\text{v})$$

Putting $t = 0$ in Eq. (i),

$$20 - 10\, i(0^+) - 2\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = \frac{20 - 10\, i(0^+)}{2} = 10 \text{ A/s}$$

From Eq. (v),

$$10 = -0.03\, k_1 - 4.97\, k_2 \qquad \ldots(\text{vi})$$

Solving Eqs (iv) and (vi),

$$k_1 = 2.02$$

$$k_2 = 2.02$$

$$i(t) = 2.02\, e^{-0.03t} - 2.02\, e^{-4.97t} \quad \text{for } t > 0$$

Example 6.57 *In the network of Fig. 6.173, the switch is closed and a steady state is reached in the network. At $t = 0$, the switch is opened. Find the expression for the current $i_2(t)$ in the inductor.*

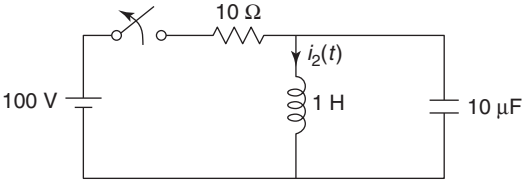


Fig. 6.173

Solution At $t = 0^-$, the network is shown in Fig. 6.174.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

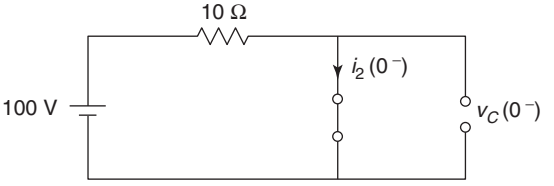


Fig. 6.174

$$i_2(0^-) = \frac{100}{10} = 10 \text{ A}$$

$$v_C(0^-) = 0$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_2(0^+) = 10 \text{ A}$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.175.
Writing the KVL equation for $t > 0$,

$$-1\frac{di_2}{dt} - \frac{1}{10 \times 10^{-6}} \int_0^t i\, dt = 0 \qquad \ldots(\text{i})$$

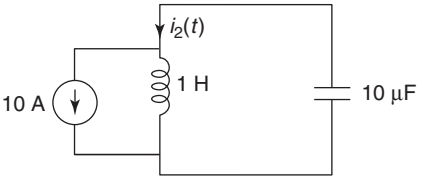


Fig. 6.175

Differentiating Eq. (i),

$$-\frac{d^2 i_2}{dt^2} - 10^5 i = 0$$

$$\frac{d^2 i_2}{dt^2} + 10^5 i = 0$$

$$(D^2 + 10^5) i = 0$$

$$D_1 = j\,316,\ D_2 = -j\,316$$

The solution of this differential equation is given by,

$$i_2(t) = k_1 \cos 316t + k_2 \sin 316t \qquad \ldots(ii)$$

Differentiating Eq. (ii),

$$\frac{di_2}{dt} = -316\ k_1 \sin 316t + 316\ k_2 \cos 316t \qquad \ldots(iii)$$

At $t = 0, i_2(0) = 10$ A

$$10 = k_1 \qquad \ldots(iv)$$

$$\frac{di_2}{dt}(0) = 316\ k_2 \qquad \ldots(v)$$

Putting $t = 0$ in Eq. (i),

$$-\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 0$$

From Eq. (v),

$$0 = 316\ k_2$$

$$k_2 = 0$$

$$i_2(t) = 10 \cos 316t \quad \text{for } t > 0$$

Example 6.58 *In the network of Fig. 6.176, capacitor C has an initial voltage $v_c(0^-)$ of 10 V and at the same instant, current in the inductor L is zero. The switch is closed at time $t = 0$. Obtain the expression for the voltage $v(t)$ across the inductor L.*

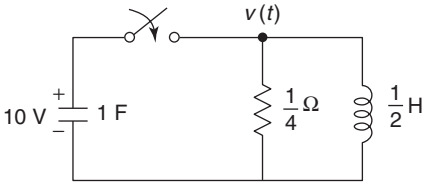


Fig. 6.176

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = v_C(0^-) = 10 \text{ V}$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = v_C(0^+) = 10 \text{ V}$$

For $t > 0$, the network is shown in Fig. 6.177.
Writing the KCL equation for $t > 0$,

$$1\frac{dv}{dt} + \frac{v}{\frac{1}{4}} + \frac{1}{\frac{1}{2}}\int_0^t v\,dt = 0 \qquad \ldots(\text{i})$$

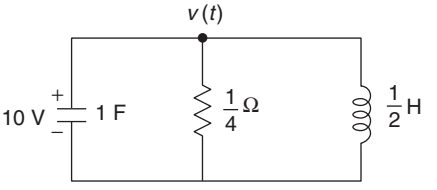


Fig. 6.177

Differentiating Eq. (i),

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 2v = 0$$
$$(D^2 + 4D + 2)v = 0$$
$$D_1 = -1,\ D_2 = -3$$

The solution of this differential equation is given by,

$$v(t) = k_1 e^{-t} + k_2 e^{-3t} \qquad \ldots(\text{ii})$$

Differentiating Eq. (iii),

$$\frac{dv}{dt} = -k_1 e^{-t} - 3k_2 e^{-3t} \qquad \ldots(\text{iii})$$

At $t = 0$, $v(0) = 10$ V

$$10 = k_1 + k_2 \qquad \ldots(\text{iv})$$

$$\frac{dv}{dt}(0) = -k_1 - 3k_2 \qquad \ldots(\text{v})$$

Putting $t = 0$ in Eq. (i),

$$\frac{dv}{dt}(0^+) + 4v(0) + 0 = 0$$
$$\frac{dv}{dt}(0^+) = -40 \text{ V/s}$$

From Eq. (v),

$$-40 = -k_1 - 3k_2 \qquad \ldots(\text{vi})$$

Solving Eqs (iv) and (vi),

$$k_1 = -5$$
$$k_2 = 15$$
$$v(t) = -5e^{-t} + 15e^{-3t} \qquad \text{for } t > 0$$

Example 6.59 *In the network of Fig. 6.178, the switch is opened at $t = 0$ obtain the expression for $v(t)$. Assume zero initial conditions.*

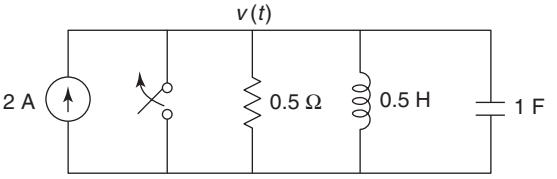


Fig. 6.178

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor can not change instantaneously,

$$i_L(0^+) = 0$$

$$v(0^+) = v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.179.
Writing the KCL equation for $t > 0$,

$$\frac{v}{0.5} + \frac{1}{0.5}\int_0^t v\,dt + 1\frac{dv}{dt} = 2 \qquad \ldots(i)$$

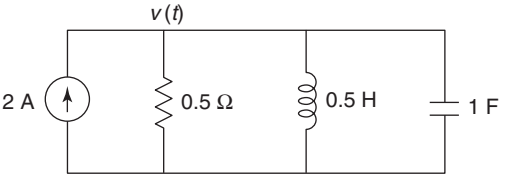


Fig. 6.179

Differentiating Eq. (i),

$$2\frac{dv}{dt} + 2v + \frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + 2v = 0$$

$$(D^2 + 2D + 2)v = 0$$

$$D_1 = -1 + j1, \quad D_2 = -1 - j1$$

The solution of this differential equation is given by,

$$v(t) = e^{-t}(k_1 \cos t + k_2 \sin t) \qquad \ldots \text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{dv}{dt} = -e^{-t}(k_1 \cos t + k_2 \sin t) + e^{-t}(-k_1 \sin t + k_2 \cos t)$$

$$= e^{-t}[-k_1(\cos t + \sin t) + k_2(\cos t - \sin t)] \qquad \ldots(iii)$$

At $t = 0$, $v(0) = 0$

$$0 = k_1 \qquad \ldots(iv)$$

$$\frac{dv}{dt}(0) = -k_1 + k_2 \qquad \ldots(v)$$

Putting $t = 0$ in Eq. (i),

$$2v(0) + 0 + \frac{dv}{dt}(0) = 2$$

$$\frac{dv}{dt}(0) = 2\ V/s$$

From Eq. (v),

$$2 = -k_1 + k_2 \qquad \ldots(vi)$$

Solving Eq. (iv) and (vi),

$$k_1 = 0$$
$$k_2 = 2$$
$$v(t) = 2\,e^{-t}\sin t \qquad \text{for } t > 0$$

Example 6.60 *The network shown in Fig. 6.180, a sinusoidal voltage $v = 150\ sin(200t + \phi)$ is applied at $\phi = 30°$. Determine the current i(t).*

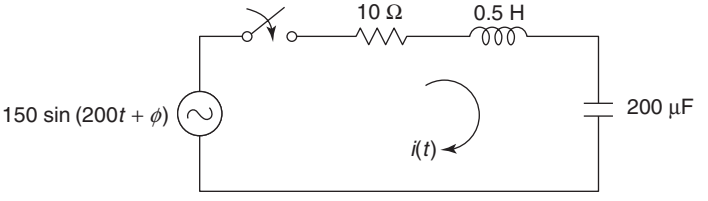


Fig. 6.180

Solution Writing the KVL equation for $t > 0$,

$$150\sin(200t + 30°) - 10i - 0.5\frac{di}{dt} - \frac{1}{200\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots(i)$$

Differentiating Eq. (i),

$$30000\cos(200t + 30°) - 10\frac{di}{dt} - 0.5\frac{d^2i}{dt^2} - 5000i = 0$$
$$\frac{d^2i}{dt^2} + 20\frac{di}{dt} + 10000i = 60000\cos(200t + 30°)$$
$$(D^2 + 20D + 10000)i = 60000\cos(200t + 30°) \qquad \ldots(ii)$$

The roots of the characteristic equation are

$$D_1 = -10 + j\ 99.5, \quad D_2 = -10 - j\ 99.5$$

The complimentary function is

$$i_C = e^{-10t}(K_1\cos 99.5t + K_2\sin 99.5t)$$

Let the particular function be

$$i_P = A\cos(200t + 30°) + B\sin(200t + 30°)$$
$$i_P' = -200A\sin(200t + 30°) + 200B\cos(200t + 30°)$$
$$i_P'' = -40000A\cos(200t + 30°) - 40000B\sin(200t + 30°)$$

Substituting these values in Eq. (ii),

$$-40000\ A\cos(200t+30°)-40000\ B\sin(200t+30°)+20[-200\ A\sin(200t+30°)+200\ B\cos(200t+30°)]$$
$$+10000[A\cos(200t+30°)+B\sin(200t+30°)]=60000\cos(200t+30°)$$

$$(-40000\ B-4000\ A+10000\ B)\sin(200t+30°)+(-40000\ A+4000\ B+10000\ A)\cos(200t+30°)$$
$$=60000\cos(200t+30°)$$

Equating the coefficients,

$$-40000\ B-4000\ A+10000\ B=0$$
$$-40000\ A+4000\ B+10000\ A=60000$$

Solving these equations,

$$A=-1.97$$
$$B=0.26$$
$$i_P=-1.97\cos(200t+30°)+0.26\sin(200t+30°)$$

Let $A\sin\phi=-1.97$

and $A\cos\phi=0.26$

$$A^2\sin^2\phi+A^2\cos^2\phi=(-1.97)^2+(0.26)^2=3.95$$
$$A=1.987$$

and $$\phi=\tan^{-1}\left(\frac{-1.97}{0.26}\right)=-82.48°$$

$$i_P=1.987\sin(-82.48°)\cos(200t+30°)+1.987\cos(-82.48°)\sin(200t+30°)$$
$$=1.987\sin(200t+30°-82.48°)$$
$$=1.987\sin(200t-52.48°)$$

The solution of the differential equation is given by,

$$i(t)=e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+1.987\sin(200t-52.48°) \quad \ldots(iii)$$

Differentiating Eq. (iii),

$$\frac{di}{dt}=e^{-10t}(-99.5\ k_1\sin 99.5t+99.5\ k_2\cos 99.5t)$$
$$-10\ e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+(1.987)(200)\cos(200t-52.48°)$$

At $t=0$, $i(0)=0$

$$0=k_1+1.987\sin(-52.48°)$$
$$k_1=1.58 \quad \ldots(iv)$$

$$\frac{di}{dt}(0)=99.5\ k_2-10\ k_1+(1.987)(200)\cos(-52.48°)$$
$$=99.5\ k_2-10(1.58)+242.03$$
$$=99.5\ k_2+226.23 \quad \ldots(v)$$

Putting $t = 0$ in Eq. (i),

$$150\sin(30°) - 10(0) - 0.5\frac{di}{dt}(0) - 0 = 0$$

$$\frac{di}{dt}(0) = 150 \text{ A/s}$$

From Eq. (v),

$$150 = 99.5\, k_2 + 226.23$$

$$k_2 = -0.77$$

$$i(t) = e^{-10t}(1.58\cos 99.5t - 0.77\sin 99.5t) + 1.987\sin(200t - 52.48°) \quad \text{for } t > 0$$

Example 6.61 *The switch in the network of Fig. 6.181 is opened at t = 0. Find i (t) for t > 0 if,*

(a) $L = \frac{1}{2}$ *H and* $C = 1$ *F* *(b)* $L = 1$ *H and* $C = 1$ *F* *(c)* $L = 5$ *H and* $C = 1$ *F*

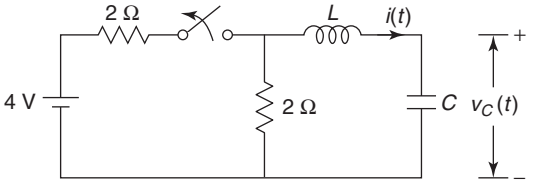


Fig. 6.181

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an pen circuit.

$$v_C(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

$$i(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 2 \text{ V}$$

$$i(0^+) = 0$$

Case I *When* $R = 2\ \Omega$, $L = \frac{1}{2}$ *H*, $C = 1$ *F*

$$\alpha = \frac{R}{2L} = \frac{2}{2 \times \frac{1}{2}} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times 1}} = \frac{1}{\sqrt{0.5}} = 1.414$$

$$\alpha > \omega_0$$

This indicates an overdamped case.

$$i(t) = A_1 e^{s_1 t} - A_2 e^{s_2 t}$$

where, $$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2 - \sqrt{4-2} = -2 - \sqrt{2} = -3.414$$

and $$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2 + \sqrt{2} = -0.586$$

$$i(t) = k_1 e^{-3.414t} + k_2 e^{-0.586t}$$

At $t = 0$, $i(0) = 0$

$$k_1 + k_2 = 0 \qquad \ldots(i)$$

Also $v_L(0^+) + v_C(0^+) + v_R(0^+) = 0$

$$v_L(0^+) = -v_R(0^+) - v_C(0^+) = -2i(0^+) - v_C(0^+) = -2 \text{ V} \qquad \ldots(ii)$$

$$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{0.5} = -4 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$-3.414\, k_1 - 0.586\, k_2 = -4 \qquad \ldots(iii)$$

Solving Eqs (i) and (iii), we get

$$k_1 = 1.414 \quad \text{and} \quad k_2 = -1.414$$

$$i(t) = 1.414(e^{-3.414t} - e^{-0.586t}) \qquad \text{for } t > 0$$

Case II *When* $R = 2\ \Omega$, $L = 1\ H$, $C = 1\ F$

$$\alpha = \frac{R}{2L} = \frac{2}{2 \times 1} = \frac{2}{2} = 1$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1}} = 1$$

$$\alpha = \omega_0$$

This indicates a critically damped case.

$$i(t) = e^{-\alpha t}(k_1 + k_2 t) = e^{-t}(k_1 + k_2 t)$$

At $t = 0$, $i(0) = 0$

$$k_1 = 0$$

Also, $$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{1} = -2 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$\left.\frac{di}{dt}\right|_{t=0} = -k_1 + k_2 = -2$$

$$k_2 = -2$$

$$i(t) = -2t\,e^{-t} \qquad \text{for } t > 0$$

Case III *When* $R = 2\ \Omega$, $L = 5\ H$, $C = 1\ F$

$$\alpha = \frac{R}{2L} = \frac{2}{10} = 0.2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5}} = 0.447$$

$$\alpha < \omega_0$$

This indicates an underdamped case.

$$i\,(t) = e^{-\alpha t}\,(B_1 \cos\,\omega_d\,t + B_2 \sin\,\omega_d\,t)$$

where,

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{(0.447)^2 - (0.2)^2} = 0.4$$

$$s_{1,2} = -\alpha \pm j\omega_d = -0.2 \pm j0.4$$

$$i(t) = e^{-0.2t}\,(B_1 \cos 0.4t + B_2 \sin 0.4t)$$

Applying the initial condition,

$$i(0^+) = 0$$

and

$$\frac{di}{dt}(0^+) = -\frac{v_L(0^+)}{L} = -\frac{2}{5}$$

$$B_1 = i(0) = 0$$

$$B_2 = -1$$

$$i(t) = -e^{-0.2t} \sin 0.4t \qquad \text{for } t > 0$$

Exercises

6.1 The switch in Fig. 6.182 is moved from the position a to b at $t = 0$, the network having been in steady state in the position a. Determine $i_1(0^+), i_2(0^+), i_3(0^+), \frac{di_2}{dt}(0^+)$ and $\frac{di_3}{dt}(0^+)$.

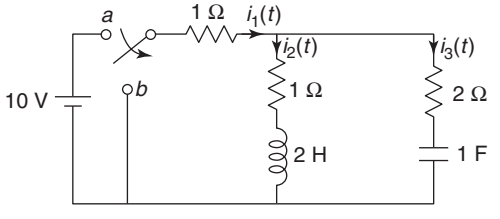


Fig. 6.182

[1.66 A, 5 A, –3.33 A, –3.33 A/s, 2.22 A/s]

6.2 The switch K is closed at $t = 0$ in the network shown in Fig. 6.183. Determine $i(0^+), \frac{di}{dt}(0^+)$ and $\frac{d^2i}{dt^2}(0^+)$.

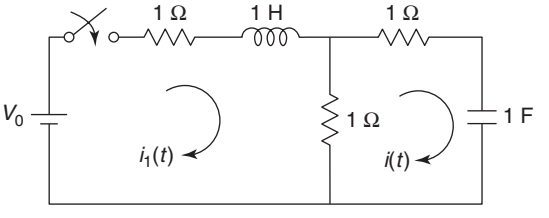


Fig. 6.183

$$\left[0, \frac{1}{2}V_0, -V_0\right]$$

6.3 In the network of Fig. 6.184, the switch K is closed at $t = 0$. At $t = 0^-$, all capacitor voltages and inductor currents are zero. Find $v_1, \frac{dv_1}{dt}, v_2, \frac{dv_2}{dt}, v_3$ and $\frac{dv_3}{dt}$ at $t = 0^+$.

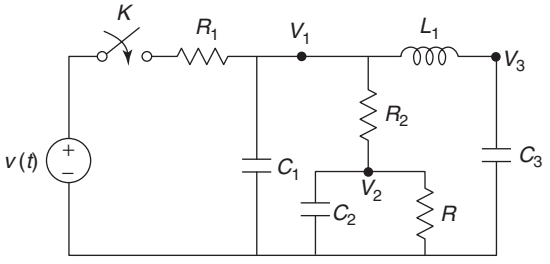


Fig. 6.184

$$\left[0, \frac{1}{C}\frac{v(0^+)}{R_1}, 0, 0, 0, 0\right]$$

6.4 In the network at Fig. 6.185, the capacitor C_1 is charged to voltage 1000 V and the switch K is closed at $t = 0$. Find $\frac{d^2 i_2}{dt^2}$ at $t = 0^+$.

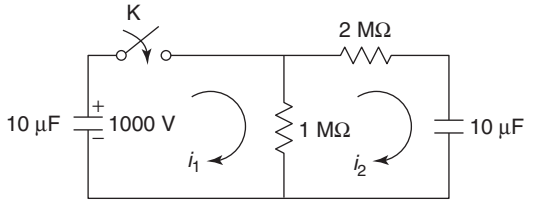


Fig. 6.185

$$\left[\frac{17}{400000}\text{A/s}^2\right]$$

6.5 In the network shown in Fig. 6.186, switch is closed at $t = 0$. Obtain the current $i_2(t)$.

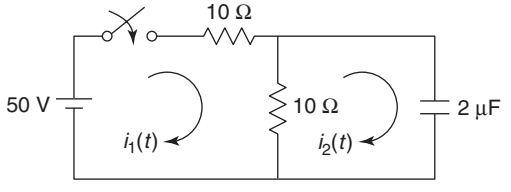


Fig. 6.186

$[i_2(t) = 5\ e^{-100000t}]$

6.6 The network shown in Fig. 6.187 is under steady-state when the switch is closed. At $t = 0$, it is opened. Obtain an expression for $i(t)$.

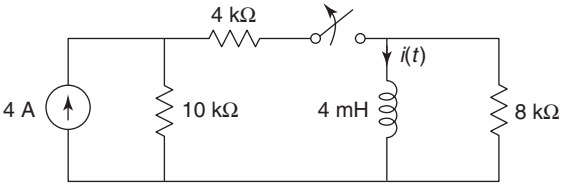


Fig. 6.187

$[i(t) = 2.857\ e^{-2\times10^6\ t}]$

6.7 The switch in Fig. 6.188 is open for a long time and closes at $t = 0$. Determine $i(t)$ for $t > 0$.

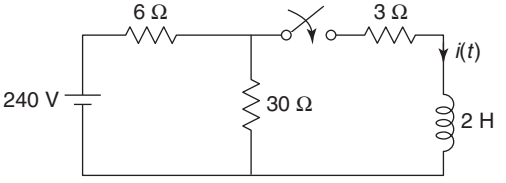


Fig. 6.188

$[i(t) = 25(1 - e^{-4t})]$

6.8 In the network shown in Fig. 6.189, the steady state is reached with the switch open. At $t = 0$, the switch is closed Find $v_C(t)$ for $t > 0$.

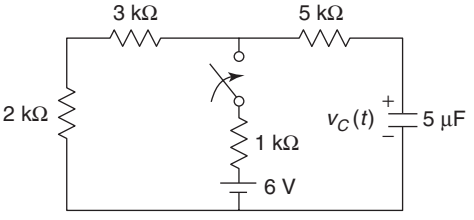


Fig. 6.189

$[v_C(t) = 5e^{-20t}]$

6.9 The circuit shown in Fig. 6.190 has acquired steady state before switching at $t = 0$.

(i) Obtain $v_C(0^+)$, $v_C(0^-)$, $i(0^+)$ and $i(0^-)$.
(ii) Obtain time constant for $t > 0$.
(iii) Find current $i(t)$ for $t > 0$.

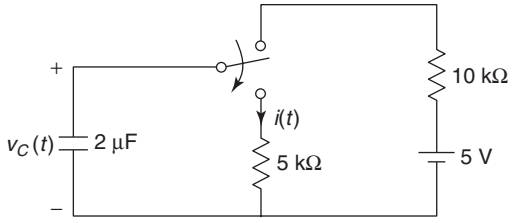


Fig. 6.190

[(i) 5 V, 5 V, 1 mA, 0, (ii) 0.01 s, (iii) $e^{-100\,t}$ mA]

6.10 In the network shown in Fig. 6.191, the switch is initially at the position 1 for a long time. At $t = 0$, the switch is changed to the position 2. Find current $i\,(t)$ for $t > 0$.

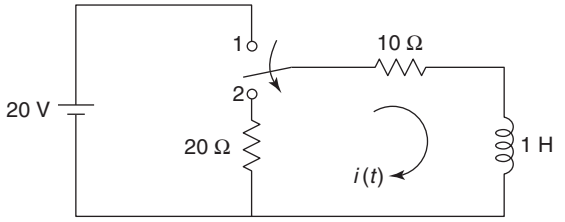


Fig. 6.191

[$i\,(t) = 2\,e^{-30t}$]

6.11 In the network shown in Fig. 6.192, the switch is closed at $t = 0$. Find $v\,(t)$ for $t > 0$.

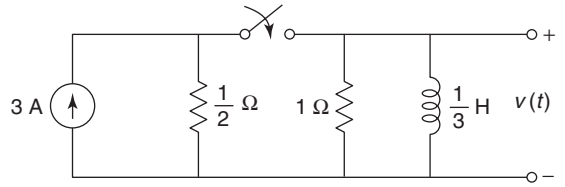


Fig. 6.192

[$v(t) = e^{-t}$]

6.12 In the network shown in Fig. 6.193, the switch is in the position 1 for a long time and at $t = 0$, the switch is moved to the position . Find $v\,(t)$ for $t > 0$.

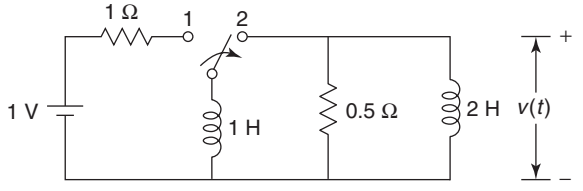


Fig. 6.193

[$v\,(t) = -0.5\,e^{-\frac{3}{4}t}$]

6.13 In Fig. 6.194, the switch is open until time $t = 100$ seconds and is closed for all times thereafter. Find $v\,(t)$ for all times greater than 100 if $v\,(100) = -3$ V.

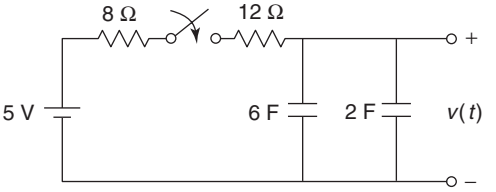


Fig. 6.194

$$\left[v(t) = 5 - 8e^{-\frac{(t-100)}{160}}\right]$$

6.14 A series *RL* circuit shown in Fig. 6.195 has a constant voltage V applied at $t = 0$. At what time does $v_R = v_L$.

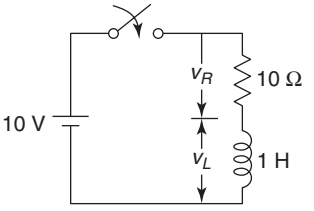


Fig. 6.195

[0.0693 s]

6.15 In the circuit shown in Fig. 6.196, at time $t = 0$, the voltage across the capacitor is zero and the switch is moved to the position y. The switch is kept at position y for 20 seconds and then moved to position z and kept in that position thereafter. Find the voltage across the capacitor at $t = 30$ seconds.

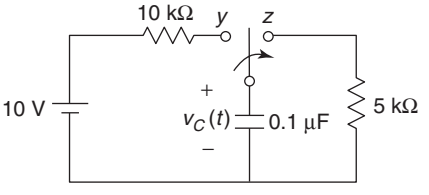


Fig. 6.196

[0]

6.16 Determine whether *RLC* series circuit shown in Fig 6.197 is underdamped, overdamped or

critically damped. Also, find $v_L(0^+)$, $\frac{di}{dt}(0^+)$ and $i(\infty)$.

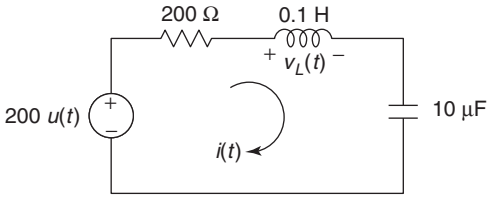


Fig. 6.197

[critically damped, 200 V, 2000 A/s, 0]

6.17 Determine whether *RLC* circuit of Fig 6.198 is underdamped, overdamped or critically damped. Also find $v_L(0^+)$, $\frac{di}{dt}(0^+)$, $\frac{d^2v}{dt^2}(0^+)$ if $v(t) = u(t)$.

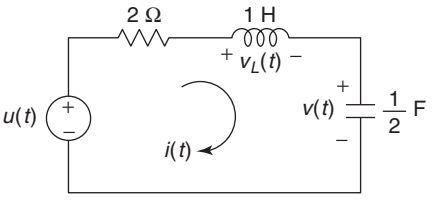


Fig. 6.198

[underdamped 1 V, 1 A/s, 2 V/s^2]

Objective-Type Questions

6.1 The voltages v_{C_1}, v_{C_2} and v_{C_3} across the capacitors in the circuit in Fig. 6.199 under steady state are respectively

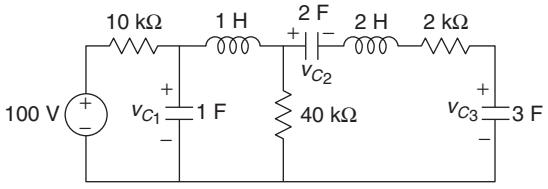


Fig. 6.199

(a) 80 V, 32 V, 48 V
(b) 80 V, 48 V, 32 V
(c) 20 V, 8 V, 12 V
(d) 20 V, 12 V, 8 V

6.2 In the circuit of Fig. 6.200, the voltage $v(t)$ is

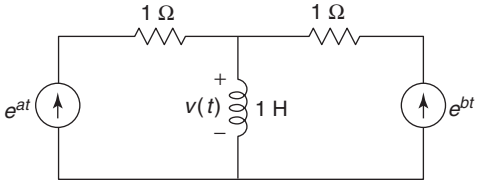


Fig. 6.200

(a) $e^{at} - e^{bt}$ (b) $e^{at} + e^{bt}$
(c) $ae^{dt} - be^{bt}$ (d) $ae^{at} + be^{bt}$

6.3 The differential equation for the current $i(t)$ in the circuit of Fig. 6.201 is

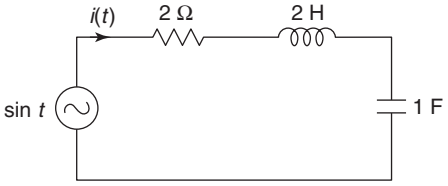


Fig. 6.201

(a) $2\frac{d^2i}{dt^2} + 2\frac{di}{dt} + i(t) = \sin t$

(b) $\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 2i(t) = \cos t$

(c) $2\frac{d^2i}{dt^2} + 2\frac{di}{dt} + i(t) = \cos t$

(d) $\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 2i(t) = \sin t$

6.4 At $t = 0^+$, the current i_1 in Fig. 6.202 is

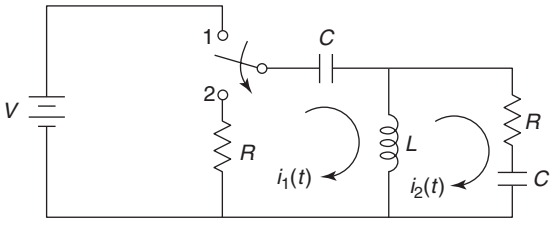


Fig. 6.202

(a) $-\frac{V}{2R}$ (b) $-\frac{V}{R}$

(c) $-\frac{V}{4R}$ (d) zero

6.5 For the circuit shown in Fig. 6.203, the time constant $RC = 1$ ms. The input voltage is $v_i(t) = \sqrt{2}\sin 10^3 t$. The output voltage $v_0(t)$ is equal to

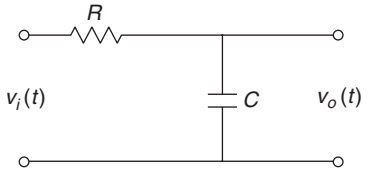


Fig. 6.203

(a) $\sin(10^3 - 45°)$ (b) $\sin(10^3 t + 45°)$

(c) $\sin(10^3 t - 53)$ (d) $\sin(10^3 t + 53°)$

6.6 For the *RL* circuit shown in Fig. 6.204, the input voltage $v_i(t) = u(t)$. The current $i(t)$ is

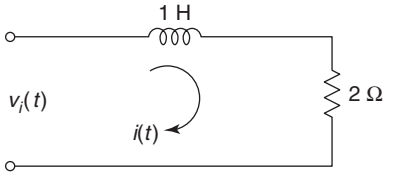
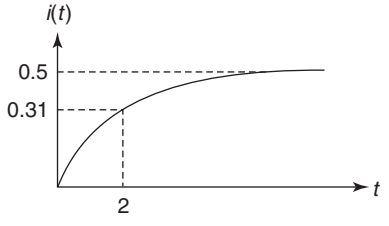
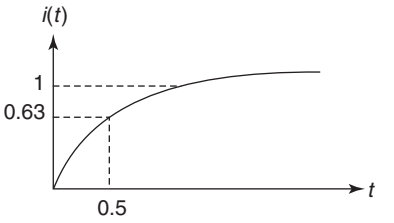


Fig. 6.204

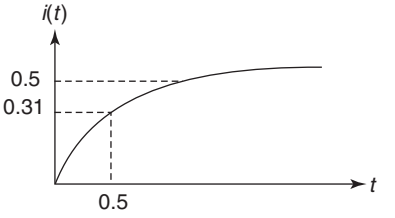
(a)



(b)



(c)



(d)

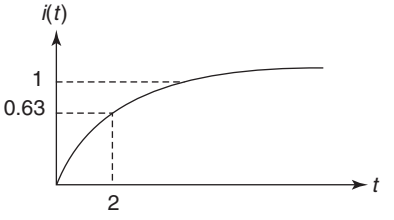


Fig. 6.205

6.7 The condition on R, L and C such that the step response $v(t)$ in Fig. 6.206 has no oscillations is

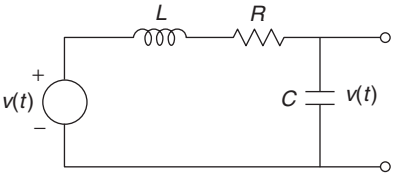


Fig. 6.206

(a) $R \geq \frac{1}{2}\sqrt{\frac{L}{C}}$ (b) $R \geq \sqrt{\frac{L}{C}}$

(c) $R \geq 2\sqrt{\frac{L}{C}}$ (d) $R = \frac{1}{\sqrt{LC}}$

6.8 The switch S in Fig. 6.207 closed at $t = 0$. If $v_2(0) = 10$ V and $v_g(0) = 0$ respectively, the voltages across capacitors in steady state will be

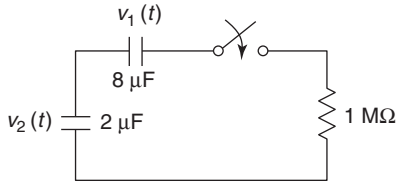


Fig. 6.207

(a) $v_2(\infty) = v_1(\infty) = 0$

(b) $v_2(\infty) = 2$ V, $v_1(\infty) = 8$ V

(c) $v_2(\infty) = v_1(\infty) = 8$ V

(d) $v_2(\infty) = 8$ V, $v_1(\infty) = 2$ V

6.9 The time constant of the network shown in Fig. 6.208 is

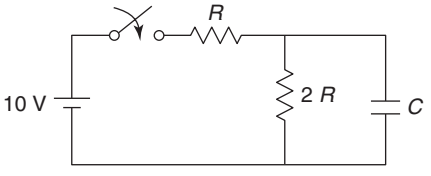


Fig. 6.208

(a) $2\,RC$ (b) $3\,RC$

(c) $\frac{1}{2}RC$ (d) $\frac{2}{3}RC$

6.10 In the series *RC* circuit shown in Fig. 6.209, the voltage across *C* starts increasing when the dc source is switched on. The rate of increase of voltage across *C* at the instant just after the switch is closed i.e., at $t = 0^+$ will be

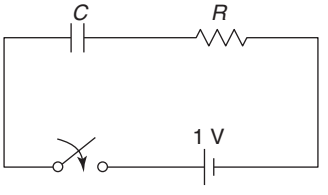


Fig. 6.209

(a) zero (b) infinity

(c) RC (d) $\frac{1}{RC}$

6.11 The $v - i$ characteristic as seen from the terminal pair ($A - B$) of the network of Fig. 6.210(a) is shown in Fig. 6.210(b). If an inductance of value 6 mH is connected across the terminal pair, the time constant of the system will be

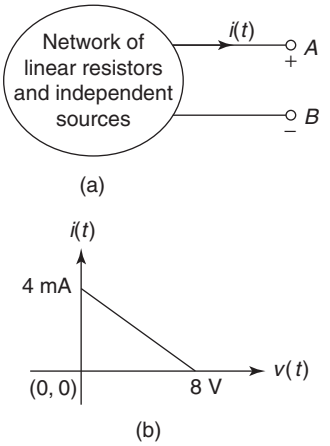


Fig. 6.210

(a) 3 μs

(b) 12 s

(c) 32 s

(d) unknown, unless actual network is specified

6.12 In the network shown in Fig. 6.211, the circuit was initially in the steady-state condition with the switch *K* closed. At the instant when the switch is opened, the rate of decay of current through inductance will be

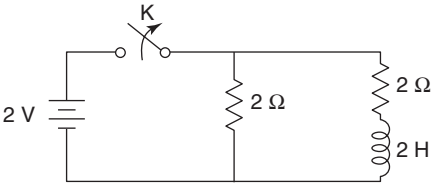


Fig. 6.211

(a) zero (b) 0.5 A/s

(c) 1 A/s (d) 2 A/s

6.13 A step function voltage is applied to an *RLC* series circuit having $R = 2\ \Omega$, $L = 1$ H and $C = 1$ F. The transient current response of the circuit would be

(a) over damped (b) critically damped

(c) under damped (d) none of these

Answers to Objective-Type Questions

6.1 (b)	6.2 (d)	6.3 (c)	6.4 (d)	6.5 (a)	6.6 (b)	6.7 (c)
6.8 (d)	6.9 (d)	6.10 (d)	6.11 (a)	6.12 (d)	6.13 (b)	

7

Laplace Transform and Its Application

7.1 INTRODUCTION

Time-domain analysis is the conventional method of analysing a network. For a simple network with first-order differential equation of network variable, this method is very useful. But as the order of network variable equation increases, this method of analysis becomes very tedious. For such applications, frequency domain analysis using Laplace transform is very convenient. Time-domain analysis, also known as *classical method,* is difficult to apply to a differential equation with excitation functions which contain derivatives. Laplace transform methods prove to be superior. The Laplace transform method has the following advantages:

(1) Solution of differential equations is a systematic procedure.
(2) Initial conditions are automatically incorporated.
(3) It gives the complete solution, i.e., both complementary and particular solution in one step.

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

7.2 LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t$$

where s is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^\infty |\, f(t)\, |\, e^{-\sigma t}\, \mathrm{d}t < \infty$$

where σ is real and positive.
The inverse Laplace transform $L^{-1}\{F(s)\}$ is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds$$

7.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

1. Constant Function k

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty ke^{-st}dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}$$

2. Function t^n

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st}dt$$

Putting $st = x,\ dt = \frac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{dx}{s} = s^{\frac{1}{n+1}}\int_0^\infty x^n e^{-x}dx = \frac{\sqrt{n+1}}{s^{n+1}},\ s>0,\ n+1>0$$

If n is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. Unit-Step Function

The unit-step function (Fig 7.1) is defined by the equation,

$$u(t) = 1 \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of unit step function is

$$L\{u(t)\} = \int_0^\infty 1\cdot e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Fig. 7.1 *Unit-step function*

4. Delayed or Shifted Unit-Step Function

The delayed or shifted unit-step function (Fig 7.2) is defined by the equation

$$u(t-a) = 1 \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $u(t-a)$ is

$$L\{u(t-a)\} = \int_a^\infty 1\cdot e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

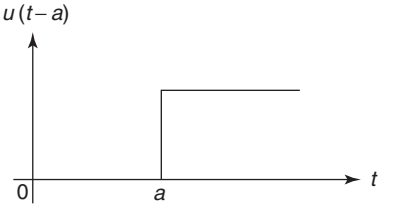


Fig. 7.2 *Shifted unit-step function*

5. Unit-Ramp Function

The unit-ramp function (Fig 7.3) is defined by the equation

$$r(t) = t \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^\infty t\, e^{-st} = \frac{1}{s^2}$$

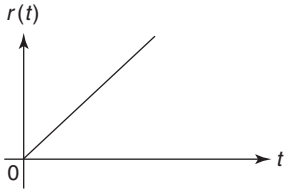


Fig. 7.3 *Unit-ramp function*

6. Delayed Unit-Ramp Function

The delayed unit-ramp function (Fig 7.4) is defined by the equation

$$r\,(t-a) = t \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $r\,(t-a)$ is

$$L\{r(t-a)\} = \int_a^\infty t\, e^{-st}\, dt = \frac{e^{-as}}{s^2}$$

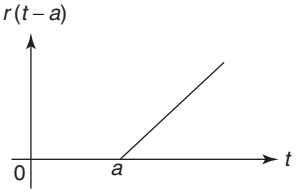


Fig. 7.4 *Delayed unit-ramp function*

7. Unit-Impulse Function

The unit-impulse function (Fig 7.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad t = 0$$

The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = 1$$

Fig. 7.5 *Unit-impulse function*

8. Exponential Function (e^{at})

The Laplace transform of the exponential function (Fig 7.6) is

$$L\{e^{at}\} = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

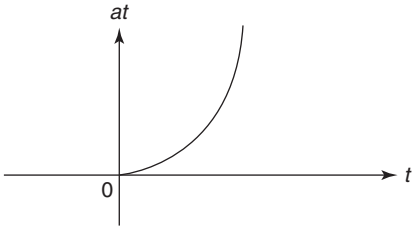


Fig. 7.6 *Exponential function*

9. Sine Function

We know that $\sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin \omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

10. Cosine Function

We know that $\cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos \omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

11. Hyperbolic sine function

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2 - \omega^2}$$

12. Hyperbolic cosine function

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2 - \omega^2}$$

13. Exponentially Damped Function

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t)\, dt\right\} = \int_{0^-}^{\infty} f(t) e^{-at}\, e^{-st}\, dt = \int_{0^-}^{\infty} f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s + a)$ in place of s in the transform of $f(t)$.

$$L\left\{e^{-at} \sin \omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \sinh \omega t\right\} = \frac{\omega}{(s+a)^2 - \omega^2}$$

$$L\left\{e^{-at} \cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \cosh \omega t\right\} = \frac{s+a}{(s+a)^2 - \omega^2}$$

7.4 PROPERTIES OF LAPLACE TRANSFORM

7.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$
where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^{\infty} \{af_1(t) + bf_2(t)\} e^{-st} dt = a\int_0^{\infty} f_1(t) e^{-st} dt + b\int_0^{\infty} f_2(t) e^{-st}\, dt = aF_1(s) + bF_2(s)$$

7.4.2 Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{f(at)\} = \int_0^{\infty} f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \dfrac{dx}{a}$

$$L\{(f(at)\} = \int_0^\infty f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^\infty f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

7.4.3 Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\, d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s+a)t}\, dt = F(s+a)$$

7.4.4 Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as} F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st}\, dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x) e^{-s(a+x)}\, dx = e^{-as}\int_0^\infty f(x) e^{-sx}\, dx = e^{-as}\int_0^\infty f(t)\, e^{-st}\, dt = e^{-as} F(s)$$

7.4.5 Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\, f(t)\} = -\dfrac{d}{ds} F(s)$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Differentiating both the sides w.r.t s using DUIS,

$$\frac{d}{ds} F(s) = \frac{d}{ds}\int_0^\infty f(t) e^{-st} dt = \int_0^\infty \frac{\partial}{\partial s} f(t) e^{-st}\, dt$$

$$= \int_0^\infty (-t)\, f(t) e^{-st}\, dt = \int_0^\infty \{-t\, f(t)\} e^{-st} dt = -L\{t\, f(t)\}$$

$$L\{t\, f(t)\} = (-1)\frac{d}{ds} F(s)$$

7.4.6 Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty f(t)e^{-st}dt\,ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}ds\right]dt = \int_0^\infty \left[\frac{1}{-t}f(t)e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$

7.4.7 Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$ then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

In general,

$$L\{f''(t)\} = s''F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{(n-1)}(0)$$

Proof

$$L\{f'(t)\} = \int_0^\infty f'(t)\,e^{-st}\,dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty (-s)f(t)e^{-st}dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\,t\} = -f' + sL\{f'\,t\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right] = -f'(0) - sf(0) + s^2L\{f(t)\}$$

In general, $L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{n-1}(0)$

7.4.8 Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

Proof

$$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \int_0^t f(t)\,dt \Big\} e^{-st}\,dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right]dt = \int_0^\infty \frac{1}{s} f(t)e^{-st}dt = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

7.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \int_0^\infty f'(t)e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to\infty}[f'(t)e^{-st}]dt + f(0) = 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

7.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to 0} \int_0^\infty f'(t)\,e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to 0}\left[f'(t)e^{-st}\right]dt + f(0) = \int_0^\infty f'(t)dt + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) = \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

7.5 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion

7.5.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 7.1 *Find the inverse Laplace transform of* $\frac{s^2-3s+4}{s^3}$.

Solution

$$F(s)=\frac{s^2-3s+4}{s^3}=\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}$$

$$L^{-1}\{F(s)\}=1-3t+2t^2$$

Example 7.2 *Find the inverse Laplace transform of* $\frac{3s+4}{s^2+9}$.

Solution

$$F(s)=\frac{3s+4}{s^2+9}=\frac{3s}{s^2+9}+\frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\}=3\cos 3t+\frac{4}{3}\sin 3t$$

Example 7.3 *Find the inverse Laplace transform of* $\frac{4s+15}{16s^2-25}$.

Solution

$$F(s)=\frac{4s+15}{16s^2-25}=\frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}=\frac{1}{4}\frac{s}{s^2-\frac{25}{16}}+\frac{15}{16}\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\}=\frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

Example 7.4 *Find the inverse Laplace transform of* $\frac{2s+2}{s^2+2s+10}$.

Solution

$$F(s)=\frac{2s+2}{s^2+2s+10}=\frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\}=2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}=2e^{-t}\cos 3t$$

Example 7.5 *Find the inverse Laplace transform of* $\frac{3s+7}{s^2-2s-3}$.

Solution

$$F(s)=\frac{3s+7}{s^2-2s-3}=\frac{3(s-1)+10}{(s-1)^2-4}=3\frac{(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}$$

$$L^{-1}\{F(s)\}=3e^{t}L^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^{t}L^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^{t}\cosh 2t+5e^{t}\sinh 2t$$

7.5.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\frac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be

divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I Factors are linear and distinct,

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II Factors are linear and repeated,

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \ldots + \frac{B_n}{(s+b)^n}$$

Case III Factors are quadratic and distinct,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial-fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV Factors are quadratic are repeated,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial-fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{s^2+cs+d} + \frac{C_2s+D_2}{(s^2+cs+d)^2} + \ldots + \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 7.6 *Find the inverse Laplace transform of* $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$A = sF(s)\big|_{s=0} = \left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0} = \frac{2}{3}$$

$$B = (s+1)F(s)\big|_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$C = (s+3)F(s)\big|_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Example 7.7 *Find the inverse Laplace transform of* $\dfrac{s+2}{s^2(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s+2 = As(s+3) + B(s+3) + Cs^2$$

$$= As^2 + 3As + Bs + 3B + Cs^2$$

$$= (A+C)s^2 + (3A+B)s + 3B$$

Comparing coefficients of s^2, s^1 and s^0,

$$A + C = 0$$

$$3A + B = 1$$

$$3B = 2$$

Solving these equations,

$$A = \frac{1}{9},\ B = \frac{2}{3},\ C = -\frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7.8 *Find the inverse Laplace transform of* $\dfrac{s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

$$F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\begin{aligned} 5s^2 - 15s - 11 &= A(s-2)^2 + B(s+1)(s-2) + C(s+1) \\ &= A(s^2 - 4s + 4) + B(s^2 - s - 2) + C(s+1) \\ &= As^2 - 4As + 4A + Bs^2 - Bs - 2B + Cs + c \\ &= (A+B)s^2 - (4A + B - C)s + (4A - 2B + C) \end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned} A + B &= 5 \\ 4A + B - C &= 15 \\ 4A - 2B + C &= -11 \end{aligned}$$

Solving these equations,

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= -7 \end{aligned}$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\} = e^{-t} + 4e^{2t} - 7te^{2t}$$

Example 7.9 *Find the inverse Laplace transform of* $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

$$F(s) = \frac{3s+1}{(s+1)(s^2+2)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$\begin{aligned} 3s+1 &= A(s^2+2) + (Bs+C)(s+1) \\ &= As^2 + 2A + Bs^2 + Bs + Cs + C \\ &= (A+B)s^2 + (B+C)s + (2A+C) \end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned} A + B &= 0 \\ B + C &= 3 \\ 2A + C &= 1 \end{aligned}$$

Solving these equations,

$$A = -\frac{2}{3},\ B = \frac{2}{3},\ C = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\} = -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2t} + \frac{7}{3\sqrt{2}}\sin\sqrt{2t}$$

Example 7.10 *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

$$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

7.6 THE TRANSFORMED CIRCUIT

Voltage–current relationships of network elements can also be represented in the frequency domain.

1. Resistor For the resistor, the *v–i* relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 7.7.

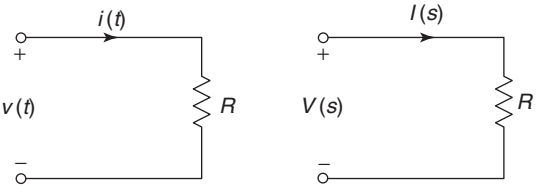


Fig. 7.7 *Resistor*

2. Inductor For the inductor, the *v–i* relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 7.8.

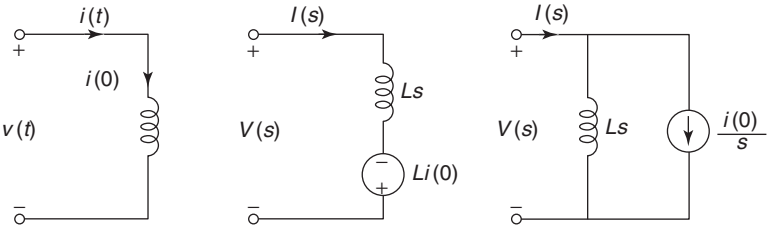


Fig. 7.8 *Inductor*

3. Capacitor For capacitor, the *v*–*i* relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 7.9.

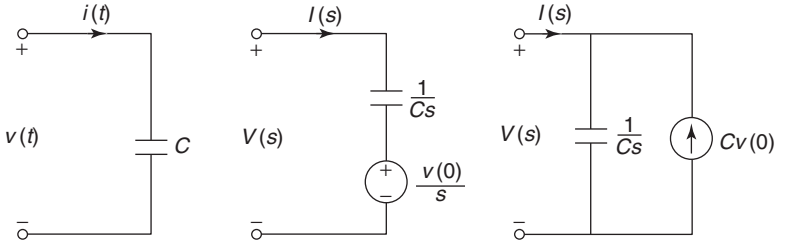


Fig. 7.9 *Capacitor*

7.7 RESISTOR–INDUCTOR CIRCUIT

Consider a series *RL* circuit as shown in Fig. 7.10. The switch is closed at time $t = 0$.

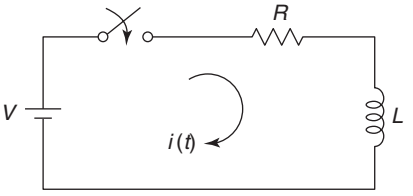


Fig. 7.10 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.11.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{\dfrac{V}{L}}{s\left(s + \dfrac{R}{L}\right)}$$

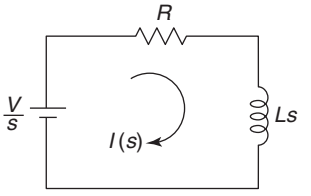


Fig. 7.11 *Transformed network*

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \dfrac{R}{L}}$$

$$A = sI(s)\,|_{s=0} = s \times \left.\frac{\dfrac{V}{L}}{s\left(s + \dfrac{R}{L}\right)}\right|_{s=0} = \frac{V}{R}$$

$$B = \left(s + \frac{R}{L}\right) I(s)\Big|_{s=-\frac{R}{L}} = \left(s + \frac{R}{L}\right) \times \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}\Bigg|_{s=-\frac{R}{L}} = -\frac{V}{R}$$

$$I(s) = \frac{\frac{V}{R}}{s} + \frac{\left(-\frac{V}{R}\right)}{s + \frac{R}{L}}$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t}$$

$$= \frac{V}{R}\left[1 - e^{-\frac{R}{L}t}\right] \quad \text{for } t > 0$$

Example 7.11 *In the network of Fig. 7.12, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.*

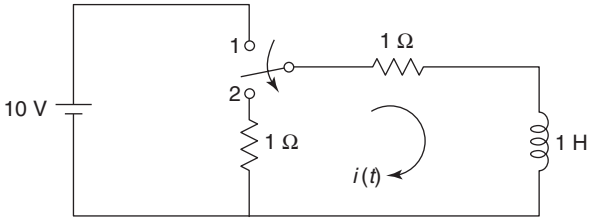


Fig. 7.12

Solution At $t = 0^-$, the network is shown in Fig 7.13. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{10}{1} = 10 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i\,(0^+) = 10 \text{ A}$$

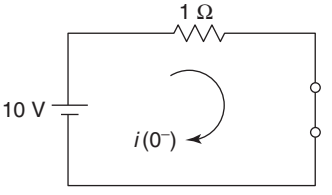


Fig. 7.13

For $t > 0$, the transformed network is shown in Fig. 7.14.
Applying KVL to the mesh for $t > 0$,

$$-I(s) - I(s) - sI(s) + 10 = 0$$

$$I(s)(s + 2) = 10$$

$$I(s) = \frac{10}{s + 2}$$

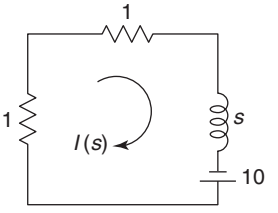


Fig. 7.14

Taking inverse Laplace transform,

$$i\,(t) = 10e^{-2t} \quad \text{for } t > 0$$

Example 7.12 *The network of Fig. 7.15 was initially in the steady state with the switch in the position a. At t = 0, the switch goes from a to b. Find an expression for voltage v (t) for t > 0.*

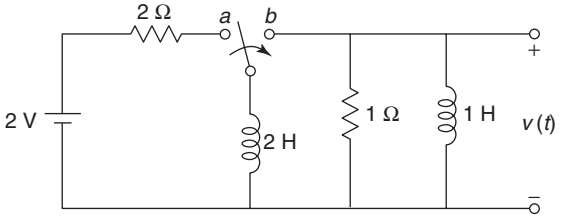


Fig. 7.15

Solution At $t = 0^-$, the network is shown in Fig 7.16. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

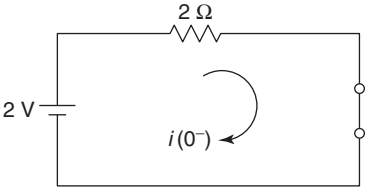


Fig. 7.16

$$i(0^-) = \frac{2}{2} = 1 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.17.

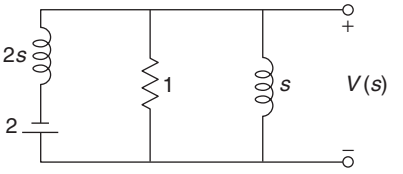


Fig. 7.17

Applying KCL at the node for $t > 0$,

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

$$V(s)\left(1+\frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v\,(t) = -\,e^{-1.5\,t} \qquad \text{for } t > 0$$

Example 7.13 *In the network of Fig. 7.18, the switch is opened at t = 0. Find i(t).*

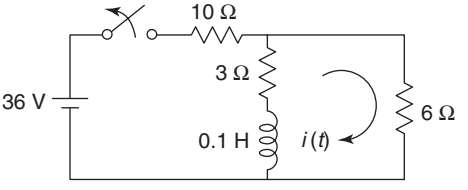


Fig. 7.18

Solution At $t = 0^-$, the network is shown in Fig. 7.19. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the inductor acts as a short circuit.

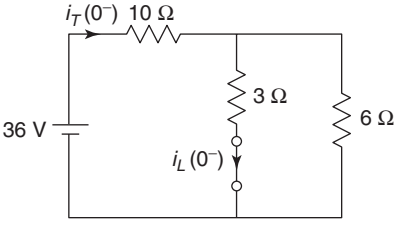


Fig. 7.19

$$i_T(0^-) = \frac{36}{10 + (3 \,\|\, 6)} = \frac{36}{10+2} = 3 \text{ A}$$

$$i_L(0^-) = 3 \times \frac{6}{6+3} = 2 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.20

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\, I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = -0.2$$

$$I(s) = \frac{-0.2}{0.1s + 9} = \frac{-2}{s + 90}$$

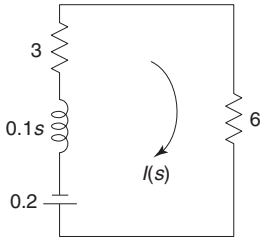


Fig. 7.20

Taking inverse Laplace transform,

$$i(t) = -2e^{-90t} \qquad \text{for } t > 0$$

Example 7.14 *The network shown in Fig. 7.21 has acquired steady-state with the switch closed for $t < 0$. At $t = 0$, the switch is opened. Obtain i (t) for $t > 0$.*

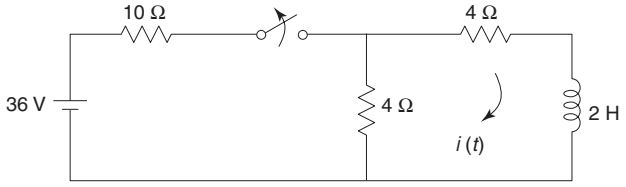


Fig. 7.21

Solution At $t = 0^-$, the network is shown in Fig 7.22. At $t = 0^-$, the switch is closed and the network has acquired steady-state. Hence, the inductor acts as a short circuit.

$$i_T(0^-) = \frac{36}{10 + (4 \,||\, 4)} = \frac{36}{10 + 2} = 3 \text{ A}$$

$$i(0^-) = 3 \times \frac{4}{4 + 4} = 1.5 \text{ A}$$

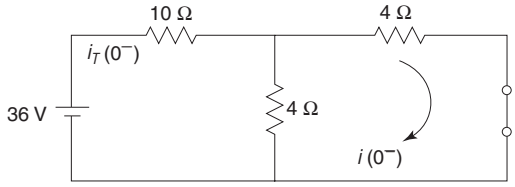


Fig. 7.22

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.5 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.23.

Applying KVL to the mesh for $t > 0$,

$$-4I(s) - 4I(s) - 2sI(s) + 3 = 0$$

$$8I(s) + 2sI(s) = 3$$

$$I(s) = \frac{3}{2s + 8} = \frac{1.5}{s + 4}$$

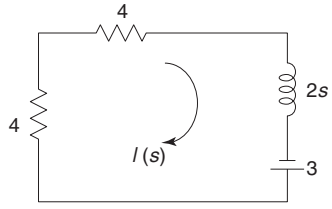


Fig. 7.23

Taking the inverse Laplace transform,

$$i(t) = 1.5e^{-4t} \qquad \text{for } t > 0$$

Example 7.15 *In the network shown in Fig. 7.24, the switch is closed at t = 0, the steady-state being reached before t = 0. Determine current through inductor of 3 H.*

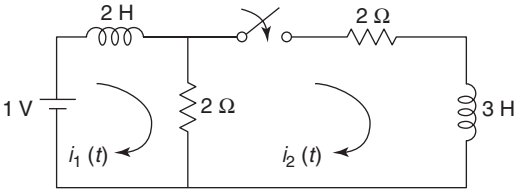


Fig. 7.24

Solution At $t = 0^-$, the network is shown in Fig. 7.25. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

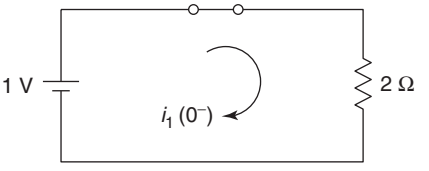


Fig. 7.25

$$i_1(0^-) = \frac{1}{2} \text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2} \text{ A}$$

$$i_2(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 17.26. Applying KVL to Mesh 1,

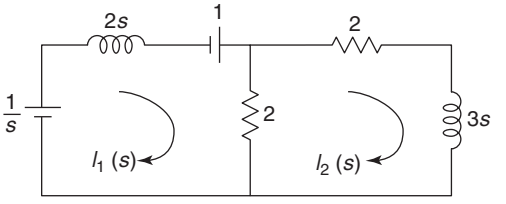


Fig. 7.26

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2 + 2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2(s) = 0$$

$$-2I_1(s) + (4 + 3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3s^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

$$C = \left(s+\frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

Example 7.16 *In the network of Fig. 7.27, the switch is closed at t = 0 with the network previously unenergised. Determine currents $i_1(t)$.*

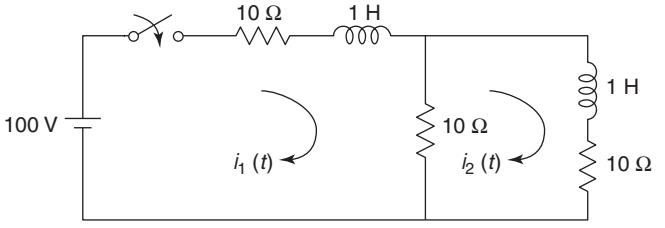


Fig. 7.27

Solution For $t > 0$, the transformed network is shown in Fig. 7.28.

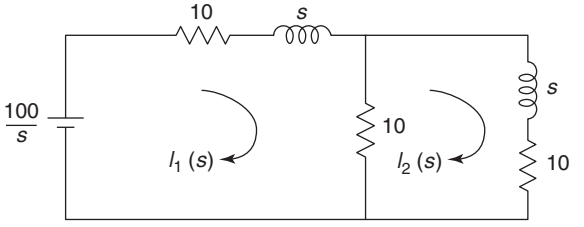


Fig. 7.28

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\,I_2(s) - 10\,I_2(s) = 0$$

$$-10\,I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = s\, I_1(s)\,|_{s=0} = \left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+10) I_1(s)\,|_{s=-10} = \left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30) I_1(s)\,|_{s=-30} = \left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30} = -\frac{5}{3}$$

$$I_1(s) = \frac{20}{3}\frac{1}{s} - \frac{5}{s+10} - \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{1000}{s}}{(s+20)^2 - 100} = \frac{1000}{s(s^2+40s+300)} = \frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = sI_2(s)\,|_{s=0} = \left.\frac{1000}{(s+10)(s+30)}\right|_{s=0} = \frac{10}{3}$$

$$B = (s+10) I_2(s)\,|_{s=-10} = \left.\frac{1000}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30) I_2(s)\,|_{s=-30} = \left.\frac{1000}{s(s+10)}\right|_{s=-30} = \frac{5}{3}$$

$$I_2(s) = \frac{10}{3}\frac{1}{s} - \frac{5}{s+10} + \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t} \qquad \text{for } t > 0$$

7.8 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 7.29. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 7.30.

Applying KVL to the mesh,

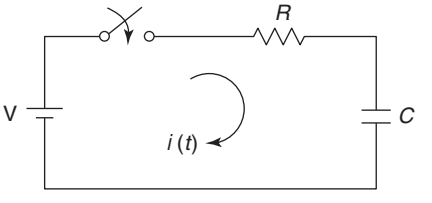


Fig. 7.29 *RC circuit*

$$\frac{V}{s} - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$\left(R + \frac{1}{Cs}\right) I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{R + \frac{1}{Cs}} = \frac{\frac{V}{s}}{\frac{RCs+1}{Cs}} = \frac{\frac{V}{R}}{s + \frac{1}{RC}}$$

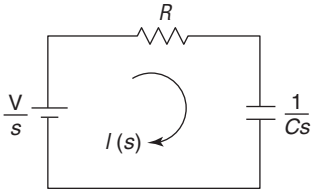


Fig. 7.30 *Transformed network*

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

Example 7.17 *In the network of Fig. 7.31, the switch is moved from a to b at t = 0. Determine i (t) and v_c (t).*

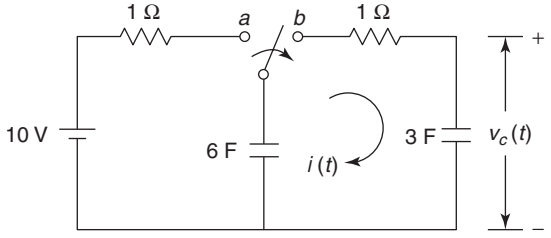


Fig. 7.31

Solution At $t = 0^-$, the network is shown in Fig. 7.32. At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,F}(0^-) = 10 \text{ V}$$
$$i(0^-) = 0$$
$$v_{3\,F}(0^-) = 0$$

Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,F}(0^+) = 10 \text{ V}$$
$$v_{3\,F}(0^+) = 0$$

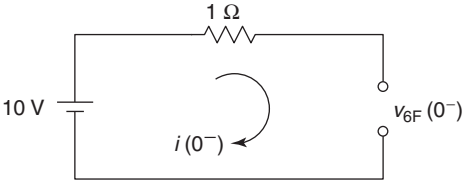


Fig. 7.32

For $t > 0$, the transformed network is shown in 7.33.

Applying KVL to the mesh for $t > 0$,

$$\frac{10}{s} - \frac{1}{6s} I(s) - I(s) - \frac{1}{3s} I(s) = 0$$

$$\frac{1}{6s} I(s) + I(s) + \frac{1}{3s} I(s) = \frac{10}{s}$$

$$I(s) = \frac{10}{s\left(1 + \frac{1}{6s} + \frac{1}{3s}\right)} = \frac{60}{6s+3} = \frac{10}{s+0.5}$$

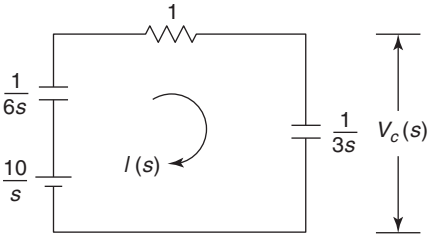


Fig. 7.33

Taking the inverse Laplace transform,

$$i(t) = 10e^{-0.5t} \qquad \text{for } t > 0$$

Voltage across the 3 F capacitor is given by

$$V_c(s) = \frac{1}{3s} I(s) = \frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\big|_{s=0} = \frac{10}{3(s+0.5)}\bigg|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\big|_{s=-0.5} = \frac{10}{3s}\bigg|_{s=-0.5} = -\frac{20}{3}$$

$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1 - e^{-0.5t}) \qquad \text{for } t > 0$$

Example 7.18 *The switch in the network shown in Fig. 7.34 is closed at t = 0. Determine the voltage cross the capacitor.*

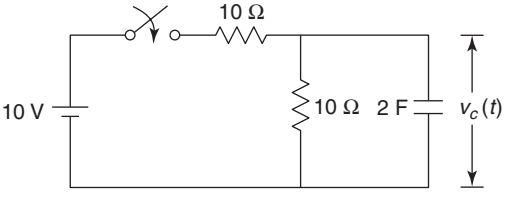


Fig. 7.34

Solution At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.35.

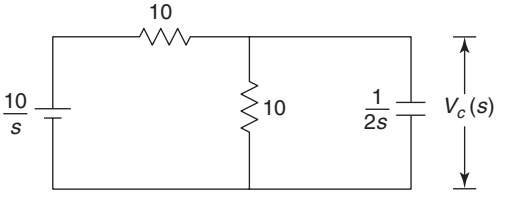


Fig. 7.35

Applying KCL at the node for $t > 0$,

$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.1}$$

$$A = sV_c(s)\big|_{s=0} = \frac{0.5}{s+0.1}\bigg|_{s=0} = \frac{0.5}{0.1} = 5$$

$$B = (s+0.1)V_c(s)\big|_{s=-0.1} = \frac{0.5}{s}\bigg|_{s=-0.1} = -\frac{0.5}{0.1} = -5$$

$$V_c(s) = \frac{5}{s} - \frac{5}{s+0.1}$$

Taking inverse Laplace transform,

$$v_c(t) = 5 - 5e^{-0.1t} \qquad \text{for } t > 0$$

Example 7.19 *In the network of Fig. 7.36, the switch is closed for a long time and at t =0, the switch is opened. Determine the current through the capacitor.*

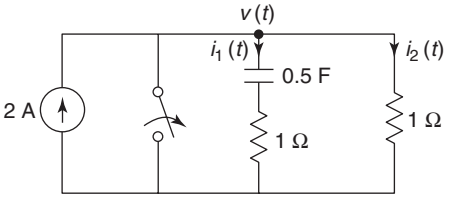


Fig. 7.36

Solution At $t = 0^-$, the network is shown in Fig. 7.37. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v_c(0^-) = 0$$

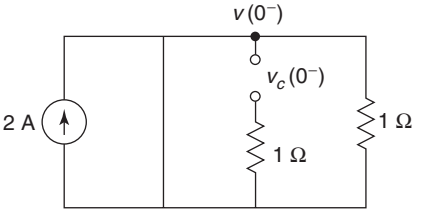


Fig. 7.37

Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.38.

Applying KVL to two parallel branches,

$$\frac{2}{s} I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

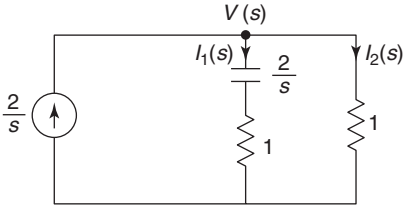


Fig. 7.38

$$\frac{2}{s}I_1(s)+I_1(s)=\frac{2}{s}-I_1(s)$$

$$\frac{2}{s}I_1(s)+2I_1(s)=\frac{2}{s}$$

$$I_1(s)=\frac{\frac{2}{s}}{\frac{2}{s}+2}=\frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t)=e^{-t} \qquad \text{for } t>0$$

Example 7.20 *In the network of Fig. 7.39, the switch is moved from a to b, at t = 0. Find v(t).*

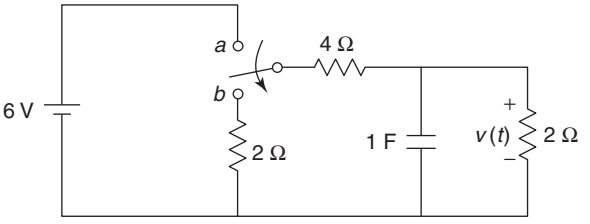


Fig. 7.39

Solution At $t = 0^-$, the network is shown in Fig 7.40. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v(0^-)=6\times\frac{2}{4+2}=2\text{ V}$$

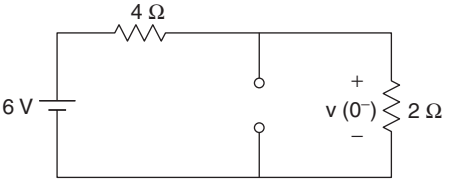


Fig. 7.40

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+)=2\text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.41.

Applying KCL at the node for $t > 0$,

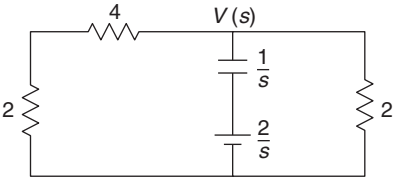


Fig. 7.41

$$\frac{V(s)}{6}+\frac{V(s)-\frac{2}{s}}{\frac{1}{s}}+\frac{V(s)}{2}=0$$

$$V(s)\left(\frac{2}{3}+s\right)=2$$

$$V(s)=\frac{2}{s+\frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t)=2e^{-\frac{2}{3}t} \qquad \text{for } t>0$$

Example 7.21 *The network shown in Fig. 7.42 has acquired steady-state at $t < 0$ with the switch open. The switch is closed at $t = 0$. Determine $v(t)$.*

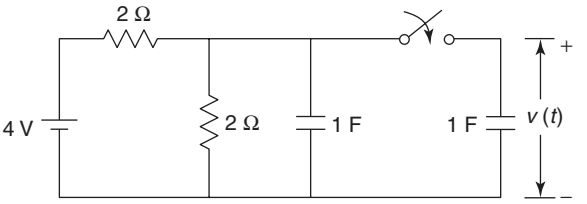


Fig. 7.42

Solution At $t = 0^-$, the network is shown in Fig 7.43. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

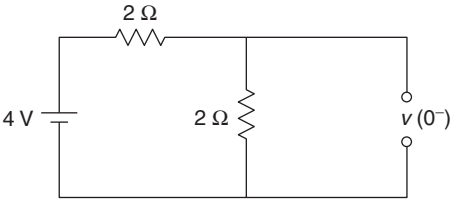


Fig. 7.43

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.44.

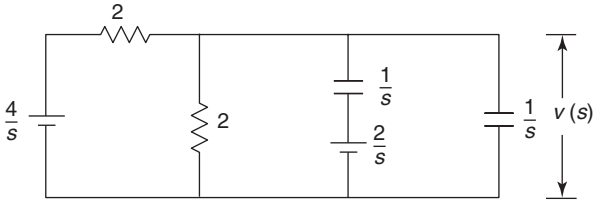


Fig. 7.44

Applying KCL at the node for $t > 0$,

$$\frac{V(s) - \frac{4}{s}}{2} + \frac{V(s)}{2} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{\frac{1}{s}} = 0$$

$$2sV(s) + V(s) = \frac{2}{s} + 2$$

$$V(s) = \frac{\frac{2}{s} + 2}{2s+1} = \frac{2s+2}{s(2s+1)} = \frac{2}{s} - \frac{2}{2s+1} = \frac{2}{s} - \frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t) = 2 - e^{-0.5t} \qquad \text{for } t > 0$$

7.9 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit shown in Fig. 7.45. The switch is closed at time $t = 0$.

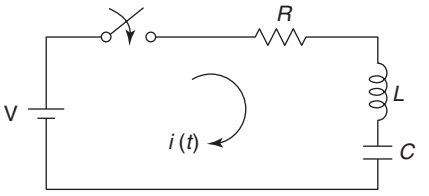


Fig. 7.45 *RLC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.46.
Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + Ls + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$\left(\frac{LCs^2 + RCs + 1}{Cs}\right)I(s) = \frac{V}{s}$$

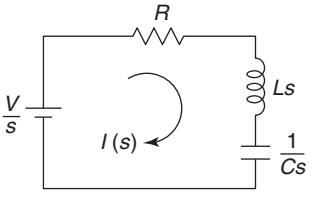


Fig. 7.46 *Transformed network*

$$I(s) = \frac{\frac{V}{s}}{\frac{LCs^2 + RCs + 1}{Cs}} = \frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)}$$

where s_1 and s_2 are the roots of the equation $s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $\alpha = \frac{R}{2L}$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $\beta = \sqrt{\alpha^2 - \omega_0^2}$

By partial-fraction expansion, of $I(s)$,

$$I(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$$A = (s - s_1)\, I(s)\big|_{s=s_1} = \frac{\frac{V}{L}}{s_1 - s_2}$$

$$B = (s - s_2)\, I(s)\big|_{s=s_2} = \frac{\frac{V}{L}}{s_2 - s_1} = -\frac{\frac{V}{L}}{s_1 - s_2}$$

$$I(s) = \frac{V}{L(s_1 - s_2)}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{L(s_1 - s_2)}\left[e^{s_1 t} - e^{s_2 t}\right] = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation. Now depending upon the values of s_1 and s_2, we have 3 cases of the response.

Case I When the roots are real and unequal, it gives an overdamped response.

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$\alpha > \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\left(k_1 e^{\beta t} + k_2 e^{-\beta t}\right)$$

or $$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad \text{for } t > 0$$

Case II When the roots are real and equal, it gives a critically damped response.

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\alpha = \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\,(k_1 + k_2\, t) \qquad \text{for } t > 0$$

Case III When the roots are complex conjugate, it gives an underdamped response.

$$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

$$\alpha < \omega_0$$

In this case, the solution is given by

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Let $$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$$

where $$j = \sqrt{-1}$$

and $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hence

$$i(t) = e^{-\alpha t}\left(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t}\right)$$
$$= e^{-\alpha t}\left[(k_1 + k_2)\left\{\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right\} + j(k_1 - k_2)\left\{\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right\}\right]$$
$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \qquad \text{for } t > 0$$

Example 7.22 *The switch in Fig. 7.47 is opened at time t = 0. Determine the voltage v(t) for t > 0.*

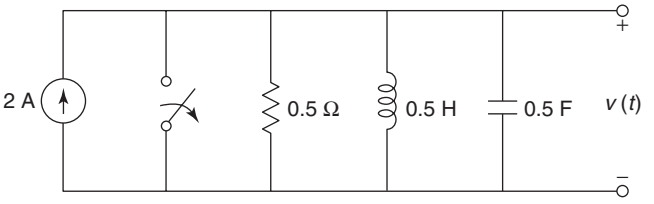


Fig. 7.47

Solution At $t = 0^-$, the network is shown in Fig. 7.48. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

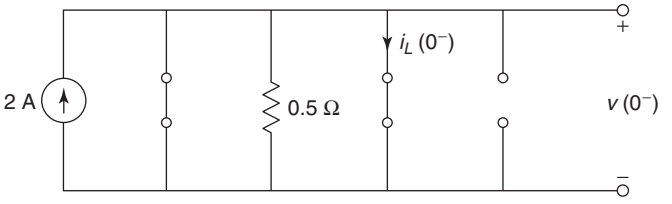


Fig. 7.48

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.49.

Applying KCL at the node for $t > 0$,

$$\frac{V(s)}{0.5} + \frac{V(s)}{0.5s} + \frac{V(s)}{\frac{1}{0.5s}} = \frac{2}{s}$$

$$2V(s) + \frac{2}{s}V(s) + 0.5sV(s) = \frac{2}{s}$$

$$V(s) = \frac{\frac{2}{s}}{\frac{2}{s} + 0.5s + 2} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

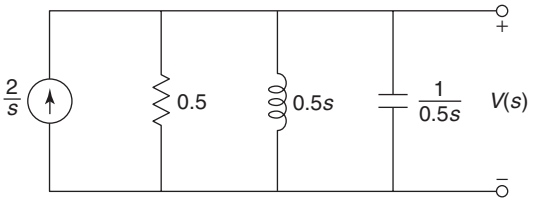


Fig. 7.49

Taking inverse Laplace transform,

$$v(t) = 4t\ e^{-2t} \qquad \text{for } t > 0$$

Example 7.23 *In the network of Fig. 7.50, the switch is closed and steady-state is attained. At $t = 0$, switch is opened. Determine the current through the inductor.*

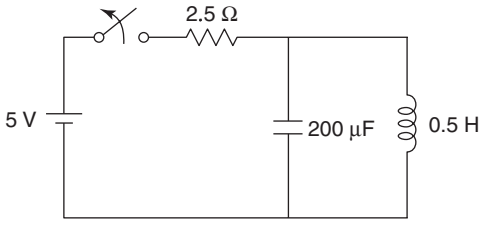


Fig. 7.50

Solution At $t = 0^-$, the network is shown in Fig. 7.51. At $t = 0^-$, the switch is closed and steady-state condition is attained. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit. Current through inductor is same as the current through the resistor.

$$i_L(0^-) = \frac{5}{2.5} = 2 \text{ A}$$

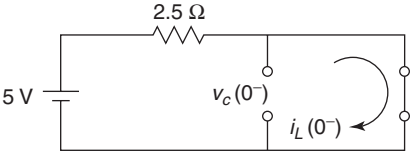


Fig. 7.51

Voltage across the capacitor is zero as it is connected in parallel with a short.

$$v_c(0^-) = 0$$

Since voltage across the capacitor and current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.52. Applying KVL to the mesh for $t > 0$,

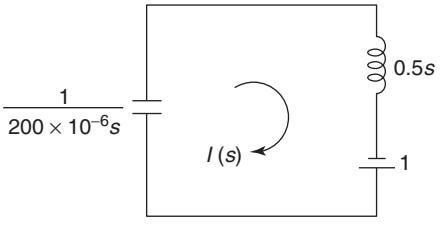


Fig. 7.52

$$-\frac{1}{200 \times 10^{-6} s} I(s) - 0.5s\, I(s) + 1 = 0$$

$$0.5s\, I(s) - 1 + 5000 \frac{I(s)}{s} = 0$$

$$I(s) = \frac{1}{0.5s + \frac{5000}{s}} = \frac{2s}{s^2 + 10000}$$

Taking inverse Laplace transform,

$$i(t) = 2 \cos 100t \qquad \text{for } t > 0$$

Example 7.24 *In the network shown in Fig. 7.53, the switch is opened at t =0. Steady-state condition is achieved before t = 0. Find i(t).*

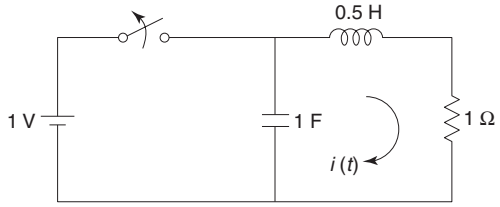


Fig. 7.53

Solution At $t = 0^-$, the network is shown in Fig 7.54. At $t = 0^-$, the switch is closed and steady-state condition is achieved. Hence, the capacitor acts as an open circuit and the inductor acts as a short circuit.

$$v_c(0^-) = 1 \text{ V}$$
$$i(0^-) = 1 \text{ A}$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 1 \text{ V}$$
$$i(0^+) = 1 \text{ A}$$

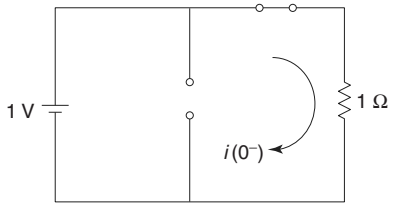


Fig. 7.54

For $t > 0$, the transformed network is shown in Fig. 7.55.

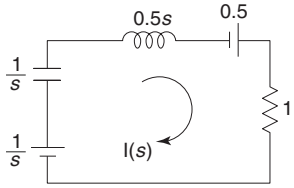


Fig. 7.55

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s} - \frac{1}{s} I(s) - 0.5s\, I(s) + 0.5 - I(s) = 0$$

$$0.5 + \frac{1}{s} = \frac{1}{s} I(s) + 0.5s\, I(s) + I(s)$$

$$I(s)\left[1 + \frac{1}{s} + 0.5s\right] = 0.5 + \frac{1}{s}$$

$$I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$i(t) = e^{-t}\cos t + e^{-t}\sin t \qquad \text{for } t > 0$$

Example 7.25 *In the network shown in Fig. 7.56, the switch is closed at t = 0. Find the currents $i_1(t)$ and $i_2(t)$ when initial current through the inductor is zero and initial voltage on the capacitor is 4 V.*

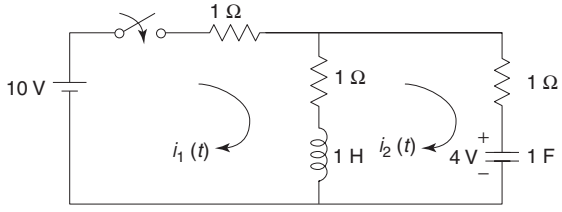


Fig. 7.56

Solution For $t > 0$, the transformed network is shown in Fig. 7.57.

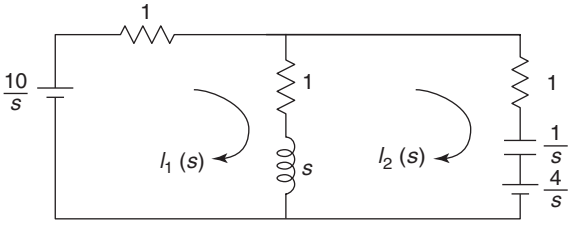


Fig. 7.57

Applying KVL to Mesh 1,

$$\frac{10}{s} - I_1(s) - (1+s)\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+2)I_1(s) - (s+1)I_2(s) = \frac{10}{s}$$

Applying KVL to Mesh 2,

$$-(s+1)\left[I_2(s) - I_1(s)\right] - I_2(s) - \frac{1}{s}I_2(s) - \frac{4}{s} = 0$$

$$-(s+1)I_1(s) + \left(s+2+\frac{1}{s}\right)I_2(s) = -\frac{4}{s}$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{10}{s} & -(s+1) \\ -\frac{4}{s} & s+2+\frac{1}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{\left(\frac{10}{s}\right)\left(\frac{s^2+2s+1}{s}\right) - (s+1)\left(\frac{4}{s}\right)}{(s+2)\left(\frac{s^2+2s+1}{s}\right) - (s+1)^2} = \frac{\frac{10}{s^2}(s+1)^2 - (s+1)\frac{4}{s}}{(s+2)\frac{(s+1)^2}{s} - (s+1)^2}$$

$$= \frac{\frac{10}{s^2}(s+1) - \frac{4}{s}}{(s+2)\frac{(s+1)}{s} - (s+1)} = \frac{3s+5}{s(s+1)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+1}$$

$$A = sI_1(s)\,|_{s=0} = \left.\frac{3s+5}{s+1}\right|_{s=0} = 5$$

$$B = (s+1)I_1(s)\,|_{s=-1} = \left.\frac{3s+5}{s}\right|_{s=-1} = -2$$

$$I_1(s) = \frac{5}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform,

$$i_1(t) = 5 - 2e^{-t} \qquad \text{for } t > 0$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+2 & \frac{10}{s} \\ -(s+1) & -\frac{4}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{3s+1}{(s+1)^2} = \frac{3s+3-2}{(s+1)^2} = \frac{3(s+1)-2}{(s+1)^2} = \frac{3}{s+1} - \frac{2}{(s+1)^2}$$

Taking inverse Laplace transform,

$$i_2(t) = 3e^{-t} - 2te^{-t} \qquad \text{for } t > 0$$

7.10 RESPONSE OF RL CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RL* circuit shown in Fig. 7.58. When the switch is closed at $t = 0$, $i(0^-) = i(0^+) = 0$.

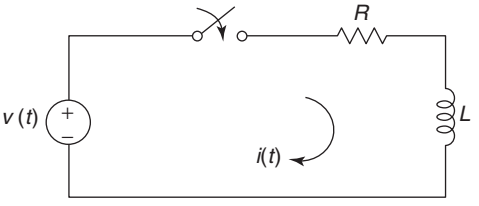


Fig. 7.58 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.59.
Applying KVL to the mesh,

$$V(s) - R\,I(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{V(s)}{R+Ls} = \frac{1}{L}\frac{V(s)}{s+\frac{R}{L}}$$

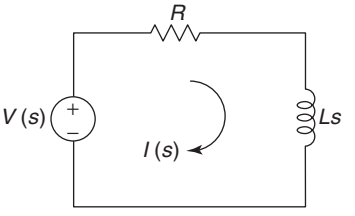


Fig. 7.59 *Transformed network*

(a) When the unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{1}{L}\frac{\frac{1}{s}}{s+\frac{R}{L}}$$

$$= \frac{1}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{1}{L}\left(\frac{A}{s} + \frac{B}{s+\frac{R}{L}}\right)$$

$$A = s\,I(s)\big|_{s=0} = \left.\frac{1}{s+\frac{R}{L}}\right|_{s=0} = \frac{L}{R}$$

$$B = \left(s+\frac{R}{L}\right)I(s)\big|_{s=-\frac{R}{L}} = \left.\frac{1}{s}\right|_{s=-\frac{R}{L}} = -\frac{L}{R}$$

$$I(s) = \frac{1}{L}\left(\frac{L}{R}\frac{1}{s} - \frac{L}{R}\frac{1}{s+\frac{R}{L}}\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t \qquad \text{for } t > 0$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)}$$

By partial-faction expansion,

$$\frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+\frac{R}{L}}$$

$$\frac{1}{L} = As\left(s+\frac{R}{L}\right) + B\left(s+\frac{R}{L}\right) + Cs^2$$

Putting $s = 0$,

$$B = \frac{1}{R}$$

Putting $s = -\frac{R}{L}$,

$$C = \frac{L}{R^2}$$

Comparing coefficients of s^2,

$$A + C = 0$$

$$A = -C = -\frac{L}{R^2}$$

$$I(s) = -\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L}{R^2}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = -\frac{L}{R^2} + \frac{1}{R}t + \frac{L}{R^2}e^{-\left(\frac{R}{L}\right)t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(c) When unit impulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{1}{L}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{L}e^{-\left(\frac{R}{L}\right)t} \qquad \text{for } t > 0$$

Example 7.26 *At t = 0, unit pulse voltage of unit width is applied to a series RL circuit as shown in Fig. 7.60. Obtain an expression for i(t).*

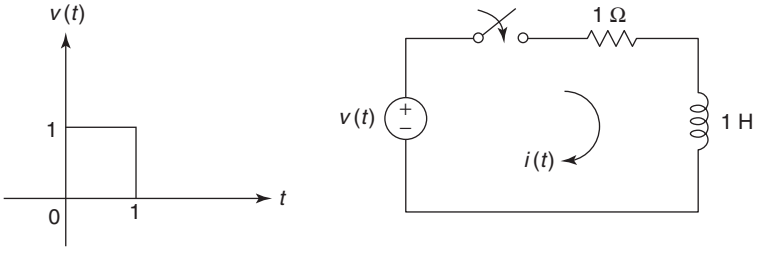


Fig. 7.60

Solution

$$v(t) = u(t) - u(t-1)$$

$$V(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

For $t > 0$, the transformed network is shown in Fig. 7.61.
Applying KVL to the mesh,

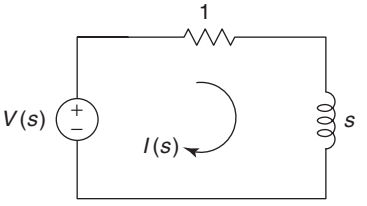


Fig. 7.61

$$V(s) - I(s) - sI(s) = 0$$

$$I(s) = \frac{V(s)}{s+1}$$

$$= \frac{1-e^{-s}}{s(s+1)}$$

$$= \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$

Taking inverse Laplace transform,

$$i(t) = u(t) - e^{-t}u(t) - u(t-1) + e^{-(t-1)}u(t-1)$$

$$= (1-e^{-t})u(t) - [1-e^{-(t-1)}]u(t-1) \qquad \text{for } t > 0$$

Example 7.27 *For the network shown in Fig. 7.62, determine the current i(t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

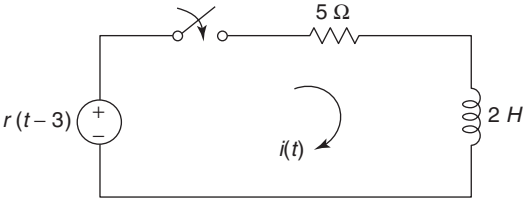


Fig. 7.62

Solution For $t > 0$, the transformed network is shown in Fig. 7.63.
Applying KVL to the mesh for $t > 0$,

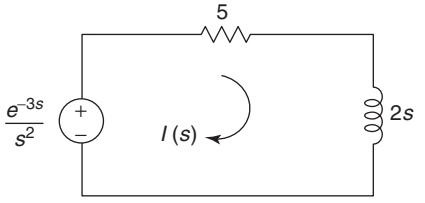


Fig. 7.63

$$\frac{e^{-3s}}{s^2} - 5\,I(s) - 2s\,I(s) = 0$$

$$5\,I(s) + 2s\,I(s) = \frac{e^{-3s}}{s^2}$$

$$I(s) = \frac{e^{-3s}}{s^2(2s+5)} = \frac{0.5\,e^{-3s}}{s^2(s+2.5)}$$

By partial-fraction expansion,

$$\frac{0.5}{s^2(s+2.5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2.5}$$

$$0.5 = As(s+2.5) + B(s+2.5) + Cs^2$$

$$= As^2 + 2.5As + Bs + 2.5B + Cs^2$$

$$= (A+C)s^2 + (2.5A+B)s + 2.5\,B$$

Comparing coefficients of s^2, s and s^0,

$$A + C = 0$$

$$2.5A + B = 0$$

$$2.5B = 0.5$$

Solving these equations,

$$A = -0.08$$
$$B = 0.2$$
$$C = 0.08$$
$$I(s) = e^{-3s}\left(-\frac{0.08}{s} + \frac{0.2}{s^2} + \frac{0.08}{s+2.5}\right)$$
$$= -0.08\frac{e^{-3s}}{s} + 0.2\frac{e^{-3s}}{s^2} + 0.08\frac{e^{-3s}}{s+2.5}$$

Taking inverse Laplace transform,

$$i(t) = -0.08u(t-3) + 0.2r(t-3) + 0.08e^{-2.5(t-3)}u(t-3)$$

Example 7.28 *Determine the expression for $v_L(t)$ in the network shown in Fig. 7.64. Find $v_L(t)$ when (i) $v_s(t) = \delta(t)$, and (ii) $v_s(t) = e^{-t}u(t)$.*

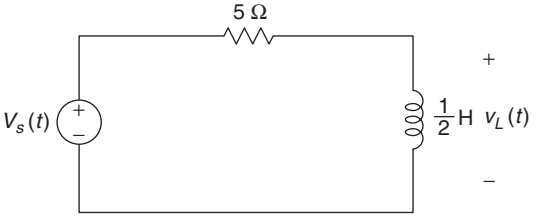


Fig. 7.64

Solution For $t > 0$, the transformed network is shown in Fig. 7.65.
By voltage-division rule,

$$V_L(s) = V_s(s) \times \frac{\frac{s}{2}}{\frac{s}{2}+5} = \frac{s}{s+10}V_s(s)$$

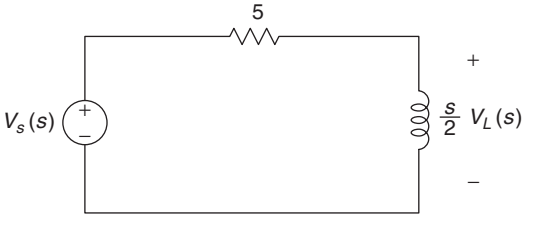


Fig. 7.65

(a) For impulse input,

$$V_s(s) = 1$$
$$V_L(s) = \frac{s}{s+10} = \frac{s+10-10}{s+10} = 1 - \frac{10}{s+10}$$

Taking inverse Laplace transform,

$$V_L(t) = \delta(t) - 10e^{-10t}u(t) \qquad \text{for } t > 0$$

(b) For $v_s(t) = e^{-t}u(t)$,

$$V_s(s) = \frac{1}{s+1}$$
$$V_L(s) = \frac{s}{(s+10)(s+1)}$$

By partial-fraction expansion,

$$V_L(s) = \frac{A}{s+10} + \frac{B}{s+1}$$

$$A = (s+10)V_L(s)\big|_{s=-10} = \frac{s}{s+1}\bigg|_{s=-10} = \frac{10}{9}$$

$$B = (s+1)V_L(s)\big|_{s=-1} = \frac{s}{s+10}\bigg|_{s=-1} = -\frac{1}{9}$$

$$V_L(s) = \frac{10}{9}\frac{1}{s+10} - \frac{1}{9}\frac{1}{s+1}$$

Taking inverse Laplace transform,

$$v_L(t) = \frac{10}{9}e^{-10t}u(t) - \frac{1}{9}e^{-t}u(t)$$

$$= \left(\frac{10}{9}e^{-10t} - \frac{1}{9}e^{-t}\right)u(t) \qquad \text{for } t > 0$$

Example 7.29 *For the network shown in Fig. 7.66, determine the current i (t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

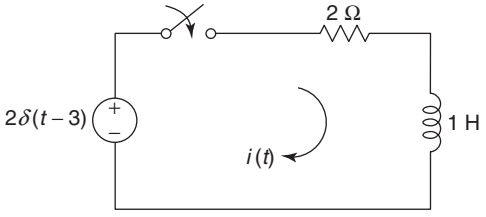


Fig. 7.66

Solution For $t > 0$, the transformed network is shown in Fig. 7.67.

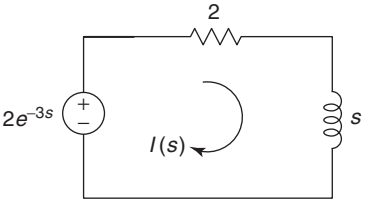


Fig. 7.67

Applying KVL to the mesh for $t > 0$,

$$2e^{-3s} - 2I(s) - sI(s) = 0$$

$$2I(s) + sI(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = 2e^{-2(t-3)}u(t-3) \qquad \text{for } t > 0$$

Example 7.30 *Determine the current i(t) in the network shown in Fig. 7.68, when the switch is closed at t = 0.*

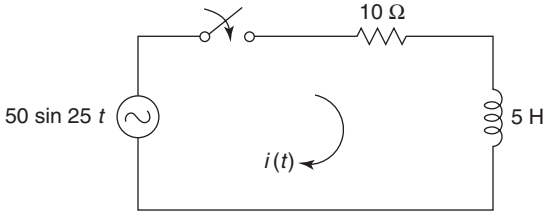


Fig. 7.68

Solution For $t > 0$, the transformed network is shown in Fig. 7.69.
Applying KVL to the mesh for $t > 0$,

$$\frac{1250}{s^2+625} - 10I(s) - 5sI(s) = 0$$

$$I(s) = \frac{250}{(s^2+625)(s+2)}$$

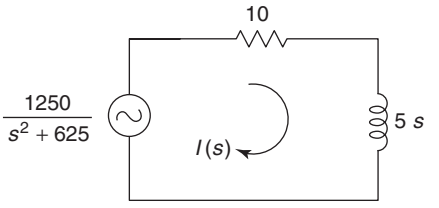


Fig. 7.69

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+625} + \frac{C}{s+2}$$

$$250 = (As+B)(s+2) + C(s^2+625)$$

$$= (A+C)s^2 + (2A+B)s + (2B+625C)$$

Comparing coefficients,

$$A + C = 0$$

$$2A + B = 0$$

$$2B + 625C = 250$$

Solving the equations,

$$A = -0.397$$

$$B = 0.795$$

$$C = 0.397$$

$$I(s) = \frac{-0.397s + 0.795}{s^2+625} + \frac{0.397}{s+2} = -\frac{0.397s}{s^2+625} + \frac{0.795}{s^2+625} + \frac{0.397}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = -0.397\cos 25t + 0.032\sin 25t + 0.397e^{-2t} \qquad \text{for } t > 0$$

Example 7.31 *Find impulse response of the current i(t) in the network shown in Fig. 7.70.*

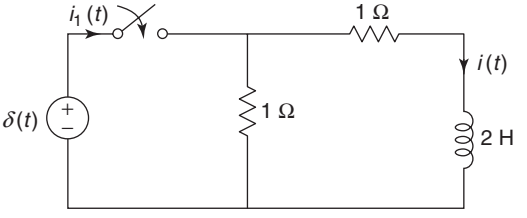


Fig. 7.70

Solution The transformed network is shown in Fig. 7.71.

$$Z(s) = \frac{1(2s+1)}{2s+1+1} = \frac{2s+1}{2s+2}$$

$$I_1(s) = \frac{V(s)}{Z(s)} = \frac{1}{\frac{2s+1}{2s+2}} = \frac{2s+2}{2s+1}$$

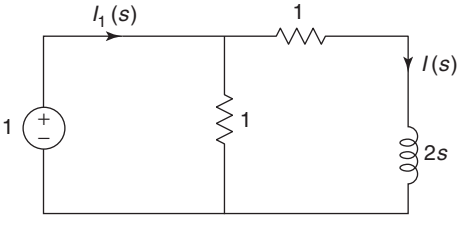


Fig. 7.71

By current-division rule,

$$I(s) = I_1(s) \times \frac{1}{2s+2} = \frac{1}{2s+2} \times \frac{2s+2}{2s+1} = \frac{1}{2s+1} = \frac{1}{2}\frac{1}{s+0.5}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}e^{-0.5t}u(t) \qquad \text{for } t > 0$$

Example 7.32 *The network shown in Fig. 7.72 is at rest for $t < 0$. If the voltage $v(t) = u(t)\cos t + A\delta(t)$ is applied to the network, determine the value of A so that there is no transient term in the current response i(t).*

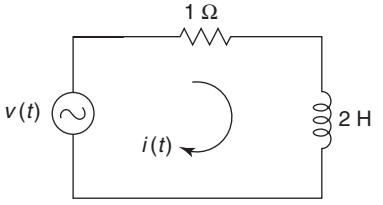


Fig. 7.72

$$v(t) = u(t)\cos t + A\delta(t)$$

$$V(s) = \frac{s}{s^2+1} + A$$

Solution For $t > 0$, the transformed network is shown in Fig. 7.73. Applying KVL to the mesh for $t > 0$,

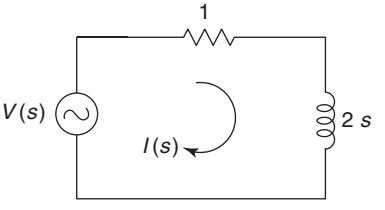


Fig. 7.73

$$V(s) = 2sI(s) + I(s) = \frac{s}{s^2+1} + A$$

$$I(s) = \frac{s + A(s^2+1)}{2\left(s+\frac{1}{2}\right)(s^2+1)} = \frac{K_1}{s+\frac{1}{2}} + \frac{K_2 s + K_3}{s^2+1}$$

The transient part of the response is given by the first term. Hence, for the transient term to vanish, $K_1 = 0$.

$$K_1 = \left(s+\frac{1}{2}\right)I(s)\Big|_{s=-\frac{1}{2}} = \frac{\frac{-1}{2} + A\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)}$$

When $K_1 = 0$

$$\frac{5}{4}A = \frac{1}{2}$$

$$A = \frac{2}{5} = 0.4$$

7.11 RESPONSE OF RC CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RC* circuit as shown in Fig. 7.74.

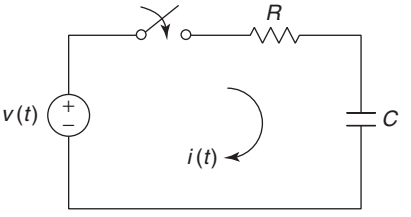


Fig. 7.74 *RC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.75.
Applying KVL to the mesh,

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{\frac{1}{Cs} + R} = \frac{sV(s)}{R\left(s + \frac{1}{RC}\right)}$$

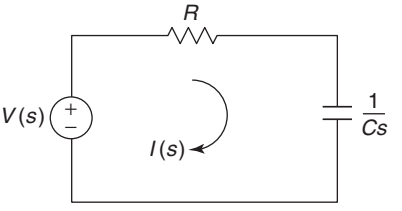


Fig. 7.75 *Transformed network*

(a) When unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{s \times \frac{1}{s}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{s \times \frac{1}{s^2}}{R\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{R}}{s\left(s + \frac{1}{RC}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}}$$

$$A = s\,I(s)\big|_{s=0} = \left.\frac{\frac{1}{R}}{s + \frac{1}{RC}}\right|_{s=0} = C$$

$$B = \left(s + \frac{1}{RC}\right) I(s)\Big|_{s=-\frac{1}{RC}} = \left.\frac{\frac{1}{R}}{s}\right|_{s=-\frac{1}{RC}} = -C$$

$$I(s) = \frac{C}{s} - \frac{C}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = C - Ce^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(c) When unit inpulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{s}{R\left(s + \frac{1}{RC}\right)} = \frac{s + \frac{1}{RC} - \frac{1}{RC}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left(1 - \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC}e^{-\frac{1}{RC}t}\right] \qquad \text{for } t > 0$$

Example 7.33 *A rectangular voltage pulse of unit height and T-seconds duration is applied to a series RC network at t = 0. Obtain the expression for the current i(t). Assume the capacitor to be initially uncharged.*

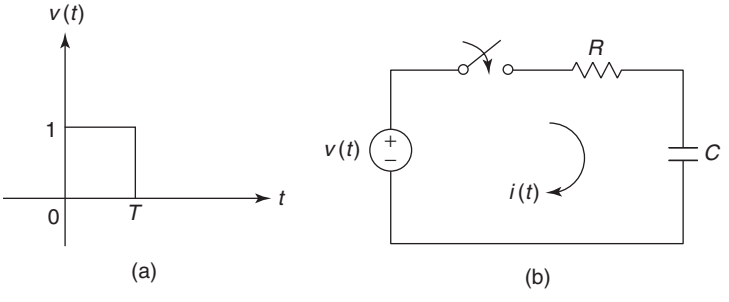


Fig. 7.76

Solution

$$v(t) = u(t) - u(t - T)$$

$$V(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1 - e^{-sT}}{s}$$

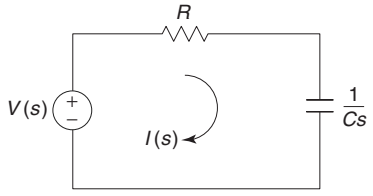


Fig. 7.77

For $t > 0$, the transformed network is shown in Fig. 7.77.
Applying KVL to the mesh for $t > 0$,

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}} = \frac{\frac{1}{R} s}{s + \frac{1}{RC}} V(s) = \frac{1 - e^{-sT}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{1}{s + \frac{1}{RC}} - \frac{e^{-sT}}{s + \frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[e^{-\left(\frac{1}{RC}\right)t} u(t) - e^{-\left(\frac{1}{RC}\right)(t-T)} u(t - T)\right] \qquad \text{for } t > 0$$

Example 7.34 *For the network shown in Fig. 7.78, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

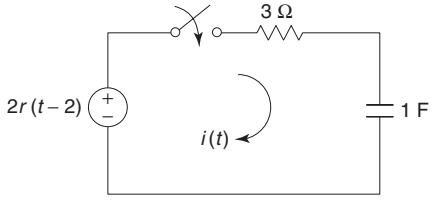


Fig. 7.78

Solution For $t > 0$, the transformed network is shown in Fig. 7.79.
Applying KVL to the mesh for $t > 0$,

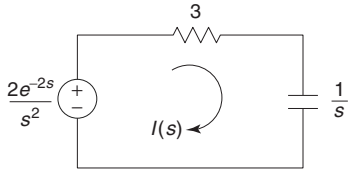


Fig. 7.79

$$\frac{2e^{-2s}}{s^2} - 3\, I(s) - \frac{1}{s} I(s) = 0$$

$$\left(3 + \frac{1}{s}\right) I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s^2\left(3 + \frac{1}{s}\right)} = \frac{0.67e^{-2s}}{s(s + 0.33)}$$

By partial-fraction expansion,

$$\frac{0.67}{s(s+0.33)} = \frac{A}{s} + \frac{B}{s+0.33}$$

$$A = \left.\frac{0.67}{s+0.33}\right|_{s=0} = 2$$

$$B = \left.\frac{0.67}{s}\right|_{s=-0.33} = -2$$

$$I(s) = e^{-2s}\left(\frac{2}{s} - \frac{2}{s+0.33}\right) = 2\frac{e^{-2s}}{s} - 2\frac{e^{-2s}}{s+0.33}$$

Taking inverse Laplace transform,

$$i(t) = 2u(t-2) - 2e^{-0.33(t-2)}u(t-2) \qquad \text{for } t > 0$$

Example 7.35 *For the network shown in Fig. 7.80, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

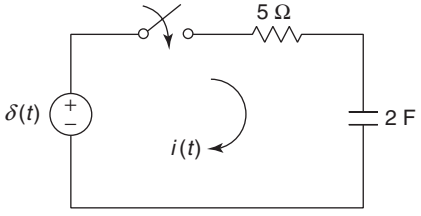


Fig. 7.80

Solution For $t > 0$, the transformed network is shown in Fig. 7.81.
Applying KVL to the mesh for $t > 0$,

$$1 - 5\,I(s) - \frac{1}{2s}I(s) = 0$$

$$\left(5 + \frac{1}{2s}\right)I(s) = 1$$

$$I(s) = \frac{1}{5 + \frac{1}{2s}}$$

$$= \frac{2s}{10s+1}$$

$$= \frac{0.2s}{s+0.1}$$

$$= \frac{0.2(s+0.1-0.1)}{s+0.1}$$

$$= 0.2\left(1 - \frac{0.1}{s+0.1}\right)$$

$$= 0.2 - \frac{0.02}{s+0.1}$$

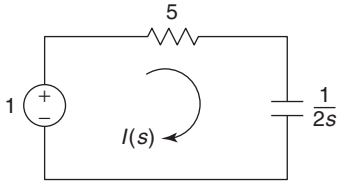


Fig. 7.81

Taking inverse Laplace transform,

$$i(t) = 0.2\delta(t) - 0.02\,e^{-0.1t}u(t)$$

Example 7.36 *For the network shown in Fig. 7.82, find the response $v_0(t)$.*

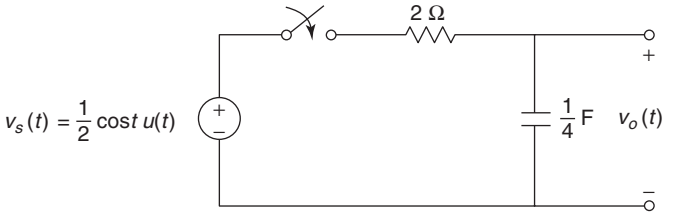


Fig. 7.82

Solution For $t > 0$, the transformed network is shown in Fig. 7.83.

$$V_s(s) = \frac{1}{2}\frac{s}{s^2+1}$$

By voltage-division rule,

$$V_o(s) = V_s(s) \times \frac{\frac{4}{s}}{2+\frac{4}{s}} = \frac{2V_s(s)}{s+2} = \frac{s}{(s^2+1)(s+2)}$$

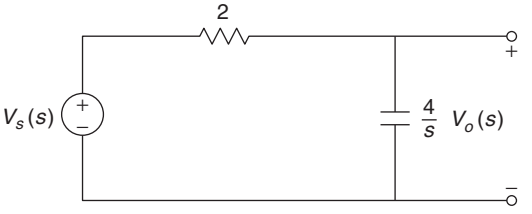


Fig. 7.83

By partial-fraction expansion,

$$V_o(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

$$s = (As+B)(s+2) + c(s^2+1)$$

$$s = (A+C)s^2 + (2A+B)s + (2B+C)$$

Comparing coefficient of s^2, s and s^0,

$$A + C = 0$$

$$2A + B = 1$$

$$2B + C = 0$$

Solving the equations,

$$A = 0.4, \quad B = 0.2, \quad C = -0.4$$

$$V_o(s) = \frac{0.4s+0.2}{s^2+1} - \frac{0.4}{s+2} = \frac{0.4s}{s^2+1} + \frac{0.2}{s^2+1} - \frac{0.4}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \qquad \text{for } t > 0$$

Example 7.37 *Find the impulse response of the voltage across the capacitor in the network shown in Fig. 7.84. Also determine response $v_c(t)$ for step input.*

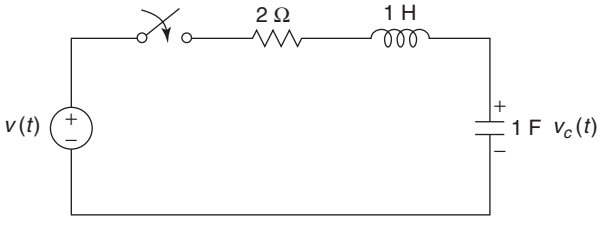


Fig. 7.84

Solution For $t > 0$, the transformed network is shown in Fig. 7.85.

By voltage-division rule,

$$V_c(s) = V(s) \times \frac{\frac{1}{s}}{2 + s + \frac{1}{s}}$$

$$= \frac{V(s)}{s^2 + 2s + 1} = \frac{V(s)}{(s+1)^2}$$

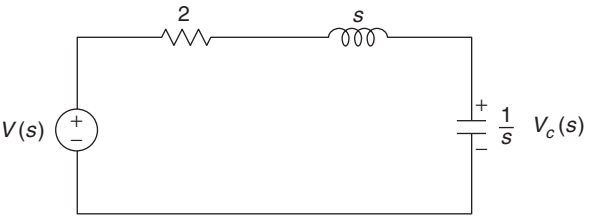


Fig. 7.85

(a) For impulse input,

$$V(s) = 1$$

$$V_c(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$v_c(t) = te^{-t}u(t)$ for $t > 0$

(b) For step input,

$$V(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(s+1)^2}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$

$$= A(s^2 + 2s + 1) + B(s^2 + s) + Cs$$

$$= (A + B)s^2 + (2A + B + C)s + A$$

Comparing coefficient of s^2, s^1 and s^0,

$$A = 1$$

$$A + B = 0$$

$$B = -A = -1$$

$$2A + B + C = 0$$

$$C = -2A - B = -2 + 1 = -1$$

$$V_c(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$

$$= (1 - e^{-t} - te^{-t})ut \qquad \text{for } t > 0$$

Example 7.38 *For the network shown in Fig. 7.86, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

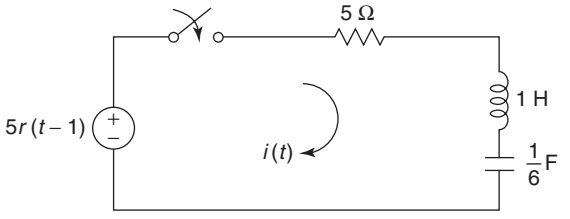


Fig. 7.86

Solution For $t > 0$, the transformed network is shown in Fig. 7.87.

Applying KVL to the mesh for $t > 0$,

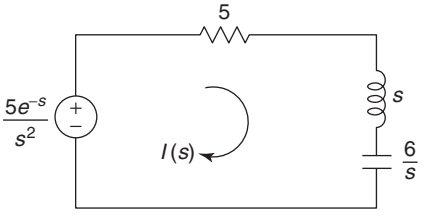


Fig. 7.87

$$\frac{5e^{-s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2+5s+6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right] = \frac{5}{6}\frac{e^{-s}}{s} + \frac{5}{3}\frac{e^{-s}}{s+3} - \frac{5}{2}\frac{e^{-s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1) \qquad \text{for } t > 0$$

Example 7.39 *For the network shown in Fig. 7.88, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions.*

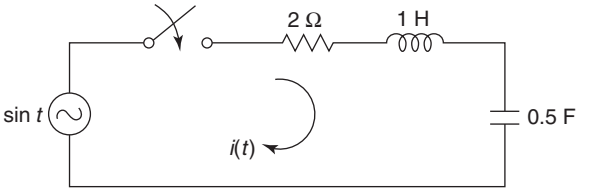


Fig. 7.88

Solution For $t > 0$, the transformed network is shown in Fig. 7.89.
Applying KVL to the mesh for $t > 0$,

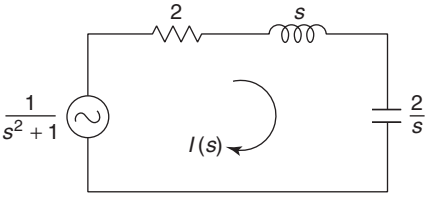


Fig. 7.89

$$\frac{1}{s^2+1} - 2I(s) - s\,I(s) - \frac{2}{s}I(s) = 0$$

$$\left(2 + s + \frac{2}{s}\right)I(s) = \frac{1}{s^2+1}$$

$$I(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing coefficients of s^3, s^2, s^1 and s^0,

$$A + C = 0$$

$$2A + B + D = 0$$

$$2A + 2B + C = 1$$

$$2B + D = 0$$

Solving these equations,

$$A = 0.2,\ B = 0.4,\ C = -0.2,\ D = -0.8$$

$$I(s) = \frac{0.2s+0.4}{s^2+1} - \frac{0.2s+0.8}{s^2+2s+2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2s+0.2+0.6}{(s+1)^2+(1)^2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2(s+1)}{(s+1)^2+1} - \frac{0.6}{(s+1)^2+1}$$

Taking inverse Laplace transform,

$$i(t) = 0.2\cos t + 0.4\sin t - 0.2\,e^{-t}\cos t - 0.6\,e^{-t}\sin t$$

$$= 0.2\cos t + 0.4\sin t - e^{-t}(0.2\cos t + 0.6\sin t) \qquad \text{for } t > 0$$

Example 7.40 *For the network shown in Fig. 7.90, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions in the network elements.*

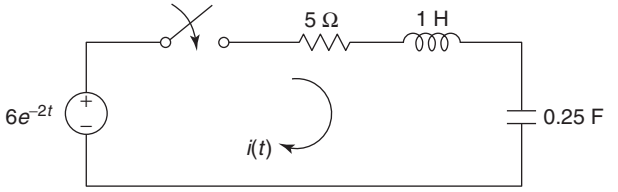


Fig. 7.90

Solution For $t > 0$, the transformed network is shown in Fig. 7.91.
Applying KVL to the mesh for $t > 0$,

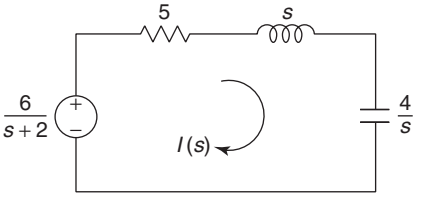


Fig. 7.91

$$\frac{6}{s+2} - 5I(s) - s\,I(s) - \frac{4}{s}I(s) = 0$$

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6s}{(s+2)(s^2+5s+4)}$$

$$= \frac{6s}{(s+2)(s+1)(s+4)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = (s+2)I(s)\,|_{s=-2} = \left.\frac{6s}{(s+1)(s+4)}\right|_{s=-2} = 6$$

$$B = (s+1)I(s)\,|_{s=-1} = \left.\frac{6s}{(s+2)(s+4)}\right|_{s=-1} = -2$$

$$C = (s+4)I(s)\,|_{s=-4} = \left.\frac{6s}{(s+2)(s+1)}\right|_{s=-4} = -4$$

$$I(s) = \frac{6}{s+2} - \frac{2}{s+1} - \frac{4}{s+4}$$

Taking inverse Laplace transform,

$$i(t) = 6e^{-2t}u(t) - 2e^{-t}u(t) - 4\,e^{-4t}u(t) \qquad \text{for } t > 0$$

Example 7.41 *The network shown has zero initial conditions. A voltage $v_i(t) = \delta(t)$ applied to two terminal network produces voltage $v_o(t) = [e^{-2t} + e^{-3t}]\,u(t)$. What should be $v_i(t)$ to give $v_o(t) = t\,e^{-2t}\,u(t)$?*

Fig. 7.92

Solution For $v_i(t) = \delta(t)$,

$$V_i(s) = 1$$
$$v_o(t) = [e^{-2t} + e^{-3t}]u(t)$$
$$V_o(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

System function $H(s) = \dfrac{V_o(s)}{V_i(s)}$

$$= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)} \qquad \ldots(i)$$

For $v_o(t) = te^{-2t}u(t)$,

$$V_o(s) = \frac{1}{(s+2)^2}$$

From Eq. (i),

$$V_i(s) = \frac{V_o(s)}{H(s)} = \frac{1}{(s+2)^2} \times \frac{(s+2)(s+3)}{2s+5} = \frac{(s+3)}{2(s+2.5)(s+2)}$$

By partial-fraction expansion,

$$V_i(s) = \frac{A}{s+2} + \frac{B}{s+2.5}$$
$$A = 1$$
$$B = -0.5$$
$$V_i(s) = \frac{1}{s+2} - \frac{0.5}{s+2.5}$$

Taking inverse Laplace transform,

$$v_i(t) = e^{-2t} - 0.5e^{-2.5t} \qquad \text{for } t > 0$$

Example 7.42 *A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal.*

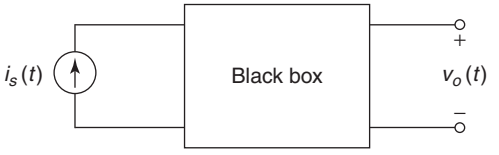


Fig. 7.93

Solution

$$v_o(t) = 2e^{-t} - e^{-3t}$$
$$V_o(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t) = \delta(t)$,

$$I_s(s) = 1$$

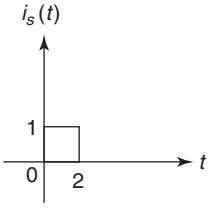


Fig. 7.94

$$V_o(s) = Z(s)\, I_s(s) \qquad \ldots(i)$$

$$Z(s) = \frac{V_o(s)}{I_s(s)} = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t)$ is a pulse of 1 A height and a duration of 2 seconds then,

$$i_s(t) = u(t) - u(t-2)$$

$$I_s(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

From Eq. (i),

$$V_o(s) = \left[\frac{2}{s+1} - \frac{1}{s+3}\right]\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{2}{s(s+1)} - \frac{1}{s(s+3)} - \frac{2e^{-2s}}{s(s+1)} + \frac{e^{-2s}}{s(s+3)}$$

$$= 2\left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right] - 2e^{-2s}\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{e^{-2s}}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace transform,

$$v(t) = 2[u(t) - e^{-t}u(t)] - \frac{1}{3}[u(t) - e^{-3t}u(t)] - 2[u(t-2) - e^{-(t-2)}u(t-2)] + \frac{1}{3}[u(t-2) - e^{-3(t-2)}u(t-2)]$$

for $t > 0$

Exercises

7.1 For the network shown in Fig 7.95, the switch is closed at $t = 0$. Find the current i1(t) for $t > 0$.

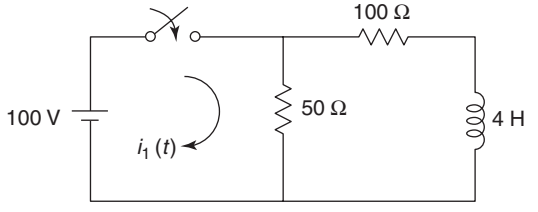


Fig. 7.95

$[i_1(t) = 3 - e^{-25\,t}]$

7.2 Determine the current $i(t)$ in the network of Fig. 7.96, when the switch is closed at $t = 0$. The inductor is initially unenergized.

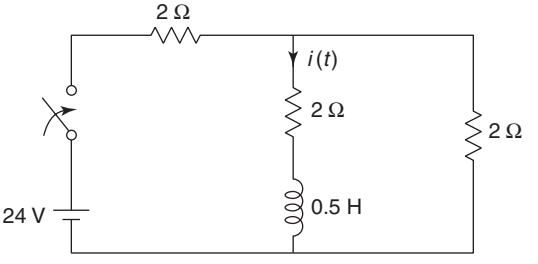


Fig. 7.96

$[i(t) = 4(1 - e^{-6t})]$

7.3 In the network of Fig. 7.97, after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.

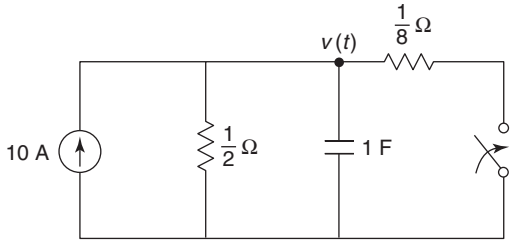


Fig. 7.97

$[v(t) = 1 + 4\ e^{-10t}]$

7.4 The circuit of Fig. 7.98, has been in the condition shown for a long time. At $t = 0$, switch is closed. Find $v(t)$ for $t > 0$.

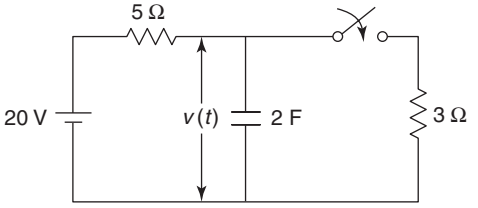


Fig. 7.98

$[v(t) = 7.5 + 12.5\ e^{-(4/15)t}]$

7.5 Figure 7.99 shows a circuit which is in the steady-state with the switch open. At $t = 0$, the switch is closed. Determine the current $i(t)$. Find its value at $t = 0.114$ μ seconds.

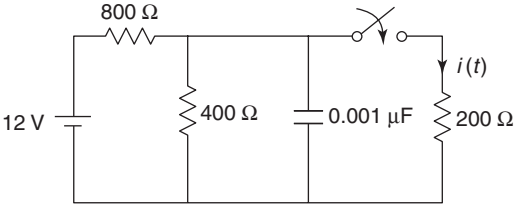


Fig. 7.99

$[i(t) = 0.00857 + 0.01143\ e^{-8.75 \times 10^6 t}, 0.013\ \text{A}]$

7.6 Find $i(t)$ for the network shown in Fig. 7.100.

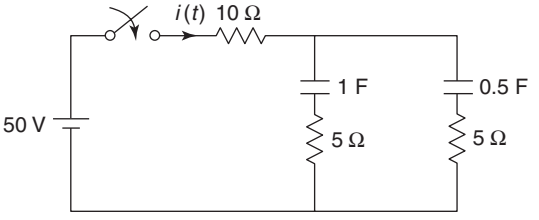


Fig. 7.100

$[i(t) = 0.125\ e^{-0.308t} + 3.875\ e^{-0.052t}]$

7.7 Determine $v(t)$ in the network of Fig. 7.101 where $i_L(0^-) = 15$ A and $v_c(0^-) = 5$ V.

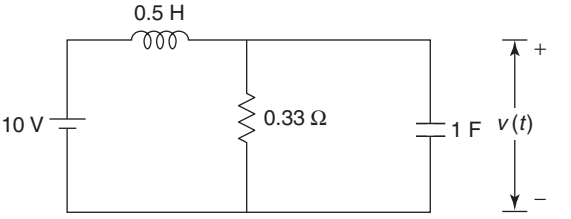


Fig. 7.101

$[v(t) = 10 - 10e^{-t} + 5e^{-2t}]$

7.8 The network shown in Fig. 7.102 has acquired steady state with the switch at position 1 for $t < 0$. At $t = 0$, the switch is thrown to the position 2. Find $v(t)$ for $t > 0$.

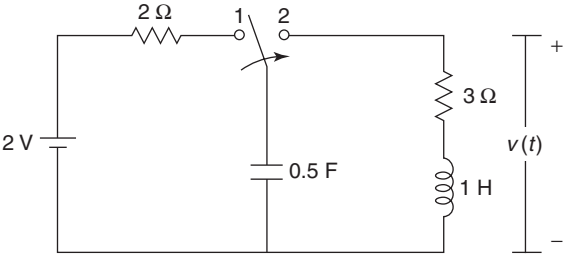


Fig. 7.102

$[v(t) = 4e^{-t} - 2\ e^{-2t}]$

7.9 In the network shown in Fig. 7.103, the switch is closed at $t = 0$. Find current $i_1(t)$ for $t > 0$.

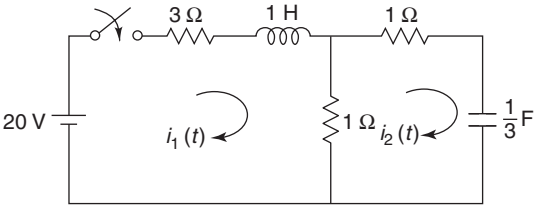


Fig. 7.103

$[i_1(t) = 5 + 5e^{-2t} - 10e^{-3t}]$

7.10 In the network shown in Fig. 7.104, the switch is closed at $t = 0$. Find the current through the 30 Ω resistor.

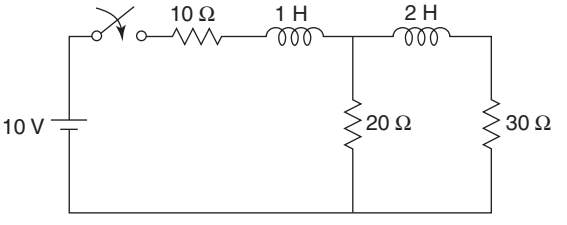


Fig. 7.104

$[i(t) = 0.1818 - 0.265\ e^{-13.14t} + 0.083\ e^{-41.86t}]$

7.11 The network shown in Fig. 7.105 is in steady state with s_1 closed and s_2 open. At $t = 0$, s_1 is opened and s_2 is closed. Find the current through the capacitor.

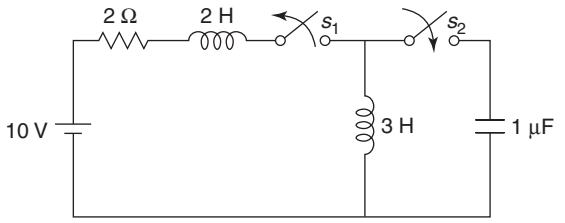


Fig. 7.105

$[i(t) = 5 \cos (0.577 \times 10^3\, t)]$

7.12 In the network shown in Fig. 7.106, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

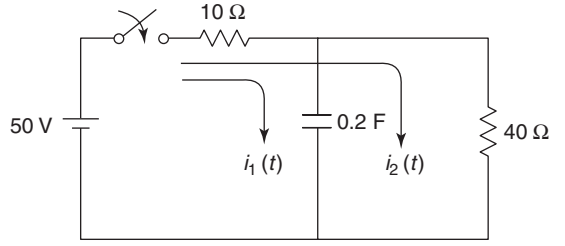


Fig. 7.106

$[i_1(t) = 5\, e^{-0.625t},\ i_2(t) = 1 - e^{-0.625t}\ ]$

7.13 For the network shown in Fig. 7.107, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

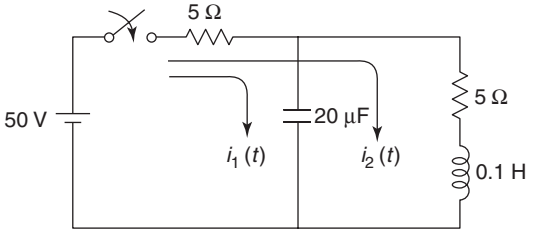


Fig. 7.107

$$\begin{bmatrix} i_1(t) = 0.101e^{-100.5t} + 10.05e^{-9949.5t} \\ i_2(t) = 5 - 5.05e^{-100.5t} + 0.05e^{-9949.5t} \end{bmatrix}$$

7.14 In the network shown in Fig. 7.108, the switch is opened at $t = 0$, the steady state having been established previously. Find $i(t)$ for $t > 0$.

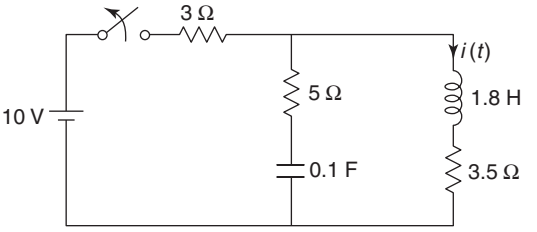


Fig. 7.108

$[i(t) = 1.5124e^{-2.22t} + 3.049e^{-2.5t}]$

7.15 Find the current $i(t)$ in the network of Fig. 7.109, if the switch is closed at $t = 0$. Assume initial conditions to be zero.

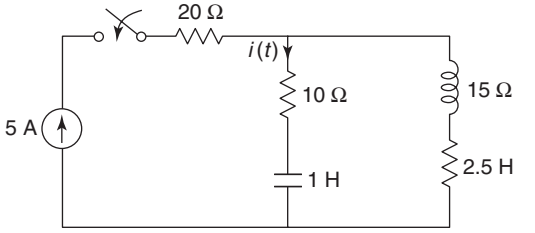


Fig. 7.109

$[i(t) = 3 + 0.57e^{-7.14\,t}]$

7.16 In the network shown in Fig. 7.110, find the voltage $v(t)$ for $t > 0$.

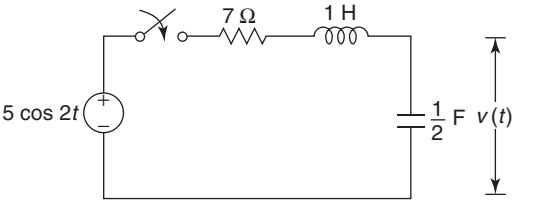


Fig. 7.110

$$\left[v(t) = -\frac{6}{5}e^{-t} + \frac{9}{10}e^{-6t} + \frac{3}{10}\cos 2t + \frac{21}{10}\sin 2t \right]$$

7.17 For the network shown in Fig. 7.111, determine $v(t)$ when the input is

(i) an impulse function $[e^{-t}\, u(t)]$

(ii) $i(t) = 4e^{-t}\, u(t)$ $[4t\, e^{-t}\, u(t)]$

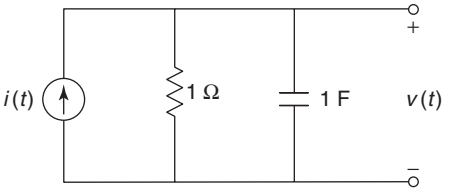


Fig. 7.111

7.18 For a unit-ramp input shown in Fig. 7.112, find the response $v_c(t)$ for $t > 0$.

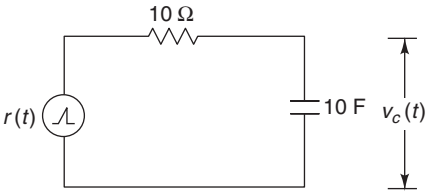


Fig. 7.112

$[v_c(t) = -100\ u(t) + 100e^{-0.01t}\ u(t) + tu(t)]$

Objective-Type Questions

7.1 If the Laplace transform of the voltage across a capacitor of value $\frac{1}{2}$ F is

$$V_c(s) = \frac{1}{s^2 + 1}$$

the value of the current through the capacitor at $t = 0^+$ is

(a) 0 (b) 2 A

(c) $\frac{1}{2}$ A (d) 1 A

7.2 The response of an initially relaxed linear constant parameter network to a unit impulse applied at $t = 0$ is $4\,e^{-2t}\,u(t)$. The response of this network to a unit-step function will be

(a) $2[1 - e^{-2t}]\,u(t)$

(b) $4[e^{-t} - e^{-2t}]\,u(t)$

(c) $\sin 2t$

(d) $(1 - 4\,e^{-4t})\,u(t)$

7.3 The Laplace transform of a unit-ramp function starting at $t = a$ is

(a) $\frac{1}{(s+a)^2}$ (b) $\frac{e^{-as}}{(s+a)^2}$

(c) $\frac{e^{-as}}{s^2}$ (d) $\frac{a}{s^2}$

7.4 The Laplace transform of $e^{at} \cos \alpha t$ is equal to

(a) $\frac{s-\alpha}{(s-\alpha)^2 + \alpha^2}$

(b) $\frac{s+\alpha}{(s-\alpha)^2 + \alpha^2}$

(c) $\frac{1}{(s-\alpha)^2}$

(d) none of the above

7.5 The circuit shown in Fig. 7.113 has initial current $i(0^-) = 1$ A through the inductor and an initial voltage $v_c(0^-) = -1$ V across the capacitor. For input $v(t) = u(t)$, the Laplace transform of the current $i(t)$ for $t \geq 0$ is

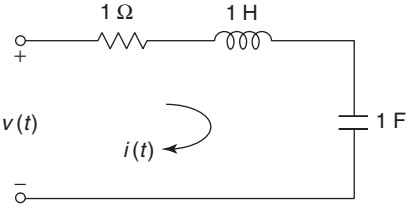


Fig. 7.113

(a) $\frac{s}{s^2 + s + 1}$ (b) $\frac{s+2}{s^2 + s + 1}$

(c) $\frac{s-2}{s^2 + s + 1}$ (d) $\frac{s-2}{s^2 + s + 1}$

7.6 A square pulse of 3 volts amplitude is applied to an *RC* circuit shown in Fig. 7.114. The capacitor is initially uncharged. The output voltage v_0 at time $t = 2$ seconds is

(a) 3 V (b) –3 V

(c) 4 V (d) –4 V

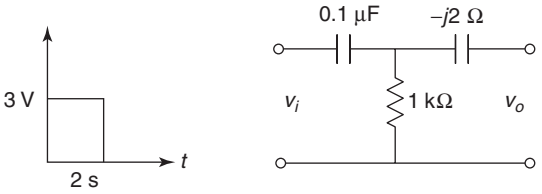


Fig. 7.114

7.7 A 2 mH inductor with some initial current can be represented as shown in Fig. 7.115. The value of the initial current is

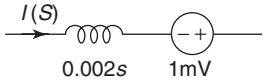


Fig. 7.115

(a) 0.5 A (b) 2 A
(c) 1 A (d) 0

7.8 A current impulse 5 $\delta(t)$ is forced through a capacitor C. The voltage $v_c(t)$ across the capacitor is given by

(a) $5t$ (b) $5u(t) - C$
(c) $\frac{5}{C}t$ (d) $\frac{5u(t)}{C}$

7.9 In the circuit shown in Fig. 7.116, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be

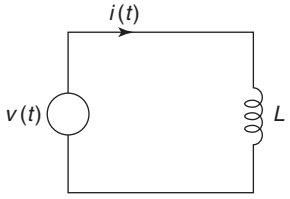


Fig. 7.116

(a) a constant voltage
(b) a linearly increasing voltage
(c) an ideal impulse
(d) as exponential increasing voltage

7.10 When a unit-impulse voltage is applied to an inductor of 1 H, the energy supplied by the source is

(a) ∞ (b) 1 J
(c) $\frac{1}{2}$ J (d) 0

Answers to Objective-Type Questions

7.1 (c) 7.2 (a) 7.3 (c) 7.4 (a) 7.5 (b) 7.6 (b)
7.7 (a) 7.8 (d) 7.9 (c) 7.10 (c)

8

Network Functions

8.1 INTRODUCTION

A network function gives the relation between currents or voltages at different parts of the network. It is broadly classified as *driving point* and *transfer function*. It is associated with terminals and ports.

Any network may be represented schematically by a rectangular box. Terminals are needed to connect any network to any other network or for taking some measurements. Two such associated terminals are called *terminal pair* or *port*. If there is only one pair of terminals in the network, it is called a one-port network. If there are two pairs of terminals, it is called a two-port network. The port to which energy source is connected is called the *input port*. The port to which load is connected is known as the *output port*. One such network having only one pair of terminals (1 – 1′) is shown in Fig. 8.1 (a) and is called *one-port network*. Figure 8.1 (b) shows a two-port network with two pairs of terminals. The terminals 1 – 1′ together constitute a port. Similarly, the terminals 2 – 2′ constitute another port.

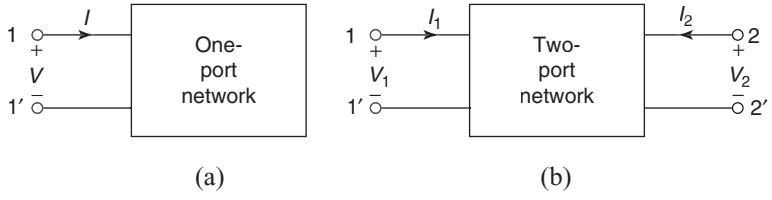


Fig. 8.1 *(a) One-port network (b) Two-port network*

A voltage and current are assigned to each of the two ports. V_1 and I_1 are assigned to the input port, whereas V_2 and I_2 are assigned to the output port. It is also assumed that currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2 respectively.

8.2 DRIVING-POINT FUNCTIONS

If excitation and response are measured at the same ports, the network function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function (and its reciprocal) can be defined.

1. ***Driving-point Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at the same port. It is denoted by $Z(s)$.

$$Z(s) = \frac{V(s)}{I(s)}$$

2. ***Driving-point Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at the same port. It is denoted by $Y(s)$.

$$Y(s) = \frac{I(s)}{V(s)}$$

For a two-port network, the driving-point impedance function and driving-point admittance function at port 1 are

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

Similarly, at port 2,

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

8.3 TRANSFER FUNCTIONS

The transfer function is used to describe networks which have at least two ports. It relates a voltage or current at one port to the voltage or current at another port. These functions are also defined as the ratio of a response transform to an excitation transform. Thus, there are four possible forms of transfer functions.

1. ***Voltage Transfer Function*** It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$.

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_{21}(s) = \frac{V_1(s)}{V_2(s)}$$

2. ***Current Transfer Function*** It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$.

$$\alpha_{12}(s) = \frac{I_2(s)}{I_1(s)}$$

$$\alpha_{21}(s) = \frac{I_1(s)}{I_2(s)}$$

3. *Transfer Impedance Function* It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted by $Z(s)$.

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)}$$

$$Z_{21}(s) = \frac{V_1(s)}{I_2(s)}$$

4. *Transfer Admittance Function* It is defined as the ratio of the current transform at one port to the voltage transform at another port. It denoted by Y (s).

$$Y_{12}(s) = \frac{I_2(s)}{V_1(s)}$$

$$Y_{21}(s) = \frac{I_1(s)}{V_2(s)}$$

Example 8. 1 *Determine the driving-point impedance function of a one-port network shown in Fig. 8.2.*

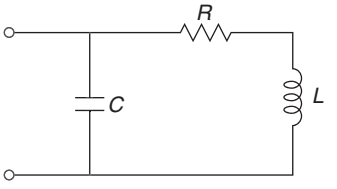


Fig. 8.2

Solution The transformed network is shown in Fig. 8.3.

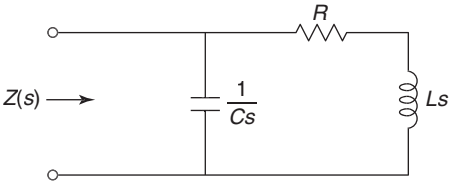


Fig. 8.3

$$Z(s) = \frac{\frac{1}{Cs}(R+Ls)}{\frac{1}{Cs}+(R+Ls)} = \frac{R+Ls}{LCs^2+RCs+1} = \frac{1}{C}\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example 8.2 *Determine the driving-point impedance of the network shown in Fig. 8.4.*

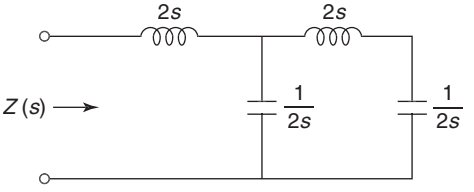


Fig. 8.4

Solution

$$Z(s) = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{1}{2s}+2s+\frac{1}{2s}} = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{2+4s^2}{2s}} = 2s + \frac{2s+\frac{1}{2s}}{2+4s^2} = \frac{4s+8s^3+2s+\frac{1}{2s}}{2+4s^2} = \frac{16s^4+12s^2+1}{8s^3+4s}$$

Example 8.3 *Determine the driving-point impedance of the network shown in Fig. 8.5.*

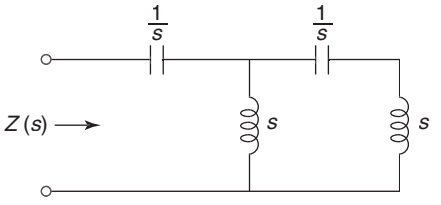


Fig. 8.5

Solution

$$Z(s)=\frac{1}{s}+\frac{s\left(\frac{1}{s}+s\right)}{s+\frac{1}{s}+s}=\frac{1}{s}+\frac{(1+s^2)s}{2s^2+1}=\frac{1}{s}+\frac{s+s^3}{2s^2+1}=\frac{s^4+3s^2+1}{2s^3+s}$$

Example 8.4 *Find the driving-point admittance function of the network shown in Fig. 8.6.*

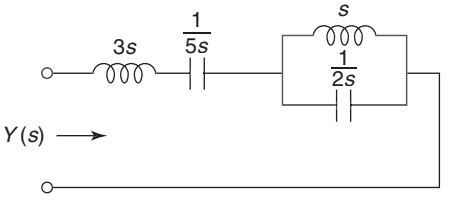


Fig. 8.6

Solution

$$Z(s)=3s+\frac{1}{5s}+\frac{s\left(\frac{1}{2s}\right)}{s+\frac{1}{2s}}=3s+\frac{1}{5s}+\frac{s}{2s^2+1}=\frac{30s^4+15s^2+2s^2+1+5s^2}{5s\,(2s^2+1)}=\frac{30s^4+22s^2+1}{5s\,(2s^2+1)}$$

$$Y(s)=\frac{1}{Z(s)}=\frac{5s\,(2s^2+1)}{30s^4+22s^2+1}$$

Example 8.5 *Find the transfer impedance function $Z_{12}(s)$ for the network shown in Fig. 8.7.*

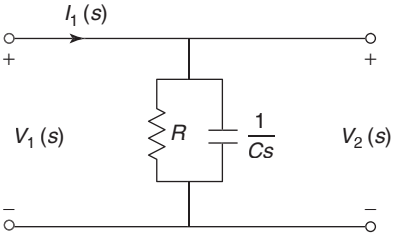


Fig. 8.7

Solution

$$V_2(s) = I_1(s)\frac{R\left(\frac{1}{Cs}\right)}{R+\frac{1}{Cs}}$$

$$\frac{V_2(s)}{I_1(s)} = \frac{R}{RCs+1}$$

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{C\left(s+\frac{1}{RC}\right)}$$

Example 8.6 *Find voltage transfer function of the two-port network shown in Fig. 8.8.*

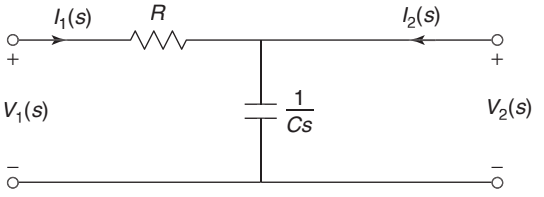


Fig. 8.8

Solution By voltage division rule,

$$V_2(s) = V_1(s)\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = V_1(s)\frac{1}{RCs+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Voltage transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

8.4 ANALYSIS OF LADDER NETWORKS

The network functions of a ladder network can be obtained by a simple method. This method depends upon the relationships that exist between the branch currents and node voltages of the ladder network. Consider the network shown in Fig. 8.9 where all the impedances are connected in series branches and all the admittances are connected in parallel branches.

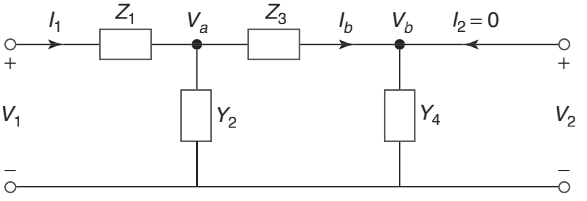


Fig. 8.9 *Ladder network*

Analysis is done by writing the set of equations. In writing these equations, we begin at the port 2 of the ladder and work towards the port 1.

$$V_b = V_2$$
$$I_b = Y_4\, V_2$$
$$V_a = Z_3\, I_b + V_2 = (Z_3 Y_4 + 1)\, V_2$$
$$I_1 = Y_2\, V_a + I_b = [Y_2\,(Z_3\, Y_4 + 1) + Y_4]\, V_2$$
$$V_1 = Z_1\, I_1 + V_a = [Z_1\,\{Y_2\,(Z_3\, Y_4 + 1) + Y_4\} + (Z_3\, Y_4 + 1)]\, V_2$$

Thus, each succeeding equation takes into account one new impedance or admittance. Except the first two equations, each subsequent equation is obtained by multiplying the equation just preceding it by imittance (either impedance or admittance) that is next down the line and then adding to this product the equation twice preceding it. After writing these equations, we can obtain any network function.

Example 8.7 *For the network shown in Fig. 8.10, determine transfer function* $\frac{V_2}{V_1}$.

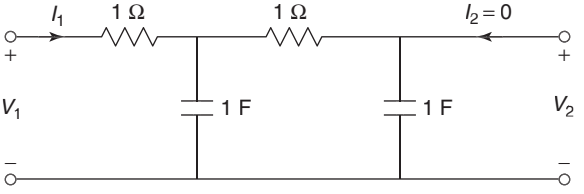


Fig. 8.10

Solution The transformed network is shown in Fig. 8.11.

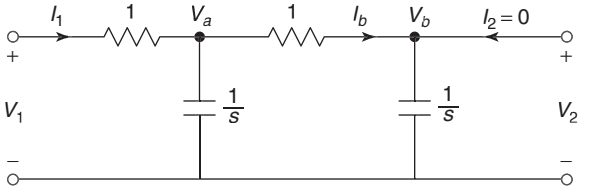


Fig. 8.11

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_a = 1I_b + V_2$$

$$= sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

$$V_1 = 1I_1 + V_a = (s^2 + 2s)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 8.8 *For the network shown in Fig. 8.12, determine the voltage transfer function* $\frac{V_2}{V_1}$.

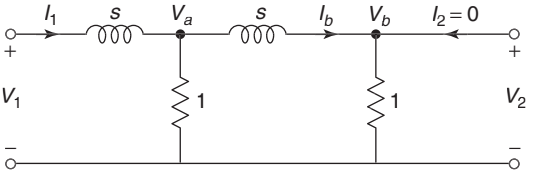


Fig. 8.12

Solution

$$V_b = V_2$$

$$I_b = \frac{V_2}{1} = V_2$$

$$V_a = sI_b + V_2 = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{1} + I_b = (s+1)V_2 + V_2 = (s+2)V_2$$

$$V_1 = sI_1 + V_a = s(s+2)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 8.9 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$ *and* $\frac{V_2}{I_1}$ *for the network shown in Fig. 8.13.*

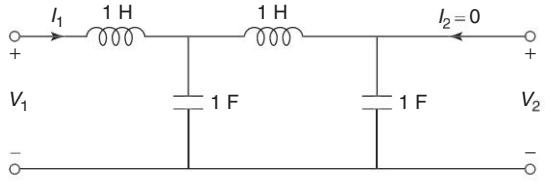


Fig. 8.13

Solution The transformed network is shown in Fig. 8.14.

$$V_b = V_2$$
$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$
$$V_a = sI_b + V_2 = s(sV_2) + V_2 = (s^2 + 1)V_2$$

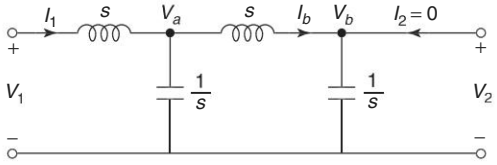


Fig. 8.14

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s^2 + 1)V_2 + sV_2 = (s^3 + 2s)V_2$$
$$V_1 = sI_1 + V_a = s(s^3 + 2s)V_2 + (s^2 + 1)V_2 = (s^4 + 2s^2 + s^2 + 1)V_2 = (s^4 + 3s^2 + 1)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{s^4 + 3s^2 + 1}{s^3 + 2s}$$
$$\frac{V_2}{V_1} = \frac{1}{s^4 + 3s^2 + 1}$$
$$\frac{V_2}{I_1} = \frac{1}{s^3 + 2s}$$

Example 8.10 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$, *and* $\frac{V_2}{I_1}$ *for the network in Fig. 8.15.*

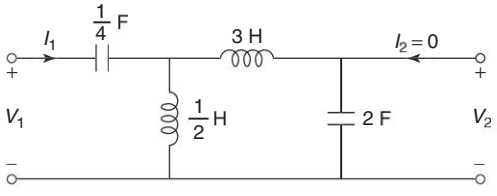


Fig. 8.15

Solution The transformed network is shown in Fig. 8.16.

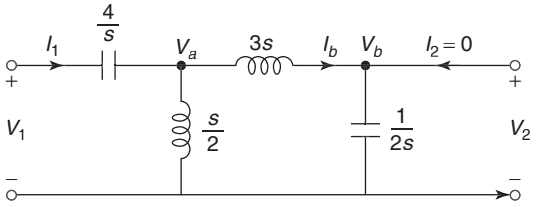


Fig. 8.16

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{2s}} = 2sV_2$$

$$V_a = 3sI_b + V_2 = 3s(2sV_2) + V_2 = (6s^2 + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{s}{2}} + I_b = \frac{2}{s}(6s^2 + 1)V_2 + 2sV_2 = \left(\frac{14s^2 + 2}{s}\right)V_2$$

$$V_1 = \frac{4}{s}I_1 + V_a = \frac{4}{s}\left(\frac{14s^2 + 2}{s}\right)V_2 + (6s^2 + 1)V_2 = \left(\frac{6s^4 + 57s^2 + 8}{s^2}\right)V_2$$

Hence,

$$\frac{V_1}{I_1} = \frac{6s^4 + 57s^2 + 8}{14s^3 + 2s}$$

$$\frac{V_2}{V_1} = \frac{s^2}{6s^4 + 57s^2 + 8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2 + 2}$$

Example 8.11 *For the ladder network of Fig. 8.17, find the driving point-impedance at the 1 – 1′ terminal with 2 – 2′ open.*

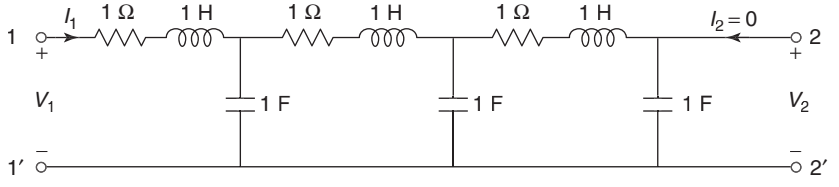


Fig. 8.17

Solution The transformed network is shown in Fig. 8.18.

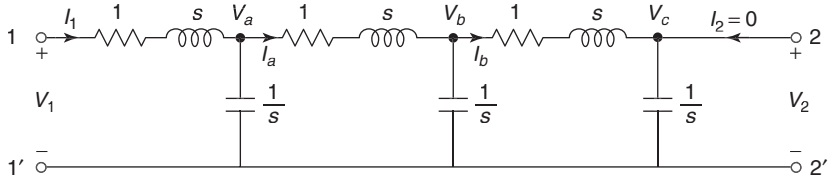


Fig. 8.18

$$V_c = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_b = (s+1)\,I_b + V_2 = (s+1)sV_2 + V_2 = (s^2 + s + 1)V_2$$

$$I_a = \frac{V_b}{\frac{1}{s}} + I_b = sV_b + I_b = s(s^2 + s + 1)V_2 + sV_2 = (s^3 + s^2 + 2s)V_2$$

$$V_a = (s+1)\,I_a + V_b = (s+1)(s^3 + s^2 + 2s)V_2 + (s^2 + s + 1)V_2 = (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(s^4 + 2s^3 + 4s^2 + 3s + 1)V_2 + (s^3 + s^2 + 2s)V_2$$

$$= (s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2$$

$$V_1 = (s+1)\,I_1 + V_a = (s+1)(s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2 + (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$= (s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1)V_2$$

Hence, $$Z_{11} = \frac{V_1}{I_1} = \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

Example 8.12 *Determine the voltage transfer function* $\frac{V_2}{V_1}$ *for the network shown in Fig. 8.19.*

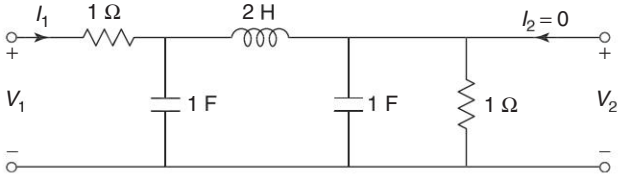


Fig. 8.19

Solution The transformed network is shown in Fig. 8.20.

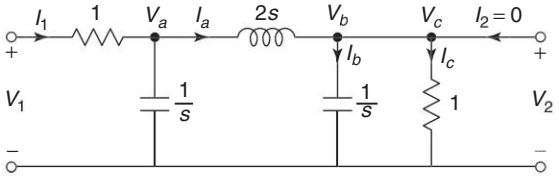


Fig. 8.20

$$V_c = V_b = V_2$$

$$I_a = I_b + I_c = \frac{V_2}{\frac{1}{s}} + \frac{V_2}{1} = sV_2 + V_2 = (s+1)V_2$$

$$V_a = 2s\,I_a + V_2 = 2s(s+1)V_2 + V_2$$

$$= (2s^2 + 2s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(2s^2 + 2s + 1)V_2 + (s+1)V_2 = (2s^3 + 2s^2 + 2s + 1)V_2$$

$$V_1 = 1I_1 + V_a = (2s^3 + 2s^2 + 2s + 1)V_2 + (2s^2 + 2s + 1)V_2 = (2s^3 + 4s^2 + 4s + 2)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{2s^3 + 4s^2 + 4s + 2}$$

Example 8.13 *For the network shown in Fig. 8.21, determine the transfer function* $\frac{I_2}{V_1}$.

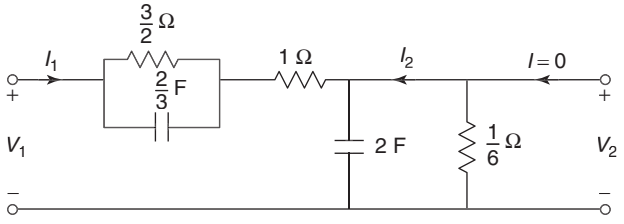


Fig. 8.21

Solution The transformed network is shown in Fig. 8.22.

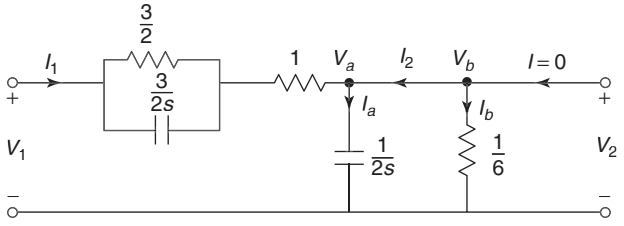


Fig. 8.22

$$V_b = V_a = V_2$$

$$I_1 = I_a + I_b = \frac{V_2}{\frac{1}{2s}} + \frac{V_2}{\frac{1}{6}} = 2sV_2 + 6V_2 = (2s+6)V_2$$

$$V_1 = \left(\frac{\frac{3}{2}\times\frac{3}{2s}}{\frac{3}{2}+\frac{3}{2s}}+1\right)I_1 + V_2 = \left(\frac{9}{6s+6}+1\right)I_1 + V_2 = \frac{(6s+15)}{6s+6}(2s+6)V_2 + V_2 = \left[\frac{(2s+5)(s+3)}{(s+1)}+1\right]V_2$$

$$= \left[\frac{(2s+5)(s+3)+(s+1)}{s+1}\right]V_2 = \left(\frac{2s^2+6s+5s+15+s+1}{s+1}\right)V_2 = \left(\frac{2s^2+12s+16}{s+1}\right)V_2$$

$$= \frac{2(s^2+6s+8)}{s+1}V_2 = \frac{2(s+4)(s+2)}{s+1}V_2$$

Also, $I_2 = -I_b = -6V_2$

$$\frac{I_2}{V_1} = -\frac{3(s+1)}{(s+4)(s+2)}$$

Example 8.14 *For the network shown in Fig. 8.23, compute* $\alpha_{12}(s) = \frac{I_2}{I_1}$ *and* $Z_{12}(s) = \frac{V_2}{I_1}$.

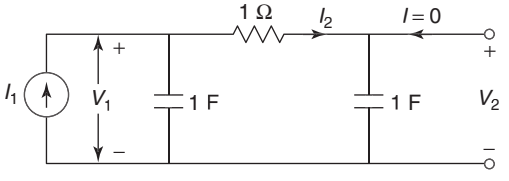


Fig. 8.23

Solution The transformed network is shown in Fig. 8.24.

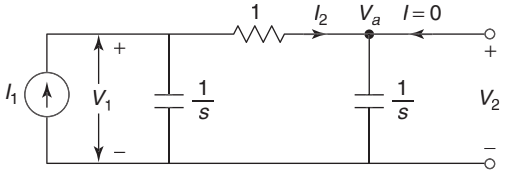


Fig. 8.24

$$V_a = V_2$$

$$I_2 = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_1 = 1I_2 + V_a = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_1}{\frac{1}{s}} + I_2 = sV_1 + I_2 = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

Hence, $\alpha_{12}(s) = \dfrac{I_2}{I_1} = \dfrac{1}{s+2}$

and $Z_{12}(s) = \dfrac{V_2}{I_1} = \dfrac{1}{s^2+2s}$

Example 8.15 *Determine the voltage ratio* $\dfrac{V_2}{V_1}$, *current ratio* $\dfrac{I_2}{I_1}$, *transfer impedance* $\dfrac{V_2}{I_1}$ *and driving- point impedance* $\dfrac{V_1}{I_1}$ *for the network shown in Fig. 8.25.*

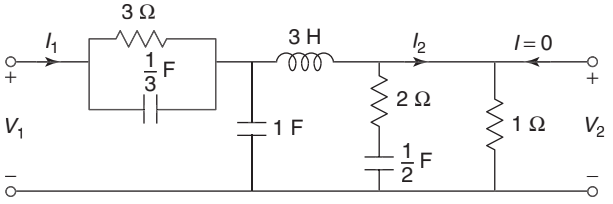


Fig. 8.25

Solution The transformed network is shown in Fig. 8.26.

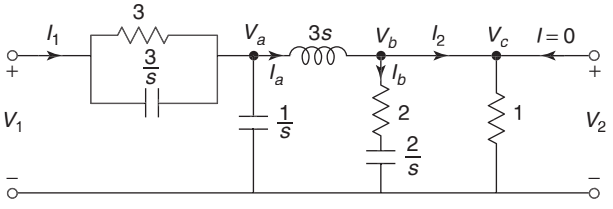


Fig. 8.26

$$V_c = V_b = V_2$$

$$I_2 = \frac{V_2}{1} = V_2$$

$$I_a = I_b + I_2 = \frac{V_2}{2 + \frac{2}{s}} + \frac{V_2}{1} = \frac{s}{2s+2}V_2 + V_2 = \left(\frac{3s+2}{2s+2}\right)V_2$$

$$V_a = 3s\,I_a + V_2 = \frac{3s(3s+2)}{2s+2}V_2 + V_2 = \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = \frac{s(9s^2+8s+2)}{2s+2}V_2 + \left(\frac{3s+2}{2s+2}\right)V_2 = \left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2$$

$$V_1 = \left(\frac{3\times\frac{3}{s}}{3+\frac{3}{s}}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)\left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2 + \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$= \left[\frac{27s^3+24s^2+15s+6+9s^3+8s^2+2s+9s^2+8s+2}{(s+1)(2s+2)}\right]V_2 = \frac{(36s^3+41s^2+25s+8)}{(s+1)(2s+2)}V_2$$

$$= \left(\frac{36s^3+41s^2+25s+8}{2s^2+4s+2}\right)V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{2s^2+4s+2}{36s^3+41s^2+25s+8}$$

$$\frac{I_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_1}{I_1} = \frac{36s^3+41s^2+25s+8}{(s+1)(9s^3+8s^2+5s+2)} = \frac{36s^3+41s^2+25s+8}{9s^4+17s^3+13s^2+7s+2}$$

Example 8.16 *For the resistive two-port network of Fig. 8.27, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

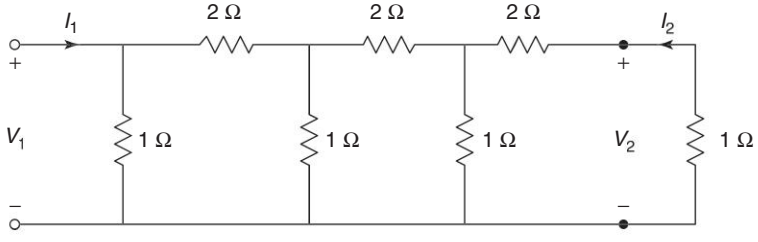


Fig. 8.27

Solution The network is redrawn as shown in Fig. 8.28.

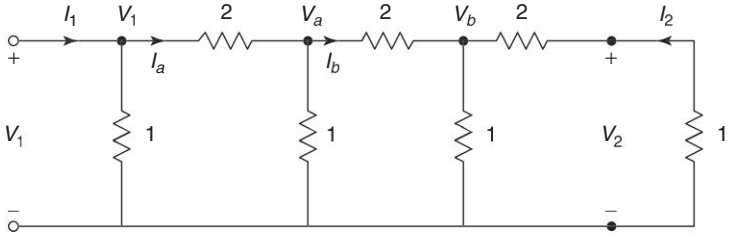


Fig. 8.28

$$I_2 = -\frac{V_2}{1} = -V_2$$

$$V_b = -3I_2 = 3V_2$$

$$I_b = \frac{V_b}{1} + \frac{V_b}{3} = \frac{4}{3}V_b = 4V_2$$

$$V_a = 2I_b + V_b = 8V_2 + 3V_2 = 11V_2$$

$$I_a = \frac{V_a}{1} + I_b = 11V_2 + 4V_2 = 15V_2$$

$$V_1 = 2I_a + V_a = 30V_2 + 11V_2 = 41V_2$$

$$I_1 = \frac{V_1}{1} + I_a = 41V_2 + 15V_2 = 56V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{1}{41}$$

$$\frac{V_2}{I_1} = \frac{1}{56}\ \Omega$$

$$\frac{I_2}{V_1} = -\frac{1}{41}\ \text{℧}$$

$$\frac{I_2}{I_1} = -\frac{1}{56}$$

Example 8.17 *Find the network function* $\frac{V_2}{V_1}$ *for the network shown in Fig. 8.29.*

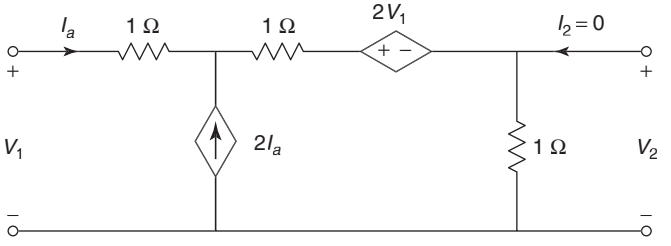


Fig. 8.29

Solution The network is redrawn as shown in Fig. 8.30.

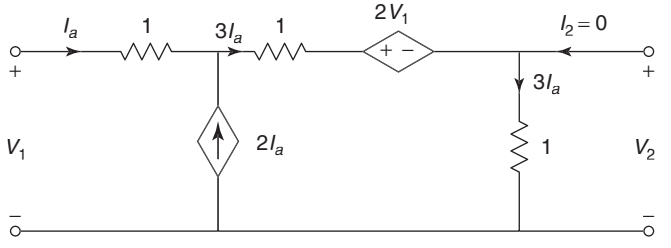


Fig. 8.30

From Fig. 8.30,

$$V_2 = 1\,(3\,I_a) = 3\,I_a$$

Applying KVL to the outermost loop,

$$V_1 - 1\,(I_a) - 1\,(3I_a) - 2\,V_1 - 1\,(3I_a) = 0$$

$$V_1 = -7I_a$$

Hence,

$$\frac{V_2}{V_1} = -\frac{3}{7}$$

Example 8.18 *Find the network function* $\frac{I_2}{I_1}$ *for the network shown in Fig. 8.31.*

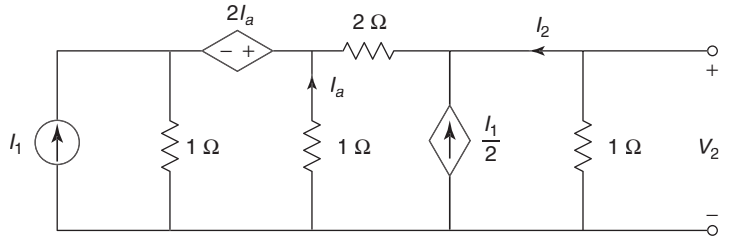


Fig. 8.31

Solution The network is redrawn as shown in Fig. 8.32.

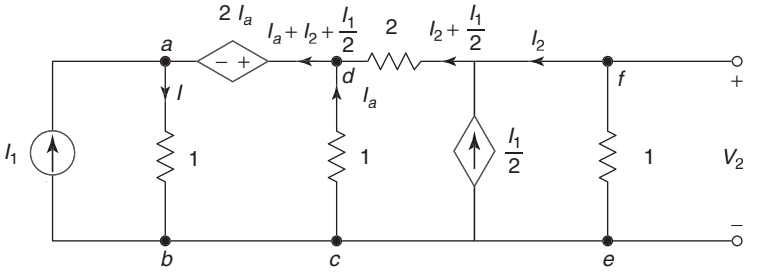


Fig. 8.32

From Fig. 8.32,

$$I = I_1 + I_a + I_2 + \frac{I_1}{2}$$

$$= \frac{3}{2} I_1 + I_a + I_2 \qquad \text{...(i)}$$

Applying KVL to the loop *abcda*,

$$-1I - 1I_a - 2I_a = 0$$

$$-I - 3I_a = 0$$

$$I + 3I_a = 0 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$\frac{3}{2}I_1 + I_a + I_2 + 3I_a = 0$$

$$\frac{3}{2}I_1 + I_2 + 4I_a = 0 \qquad \text{...(iii)}$$

Applying KVL to the loop *dcefd,*

$$1I_a - 1I_2 - 2\left(I_2 + \frac{I_1}{2}\right) = 0$$

$$I_a - 3I_2 - I_1 = 0$$

$$I_a = 3I_2 + I_1 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (iii),

$$\frac{3}{2}I_1 + I_2 + 4(3I_2 + I_1) = 0$$

$$\frac{3}{2}I_1 + I_2 + 12I_2 + 4I_1 = 0$$

$$\frac{11}{2}I_1 + 13I_2 = 0$$

$$13I_2 = -\frac{11}{2}I_1$$

Hence,

$$\frac{I_2}{I_1} = -\frac{11}{26}$$

8.5 ANALYSIS OF NON-LADDER NETWORKS

The above method is applicable for ladder networks. There are other network configurations to which the technique described is not applicable. Figure 8.33 shows one such network.

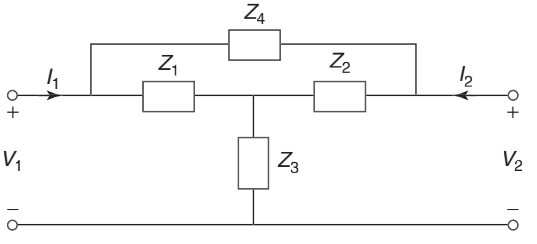


Fig. 8.33 *Non-ladder network*

For such a type of network, it is necessary to express the network functions as a quotient of determinants, formulated on KVL and KCL basis.

Example 8.19 *For the resistive bridged T network shown in Fig. 8.34, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

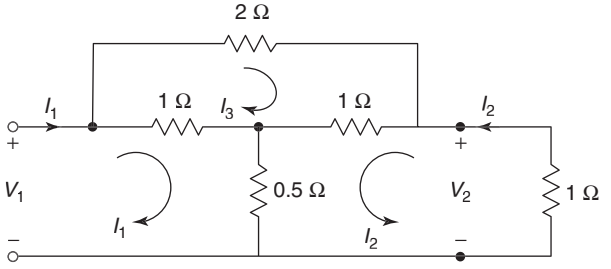


Fig. 8.34

Solution Applying KVL to Mesh 1,

$$V_1 = 1.5I_1 + 0.5I_2 - I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = 0.5I_1 + 2.5I_2 + I_3 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -I_1 + I_2 + 4I_3 \quad \text{...(iii)}$$

Writing these equations in matrix form,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad \text{...(iv)}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & 0.5 & -1 \\ 0 & 2.5 & 1 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{V_1(10-1)}{9} = V_1 \quad \text{...(iv)}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 1.5 & V_1 & -1 \\ 0.5 & 0 & 1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{-V_1(2+1)}{9} = -\frac{1}{3}V_1 \quad \text{...(v)}$$

From Fig. 8.34, $V_2 = -1\,(I_2) = -I_2$

From Eq. (v), $V_1 = -3I_2$

From Eqs. (iv) and (v), $I_2 = -\frac{1}{3}V_1 = -\frac{1}{3}I_1$

$I_1 = -3I_2$

Hence, $\frac{I_2}{V_1} = -\frac{1}{3}$ ℧

$$\frac{I_2}{I_1} = \frac{-\frac{1}{3}V_1}{V_1} = -\frac{1}{3}$$

$$\frac{V_2}{V_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}$$

$$\frac{V_2}{I_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}\,\Omega$$

Example 8.20 *For the network of Fig. 8.35, find Z_{11}, Z_{12} and G_{12}.*

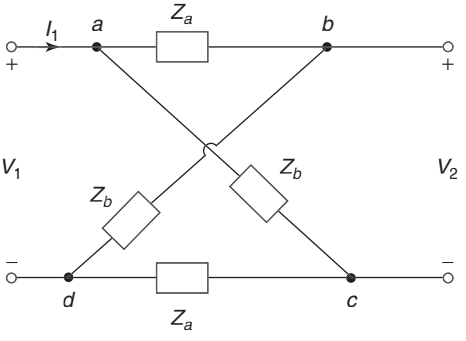


Fig. 8.35

Solution The network can be redrawn as shown in Fig. 8.36. Since the network consists of two identical impedances connected in parallel, the current in I_1 divides equally in each branch.

$$V_1 = (Z_a + Z_b)\frac{I_1}{2}$$

$$Z_{11} = \frac{V_1}{I_1} = \frac{Z_a + Z_b}{2}$$

$$V_2 = Z_b\frac{I_1}{2} - Z_a\left(\frac{I_1}{2}\right) = (Z_b - Z_a)\frac{I_1}{2}$$

$$Z_{12} = \frac{V_2}{I_1} = \frac{Z_b - Z_a}{2}$$

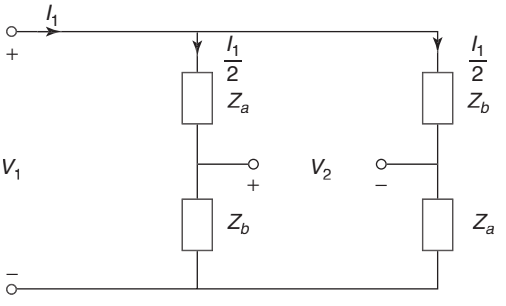


Fig. 8.36

By voltage-division rule,

$$V_2 = \frac{Z_b}{Z_a + Z_b}V_1 - \frac{Z_a}{Z_a + Z_b}V_1 = \frac{Z_b - Z_a}{Z_a + Z_b}V_1$$

$$G_{12} = \frac{V_2}{V_1} = \frac{Z_b - Z_a}{Z_a + Z_b}$$

Example 8.21 *For the network shown in Fig. 8.37, determine Z_{11} (s), G_{12} (s) and Z_{12} (s).*

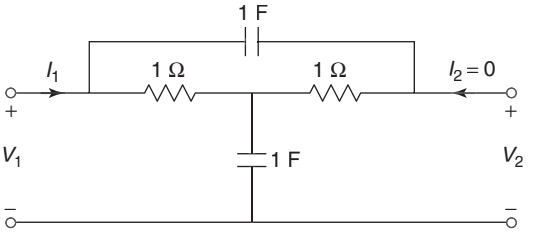


Fig. 8.37

Solution The transformed network is shown in Fig. 8.38.

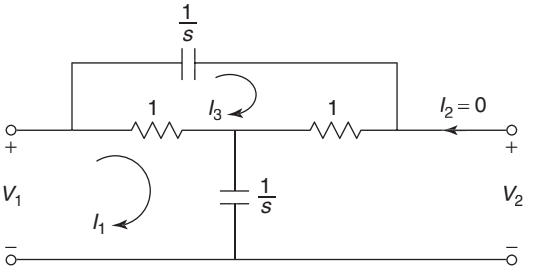


Fig. 8.38

Applying KVL to Mesh 1,

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{s} I_1 + I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + \left(2+\frac{1}{s}\right) I_3 = 0$$

$$I_3 = \left(\frac{s}{2s+1}\right) I_1 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eqs. (i) and (ii),

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - \left(\frac{s}{2s+1}\right) I_1 = \left(\frac{s+1}{s} - \frac{s}{2s+1}\right) I_1 = \left[\frac{s^2+3s+1}{s(2s+1)}\right] I_1$$

$$V_2 = \frac{1}{s} I_1 + \frac{s}{2s+1} I_1 = \left[\frac{s^2+2s+1}{s(2s+1)}\right] I_1$$

Hence,

$$Z_{11}(s) = \frac{V_1}{I_1} = \frac{s^2+3s+1}{s(2s+1)}$$

$$Z_{12}(s) = \frac{V_2}{I_1} = \frac{s^2+2s+1}{s(2s+1)}$$

$$G_{12}(s) = \frac{V_2}{V_1} = \frac{s^2+2s+1}{s^2+3s+1}$$

Example 8.22 *For the network shown in Fig.8.39, find the driving-point admittance Y_{11} and transfer admittance Y_{12}.*

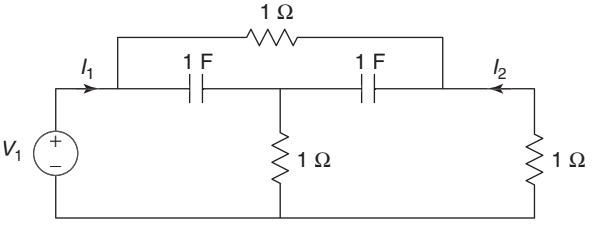


Fig. 8.39

Solution The transformed network is shown in Fig. 8.40.

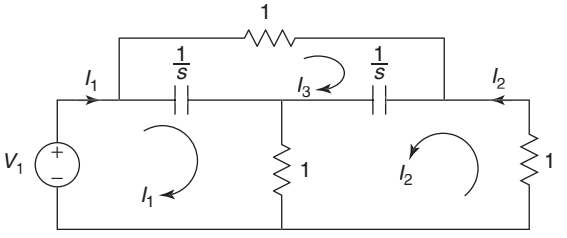


Fig. 8.40

Applying KVL to Mesh 1,

$$V_1 = \left(\frac{1}{s}+1\right) I_1 + I_2 - \frac{1}{s} I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = I_1 + \left(2+\frac{1}{s}\right) I_2 + \frac{1}{s} I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -\frac{1}{s} I_1 + \frac{1}{s} I_2 + \left(\frac{2}{s}+1\right) I_3 \qquad \text{...(iii)}$$

Writing these equations in matrix from,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = \left(\frac{1}{s}+1\right)\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right]-1\left[(1)\left(\frac{2}{s}+1\right)+\frac{1}{s^2}\right]-\frac{1}{s}\left[(1)\left(\frac{1}{s}\right)+\left(\frac{1}{s}\right)\left(2+\frac{1}{s}\right)\right]$$

$$= \frac{s^2+5s+2}{s^2}$$

$$\Delta_1 = \begin{vmatrix} V_1 & 1 & -\frac{1}{s} \\ 0 & 2+\frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = V_1\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right] = V_1\left(\frac{2s^2+5s+1}{s^2}\right)$$

$$I_1 = V_1\left(\frac{2s^2+5s+1}{s^2+5s+2}\right)$$

Hence,
$$Y_{11} = \frac{I_1}{V_1} = \frac{2s^2+5s+1}{s^2+5s+2}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{s}+1 & V_1 & -\frac{1}{s} \\ 1 & 0 & \frac{1}{s} \\ -\frac{1}{s} & 0 & \frac{2}{s}+1 \end{vmatrix} = -V_1\left(\frac{2}{s}+1+\frac{1}{s^2}\right) = -V_1\left(\frac{s^2+2s+1}{s^2}\right)$$

$$I_2 = -V_1\left(\frac{s^2+2s+1}{s^2+5s+2}\right)$$

Hence,
$$Y_{12} = \frac{I_2}{V_1} = -\frac{s^2+2s+1}{s^2+5s+2}$$

8.6 POLES AND ZEROS OF NETWORK FUNCTIONS

The network function $F(s)$ can be written as ratio of two polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}{b_m s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are the coefficients of the polynomials $N(s)$ and $D(s)$. These are real and positive for networks of passive elements. Let $N(s) = 0$ have n roots as $z_1, z_2, \ldots\ldots, z_n$ and $D(s) = 0$ have m roots as $p_1, p_2, \ldots\ldots, p_m$. Then $F(s)$ can be written as

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

where $\frac{a_n}{b_m}$ is a constant called *scale factor* and $z_1, z_2, \ldots, z_n, p_1, p_2, \ldots, p_m$ are complex frequencies. When the variable s has the values $z_1, z_2, \ldots, z_n$, the network function becomes zero; such complex frequencies are known as the zeros of the network function. When the variable s has values $p_1, p_2, \ldots, p_m$, the network function becomes infinite; such complex frequencies are known as the poles of the network function. A network function is completely specified by its poles, zeros and the scale factor.

If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles or multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

The total number of zeros is equal to the total number of poles. For any network function, poles and zeros at zero and infinity are taken into account in addition to finite poles and zeros.

Poles and zeros are critical frequencies. The network function becomes infinite at poles, while the network function becomes zero at zeros. The network function has a finite, non-zero value at other frequencies.

Poles and zeros provide a representation of a network function as shown in Fig. 8.41. The zeros are shown by circles and the poles by crosses. This diagram is referred to as pole-zero plot.

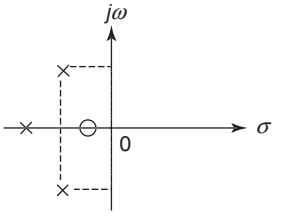


Fig. 8.41 *Pole-zero plot*

8.7 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR DRIVING-POINT FUNCTIONS [COMMON FACTORS IN *N*(*s*) AND *D*(*s*) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real and positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of all poles and zeros, must be negative or zero, i.e., the poles and zeros must lie in left half of s plane.

(4) If the real part of pole or zero is zero, then that pole or zero must be simple.

(5) The polynomials $N(s)$ and $D(s)$ may not have missing terms between those of highest and lowest degree, unless all even or all odd terms are missing.

(6) The degree of $N(s)$ and $D(s)$ may differ by either zero or one only. This condition prevents multiple poles and zeros at $s = \infty$.

(7) The terms of lowest degree in $N(s)$ and $D(s)$ may differ in degree by one at most. This condition prevents multiple poles and zeros at $s = 0$.

8.8 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR TRANSFER FUNCTIONS [COMMON FACTORS IN *N*(S) AND *D*(S) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real, and those for $D(s)$ must be positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of poles must be negative or zero. If the real part is zero, then that pole must be simple.

(4) The polynomial $D(s)$ may not have any missing terms between that of highest and lowest degree, unless all even or all odd terms are missing.

(5) The polynomial $N(s)$ may have terms missing between the terms of lowest and highest degree, and some of the coefficients may be negative.
(6) The degree of $N(s)$ may be as small as zero, independent of the degree of $D(s)$.
(7) For voltage and current transfer functions, the maximum degree of $N(s)$ is the degree of $D(s)$.
(8) For transfer impedance and admittance functions, the maximum degree of $N(s)$ is the degree of $D(s)$ plus one.

Example 8.23

Test whether the following represent driving-point immittance functions.

(a) $\dfrac{5s^4 + 3s^2 - 2s + 1}{s^3 + 6s + 20}$ (b) $\dfrac{s^3 + s^2 + 5s + 2}{s^4 + 6s^3 + 9s^2}$ (c) $\dfrac{s^2 + 3s + 2}{s^2 + 6s + 2}$

Solution

(a) The numerator and denominator polynomials have a missing term between those of highest and lowest degree and one of the coefficient is negative in numerator polynomial. Hence, the function does not represent a driving-point immittance function.
(b) The term of lowest degree in numerator and denominator polynomials differ in degree by two. Hence, the function does not represent a driving-point immittance function.
(c) The function satisfies all the necessary conditions. Hence, it represents a driving-point immittance function.

Example 8.24

Test whether the following represent transfer functions.

(a) $G_{21} = \dfrac{3s + 2}{5s^3 + 4s^2 + 1}$ (b) $\alpha_{12} = \dfrac{2s^2 + 5s + 1}{s + 7}$ (c) $Z_{21} = \dfrac{1}{s^3 + 2s}$

Solution

(a) The polynomial $D(s)$ has a missing term between that of highest and lowest degree. Hence, the function does not represent a transfer function.
(b) The degree of $N(s)$ is greater than $D(s)$. Hence the function does not represent a transfer function.
(c) The function satisfies all the necessary conditions. Hence, it represents a transfer function.

Example 8.25

Obtain the pole-zero plot of the following functions.

(a) $F(s) = \dfrac{s(s + 2)}{(s + 1)(s + 3)}$ (b) $F(s) = \dfrac{s(s + 1)}{(s + 2)^2 (s + 3)}$

(c) $F(s) = \dfrac{s(s + 2)}{(s + 1 + j1)(s + 1 - j1)}$ (d) $F(s) = \dfrac{(s + 1)^2 (s + 5)}{(s + 2)(s + 3 + j2)(s + 3 - j2)}$

(e) $F(s) = \dfrac{s^2 + 4}{(s + 2)(s^2 + 9)}$

Solution

(a) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1$ and $s = -3$.
The pole-zero plot is shown in Fig. 8.42.

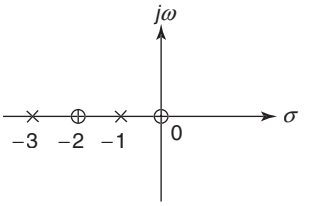


Fig. 8.42

(b) The function $F(s)$ has zeros at $s = 0$ and $s = -1$ and poles at $s = -2, -2$ and $s = -3$.
The pole-zero plot is shown in Fig. 8.43.

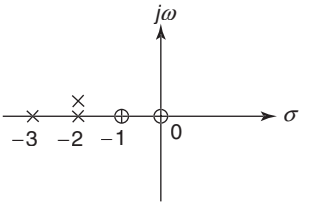


Fig. 8.43

(c) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1 - j1$ and $s = -1 + j1$.
The pole-zero plot is shown in Fig. 8.44.

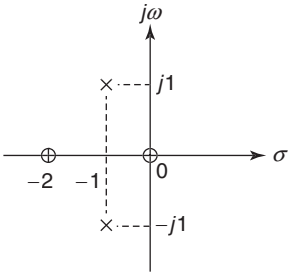


Fig. 8.44

(d) The function $F(s)$ has zeros at $s = -1, -1$ and $s = -5$ and poles at $s = -2$, $s = -3 + j2$ and $s = -3 - j2$. The pole-zero plot is shown in Fig. 8.45.

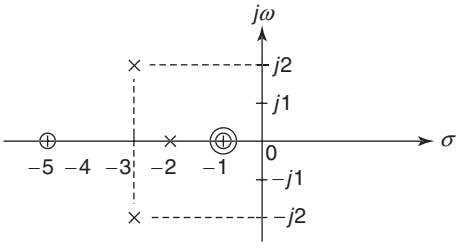


Fig. 8.45

(e) The function $F(s)$ has zeros at $s = j2$ and $s = -j2$ and poles at $s = -2$, $s = j3$ and $s = -j3$. The pole-zero plot is shown in Fig. 8.46.

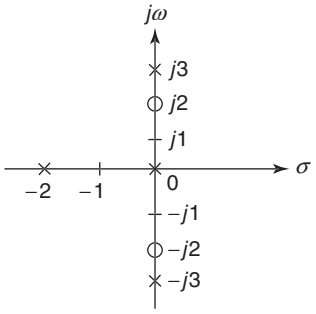


Fig. 8.46

Example 8.26 *Find poles and zeros of the impedance of the network shown in Fig. 8.47 and plot them on the s-plane.*

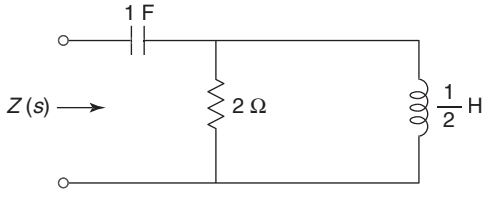


Fig. 8.47

Solution The transformed network is shown in Fig. 8.48.

$$Z(s) = \frac{1}{s} + \frac{\frac{s}{2} \times 2}{\frac{s}{2} + 2} = \frac{1}{s} + \frac{2s}{s+4} = \frac{2s^2 + s + 4}{s(s+4)} = \frac{2(s^2 + 0.5s + 2)}{s(s+4)}$$

$$= \frac{2(s + 0.25 + j1.4)(s + 0.25 - j1.4)}{s(s+4)}$$

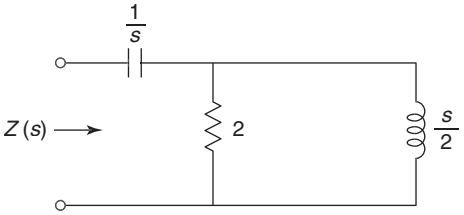


Fig. 8.48

The function $Z(s)$ has zeros at $s = -0.25 + j1.4$ and $s = -0.25 - j1.4$ and poles at $s = 0$ and $s = -4$ as shown in Fig. 8.49.

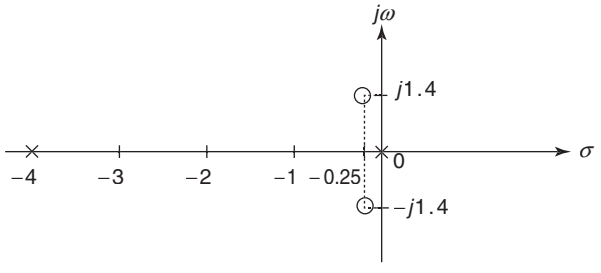


Fig. 8.49

Example 8.27 *Determine the poles and zeros of the impedance function Z (s) in the network shown in Fig. 8.50.*

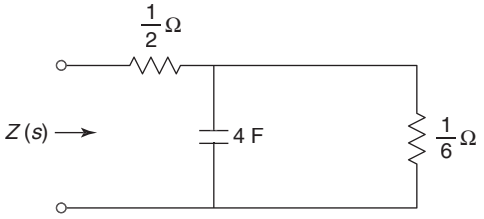


Fig. 8.50

Solution The transformed network is shown in Fig. 8.51.

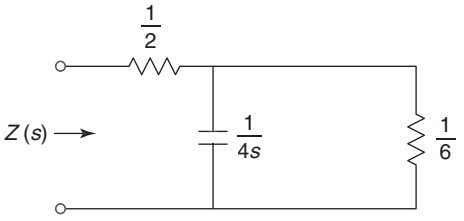


Fig. 8.51

$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4s} \times \frac{1}{6}}{\frac{1}{4s} + \frac{1}{6}} = \frac{1}{2} + \frac{1}{4s+6} = \frac{4s+8}{2(4s+6)} = \frac{s+2}{2s+3} = \frac{0.5(s+2)}{s+1.5}$$

The function $Z(s)$ has zero at $s = -2$ and pole at $s = -1.5$.

Example 8.28 *Determine Z(s) in the network shown in Fig. 8.52. Find poles and zeros of Z(s) and plot them on s-plane.*

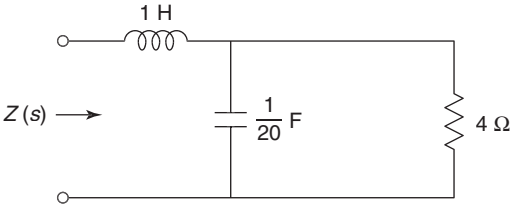


Fig. 8.52

Solution The transformed network is shown in Fig. 8.53.

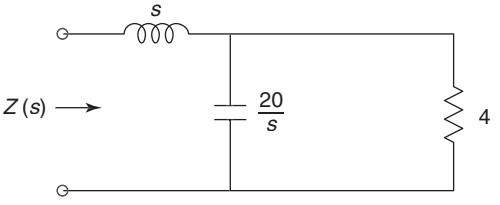


Fig. 8.53

$$Z(s) = s + \frac{\frac{20}{s} \times 4}{\frac{20}{s} + 4} = s + \frac{80}{4s + 20} = s + \frac{20}{s + 5} = \frac{s(s+5) + 20}{s+5} = \frac{s^2 + 5s + 20}{s+5}$$

$$= \frac{(s + 2.5 + j3.71)(s + 2.5 - j3.71)}{s+5}$$

The function $Z(s)$ has zeros at $s = -2.5 + j3.71$ and $s = -2.5 - j3.71$ and pole at $s = -5$.
The pole-zero diagram is shown in Fig. 8.54.

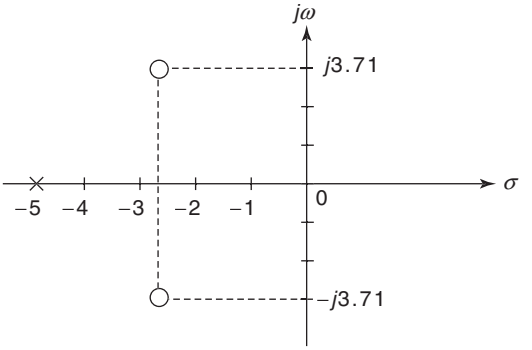


Fig. 8.54

Example 8.29 *For the network shown in Fig. 8.55, plot poles and zeros of function* $\frac{I_0}{I_i}$.

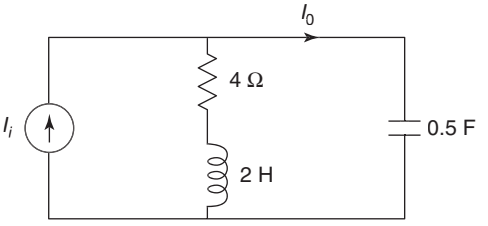


Fig. 8.55

Solution The transformed network is shown in Fig. 8.56.
By current-division rule,

$$I_0 = I_i \left(\frac{4 + 2s}{4 + 2s + \frac{2}{s}} \right)$$

$$\frac{I_0}{I_i} = \frac{s(4+2s)}{4s + 2s^2 + 2} = \frac{s(s+2)}{s^2 + 2s + 1} = \frac{s(s+2)}{(s+1)(s+1)}$$

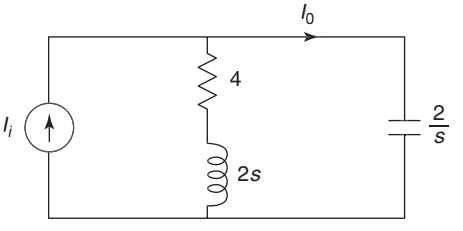


Fig. 8.56

The function has zeros at $s = 0$ and $s = -2$ and double poles at $s = -1$.
The pole-zero diagram is shown in Fig. 8.57.

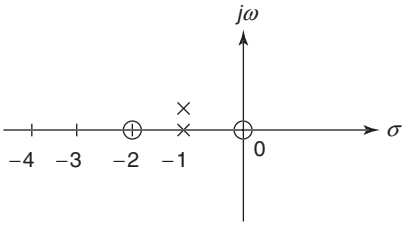


Fig. 8.57

Example 8.30 *Draw the pole-zero diagram of* $\frac{I_2}{I_1}$ *for the network shown in Fig. 8.58.*

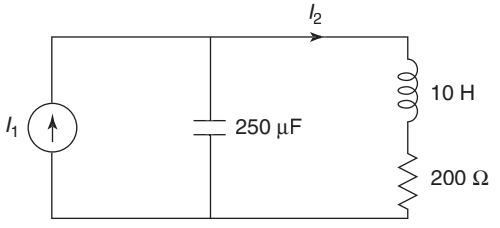


Fig. 8.58

Solution The transformed network is shown in Fig. 8.59.

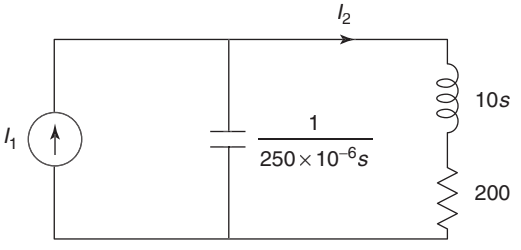


Fig. 8.59

By current-division rule,

$$I_2 = I_1 \frac{\dfrac{1}{250\times10^{-6}s}}{\dfrac{1}{250\times10^{-6}s} + 10s + 200}$$

$$\frac{I_2}{I_1} = \frac{400}{s^2 + 20s + 400} = \frac{400}{(s + 10 - j17.32)(s + 10 + j17.32)}$$

The function has no zero and poles at $s = -10 + j17.32$ and $s = -10 - j17.32$.
The pole-zero diagram is shown in Fig. 8.60.

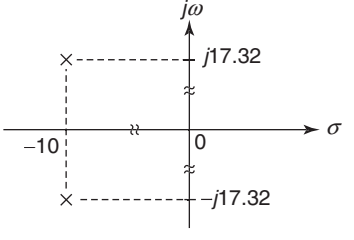


Fig. 8.60

Example 8.31 *For the network shown in Fig. 8.61, draw pole-zero plot of* $\frac{V_c}{V_1}$.

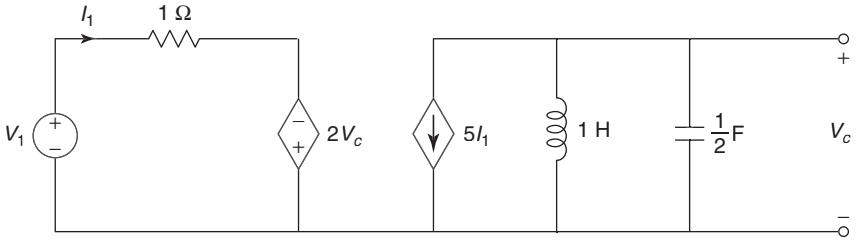


Fig. 8.61

Solution The transformed network is shown in Fig. 8.62.

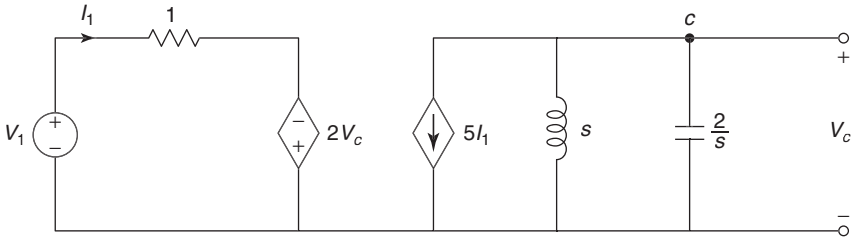


Fig. 8.62

Applying KVL to the left loop,

$$V_1 - 1I_1 + 2V_c = 0$$

$$I_1 = V_1 + 2V_c$$

Applying KCL at Node *C*,

$$5I_1 + \frac{V_c}{s} + \frac{V_c}{\frac{2}{s}} = 0$$

$$5(V_1 + 2V_c) + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$5V_1 + 10V_c + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$V_c\left(\frac{20s + 2 + s^2}{2s}\right) = -5V_1$$

$$\frac{V_c}{V_1} = -\frac{10s}{s^2 + 20s + 2} = -\frac{10s}{(s + 0.1)(s + 19.9)}$$

The function has zero at $s = 0$ and poles at $s = -0.1$ and $s = -19.9$.
The pole-zero diagram is shown in Fig. 8.63.

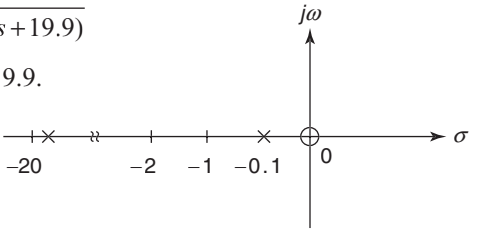


Fig. 8.63

Example 8.32 *Find the driving point admittance function and draw pole-zero plot for the network shown in Fig. 8.64.*

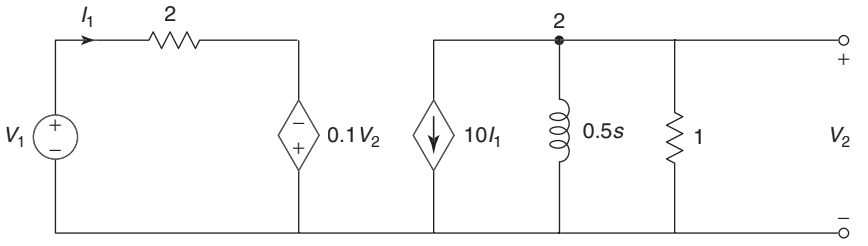


Fig. 8.64

Solution Applying KVL to the left loop,

$$V_1 - 2\,I_1 + 0.1\,V_2 = 0$$

$$I_1 = \frac{V_1 + 0.1\,V_2}{2} \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$10\,I_1 + \frac{V_2}{0.5s} + \frac{V_2}{1} = 0$$

$$10\,I_1 + \frac{2}{s}V_2 + V_2 = 0$$

$$10\,I_1 + \left(\frac{2}{s} + 1\right)V_2 = 0$$

$$10\,I_1 + \left(\frac{2+s}{s}\right)V_2 = 0$$

$$\left(\frac{s+2}{s}\right)V_2 = -10\,I_1$$

$$V_2 = -\frac{10s}{s+2}\,I_1 \qquad \text{...(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{V_1 + 0.1\left(-\dfrac{10s}{s+2}\right)I_1}{2} = 0.5V_1 + 0.05\left(-\frac{10s}{s+2}\right)I_1$$

$$I_1\left(1 + \frac{0.5s}{s+2}\right) = 0.5V_1$$

Hence,

$$Y_{11} = \frac{I_1}{V_1} = \frac{0.5}{1 + \dfrac{0.5s}{s+2}} = \frac{0.5\,(s+2)}{s + 0.5s + 2} = \frac{0.5s+1}{1.5s+2}$$

The function has zero at $s = -2$ and pole at $s = -1.33$. The pole-zero diagram is shown in Fig 8.65.

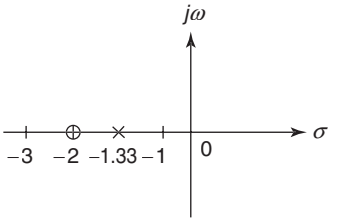


Fig. 8.65

Example 8.33 *For the network shown in Fig. 8.66, determine* $\frac{V_2}{I_g}$. *Plot the pole-zero diagram of* $\frac{V_2}{I_g}$.

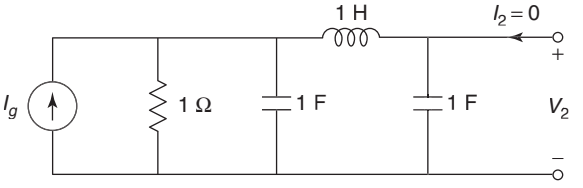


Fig. 8.66

Solution The transformed network is shown in Fig. 8.67.

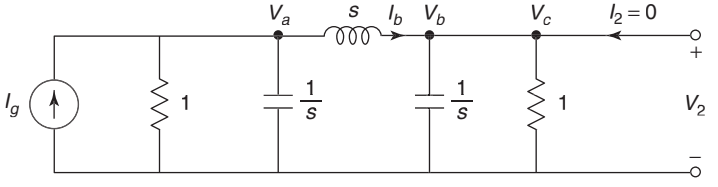


Fig. 8.67

$$V_c = V_b = V_2$$

$$I_b = \frac{V_b}{\frac{1}{s}} + \frac{V_c}{1} = sV_2 + V_2 = (s+1)V_2$$

$$V_a = sI_b + V_b = s(s+1)V_2 + V_2 = (s^2 + s + 1)V_2$$

$$I_g = \frac{V_a}{1} + \frac{V_a}{\frac{1}{s}} + I_b = (s^2 + s + 1)V_2 + s(s^2 + s + 1)V_2 + (s+1)V_2 = (s^3 + 2s^2 + 3s + 2)V_2$$

Hence, $$\frac{V_2}{I_g} = \frac{1}{s^3 + 2s^2 + 3s + 2}$$

The function has no zeros. It has poles at $s = -1$, $s = -0.5 + j1.32$ and $s = -0.5 - j1.32$,
The pole-zero diagram is shown in Fig. 8.68.

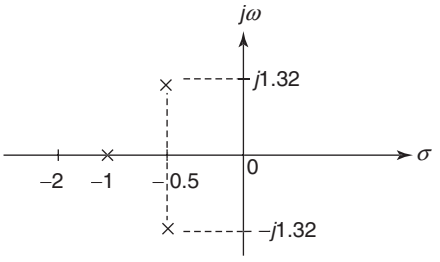


Fig. 8.68

Example 8.34 *For the transfer function* $H(s) = \frac{V_0}{V_i} = \frac{10}{s^2 + 3s + 10}$, *realise the function using the network shown in Fig. 8.69. Find L and C when R = 5 Ω.*

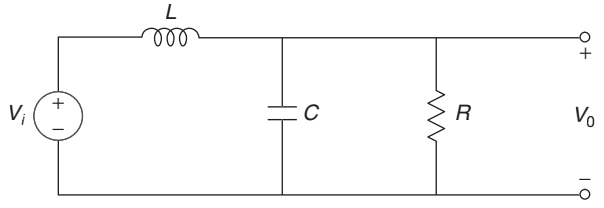


Fig. 8.69

Solution The transformed network is shown in Fig. 8.70.

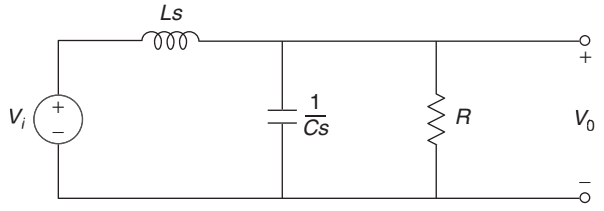


Fig. 8.70

Simplifying the network as shown in Fig. 8.71,

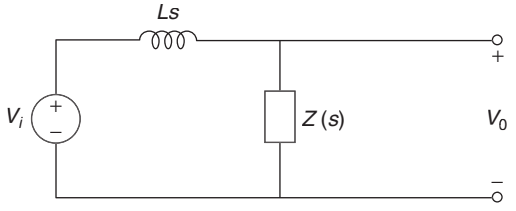


Fig. 8.71

$$Z(s) = \frac{R \times \frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{R}{RCs + 1}$$

$$V_0 = V_i \frac{\dfrac{R}{RCs+1}}{Ls + \dfrac{R}{RCs+1}}$$

$$\frac{V_0}{V_i} = \frac{R}{RLC\,s^2 + Ls + R} = \frac{\dfrac{1}{LC}}{s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}} \quad \text{...(i)}$$

But $$\frac{V_0}{V_i} = \frac{10}{s^2 + 3s + 10} \quad \text{...(ii)}$$

and $$R = 5\ \Omega$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{RC} = 3$$

$$\frac{1}{LC} = 10$$

Solving the above equations,

$$L = 1.5\ \text{H}$$

$$C = \frac{1}{15}\ \text{F}$$

Example 8.35 *Obtain the impedance function Z(s) for which pole-zero diagram is shown in Fig. 8.72.*

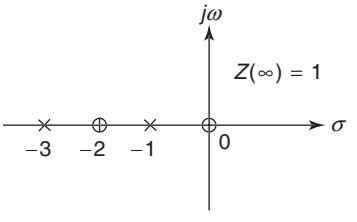


Fig. 8.72

Solution The function $Z(s)$ has poles at $s = -1$ and $s = -3$ and zeros at $s = 0$ and $s = -2$.

$$Z(s) = H\frac{s(s+2)}{(s+1)(s+3)} = H\frac{s^2\left(1+\dfrac{2}{s}\right)}{s^2\left(1+\dfrac{1}{s}\right)\left(1+\dfrac{3}{s}\right)}$$

For $s = \infty$,

$$Z(\infty) = H\frac{1}{(1)(1)} = H$$

When $$Z(\infty) = 1,$$

$$H = 1$$

$$Z(s) = \frac{s(s+2)}{(s+1)(s+3)}$$

Example 8.36 *Obtain the admittance function Y(s) for which the pole-zero diagram is shown in Fig. 8.73.*

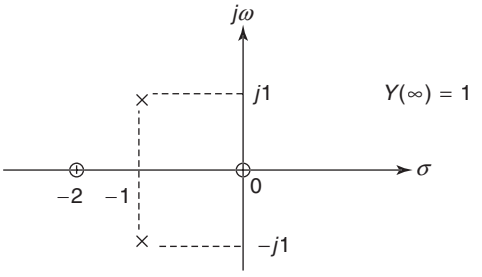


Fig. 8.73

Solution The function $Y(s)$ has poles at $s = -1 + j1$ and $s = -1 - j1$ and zeros at $s = 0$ and $s = -2$.

$$Y(s) = H\frac{s(s+2)}{(s+1+j1)(s+1-j1)} = H\frac{s(s+2)}{(s+1)^2-(j1)^2} = H\frac{s(s+2)}{s^2+2s+2} = H\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

For $s = \infty$,

$$Y(\infty) = H\frac{(1)}{(1)} = H$$

When $Y(\infty) = 1$,

$$H = 1$$

$$Y(s) = \frac{s(s+2)}{s^2+2s+2}$$

Example 8.37 *A network and its pole-zero configuration are shown in Fig. 8.74. Determine the values of R, L and C if Z (j0) = 1.*

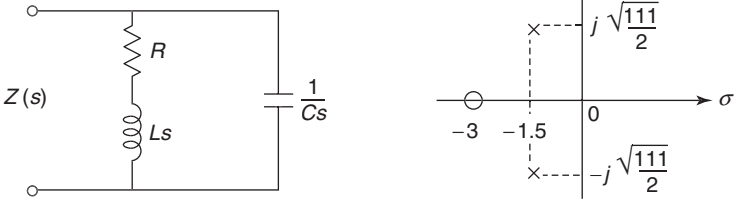


Fig. 8.74

Solution

$$Z(s) = \frac{(Ls+R)\frac{1}{Cs}}{(Ls+R)+\frac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}} \quad ...(i)$$

From the pole-zero diagram, zero is at $s = -3$ and poles are at $s = -1.5 + j\frac{\sqrt{111}}{2}$ and $s = -1.5 - j\frac{\sqrt{111}}{2}$

$$Z(s) = H\frac{s+3}{\left(s+1.5+j\frac{\sqrt{111}}{2}\right)\left(s+1.5-j\frac{\sqrt{111}}{2}\right)}$$

$$Z(s) = H\frac{s+3}{(s+1.5)^2 - \left(j\frac{\sqrt{111}}{2}\right)^2} = H\frac{s+3}{s^2+3s+30}$$

When $Z(j0) = 1$,

$$1 = H\left(\frac{3}{30}\right)$$

$$H = 10$$

$$Z(s) = \frac{10(s+3)}{s^2+3s+30} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 3$$

$$\frac{1}{C} = 10$$

$$\frac{1}{LC} = 30$$

Solving the above equations,

$$C = \frac{1}{10}\text{F}$$

$$L = \frac{1}{3}\text{H}$$

$$R = 1\,\Omega$$

Example 8.38 *A network is shown in Fig. 8.75. The poles and zeros of the driving-point function Z(s) of this network are at the following places:*

Poles at $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Zero at –1

If Z (j0) = 1, determine the values of R, L and C.

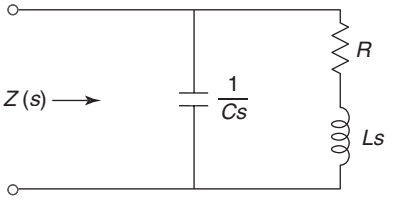


Fig. 8.75

Solution

$$Z(s) = \frac{(Ls+R)\frac{1}{Cs}}{Ls+R+\frac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}} \quad ...(i)$$

The poles are at $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ and zero is at –1.

$$Z(s) = H\frac{s+1}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} = H\frac{s+1}{\left(s+\frac{1}{2}\right)^2-\left(j\frac{\sqrt{3}}{2}\right)^2} = H\frac{s+1}{s^2+s+1}$$

When $Z(j0) = 1$,

$$1 = H\frac{(1)}{(1)}$$

$$H = 1$$

$$Z(s) = \frac{s+1}{s^2+s+1} \quad ...(ii)$$

Comparing Eq. (ii) with Eq. (i),

$$C = 1$$

$$\frac{R}{L} = 1$$

$$\frac{1}{LC} = 1$$

Solving the above equations,

$C = 1$ F

$L = 1$ H

$R = 1\ \Omega$

Example 8.39 *The pole-zero diagram of the driving-point impedance function of the network of Fig. 8.76 is shown below. At dc, the input impedance is resistive and equal to 2 W. Determine the values of R, L and C.*

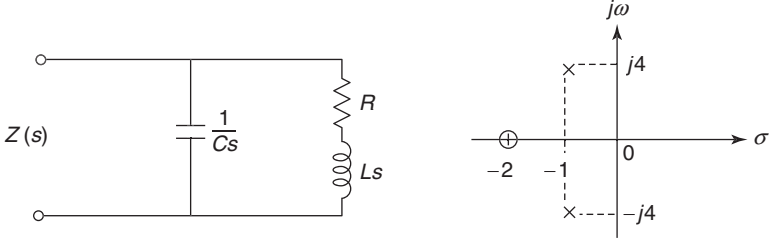


Fig. 8.76

Solution

$$Z(s) = \frac{(Ls+R)\frac{1}{Cs}}{Ls+R+\frac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}} \quad ...(i)$$

From the pole-zero diagram, zero is at $s = –2$ and poles are at $s = –1 + j4$ and $s = –1 – j4$.

$$Z(s) = H\frac{s+2}{(s+1+j4)(s+1-j4)} = H\frac{s+2}{(s+1)^2-(j4)^2} = H\frac{s+2}{s^2+2s+17}$$

At dc, i.e., $\omega = 0, Z(j0) = 2$

$$2 = H\frac{2}{17}$$

$$H = 17$$

$$Z(s) = 17\frac{s+2}{s^2+2s+17} \qquad ...(ii)$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 17$$

$$\frac{R}{L} = 2$$

$$\frac{1}{LC} = 17$$

Solving the above equations,

$$C = \frac{1}{17}\text{F}$$

$$L = 1\text{ H}$$

$$R = 2\ \Omega$$

Example 8.40 *The network shown in Fig. 8.77 has the driving-point admittance Y (s) of the form*

$$Y(s) = H\frac{(s-s_1)(s-s_2)}{(s-s_3)}$$

(*a*) *Express s_1, s_2, s_3 in terms of R, L and C.*
(*b*) *When $s_1 = -10 + j10^4$, $s_2 = -10 - j10^4$ and $Y(j0) = 10^{-1}$ mho, find the values of R, L and C and determine the value of s_3.*

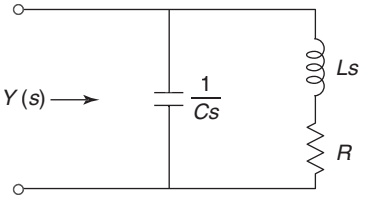


Fig. 8.77

Solution

(a) $$Y(s) = Cs + \frac{1}{Ls+R} = \frac{(Ls+R)Cs+1}{Ls+R} = \frac{LCs^2+RCs+1}{Ls+R} = \frac{C\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s+\frac{R}{L}} \qquad ...(i)$$

But $$Y(s) = \frac{H(s-s_1)(s-s_2)}{(s-s_3)}$$

where
$$s_1, s_2 = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_3 = -\frac{R}{L}$$

(b) When
$$s_1 = -10 + j10^4$$
$$s_2 = -10 - j10^4$$

$$Y(s) = H\frac{(s+10-j10^4)(s+10+j10^4)}{s-s_3} = H\frac{s^2+20s+10^8}{s-s_3} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 20$$
$$s_3 = -20$$
$$Y(s) = H\frac{(s^2+20s+10^8)}{(s+20)}$$

At $s = j0$,

$$Y(j0) = H\frac{(10^8)}{20} = 10^{-1}$$
$$H = 0.02\times10^{-6}$$
$$Y(s) = 0.02\times10^{-6}\frac{(s^2+20s+10^8)}{(s+20)} \quad \text{...(iii)}$$

Comparing Eq. (iii) with Eq. (i),

$$C = 0.02\times10^{-6}\,\text{F} = 0.02\,\mu\text{F}$$
$$\frac{1}{LC} = 10^8$$
$$L = \frac{1}{2}\,\text{H}$$
$$\frac{R}{L} = 20$$
$$R = 10\,\Omega$$

Example 8.41 *A network and pole-zero diagram for driving-point impedance Z(s) are shown in Fig. 8.78. Calculate the values of the parameters R, L, G and C if Z(j0) = 1.*

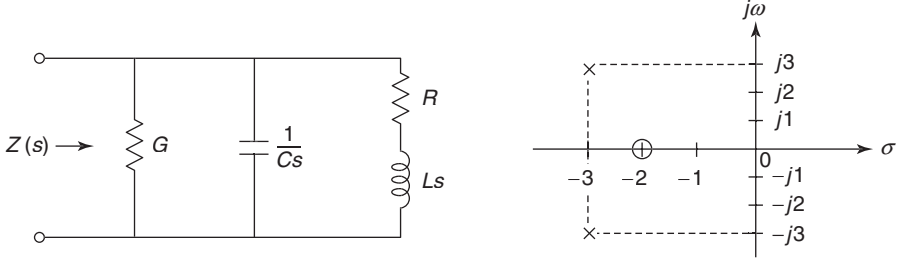


Fig. 8.78

Solution It is easier to calculate $Y(s)$ and then invert it to obtain $Z(s)$.

$$Y(s) = G + Cs + \frac{1}{Ls+R} = \frac{(G+Cs)(Ls+R)+1}{Ls+R} = \frac{LCs^2 + (GL+RC)\,s + 1 + GR}{Ls+R}$$

$$Z(s) = \frac{1}{Y(s)} = \frac{Ls+R}{LCs^2 + (GL+RC)\,s + 1 + GR} = \frac{\frac{1}{C}\left(s + \frac{R}{L}\right)}{s^2 + \left(\frac{G}{C} + \frac{R}{L}\right)s + \left(\frac{1+GR}{LC}\right)} \quad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -2$ and poles are at $s = -3 \pm j3$.

$$Z(s) = H\frac{(s+2)}{(s+3-j3)(s+3+j3)} = H\frac{(s+2)}{(s+3)^2 - (j3)^2} = H\frac{s+2}{s^2+6s+18}$$

When $Z(j0) = 1$,

$$1 = H\frac{2}{18}$$

$$H = 9$$

$$Z(s) = \frac{9\,(s+2)}{(s^2+6s+18)} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 9$$

$$\frac{R}{L} = 2$$

$$\frac{G}{C} + \frac{R}{L} = 6$$

$$\frac{1+GR}{LC} = 18$$

Solving the above equation,

$$C = \frac{1}{9}\text{ F}$$

$$L = \frac{9}{10}\text{ H}$$

$$G = \frac{4}{9}\,\text{℧}$$

$$R = \frac{9}{5}\,\Omega$$

Example 8.42 *A series R-L-C circuit has its driving-point admittance and pole-zero diagram is shown in Fig. 8.79. Find the values of R, L and C.*

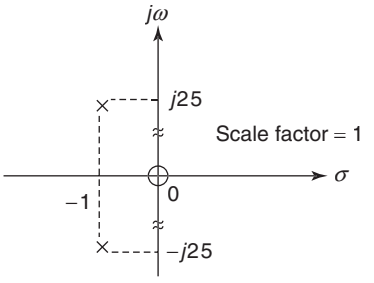


Fig. 8.79

Solution The function $Y(s)$ has poles at $s = -1 + j25$ and $s = -1 - j25$ and zero at $s = 0$.

$$Y(s) = H\frac{s}{(s+1+j25)(s+1-j25)} = H\frac{s}{(s+1)^2-(j25)^2} = H\frac{s}{s^2+2s+626}$$

Scale factor $H = 1$

$$Y(s) = \frac{s}{s^2+2s+626} \quad \text{...(i)}$$

For a series *RLC* circuit,

$$Z(s) = R + Ls + \frac{1}{Cs} = \frac{LCs^2+RCs+1}{Cs} = \frac{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{s}{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \quad \text{...(ii)}$$

Comparing Eq. (i) with Eq. (ii),

$$L = 1\text{ H}$$
$$\frac{1}{LC} = 626$$
$$C = \frac{1}{626}\text{ F}$$
$$\frac{R}{L} = 2$$
$$R = 2\ \Omega$$

8.9 TIME-DOMAIN BEHAVIOUR FROM THE POLE-ZERO PLOT

The time-domain behaviour of a system can be determined from the pole-zero plot. Consider a network function

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

The poles of this function determine the time-domain behaviour of $f(t)$.The function $f(t)$ can be determined from the knowledge of the poles, the zeros and the scale factor H. Figure 8.80 shows some pole locations and the corresponding time-domain response.

(i) When pole is at origin, i.e., at $s = 0$, the function $f(t)$ represents steady-state response of the circuit i.e., dc value. (Fig. 8.80)

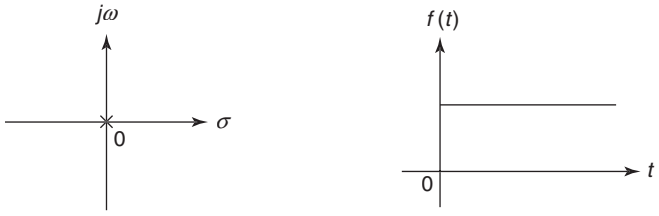


Fig. 8.80 *Pole at origin*

(ii) When pole lies in the left half of the s-plane, the response decreases exponentially. (Fig. 8.81)

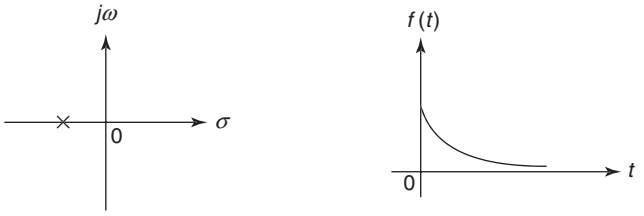


Fig. 8.81 *Pole in left half of the s-plane*

(iii) When pole lies in the right half of the s-plane, the response increases exponentially. A pole in the right-half plane gives rise to unbounded response and unstable system. (Fig. 8.82)

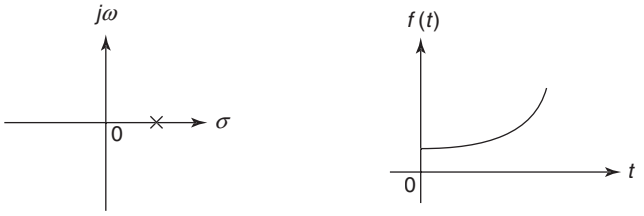


Fig. 8.82 *Pole in right half of the s-plane*

(iv) For $s = 0 + j\omega_n$, the response becomes $f(t) = Ae^{\pm j\omega_n t} = \text{A}(\cos\ \omega_n t \pm j\ \sin\ \omega_n t)$.The exponential response $e^{\pm j\omega_n t}$ may be interpreted as a rotating phasor of unit length. A positive sign of exponential $e^{j\omega_n t}$ indicates counterclockwise rotation, while a negative sign of exponential $e^{-j\omega_n t}$ indicates clockwise rotation. The variation of exponential function $e^{j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state. (Fig. 8.83)

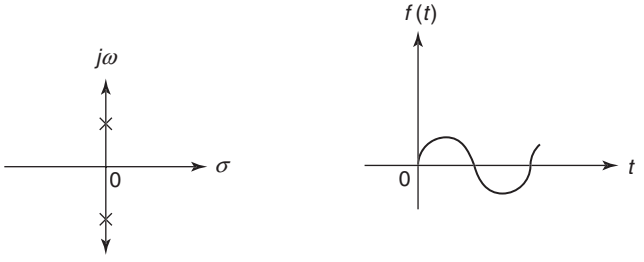


Fig. 8.83 *Poles on jω-axis*

(v) For $s = \sigma_n + j\omega_n$, the response becomes $f(t) = Ae^{st} = Ae^{(\sigma_n + j\omega_n t)} = Ae^{\sigma_n t}\, e^{j\omega_n t}$. The response $e^{\sigma_n t}$ is an exponentially increasing or decreasing function. The response $e^{j\omega_n t}$ is a sinusoidal function. Hence, the response of the product of these responses will be over damped sinusoids or under damped sinusoids (Fig. 8.84).

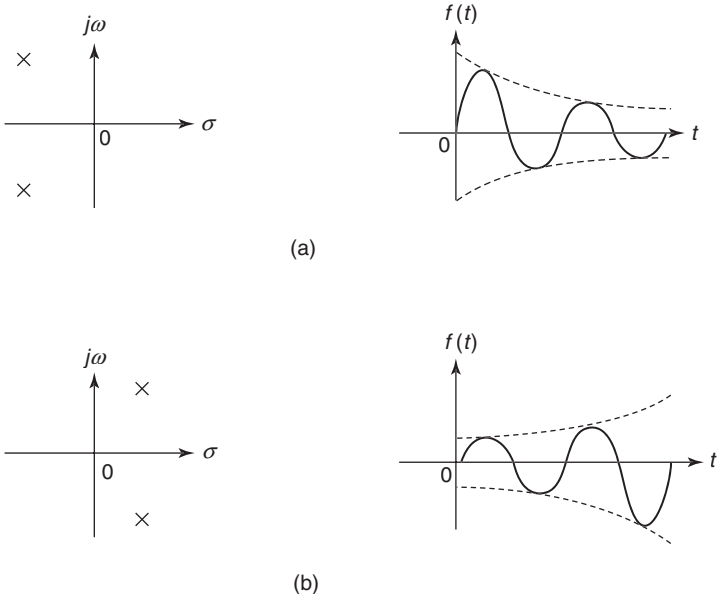


Fig. 8.84 *(a) Complex conjugate poles in left half of the S-plane*
(b) Complex conjugate poles in right half of the S-plane

(vi) The real part s of the pole is the displacement of the pole from the imaginary axis. Since σ is also the damping factor, a greater value of σ (i.e., a greater displacement of the pole from the imaginary axis) means that response decays more rapidly with time. The poles with greater displacement from the real axis correspond to higher frequency of oscillation (Fig. 8.85).

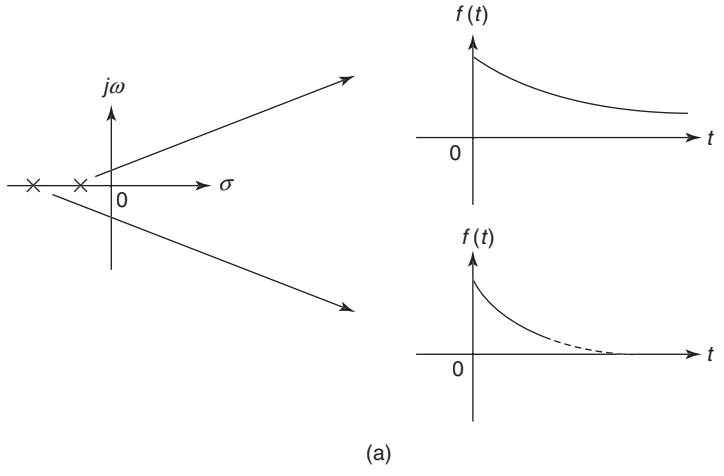
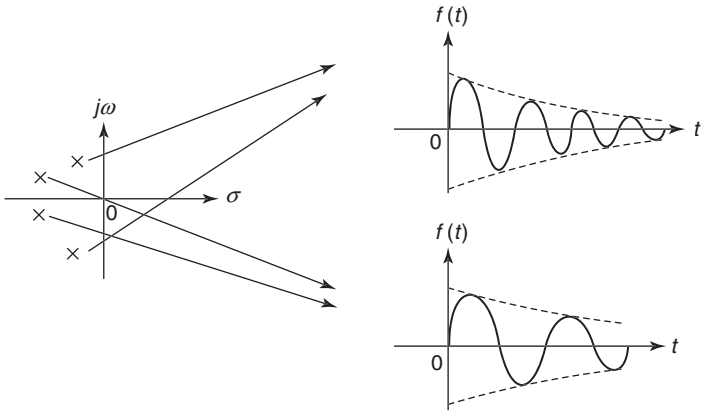


Fig. 8.85 *Nature of response with different positions of poles*



(b)

Fig.8.85 *(Continued)*

8.9.1 Stability of the Network

Stability of the network is directly related to the location of poles in the *s*-plane.

(i) When all the poles lie in the left half of the *s*-plane, the network is said to be stable.
(ii) When the poles lie in the right half of the *s*-plane, the network is said to be unstable.
(iii) When the poles lie on the $j\omega$ axis, the network is said to be marginally stable.
(iv) When there are multiple poles on the $j\omega$ axis, the network is said to be unstable.
(v) When the poles move away from $j\omega$ axis towards the left half of the *s*-plane, the relative stability of the network improves.

8.10 GRAPHICAL METHOD FOR DETERMINATION OF RESIDUE

Consider a network function,

$$F(s) = H\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

By partial fraction expansion,

$$F(s) = \frac{K_1}{(s-p_1)} + \frac{K_2}{(s-p_2)} + \cdots + \frac{K_m}{(s-p_m)}$$

The residue K_i is given by

$$K_i = (s-p_i)\,F(s)\big|_{s\to p_i} = H\frac{(p_i-z_1)(p_i-z_2)\cdots(p_i-z_n)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_m)}$$

Each term $(p_i - z_i)$ represents a phasor drawn from zero z_i to pole p_i.
Each term $(p_i - p_k)$, $i \neq k$, represents a phasor drawn from other poles to the pole p_i.

$$K_i = H\frac{\text{Product of phasors (polar form) from each zero to } p_i}{\text{Product of phasors (polar form) from other poles to } p_i}$$

The residues can be obtained by graphical method in the following way:

(1) Draw the pole-zero diagram for the given network function.
(2) Measure the distance from each of the other poles to a given pole.
(3) Measure the distance from each of the other zeros to a given pole.
(4) Measure the angle from each of the other poles to a given pole.
(5) Measure the angle from each of the other zeros to a given pole.
(6) Substitute these values in the required residue equation.

The graphical method can be used if poles are simple and complex. But it cannot be used when there are multiple poles.

Example 8.43 *The current I(s) in a network is given by* $I(s) = \dfrac{2s}{(s+1)(s+2)}$. *Plot the pole-zero pattern in the s-plane and hence obtain i(t).*

Solution Poles are at $s = -1$ and $s = -2$ and zero is at $s = 0$. The pole-zero plot is shown in Fig. 8.86.

By partial-fraction expansion,

$$I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

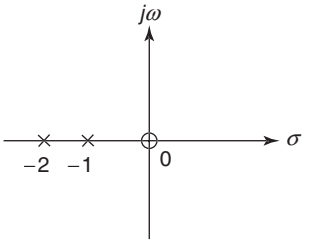


Fig. 8.86

The coefficients K_1 and K_2, often referred as residues, can be evaluated from he pole-zero diagram. From Fig. 8.87,

$$K_1 = H\,\frac{\text{Phasor from zero at origin to pole at } A}{\text{Phasor from pole at } B \text{ to pole at } A} = 2\left\{\frac{1\angle 180^\circ}{1\angle 0^\circ}\right\} = 2\angle 180^\circ = -2$$

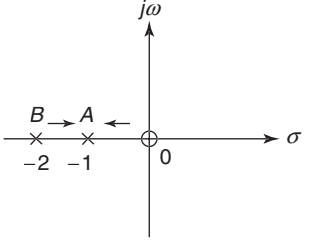


Fig. 8.87

From Fig. 8.88,

$$K_2 = H\,\frac{\text{Phasor from zero at origin to pole at } B}{\text{Phasor from pole at } A \text{ to pole at } B} = 2\left\{\frac{2\angle 180^\circ}{1\angle 180^\circ}\right\} = 4$$

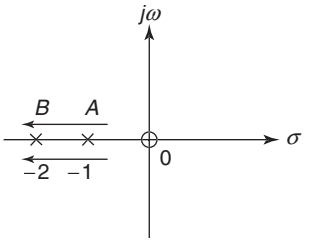


Fig. 8.88

$$I(s) = -\frac{2}{s+1} + \frac{4}{s+2}$$

Taking inverse Laplace transform,

$$i\,(t) = -2e^{-t} + 4e^{-2t}$$

Example 8.44 *The voltage V(s) of a network is given by*

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)}$$

Plot its pole-zero diagram and hence obtain v (t).

Solution

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)} = \frac{3s}{(s+2)(s+1+j1)(s+1-j1)}$$

Poles are at $s = -2$ and $s = -1 \pm j1$ and zero is at $s = 0$ as shown in Fig. 8.89.

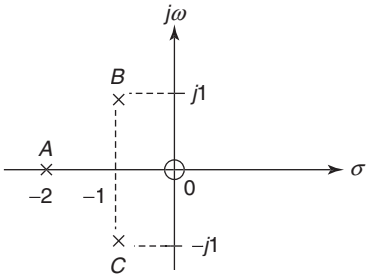


Fig. 8.89

By partial-fraction expansion,

$$V(s) = \frac{K_1}{s+2} + \frac{K_2}{s+1-j1} + \frac{K_2^*}{s+1+j1}$$

The coefficients K_1, K_2 and K_2^* can be evaluated from the pole-zero diagram.
From Fig. 8.90,

$$K_1 = \frac{3\,(\overline{OA})}{(\overline{BA})\,(\overline{CA})} = 3\left[\frac{2\angle 180°}{(\sqrt{2}\angle -135°)\,(\sqrt{2}\angle 135°)}\right] = 3\sqrt{180°} = -3$$

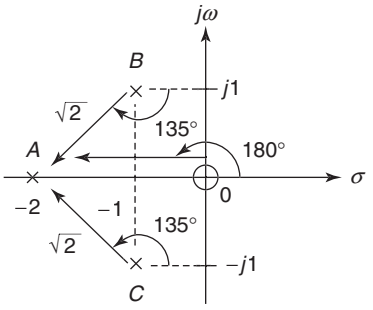


Fig. 8.90

From Fig. 8.91,

$$K_2 = \frac{3\,(\overline{OB})}{(\overline{AB})\,(\overline{CB})} = 3\left[\frac{(\sqrt{2}\angle 135°)}{(\sqrt{2}\angle 45°)\,(2\angle 90°)}\right] = \frac{3}{2}$$

$$K_2^* = \frac{3}{2}$$

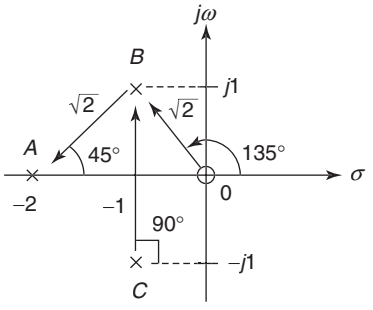


Fig. 8.91

$$V(s) = -\frac{3}{(s+2)} + \frac{\frac{3}{2}}{(s+1-j1)} + \frac{\frac{3}{2}}{(s+1+j1)}$$

Taking inverse Laplace transform,

$$v(t) = -3e^{-2t} + \frac{3}{2}\left[e^{(-1+j1)t} + e^{(-1-j1)t}\right] = -3e^{-2t} + 2\times\frac{3}{2}e^{-t}\left(\frac{e^{j1}+e^{-j1}}{2}\right) = -3e^{-2t} + 3e^{-t}\cos t$$

Example 8.45 *Find the function v(t) using the pole-zero plot of following function:*

$$V(s) = \frac{(s+2)(s+6)}{(s+1)(s+5)}$$

Solution If the degree of the numerator is greater or equal to the degree of the denominator, we have to divide the numerator by the denominator such that the remainder can be expanded into partial fractions.

$$V(s) = \frac{s^2 + 8s + 12}{s^2 + 6s + 5} = 1 + \frac{2s + 7}{s^2 + 6s + 5} = 1 + \frac{2(s + 3.5)}{(s + 1)(s + 5)}$$

By partial fraction expansion,

$$V(s) = 1 + \frac{K_1}{s + 1} + \frac{K_2}{s + 5}$$

K_1 and K_2 can be evaluated from the pole-zero diagram shown in Fig. 8.92 and Fig. 8.93.

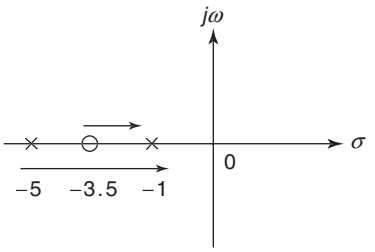


Fig. 8.92

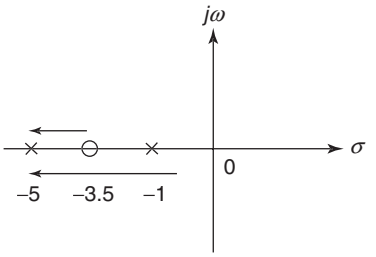


Fig. 8.93

From Fig. 8.92

$$K_1 = 2\left(\frac{2.5\angle 0°}{4\angle 0°}\right) = \frac{5}{4}$$

From Fig. 8.93

$$K_2 = 2\left(\frac{1.5\angle 180°}{4\angle 180°}\right) = \frac{3}{4}$$

$$V(s) = 1 + \frac{\frac{5}{4}}{s + 1} + \frac{\frac{3}{4}}{s + 5}$$

Taking inverse Laplace transform,

$$v(t) = \delta(t) + \frac{5}{4}e^{-t} + \frac{3}{4}e^{-5t}$$

Example 8.46 *The pole-zero plot of the driving-point impedance of a network is shown in Fig. 8.94. Find the time-domain response.*

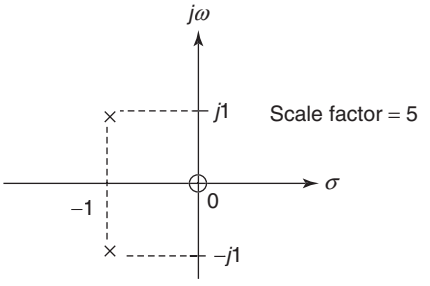


Fig. 8.94

Solution The function $Z(s)$ has poles at $s = -1 + j1$ and $s = -1 - j1$ and zero at $s = 0$.

$$Z(s) = H\frac{s}{(s+1+j1)(s+1-j1)}$$

Scale factor $H = 5$

$$Z(s) = \frac{5s}{(s+1+j1)(s+1-j1)}$$

By partial fraction expansion,

$$Z(s) = \frac{K_1}{s+1+j1} + \frac{K_1^*}{s+1-j1}$$

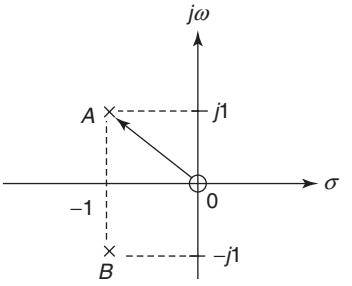


Fig. 8.95

The coefficients K_1 and K_1^* can be evaluated from the pole-zero diagram. From Fig. 8.95,

$$K_1 = \frac{5\,(\overline{OA})}{(\overline{BA})} = \frac{5\,(\sqrt{2}\angle 135°)}{2\angle 90°} = 3.54\angle 45°$$

$$K_1^* = 3.54\angle -45°$$

$$Z(s) = \frac{3.54\angle 45°}{s+1+j1} + \frac{3.54\angle -45°}{s+1-j1}$$

Taking inverse Laplace transform,

$$z(t) = 3.54 \angle 45°\, e^{(-1-j1)t} + 3.54 \angle -45° e^{(-1+j1)t}$$

Example 8.47 *Evaluate amplitude and phase of the network function* $F(s) = \dfrac{4s}{s^2+2s+2}$ *from the pole-zero plot at* $s = j2$.

Solution

$$F(s) = \frac{4s}{s^2+2s+2} = \frac{4s}{(s+1+j1)(s+1-j1)}$$

The pole-zero plot is shown in Fig. 8.96.

At $s = j2$,

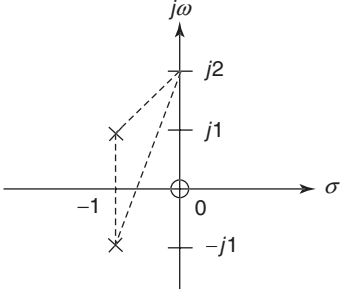


Fig. 8.96

$$|F(j2)| = \frac{\text{Product of phasor magnitudes from all zero to } j2}{\text{Product of phasor magnitudes from all poles to } j2} = \frac{2}{(\sqrt{2})\,(\sqrt{10})} = 0.447$$

$$\phi(\omega) = \tan^{-1}\left(\frac{2}{0}\right) - \tan^{-1}\left(\frac{3}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = 90° - 71.56° - 45° = -26.56°$$

Example 8.48

Using the pole-zero plot, find magnitude and phase of the function

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)} \text{ at } s = j4.$$

Solution

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

The pole-zero plot is shown in Fig. 8.97.

At $s = j4$,

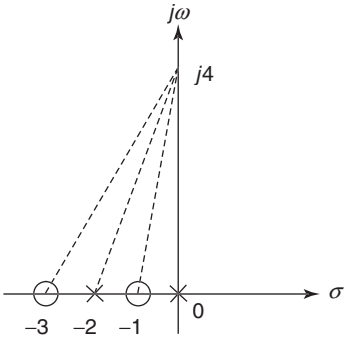


Fig. 8.97

$$|F(j4)| = \frac{\text{Product of phasor magnitudes from all zeros to } j4}{\text{product of phasor magnitudes from all poles to } j4} = \frac{(5)(\sqrt{17})}{(\sqrt{20})(4)} = 1.15$$

$$\phi(\omega) = \tan^{-1}\left(\frac{4}{1}\right) + \tan^{-1}\left(\frac{4}{3}\right) - \tan^{1}\left(\frac{4}{0}\right) - \tan^{-1}\left(\frac{4}{2}\right) = 75.96° + 53.13° - 90° - 63.43° = -24.34°$$

Example 8.49

Plot amplitude and phase response for

$$F(s) = \frac{s}{s+10}$$

Solution

$$F(j\omega) = \frac{j\omega}{j\omega + 10}$$

$$|F(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 100}}$$

ω	$\lvert F(j\omega)\rvert$
0	0
10	0.707
100	0.995
1000	1

The amplitude response is shown in Fig. 8.98.

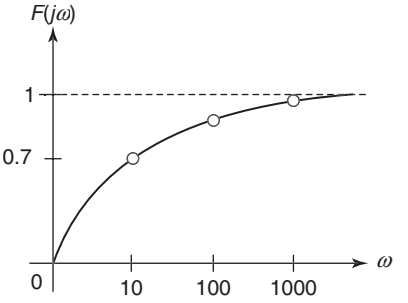


Fig. 8.98

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) = 90° - \tan^{-1}\left(\frac{\omega}{10}\right)$$

ω	$\phi(\omega)$
0	90°
10	45°
100	5.7°
1000	0°

The phase response is shown in Fig. 8.99

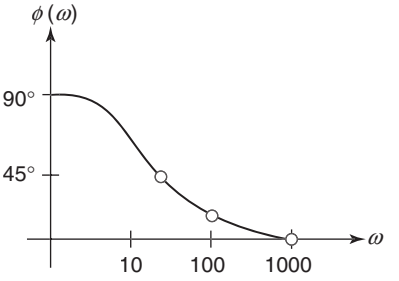


Fig. 8.99

Example 8.50 *Sketch amplitude and phase response for* $F(s) = \dfrac{s+10}{s-10}$

Solution

$$F(j\omega) = \frac{j\omega + 10}{j\omega - 10}$$

$$|F(f\omega)| = \frac{\sqrt{\omega^2 + 100}}{\sqrt{\omega^2 + 100}}$$

For all ω, magnitude is unity.
The amplitude response is shown in Fig. 8.100.

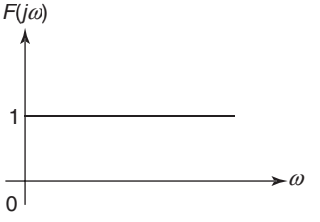


Fig. 8.100

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(-\frac{\omega}{10}\right) = 2\tan^{-1}\left(\frac{\omega}{10}\right)$$

The phase response is shown in Fig. 8.101.

ω	$\phi(\omega)$
0	0°
10	90°
100	168.6°
1000	178.9°

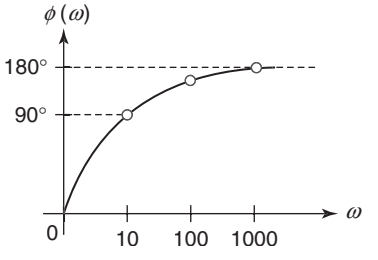


Fig. 8.101

Exercises

8.1 Determine the driving-point impedance $\frac{V_1}{I_1}$, transfer impedance $\frac{V_2}{I_1}$ and voltage transfer ratio $\frac{V_2}{V_1}$ for the network shown in Fig. 8.102.

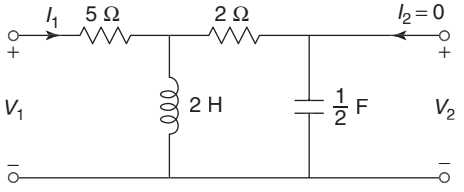


Fig. 8.102

$$\left[\frac{V_1}{I_1}=\frac{7s^2+7s+5}{s2+s+1};\frac{V_2}{I_1}=\frac{2s}{s^2+s+1};\ \frac{V_2}{V_1}=\frac{2s}{7s^2+7s+5}\right]$$

8.2 For the network shown in Fig. 8.103, determine $\frac{V_2}{V_1}$ and $\frac{V_2}{I_1}$.

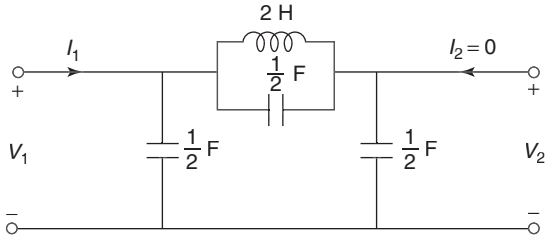


Fig. 8.103

$$\left[\frac{V_2}{V_1}=\frac{s^2+1}{2s^2+1};\frac{V_2}{I_1}=\frac{2s^2+2}{s(3s^2+2)}\right]$$

8.3 Find the open-circuit transfer impedance Z_{21} and open-circuit voltage ratio G_{21} for the ladder network shown in Fig. 8.104.

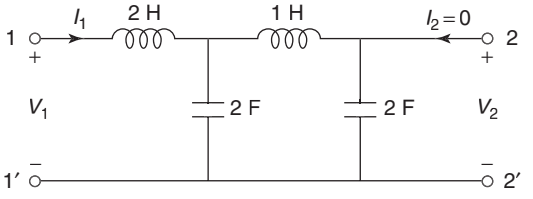


Fig. 8.104

$$\left[Z_{21}=\frac{1}{2s^3+3s},G_{21}=\frac{1}{4s^4+7s^2+1}\right]$$

8.4 For the two-port network shown in Fig. 8.105, determine Z_{11}, Z_{21} and voltage transfer ratio $G_{21}(s)$.

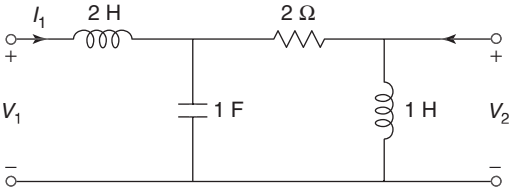


Fig. 8.105

$$\left[Z_{11}=\frac{2s^3+4s^2+3s+2}{s^2+2s+1},Z_{21}=\frac{s}{s^2+2s+1},\ G_{21}=\frac{s}{2s^3+4s^2+3s+2}\right]$$

8.5 Draw the pole-zero diagram of the following network functions:

(i) $F(s)=\dfrac{s^2+4}{s^2+6s+4}$

(ii) $F(s)=\dfrac{5s-12}{s^2+4s+13}$

(iii) $F(s)=\dfrac{s+1}{(s^2+2s+2)^2}$

(iv) $F(s)=\dfrac{s(s^2+5)}{s^4+2s^2+1}$

(v) $F(s) = \dfrac{s^2 + s + 2}{s^4 + 5s^3 + 6s^2}$

(vi) $F(s) = \dfrac{s^2 - s}{s^3 + 2s^2 - s - 2}$

(vii) $F(s) = \dfrac{s^2 + 3s + 2}{s^2 + 3s}$

(viii) $F(s) = \dfrac{(s^2 + 4)(s + 1)}{(s^2 + 1)(s^2 + 2s + 5)}$

8.6 For the network shown in Fig. 8.106, draw the pole-zero plot of the impedance function $Z(s)$.

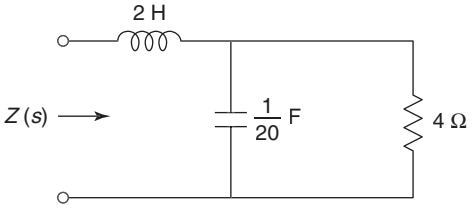


Fig. 8.106

$$Z(s) = \left[\frac{(s + 2.5 - j1.94)(s + 2.5 + j1.94)}{s + 5}\right]$$

8.7 For the network shown in Fig. 8.107, draw the pole-zero plot of driving-point impedance function $Z(s)$.

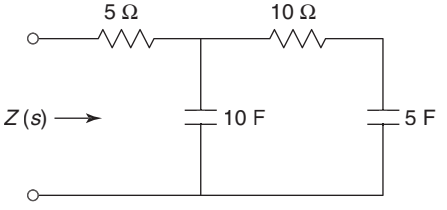


Fig. 8.107

$$\left[Z(s) = \frac{5(s + 0.01)(s + 0.04)}{s(s + 0.03)}\right]$$

8.8 Find the driving-point impedance of the network shown in Fig. 8.108. Also, find poles and zeros.

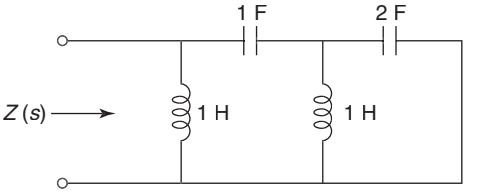


Fig. 8.108

$$\left[Z(s) = \frac{1.5s(s^2 + 0.33)}{(s^2 + 1.707)(s^2 + 0.293)}\right]$$

8.9 Find network functions $\dfrac{V_2}{V_1}$ and $\dfrac{V_1}{I_1}$ for the network shown in Fig. 8.109 and plot poles and zeros of $\dfrac{V_2(s)}{V_1(s)}$.

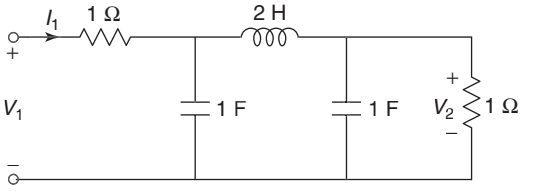


Fig. 8.109

$$\left[\frac{V_2}{V_1} = \frac{1}{2(s^3 + 2s^2 + 2s + 1)}, \frac{V_1}{I_1} = \frac{2(s^3 + 2s^2 + 2s + 1)}{2s^3 + 2s^2 + 2s + 1}\right]$$

8.10 For the network shown in Fig. 8.110, determine $\dfrac{V_1}{I_1}$ *and* $\dfrac{V_2}{I_1}$. Plot the poles and zeros of $\dfrac{V_2}{I_1}$.

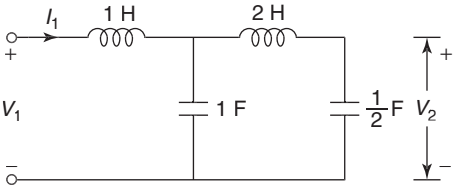


Fig. 8.110

$$\left[\frac{V_1}{I_1} = \frac{2s^4 + 5s^2 + 2}{2s^3 + 3s}, \frac{V_2}{I_1} = \frac{2}{2s^3 + 3s}\right]$$

8.11 For the network shown in Fig. 8.111, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the pole and zeros for $\frac{V_2}{V_1}$.

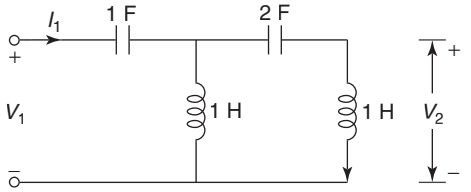


Fig. 8.111

$$\left[\frac{V_1}{I_1} = \frac{s^4 + 3s^2 + 1}{2s^3 + s}\right]$$

8.12 For the network shown in Fig. 8.112, plot the poles and zeros of transfer impedance function.

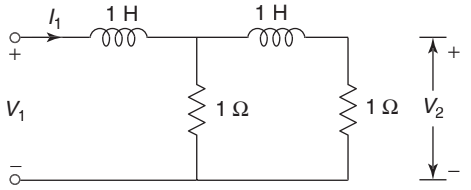


Fig. 8.112

$$\left[\frac{V_2}{I_1} = \frac{1}{s+2}\right]$$

8.13 For the network shown in Fig. 8.113, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the poles and zeros of transfer impedance function.

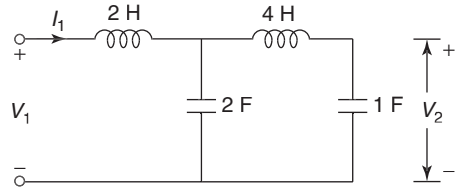


Fig. 8.113

$$\left[\frac{V_1}{I_1} = \frac{16s^4 + 10s^2 + 1}{8s^3 + 3s}, \frac{V_2}{I_1} = \frac{1}{8s^3 + 3s}, \frac{V_2}{V_1} = \frac{1}{16s^4 + 10s^2 + 1}\right]$$

8.14 Obtain the impedance function for which the pole-zero diagram is shown in Fig. 8.114.

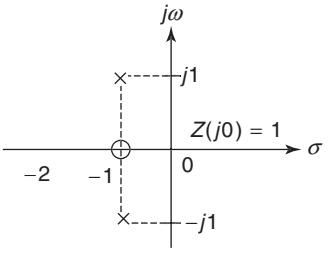


Fig. 8.114

$$\left[Z(s) = \frac{2(s+1)}{s^2 + 2s + 2}\right]$$

8.15 For the network shown in Fig. 8.115, poles and zeros of driving point function $Z(s)$ are,

Poles: $(-1 \pm j4)$; zero: -2

If $Z(j0) = 1$, find the values of R, L and C.

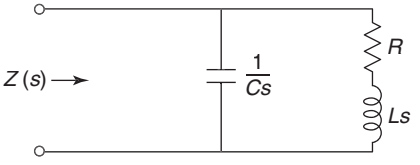


Fig. 8.115

$$\left[1\Omega, 0.5\text{ H}, \frac{2}{17}\text{ F}\right]$$

8.16 For the two-port network shown in Fig. 8.116, find R_1, R_2 and C. $\frac{V_2}{V_1} = \frac{0.2}{s^2 + 3s + 2}$

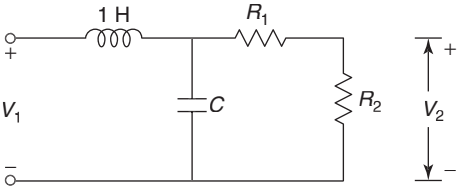


Fig. 8.116

$$\left[\frac{3}{5}\ \Omega, \frac{1}{15}\ \Omega, 0.5\text{ F}\right]$$

8.17 For the given network function, draw the pole-zero diagram and hence obtain the time domain voltage.

$$V(s) = \frac{5\,(s+5)}{(s+2)(s+7)}$$

$$[v(t) = 3e^{-2t} + 2e^{-7t}]$$

8.18 A transfer function is given by $Y(s) = \dfrac{10s}{(s+5+j15)(s+5-j15)}$. Find time-domain response using graphical method.

$$\left[5.26\angle 18.4^\circ\, e^{-(5+j15)t} + 5.26\angle -18.4^\circ\, e^{-(5-j15)t}\right]$$

Objective-Type Questions

8.1 Of the four networks N_1, N_2, N_3 and N_4 of Fig. 8.117, the networks having identical driving-point functions are

(a) N_1 and N_2 (b) N_2 and N_4
(c) N_1 and N_3 (d) N_1 and N_4

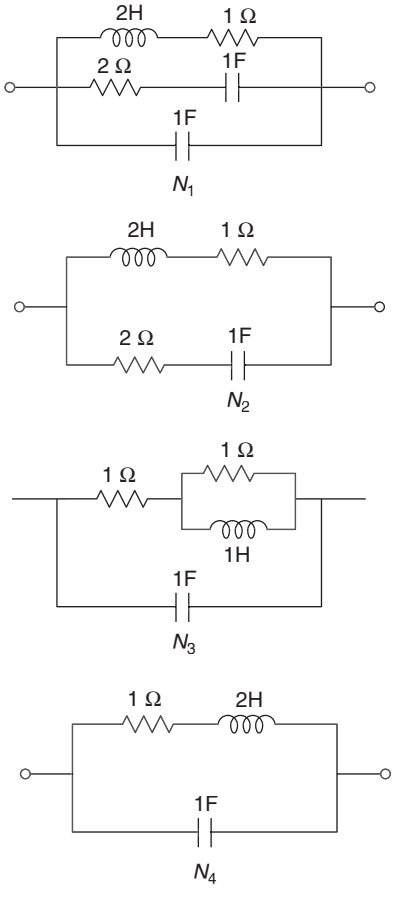


Fig. 8.117

8.2 The driving-point impedance $Z(s)$ of a network has the pole-zero locations as shown in Fig. 8.118. If $Z(0) = 3$, then Z(s) is

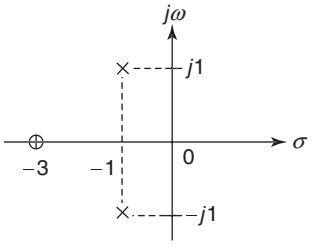


Fig. 8.118

(a) $\dfrac{3(s+3)}{s^2+2s+3}$ (b) $\dfrac{2(s+3)}{s^2+2s+2}$

(c) $\dfrac{3(s-3)}{s^2-2s-2}$ (d) $\dfrac{2(s-3)}{s^2-2s-3}$

8.3 For the circuit shown in Fig. 8.119, the initial conditions are zero. Its transfer function $H(s) = \dfrac{V_0(s)}{V_1(s)}$ is

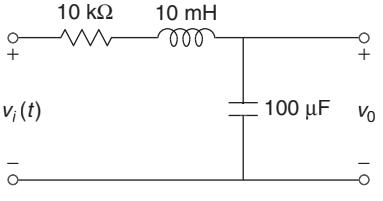


Fig. 8.119

(a) $\dfrac{1}{s^2+10^6 s+10^6}$ (b) $\dfrac{10^6}{s^2+10^3 s+10^6}$

(c) $\dfrac{10^3}{s^2+10^3 s+10^6}$ (d) $\dfrac{10^6}{s^2+10^6 s+10^6}$

8.4 In Fig. 8.120, assume that all the capacitors are initially uncharged. If $v_i(t) = 10\ u(t)$, then $v_0(t)$ is given by

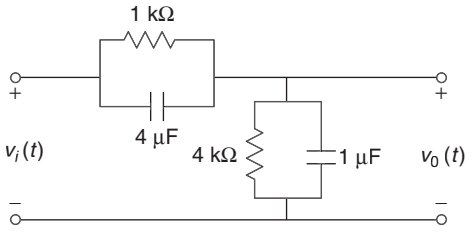


Fig. 8.120

(a) $8\ e^{-0.004\,t}$ (b) $8\ (1-e^{-0.004\,t})$

(c) $8\ u(t)$ (d) 8

8.5 A system is represented by the transfer function $\dfrac{10}{(s+1)(s+2)}$. The dc gain of this system is

(a) 1 (b) 2

(c) 5 (d) 10

8.6 Which one of the following is the ratio $\dfrac{V_{24}}{V_{13}}$ of the network shown in Fig. 8.121.

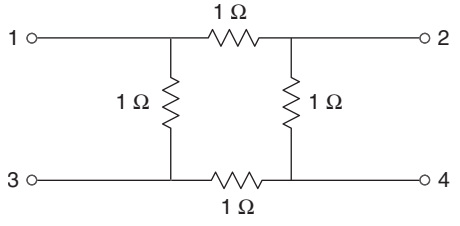


Fig. 8.121

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{3}{4}$ (d) $\dfrac{4}{3}$

8.7 A network has response with time as shown in Fig. 8.122. Which one of the following diagrams represents the location of the poles of this network?

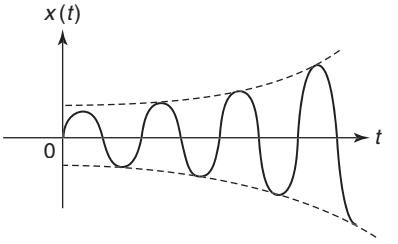


Fig. 8.122

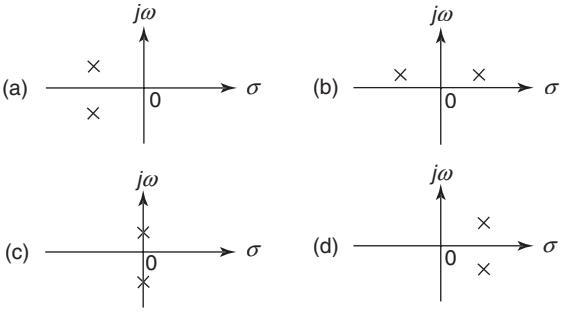


Fig. 8.123

8.8 The transfer function of a low-pass *RC* network is

(a) $(RCs)\,(1+RCs)$ (b) $\dfrac{1}{1+RCs}$

(c) $\dfrac{RCs}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

8.9 The driving-point admittance function of the network shown in Fig. 8.124 has a

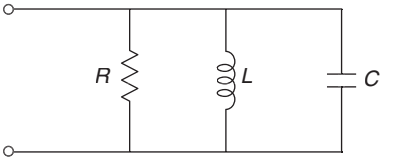


Fig. 8.124

(a) pole at $s = 0$ and zero at $s = \infty$
(b) pole at $s = 0$ and pole at $s = \infty$
(c) pole at $s = \infty$ and zero at $s = 0$
(d) pole at $s = \infty$ and zero at $s = \infty$

8.10 The transfer function $Y_{12}(s) = \dfrac{I_2(s)}{V_1(s)}$ for the network shown in Fig. 8.125 is

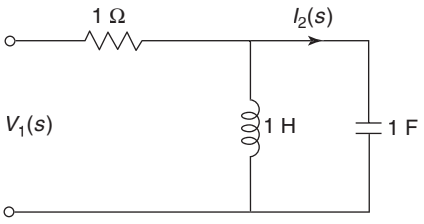


Fig. 8.125

(a) $\dfrac{s^2}{s^2+s+1}$ (b) $\dfrac{s}{s+1}$

(c) $\dfrac{1}{s+1}$ (d) $\dfrac{s+1}{s^2+1}$

8.11 As the poles of a network shift away from the x axis, the response

(a) remains constant

(b) becomes less oscillating

(c) becomes more oscillating

(d) none of these

Answers to Objective-Type Questions

8.1. (c) 8.2. (b) 8.3. (d) 8.4. (c) 8.5. (c) 8.6. (a) 8.7. (d)

8.8. (b) 8.9. (a) 8.10. (a) 8.11. (c)

9

Network Synthesis

9.1 INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

9.2 HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

   are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.
2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

Example 9.1 *State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Example 9.2 *Test whether the polynomial* $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s \overline{)\, s^4 + 5s^2 + 4 \;(\, s}$$
$$\underline{s^4 + 3s^2}$$
$$2s^2 + 4 \overline{)\, s^3 + 3s \;(\, \frac{1}{2}s}$$
$$\underline{s^3 + 2s}$$
$$s \overline{)\, 2s^2 + 4 \;(\, 2s}$$
$$\underline{2s^2}$$
$$4 \overline{)\, s \;(\, \frac{1}{4}s}$$
$$\underline{s}$$
$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 9.3 *Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\; n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2 \overline{)} s^3 + 5s \left(\frac{1}{4}s$$

$$s^3 + \frac{2}{4}s$$

$$\frac{9}{2}s \overline{)} 4s^2 + 2 \left(\frac{8}{9}s$$

$$4s^2$$

$$2 \overline{)} \frac{9}{2}s \left(\frac{9}{4}s$$

$$\frac{9}{2}s$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 9.4 *Test whether the polynomial $P(s) = s^4 + s^3 + 3s^2 + 2s + 12$ is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 3s^2 + 12$

Odd part of $P(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 2s \overline{)} s^4 + 3s^2 + 12 \left(s$$

$$s^4 + 2s^2$$

$$s^2 + 12 \overline{)} s^3 + 2s \left(s$$

$$s^3 + 12s$$

$$-10s \overline{)} s^2 + 12 \left(-\frac{1}{10}s$$

$$s^2$$

$$12 \overline{)} -10s \left(-\frac{10}{12}s$$

$$-10s$$

$$0$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 9.5 *Prove that polynomial* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is not Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 2s^2 + 2$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{r} s^3 + 3s \overline{)s^4 + 2s^2 + 2} \ (s \\ \underline{s^4 + 3s^2} \\ -s^2 + 2 \overline{)s^3 + 3s} \ (-s \\ \underline{s^3 - 2s} \\ 5s \overline{)-s^2 + 2} \ \left(-\frac{1}{5}s\right. \\ \underline{-s^2} \\ 2 \overline{)5s} \ \left(\frac{5}{2}s\right. \\ \underline{5s} \\ 0 \end{array}$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 9.6 *Prove that polynomial* $P(s) = 2s^4 + 5s^3 + 6s^2 + 3s + 1$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^4 + 6s^2 + 1$

Odd part of $P(s) = n(s) = 5s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{r} 5s^3 + 3s \overline{)2s^4 + 6s^2 + 1} \ \left(\frac{2}{5}s\right. \\ \underline{2s^4 + \frac{6}{5}s^2} \\ \frac{24}{5}s^2 + 1 \overline{)5s^3 + 3s} \ \left(\frac{25}{24}s\right. \\ \underline{5s^3 + \frac{25}{24}s} \end{array}$$

$$\frac{47}{24}s\overline{)\frac{24}{5}s^2+1}\left(\frac{576}{235}s\right.$$
$$\underline{\frac{24}{5}s^2}$$
$$1\overline{)\frac{47}{24}s}\left(\frac{24}{47}s\right.$$
$$\underline{\frac{47}{24}s}$$
$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 9.7 *Test whether the polynomial* $P(s)=s^4+7s^3+6s^2+21s+8$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+6s^2+8$

Odd part of $P(s)=n(s)=7s^3+21s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$7s^3+21s\overline{)s^4+6s^2+8}\left(\frac{1}{7}s\right.$$
$$\underline{s^4+3s^2}$$
$$3s^2+8\overline{)7s^3+21s}\left(\frac{7}{3}s\right.$$
$$\underline{7s^3+\frac{56}{3}s}$$
$$\frac{7}{3}s\overline{)3s^2+8}\left(\frac{9}{7}s\right.$$
$$\underline{3s^2}$$
$$8\overline{)\frac{7}{3}s}\left(\frac{7}{24}s\right.$$
$$\underline{\frac{7}{3}s}$$
$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 9.8 *Check whether* $P(s)=s^4+5s^3+5s^2+4s+10$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+5s^2+10$

Odd part of $P(s)=n(s)=5s^3+4s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{r}
\left(\frac{1}{5}s\right. \\
5s^3 + 4s\Big)\, s^4 + 5s^2 + 10 \\
\underline{s^4 + \frac{4}{5}s^2} \\
\left(\frac{25}{21}s\right. \\
\frac{21}{5}s^2 + 10\Big)\, 5s^3 + 4s \\
\underline{5s^3 + \frac{250}{21}s} \\
\left(-\frac{441}{830}s\right. \\
-\frac{166}{21}s\Big)\, \frac{21}{5}s^2 + 10 \\
\underline{\frac{21}{5}s^2} \\
\left(-\frac{166}{210}s\right. \\
10\Big)\, -\frac{166}{21}s \\
\underline{-\frac{166}{21}s} \\
0
\end{array}$$

Since the last two quotient terms are negative, the polynomial $P(s)$ is not Hurwitz.

Example 9.9 *Test whether the polynomial $s^5 + 3s^3 + 2s$ is Hurwitz.*

Solution Since the given polynomial contains odd functions only, it is not possible to perform a continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 9s^2 + 2$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$\begin{array}{r}
\left(\frac{1}{5}s\right. \\
5s^4 + 9s^2 + 2\Big)\, s^5 + 3s^3 + 2s \\
\underline{s^5 + \frac{9}{5}s^3 + \frac{2}{5}s} \\
\left(\frac{25}{6}s\right. \\
\frac{6}{5}s^3 + \frac{8}{5}s\Big)\, 5s^4 + 9s^2 + 2
\end{array}$$

$$
\begin{array}{r}
5s^4+\frac{20}{3}s^2 \\
\hline
\frac{7}{3}s^2+2\Big)\frac{6}{5}s^3+\frac{8}{5}s\Big(\frac{18}{35}s \\
\frac{6}{5}s^3+\frac{36}{35}s \\
\hline
\frac{20}{35}s\Big)\frac{7}{3}s^2+2\Big(\frac{49}{12}s \\
\frac{7}{3}s^2 \\
\hline
2\Big)\frac{20}{35}s\Big(\frac{10}{35}s \\
\frac{20}{35}s \\
\hline
0
\end{array}
$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 9.10 *Test whether the polynomial P(s) is Hurwitz.*

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$
\begin{array}{r}
5s^4+3s^2+1\Big)s^5+\ s^3+s\Big(\frac{1}{5}s \\
s^5+\frac{3}{5}s^3+\frac{1}{5}s \\
\hline
\frac{2}{5}s^3+\frac{4}{5}s\Big)5s^4+\ 3s^2+1\Big(\frac{25}{2}s \\
5s^4+10s^2 \\
\hline
-7s^2+1\Big)\frac{2}{5}s^3+\ \frac{4}{5}s\Big(-\frac{2}{35}s \\
\frac{2}{5}s^3-\frac{2}{35}s \\
\hline
\end{array}
$$

$$\frac{26}{35}s\Big) -7s^2+1\Big(-\frac{245}{26}s$$

$$\underline{-7s^2}$$

$$1\Big)\frac{26}{35}s\Big(\frac{26}{35}s$$

$$\underline{\frac{26}{35}s}$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

Example 9.11 *Test the polynomial P(s) of Hurwitz property.*

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4$$

Solution Even part of $P(s) = m(s) = s^6 + 8s^4 + 17s^2 + 4$

Odd part of $P(s) = n(s) = 3s^5 + 15s^3 + 12s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5+15s^3+12s\Big)s^6+8s^4+17s^2+4\ \Big(\frac{1}{3}s$$

$$\underline{s^6+5s^4+4s^2}$$

$$3s^4+13s^2+4\Big)3s^5+15s^3+12s\Big(s$$

$$\underline{3s^5+13s^3+4s}$$

$$2s^3+8s\Big)3s^4+13s^2+4\Big(\frac{3}{2}s$$

$$\underline{3s^4+12s^2}$$

$$s^2+4\Big)2s^3+8s\Big(2s$$

$$\underline{2s^3+8s}$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor (s^2+4).

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4 = (s^2+4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\frac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\frac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let $$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$

$$m_2(s) = s^4 + 4s^2 + 1$$

$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$\begin{array}{r}
\frac{1}{3}s \\
3s^3+3s \;\big)\; s^4+4s^2+1 \;\big(\\
\underline{s^4+\ s^2} \\
3s^2+1 \;\big)\; 3s^3+3s \;\big(\; s \\
\underline{3s^3+s} \\
2s \;\big)\; 3s^2+1 \;\big(\; \frac{3}{2}s \\
\underline{3s^2} \\
1 \;\big)\; 2s \;\big(\; 2s \\
\underline{2s} \\
0
\end{array}$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.

Hence, $P(s) = (s^2+4)(s^4+3s^3+4s^2+3s+1)$ is Hurwitz.

Example 9.12 *Test whether the polynomial* $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + s^4 + 8s^2 + 4$

Odd part of $P(s) = n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$$Q(s) = \frac{n(s)}{m(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
2s^6+s^4+8s^2+4\,\big)\,s^7+2s^5+4s^3+8s\,\big(\frac{1}{2}s \\
\quad\underline{s^7+\frac{1}{2}s^5+4s^3+2s} \\
\quad\quad \frac{3}{2}s^5 \qquad +6s\,\big)\,2s^6+s^4+8s^2+4\,\big(\frac{4}{3}s \\
\quad\quad\quad\underline{2s^6 \qquad +8s^2} \\
\quad\quad\quad\quad s^4 \qquad +4\,\big)\,\frac{3}{2}s^5+6s\,\big(\frac{3}{2}s \\
\quad\quad\quad\quad\quad\underline{\frac{3}{2}s^5+6s} \\
\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

Since the division has terminated abruptly it indicates a common factor $s^4 + 4$. The polynomial can be written as

$$P(s) = (s^4 + 4)(s^3 + 2s^2 + 2s + 1)$$

If both the factor are Hurwitz, $P(s)$ will be Hurwitz.
In the polynomial $(s^4 + 4)$, the terms s^3, s^2 and s are missing. Hence, it is not Hurwitz.
Therefore, $P(s)$ is not Hurwitz.

Example 9.13 *Test whether the polynomial* $2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + 13s^4 + 56s^2 + 25$

Odd part of $P(s) = n(s) = s^5 + 6s^3 + 25s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^5+6s^3+25s)2s^6+13s^4+56s^2+25(2s \\
\quad\underline{2s^6+12s^4+50s^2} \\
\quad\quad s^4+6s^2 \quad +25)s^5+6s^3+25s(s \\
\quad\quad\quad\underline{s^5+6s^3+25s} \\
\quad\quad\quad\quad 0
\end{array}
$$

The division has terminated abruptly.

$$P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25 = (s^4 + 6s^2 + 25)\,(2s^2 + s + 1)$$

Let $P_1(s) = s^4 + 6s^2 + 25$

Since $P_1(s)$ contains only even functions, we have to find the continued fraction expansion of $\frac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 4s^3 + 12s$$

By continued fraction expansion,

$$\begin{array}{l}
4s^3+12s\,\big)\,s^4+6s^2+25\,\big(\,\frac{1}{4}s \\
\quad\underline{s^4+3s^2} \\
\quad\quad 3s^2+25\,\big)\,4s^3+\;\;12s\,\big(\,\frac{4}{3}s \\
\quad\quad\quad\underline{4s^3+\frac{100}{3}s} \\
\quad\quad\quad\quad -\frac{64}{3}s\,\big)\,3s^2+25\,\big(-\frac{9}{64}s \\
\quad\quad\quad\quad\quad\underline{3s^2} \\
\quad\quad\quad\quad\quad\quad 25\,\big)-\frac{64}{3}s\,\big(-\frac{64}{75}s \\
\quad\quad\quad\quad\quad\quad\quad\underline{-\frac{64}{3}s} \\
\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}$$

Since two of the quotient terms are negative, $P_1(s)$ is not Hurwitz.
We need not test the other factor $(2s^2 + s + 1)$ for being Hurwitz.
Hence, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\quad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

s^n	a_n	a_{n-2}	a_{n-4}	…
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	…
s^{n-2}	b_n	b_{n-1}	b_{n-2}	…
s^{n-3}	c_n	c_{n-1}		…
.	.			
.	.			
.	.			
.	.			
s^1	.			
s^0	.			

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where

$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Example 9.14 *Test whether* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

s^4	1	6	8
s^3	7	21	
s^2	3	8	
s^1	$\frac{7}{3}$	0	
s^0	8		

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Example 9.15 *Determine whether* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

s^4	1	2	2
s^3	1	3	
s^2	−1	2	
s^1	5	0	
s^0	2		

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 9.16 *Test whether* $P(s) = s^5 + 2s^4 + 4s^3 + 6s^2 + 2s + 5$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	4	2
s^4	2	6	5
s^3	1	−0.5	
s^2	7	5	
s^1	−1.21		
s^0	5		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 9.17 *Test whether the polynomial $P(s) = s^5 + s^3 + s$ is Hurwitz.*

Solution The given polynomial contains odd functions only.

$$P'(s) = 5s^4 + 3s^2 + 1$$

The Routh array is given by,

s^5	1	1	1
s^4	5	3	1
s^3	0.4	0.8	
s^2	−7	1	
s^1	0.86		
s^0	1		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 9.18 *Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.*

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

s^8	1	5	2	3	1
s^7	8	30	8	6	0
s^6	1.25	1	2.25	1	
s^5	23.6	−6.4	−0.4	0	
s^4	1.33	2.27	1		
s^3	−46.6	−18.14	0		
s^2	1.75	1			
s^1	8.49				
s^0	1				

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 9.19 *Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.*

Solution The Routh array is given by,

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	0	0	
s^0			

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.
(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.
(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	96	0	
s^0	48		

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Example 9.20 *Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution The Routh array is given by,

s^6	2	13	56	25
s^5	1	6	25	
s^4	1	6	25	
s^3	0	0	0	
s^2				
s^1				
s^0				

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

s^6	2	13	56	25
s^5	1	6	25	
s^4	1	6	25	
s^3	4	12		
s^2	3	25		
s^1	–21.3			
s^0	25			

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 9.21 *Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.*

Solution The Routh array is given by,

s^4	1	a	3
s^3	1	2	
s^2	$a-2$	3	
s^1	$\frac{2a-7}{a-2}$		
s^0	3		

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$
$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$
$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \frac{7}{2}$.

Example 9.22 *Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.*

Solution The Routh array is given by,

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3}$	0
s^0	k	

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e., $$\frac{6-k}{3} > 0$$

$$6 - k > 0$$

i.e., $k < 6$ and $k > 0$
Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

9.3 POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.

(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,
Re $F(s) \geq 0$ for Re$(s) \geq 0$

9.3.1 Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\frac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

9.3.2 Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. Re $F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s - j\omega_0} + \frac{K_1^*}{s + j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} = \frac{m_1 + n_1}{m_2 + n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s) = \frac{m_1 + n_1}{m_2 + n_2}\frac{m_2 - n_2}{m_2 - n_2} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} + \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$\text{Od } F(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\big|_{s=j\omega} = j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.

The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2 - n_2^2\big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1 m_2 - n_1 n_2\big|_{s=j\omega} = A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

Example 9.23 *Test whether* $F(s) = \dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s = -1$ and zero at $s = -3$ as shown in Fig. 9.1.

Thus, pole and zero are in the left half of the s-plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

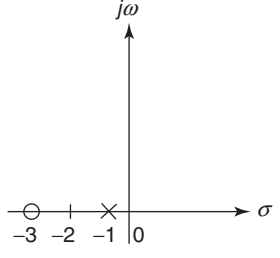


Fig. 9.1

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (3)(1) - (s)(s) \,|_{s=j\omega} = 3 - s^2 \,|_{s=j\omega} = 3 + \omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.24

Test whether $F(s) = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14} = \dfrac{(s+5)(s+1)}{(s+7)(s+2)}$

The function $F(s)$ has poles at $s = -7$ and $s = -2$ and zeros at $s = -5$ an $s = -1$ as shown in Fig. 9.2.

Thus, all the poles and zeros are in the left half of the s plane.

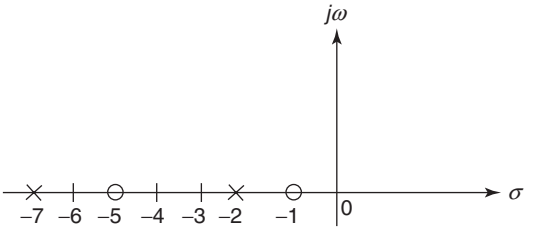


Fig. 9.2

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2 + 5)(s^2 + 14) - (6s)(9s) \,|_{s=j\omega} = s^4 - 35s^2 + 70 \,|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.25

Test whether $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)} = \dfrac{s^3 + 8s^2 + 15s}{s^2 + 5s + 4}$

The function $F(s)$ has poles at $s = -1$ and $s = -4$ and zeros at $s = 0$, $s = -3$ and $s = -5$ as shown in Fig. 9.3.

Thus, all the poles and zeros are in the left half of the s plane.

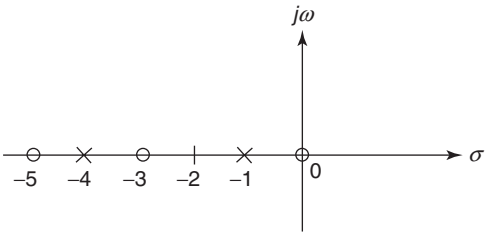


Fig. 9.3

(b) There is no pole on the $j\omega$ axis, hence the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 8s^2$

Odd part of $N(s) = n_1 = s^3 + 15s$

Even part of $D(s) = m_2 = s^2 + 4$

Odd part of $D(s) = n_2 = 5s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (8s^2)(s^2+4) - (s^3+15s)(5s)\,|_{s=j\omega} = 3s^4 - 43s^2\,|_{s=j\omega} = 3\omega^4 + 43\omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.26

Test whether $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 9.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

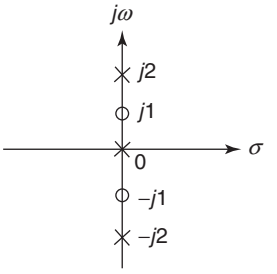


Fig. 9.4

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1 = s\,F(s)\,|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)\,|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+1)(0) - (0)(s^3+4s)\,|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.27

Test whether $F(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \dfrac{2s^3 + 2s^2 + 3s + 2}{(s + j1)(s - j1)}$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 2s^2 + 2$

Odd part of $N(s) = n(s) = 2s^3 + 3s$

By continued fraction expansion,

$$\begin{array}{r} s \\ 2s^2+2 \;\overline{)\; 2s^3+3s} \\ \underline{2s^3+2s} \\ s \\ 2s^2+2 \;\overline{)\; 2s} \\ \underline{2s^2} \\ 2 \\ s \;\overline{)\; \left(\frac{1}{2}s\right.} \\ \underline{s} \\ 0 \end{array}$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s = -j1$ and $s = j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$s^2+1\Big)2s^3+2s^2+3s+2\Big(2s+2$$
$$\underline{2s^3\quad\quad+2s}$$
$$2s^2+\ s+2$$
$$\underline{2s^2\quad\quad+2}$$
$$s$$

$$F(s)=2s+2+\frac{s}{s^2+1}$$

By partial-fraction expansion,

$$F(s)=2s+2+\frac{K_1}{s+j1}+\frac{K_1^*}{s-j1}$$

$$K_1=(s+j1)F(s)\,|_{s=-j1}=\frac{-j1}{-j1-j1}=\frac{1}{2}$$

$$K_1^*=K_1=\frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s)=m_1=2s^2+2$

Odd part of $N(s)=n_1=2s^3+3s$

Even part of $D(s)=m_2=s^2+1$

Odd part of $D(s)=n_2=0$

$$A(\omega^2)=m_1m_2-n_1n_2\,|_{s=j\omega}=(2s^2+2)(s^2+1)-(2s^3+3s)(0)\,|_{s=j\omega}=2s^4+4s^2+2\;|_{s=j\omega}=2(\omega^4-2\omega^2+1)$$
$$=2(\omega^2-1)^2$$

$A(\omega^2)\geq 0$ for all $\omega\geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.28 *Test whether* $F(s)=\dfrac{s^3+6s^2+7s+3}{s^2+2s+1}$ *is positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s^3+6s^2+7s+3}{s^2+2s+1}=\dfrac{s^3+6s^2+7s+3}{(s+1)(s+1)}$

Since a numerator polynomial cannot be easily factorized, we will test whether $N(s)$ is Hurwitz.

Even part of $N(s)=m(s)=6s^2+3$

Odd part of $N(s)=n(s)=s^3+7s$

By continued fraction expansion,

$$
\begin{array}{r}
6s^2+3\;\Big)\,s^3+7s\;\Big(\,\frac{1}{6}s \\
\underline{s^3+0.5s} \\
6.5s\;\Big)\,6s^2+3\;\Big(\,0.92s \\
\underline{6s^2} \\
3\;\Big)\,6.5s\;\Big(\,2.17s \\
\underline{6.5s} \\
0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that the zeros are in the left half of the s plane.

The function $F(s)$ has a double pole at $s = -1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 6s^2 + 3$

Odd part of $N(s) = n_1 = s^3 + 7s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 2s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (6s^2+3)(s^2+1) - (s^3+7s)(2s)\,|_{s=j\omega} = \; 4s^4 - 5s^2 + 3\,|_{s=j\omega} = 4\omega^4 + 5\omega^2 + 3$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 9.29

Test whether $F(s) = \dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s) = \frac{N(s)}{D(s)} = \frac{s^2+s+6}{s^2+s+1} = \frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s = -\frac{1}{2} \pm j\frac{\sqrt{23}}{2}$ and poles at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 6$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1m_2 - n_1n_2\,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s)\,|_{s=j\omega} = s^4 + 6s^2 + 6\,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$
This condition is not satisfied.
Hence, the function $F(s)$ is not positve real.

Example 9.30 *Test whether* $F(s) = \dfrac{s^2+4}{s^3+3s^2+3s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+4}{s^3+3s^2+3s+1} = \dfrac{(s+j2)(s-j2)}{(s+1)^3}$

The function $F(s)$ has two zeros at $s = \pm j2$ and three poles at $s = -1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 4$
Odd part of $N(s) = n_1 = 0$
Even part of $D(s) = m_2 = 3s^2 + 1$
Odd part of $D(s) = n_2 = s^3 + 3s$

$$A(\omega^2) = m_1m_2 - n_1n_2\,|_{s=j\omega} = (s^2+4)(3s^2+1) - (0)(s^3+3s)\,|_{s=j\omega} = 3s^4 + 13s^2 + 4\,|_{s=j\omega} = 3\omega^4 - 13\omega^2 + 4$$

For $\omega = 1$, $A(\omega)^2 = 3 - 13 + 4 = -6$
This condition is not satisfied.
Hence, the function $F(s)$ is not positive real.

Example 9.31 *Test whether* $F(s) = \dfrac{s^3+5s}{s^4+2s^2+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+5s}{s^4+2s^2+1} = \dfrac{s(s^2+5)}{(s^2+1)^2} = \dfrac{s(s+j\sqrt{5})(s-j\sqrt{5})}{(s\pm j1)(s\pm j1)}$

The function $F(s)$ has zeros at $s = 0$, $s = \pm j\sqrt{5}$ and two poles at $s = j1$ and two poles at $s = -j1$.

Thus, poles on the $j\omega$ axis are not simple.
Hence, the function $F(s)$ not positive real.

Example 9.32 *Test whether* $F(s) = \dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$ *is positive real function.*

Solution

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$$

Here, it is easier to prove that $N(s)$ and $D(s)$ are Hurwitz.

By Routh array,

s^4	1	1	2
s^3	3	1	
s^2	$\frac{2}{3}$	2	
s^1	-8		
s^0	2		

Since there is a sign change in the first column of the array, $N(s)$ is not Hurwitz. Thus, all the zeros are not in the left half of the s plane. The remaining two tests need not be carried out.
Hence, the function $F(s)$ is not positive real.

9.4 ELEMENTARY SYNTHESIS CONCEPTS

We know that impedances and admittances of passive networks are positive real functions. Hence, addition of impedances of the two passive networks gives a function which is also a positive real function. Thus, $Z(s) = Z_1(s) + Z_2(s)$ is a positive real function, if $Z_1(s)$ and $Z_2(s)$ are positive real functions. Similarly, $Y(s) = Y_1(s) + Y_2(s)$ is a positive real function, if $Y_1(s)$ and $Y_2(s)$ are positive real functions. There is a special terminology for synthesis procedure. We have,

$$Z(s) = Z_1(s) + Z_2(s)$$
$$Z_2(s) = Z(s) - Z_1(s)$$

Here, $Z_1(s)$ is said to have been removed from $Z(s)$ in forming the new function $Z_2(s)$ as shown in Fig. 9.5. If the removed network is associated with the pole or zero of the original network impedance then that pole or zero is also said to have been removed.
There are four important removal operations.

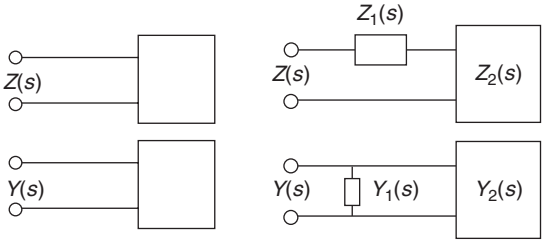


Fig. 9.5 *Network interpretation of the removal of impedance and admittance*

9.4.1 Removal of a Pole at Infinity

Consider an impedance function $Z(s)$ having a pole at infinity which means that the numerator polynomial is one degree greater than the degree of the denominator polynomial.

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Hs + \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

where $H = \frac{a_{n+1}}{b_n}$

Let $Z_1(s) = Hs$

and $Z_2(s) = \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Z(s) - Hs$

$Z_1(s) = Hs$ represents impedance of an inductor of value H. Hence, the removal of a pole at infinity corresponds to the removal of an inductor from the network of Fig. 9.6(a).

If the given function is an admittance function $Y(s)$, then $Y_1(s) = Hs$ represents the admittance of a capacitor $Y_C(s) = Cs$. The network for $Y_1(s)$ is a capacitor of value $C = H$ as shown in Fig. 9.6(b).

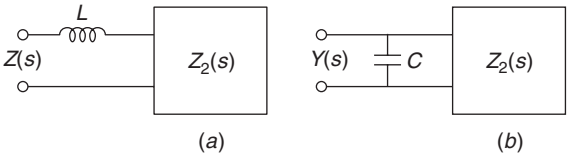


Fig. 9.6 *Network interpretation of the removal of a pole at infinity*

9.4.2 Removal of a Pole at Origin

If $Z(s)$ has a pole at the origin then it may be written as

$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_m s^m} = \frac{K_0}{s} + \frac{d_1 + d_2 s + \ldots + d_n s^{n-1}}{b_1 + b_2 s + \ldots + b_m s^{m-1}} = Z_1(s) + Z_2(s)$$

where $K_0 = \dfrac{a_0}{b_1}$

$Z_1(s) = \dfrac{K_0}{s}$ represents the impedance of a capacitor of value $\dfrac{1}{K_0}$.

If the given function is an admittance function $Y(s)$ then removal of $Y_1(s) = \dfrac{K_0}{s}$ corresponds to an inductor of value $\dfrac{1}{K_0}$.

Thus, removal of a pole from the impedance function $Z(s)$ at the origin corresponds to the removal of a capacitor, and from admittance function $Y(s)$ corresponds to removal of an inductor as shown in Fig. 9.7.

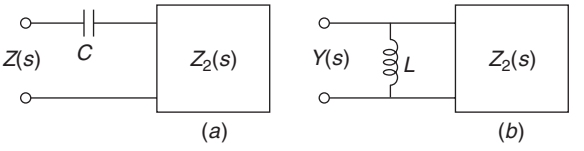


Fig. 9.7 *Network interpretation of the removal of a pole at origin*

9.4.3 Removal of Conjugate Imaginary Poles

If $Z(s)$ contains poles on the imaginary axis, i.e., at $s = \pm j\omega_1$ then $Z(s)$ will have factors $(s + j\omega_1)\,(s - j\omega_1) = s^2 + \omega_1^2$ in the denominator polynomial

$$Z(s) = \frac{p(s)}{(s^2 + \omega_1^2)\, q_1(s)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s + j\omega_1} + \frac{K_1^*}{s - j\omega_1} + Z_2(s)$$

For a positive real function, $j\omega$ axis poles must themselves be conjugate and must have equal, positive and real residues.

$$K_1 = K_1^*$$

Hence,
$$Z(s) = \frac{2K_1 s}{s^2 + \omega_1^2} + Z_2(s)$$

Thus,
$$Z_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}} = \frac{1}{Y_a + Y_b}$$

where $Y_a = \dfrac{s}{2K_1}$ is the admittance of a capacitor of value $C = \dfrac{1}{2K_1}$

and $Y_b = \dfrac{\omega_1^2}{2K_1 s}$ is the admittance of an inductor of value $L = \dfrac{2K_1}{\omega_1^2}$

If the given function is an admittance function $Y(s)$ then

$$Y_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{Z_a + Z_b} = \frac{1}{\frac{s}{2K_1} + \frac{\omega_1^2}{2K_1 s}}$$

where $Z_a = \frac{s}{2K_1}$ is the impedance of an inductor of value $L = \frac{1}{2K_1}$ and $Z_b = \frac{\omega_1^2}{2K_1 s}$ is the impedance of a capacitor of value $C = \frac{2K_1}{\omega_1^2}$.

Thus, removal of conjugate imaginary poles from impedance function $Z(s)$ corresponds to the removal of the parallel combination of $L - C$ and from admittance function $Y(s)$ corresponds to removal of series combination of $L - C$ as shown in Fig. 9.8.

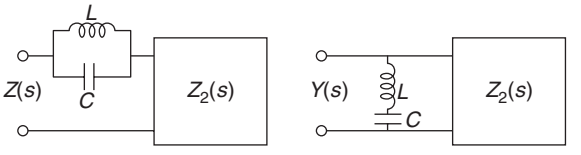


Fig. 9.8 *Network interpretation of the removal of conjugate imaginary poles*

9.4.4 Removal of a Constant

If a real number R_1 is subtracted from $Z(s)$ such that

$$Z_2(s) = Z(s) - R_1$$
$$Z(s) = R_1 + Z_2(s)$$

then R_1 represents a resistor.

If the given function is an admittance function $Y(s)$, then removal of $Y_1(s) = R_1$ represents a conductance of value R_1.

Thus, removal of a constant from impedance function $Z(s)$ corresponds to the removal of a resistance, and from admittance function $Y(s)$ corresponds to removal of a conductance.

Example 9.33 *Synthesize the impedance function* $Z(s) = \frac{s^3 + 4s}{s^2 + 2}$.

Solution

By long division of $Z(s)$,

$$s^2 + 2 \overline{)\, s^3 + 4s \,(\, s}$$
$$\underline{s^3 + 2s}$$
$$2s$$

$$Z(s) = s + \frac{2s}{s^2 + 2} = Z_1(s) + Z_2(s)$$

$Z_1(s) = s$ represents impedance of an inductor of value 1 H.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s^2 + 2}{2s} = \frac{s^2}{2s} + \frac{2}{2s} = \frac{1}{2}s + \frac{1}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = \frac{1}{2}s$ represents the admittance of a capacitor of value $\frac{1}{2}$ F.

$Y_4(s) = \frac{1}{s}$ represents the admittance of an inductor of value 1 H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.9.

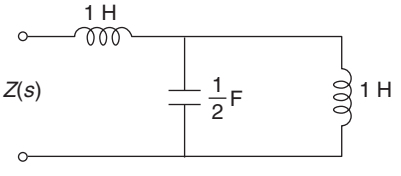


Fig. 9.9

Example 9.34 *Realise the network having impedance function*

$$Z(s) = \frac{s^2 + 2s + 10}{s(s+5)}$$

Solution

By long division of $Z(s)$,

$$s^2 + 5s \;\Big)\; s^2 + 2s + 10 \;\Big(\; \frac{2}{s}$$

$$\underline{\qquad 2s + 10}$$

$$s^2$$

$$Z(s) = \frac{2}{s} + \frac{s^2}{s^2 + 5s} = \frac{2}{s} + \frac{s}{s+5} = Z_1(s) + Z_2(s)$$

$Z_1(s) = \frac{2}{s}$ represents the impedance of capacitor of value $\frac{1}{2}$F.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s+5}{s} = 1 + \frac{5}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = 1$ represents the admittance of a resistor of value 1 Ω.

$Y_4(s) = \frac{5}{s}$ represents the admittance of an inductor of value $\frac{1}{5}$H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.10.

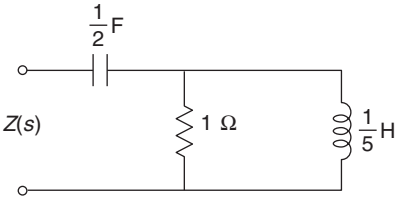


Fig. 9.10

Example 9.35 *Realise the network having impedance function* $Z(s) = \frac{6s^3 + 5s^2 + 6s + 4}{2s^3 + 2s}$.

Solution By long division of $Z(s)$,

$$2s^3 + 2s \;\Big)\; 6s^3 + 5s^2 + 6s + 4 \;\Big(\; 3$$

$$\underline{\qquad 6s^3 \qquad + 6s}$$

$$5s^2 \qquad + 4$$

$$Z(s) = 3 + \frac{5s^2 + 4}{2s^3 + 2s} = Z_1(s) + Z_2(s)$$

$Z_1(s) = 3$ represents the impedance of a resistor of value 3 Ω.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{2s^3 + 2s}{5s^2 + 4}$$

By long division of $Y_2(s)$,

$$5s^2+4\;\Big)\,2s^3+2s\;\Big(\,\frac{2}{5}s$$

$$\underline{2s^3+\frac{8}{5}s}$$

$$\frac{2}{5}s$$

$$Y_2(s)=\frac{2}{5}s+\frac{\frac{2}{5}s}{5s^2+4}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{2}{5}s$ represents the admittance of a capacitor of value $\frac{2}{5}$F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{5s^2+4}{\frac{2}{5}s}=\frac{25s^2+20}{2s}=\frac{25}{2}s+\frac{10}{s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{25}{2}s$ represents the impedance of an inductor of value $\frac{25}{2}$H.

$Z_6(s)=\frac{10}{s}$ represents the impedance of a capacitor of value $\frac{1}{10}$F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.11.

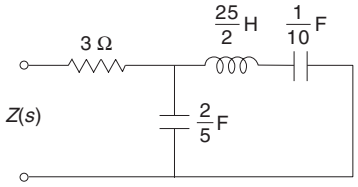


Fig. 9.11

Example 9.36

Realise the network having impedance function

$$Z(s)=\frac{s^4+10s^2+7}{s^3+2s}$$

Solution By long division of $Z(s)$,

$$s^3+2s\;\Big)\,s^4+10s^2+7\;\Big(\,s$$

$$\underline{s^4+2s^2}$$

$$8s^2+7$$

$$Z(s)=s+\frac{8s^2+7}{s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents the impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^3+2s}{8s^2+7}$$

By long division of $Y_2(s)$,

$$8s^2+7\Big)s^3+2s\Big(\frac{1}{8}s$$

$$\frac{s^3+\frac{7}{8}s}{\frac{9}{8}s}$$

$$Y_2(s)=\frac{1}{8}s+\frac{\frac{9}{8}s}{8s^2+7}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{1}{8}s$ represents the admittance of a capacitor of value $\frac{1}{8}$F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{8s^2+7}{\frac{9}{8}s}=\frac{64}{9}s+\frac{56}{9s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{64}{9}s$ represents the impedance of an inductor of value $\frac{64}{9}$H.

$Z_6(s)=\frac{56}{9s}$ represents the impedance of a capacitor of value $\frac{9}{56}$F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.12.

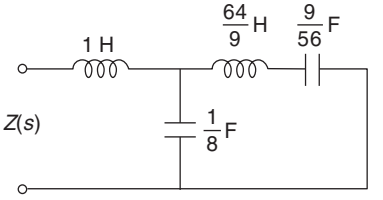


Fig. 9.12

Example 9.37 *Realise the network having admittance function* $Y(s)=\frac{4s^2+6s}{s+1}$.

Solution By long division of $Y(s)$,

$$s+1\Big)4s^2+6s\Big(4s$$

$$\frac{4s^2+4s}{2s}$$

$$Y(s)=4s+\frac{2s}{s+1}=Y_1(s)+Y_2(s)$$

$Y_1(s)=4s$ represents the admittance of a capacitor of value 4 F.

$$Z_2(s)=\frac{1}{Y_2(s)}=\frac{s+1}{2s}=\frac{1}{2}+\frac{1}{2s}=Z_3(s)+Z_4(s)$$

$Z_3(s) = \frac{1}{2}$ represents the impedance of a resistor of value $\frac{1}{2}\,\Omega$.

$Z_4(s) = \frac{1}{2s}$ represents the impedance of a capacitor of value 2 F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.13.

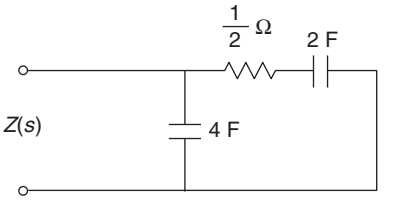


Fig. 9.13

Example 9.38 *Realise the admittance function* $Y(s) = \frac{3+5s}{4+2s}$.

Solution By long division,

$$4+2s \;\Big)\; 3+5s \;\Big(\; \frac{3}{4}$$

$$\underline{3+\frac{3}{2}s}$$

$$\frac{7}{2}s$$

$$Y(s) = \frac{3}{4} + \frac{\frac{7}{2}s}{4+2s} = Y_1(s) + Y_2(s)$$

$Y_1(s) = \frac{3}{4}$ represents the admittance of a resistor of value $\frac{4}{3}\,\Omega$.

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{4+2s}{\frac{7}{2}s} = \frac{8+4s}{7s} = \frac{8}{7s} + \frac{4}{7} = Z_3(s) + Z_4(s)$$

$Z_3(s) = \frac{8}{7s}$ represents the impedance of a capacitor of value $\frac{7}{8}$ F.

$Z_4(s) = \frac{4}{7}$ represents the impedance of a resistor of value $\frac{4}{7}\,\Omega$.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.14.

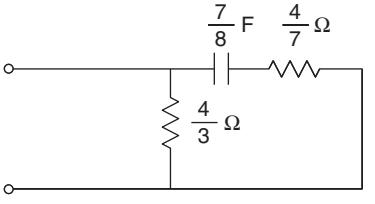


Fig. 9.14

9.5 REALISATION OF *LC* FUNCTIONS

LC driving point immitance functions have the following properties.

1. It is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$-axis.
3. The poles and zeros interlace on the $j\omega$-axis.
4. There must be either a zero or a pole at the origin and infinity.
5. The difference between any two successive powers of numerator and denominator polynomials is at most two. There cannot be any missing terms.

6. The highest powers of numerator and denominator polynomials must differ by unity; the lowest powers also differ by unity.

There are a number of methods of realising an *LC* function. But we will study only four basic forms—Foster I, Foster II, Cauer I and Cauer II forms. The Foster forms are obtained by partial-fraction expansion of $F(s)$, and the Cauer forms are obtained by continued fraction expansion of $F(s)$.

9.5.1 Foster Realisation

Consider a general *LC* function $F(s)$ given by

$$F(s) = \frac{H(s^2 + \omega_1^2)(s^2 + \omega_3^2)...}{s(s^2 + \omega_2^2)(s^2 + \omega_4^2)...}$$

where $0 \le \omega_1^2 < \omega_2^2 < \omega_3^2 \ldots$ and H is positive.

By partial-fraction expansion of $F(s)$,

$$F(s) = \frac{K_0}{s} + \frac{K_2}{s + j\omega_2} + \frac{K_2}{s - j\omega_2} + ... + K_\infty s$$

Combining terms with conjugate poles,

$$F(s) = \frac{K_0}{s} + \frac{2K_2 s}{s + \omega_2^2} ... + K_\infty s$$

where K_0, K_i and K_∞ are the residues of $F(s)$ at the origin, at $j\omega_i$ and at infinity respectively.
These residues are given by

$$K_0 = sF(s)\big|_{s=0}$$

$$K_i = \left.\frac{\left(s^2 + \omega_i^2\right)F(s)}{2s}\right|_{s^2 = -\omega_i^2}$$

$$K_\infty = \left.\frac{F(s)}{s}\right|_{s \to \infty}$$

Foster I Form If $F(s)$ represents an impedance function, it gives a series connection of impedances.

$$F(s) = Z(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2 + \omega_2^2} \ldots + K_\infty s = Z_1(s) + Z_2(s) + \ldots + Z_n(s)$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_0}$ farad.

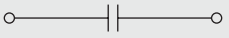
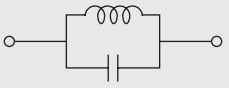
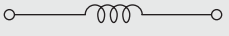
The last term $K_\infty s$ represents the impedance of an inductor of K_∞ henry.

The remaining terms, i.e., $\frac{2K_i s}{s^2 + \omega_i^2}$ represent the impedance of a parallel combination of capacitor C_i and inductor L_i. For parallel combination of L_i and C_i,

$$Z(s) = \frac{1}{C_i s + \frac{1}{L_i s}} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}} = \frac{2K_i s}{s^2 + \omega_i^2}$$

$$C_i = \frac{1}{2K_i} \text{ and } L_i = \frac{2K_i}{\omega_i^2}$$

Table 9.1 *Realisation of Foster-I form of LC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	 $C_0 = \frac{1}{K_0}$
$\frac{2K_i s}{s^2 + \omega_i^2} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}}$	$L_i = \frac{2K_i}{\omega_i^2}$  $C_i = \frac{1}{2K_i}$
$K_\infty s = Ls$	 $L_\infty = K_\infty$

The network corresponding to Foster I form is shown in Fig. 9.15.

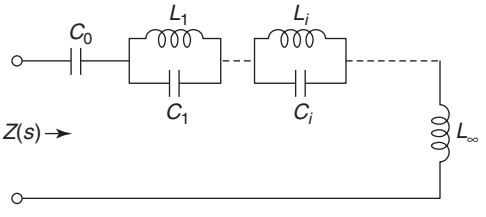


Fig. 9.15 *Foster-I form of LC network*

If $Z(s)$ has no pole at the origin then capacitor C_0 is not present in the network. Similarly, if there is no pole at ∞, inductor L_∞ is not present in the network.

Foster II Form If $F(s)$ represents an admittance function, it gives the parallel combination of admittances.

$$F(s) = Y(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2 + \omega_2^2} + \ldots + K_\infty s = Y_1(s) + Y_2(s) + \ldots Y_n(s)$$

The first term $\frac{K_0}{s}$ represents the admittance of an inductor of $\frac{1}{K_0}$ henry.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farad.

The remaining terms, i.e., $\frac{2K_i s}{s^2 + \omega_i^2}$ represent the admittance of a series combination of an inductor L_i and a capacitor C_i.

For series combination of L_i and C_i,

$$Y(s) = \frac{1}{L_i s + \frac{1}{C_i s}} = \frac{\left(\frac{1}{L_i}\right)s}{s^2 + \frac{1}{L_i C_i}} = \frac{2K_i s}{s^2 + \omega_i^2}$$

$$L_i = \frac{1}{2K_i} \quad \text{and} \quad C_i = \frac{2K_i}{\omega_i^2}$$

Table 9.2 *Realisation of Foster-II form of LC network*

Admittance function	Element
$\frac{K_0}{s} = \frac{1}{L_0 s}$	$L_0 = \frac{1}{K_0}$
$\frac{2K_i s}{s^2 + \omega_i^2} = \frac{\left(\frac{1}{L_i}\right)s}{s^2 + \frac{1}{L_i C_i}}$	$L_i = \frac{1}{2K_i}$, $C_i = \frac{2K_i}{\omega_i^2}$
$K_\infty s = Cs$	$C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 9.16.

If $Y(s)$ has no pole at the origin then inductor L_0 is not present. Similarly, if there is no pole at infinity, capacitor C_∞ is not present.

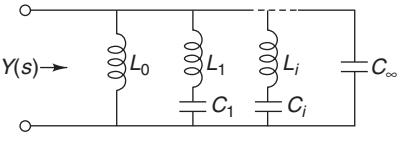


Fig. 9.16 *Foster-II form of LC network*

9.5.2 Cauer Realisation or Ladder Realisation

Cauer I Form Since the numerator and denominator polynomials of an *LC* function always differ in degrees by unity, there is always a zero or a pole at $s = \infty$. The Cauer I Form is obtained by successive removal of a pole or a zero at infinity from the function.

Consider an impedance function $Z(s)$ having a pole at infinity.

By removing the pole at infinity, we get

$$Z_2(s) = Z(s) - L_1 s$$

Now, $Z_2(s)$ has a zero at $s = \infty$. If we invert $Z_2(s)$, $Y_2(s)$ will have a pole at $s = \infty$.

By removing this pole,

$$Y_3(s) = Y_2(s) - C_2 s$$

Now $Y_3(s)$ has a zero at $s = \infty$, which we can invert and remove. This process continues until the remainder is zero. Each time we remove a pole, we remove an inductor or a capacitor depending on whether the function is an impedance or an admittance. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = L_1 s + \cfrac{1}{C_2 s + \cfrac{1}{L_3 s + \cfrac{1}{C_4 s + \ldots}}}$$

Thus, the final structure is a ladder network whose series arms are inductors and shunt arms are capacitors. The Cauer I network is shown in Fig. 9.17.

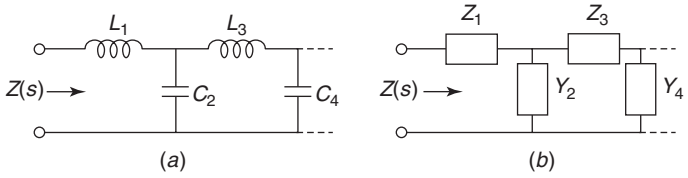


Fig. 9.17 *Cauer I form of LC network*

If the impedance function has zero at infinity, i.e., if degree of numerator is less than that of its denominator by unity, the function is first inverted and continued fraction expansion proceeds as usual. In this case, the first element is a capacitor as shown in Fig. 9.18.

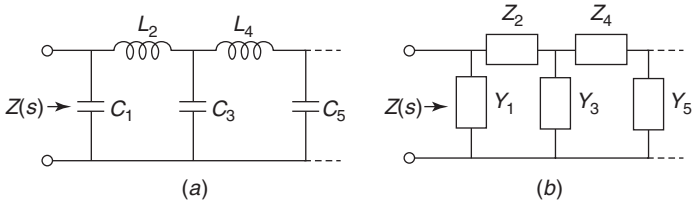


Fig. 9.18 *Cauer-I form of LC network*

Cauer II Form Since the lowest degrees of numerator and denominator polynomials of *LC* function must differ by unity, there is always a zero or a pole at $s = 0$. The Cauer II form is obtained by successive removal of a pole or a zero at $s = 0$ from the function.

In this method, continued fraction expansion of $Z(s)$ is carried out in terms of poles at the origin by removal of the pole at the origin, inverting the resultant function to create a pole at the origin which is removed and this process is continued until the remainder is zero. To do this, we arrange both numerator and denominator polynomials in ascending order and divide the lowest power of the denominator into the lowest power of the numerator. Then we invert the remainder and divide again. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \cfrac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{L_2 s} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{L_4 s} + \ldots}}}$$

Thus, the final structure is a ladder network whose first element is a series capacitor and second element is a shunt inductor as shown in Fig. 9.19.

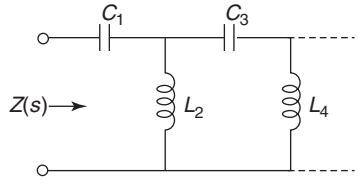


Fig. 9.19 *Cauer II form of LC network*

If the impedance function has a zero at the origin then the first element is a shunt inductor and the second element is a series capacitor as shown in Fig. 9.20.

Thus, the *LC* function $F(s)$ can be realised in four different forms. All these forms have the same number of elements and the number is equal to the number of poles and zeros of $F(s)$ includng any at infinity.

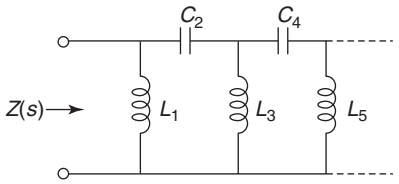


Fig. 9.20 *Cauer–II form of LC network*

Example 9.39 *State whether the following functions are driving point immittance of LC networks are not:*

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$ (b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

Solution

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$

The function $Z(s)$ has poles at $s = \pm j1$ and $s = \pm j\sqrt{3}$ and zeros at $s = 0$ and $s = \pm j2$ as shown in Fig. 9.21. Since the poles and zeros do not interlace on the $j\omega$ axis, the function $Z(s)$ is not an *LC* impedance function.

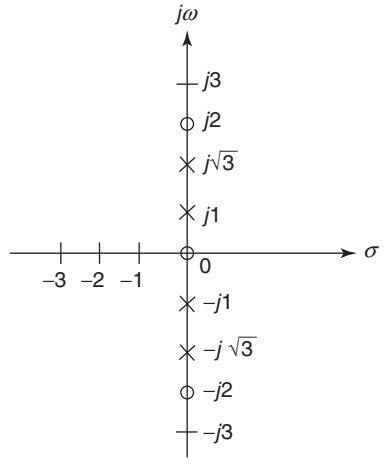


Fig. 9.21

(b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

The function $Z(s)$ has poles at $s = 0$ and $s = \pm j\sqrt{2}$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 9.22. The poles and zeros are simple and lie on the $j\omega$-axis. The poles and zeros interlace on the $j\omega$ axis. Hence, the function $Z(s)$ is an *LC* impedance function.

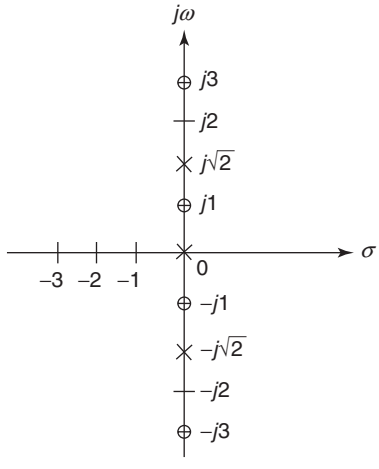


Fig. 9.22

Example 9.40 *Realise the Foster and Cauer forms of the following impedance function*

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = \pm j2$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 9.23.

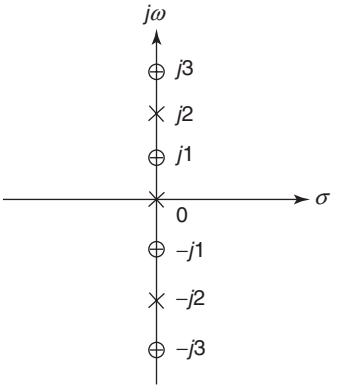


Fig. 9.23

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the $j\omega$ axis. Poles and zeros are interlaced. Hence, the given function is an *LC* function.

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{4s^4+40s^2+36}{s^3+4s}$$

$$s^3+4s)\ 4s^4+40s^2+36\ (4s$$

$$\frac{4s^4+16s^2}{24s^2+36}$$

$$Z(s) = 4s + \frac{24s^2+36}{s^3+4s} = 4s + \frac{24s^2+36}{s(s^2+4)}$$

By partial-fraction expansion,

$$Z(s) = 4s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = 4s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+4}$$

where

$$K_0 = sZ(s)\big|_{s=0} = \frac{4(1)(9)}{4} = 9$$

$$K_1 = \left.\frac{(s^2+4)Z(s)}{2s}\right|_{s^2=-4} = \frac{4(-4+1)(-4+9)}{2(-4)} = \frac{15}{2}$$

$$Z(s) = 4s + \frac{9}{s} + \frac{15s}{s^2+4}$$

The first term represents the impedance of an inductor of 4 H. The second term represents the impedance of a capacitor of $\frac{1}{9}$ F. The third term represents the impedance of a parallel *LC* network.

For a parallel *LC* network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = \frac{1}{15}\text{F}$$

$$L = \frac{15}{4}\text{H}$$

The network is shown in Fig. 9.24.

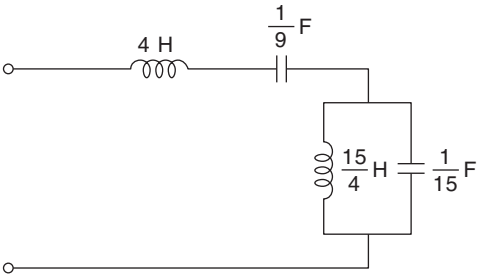


Fig. 9.24

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where

$$K_1 = \frac{(s^2+1)}{2s} Y(s)\Big|_{s^2=-1} = \frac{(-1+4)}{8(-1+9)} = \frac{3}{64}$$

$$K_2 = \frac{(s^2+9)}{2s} Y(s)\Big|_{s^2=-9} = \frac{(-9+4)}{8(-9+1)} = \frac{5}{64}$$

$$Y(s) = \frac{\left(\frac{3}{32}\right)s}{s^2+1} + \frac{\left(\frac{5}{32}\right)s}{s^2+9}$$

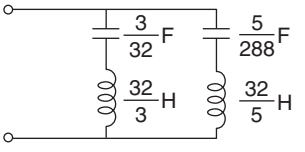
These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{32}{3}\text{H} \qquad C_1 = \frac{3}{32}\text{F}$$

$$L_2 = \frac{32}{5}\text{H} \qquad C_2 = \frac{5}{288}\text{F}$$



The network is shown in Fig. 9.25.

Fig. 9.25

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{4s^4 + 40s^2 + 36}{s^3 + 4s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$s^3+4s\,\Big)4s^4+40s^2+36\,\Big(4s \leftarrow Z$$

$$\underline{4s^4+16s^2}$$

$$24s^2+36\,\Big)s^3+4s\,\Big(\frac{1}{24}s \leftarrow Y$$

$$\underline{s^3+\frac{3}{2}s}$$

$$\frac{5}{2}s\,\Big)24s^2+36\,\Big(\frac{48}{5}s \leftarrow Z$$

$$\underline{24s^2}$$

$$36\,\Big)\frac{5}{2}s\,\Big(\frac{5}{72}s \leftarrow Y$$

$$\underline{\frac{5}{2}s}$$

$$0$$

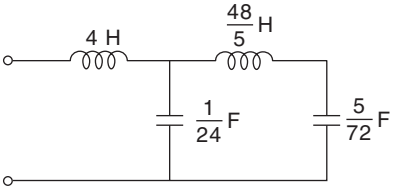


Fig. 9.26

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.26.

Cauer II Form The Cauer II form is obtained from partial-fraction expansion about pole at origin.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

The function $Z(s)$ has a pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s)=\frac{36+40s^2+4s^4}{4s+s^3}$$

By continued fraction expansion,

$$4s+s^3\,\Big)36+40s^2+4s^4\,\Big(\frac{9}{s} \leftarrow \mathrm{Z}$$

$$\underline{36+\ \ 9s^2}$$

$$31s^2+4s^4\,\Big)4s+s^3\,\Big(\frac{4}{31s} \leftarrow Y$$

$$\underline{4s+\frac{16}{31}s^3}$$

$$\frac{15}{31}s^3\,\Big)31s^2+4s^4\,\Big(\frac{961}{15s} \leftarrow Z$$

$$\underline{31s^2}$$

$$4s^4 \Big) \frac{15}{31}s^3 \Big(\frac{15}{124s} \leftarrow Y$$

$$\frac{\frac{15}{31}s^3}{0}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.27.

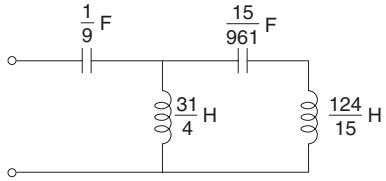


Fig. 9.27

Example 9.41

Realise Foster forms of the LC impedance function

$$Z(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$$

Solution

Foster I Form The Foster I form is obtained by continued-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} = \frac{s^4+4s^2+3}{s^3+2s}$$

$$s^3 + 2s \Big) s^4 + 4s^2 + 3 \Big(s$$

$$\frac{s^4 + 2s^2}{2s^2 + 3}$$

$$Z(s) = s + \frac{2s^2+3}{s^3+2s} = s + \frac{2s^2+3}{s(s^2+2)}$$

By partial-fraction expansion,

$$Z(s) = s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+2}$$

where

$$K_0 = sZ(s)|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_1 = \frac{(s^2+2)}{2s} Z(s)\Big|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)} = \frac{1}{4}$$

$$Z(s) = s + \frac{\left(\frac{3}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)s}{s^2+2}$$

The first term represents the impedance of an inductor of 1 H. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of a parallel LC network.

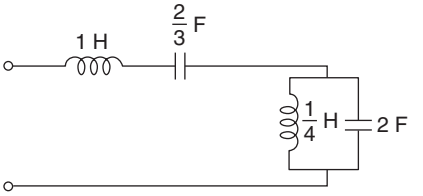
For a parallel *LC* network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = 2\text{F}$$

$$L = \frac{1}{4}\text{H}$$



The network is shown in Fig. 9.28.

Fig. 9.28

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j\sqrt{3}} + \frac{K_2^*}{s-j\sqrt{3}} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+3}$$

where

$$K_1 = \frac{(s^2+1)}{2s} Y(s)\Big|_{s^2=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = \frac{(s^2+3)}{2s} Y(s)\Big|_{s^2=-3} = \frac{-3+2}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = \frac{\left(\frac{1}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

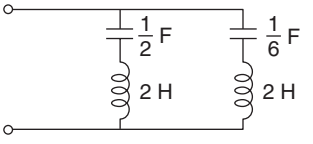
These two terms represent admittance of a series *LC* network. For a series *LC* network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = 2\text{H}, \quad C_1 = \frac{1}{2}\text{F}$$

$$L_2 = 2\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$



The network is shown in Fig. 9.29.

Fig. 9.29

Example 9.42 *Realise Foster forms of the following LC impedance function:*

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+j\sqrt{2}} + \frac{K_1^*}{s-j\sqrt{2}} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2} = \frac{K_0}{s} + \frac{2K_1 s}{s^2+2} + \frac{2K_2 s}{s^2+4}$$

where

$$K_0 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_1 = \frac{(s^2+2)}{2s} Z(s)\bigg|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{1}{8}$$

$$K_2 = \frac{(s^2+4)}{2s} Z(s)\bigg|_{s^2=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{3}{16}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\left(\frac{1}{4}\right)s}{s^2+2} + \frac{\left(\frac{3}{8}\right)s}{s^2+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The other two terms represent the impedance of a parallel *LC* network.

For a parallel *LC* network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C_1 = 4\text{F}, \quad L_1 = \frac{1}{8}\text{H}$$

$$C_2 = \frac{8}{3}\text{F}, \quad L_2 = \frac{3}{32}\text{H}$$

The network is shown in Fig. 9.30.

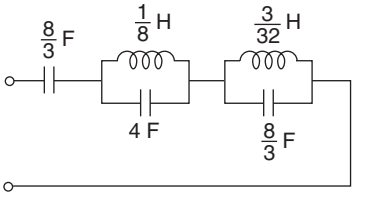


Fig. 9.30

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)} = \frac{s^5+6s^3+8s}{s^4+4s^2+3}$$

Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$s^4+4s^2+3)s^5+6s^3+8s(s$$
$$\underline{s^5+4s^3+3s}$$
$$2s^3+5s$$

$$Y(s) = s + \frac{2s^3+5s}{s^4+4s^2+3} = s + \frac{2s^3+5s}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s) = s + \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j\sqrt{3}} + \frac{K_2^*}{s-j\sqrt{3}} = s + \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+3}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)(-1+4)}{2(-1+3)} = \frac{3}{4}$$

$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{(-3+2)(-3+4)}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = s + \frac{\left(\frac{3}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

The first term represents the admittance of capacitor of 1 F. The other two terms represent admittance of a series *LC* network. For a series *LC* network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{2}{3}\text{H}, \quad C_1 = \frac{3}{2}\text{F}$$

$$L_2 = 2\,\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$

The network is shown in Fig. 9.31.

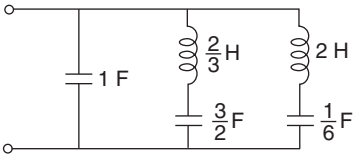


Fig. 9.31

Example 9.43 *Realise Cauer forms of the following LC impedance function:*

$$Z(s) = \frac{10s^4+12s^2+1}{2s^3+2s}$$

Solution

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{10s^4+12s^2+1}{2s^3+2s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$\begin{array}{l} 2s^3+2s\,)\,10s^4+12s^2+1\,(5s \leftarrow Z \\ \quad \underline{10s^4+10s^2} \\ \qquad 2s^2+1\,)\,2s^3+2s\,(s \leftarrow Y \\ \qquad\quad \underline{2s^3+s} \\ \qquad\qquad s\,)\,2s^2+1\,(2s \leftarrow Z \\ \qquad\qquad\quad \underline{2s^2} \\ \qquad\qquad\qquad 1\,)\,s\,(s \leftarrow Y \\ \qquad\qquad\qquad\quad \frac{s}{0} \end{array}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.32.

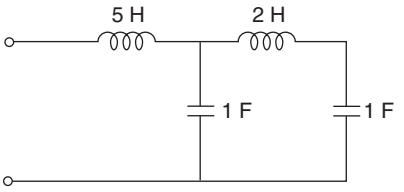


Fig. 9.32

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin.

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

The function $Z(s)$ has a pole at the origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s) = \frac{1 + 12s^2 + 10s^4}{2s + 2s^3}$$

By continued fraction expansion of $Z(s)$,

$$2s + 2s^3 \overline{)1 + 12s^2 + 10s^4} \left(\frac{1}{2s} \leftarrow \text{Z} \right.$$

$$\underline{1 + \quad s^2}$$

$$11s^2 + 10s^4 \overline{)2s + 2s^3} \left(\frac{2}{11s} \leftarrow Y \right.$$

$$\underline{2s + \frac{20}{11}s^3}$$

$$\frac{2}{11}s^3 \overline{)11s^2 + 10s^4} \left(\frac{121}{2s} \leftarrow Z \right.$$

$$\underline{11s^2}$$

$$10s^4 \overline{)\frac{2}{11}s^3} \left(\frac{2}{110s} \leftarrow Y \right.$$

$$\underline{\frac{2}{11}s^3}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in Cauer or ladder realisation. The network is shown in Fig. 9.33.

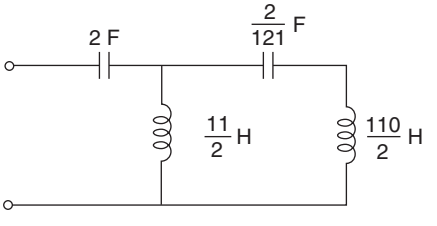


Fig. 9.33

Example 9.44 *Realise the following network function in Cauer I form:*

$$Z(s) = \frac{6s^4 + 42s^2 + 48}{s^5 + 18s^3 + 48s}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates the presence of a zero at infinity. The admittance function $Y(s)$ has a pole at infinity. Hence, the continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^5 + 18s^3 + 48s}{6s^4 + 42s^2 + 48}$$

By continued fraction expansion

$$6s^4 + 42s^2 + s^2 \Big) s^5 + 18s^3 + 48s \Big(\frac{1}{6}s \leftarrow Y$$

$$\underline{s^5 + 7s^3 + 8s}$$

$$11s^3 + 40s \Big) 6s^4 + 42s^2 + 48 \Big(\frac{6}{11}s \leftarrow Z$$

$$\underline{6s^4 + \frac{240}{11}s^2}$$

$$\frac{222}{11}s^2 + 30 \Big) 11s^3 + 40s \Big(\frac{121}{222}s \leftarrow Y$$

$$\underline{11s^3 + \frac{5808}{222}s}$$

$$\frac{3072}{222}s \Big) \frac{222}{11}s^2 + 48 \Big(\frac{49284}{33792}s \leftarrow Z$$

$$\underline{\frac{222}{11}s^2}$$

$$48 \Big) \frac{3072}{222}s \Big(\frac{128}{444}s \leftarrow Y$$

$$\underline{\frac{3072}{222}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.34.

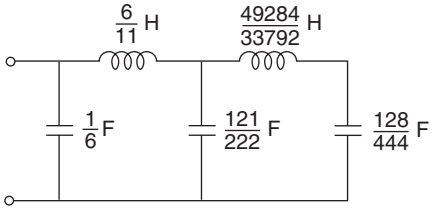


Fig. 9.34

Example 9.45 *Realise Cauer II form of the function:*

$$Z_{LC}(s) = \frac{s(s^4 + 3s^2 + 1)}{3s^4 + 4s^2 + 1}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin. The given function has a zero at the origin. The admittance function $Y(s)$ has a pole at origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y_{LC}(s) = \frac{3s^4 + 4s^2 + 1}{s^5 + 3s^3 + s} = \frac{1 + 4s^2 + 3s^4}{s + 3s^3 + s^5}$$

By continued fraction expansion of $Y(s)$, we have

$$s + 3s^3 + s^5 \overline{)1 + 4s^2 + 3s^4} \left(\frac{1}{s} \leftarrow Y\right.$$

$$\underline{1 + 3s^2 + s^4}$$

$$s^2 + 2s^4 \overline{)s + 3s^3 + s^5} \left(\frac{1}{s} \leftarrow Z\right.$$

$$\underline{s + 2s^3}$$

$$s^3 + s^5 \overline{)s^2 + 2s^4} \left(\frac{1}{s} \leftarrow Y\right.$$

$$\underline{s^2 + s^4}$$

$$s^4 \overline{)s^3 + s^5} \left(\frac{1}{s} \leftarrow Z\right.$$

$$\underline{s^3}$$

$$s^5 \overline{)s^4} \left(\frac{1}{s} \leftarrow Y\right.$$

$$\frac{s^4}{0}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.35.

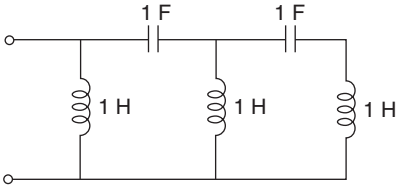


Fig. 9.35

Example 9.46 *Obtain the Cauer I form of realisation for the function*

$$Z_{LC}(s) = \frac{s^5 + 7s^3 + 10s}{s^4 + 5s^2 + 4}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z_{LC}(s)$ about pole at infinity.

$$
\begin{array}{r}
s^4+5s^2+4\,\big)\,s^5+7s^3+10s\,\big(\;s\leftarrow Z \\
\underline{s^5+5s^3+\;\;4s\;\;\;

By continued fraction expansion of $Y(s)$,

$$
\begin{array}{l}
2s+s^3\,\big)\,3+4s^2+s^4\,\Big(\dfrac{3}{2s} \leftarrow Y \\
\quad \underline{3+\dfrac{3}{2}s^2} \\
\quad\quad \dfrac{5}{2}s^2+s^4\,\big)\,2s+s^3\,\Big(\dfrac{4}{5s} \leftarrow Z \\
\quad\quad\quad \underline{2s+\dfrac{4}{5}s^3} \\
\quad\quad\quad\quad \dfrac{s^3}{5}\,\big)\,\dfrac{5}{2}s^2+s^4\,\Big(\dfrac{25}{2s} \leftarrow Y \\
\quad\quad\quad\quad\quad \underline{\dfrac{5}{2}s^2} \\
\quad\quad\quad\quad\quad\quad s^4\,\big)\,\dfrac{1}{5}s^3\,\Big(\dfrac{1}{5s} \leftarrow Z \\
\quad\quad\quad\quad\quad\quad\quad \underline{\dfrac{1}{5}s^3} \\
\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 9.37.

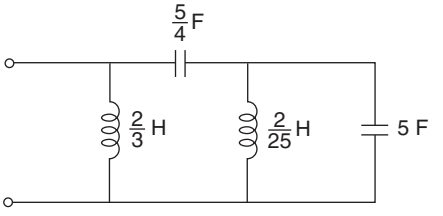


Fig. 9.37

9.6 REALISATION OF *RC* FUNCTIONS

RC driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency nearest to the origin is a pole.
4. The highest critical frequency farthest to the origin is a zero.
5. Residues evaluated at the poles of $Z_{RC}(s)$ are real and positive.
6. The slope $\dfrac{d}{d\sigma}Z_{RC}$ is negative.
7. $Z_{RC}(\infty) < Z_{RC}(0)$.

RC functions can also be realised in four different ways. The impedance function of RC networks is given by,

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)\ldots}{s(s+\sigma_2)\ldots}$$

9.6.1 Foster Realisation

Foster I Form The Foster I form is obtained by partial-fraction expansion of $Z(s)$.

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+\sigma_1} + \frac{K_2}{s+\sigma_2} + \ldots + K_\infty$$

where K_0, K_1, K_2, … K_∞ are residues of $Z(s)$.

$$K_o = sZ(s)\big|_{s=0}$$

$$K_i = (s+\sigma_i)Z(s)\big|s = -\sigma_i$$

$$K_\infty = \left.\frac{Z(s)}{s}\right|_{s\to\infty}$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_o}$ farads.

The last term K_∞ represents the impedance of a resistor of K_∞ ohms.

The remaining terms, i.e., $\frac{K_i}{s+\sigma_i}$ represent the impedance of the parallel combination of resistor R_i and capacitor C_i. For parallel combinaton of R_i and C_i,

$$Z(s) = \frac{R_i\left(\frac{1}{C_i s}\right)}{R_i + \frac{1}{C_i s}} = \frac{K_i}{s+\sigma_i}$$

$$R_i = \frac{K_i}{\sigma_i} \text{ and } C_i = \frac{1}{K_i}$$

Table 9.3 *Realisation of Foster-I form of RC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	$C_0 = \frac{1}{K_0}$
	$Ri = \frac{K_i}{\sigma_i}$
$\frac{K_i}{s+\sigma_i} = \frac{(R_i)\left(\frac{1}{C_i s}\right)}{R_i + \frac{1}{C_i s}}$	$C_i = \frac{1}{K_i}$
$K_\infty = R_\infty$	$R_\infty = K_\infty$

The network corresponding to the Foster-I form is shown in Fig. 9.38.

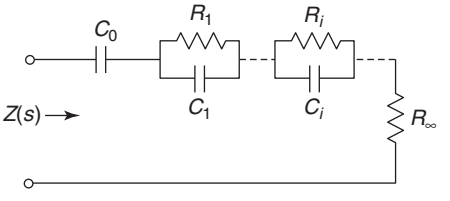


Fig. 9.38 *Foster-I form of RC network*

Foster II Form The Foster-II form is obtained by partial fraction expansion of $Y(s)$. Since $Y(s) = \frac{1}{Z(s)}$ has negative residue at its pole, Foster II form is obtained by expanding $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{K_o}{s} + \sum_{i=1}^{n} \frac{K_i}{(s+\sigma_i)} + K_\infty$$

Multiplying this equation by s,

$$Y(s) = K_o + \sum_{i=1}^{n} \frac{K_i s}{(s+\sigma_i)} + K_\infty s$$

The first term K_o represents the conductance of a resistor of $\frac{1}{K_o}$ ohms.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farads.

The remaining terms, i.e., $\frac{K_i s}{s+\sigma_i}$ represent the admittance of series combination of resistor R_i and capacitor C_i with $R_i = \frac{1}{K_i}$ ohms and $C_i = \frac{K_i}{\sigma_i}$ farads.

Table 9.4 *Realisation of Foster II form of RC network*

Admittance function	Element
$K_0 = \frac{1}{R_0}$	$R_o = \frac{1}{K_o}$
$\frac{K_i}{s+\omega_i} = \frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_i C_i}}$	$R_i = \frac{1}{K_i}$, $C_i = \frac{K_i}{\sigma_i}$
$K_\infty s = C_\infty s$	$C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 9.39.

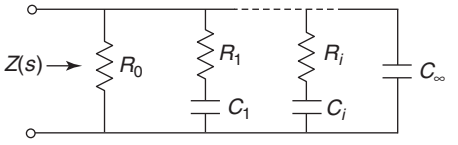


Fig. 9.39 *Foster II form of RC network*

9.6.2 Cauer Realisation

Cauer I Form The Cauer I form is obtained by removal of the pole from the impedance function $Z(s)$ at $s = \infty$. This is the same as a continued fraction expansion of the impedance function about infinity. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \ldots}}}$$

The network is shown in Fig. 9.40.

In the network shown in Fig. 9.40, if $Z(s)$ has a zero at $s = \infty$, the first element is the capacitor C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

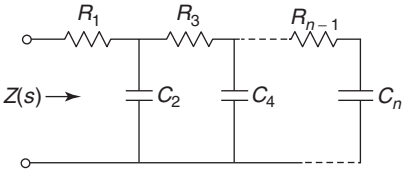


Fig. 9.40 *Cauer-I form of RC network*

Cauer II Form The Cauer II form is obtained by removal of the pole from the impedance function at the origin. This is the same as a continued fraction expansion of an impedance function about the origin.

If the given impedance function has a pole at the origin, it is removed as a capacitor C_1. The reciprocal of the remainder function has a minimum value at $s = 0$ which is removed as a constant of resistor R_2. If the original impedance has no pole at the origin, then the first capacitor is absent and the process is repeated with the removal of the constant corresponding to the resistor R_2.

The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_2} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{R_4} + \ldots}}}$$

The network is shown in Fig. 9.41.

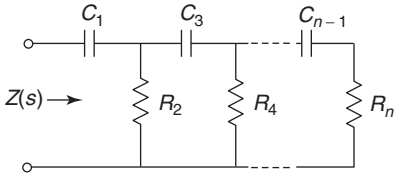


Fig. 9.41 *Caver-II form of RC network*

In the network shown in Fig. 9.41, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Example 9.48

Determine whether following functions are RC impedance function or not.

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$ *(b)* $\dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

Solution

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

The function Z(s) has poles at $s = 0$ and $s = -3$ and zeros at $s = -2$ and $s = -4$ as shown in Fig. 9.42. The poles and zeros are simple and located on the negative real axis of the s plane. The poles and zeros are interlaced. The lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an *RC* impedance function.

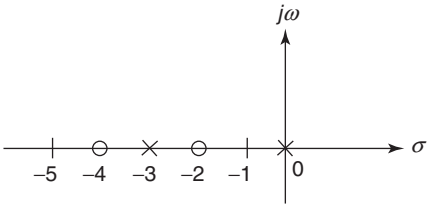


Fig. 9.42

(b) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

The function $Z(s)$ has poles at $s = -2$ and $s = -6$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 9.43. The poles and zeros are simple and located on the negative real axis of the s-plane. The poles and zeros are interlaced. But the lowest critical frequency nearest to the origin is not a pole, but zero. Hence, the function $Z(s)$ is not an *RC* impedance function.

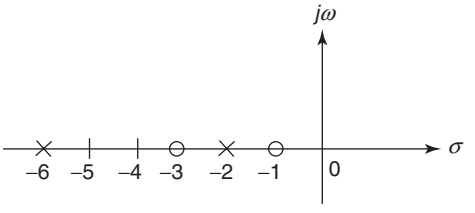


Fig. 9.43

Example 9.49 *Realise the Foster and Cauer forms of the impedance function*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 9.44.

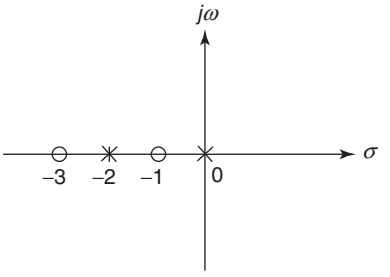


Fig. 9.44

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the negative real axis. The poles and zeros are interlaced and the lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an *RC* function.

Foster I Form The Foster I form is obtained by partial fraction expansion of impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{s^2 + 4s + 3}{s^2 + 2s}$$

$$s^2 + 2s \overline{)\, s^2 + 4s + 3 \,(\, 1}$$

$$\underline{\quad s^2 + 2s \quad}$$

$$2s + 3$$

$$Z(s) = 1 + \frac{2s+3}{s^2+2s} = 1 + \frac{2s+3}{s(s+2)}$$

By partial-fraction expansion,

$$Z(s) = 1 + \frac{K_1}{s} + \frac{K_2}{s+2}$$

where

$$K_1 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_2 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{-2} = \frac{1}{2}$$

$$Z(s) = 1 + \frac{\frac{3}{2}}{s} + \frac{\frac{1}{2}}{s+2}$$

The first term represents the impedance of a resistor of 1 Ω. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R = \frac{1}{4}\ \Omega$$

$$C = 2\ \text{F}$$

The network is shown in Fig. 9.45.

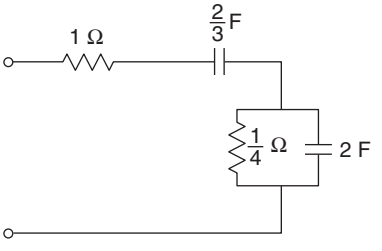


Fig. 9.45

Foster II Form The Foster II form is obtained by the partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s+2)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{s+2}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{1}{2}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

$$Y(s) = \frac{\frac{1}{2}s}{s+1} + \frac{\frac{1}{2}s}{s+3}$$

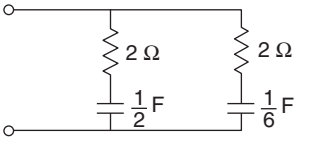
These two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 2\ \Omega, \quad C_1 = \frac{1}{2}\ \text{F}$$

$$R_2 = 2\ \Omega, \quad C_2 = \frac{1}{6}\ \text{F}$$



The network is shown in Fig. 9.46.

Fig. 9.46

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{s^2 + 4s + 3}{s^2 + 2s}$$

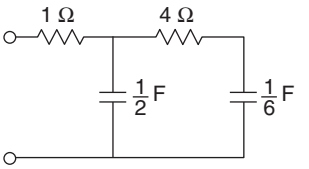
By continued fraction expansion,

$$s^2 + 2s \,\Big)\, s^2 + 4s + 3 \,\Big(\, 1 \leftarrow Z$$

$$\underline{s^2 + 2s}$$

$$2s + 3 \,\Big)\, s^2 + 2s \,\Big(\, \frac{1}{2}s \leftarrow Y$$

$$
\begin{array}{r}
s^2+\frac{3}{2}s \\
\hline
\frac{1}{2}s\;\Big)\,2s+3\;\Big(\;4 \leftarrow Z \\
\underline{2s} \\
3\;\Big)\,\frac{1}{2}s\;\Big(\;\frac{1}{6}s \leftarrow Y \\
\underline{\frac{1}{2}s} \\
0
\end{array}
$$



The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 9.47.

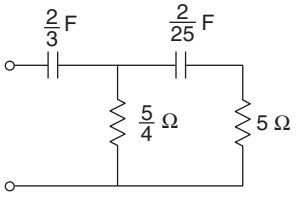
Fig. 9.47

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s) = \frac{3+4s+s^2}{2s+s^2}$$

By continued fraction expansion,

$$
\begin{array}{r}
2s+s^2\;\Big)\,3+4s+s^2\;\Big(\;\frac{3}{2s} \leftarrow Z \\
\underline{3+\frac{3}{2}s} \\
\frac{5}{2}s+s^2\;\Big)\,2s+s^2\;\Big(\;\frac{4}{5} \leftarrow Y \\
\underline{2s+\frac{4}{5}s^2} \\
\frac{1}{5}s^2\;\Big)\,\frac{5}{2}s+s^2\;\Big(\;\frac{25}{2s} \leftarrow Z \\
\underline{\frac{5}{2}s} \\
s^2\;\Big)\,\frac{1}{5}s^2\;\Big(\;\frac{1}{5} \leftarrow Y \\
\underline{\frac{1}{5}s^2} \\
0
\end{array}
$$



The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 9.48.

Fig. 9.48

Example 9.50 *Determine the Foster form of realisation of the RC impedance function.*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where

$$K_0 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{(-2)(-2+4)} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$$

$$K_2 = (s+4)Z(s)\big|_{s=-4} = \frac{(-4+1)(-4+3)}{(-4)(-4+2)} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{3}{8}}{s+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{1}{8}\,\Omega, \quad C_1 = 4\text{ F}$$

$$R_2 = \frac{3}{32}\,\Omega, \quad C_2 = \frac{8}{3}\text{ F}$$

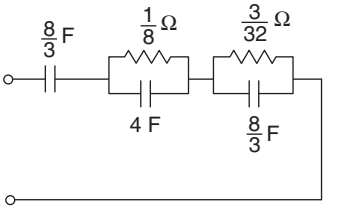


Fig. 9.49

The network is shown in Fig. 9.49.

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2+6s+8}{s^2+4s+3}$$

Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$s^2+4s+3\Big)\ s^2+6s+8\big(1$$

$$\frac{s^2+4s+3}{2s+5}$$

$$\frac{Y(s)}{s}=1+\frac{2s+5}{s^2+4s+3}=1+\frac{2s+5}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s}=1+\frac{K_1}{s+1}+\frac{K_2}{s+3}$$

where

$$K_1=(s+1)\frac{Y(s)}{s}\bigg|_{s=-1}=\frac{(-1+2)(-1+4)}{(-1+3)}=\frac{(1)(3)}{2}=\frac{3}{2}$$

$$K_2=(s+3)\frac{Y(s)}{s}\bigg|_{s=-3}=\frac{(-3+2)(-3+4)}{(-3+1)}=\frac{(-1)(1)}{(-2)}=\frac{1}{2}$$

$$\frac{Y(s)}{s}=1+\frac{\frac{3}{2}}{s+1}+\frac{\frac{1}{2}}{s+3}$$

$$Y(s)=s+\frac{\frac{3}{2}s}{s+1}+\frac{\frac{1}{2}s}{s+3}$$

The first term represents the admittance of a capacitor of 1 F. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s)=\frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1=\frac{2}{3}\,\Omega,\quad C_1=\frac{3}{2}\,\text{F}$$

$$R_2=2\,\Omega,\quad C_2=\frac{1}{6}\,\text{F}$$

The network is shown in Fig. 9.50.

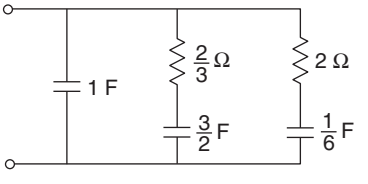


Fig. 9.50

Example 9.51

Realise Foster forms of the following RC impedance function

$$Z(s)=\frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$Z(s) = \frac{2s^2 + 12s + 16}{s^2 + 4s + 3}$$

$$s^2 + 4s + 3 \overline{)2s^2 + 12s + 16} \; (2$$

$$\underline{2s^2 + 8s + 6}$$

$$4s + 10$$

$$Z(s) = 2 + \frac{4s + 10}{s^2 + 4s + 3} = 2 + \frac{4s + 10}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$Z(s) = 2 + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)Z(s)\big|_{s=-1} = \frac{2(-1+2)(-1+4)}{(-1+3)} = 3$$

$$K_2 = (s+3)Z(s)\big|_{s=-3} = \frac{2(-3+2)(-3+4)}{(-3+1)} = 1$$

$$Z(s) = 2 + \frac{3}{s+1} + \frac{1}{s+3}$$

The first term represents the impedance of a resistor of 2 Ω. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\dfrac{1}{C_i}}{s + \dfrac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 3\ \Omega, \quad C_1 = \frac{1}{3}\ \text{F}$$

$$R_2 = \frac{1}{3}\ \Omega, \quad C_2 = 1\ \text{F}$$

The network is shown in Fig. 9.51.

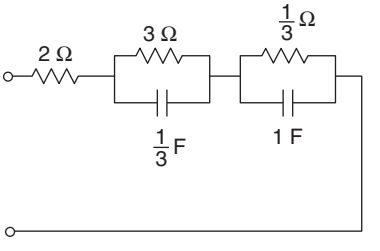


Fig. 9.51

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{(s+1)(s+3)}{2(s+2)(s+4)}$$

$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{2s(s+2)(s+4)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where

$$K_0 = s\frac{Y(s)}{s}\bigg|_{s=0} = \frac{(1)(3)}{(2)(2)(4)} = \frac{3}{16}$$

$$K_1 = (s+2)\frac{Y(s)}{s}\bigg|_{s=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{(-1)(1)}{2(-2)(2)} = \frac{1}{8}$$

$$K_2 = (s+4)\frac{Y(s)}{s}\bigg|_{s=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{(-3)(-1)}{2(-4)(-2)} = \frac{3}{16}$$

$$\frac{Y(s)}{s} = \frac{\frac{3}{16}}{s} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{16}}{s+4}$$

$$Y(s) = \frac{3}{16} + \frac{\frac{1}{8}s}{s+2} + \frac{\frac{3}{16}s}{s+4}$$

The first term represents the admittance of a resistor of $\frac{16}{3}\ \Omega$. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 8\ \Omega, \qquad C_1 = \frac{1}{16}\ \text{F}$$

$$R_2 = \frac{16}{3}\ \Omega, \quad C_2 = \frac{3}{64}\ \text{F}$$

The network is shown in Fig. 9.52.

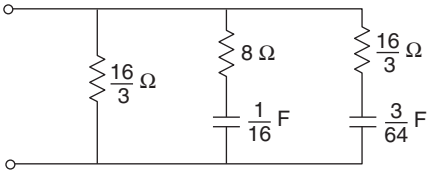


Fig. 9.52

Example 9.52

Obtain the Cauer forms of the RC impedance function

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$\begin{array}{l}
2s^2+8s+6\,\big)\,s^2+8s+12\,\big(\,\frac{1}{2} \leftarrow Z \\
\underline{s^2+4s+3} \\
4s+9\,\big)\,2s^2+8s+6\,\big(\,\frac{1}{2}s \leftarrow Y \\
\underline{2s^2+\frac{9}{2}s} \\
\frac{7}{2}s+6\,\big)\,4s+9\,\big(\,\frac{8}{7} \leftarrow Z \\
\underline{4s+\frac{48}{7}}
\end{array}$$

$$\frac{15}{7}\Big)\frac{7}{2}s+6\Big(\frac{49}{30}s \leftarrow Y$$

$$\frac{\frac{7}{2}s}{6}\Big)\frac{15}{7}\Big(\frac{5}{14} \leftarrow Z$$

$$\frac{\frac{15}{7}}{0}$$

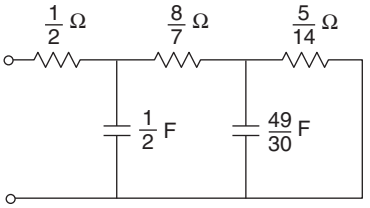


Fig. 9.53

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.53.

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s)=\frac{12+8s+s^2}{6+8s+2s^2}$$

By continued fraction expansion,

$$6+8s+2s^2\Big)12+8s+s^2\Big(2$$

$$\frac{12+16s+4s^2}{-8s-3s^2}$$

Since negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Big)6+8s+2s^2\Big(\frac{1}{2} \leftarrow Y$$

$$\frac{6+4s+\frac{1}{2}s^2}{4s+\frac{3}{2}s^2}\Big)12+8s+s^2\Big(\frac{3}{s} \leftarrow Z$$

$$\frac{12+\frac{9}{2}s}{\frac{7}{2}s+s^2}\Big)4s+\frac{3}{2}+s^2\Big(\frac{8}{7} \leftarrow Y$$

$$\frac{4s+\frac{8}{7}s^2}{\frac{5}{14}s^2}\Big)\frac{7}{2}s+s^2\Big(\frac{49}{5s} \leftarrow Z$$

$$\begin{array}{r} \frac{7}{2}s \\ \hline s^2\Big)\frac{5}{14}s^2\Big(\frac{5}{14} \leftarrow Y \\ \frac{5}{14}s \\ \hline 0 \end{array}$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.54.

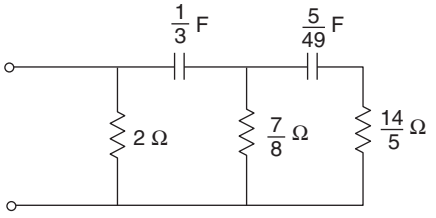


Fig. 9.54

Example 9.53 *Realise the RC impedance in Cauer I and Foster I forms*

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates presence of a zero at infinity. Hence, the admittance function $Y(s)$ has a pole at infinity.

$$Y(s) = \frac{s^2+8s+12}{s+4}$$

By continued fraction expansion,

$$\begin{array}{r} s \\ \hline s+4\Big)s^2+8s+12\Big(s \leftarrow Y \\ s^2+4s \\ \hline 4s+12\Big)s+4\Big(\frac{1}{4} \leftarrow Z \\ s+3 \\ \hline 1\Big)4s+12\Big(4s \leftarrow Y \\ 4s \\ \hline 12\Big)1\Big(\frac{1}{12} \leftarrow Z \\ \frac{1}{0} \end{array}$$

The impedances are connected in series branches, whereas the admittances are connected in parallel branches. The network is shown in Fig. 9.55.

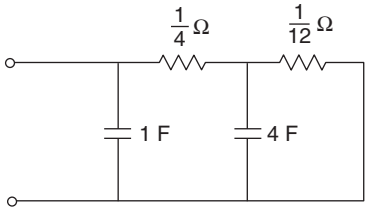


Fig. 9.55

Foster I Form The Foster I form is obtained by partial fraction expansion of $Z(s)$.

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

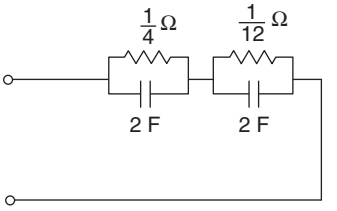
By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+4)}{(-2+6)} = \frac{1}{2}$$
$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{(-6+4)}{(-6+2)} = \frac{1}{2}$$
$$Z(s) = \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+6}$$

These two terms represent the impedance of a parallel *RC* circuit for which
$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,
$$R_1 = \frac{1}{4}\,\Omega, \quad C_1 = 2\text{ F}$$
$$R_2 = \frac{1}{12}\,\Omega, \quad C_2 = 2\text{ F}$$



The network is shown in Fig. 9.56.

Fig. 9.56

Example 9.54 *The RC driving-point impedance function is given as* $Z(s) = H\dfrac{(s+1)(s+4)}{s(s+3)}$ *Realise the impedance function in the ladder form, given* $Z(-2) = 1$.

Solution Putting $s = -2$,
$$Z(-2) = H\frac{(-2+1)(-2+4)}{(-2)(-2+3)} = H$$
$$H = 1$$
$$Z(s) = \frac{(s+1)(s+4)}{s(s+3)}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,
$$s^2+3s\,\overline{)\,s^2+5s+4\,(}\;1 \leftarrow Z$$
$$\frac{s^2+3s}{2s+4\,\overline{)\,s^2+3s\,(}}\;\frac{1}{2}s \leftarrow Y$$
$$\frac{s^2+2s}{s\,\overline{)\,2s+4\,(}}\;2 \leftarrow Z$$
$$\frac{2s}{4\,\overline{)\,s\,(}}\;\frac{1}{4}s \leftarrow Y$$
$$\frac{s}{0}$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 9.57.

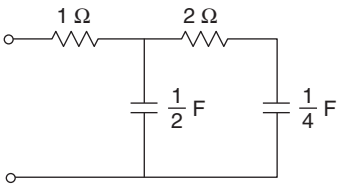


Fig. 9.57

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s) = \frac{4 + 5s + s^2}{3s + s^2}$$

By continued fraction expansion,

$$3s + s^2 \overline{)4 + 5s + s^2} \left(\frac{4}{3s} \leftarrow Z \right.$$

$$\underline{4 + \frac{4}{3}s}$$

$$\frac{11}{3}s + s^2 \overline{)3s + s^2} \left(\frac{9}{11} \leftarrow Y \right.$$

$$\underline{3s + \frac{9}{11}s^2}$$

$$\frac{2}{11}s^2 \overline{)\frac{11}{3}s + s^2} \left(\frac{121}{6s} \leftarrow Z \right.$$

$$\underline{\frac{11}{3}s}$$

$$s^2 \overline{)\frac{2}{11}s^2} \left(\frac{2}{11} \leftarrow Y \right.$$

$$\underline{\frac{2}{11}s^2}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.58.

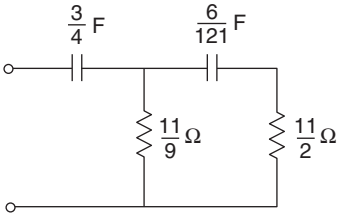


Fig. 9.58

Example 9.55 *An impedance function has the pole-zero diagram as shown in Fig. 9.59. Find the impedance function such that* $Z(-4) = \frac{3}{4}$ *and realise in Cauer I and Foster II forms.*

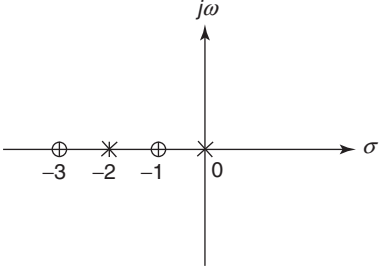


Fig. 9.59

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$.

$$Z(s) = H\frac{(s+1)(s+3)}{s(s+2)}$$

Putting $s = -4$,

$$Z(-4) = H\frac{(-4+1)(-4+3)}{(-4)(-4+2)} = H\frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}\text{H}$$

$$\frac{3}{4} = \frac{3}{8}\text{H}$$

$$H = 2$$

$$Z(s) = \frac{2(s+1)(s+3)}{s(s+2)} = \frac{2s^2+8s+6}{s^2+2s}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,

$$s^2+2s\,\Big)2s^2+8s+6\,\Big(\,2 \leftarrow Z$$

$$\underline{2s^2+4s}$$

$$4s+6\,\Big)s^2+2s\,\Big(\,\frac{1}{4}s \leftarrow Y$$

$$\underline{s^2+\frac{3}{2}s}$$

$$\frac{1}{2}s\,\Big)4s+6\,\Big(\,8 \leftarrow Z$$

$$\underline{4s}$$

$$6\,\Big)\frac{1}{2}s\,\Big(\,\frac{1}{12}s \leftarrow Y$$

$$\underline{\frac{1}{2}s}$$

$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 9.60.

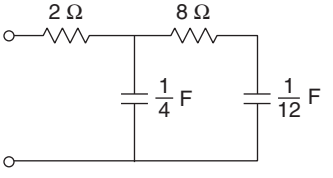


Fig. 9.60

Foster II Form The Foster II form is obtained by partial fraction expansion of $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{s+2}{2(s+1)(s+3)}$$

By partial fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{2(-3+1)} = \frac{-1}{2(-2)} = \frac{1}{4}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{4}}{s+3}$$

$$Y(s) = \frac{\frac{1}{4}s}{s+1} + \frac{\frac{1}{4}s}{s+3}$$

Two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 4\ \Omega, \quad C_1 = \frac{1}{4}\ \text{F}$$

$$R_2 = 4\ \Omega, \quad C_2 = \frac{1}{12}\ \text{F}$$

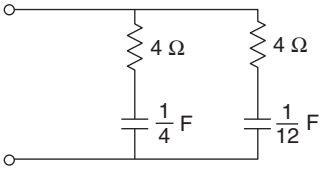


Fig. 9.61

The network is shown in Fig. 9.61.

9.7 REALISATION OF *RL* FUNCTIONS

RL driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the *s* plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.
4. The highest critical frequency is a pole which may be at infinity.
5. Residues evaluated at the poles of $Z_{RL}(s)$ are real and negative while that of $\frac{Z_{RL}(s)}{s}$ are real and positive.
6. The slope $\frac{d}{d\sigma} Z_{RL}$ is positive.
7. $Z_{RL}(0) < Z_{RL}(\infty)$.

The admittance of an inductor is similar to the impedance of a capacitor. Hence, properties of an *RL* admittance are identical to those of an *RC* impedance and vice-versa, i.e.,

$$Z_{RC}(s) = Y_{RL}(s)$$
$$Z_{RL}(s) = Y_{RC}(s)$$

An *RL* admittance can be considered as the dual of an *RC* impedance and vice-versa.

Example 9.56 *Indicate which of the following functions are either RL, RC or LC impedance functions.*

(a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$ *(b)* $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$ *(d)* $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+16)}$

Solution (a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(b) $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(d) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

Example 9.57 *Realise following RL impedance function in Foster-I and Foster-II form.*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where

$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(-2+6)} = -\frac{1}{2}$$

$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6+2)} = -\frac{15}{2}$$

Since residues of $Z(s)$ are negative, partial fraction expansion of $\frac{Z(s)}{s}$ is carried out.

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

By partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where

$$K_0 = s\frac{Z(s)}{s}\bigg|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$

$$K_1 = (s+2)\frac{Z(s)}{s}\bigg|_{s=-2} = \frac{2(-2+1)(-2+3)}{(2)(-2+6)} = \frac{1}{4}$$

$$K_2 = (s+6)\frac{Z(s)}{s}\bigg|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6)(-6+2)} = \frac{5}{4}$$

$$\frac{Z(s)}{s} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{5}{4}}{s+6}$$

$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4}s}{s+2} + \frac{\frac{5}{4}s}{s+6}$$

The first term represents the impedance of the resistor of $\frac{1}{2}\,\Omega$. The other two terms represent the impedance of the parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad L_1 = \frac{1}{8}\,\text{H}$$

$$R_2 = \frac{5}{4}\,\Omega, \quad L_2 = \frac{5}{24}\,\text{H}$$

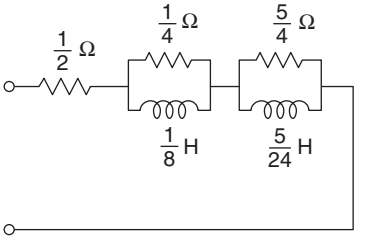


Fig. 9.62

The network is shown in Fig. 9.62.

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$2s^2+8s+6\overline{)s^2+8s+12}\left(\frac{1}{2}\right.$$

$$\frac{s^2+4s+3}{4s+9}$$

$$Y(s)=\frac{1}{2}+\frac{4s+9}{2s^2+8s+6}=\frac{1}{2}+\frac{4s+9}{2(s+1)(s+3)}$$

By partial-fraction expansion,

$$Y_1(s)=\frac{4s+9}{2(s+1)(s+3)}=\frac{K_0}{s+1}+\frac{K_1}{s+3}$$

where

$$K_0=(s+1)Y_1(s)\big|_{s=-1}=\frac{(-4+9)}{2(-1+3)}=\frac{5}{4}$$

$$K_1=(s+3)Y_1(s)\big|_{s=-3}=\frac{(-12+9)}{2(-3+1)}=\frac{3}{4}$$

$$Y(s)=\frac{1}{2}+\frac{\frac{5}{4}}{s+1}+\frac{\frac{3}{4}}{s+3}$$

The first term represents the admittance of a resistor of 2 Ω. The other two terms represent the admittance of a series *RL* circuit. For a series *RL* circuit,

$$Y_{RL}(s)=\frac{\frac{1}{L_i}}{s+\frac{R_i}{L_i}}$$

By direct comparison,

$$R_1=\frac{4}{5}\,\Omega,\quad L_1=\frac{4}{5}\,\text{H}$$

$$R_2=4\,\Omega,\quad L_2=\frac{4}{3}\,\text{H}$$

The network is shown in Fig. 9.63.

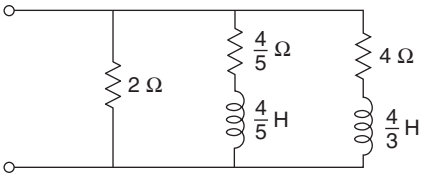


Fig. 9.63

Example 9.58 *Find the Foster forms of the RL impedance function:*

$$Z(s)=\frac{(s+1)(s+4)}{(s+5)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of impedance function $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s}=\frac{(s+1)(s+4)}{s(s+5)(s+3)}$$

By partial-faction expansion,

$$\frac{Z(s)}{s}=\frac{K_0}{s}+\frac{K_1}{s+3}+\frac{K_2}{s+5}$$

where

$$K_0=s\frac{Z(s)}{s}\bigg|_{s=0}=\frac{(1)(4)}{(5)(3)}=\frac{4}{15}$$

$$K_1 = (s+3)\frac{Z(s)}{s}\bigg|_{s=-3} = \frac{(-3+1)(-3+4)}{(-3)(-3+5)} = \frac{(-2)(1)}{(-3)(2)} = \frac{1}{3}$$

$$K_2 = (s+5)\frac{Z(s)}{s}\bigg|_{s=-5} = \frac{(-5+1)(-5+4)}{(-5)(-5+3)} = \frac{(-4)(-1)}{(-5)(-2)} = \frac{2}{5}$$

$$\frac{Z(s)}{s} = \frac{\frac{4}{15}}{s} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{5}}{s+5}$$

$$Z(s) = \frac{4}{15} + \frac{\frac{1}{3}s}{s+3} + \frac{\frac{2}{5}s}{s+5}$$

The first term represents the impedance of the resistor of $\frac{4}{15}\,\Omega$. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{3}\,\Omega, \quad L_1 = \frac{1}{9}\,\text{H}$$

$$R_2 = \frac{2}{5}\,\Omega, \quad L_2 = \frac{2}{25}\,\text{H}$$

The network is shown in Fig. 9.64.

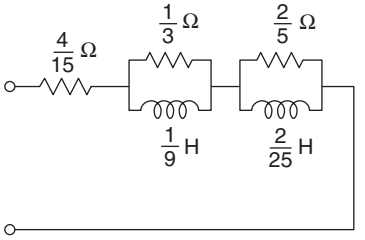


Fig. 9.64

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+5)(s+3)}{(s+1)(s+4)} = \frac{s^2+8s+15}{s^2+5s+4}$$

$$s^2+5s+4 \overline{)\, s^2+8s+15 \,(1}$$

$$\underline{s^2+5s+4}$$

$$3s+11$$

$$Y(s) = 1 + \frac{3s+11}{(s+1)(s+4)}$$

By partial-fraction expansion,

$$Y_1(s) = \frac{K_0}{s+1} + \frac{K_1}{s+4}$$

where

$$K_0 = (s+1)Y_1(s)\big|_{s=-1} = \frac{(-3+11)}{(-1+4)} = \frac{8}{3}$$

$$K_1 = (s+4)Y_1(s)\big|_{s=-4} = \frac{(-12+11)}{(-4+1)} = \frac{1}{3}$$

$$Y_1(s) = \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

$$Y(s) = 1 + \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

The first term represents the admittance of a resistor of 1 Ω. The other two terms represent the admittance of a series *RL* circuit.
For a series *RL* circuit,

$$Y_{RL}(s) = \frac{\frac{1}{L_i}}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{3}{8}\,\Omega, \quad L_1 = \frac{3}{8}\,\text{H}$$

$$R_2 = 12\,\Omega, \quad L_2 = 3\,\text{H}$$

The network is shown in Fig. 9.65.

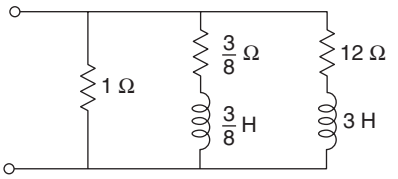


Fig. 9.65

Example 9.59

Find the Cauer forms of the RL impedance function:

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by a continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{2s^2+8s+6}{s^2+8s+12}$$

By continued fraction expansion,

$$s^2+8s+12\overline{)2s^2+8s+6}(2 \leftarrow Z$$

$$\underline{2s^2+16s+24}$$

$$-8s-18$$

Since a negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\overline{)s^2+8s+12}\left(\frac{1}{2} \leftarrow Y\right.$$

$$\underline{s^2+4s+3}$$

$$4s+9\overline{)2s^2+8s+6}\left(\frac{1}{2}s \leftarrow Z\right.$$

$$\underline{2s^2+\frac{9}{2}s}$$

$$\frac{7}{2}s+6\overline{)4s+9}\left(\frac{8}{7} \leftarrow Y\right.$$

$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\overline{)\frac{7}{2}s+6}\left(\frac{49}{30}s \leftarrow Z\right.$$

$$\underline{\frac{7}{2}s}$$

$$6\overline{)\frac{15}{7}}\left(\frac{15}{42} \leftarrow Y\right.$$

$$\underline{\frac{15}{7}}$$

$$0$$

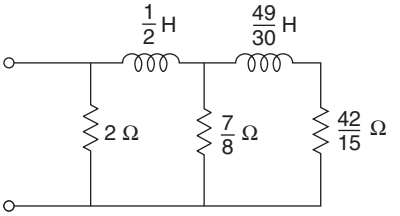


Fig. 9.66

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.66.

Cauer II Form The Cauer II form is obtained from a continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s) = \frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\overline{)6+8s+2s^2}\left(\frac{1}{2} \leftarrow Z\right.$$

$$\underline{6+4s+\frac{1}{2}s^2}$$

$$4s+\frac{3}{2}s^2\overline{)12+8s+s^2}\left(\frac{3}{s} \leftarrow Y\right.$$

$$\underline{12+\frac{9}{2}s}$$

$$\frac{7}{2}s+s^2\overline{)4s+\frac{3}{2}s^2}\left(\frac{8}{7} \leftarrow Z\right.$$

$$\underline{4s+\frac{8}{7}s^2}$$

$$\frac{5}{14}s^2\overline{)\frac{7}{2}s+s^2}\left(\frac{98}{10s} \leftarrow Y\right.$$

$$\underline{\frac{7}{2}s}$$

$$s^2\overline{)\frac{5}{14}s^2}\left(\frac{5}{14} \leftarrow Z\right.$$

$$\underline{\frac{5}{14}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.67.

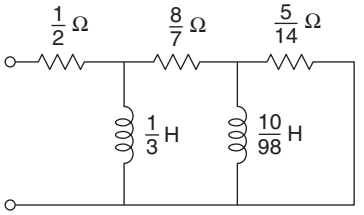


Fig. 9.67

Example 9.60 *Obtain the Foster I and Cauer I forms of the RL impedance function.*

$$Z(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial fraction expansion of $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s} = \frac{(s+4)(s+8)}{(s+1)(s+6)}$$

Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$s^2 + 7s + 6)s^2 + 12s + 32(1$$

$$\frac{s^2 + 7s + 6}{5s + 26}$$

$$\frac{Z(s)}{s} = 1 + \frac{5s+26}{s^2+7s+6} = 1 + \frac{5s+26}{(s+1)(s+6)}$$

By partial-fraction expansion,

$$\frac{Z(s)}{s} = 1 + \frac{K_0}{s+1} + \frac{K_1}{s+6}$$

where

$$K_0 = \left.\frac{5s+26}{s+6}\right|_{s=-1} = \frac{-5+26}{-1+6} = \frac{21}{5}$$

$$K_1 = \left.\frac{5s+26}{s+1}\right|_{s=-6} = \frac{-30+26}{-6+1} = \frac{4}{5}$$

$$\frac{Z(s)}{s} = 1 + \frac{\frac{21}{5}}{s+1} + \frac{\frac{24}{5}}{s+6}$$

$$Z(s) = s + \frac{\frac{21}{5}s}{s+1} + \frac{\frac{4}{5}s}{s+6}$$

The first term represents the impedance of the inductor of 1 H. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{21}{5}\,\Omega, \quad L_1 = \frac{21}{5}\,\text{H}$$

$$R_2 = \frac{4}{5}\,\Omega, \quad L_2 = \frac{4}{30}\,\text{H}$$

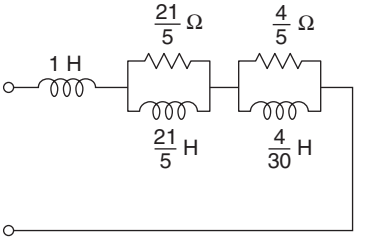


Fig. 9.68

The network is shown in Fig. 9.68.

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{s^3 + 12s^2 + 32s}{s^2 + 7s + 6}$$

By continued fraction expansion,

$$s^2 + 7s + 6 \overline{)\, s^3 + 12s^2 + 32s \,(\, s \leftarrow Z}$$

$$\underline{s^3 + 7s^2 + 6s}$$

$$5s^2 + 26s \overline{)\, s^2 + 7s + 6 \,(\, \frac{1}{5} \leftarrow Y}$$

$$\underline{s^2 + \frac{26}{5}s}$$

$$\frac{9}{5}s + 6 \overline{)\, 5s^2 + 26s \,(\, \frac{25}{9}s \leftarrow Z}$$

$$\underline{5s^2 + \frac{50}{3}s}$$

$$\frac{28}{3}s \overline{)\, \frac{9}{5}s + 6 \,(\, \frac{27}{140} \leftarrow Y}$$

$$\underline{\frac{9}{5}s}$$

$$6 \overline{)\, \frac{28}{3}s \,(\, \frac{28}{18}s \leftarrow Z}$$

$$\underline{\frac{28}{3}s}$$

$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 9.69.

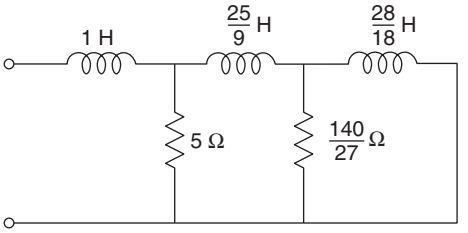


Fig. 9.69

Exercises

9.1 Test the following polynomials for Hurwitz property:

(i) $s^3 + s^2 + 2s + 2$
(ii) $s^4 + s^2 + s + 1$
(iii) $s^3 + 4s^2 + 5s + 2$
(iv) $s^4 + 7s^3 + 6s^2 + 21s + 8$
(v) $s^4 + s^3 + s + 1$
(vi) $s^7 + 3s^6 + 8s^5 + 15s^4 + 17s^3 + 12s^2 + 4s$
(vii) $s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$
(viii) $s^7 + 3s^5 + 2s^3 + s$
(ix) $s^5 + 2s^3 + s$
(x) $s^3 + 2s^2 + 4s + 2$
(xi) $s^4 + s^3 + 4s^2 + 2s + 3$
(xii) $s^5 + 8s^4 + 24s^3 + 28s^2 + 23s + 6$
(xiii) $s^7 - 2s^6 + 2s^5 + 9s^2 + 8s + 4$
(xiv) $s^7 + 3s^5 + 2s^3 + 3$
(xv) $s^5 + s^3 + s$
(xvi) $s^6 + 7s^4 + 5s^3 + s^2 + s$
(xvii) $s^4 + s^3 + 2s^2 + 3s + 2$

9.2 Determine whether the following functions are positive real:

(i) $\dfrac{s^3 + 5s}{s^4 + 2s^2 + 1}$

(ii) $\dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$

(iii) $\dfrac{2s^2 + 2s + 1}{s^3 + 2s^2 + s + 2}$

(iv) $\dfrac{s^4 + 3s^3 + s^2 + s + 2}{s^3 + s^2 + s + 1}$

(v) $\dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$

(vi) $\dfrac{s^2 + 2s + 1}{s^2 + 4s + 4}$

(vii) $\dfrac{s^3 + 2s^2 + 3s + 1}{s^3 + 2s^2 + s + 2}$

(viii) $\dfrac{s^3 + 2s^2 + s + 1}{s^2 + s + 1}$

(ix) $\dfrac{s + 4}{s^2 + 2s + 1}$

(x) $\dfrac{s^2 + 4s + 3}{s^2 + 6s + 8}$

(xi) $\dfrac{s^2 + 1}{s^3 + 4s}$

(xii) $\dfrac{s^4 + 2s^3 + 3s^2 + 1}{s^4 + s^3 + 3s^2 + 2s + 1}$

(xiii) $\dfrac{s^2 + 2s + 4}{(s+1)(s+3)}$

(xiv) $\dfrac{2s + 4}{s + 5}$

(xv) $\dfrac{s^2 + 2s}{s^2 + 1}$

(xvi) $\dfrac{s^2 + 4}{s^3 + 3s^2 + 3s + 1}$

9.3 Determine whether the following functions are *LC, RC* or *RL* function:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(iii) $Z(s) = \dfrac{(s+1)(s+8)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$

(v) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

(vi) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

(vii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(viii) $F(s) = \dfrac{(s+1)(s+2)}{s(s+3)}$

(ix) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(x) $Z(s) = \dfrac{(s+2)(s+4)}{(s+1)}$

(xi) $Y(s) = \dfrac{4(s+3)}{(s+1)(s+5)}$

(xii) $Y(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(xiii) $Z(s) = \dfrac{s(s^2+4)(s^2+16)}{(s^2+9)(s^2+25)}$

(xiv) $Z(s) = \dfrac{(s^2+1)(s^2+8)}{s(s^2+4)}$

9.4 Realise the following functions in Foster I form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $F(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(v) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

(vi) $Y(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

(vii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

9.5 Realise the following functions in Foster II form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $Z(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Y(s) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)}$

(v) $Z(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

9.6 Realise the following functions in Cauer I form:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(iii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(v) $F(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(vi) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{6(s+2)(s+4)}{s(s+3)}$

(viii) $Z(s) = \dfrac{s^3+2s}{s^4+4s^2+3}$

(ix) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(x) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

9.7 Realise the following function in Cauer II form:

(i) $F(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(ii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iv) $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(v) $F(s) = \dfrac{s^3+12s^2+32s}{s^2+7s+6}$

(vi) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(viii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

9.8 An impedance function has the pole-zero diagram as shown in Fig. 9.70 below. If $Z(-2) = 3$, synthesise the impedance function in Foster and Cauer forms.

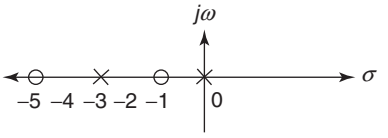


Fig. 9.70

9.9 An impedance function has the pole-zero diagram as shown in Fig. 9.71. Find the impedance function such that $Z(-4) = \dfrac{8}{3}$ and realise in Cauer forms.

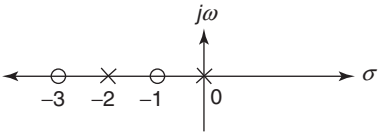


Fig. 9.71

9.10 For the realisation of a given function $F(s)$.

$$F(s) = \frac{K_0}{s} + \sum_{i=1}^{n} \frac{sK_i}{(s^2+\omega_i^2)} + sK_\infty$$

where K_0, K_i (i = 1, 2, 3, … n) and K_∞ are constants.

(i) Mention the type of function (*RC*, *RL* or *LC*)

(ii) Given that $K_0 = 6$, $K_1 = 8$, $\omega_1 = 4$, $K_2 = 10$, $\omega_2 = 8$, $K_\infty = 5$, find the component values of realised network for $F(s) = Z(s)$ and $F(s) = Y(s)$. Draw neat diagrams.

Objective-Type Questions

9.1 The necessary and sufficient condition for a rational function $F(s)$ to be the driving-point impedance of an *RC* network is that all poles and zeros should be

(a) simple and lie on the negative real axis in the s-plane
(b) complex and lie in the left half of s-plane
(c) complex and lie in the right-half of s-plane
(d) simple and lie on the positive real axis of the s-plane

9.2 The number of roots of $s^3 + 5s^2 + 7s + 3 = 0$ in the left half of s-plane is

(a) zero (b) one
(c) two (d) three

9.3 The first and the last critical frequencies of a driving-point impedance function of a passive network having two kinds of elements, are a pole and a zero respectively. The above property will be satisfied by

(a) *RL* network only
(b) *RC* network only
(c) *LC* network only
(d) *RC* as well as *RL* network

9.4 The pole-zero pattern of a particular network is shown in Fig. 9.72. It is that of an

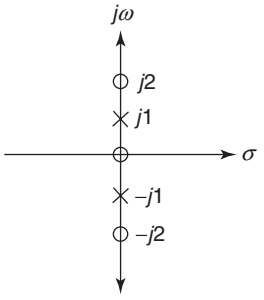


Fig. 9.72

(a) *LC* network (b) *RC* network
(c) *RL* network (d) none of these

9.5 The first critical frequency nearest to the origin of the complex frequency plane for an *RL* driving-point impedance function will be

(a) a zero in the left-half plane
(b) a zero in the right-half plane
(c) a pole in the left-half plane
(d) a pole in the right-half plane

9.6 Consider the following polynomials:
$P_1 = s^8 + 2s^6 + 4s^4$
$P_2 = s^6 - 3s^2 + 2s^2 + 1$
$P_3 = s^4 + 3s^3 + 3s^2 + 2s + 1$
$P_4 = s^7 + 2s^6 + 2s^4 + 4s^3 + 8s^2 + 8s + 4$
which one of these polymials is not Hurwitz?

(a) P_1 (b) P_2
(c) P_3 (d) P_4

9.7 For very high frequencies, the driving-point admittance function, $Y(s) = \dfrac{4(s+1)(s+3)}{s(s+2)(s+4)}$ behaves as

(a) a resistance of $\dfrac{3}{2}\,\Omega$
(b) a capacitance of 4 F
(c) an inductance of $\dfrac{1}{4}$ H
(d) an inductance of 4 H

9.8 The driving-point impedance $Z(s) = \dfrac{s+3}{s+4}$ behaves as

(a) a resistance of 0.75 Ω at low frequencies
(b) a resistance of 1 Ω at high frequencies
(c) both (a) and (b) above
(d) none of the above

9.9 An *RC* driving-point impedance function has zeros at $s = -2$ and $s = -5$. The admissible poles for the functions would be

(a) $s = 0, s = -6$ (b) $s = -1, s = -3$
(c) $s = 0, s = -1$ (d) $s = -3, s = -4$

9.10 Consider the following from the point of view of possible realisation as driving-point impedances using passive elements:

1. $\dfrac{1}{s(s+5)}$ 2. $\dfrac{s+3}{s^2(s+5)}$

3. $\dfrac{s^2+3}{s^2(s^2+5)}$ 4. $\dfrac{s+5}{s(s+5)}$

Of these, the realisable are

(a) 1, 2 and 4 (b) 1, 2 and 3
(c) 3 and 4 (d) none of these

9.11 The poles and zeros of a driving-point function of a network are simple and interlace on the negative real axis with a pole closest to the origin. It can be realised

(a) by an *LC* network
(b) as an *RC* driving point impedance
(c) as an *RC* driving point admittance
(d) only by an *RLC* network

9.12 If $F_1(s)$ and $F_2(s)$ are two positive real functions then the function which is always positive real, is

(a) $F_1(s)\,F_2(s)$ (b) $\dfrac{F_1(s)}{F_2(s)}$

(c) $\dfrac{F_1(s)F_2(s)}{F_1(s)+F_2(s)}$ (d) $F_1(s) - F_2(s)$

9.13 The circuit shown in Fig. 9.73 is

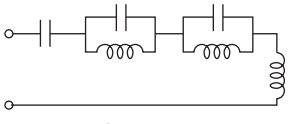


Fig. 9.73

(a) Cauer I form (b) Foster I form
(c) Cauer II form (d) Foster II form

9.14 For an *RC* driving-point impedance function, the poles and zeros

(a) should alternate on the real axis
(b) should alternate only on the negative real axis
(c) should alternate on the imaginary axis
(d) can lie anywhere on the left half-plane

Answers to Objective-Type Questions

9.1 (a)	9.2 (a)	9.3 (b)	9.4 (a)	9.5 (a)	9.6 (b)	9.7 (c)
9.8 (c)	9.9 (b)	9.10 (a)	9.11 (b)	9.12 (c)	9.13 (b)	9.14 (b)

10

Two-Port Networks

10.1 INTRODUCTION

A two-port network has two pairs of terminals, one pair at the input known as *input port* and one pair at the output known as *output port* as shown in Fig. 10.1. There are four variables V_1, V_2, I_1 and I_2 associated with a two-port network. Two of these variables can be expressed in terms of the other two variables. Thus, there will be two dependent variables and two independent variables. The number of possible combinations generated by four variables taken two at a time is ${}^4C_2$, i.e., six. There are six possible sets of equations describing a two-port network.

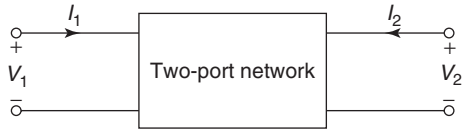


Fig. 10.1 *Two-port network*

Table 10.1 *Two–port parameters*

Parameter	*Variables*		*Equation*
	Express	*In terms of*	
Open-Circuit Impedance	V_1, V_2	I_1, I_2	$V_1 = Z_{11} I_1 + Z_{12} I_2$ $V_2 = Z_{21} I_1 + Z_{22} I_2$
Short-Circuit Admittance	I_1, I_2	V_1, V_2	$I_1 = Y_{11} V_1 + Y_{12} V_2$ $I_2 = Y_{21} V_1 + Y_{22} V_2$
Transmission	V_1, I_1	V_2, I_2	$V_1 = AV_2 - BI_2$ $I_1 = CV_2 - DI_2$
Inverse Transmission	V_2, I_2	V_1, I_1	$V_2 = A' V_1 - B' I_1$ $I_2 = C' V_1 - D' I_1$
Hybrid	V_1, I_2	I_1, V_2	$V_1 = h_{11} I_1 + h_{12} V_2$ $I_2 = h_{21} I_1 + h_{22} V_2$
Inverse Hybrid	I_1, V_2	V_1, I_2	$I_1 = g_{11} V_1 + g_{12} I_2$ $V_2 = g_{21} V_1 + g_{22} I_2$

10.2 OPEN-CIRCUIT IMPEDANCE PARAMETERS (Z PARAMETERS)

The Z parameters of a two-port network may be defined by expressing two-port voltages V_1 and V_2 in terms of two-port currents I_1 and I_2.

$$(V_1, V_2) = f(I_1, I_2)$$
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$[V] = [Z][I]$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero.

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

where Z_{11} is the driving-point impedance with the output port open-circuited. It is also called *open-circuit input impedance*.

Similarly,

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

where Z_{21} is the transfer impedance with the output port open-circuited. It is also called *open-circuit forward transfer impedance*.

Case 2 When input port is open-circuited, i.e., $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where Z_{12} is the transfer impedance with the input port open-circuited. It is also called *open-circuit reverse transfer impedance*.

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

where Z_{22} is the open-circuit driving-point impedance with the input port open-circuited. It is also called *open circuit output impedance*.

As these impedance parameters are measured with either the input or output port open-circuited, these are called *open-circuit impedance parameters*.

The equivalent circuit of the two-port network in terms of Z parameters is shown in Fig. 10.2.

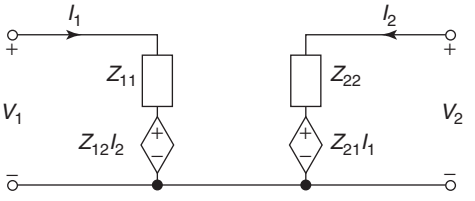


Fig. 10.2 *Equivalent circuit of the two-port network in terms of Z parameter*

10.2.1 Condition for Reciprocity

A network is said to be reciprocal if the ratio of excitation at one port to response at the other port is same if excitation and response are interchanged.

(a) As shown in Fig. 10.3, voltage V_s is applied at the input port with the output port short-circuited.

Fig. 10.3 *Network for deriving condition for reciprocity*

i.e.,
$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

From the *Z*-parameter equations,
$$V_s = Z_{11}\, I_1 - Z_{12}\, I_2'$$
$$0 = Z_{21}\, I_1 - Z_{22}\, I_2'$$
$$I_1 = \frac{Z_{22}}{Z_{21}} I_2'$$
$$V_s = Z_{11} \frac{Z_{22}}{Z_{21}} I_2' - Z_{12}\, I_2'$$
$$\frac{V_s}{I_2'} = \frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{21}}$$

(b) As shown in Fig. 10.4, voltage V_s is applied at the output port with input port short-circuited.

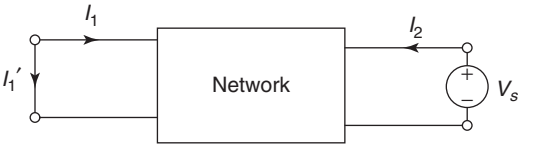


Fig. 10.4 *Network for deriving condition for reciprocity*

i.e.,
$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1 = -I_1'$$

From the *Z*-parameter equations,
$$0 = -Z_{11}\, I_1' + Z_{12}\, I_2$$
$$V_s = -Z_{21}\, I_1' + Z_{22}\, I_2$$
$$I_2 = \frac{Z_{11}}{Z_{12}} I_1'$$
$$V_s = -Z_{21} I_1' + Z_{22} \frac{Z_{11}}{Z_{12}} I_1'$$
$$\frac{V_s}{I_1'} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{12}}$$

Hence, for the network to be reciprocal,
$$\frac{V_s}{I_1'} = \frac{V_s}{I_2'}$$

i.e.,
$$Z_{12} = Z_{21}$$

10.2.2 Condition for Symmetry

For a network to be symmetrical, the voltage-to-current ratio at one port should be the same as the voltage-to-current ratio at the other port with one of the ports open-circuited.

(a) When the output port is open-circuited, i.e., $I_2 = 0$
From the *Z*-parameter equation,
$$V_s = Z_{11}\, I_1$$
$$\frac{V_s}{I_1} = Z_{11}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$
From the *Z*-parameter equation,
$$V_s = Z_{22}\, I_2$$
$$\frac{V_s}{I_2} = Z_{22}$$

Hence, for the network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $$Z_{11} = Z_{22}$$

Example 10.1 *Test results for a two-port network are (a) $I_1 = 0.1 \angle 0°$ A, $V_1 = 5.2 \angle 50°$ V, $V_2 = 4.1 \angle -25°$ V with Port 2 open-circuited (b) $I_2 = 0.1 \angle 0°$A, $V_1 = 3.1 \angle -80°$ V, $V_2 = 4.2 \angle 60°$ V, with Port 1 open-circuited. Find Z parameters.*

Solution

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{5.2\angle 50°}{0.1\angle 0°} = 52 \angle 50° \ \Omega, \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{3.1\angle -80°}{0.1\angle 0°} = 31 \angle -80° \ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{4.1\angle -25°}{0.1\angle 0°} = 41 \angle -25° \ \Omega, \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{4.2\angle 60°}{0.1\angle 0°} = 42 \angle 60° \ \Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 52\angle 50° & 31\angle -80° \\ 41\angle -25° & 42\angle 60° \end{bmatrix}$$

Example 10.2 *Find the Z parameters for the network shown in Fig. 10.5.*

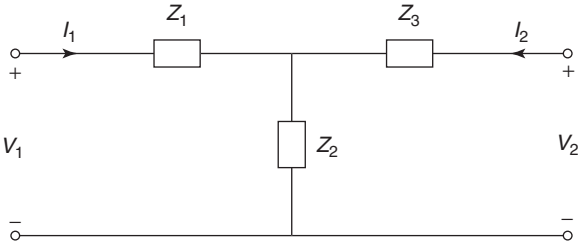


Fig. 10.5

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.

Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)\, I_1$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2$$

Also $$V_2 = Z_2\, I_1$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2$$

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$.

Applying KVL to Mesh 2,

$$V_2 = (Z_2 + Z_3)\, I_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3$$

Also

$$V_1 = Z_2\, I_2$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_2 \\ Z_2 & Z_2 + Z_3 \end{bmatrix}$$

Second Method

The network is redrawn as shown in Fig. 10.6.

Applying KVL to Mesh 1,

$$V_1 = Z_1 I_1 + Z_2(I_1 + I_2)$$
$$= (Z_1 + Z_2) I_1 + Z_2 I_2 \quad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$V_2 = Z_3 I_2 + Z_2(I_1 + I_2)$$
$$= Z_2 I_1 + (Z_2 + Z_3) I_2 \quad \ldots(\text{ii})$$

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_2 \\ Z_2 & Z_2 + Z_3 \end{bmatrix}$$

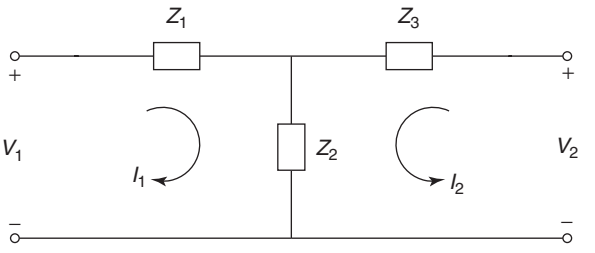


Fig. 10.6

Example 10.3

Find Z-parameter for the network shown in Fig. 10.7.

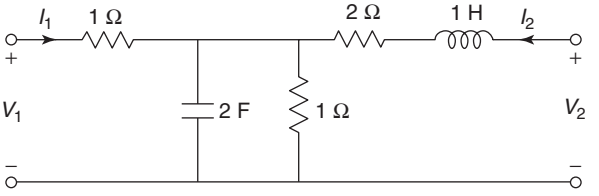


Fig. 10.7

Solution The transformed network is shown in Fig. 10.8.

$$Z_1 = 1$$

$$Z_2 = \frac{\left(\frac{1}{2s}\right)(1)}{\frac{1}{2s} + 1} = \frac{1}{2s+1}$$

$$Z_3 = s + 2$$

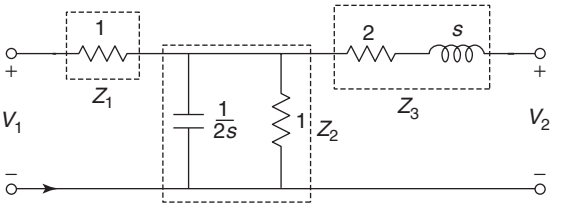


Fig. 10.8

From definition of *Z*-parameters,

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2 = 1 + \frac{1}{2s+1} = \frac{2s+2}{2s+1},$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2 = \frac{1}{2s+1}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2 = \frac{1}{2s+1},$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3 = \frac{1}{2s+1} + s + 2 = \frac{2s^2 + 5s + 3}{2s+1}$$

Example 10.4 *Find Z-parameters for the network shown in Fig. 10.9.*

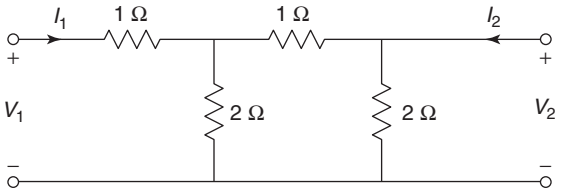


Fig. 10.9

Solution The network is redrawn as shown in Fig. 10.10.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots(iii)$$

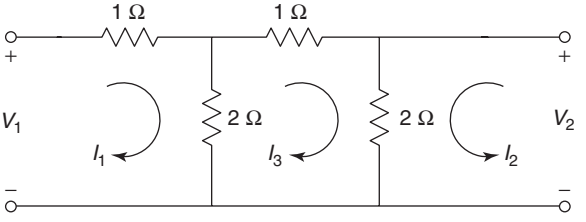


Fig. 10.10

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \frac{4}{5}I_1 + \frac{4}{5}I_2$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + \frac{4}{5}I_1 - \frac{4}{5}I_2$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2 \qquad \ldots(v)$$

Comparing Eqs (iv) and (v) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{6}{5} \end{bmatrix}$$

Example 10.5 *Find the Z-parameters for the network shown in Fig. 10.11.*

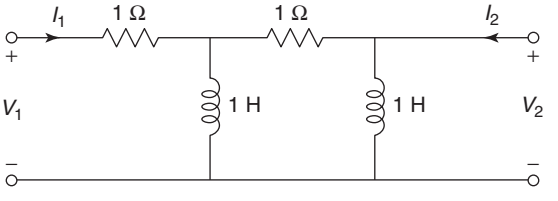


Fig. 10.11

Solution The transformed network is shown in Fig. 10.12.

Applying KVL to Mesh 1,

$$V_1 = (s+1)I_1 - sI_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = sI_2 + sI_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-sI_1 + sI_2 + (2s+1)I_3 = 0$$

$$I_3 = \frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2 \quad \ldots(iii)$$

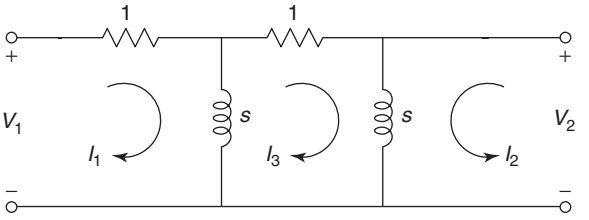


Fig. 10.12

Substituting Eq. (iii) in Eq. (i),

$$V_1 = (s+1)I_1 - s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2+3s+1}{2s+1}\right)I_1 + \left(\frac{s^2}{2s+1}\right)I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = sI_2 + s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2}{2s+1}\right)I_1 + \left(\frac{s^2+s}{2s+1}\right)I_2 \quad \ldots(v)$$

Comparing Eqs (iv) and (v) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{2s+1} & \dfrac{s^2}{2s+1} \\ \dfrac{s^2}{2s+1} & \dfrac{s^2+s}{2s+1} \end{bmatrix}$$

Example 10.6 *Find the open-circuit impedance parameters for the network shown in Fig. 10.13. Determine whether the network is symmetrical and reciprocal.*

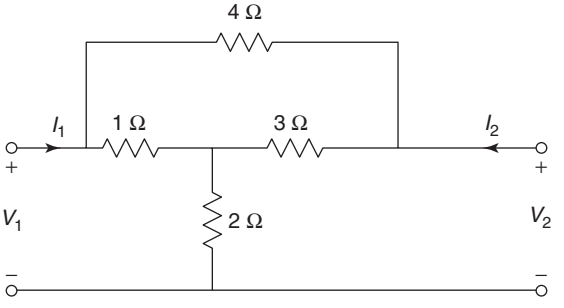


Fig. 10.13

Solution The network is redrawn as shown in Fig. 10.14.

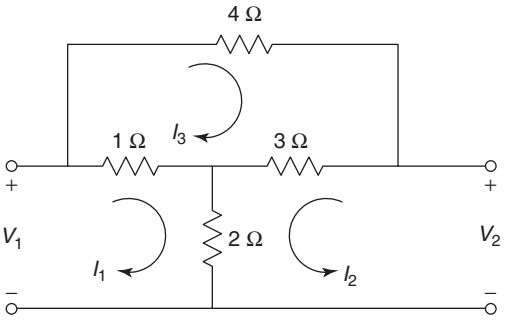


Fig. 10.14

Applying KVL to Mesh 1,

$$V_1 - 1(I_1 - I_3) - 2(I_1 + I_2) = 0$$

$$V_1 = 3I_1 + 2I_2 - I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 - 3(I_2 + I_3) - 2(I_1 + I_2) = 0$$

$$V_2 = 2I_1 + 5I_2 + 3I_3 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-4I_3 - 3(I_2 + I_3) - 1(I_3 - I_1) = 0$$

$$I_1 - 3I_2 + 8I_3 = 0$$

$$I_3 = \frac{1}{8}I_1 - \frac{3}{8}I_2 \qquad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{23}{8}I_1 + \frac{19}{8}I_2 \qquad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 5I_2 + 3\left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{19}{8}I_1 + \frac{31}{8}I_2 \qquad \ldots(v)$$

Comparing Eqs (iv) and (v) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{23}{8} & \frac{19}{8} \\ \frac{19}{8} & \frac{31}{8} \end{bmatrix}$$

Since $Z_{11} \neq Z_{22}$, the network is not symmetrical.
Since $Z_{12} = Z_{21}$, the network is reciprocal.

10.3 SHORT-CIRCUIT ADMITTANCE PARAMETERS (Y PARAMETERS)

The *Y* parameters of a two-port network may be defined by expressing the two-port currents I_1 and I_2 in terms of the two-port voltages V_1 and V_2.

$$(I_1, I_2) = f(V_1, V_2)$$

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

In matrix form, we can write

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$[I] = [Y][V]$$

The individual Y parameters for a given network can be defined by setting each of the port voltages equal to zero.

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

where Y_{11} is the driving-point admittance with the output port short-circuited. It is also called *short-circuit input admittance*.

Similarly,

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

where Y_{21} is the transfer admittance with the output port short-circuited. It is also called *short-circuit forward transfer admittance*.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

where Y_{12} is the transfer admittance with the input port short-circuited. It is also called *short-circuit reverse transfer admittance*.

Similarly,

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

where Y_{22} is the short-circuit driving-point admittance with the input port short-circuited. It is also called the *short circuit output admittance*.

As these admittance parameters are measured with either input or output port short-circuited, these are called *short-circuit admittance parameters*.

The equivalent circuit of the two-port network in terms of Y parameters is shown in Fig. 10.15.

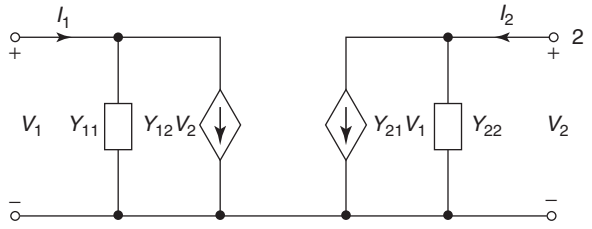


Fig. 10.15 *Equivalent circuit of the two-port network in terms of Y-parameters*

10.3.1 Condition for Reciprocity

(a) As shown in Fig. 10.16, voltage V_s is applied at input port with the output port short-circuited.

i.e, $V_1 = V_s$

$V_2 = 0$

$I_2 = -I_2'$

From the Y-parameter equation,

$$-I_2' = Y_{21} V_s$$

$$\frac{I_2}{V_s} = -Y_{21}$$

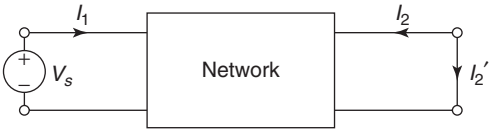


Fig. 10.16 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 10.17, voltage V_s is applied at output port with the input port short-circuited.

i.e, $V_2 = V_s$

$V_1 = 0$

$I_1 = -I_1'$

From the *Y*-parameter equation,

$$-I_1' = Y_{12}\, V_s$$

$$\frac{I_1'}{V_s} = -Y_{12}$$

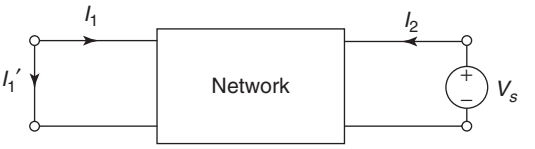


Fig. 10.17 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,

$$\frac{I_2'}{V_s} = \frac{I_1'}{V_s}$$

i.e, $Y_{12} = Y_{21}$

10.3.2 Condition for Symmetry

(a) When the output port is short-circuited, i.e., $V_2 = 0$.
From the *Y*-parameter equation,

$$I_1 = Y_{11}\, V_s$$

$$\frac{V_s}{I_1} = \frac{1}{Y_{11}}$$

(b) When the input port is short-circuited, i.e., $V_1 = 0$.
From the *Y*-parameter equation,

$$I_2 = Y_{22}\, V_s$$

$$\frac{V_s}{I_2} = \frac{1}{Y_{22}}$$

Hence, for the network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $Y_{11} = Y_{22}$

Example 10.7 *Test results for a two-port network are*

(a) Port 2 short-circuited: $V_1 = 50\angle 0°\, V, I_1 = 2.1\angle -30°\, A, I_2 = -1.1\angle -20°\, A$

(b) Port 1 short-circuited: $V_2 = 50\angle 0°\, V, I_2 = 3\angle -15°\, A, I_1 = -1.1\angle -20°\, A.$

Find Y-parameters.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.1\angle -30°}{50\angle 0°} = 0.042\angle -30°\ \mho, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-1.1\angle -20°}{50\angle 0°} = -0.022\angle -20°\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-1.1\angle -20°}{50\angle 0°} = -0.022\angle -20°\ \mho, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3\angle -15°}{50\angle 0°} = 0.06\angle -15°\ \mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.042\angle -30° & -0.022\angle -20° \\ -0.022\angle -20° & 0.06\angle -15° \end{bmatrix}$$

Example 10.8 *Find Y-parameters for the network shown in Fig. 10.18.*

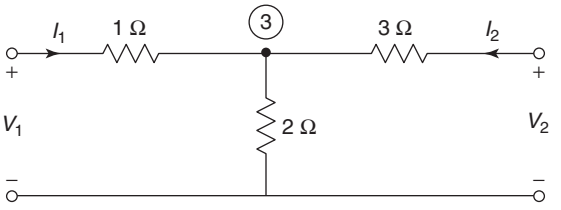


Fig. 10.18

Solution

First Method

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 10.19,

$$R_{eq} = 1 + \frac{2 \times 3}{2+3} = 1 + \frac{6}{5} = \frac{11}{5}\ \Omega$$

Now, $$V_1 = \frac{11}{5} I_1$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{5}{11}\ \text{℧}$$

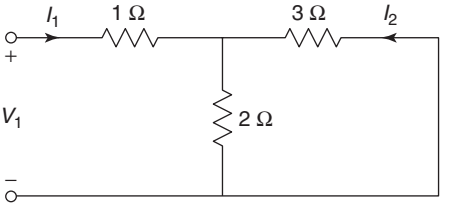


Fig. 10.19

Also, $$I_2 = \frac{2}{5}(-I_1) = -\frac{2}{5} \times \frac{5}{11} V_1 = -\frac{2}{11} V_1$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{2}{11}\ \text{℧}$$

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 10.20,

$$R_{eq} = 3 + \frac{1 \times 2}{1+2} = 3 + \frac{2}{3} = \frac{11}{3}\ \Omega$$

Now $$V_2 = \frac{11}{3} I_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3}{11}\ \text{℧}$$

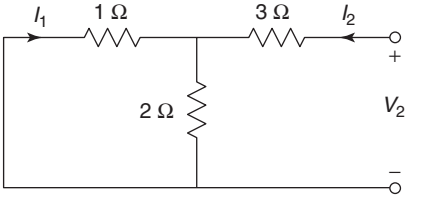


Fig. 10.20

Also $$I_1 = \frac{2}{3}(-I_2) = -\frac{2}{3} \times \frac{3}{11} V_2 = -\frac{2}{11} V_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{2}{11}\ \text{℧}$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Second Method (Refer Fig. 10.18)

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \ldots(i)$$

$$I_2 = \frac{V_2 - V_3}{3}$$

$$= \frac{V_2}{3} - \frac{V_3}{3} \quad \ldots(ii)$$

Applying KCL at Node 3,

$$I_1 + I_2 = \frac{V_3}{2} \quad \ldots(iii)$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$V_1 - V_3 + \frac{V_2}{3} - \frac{V_3}{3} = \frac{V_3}{2}$$

$$V_1 + \frac{V_2}{3} = \frac{11}{6} V_3$$

$$V_3 = \frac{6}{11} V_1 + \frac{2}{11} V_2 \quad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = V_1 - \frac{6}{11} V_1 - \frac{2}{11} V_2$$

$$= \frac{5}{11} V_1 - \frac{2}{11} V_2 \quad \ldots(v)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{V_2}{3} - \frac{1}{3}\left(\frac{6}{11} V_1 + \frac{2}{11} V_2\right)$$

$$= -\frac{2}{11} V_1 + \frac{3}{11} V_2 \quad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Example 10.9 *Find Y-parameters of the network shown in Fig. 10.21.*

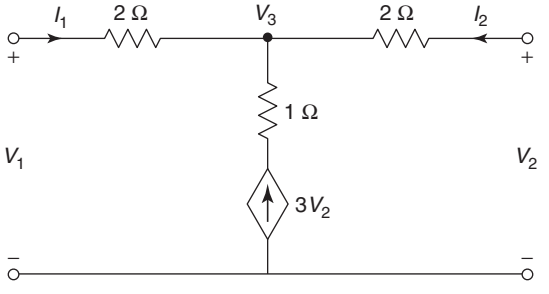


Fig. 10.21

Solution From Fig. 10.21,

$$I_1 = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2} V_1 - \frac{1}{2} V_3 \quad \ldots(i)$$

$$I_2 = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \quad \ldots(ii)$$

Applying KCL at Node 3,

$$I_1 + I_2 + 3V_2 = 0 \quad \ldots(iii)$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} + 3V_2 = 0$$

$$2V_3 = V_1 + 7V_2$$

$$V_3 = \frac{1}{2}V_1 + \frac{7}{2}V_2 \quad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = \frac{1}{2}V_1 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= \frac{1}{4}V_1 - \frac{7}{4}V_2 \quad \ldots(v)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{1}{2}V_2 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= -\frac{1}{4}V_1 - \frac{5}{4}V_2 \quad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}$$

Example 10.10 *Determine Y-parameters for the network shown in Fig. 10.22. Determine whether the network is symmetrical and reciprocal.*

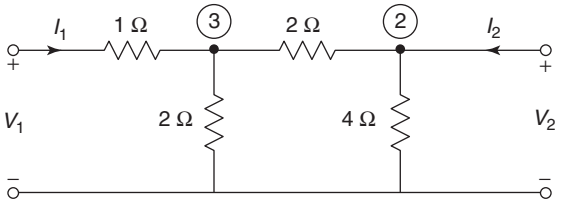


Fig. 10.22

Solution From Fig. 10.31,

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \quad \ldots(i)$$

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{2} + \frac{V_3 - V_2}{2}$$

$$= V_3 - \frac{V_2}{2} \quad \ldots(ii)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{4} + \frac{V_2 - V_3}{2}$$

$$= \frac{3}{4}V_2 - \frac{V_3}{2} \qquad \ldots(iii)$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = V_3 - \frac{V_2}{2}$$

$$V_3 = \frac{V_1}{2} + \frac{V_2}{4} \qquad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = \frac{V_1}{2} + \frac{V_2}{4} - \frac{V_2}{2}$$

$$= \frac{V_1}{2} - \frac{V_2}{4} \qquad \ldots(v)$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = \frac{3}{4}V_2 - \frac{1}{2}\left(\frac{V_1}{2} + \frac{V_2}{4}\right)$$

$$= -\frac{V_1}{4} + \frac{5V_2}{8} \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{8} \end{bmatrix}$$

Since $Y_{11} \neq Y_{22}$, the network is not symmetrical.
Since $Y_{12} = Y_{21}$, the network is reciprocal.

Example 10.11 *Determine the short-circuit admittance parameters for the network shown in Fig. 10.23.*

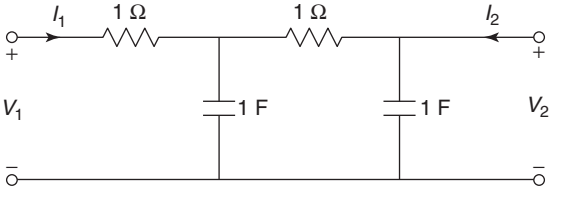


Fig. 10.23

Solution The transformed network is shown in Fig. 10.24. From Fig. 10.24,

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \ldots(i)$$

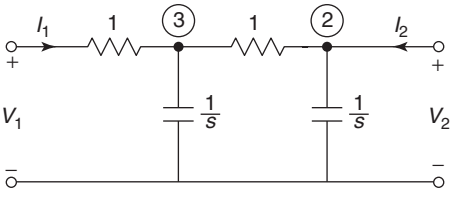


Fig. 10.24

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{\frac{1}{s}} + \frac{(V_3 - V_2)}{1}$$

$$= (s+1)V_3 - V_2 \quad \ldots(ii)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + \frac{(V_2 - V_3)}{1}$$

$$= (s+1)V_2 - V_3 \quad \ldots(iii)$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = (s+1)V_3 - V_2$$

$$(s+2)V_3 = V_1 + V_2$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \quad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = (s+1)\left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right) - V_2$$

$$= \frac{s+1}{s+2}V_1 - \frac{1}{s+2}V_2 \quad \ldots(v)$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = (s+1)V_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\frac{1}{s+2}V_1 + \frac{s^2+3s+1}{s+2}V_2 \quad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{1}{s+2} \\ -\frac{1}{s+2} & \frac{s^2+3s+1}{s+2} \end{bmatrix}$$

Example 10.12 *Determine Y-parameters for the network shown in Fig. 10.25.*

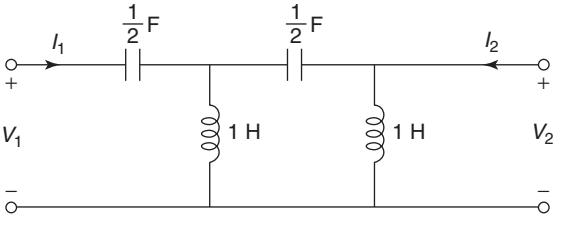


Fig. 10.25

Solution The transformed network is shown in Fig. 10.26.
From Fig. 10.26,

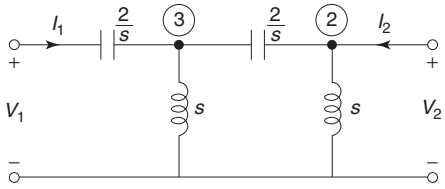


Fig. 10.26

$$I_1 = \frac{V_1 - V_3}{\frac{2}{s}}$$

$$= \frac{s}{2}V_1 - \frac{s}{2}V_3 \qquad \ldots(i)$$

Applying KCL at Node 3,

$$\frac{s}{2}(V_1 - V_3) = \frac{V_3}{s} + \frac{s}{2}(V_3 - V_2)$$

$$\frac{s}{2}V_3 + \frac{1}{s}V_3 + \frac{s}{2}V_3 = \frac{s}{2}V_1 + \frac{s}{2}V_2$$

$$V_3 = \frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2 \qquad \ldots(ii)$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s}{2}V_1 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= \left[\frac{s}{2} - \frac{s^3}{4(s^2+1)}\right]V_1 - \frac{s^3}{4(s^2+1)}V_2$$

$$= \frac{s^3+2s}{4(s^2+1)}V_1 - \frac{s^3}{4(s^2+1)}V_2 \qquad \ldots(iii)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{s} + \frac{s}{2}(V_2 - V_3)$$

$$= \frac{s^2+2}{2s}V_2 - \frac{s}{2}V_3 \qquad \ldots(iv)$$

Substituting Eq. (ii) in Eq. (iv),

$$I_2 = \frac{s^2+2}{2s}V_2 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \left[\frac{s^2+2}{2s} - \frac{s^3}{4(s^2+1)}\right]V_2$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2 \qquad \ldots(v)$$

Comparing Eqs (iii) and (v) with *Y*-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{s^3+2s}{4(s^2+1)} & -\frac{s^3}{4(s^2+1)} \\ -\frac{s^3}{4(s^2+1)} & \frac{s^4+6s^2+4}{4s(s^2+1)} \end{bmatrix}$$

Example 10.13 *Obtain Y-parameters of the network shown in Fig. 10.27.*

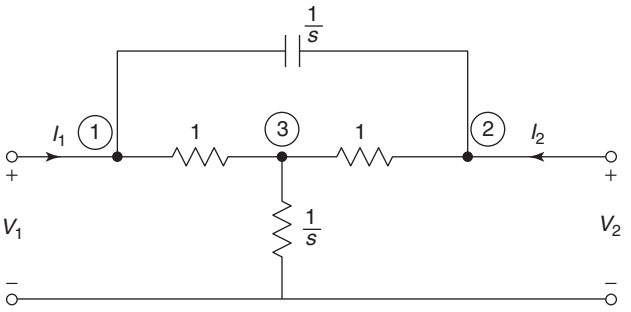


Fig. 10.27

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1 - V_3}{1} + \frac{V_1 - V_2}{\frac{1}{s}}$$

$$= (s+1)V_1 - sV_2 - V_3 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2 - V_3}{1} + \frac{V_2 - V_1}{\frac{1}{s}}$$

$$= (s+1)V_2 - sV_1 - V_3 \qquad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3}{\frac{1}{s}} + \frac{V_3 - V_1}{1} + \frac{V_3 - V_2}{1} = 0$$

$$(s+2)V_3 - V_1 - V_2 = 0$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \qquad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$I_1 = (s+1)V_1 - sV_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_1 - \left[\frac{s(s+2)+1}{(s+2)}\right]V_2$$

$$= \left(\frac{s^2+3s+1}{s+2}\right)V_1 - \left(\frac{s^2+2s+1}{s+2}\right)V_2 \qquad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$I_2 = (s+1)V_2 - sV_1 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\left[\frac{s(s+2)+1}{(s+2)}\right]V_1 + \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_2$$

$$= -\left(\frac{s^2+2s+1}{s+2}\right)V_1 + \left(\frac{s^2+3s+1}{s+2}\right)V_2 \quad \ldots(v)$$

Comparing Eqs (iv) and (v) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{s+2} & -\dfrac{(s^2+2s+1)}{s+2} \\ -\dfrac{(s^2+2s+1)}{s+2} & \dfrac{s^2+3s+1}{s+2} \end{bmatrix}$$

10.4 TRANSMISSION PARAMETERS (ABCD PARAMETERS)

The transmission parameters or chain parameters or *ABCD* parameters serve to relate the voltage and current at the input port to voltage and current at the output port. In equation form,

$$(V_1, I_1) = f(V_2, -I_2)$$

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Here, the negative sign is used with I_2 and not for parameters B and D. The reason the current I_2 carries a negative sign is that in transmission field, the output current is assumed to be coming out of the output port instead of going into the port.

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called transmission matrix.

For a given network, these parameters are determined as follows:

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

where A is the reverse voltage gain with the output port open-circuited.

Similarly,
$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

where C is the transfer admittance with the output port open-circuited.

Case 2 When output port is short-circuited, i.e., $V_2 = 0$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

where B is the transfer impedance with the output port short-circuited.

Similarly,
$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

where D is the reverse current gain with the output port short-circuited.

10.4.1 Condition for Reciprocity

(a) As shown in Fig. 10.28, voltage V_s is applied at the input port with the output port short-circuited.

i.e.,
$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

From the transmission parameter equations,

$$V_s = B\, I_2'$$
$$\frac{V_s}{I_2'} = B$$

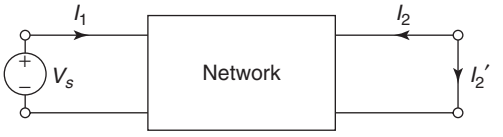


Fig. 10.28 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 10.29, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,
$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1' = -I_1$$

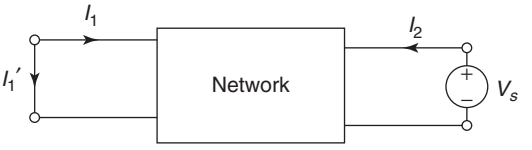


Fig. 10.29 *Network for deriving condition for reciprocity*

From the transmission parameter equations,

$$0 = AV_s - BI_2$$
$$-I_1' = CV_s - DI_2$$
$$I_2 = \frac{A}{B} V_s$$
$$-I_1' = CV_s - \frac{AD}{B} V_s$$
$$\frac{V_s}{I_1'} = \frac{B}{AD - BC}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e.,
$$B = \frac{B}{AD - BC}$$

i.e.,
$$AD - BC = 1$$

10.4.2 Condition for Symmetry

(a) When the output port is open-circuited, i.e., $I_2 = 0$.
From the transmission-parameter equations,

$$V_s = AV_2$$
$$I_1 = CV_2$$
$$\frac{V_s}{I_1} = \frac{A}{C}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$.
From the transmission parameter equation,

$$CV_s = DI_2$$
$$\frac{V_s}{I_2} = \frac{D}{C}$$

Hence, for network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $$A = D$$

Example 10.14 *Find the transmission parameters for the network shown in Fig. 10.30.*

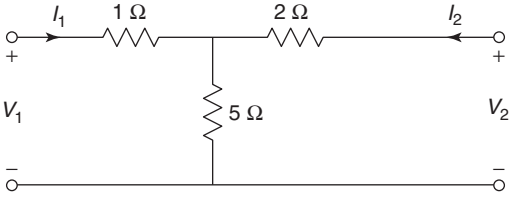


Fig. 10.30

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.

$$V_1 = 6I_1$$

and $$V_2 = 5I_1$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{6I_1}{5I_1} = \frac{6}{5}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{5}\,\text{℧}$$

Case 2 When the output port is short-circuited, i.e., $V_2 = 0$, as shown in Fig. 10.31,

$$R_{eq} = 1 + \frac{5\times 2}{5+2} = 1 + \frac{10}{7} = \frac{17}{7}\,\Omega$$

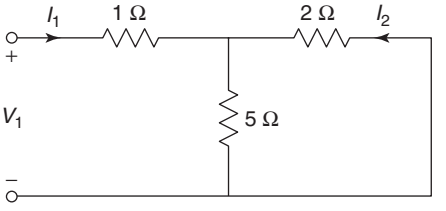


Fig. 10.31

Now $$V_1 = \frac{17}{7} I_1$$

and $$I_2 = \frac{5}{7}(-I_1) = -\frac{5}{7} I_1$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{\frac{17}{7} I_1}{-\frac{5}{7} I_1} = \frac{17}{5}\,\Omega$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{7}{5}$$

Hence, the transmission parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Second Method (Refer Fig. 10.37)
Applying KVL to Mesh 1,

$$V_1 = 6I_1 + 5I_2 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 5I_1 + 7I_2 \quad \ldots(ii)$$

Hence,

$$5I_1 = V_2 - 7I_2$$

$$I_1 = \frac{1}{5}V_2 - \frac{7}{5}I_2 \quad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 6\left(\frac{1}{5}V_2 - \frac{7}{5}I_2\right) + 5I_2$$

$$= \frac{6}{5}V_2 - \frac{17}{5}I_2 \quad \ldots(iv)$$

Comparing Eqs (iii) and (iv) with transmission parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Example 10.15

Obtain ABCD parameters for the network shown in Fig.10.32.

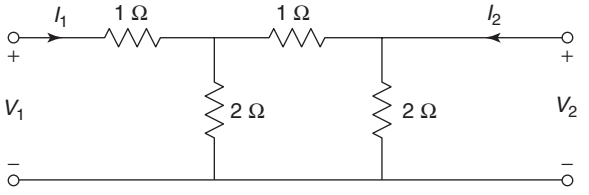


Fig. 10.32

Solution The network is redrawn as shown in Fig.10.33.
Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - I_3 - 2(I_3 + I_2) = 0$$

$$5I_3 = 2I_1 - 2I_2$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \quad \ldots(iii)$$

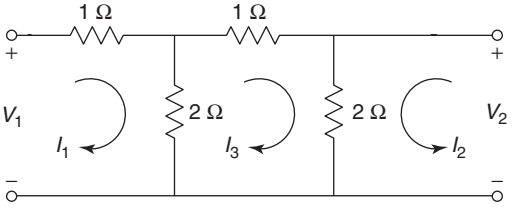


Fig. 10.33

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2$$

$$\frac{4}{5}I_1 = V_2 - \frac{6}{5}I_2$$

$$I_1 = \frac{5}{4}V_2 - \frac{3}{2}I_2 \qquad \ldots(v)$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{11}{5}\left(\frac{5}{4}V_2 - \frac{3}{2}I_2\right) + \frac{4}{5}I_2$$

$$= \frac{11}{4}V_2 - \frac{5}{2}I_2 \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{11}{4} & \frac{5}{2} \\ \frac{5}{4} & \frac{3}{2} \end{bmatrix}$$

Example 10.16 *Determine the transmission parameters for the network shown in Fig. 10.34.*

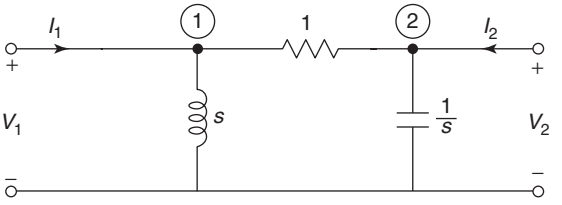


Fig. 10.34

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1}{s} + (V_1 - V_2)$$

$$= \frac{s+1}{s}V_1 - V_2 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + (V_2 - V_1)$$

$$= (s+1)V_2 - V_1$$

$$V_1 = (s+1)V_2 - I_2 \qquad \ldots(ii)$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s+1}{s}[(s+1)V_2 - I_2] - V_2$$
$$= \left[\frac{(s+1)^2}{s} - 1\right]V_2 - \frac{s+1}{s}I_2$$
$$= \left(\frac{s^2+s+1}{s}\right)V_2 - \left(\frac{s+1}{s}\right)I_2 \quad \ldots(iii)$$

Comparing Eqs (ii) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ \frac{s^2+s+1}{s} & \frac{s+1}{s} \end{bmatrix}$$

Example 10.17 *Find transmission parameters for the two-port network shown in Fig. 10.35.*

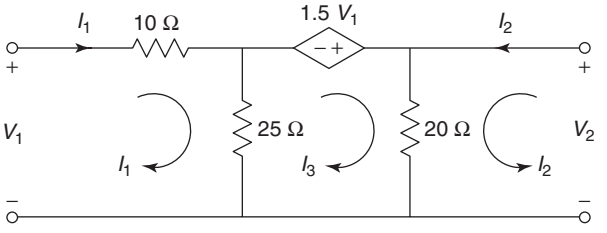


Fig. 10.35

Solution Applying KVL to Mesh 1,

$$V_1 = 10I_1 + 25(I_1 - I_3)$$
$$= 35I_1 - 25I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 20(I_2 + I_3)$$
$$= 20I_2 + 20I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-25(I_3 - I_1) + 1.5V_1 - 20(I_2 + I_3) = 0$$
$$-25I_3 + 25I_1 + 1.5(35I_1 - 25I_3) - 20I_2 - 20I_3 = 0$$
$$-25I_3 + 25I_1 + 52.5I_1 - 37.5I_3 - 20I_2 - 20I_3 = 0$$
$$82.5I_3 = 77.5I_1 - 20I_2$$
$$I_3 = 0.94I_1 - 0.24I_2 \quad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 35I_1 - 25(0.94I_1 - 0.24I_2)$$
$$= 11.5I_1 + 6I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 20I_2 + 20(0.94I_1 - 0.24I_2)$$
$$= 18.8I_1 + 15.2I_2 \quad \ldots(v)$$

From Eq. (v),

$$I_1 = 0.053V_2 - 0.81I_2 \quad \ldots(vi)$$

Substituting Eq. (vi) in Eq. (iv),

$$V_1 = 11.5(0.053\,V_2 - 0.81I_2) + 6I_2$$
$$= 0.61\,V_2 - 3.32I_2 \quad \ldots(\text{vii})$$

Comparing Eqs (vi) and (vii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.61 & -3.32 \\ 0.053 & -0.81 \end{bmatrix}$$

10.5 INVERSE TRANSMISSION PARAMETERS (A′B′C′D′ PARAMETERS)

The inverse transmission parameters serve to relate the voltage and current at the outport port to the voltage and current at the input port. In equation form,

$$(V_2, I_2) = f(V_1, -I_1)$$
$$V_2 = A'\,V_1 - B'I_1$$
$$I_2 = C'\,V_1 - D'I_1$$

In matrix form, we can write

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

where matrix $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$ is called the *inverse transmission matrix*.

For a given network, these parameters are determined as follows:

Case 1 When the input port is open-circuited, i.e., $I_1 = 0$

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0}$$

where A' is the forward voltage gain with the input port open-circuited.

Similarly,
$$C' = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

where C' is the transfer admittance with the input port open-circuited.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$B' = -\left.\frac{V_2}{I_1}\right|_{V_1=0}$$

where B' is the transfer impedance with the input port short-circuited.

Similarly,

$$D' = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

where D' is the forward current gain with the input port short-circuited.

10.5.1 Condition for Reciprocity

(a) As shown in Fig. 10.36, voltage V_s is applied at input port with the output port short-circuited.

i.e., $V_1 = V_s$

$V_2 = 0$

$I_2 = -I_2'$

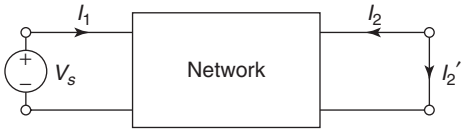


Fig. 10.36 *Network for deriving condition for reciprocity*

From the inverse transmission parameter equations,

$$0 = A' V_s - B' I_1$$
$$-I_2' = C' V_s - D' I_1$$
$$\frac{I_2'}{V_s} = \frac{A'D' - B'C'}{B'}$$

(b) As shown in Fig. 10.37, voltage V_s is applied at the output port with the input port short-circuited.

i.e., $V_2 = V_s$

$V_1 = 0$

$I_1 = -I_1'$

From the inverse transmission parameter equations,

$$V_s = B' I_1'$$
$$I_2 = \frac{A'}{B'} V_s$$
$$\frac{I_1'}{V_s} = \frac{1}{B'}$$

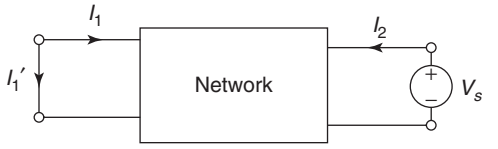


Fig. 10.37 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,

$$\frac{I_2'}{V_s} = \frac{I_1'}{V_s}$$

i.e., $A'D' - B'C' = 1$

10.5.2 Condition for Symmetry

The condition for symmetry is obtained from the *Z*-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 0 = \frac{D'}{C'}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 0 = \frac{A'}{C'}$$

For symmetrical network

$$Z_{11} = Z_{22}$$
$$A' = D'$$

Example 10.18 *Find the inverse transmission parameters for the network shown in Fig. 10.38.*

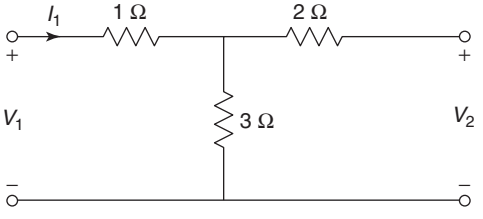


Fig. 10.38

Solution

First method

Case 1 When the input port is open-circuited, i.e., $I_1 = 0$

$$V_1 = 3I_2$$
$$V_2 = 5I_2$$

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0} = \frac{5}{3}$$

$$C' = \left.\frac{I_2}{V_1}\right|_{I_1=0} = \frac{1}{3}\,\mho$$

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 10.39.
By current division rule,

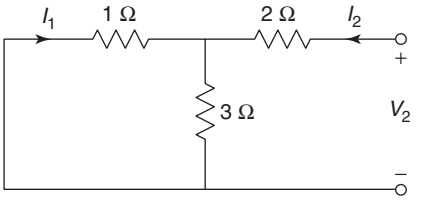


Fig. 10.39

$$I_1 = -\frac{3}{4} I_2$$

$$D' = -\left.\frac{I_2}{I_1}\right|_{V_1=0} = \frac{4}{3}$$

$$R_{eq} = 2 + \frac{3\times 1}{3+1} = 2 + \frac{3}{4} = \frac{11}{4}\,\Omega$$

$$V_2 = \frac{11}{4} I_2 = \frac{11}{4}\left(-\frac{4}{3}\right) I_1 = -\frac{11}{3} I_1$$

$$B' = -\left.\frac{V_2}{I_1}\right|_{V_1=0} = \frac{11}{3}\Omega$$

Second Method Refer Fig. 10.38.
Applying KVL to Mesh 1,

$$V_1 = 4I_1 + 3I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 3I_1 + 5I_2 \qquad \text{...(ii)}$$

Hence, $3I_2 = V_1 - 4I_1$

$$I_2 = \frac{1}{3} V_1 - \frac{4}{3} I_1 \qquad \text{…(iii)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 3I_1 + 5\left(\frac{1}{3} V_1 - \frac{4}{3} I_1\right)$$

$$= \frac{5}{3} V_1 - \frac{11}{3} I_1 \qquad \text{…(iv)}$$

Comparing Eqs (iii) and (iv) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{11}{3} \\ \frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

Example 10.19

Find the inverse transmission parameters for the network shown in Fig. 10.40

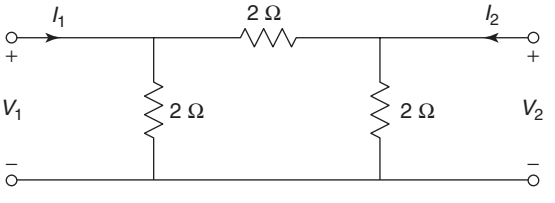


Fig. 10.40

Solution The network is redrawn as shown in Fig. 10.41.

Applying KVL to Mesh 1,

$$V_1 = 2I_1 - 2I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2I_1 + 2I_2 + 6I_3 = 0$$

$$I_3 = \frac{1}{3}I_1 - \frac{1}{3}I_2 \quad \ldots(iii)$$

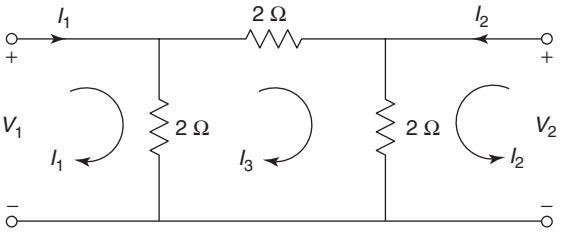


Fig. 10.41

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 2I_1 - 2\left(\frac{1}{3}I_1 - \frac{1}{3}I_2\right)$$

$$= \frac{4}{3}I_1 + \frac{2}{3}I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{1}{3}I_1 - \frac{1}{3}I_2\right)$$

$$= \frac{2}{3}I_1 + \frac{4}{3}I_2 \quad \ldots(v)$$

Rewriting Eq. (iv),

$$\frac{2}{3}I_2 = V_1 - \frac{4}{3}I_1$$

$$I_2 = \frac{3}{2}V_1 - 2I_1 \quad \ldots(vi)$$

Substituting Eq. (vi) in Eq. (v),

$$V_2 = \frac{2}{3}I_1 + \frac{4}{3}\left(\frac{3}{2}V_1 - 2I_1\right)$$

$$= 2V_1 - 2I_1 \quad \ldots(vii)$$

Comparing Eq. (vi) and (vii) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} & 2 \end{bmatrix}$$

Example 10.20 *Find the inverse transmission parameters for the network shown in Fig. 10.42.*

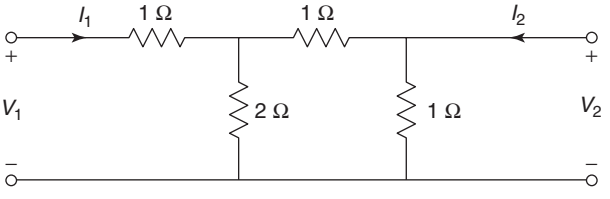


Fig. 10.42

Solution The network is redrawn as shown in Fig. 10.43.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = I_2 + I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2I_1 + I_2 + 4I_3 = 0$$

$$I_3 = \frac{1}{2}I_1 - \frac{1}{4}I_2 \quad \ldots(iii)$$

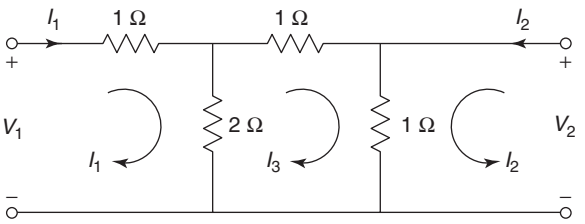


Fig. 10.43

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{1}{2}I_1 - \frac{1}{4}I_2\right)$$

$$= 2I_1 + \frac{1}{2}I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = I_2 + \frac{1}{2}I_1 - \frac{1}{4}I_2$$

$$= \frac{1}{2}I_1 + \frac{3}{4}I_2 \quad \ldots(v)$$

Rewriting Eq. (iv),

$$\frac{1}{2}I_2 = V_1 - 2I_1$$

$$I_2 = 2V_1 - 4I_1 \quad \ldots(vi)$$

Substituting the Eq. (vi) in Eq. (v),

$$V_2 = \frac{1}{2}I_1 + \frac{3}{4}(2V_1 - 4I_1)$$

$$= \frac{3}{2}V_1 - \frac{5}{2}I_1 \quad \ldots(vii)$$

Comparing Eqs (vi) and (vii) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ 2 & 4 \end{bmatrix}$$

10.6 HYBRID PARAMETERS (*h* PARAMETERS)

The hybrid parameters of a two-port network may be defined by expressing the voltage of input port V_1 and current of output port I_2 in terms of current of input port I_1 and voltage of output port V_2.

$$(V_1, I_2) = f(I_1, V_2)$$

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$

$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The individual *h* parameters can be defined by setting $I_1 = 0$ and $V_2 = 0$.

Case 1 When the output port is short-circuited i.e., $V_2 = 0$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

where h_{11} is the short-circuit input impedance.

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

where h_{21} is the short-circuit forward current gain.

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

where h_{12} is the open-circuit reverse voltage gain.

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

where h_{22} is the open-circuit output admittance.

Since h parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called *hybrid parameters*.

The equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 10.44.

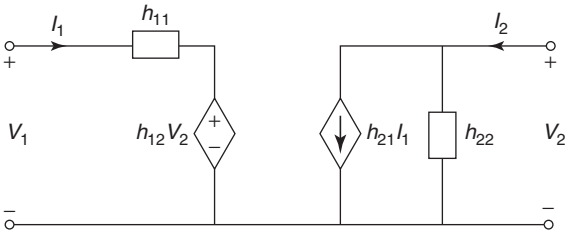


Fig. 10.44 *Equivalent circuit of the two-port network in terms of h-parameters*

10.6.1 Condition for Reciprocity

(a) As shown in Fig. 10.45, voltage V_s is applied at the input port and the output port is short-circuited.

i.e., $V_1 = V_s$

$V_2 = 0$

$I_2' = -I_2$

From the h-parameter equations,

$V_s = h_{11}\, I_1$

$-I_2' = h_{21}\, I_1$

$$\frac{V_s}{I_2'} = -\frac{h_{11}}{h_{21}}$$

Fig. 10.45 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 10.46, voltage V_s is applied at the output port with the input port short-circuited.

i.e., $V_1 = 0$

$V_2 = V_s$

$I_1 = -I_1'$

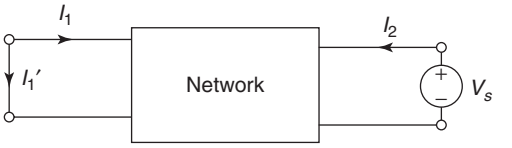


Fig. 10.46 *Network for deriving condition for reciprocity*

From the *h*-parameter equations,

$$0 = h_{11} I_1 + h_{12} V_s$$
$$h_{12} V_s = -h_{11} I_1 = h_{11} I_1'$$
$$\frac{V_s}{I_1'} = \frac{h_{11}}{h_{12}}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e., $$h_{21} = -h_{12}$$

10.6.2 Condition for Symmetry

The condition for symmetry is obtained from the *Z*-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = \left.\frac{h_{11} I_1 + h_{12} V_2}{I_1}\right|_{I_2 = 0} = h_{11} + h_{12} \frac{V_2}{I_1}$$

But with $I_2 = 0$,

$$0 = h_{21} I_1 + h_{22} V_2$$
$$\frac{V_2}{I_1} = -\frac{h_{21}}{h_{22}}$$
$$Z_{11} = h_{11} - \frac{h_{12} h_{21}}{h_{22}} = \frac{h_{11} h_{22} - h_{12} h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

where $$\Delta h = h_{11} h_{22} - h_{12} h_{21}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

With $I_1 = 0$,

$$I_2 = h_{22} V_2$$
$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = \frac{1}{h_{22}}$$

For a symmetrical network, $$Z_{11} = Z_{22}$$

i.e., $$\frac{\Delta h}{h_{22}} = \frac{1}{h_{22}}$$

i.e., $$\Delta h = 1$$

i.e., $$h_{11} h_{22} - h_{12} h_{21} = 1$$

Example 10.21 *In the two-port network shown in Fig. 10.47, compute h-parameters from the following data:*

(a) With the output port short-circuited: $V_1 = 25\ V$, $I_1 = 1\ A$, $I_2 = 2\ A$

(b) With the input port open-circuited: $V_1 = 10\ V$, $V_2 = 50\ V$, $I_2 = 2\ A$

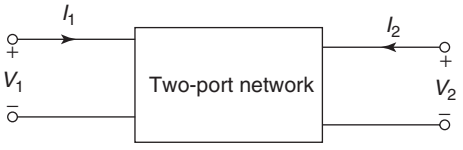


Fig. 10.47

Solution

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{25}{1} = 25\ \Omega, \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{10}{50} = 0.2$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{2}{1} = 2, \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2}{50} = 0.04\ \mho$$

Hence, the *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 25 & 0.2 \\ 2 & 0.04 \end{bmatrix}$$

Example 10.22

Determine hybrid parameters for the network of Fig. 10.48. Determine whether the network is reciprocal.

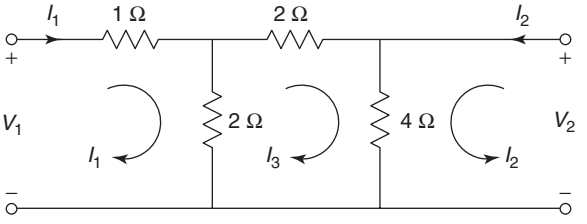


Fig. 10.48

Solution

First Method

Case 1 When Port 2 is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 10.49,

$$R_{eq} = 1 + \frac{2\times 2}{2+2} = 2\ \Omega$$

Now, $V_1 = 2I_1$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 2\ \Omega$$

Also, $I_2 = -I_1 \times \frac{2}{2+2} = -\frac{I_1}{2}$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\frac{1}{2}$$

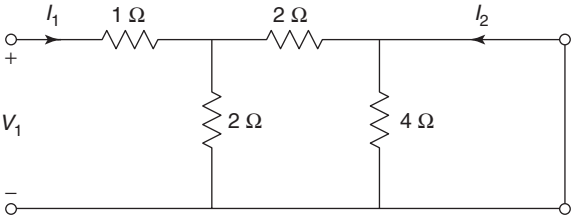


Fig. 10.49

Case 2 When Port 1 is open-circuited, i.e., $I_1 = 0$ as shown in Fig. 10.50,

$$R_{eq} = \frac{(2+2)\times 4}{2+2+4} = 2\ \Omega$$

$$V_1 = 2I_y$$

$$I_y = \frac{I_2}{2}$$

$$V_2 = 4I_x$$

$$I_x = \frac{I_2}{2}$$

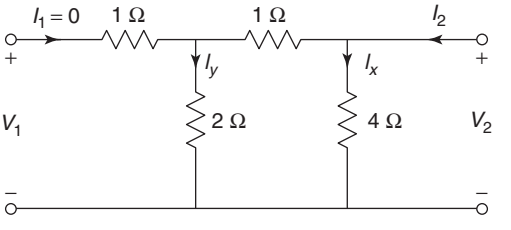


Fig. 10.50

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{2I_y}{4I_x} = \frac{2\times\frac{I_2}{2}}{4\times\frac{I_2}{2}} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2I_x}{4I_x} = \frac{1}{2}\,\mho$$

Hence, the *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Second Method (Refer Fig. 10.48)
Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 4I_2 + 4I_3 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 2I_3 - 4(I_3 + I_2) = 0$$

$$8I_3 = 2I_1 - 4I_2$$

$$I_3 = \frac{I_1}{4} - \frac{I_2}{2} \qquad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= \frac{5}{2}I_1 + I_2 \qquad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 4I_2 + 4\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= 4I_2 + I_1 - 2I_2$$

$$= I_1 + 2I_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{2}V_2 \qquad \ldots(v)$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{5}{2}I_1 - \frac{1}{2}I_1 + \frac{1}{2}V_2$$

$$= 2I_1 + \frac{1}{2}V_2 \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *h*-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since $h_{12} = -h_{21}$, the network is reciprocal.

Example 10.23 *Find h-parameters for the network shown in Fig. 10.51.*

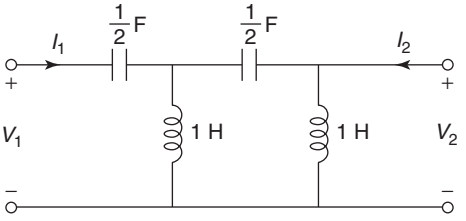


Fig. 10.51

Solution As solved in Example 10.12, derive the equations for I_1 and I_2 in terms of V_1 and V_2.

$$I_1 = \frac{s^3 + 2s}{4(s^2+1)} V_1 - \frac{s^3}{4(s^2+1)} V_2 \quad \ldots(i)$$

$$I_2 = -\frac{s^3}{4(s^2+1)} V_1 + \frac{s^4 + 6s^2 + 4}{4s(s^2+1)} V_2 \quad \ldots(ii)$$

From Eq. (i),

$$V_1 = \frac{4(s^2+1)}{s(s^2+2)} I_1 + \frac{s^2}{s^2+2} V_2 \quad \ldots(iii)$$

Substituting Eq. (iii) in Eq (ii),

$$I_2 = -\frac{s^3}{4(s^2+1)}\left[\frac{4(s^2+1)}{s(s^2+2)} I_1 + \frac{s^2}{s^2+2} V_2\right] + \frac{s^4 + 6s^2 + 4}{4s(s^2+1)} V_2$$

$$= -\frac{s^2}{s^2+2} I_1 + \frac{2(s^2+1)}{s(s^2+2)} V_2 \quad \ldots(iv)$$

Comparing Eqs (iii) and (iv) with *h*-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{4(s^2+1)}{s(s^2+2)} & \dfrac{s^2}{s^2+2} \\ -\dfrac{s^2}{s^2+2} & \dfrac{2(s^2+1)}{s(s^2+2)} \end{bmatrix}$$

10.7 INVERSE HYBRID PARAMETERS (*g* PARAMETERS)

The inverse hybrid parameters of a two-port network may be defined by expressing the current of the input port I_1 and voltage of the output port V_2 in terms of the voltage of the input port V_1 and the current of the output port I_2.

$$(I_1, V_2) = f(V_1, I_2)$$

$$I_1 = g_{11} V_1 + g_{12} I_2$$

$$V_2 = g_{21} V_1 + g_{22} I_2$$

In matrix form, we can write

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

The individual *g* parameters can be defined by setting $V_1 = 0$ and $I_2 = 0$.

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}$$

where g_{11} is the open-circuit input admittance.

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

where g_{21} is the open-circuit forward voltage gain.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

where g_{12} is the short-circuit reverse current gain.

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

where g_{22} is the short-circuit output impedance.

The equivalent circuit of a two-port network in terms of inverse hybrid parameters is shown in Fig. 10.52.

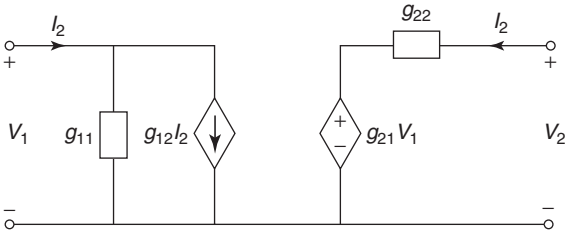


Fig. 10.52 *Equivalent circuit of two-port network in terms of g-parameters*

10.7.1 Condition for Reciprocity

(a) As shown in Fig. 10.53, voltage V_s is applied at the input port and the output port is short-circuited, i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

From the *g*-parameter equation,

$$g_{21} V_s = g_{22} I_2'$$
$$\frac{V_s}{I_2'} = \frac{g_{22}}{g_{21}}$$

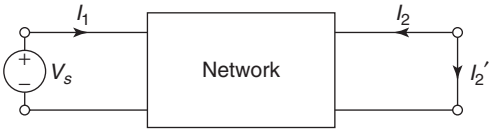


Fig. 10.53 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 10.54, voltage V_s is applied at the output port with the input port short-circuited, i.e.,

$$V_1 = 0$$
$$V_2 = V_s$$
$$I_1 = -I_1'$$

From the *g*-parameter equations,

$$-I_1' = g_{12} I_2$$
$$V_s = g_{22} I_2$$
$$\frac{V_s}{I_1'} = -\frac{g_{22}}{g_{12}}$$

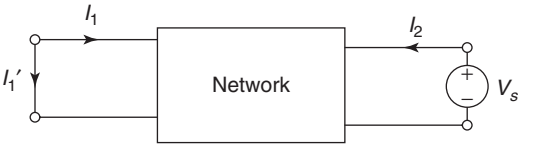


Fig. 10.54 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e., $$g_{21} = -g_{12}$$

10.7.2 Condition for Symmetry

The condition for symmetry is obtained from the *Z*-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{1}{g_{11}}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$0 = g_{11}\, V_1 + g_{12}\, I_2$$

$$V_1 = -\frac{g_{12}}{g_{11}} I_2$$

$$V_2 = g_{21}\left(-\frac{g_{12}}{g_{11}}\right) I_2 + g_{22}\, I_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}$$

For a symmetrical network, $Z_{11} = Z_{22}$.

i.e., $$\frac{1}{g_{11}} = \frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}$$

i.e., $$g_{11}g_{22} - g_{12}g_{21} = 1$$

Table 10.2 *Conditions for reciprocity and symmetry*

Parameter	Condition for Reciprocity	Condition for Symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{22}$
T	$AD - BC = 1$	$A = D$
T'	$A'D' - B'C' = 1$	$A' = D'$
h	$h_{12} = -h_{21}$	$h_{11}\, h_{22} - h_{12}\, h_{21} = 1$
g	$g_{12} = -g_{21}$	$g_{11}\, g_{22} - g_{12}\, g_{21} = 1$

Example 10.24 *Find g-parameters for the network shown in Fig. 10.55.*

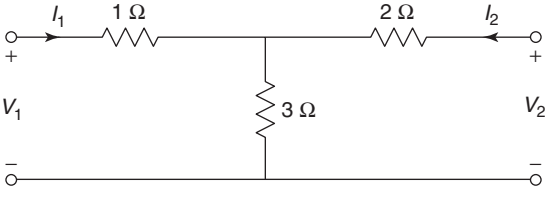


Fig. 10.55

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

Applying KVL to Mesh 1,

$$V_1 = 4\, I_1$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{1}{4}\ \text{℧}$$

Also $V_2 = 3\, I_1$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{3\, I_1}{4\, I_1} = \frac{3}{4}$$

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 10.56.

By current division rule,

$$I_1 = -\frac{3}{4}\, I_2$$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} = -\frac{3}{4}$$

$$R_{eq} = 2 + \frac{3\times 1}{3+1} = 2 + \frac{3}{4} = \frac{11}{4}\ \Omega$$

$$V_2 = \frac{11}{4}\, I_2$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \frac{11}{4}\ \Omega$$

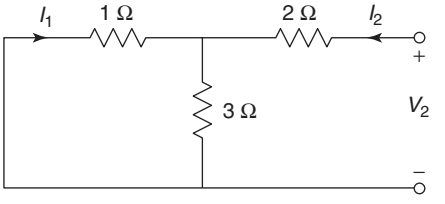


Fig. 10.56

Second Method (Refer Fig. 10.55).

Applying KVL to Mesh 1,

$$V_1 = 4\, I_1 + 3\, I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 3\, I_1 + 5\, I_2 \qquad \ldots(ii)$$

Hence, $4\, I_1 = V_1 - 3\, I_2$

$$I_1 = \frac{1}{4} V_1 - \frac{3}{4} I_2 \qquad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 3\left(\frac{1}{4} V_1 - \frac{3}{4} I_2\right) + 5\, I_2$$

$$= \frac{3}{4} V_1 + \frac{11}{4} I_2 \qquad \ldots(iv)$$

Comparing Eqs (iii) and (iv) with *g*-parameter equations,

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} \end{bmatrix}$$

Example 10.25 *Find g-parameters for the network shown in Fig. 10.57.*

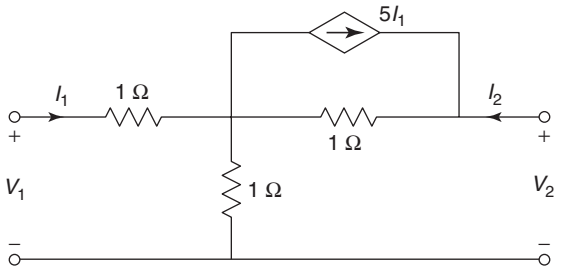


Fig. 10.57

Solution The network is redrawn using source transformation as shown in Fig. 10.58.

Applying KVL to Mesh 1,

$$V_1 - 1\,I_1 - 1\,(I_1 + I_2) = 0$$

$$V_1 = 2\,I_1 + I_2 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 - 5\,I_1 - 1\,I_2 - 1\,(I_2 + I_1) = 0$$

$$V_2 = 6\,I_1 + 2\,I_2 \quad \ldots(ii)$$

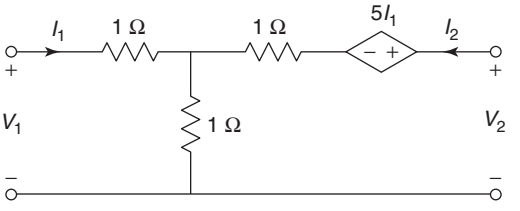


Fig. 10.58

Hence, $2\,I_1 = V_1 - I_2$

$$I_1 = \frac{1}{2}V_1 - \frac{1}{2}I_2 \quad \ldots(iii)$$

Substituting Eq (iii) in Eq. (ii),

$$V_2 = 6\left(\frac{1}{2}V_1 - \frac{1}{2}I_2\right) + 2\,I_2$$

$$= 3\,V_1 - I_2 \quad \ldots(iv)$$

Comparing Eqs (iii) and (iv) the *g*- parameter equations,

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

10.8 INTER-RELATIONSHIPS BETWEEN THE PARAMETERS

When it is required to find out two or more parameters of a particular network then finding each parameter will be tedious. But if we find a particular parameter then the other parameters can be found if the inter-relationship between them is known.

10.8.1 Z-parameters in Terms of Other Parameters

1. Z-parameters in Terms of Y-parameters We know that

$$I_1 = Y_{11}\,V_1 + Y_{12}\,V_2$$

$$I_2 = Y_{21}\,V_1 + Y_{22}\,V_2$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix}}{\begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}} = \frac{Y_{22}I_1 - Y_{12}I_2}{Y_{11}Y_{22} - Y_{12}Y_{21}} = \frac{Y_{22}}{\Delta Y}I_1 - \frac{Y_{12}}{\Delta Y}I_2$$

where $\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21}$

Comparing with $V_1 = Z_{11}I_1 + Z_{12}, I_2,$

$$Z_{11} = \frac{Y_{22}}{\Delta Y}$$

$$Z_{12} = -\frac{Y_{12}}{\Delta Y}$$

Also, $$V_2 = \frac{\begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix}}{\Delta Y} = \frac{Y_{11}}{\Delta Y} I_2 - \frac{Y_{21}}{\Delta Y} I_1$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{22} = \frac{Y_{11}}{\Delta Y}$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y}$$

2. *Z-parameter in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the second equation,

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{21} = \frac{1}{C}$$

$$Z_{22} = \frac{D}{C}$$

Also, $$V_1 = A\left[\frac{1}{C} I_1 + \frac{D}{C} I_2\right] - BI_2 = \frac{A}{C} I_1 + \left[\frac{AD}{C} - B\right] I_2 = \frac{A}{C} I_1 + \left[\frac{AD - BC}{C}\right] I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2,$

$$Z_{11} = \frac{A}{C}$$

$$Z_{12} = \frac{AD - BC}{C}$$

3. *Z-parameters in Terms of A'B'C'D' Parameters* We know that

$$V_2 = A'V_1 - B'\, I_1$$

$$I_2 = C'V_1 - D'\, I_1$$

Rewriting the second equation,

$$V_1 = \frac{D'}{C'} I_1 + \frac{1}{C'} I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2,$

$$Z_{11} = \frac{D'}{C'}$$

$$Z_{12} = \frac{1}{C'}$$

Also, $$V_2 = A'\left[\frac{D'}{C'} I_1 + \frac{1}{C'} I_2\right] - B' I_1 = \left[\frac{A'D' - B'C'}{C'}\right] I_1 + \frac{A'}{C'} I_2$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$,

$$Z_{21} = \frac{A'D' - B'C'}{C'} = \frac{\Delta T'}{C'}$$

$$Z_{22} = \frac{A'}{C'}$$

4. Z-parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

Rewriting the second equation,

$$V_2 = -\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2$$

Comparing with

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$$

$$Z_{21} = -\frac{h_{21}}{h_{22}}$$

$$Z_{22} = \frac{1}{h_{22}}$$

Also, $$V_1 = h_{11} I_1 + h_{12}\left[-\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2\right] = h_{11} I_1 + \frac{h_{12}}{h_{22}} I_2 - \frac{h_{12}h_{21}}{h_{22}} I_1 = \left[\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right] I_1 + \frac{h_{12}}{h_{22}} I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$,

$$Z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

$$Z_{12} = \frac{h_{12}}{h_{22}}$$

5. Z-parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11} V_1 + g_{12}\, I_2$$

$$V_2 = g_{21} V_1 + g_{22}\, I_2$$

Rewriting the first equation,

$$V_1 = \frac{1}{g_{11}} I_1 - \frac{g_{12}}{g_{11}} I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$,

$$Z_{11} = \frac{1}{g_{11}}$$

$$Z_{12} = -\frac{g_{12}}{g_{11}}$$

Also $$V_2 = g_{21}\left[\frac{1}{g_{11}} I_1 - \frac{g_{12}}{g_{11}} I_2\right] + g_{22}\, I_2 = \frac{g_{21}}{g_{11}} I_1 + \left[\frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}\right] I_2$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$,

$$Z_{21} = \frac{g_{21}}{g_{11}}$$

$$Z_{22} = \frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{11}} = \frac{\Delta g}{g_{11}}$$

10.8.2 *Y*-parameters in Terms of Other Parameters

1. Y-parameters in terms of Z-parameters We know that

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

By Cramer's rule,

$$I_1 = \frac{\begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}} = \frac{Z_{22}\, V_1 - Z_{12}\, V_2}{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}} = \frac{Z_{22}}{\Delta Z} V_1 - \frac{Z_{12}}{\Delta Z} V_2$$

where $\Delta Z = Z_{11}\, Z_{22} - Z_{12}\, Z_{21}$

Comparing with $I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$,

$$Y_{11} = \frac{Z_{22}}{\Delta Z}$$

$$Y_{12} = -\frac{Z_{12}}{\Delta Z}$$

Also, $$I_2 = \frac{\begin{vmatrix} Z_{11} & V_1 \\ Z_{21} & V_2 \end{vmatrix}}{\Delta Z} = \frac{Z_{11} V_2 - Z_{12} V_1}{\Delta Z} = -\frac{Z_{21}}{\Delta Z} V_1 + \frac{Z_{11}}{\Delta Z} V_2$$

Comparing with $I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$,

$$Y_{21} = -\frac{Z_{21}}{\Delta Z}$$

$$Y_{22} = \frac{Z_{11}}{\Delta Z}$$

2. Y-parameters in Terms of ABCD Parameters We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the first equation,

$$I_2 = -\frac{1}{B} V_1 + \frac{A}{B} V_2$$

Comparing with $I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$,

$$Y_{21} = -\frac{1}{B}$$

$$Y_{22} = \frac{A}{B}$$

Also, $$I_1 = CV_2 - D\left[-\frac{1}{B} V_1 + \frac{A}{B} V_2\right] = \frac{D}{B} V_1 + \left[\frac{BC - AD}{B}\right] V_2$$

Comparing with $I_1 = Y_{11}V_1 + Y_{12}V_2,$

$$Y_{11} = \frac{D}{B}$$

$$Y_{12} = \frac{BC - AD}{B} = -\frac{AD - BC}{B} = -\frac{\Delta T}{B}$$

3. Y-parameters in Terms of A'B'C'D' Parameters We know that

$$V_2 = A'V_1 - B'I_1$$

$$I_2 = C'V_1 - D'I_1$$

Rewriting the first equation,

$$I_1 = \frac{A'}{B'}V_1 - \frac{1}{B'}V_2$$

Comparing with $I_1 = Y_{11}V_1 + Y_{12}V_2,$

$$Y_{11} = \frac{A'}{B'}$$

$$Y_{12} = -\frac{1}{B'}$$

Also, $$I_2 = C'V_1 - D'\left[\frac{A'}{B'}V_1 - \frac{1}{B'}V_2\right] = -\left[\frac{A'D' - B'C'}{B'}\right]V_1 + \frac{D'}{B'}V_2$$

Comparing with $I_2 = Y_{21}V_1 + Y_{22}V_2,$

$$Y_{21} = -\left[\frac{A'D' - B'C'}{B'}\right] = -\frac{\Delta T'}{B'}$$

and $$Y_{22} = \frac{D'}{B'}$$

4. Y-parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

Rewriting the first equation,

$$I_1 = \frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2$$

Comparing with $I_1 = Y_{11}V_1 + Y_{12}V_2,$

$$Y_{11} = \frac{1}{h_{11}}$$

$$Y_{12} = -\frac{h_{12}}{h_{11}}$$

Also $$I_2 = h_{21}\left[\frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2\right] + h_{22}V_2 = \frac{h_{21}}{h_{11}}V_1 + \left[\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}}\right]V_2$$

Comparing with $I_2 = Y_{21}V_1 + Y_{22}V_2,$

$$Y_{21} = \frac{h_{21}}{h_{11}}$$

$$Y_{22} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}} = \frac{\Delta h}{h_{11}}$$

5. Y-parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11} V_1 + g_{12} I_2$$
$$V_2 = g_{21} V_1 + g_{22} I_2$$

Rewriting the second equation,

$$I_2 = -\frac{g_{21}}{g_{22}} V_1 + \frac{1}{g_{22}} V_2$$

Comparing with $I_2 = Y_{21} V_1 + Y_{22} V_2$,

$$Y_{21} = -\frac{g_{21}}{g_{22}}$$
$$Y_{22} = \frac{1}{g_{22}}$$

Also, $$I_1 = g_{11} V_1 + g_{12}\left[-\frac{g_{21}}{g_{22}} V_1 + \frac{1}{g_{22}} V_2\right] = \left[\frac{g_{11} g_{22} - g_{12} g_{21}}{g_{22}}\right] V_1 + \frac{g_{12}}{g_{22}} V_2$$

Comparing with $I_1 = Y_{11} V_1 + Y_{12} V_2$,

$$Y_{11} = \frac{g_{11} g_{22} - g_{12} g_{21}}{g_{22}} = \frac{\Delta g}{g_{22}}$$
$$Y_{12} = \frac{g_{12}}{g_{22}}$$

10.8.3 *ABCD* Parameters in Terms of Other Parameters

1. ABCD Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the second equation,

$$I_1 = \frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2$$

Comparing with $I_1 = C V_2 - D I_2$,

$$C = \frac{1}{Z_{21}}$$
$$D = \frac{Z_{22}}{Z_{21}}$$

Also, $$V_1 = Z_{11}\left[\frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2\right] + Z_{12} I_2 = \frac{Z_{11}}{Z_{21}} V_2 - \frac{Z_{22} Z_{11}}{Z_{21}} I_2 + Z_{12} I_2$$
$$= \frac{Z_{11}}{Z_{21}} V_2 - \left[\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}}\right] I_2$$

Comparing with $V_1 = A V_2 - B I_2$,

$$A = \frac{Z_{11}}{Z_{21}}$$
$$B = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}} = \frac{\Delta Z}{Z_{21}}$$

2. *ABCD Parameters in terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Rewriting the second equation,

$$V_1 = -\frac{Y_{22}}{Y_{21}}V_2 + \frac{1}{Y_{21}}I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = -\frac{Y_{22}}{Y_{21}}$$

$$B = -\frac{1}{Y_{21}}$$

Also,

$$I_1 = Y_{11}\left[-\frac{Y_{22}}{Y_{21}}V_2 + \frac{1}{Y_{21}}I_2\right] + Y_{12}V_2 = \left[\frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}}\right]V_2 + \frac{Y_{11}}{Y_{21}}I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{Y_{12}Y_{21} - Y_{11}Y_{22}}{Y_{21}} = -\frac{\Delta Y}{Y_{21}}$$

$$D = -\frac{Y_{11}}{Y_{21}}$$

3. *ABCD Parameters in terms of A'B'C'D' Parameters* We know that

$$V_2 = A'V_1 - B'I_1$$
$$I_2 = C'V_1 - D'I_1$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} V_2 & B' \\ I_2 & D' \end{vmatrix}}{\begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix}} = \frac{D'}{\Delta T'}V_2 - \frac{B'}{\Delta T'}I_2$$

where $\Delta T' = A'D' - B'C'$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{D'}{\Delta T'}$$

$$B = \frac{B'}{\Delta T'}$$

Also,

$$-I_1 = \frac{\begin{vmatrix} A' & V_2 \\ C' & I_2 \end{vmatrix}}{\Delta T'} = -\frac{C'}{\Delta T'}V_2 + \frac{A'}{\Delta T'}I_2$$
$$I_1 = \frac{C'}{\Delta T'}V_2 - \frac{A'}{\Delta T'}I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{C'}{\Delta T'}$$

$$D = \frac{A'}{\Delta T'}$$

4. ABCD Parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$
$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

Rewriting the second equation,

$$I_1 = -\frac{h_{22}}{h_{21}} V_2 + \frac{1}{h_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = -\frac{h_{22}}{h_{21}}$$

$$D = -\frac{1}{h_{21}}$$

Also, $$V_1 = h_{11}\left[\frac{1}{h_{21}} I_2 - \frac{h_{22}}{h_{21}} V_2\right] + h_{12}\, V_2 = \left[\frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}}\right] V_2 + \frac{h_{11}}{h_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}} = -\frac{\Delta h}{h_{21}}$$

$$B = -\frac{h_{11}}{h_{21}}$$

5. ABCD Parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11}\, V_1 + g_{12}\, I_2$$
$$V_2 = g_{21}\, V_1 + g_{22}\, I_2$$

Rewriting the second equation,

$$V_1 = \frac{1}{g_{21}} V_2 - \frac{g_{22}}{g_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{1}{g_{21}}$$

$$B = \frac{g_{22}}{g_{21}}$$

Also, $$I_1 = g_{11}\left[\frac{1}{g_{21}} V_2 - \frac{g_{22}}{g_{21}} I_2\right] + g_{12}\, I_2 = \frac{g_{11}}{g_{21}} V_2 - \left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{21}}\right] I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{g_{11}}{g_{21}}$$

$$D = \frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{21}} = \frac{\Delta g}{g_{21}}$$

10.8.4 *A'B'C'D'* Parameters in Terms of Other Parameters

1. A'B'C'D' Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Rewriting the first equation,

$$I_2 = \frac{1}{Z_{12}} V_1 - \frac{Z_{11}}{Z_{12}} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{1}{Z_{12}}$$

$$D' = \frac{Z_{11}}{Z_{12}}$$

Also,

$$V_2 = Z_{21} I_1 + Z_{22}\left[\frac{1}{Z_{12}} V_1 - \frac{Z_{11}}{Z_{12}} I_1\right] = Z_{12} I_1 + \frac{Z_{22}}{Z_{12}} V_1 - \frac{Z_{22} Z_{11}}{Z_{12}} I_1$$

$$= \frac{Z_{22}}{Z_{12}} V_1 - \left[\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{12}}\right] I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{Z_{22}}{Z_{12}}$$

$$B' = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{12}} = \frac{\Delta Z}{Z_{12}}$$

2. *A'B'C'D' Parameters in Terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the first equation,

$$V_2 = -\frac{Y_{11}}{Y_{12}} V_1 + \frac{1}{Y_{12}} I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = -\frac{Y_{11}}{Y_{12}}$$

$$B' = -\frac{1}{Y_{12}}$$

Also,

$$I_2 = Y_{21} V_1 + Y_{22}\left[-\frac{Y_{11}}{Y_{12}} V_1 + \frac{1}{Y_{12}} I_1\right] = \left[\frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{12}}\right] V_1 + \frac{Y_{22}}{Y_{12}} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{12}} = -\frac{\Delta Y}{Y_{12}}$$

$$D' = -\frac{Y_{22}}{Y_{12}}$$

3. *A' B' C' D' Parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

By Cramer's rule,

$$V_2 = \frac{\begin{vmatrix} V_1 & B \\ I_1 & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{D}{\Delta T} V_2 - \frac{B}{\Delta T} I_2$$

where $\Delta T = AD - BC$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{D}{\Delta T}$$

$$B' = \frac{B}{\Delta T}$$

Also,
$$-I_2 = \frac{\begin{vmatrix} A & V_1 \\ C & I_1 \end{vmatrix}}{\Delta T} = -\frac{C}{\Delta T} V_1 + \frac{A}{\Delta T} I_1$$

$$I_2 = \frac{C}{\Delta T} V_1 - \frac{A}{\Delta T} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{C}{\Delta T}$$

$$D' = \frac{A}{\Delta T}$$

4. *A′ B′ C′ D′ Parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

Rewriting the first equation,

$$V_2 = \frac{1}{h_{12}} V_1 - \frac{h_{11}}{h_{12}} I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{1}{h_{12}}$$

$$B' = \frac{h_{11}}{h_{12}}$$

Also,
$$I_2 = h_{21} I_1 + h_{22}\left[\frac{1}{h_{12}} V_1 - \frac{h_{11}}{h_{12}} I_1\right] = \frac{h_{22}}{h_{12}} V_1 + \left[\frac{h_{11} h_{22} - h_{12} h_{21}}{h_{12}}\right] I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{h_{22}}{h_{12}}$$

$$D' = \left[\frac{h_{11} h_{22} - h_{12} h_{21}}{h_{12}}\right] = \frac{\Delta h}{h_{12}}$$

5. *A′ B′ C′ D′ parameters in Terms of Inverse Hybrid Parameters* We know that

$$I_1 = g_{11} V_1 + g_{12} I_2$$
$$V_2 = g_{21} V_1 + g_{22} I_2$$

Rewriting the first equation,

$$I_2 = -\frac{g_{11}}{g_{12}} V_1 + \frac{1}{g_{12}} I_1$$

Comparing with $I_2 = C' V_1 - D' I_1$,

$$C = -\frac{g_{11}}{g_{12}}$$

$$D' = -\frac{1}{g_{12}}$$

Also, $$V_2 = g_{21} V_1 + g_{22}\left[-\frac{g_{11}}{g_{12}} V_1 + \frac{1}{g_{12}} I_1\right] = -\left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{12}}\right] V_2 + \frac{g_{22}}{g_{12}} I_1$$

Comparing with $V_2 = A' V_1 - B' I_1$,

$$A' = -\left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{12}}\right] = -\frac{\Delta g}{g_{12}}$$

$$B' = \frac{g_{22}}{g_{12}}$$

10.8.5 Hybrid Parameters in Terms of Other Parameters

1. ***Hybrid Parameters in terms of Z-parameters*** We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the second equation,

$$I_2 = -\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2$,

$$h_{21} = -\frac{Z_{21}}{Z_{22}}$$

$$h_{22} = \frac{1}{Z_{22}}$$

Also, $$V_1 = Z_{11} I_1 + Z_{12}\left[-\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2\right] = \left[\frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}}\right] I_1 + \frac{Z_{12}}{Z_{22}} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}} = \frac{\Delta Z}{Z_{22}}$$

$$h_{12} = \frac{Z_{12}}{Z_{22}}$$

2. ***Hybrid Parameters in terms of Y-parameters*** We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the first equation,

$$V_1 = \frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{1}{Y_{11}}$$

$$h_{12} = -\frac{Y_{12}}{Y_{11}}$$

Also, $$I_2 = Y_{21}\left[\frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2\right] + Y_{22} V_2 = \left[\frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}}\right] V_2 + \frac{Y_{21}}{Y_{11}} I_1$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2$,

$$h_{21} = \frac{Y_{22}}{Y_{11}}$$

$$h_{22} = \frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}} = \frac{\Delta Y}{Y_{11}}$$

3. Hybrid Parameters in Terms of ABCD Parameters We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the second equation,

$$I_2 = -\frac{1}{D} I_1 + \frac{C}{D} V_2$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2$,

$$h_{21} = -\frac{1}{D}$$

$$h_{22} = \frac{C}{D}$$

Also, $$V_1 = AV_2 - B\left[-\frac{1}{D} I_1 + \frac{C}{D} V_2\right] = \frac{B}{D} I_1 + \left[\frac{AD - BC}{D}\right] V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{B}{D}$$

$$h_{12} = \frac{AD - BC}{D} = \frac{\Delta T}{D}$$

4. Hybrid Parameters in Terms of A'B'C'D' Parameters We know that

$$V_2 = A' V_1 - B' I_1$$

$$I_2 = C' V_1 - D' I_1$$

Rewriting the first equation,

$$V_1 = \frac{B'}{A'} I_1 + \frac{1}{A'} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2$,

$$h_{11} = \frac{B'}{A'}$$

$$h_{12} = \frac{1}{A'}$$

Also, $$I_2 = C'\left[\frac{B'}{A'} I_1 + \frac{1}{A'} V_2\right] - D' I_1 = -\left[\frac{A'D' - B'C'}{A'}\right] I_1 + \frac{C'}{A'} V_2$$

Comparing with $$I_2 = h_{21} I_1 + h_{22} V_2,$$

$$h_{21} = -\left[\frac{A'D' - B'C'}{A'}\right] = -\frac{\Delta T'}{A'}$$

$$h_{22} = \frac{C'}{A'}$$

5. Hybrid Parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11} V_1 + g_{12} I_2$$
$$V_2 = g_{21} V_1 + g_{22} I_2$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} I_1 & g_{12} \\ V_2 & g_{22} \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}} = \frac{g_{22}}{\Delta g} I_1 - \frac{g_{12}}{\Delta g} V_2$$

where $$\Delta g = g_{11} g_{22} - g_{12} g_{21}$$

Comparing with $$V_1 = h_{11} I_1 + h_{12} V_2,$$

$$h_{11} = \frac{g_{22}}{\Delta g}$$

$$h_{12} = -\frac{g_{12}}{\Delta g}$$

Also, $$I_2 = \frac{\begin{vmatrix} g_{11} & I_1 \\ g_{21} & V_2 \end{vmatrix}}{\Delta g} = -\frac{g_{21}}{\Delta g} I_1 + \frac{g_{11}}{\Delta g} V_2$$

Comparing with $$I_2 = h_{21} I_1 + h_{22} V_2,$$

$$h_{21} = -\frac{g_{21}}{\Delta g}$$

$$h_{22} = \frac{g_{11}}{\Delta g}$$

10.8.6 Inverse Hybrid Parameters in Terms of Other Parameters

1. Inverse Hybrid Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the first equation,

$$I_1 = \frac{1}{Z_{11}} V_1 - \frac{Z_{12}}{Z_{11}} I_2$$

Comparing with $$I_1 = g_{11} V_1 + g_{12} I_2,$$

$$g_{11} = \frac{1}{Z_{11}}$$

$$g_{12} = -\frac{Z_{12}}{Z_{11}}$$

Also, $$V_2 = Z_{21}\left[\frac{1}{Z_{11}}V_1 - \frac{Z_{12}}{Z_{11}}I_2\right] + Z_{22}\,I_2 = \frac{Z_{21}}{Z_{11}}V_1 + \left[\frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}}\right]I_2$$

Comparing with $V_2 = g_{21}V_1 + g_{22}\,I_2$,

$$g_{21} = \frac{Z_{21}}{Z_{11}}$$

$$g_{22} = \frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} = \frac{\Delta Z}{Z_{11}}$$

2. *Inverse Hybrid Parameters in Terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Rewriting the second equation,

$$V_2 = -\frac{Y_{21}}{Y_{22}}V_1 + \frac{1}{Y_{22}}I_2$$

Comparing with $V_2 = g_{21}V_1 + g_{22}\,I_2$,

$$g_{21} = -\frac{Y_{21}}{Y_{22}}$$

$$g_{22} = \frac{1}{Y_{22}}$$

Also, $$I_1 = Y_{11}V_1 + Y_{12}\left[-\frac{Y_{21}}{Y_{22}}V_1 + \frac{1}{Y_{22}}I_2\right] = \left[\frac{Y_{11}\,Y_{22} - Y_{12}\,Y_{21}}{Y_{22}}\right]V_1 + \frac{Y_{12}}{Y_{22}}I_2$$

Comparing with $I_1 = g_{11}V_1 + g_{12}\,I_2$,

$$g_{11} = \frac{Y_{11}\,Y_{22} - Y_{12}\,Y_{21}}{Y_{22}} = \frac{\Delta Y}{Y_{22}}$$

$$g_{12} = \frac{Y_{12}}{Y_{22}}$$

3. *Inverse Hybrid Parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the first equation,

$$V_2 = \frac{1}{A}V_1 + \frac{B}{A}I_2$$

Comparing with $V_2 = g_{21}V_1 + g_{22}\,I_2$,

$$g_{21} = \frac{1}{A}$$

$$g_{22} = \frac{B}{A}$$

Also, $$I_1 = C\left[\frac{1}{A}V_1 + \frac{B}{A}I_2\right] - DI_2 = \frac{C}{A}V_1 - \left[\frac{AD - BC}{A}\right]I_2$$

Comparing with $I_1 = g_{11} V_1 + g_{12} I_2,$

$$g_{11} = \frac{C}{A}$$

$$g_{12} = -\left[\frac{AD - BC}{A}\right] = -\frac{\Delta T}{A}$$

4. Inverse Hybrid Parameters in Terms of A'B'C'D' Parameters We know that

$$V_2 = A' V_1 - B' I_1$$

$$I_2 = C' V_1 - D' I_1$$

Rewriting the second equation,

$$I_1 = \frac{C'}{D'} V_1 - \frac{1}{D'} I_2$$

Comparing with $I_1 = g_{11} V_1 + g_{12} I_2,$

$$g_{11} = \frac{C'}{D'}$$

$$g_{12} = -\frac{1}{D'}$$

Also, $$V_2 = A' V_1 - B'\left[\frac{C'}{D'} V_1 - \frac{1}{D'} I_2\right] = \left[\frac{A'D' - B'C'}{D'}\right] V_1 + \frac{B'}{D'} I_2$$

Comparing with $V_2 = g_{21} V_1 + g_{22} I_2,$

$$g_{21} = \left[\frac{A'D' - B'C'}{D'}\right] = \frac{\Delta T'}{D'}$$

$$g_{22} = \frac{B'}{D'}$$

5. Inverse Hybrid Parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

By Cramer's rule,

$$I_1 = \frac{\begin{vmatrix} V_1 & h_{12} \\ I_2 & h_{22} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}} = \frac{h_{22}}{\Delta h} V_1 - \frac{h_{12}}{\Delta h} I_2$$

where $\Delta h = h_{11} h_{22} - h_{12} h_{21}$

Comparing with $I_1 = g_{11} V_1 + g_{12} I_2,$

$$g_{11} = \frac{h_{22}}{\Delta h}$$

$$g_{12} = -\frac{h_{12}}{\Delta h}$$

Also, $$V_2 = \frac{\begin{vmatrix} h_{11} & V_1 \\ h_{21} & I_2 \end{vmatrix}}{\Delta h} = -\frac{h_{21}}{\Delta h} V_1 + \frac{h_{11}}{\Delta h} I_2$$

Comparing with $V_2 = g_{21} V_1 + g_{22} I_2,$

$$g_{21} = -\frac{h_{21}}{\Delta h}$$

$$g_{22} = \frac{h_{11}}{\Delta h}$$

Table 10.3 *Inter-relationship between parameters*

$$\Delta X = X_{11} X_{22} - X_{12} X_{21}$$

	In terms of											
	[Z]		[Y]		[T]		[T′]		[h]		[g]	
[Z]	Z_{11}	Z_{12}	$\frac{Y_{22}}{\Delta Y}$	$-\frac{Y_{12}}{\Delta Y}$	$\frac{A}{C}$	$\frac{\Delta T}{C}$	$\frac{D'}{C'}$	$\frac{1}{C'}$	$\frac{\Delta h}{h_{22}}$	$\frac{h_{12}}{h_{22}}$	$\frac{1}{g_{11}}$	$-\frac{g_{12}}{g_{11}}$
	Z_{21}	Z_{22}	$-\frac{Y_{21}}{\Delta Y}$	$\frac{Y_{11}}{\Delta Y}$	$\frac{1}{C}$	$\frac{D}{C}$	$\frac{\Delta T'}{C'}$	$\frac{A'}{C'}$	$-\frac{h_{21}}{h_{22}}$	$\frac{1}{h_{22}}$	$\frac{g_{21}}{g_{11}}$	$\frac{\Delta g}{g_{11}}$
[Y]	$\frac{Z_{22}}{\Delta Z}$	$-\frac{Z_{12}}{\Delta Z}$	Y_{11}	Y_{12}	$\frac{D}{B}$	$-\frac{\Delta T}{B}$	$\frac{A'}{B'}$	$-\frac{1}{B'}$	$\frac{1}{h_{11}}$	$-\frac{h_{12}}{h_{11}}$	$\frac{\Delta g}{g_{22}}$	$\frac{g_{12}}{g_{22}}$
	$-\frac{Z_{21}}{\Delta Z}$	$\frac{Z_{11}}{\Delta Z}$	Y_{21}	Y_{22}	$-\frac{1}{B}$	$\frac{A}{B}$	$\frac{\Delta T'}{B'}$	$\frac{D'}{B'}$	$\frac{h_{21}}{h_{11}}$	$\frac{\Delta h}{h_{11}}$	$-\frac{g_{21}}{g_{22}}$	$\frac{1}{g_{22}}$
[T]	$\frac{Z_{11}}{Z_{21}}$	$\frac{\Delta Z}{Z_{21}}$	$-\frac{Y_{22}}{Y_{21}}$	$-\frac{1}{Y_{21}}$	A	B	$\frac{D'}{\Delta T'}$	$\frac{B'}{\Delta T'}$	$-\frac{\Delta h}{h_{21}}$	$-\frac{h_{11}}{h_{21}}$	$\frac{1}{g_{21}}$	$\frac{g_{22}}{g_{21}}$
	$\frac{1}{Z_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$-\frac{\Delta Y}{Y_{21}}$	$-\frac{Y_{11}}{Y_{21}}$	C	D	$\frac{C'}{\Delta T'}$	$\frac{A'}{\Delta T'}$	$-\frac{h_{22}}{h_{21}}$	$-\frac{1}{h_{21}}$	$\frac{g_{11}}{g_{21}}$	$\frac{\Delta g}{g_{21}}$
[T′]	$\frac{Z_{22}}{Z_{12}}$	$\frac{\Delta Z}{Z_{12}}$	$-\frac{Y_{11}}{Y_{12}}$	$-\frac{1}{Y_{12}}$	$\frac{D}{\Delta T}$	$\frac{B}{\Delta T}$	A'	B'	$\frac{1}{h_{12}}$	$\frac{h_{11}}{h_{12}}$	$-\frac{\Delta g}{g_{12}}$	$-\frac{g_{22}}{g_{12}}$
	$\frac{1}{Z_{12}}$	$\frac{Z_{11}}{Z_{12}}$	$-\frac{\Delta Y}{Y_{12}}$	$-\frac{Y_{22}}{Y_{12}}$	$\frac{C}{\Delta T}$	$\frac{A}{\Delta T}$	C'	D'	$\frac{h_{22}}{h_{12}}$	$\frac{\Delta h}{h_{12}}$	$-\frac{g_{11}}{g_{12}}$	$-\frac{1}{g_{12}}$
[h]	$\frac{\Delta Z}{Z_{22}}$	$\frac{Z_{12}}{Z_{22}}$	$\frac{1}{Y_{11}}$	$-\frac{Y_{12}}{Y_{11}}$	$\frac{B}{D}$	$\frac{\Delta T}{D}$	$\frac{B'}{A'}$	$\frac{1}{A'}$	h_{11}	h_{12}	$\frac{g_{22}}{\Delta g}$	$-\frac{g_{12}}{\Delta g}$
	$-\frac{Z_{21}}{Z_{22}}$	$\frac{1}{Z_{22}}$	$\frac{Y_{21}}{Y_{11}}$	$\frac{\Delta Y}{Y_{11}}$	$-\frac{1}{D}$	$\frac{C}{D}$	$-\frac{\Delta T'}{A'}$	$\frac{C'}{A'}$	h_{21}	h_{22}	$-\frac{g_{21}}{\Delta g}$	$\frac{g_{11}}{\Delta g}$
[g]	$\frac{1}{Z_{11}}$	$-\frac{Z_{12}}{Z_{11}}$	$\frac{\Delta Y}{Y_{22}}$	$\frac{Y_{12}}{Y_{22}}$	$\frac{C}{A}$	$-\frac{\Delta T}{A}$	$\frac{C'}{D'}$	$-\frac{1}{D'}$	$\frac{h_{22}}{\Delta h}$	$-\frac{h_{12}}{\Delta h}$	g_{11}	g_{12}
	$\frac{Z_{21}}{Z_{11}}$	$\frac{\Delta Z}{Z_{11}}$	$-\frac{Y_{21}}{Y_{22}}$	$\frac{1}{Y_{22}}$	$\frac{1}{A}$	$\frac{B}{A}$	$\frac{\Delta T'}{D'}$	$\frac{B'}{D'}$	$-\frac{h_{21}}{\Delta h}$	$\frac{h_{11}}{\Delta h}$	g_{21}	g_{22}

Example 10.26 *The Z parameters of a two-port network are $Z_{11} = 20\ \Omega$, $Z_{22} = 30\ \Omega$, $Z_{12} = Z_{21} = 10\ \Omega$. Find Y and ABCD parameters.*

Solution

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (20)(30) - (10)(10) = 500$$

***Y*-parameters**

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{30}{500} = \frac{3}{50}\ \text{℧}, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \text{℧}$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \text{℧}, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{20}{500} = \frac{2}{50}\ \text{℧}$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{50} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{2}{50} \end{bmatrix}$$

***ABCD* parameters**

$$A = \frac{Z_{11}}{Z_{21}} = \frac{20}{10} = 2, \qquad B = \frac{\Delta Z}{Z_{21}} = \frac{500}{10} = 50$$

$$C = \frac{1}{Z_{21}} = \frac{1}{10} = 0.1, \qquad D = \frac{Z_{22}}{Z_{21}} = \frac{30}{10} = 3$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2 & 50 \\ 0.1 & 3 \end{bmatrix}$$

Example 10.27 *Currents I_1 and I_2 entering at Port 1 and Port 2 respectively of a two-port network are given by the following equations:*

$$I_1 = 0.5V_1 - 0.2V_2$$

$$I_2 = -0.2V_1 + V_2$$

Find Y, Z and ABCD parameters for the network.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.5\ \text{℧}, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -0.2\ \text{℧}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -0.2\ \text{℧}, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 1\ \text{℧}$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 1 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (0.5)(1) - (-0.2)(-0.2) = 0.46$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{1}{0.46} = 2.174\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{0.5}{0.46} = 1.087\ \Omega$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2.174 & 0.434 \\ 0.434 & 1.087 \end{bmatrix}$$

***ABCD* parameters**

$$A = -\frac{Y_{22}}{Y_{21}} = -\frac{1}{-0.2} = 5, \qquad B = -\frac{1}{Y_{21}} = -\frac{1}{-0.2} = 5$$

$$C = -\frac{\Delta Y}{Y_{21}} = -\frac{0.46}{-0.2} = 2.3, \qquad D = -\frac{Y_{11}}{Y_{21}} = -\frac{0.5}{-0.2} = 2.5$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2.3 & 2.5 \end{bmatrix}$$

Example 10.28

Using the relation $Y = Z^{-1}$, show that $|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$.

Solution We know that

$$Y = Z^{-1}$$

i.e.,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{bmatrix}$$

$$|Z| = Z_{11} Z_{22} - Z_{12} Z_{21}$$

$$\frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right) = \frac{1}{2}\left(\frac{Z_{22}}{\frac{Z_{22}}{\Delta Z}} + \frac{Z_{11}}{\frac{Z_{11}}{\Delta Z}}\right) = \frac{1}{2}(\Delta Z + \Delta Z) = \frac{1}{2}(2\Delta Z) = \Delta Z = Z_{11} Z_{12} - Z_{12} Z_{21}$$

$$|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$$

Example 10.29

For the network shown in Fig. 10.59, find Z and Y-parameters.

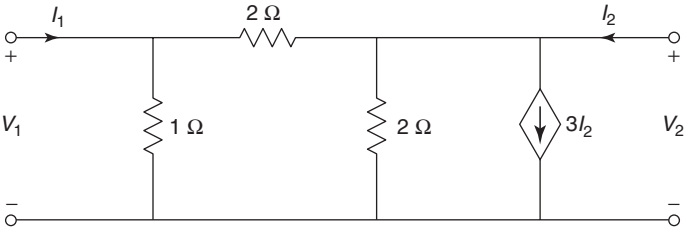


Fig. 10.59

Solution The network is redrawn by source transformation technique as shown in Fig. 10.60.

Applying KVL to Mesh 1,

$$V_1 = I_1 - I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2(I_2 + I_3) - 6I_2$$
$$= -4I_2 + 2I_3 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-(I_3 - I_1) - 2I_3 - 2(I_2 + I_3) + 6I_2 = 0$$
$$5I_3 = I_1 + 4I_2$$
$$I_3 = \frac{1}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots(iii)$$

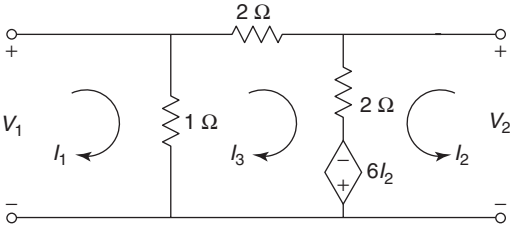


Fig. 10.60

Substituting Eq. (iii) in Eq. (i),

$$V_1 = I_1 - \frac{1}{5}I_1 - \frac{4}{5}I_2$$
$$= \frac{4}{5}I_1 - \frac{4}{5}I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = -4I_2 + 2\left(\frac{1}{5}I_1 + \frac{4}{5}I_2\right)$$
$$= \frac{2}{5}I_1 - \frac{12}{5}I_2 \quad \ldots(v)$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{12}{5} \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = \left(\frac{4}{5}\right)\left(-\frac{12}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{2}{5}\right) = -\frac{40}{25} = -\frac{8}{5}$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{-\frac{12}{5}}{-\frac{8}{5}} = \frac{3}{2}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = \frac{-\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = \frac{-\frac{2}{5}}{-\frac{8}{5}} = \frac{1}{4}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Example 10.30 *Find Z and h-parameters for the network shown in Fig. 10.61.*

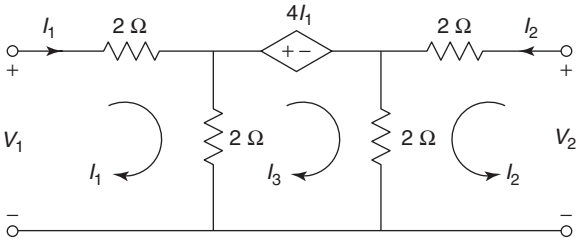


Fig. 10.61

Solution Applying KVL to Mesh 1,

$$V_1 = 2I_1 + 2(I_1 - I_3)$$
$$= 4I_1 - 2I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2(I_2 + I_3)$$
$$= 4I_2 + 2I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 4I_1 - 2(I_3 + I_2) = 0$$

$$I_1 + I_2 = -2I_3 \quad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 4I_1 + I_1 + I_2$$

$$= 5I_1 + I_2 \quad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 4I_2 - I_1 - I_2$$

$$= -I_1 + 3I_2 \quad \ldots(v)$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

***h*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (5)(3) - (1)(-1) = 15 + 1 = 16$$

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{16}{3}\,\Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{3}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = \frac{1}{3}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{3}\,\mho$$

Hence, the *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{22} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{16}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Example 10.31

Find Y and Z-parameters for the network shown in Fig. 10.62.

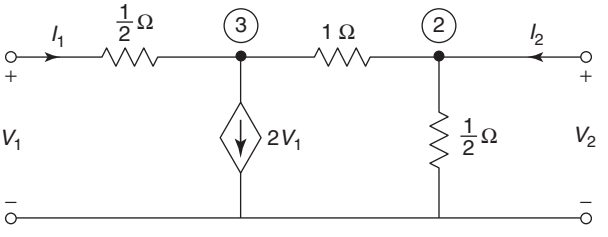


Fig. 10.62

Solution Applying KCL at Node 3,

$$2(V_1 - V_3) = 2V_1 + (V_3 - V_2)$$

$$V_3 = \frac{V_2}{3} \quad \ldots(i)$$

Now,

$$I_1 = 2V_1 + (V_3 - V_2)$$

$$= 2V_1 + \frac{V_2}{3} - V_2$$

$$= 2V_1 - \frac{2}{3}V_2 \quad \ldots(ii)$$

$$I_2 = 2V_2 + (V_2 - V_3)$$
$$= 3V_2 - \frac{V_2}{3}$$
$$= \frac{8}{3}V_2 \qquad \ldots(iii)$$

Comparing Eqs (ii) and (iii) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

Z-parameters

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (2)\left(\frac{8}{3}\right) - 0 = \frac{16}{3}$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{\frac{16}{3}} = \frac{1}{2}\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{\left(-\frac{2}{3}\right)}{\frac{16}{3}} = \frac{1}{8}\Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{0}{\frac{16}{3}} = 0, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{2}{\frac{16}{3}} = \frac{3}{8}\Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 10.32 *For the network shown in Fig. 10.63, find Y and Z-parameters.*

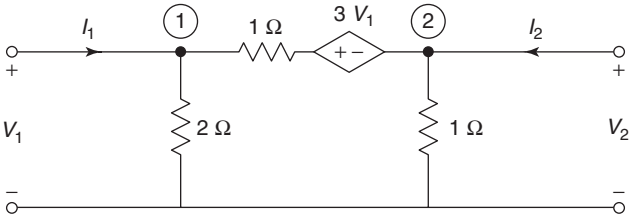


Fig. 10.63

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{2} + \frac{V_1 - 3V_1 - V_2}{1}$$
$$= -\frac{3}{2}V_1 - V_2 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{1} + \frac{V_2 + 3V_1 - V_1}{1} \qquad \ldots(ii)$$
$$= 2V_1 + 2V_2$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -1 \\ 2 & 2 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = \left(-\frac{3}{2}\right)(2) - (-1)(2) = -3 + 2 = -1$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{2}{-1} = -2\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-1)}{(-1)} = -1\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{2}{(-1)} = 2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{-\frac{3}{2}}{-1} = \frac{3}{2}\ \Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & \frac{3}{2} \end{bmatrix}$$

Example 10.33 *Find Z-parameters for the network shown in Fig. 10.64. Hence, find Y and h-parameters.*

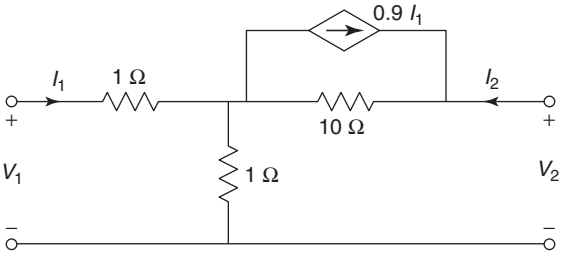


Fig. 10.64

Solution The network is redrawn by source transformation technique as shown in Fig. 10.65.

Applying KVL to Mesh 1,

$$V_1 = 2I_1 + I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 9I_1 + 10I_2 + 1(I_1 + I_2)$$
$$= 10I_1 + 11I_2 \qquad \ldots(ii)$$

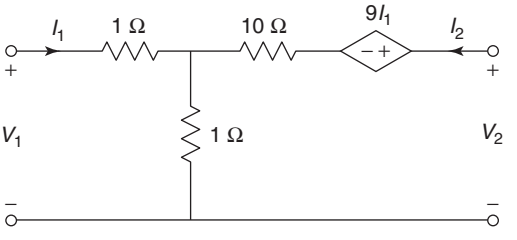


Fig. 10.65

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 10 & 11 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (2)(11) - (1)(10) = 22 - 10 = 12$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{11}{12}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{1}{12}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{12} = -\frac{5}{6}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{2}{12} = \frac{1}{6}\ \mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{12} & -\frac{1}{12} \\ -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

***h*-parameters**

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{12}{11}\,\Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{11}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = -\frac{10}{11}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{11}\,\mho$$

Hence, *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{12}{11} & \frac{1}{11} \\ -\frac{10}{11} & \frac{1}{11} \end{bmatrix}$$

Example 10.34 *Find Y and Z-parameters of the network shown in Fig. 10.66.*

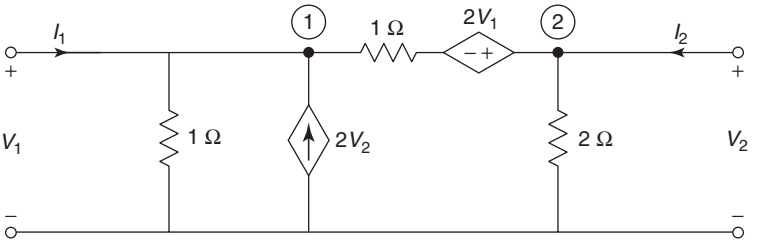


Fig. 10.66

Solution Applying KCL at Node 1,

$$I_1 + 2V_2 = \frac{V_1}{1} + \frac{3V_1 - V_2}{1}$$

$$I_1 = 4V_1 - 3V_2 \qquad \ldots(\text{i})$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + \frac{V_2 - 2V_1 - V_1}{1}$$

$$= -3V_1 + 1.5V_2 \qquad \ldots(\text{ii})$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 1.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (4)(1.5) - (-3)(-3) = -3$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{1.5}{3} = -0.5\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = -\frac{4}{3} = -\frac{4}{3}\,\Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ -1 & -\frac{4}{3} \end{bmatrix}$$

Example 10.35 *Determine Y and Z-parameters for the network shown in Fig. 10.67.*

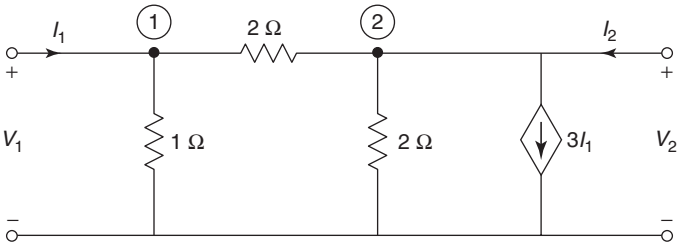


Fig. 10.67

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$
$$= 1.5V_1 - 0.5V_2 \quad \ldots(i)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + 3I_1 + \frac{V_2 - V_1}{2}$$
$$= \frac{V_2}{2} + 3(1.5V_1 - 0.5V_2) + \frac{V_2 - V_1}{2}$$
$$= 0.5V_2 + 4.5V_1 - 1.5V_2 + 0.5V_2 - 0.5V_1$$
$$= 4V_1 - 0.5V_2 \quad \ldots(ii)$$

Comparing Eqs (i) and (ii) with the *Y*-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (1.5)(-0.5) - (-0.5)(4) = 1.25$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{0.5}{1.25} = -0.4\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{0.5}{1.25} = 0.4\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{4}{1.25} = -3.2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{1.5}{1.25} = 1.2\ \Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix}$$

Example 10.36 *Determine the Y and Z-parameters for the network shown in Fig. 10.68.*

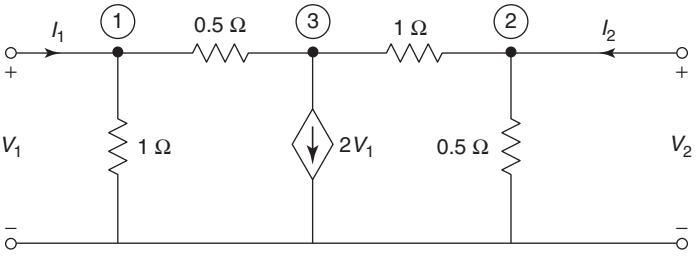


Fig. 10.68

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_3}{0.5}$$
$$= 3V_1 - 2V_3 \quad \ldots(i)$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{0.5} + \frac{V_2 - V_3}{1}$$
$$= 3V_2 - V_3 \quad \ldots(ii)$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{0.5} + 2V_1 + \frac{V_3 - V_2}{1} = 0$$
$$V_3 = \frac{1}{3}V_2 \quad \ldots(iii)$$

Substituting Eq. (iii) in Eqs (i) and (ii),

$$I_1 = 3V_1 - \frac{2}{3}V_2 \quad \ldots(iv)$$

$$I_2 = 0V_1 + \frac{8}{3}V_2 \quad \ldots(v)$$

Comparing Eqs (iv) and (v) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (3)\left(\frac{8}{3}\right) - 0 = 8$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{8} = \frac{1}{3}\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega$$

$$Z_{21} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{Y\Delta} = \frac{3}{8}\,\Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 10.37 *Determine Z and Y-parameters of the network shown in Fig. 10.69.*

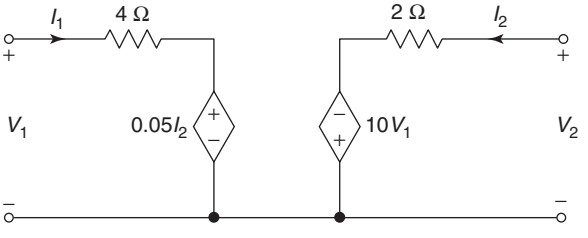


Fig. 10.69

Solution Applying KVL to Mesh 1,

$$V_1 - 4I_1 - 0.05\,I_2 = 0$$
$$V_1 = 4I_1 + 0.05I_2 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 - 2I_2 + 10V_1 = 0$$
$$V_2 = 2I_2 - 10V_1 \quad \ldots(ii)$$

Substituting Eq. (i) in Eq. (ii),

$$V_2 = 2I_2 - 40I_1 - 0.5I_2$$
$$= -40I_1 + 1.5I_2 \quad \ldots(iii)$$

Comparing Eqs (i) and (iii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0.05 \\ -40 & 1.5 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (4)(1.5) - (0.05)(-40) = 8$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{1.5}{8}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{0.05}{8}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-40)}{8} = \frac{40}{8}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{4}{8}\,\mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1.5}{8} & -\frac{0.05}{8} \\ \frac{40}{8} & \frac{4}{8} \end{bmatrix}$$

Example 10.38

Determine Z and Y-parameters of the network shown in Fig. 10.70.

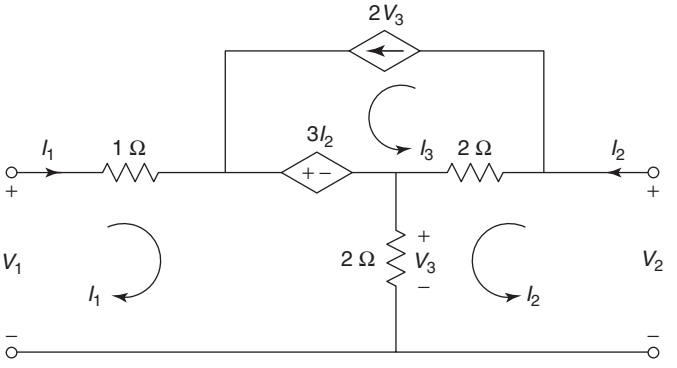


Fig. 10.70

Solution Applying KVL to Mesh 1,

$$V_1 - 1I_1 - 3I_2 - 2(I_1 + I_2) = 0$$
$$V_1 = 3I_1 + 5I_2 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 - 2(I_2 - I_3) - 2(I_1 + I_2) = 0$$
$$V_2 - 2I_2 + 2I_3 - 2I_1 - 2I_2 = 0$$
$$V_2 = 2I_1 + 4I_2 - 2I_3 \quad \ldots(ii)$$

Writing equation for Mesh 3,

$$I_3 = 2V_3 \qquad \text{...(iii)}$$

From Fig. 10.70,

$$V_3 = 2(I_1 + I_2)$$
$$I_3 = 2V_3 = 4I_1 + 4I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$V_2 = -6I_1 - 4I_2 \qquad \text{...(v)}$$

Comparing Eqs (i) and (v) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -6 & -4 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (3)(-4) - (5)(-6) = 18$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = -\frac{4}{18} = -\frac{2}{9}\,\text{℧}, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{5}{18}\,\text{℧}$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-6)}{18} = \frac{1}{3}\,\text{℧}, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{3}{18}\,\text{℧}$$

Hence, *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & -\frac{5}{18} \\ \frac{1}{3} & \frac{3}{18} \end{bmatrix}$$

10.9 INTERCONNECTION OF TWO-PORT NETWORKS

We shall now discuss the various types of interconnections of two-port networks, namely, cascade, parallel, series, series-parallel and parallel-series. We shall derive the relation between the input and output quantities of the combined two-port networks.

10.9.1 Cascade Connection

1. Transmission Parameter Representation Figure 10.71 shows two-port networks connected in cascade. In the cascade connection, the output port of the first network becomes the input port of the second network. Since it is assumed that input and output currents are positive when they enter the network, we have

$$I_1' = -I_2$$

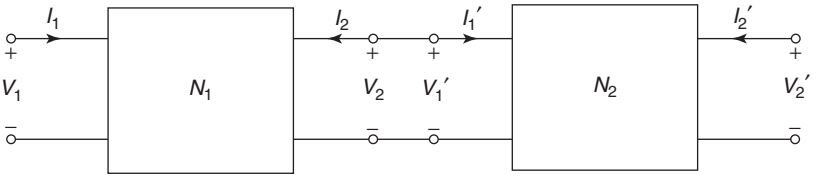


Fig. 10.71 *Cascade Connection*

Let A_1, B_1, C_1, D_1 be the transmission parameters of the network N_1 and A_2, B_2, C_2, D_2 be the transmission parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad \text{...(10.1)}$$

For the network N_2,

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad \text{...(10.2)}$$

Since $V_1' = V_2$ and $I_2' = -I_2$, we can write

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad \text{...(10.3)}$$

Combining Eqs (10.1) and (10.3),

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad \text{...(10.4)}$$

Equation 10.4 shows that the resultant *ABCD* matrix of the cascade connection is the product of the individual *ABCD* matrices.

2. *Inverse Transmission Parameter Representation* Figure 10.72 shows two-port networks connected in cascade. Since it is assumed that input and output currents are positive when they enter the network, we have

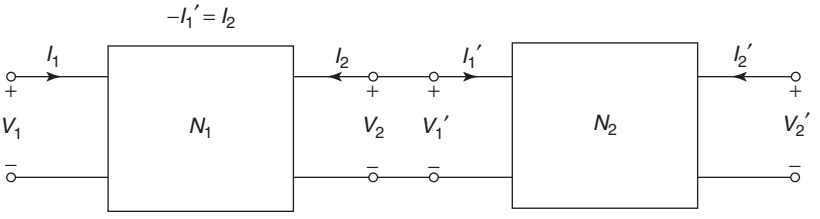


Fig. 10.72 *Cascade connection*

Let A_1', B_1', C_1', D_1' be the transmission parameters of the network N_1 and A_2', B_2', C_2', D_2' be the transmission parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} \quad \text{...(10.5)}$$

For the network N_2,

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} V_2' \\ -I_1' \end{bmatrix} \quad \text{...(10.6)}$$

Since $V_1' = V_2$ and $-I_1' = I_2$, we can write

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad \text{...(10.7)}$$

Combining equations (10.5) and (10.7),

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \quad \text{...(10.8)}$$

Equation 10.8 shows that the resultant $A'B'C'D'$ matrix of the cascade connection is the product of the individual $A'B'C'D'$ matrices.

Example 10.39 *Two identical sections of the network shown in Fig. 10.73 are connected in cascade. Obtain the transmission parameters of the overall connection.*

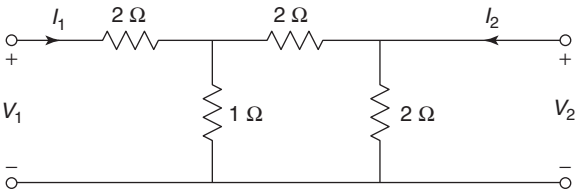


Fig. 10.73

Solution The network is redrawn as shown in Fig. 10.74.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - I_3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{1}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots(iii)$$

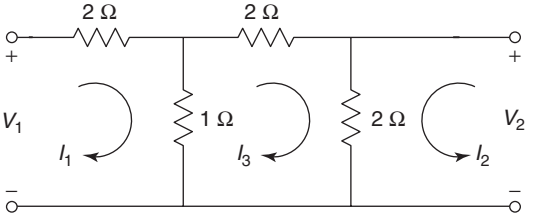


Fig. 10.74

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{14}{5}I_1 + \frac{2}{5}I_2 \qquad \ldots(iv)$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{2}{5}I_1 + \frac{6}{5}I_2$$

$$I_1 = \frac{5}{2}V_2 - 3I_2 \qquad \ldots(v)$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{14}{5}\left(\frac{5}{2}V_2 - 3I_2\right) + \frac{2}{5}I_2$$

$$= 7V_2 - 8I_2 \qquad \ldots(vi)$$

Comparing the Eqs (vi) and (v) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}$$

Hence, transmission parameters of the overall cascaded network are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix} = \begin{bmatrix} 69 & 80 \\ 25 & 29 \end{bmatrix}$$

Example 10.40 *Determine ABCD parameters for the ladder network shown in Fig. 10.75.*

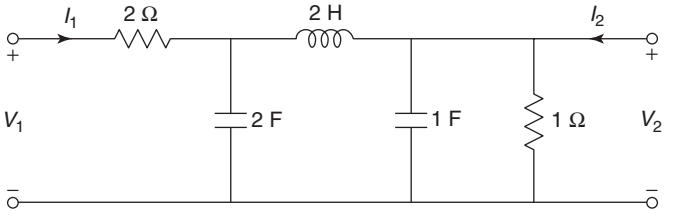


Fig. 10.75

Solution The above network can be considered as a cascade connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 10.76.

Applying KVL to Mesh 1,

$$V_1 = \left(2 + \frac{1}{2s}\right) I_1 + \frac{1}{2s} I_2 \qquad \ldots(i)$$

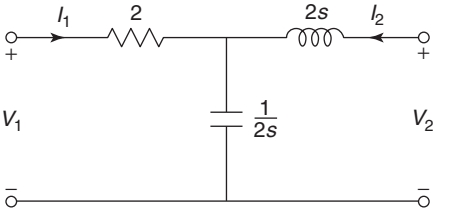


Fig. 10.76

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{2s} I_1 + \left(2s + \frac{1}{2s}\right) I_2 \qquad \ldots(ii)$$

From Eq. (ii),

$$I_1 = 2s\, V_2 - (4s^2 + 1)\, I_2 \qquad ...(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = \left(2 + \frac{1}{2s}\right)[2s\, V_2 - (4s^2 + 1) I_2] + \frac{1}{2s} I_2$$

$$= (4s + 1) V_2 - (8s^2 + 2s + 2) I_2 \qquad \ldots(iv)$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 4s+1 & 8s^2+2s+2 \\ 2s & 4s^2+1 \end{bmatrix}$$

The network N_2 is shown in Fig. 10.77.

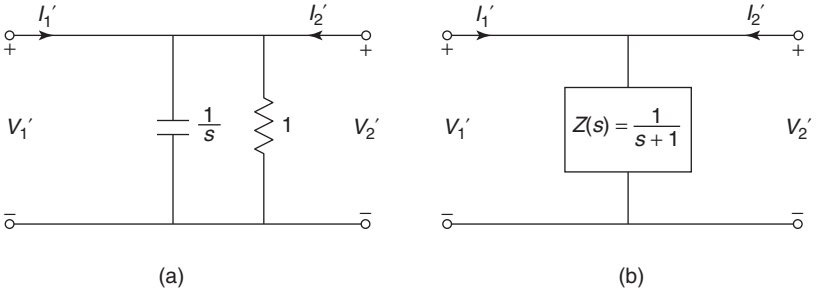


Fig. 10.77

Applying KVL to Mesh 1,

$$V_1' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \ldots(ii)$$

From Eq. (ii),

$$I_1' = (s+1)V_2' - I_2' \quad \ldots(iii)$$

Also,

$$V_1' = V_2' \quad \ldots(iv)$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}$$

Hence, overall *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4s+1 & 8s^2+2s+2 \\ 2s & 4s^2+1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} 8s^3+10s^2+8s+3 & 8s^2+2s+2 \\ 4s^3+4s^2+3s+1 & 4s^2+1 \end{bmatrix}$$

10.9.2 Parallel Connection

Figure 10.78 shows two-port networks connected in parallel. In the parallel connection, the two networks have the same input voltages and the same output voltages.

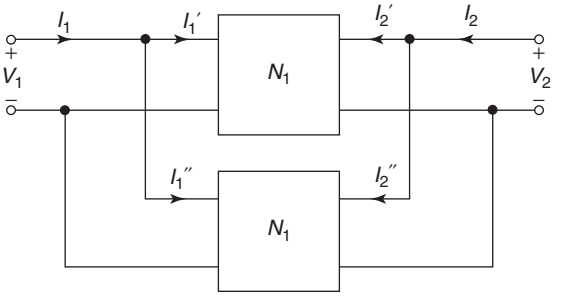


Fig. 10.78 *Parallel connection*

Let $Y_{11}', Y_{12}', Y_{21}', Y_{22}'$ be the *Y*-parameters of the network N_1 and $Y_1'', Y_{12}'', Y_{21}'', Y_{22}''$ be the *Y*-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the combined network, $I_1 = I_1' + I_1''$ and $I_2 = I_2' + I_2''$.

Hence,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1' + I_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant *Y*-parameter matrix for parallel connected networks is the sum of *Y* matrices of each individual two-port networks.

Example 10.41 *Determine Y-parameters for the network shown in Fig. 10.79.*

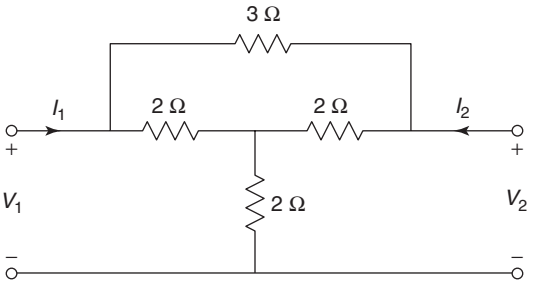


Fig. 10.79

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 10.80.

Applying KCL at Node 3,

$$I_1' + I_2' = \frac{V_3}{2} \qquad \ldots(i)$$

From Fig. 10.80,

$$I_1' = \frac{V_1 - V_3}{2} \qquad \ldots(ii)$$

$$I_2' = \frac{V_2 - V_3}{2} \qquad \ldots(iii)$$

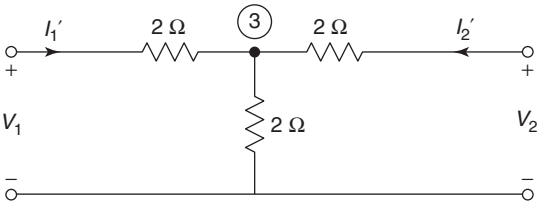


Fig. 10.80

Substituting Eqs (ii) and (iii) in Eq (i),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} = \frac{V_3}{2}$$

$$3V_3 = V_1 + V_2$$

$$V_3 = \frac{V_1}{3} + \frac{V_2}{3} \qquad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= \frac{1}{3}V_1 - \frac{1}{6}V_2 \qquad \ldots(v)$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= -\frac{1}{6}V_1 + \frac{1}{3}V_2 \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

The network N_2 is shown in Fig. 10.81.

$$I_1'' = -I_2'' = \frac{V_1 - V_2}{3} = \frac{1}{3}V_1 - \frac{1}{3}V_2$$

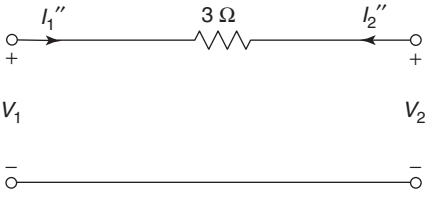


Fig. 10.81

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The overall *Y*-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3} & -\frac{1}{6}-\frac{1}{3} \\ -\frac{1}{6}-\frac{1}{3} & \frac{1}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

Example 10.42

Find Y-parameters for the network shown in Fig. 10.82

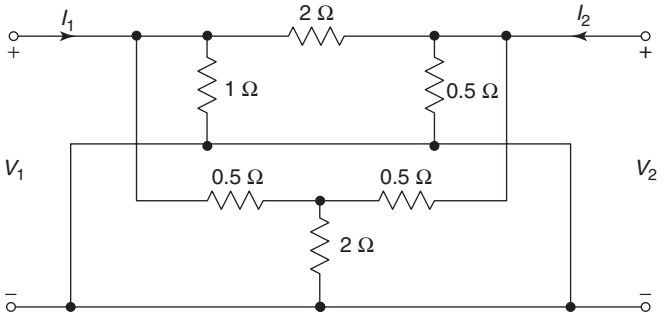


Fig. 10.82

Solution The above network can be considered as a parallel combination of two networks N_1 and N_2. The network N_1 is shown in Fig. 10.83.
Applying KCL at Node 1,

$$I_1' = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$

$$= \frac{3}{2}V_1 - \frac{1}{2}V_2 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$I_2' = \frac{V_2}{0.5} + \frac{V_2 - V_1}{2}$$

$$= -\frac{1}{2}V_1 + \frac{5}{2}V_2 \qquad \ldots(ii)$$

Comparing Eqs (i) and (ii) with *Y*-parameter equation,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

The network N_2 is shown in Fig. 10.84.

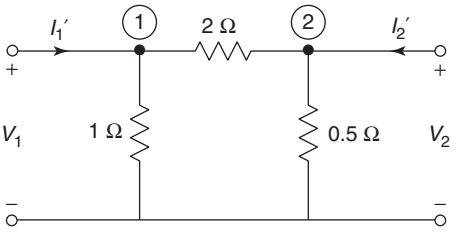


Fig. 10.83

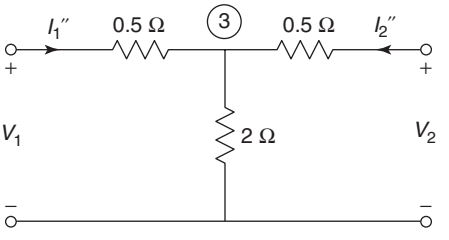


Fig. 10.84

Applying KCL at Node 3,

$$I_1'' + I_2'' = \frac{V_3}{2} \quad \ldots(i)$$

where

$$I_1'' = \frac{V_1 - V_3}{0.5} = 2V_1 - 2V_3$$

$$I_2'' = \frac{V_2 - V_3}{0.5} = 2V_2 - 2V_3$$

Substituting I_1'' and I_2'' in Eq (i),

$$2V_1 - 2V_3 + 2V_2 - 2V_3 = 0.5V_3$$

$$4.5V_3 = 2V_1 + 2V_2$$

$$V_3 = \frac{4}{9}V_1 + \frac{4}{9}V_2 \quad \ldots(ii)$$

$$I_1'' = 2V_1 - 2V_3 = 2V_1 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right) = \frac{10}{9}V_1 - \frac{8}{9}V_2 \quad \ldots(iii)$$

and

$$I_2'' = 2V_2 - 2V_3 = 2V_2 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right)$$

$$= -\frac{8}{9}V_1 + \frac{10}{9}V_2 \quad \ldots(iv)$$

Comparing Eqs (iii) and (iv) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & -\frac{8}{9} \\ -\frac{8}{9} & \frac{10}{9} \end{bmatrix}$$

Hence, overall *Y*-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{10}{9} & -\frac{1}{2} - \frac{8}{9} \\ -\frac{1}{2} - \frac{8}{9} & \frac{5}{2} + \frac{10}{9} \end{bmatrix} = \begin{bmatrix} \frac{47}{18} & -\frac{25}{18} \\ -\frac{25}{18} & \frac{65}{18} \end{bmatrix}$$

Example 10.43 *Find Y-parameters for the network shown in Fig. 10.85.*

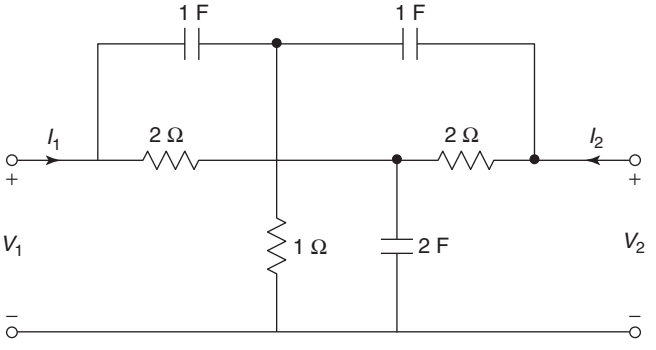


Fig. 10.85

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 10.86.

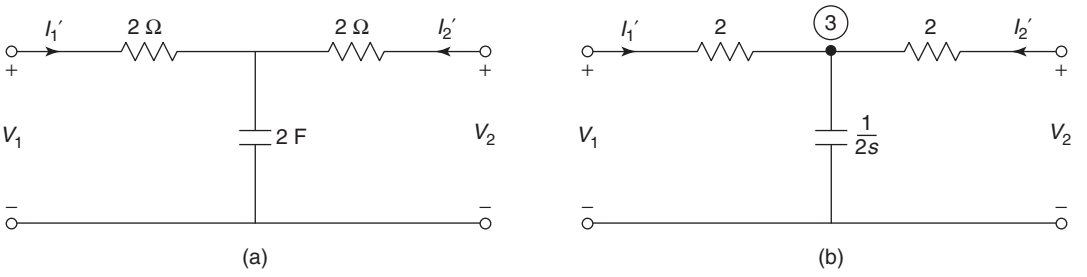


Fig. 10.86

Applying KCL at Node 3,

$$I_1' + I_2' = 2s\,V_3 \qquad \ldots(i)$$

From Fig. 10.86,

$$I_1' = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2}V_1 - \frac{1}{2}V_3 \qquad \ldots(ii)$$

$$I_2' = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \qquad \ldots(iii)$$

Substituting Eq. (ii) and Eq. (iii) in Eq. (i),

$$\frac{V_1}{2} - \frac{V_3}{2} + \frac{V_2}{2} - \frac{V_3}{2} = (2s)\,V_3$$

$$(2s+1)V_3 = \frac{V_1}{2} + \frac{V_2}{2}$$

$$V_3 = \frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2 \qquad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2\right]$$

$$= \left(\frac{4s+1}{8s+4}\right)V_1 - \left(\frac{1}{8s+4}\right)V_2 \qquad \ldots(v)$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 - \frac{1}{2(2s+1)}V_2\right]$$

$$= -\left(\frac{1}{8s+4}\right)V_1 + \left(\frac{4s+1}{8s+4}\right)V_2 \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \dfrac{4s+1}{8s+4} & -\dfrac{1}{8s+4} \\ -\dfrac{1}{8s+4} & \dfrac{4s+1}{8s+4} \end{bmatrix}$$

The network N_2 is shown in Fig. 10.87.

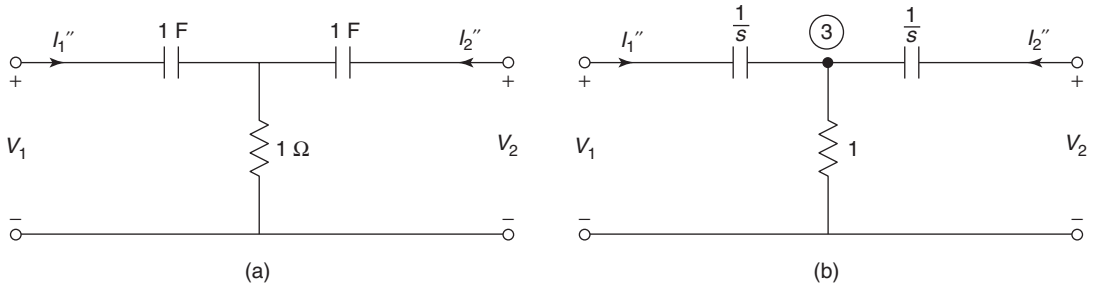


Fig. 10.87

Applying KCL at Node 3,

$$I_1'' + I_2'' = V_3 \qquad \ldots(i)$$

From Fig. 10.87,

$$I_1'' = \frac{V_1 - V_3}{\frac{1}{s}}$$

$$= sV_1 - sV_3 \qquad \ldots(ii)$$

$$I_2' = \frac{V_2 - V_3}{\frac{1}{s}}$$

$$= sV_2 - sV_3 \qquad \ldots(iii)$$

Substituting Eqs (ii) and (iii) in Eq. (i),

$$sV_1 - sV_3 + sV_2 - sV_3 = V_3$$

$$(2s+1)V_3 = sV_1 + sV_2$$

$$V_3 = \left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2 \qquad \ldots(iv)$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1'' = sV_1 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \frac{s}{(2s+1)}V_2\right]$$

$$= \left[\frac{s(s+1)}{2s+1}\right]V_1 - \left(\frac{s^2}{2s+1}\right)V_2 \qquad \ldots(v)$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2'' = sV_2 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2\right]$$

$$= -\left(\frac{s^2}{2s+1}\right)V_1 + \left[\frac{s(s+1)}{2s+1}\right]V_2 \qquad \ldots(vi)$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{s(s+1)}{2s+1} & -\left(\frac{s^2}{2s+1}\right) \\ -\left(\frac{s^2}{2s+1}\right) & \frac{s(s+1)}{2s+1} \end{bmatrix}$$

Hence, the overall *Y*-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \dfrac{4s^2+8s+1}{4(2s+1)} & -\dfrac{(4s^2+1)}{4(2s+1)} \\ -\dfrac{(4s^2+1)}{4(2s+1)} & \dfrac{4s^2+8s+1}{4(2s+1)} \end{bmatrix}$$

10.9.3 Series Connection

Figure 10.88 shows two-port networks connected in series. In a series connection, both the networks carry the same input current. Their output currents are also equal.

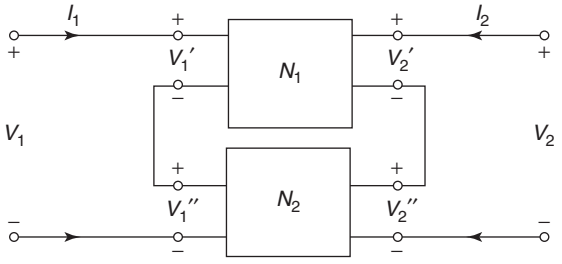


Fig. 10.88 *Series connection*

Let $Z_{11}', Z_{12}', Z_{21}', Z_{22}'$ be the *Z*-parameters of the network N_1 and $Z_{11}'', Z_{12}'', Z_{21}'', Z_{22}''$ be the *Z*-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the combined network $V_1 = V_1' + V_1''$ and $V_2 = V_2' + V_2''$.

Hence,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1' + V_1'' \\ V_2' + V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus, the resultant *Z*-parameter matrix for the series-connected networks is the sum of *Z* matrices of each individual two-port network.

Example 10.44 *Two identical sections of the network shown in Fig. 10.89 are connected in series. Obtain Z-parameters of the overall connection.*

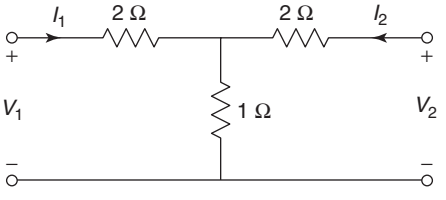


Fig. 10.89

Solution

Applying KVL to Mesh 1,

$$V_1 = 3I_1 + I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = I_1 + 3I_2 \qquad \ldots(ii)$$

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Hence, *Z*-parameters of the overall connection are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$$

Example 10.45

Determine Z-parameters for the network shown in Fig. 10.90.

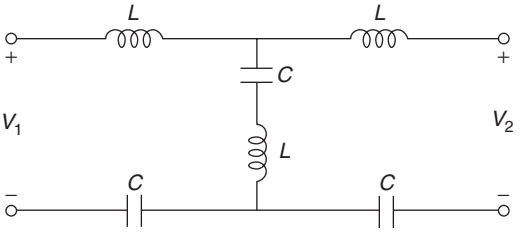


Fig. 10.90

Solution The above network can be considered as a series connection of two networks, N_1 and N_2.

The network N_1 is shown in Fig. 10.91.

Applying KVL to Mesh 1,

$$V_1' = \left(Ls + \frac{1}{Cs}\right)I_1 + \left(\frac{1}{Cs}\right)I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2' = \left(\frac{1}{Cs}\right)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots(ii)$$

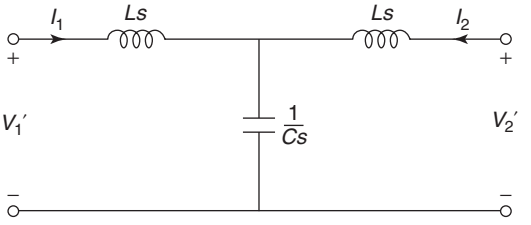


Fig. 10.91

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11}' & Z_{12}' \\ Z_{21}' & Z_{22}' \end{bmatrix} = \begin{bmatrix} Ls + \frac{1}{Cs} & \frac{1}{Cs} \\ \frac{1}{Cs} & Ls + \frac{1}{Cs} \end{bmatrix}$$

The network N_2 is shown in Fig. 10.92.

Applying KVL to Mesh 1,

$$V_1'' = \left(Ls + \frac{1}{Cs}\right)I_1 + (Ls)I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2'' = (Ls)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots(ii)$$

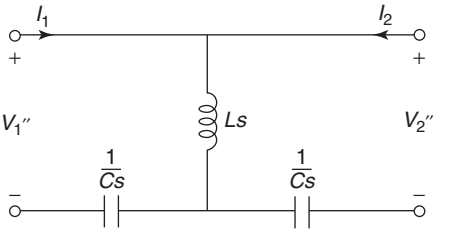


Fig. 10.92

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} Ls + \dfrac{1}{Cs} & Ls \\ Ls & Ls + \dfrac{1}{Cs} \end{bmatrix}$$

Hence, the overall *Z*-parameters of the network are,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix} = \begin{bmatrix} 2Ls + \dfrac{2}{Cs} & Ls + \dfrac{1}{Cs} \\ Ls + \dfrac{1}{Cs} & 2Ls + \dfrac{2}{Cs} \end{bmatrix} = \left(Ls + \frac{1}{Cs} \right) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

10.9.4 Series-Parallel Connection

Figure 10.93 shows two networks connected in series-parallel. Here, the input ports of two networks are connected in series and the output ports are connected in parallel.

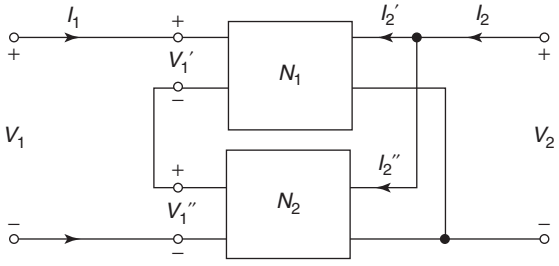


Fig. 10.93 *Series-parallel connection*

Let $h_{11}', h_{12}', h_{21}', h_{22}'$ be the *h*-parameters of the network N_1 and $h_{11}'', h_{12}'', h_{21}'', h_{22}''$ be the *h*-parameters of the network N_2.
For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the combined network, $V_1 = V_1' + V_1''$ and $I_2 = I_2' + I_2''$

Hence,
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1' + V_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}' + h_{11}'' & h_{12}' + h_{12}'' \\ h_{21}' + h_{21}'' & h_{22}' + h_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant *h*-parameter matrix is the sum of *h*-parameter matrices of each individual two-port networks.

Example 10.46 *Determine h-parameters for the network shown in Fig. 10.94.*

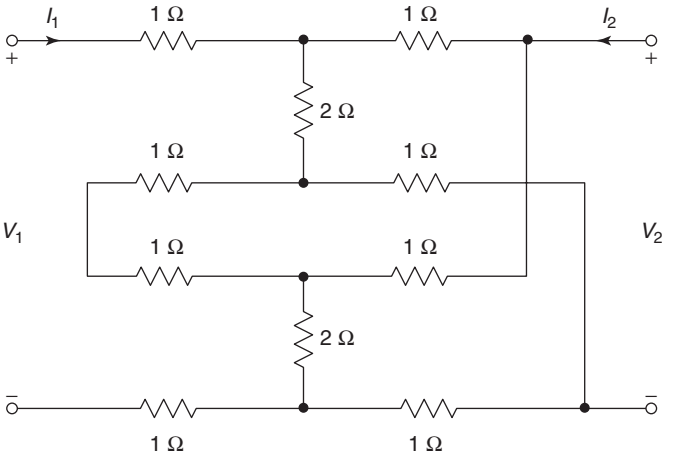


Fig. 10.94

Solution The above network can be considered as a series-parallel connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 10.95.

Applying KVL to Mesh 1,

$$V_1 = 4I_1 + 2I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 4I_2 \qquad \ldots(ii)$$

Rewriting Eq. (ii)

$$4I_2 = -2I_1 + V_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{4}V_2 \qquad \ldots(iii)$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 4I_1 + 2\left(-\frac{1}{2}I_1 + \frac{1}{4}V_2\right)$$

$$= 3I_1 + \frac{1}{2}V_2 \qquad \ldots(iv)$$

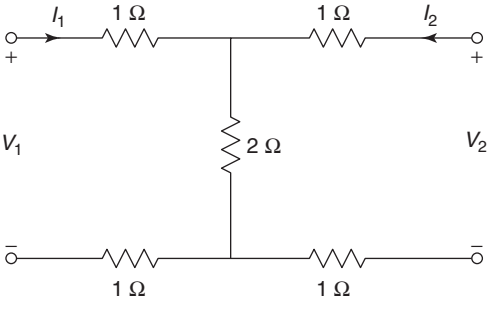


Fig. 10.95

Comparing Eqs (iii) and (iv) with *h*-parameters equations,

$$\begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

For network N_2, *h*-parameters will be same as the two networks are identical.

$$\begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Hence, the overall *h*-parameters of the network are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} + \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

10.9.5 Parallel–Series Connection

Figure 10.96 shows two networks connected in parallel–series. Here the input ports of two networks are connected in parallel and the output ports are connected in series.

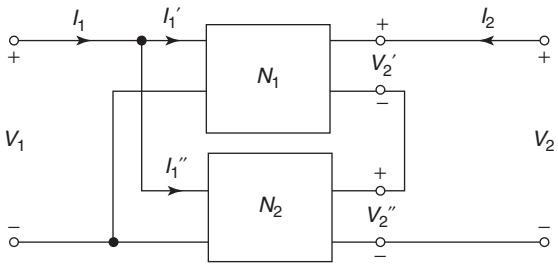


Fig. 10.96 *Parallel–series connection*

Let $g_{11}', g_{12}', g_{21}', g_{22}'$ be the g-parameters of the network N_1 and $g_{11}'', g_{12}'', g_{21}'', g_{22}''$ be the g-parameters of the network N_2,

For the network N_1,

$$\begin{bmatrix} I_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} I_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

For the combined network, $I_1 = I_1' + I_1''$ and $V_2 = V_2' + V_2''$

Hence,

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1' + I_1'' \\ V_2' + V_2'' \end{bmatrix} = \begin{bmatrix} g_{11}' + g_{11}'' & g_{12}' + g_{12}'' \\ g_{21}' + g_{21}'' & g_{22}' + g_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

Thus, the resultant g-parameter matrix is the sum of the g-parameter matrices of each individual two-port network.

Example 10.47 *Determine the g-parameters for the network shown in Fig. 10.97.*

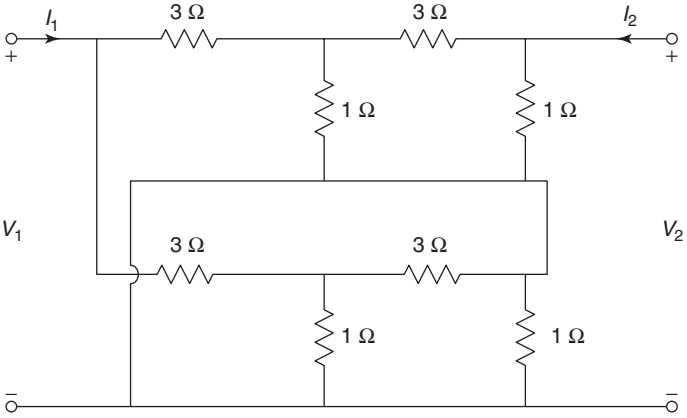


Fig. 10.97

Solution The above network can be considered as a parallel series connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 10.98.

Applying KVL to Mesh 1,

$$V_1 = 4I_1 - I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = I_2 + I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-I_1 + I_2 + 5I_3 = 0$$

$$I_3 = \frac{1}{5}I_1 - \frac{1}{5}I_2 \quad \ldots(iii)$$

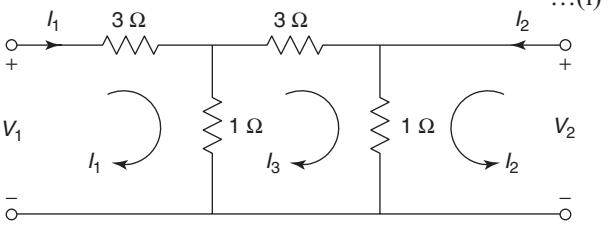


Fig. 10.98

Substituting Eq (iii) in Eq (i),

$$V_1 = 4I_1 - \left(\frac{1}{5}I_1 - \frac{1}{5}I_2\right)$$

$$= \frac{19}{5}I_1 + \frac{1}{5}I_2 \quad \ldots(iv)$$

Substituting Eq (iii) in Eq (ii),

$$V_2 = I_2 + \left(\frac{1}{5}I_1 - \frac{1}{5}I_2\right)$$

$$= \frac{1}{5}I_1 + \frac{4}{5}I_2 \quad \ldots(v)$$

Rewriting Eq (iv),

$$\frac{19}{5}I_1 = V_1 - \frac{1}{5}I_2$$

$$I_1 = \frac{5}{19}V_1 - \frac{1}{19}I_2 \quad \ldots(vi)$$

Substituting Eq (vi) in Eq (v),

$$V_2 = \frac{1}{5}\left(\frac{5}{19}V_1 - \frac{1}{19}I_2\right) + \frac{4}{5}I_2$$

$$= \frac{1}{19}V_1 + \frac{15}{19}I_2 \quad \ldots(vii)$$

Comparing Eqs (vi) and (vii) with *g*-parameters equation,

$$\begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix}$$

For the network N_2, *g*-parameters will be same as the two networks are identical.

$$\begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix}$$

Hence, the overall *g*-parameters of the network are

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} + \begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix} + \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix} = \begin{bmatrix} \frac{10}{19} & -\frac{2}{19} \\ \frac{2}{19} & \frac{30}{19} \end{bmatrix}$$

10.10 *T*-NETWORK

Any two-port network can be represented by an equivalent *T* network as shown in Fig. 10.99.
The elements of the equivalent *T* network may be expressed in terms of *Z*-parameters.
Applying KVL to Mesh 1,

$$V_1 = Z_A I_1 + Z_C(I_1 + I_2)$$
$$= (Z_A + Z_C)I_1 + Z_C I_2 \quad ...(10.9)$$

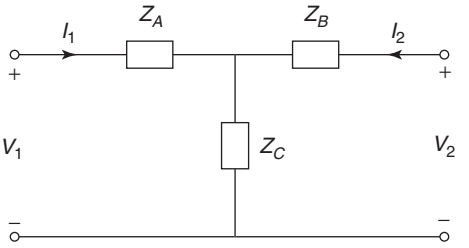


Fig. 10.99 *T-Network*

Applying KVL to Mesh 2,

$$V_2 = Z_B I_2 + Z_C(I_2 + I_1)$$
$$= Z_C I_1 + (Z_B + Z_C)I_2 \quad ...(10.10)$$

Comparing Eqs (10.9) and (10.10) with *Z*-parameter equations,

$$Z_{11} = Z_A + Z_C$$
$$Z_{12} = Z_C$$
$$Z_{21} = Z_C$$
$$Z_{22} = Z_B + Z_C$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12} = Z_{11} - Z_{21}$$
$$Z_B = Z_{22} - ZZ_{21} = Z_{22} - Z_{12}$$
$$Z_C = Z_{12} = Z_{21}$$

10.11 *PI* (π)-NETWORK

Any two-port network can be represented by an equivalent pi (π) network as shown in Fig. 10.100.
Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B(V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \quad ...(10.11)$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B(V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \quad ...(10.12)$$

Comparing Eqs (10.11) and (10.12) with *Y*-parameter equations,

$$Y_{11} = Y_A + Y_B$$
$$Y_{12} = -Y_B$$
$$Y_{21} = -Y_B$$
$$Y_{22} = Y_B + Y_C$$

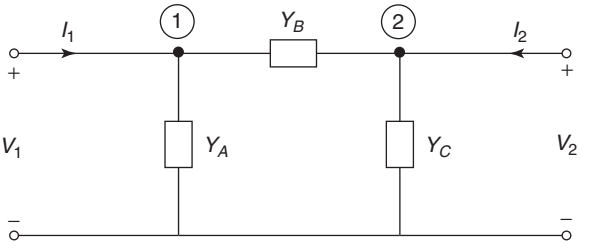


Fig. 10.100 *π-network*

Solving the above equations,

$$Y_A = Y_{11} + Y_{12} = Y_{11} + Y_{21}$$
$$Y_B = -Y_{12} = -Y_{21}$$
$$Y_C = Y_{22} + Y_{12} = Y_{22} + Y_{21}$$

Example 10.48 *The Z-parameters of a two-port network are: $Z_{11} = 10\ \Omega$, $Z_{12} = Z_{21} = 5\ \Omega$, $Z_{22} = 20\ \Omega$. Find the equivalent T-network.*

Solution The *T*-network is shown in Fig. 10.101

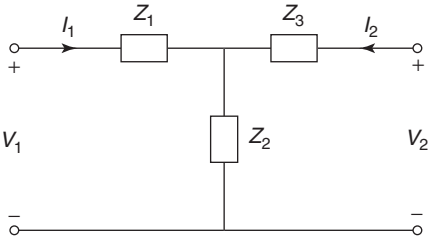


Fig. 10.101

Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)I_1 + Z_2 I_2 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = Z_2 I_1 + (Z_2 + Z_3)I_2 \qquad \ldots(ii)$$

Comparing Eqs (i) and (ii) with Z parameter equations,

$$Z_{11} = Z_1 + Z_2 = 10$$
$$Z_{12} = Z_2 = 5$$
$$Z_{21} = Z_2 = 5$$
$$Z_{22} = Z_2 + Z_3 = 20$$

Solving the above equations,

$$Z_1 = 5\ \Omega$$
$$Z_2 = 5\ \Omega$$
$$Z_3 = 15\ \Omega$$

Example 10.49 *Admittance parameters of a pi network are $Y_{11} = 0.09$ ℧, $Y_{12} = Y_{21} = -0.05$ ℧ and $Y_{22} = 0.07$ ℧. Find the values of R_a, R_b and R_c.*

Solution The pi network is shown in Fig. 10.102.

Applying KCL at Node, 1,

$$I_1 = \frac{V_1}{R_a} + \frac{V_1 - V_2}{R_b}$$
$$= \left(\frac{1}{R_a} + \frac{1}{R_b}\right)V_1 - \frac{1}{R_b}V_2 \qquad \ldots(i)$$

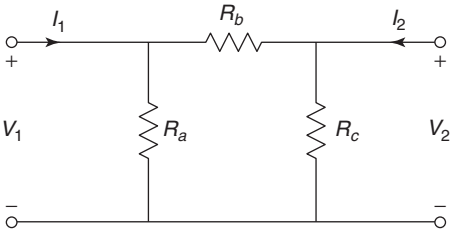


Fig. 10.102

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{R_c} + \frac{V_2 - V_1}{R_b}$$
$$= -\frac{1}{R_b}V_1 + \left(\frac{1}{R_B} + \frac{1}{R_c}\right)V_2 \qquad \ldots(ii)$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$Y_{11} = \frac{1}{R_a} + \frac{1}{R_b} = 0.09$$
$$Y_{12} = -\frac{1}{R_b} = -0.05$$
$$Y_{21} = -\frac{1}{R_b} = -0.05$$
$$Y_{22} = \frac{1}{R_b} + \frac{1}{R_c} = 0.07$$

Solving the above equations,

$$R_a = 25\ \Omega$$
$$R_b = 20\ \Omega$$
$$R_c = 50\ \Omega$$

Example 10.50 *Find the parameters YA, YB and YC of the equivalent p network as shown in Fig. 10.103 to represent a two-terminal pair network for which the following measurements were taken:*

(a) With terminal 2 short-circuited, a voltage of 10 ∠ 0° V applied at terminal pair I resulted in $I_1 = 2.5 \angle 0°$ A and $I_2 = -0.5 \angle 0°$ A.

(b) With terminal 1 short-circuited, the same voltage at terminal pair 2 resulted in $I_2 = 1.5 \angle 0°$ A and $I_1 = -1.1 \angle -20°$ A.

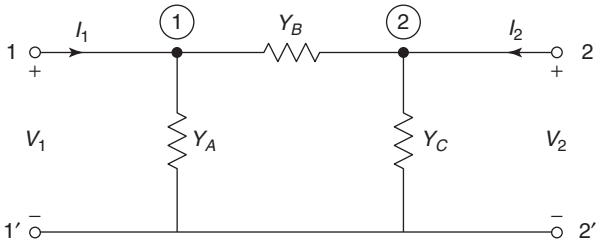


Fig. 10.103

Solution Since measurements were taken with either of the terminal pairs short-circuited, we have to calculate *Y*-parameters first.

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.5\angle 0°}{10\angle 0°} = 0.25\ \text{℧}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-0.5\angle 0°}{10\angle 0°} = -0.05\ \text{℧}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1.5\angle 0°}{10\angle 0°} = 0.15\ \text{℧}$$

Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B(V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \quad \ldots(\text{i})$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B(V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \quad \ldots(\text{ii})$$

Comparing Eqs (i) and (ii) with the *Y*-parameter equation,

$$Y_{11} = Y_A + Y_B = 0.25$$
$$Y_{12} = Y_{21} = -Y_B = -0.05$$
$$Y_{22} = Y_B + Y_C = 0.15$$

Solving the above equation,

$$Y_A = 0.20\ \text{℧}$$
$$Y_B = 0.05\ \text{℧}$$
$$Y_C = 0.10\ \text{℧}$$

Example 10.51 *A network has two input terminals (a, b) and two output terminals (c, d) as shown in Fig. 10.104. The input impedance with c and d open-circuited is (250 + j100) ohms and with c and d short-circuited is (400 + j 300) ohms. The impedance across c and d with a and b open-circuited is 200 ohms. Determine the equivalent T-network parameters.*

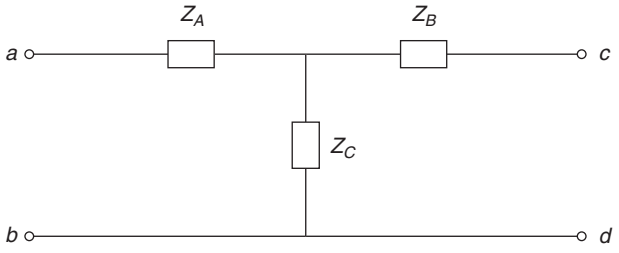


Fig. 10.104

Solution The input impedance with *c* and *d* open-circuited is

$$Z_A + Z_C = 250 + j100 \quad \ldots(i)$$

The input impedance with *c* and *d* short-circuited is,

$$Z_A + \frac{Z_B Z_C}{Z_B + Z_C} = 400 + j300 \quad \ldots(ii)$$

The impedance across *c* and *d* with *a* and *b* open-circuited is

$$Z_B + Z_C = 200 \quad \ldots(iii)$$

Subtracting Eq. (i) from (ii),

$$\frac{Z_B Z_C}{Z_B + Z_C} - Z_C = 150 + j200 \quad \ldots(iv)$$

From Eq. (iii),

$$Z_B = 200 - Z_C \quad \ldots(v)$$

Subtracting the value of Z_B in the equation (iv) and simplifying,

$$Z_C = (100 - j200)\ \Omega \quad \ldots(vi)$$

From Eqs (i) and (vi),

$$Z_A = (150 + j300)\ \Omega$$

From Eqs (iii) and (vi),

$$Z_B = (100 + j200)\ \Omega$$

Example 10.52 *Find the equivalent π-network for the T-network shown in Fig. 10.105.*

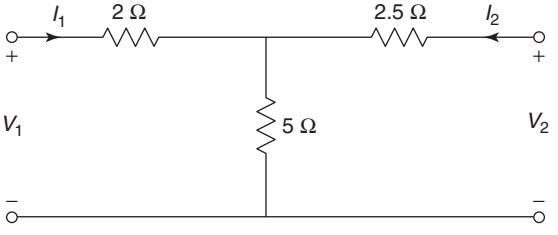


Fig. 10.105

Solution Figure 10.106 shows *T*-network and π-network.

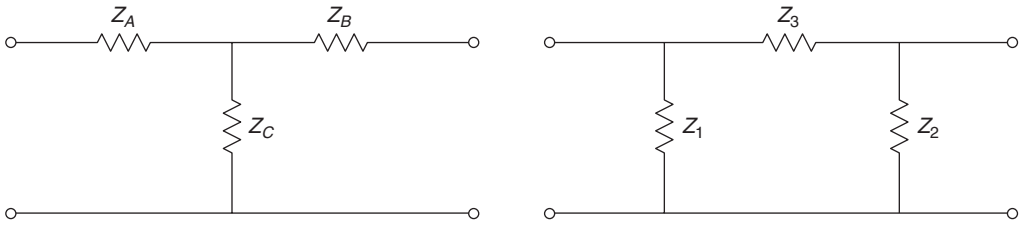


Fig. 10.106

For converting a *T*-network (star network) into an equivalent π-network (delta network), we can use star-delta transformation technique.

$$Z_1 = Z_A + Z_C + \frac{Z_A Z_C}{Z_B} = 2 + 5 + \frac{2 \times 5}{2.5} = 11\,\Omega$$

$$Z_3 = Z_A + Z_B + \frac{Z_A Z_B}{Z_C} = 2 + 2.5 + \frac{2 \times 2.5}{5} = 5.5\,\Omega$$

$$Z_2 = Z_B + Z_C + \frac{Z_B Z_C}{Z_A} = 2.5 + 5 + \frac{2.5 \times 5}{2} = 13.75\,\Omega$$

The equivalent π-network is shown in Fig. 10.107.

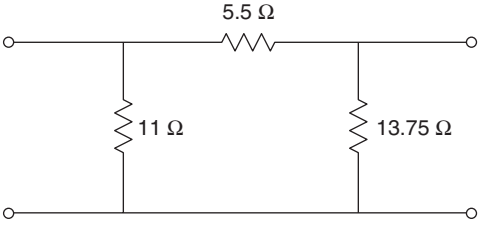


Fig. 10.107

Example 10.53 *For the network shown in Fig. 10.108. Find the equivalent T-network.*

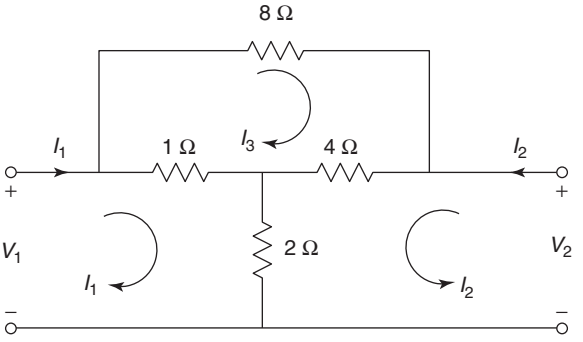


Fig. 10.108

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 - I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 6I_2 + 4I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$13I_3 - I_1 + 4I_2 = 0$$

$$I_3 = \frac{1}{13} I_1 - \frac{4}{13} I_2 \quad \ldots(iii)$$

Substituting the Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \frac{1}{13}I_1 + \frac{4}{13}I_2$$

$$= \frac{38}{13}I_1 + \frac{30}{13}I_2 \qquad \ldots(iv)$$

Substituting the Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 6I_2 + 4\left(\frac{1}{13}I_1 - \frac{4}{13}I_2\right)$$

$$= \frac{30}{13}I_1 + \frac{62}{13}I_2 \qquad \ldots(v)$$

The *T*-network is shown in Fig. 10.109.

Applying KVL to Mesh 1,

$$V_1 = (Z_A + Z_C)I_1 + Z_C I_2 \qquad \ldots(vi)$$

Applying KVL to Mesh 2,

$$V_2 = Z_C I_1 + (Z_B + Z_C)I_2 \qquad \ldots(vii)$$

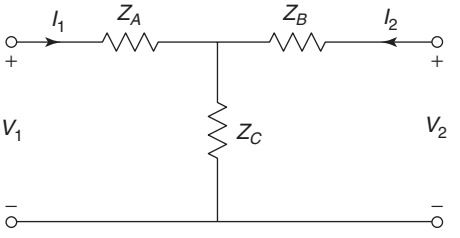


Fig. 10.109

Comparing Eqs (iv) and (v) with Eqs (vi) and (vii),

$$Z_A + Z_C = \frac{38}{13}$$

$$Z_C = \frac{30}{13}$$

$$Z_B + Z_C = \frac{62}{13}$$

Solving the above equations,

$$Z_A = \frac{8}{13}\Omega$$

$$Z_B = \frac{32}{13}\Omega$$

$$Z_C = \frac{30}{13}\Omega$$

10.12 LATTICE NETWORKS

A lattice network is one of the common two-port networks, shown in Fig. 10.110. It is used in filter sections and is also used as attenuator. This network can be represented in terms of *z*-parameters.

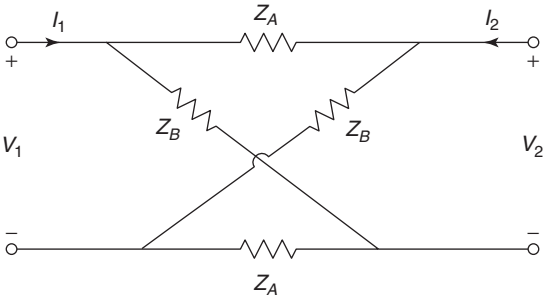


Fig. 10.110 *Lattice network*

The lattice network can be redrawn as a bridge network as shown in Fig. 10.111. This lattice network is symmetric and reciprocal. The current I_1 divides equally between the two arms of the bridge.

When the output port is open-circuited, i.e., $I_2 = 0$

$$V_1 = \frac{I_1}{2}(Z_A + Z_B)$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{Z_A + Z_B}{2}$$

Also

$$V_2 = \frac{I_1}{2}Z_B - \frac{I_1}{2}Z_A = \frac{I_1}{2}(Z_B - Z_A)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{Z_B - Z_A}{2}$$

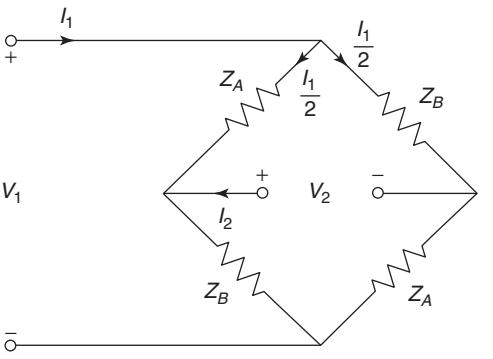


Fig. 10.111 *Bridge network*

Since the network is symmetric,

$$Z_{11} = Z_{22} = \frac{Z_A + Z_B}{2}$$

$$Z_{12} = Z_{21} = \frac{Z_B - Z_A}{2}$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12}$$

$$Z_B = Z_{11} + Z_{12}$$

The lattice network can be represented in terms of other two-port network parameters, with the help of inter-relationship formulae of various parameters.

Example 10.54

Find the lattice equivalent of a symmetrical T network shown in Fig. 10.112.

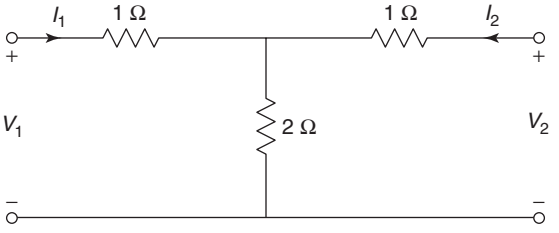


Fig. 10.112

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 3I_2 \quad \ldots(ii)$$

Comparing Eqs (i) and (ii) with *Z*-parameter equations,

$$Z_{11} = 3\ \Omega$$

$$Z_{12} = 2\ \Omega$$

$$Z_{21} = 2\ \Omega$$

$$Z_{22} = 3\ \Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 3 - 2 = 1\,\Omega$$
$$Z_B = Z_{11} + Z_{12} = 3 + 2 = 5\,\Omega$$

The lattice network is shown in Fig. 10.113.

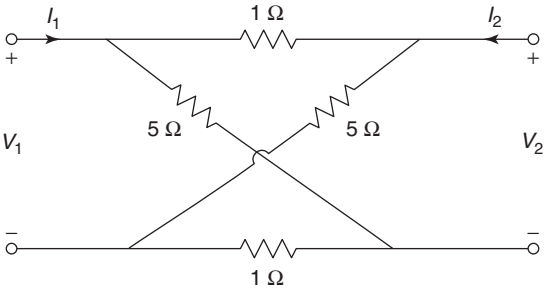


Fig. 10.113

Example 10.55 *Find the lattice equivalent of a symmetric π-network shown in Fig. 10.114.*

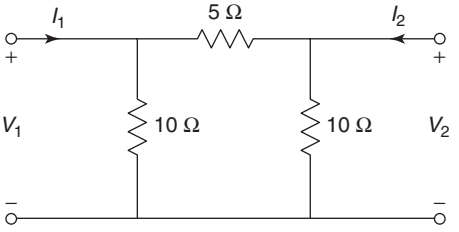


Fig. 10.114

Solution The network is redrawn as shown in Fig. 10.115.

Applying KVL to Mesh 1,

$$V_1 = 10\,I_1 - 10\,I_3 \quad \ldots(i)$$

Applying KVL to Mesh 2,

$$V_2 = 10\,I_2 + 10\,I_3 \quad \ldots(ii)$$

Applying KVL to Mesh 3,

$$-10\,I_1 + 10\,I_2 + 25\,I_3 = 0$$

$$I_3 = \frac{2}{5}I - \frac{2}{5}I_2 \quad \ldots(iii)$$

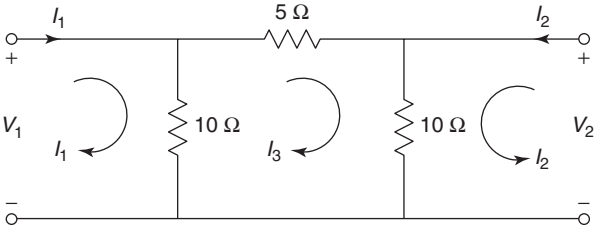


Fig. 10.115

Substituting Eq (iii) in Eq (i),

$$V_1 = 10\,I_1 - 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 6\,I_1 + 4\,I_2 \quad \ldots(iv)$$

Substituting Eq (iii) in Eq (ii),

$$V_2 = 10\,I_2 + 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 4\,I_1 - 6\,I_2 \quad \ldots(v)$$

Comparing the Eqs (iv) and (v) with Z-parameter equations,

$$Z_{11} = 6\ \Omega$$
$$Z_{12} = 4\ \Omega$$
$$Z_{21} = 4\ \Omega$$
$$Z_{22} = 6\ \Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 6 - 4 = 2\ \Omega$$
$$Z_B = Z_{11} + Z_{12} = 6 + 4 = 10\ \Omega$$

The lattice network is shown in Fig. 10.116.

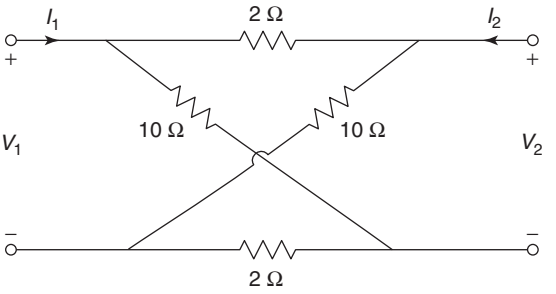


Fig. 10.116

10.13 TERMINATED TWO-PORT NETWORKS

10.13.1 Driving-Point Impedance at Input Port

A two-port network is shown in Fig. 10.117. The output port of the network is terminated in load impedance Z_L. The input impedance of this network can be expressed in terms of parameters of two-port network parameters.

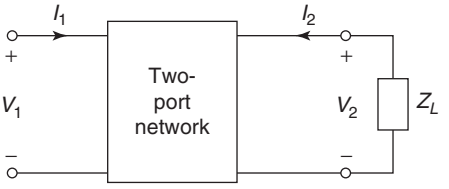


Fig. 10.117 *Terminated two-port network*

1. Input Impedance in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

From Fig. 10.117,

$$V_2 = -Z_L I_2$$
$$-I_2 Z_L = Z_{21} I_1 + Z_{22} I_2$$
$$I_2 = -\frac{Z_{21}}{Z_{22} + Z_L} I_1$$
$$Z_{in} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\left(-\frac{Z_{21}}{Z_{22} + Z_L}\right) = \frac{Z_{11}Z_{22} + Z_{11}Z_L - Z_{12}Z_{21}}{Z_{22} + Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \lim_{Z_L \to \infty} \frac{\frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_L} + Z_{11}}{\frac{Z_{22}}{Z_L} + 1} = Z_{11}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{22}}$$

2. *Input Impedance in Terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

From Fig. 10.117,

$$V_2 = -Z_L I_2$$
$$I_2 = -\frac{V_2}{Z_L} = -Y_L V_2 \qquad \text{where } Y_L = \frac{1}{Z_L}$$
$$-Y_L V_2 = Y_{21}V_1 + Y_{22}V_2$$
$$V_2 = -\frac{Y_{21}}{Y_{22} + Y_L}V_1$$
$$I_1 = Y_{11}V_1 + Y_{12}\left(-\frac{Y_{21}}{Y_{22} + Y_L}\right)V_1 = Y_{11}V_1 - \frac{Y_{21}Y_{12}}{Y_{22} + Y_L}V_1 = \frac{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}{Y_{22} + Y_L}V_1$$
$$Z_{in} = \frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}$$

When output port is open-circuited, i.e., $Y_L = 0$

$$Z_{in} = \frac{Y_{22}}{Y_{11}Y_{22} - Y_{12}Y_{21}}$$

When output port is short-circuited, i.e., $Y_L = \infty$,

$$Z_{in} = \lim_{Y_L \to \infty} \frac{\frac{Y_{22}}{Y_L} + 1}{\frac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_L} + Y_{11}} = \frac{1}{Y_{11}}$$

3. *Input Impedance in Terms of Transmission Parameters* We know that

$$V_1 = AV_2 - BV_2$$
$$I_1 = CV_2 - DI_2$$

From Fig. 10.117,

$$V_2 = -Z_L I_2$$
$$I_1 = -CZ_L I_2 - DI_2 = -(CZ_L + D)I_2$$
$$I_2 = -\frac{I_1}{CZ_L + D}$$
$$V_1 = AZ_L I_2 - B = \left(-\frac{I_1}{CZ_L + D}\right) = \left(\frac{AZ_L + B}{CZ_L + D}\right)I_1$$
$$Z_{in} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \frac{A}{C}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = \frac{B}{D}$$

4. Input Impedance in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$
$$V_2 = -Z_L I_2$$
$$I_2 = h_{21}I_1 - h_{22}Z_L I_2$$
$$I_2 = \frac{h_{21}}{1 + h_{22}Z_L} I_L$$
$$V_2 = -\frac{h_{21}Z_L}{1 + h_{22}Z_L} I_1$$

Substituting the value of V_2 in V_1,

$$V_1 = h_{11}I_1 + h_{12}\left[\frac{-h_{21}Z_L}{1 + h_{22}Z_L} I_L\right] = \left[\frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1 + h_{22}Z_L}\right] I_1$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1 + h_{22}Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = h_{11}$$

10.13.2 Driving-Point Impedance at Output Port

A two-port network is shown in Fig. 10.118. The input port is terminated in load impedance Z_L. The output impedance of this network can be expressed in terms of two port network parameters.

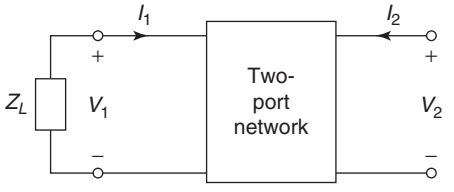


Fig. 10.118 *Terminated two-port network*

1. Output Impedance in terms of Z-parameters We know that

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

From Fig. 10.118,

$$V_1 = -Z_L I_1$$

$$-I_1 Z_1 = Z_{11} I_1 + Z_{12} I_2$$

$$I_1 = \left(-\frac{Z_{12}}{Z_L + Z_{11}} \right) I_2$$

$$V_2 = Z_{21} \left(-\frac{Z_{12}}{Z_L + Z_{11}} \right) I_2 + Z_{22} I_2 = I_2 \left(Z_{22} - \frac{Z_{21} Z_{12}}{Z_L + Z_{11}} \right) = \left(\frac{Z_{11} Z_{22} - Z_{12} Z_{21} + Z_{22} Z_L}{Z_{11} + Z_L} \right) I_2$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Z_{11} Z_{22} - Z_{12} Z_{21} + Z_{22} Z_L}{Z_{11} + Z_L}$$

If the input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = Z_{22}$$

If the input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{11}}$$

2. *Output Impedance in Terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

From Fig. 10.118,

$$V_1 = -Z_L I_1$$

$$I_1 = -\frac{V_1}{Z_L} = -Y_L V_1$$

$$-Y_L V_1 = Y_{11} V_1 + Y_{12} V_2$$

$$V_1 = \left(-\frac{Y_{12}}{Y_L + Y_{11}} \right) V_2$$

$$I_2 = Y_{21} \left(-\frac{Y_{12}}{Y_L + Y_{11}} \right) V_2 + Y_{22} V_2 = V_2 \left[Y_{22} - \frac{Y_{21} Y_{12}}{Y_L + Y_{11}} \right] = V_2 \left[\frac{Y_{11} Y_{22} - Y_{12} Y_{21} + Y_L Y_{22}}{Y_L + Y_{11}} \right]$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Y_L + Y_{11}}{Y_{11} Y_{22} - Y_{12} Y_{21} + Y_L Y_{22}}$$

If input port is open-circuited, i.e., $Y_L = 0$,

$$Z_0 = \frac{Y_{11}}{Y_{11} Y_{22} - Y_{12} Y_{21}}$$

If input port is short-circuited, i.e., $Y_L = \infty$,

$$Z_0 = \frac{1}{Y_{22}}$$

3. *Output Impedance in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

From Fig. 10.118,

$$V_1 = -Z_L I_1$$
$$\frac{V_1}{I_1} = -Z_L = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$
$$V_2(CZ_L + A) = I_2(DZ_L + B)$$
$$Z_0 = \frac{V_2}{I_2} = \frac{DZ_L + B}{CZ_L + A}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{D}{C}$$

If input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{B}{A}$$

4. *Output Impedance in Terms of h-parameters* We know that

$$V_1 = h_{11} I_1 + h_{12} V_{12}$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

From Fig. 10.118,

$$V_1 = -Z_L I_1$$
$$-I_1 Z_L = h_{11} I_1 + h_{22} V_2$$
$$I_1 = \left(-\frac{h_{12}}{h_{11} + Z_L} \right) V_2$$
$$I_2 = h_{21} \left(-\frac{h_{12}}{h_{11} + Z_L} \right) V_2 + h_{22} V_2 = V_2 \left[\frac{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}{h_{11} + Z_L} \right]$$
$$Z_0 = \frac{V_2}{I_2} = \frac{h_{11} + Z_L}{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{1}{h_{22}}$$

If input port is short-circuited i.e., $Z_L = 0$,

$$Z_0 = \frac{h_{11}}{h_{11}h_{22} - h_{12}h_{21}}$$

Example 10.56 *Measurements were made on a two-terminal network shown in Fig. 10.119.*

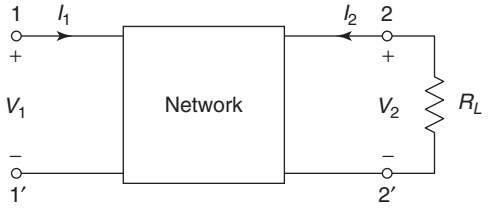


Fig. 10.119

(a) With terminal pair 2 open, a voltage of 100 ∠0° V applied to terminal pair 1 resulted in

$$I_1 = 10\ \angle 0^\circ\ A, \quad V_2 = 25\ \angle 0^\circ\ V$$

(b) With terminal pair 1 open, the same voltage applied to terminal pair 2 resulted in

$$I_2 = 20\ \angle 0^\circ\ A, \quad V_1 = 50\ \angle 0^\circ\ V$$

Write mesh equations for this network. What will be the voltage across a 10-Ω resistor connected across Terminal pair 2 if a 100 ∠0° V is connected across terminal pair 1?

Solution Since measurements were done with either of the terminal pairs open-circuited, we have to calculate *Z*-parameters first.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{100\angle 0^\circ}{10\angle 0^\circ} = 10\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{25\angle 0^\circ}{10\angle 0^\circ} = 2.5\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{100\angle 0^\circ}{20\angle 0^\circ} = 5\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{50\angle 0^\circ}{20\angle 0^\circ} = 2.5\ \Omega$$

Putting these values in *Z*-parameter equations,

$$V_1 = 10I_1 + 2.5I_2 \qquad \ldots(i)$$

$$V_2 = 2.5I_1 + 5I_2 \qquad \ldots(ii)$$

When a 10-Ω resistor is connected across terminal pair 1,

$$V_1 = 100\ \angle 0^\circ\ V$$

$$V_2 = -R_L I_2 = -10I_2$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$100 = 10I_1 + 2.5I_2$$

and

$$-10I_2 = 2.5I_1 + 5I_2$$

$$2.5I_1 = -15I_2$$

$$I_1 = -6I_2$$

$$100 = -60I_2 + 2.5I_2$$

$$I_2 = -\frac{100}{57.5} = -1.74\ \text{A}$$

Voltage across the resistor $= -I_2 R_L = -10(-1.74) = 17.4$ V

Example 10.57 *The Z-parameters of a two-port network shown in Fig. 10.120 are $Z_{11} = Z_{22} = 10\ \Omega$, $Z_{21} = Z_{22} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, I_2, V_2 and input impedance.*

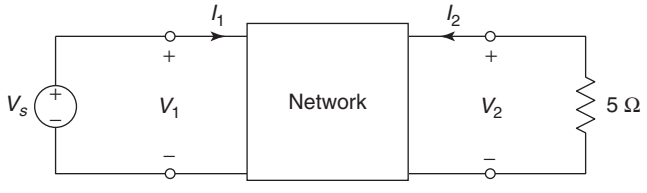


Fig. 10.120

Solution

$$V_1 = V_s = 20 \text{ V}$$
$$V_2 = -20I_2$$

The two-port network can be represented in terms of *Z*-parameters.

$$V_1 = 10I_1 + 4I_2 \quad \ldots(\text{i})$$
$$V_2 = 4I_1 + 10I_2 \quad \ldots(\text{ii})$$
$$-20I_2 = 4I_1 + 10I_2$$
$$4I_1 = -30I_2$$
$$I_1 = -7.5I_2$$

Substituting the value of I_1 in Eq. (i),

$$V_1 = 10(-7.5I_2) + 4I_2 = -71I_2$$
$$20 = -71I_2$$
$$I_2 = -0.28 \text{ A}$$
$$I_1 = -7.5(-0.28) = 2.1 \text{ A}$$
$$V_2 = -20(-0.28) = 56 \text{ V}$$

Input impedance $Z_i = \dfrac{V_1}{I_1} = \dfrac{20}{2.1} = 9.52\ \Omega$

Example 10.58 *The Z-parameters of a two-port network shown in Fig. 10.121 are, Z_{11} = 2 Ω, Z_{12} = 1 Ω, Z_{21} = 2 Ω, Z_{22} = 5 Ω. Calculate the voltage ratio $\dfrac{V_2}{V_s}$, current ratio $-\dfrac{I_2}{I_1}$ and input impedance $\dfrac{V_1}{I_1}$.*

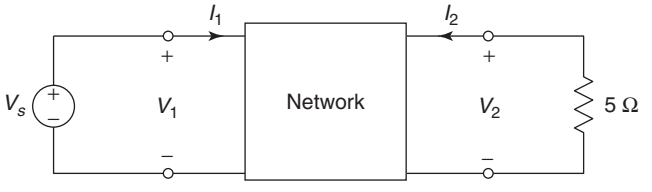


Fig. 10.121

Solution The two-port network can be represented in terms of *Z*-parameters.

$$V_1 = 2I_1 + I_2 \quad \ldots(\text{i})$$
$$V_2 = 2I_1 + 5I_2 \quad \ldots(\text{ii})$$

When the 5 Ω resistor is connected across port-2,

$$V_2 = -5I_2 \quad \ldots(\text{iii})$$

Applying KVL to the input port,

$$V_s - 1I_1 - V_1 = 0$$
$$V_1 = V_s + I_1 \quad \ldots(\text{iv})$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$V_s - I_1 = 2I_1 + I_2$$
$$V_s = 3I_1 + I_2 \quad \ldots(\text{v})$$

and
$$-5I_2 = 2I_1 + 5I_2$$
$$0 = 2I_1 + 10I_2 \quad \ldots(vi)$$

Solving Eqs (v) and (vi), we get

$$I_1 = \frac{\begin{vmatrix} V_s & 1 \\ 0 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = \frac{5}{14}V_s$$

$$I_2 = \frac{\begin{vmatrix} 3 & V_s \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = -\frac{1}{14}V_s$$

$$-\frac{I_2}{I_1} = \frac{1}{5}$$

$$V_2 = 2I_1 + 5I_2 = 2\left(\frac{5}{14}V_s\right) + 5\left(-\frac{1}{14}V_s\right) = \frac{5}{14}V_s$$

$$\frac{V_2}{V_s} = \frac{5}{14}$$

$$V_1 = 2I_1 + I_2 = 2\left(\frac{5}{14}V_s\right) - \frac{1}{14}V_s = \frac{9}{14}V_s$$

$$\frac{V_1}{I_1} = \frac{9}{5}\Omega$$

Example 10.59 *The following equations give the voltages V_1 and V_2 at the two ports of a two-port network shown in Fig. 10.122.*

$$V_1 = 5I_1 + 2I_2$$
$$V_2 = 2I_1 + I_2$$

A load resistor of 3 Ω is connected across port 2. Calculate the input impedance.

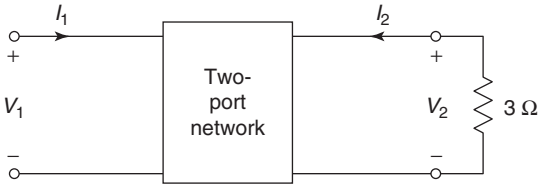


Fig. 10.122

Solution From Fig. 10.122,

$$V_2 = -3I_2 \quad \ldots(i)$$

Substituting Eq. (i) in the given equation,

$$-3I_2 = 2I_1 + I_2$$
$$I_2 = -\frac{I_1}{2} \quad (ii)$$

Substituting the Eq. (ii) in the given equation.

$$V_1 = 5I_1 - I_1 = 4I_1$$

Input impedance
$$Z_i = \frac{V_1}{I_1} = 4\ \Omega$$

Example 10.60 *The y-parameters for a two-port network shown in Fig. 10.123 are given as, $Y_{11} = 4$ ℧, $Y_{22} = 5$ ℧, $Y_{12} = Y_{21} = 4$ ℧. If a resistor of 1 Ω is connected across port-1 of the network then find the output impedance.*

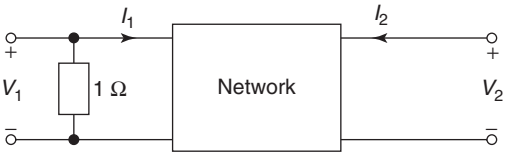


Fig. 10.123

Solution The two-port network can be represented in terms of *Y*-parameters.

$$I_1 = 4V_1 + 4V_2 \quad \ldots(i)$$

$$I_2 = 4V_1 + 5V_2 \quad \ldots(ii)$$

When the 1-Ω resistor is connected across port-1 of the network,

$$V_1 = -1I_1 = -I_1$$

$$I_1 = -V_1$$

Substituting value of I_1 in Eq (i),

$$-V_1 = 4V_1 + 4V_2$$

$$-5V_1 = 4V_2$$

$$V_1 = -\frac{4}{5}V_2$$

Substituting value of V_1 in Eq (ii),

$$I_2 = 4\left(-\frac{4}{5}V_2\right) + 5V_2 = \frac{9}{5}V_2$$

Output impedance $Z_0 = \dfrac{V_2}{I_2} = \dfrac{5}{9}\,\Omega$

Example 10.61 *The following equation gives the voltage and current at the input port of a two-port network shown in Fig. 10.124.*

$$V_1 = 5V_2 - 3I_2$$

$$I_1 = 6V_2 - 2I_2$$

A load resistance of 5 Ω is connected across the output port. Calculate the input impedance.

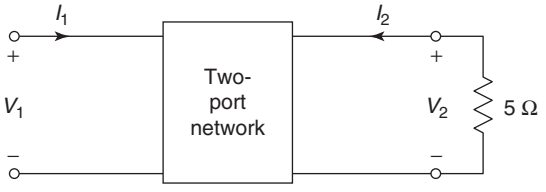


Fig. 10.124

Solution From Fig. 10.124,

$$V_2 = -5I_2$$

Substituting the value of V_2 in the given equations,

$$V_1 = 5(-5I_2) - 3I_2 = -28I_2$$

$$I_1 = 6(-5I_2) - 2I_2 = -32I_2$$

Input impedance $Z_i = \dfrac{V_1}{I_1} = \dfrac{-28I_2}{-32I_2} = \dfrac{7}{8}\,\Omega$

Example 10.62 *The ABCD parameters of a two-port network shown in Fig. 10.125 are A = 2.5, B = 4 Ω, C = 1 ℧, D = 2. What must be the input voltage V_1 applied for the output voltage V_2 to be 10 V across the load of 10 Ω connected at Port 2?*

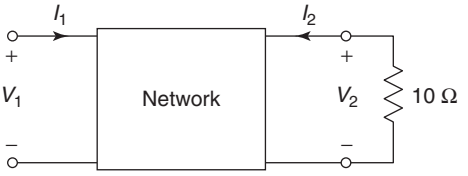


Fig. 10.125

Solution The two-port network can be represented in terms of *ABCD* parameters.

$$V_1 = 2.5\,V_2 - 4I_2 \quad \ldots(i)$$

$$I_1 = V_2 - 2I_2 \quad \ldots(ii)$$

When the 10 Ω resistor is connected across Port 2,

$$V_2 = -10I_2 = 10 \quad \ldots(iii)$$

$$I_2 = -1\text{A}$$

$$V_1 = 2.5(10) - 4(-1) = 29\text{ V}$$

Example 10.63 *The h-parameters of a two-port network shown in Fig. 10.126 are h_{11} = 4 Ω, h_{12} = 1, h_{21} = 1, h_{22} = 0.5 ℧. Calculate the output voltage V_2 when the output port is terminated in a 3 Ω resistance and a 1 V is applied at the input port.*

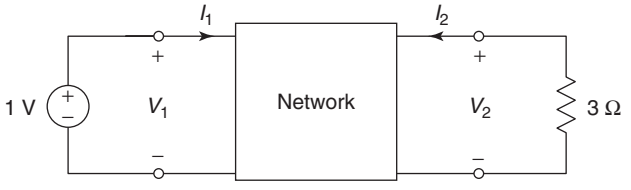


Fig. 10.126

Solution

$$V_1 = 1\text{ V}$$

$$V_2 = -3\,I_2$$

The two-port network can be represented in terms of *h*-parameters.

$$V_1 = 4I_1 + V_2 \quad \ldots(i)$$

$$I_2 = I_1 + 0.5\,V_2 \quad \ldots(ii)$$

$$I_2 = I_1 + 0.5(-3I_2)$$

$$2.5\,I_2 = I_1$$

Substituting the value of V_1 and I_1 in Eq (i),

$$1 = 4(2.5\,I_2) - 3\,I_2$$

$$1 = 7I_2$$

$$I_2 = \frac{1}{7}\text{A}$$

$$V_2 = -3\left(\frac{1}{7}\right) = -\frac{3}{7}\text{V}$$

Example 10.64 *The h-parameters of a two-port network shown in Fig. 10.127 are h_{11} = 1 Ω, $h_{12} = -h_{21} = 2$, h_{22} = 1 ℧. The power absorbed by a load resistance of 1 Ω connected across port-2 is 100W. The network is excited by a voltage source of generated voltage V_s and internal resistance 2 Ω. Calculate the value of V_s.*

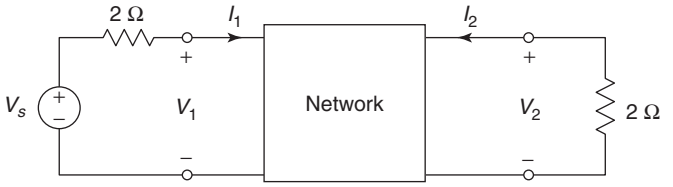


Fig. 10.127

Solution The two-port network can be represented in terms of *h*-parameters.

$$V_1 = I_1 + 2V_2 \quad \ldots(i)$$

$$I_2 = -2I_1 + V_2 \quad \ldots(ii)$$

When the 1 Ω resister is connected across port-2,

$$\frac{V_2^2}{1} = 100$$

$$V_2 = 10 \text{ V}$$

$$V_2 = -1I_2 = 10$$

$$I_2 = -10 \text{ A}$$

Substituting values of I_2 and V_2 in Eq (ii),

$$-10 = -2I_1 + 10$$

$$I_1 = 10 \text{ A}$$

Applying KVL to the input port,

$$V_s - 2I_1 - V_1 = 0$$

$$V_s - 2I_1 - (I_1 + 2V_2) = 0$$

$$V_s - 3I_1 - 2V_2 = 0$$

$$V_s = 3I_1 + 2V_2 = 3(10) + 2(10) = 50 \text{ V}$$

Exercises

10.1 Determine *Z*-parameters for the network shown in Fig. 10.128.

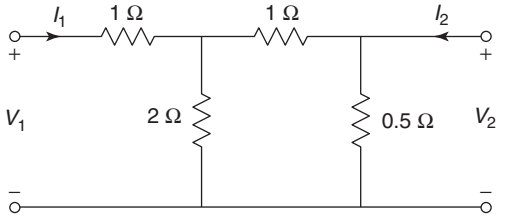


Fig. 10.128

$$Z = \begin{bmatrix} \frac{13}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} \end{bmatrix}$$

10.2 Find *Z*-parameters for the network shown in Fig. 10.129.

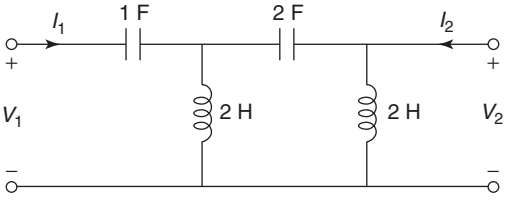


Fig. 10.129

$$Z = \begin{bmatrix} \frac{4s^4 + 6s^2 + 1}{4s^3 + s} & \frac{4s^3}{4s^2 + 1} \\ \frac{4s^3}{4s^2 + 1} & \frac{4s^3 + 2s}{4s^2 + 1} \end{bmatrix}$$

10.3 Find *Y*-parameters of the network shown in Fig. 10.130.

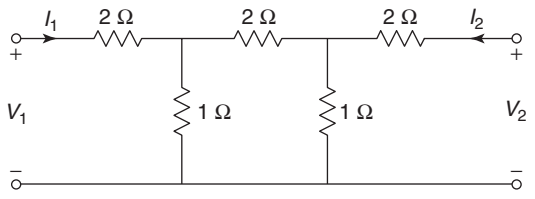


Fig. 10.130

$$Y = \begin{bmatrix} 0.36 & -0.033 \\ -0.033 & -0.36 \end{bmatrix}$$

10.4 Find *Y*-parameters for the network shown in Fig. 10.131.

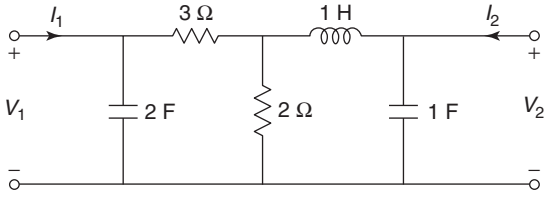


Fig. 10.131

$$Y = \begin{bmatrix} \dfrac{10s^2+13s+2}{5s+6} & -\dfrac{2}{5s+6} \\ -\dfrac{2}{5s+6} & \dfrac{5s^2+6s+5}{5s+6} \end{bmatrix}$$

10.5 Find *Y*-parameters for the network shown in Fig. 10.132.

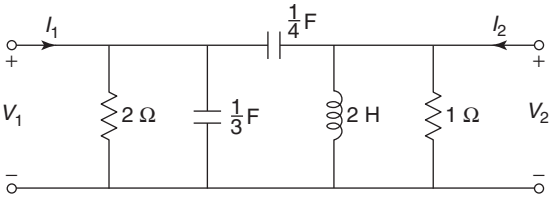


Fig. 10.132

$$Y = \begin{bmatrix} \dfrac{7s+6}{12} & -\dfrac{s}{4} \\ -\dfrac{s}{4} & \dfrac{s^2+4s+2}{4s} \end{bmatrix}$$

10.6 Find *Y*-parameters for the network shown in Fig. 10.133.

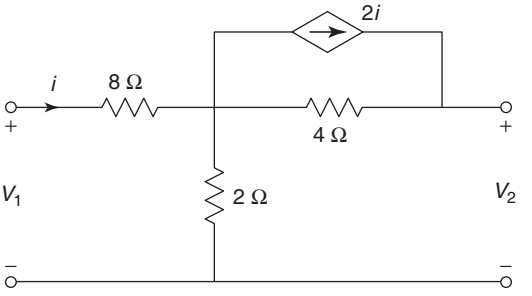


Fig. 10.133

$$Y = \begin{bmatrix} \dfrac{3}{20} & -\dfrac{1}{20} \\ -\dfrac{1}{4} & \dfrac{1}{4} \end{bmatrix}$$

10.7 Show the *ABCD* parameters of the network shown in Fig. 10.134.

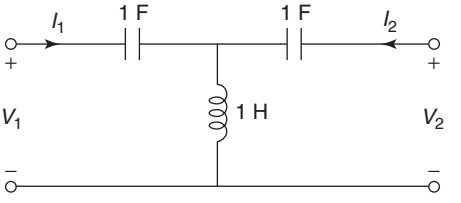


Fig. 10.134

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{1+s^2}{s^2} & \dfrac{1+2s^2}{s^3} \\ \dfrac{1}{s} & \dfrac{1+s^2}{s^2} \end{bmatrix}$$

10.8 Find *ABCD* parameters for the network shown in Fig. 10.135.

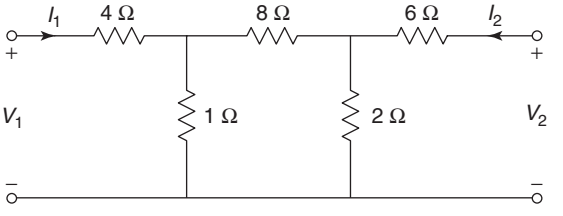


Fig. 10.135

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 27 & 206 \\ \dfrac{11}{2} & 42 \end{bmatrix}$$

10.9 For the network shown in Fig. 10.136, determine parameter h_{21}.

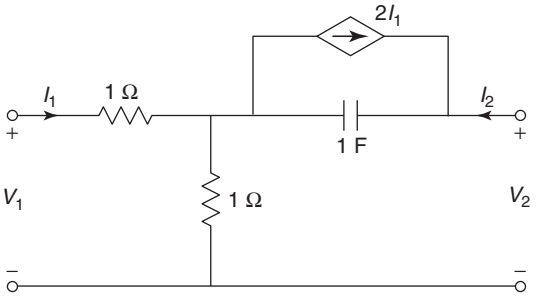


Fig. 10.136

$$\left[h_{21} = \frac{-(2+s)}{1+s}\right]$$

10.10 Determine Y and Z-parameters for the network shown in Fig. 10.137.

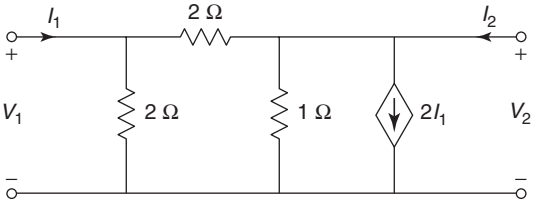


Fig. 10.137

$$\left[Y_{11} = 1\,\text{℧}, Y_{12} = -0.5\,\text{℧}, Y_{21} = 1.5\,\text{℧}, Y_{22} = 0.5\,\text{℧}\right.$$
$$\left. Z_{11} = \frac{2}{5}\Omega, Z_{12} = \frac{2}{5}\Omega, Z_{21} = -\frac{6}{5}\Omega, Z_{22} = \frac{4}{5}\Omega\right]$$

10.11 For the bridged T, R-C network shown in Fig. 10.138 determine Y-parameters using interconnections of two-port networks.

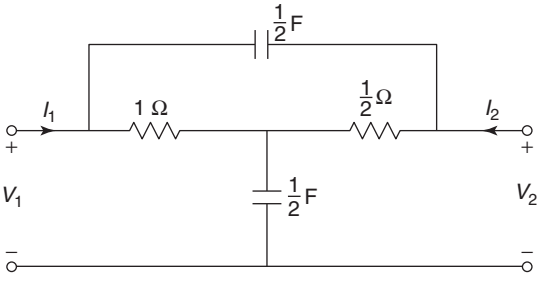


Fig. 10.138

$$Y = \begin{bmatrix} \dfrac{s^2+8s+8}{2(s+6)} & -\dfrac{(s^2+6s+8)}{2(s+6)} \\ -\dfrac{(s^2+6s+8)}{2(s+6)} & \dfrac{s^2+10s+8}{2(s+6)} \end{bmatrix}$$

10.12 For the network of Fig. 10.139, find Y-parameters using interconnection of two-port networks.

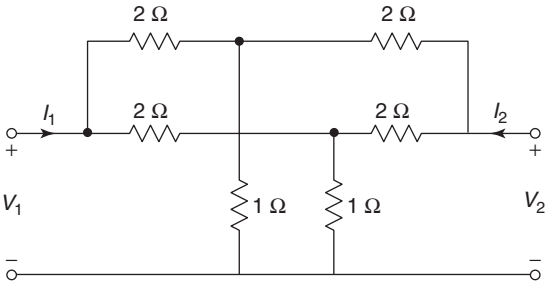


Fig. 10.139

$$Y = \begin{bmatrix} \dfrac{3}{4} & -\dfrac{1}{4} \\ -\dfrac{1}{4} & \dfrac{3}{4} \end{bmatrix}$$

10.13 Two identical sections of the network shown in Fig. 10.140 are connected in parallel. Obtain Y-parameters of the connection.

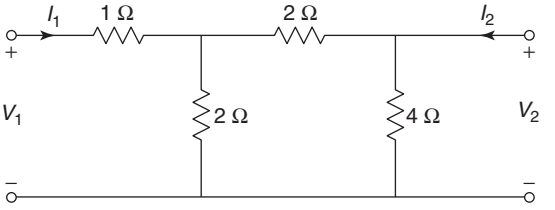


Fig. 10.140

$$Y = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{5}{4} \end{bmatrix}$$

10.14 Determine Y-parameters using interconnection of two-port networks for the network shown in Fig. 10.141.

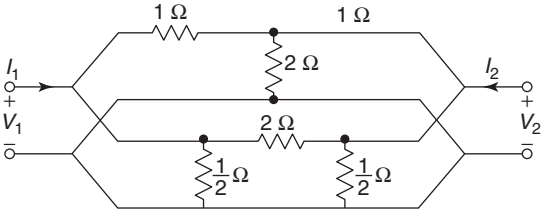


Fig. 10.141

$$Y = \begin{bmatrix} 3.1 & -0.9 \\ -0.9 & 3.1 \end{bmatrix}$$

10.15 Determine the transmission parameters of the network shown in Fig. 10.142 using the concept of interconnection of two two-port networks.

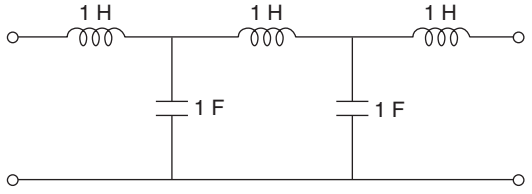


Fig. 10.142

$$\begin{bmatrix} 1+3s^2+s^4 & 3s+4s^3+s^5 \\ 2s+s^3 & 1+3s^2+s^4 \end{bmatrix}$$

10.16 Two networks shown in Fig. 10.143 are connected in series. Obtain the *Z*-parameters of the resulting network.

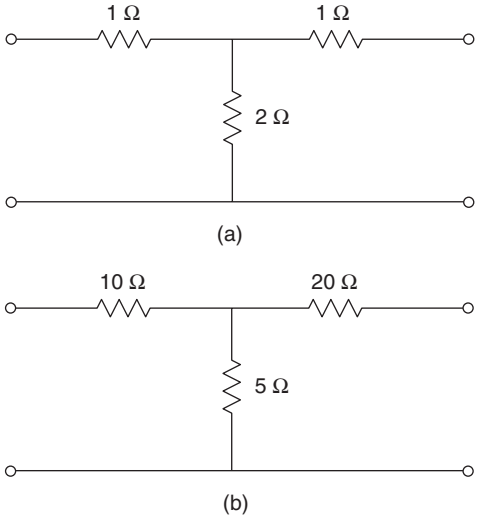


Fig. 10.143

$$\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$$

10.17 Two identical sections of the network shown in Fig. 10.144 are connected in series-parallel. Determine the *h*-parameters of the overall network.

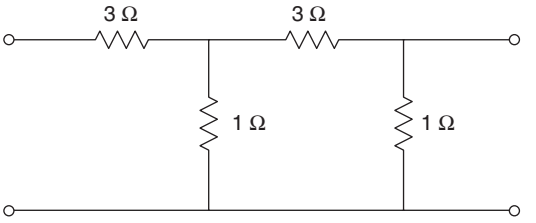


Fig. 10.144

$$\begin{bmatrix} \frac{15}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

10.18 The *h*-parameters of a two-port network shown in Fig. 10.145 are $h_{11} = 2\ \Omega$, $h_{12} = 4$, $h_{21} = -5$, $h_{22} = 2\ \mho$. Determine the supply voltage V_s if the power dissipated in the load resistor of 4 Ω is 25 W and $R_s = 2\ \Omega$.

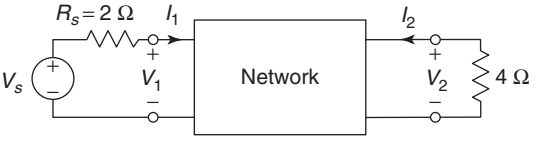


Fig. 10.145

[58 V]

10.19 The *Z*-parameters of a two-port network are $Z_{11} = 2.1\ \Omega$, $Z_{12} = Z_{21} = 0.6\ \Omega$, $Z_{22} = 1.6\ \Omega$. A resistor of 2 Ω is connected across port 2. What voltage must be applied at port 1 to produce a current of 0.5 A in the 2 Ω resistor.

[6 V]

10.20 If a two-port network has $Z_{11} = 25\ \Omega$, $Z_{12} = Z_{21} = 20\ \Omega$, $Z_{22} = 50\ \Omega$, find the equivalent T-network.

[10 Ω, 30 Ω, 20 Ω]

Objective-Type Questions

10.1 The open-circuit impedance matrix of the two-port network shown in Fig. 10.146 is

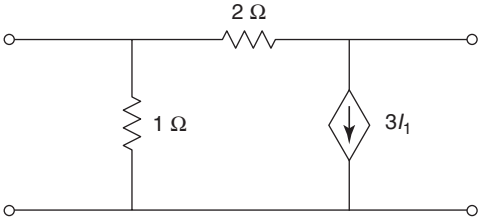


Fig. 10.146

(a) $\begin{bmatrix} -2 & 1 \\ -8 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -8 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

10.2 Two two-port networks are connected in cascade. The combination is to be represented as a single two-port network. The parameters are obtained by multiplying the individual

(a) z-parameter matrix
(b) h-parameter matrix
(c) y-parameter matrix
(d) $ABCD$ parameter matrix

10.3 For a two-port network to be reciprocal

(a) $z_{11} = z_{22}$ (b) $y_{21} = y_{12}$
(c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

10.4 The short-circuit admittance matrix of a two-port network is $\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. The two-port network is

(a) non-reciprocal and passive
(b) non-reciprocal and active
(c) reciprocal and passive
(d) reciprocal and active

10.5 A two-port network is shown in Fig. 10.147. The parameter h_{21} for this network can be given by

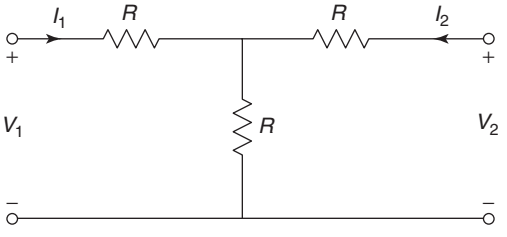


Fig. 10.147

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$
(c) $-\frac{3}{2}$ (d) $\frac{3}{2}$

10.6 The admittance parameter Y_{12} in the two-port network in Fig. 10.148.

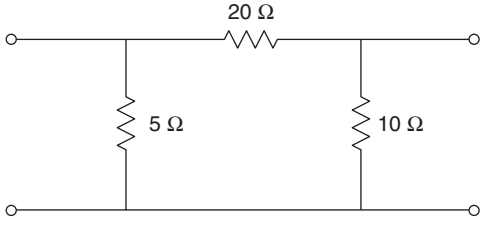


Fig. 10.148

(a) –0.2 mho (b) 0.1 mho
(c) –0.05 mho (d) 0.05 mho

10.7 The Z-parameters Z_{11} and Z_{21} for the two-port network in Fig. 10.149 are,

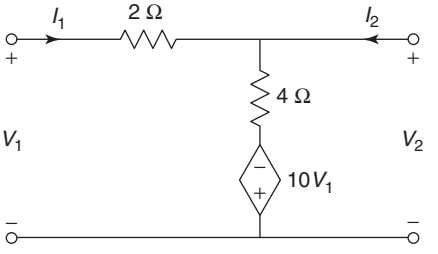


Fig. 10.149

(a) $-\frac{6}{11}\Omega, \frac{16}{11}\Omega$ (b) $\frac{6}{11}\Omega, \frac{4}{11}\Omega$
(c) $\frac{6}{11}\Omega, -\frac{16}{11}\Omega$ (d) $\frac{4}{11}\Omega, \frac{4}{11}\Omega$

10.8 The impedance parameters Z_{11} and Z_{12} of a two-port network in Fig. 10.150.

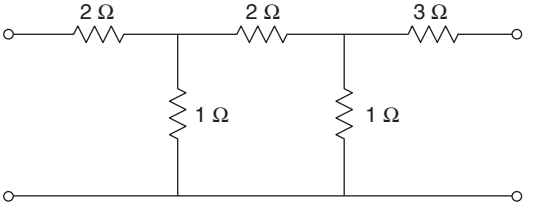


Fig. 10.150

(a) 2.75 Ω, 0.25 Ω (b) 3 Ω, 0.5 Ω
(c) 3 Ω, 0.25 Ω (d) 2.25 Ω, 0.5 Ω

10.9 The h parameters of the circuit shown in Fig. 10.151.

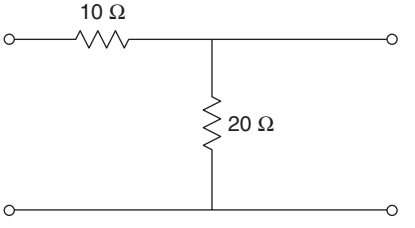


Fig. 10.151

(a) $\begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & -1 \\ 1 & 0.05 \end{bmatrix}$
(c) $\begin{bmatrix} 30 & 20 \\ 20 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 1 \\ -1 & 0.05 \end{bmatrix}$

10.10 A two-port network is represented by *ABCD* parameters given by $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$. If port 2 is terminated by R_L, then the input impedance seen at port 1 is given by

(a) $\dfrac{A + BR_L}{C + DR_L}$ (b) $\dfrac{AR_L + C}{BR_L + D}$

(c) $\dfrac{DR_L + A}{BR_L + C}$ (d) $\dfrac{B + AR_L}{D + CR_L}$

10.11 In the two-port network shown in Fig. 10.152, Z_{12} and Z_{21} are respectively

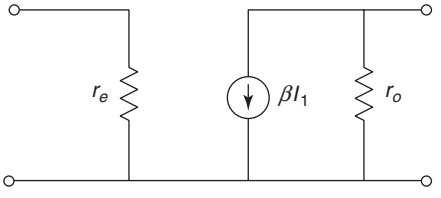


Fig. 10.152

(a) r_e and βr_0 (b) 0 and $-\beta r_0$

(c) 0 and βr_0 (d) r_e and $-\beta r_0$

10.12 If a two-port network is passive, then we have, with the usual notation, the following relationship for symmetrical network

(a) $h_{12} = h_{21}$
(b) $h_{12} = -h_{21}$
(c) $h_{11} = h_{22}$
(d) $h_{11}h_{22} - h_{12}h_{21} = 1$

10.13 A two-port network is defined by the following pair of equations $I_1 = 2V_1 + V_2$ and $I_2 = V_1 + V_2$. Its impedance parameters (Z_{11}, Z_{12}, Z_{21}, Z_{22}) are given by

(a) 2, 1, 1, 1 (b) 1, –1, –1, 2

(c) 1, 1, 1, 2 (d) 2, –1, –1, 1

10.14 A two-port network has transmission parameters $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The input impedance of the network at port 1 will be

(a) $\dfrac{A}{C}$ (b) $\dfrac{AD}{BC}$

(c) $\dfrac{AB}{DC}$ (d) $\dfrac{D}{C}$

10.15 A two-port network is symmetrical if

(a) $Z_{11}Z_{22} - Z_{12}Z_{21} = 1$
(b) $AD - BC = 1$
(c) $h_{11}h_{22} - h_{12}h_{21} = 1$
(d) $Y_{11}Y_{22} - Y_{12}Y_{21} = 1$

10.16 For the network shown in Fig. 10.153 admittance parameters are $Y_{11} = 8$ mho, $Y_{12} = Y_{21} = -6$ mho and $Y_{22} = 6$ mho. The value of Y_A, Y_B and Y_C (in mho) will be respectively

(a) 2, 6, –6 (b) 2,6,0

(c) 2,0,6 (d) 2,6,8

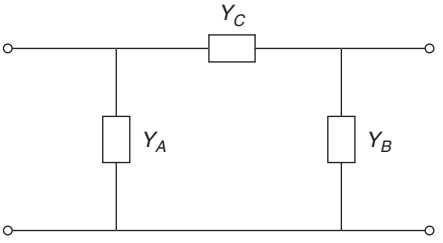


Fig. 10.153

10.17 The impedance matrices of two two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 3 & 25 \end{bmatrix}$. If these two networks are connected in series, the impedance matrix of the resulting two-port network will be

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$ (d) inderminate

10.18 If the π network and T network are equivalent, then the values of R_1, R_2 and R_3 (in ohms) will be respectively

(a) 6, 6, 6 (b) 6, 6, 9

(c) 9, 6, 9 (d) 6, 9, 6

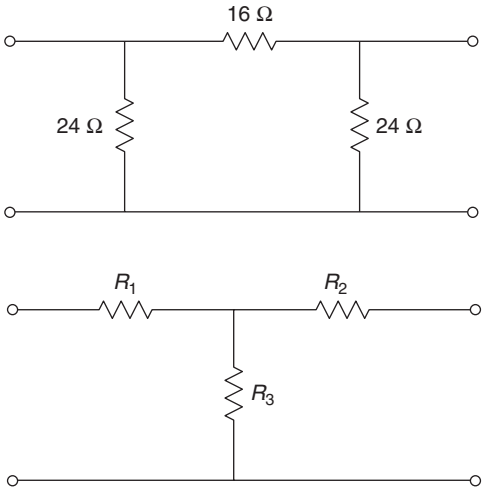


Fig. 10.154

10.19 For a two-port symmetrical bilateral network, if $A = 3$ and $B = 1$, the value of the parameter C will be

(a) 4 (b) 6

(c) 8 (d) 16

10.20 The impedance matrix for the network shown in Fig. 10.155 is

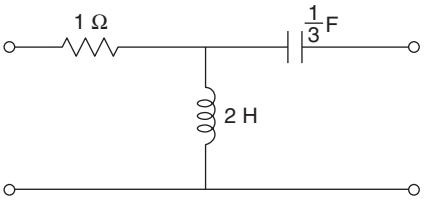


Fig. 10.155

(a) $\begin{bmatrix} 2s+1 & 2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$ (b) $\begin{bmatrix} 2s+1 & -2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$

(c) $\begin{bmatrix} 2s+1 & 2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$ (d) $\begin{bmatrix} 2s+\frac{3}{2} & -2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$

10.21 With the usual notations, a two-port resistive network satisfies the conditions $A = D = \frac{3}{2}B = \frac{4}{3}C$. The Z_{11} of the network is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$

(c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Answers to Objective-Type Questions

10.1. (a)	10.2. (d)	10.3. (b) , (c)	10.4. (b)	10.5. (a)	10.6. (c)	10.7. (c)
10.8. (a)	10.9. (d)	10.10. (d)	10.11. (b)	10.12. (d)	10.13. (b)	10.14. (a)
10.15. (c)	10.16. (c)	10.17. (b)	10.18. (a)	10.19. (c)	10.20. (a)	10.21. (b)

11

Filters and Attenuators

11.1 INTRODUCTION

Filters are frequency-selective networks that attenuate signals at some frequency and allow others to pass with or without attenuation. A filter is constructed from purely reactive elements. Ideally, filters should produce no attenuation in the desired band, called *pass band* and should provide attenuation at all other frequencies called *attenuation band* or *stop band*. The frequency which separates the pass band and the stop band is called *cut-off frequency*. Filter networks are widely used in communication systems to separate various channels in carrier-frequency telephone circuits.

11.2 CLASSIFICATION OF FILTERS

On the basis of frequency characteristics, filters are classified into four categories:

(i) Low-pass filter
(ii) High-pass filter
(iii) Band-pass filter
(iv) Band-stop filter

A *low-pass filter* allows all frequencies up to a certain cut-off frequency to pass through it and attenuates all the other frequencies above the cut-off frequency.

A *high-pass filter* attenuates all the frequency below the cut-off frequency and allows all other frequencies above the cut-off frequency to pass through it.

A *band-pass filter* allows a limited band of frequencies to pass through it and attenuates all other frequencies below or above the frequency band.

A *band-stop filter* attenuates a limited band of frequencies but allows all other frequencies to pass through it.

11.3 T-NETWORK

Figure 11.1 shows a *T* network.

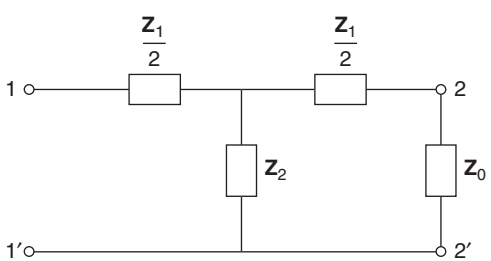


Fig. 11.1 *T-network*

11.3.1 Characteristic Impedance

For a *T* network, the value of input impedance when it is terminated by characteristic impedance $\mathbf{Z}_0$, is given by

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}$$

But $\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$

$$\mathbf{Z}_0 = \frac{\mathbf{Z}_1}{2} + \frac{2\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_2\mathbf{Z}_0}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0}{2(\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0)}$$

$$2\mathbf{Z}_1\mathbf{Z}_0 + 4\mathbf{Z}_2\mathbf{Z}_0 + 4\mathbf{Z}_0^2 = \mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0$$

$$4\mathbf{Z}_0^2 = \mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0^2 = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Hence, characteristic impedance for symmetrical *T* section is given by,

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $\mathbf{Z}_{oc} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 = \frac{\mathbf{Z}_1 + 2Z_2}{2}$

Short-circuit impedance $\mathbf{Z}_{sc} = \frac{\mathbf{Z}_1}{2} + \frac{\frac{\mathbf{Z}_1}{2}\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + 2\mathbf{Z}_2} = \frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}$

$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \left(\frac{\mathbf{Z}_1 + 2\mathbf{Z}_2}{2}\right)\left(\frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}\right) = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2 = \mathbf{Z}_{0T}^2$$

$$\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

11.3.2 Propagation Constant

The propagation constant γ of the network in Fig. 11.2 is given by,

$$\gamma = \log_e \frac{I_1}{I_2}$$

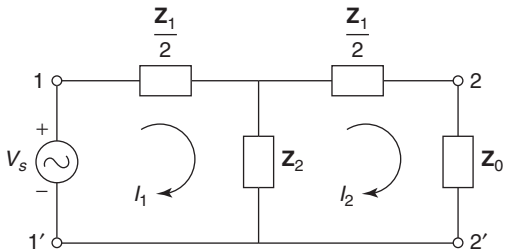


Fig. 11.2 *T-network*

Applying KVL to Mesh1,

$$V_s - \frac{\mathbf{Z}_1}{2} I_1 - \mathbf{Z}_2 (I_1 - I_2) = 0$$

$$V_s = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 \right) I_1 - \mathbf{Z}_2 I_2$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2 (I_2 - I_1) - \frac{\mathbf{Z}_1}{2} I_2 - \mathbf{Z}_0 I_2 = 0$$

$$-\mathbf{Z}_2 I_1 + \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2 = 0$$

$$\mathbf{Z}_2 I_1 = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2$$

$$\frac{I_1}{I_2} = \frac{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}{\mathbf{Z}_2} = e^{\gamma}$$

$$\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 = \mathbf{Z}_2 e^{\gamma}$$

$$\mathbf{Z}_0 = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

The characteristic impedance of the *T*-network is given by,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

$$\sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2} = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

$$\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2 = \mathbf{Z}_2^2 (e^{\gamma} - 1)^2 + \frac{\mathbf{Z}_1^2}{4} - \mathbf{Z}_1 \mathbf{Z}_2 (e^{\gamma} - 1)$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 (1 + e^{\gamma} - 1) = 0$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 e^{\gamma} = 0$$

$$\mathbf{Z}_2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 e^{\gamma} = 0$$

$$(e^{\gamma} - 1)^2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2} e^{\gamma}$$

$$e^{2\gamma} - 2e^{\gamma} + 1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2 e^{-\gamma}}$$

$$e^{-\gamma} (e^{2\gamma} - 2e^{\gamma} + 1) = \frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$e^{\gamma} + e^{-\gamma} - 2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

Dividing both the sides by 2,

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\cosh\gamma = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\sinh\gamma = \sqrt{\cosh^2\gamma - 1} = \sqrt{\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2 - 1} = \sqrt{1 + \frac{\mathbf{Z}_1}{\mathbf{Z}_2} + \frac{\mathbf{Z}_1^2}{4\mathbf{Z}_2^2} - 1} = \sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2} + \left(\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2}$$

$$= \frac{1}{\mathbf{Z}_2}\sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2}$$

$$\tanh\gamma = \frac{\sinh\gamma}{\cosh\gamma} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2 + \frac{\mathbf{Z}_1}{2}}$$

But $\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$

and $\mathbf{Z}_{oc} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2$

$$\tanh\gamma = \sqrt{\frac{\mathbf{Z}_{sc}}{\mathbf{Z}_{oc}}}$$

Also, $\sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}(\cosh\gamma - 1)} = \sqrt{\frac{1}{2}\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2} - 1\right)} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$

11.4 π NETWORK

Figure 11.3 shows a π-network.

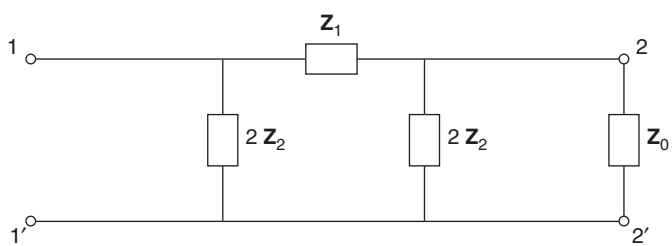


Fig. 11.3 *π-network*

11.4.1 Characteristic Impedance

For a π network, the value of input impedance when it is terminated by impedance $\mathbf{Z}_0$, is given by,

$$\mathbf{Z}_{\text{in}} = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0}\right)}{\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_2}$$

But $$\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$$

$$\mathbf{Z}_0 = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0}\right)}{\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_2}$$

$$\mathbf{Z}_0\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0^2}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_0\mathbf{Z}_2 = \frac{2\mathbf{Z}_2(2\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_0\mathbf{Z}_1 + 2\mathbf{Z}_0\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_0}$$

$$2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_0^2 + 2\mathbf{Z}_0\mathbf{Z}_2^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 + 2\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2 + 2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_0\mathbf{Z}_2^2$$

$$\mathbf{Z}_1\mathbf{Z}_0^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2$$

$$\mathbf{Z}_0^2(\mathbf{Z}_1 + 4\mathbf{Z}_2) = 4\mathbf{Z}_1\mathbf{Z}_2^2$$

$$\mathbf{Z}_0^2 = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$$

$$\mathbf{Z}_0 = \sqrt{\frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}}$$

Dividing both the sides by $4Z_2$,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}$$

Hence, characteristic impedance of a symmetrical π network is given by

$$\mathbf{Z}_{0\pi} = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}}$$

But $$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $\mathbf{Z}_{oc} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_1 + 2\mathbf{Z}_2} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$

Short-circuit impedance $\mathbf{Z}_{sc} = \dfrac{2\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_2 + \mathbf{Z}_1}$

$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\mathbf{Z}_{0\pi} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

11.4.2 Propagation Constant

The propagation constant of a symmetrical π network is same as that of a symmetrical T network.

11.5 CHARACTERISTIC OF FILTERS

A filter transmits or passes desired range of frequencies without loss and attenuates all undesired frequencies.

The propagation constant

$$\gamma = \alpha + j\beta$$

where α is attenuation constant and β is the phase constant. We know that

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\sinh\left(\frac{\alpha + j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Depending upon the type of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, there are two cases:

Case (i) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are same type of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ and $\sinh\frac{\gamma}{2}$ are real.

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = 0$$

$$\sin\frac{\beta}{2} = 0 \qquad \text{if } \beta = 0 \text{ or } n\pi \text{ where } n = 0, 1, 2, \ldots \quad \left[\because \cosh\frac{\alpha}{2} \text{ cannot be zero}\right]$$

$$\cos\frac{\beta}{2} = 1$$

$$\sinh\frac{\alpha}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\alpha = 2\sinh^{-1}\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Case (ii) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative, i.e. $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < 0$ and $\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$ is purely imaginary.

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0$$

$$j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = j\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

The two conditions of operation of the filter, viz. the pass band and stop band are mathematically obtained from these equations. Both the above equations must be satisfied simultaneously by α and β. Two conditions may arise

(a) $\sinh\frac{\alpha}{2}=0$ i.e $\alpha=0$ when $\beta\neq 0$

Since $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\sin\frac{\beta}{2}$ is real,

$$\sin\frac{\beta}{2}=\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right| \qquad \text{as } \cosh\frac{\alpha}{2}=1$$

This signifies the region of zero attenuation or pass band which is limited by the upper limit of sine terms.

$$\sin\frac{\beta}{2}=1$$

$$-1<\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}<0$$

The phase angle in the pass band is given by,

$$\frac{\beta}{2}=\sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\beta=2\sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

(b) $$\cos\frac{\beta}{2}=0$$

$$\sin\frac{\beta}{2}=\pm 1$$

Since $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\cosh\frac{\alpha}{2}$ is real,

$$\cosh\frac{\alpha}{2}=\left|\sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2}}\right|$$

This signifies a stop band since $\alpha\neq 0$.

$$\frac{\alpha}{2}=\cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\alpha=2\cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

and $$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}<-1$$

11.6 CONSTANT-*k* LOW PASS FILTER

A T or π network is said to be of the constant *k* type if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances satisfying the relation

$$\mathbf{Z}_1\mathbf{Z}_2=k^2$$

where k is a constant, independent of frequency. k is often referred to as *design impedance* or *nominal impedance* of the constant-k filter. The constant-k, T or π-type filter is also known as the prototype filter because other complex networks can be derived from it. Figure 11.4 shows constant-k, T and π-section filters.

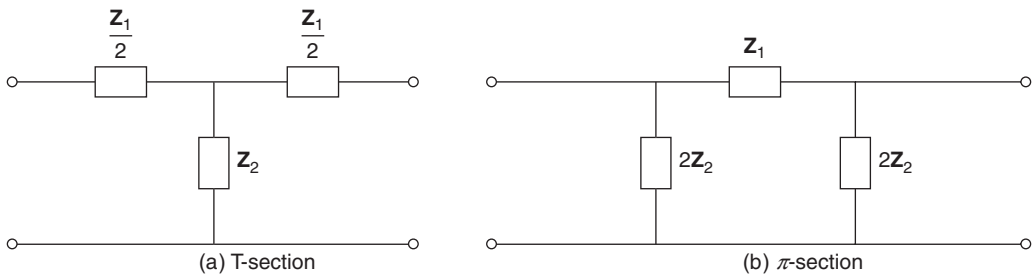


Fig. 11.4 *Constant–k Filter*

In constant-k low pass filter,

$$\mathbf{Z}_1 = j\omega L$$

$$\mathbf{Z}_2 = -j\frac{1}{\omega c} = \frac{1}{j\omega c}$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{(j\omega L)\left(\frac{1}{j\omega c}\right)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$$

$$\mathbf{Z}_1 = 0$$

$$j\omega L = 0$$

$$\omega = 0$$

$$f = 0$$

(ii) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$j\omega L = \frac{4j}{\omega C}$$

$$\omega^2 LC = 4$$

$$\omega^2 = \frac{4}{LC}$$

$$\omega = \omega_c = \frac{2}{\sqrt{LC}}$$

$$f = f_c = \frac{1}{\pi\sqrt{LC}}$$

Hence, the pass band starts at $f = 0$ and continues up to the cut-off frequency f_c. All the frequencies above f_c are in the attenuation or stop band.

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $$\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\sqrt{\frac{\omega^2 LC}{4}} = 2\cosh^{-1}\sqrt{\frac{\omega^2}{\omega_c{}^2}}$$

$$= 2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

The attenuation constant α is zero throughout the pass band but increases gradually from the cut-off frequency. The variation of α is plotted in Fig. 11.5.

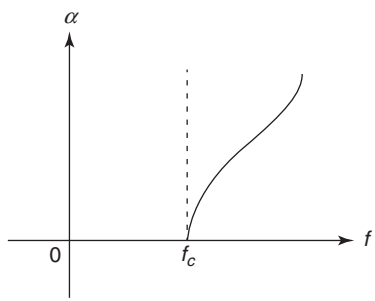


Fig. 11.5 *Variation of α with frequency*

4. Phase Constant

In pass band,

$$\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f}{f_c}\right)$$

In stop band, $\beta = \pi$.

The phase constant β is zero at zero frequency and increases gradually through the pass band, reaches π at cut-off frequency f_c and remains at π for all frequency beyond f_c. The variation of β is plotted in Fig. 11.6.

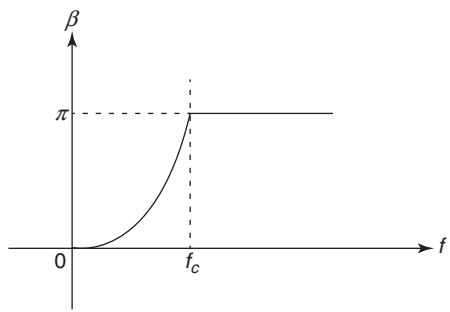


Fig. 11.6 *Variation of β with frequency*

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)} = k\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} = k\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}}$$

The plot of characteristic impedance versus frequency is shown in Fig. 11.7.

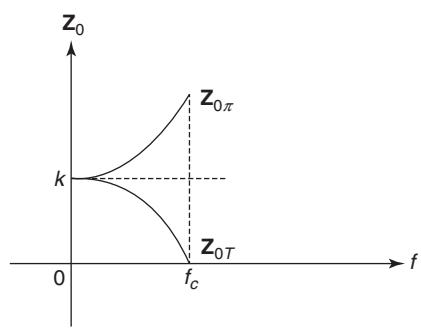


Fig. 11.7 *Variation of characteristic impedance with frequency*

The characteristic impedance $\mathbf{Z}_{0T}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0T} = 0$. For $f > f_c$, $\mathbf{Z}_{0T}$ is imaginary in the stop band, rising to infinite reactance at infinite frequency. $\mathbf{Z}_{0\pi}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0\pi}$ is infinite and for $f > f_c$, $\mathbf{Z}_{0\pi}$ is imaginary.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{\pi f_c}$$

$$C = \frac{1}{\pi f_c k}$$

The constant-k, T and π section filters are shown in Fig. 11.8.

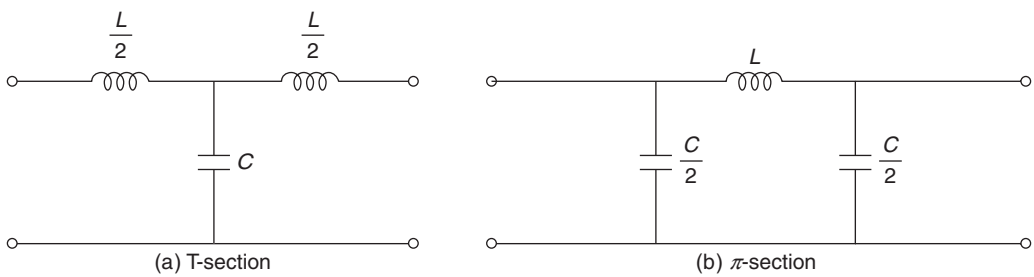


Fig. 11.8 *Constant-k low-pass filter*

Example 11.1 *Find the nominal impedance, cut-off frequency and pass band for the network shown in Fig. 11.9.*

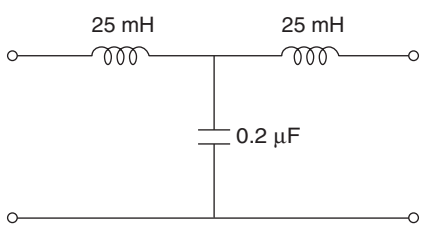


Fig. 11.9

Solution The network is a low-pass filter.

$$\frac{L}{2} = 25 \text{ mH}, \quad C = 0.2 \text{ μF}$$

$$L = 50 \text{ mH}$$

(a) Nominal Impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50\times10^{-3}}{0.2\times10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 3.18\ \text{kHz}$$

(c) Pass band
The pass band is from zero to 3.18 kHz.

Example 11.2 *A low-pass filter is composed of a symmetrical π section. Each series branch is a 0.02 H inductor and shunt branch is a 2 μF capacitor. Find (a) cut-off frequency, (b) nominal impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase shift constant at 200 Hz and 2000 Hz.*

Solution

$$L = 0.02\ \text{H}, \quad \frac{C}{2} = 2\ \mu\text{F}$$

$$C = 4\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{0.02\times4\times10^{-6}}} = 1125.4\ \text{Hz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.02}{4\times10^{-6}}} = 70.71\ \Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1-\left(\frac{200}{1125.4}\right)^2}} = 71.85\ \Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1-\left(\frac{2000}{1125.4}\right)^2}} = j48.13\ \Omega$$

(d) Attenuation at 200 Hz and 2000 Hz
The 200 Hz frequency lies in the pass band.

$$\alpha = 0$$

At 2000 Hz,

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{2000}{1125.4}\right) = 2.35$$

(e) Phase shift constant at 200 Hz and 2000 Hz
At 200 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right) = 2\sin^{-1}\left(\frac{200}{1125.4}\right) = 20.47°$$

2000 Hz frequency lies in the attenuation band.

$$\beta = 180°$$

Example 11.3 *A π section filter network consists of a series arm inductor of 20 mH and two shunt-arm capacitors of 0.16 μF each.Calculate the cut-off frequency, and attenuation and phase shift at 15 kHz. What is the value of the nominal impedance in the pass band?*

Solution $L = 20$ mH, $\dfrac{C}{2} = 0.16$ μF

$C = 0.32$ μF

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{20\times10^{-3}\times0.32\times10^{-6}}} = 3.98 \text{ kHz}$$

(b) Attenuation at 15 kHz

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{15\times10^3}{3.98\times10^3}\right) = 4$$

(c) Phase shift at 15 kHz

15 kHz frequency lies in the attenuation band.

$$\beta = 180°$$

(d) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{20\times10^{-3}}{0.32\times10^{-6}}} = 250\ \Omega$$

Example 11.4 *Each of the two series elements of a T-section low-pass filter consists of an inductor of 60 mH having negligible resistance and a shunt element having a capacitance of 0.2 μF. Calculate (a) the cut-off frequency, (b) nominal impedance, and (c) characteristic impedance at frequencies of 1 kHz and 5 kHz.*

Solution $L = 60$ mH, $C = 0.2$ μF

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{60\times10^{-3}\times0.2\times10^{-6}}} = 2.91 \text{ kHz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{60\times10^{-3}}{0.2\times10^{-6}}} = 547.72\ \Omega$$

(c) Characteristic impedance at 1 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{1\times10^3}{2.91\times10^3}\right)^2} = 514.36\,\Omega$$

Characteristic impedance at 5 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{5\times10^3}{2.91\times10^3}\right)^2} = j765.29\ \Omega$$

Example 11.5 *Design a constant-k low-pass T and π section filters having cut-off frequency of 4 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_c = 4\text{ kHz}, \quad k = 500\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi\times4\times10^3} = 39.79\text{ mH}$$

$$\text{C} = \frac{1}{\pi f_c k} = \frac{1}{\pi\times4\times10^3\times500} = 0.16\ \mu\text{F}$$

The *T* section consists of an inductor of $\frac{L}{2}$, i.e. 19.9 mH in each series branch and a capacitor of 0.16 μF in the shunt branch as shown in Fig. 11.10 (a). The *π*-section consists of an inductor of 39.79 mH in the series branch and a capacitor of $\frac{C}{2}$, i.e. 0.08 μF in each shunt branch as shown in Fig. 11.10 (b).

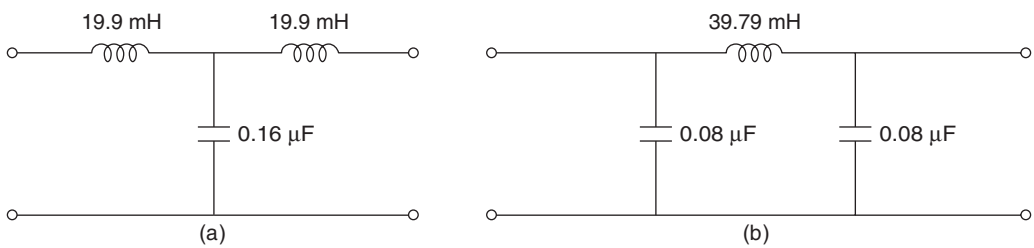


Fig. 11.10

Example 11.6 *Design a constant-k, T section low-pass filter is having 2.5 kHz cut-off frequency and nominal impedance of 700 Ω. Also find the frequency at which this filter produces an attenuation of 19.1 dB. Find its characteristic impedances and phase constant at pass band and stop or attenuation band.*

Solution

$$f_c = 2.5\text{ kHz}, \quad k = 700\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{700}{\pi\times2.5\times10^3} = 89.13\text{ mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi\times2.5\times10^3\times700} = 0.18\ \mu\text{F}$$

The *T*-section filter consists of an inductor of $\frac{L}{2}$, i.e. 44.57 mH in each series arm and a capacitor of 0.18 μF in the shunt arm as shown in Fig. 11.11.

$$\alpha = 19.1 \text{ dB} = 2.197 \text{ nepers}$$

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_C}\right)$$

$$2.197 = 2\cosh^{-1}\left(\frac{f}{2.5\times10^3}\right)$$

$$f = 4.17 \text{ kHz}$$

β is imaginary in pass band.

In attenuation band, $\beta = 180°$

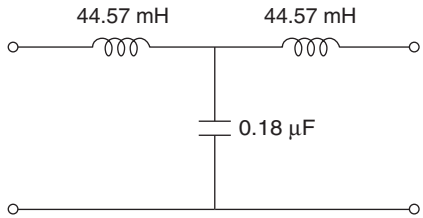


Fig. 11.11

11.7 CONSTANT-*k* HIGH-PASS FILTER

A constant-*k* high-pass filter is obtained by changing the positions of series and shunt arm of the constant-*k* low-pass filter, Figure 11.12 shows a constant-*k*, *T* and π section, high-pass filter.

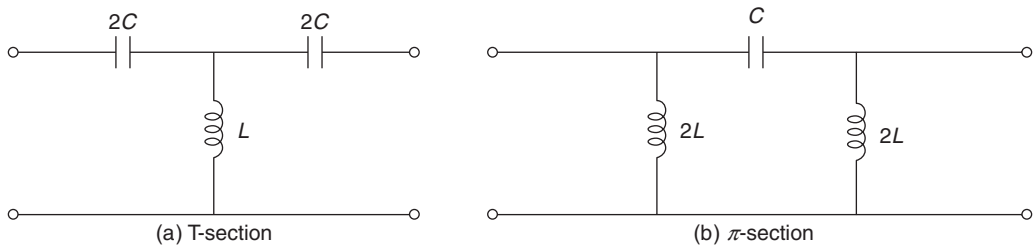


Fig. 11.12 *Constant-k high-pass filter*

In a constant-*k* high-pass filter

$$\mathbf{Z}_1 = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

$$\mathbf{Z}_2 = j\omega L$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\left(\frac{1}{j\omega C}\right)(j\omega L)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$

$$\mathbf{Z}_1 = 0$$

$$\frac{1}{j\omega C} = 0$$

$$\omega = \infty$$

$$f = \infty$$

(ii) When $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$-j\frac{1}{\omega_c} = 4j\omega L$$

$$\omega^2 LC = \frac{1}{4}$$

$$\omega^2 = \frac{1}{4LC}$$

$$\omega = \omega_c = \frac{1}{2\sqrt{LC}}$$

$$f = f_c = \frac{1}{4\pi\sqrt{LC}}$$

Hence, the filter passes all the frequencies beyond f_c. The pass band starts at $f = f_c$ and continues up to infinite frequency. All the frequencies below the cut-off frequency lie in the attenuation or stop band.

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\left(\frac{f_c}{f}\right)$

The attenuation constant α decreases gradually to zero at the cut-off frequency and remains at zero through the pass band. The variation of α is plotted in Fig. 11.13.

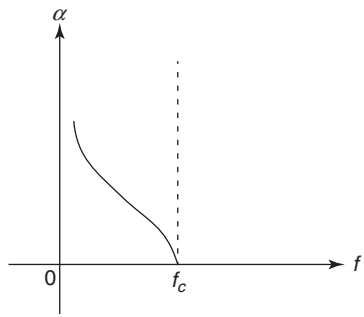


Fig. 11.13 *Variation of α with frequency*

4. Phase Constant

In pass band, $\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f_c}{f}\right)$

In stop band, $\beta = \pi$.

The phase constant β remains at constant value π in the stop band and decreases to $-\pi$ in at f_c and reaches zero value gradually as f increases in the pass band. The variation of β is plotted in Fig. 11.14.

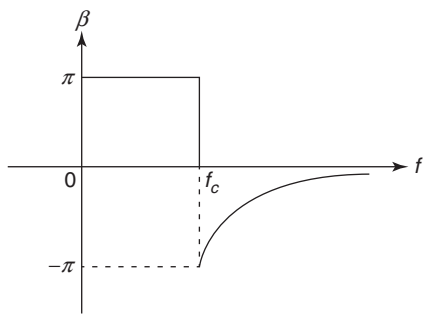


Fig. 11.14 *Variation of β with frequency*

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)} = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

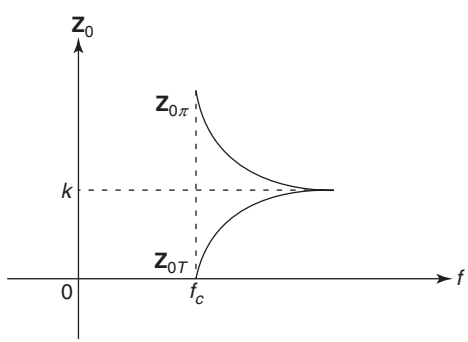


Fig. 11.15 *Variation of characteristic impedance with frequency*

The variation of characteristic impedance is plotted in Fig. 11.15.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{4\pi f_c}$$

$$C = \frac{1}{4\pi f_c k}$$

Example 11.7 *Find the characteristic impedance, cut-off frequency and pass band for the network shown in Fig. 11.16.*

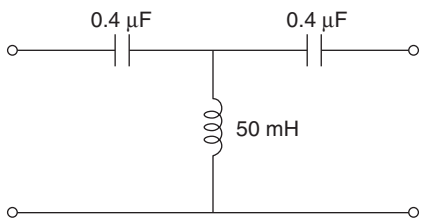


Fig. 11.16

Solution The network is a high-pass filter.

$$2C = 0.4\ \mu\text{F}, \quad L = 50\ \text{mH}$$
$$C = 0.2\ \mu\text{F}$$

(a) Characteristic impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50\times10^{-3}}{0.2\times10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 795.77\ \text{Hz}$$

(c) Pass band

The pass band is from 795.77 Hz to infinite frequency.

Example 11.8 *A high-pass filter section is constructed from two capacitors of 1 μF each and a 15 mH inductance. Find (a) cut-off frequency, (b) infinite frequency characteristic impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase constant at 200 Hz and 2000 Hz.*

Solution

$$L = 15\ \text{mH}, \quad 2C = 1\ \mu\text{F}$$
$$C = 0.5\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{15\times10^{-3}\times0.5\times10^{-6}}} = 918.88 \text{ Hz}$$

(b) Infinite-frequency characteristic impedance

At $f = \infty$, $$\mathbf{Z}_0 = k = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.5\times10^{-3}}{0.5\times10^{-6}}} = 173.21\ \Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = j776.72\ \Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = 153.85\ \Omega$$

(d) Attenuation at 200 Hz

$$\alpha = 2\cosh^{-1}\left(\frac{f_c}{f}\right) = 2\cosh^{-1}\left(\frac{918.88}{200}\right) = 4.41$$

The frequency of 2000 Hz lies in the pass band.

$$\alpha = 0$$

(e) Phase constant

The frequency 200 Hz lies in the attenuation band.

$$\beta = \pi \text{ radians}$$

At 2000 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{918.88}{2000}\right) = 54.7°$$

Example 11.9 *Design a constant-k high-pass T and π sections filters having a cut-off frequency of 2000 Hz and infinite frequency characteristic impedance of 300 Ω.*

Solution

$$f_c = 2000 \text{ Hz}, \quad k = 300\ \Omega$$

$$L = \frac{k}{4\pi f_c} = \frac{300}{4\pi\times2000} = 11.94 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi\times2000\times300} = 0.13\ \mu\text{F}$$

The *T*-section filter consists of a capacitor of 2*C*, i.e 0.26 μF in each series arm and an inductor of 11.94 mH in shunt arm as shown in Fig. 11.17 (a).

The *π*-section filter consists of a capacitor of 0.13 μF in the series arm and an inductor of 2*L*, i.e. 23.88 mH in each shunt arm as shown in Fig. 11.17 (b).

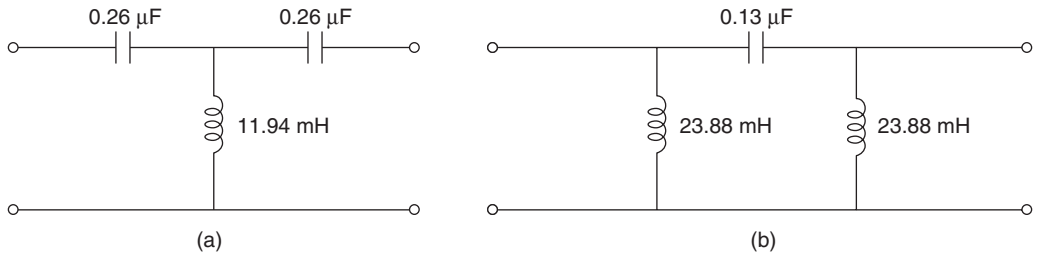


Fig. 11.17

Example 11.10 *Design a T-section constant-k high-pass filter having a cut-off frequency of 10 kHz and a design impedance of 600 Ω. Find its characteristic impedance and phase constant at 25 kHz.*

Solution $f_c = 10$ kHz, $k = 600\ \Omega$

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 10 \times 10^3} = 4.77 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 10 \times 10^3 \times 600} = 0.013\ \mu\text{F}$$

The *T*-section filter consists of a capacitor of $2C$, i.e. 0.026 μF in each series arm and an inductor of 4.77 mH in shunt arm as shown in Fig. 11.18.

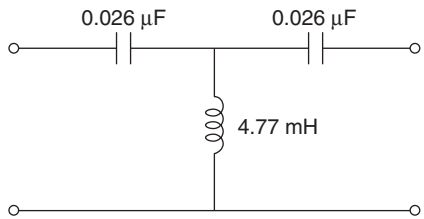


Fig. 11.18

(a) Characteristic impedance at 25 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 600\sqrt{1-\left(\frac{10\times10^3}{25\times10^3}\right)^2} = 549.91\ \Omega$$

(b) Phase constant at 25 kHz

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{10\times10^3}{25\times10^3}\right) = 47.16°$$

The frequency of 25 kHz lies in the attenuation band.

In the attenuation band, $\beta = 180° = \pi$ radians

11.8 BAND-PASS FILTER

A band-pass filter attenuates all the frequencies below a lower cut-off frequency and above an upper cut-off frequency. It passes a band of frequencies without attenuation. A band-pass filter is obtained by using a low pass filter followed by a high-pass filter.

Figure 11.19 shows a band-pass filter. The series arm is a series resonant circuit comprising L_1 and C_1 while its shunt arm is formed by a parallel resonant circuit L_2 and C_2. The resonant frequency of series arm and shunt arm are made equal.

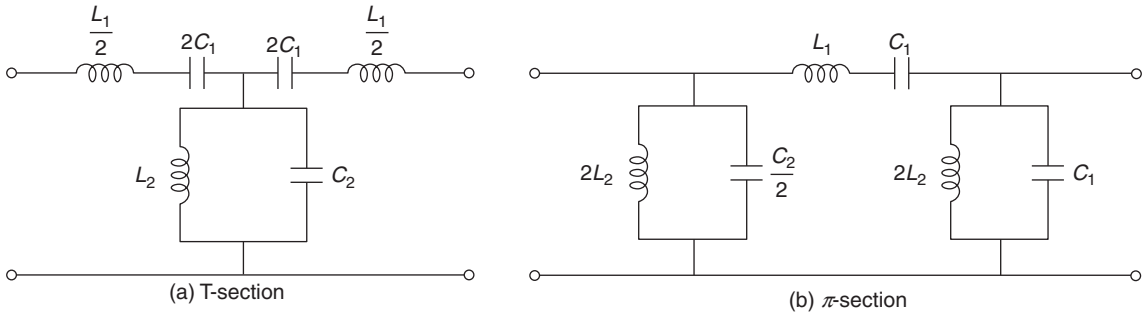


Fig. 11.19 *Band-pass filter*

For series arm,

$$\omega_0 \frac{L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

For shunt arm,

$$\frac{1}{\omega_0 C_2} = \omega_0 L_2$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

For series arm,

$$\mathbf{Z}_1 = j\omega L_1 - \frac{j}{\omega c_1} = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)$$

For shunt arm,

$$\mathbf{Z}_2 = \frac{j\omega L_2 \dfrac{1}{j\omega c_2}}{j\omega L_2 + \dfrac{1}{j\omega c_2}} = \frac{j\omega L_2}{1 - \omega^2 L_2 C_2}$$

$$\mathbf{Z}_1 \mathbf{Z}_2 = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)\left(\frac{j\omega L_2}{1 - \omega^2 L_2 C_2}\right) = -\frac{L_2}{C_1}\left(\frac{\omega^2 L_1 C_1 - 1}{1 - \omega^2 L_2 C_2}\right) = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

where k is constant.

For constant k filter, at cut-off frequency,

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1^2 = -4\mathbf{Z}_1 \mathbf{Z}_2 = -4k^2$$

$$\mathbf{Z}_1 = \pm j2k$$

i.e. the value of $\mathbf{Z}_1$ at lower cut-off frequency is equal to the negative value of $\mathbf{Z}_1$ at the upper cut-off frequency.

$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -\left(\frac{1}{j\omega_2 C_1} + j\omega_2 L_1\right)$$

$$1 - \omega_1^2 L_1 C_1 = \frac{\omega_1}{\omega_2}\left(\omega_2^2 L_1 C_1 - 1\right)$$

But
$$\omega_0^2 = \frac{1}{L_1 C_1}$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{\omega_1}{\omega_2}\left(\frac{\omega_2^2}{\omega_0^2} - 1\right)$$
$$(\omega_0^2 - \omega_1^2)\omega_2 = \omega_1(\omega_2^2 - \omega_0^2)$$
$$\omega_0^2\omega_2 - \omega_1^2\omega_2 = \omega_1\omega_2^2 - \omega_1\omega_0^2$$
$$\omega_0^2(\omega_1 + \omega_2) = \omega_1\omega_2(\omega_2 + \omega_1)$$
$$\omega_0^2 = \omega_1\omega_2$$
$$\omega_0 = \sqrt{\omega_1\omega_2}$$
$$f_0 = \sqrt{f_1 f_2}$$

Thus, resonant frequency is the geometric mean of the cutoff frequencies. The variation of attenuation, phase constant and characteristic impedance with frequency are shown in Fig. 11.20.

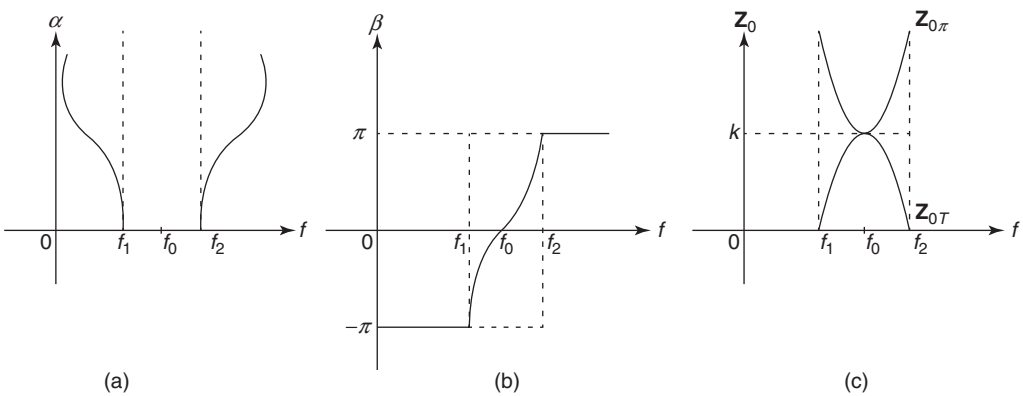


Fig. 11.20 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant–k band-pass filter*

Design of Filter

If the filter is terminated in a load resistance $R = k$ then at lower cut-off frequency,

$$\mathbf{Z}_1 = -2jk$$
$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -2jk$$
$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2k$$
$$1 - \omega_1^2 L_1 C_1 = 2k\omega_1 C_1$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = 2k\omega_1 C_1 \qquad \left(\because\ \omega_0^2 = \frac{1}{L_1 C_1}\right)$$
$$1 - \left(\frac{f_1}{f_0}\right)^2 = 4\pi k f_1 C_1$$

$$1-\frac{f_1^2}{f_1 f_2}=4\pi k f_1 C_1 \qquad \left(\because f_0=\sqrt{f_1 f_2}\right)$$

$$f_2-f_1=4\pi k f_1 f_2 C_1$$

$$C_1=\frac{f_2-f_1}{4\pi\, k f_1 f_2}$$

$$L_1=\frac{1}{\omega_0^2 C_1}=\frac{4\pi\, k f_1 f_2}{\omega_0^2(f_2-f_1)}=\frac{4\pi\, k f_1 f_2}{4\pi^2\, f_0^2\,(f_2-f_1)}=\frac{k}{\pi(f_2-f_1)}$$

For shunt arm,

$$\mathbf{Z}_1\mathbf{Z}_2=\frac{L_2}{C_1}=\frac{L_1}{C_2}=k^2$$

$$L_2=C_1k^2=\frac{(f_2-f_1)k}{4\pi\, f_1 f_2}$$

$$C_2=\frac{L_1}{k^2}=\frac{1}{\pi(f_2-f_1)k}$$

Example 11.11 *In a constant-k band-pass filter, the ratio of capacitances in the shunt and series arms is 100:1. The resonant frequency of both arms is 1000 Hz. Find the bandwidth,*

Solution

$$\frac{C_2}{C_1}=\frac{100}{1}=100, \quad f_0=1000\text{ Hz}$$

$$C_2=\frac{1}{\pi k(f_2-f_1)}$$

$$C_1=\frac{f_2-f_1}{4\pi k f_1 f_2}$$

$$\frac{C_2}{C_1}=\frac{1}{\pi k(f_2-f_1)}\,\frac{4\pi k f_1 f_2}{f_2-f_1}$$

$$100=\frac{4 f_1 f_2}{(f_2-f_1)^2}$$

$$100=\frac{4 f_0^2}{(f_2-f_1)^2}$$

$$f_2-f_1=\frac{2 f_0}{\sqrt{100}}=\frac{2\times1000}{\sqrt{100}}=200$$

$$BW=f_2-f_1=200\text{ Hz}$$

Example 11.12 *Design a band pass constant-k filter with cut-off frequency of 4 kHz and 10 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_1=4\text{ kHz}, \quad f_2=10\text{ kHz}, \quad k=500\ \Omega$$

$$L_1=\frac{k}{\pi(f_2-f_1)}=\frac{500}{\pi(10\times10^3-4\times10^3)}=26.53\text{ mH}$$

$$C_1=\frac{f_2-f_1}{4\pi k f_1 f_2}=\frac{10\times10^3-4\times10^3}{4\pi\times500\times4\times10^3\times10\times10^3}=0.024\ \mu\text{F}$$

$$L_2 = \frac{(f_2 - f_1)k}{4\pi f_1 f_2} = \frac{(10\times10^3 - 4\times10^3)\,500}{4\pi\times4\times10^3\times10\times10^3} = 5.97 \text{ mH}$$

$$C_2 = \frac{1}{\pi(f_2 - f_1)k} = \frac{1}{\pi\times(10\times10^3 - 4\times10^3)\times500} = 0.11\,\mu\text{F}$$

The *T*-section filter consists of series combination of an inductor of $\frac{L_1}{2}$, i.e. 13.27 mH and a capacitor of 2 C_1, i.e. 0.048 μF in each series arm and a parallel combination of an inductor of 5.97 mH and a capacitor of 0.11 μF in shunt arm as shown in Fig. 11.21.

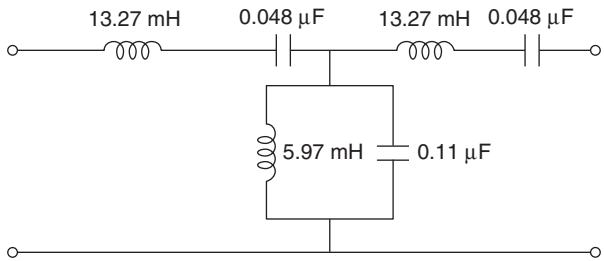


Fig. 11.21

The π-section filter consists of a series combination of an inductor 26.53 mH and a capacitor of 0.024 μF in the series arm and a parallel combination of an inductor of $2L_2$, i.e. 11.94 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.055 μF in each shunt arm as shown in Fig. 11.22.

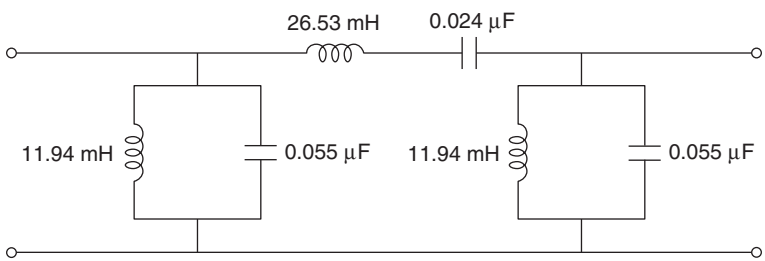


Fig. 11.22

11.9 BAND-STOP FILTER

A band-stop filter attenuates a specified band of frequencies and allows all frequencies below and above this band. A band-stop filter is realised by connecting a low-pass filter in parallel with a high-pass filter. Figure 11.23 shows a band-stop filter.

As in the band-pass filter, the series and shunt arms are chosen to resonate at same frequency ω_0.

For series arm,

$$\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

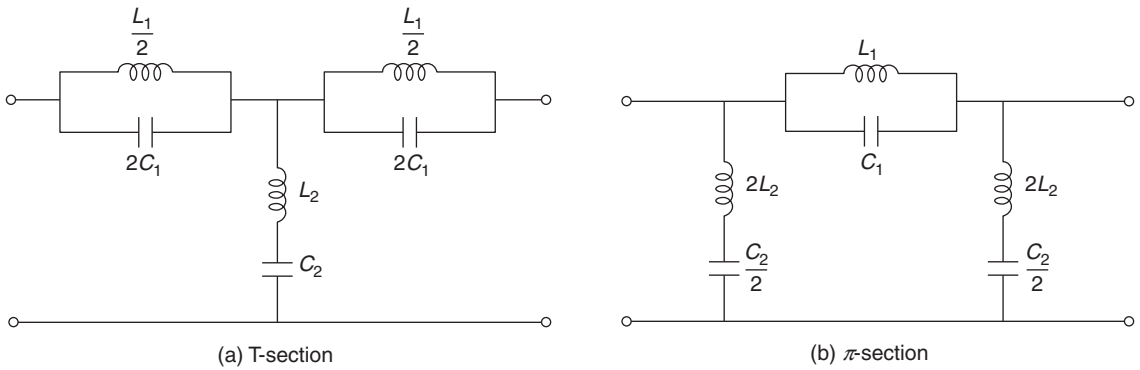


Fig. 11.23 *Band-stop filter*

For shunt arm,

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2}$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

Similarly,
$$\mathbf{Z}_1 \mathbf{Z}_2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} = k^2$$

and
$$f_0 = \sqrt{f_1 f_2}$$

At cut-off frequencies,
$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1 \mathbf{Z}_2 = -4\mathbf{Z}_2^2 = k^2$$

$$\mathbf{Z}_2 = \pm j\frac{k}{2}$$

The variation of attenuation, phase shift and characteristic impedance with frequency are shown in Fig. 11.24.

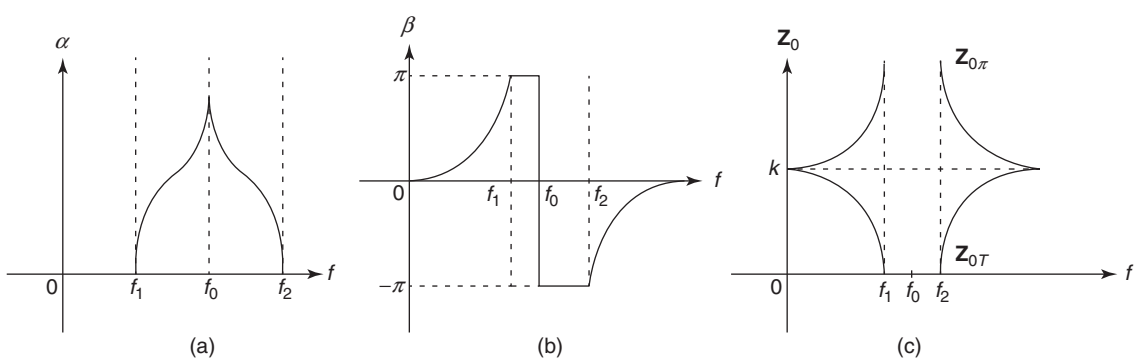


Fig 11.24 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant-k band-stop filter.*

Design of Filter

If the load is terminated in load resistance $R = k$ then at lower cut-off frequency,

$$\mathbf{Z}_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = j\frac{k}{2}$$

$$\frac{1}{\omega_1 C_2} - \omega_1 L_2 = \frac{k}{2}$$

$$1 - \omega_1^2 L_2 C_2 = \omega_1 C_2 \frac{k}{2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{k}{2}\omega_1 C_2 \qquad \left(\because \omega_0^2 = \frac{1}{L_2 C_2}\right)$$

$$1 - \left(\frac{f_1}{f_0}\right)^2 = k\pi f_1 C_2$$

$$C_2 = \frac{1}{k\pi f_1}\left[1 - \left(\frac{f_1}{f_0}\right)^2\right] = \frac{1}{k\pi}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = \frac{1}{k\pi}\left(\frac{f_2 - f_1}{f_1 f_2}\right) = \frac{f_2 - f_1}{\pi k f_1 f_2}$$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{\pi k f_1 f_2}{\omega_0^2 (f_2 - f_1)} = \frac{\pi k f_1 f_2}{4\pi^2 f_0^2 (f_2 - f_1)} = \frac{k}{4\pi(f_2 - f_1)}$$

$$k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1}$$

$$L_1 = k^2 C_2 = \frac{k(f_2 - f_1)}{\pi f_1 f_2}$$

$$C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi\, k(f_2 - f_1)}$$

Example 11.13 *Design a constant-k band-stop filter having cut-off frequencies at 2000 Hz and 5000 Hz and characteristic resistance of 600 Ω.*

Solution

$$f_1 = 2000 \text{ Hz}, \quad f_2 = 5000 \text{ Hz}, \quad k = 600\ \Omega$$

$$L_1 = \frac{k(f_2 - f_1)}{\pi f_1 f_2} = \frac{600(5000 - 2000)}{\pi \times 2000 \times 5000)} = 57.3 \text{ mH}$$

$$C_1 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{1}{4\pi \times 600 \times (5000 - 2000)} = 0.044\ \mu\text{F}$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)} = \frac{600}{4\pi \times (5000 - 2000)} = 15.92 \text{ mH}$$

$$C_2 = \frac{f_2 - f_1}{\pi k f_1 f_2} = \frac{5000 - 2000}{\pi \times 600 \times 2000 \times 5000} = 0.16\ \mu\text{F}$$

The *T*-section filter consists of a parallel combination of $\frac{L_1}{2}$, i.e. 28.65 mH and a capacitor of 2 C_1, i.e. 0.088 μF in each series arm and a series combination of an inductor of 11.92 mH and a capacitor of 0.16 μF in shunt arm as shown in Fig. 11.25.

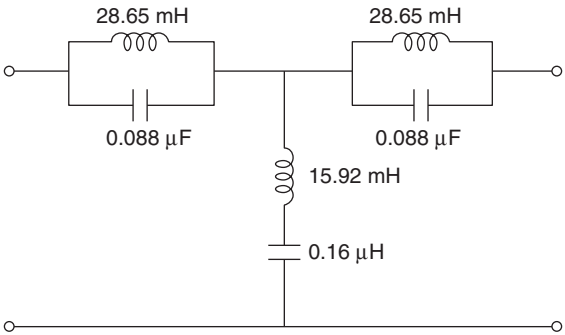


Fig. 11.25

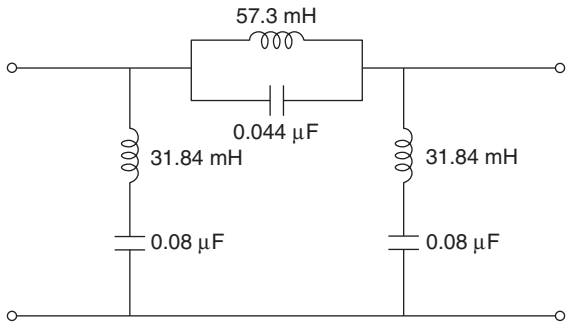


Fig. 11.26

The π-section filter consists of a parallel combination of an inductor of 57.3 mH and a capacitor of 0.044 μF in the series arm and a series combination of an inductor of $2L_2$, i.e. 31.84 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.08 μF in each shunt arm as shown in Fig. 11.26.

11.10 || *m*-DERIVED FILTERS

There are two disadvantages of constant-*k* filters.

(i) The attenuation is not sharp in the stop band.
(ii) The characteristic impedance varies widely in the pass band.

If the constant-*k* filter is regarded as prototype, it is possible to design a filter to have rapid attenuation in the stop band and the same characteristic impedance as the prototype throughout the pass band and stop band. Such a filter is called an *m- derived filter*.

Constant-*k* and *m*-derived *T* sections are shown in Fig. 11.27. The constant-*k T*-network is identical to that of the *m*-derived *T*-network except that the series arm is multiplier by *m*.

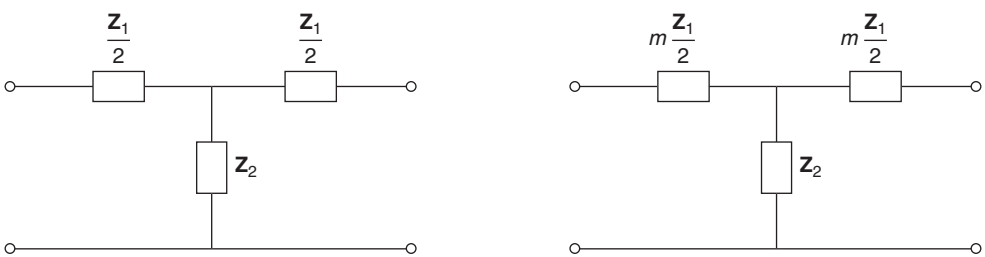


Fig. 11.27 *Constant-k and m-derived T sections*

For the constant-*k T* section,

$$\mathbf{Z}_{OT} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

For the *m*-derived *T* section,

$$\mathbf{Z}_{OT} = \sqrt{\frac{m^2 \mathbf{Z}_1^2}{4} + m \mathbf{Z}_1 \mathbf{Z}_2'}$$

Equating the characteristic impedance of the networks,

$$\sqrt{\frac{\mathbf{Z}_1^2}{4}+\mathbf{Z}_1\,\mathbf{Z}_2}=\sqrt{\frac{m^2\mathbf{Z}_1^2}{4}+m\mathbf{Z}_1\,\mathbf{Z}_2'}$$

$$\frac{\mathbf{Z}_1^2}{4}+\mathbf{Z}_1\,\mathbf{Z}_2=\frac{m^2\mathbf{Z}_1^2}{4}+m\mathbf{Z}_1\,\mathbf{Z}_2'$$

$$m\mathbf{Z}_1\,\mathbf{Z}_2'=\frac{\mathbf{Z}_1^2}{4}(1-m^2)+\mathbf{Z}_1\,\mathbf{Z}_2$$

$$\mathbf{Z}_2'=\left(\frac{1-m^2}{4m}\right)\mathbf{Z}_1+\frac{\mathbf{Z}_2}{m}$$

Hence, $\mathbf{Z}_2'$ must be an impedance composed of two series impedances $\left(\frac{1-m^2}{4m}\right)\mathbf{Z}_1$ and $\frac{\mathbf{Z}_2}{m}$. The term $\frac{1-m^2}{4m}$ should be positive to realise the impedance $\mathbf{Z}_2'$ physically, i.e. $0 < m < 1$. Thus, an *m*-derived filter can be obtained from constant-*k* filter by modifying its series and shunt impedances as shown in Fig. 11.28.

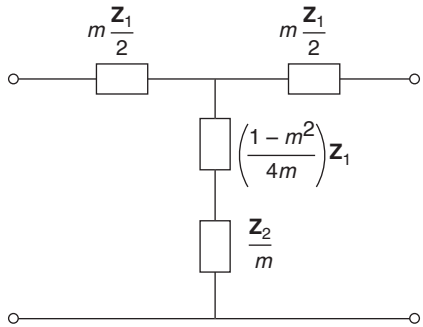


Fig. 11.28 *m-derived T-section filter*

Similarly, the constant-*k* π-section filter can be modified to *m*-derived π section filter. A constant-*k* and *m*-derived π sections are shown in Fig. 11.29.

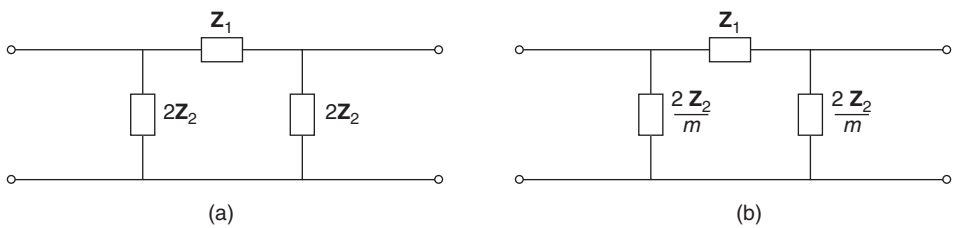


Fig. 11.29 *(a) Constant-k section (b) m-derived π section*

For the constant-*k* π section,

$$\mathbf{Z}_{O\pi}=\sqrt{\frac{\mathbf{Z}_1\,\mathbf{Z}_2}{1+\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}$$

For the *m*-derived π section,

$$\mathbf{Z}_{O\pi}=\sqrt{\frac{\mathbf{Z}_1'\dfrac{\mathbf{Z}_2}{m}}{1+\dfrac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}}}$$

Equating the characteristic impedance of the networks,

$$\sqrt{\frac{\mathbf{Z}_1\,\mathbf{Z}_2}{1+\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}=\sqrt{\frac{\mathbf{Z}_1'\dfrac{\mathbf{Z}_2}{m}}{1+\dfrac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}}}$$

$$\frac{\mathbf{Z}_1 \mathbf{Z}_2}{1+\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}} = \frac{\mathbf{Z}_1' \dfrac{\mathbf{Z}_2}{m}}{1+\dfrac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}}$$

$$\mathbf{Z}_1 \mathbf{Z}_2 \left(1+\frac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}\right) = \mathbf{Z}_1' \frac{\mathbf{Z}_2}{m}\left(1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)$$

$$\mathbf{Z}_1 \mathbf{Z}_2 + \frac{m\mathbf{Z}_1 \mathbf{Z}_1'}{4} = \frac{\mathbf{Z}_1' \mathbf{Z}_2}{m} + \frac{\mathbf{Z}_1 \mathbf{Z}_1'}{4m}$$

$$4\mathbf{Z}_1 \mathbf{Z}_2 + m\mathbf{Z}_1 \mathbf{Z}_1' = \frac{4\mathbf{Z}_1' \mathbf{Z}_2 + \mathbf{Z}_1 \mathbf{Z}_1'}{m}$$

$$\mathbf{Z}_1'\left(\frac{\mathbf{Z}_1}{m}+\frac{4\mathbf{Z}_2}{m}-m\mathbf{Z}_1\right) = 4\mathbf{Z}_1 \mathbf{Z}_2$$

$$\mathbf{Z}_1' = \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\dfrac{\mathbf{Z}_1}{4m}+\dfrac{\mathbf{Z}_2}{4m}-\dfrac{m\mathbf{Z}_1}{4}}$$

$$= \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\dfrac{\mathbf{Z}_1}{4m}(1-m^2)+\dfrac{\mathbf{Z}_2}{m}} = \frac{\mathbf{Z}_1 \mathbf{Z}_2 \dfrac{4m^2}{(1-m^2)}}{m\mathbf{Z}_1+\dfrac{4m^2}{m(1-m)}\mathbf{Z}_2}$$

$$= \frac{m\mathbf{Z}_1 \dfrac{4m}{1-m^2}\mathbf{Z}_2}{m\mathbf{Z}_1+\dfrac{4m}{1-m^2}\mathbf{Z}_2}$$

Hence, $\mathbf{Z}_1'$ must be an impedance composed of two parallel impedances $m\mathbf{Z}_1$ and $\left(\dfrac{4m}{1-m^2}\right)\mathbf{Z}_2$. Thus, an *m*-derived filter can be obtained from a constant-*k* filter by modifying its series and shunt impedances as shown in Fig. 11.30.

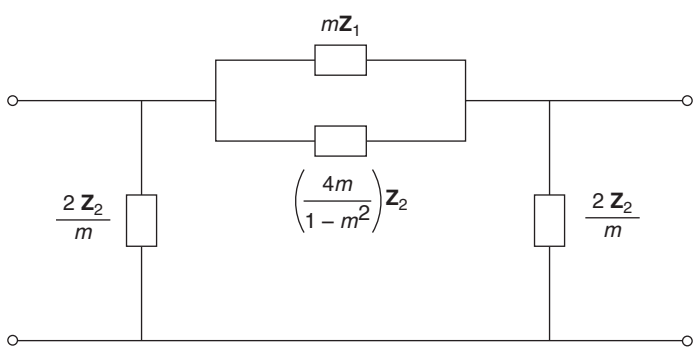


Fig. 11.30 *m-derived π-section filter*

11.11 *m*-DERIVED LOW-PASS FILTER

Figure 11.31 shows an *m*-derived low-pass *T* and π filter. At a particular frequency, the shunt arm of the *T* section or series arm of the π section will be in resonance giving a short circuit to the input for the *T* section and open circuit for the π section causing infinite attenuation.

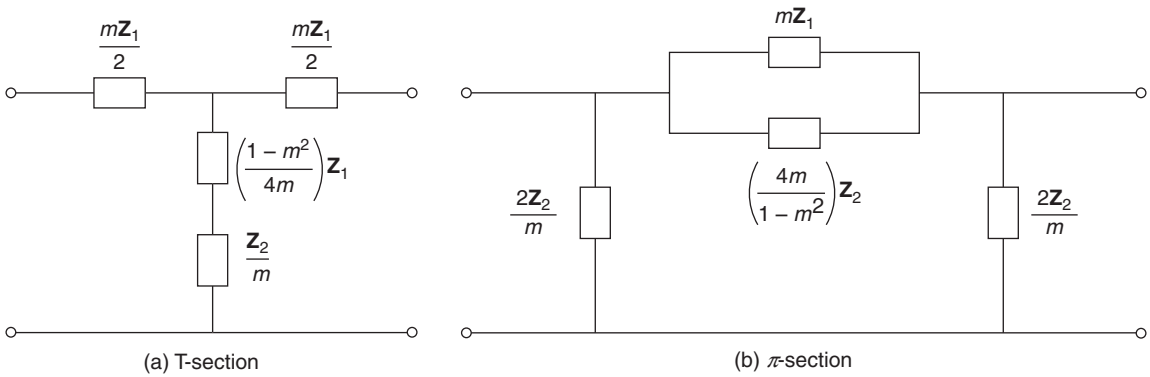


Fig. 11.31 *m-derived low-pass filter*

Let $\mathbf{Z}_1 = j\omega L$

$$\mathbf{Z}_2 = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

At resonant frequency,

$$\left(\frac{1-m^2}{4m}\right)\omega_0 L = \frac{1}{m\,\omega_0 C}$$

$$\omega_0{}^2 = \frac{4}{(1-m^2)\,LC}$$

$$\omega_0 = \frac{2}{\sqrt{LC(1-m^2)}}$$

$$f_0 = \frac{1}{\pi\sqrt{LC(1-m^2)}} = f_\infty$$

For low pass filter,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$

$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}$$

For sharp cut-off, f_∞ should be near to f_C. For smaller value of m, f_∞ comes close to f_c. The *m*-derived *T*-section low-pass filter is shown in Fig. 11.32.

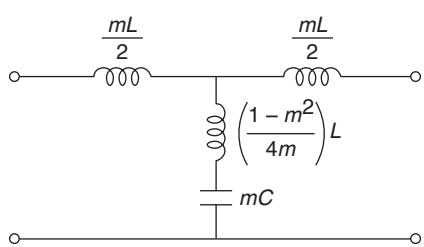


Fig. 11.32 *m-derived T-section low-pass filter*

Similarly, for *m*-derived π-section, the inductance and capacitance in the series arm constitute a resonant circuit.

At resonant frequency,

$$m\omega_0 L = \frac{1}{\left(\frac{1-m^2}{4m}\right)\omega_0 C}$$

$$\omega_0^2 = \frac{4}{LC(1-m^2)}$$

$$\omega_0 = \frac{4}{\sqrt{LC(1-m^2)}}$$

$$f_0 = \frac{1}{\pi\sqrt{LC(1-m^2)}} = f_\infty$$

For low-pass filter,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$

For both *m*-derived low-pass networks, $f_\infty > f_c$ for positive values of m $(0 < m < 1)$.

The *m*-derived π section low-pass filter is shown in Fig. 11.33.

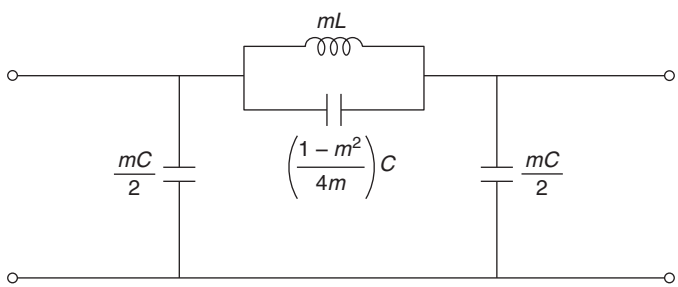


Fig. 11.33 *m-derived π-section low-pass filter*

The variation of attenuation with frequency is shown in Fig. 11.34. It is seen that the attenuation rises abruptly becomes infinite at f_∞ and then decreases for the *m*-derived filter.

Thus, an *m*-derived filter gives a sharp cut-off. The decrease in attenuation for frequencies higher than f_∞ can be avoided by connecting an *m*-derived filter and a constant-*k* filter in cascade.

The variation of β with frequency is shown in Fig. 11.35.

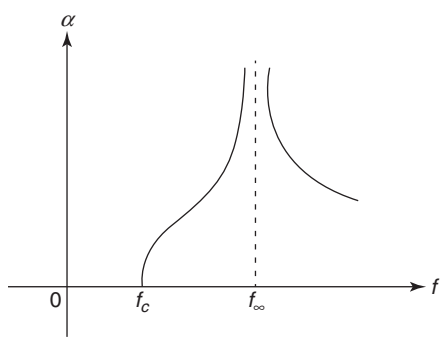


Fig. 11.34 *Variation of α with frequency*

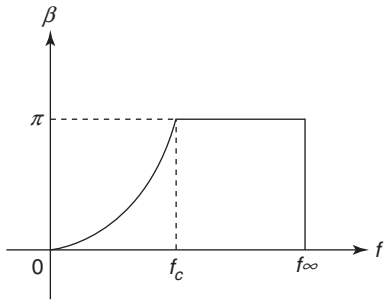


Fig. 11.35 *Variation of β with frequency*

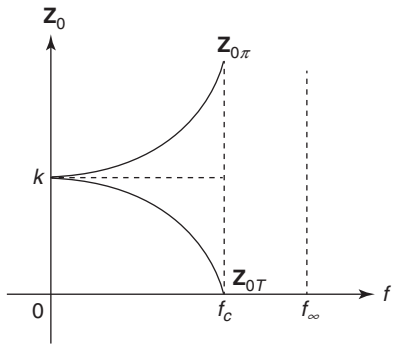


Fig. 11.36 *Variation in characteristic impedance with frequency.*

The variation of characteristic impedance with frequency is shown in Fig. 11.36. The characteristic impedance of an *m*-derived filter also varies with frequency. The impedance matching can be achieved by using half-section terminations.

Example 11.14 *Design an m-derived T-section low-pass filter having cut-off frequency of 800 Hz, design impedance of 500 Ω and frequency of attenuation of 1000 Hz.*

Solution $k = 500\ \Omega, \quad f_c = 800\ \text{Hz}, \quad f_\infty = 1000\ \text{Hz}$

For a constant-*k* *T*-section low-pass filter

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi \times 800} = 198.94\ \text{mH}$$

$$C = \frac{1}{\pi f_c\, k} = \frac{1}{\pi \times 800 \times 500} = 0.8\ \mu\text{F}$$

For an *m*-derived *T*-section low-pass filter

$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1-\left(\frac{800}{1000}\right)^2} = 0.6$$

$$\frac{mL}{2} = \frac{0.6 \times 198.94 \times 10^{-3}}{2} = 59.68\ \text{mH}$$

$$mC = 0.6 \times 0.8 \times 10^{-6} = 0.48\ \mu\text{F}$$

$$\left(\frac{1-m^2}{4m}\right)L = \left[\frac{1-(0.6)^2}{4(0.6)}\right] \times 198.94 \times 10^{-3} = 53.05\ \text{mH}$$

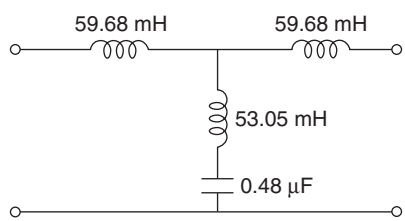


Fig. 11.37

The *m*-derived *T*-section low-pass filter is shown in Fig. 11.37.

Example 11.15 *Design an m-derived π-section low-pass filter having cut-off frequency of 1500 Hz, design impedance of 500 Ω and infinite attenuation frequency of 2000 Hz.*

Solution $f_C = 1500\ \text{Hz}, \quad k = 500\ \Omega, \quad f_\infty = 2000\ \text{Hz}$

For constant-*k* *π*-section low-pass filter,

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi \times 1500} = 106.1\ \text{mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi \times 1500 \times 500} = 0.424\ \mu\text{F}$$

For *m*-derived π-section low-pass filter,

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{1500}{2000}\right)^2} = 0.66$$

$$\frac{mC}{2} = \frac{0.66 \times 0.424 \times 10^{-6}}{2} = 0.14\ \mu\text{F}$$

$$mL = 0.66 \times 106.1 \times 10^{-3} = 70.03\ \text{mH}$$

$$\left(\frac{1 - m^2}{4m}\right) C = \frac{1 - (0.66)^2}{4 \times 0.66} \times 0.424 \times 10^{-6} = 0.09\ \mu\text{F}$$

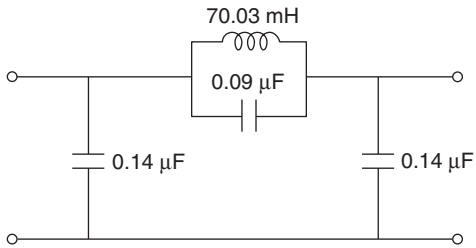


Fig. 11.38

The *m*-derived low-pass filter is shown in Fig. 11.38.

11.12 || *m*-DERIVED HIGH-PASS FILTER

Figure 11.39 shows *m*-derived high-pass *T* and π filter.

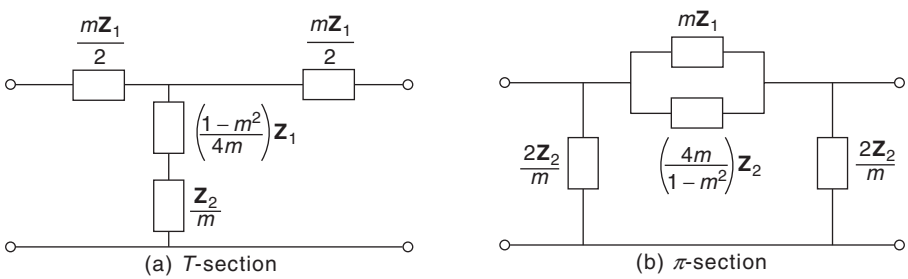


Fig. 11.39 *m-derived high-pass filter*

Let $\mathbf{Z}_1 = \dfrac{1}{j\omega c}$

$\mathbf{Z}_2 = j\omega L$

At resonant frequency,

$$\omega_0 \frac{L}{m} = \frac{1}{\omega_0 \left(\dfrac{4m}{1 - m^2}\right) C}$$

$$\omega_0^2 = \frac{1}{\dfrac{L}{m}\left(\dfrac{4m}{1 - m^2}\right) C} = \frac{1 - m^2}{4LC}$$

$$\omega_0 = \frac{\sqrt{1 - m^2}}{2\sqrt{LC}}$$

$$f_0 = \frac{\sqrt{1 - m^2}}{4\pi\sqrt{LC}} = f_\infty$$

For a high-pass filter,

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$$

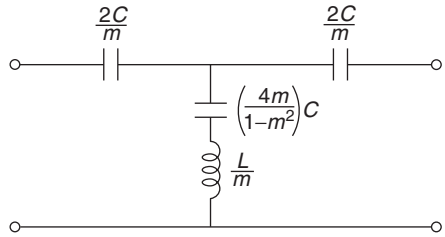


Fig. 11.40 *m-derived T-section high-pass filter*

The *m*-derived *T*-section high-pass filter is shown in Fig. 11.40.

Similarly, for an *m*-derived π section, the resonant circuit is formed by the series arm inductance and capacitance.

At resonant frequency,

$$\left(\frac{4m}{1-m^2}\right)\omega_0 L = \frac{1}{\frac{\omega_0}{m}C}$$

$$\omega_0^2 = \frac{1-m^2}{4LC}$$

$$\omega_0 = \frac{\sqrt{1-m^2}}{2\sqrt{LC}}$$

$$f_0 = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}} = f_\infty$$

For a high-pass filter,

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$$

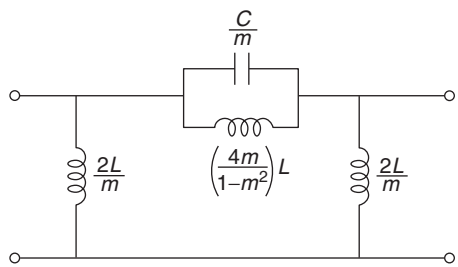


Fig. 11.41 *m-derived π-section high-pass filter*

The *m*-derived π section high-pass filter is shown in Fig. 11.41. The variation of attenuation phase constant and characteristic impedance with frequency are shown in Fig. 11.42 for an *m*-derived high-pass filter. The attenuation rises abruptly, becomes infinite at f_∞ and then decreases, thus giving a sharp cut-off.

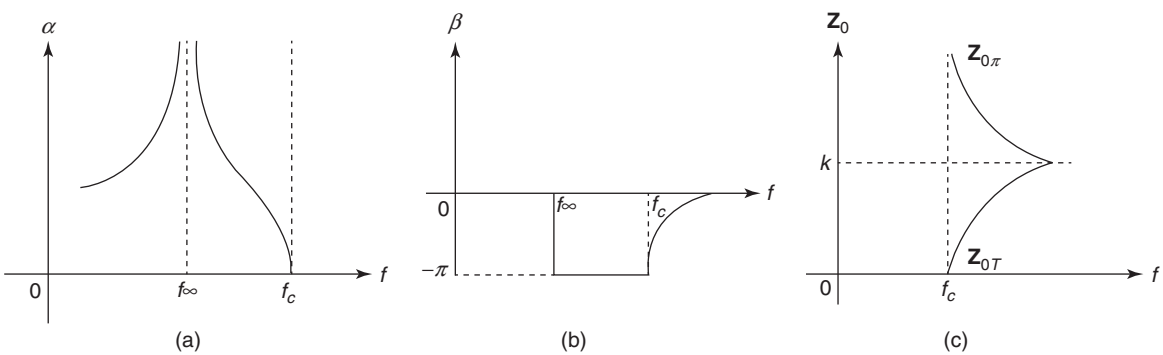


Fig. 11.42 *Variation of (a) attenuation, (b) phase constant, (c) characteristic impedance of m-derived high-pass filter.*

The use of *m*-derived and constant-*k* high pass filters in series gives a sharp cut-off and high attenuation in the entire stop band. The half-section termination provides impedance matching.

Example 11.16 *Design an m-derived T-section high-pass filter with a cutoff frequency of 2 kHz, design impedance of 700 Ω and m = 0.6.*

Solution $f_C = 2$ kHz, $k = 700\ \Omega$, $m = 0.6$

For constant-*k* *T*-section high-pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{700}{4\pi \times 2 \times 10^3} = 27.85 \text{ mH}$$

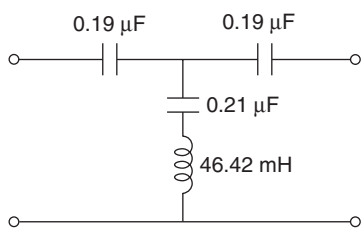
$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 2000 \times 700} = 0.057\ \mu\text{F}$$

For an *m*-derived *T*-section high-pass filter,

$$\frac{L}{m} = \frac{27.85 \times 10^{-3}}{0.6} = 46.42 \text{ mH}$$

$$\frac{2C}{m} = \frac{2 \times 0.057 \times 10^{-6}}{0.6} = 0.19\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)C = \frac{4 \times 0.6}{1-(0.6)^2} \times 0.057 \times 10^{-6} = 0.21\ \mu\text{F}$$



The *m*-derived high-pass filter is shown in Fig. 11.43. **Fig. 11.43**

Example 11.17 *Design the π-section of an m-derived high-pass filter having a design impedance of 600 Ω, cut-off frequency of 4 kHz and an infinite attenuation at 3.6 kHz.*

Solution $k = 600\ \Omega$, $f_c = 4$ kHz, $f_\infty = 3.6$ kHz

For a constant-*k* *π*-section high-pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 4 \times 10^3} = 11.94 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 4 \times 10^3 \times 600} = 0.033\ \mu\text{F}$$

For an *m*-derived *π*-section high-pass filter,

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1-\left(\frac{3.6 \times 10^3}{4 \times 10^3}\right)^2} = 0.436$$

$$\frac{2L}{m} = \frac{2 \times 11.94 \times 10^{-3}}{0.436} = 54.77 \text{ mH}$$

$$\frac{C}{m} = \frac{0.033 \times 10^{-6}}{0.436} = 0.076\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)L = \frac{4\times 0.436}{1-(0.436)^2}\times 11.94\times 10^{-3} = 25.71 \text{ mH}$$

The m-derived π-section high-pass filter is shown in Fig. 11.44.

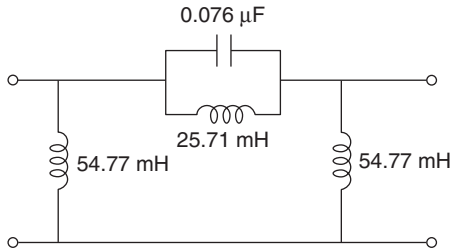


Fig. 11.44

11.13 TERMINATING HALF SECTIONS

A filter is composed of a number of sections. As the characteristic impedance of an equivalent T or π section does not match with each other, a half section is used for impedance matching between T and π sections.

The input impedance of the half section is same as the characteristic impedance of a T section, while the output impedance of the half section is same as the characteristic impedance of the π section. Hence, matching can be obtained if a half section is connected in between a T and π section.

11.13.1 Constant-*k* Half Sections

Figure 11.45(a) shows a constant-k T-section filter. If this section is bisected longitudinally, a half section is obtained as shown in Fig. 11.45 (b).

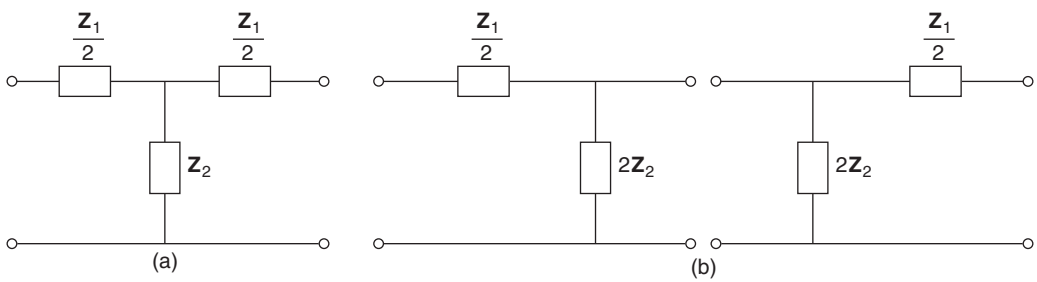


Fig. 11.45 *(a) Constant-k T-section (b) constant-k half T-section*

Similarly, a constant-k π section can be bisected into two half sections as shown in Fig. 11.46.

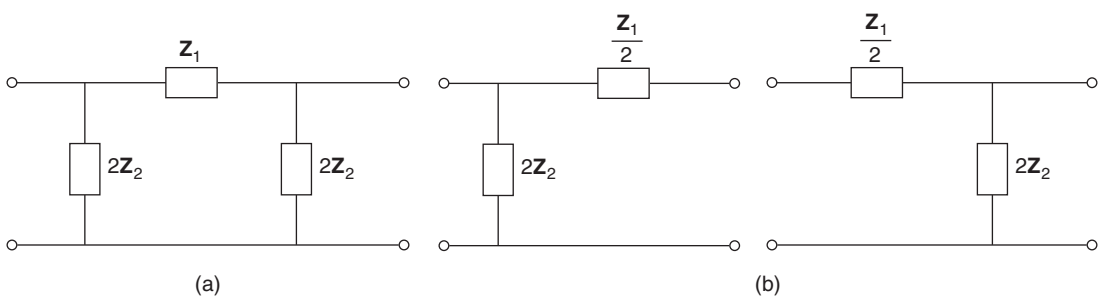


Fig. 11.46 *(a) Constant-k π-section (b) constant-k half π-section*

Figure 11.47 shows a constant-k half section. The image impedance of the section as seen from terminals 1-2 and terminals 3-4 can be found from the open circuit and short-circuit impedances.

$$\mathbf{Z}_{12} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\frac{(2\mathbf{Z}_2)\left(2\mathbf{Z}_2 \dfrac{\mathbf{Z}_1}{2}\right)}{2\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1}{2}}} = \sqrt{\frac{4\mathbf{Z}_1\,\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}} = \mathbf{Z}_{0\pi}$$

$$\mathbf{Z}_{34} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\left(\frac{\mathbf{Z}_1}{2} + 2\mathbf{Z}_2\right)\left(\frac{\mathbf{Z}_1}{2}\right)}$$

$$= \sqrt{\mathbf{Z}_1\,\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)} = \mathbf{Z}_{0T}$$

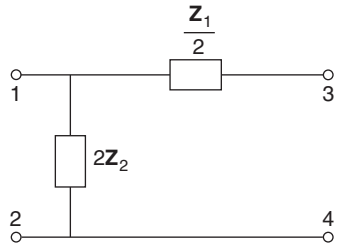


Fig. 11.47 *Constant-k half section*

Thus, the half section has the impedance characteristics of a π section between terminals 1-2 and that of a T section between terminals 3-4. Hence, this half section can be used to match a π section to a T section. It can also be used to match a filter section to a terminating impedance which differs from the characteristic impedance of a π section.

11.13.2 *m*-derived Half Sections

Figure 11.48 shows and *m*-derived half section. The image impedance of the section as seen from terminals 1-2 and terminals 3-4 can be found from the open circuit and short-circuit impedances.

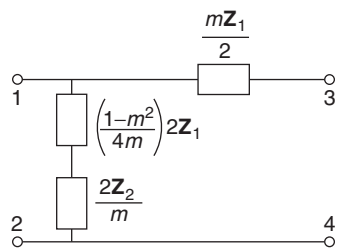


Fig. 11.48 *m-derived half section*

$$\mathbf{Z}_{12} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)\left(m\dfrac{\mathbf{Z}_1}{2}\right)}{\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)^2\left(\dfrac{m\mathbf{Z}_1}{2}\right)}{\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)^2\left(\dfrac{m\mathbf{Z}_1}{2}\right)^2}{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}\right)\left(\dfrac{m\mathbf{Z}_1}{2}\right)}}$$

$$= \sqrt{\frac{\left[\left(\dfrac{1-m^2}{4}\right)\mathbf{Z}_1^2 + \mathbf{Z}_1\,\mathbf{Z}_2\right]^2}{\dfrac{\mathbf{Z}_1^2}{4} - \dfrac{m^2\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{m^2\mathbf{Z}_1^2}{4}}}$$

$$= \frac{\mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{(1-m^2)}{4}\mathbf{Z}_1^2}{\sqrt{\mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}}$$

$$= \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_{0T}} \left[1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} (1-m)^2 \right]$$

$$= \mathbf{Z}_{0\pi} \left[1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} (1-m)^2 \right]$$

$$\mathbf{Z}_{34} = \sqrt{\mathbf{Z}_{oc}\, \mathbf{Z}_{sc}}$$

$$= \sqrt{\left[\frac{m\mathbf{Z}_1}{2} + \left(\frac{1-m^2}{2m} \right) \mathbf{Z}_1 + \frac{2\mathbf{Z}_2}{m} \right] \left(\frac{m\mathbf{Z}_1}{2} \right)}$$

$$= \sqrt{\frac{m^2 \mathbf{Z}_1^2}{4} + \frac{\mathbf{Z}_1^2}{4} - \frac{m^2 \mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

$$= \sqrt{\mathbf{Z}_1 \mathbf{Z}_2 \left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} \right)}$$

$$= \mathbf{Z}_{0T}$$

Hence, the terminals 3-4 of the m-derived half section can be used to match the impedance of a constant-*k* *T*-section filter.

Example 11.18 *Find the values of shunt and series element of each half section of a constant-k T-section high-pass filter. The termination is 600 Ω and the filter is cut off below 20000 Hz.*

Solution $k = 600\ \Omega, \quad f_c = 20000$ Hz

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 20000} = 2.39 \text{ mH}$$

$$C = \frac{1}{4\pi f_c\, k} = \frac{1}{4\pi \times 20000 \times 600} = 0.0066\ \mu\text{F}$$

The half section consists of a capacitor of $2C$, i.e. 0.0133 μF in the series branch and an inductor of $2L$, i.e. 4.78 mH in the shunt branch as shown in Fig. 11.49.

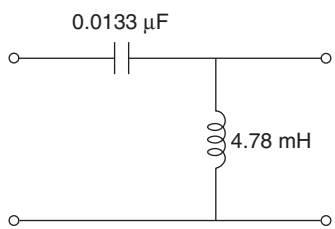


Fig. 11.49

Example 11.19 *A prototype high-pass filter with $L = 4.77$ mH and $C = 0.0133\ \mu$F has a cut-off frequency of 10 kHz and design impedance of 600 Ω. Design a suitable terminating half section with $m = 0.6$ such as to operate with same cut-off frequency.*

Solution $L = 4.77$ mH, $C = 0.0133$ μF, $f_c = 10$ kHz, $k = 600\ \Omega$

For *m*-derived *T*-section high-pass filter

$$\frac{L}{m} = \frac{4.77 \times 10^{-3}}{0.6} = 7.95 \text{ mH}$$

$$\frac{2C}{m} = \frac{2 \times 0.0133 \times 10^{-6}}{0.6} = 0.044\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2} \right) C = \frac{4 \times 0.6}{1-(0.6)^2} \times 0.0133 \times 10^{-6} = 0.05\ \mu\text{F}$$

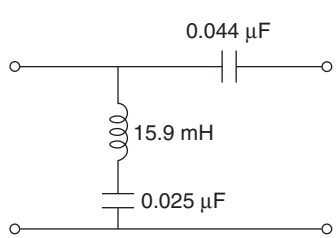


Fig. 11.50

The *m*-derived *T*-half section is shown in Fig. 11.50.
For *m*-derived π-section high-pass filter,

$$\frac{2L}{m} = \frac{2 \times 4.77 \times 10^{-3}}{0.6} = 15.9 \text{ mH}$$

$$\frac{C}{m} = \frac{0.0133 \times 10^{-6}}{0.6} = 0.022 \ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)L = \frac{4 \times 0.6}{1-(0.6)^2} \times 4.77 \times 10^{-3} = 17.89 \text{ mH}$$

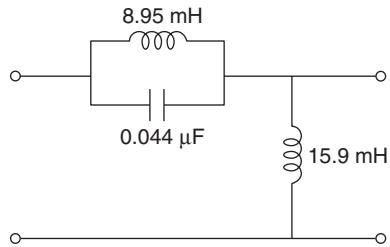


Fig. 11.51

The *m*-derived π-half section is shown in Fig. 11.51.

11.14 COMPOSITE FILTER

A constant-*k* filter does not give a sharp cut-off. An *m*-derived filter section has a sharp cut off, but its attenuation decreases for frequencies beyond f_∞. Moreover, two terminating half sections, one at each end, are needed for impedance matching. A filter composed of all the above sections is known as a *composite filter*. Figure 11.52 shows a composite low-pass filter.

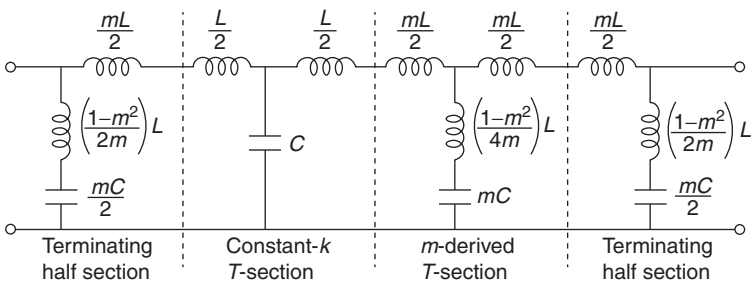


Fig. 11.52 *Composite low-pass filter*

A composite filter consists of the following components:

(i) One or more constant-*k* sections to produce cut-off frequency, i.e. transition from pass band to the stop band at a specified frequency f_c.
(ii) One or more *m*-derived sections to give infinite attenuation at a frequency f_∞ near to f_c.
(iii) Two terminating *m*-derived half sections with $m = 0.6$ to provide matching with the source and the load.

Figure 11.53 shows the attenuation versus frequency characteristics of a constant-*k* low-pass filter, *m*-derived low-pass filter and a composite filter.

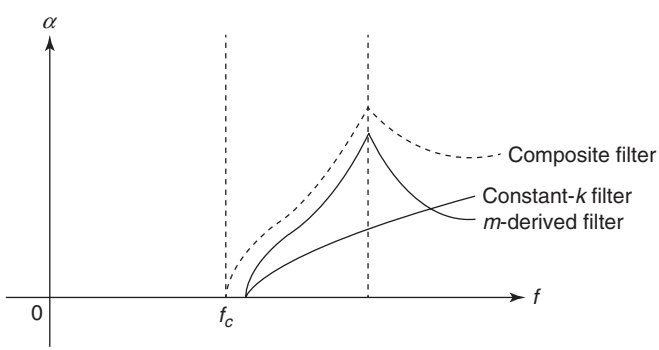


Fig. 11.53 *Attenuation characteristics*

For a composite filter, the attenuation rises very rapidly with frequency in the range f_c to f_∞ and falls only marginally with frequency.

Example 11.20 *Design a composite low-pass filter to have a cut-off frequency of 1000 Hz and a characteristic impedance of 600 Ω. Use one constant-k T section, one m-derived T section and two terminating half sections with m = 0.6. The frequency of infinite attenuation is 1050 Hz.*

Solution $f_c = 1000$ Hz, $k = 600\ \Omega$, $f_\infty = 1050$ Hz

For constant-*k* *T* section,

$$L = \frac{k}{\pi f_c} = \frac{600}{\pi \times 1000} = 0.19 \text{ H}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi \times 1000 \times 600} = 0.53\ \mu\text{F}$$

Hence, the constant-*k* *T* section filter consists of an inductor of $\frac{L_1}{2}$, i.e. 0.095 H in series arm and a capacitor C_1, i.e. 0.53 μF in the shunt arm.
For the *m*-derived full *T* section,

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{1000}{1050}\right)^2} = 0.305$$

$$\frac{mL}{2} = \frac{0.305 \times 0.19}{2} = 0.029 \text{ H}$$

$$mC = 0.305 \times 0.53 \times 10^{-6} = 0.162\ \mu\text{F}$$

$$\left(\frac{1 - m^2}{4m}\right) L = \frac{1 - (0.305)^2}{4 \times 0.305} \times 0.19 = 0.14 \text{ H}$$

For each terminating half section with $m = 0.6$,

$$\frac{mL}{2} = \frac{0.6 \times 0.19}{2} = 0.057 \text{ H}$$

$$\frac{mC}{2} = \frac{0.6 \times 0.53 \times 10^{-6}}{2} = 0.159\ \mu\text{F}$$

$$\left(\frac{1 - m^2}{2m}\right) L = \frac{1 - (0.6)^2}{2 \times 0.6} \times 0.19 = 0.101 \text{ H}$$

The composite low-pass filter is shown in Fig. 11.54.

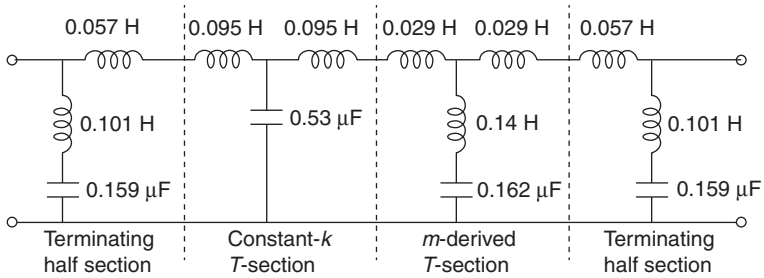


Fig. 11.54

Example 11.21 *Design a composite high-pass filter having a characteristic impedance of 600 Ω and a cut-off frequency of 1000 Hz. Use one a constant-k T section, one m-derived T section with m = 0.2 and two m-derived terminating half sections with m = 0.6.*

Solution $k = 600\ \Omega, \quad f_c = 1000\ \text{Hz}$

For a constant-*k T* section,

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 1000} = 0.0477\ \text{H}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 1000 \times 600} = 0.133\ \mu\text{F}$$

Hence, the constant-*k T* section filter consists of an inductor of 0.0477 H in the shunt branch and a capacitor of 2*C*, i.e. 0.266 μF in each series branch.

For an *m*-derived *T* section with $m = 0.2$,

$$\frac{2C}{m} = \frac{2 \times 0.133 \times 10^{-6}}{0.2} = 1.33\ \mu\text{F}$$

$$\frac{L}{m} = \frac{0.0477}{0.2} = 0.239\ \text{H}$$

$$\left(\frac{4m}{1-m^2}\right)C = \frac{4 \times 0.2}{1-(0.2)^2} \times 0.133 \times 10^{-6} = 0.11\ \mu\text{F}$$

For each terminating half section with $m = 0.6$,

$$\frac{2C}{m} = \frac{2 \times 0.133 \times 10^{-6}}{0.6} = 0.44\ \mu\text{F}$$

$$\frac{2L}{m} = \frac{2 \times 0.0477}{0.6} = 0.159\ \text{H}$$

$$\left(\frac{2m}{1-m^2}\right)C = \frac{2 \times 0.6}{1-(0.6)^2} \times 0.133 \times 10^{-6} = 0.25\ \mu\text{F}$$

The composite high-pass filter is shown in Fig. 11.55.

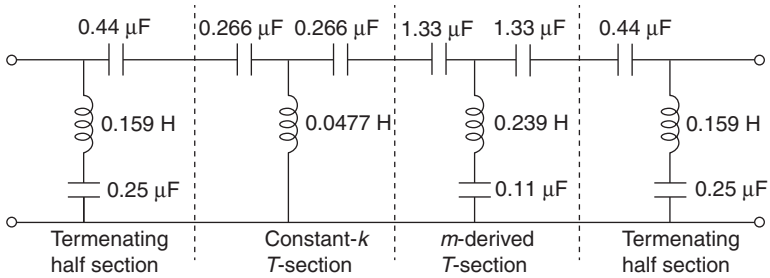


Fig. 11.55

11.15 ATTENUATOR

An attenuator is a two-port resistive network. It is used to reduce the signal level when used between a generator and load. Attenuators may be symmetrical or asymmetrical. They may provide fixed or variable attenuation. A fixed attenuator is also called a pad. The attenuation is expressed in decibels (dB).

$$\text{Attenuation in } dB = 10 \log_{10} \frac{P_1}{P_2}$$

where P_1 is the input power and P_2 is the output power.
For a properly matched network,

$$P_1 = I_1^2 R_0 = \frac{V_1^2}{R_0}$$

$$P_2 = I_2^2 R_0 = \frac{V_2^2}{R_0}$$

where R_0 is the characteristic resistance of the network.

Hence, $$\text{attenuation in } dB = 20 \log_{10} \frac{V_1}{V_2} = 20 \log_{10} \frac{I_1}{I_2}$$

Attenuation is also expressed in nepers as the natural logarithm of the voltage or current ratio.

$$\text{Attenuation in nepers} = \ln \frac{V_1}{V_2} = \ln \frac{I_1}{I_2} = \frac{1}{2} \ln \frac{P_1}{P_2}$$

If $$\frac{V_1}{V_2} = \frac{I_1}{I_2} = N \text{ then } \frac{P_1}{P_2} = N^2$$

$$\text{Attenuation in } dB = 20 \log_{10} N$$

$$N = \text{antilog}\left(\frac{dB}{20}\right)$$

where N is the attenuation in nepers.

Relation between Decibel and Neper

$$\text{Attenuation in nepers} = \ln\frac{V_1}{v_2}$$

$$\text{Attenuation in } dB = 20\log_{10}\frac{V_1}{V_2}$$

Changing the base of the algorithm,

$$\text{attenuation in } dB = 20\frac{\ln\frac{V_1}{V_2}}{\ln 10} = \frac{20}{2.303}\ln\frac{V_1}{V_2} = 8.686 \text{ (attenuation in neper)}$$

$$\text{Attenuation in nepers} = \frac{1}{8.686}\text{ (attenuation in } dB) = 0.115 \text{ (attenuation in } dB)$$

11.16 LATTICE ATTENUATOR

A lattice is a general type of symmetrical and balanced network. Any symmetrical balanced or unbalanced network can be transformed into an equivalent lattice network. Figure 11.56 shows a lattice network terminated through a characteristic resistance R_0.

The elements of a lattice attenuator can be specified in terms of characteristic impedance and propagation constant.

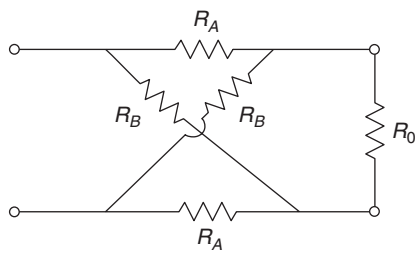


Fig. 11.56 *Lattice attenuator*

We know that $Z_0 = \sqrt{Z_{oc}\, Z_{sc}}$

$$Z_{oc} = \frac{R_A + R_B}{2}$$

$$Z_{sc} = \frac{2R_A R_B}{R_A + R_B}$$

$$Z_0 = R_0 = \sqrt{Z_{oc}\, Z_{sc}} = \sqrt{\left(\frac{R_A + R_B}{2}\right)\left(\frac{2R_A R_B}{R_A + R_B}\right)} = \sqrt{R_A R_B}$$

Redrawing the network as shown in Fig. 11.57,

Applying KVL to the loop,

$$V_1 = R_0 I_1 = R_A(I_1 - I) + R_0 I_2 + R_A(I + I_2)$$

$$R_0 I_1 = R_A(I_1 + I_2) + R_0 I_2$$

$$(R_0 - R_A)I_1 = (R_A + R_0)I_2$$

$$\frac{I_1}{I_2} = \frac{R_A + R_0}{R_0 - R_A} = \frac{1 + \frac{R_A}{R_0}}{1 - \frac{R_A}{R_0}}$$

$$N = e^{\alpha} = \frac{I_1}{I_2} = \frac{1 + \frac{R_A}{R_0}}{1 - \frac{R_A}{R_0}} = \frac{1 + \sqrt{\frac{R_A}{R_0}}}{1 - \sqrt{\frac{R_A}{R_0}}}$$

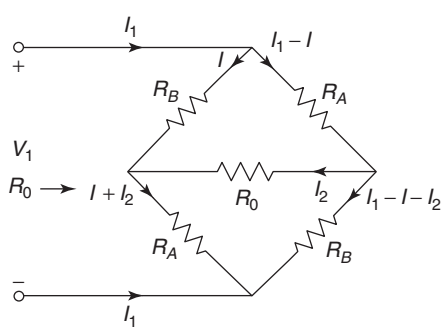


Fig. 11.57 *Modified network*

$$\alpha = \log\left(\frac{1+\sqrt{\frac{R_A}{R_0}}}{1-\sqrt{\frac{R_A}{R_0}}}\right)$$

We can express R_A and R_B in terms of attenuation.

$$N\left(1-\frac{R_A}{R_0}\right) = 1+\frac{R_A}{R_0}$$

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

Similarly, $R_B = R_0\left(\frac{N+1}{N-1}\right)$

These equations can be used to find the equivalent lattice network for *T* and π networks using the bisection theorem.

Bisection Theorem

A network is said to have been bisected when the open-circuited and short-circuited input impedances of the two bisected networks are equivalent. Also, the square root of the product of these impedances is the characteristic impedance of the original whole network.

The bisection theorem states that '*any symmetrical balanced or unbalanced network can be transformed into an equivalent lattice network. The series arm of the lattice is equal to the short-circuited impedance of the bisected network and the diagonal arm is equal to the open-circuited impedance of the bisected network'.*

Example 11.22 *Design a lattice attenuator if the characteristic resistance is 200 Ω and the attenuation is 20 dB.*

Solution $N(dB) = 20\,dB, \quad R_0 = 200\ \Omega$

$$N(dB) = 20\log_{10} N$$

$$20 = 20\log_{10} N$$

$$N = 10$$

$$R_A = R_0\left(\frac{N-1}{N+1}\right) = 200\left(\frac{10-1}{10+1}\right) = 163.6\ \Omega$$

$$R_B = R_0\left(\frac{N+1}{N-1}\right) = 200\left(\frac{10+1}{10-1}\right) = 244.5\ \Omega$$

The lattice attenuator is shown in Fig. 11.58.

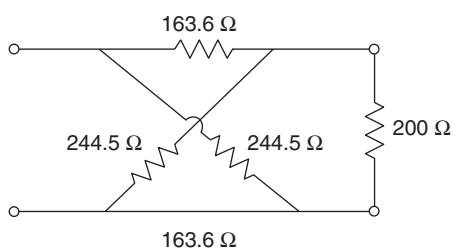


Fig. 11.58

11.17 *T*-TYPE ATTENUATOR

A *T*-type attenuator is a symmetrical but unbalanced network and is shown in Fig. 11.59 (a). Its bisected network is shown in Fig. 11.59 (b).

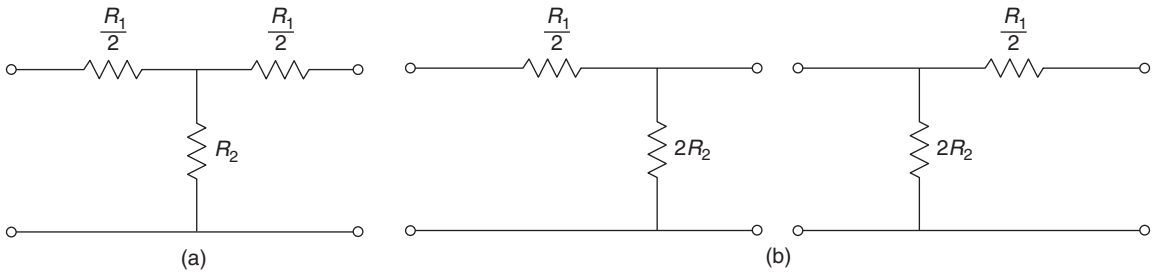


Fig. 11.59 *(a) T-type attenuator (b) Bisection of T-type attenuator*

From the bisection theorem, the series arm of the lattice network is equal to the short-circuited input resistance of the bisected network and the diagonal arm is equal to the open-circuited input resistance of the bisected network.

$$R_A = \frac{R_1}{2}$$

$$R_B = \frac{R_1}{2} + 2R_2$$

From lattice network,

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

and

$$R_B = R_0\left(\frac{N+1}{N-1}\right)$$

Solving these equations,

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right)$$

$$2R_2 = R_0\left(\frac{N+1}{N-1} - \frac{N-1}{N+1}\right)$$

$$R_2 = \frac{R_0}{2}\left(\frac{4N}{N^2-1}\right) = \frac{2N\,R_0}{N^2-1}$$

Example 11.23 *If the ratio of R_1 and R_2 of a symmetrical T network is $\frac{1}{4}$, find the ratio of the input current to the output current. Also, calculate the attenuation in dB.*

Solution

$$\frac{R_1}{R_2} = \frac{1}{4}$$

For a *T*-type attenuator,

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) \quad \text{(i)}$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) \quad \text{(ii)}$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\dfrac{2N}{N^2-1}}{2\left(\dfrac{N-1}{N+1}\right)} = \frac{1}{4}$$

$$\frac{2N}{N^2-1} = \frac{1}{2}\left(\frac{N-1}{N+1}\right)$$

$$\frac{2N}{(N+1)(N-1)} = \frac{1}{2}\left(\frac{N-1}{N+1}\right)$$

$$4N = (N-1)^2$$

$$N^2 - 6N + 1 = 0$$

$$N = 5.83 \quad \text{or} \quad N = 0.172$$

$$N \text{ in } dB = 20\log_{10} 5.83 = 15.31\ dB$$

or $$N \text{ in } dB = 20\log_{10} 0.172 = -15.29\ dB$$

Since attenuation cannot be negative,

$$N = 5.83 \text{ nepers} = 15.31\ dB$$

$$\frac{I_1}{I_2} = N = 5.83$$

Example 11.24 *An attenuator is composed of symmetrical T-section having a series arm of 175 Ω and a shunt arm of 350 Ω. Find the characteristic impedance and attenuation in dB.*

Solution

$$\frac{R_1}{2} = 175\ \Omega, \quad R_1 = 350\ \Omega, \quad R_2 = 350\ \Omega$$

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) = 350 \qquad \ldots\text{(i)}$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 350 \qquad \ldots\text{(ii)}$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\dfrac{2N}{N^2-1}}{2\left(\dfrac{N-1}{N+1}\right)} = 1$$

$$\frac{2N}{N^2-1} = 2\left(\frac{N-1}{N+1}\right)$$

$$\frac{N}{(N+1)(N-1)} = \frac{N-1}{N+1}$$

$$N = (N-1)^2$$

$$N^2 - 3N + 1 = 0$$

$$N = 2.618 \quad \text{or} \quad N = 0.3819$$

$$N \text{ in } dB = 20\log_{10} 2.618 = 8.36\ dB$$

$$\text{or } N \text{ in } dB = 20\log_{10} 0.3819 = -8.36\ dB$$

Since attenuation cannot be negative,

$$N = 2.618 \text{ nepers} = 8.36\ dB$$

Putting $N = 2.618$ in Eq. (i),

$$350 = 2R_0\left(\frac{2.618-1}{2.618+1}\right)$$

$$R_0 = 391.32\ \Omega$$

Example 11.25 *Design a T-type attenuator to give an attenuation of 60 dB and characteristic resistance of 500 Ω.*

Solution

$$N(dB) = 60\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$60 = 20\log_{10} N$$

$$N = 1000$$

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) = 2\times 500\left(\frac{1000-1}{1000+1}\right) = 998\ \Omega$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 500\left[\frac{2\times 1000}{(1000)^2-1}\right] = 1\ \Omega$$

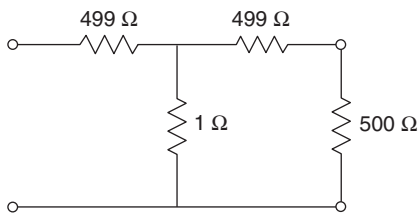


Fig. 11.60

The *T*-type attenuator is shown in Fig. 11.60.

11.18 π-TYPE ATTENUATOR

A symmetrical π-type attenuator is shown in Fig. 11.61 (a). Its bisected network is shown in Fig. 11.61 (b).

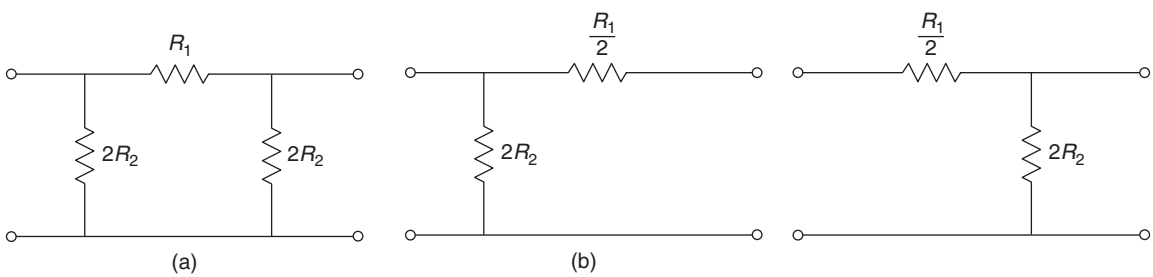


Fig. 11.61 *(a) π-attenuator (b) Bisection of π-type attenuator*

From the bisection theorem, the series arm of the lattice network is equal to the short-circuited input resistance of the bisected network and the diagonal arm is equal to the open-circuit input resistance of the bisected network.

$$R_A = \frac{R_1\ R_2}{\frac{R_1}{2} + 2R_2}$$

$$R_B = 2R_2$$

From lattice network,

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

$$R_B = R_0\left(\frac{N+1}{N-1}\right)$$

Solving these equations,

$$R_1 = \frac{2R_A R_B}{R_B - R_A} = R_0\left(\frac{N^2-1}{2N}\right)$$

$$R_2 = \frac{R_B}{2} = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right)$$

Example 11.26 *An attenuator is composed of a symmetrical π-section having series arm of 275 Ω and each shunt arm of 450 Ω. Find the characteristic impedance and attenuation in dB.*

Solution

$$R_1 = 275\ \Omega, \quad 2R_2 = 450\ \Omega, \quad R_2 = 225\ \Omega$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 275 \qquad \ldots(i)$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = 225 \qquad \ldots(ii)$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\frac{N^2-1}{2N}}{\frac{1}{2}\left(\frac{N+1}{N-1}\right)} = \frac{275}{225}$$

$$\frac{(N+1)(N-1)}{2N} = 0.611\left(\frac{N+1}{N-1}\right)$$

$$(N-1)^2 = 2N(0.611)$$

$$N^2 - 3.222N + 1 = 0$$

$$N = 2.872 \quad \text{or} \quad N = 0.348$$

$$N \text{ in } dB = 20\log_{10} 2.872 = 9.16\ dB$$

or

$$N \text{ in } dB = 20\log_{10} 0.348 = -9.16\ dB$$

Since attenuation cannot be negative,

$$N = 2.872 \text{ neper} = 9.16\ dB$$

Putting $N = 2.872$ in Eq. (i),

$$275 = R_0\left[\frac{(2.872)^2 - 1}{2(2.872)}\right]$$

$$R_0 = 217.92\ \Omega$$

Example 11.27 *Design a π-type attenuator to give attenuation of 20 dB and characteristic resistance of 500 Ω.*

Solution

$$N(dB) = 20\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$20 = 20\log_{10} N$$

$$N = 10$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 500\left[\frac{(10)^2-1}{2\times 10}\right] = 2475\ \Omega$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = \frac{500}{2}\left(\frac{10+1}{10-1}\right) = 305.5\ \Omega$$

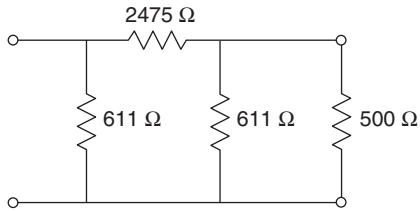


Fig. 11.62

The π-type attenuator is shown in Fig. 11.62.

11.19 LADDER-TYPE ATTENUATOR

When a number of symmetrical *T*-or is π-type attenuators are connected in cascade to provide the attenuation in steps, the resultant network is termed **ladder-type attenuator.** Figure 11.63 shows a ladder-type attenuator with three identical and symmetrical π-sections connected in cascade.

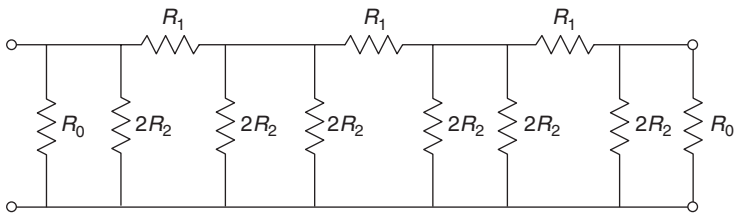


Fig. 11.63 *Ladder-type attenuator*

The output terminals are terminated with resistance R_0 which is the characteristic resistance of the network. The source end also has resistance R_0 so that it is a matched network. The net attenuation is the sum of the attenuation of individual sections between the input and output terminals.

Example 11.28 *Design a ladder-type attenuator for a load resistance of 500 Ω and an attenuation of 3 dB per step.*

Solution

$$N(dB) = 3\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$3 = 20\log_{10} N$$

$$N = 1.413$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 500\left[\frac{(1.413)^2-1}{2\times 1.413}\right] = 176.32\ \Omega$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = \frac{500}{2}\left(\frac{1.413+1}{1.413-1}\right) = 1460.65\ \Omega$$

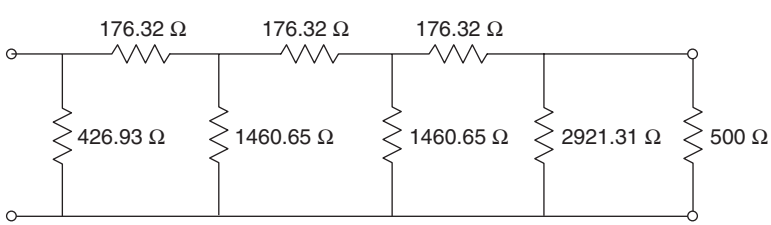


Fig. 11.64

Exercises

11.1 A *T*-section low-pass filter has a series inductance of 80 mH and a shunt capacitance of 0.022 μF. Find the cut-off frequency and nominal impedance. Also design the equivalent π-section.

$$\left[\begin{array}{r} f_c = 7.587 \text{ kHz}, \quad k = 1.907 \text{ k}\Omega, L = 80 \text{ mH}, \\ \frac{C}{2} = 0.011 \ \mu\text{F} \end{array}\right]$$

11.2 Calculate the frequency of a constant-*k*, *T*-section low-pass filter having a cut-off frequency of 1000 Hz at which it has an attenuation of 10 *dB*.

$$[f = 1170 \text{ Hz}]$$

11.3 Design a low-pass constant-*k* (a) *T*-section, and (b) π-section filter with $f_c = 6$ kHz and $R_O = 500\ \Omega$. Calculate α and β for the filters for $f = 10$ kHz. Also determine the frequency at which the attenuation is 10 *dB*.

$$\left[\begin{array}{r} \alpha = 2.2 \text{ neper}, \quad \beta = 1.46 \text{ radians}, \\ \beta = \pi \text{ at } 10 \text{ kHz} T \rightarrow \frac{L}{2} = 13.25 \text{ mH}, \\ C = 0.106 \ \mu\text{F} \pi \rightarrow L = 26.5 \text{ mH}, \\ \frac{C}{2} = 0.053 \ \mu\text{F} \\ f = 7 \text{ kHz} \end{array}\right]$$

11.4 Design a π-section constant-*k* high-pass filter having a cut-off frequency $f_c = 8$ kHz and nominal characteristic impedance $k = 600\ \Omega$. Find (a) its characteristic impedance at a frequency of 12 kHz, (b) phase constant at frequency of 12 kHz, (c) attenuation at frequency of 800 Hz.

$$\left[\begin{array}{r} L = 5.971 \text{ mH}, \quad C = 0.016 \ \mu\text{F}, \\ Z_{O\pi} = 805\ \Omega, \quad \beta = 83.6°, \quad \alpha = 5 \text{ nepers} \end{array}\right]$$

11.5 Design a band-pass constant-*k* filter with $f_1 = 2$ kHz and $f_2 = 3$ kHz and $k = 500\ \Omega$.

$$\left[\begin{array}{l} L_1 = 159 \text{ mH}, \quad C_1 = 0.026 \ \mu\text{F}, \\ L_2 = 20.8 \text{ mH}, \quad C_2 = 0.637 \ \mu\text{F} \end{array}\right]$$

11.6 Design a band-stop constant-*k* filter with cut-off frequencies of 8 kHz and 12 kHz and nominal characteristic impedance of 500 Ω.

$$\left[\begin{array}{l} L_1 = 6.6 \text{ mH}, \quad C_1 = 0.0398 \ \mu\text{F}, \\ L_2 = 9.9 \text{ mH}, \quad C_2 = 0.0265 \ \mu\text{F} \end{array}\right]$$

11.7 Design an *m*-derived *T*-section low-pass filter with cut-off frequency $f_c = 1$ kHz, design impedance of 600 Ω and frequency of infinite attenuation of 1050 Hz.

$$[L = 191.08 \text{ mH}, \quad C = 0.53 \ \mu\text{F}, \quad m = 0.3]$$

11.8 Design an *m*-derived high-pass *T*-section filter having a design impedance of 600 Ω, cut-off frequency of 5 kHz and $m = 0.35$. Also determine the frequency of infinite attenuation.

$$\left[\begin{array}{l} \frac{2C}{m} = 0.148 \ \mu\text{F}, \quad \frac{L}{m} = 27.28 \text{ mH}, \\ \frac{4m}{1-m^2} C = 0.0414 \ \mu\text{F}, \quad f_\infty = 4.684 \text{ kHz} \end{array}\right]$$

11.9 In a constant-*k* band-pass filter, the ratio of capacitance in the shunt and series arms is 50:1 and the resonant frequency of both arms is 1000 Hz. Find the bandwidth of the filter.

[282.84 Hz]

11.10 Design a symmetrical lattice attenuator to have characteristic impedance of 800 Ω and attenuation of 20 *dB*.

[654.55 Ω, 977.78 Ω]

11.11 Design a *T*-type attenuator to have a characteristic resistance of 200 Ω and attenuation of 20 *dB*.

[163.6 Ω, 40.4 Ω]

11.12 Design a π-type attenuator to give 20 *dB* attenuation and characteristic impedance of 100 Ω.

[495 Ω, 122.22 Ω]

Objective–Type Questions

11.1 A 500 Ω is resistance attenuator is to have an attenuation of 20 db. The resistances of the series and shunt arms of the lattice network, respectively are
(a) 400 Ω, 500 Ω approxi
(b) 410 Ω, 510 Ω approx
(c) 410 Ω, 610 Ω approx
(d) none of the above

11.2 2. For the design of low-pass prototype filter Fig. 11.65 of with load resistance $R_L = 1\ \Omega$ and angular frequency $\omega = 1$ rad/s the values of L and C would be

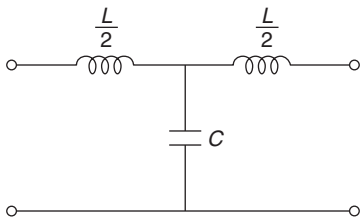


Fig. 11.65

(a) 1 H and I F
(b) 1 H and 2 F
(c) 2 H and I F
(d) 2 H and 2 F

11.3 The passband of a typical filter network with Z_1 and Z_2 as the series and shunt-arm impedances is characerised by

(a) $-1 < \frac{Z_1}{4Z_2} < 0$ (b) $-1 < \frac{Z_1}{4Z_2} < 1$

(c) $0 < \frac{Z_1}{4Z_2} < 1$ (d) none of the above

11.4 In an m-derived low-pass filter, the value of m is

(a) $\sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$ (b) $\sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}$

(c) $\sqrt{1+\left(\frac{f_\infty}{f_c}\right)^2}$ (d) $\sqrt{1+\left(\frac{f_c}{f_\infty}\right)^2}$

11.5 For the lattice-type attenuater shown in Fig. 11.66, the characteristic impedance R_0 is

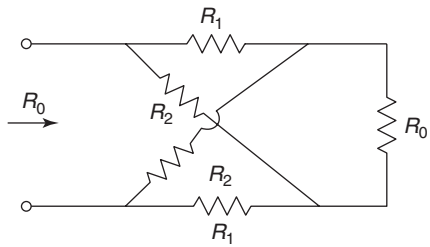


Fig. 11.66

(a) $\frac{R_1 + R_2}{2}$ (b) $\frac{2R_1R_2}{R_1 + R_2}$

(c) $\sqrt{R_1R_2}$ (d) $R_1 + \frac{R_2}{0}$

11.6 A T-type attenuator is designed for an attenuation of 40 dB and terminating resistance of 75 ohms. Which of the following values represent full series arm R_1 and shunt arm R_2?

(a) $R_1 \simeq 147\ \Omega$ (b) $R_1 \simeq 153\ \Omega$

(c) $R \simeq 1.5\ \Omega$ (d) $R_2 \simeq 3750\ \Omega$

Select the correct answer from the codes given below:
(a) 1 and 3 (b) 1 and 4
(c) 2 and 3 (d) 2 and 4

11.7 The application of the bisection theorem finds out an equivalent lattice network if the original network is
(a) symmetrical and balanced only
(b) unsymmetrical and balanced only
(c) symmetrical, balanced and unbalanced only
(d) symmetrical and unbalanced only

11.8 The attenuation of an attenuator is 20 dB. Its attenuation constant N is
(a) 1 (b) 2
(c) 10 (d) 20

11.9 Z_1 and Z_2 are the total series and shunt impedances of a T or π-filter. Consider the following zones of operation of filters and the conditions on the impedances

(a) Pass band 1. $\frac{Z_1}{4Z_2} < -1$

(b) Stop band 2. $\frac{Z_1}{4Z_2} = -1$

(c) Transition band 3. $-1 < \frac{Z_1}{4Z_2} < 0$

Tick the correct combination:

	A	B	C
(a)	1	2	3
(b)	3	1	2
(c)	3	2	1
(d)	2	3	1

11.10 Tick out the correct statement in case of a filter.

(a) Characteristic impedance is resistive in stop band

(b) Characteristic impedance is reactive in pass band

(c) Characteristic impedance is resistive in pass band

(d) None of the above

11.11 If L is the total series inductance and C the total shunt capacitacne of a T or π-type low-page filter, the pass band frequency range of the filter is given as

(a) 0 to $\frac{1}{2\pi\sqrt{L_C}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{L_C}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{L_C}}$ Hz

(d) 0 to $\frac{1}{4\pi\sqrt{L_C}}$ Hz

11.12 If C is the total series capatance, L is the total shunt inductance of a T or π-type high-pass filter, the frequency range for the stop band of the filter is

(a) 0 to $\frac{1}{2\pi\sqrt{L_C}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{L_C}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{L_C}}$ Hz

(d) 0 to $\frac{2}{4\pi\sqrt{L_C}}$ Hz

11.13 If f_1 and f_2 are the lower and uppper cut-off frequencies of the band pass filter, the series impedance Z_1 is

(a) capacitive at f_1

(b) inductive at f_1

(c) resistive at f_2

(d) none of the above

11.14 The phase constant β of a filter during stop band is

(a) Zero radian (b) 2

(c) π (d) 2π

Answers to Objective-Type Questions

11.1 (c)	11.2 (d)	11.3 (a)	11.4 (b)	11.5 (c)	11.6 (a)	11.7 (c)
11.8 (c)	11.9 (b)	11.10 (c)	11.11 (b)	11.12 (d)	11.13 (a)	11.14 (c)

12

Transmission Lines

12.1 INTRODUCTION

The electrical lines used to transmit electrical waves along them are called *transmission lines*. A transmission line consists of a pair of wires which are uniform throughout their whole length, e.g., telephone wires, electrical power lines, etc. Transmission-line parameters like resistance, inductance and capacitance are not physically separable. These parameters are distributed along the whole length of the transmission line. Transmission lines are needed for connecting various circuit elements and systems together. Open-wire and coaxial lines are commonly used for circuits operating at low frequencies, whereas strip line, microstrip line and waveguides are employed at radio and microwave frequencies.

12.2 TYPES OF TRANSMISSION LINES

There are various types of transmission lines.

1. ***Open-wire Line*** Open-wire lines or parallel-wire lines are the simplest type of transmission lines. They consist of two parallel conductors between the generator and the load. These lines are used for transmission of electric power, telegraphy, telephony signals, etc.
2. ***Cables*** A cable may consist of a large number of pairs of wires. Each pair consists of two wires which are insulated from each other by oil-impregnated insulating paper wrapped around each other and then twisted. A communication cable may contain hundreds of such pairs, whereas a power cable may contain only two or three heavy conductors, insulated and protected heavily from each other.
3. ***Coaxial Cables*** A coaxial cable is formed by placing one solid conductor of smaller diameter axially inside another hollow conductor of larger diameter and of a certain thickness. The inner conductor is supported by a solid disc of a certain dielectric material, the remaining space containing air. Alternatively, the complete cable may be filled up by a solid dielectric material.
4. ***Waveguides*** The waveguides are used to transmit the electrical waves at microwave frequencies. These are the hollow conducting tubes having uniform cross section. The energy is transmitted from inner walls of the tube by the phenomenon of total internal reflection.

12.3 PRIMARY CONSTANTS OF A TRANSMISSION LINE

The four electric circuit parameters R, L, G and C associated with the transmission lines are referred to as primary constants of the transmission lines.

1. ***Resistance*** The series resistance is contributed by the conductors themselves and depends upon their diameter and length. The resistance is uniformly distributed along the length of the transmission line. It is denoted by R and measured in ohms per unit length of transmission line.
2. ***Inductance*** The series inductance is contributed by the magnetic field of each conductor carrying a current. The inductance is also uniformly distributed along the length of the transmission line. It is denoted by L and measured in henries per unit length of the transmission line.
3. ***Capacitance*** The shunt capacitance is contributed by the two conductors, placed parallel or twisted and separated by a dielectric. The capacitance is also distributed along the length of the transmission line. It is denoted by C and measured in farads per unit length of the transmission line.
4. ***Conductance*** The dielectric in between the conductor is not perfect. Hence, a small conduction current or leakage current flows in between the wires. This results in the shunt conductance. It exists between the conductor and distributed along the length of the transmission line. It is denoted by G and measured in mho per unit length of the transmission line.

12.4 EQUIVALENT CIRCUIT REPRESENTATION OF TRANSMISSION LINE

A transmission line has four electrical parameters: resistance (R), inductance (L), capacitance (C) and conductance (G). These electrical parameters are uniform and continuously distributed along the line. The equivalent circuit of a transmission line is drawn in terms of its primary constants for a differential length Δx and arranged in the form of T or π configurations as shown in Fig.12.1 (a) and (b).

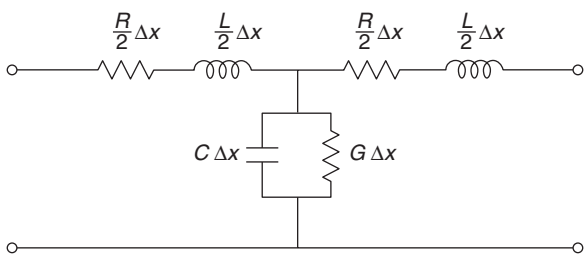


Fig. 12.1(a) *Equivalent T-section representation of a transmission line*

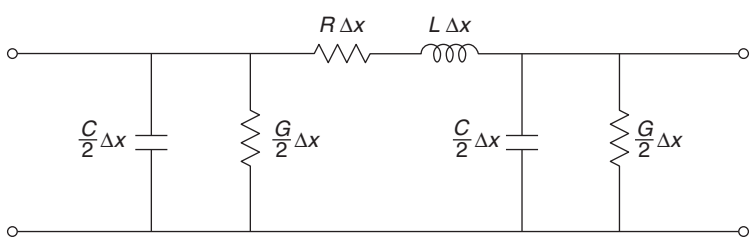


Fig. 12.1(b) *Equivalent π-section representation of a transmission line*

If the length Δx is very small and an infinitely large number of such sections are connected in cascade, the circuit behaves exactly as the transmission line of infinite length. The resulting equivalent circuit of a transmission line is shown in Fig. 12.2 and each identical section in this representation corresponds to a differential length Δx of the transmission line.

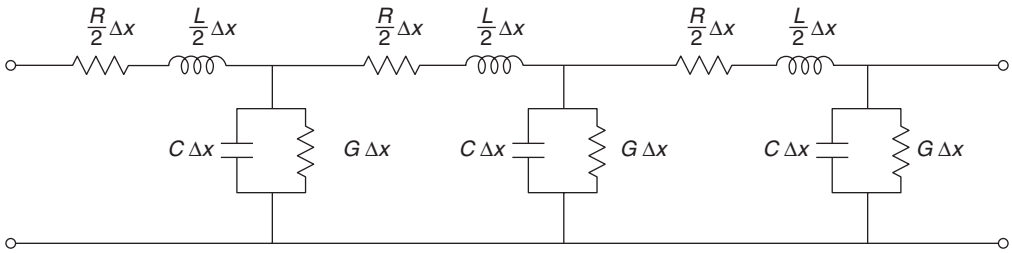


Fig. 12.2 *General equivalent circuit representation of a transmission line*

The circuit of Fig. 12.1(a) may be represented in terms of **Z** and **Y** as shown in Fig. 12.3 where,

$$\mathbf{Z} = R + j\omega L$$
$$\mathbf{Y} = G + j\omega C$$

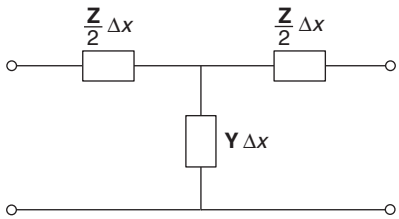


Fig. 12.3 *Equivalent T-network in terms of Z and Y*

12.5 SECONDARY CONSTANTS OF A TRANSMISSION LINE

12.5.1 Characteristic Impedance

Consider a transmission line of differential length Δx terminated in its characteristic impedance $\mathbf{Z}_0$. This line is a symmetrical network and hence can be represented by the equivalent T-network as shown in Fig.12.4.

The input impedance $\mathbf{Z}_{in}$ of the equivalent T-section is

$$\mathbf{Z}_{in} = \frac{\mathbf{Z}}{2}\Delta x + \frac{\left(\frac{\mathbf{Z}}{2}\Delta x + \mathbf{Z}_0\right)\frac{1}{\mathbf{Y}\,\Delta x}}{\frac{\mathbf{Z}}{2}\Delta x + \mathbf{Z}_0 + \frac{1}{\mathbf{Y}\,\Delta x}}$$

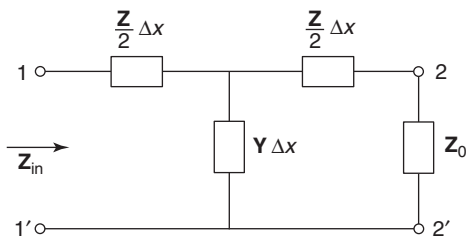


Fig. 12.4 *Equivalent T-section terminated by* $\mathbf{Z}_0$

If $\mathbf{Z}_{in} = \mathbf{Z}_0$,

$$\mathbf{Z}_0 = \frac{\mathbf{Z}}{2}\Delta x + \frac{\left(\frac{\mathbf{Z}}{2}\Delta x + \mathbf{Z}_0\right)\left(\frac{1}{\mathbf{Y}\,\Delta x}\right)}{\frac{\mathbf{Z}}{2}\Delta x + \mathbf{Z}_0 + \frac{1}{\mathbf{Y}\,\Delta x}}$$

$$\mathbf{Z}_0\frac{\mathbf{Z}}{2}\Delta x + \mathbf{Z}_0^2 + \frac{\mathbf{Z}_0}{\mathbf{Y}\,\Delta x} = \left(\frac{\mathbf{Z}}{2}\Delta x\right)^2 + \frac{\mathbf{Z}}{2}\mathbf{Z}_0\,\Delta x + \frac{\mathbf{Z}}{2\mathbf{Y}} + \frac{\mathbf{Z}}{2\mathbf{Y}} + \frac{\mathbf{Z}_0}{\mathbf{Y}\,\Delta x}$$

$$\mathbf{Z}_0^2 = \frac{\mathbf{Z}}{\mathbf{Y}} + \frac{\mathbf{Z}^2}{4}(\Delta x)^2$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}} + \frac{\mathbf{Z}^2}{4}(\Delta x)^2} = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}\left[1 + \frac{\mathbf{ZY}}{4}(\Delta x)^2\right]}$$

When Δx approaches zero,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}}$$

In terms of primary constants,

$$\mathbf{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

12.5.2 Propagation Constant

Consider a transmission line of unit length terminated in its characteristic impendence $\mathbf{Z}_0$. Let this section be represented by its equivalent *T*-section as shown in Fig.12.5.

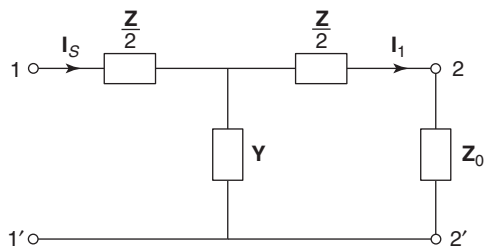


Fig. 12.5 *Equivalent T-section of transmission line of unit length*

By current division rule,

$$\mathbf{I}_1 = \mathbf{I}_S \frac{\frac{1}{\mathbf{Y}}}{\frac{1}{\mathbf{Y}} + \frac{\mathbf{Z}}{2} + \mathbf{Z}_0}$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_S} = \frac{\frac{1}{\mathbf{Y}}}{\frac{1}{\mathbf{Y}} + \frac{\mathbf{Z}}{2} + \mathbf{Z}_0}$$

$$\frac{\mathbf{I}_S}{\mathbf{I}_1} = \frac{\frac{1}{\mathbf{Y}} + \frac{\mathbf{Z}}{2} + \mathbf{Z}_0}{\frac{1}{\mathbf{Y}}}$$

$$\text{Let } \frac{\mathbf{I}_S}{\mathbf{I}_1} = e^{\gamma} = \frac{\frac{1}{\mathbf{Y}} + \frac{\mathbf{Z}}{2} + \mathbf{Z}_0}{\frac{1}{\mathbf{Y}}}$$

$$e^{\gamma} = 1 + \frac{\mathbf{ZY}}{2} + \mathbf{Z}_0\mathbf{Y} \qquad \ldots(12.1)$$

where γ is the propagation constant per unit length.

When *n* sections are connected in cascade for a particular length of a transmission line as shown in Fig. 12.6, the ratio of input to output currents for each section may be multiplied together to yield the ratio of input current $\mathbf{I}_S$ to the output current $\mathbf{I}_0$ of the entire line.

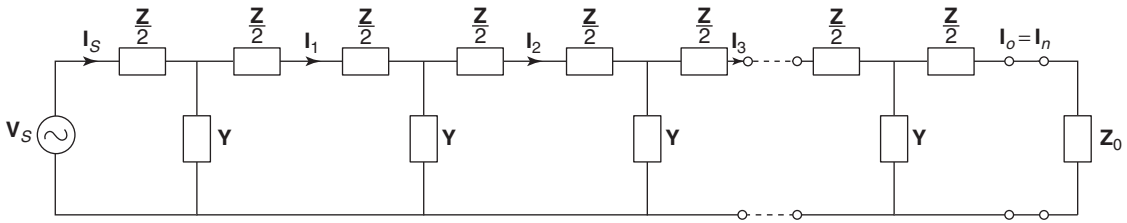


Fig. 12.6 *Cascaded n identical sections of a transmission line*

$$\frac{\mathbf{I}_S}{\mathbf{I}_1} \times \frac{\mathbf{I}_1}{\mathbf{I}_2} \times \frac{\mathbf{I}_2}{\mathbf{I}_3} \times \cdots \times \frac{\mathbf{I}_{n-1}}{\mathbf{I}_0} = \frac{\mathbf{I}_S}{\mathbf{I}_0}$$

$$\frac{\mathbf{I}_S}{\mathbf{I}_0} = e^{n\gamma}$$

For transmission line of length l, the ratio of input current to the output current is given as

$$\frac{\mathbf{I}_S}{\mathbf{I}_0} = e^{\gamma l}$$

For transmission line of length Δx, Eq. (12.1) can be written as,

$$e^{\gamma \Delta x} = 1 + \frac{\mathbf{ZY}}{2}(\Delta x)^2 + \mathbf{Z}_0 \mathbf{Y} \Delta x$$

$$= 1 + \frac{\mathbf{ZY}}{2}(\Delta x)^2 + \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}\left[1 + \frac{\mathbf{ZY}}{4}(\Delta x)^2\right]}\mathbf{Y}\Delta x$$

$$= 1 + \frac{\mathbf{ZY}}{2}(\Delta x)^2 + \sqrt{\mathbf{ZY}}\Delta x\sqrt{1 + \frac{\mathbf{ZY}}{4}(\Delta x)^2} \qquad \ldots(12.2)$$

The term $\sqrt{1 + \frac{\mathbf{ZY}}{4}(\Delta x)^2}$ may be expanded using binomial expansion.

$$\left[1 + \frac{\mathbf{ZY}}{4}(\Delta x)^2\right]^{\frac{1}{2}} = 1 + \frac{1}{2} \cdot \frac{\mathbf{ZY}}{4}(\Delta x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left[\frac{\mathbf{ZY}}{4}(\Delta x)^2\right] + \cdots$$

$$= 1 + \frac{\mathbf{ZY}}{8}(\Delta x)^2 - \frac{\mathbf{Z}^2\mathbf{Y}^2}{128}(\Delta x)^4 + \cdots$$

Similarly, the term $e^{\gamma \Delta x}$ may be expanded as,

$$e^{\gamma \Delta x} = 1 + \gamma \Delta x + \frac{\gamma^2 (\Delta x)^2}{2} + \frac{\gamma^3 (\Delta x)^3}{6} + \cdots$$

Substituting in Eq. (12.2),

$$1+\gamma\Delta x+\frac{\gamma^2(\Delta x)^2}{2}+\frac{\gamma^3(\Delta x)^3}{6}+\cdots=1+\frac{\mathbf{ZY}}{2}(\Delta x)^2+\sqrt{\mathbf{ZY}}\Delta x\left[1+\frac{\mathbf{ZY}}{8}(\Delta x)^2-\frac{\mathbf{Z}^2\mathbf{Y}^2}{128}(\Delta x)^4+\cdots\right]$$

$$=1+\sqrt{\mathbf{ZY}}(\Delta x)+\frac{(\sqrt{\mathbf{ZY}})^2}{2}(\Delta x)^2+\frac{(\sqrt{\mathbf{ZY}})^3}{8}(\Delta x)^3+\cdots$$

Cancelling 1 from both the sides and dividing by Δx,

$$\gamma+\frac{\gamma^2\Delta x}{2}+\frac{\gamma^3(\Delta x)^2}{6}+\cdots=\sqrt{\mathbf{ZY}}+\frac{(\sqrt{\mathbf{ZY}})^2}{2}\Delta x+\frac{(\sqrt{\mathbf{ZY}})^3}{8}(\Delta x)^2+\cdots$$

When Δx approaches zero,

$$\gamma=\sqrt{\mathbf{ZY}}$$

In terms of primary constants,

$$\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}=\alpha+j\beta$$

where α is the attenuation constant and β is the phase shift constant.

The phase shift constant β indicates the amount by which the phase of an input current changes in a unit distance. In a distance equal to one wavelength λ the phase shift is 2π radians.

$$\lambda=\frac{2\pi}{\beta}$$

The phase velocity of propagation is given by

$$v_p=f\lambda=\frac{\omega}{2\pi}\frac{2\pi}{\beta}=\frac{\omega}{\beta}$$

Example 12.1 *The constants of a transmission line are*

$R = 6\ \Omega/\text{km}$, $L = 2.2$ mH/km

$G = 0.25\times10^{-6}$ ℧/km, $C = 0.005\times10^{-6}$ F/km

Determine the characteristic impedance and propagation constant, attenuation constant and phase shift constant at 1 kHz.

Solution

$$\mathbf{Z}=R+j\omega L=6+j2\pi\times1000\times2.2\times10^{-3}=15.07\angle66.54°\ \Omega/\text{km}$$

$$\mathbf{Y}=G+j\omega C=0.25\times10^{-6}+j2\pi\times1000\times0.005\times10^{-6}=3.14\times10^{-5}\angle89.54°\ ℧/\text{km}$$

$$\mathbf{Z}_0=\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}}=\sqrt{\frac{15.07\angle66.54°}{3.14\times10^{-5}\angle89.54°}}=692.77\angle-11.5°\ \Omega$$

$$\gamma=\sqrt{\mathbf{ZY}}=\sqrt{(15.07\angle66.54°)(3.14\times10^{-5}\angle89.54°)}=0.022\angle78.04°\ /\text{km}=(4.55\times10^{-3}+j0.022)\ /\text{km}$$

$\alpha = 4.55\times10^{-3}$ neper/km

$\beta = 0.022$ rad/km

Example 12.2 *The values of primary constants of an open-wire line per km are R = 10 Ω, L = 3.5 mH, C = 0.008 μF and G = 0.7 μ℧. For a signal frequency of 1000 Hz, calculate the characteristic impedance $\mathbf{Z}_0$, propagation constant γ, attenuation constant α, phase constant β, wavelength λ and phase velocity v_p.*

Solution

$$R = 10\ \Omega/\text{km}, L = 3.5\ \text{mH/km}, C = 0.008\ \mu\text{F/km}$$
$$G = 0.7\ \mu℧/\text{km}, f = 1000\ \text{Hz}$$

$$\mathbf{Z} = R + j\omega L = 10 + j2\pi \times 1000 \times 3.5 \times 10^{-3} = 24.16\angle 65.55°\ \Omega/\text{km}$$
$$\mathbf{Y} = G + j\omega C = 0.7 \times 10^{-6} + j2\pi \times 1000 \times 0.008 \times 10^{-6} = 50.27 \times 10^{-6}\angle 89.2°\ ℧/\text{km}$$
$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{24.16\angle 65.66°}{50.27 \times 10^{-6}\angle 89.2°}} = 693.26\angle -11.77°\ \Omega$$
$$\gamma = \sqrt{\mathbf{ZY}} = \sqrt{(24.16\angle 65.66°)(50.27 \times 10^{-6}\angle 89.2°)} = 0.035\angle 77.43° = (7.62 \times 10^{-3} + j0.034)/\text{km}$$

$$\alpha = 7.62 \times 10^{-3}\ \text{neper/km}$$
$$\beta = 0.034\ \text{rad/km}$$
$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.034} = 184.8\ \text{km}$$
$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 1000}{0.034} = 184.8 \times 10^3\ \text{km/s}$$

Example 12.3 *A low-frequency transmission line has the following primary constants per km: R = 6Ω, L = 2.2 mH, C = 0.005 μF, G = 0.05 μ℧. Calculate its secondary constants Z_0, α and β at a frequency of 1 kHz. If the line length is 100 km, determine the attenuation and phase shift suffered by the signal. Evaluate the phase velocity of the signal.*

Solution

$$R = 6\ \Omega/\text{km}, L = 2.2\ \text{mH/km}, C = 0.005\ \mu\text{F/km},$$
$$G = 0.05\ \mu℧/\text{km}, f = 1\ \text{kHz}$$

$$\mathbf{Z} = R + j\omega L = 6 + j2\pi \times 1000 \times 2.2 \times 10^{-3} = 15.07\angle 66.54°\Omega/\text{km}$$
$$\mathbf{Y} = G + j\omega C = 0.05 \times 10^{-6} + j2\pi \times 1000 \times 0.005 \times 10^{-6} = 31.42 \times 10^{-6}\angle 89.5°\ ℧/\text{km}$$
$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{15.07\angle 66.54°}{31.42 \times 10^{-6}\angle 89.91°}} = 692.55\angle -11.69°\ \Omega$$
$$\gamma = \sqrt{\mathbf{ZY}} = \sqrt{(15.07\angle 66.54°)(31.42 \times 10^{-6}\angle 89.91°} = 21.76 \times 10^{-3}\angle 78.23°\ /\text{km}$$
$$= (4.44 \times 10^{-3} + j0.0213)/\text{km}$$

$$\alpha = 4.44 \times 10^{-3}\ \text{neper/km}$$
$$\beta = 0.0213\ \text{rad/km}$$

For $l = 100$ km,

$$\alpha = 4.44 \times 10^{-3} \times 100 = 0.44 \text{ neper}$$

$$\beta = 0.0213 \times 100 = 2.13 \text{ rad}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 1000}{0.0213} = 294.99 \text{ km/s}$$

Example 12.4 *The characteristic impedance of a uniform transmission line is 2309.6 Ω at a frequency of 800 MHz. At this frequency, the propagation constant is 0.054 (0.0366 + j 0.99). Determine R and L.*

Solution

$$\mathbf{Z}_0 = 2309.6\ \Omega,\ f = 800 \text{ MHz}$$

$$\gamma = 0.054(0.0366 + j0.99) = 1.98 \times 10^{-3} + j0.053$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \ldots(\text{i})$$

$$\gamma = \sqrt{\mathbf{ZY}} = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \ldots(\text{ii})$$

Multiplying Eq. (i) and (ii),

$$\mathbf{Z}_0 \gamma = R + j\omega L$$

$$R + j\omega L = 2309.6(1.98 \times 10^{-3} + j0.053) = 4.56 + j123.47$$

$$R = 4.56\ \Omega/\text{unit length}$$

$$\omega \text{L} = 123.47$$

$$2\pi \times 800 \times 10^6 \times L = 123.47$$

$$L = 0.0245\ \mu\text{H/ per unit length}$$

Example 12.5 *A transmission line has a characteristic impedance of 710 ∠ −16° Ω at 1 kHz. At this frequency, attenuation and phase shift is found to be 0.01 neper and 0.035 radian per km respectively. Calculate the primary constants of the line.*

Solution

$$\mathbf{Z}_0 = 710 \angle -16°\ \Omega,\ \alpha = 0.01 \text{ neper},$$
$$\beta = 0.035 \text{ rad}, f = 1 \text{ kHZ}$$

$$\gamma = \alpha + j\beta = 0.01 + j\ 0.035 = 0.0364 \angle 74.05°/\text{km}$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \ldots(\text{i})$$

$$\gamma = \sqrt{\mathbf{ZY}} = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \ldots(\text{ii})$$

Multiplying Eq. (i) and (ii),

$$R + j\omega L = \mathbf{Z}_0\ \gamma = (710\angle -16°)(0.0364\ \angle\ 74.05°) = 25.84\angle 58.05° = 13.68 + j21.93\ \Omega$$

$$R = 13.68\ \Omega/\text{km}$$

$$\omega L = 21.93$$

$$2\pi \times 10^3 \times L = 21.93$$

$$L = 3.49\ \text{mH/km}$$

Dividing Eq. (ii) by Eq. (i),

$$\sqrt{(R + j\omega L)(G + j\omega C)} \times \sqrt{\frac{G + j\omega C}{R + j\omega L}} = \frac{0.0364\angle 74.05°}{710\angle -16°}$$

$$G + j\omega C = 5.13 \times 10^{-5}\angle 90.05° = 0 + j5.13 \times 10^{-5}\ \text{℧}$$

$$G = 0$$

$$\omega C = 5.13 \times 10^{-5}$$

$$2\pi \times 10^3 \times C = 5.13 \times 10^{-5}$$

$$C = 8.16\ \text{nF/km}$$

Example 12.6 *Find the attenuation and phase shift constant of a wave propagating along the line whose propagation constant is 1.048 × 10⁻⁴ ∠88.8°/m.*

Solution

$$\gamma = 1.048 \times 10^{-4}\angle 88.8 = (2.195 \times 10^{-6} + j1.048 \times 10^{-4})/\text{m}$$

Comparing with $\gamma = \alpha + j\beta$,

$$\alpha = 2.195 \times 10^{-6}\ \text{neper/m}$$

$$\beta = 1.048 \times 10^{-4}\ \text{rad/m}$$

Example 12.7 *A 12 km open wire line is terminated in its characteristic impedance. At a certain frequency, the signal voltage measured at a distance of 1 km reduces to 90% of the sending end voltage. Determine voltage at the receiving end in terms of the sending end voltage V_S.*

Solution

Signal voltage v at 1 km = 0.9 $\mathbf{V}_S$

$$v = \mathbf{V}_S\ e^{-\gamma l}$$

$$0.9\mathbf{V}_S = \mathbf{V}_S\ e^{-\gamma}$$

$$e^{-\gamma} = 0.9$$

When l = 12 km, $v = \mathbf{V}_R$

$$\mathbf{V}_R = \mathbf{V}_S\ e^{-\gamma \times 12}$$

$$= \mathbf{V}_S\ (e^{-\gamma})^{12}$$

$$= \mathbf{V}_S (0.9)^{12}$$

$$= 0.282\ \mathbf{V}_S$$

Example 12.8 *A coaxial cable has a characteristic impedance of 50 Ω and an attenuation constant of 1.2 dB/km. A 20 km length of this cable is terminated in its characteristic impedance and the input power to the line is 0.5W. Find (a) the output power, and (b) the output current.*

Solution

$$\mathbf{Z}_0 = 50\ \Omega,\ \alpha = 1.2\ \text{dB/km},\ l = 20\ \text{km},\ P_S = 0.5\ \text{W}.$$

(a)

$$\alpha = 1.2\ \text{dB/km} = 1.2 \times 20 \times 0.115\ \text{nepers} = 2.76\ \text{nepers}$$

$$\ln \frac{\mathbf{I}_S}{\mathbf{I}_R} = 2.76$$

$$\frac{\mathbf{I}_S}{\mathbf{I}_R} = e^{2.76} = 15.8$$

$$\frac{P_S}{P_R} = \left(\frac{\mathbf{I}_S}{\mathbf{I}_R}\right)^2$$

$$\frac{0.5}{P_R} = (15.8)^2$$

$$P_R = 2\ \text{mW}$$

(b)

$$\mathbf{Z}_0 = 50\ \Omega$$

$$P_S = |\mathbf{I}_S|^2 |\mathbf{Z}_o|$$

$$0.5 = |\mathbf{I}_S|^2 (50)$$

$$\mathbf{I}_S = 0.1\ \text{A}$$

$$\frac{\mathbf{I}_S}{\mathbf{I}_R} = 15.8$$

$$\frac{0.1}{\mathbf{I}_R} = 15.8$$

$$\mathbf{I}_R = 6.33\ \text{mA}$$

12.6 INFINITE LINE

The propagation of electric energy along a uniform and symmetrical line may be considered in terms of the results obtained for an imaginary line of infinite length shown in Fig. 12.7.

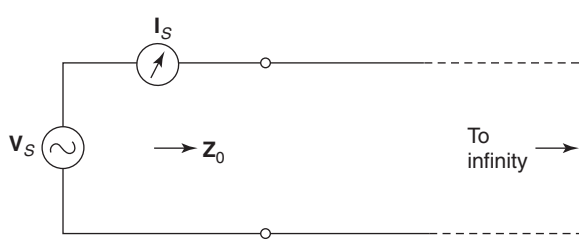


Fig. 12.7 *Infinite line*

A signal voltage $\mathbf{V}_S$ is applied at the sending end of the transmission line which sends a current $\mathbf{I}_S$ into the line as shown in Fig.12.7. The current $\mathbf{I}_S$ depends upon the leakage conductance and capacitance between the line conductors.

The ratio of the voltage $\mathbf{V}_S$ and the current $\mathbf{I}_S$ gives the input impedance of the line. This input impedance of the infinite line is called the characteristic impedance of the transmission line and is denoted by $\mathbf{Z}_0$. The characteristic impedance is a phasor quantity having a magnitude of $|\mathbf{Z}_0|$ and an angle ϕ. Both magnitude and angle of the characteristic impedance vary with the frequency.

Since the line has an infinite length, no waves can ever reach the receiving end and hence there is no possibility of the reflection at the receiving end. There cannot be any reflected waves, returning to the sending end.

12.7 TERMINATED TRANSMISSION LINES

When a finite length of line is connected with infinite line of same characteristic impedance, their total impedance is the same as that of infinite line itself because they together make an infinite line (Fig. 12.8).

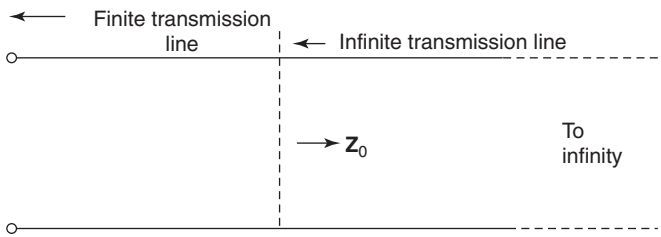


Fig. 12.8 *Terminated transmission line*

The input impedance of an infinite line is $\mathbf{Z}_0$. Hence, the finite line has an input impedance $\mathbf{Z}_0$ when it is terminated in $\mathbf{Z}_0$, or a finite line terminated by its impedance $\mathbf{Z}_0$ behaves as an infinite line (Fig. 12.9).

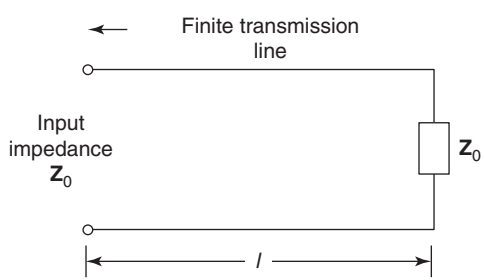


Fig. 12.9 *Equivalent circuit of an infinite line*

12.8 GENERAL SOLUTION OF A TRANSMISSION LINE

Consider a transmission line of length l which is made up of cascaded infinitesimal T-sections. One such section of length ds and carrying current $\mathbf{I}$ is shown in Fig.12.10.

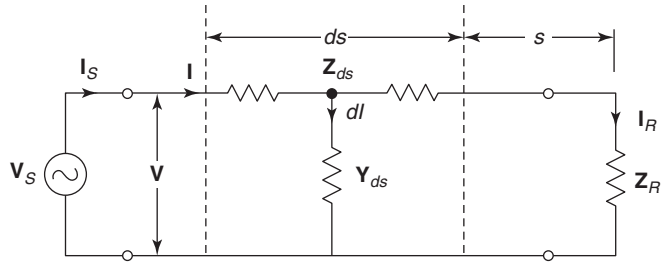


Fig. 12.10 *Infinitesimal T-section of a long line*

Since the series line impedance is **Z** ohms per unit length, the series impedance of the element is **Z***ds* ohms, and the voltage drop in the length *ds* is

$$d\mathbf{V} = \mathbf{I}\mathbf{Z}ds$$

$$\frac{d\mathbf{V}}{ds} = \mathbf{I}\mathbf{Z} \qquad \ldots(12.3)$$

Since the shunt line admittance is **Y** mhos per unit, the shunt admittance of the element is **Y***ds* mhos, and the current flowing across the line or from one conductor to the other is

$$d\mathbf{I} = \mathbf{V}\mathbf{Y}ds$$

$$\frac{d\mathbf{I}}{ds} = \mathbf{V}\mathbf{Y} \qquad \ldots(12.4)$$

Differentiating Eq. (12.3) and (12.4), w.r.t. *s*,

$$\frac{d^2\mathbf{V}}{ds^2} = \mathbf{Z}\frac{d\mathbf{I}}{ds} \qquad \ldots(12.5)$$

and

$$\frac{d^2\mathbf{I}}{ds^2} = \mathbf{Y}\frac{d\mathbf{V}}{ds} \qquad \ldots(12.6)$$

Equations (12.5) and (12.6) are the second-order differential equations
The auxillary equation for Eq. (12.5) is

$$(m^2 - \mathbf{ZY})\mathbf{V} = 0$$

$$m = \pm\sqrt{\mathbf{ZY}}$$

The solution of this differential equation is

$$\mathbf{V} = A^{\sqrt{\mathbf{ZY}}s} + Be^{-\sqrt{\mathbf{ZY}}s} \qquad \ldots(12.7)$$

Similarly, the solution of the differential equation (12.6) is

$$\mathbf{I} = C^{\sqrt{\mathbf{ZY}}s} + De^{-\sqrt{\mathbf{ZY}}s} \qquad \ldots(12.8)$$

where *A*, *B*, *C* and *D* are arbitrary constants and *s* is the distance measured from the receiving end.

At the receiving end, $s = 0$ (as distance is measured from the receiving end) $\mathbf{V} = \mathbf{V}_R$ and $\mathbf{I} = \mathbf{I}_R$. Substituting in Eq. (12.7) and (12.8),

$$\mathbf{V}_R = A + B \qquad \ldots(12.9)$$

and

$$\mathbf{I}_R = C + D \qquad \ldots(12.10)$$

Differentiating Eq. (12.7) w.r.t. s,

$$\frac{d\mathbf{V}}{ds} = A\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} + B(-\sqrt{\mathbf{ZY}})e^{-\sqrt{\mathbf{ZY}}s}$$

$$\mathbf{IZ} = A\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} - B\sqrt{\mathbf{ZY}}e^{-\sqrt{\mathbf{ZY}}s}$$

$$\mathbf{I} = \frac{A}{\mathbf{Z}}\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} - \frac{B}{\mathbf{Z}}\sqrt{\mathbf{ZY}}\, e^{-\sqrt{\mathbf{ZY}}s}$$

$$= A\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}}\, e^{\sqrt{\mathbf{ZY}}s} - B\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}}\, e^{-\sqrt{\mathbf{ZY}}s} \qquad \ldots(12.11)$$

Differentiating Eq. (12.8) w.r.t. s,

$$\frac{d\mathbf{I}}{ds} = C\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} + D(-\sqrt{\mathbf{ZY}})e^{-\sqrt{\mathbf{ZY}}s}$$

$$\mathbf{VY} = C\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} - D\sqrt{\mathbf{ZY}}e^{-\sqrt{\mathbf{ZY}}s}$$

$$\mathbf{V} = \frac{C}{\mathbf{Y}}\sqrt{\mathbf{ZY}}\, e^{\sqrt{\mathbf{ZY}}s} - \frac{D}{\mathbf{Y}}\sqrt{\mathbf{ZY}}e^{-\sqrt{\mathbf{ZY}}s}$$

$$= C\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}}\, e^{\sqrt{\mathbf{ZY}}s} - D\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}}e^{-\sqrt{\mathbf{ZY}}s} \qquad \ldots(12.12)$$

At $s = 0$, $\mathbf{V} = \mathbf{V}_R$ and $\mathbf{I} = \mathbf{I}_R$.

$$\mathbf{I}_R = A\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}} - B\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}} \qquad \ldots(12.13)$$

$$\mathbf{V}_R = C\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} - D\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} \qquad \ldots(12.14)$$

Solving Eqs (12.9), (12.10), (12.13) and (12.14),

$$\left.\begin{aligned} A &= \frac{\mathbf{V}_R}{2} + \frac{\mathbf{I}_R}{2}\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \frac{\mathbf{V}_R}{2}\left(1 + \frac{\mathbf{Z}_o}{\mathbf{Z}_R}\right) \\ B &= \frac{\mathbf{V}_R}{2} + \mathbf{I}_R\sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \frac{\mathbf{V}_R}{2}\left(1 - \frac{\mathbf{Z}_o}{\mathbf{Z}_R}\right) \\ C &= \frac{\mathbf{I}_R}{2} + \frac{\mathbf{V}_R}{2}\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}} = \frac{\mathbf{I}_R}{2}\left(1 + \frac{\mathbf{Z}_R}{\mathbf{Z}_o}\right) \\ D &= \frac{\mathbf{I}_R}{2} - \frac{\mathbf{V}_R}{2}\sqrt{\frac{\mathbf{Y}}{\mathbf{Z}}} = \frac{\mathbf{I}_R}{2}\left(1 - \frac{\mathbf{Z}_R}{\mathbf{Z}_o}\right) \end{aligned}\right\} \left[\begin{aligned} &\because \mathbf{Z}_R = \frac{\mathbf{V}_R}{\mathbf{I}_R} \\ &\text{and } \mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} \end{aligned}\right]$$

Hence, the general solution of the differential equation is,

$$\mathbf{V} = \frac{\mathbf{V}_R}{R}\left[\left(1 + \frac{\mathbf{Z}_0}{\mathbf{Z}_R}\right)e^{\sqrt{\mathbf{ZY}}s} + \left(1 - \frac{\mathbf{Z}_0}{\mathbf{Z}_R}\right)e^{-\sqrt{\mathbf{ZY}}s}\right]$$

$$= \frac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\,\mathbf{Z}_R}\left[e^{\sqrt{\mathbf{ZY}}s} + \left(\frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0}\right)e^{-\sqrt{\mathbf{ZY}}s}\right]$$

$$= \frac{\mathbf{V}_R}{2\mathbf{Z}_R}\left[(\mathbf{Z}_R + \mathbf{Z}_0)e^{\sqrt{\mathbf{ZY}}s} + (\mathbf{Z}_R - \mathbf{Z}_0)e^{-\sqrt{\mathbf{ZY}}s}\right]$$

$$= \frac{\mathbf{V}_R}{2\mathbf{Z}_R}\left[\mathbf{Z}_R e^{\sqrt{\mathbf{ZY}}s} + \mathbf{Z}_0 e^{\sqrt{\mathbf{ZY}}s} + \mathbf{Z}_R e^{-\sqrt{\mathbf{ZY}}s} - \mathbf{Z}_0 e^{-\sqrt{\mathbf{ZY}}s}\right]$$

$$= \mathbf{V}_R\left(\frac{e^{\sqrt{\mathbf{ZY}}s} + e^{-\sqrt{\mathbf{ZY}}s}}{2}\right) + \frac{\mathbf{Z}_0\mathbf{V}_R}{\mathbf{Z}_R}\left(\frac{e^{\sqrt{\mathbf{ZY}}s} - e^{-\sqrt{\mathbf{ZY}}s}}{2}\right)$$

$$= \mathbf{V}_R \cosh(\sqrt{\mathbf{ZY}}s) + \mathbf{I}_R\mathbf{Z}_0 \sinh(\sqrt{\mathbf{ZY}}s)$$

$$= \mathbf{V}_R \cosh(\gamma s) + \mathbf{I}_R\mathbf{Z}_0 \sinh(\gamma s) \quad \ldots(12.15)$$

Similarly, the general solution of the differential equation is

$$\mathbf{I} = \mathbf{I}_R \cosh(\sqrt{\mathbf{ZY}}s) + \frac{\mathbf{V}_R}{\mathbf{Z}_0}\sinh(\sqrt{\mathbf{ZY}}s)$$

$$= \mathbf{I}_R \cosh(\gamma s) + \frac{\mathbf{V}_R}{\mathbf{Z}_0}\sinh(\gamma s) \quad \ldots(12.16)$$

The similar equations can be obtained in terms of sending end voltage $\mathbf{V}_S$ and $\mathbf{I}_S$. If x is the distance measured from the sending end then

$$x = l - s$$

The equations (12.15) and (12.16) can be written as

$$\mathbf{V} = \mathbf{V}_S \cosh(\gamma x) - \mathbf{I}_S\mathbf{Z}_0 \sinh(\gamma x)$$

$$\mathbf{I} = \mathbf{I}_S \cosh(\gamma x) - \frac{\mathbf{V}_S}{\mathbf{Z}_0}\sinh(\gamma x)$$

These equations can be applied for particular cases.

1. Finite Line Terminated in Z_o

At $x = l$, $\mathbf{V} = \mathbf{V}_R$, $\mathbf{I} = \mathbf{I}_R$

$$\mathbf{V}_R = \mathbf{V}_S \cosh(\gamma l) - \mathbf{I}_S\mathbf{Z}_0 \sinh(\gamma l)$$

$$\mathbf{I}_R = \mathbf{I}_S \cosh(\gamma l) - \frac{\mathbf{V}_S}{\mathbf{Z}_0}\sinh(\gamma l)$$

$$\mathbf{Z}_R = \frac{\mathbf{V}_R}{\mathbf{I}_R} = \frac{\mathbf{Z}_0\left[\mathbf{V}_S \cosh(\gamma l) - \mathbf{I}_S\mathbf{Z}_0 \sinh(\gamma l)\right]}{\mathbf{I}_S\mathbf{Z}_0 \cosh(\gamma l) - \mathbf{V}_S \sinh(\gamma l)}$$

But $\mathbf{Z}_R = \mathbf{Z}_0$.

$$\mathbf{Z}_0 = \frac{\mathbf{Z}_0\left[\mathbf{V}_S \cosh(\gamma l) - \mathbf{I}_S\mathbf{Z}_0 \sinh(\gamma l)\right]}{\mathbf{I}_S\mathbf{Z}_0 \cosh(\gamma l) - \mathbf{V}_S \sinh(\gamma l)}$$

$$\frac{\mathbf{V}_S}{\mathbf{I}_S} = \mathbf{Z}_0\frac{\cosh(\gamma l) + \sinh(\gamma l)}{\cosh(\gamma l) + \sinh(\gamma l)} = \mathbf{Z}_0$$

Hence, the input impedance of a finite line terminated in its characteristic impedance $\mathbf{Z}_0$ is also $\mathbf{Z}_0$.

2. Finite Line Open Circuited at Load End

At $x = l$, $\mathbf{I}_R = 0$

$$0 = \mathbf{I}_S \cosh(\gamma l) - \frac{\mathbf{V}_S}{\mathbf{Z}_0}\sinh(\gamma l)$$

$$\frac{\mathbf{V}_S}{\mathbf{I}_S} = \mathbf{Z}_0\frac{\cosh(\gamma l)}{\sinh(\gamma l)} = \mathbf{Z}_0 \coth(\gamma l)$$

Hence, the input impedance of a finite line open circuited at load end is, $\mathbf{Z}_0 \coth(\gamma l)$

$$\mathbf{Z}_{in} = \mathbf{Z}_{0c} = \mathbf{Z}_0 \coth(\gamma l)$$

Note:

If $l \to \infty$, $\coth(\gamma l) \to 1$.

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_{0c} = \mathbf{Z}_0$$

3. Finite Line Short Circuited at Load End

$$\text{At } x = l,\ \mathbf{V}_R = 0$$

$$0 = \mathbf{V}_S \cosh(\gamma l) - \mathbf{I}_S \mathbf{Z}_0 \sinh(\gamma l)$$

$$\frac{\mathbf{V}_S}{\mathbf{I}_S} = \mathbf{Z}_0 \tanh(\gamma l)$$

Hence, the input impedance of a finite line short circuit at load end is, $\mathbf{Z}_0 \tanh(\gamma l)$

$$\mathbf{Z}_{in} = \mathbf{Z}_{sc} = \mathbf{Z}_0 \tanh(\gamma l)$$

12.9 REFLECTION

When a transmission line is not terminated in characteristic impedance $\mathbf{Z}_0$, a part of the wave is reflected back from the receiving end. Hence, reflection phenomenon exists for a transmission line which is not terminated in $\mathbf{Z}_0$. Reflection is maximum when the transmission line is open circuited, i.e., $\mathbf{Z}_R = \infty$ or short circuited, i.e., $\mathbf{Z}_R = 0$. The reflection is zero when $\mathbf{Z}_R = \mathbf{Z}_0$.

From the general solution of a transmission line, we can write

$$\mathbf{V} = \frac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\mathbf{Z}_R}\left[e^{\sqrt{\mathbf{ZY}}s} + \left(\frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0}\right)e^{-\sqrt{\mathbf{ZY}}s}\right]$$

$$= \frac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\mathbf{Z}_R} e^{\gamma s} + \frac{\mathbf{V}_R(\mathbf{Z}_R - \mathbf{Z}_0)}{2\mathbf{Z}_R} e^{-\gamma s} \qquad \left[\because \gamma = \sqrt{\mathbf{ZY}}\right]$$

The first part of $\mathbf{V}$ varies exponentially with s and is called *incident wave,* which travels from the sending end to the receiving end. When the incident wave travels from the sending end to the receiving end, its amplitude goes on decreasing.

$$\mathbf{V}_1 = \frac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\mathbf{Z}_R} e^{\gamma s} = \text{Incident wave}$$

The second part of $\mathbf{V}$ varies exponentially with $-s$ and is called reflected wave, which travel from the receiving to the sending end. When the reflected wave travels from the receiving end towards the ending end, its amplitude goes on decreasing.

$$\mathbf{V}_2 = \frac{\mathbf{V}_R(\mathbf{Z}_R - \mathbf{Z}_0)}{2\mathbf{Z}_R} e^{-\gamma s} = \text{Reflected wave}$$

Hence, the total voltage at any point on the line is the phasor sum of incident and reflected waves.

12.10 REFLECTION COEFFICIENT

The ratio of the amplitudes of the reflected and incident voltage or current waves at the receiving end of the line is called the reflection coefficient. It is denoted by $\mathbf{\Gamma}$.

$$\mathbf{\Gamma} = \frac{\text{Reflected voltage or current at load}}{\text{Incident voltage or current at load}}$$

The reflected voltage at load is component $\mathbf{V}_2$ at the receiving end i.e. with $s = 0$.

$$\mathbf{V}_2\big|_{s=0} = \frac{\mathbf{V}_R(\mathbf{Z}_R - \mathbf{Z}_0)}{2\mathbf{Z}_R}$$

The incident voltage at load is component $\mathbf{V}_1$ at the receiving end, i.e. with $s = 0$.

$$\mathbf{V}_1\big|_{s=0} = \frac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\mathbf{Z}_R}$$

$$\boldsymbol{\Gamma} = \frac{\dfrac{\mathbf{V}_R(\mathbf{Z}_R - \mathbf{Z}_0)}{2\mathbf{Z}_R}}{\dfrac{\mathbf{V}_R(\mathbf{Z}_R + \mathbf{Z}_0)}{2\mathbf{Z}_R}} = \frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0}$$

It is a measure of mismatch between the load impedance $\mathbf{Z}_R$ and the characteristic impedance $\mathbf{Z}_0$ of the line.

When $\mathbf{Z}_R = \mathbf{Z}_0$, $\boldsymbol{\Gamma} = 0$ and there is no reflection.
When $\mathbf{Z}_R = 0$, i.e., the line is short circuited,

$$\boldsymbol{\Gamma} = -1 = 1 \angle 180°$$

and the reflection is maximum.
When $\mathbf{Z}_R = \infty$, i.e. the line is open circuited,

$$\boldsymbol{\Gamma} = \frac{1 - \dfrac{\mathbf{Z}_0}{\mathbf{Z}_R}}{1 + \dfrac{\mathbf{Z}_0}{\mathbf{Z}_R}} = 1 = 1 \angle 0°$$

and the reflection is maximum.
$\boldsymbol{\Gamma}$ is a vector quantity.

12.11 STANDING WAVES

When reflection takes place in a transmission line and the reflected component of the waves is present at some point of the transmission line, this component becomes in phase with incident waves and the two components add together. At some other point, the two components oppose each other and the resulting wave becomes very small. The plot of the resultant voltage or current waves at different points of the line is termed standing waves. The points where the resultant voltage or current wave is maximum are termed voltage or current maxima. Similarly, the points where the voltage or current wave is minimum are termed voltage or current minima. At maxima, the reflected and incident waves are in phase whereas, at minima, the reflected and incident waves are out of phase.

12.12 STANDING WAVE RATIO

The standing wave ratio is defined as the ratio of the maximum to minimum magnitudes of voltages or currents on a line having standing waves and is denoted by the symbol S.

$$S = \frac{|\mathbf{V}_{\max}|}{|\mathbf{V}_{\min}|} = \frac{|\mathbf{I}_{\max}|}{|\mathbf{I}_{\min}|}$$

When the line is not terminated properly, standing waves are produced. The ratio of $\mathbf{V}_{max}$ to $\mathbf{V}_{min}$ is termed the *Voltage Standing Wave Ratio* (VSWR). Similarly, the ratio of $\mathbf{I}_{max}$ to $\mathbf{I}_{min}$ is termed the *Current Standing Wave Ratio* (ISWR). Voltage standing wave ratio is the term most commonly employed because it is simpler to carry out voltage measurement along a line. For current measurements, we have to break the line and insert on ammeter. Hence, practically, only VSWR measurement is done and VSWR is nothing but SWR. Theoretically, the value of S lies between 1 and ∞. VSWR is a scalar quantity.

12.12.1 Relation between Standing Wave Ratio (s) and Magnitude of Reflection Coefficient (Γ)

The points of voltage maxima occurs at points at which the incident and reflected waves are in phase and add up directly. The points of voltage minima occurs at points at which the incident and reflected waves are not in phase.

Hence,
$$\mathbf{V}_{max} = |\mathbf{V}_i| + |\mathbf{V}_r|$$
$$\mathbf{V}_{min} = |\mathbf{V}_i| - |\mathbf{V}_r|$$

where $\mathbf{V}_i$ = rms value of incident voltage.

and $\mathbf{V}_r$ = rms value of reflected voltage.

$$\text{VSWR} = S = \frac{|\mathbf{V}_{max}|}{|\mathbf{V}_{min}|} = \frac{|\mathbf{V}_i| + |\mathbf{V}_r|}{|\mathbf{V}_i| - |\mathbf{V}_r|} = \frac{1 + \left|\frac{\mathbf{V}_r}{\mathbf{V}_i}\right|}{1 - \left|\frac{\mathbf{V}_r}{\mathbf{V}_i}\right|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Applying componendo and dividendo,

$$\frac{S}{1} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$
$$\frac{S-1}{S+1} = \frac{1 + |\Gamma| - 1 + |\Gamma|}{1 + |\Gamma| + 1 - |\Gamma|} = \frac{2|\Gamma|}{2} = |\Gamma|$$
$$|\Gamma| = \frac{S-1}{S+1}$$

Example 12.9 *A lossless line has a characteristic impedance of 400 Ω. Determine the standing wave ratio if the receiving end impedance is 800 Ω.*

Solution

$$\mathbf{Z}_0 = 400\ \Omega,\ \mathbf{Z}_R = 800\ \Omega$$
$$\Gamma = \frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0} = \frac{800 - 400}{800 + 400} = 0.33$$
$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$

Example 12.10 *If VSWR of a line is 1.5 then calculate its reflection coefficient.*

Solution

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = 1.5$$

$$1+|\Gamma| = 1.5 - 1.5|\Gamma|$$

$$|\Gamma| = 0.2$$

Example 12.11 *Determine* Γ *of a line for which* $Z_R = 200\ \Omega$, $\mathbf{Z}_0 = 692 \angle -12°\ \Omega$.

Solution

$$\Gamma = \frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0} = \frac{200 - (692 \angle -12°)}{200 + (692 \angle -12°)} = 0.56 \angle 172.53°$$

Example 12.12 *A radio frequency line with* $Z_0 = 70\ \Omega$ *is terminated by* $Z_L = (115 - j80)\ \Omega$ *at* $\lambda = 2.5$ *m. Find the VSWR and the maximum and minimum line impedances.*

Solution

$$\mathbf{Z}_0 = 70\ \Omega, \mathbf{Z}_R = (115 - j80)\ \Omega$$

$$\Gamma = \frac{\mathbf{Z}_R - \mathbf{Z}_0}{\mathbf{Z}_R + \mathbf{Z}_0} = \frac{115 - j80 - 70}{115 - j80 + 70} = 0.46 \angle -37.26°$$

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.46}{1-0.46} = 2.7$$

$$\mathbf{Z}_{S\,\max} = S\,\mathbf{Z}_0 = (2.7)\,(70) = 189\ \Omega$$

$$\mathbf{Z}_{S\,\min} = \frac{\mathbf{Z}_0}{S} = \frac{70}{2.7} = 25.93\ \Omega$$

12.13 SCATTERING PARAMETERS OR *S*-PARAMETERS

Z-parameters are useful in analysing series circuits while **Y**-parameters are useful in analysing parallel circuits. Similarly, transmission parameters are useful for chain or cascade circuits. However, the characterisation procedure of these parameters requires an open or short circuit at another port. This makes it very difficult to determine the parameters of a network at radio and microwave frequencies. Hence, a new set of parameters, i.e., *scattering parameters*, based on travelling waves are used.

Figure 12.11 shows a network with incident and reflected waves at its two ports.

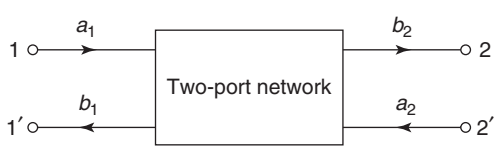


Fig. 12.11 *Two-port network with associated incident and reflection waves*

Here a_1 is an incident wave and b_1 is the reflected wave at Port 1. Similarly, a_2 and b_2 represent incident and reflected waves at Port 2 respectively. When a source is connected at Port 1, which produces the incident

wave a_1, a part of this wave is reflected back at the input due to impedance mismatching while the remaining signal is transmitted through the network. It may change in magnitude as well as in phase before emerging at Port 2. Depending on the termination at this port, a part of the signal is reflected back as input to Port 2. Hence, the reflected wave b_1 depends on incident signals a_1 and a_2 at the two ports. Similarly, the emerging wave b_2 also depends on a_1 and a_2.

Mathematically,

$$b_1, b_2 = f(a_1, a_2)$$
$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

In matrix form, we can write

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
$$[b] = [S][a]$$

The individual S parameters for a given network can be defined by terminating each of the ports by a matched load.

Case I When Port 2 is terminated by a matched load, i.e. $a_2 = 0$.

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$

where S_{11} is the reflection coefficient Γ_1 at Port 1 when Port 2 is terminated by a matched load.
Similarly,

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

where S_{21} is the forward transmission coefficient of Port 1 when Port 2 is terminated by a matched load.

Case II When the Port 1 is terminated by a matched load, i.e. $a_1 = 0$.

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}$$

where S_{12} is the reverse transmission coefficient when Port 1 is terminated by a matched load.
Similarly,

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$

where S_{22} is the reflection coefficient Γ_2 at Port 2 when the Port 1 is terminated by a matched load.

12.14 IMPEDANCE SMITH CHART

The Smith chart is basically a polar plot of the reflection coefficient Γ. Since Γ is expressed in terms of normalised impedance, the Smith chart is also an impedance locus diagram which is extensively used for the analysis of *transmission lines*, RF and microwave devices.

It comprises two sets of circles to represent real and imaginary parts of an impedance. These circles are known as *constant resistance circles* and *constant reactance circles*. As the resistive and reactive components are in circular form, the Smith chart is also known as circular chart.

An impedance $\mathbf{Z} = R + jX$ when normalised with the characteristic impedance $\mathbf{Z}_0$ is denoted by $\bar{\mathbf{Z}} = r + jx$ where $\bar{\mathbf{Z}} = \dfrac{\mathbf{Z}}{\mathbf{Z}_0}, r = \dfrac{R}{\mathbf{Z}_0}$ and $x = \dfrac{X}{\mathbf{Z}_0}$. The reflection coefficient for a normalised impedance $\bar{\mathbf{Z}}$ is given by,

$$\Gamma = \frac{\bar{\mathbf{Z}} - \mathbf{Z}_0}{\bar{\mathbf{Z}} + \mathbf{Z}_0} = \frac{\bar{\mathbf{Z}} - 1}{\bar{\mathbf{Z}} + 1}$$

By componendo-dividendo,

$$\frac{1+\Gamma}{1-\Gamma} = \frac{(\bar{\mathbf{Z}}+1)+(\bar{\mathbf{Z}}-1)}{(\bar{\mathbf{Z}}+1)-(\bar{\mathbf{Z}}-1)} = \frac{2\bar{\mathbf{Z}}}{2} = \bar{\mathbf{Z}}$$

$$\bar{\mathbf{Z}} = \frac{1+\Gamma}{1-\Gamma}$$

$$r + jx = \frac{1+(u+jv)}{1-(u+jv)}$$

$$= \frac{(1+u)+jv}{(1-u)+jv} \cdot \frac{(1-u)+jv}{(1-u)+jv}$$

$$= \frac{(1-u^2)+jv(1+u+1-u)-v^2}{(1-u)^2+v^2}$$

$$= \frac{(1-u^2-v^2)+j2v}{(1-u)^2+v^2}$$

$$= \frac{1-u^2-v^2}{(1-u)^2+v^2} + j\frac{2v}{(1-u)^2+v^2}$$

Equating real parts,

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$r[(1-u)^2+v^2] = 1-u^2-v^2$$

$$r(1-2u+u^2+v^2) = 1-u^2-v^2$$

$$u^2(1+r) - 2ur + v^2(1+r) = 1-r$$

$$u^2 - 2u\left(\frac{r}{1+r}\right) + v^2 = \frac{1-r}{1+r}$$

$$u^2 - 2u\left(\frac{r}{1+r}\right) + \left(\frac{r}{1+r}\right)^2 + v^2 = \frac{1-r}{1+r} + \left(\frac{r}{1+r}\right)^2$$

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1-r^2+r^2}{(1+r)^2} = \frac{1}{(1+r)^2} \qquad \ldots(12.17)$$

Equation (12.17) represents a family of circles with radius $\dfrac{1}{1+r}$ and centre at $\left(\dfrac{r}{1+r}, 0\right)$. These circles are known as *constant resistance circles*.

Equating imaginary parts,

$$x = \frac{2v}{(1-u)^2+v^2}$$

$$x[(1-u)^2+v^2] = 2v$$

$$x(1-2u+u^2+v^2) - 2v = 0$$

$$(u^2-2u+1)x + v^2x - 2v = 0$$

$$u^2 - 2u + 1 + v^2 - \frac{2v}{x} = 0$$

$$(u-1)^2 + v^2 - \frac{2v}{x} = 0$$

$$(u-1)^2 + v^2 - \frac{2v}{x} + \frac{1}{x^2} = \frac{1}{x^2}$$

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2} \qquad \ldots(12.18)$$

Equation (12.18) represents a family of circles with radius $\frac{1}{x}$ and centre at $\left(1, \frac{1}{x}\right)$. These circles are known as *constant reactance circles*.

12.14.1 Constant Resistance Circles

Figure 12.12 shows a family of constant resistance circles for different values of r ranging between 0 and ∞. The constant resistance circles have their centre at $\left(\frac{r}{r+1}, 0\right)$ with radii $\frac{1}{1+r}$.

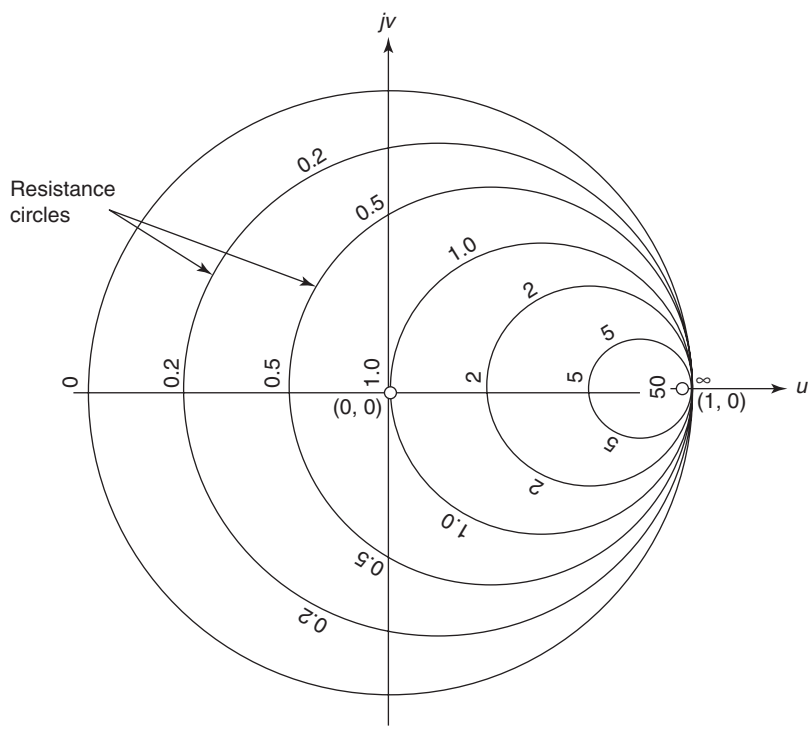


Fig. 12.12 *Constant resistance circles in complex Γ plane*

The constant resistance circles have the following characteristics:

1. The circles have their centres on the real axis.
2. All circles pass through the point (1, 0) in the complex Γ plane.
3. For $r = 0$, the centre lies at the origin of the plane, i.e., (0, 0), and it shifts to the right as r increases.

4. As r increases, the radius of the circle goes on decreasing and when $r \to \infty$, the radius approaches zero.
5. The outermost circle with centre (0, 0) and radius unity, corresponds to $r = 0$.
6. The rightmost point on the unit circle (1, 0) represents $r = 0$ as well as $r = \infty$.

12.14.2 Constant Reactance Circles

Figure 12.13 shows a family of constant reactance circles for different values of x. The constant reactance circles have their centres at $\left(1, \frac{1}{x}\right)$ with radii $\frac{1}{x}$.

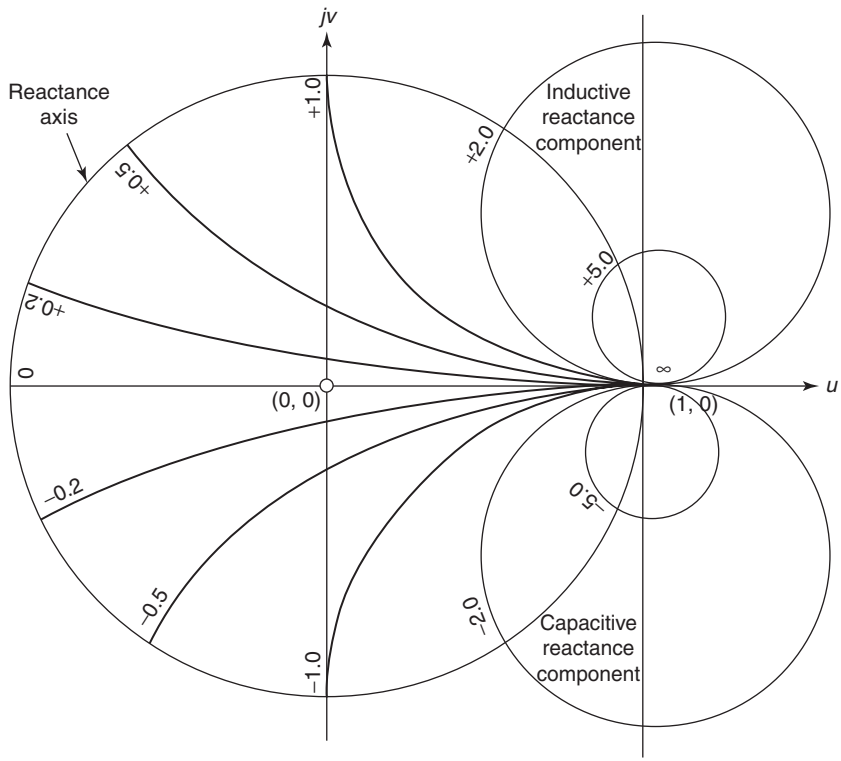


Fig. 12.13 *Constant reactance circles in the complex* Γ *plane.*

The constant reactance circles have the following characteristics:

1. The circles have their centre on the vertical axis passing through the point (1, 0).
2. For positive x, the centre lies above the real axis and for negative x, the centre lies below the real axis.
3. For $x = 0$, the centre is at $(1, \pm\infty)$ and radius is ∞. Hence, this circle represents a straight line and lies on the real axis.
4. As the magnitude of the reactance increases, the radius of the circle $\frac{1}{x}$ decreases and it approaches zero as $x \to \pm\infty$.

5. The real axis corresponds to $x = 0$ and represents purely resistance load.
6. The rightmost point on the unit circle (1, 0) represents $x = 0$ as well as $x = \pm\infty$.

12.14.3 The Smith Chart

The Smith chart is a graphical plot which is obtained by superposing the constant resistance and constant reactance circles with the unit circle in the complex plane. The Smith chart can be used for impedances as well as admittances.

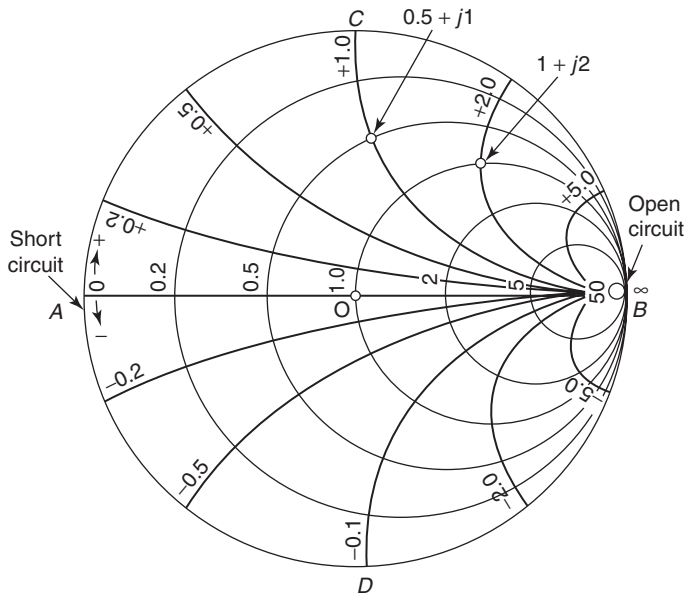


Fig. 12.14 *Smith chart*

The impedance Smith chart has the following characteristics:

1. The leftmost point A on the Smith chart corresponds to $r = 0$, $x = 0$ and hence, represents the zero impedance load or short-circuited load.
2. The right most point B on the Smith chart corresponds to $r = \infty$, $x = \infty$ and hence, represents infinite impedance load or open-circuited load.
3. The centre of the Smith chart O corresponds to $r = 1$, $x = 0$ and hence, represents the matched load.
4. The horizontal line represents pure resistance and the outermost circle represents pure reactance.
5. The uppermost point C represents a pure inductance of unit magnitude and the lower most point D represents a pure capacitance of unit magnitude.
6. The upper half of the Smith chart represents the complex inductive load and the lower half represents the complex capacitive loads.

A complete Smith chart is shown in Fig. 12.15.

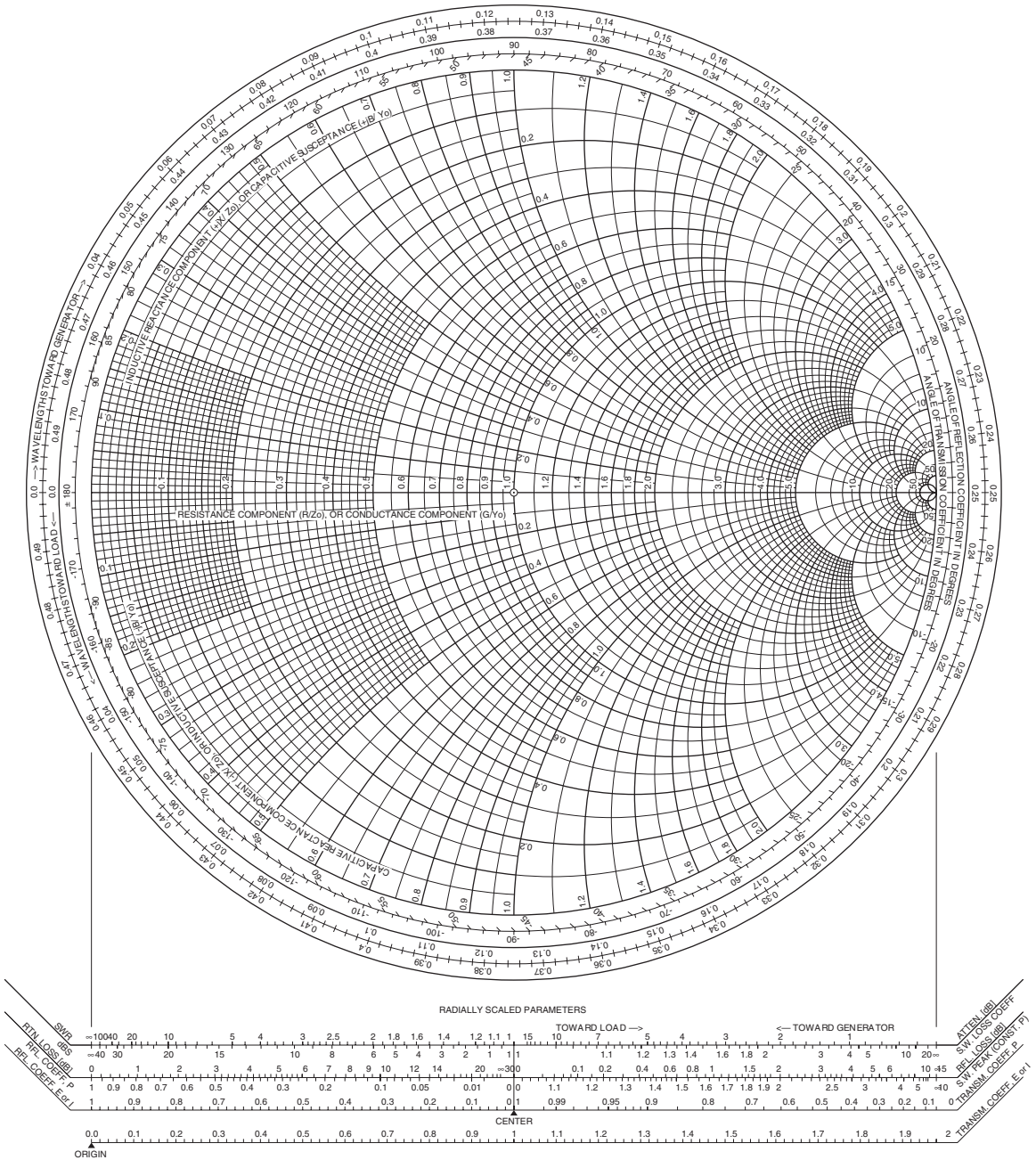


Fig. 12.15 *Complete Smith chart*

Generally, the circumference of the Smith chart is marked with the length normalised to the wavelength. Arrows indicating 'towards generator' and 'towards load' are also indicated. A clockwise rotation on the Smith chart indicates movement 'towards generator from the load'and an anticlockwise rotation on the Smith

chart indicates a movement 'towards load from the generator'. One complete rotation around the Smith chart corresponds to a phase change of 2π and represents a distance $\frac{\lambda}{2}$ on the line. The distance $\frac{\lambda}{2}$ on the line corresponds to a movement of 360° on the chart. Hence, the distance λ corresponds to a 720° movement on the chart.

$$\lambda = 720°$$

12.14.4 Constant VSWR Circles

Consider Eq. (12.17).

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}$$

If $v = 0$, the centres of all these circles lie on the horizontal real axis of the Smith chart.

$$\left(u - \frac{r}{1+r}\right)^2 = \frac{1}{(1+r)^2}$$

$$u - \frac{r}{1+r} = \pm\frac{1}{1+r}$$

$$u = \frac{r}{1+r} \pm \frac{1}{1+r} = \frac{r \pm 1}{1+r}$$

$$u = \frac{r+1}{1+r} \text{ or } u = \frac{r-1}{1+r}$$

i.e., $$u = 1 \quad \text{or} \quad u = \frac{r-1}{1+r}$$

For $$u = \frac{r-1}{1+r},$$

By componendo-dividendo,

$$\frac{1+u}{1-u} = \frac{(1+r)+(r-1)}{(1+r)-(r-1)} = \frac{2r}{2} = r$$

But $$u + jv = \Gamma$$

$$u = |\Gamma| \qquad (\because v = 0)$$

$$r = \frac{1+|\Gamma|}{1-|\Gamma|} = S$$

Thus, constant VSWR circles can be drawn with radius equal to the distance between centre of the Smith chart and point $r = S$.

The constant VSWR circles have the following characteristics:

1. All the circles have the same centre, i.e. the origin of the complex Γ plane.

2. The origin in the complex Γ plane represents $|\Gamma| = 0$ or $S = 1$. As we move radially outwards, the $|\Gamma|$ and S increases. For the outermost unity circle, $|\Gamma| = 1$ and $S = \infty$.

3. The origin corresponds to the condition $|\Gamma| = 0$ and $S = 1$, i.e. no reflection on the transmission line. For the outermost circle, i.e. $|\Gamma| = 1$, the entire power is reflected on the transmission line.

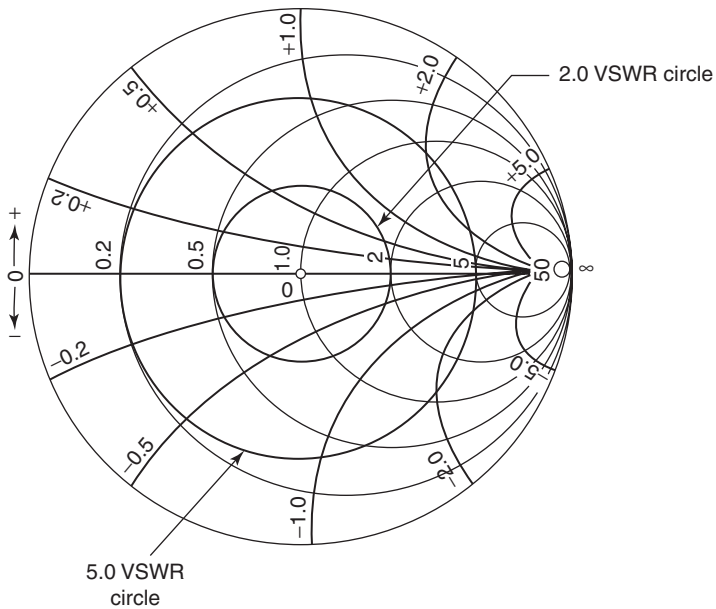


Fig. 12.16 *Smith chart with VSWR circles added*

12.14.5 Applications of Smith Chart

The Smith chart is a very useful graphical tool for solving transmission-line problems. The Smith chart finds application in finding the following quantities on a transmission line.

1. Normalised impedance
2. VSWR
3. Reflection coefficient
4. Input impedance of the line

1. Plotting Normalised Impedance

Any complex normalised impedance can be represented by a point on the Smith chart which is intersection of constant resistance circle, i.e. $r = \dfrac{R}{\mathbf{Z}_0}$ and constant reactance circle, i.e. $x = \dfrac{X}{\mathbf{Z}_0}$.

2. Measurement of VSWR

VSWR can be measured by drawing a constant VSWR circle with centre of the chart (1, 0) and radius equal to distance between the centre and point indicating the normalised impedance. The point of intersection of VSWR circle with the real axis at the right side of the centre gives VSWR for given line.

3. Measurement of Reflection Coefficient

The angle of reflection coefficient Γ can be measured by extending a line from the centre of the chart to the outer circumference of the chart through the point which indicates the normalised impedance. The point at which the extended line cuts the outer circumference gives the angle of reflection coefficient Γ.

The magnitude of reflection coefficient Γ can be obtained from Γ-scale provided at the bottom of the chart. An arc is cut by intersecting the Γ-scale with radius equal to the distance between the centre of the chart and a point indicating normalised impedance. The distance from centre of the Γ-scale to the point of intersection on the horizontal Γ-scale gives the magnitude of reflection coefficient.

4. Measurement of Input Impedance of the Line

To find the input impedance of the line, first locate the normalised impedance and extend the line up to the outer circumference of the chart. To find the input impedance, move along the outer circumference in a clockwise direction toward the generator with a distance equal to the length of the transmission line. Mark the point on the outer circumference and draw a line from the centre of the chart to the generator point. This line cuts the circle drawn corresponding to the VSWR of the line. This point gives the normalised input impedance and the actual input impedance can be obtained by multiplying this normalised impedance by $\mathbf{Z}_0$.

Example 12.13 *A lossless 75 Ω transmission line is terminated by an impedance of 150 + j150 Ω. Using the Smith chart, find (a) VSWR, and (b) reflection coefficient.*

Solution

$$\mathbf{Z}_0 = 75\ \Omega, \mathbf{Z}_L = 150 + j150\ \Omega$$

1. The normalised load impedance is given by,

$$\bar{\mathbf{Z}}_L = \frac{150 + j150}{75} = 2 + j2$$

Mark the normalised load impedance point A at the intersection of the $r = 2$ circle and the $x = 2$ circle.

2. Draw the VSWR circle with centre O and radius equal to distance OA. The VSWR circle cuts the real axis at $r = 4.2$ on the right side.

$$\text{VSWR} = 4.2$$

3. The magnitude of reflection coefficient is obtained from the linear scale shown at the bottom of the chart. Mark a point from centre at a distance equal to the distance OA. This gives the value of $|\Gamma|$ as 0.62. To find the angle of reflection coefficient, draw a line OA and extend it up to A' on the outer circumference of the chart. This line intersects the scale for the angle of reflection coefficient at 30°.

$$\Gamma = 0.62 \angle 30°$$

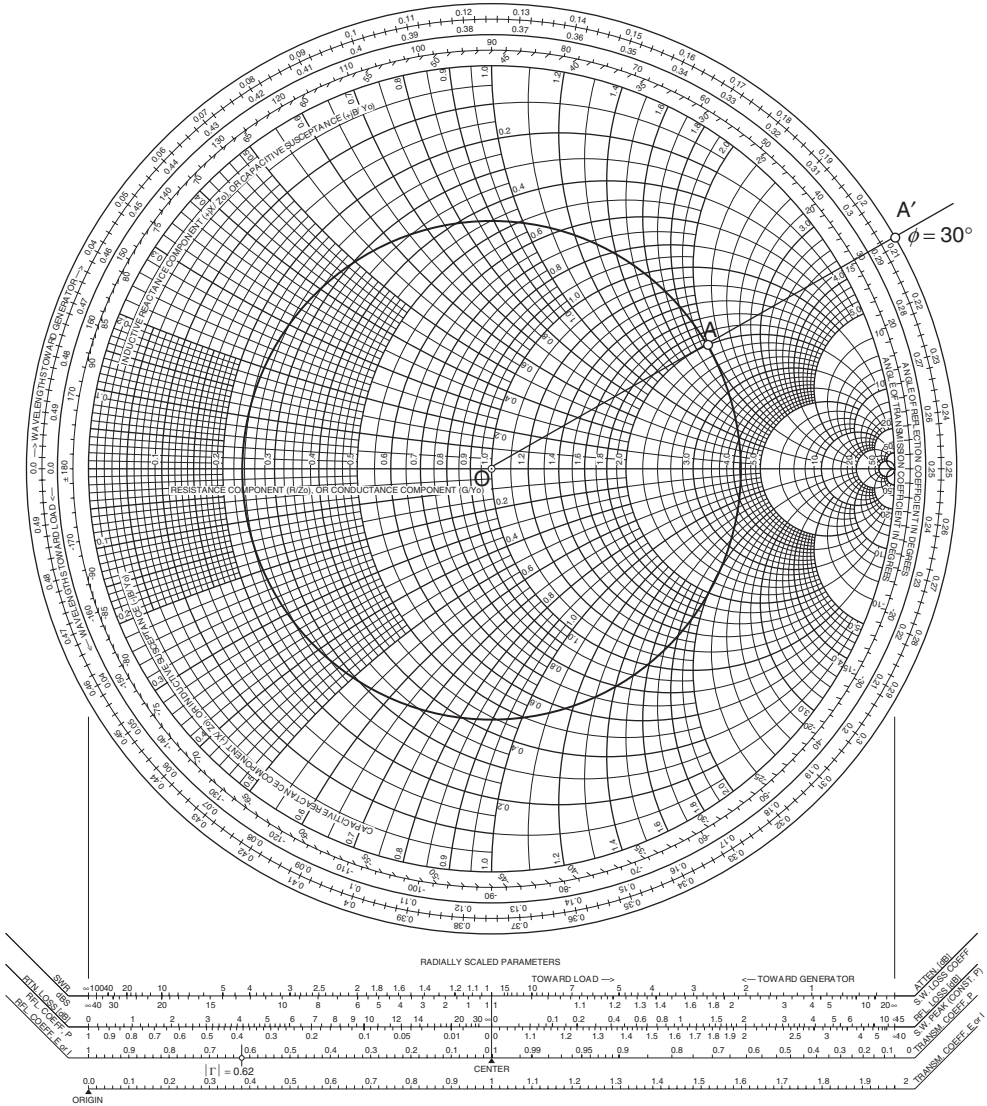


Fig. 12.17

Example 12.14 *The characteristic impedance of a high frequency-line is 100 Ω. It is terminated in an impedance of 100 + j100 Ω. Using a Smith chart, find the impedance at one-eighth wavelength away from the load end.*

Solution

$$\mathbf{Z}_0 = 100\ \Omega,\ \mathbf{Z}_L = 100 + j100\ \Omega$$

1. The normalised load impedance is given by,

$$\bar{\mathbf{Z}}_L = \frac{100 + j100}{100} = 1 + j1$$

   Mark the normalised load impedance point A at the intersection of the $r = 1$ circle and the $x = 1$ circle.

2. Draw the VSWR circle with centre O and radius equal to distance OA.

3. Draw a line OA and extend it up to A' on the outer circumference of the chart. The point A' is located at 0.163 λ on the outer circumference of the chart.
4. To find input impedance of the line $\frac{1}{8}\lambda$ away from the load, move in a clockwise direction from the point A' through a distance of $\frac{\lambda}{8}$, i.e. 0.125 λ on outer circumference and locate it as point B' at 0.163 λ + 0.125 λ = 0.288 λ. Draw the line OB' which intersects the constant VSWR circle at the point B. The point B is at the intersection of the $r = 2$ circle and the $x = -1$ circle. Hence, normalised input impedance at the point B is

$$\bar{\mathbf{Z}}_S = 2 - j1$$

The actual input impedance is

$$\mathbf{Z}_S = \mathbf{Z}_0 \bar{\mathbf{Z}}_S = 100(2 - j1) = 200 - j100\ \Omega$$

5. The complete Smith chart is shown in Fig.12.18.

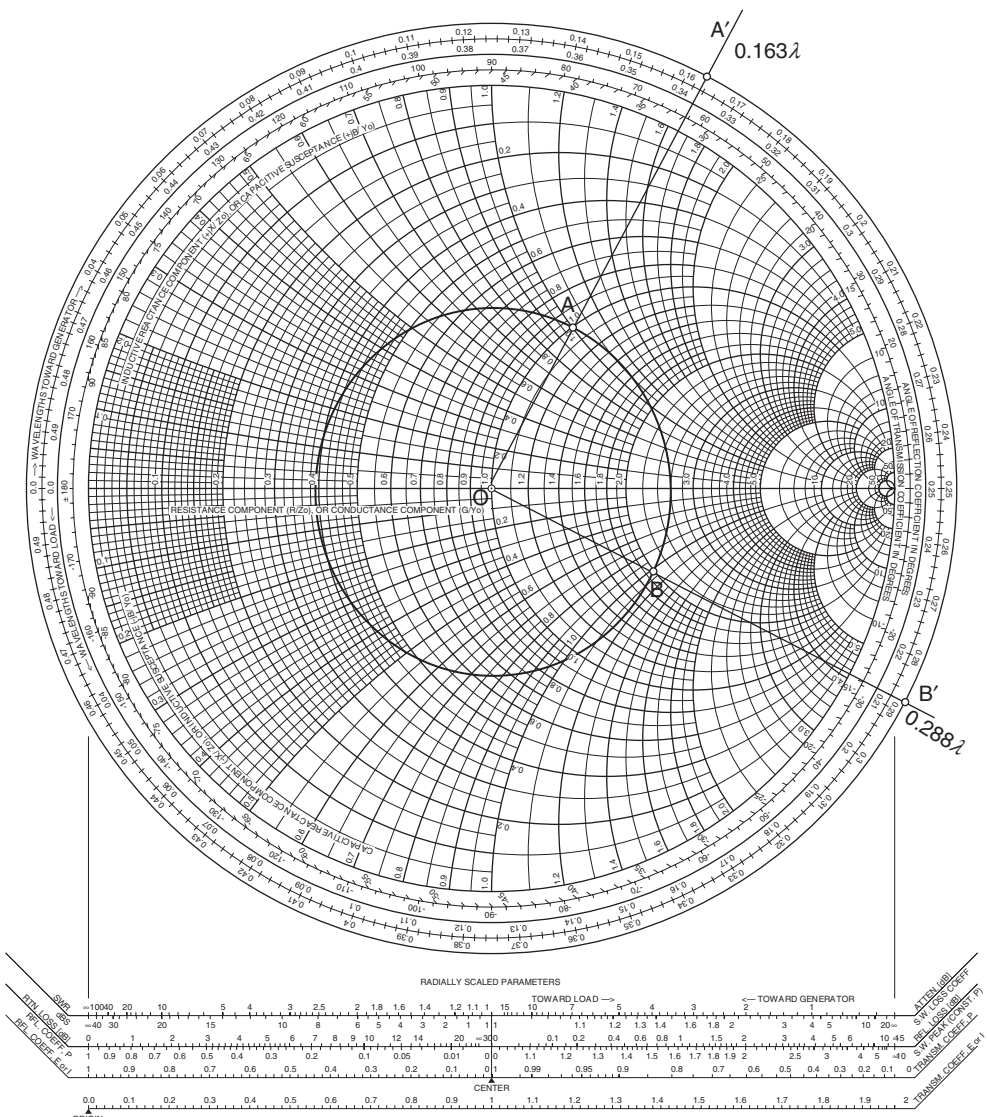


Fig. 12.18

Example 12.15 *A 50 Ω lossless transmission line is terminated in a load impedance of (25 + j50) Ω. Use Smith chart to find (a) VSWR, (b) voltage reflection coefficient, and (c) input impedance of the line given that the line is 3.3 λ long.*

Solution

$$\mathbf{Z}_0 = 50\ \Omega,\ \mathbf{Z}_L = 25 + j50\ \Omega$$

1. The normalised load impedance is given by,

$$\bar{\mathbf{Z}}_L = \frac{25 + j50}{50} = 0.5 + j1$$

Mark the normalised load impedance point A at the intersection of the $r = 0.5$ circle and the $x = 1$ circle.

2. Draw the VSWR circle with centre O and radius equal to the distance OA. The VSWR circle cuts the horizontal axis at $r = 4.2$ on the right-hand side.

$$\text{VSWR} = 4.2$$

3. Draw an arc with radius OA on the reflection coefficient line located at the lower side of the chart. This arc cuts the line at 0.62.

$$|\Gamma| = 0.62$$

Extend the line OA up to the outer circumference of the chart to get the point A'. The angle corresponding to the point A' is 83°.

$$\phi = 83°$$
$$\Gamma = |\Gamma| \angle \phi = 0.62 \angle 83°$$

4. From the point A', move a distance of 3.3 λ in the clockwise direction to get the input point. On the outer circumference, move 6 times to get a distance of 3 λ and then travel distance of 0.3 λ to reach the point B'. Draw line OB' which intersects the constant VSWR circle at point B. The point B is at the intersection of the $r = 0.28$ circle and the $x = -0.4$ circle.
Hence, normalised input impedance is

$$\bar{\mathbf{Z}}_S = 0.28 - j0.4$$

The actual input impedance is

$$\mathbf{Z}_S = \mathbf{Z}_0 \bar{\mathbf{Z}}_S = 50(0.28 - j0.4) = 14 - j20\ \Omega$$

5. The complete Smith chart is shown in Fig.12.19.

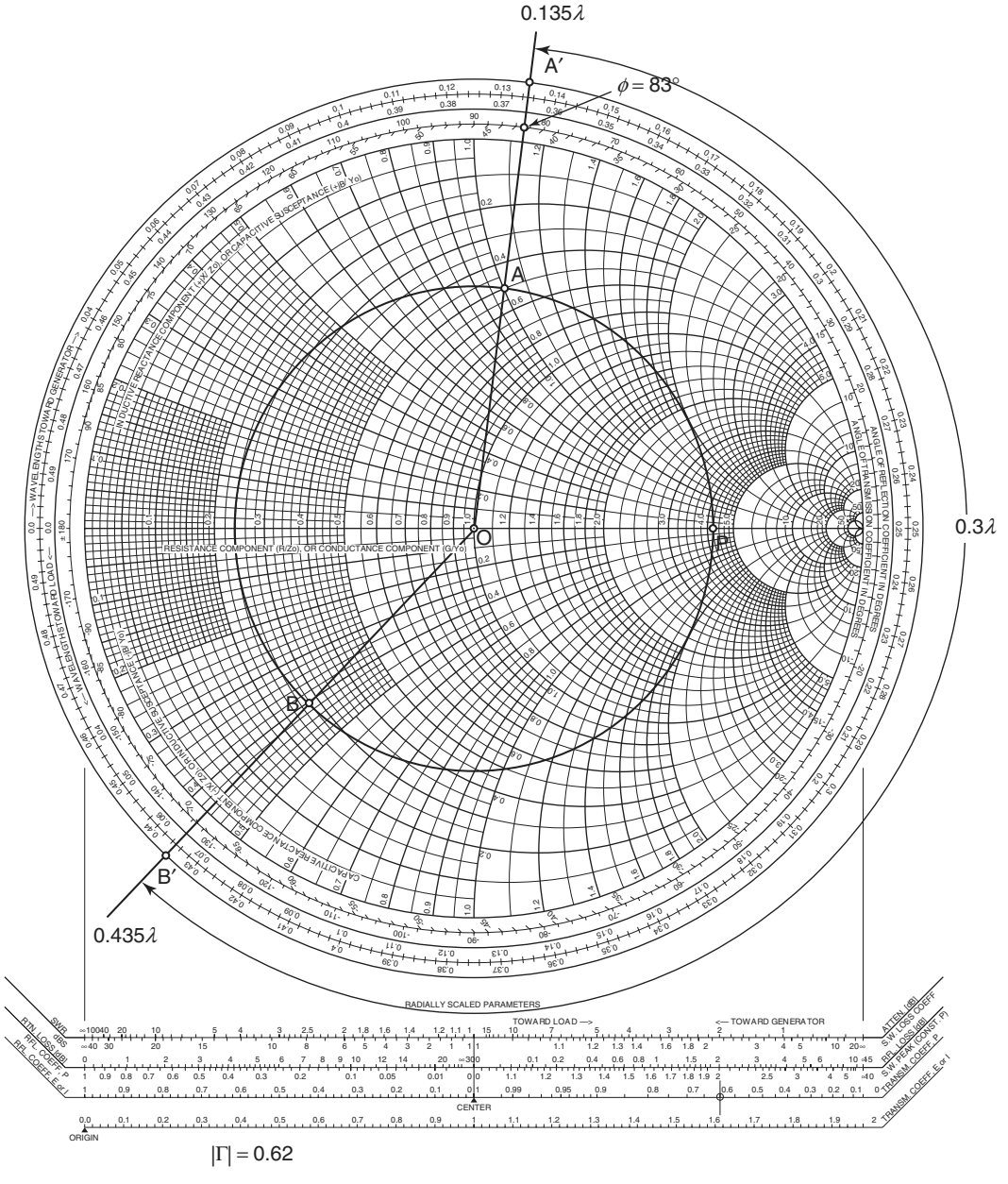

0.135λ
A′
ϕ = 83°
A
O
B
B′
0.435λ
0.3λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN
|Γ| = 0.62

Fig. 12.19

Example 12.16 *A transmission line has a characteristic impedance of 50 Ω and terminated in a load* $\mathbf{Z}_L = 75 - j100\ \Omega$. *Find the following using a Smith chart: (a) VSWR (b) Reflection coefficient (c) input impedance at a distance 0.1 λ from the load (d) location of first voltage maximum and first voltage minimum from the load.*

Solution

$$\mathbf{Z}_0 = 50\ \Omega,\ \mathbf{Z}_L = 75 - j100\ \Omega$$

1. The normalised load impedance is given by

$$\bar{\mathbf{Z}}_L = \frac{75 - j100}{50} = 1.5 - j2$$

   Mark the normalised load impedance point A at the intersection of the $r = 1.5$ and the $x = -2$ circle.

2. Draw the VSWR circle with centre O and radius equal to the distance OA. The VSWR circle cuts the horizontal axis at $r = 4.5$ on the right-hand side.

$$\text{VSWR} = 4.5$$

3. Draw a line OA and extend it up to the point A' on the outer circumference of chart. This line intersects the scale for the angle of reflection coefficient at −37°. This is the angle of reflection coefficient. The magnitude of reflection coefficient is obtained from the linear scale shown at the bottom of the chart. Mark a point from centre at a distance equal to the distance OA. This gives the value of $|\Gamma|$ as 0.64.

$$\Gamma = 0.64 \angle -37°$$

4. To find input impedance at 0.1 λ from the load, move in clockwise direction from the point A', a distance of 0.1 λ to reach the point B'. The point A' is at 0.302 λ. Hence, B' will be located at 0.302 λ + 0.1 λ = 0.402 λ.
5. Draw a line OB' which cuts the constant VSWR circle at the point B. The point B is the intersection of the $r = 0.32$ circle and the $x = -0.65$ circle. Hence, input impedance 0.1 λ away from the load is given by,

$$\bar{\mathbf{Z}}_d = 0.32 - j0.65$$

   Hence, actual impedance at 0.1 λ away from the load is given by,

$$\mathbf{Z}_d = \mathbf{Z}_0 \bar{\mathbf{Z}}_d = 50(0.32 - j0.65) = (16 - j32.5)\ \Omega$$

6. The first voltage maximum occurs to the right of the centre on the real axis at $S = 4.5$ and the first voltage minimum occurs to the left of the centre on the real axis at $S = 0.22$. To find the location of voltage maximum from the load, move in a clockwise direction from the point A' to the point P (voltage maximum point). Total distance is distance AQ + distance $QP = 0.198\ \lambda + 0.25\ \lambda = 0.448\ \lambda$. Hence, the first voltage maximum is located at 0.448 λ from the load. To find the location for voltage minimum from the load, move in a clockwise direction from the point A' to the point Q (voltage minimum point). Total distance is distance $AQ = 0.5\lambda - 0.302\ \lambda = 0.198\ \lambda$. Hence, the first voltage minimum is located at 0.18 λ from the load.
7. The complete Smith chart is shown in Fig.12.20.

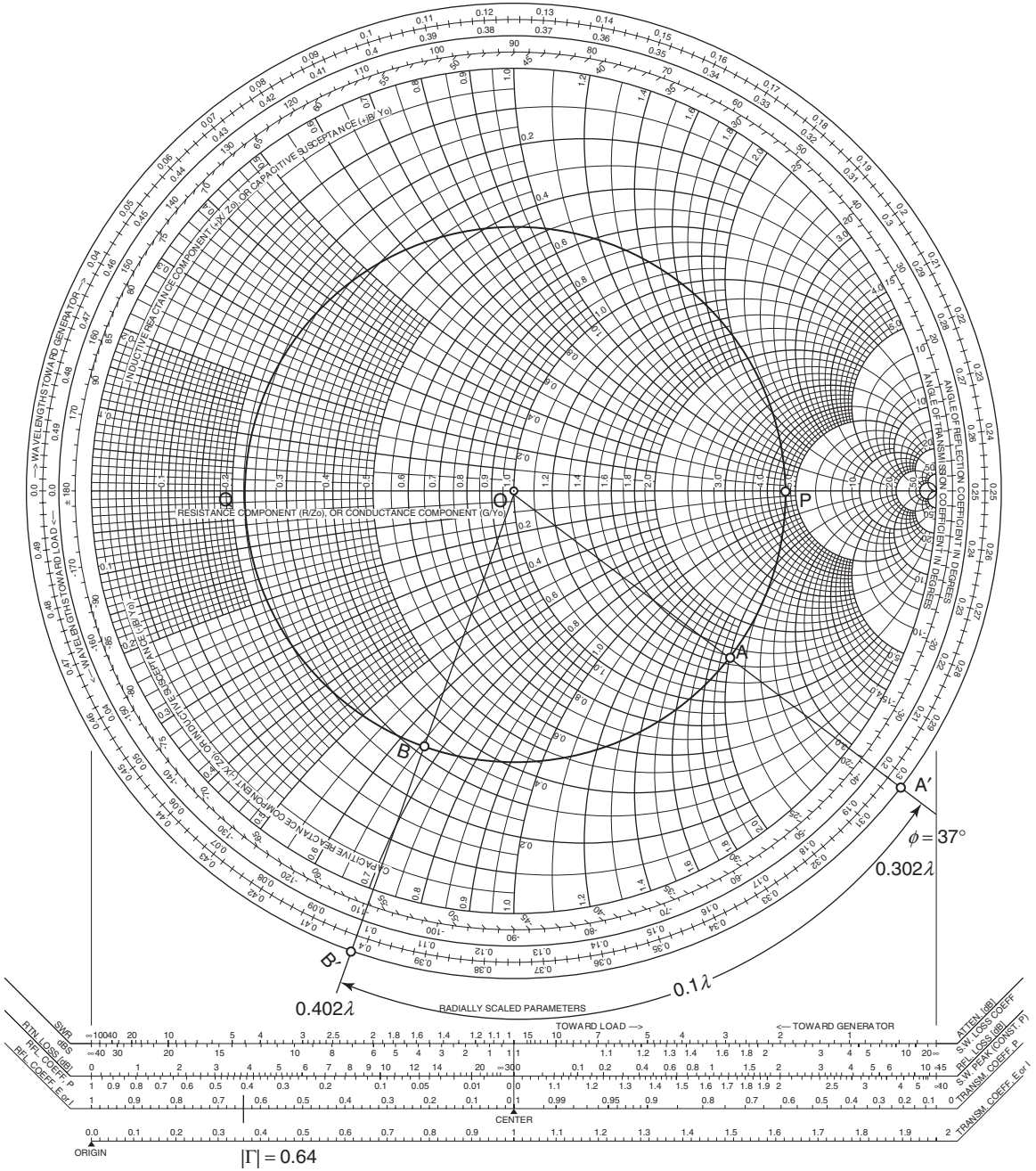

RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
O
P
A
A′
B
B′
$\phi = 37°$
0.302λ
0.1λ
0.402λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
RTN. LOSS (dB)
REFL. COEFF. E or I
ATTEN. (dB)
S.W. LOSS COEFF
REFL. LOSS (dB)
S.W. PEAK (CONST. P)
TRANSM. COEFF. E or I
CENTER
ORIGIN
$|\Gamma| = 0.64$

Fig. 12.20

Example 12.17 *A 100 m long transmission line is terminated in a load of (100 – j200) Ω. Determine the line impedance at 25 m from the load end at a frequency of 10 MHz. Assume line impedance* $\mathbf{Z}_0 =$ *100 Ω. Determine the input impedance and admittance using a Smith chart.*

Solution

$$\mathbf{Z}_0 = 100\ \Omega,\ \mathbf{Z}_L = 100 - j200\ \Omega$$

1. The normalised load impedance is given by,

$$\bar{\mathbf{Z}}_L = \frac{100 - j200}{100} = 1 - j2$$

Mark the normalised load impedance point *A* at the intersection of the $r = 1$ circle and the $x = -2$ circle.

2. Draw the VSWR circle with centre *O* and radius equal to the distance *OA*.
3.

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^6} = 30 \text{ m}$$

$$l = 25 \text{ m} = \frac{25}{30}\lambda = 0.833\ \lambda$$

4. From the point *A′*, move a distance of 0.833 λ to reach the point *B′*. Draw the line *OB′* which intersects the VSWR circle at the point *B*. The point *B* is at the intersection of the $r = 0.44$ and $x = 1.2$ circles.

Hence, the impedance of a line at a 25 m distance from the load is

$$\mathbf{Z}_S = \mathbf{Z}_0(0.44 + j1.2) = 100(0.44 + j1.2) = (44 + j120)\ \Omega$$

5. To calculate input impedance at 100 m, travel a distance of 100 m, i.e. 3.333 λ, from the point *A′*. But we reach at the same point *B′*.

Hence,

$$\mathbf{Z}_{\text{in}} = (44 + j120)\ \Omega$$

6. To find input admittance at 100 m from the load, move to the diametrically opposite point *C* from the point *B* and locate corresponding normalised admittance.

It is point $0.27 - j0.73$ on the Smith chart.

$$\bar{\mathbf{Y}}_{\text{in}} = 0.27 - j0.73.$$

Hence, actual input admittance is given by,

$$\mathbf{Y}_{\text{in}} = \mathbf{Y}_0 \bar{\mathbf{Y}}_{\text{in}} = \frac{1}{100}(0.27 - j0.73) = (2.7 - j7.3) \times 10^{-3}\ \text{℧}$$

7. The complete Smith chart is shown in Fig.12.21.

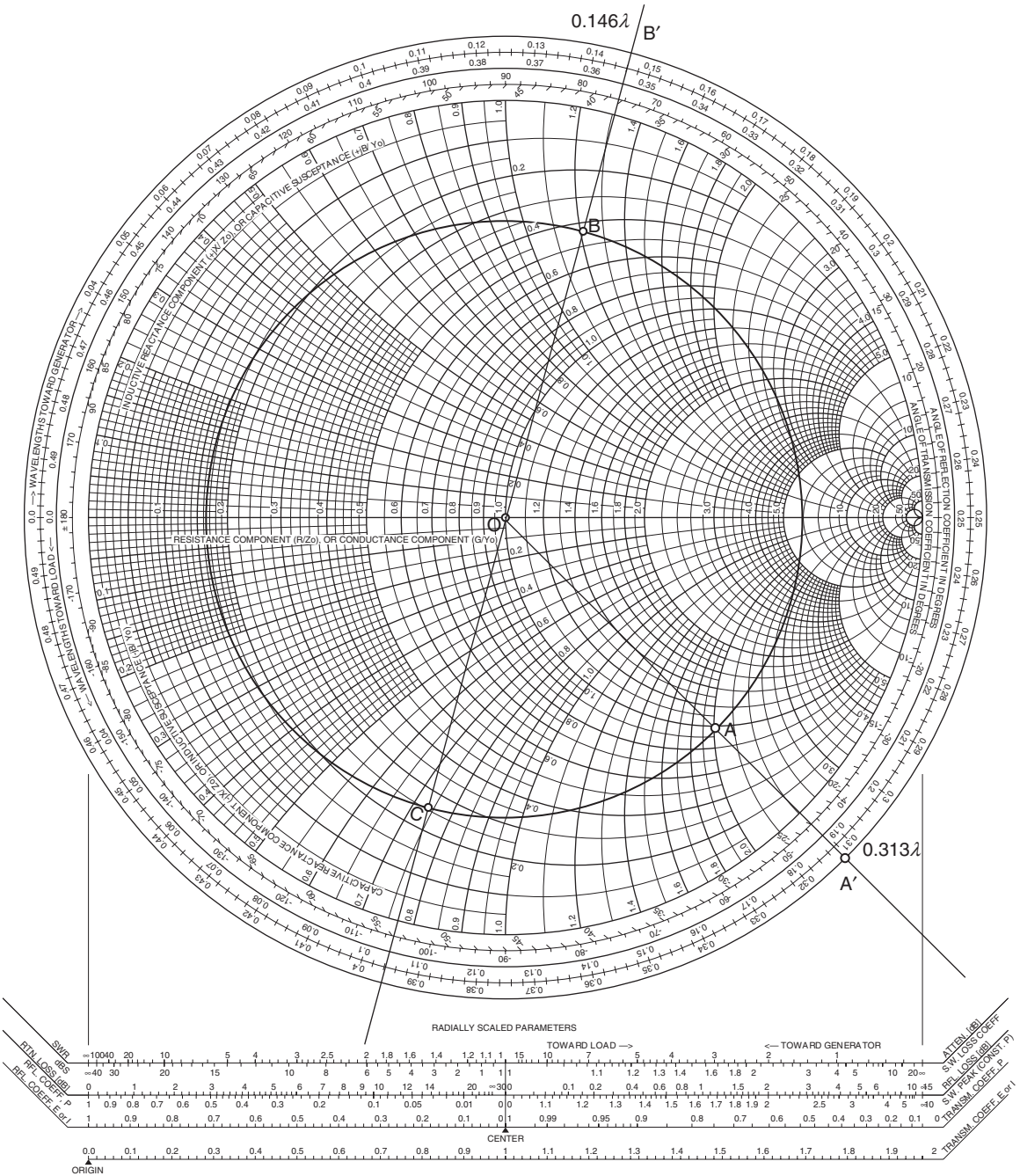

0.146λ
B′
B
O
A
C
0.313λ
A′
RESISTANCE COMPONENT (R/Z0), OR CONDUCTANCE COMPONENT (G/Y0)
INDUCTIVE REACTANCE COMPONENT (+jX/Z0), OR CAPACITIVE SUSCEPTANCE (+jB/Y0)
CAPACITIVE REACTANCE COMPONENT (-jX/Z0), OR INDUCTIVE SUSCEPTANCE (-jB/Y0)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
RTN. LOSS (dB)
REFL. COEFF. Γ
REFL. COEFF. E or I
ATTEN. (dB)
S.W. LOSS COEFF.
REFL. LOSS (dB)
S.W. PEAK (CONST. P)
TRANSM. COEFF. E or I
TRANSM. COEFF. P
CENTER
ORIGIN

Fig. 12.21

12.15 IMPEDANCE MATCHING USING SMITH CHART

When a radio frequency line is terminated in its characteristic impedance $\mathbf{Z}_0$, there is no reflection on the line. Hence, the impedance at every point on the line becomes $\mathbf{Z}_0$. The power is maximally transferred to the load under this condition. But practically, the load connected to the line may not be equal to $\mathbf{Z}_0$ giving rise to reflection. In order to avoid reflections, the impedance matching technique is used. Open or short stub lines are used for impedance matching. A *stub* is a transmission line only either short or open circuited used as reactive elements.

In the *single-stub matching technique*, a stub of suitable length is connected in parallel with a line at a certain distance from the load. The stub to be connected provides susceptance equal in magnitude but opposite in phase as compared to that of the load. Hence, antiresonance is achieved providing impedance at antiresonance equal to $\mathbf{Z}_0$. Since the stub is connected in parallel with the line, it is convenient to work with admittances.

Steps to be followed for Single-Stub Matching Analysis

1. Determine the normalised impedance of the load and mark it as point *A* on the Smith chart.
2. Draw the VSWR circle with centre *O* and radius equal to the distance *OA*. The VSWR circle cuts the horizontal axis at the right of the centre. The gives VSWR value before the use of the stub.
3. Mark the point *B* on the VSWR circle exactly diametrically opposite to the point *A*. This gives the normalised load admittance $g + jb$ corresponding to the load impedance $r + jx$.
4. From the point *B*, move towards the generator in a clockwise direction on the VSWR circle until it intersects the unity conductance circle ($g = 1$ circle). Distance travelled to at this intersection point from the load is equal to distance of the stub from the load. There will be at least two such points from the load where the conductance is unity.
5. If the admittance in the previous step is $1 + jb$ then a susceptance of $-jb$ in shunt is needed for matching.
6. Since the other end of the stub will have an open or a short, VSWR on it will be infinite. It is represented by the outermost circle of the Smith chart. Locate the desired susceptance point (i.e. $0 - jb$) on this circle and then move towards the generator in a clockwise direction until $-jb$ the point is reached from an open circuit. (i.e. zero susceptance) or a short circuit (an infinite susceptance). This distance is equal to the stub length.

Figure 12.22 shows the steps to be followed for single-stub matching using a Smith chart.

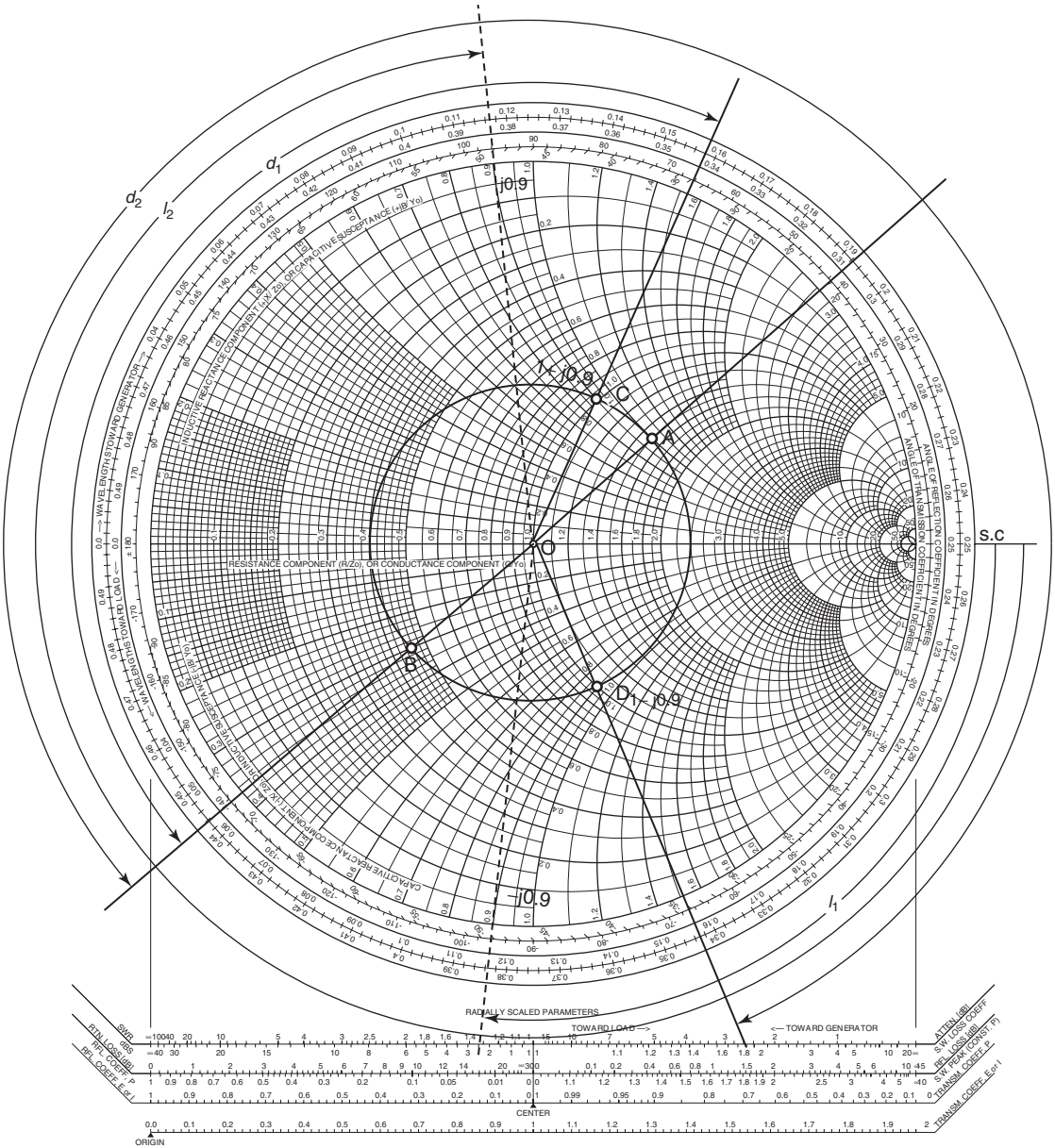
Example 12.18 *A 100 Ω lossless transmission line is to be matched with a 50 – j75 Ω load using a single stub. Calculate the stub length and its distance from the load.*

Solution

$$\mathbf{Z}_0 = 100\ \Omega,\ \mathbf{Z}_L = 50 - j75\ \Omega$$

1. The normalised load impedance is given by,

$$\bar{\mathbf{Z}}_L = \frac{50 - j75}{100} = 0.5 - j0.75$$



$A \rightarrow$ Normalised load impedance point

$B \rightarrow$ Normalised load admittance point

$C, D \rightarrow$ Intersection of VSWR circle and $g = 1$ circle

$d_1 \rightarrow$ Distance of stub from load for stub at the point C

$d_2 \rightarrow$ Distance of stub from load for stub at the point D

$l_1 \rightarrow$ Length of stub for stub at the point C

$l_2 \rightarrow$ Length of stub for stub at the point D

Fig. 12.22

Mark the normalised load impedance point A at the intersection of the $r = 0.5$ and $x = -0.75$ circles.

2. Draw the VSWR circle with centre O and radius equal to the distance OA.
3. From the load impedance point, move to the diametrically opposite point B and locate the corresponding normalised load admittance. It is the point $0.62 + j0.91$ on the Smith chart.
4. Locate the point on the VSWR circle where the real part of the admittance is unity. There are two such points, C and D, with normalised admittance values $1 + j1.3$ and $1 - j1.3$ respectively. This can also be viewed as intersection of VSWR circle and $g = 1$ circle.
5. Distance of the point C from the point B can be determined as $0.035\ \lambda$ (i.e., $0.170\ \lambda - 0.135\ \lambda$) and distance the point D from the point B is determined as $0.195\ \lambda$ (i.e., $0.33\ \lambda - 0.135\ \lambda$)
6. At the point C, susceptance is $+j1.3$. To cancel this susceptance, the stub should provide $-j1.3$ susceptance. The length of the stub can be obtained by calculating distance starting from the short-circuit end towards the generator in a clockwise direction. (Fig. 12.24)
   Length of the stub $l_1 = 0.354\ \lambda - 0.25\ \lambda = 0.104\ \lambda$
7. At the point D, susceptance is $-j1.3$. To cancel this, the stub should provide $+j1.3$ susceptance. The length of the stub can be obtained by calculating distance starting from the short-circuit end towards the generator in a clockwise direction. (Fig. 12.25)
   Length of the stub $l_2 = 0.25\ \lambda + 0.145\ \lambda = 0.395\ \lambda$

Figure 12.23 shows the position of the stub required for impedance matching. For minimising losses in the transmission line, the stub should be connected near the load end.

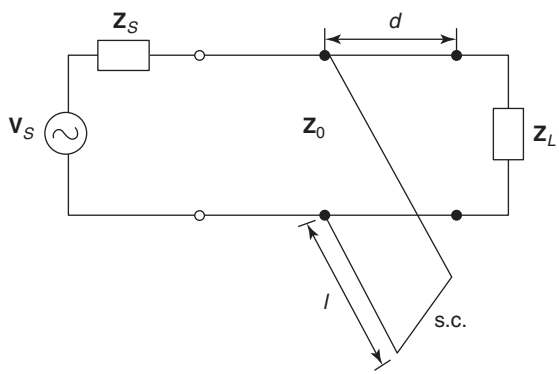


Fig. 12.23 *Position of stub required for impedance matching*

8. The complete Smith chart is shown in Figures 12.24 and 12.25.

Example 12.19 *Design a single-stub match for a load of $150 + j232.5\ \Omega$ for a $75\ \Omega$ line at 500 MHz using a Smith chart.*

Solution

$$\mathbf{Z}_0 = 75\ \Omega,\ \mathbf{Z}_L = 150 + j232.5\ \Omega$$

1. The normalised load impedance is given by

$$\bar{\mathbf{Z}}_L = \frac{150 + j232.5}{75} = 2 + j3.1$$

   Mark the normalised load impedance point A at the intersection of the $r = 2$ and $x = 3.1$ circles.
2. Draw the VSWR circle with centre O and radius equal to the distance OA.

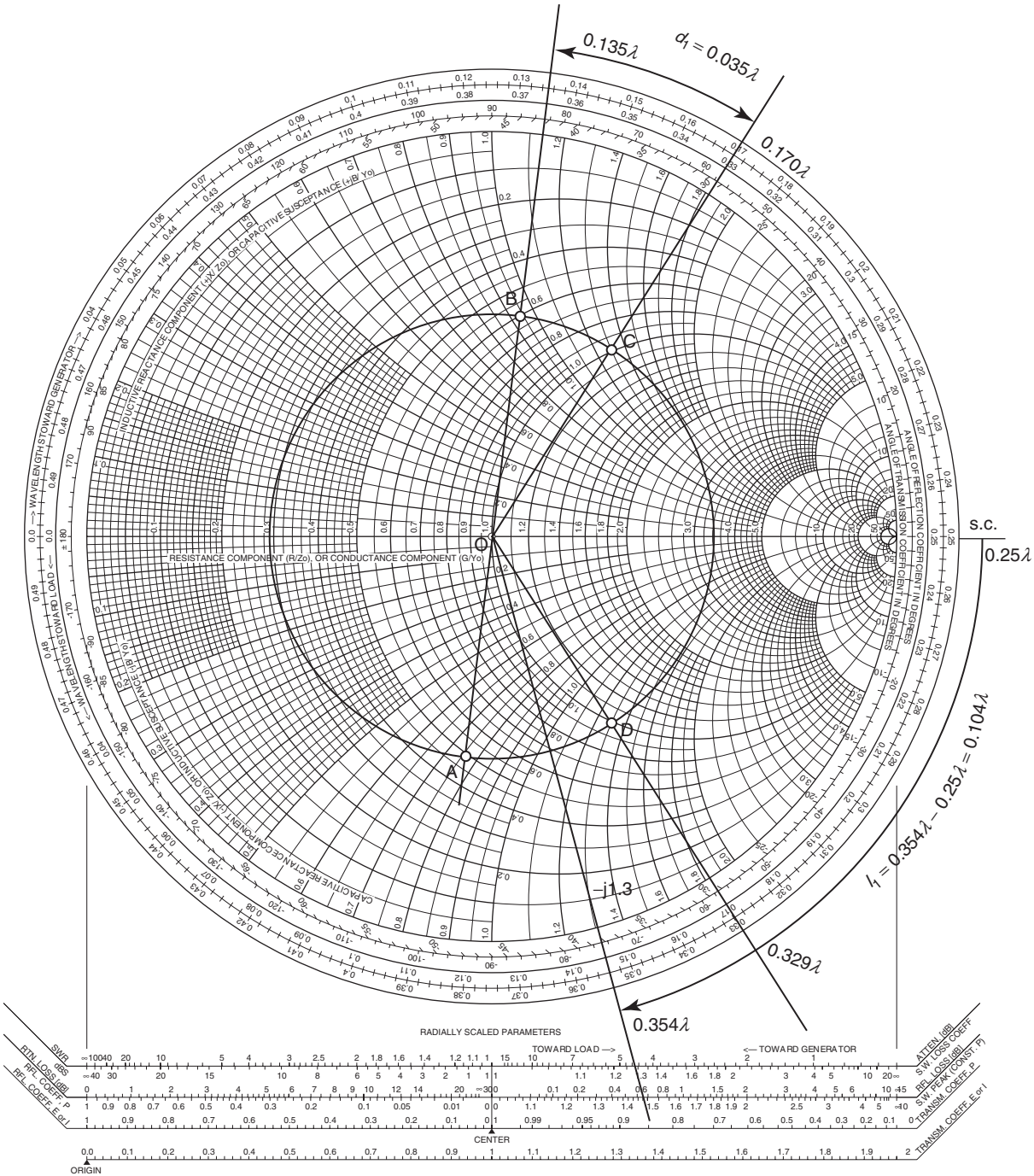

0.135λ
$\alpha_1 = 0.035\lambda$
0.170λ
s.c.
0.25λ
$l_1 = 0.354\lambda - 0.25\lambda = 0.104\lambda$
0.329λ
0.354λ
A
B
C
D
O
–j1.3
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (–jX/Zo), OR INDUCTIVE SUSCEPTANCE (–jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
RTN. LOSS (dB)
REFL. COEFF. P
REFL. COEFF. E or I
ATTEN. (dB)
S.W. LOSS COEFF.
REFL. LOSS (dB)
S.W. PEAK CONST. (F)
TRANSM. COEFF. P
TRANSM. COEFF. E or I
CENTER
ORIGIN

Fig. 12.24

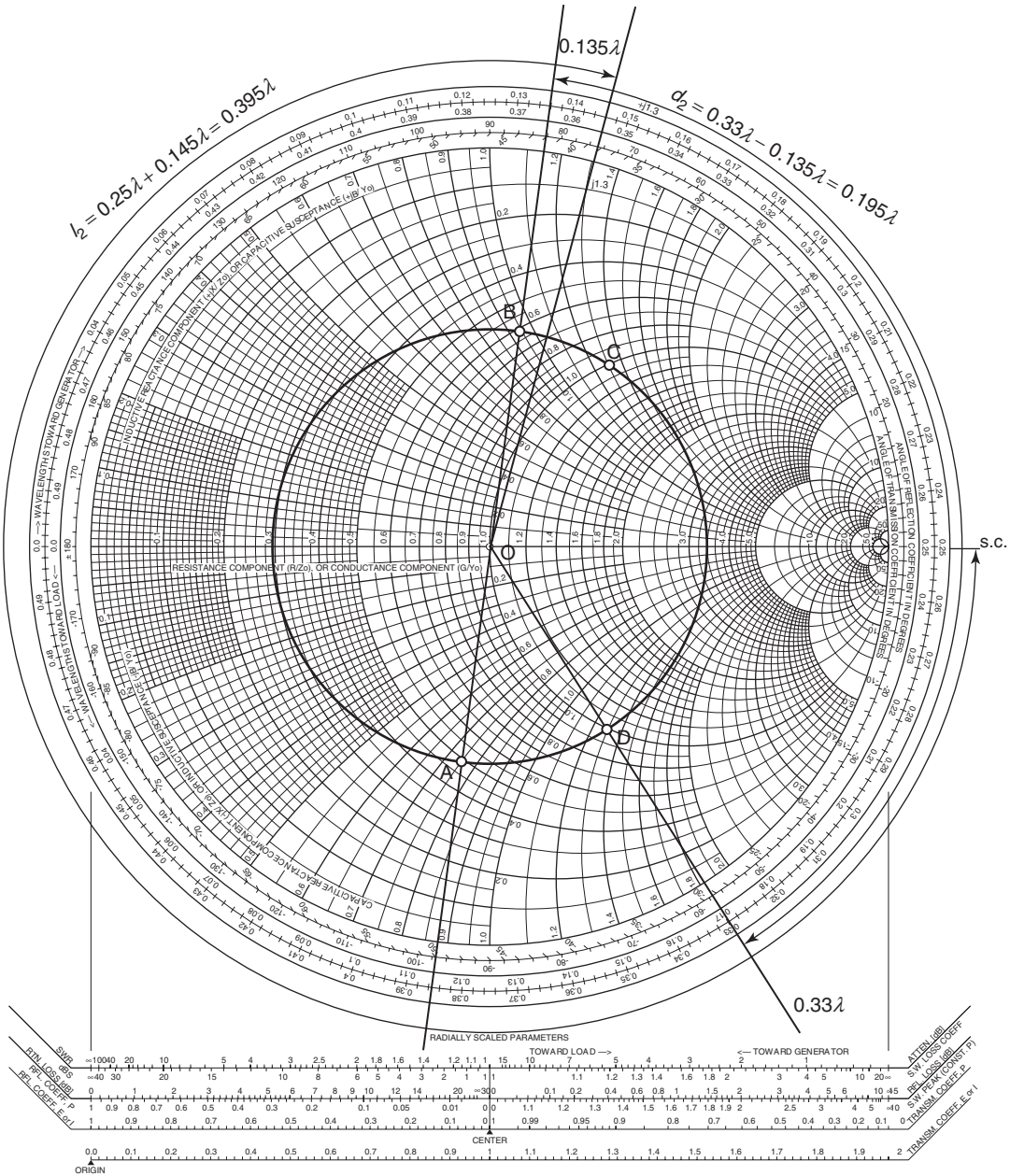


Fig. 12.25

3. From the load impedance point, move to the diametrically opposite point B and locate the corresponding normalised load admittance. It is the point 0.15 – *j*0.23 on the Smith chart.
4. The VSWR circle cuts the unity conductance circle at points *C* and *D*, which gives possible stub locations. Since the point *C* is nearest to the point *B*, the stub must be located at the point *C*.
5. Distance of the point *D* from the point *A* can be determined as 0.226 λ, i.e. [(0.5 λ – 0.464 λ) + 0.19 λ]

6. At the point *D*, the susceptance is $j2.2$. Hence, the stub should provide capacitive susceptance of $-j2.2$. The length of the stub can be obtained by calculating the distance starting from the short-circuit end towards the generator in a clockwise direction.
   Length of the stub = $0.318\ \lambda - 0.25\ \lambda = 0.068\ \lambda$
7. The complete Smith chart is shown in Fig.12.26.

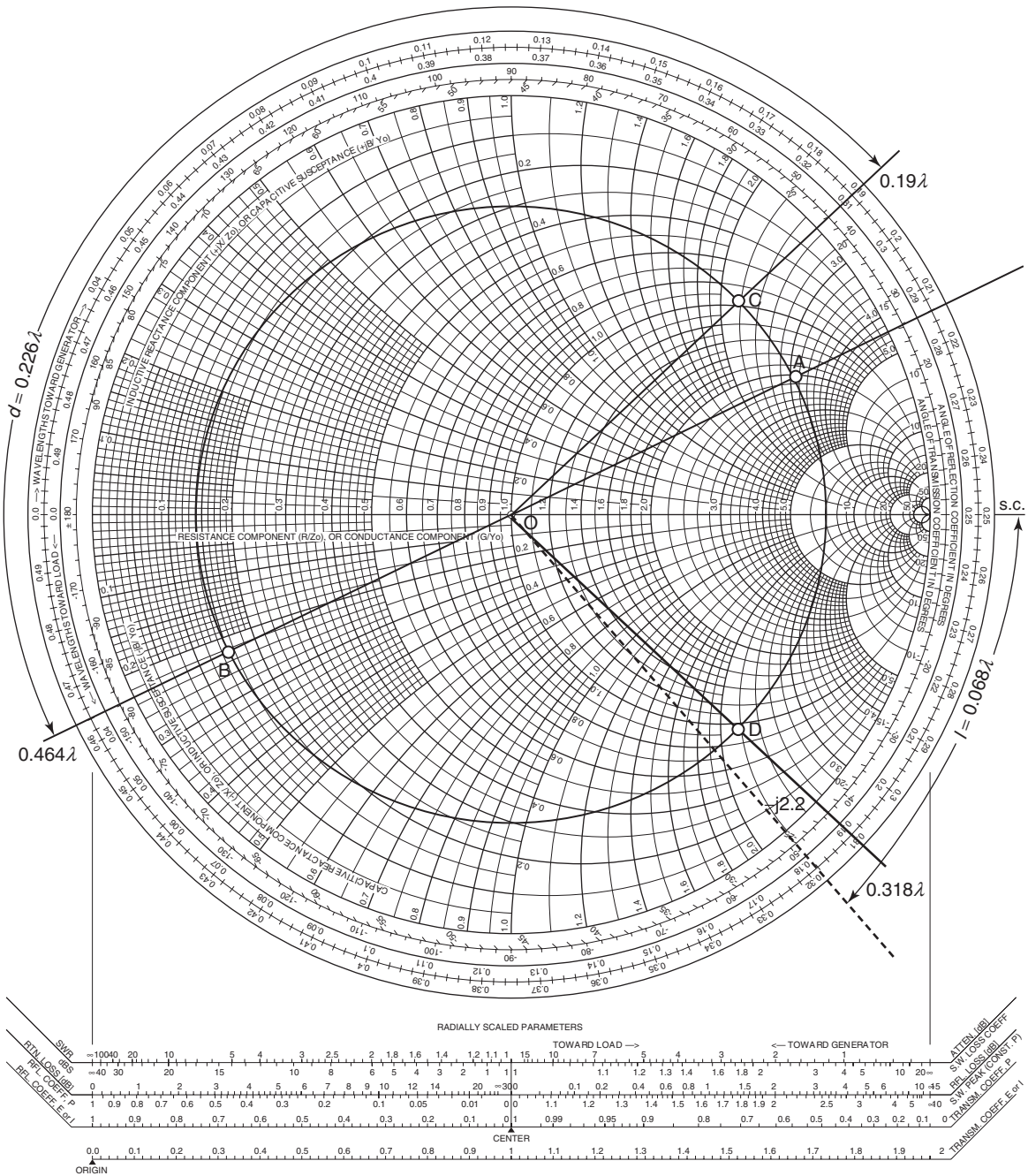


Fig. 12.26

Exercises

12.1 A transmission line has the following primary constants.
$R = 100\ \Omega$/km, $L = 0.001$ H/km, $G = 1.5\ \mu\Omega$/km, $C = 0.062\ \mu$F/km
Find the characteristic impedance and propagation constant.

[1269.82 $\angle -44.02°\ \Omega$, 0.0787 $\angle -44.02°$/km]

12.2 The characteristic impedance of a 805 m long transmission line is $94\angle -23.2°\ \Omega$, the attenuation constant is 74.5×10^{-6} Np/m and the phase-shift constant is 174×10^{-6} rad/m at 5 kHz. Calculate the line parameters R, L, G and C per metre.

[12.87 mΩ/m, 0.39 μH/m, 0, 64.09 pF/m]

12.3 A low-loss line has a characteristic impedance of 400 Ω. Determine the standing wave ratio if the receiving-end impedance is $(650 - j475)\ \Omega$.

[2.74]

12.4 If the reflection coefficient of a line is $0.3\ \angle -66°$, calculate the standing wave ratio.

[1.86]

12.5 Find the reflection coefficient of a 50 Ω transmission line when it is terminated by a load impedance of $(60 + j40)\ \Omega$.

[$0.35\ \angle 55.98°$]

12.6 A transmission line has a characteristic impedance of 300 and terminates in a load $\mathbf{Z}_L = 150 + j150\ \Omega$. Find the following using a Smith chart: (i) VSWR (ii) Reflection coefficient (iii) Input impedance at distance 0.1 λ from the load.

[2.6, $0.42\ \angle 118°$, $(414 + j336)\ \Omega$]

12.7 A 70 Ω lossless line is used at a frequency where wavelength λ equals 80 cm terminated by a load of $(140 + j91)\ \Omega$. Find the reflection coefficient and VSWR using a Smith chart.

[$0.5\ \angle 29°$, 3]

12.8 A 75 Ω lossless transmission line is to be matched with a $(100 - j\ 80)\ \Omega$ load using a single stub. Calculate the stub length and its distance from the load corresponds to the frequency of 30 MHz using a Smith chart.

[0.103 λ, 0.096 λ]

Objective-Type Questions

12.1 In a transmission line terminated with a load equal to the characteristic impedance, the reflection coefficient is

(a) 1 (b) 1 (c) 0 (d) λ

12.2 On a transmission line with standing waves, the distance between a voltage maxima and adjacent current maxima is

(a) $\frac{\lambda}{2}$ (b) $\frac{\lambda}{8}$ (c) $\frac{\lambda}{4}$ (d) λ

12.3 Stub matching eliminates standing waves on

(a) load side (b) generator side
(c) both sides of stub (d) none of these

12.4 The reflection coefficient on a lossless transmission line

(a) is always purely imaginary
(b) is always purely real
(c) is always complex
(d) none of these

12.5 VSWR in a short-circuited line equals

(a) infinity (b) unity
(c) zero (d) none

12.6 There will be no reflection in a transmission line if it is terminated by an impedance

(a) equal to characteristic impedance
(b) less than the characteristic impedance
(c) equal to twice the characteristic impedance
(d) equal to square root of its characteristic impedance

12.7 Input impedance of an open-circuited lossless line of length λ is

(a) infinity
(b) finite capacitive reactance
(c) finite inductive reactance
(d) zero

12.8 A matching stub should be

(a) nearest to load
(b) nearest to transmission end
(c) midway between both
(d) anywhere between the two

12.9 If the spacing between the wires of a transmission line is increased, its characteristic impedance will

(a) increase (b) decrease
(c) is unchanged (d) none of these

12.10 $\mathbf{Z}_0$ of a parallel wire transmission line _____with increase in distance between the conductors.

(a) increases (b) decreases
(c) does not change (d) none of these

Answers to Objective-Type Questions

12.1. (c) 12.2. (c) 12.3. (b) 12.4. (c) 12.5. (a) 12.6. (a) 12.7. (a)
12.8. (a) 12.9. (a) 12.10. (a)

Solved Question Paper December 2015

Circuits and Transmission Lines
ETC 304

Time: 3 Hours **Total Marks: 80**

N.B.: (1) Attempt question **1** and any **three** from remaining questions.
(2) All sub questions of the same question should be answered at one place only in their serial orders and not scattered.
(3) Write everything in ink only. Pencil is not allowed.
(4) Assume suitable data with justification if missing.

1. **(a)** Determine the ABCD parameters of the network shown in Fig. 1. **(5)**

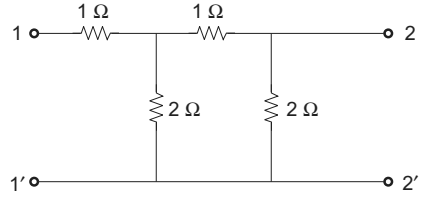


Fig. 1

Solution: Refer Example 10.15 on page 10.21.

(b) Test whether $P(s) = s^5 + 12s^4 + 60s^2 + 44s + 48$ is Hurwitz polynomial. **(5)**

Solution: Refer Example 9.19 on page 9.13.

(c) The combined inductance of two coils connected in series is 0.6 H or 0.1 H depending on relative directions of currents in the two coils. If one of the coils has a self inductance of 0.2 H, find (a) mutual inductance (b) coefficient of coupling. **(5)**

Solution: Refer Example 4.1 on page 4.7.

(d) Find Foster I and II and Cauer I and II circuits for the driving point admittance **(5)**

$$Y(s) = \frac{s^2 + 1}{s}$$

Solution: Foster-I Form

$$Z(s) = \frac{s}{s^2 + 1}$$

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$\frac{1}{C} = 1$$

$$C = 1F$$

$$\frac{1}{LC} = 1$$

$$\frac{1}{L} = 1$$

$$L = 1 \text{ H}$$

The network is shown in Fig. 2.

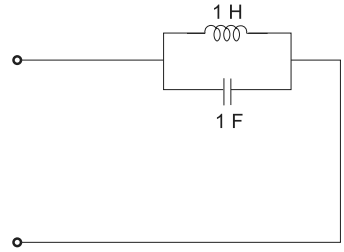


Fig. 2

Foster-II Form

$$Y(s) = \frac{s^2 + 1}{s}$$

$$= s + \frac{1}{s}$$

The first term represents admittance of a capacitor of 1F and second term represents admittance of an inductor of 1H. The network is shown in Fig. 3.

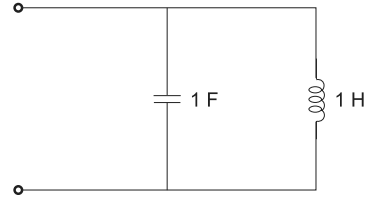


Fig. 3

Cauer-I Form

$$Y(s) = \frac{s^2 + 1}{s}$$

Since the degree of the numerator is greater than the degree of denominator by one it indicates the presence of pole at infinity.

By continued fraction expansion,

$$s \overline{)\, s^2 + 1 \,} \left(s \leftarrow Y \right.$$
$$\underline{s^2}$$
$$1 \overline{)\, s \,} \left(s \leftarrow Z \right.$$
$$\underline{s}$$
$$0$$

The network is shown in Fig. 4.

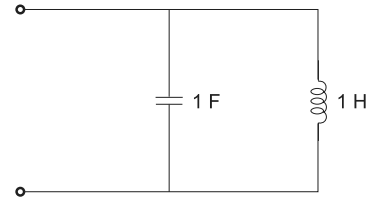


Fig. 4

Cauer-II Form

$$Y(s) = \frac{1 + s^2}{s}$$

By continued fraction expansion,

$$s \overline{)\, 1 + s^2 \,} \left(\frac{1}{s} \leftarrow Y \right.$$
$$\underline{1}$$
$$s^2 \overline{)\, s \,} \left(\frac{1}{s} \leftarrow Z \right.$$
$$\underline{s}$$
$$0$$

The network is shown in Fig. 5.

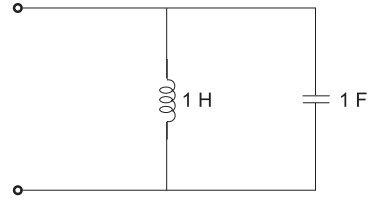


Fig. 5

2. **(a)** Find the current in the 10 Ω resistor using Thevenin's theorem for the network shown in Fig. 6. **(10)**

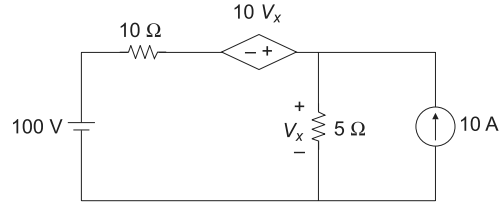


Fig. 6

Solution: Refer Example 3.57 on page 3.63.

(b) Find the value of V_x in the network shown in Fig. 7 using nodal analysis.

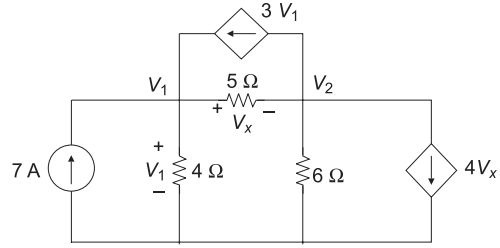


Fig. 7

Solution: From Fig. 7.

$$V_x = V_1 - V_2 \tag{i}$$

Applying KCL at Node 1,

$$7 + 3V_1 = \frac{V_1}{4} + \frac{V_1 - V_2}{5}$$

$$7 + 3V_1 = \frac{V_1}{4} + \frac{V_1}{5} - \frac{V_2}{5}$$

$$V_1\left(3 - \frac{1}{4} - \frac{1}{5}\right) + V_2\left(\frac{1}{5}\right) = -7$$

$$\frac{51}{20}V_1 + \frac{1}{5}V_2 = -7 \tag{ii}$$

Applying KCL at Node 2,

$$3V_1 + \frac{V_2 - V_1}{5} + \frac{V_2}{6} + 4V_x = 0$$

$$3V_1 + \frac{V_2}{5} - \frac{V_1}{5} + \frac{V_2}{6} + 4(V_1 - V_2) = 0$$

$$\left(3 - \frac{1}{5} + 4\right)V_1 + \left(\frac{1}{5} + \frac{1}{6} - 4\right)V_2 = 0$$

$$\frac{34}{5}V_1 - \frac{109}{30}V_2 = 0 \quad \text{(iii)}$$

Solving Eqs (ii) and (iii),

$V_1 = -2.4$ V

$V_2 = -4.48$ V

$V_x = V_1 - V_2 = -2.4 + 4.48 = 2.08$ V

(c) Check if the following polynomials are Hurwitz polynomials:

(i) $s^5 + s^3 + s$

Solution: Refer Example 9.10 on page 9.7.

(ii) $s^4 + s^3 + 2s^2 + 3s + 2$

Solution: Refer Example 9.5 on page 9.4.

3. (a) Synthesize the driving point function: **(10)**

$$F(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$ when $F(s)$ is a driving point (i) impedance (ii) admittance

Test if the circuit obtained are canonic.

Solution: Refer Example 9.41 on page 9.39.

(b) State and prove initial value theorem.

Solution: Refer Section 7.4.9 on page 7.7. **(5)**

(c) The parameters of a transmission lines are $R = 6\ \Omega$/km, $L = 2.2$ mH/km, $G = 0.25 \times 10^{-6}$ ℧/km, $C = 0.005 \times 10^{-6}$ F/km. Determine the characteristics impedance and propagation constant at a frequency of 1 GHz. **(5)**

Solution:

$$\boldsymbol{Z} = R + j\omega L = 6 + j2\pi \times 10^9 \times 2.2 \times 10^{-3} = 13823007.68 \angle 89.99°\ \Omega/\text{km}$$

$$\boldsymbol{Y} = G + j\omega C = 0.25 \times 10^{-6} + j2\pi \times 10^9 \times 0.005 \times 10^{-6} = 31.415 \angle 89.99°\ ℧/\text{km}$$

$$\boldsymbol{Z}_o = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{13823007.68 \angle 89.99°}{31.415 \angle 89.99°}} = 663.334 \angle 0°\ \Omega$$

$$\gamma = \sqrt{ZY} = \sqrt{(13823007.68 \angle 89.99°)(31.415 \angle 89.99°)} = 20.8386 \times 10^3 \angle 90°/\text{km}$$

4. (a) Determine Z and Y parameters of the network shown in Fig. 8. **(10)**

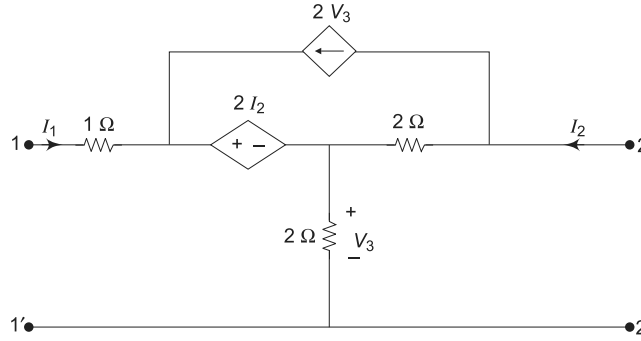


Fig. 8

Solution:

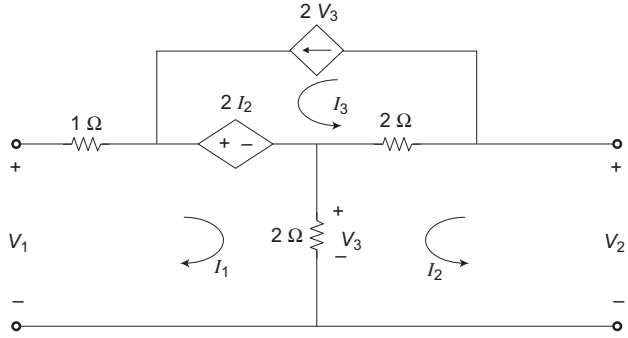


Fig. 9

From Fig. 9,

$$V_3 = 2(I_1 + I_2)$$

Applying KVL to Mesh 1,

$$V_1 - 1I_1 - 2I_2 - 2(I_1 + I_2) = 0$$
$$V_1 - I_1 - 2I_2 - 2I_1 - 2I_2 = 0$$
$$V_1 = 3I_1 + 4I_2 \quad \text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 2(I_2 - I_3) - 2(I_1 + I_2) = 0$$
$$V_2 - 2I_2 + 2I_3 - 2I_1 - 2I_2 = 0$$
$$V_2 = 2I_1 + 4I_2 - 2I_3 \quad \text{(ii)}$$

Writing equation for Mesh 3,

$$I_3 = 2V_3 = 2 \times 2\,(I_1 + I_2) = 4I_1 + 4I_2 \quad \text{(iii)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 4I_2 - 2\,(4I_1 + 4I_2)$$
$$= -6I_1 - 4I_2$$

Comparing with *Z*-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -6 & -4 \end{bmatrix}$$

$$\Delta Z = Z_{11}\,Z_{22} - Z_{12}\,Z_{21} = (3)\,(-4) - (4)\,(-6) = -12 + 24 = 12$$

***Y*-parameters**

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{-4}{12} = -\frac{1}{3}\text{℧}$$

$$Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{4}{12} = -\frac{1}{3}\text{℧}$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-6)}{12} = \frac{1}{2}\text{℧}$$

$$Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{3}{12} = \frac{1}{4}\text{℧}$$

Hence, *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

(b) Determine the voltage transfer function $\frac{V_2}{V_1}$ for the network shown in Fig. 10. **(5)**

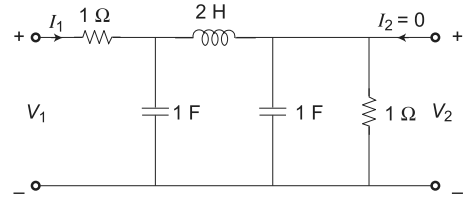


Fig. 10

Solution: Refer Example 8.12 on page 8.9.

(c) Test whether $F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ is a positive real function. **(5)**

Solution: Refer Example 9.27 on page 9.20.

5. **(a)** The network shown in Fig. 11, a steady state is reached with the switch open. At $t = 0$, the switch is closed. Determine $V_c(0^-)$, $i_1(0^+)$ $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$. **(10)**

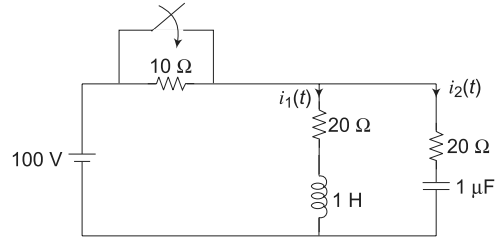


Fig. 11

Solution: Refer Example 6.22 on page 6.26.

(b) Find the voltage across the 5 Ω resistor in network shown in Fig. 12. **(5)**

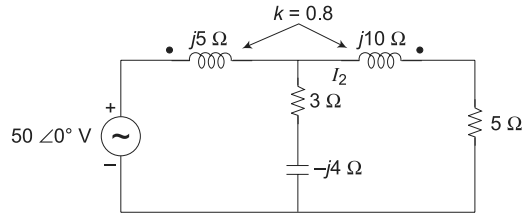


Fig. 12

Solution: Refer Example 4.28 on page 4.30.

(c) Find the function $v(t)$ using the pole-zero plot of the following function: **(5)**

$$V(s) = \frac{(s+2)(s+6)}{(s+1)(s+5)}$$

Solution: Refer Example 8.45 on page 8.45.

6. **(a)** A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal. **(10)**

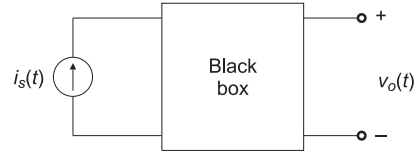


Fig. 13

Solution: Refer Example 7.42 on page 7.48.

(b) Determine the driving point impedance of the network shown in Fig. 14.

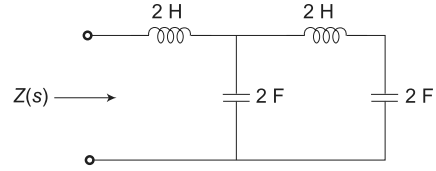


Fig. 14

Solution: Refer Example 8.2 on page 8.3.

(c) Draw the following normalised quantities on the Smith chart.

(i) $(2 + j2)\ \Omega$ (ii) $(4 - j2)\ \Omega$

(iii) $(1.0)\ \Omega$ (iv) $(j\ 1.0)\ \Omega$

Solution: Any complex normalised impedance can be represented by a point on the Smith chart which is intersection of constant resistance circle, i.e., $r = \frac{R}{\mathbf{Z}_o}$ and constant reactance circle, i.e., $x = \frac{X}{\mathbf{Z}_o}$.

$\mathbf{Z}_1 = (2 + j2)\ \Omega$, $\mathbf{Z}_2 = (4 - j2)\ \Omega$, $\mathbf{Z}_3 = 1.0\ \Omega$, $\mathbf{Z}_4 = j1.0\ \Omega$

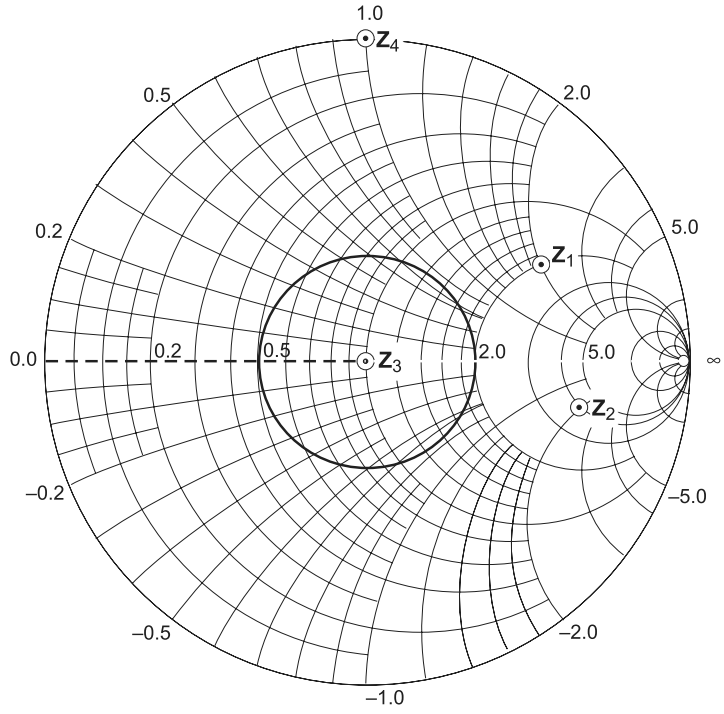


Fig. 15

Solved Question Paper May 2016

Circuits and Transmission Lines
ETC 304

Time: 3 Hours **Total Marks: 80**

N.B.: (1) Attempt question **1** and any **three** from the remaining questions. In all 4 questions are to be attempted.
(2) All sub questions of the same question should be answered at one place only in their serial orders and not scattered.
(3) Write everything in ink only. Pencil is not allowed.
(4) Assume suitable data with justification if missing.

1. **(a)** Determine Y-parameters for the network shown in Fig. 1. **(5)**

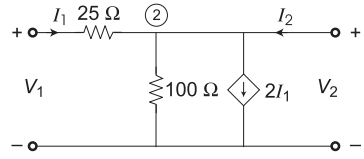


Fig. 1

Solution:

From Fig. 1,

$$I_1 = \frac{V_1 - V_2}{25}$$

$$= \frac{1}{25}V_1 - \frac{1}{25}V_2 \quad \text{...(i)}$$

Applying KCL at Node 2,

$$I_1 + I_2 = \frac{V_2}{100} + 2I_1$$

$$I_2 = I_1 + \frac{V_2}{100} \quad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$I_2 = \frac{1}{25}V_1 - \frac{1}{25}V_2 + \frac{1}{100}V_2$$

$$= \frac{1}{25}V_1 - \frac{3}{100}V_2 \quad \text{...(iii)}$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{25} & -\frac{1}{25} \\ \frac{1}{25} & -\frac{3}{100} \end{bmatrix}$$

(b) Test if $F(s) = s^4 + s^3 + 5s^2 + 3s + 4$ is a Hurwitz polynomial. **(5)**

Solution: Refer Example 9.2 on page 9.2.

(c) Two coils connected in series have self inductance 80 mH and 20 mH respectively. The total inductance of the circuit is found to be 140 mH. Determine **(5)**

(i) mutual inductance between two coils and

(ii) coefficient of coupling

Solution: $L_1 = 80$ mH, $L_2 = 20$ mH, $L_{cum} = 140$ mH

$$L_{cum} = L_1 + L_2 + 2\,M$$

$$140 = 80 + 20 + 2\,M$$

$$M = 20 \text{ mH}$$

$$M = k\sqrt{L_1 L_2}$$

$$20 = k\sqrt{(80)(20)}$$

$$k = 0.5$$

(d) Synthesize the function $Z(s) = \frac{s^2 + 2s + 2}{s^2 + s + 1}$ into a network using Cauer-I form. **(5)**

Solution: The Cauer-I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{s^2 + 2s + 2}{s^2 + s + 1}$$

By continued fraction expansion,

$$s^2 + s + 1 \;\big)\, s^2 + 2s + 2 \;\big(\, 1 \leftarrow Z$$

$$\underline{s^2 + s + 1}$$

$$s + 1 \;\big)\, s^2 + s + 1 \;\big(\, s \leftarrow Y$$

$$\underline{s^2 + s}$$

$$1 \;\big)\, s + 1 \;\big(\, s \leftarrow Z$$

$$\underline{s}$$

$$1 \;\big)\, 1 \;\big(\, 1 \leftarrow Y$$

$$\underline{1}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 2.

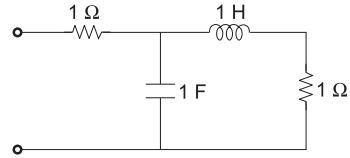


Fig. 2

2. **(a)** Find the Thevenin's equivalent across the terminals *XY* for the circuit shown in Fig. 3. **(10)**

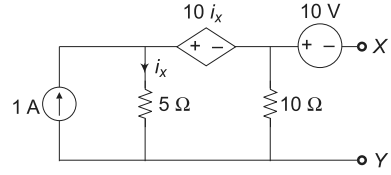


Fig. 3

Solution: Refer Example 3.52 on page 3.58.

(b) Determine the node voltage at nodes (1) and (2) of the network shown in Fig. 4 by using nodal analysis. **(5)**

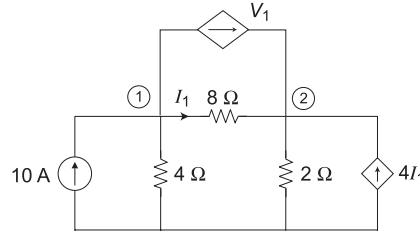


Fig. 4

Solution: From Fig. 4,

$$I_1 = \frac{V_1 - V_2}{8} \quad \text{...(i)}$$

Assume that the currents are moving away from the nodes,

Applying KCL at Node 1,

$$10 = \frac{V_1}{4} + \frac{V_1 - V_2}{8} + V_1$$

$$\left(\frac{1}{4} + \frac{1}{8} + 1\right) V_1 - \frac{1}{8} V_2 = 10$$

$$1.375\ V_1 - 0.125\ V_2 = 10 \quad \text{...(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{8} + \frac{V_2}{2} = 4I_1 + V_1$$

$$\frac{V_2 - V_1}{8} + \frac{V_2}{2} = 4\left(\frac{V_1 - V_2}{8}\right) + V_1$$

$$\left(-\frac{1}{8} - \frac{1}{2} - 1\right)V_1 + \left(\frac{1}{8} + \frac{1}{2} + \frac{1}{2}\right)V_2 = 0 \qquad \text{(iii)}$$

$$-1.625\ V_1 + 1.125\ V_2 = 0$$

Solving Eqs (ii) and (iii),

$$V_1 = 8.37 \text{ V}$$
$$V_2 = 12.09 \text{ V}$$

(c) Test whether $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ is a positive real function. **(5)**

Solution: Refer Example 9.25 on page 9.18.

3. **(a)** Synthesize the driving point function using Foster-I and Foster-II forms. **(10)**

$$Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution: The function $Z(s)$ has poles at $s = 0$ and $s = \pm j2$ and zeros at $s = \pm j1$ and $\pm j3$ as shown in Fig. 5.

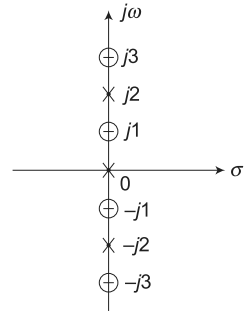


Fig. 5

From the pole-zero diagram, it is clear that the poles and zeros are simple and lie on the $j\omega$-axis. Poles and zeros are interlaced. Hence, the given function is an LC function.

Foster-I form The Foster-I form is obtained by partial fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{2s^4 + 20s^2 + 18}{s^3 + 4s}$$

$$s^3 + 4s \overline{)\, 2s^4 + 20\, s^2 + 18 \,(2s}$$
$$\underline{2s^4 + 8s^2}$$
$$12s^2 + 18$$

$$Z(s) = 2s + \frac{12s^2 + 18}{s^3 + 4s} = 2s + \frac{12s^2 + 18}{s(s^2 + 4)}$$

By partial-fraction expansion,

$$Z(s) = 2s + \frac{K_0}{s} + \frac{K_1}{s + j2} + \frac{K_1^*}{s - j2} = 2s + \frac{K_0}{s} + \frac{2K_1 s}{s^2 + 4}$$

where $$K_0 = sZ(s)\big|_{s=0} = \frac{2(1)(9)}{4} = \frac{9}{2}$$

$$K_1 = \frac{(s^2 + 4)Z(s)}{2s}\bigg|_{s^2=-4} = \frac{2(-4+1)(-4+9)}{2(-4)} = \frac{15}{4}$$

$$Z(s) = 2s + \frac{\frac{9}{2}}{s} + \frac{\left(\frac{15}{2}\right)s}{s^2 + 4}$$

The first term represents the impedance of an inductor of 2H. The second term represents the impedance of a capacitor of $\frac{2}{9}$F. The third term represents the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \left(\frac{1}{LC}\right)}$$

By direct comparison,

$$C = \frac{2}{15}\text{F}$$

$$L = \frac{15}{8}\text{H}$$

The network is shown in Fig. 6,

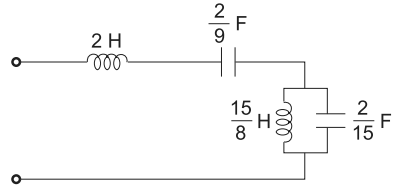


Fig. 6

Foster-II form The Foster-II form is obtained by partial-fraction of expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{2(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where $$K_1 = \frac{(s^2+1)}{2s} Y(s)\Big|_{s^2=-1} = \frac{(-1+4)}{4(-1+9)} = \frac{3}{32}$$

$$K_2 = \frac{s^2+9}{2s} Y(s)\Big|_{s^2=-9} = \frac{(-9+4)}{4(-9+1)} = \frac{5}{32}$$

$$Y(s) = \frac{\left(\frac{3}{16}\right)s}{s^2+1} + \frac{\left(\frac{5}{16}\right)s}{s^2+9}$$

These two terms represent admittance of a series *LC* network.
For a series *LC* network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{16}{3}\text{H},\ C_1 = \frac{3}{16}\text{F}$$

$$L_2 = \frac{16}{5}\text{H},\ C_2 = \frac{5}{144}\text{F}$$

The network is shown in Fig. 7.

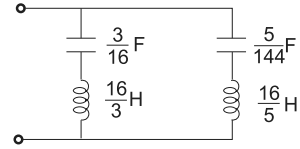


Fig. 7

(b) State and prove initial value theorem. **(5)**

Solution: Refer Section 7.4.9 on page 7.7.

(c) A transmission line has distributed parameters, R = 6 ohms/km, L = 2.2 mH/km, **(5)**

$C = 0.005$ μF/km and $G = 0.005$ μ mho/km. Determine characteristics impedance and propagation constant at 1 kHz frequency.

Solution: $\boldsymbol{Z} = R + j\omega L = 6 + j2\pi \times 1000 \times 2.2 \times 10^{-3} = 15.07\angle 66.54°\ \Omega/\text{km}$

$$\boldsymbol{Y} = G + j\omega C = 0.005 \times 10^{-6} + j2\pi \times 1000 \times 0.005 \times 10^{-6} = 3.14 \times 10^{-5}\angle 89.99°\ \text{℧/km}$$

$$\boldsymbol{Z}_0 = \sqrt{\frac{\boldsymbol{Z}}{\boldsymbol{Y}}} = \sqrt{\frac{15.07\angle 66.54°}{3.14 \times 10^{-5}\angle 89.99°}} = 692.77\angle -11.73°\Omega$$

$$\gamma = \sqrt{\boldsymbol{ZY}} = \sqrt{(15.07\angle 66.54°)(3.14 \times 10^{-5}\angle 89.99°)} = 0.022\angle 78.27°/\text{km}$$

4. (a) Find *ABCD* parameter for the two-port network shown in Fig. 8. **(10)**

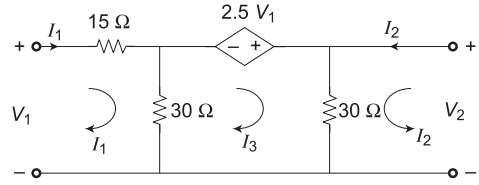


Fig. 8

Solution: Applying KVL to Mesh 1,

$$V_1 = 15I_1 + 30(I_1 - I_3)$$
$$= 45\ I_1 - 30\ I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 30(I_2 + I_3)$$
$$= 30\ I_2 + 30\ I_3 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-30(I_3 - I_1) + 2.5V_1 - 30(I_2 + I_3) = 0$$

$$-30I_3 + 30I_1 + 2.5(45I_1 - 30I_3) - 30I_2 - 30I_3 = 0$$

$$-30I_3 + 30I_1 + 112.5I_1 - 75I_3 - 30I_2 - 30I_3 = 0$$

$$135\ I_3 = 142.5I_1 - 30I_2$$
$$= 1.06\ I_1 - 0.22\ I_2 \quad \text{...(iii)}$$

Substituting Eq (iii) in Eq. (i),

$$V_1 = 45I_1 - 30(1.06I_1 - 0.22I_2)$$
$$= 13.2I_1 + 6.6I_2 \quad \text{...(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 30I_2 + 30(1.06I_1 - 0.22I_2)$$
$$= 31.8I_1 + 23.4I_2 \quad \text{...(v)}$$

From Eq. (v),

$$I_1 = 0.03\ V_2 - 0.74\ I_2 \quad \text{...(vi)}$$

Substituting Eq. (vi) in Eq. (iv),

$$V_1 = 13.2(0.03\ V_2 - 0.74I_2) + 6.6\ I_2$$
$$= 0.396\ V_2 - 3.168\ I_2 \quad \text{...(vii)}$$

Comparing Eqs (vi) and (vii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.396 & -3.168 \\ 0.03 & -0.74 \end{bmatrix}$$

(b) Find the network functions $\frac{V_1}{I_1}, \frac{V_2}{I_1}, \frac{V_2}{V_1}$ for the network shown in Fig. 9. **(5)**

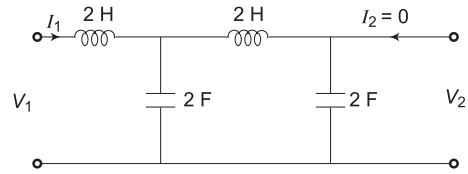


Fig. 9

Solution: The transformed network is shown in Fig. 10.

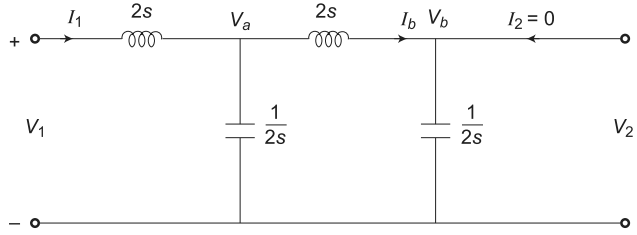


Fig. 10

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{2s}} = 2s\,V_2$$

$$V_a = 2sI_b + V_b = 2s(2sV_2) + V_2 = (4s^2 + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{2s}} + I_b = 2sV_a + I_b = 2s(4s^2 + 1)V_2 + 2sV_2 = (8s^3 + 4s)V_2$$

$$V_1 = 2sI_1 + V_a = 2s(8s^3 + 4s)V_2 + (4s^2 + 1)V_2 = (16s^4 + 12s^2 + 1)V_2$$

Hence,

$$\frac{V_1}{I_1} = \frac{16s^4 + 12s^2 + 1}{8s^3 + 4s}$$

$$\frac{V_2}{I_1} = \frac{1}{8s^3 + 4s}$$

$$\frac{V_2}{V_1} = \frac{1}{16s^4 + 12s^2 + 1}$$

(c) A transmission line has a characteristic impedance of $50 + j\,100\ \Omega$ and is terminated in a load impedance of $73 - j\,42.5\ \Omega$. Calculate: **(5)**

(i) the reflection coefficient

(ii) the standing wave ratio

Solution: $\mathbf{Z}_0 = 50 + j100\ \Omega, \qquad \mathbf{Z}_L = 73 - j\,42.5\ \Omega$

(i) $$\Gamma = \frac{\mathbf{Z}_L - \mathbf{Z}_0}{\mathbf{Z}_L + \mathbf{Z}_0} = \frac{(73 - j42.5) - (50 + j100)}{(73 - j42.5) - (50 + j100)} = 1.063 \angle -105.89°$$

(ii) $$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1.063}{1 - 1.063} = -32.746$$

5. (a) In the network shown in Fig. 11, switch k is closed at $t = 0$ and assume all initial conditions to be zero. Find $i, \frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$. **(10)**

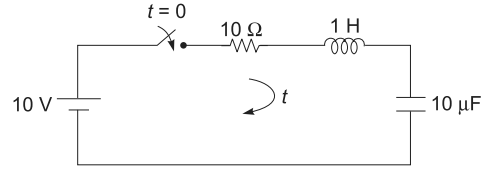


Fig. 11

Solution: Refer Example 6.3 on page 6.5.

(b) Write the KVL equations in standard form for the network shown in Fig. 12. **(5)**

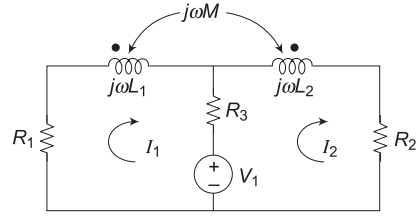


Fig. 12

Solution: The equivalent circuit in terms of dependent source is shown in Fig. 13.

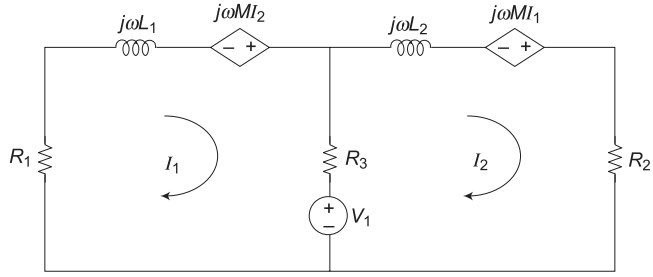


Fig. 13

Applying KVL to Mesh 1,

$$-R_1I_1 - j\omega L_1I_1 + j\omega MI_2 - R_3(I_1 - I_2) - V_1 = 0$$

$$-(R_1 + R_3 + j\omega L_1)I_1 + (R_3 + j\omega M)I_2 = V_1 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_1 - R_3(I_2 - I_1) - j\omega L_2 + j\omega MI_1 - R_2I_2 = 0$$

$$-(R_3 + j\omega M)I_1 + (R_3 + R_2 - j\omega L_2)I_2 = V_1 \quad \text{...(ii)}$$

(c) Find poles and zero of the impedance $Z(s)$ for the network shown in Fig. 14. **(5)**

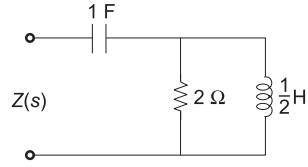


Fig. 14

Solution: Refer Example 8.26 on page 8.24.

6. **(a)** Why is the impedance matching required? Draw the following normalized quantities on the Smith chart. **(10)**

(i) $(3 + j3)\ \Omega$ (ii) $(1.0)\ \Omega$

(iii) $(2 - j1)\ \Omega$ (iv) $j\ 1.0\ \Omega$

Solution: Refer Section 12.15 on page 12.36.

Any complex normalised impedance can be represented by a point on the Smith chart which is intersection of constant resistance circle i.e., $r = \frac{R}{\mathbf{Z}_0}$ and constant reactance circle, i.e. $x = \frac{X}{\mathbf{Z}_0}$.

$$\mathbf{Z}_1 = 3 + j3\,\Omega,\ \mathbf{Z}_2 = 1\,\Omega,\ \mathbf{Z}_3 = 2 - j1\,\Omega,\ \mathbf{Z}_4 = j1\,\Omega$$

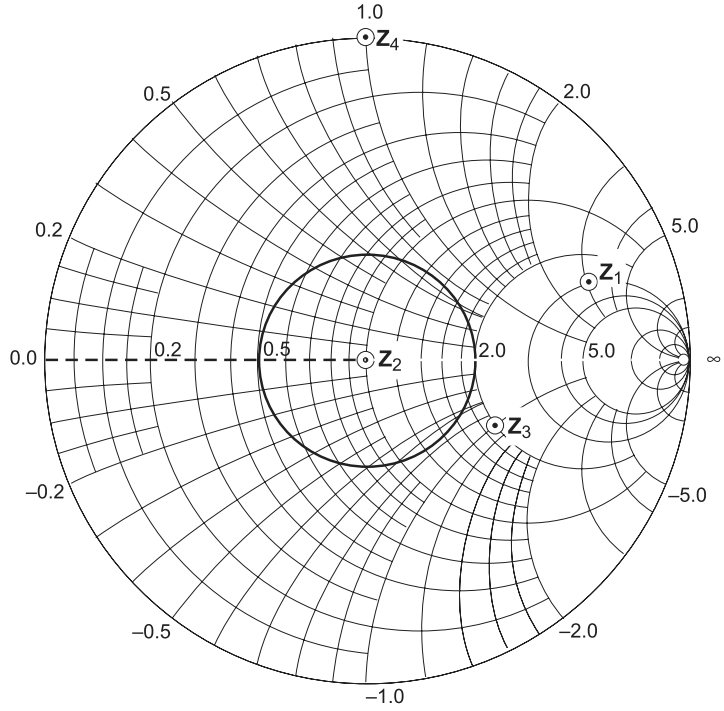


Fig. 15

(b) Write short note on time domain analysis using Laplace transform. **(5)**

Solution: Refer Section 7.1 on page 7.1.

(c) Define the following terms: **(5)**

(i) Phase velocity

(ii) Characteristic impedance

(iii) Reflection coefficients

Solution: Refer Section 12.5 on page 12.3 and Section 12.10 on Page 12.15.

Solved Question Paper December 2015

Circuit Theory
EXC 304

Time: 3 Hours **Total Marks: 80**

N.B.: (1) Question no. 1 is **compulsory**.
(2) Attempt any **three** questions out of remaining questions.
(3) **Figures** to the **right** indicate **full** marks.
(4) **Assume** suitable **data** if **required**.
(5) **Use Smith chart** for **transmission** line problem.

1. **(a)** The constants of a transmission line are: **(5)**
$R = 6\ \Omega$/km, $L = 2.2$ mH/km
$G = 0.25 \times 10^{-6}$ mho/km, $C = 0.005 \times 10^{-5}$ F/km
Determine the characteristic impedance, propagation constant, phase constant and attenuation constant at 1 kHz.

Solution: Refer Example 12.6 on page 12.6.

(b) Obtain the expression for $i(t)$ if switch is closed at $t = 0$ and $v(t)$ is $r(t)$ = ramp signal. **(5)**

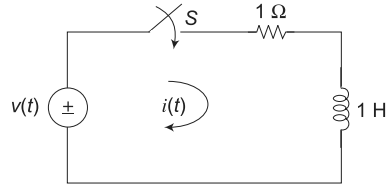


Fig. 1

Solution: For $t > 0$, the transformed network is shown in Fig. 2.

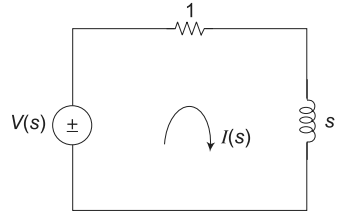


Fig. 2

Applying KVL to the mesh,

$$V(s) - I(s) - sI(s) = 0$$

$$V(s) = I(s)\,[1+ s]$$

$$I(s) = \frac{V(s)}{s+1}$$

When ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{1}{s^2(s+1)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$1 = As(s + 1) + B(s + 1) + Cs^2$$
$$= As^2 + As + Bs + B + Cs^2$$
$$= (A + C)s^2 + (A + B)s + B$$

Comparing coefficients of s^2, s and s^0,

$$A + C = 0$$
$$A + B = 0$$
$$B = 1$$

Solving these equations,

$$A = -1$$
$$B = 1$$
$$C = 1$$

$$I(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

Taking inverse Laplace transform,

$$i(t) = -1 + t + e^{-t} \qquad \text{for } t > 0$$

(c) Check whether the polynomial is Hurwitz or not by continued fraction method. **(5)**

$$F(s) = s^4 + s^3 + 4s^2 + 2s + 3$$

Solution:

$$F(s) = s^4 + s^3 + 4s^2 + 2s + 3$$

Even part of $F(s) = m(s) = s^4 + 4s^2 + 3$

Odd part of $F(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{r}
s^3 + 2s \overline{)\, s^4 + 4s^2 + 3 \,(\, s} \\
\underline{s^4 + 2s^2} \\
2s^3 + 3 \overline{)\, s^3 + 2s \,(\, \frac{1}{2}s} \\
\underline{s^3 + \frac{3s}{2}} \\
\frac{s}{2} \overline{)\, 2s^2 + 3 \,(\, 4s} \\
\underline{2s^2} \\
3 \overline{)\, \frac{s}{2} \,(\, \frac{1}{6}s} \\
\underline{\frac{s}{2}} \\
0
\end{array}
$$

Since all the quotient terms are positive, $F(s)$ is Hurwitz.

(d) Find out $\frac{V_2}{V_1}$ for the following network shown in Fig. 3. **(5)**

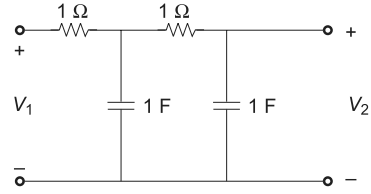


Fig. 3

Solution: Refer Example 8.7 on page 8.6.

2. **(a)** Find the voltage across 5 Ω resistor in the network shown below, if $k = 0.8$ is coefficient of coupling. **(8)**

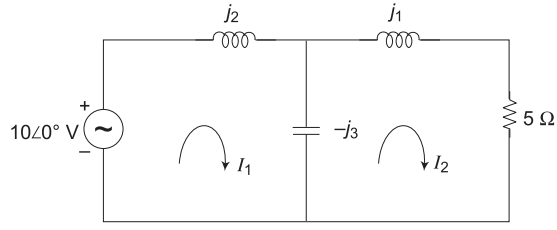


Fig. 4

Solution: For a magnetically coupled circuit,

$$X_M = k\sqrt{X_{L_1} X_{L_2}} = 0.8\sqrt{2(1)} = 1.13\ \Omega$$

The equivalent circuit in terms of dependent sources is shown in Fig. 5.

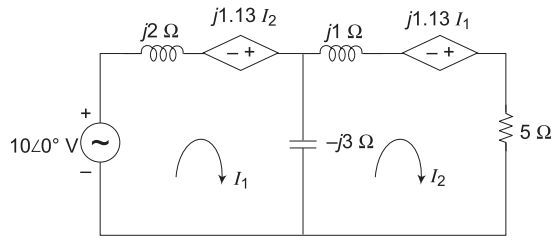


Fig. 5

Applying KVL to Mesh 1,

$$10\angle 0° - j_2 I_1 + j1.13 I_2 + 3j(I_1 - I_2) = 0$$

$$jI_1 - j1.87 I_2 = -10\angle 0°$$

$$-jI_1 + j1.87 I_2 = 10\angle 0° \quad \text{(i)}$$

Applying KVL to Mesh 2,

$$3j(I_2 - I_1) - jI_2 + j1.13 I_1 - 5I_2 = 0$$

$$-1.87 I_1 + I_2(j2 - 5) = 0$$

$$1.87 I_1 + (5 - j2) I_2 = 0 \quad \text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} -j & 1.87 \\ 1.87 & 5 - j2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$I_2 = \frac{\begin{vmatrix} -j & 10\angle 0° \\ 1.87 & 0 \end{vmatrix}}{\begin{vmatrix} -j & 1.87 \\ 1.87 & 5 - j2 \end{vmatrix}} = -2.14\angle 13.17 \text{ A}$$

$$V_{5\,\Omega} = 5I_2 = 5(-2.14\angle 13.17°) = -10.7\angle 13.17° \text{ V}$$

(b) In the circuit shown in Fig. 6, find out the expression for voltage $v(t)$ across capacitor for $t > 0$. At $t = 0$, the switch is closed. **(8)**

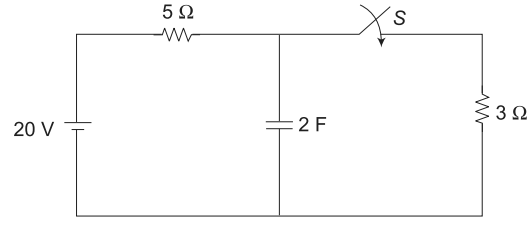


Fig. 6

Solution: At $t = 0^-$, the network is shown in Fig. 7. At $t = 0$, the network has attained steady state condition. Hence, the capacitor acts as open circuit.

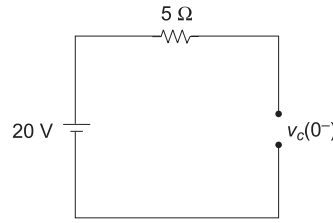


Fig. 7

$$v_c(0^-) = 20 \text{ V}$$

Since the voltage across capacitor cannot change instantaneously,

$$v_c(0^+) = 20 \text{ V}$$

For $t > 0$, the network is shown in Fig. 8.

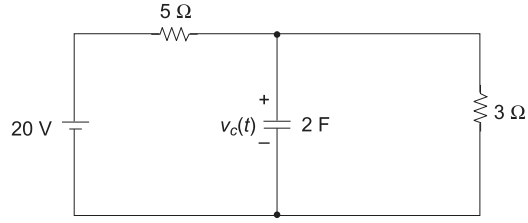


Fig. 8

Writing the KCL equation for $t > 0$,

$$\frac{v_c - 20}{5} + \frac{2dv_c}{dt} + \frac{v_c}{3} = 0$$

$$\frac{dv_c}{dt} + \frac{4v_c}{15} = 2$$

Comparing the equation with $\frac{dv}{dt} + Pv = Q$,

$$P = \frac{4}{15}, Q = 2$$

The solution of this differential equation is given by

$$v_c(t) = \frac{Q}{P} + ke^{-Pt}$$

$$= \frac{2}{4} \times 15 + ke^{-\frac{4}{15}t}$$

$$v_c(t) = 7.5 + ke^{-0.26t}$$

At $t = 0$, $v_c(0) = 20 \text{ V}$

$$20 = 7.5 + k$$

$$k = 12.5$$

$$v_c(t) = 7.5 + 12.5\, e^{-0.26t} \quad \text{for } t > 0$$

(c) Define ABCD parameters for the two port network. Hence obtain condition for symmetry. **(4)**

Solution: Refer Section 10.4 on page 10.18.

3. **(a)** Find i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$ in the circuit shown in Fig. 9. Switch is changed from position 1 to 2 at $t = 0$. **(6)**

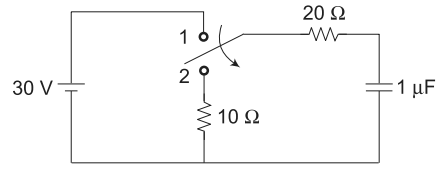


Fig. 9

Solution: Refer Example 6.8 on page 6.10.

(b) Compare and obtain Foster I and Foster II of the following RC impedance function. **(8)**

$$Z(s) = \frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution: Refer Example 9.51 on page 9.55.

(c) Obtain the Cauer from I of LC network. **(6)**

$$Z(s) = \frac{(s^2+4)(s^2+16)}{s(s^2+9)}$$

Solution: $Z(s) = \dfrac{s^4 + 20s^2 + 64}{s^3 + 9s}$

The Cauer I form is obtained from continued fraction expansion about the pole at infinity and since the degree of numerator is greater than denominator by one, it indicates presence of poles at infinity.

By continued fraction expansion,

$$\begin{array}{l}
s^3 + 9s \overline{)\, s^4 + 20s^2 + 64} \left(s \leftarrow Z \right. \\
\quad \underline{s^4 + 9s^2} \\
\quad\quad 11s^2 + 64 \overline{)\, s^3 + 9s} \left(\frac{1}{11} s \leftarrow Y \right. \\
\quad\quad\quad \underline{s^3 + \frac{64s}{11}} \\
\quad\quad\quad\quad \frac{35}{11} s \overline{)\, 11s^2 + 64} \left(\frac{121}{35} s \leftarrow Z \right. \\
\quad\quad\quad\quad\quad \underline{11s^2} \\
\quad\quad\quad\quad\quad\quad 64 \overline{)\, \frac{35s}{11}} \left(\frac{35}{704} s \leftarrow Y \right. \\
\quad\quad\quad\quad\quad\quad\quad \underline{\frac{35s}{11}} \\
\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}$$

The network is shown in Fig. 10.

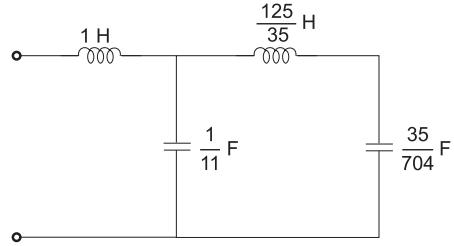


Fig. 10

4. **(a)** Derive the characteristic equation of a transmission line, also obtain $\alpha\ \beta\ \gamma$ of the transmission line. **(8)**

Solution: Refer Section 12.5 on page 12.3.

(b) Derive the relation for nominal impedance and cut off frequency for a constant *k* low pass filter. **(4)**

Solution: Refer Section 11.6 on page 11.7.

(c) A network and its pole zero diagram are shown in Fig. 11. Determine the values of *R*, *L*, *C* if $Z(0) = 1$. **(8)**

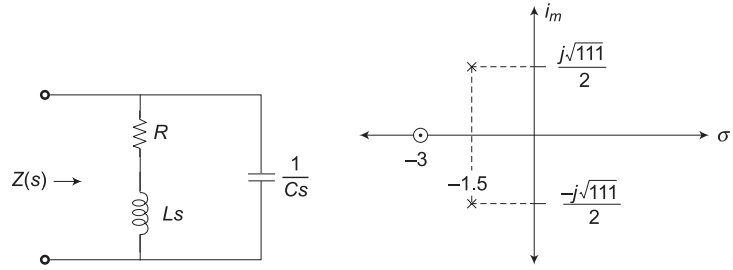


Fig. 11

Solution: Refer Example 8.37 on page 8.33.

5. **(a)** Check whether the following functions are PRF or not. **(8)**

(i) $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$

Solution: Refer Example 9.25 on page 9.18.

(ii) $F(s) = \dfrac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1}$

Solution: Refer Example 9.28 on page 9.21.

(b) Find the current I_x using superposition theorem. **(6)**

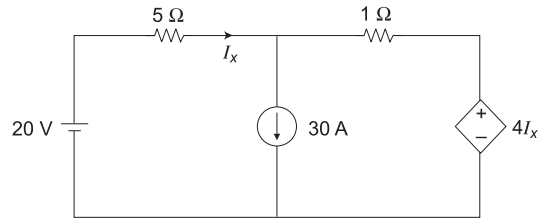


Fig. 12

Solution: Refer Example 3.14 on page 3.18.

(c) The current $I(s)$ is given by $I(s) = \dfrac{2s}{(s+1)(s+2)}$. Plot the pole zero pattern in s plane and hence, obtain $i(t)$ by finding out residues by graphical method. **(6)**

Solution: Refer Example 8.43 on page 8.43.

6. **(a)** The characteristic impedance of a high frequency line is 100 Ω. If it is terminated by a load impedance of 100 + j100 Ω. Using Smith chart find out (i) VSWR, (ii) Reflection coefficient, (iii) impedance at $\lambda/10^{th}$ of wavelength away from load, (iv) VSWR minimum and VSWR maximum away from the load. **(8)**

Solution:

$$\mathbf{Z}_0 = 100\ \Omega,\ \mathbf{Z}_L = 100 + j\ 100\ \Omega$$

1. The normalised load impedance is given by

$$\bar{\mathbf{Z}}_L = \frac{100 + j100}{100} = 1 + j1$$

Mark the normalised load impedance point A at the intersection of the $r = 1$ circle and the $x = 1$ circle.

2. Draw the VSWR with centre O and radius equal to distance OA.

$$\text{VSWR} = 2.52$$

3. Draw a line OA and extend it upto the point A′ on the outer circumference of chart. This line intersects the scale for the angle of reflection coefficient at 63°. This is the angle of reflection coefficient. The magnitude of reflection coefficient is obtained from the linear scale shown at the bottom of the chart. Mark C point from centre at a distance equal to the distance OA. This gives the value of $|\Gamma|$ as 0.445.

$$\Gamma = 0.445\angle 63°$$

4. To find input impedance of the line $\dfrac{1}{10}\lambda$ away from the load, move in a clockwise direction from the point A′ through a distance of $\dfrac{\lambda}{10}$ i.e. 0.1λ on outer circumference and locate it as point B′ at $0.162\lambda + 0.1\lambda = 0.262\lambda$. Draw the line OB′ which intersects the constant VSWR circle at the point B. The point B is at the intersection of the $r = 2.6$ circle and the $x = 0.47$ circle. Hence, normalised input impedance at the point B is

$$\bar{\mathbf{Z}}_s = 2.6 - j0.47$$

The actual impedance is

$$\mathbf{Z}_s = \mathbf{Z}_0\,\bar{\mathbf{Z}}_s = 100(2.6 - j0.47) = 260 - j47$$

5. The first voltage maximum occurs to the right of the centre on the real axis at $s = 2.6$ and the first voltage minimum occurs to the left of the centre on the real axis at $s = 0.38$. To find the location of voltage maximum from the load, move in clockwise direction from the point A′ to the point P (voltage maximum point). Total distance is distance AQ + distance QP = $0.162\lambda + 0.25\lambda = 0.412\ \lambda$. Hence, the first voltage maximum is located at 0.412λ from the load. To find the location for voltage minimum for the load, move in a clockwise direction from the point A′ to the point Q (voltage minimum point). Total distance is distance AQ = 0.162λ. Hence, the first voltage minimum is located at 0.162λ from the load.
6. The complete Smith chart is shown in Fig. 13.

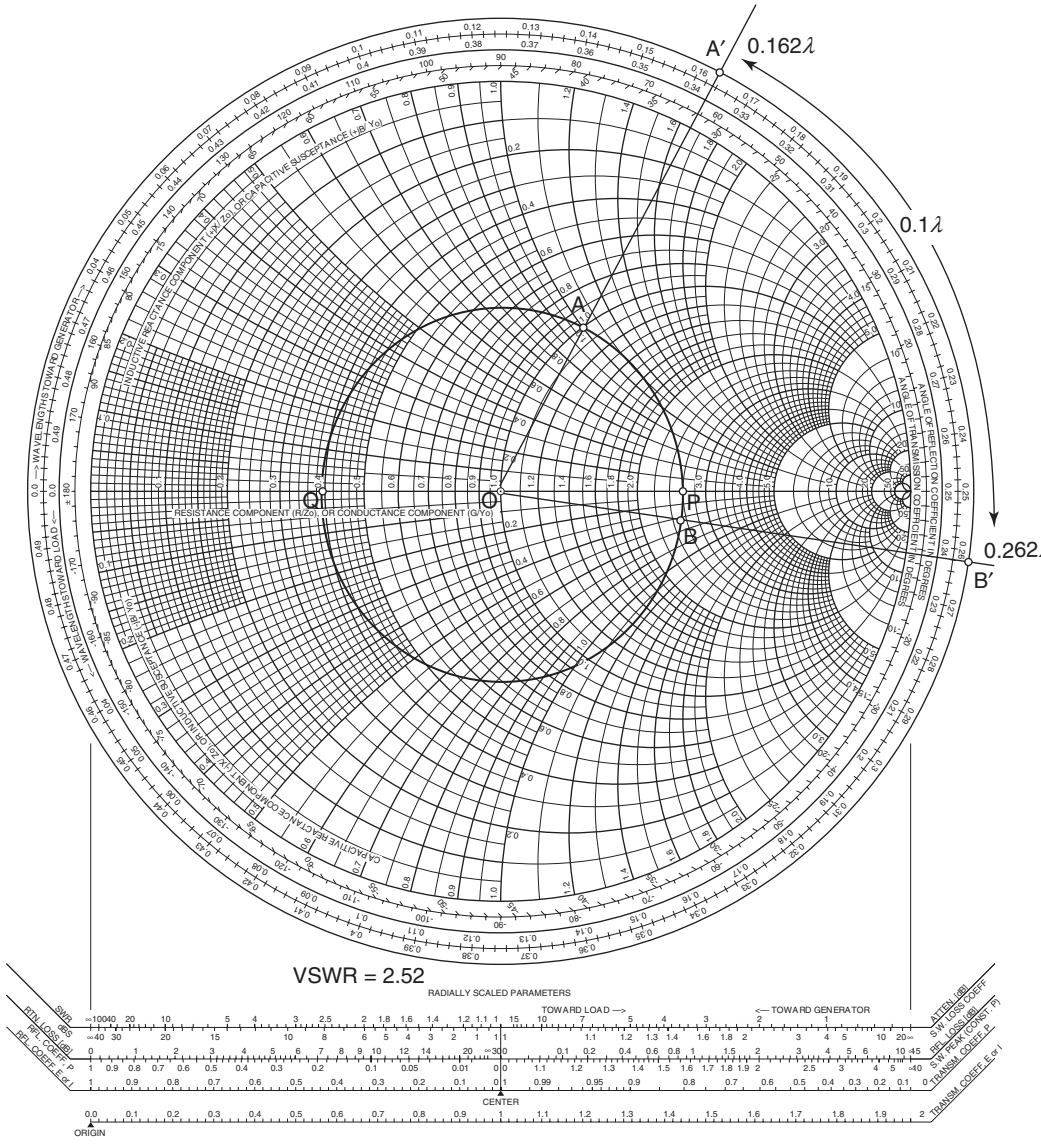


Fig. 13

(b) For the network shown in Fig. 14, find $\frac{V_1}{I_1}$ and $\frac{V_2}{I_1}$. **(6)**

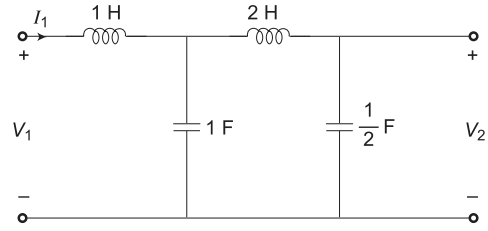


Fig. 14

Solution: The transformed network is shown in Fig. 15.

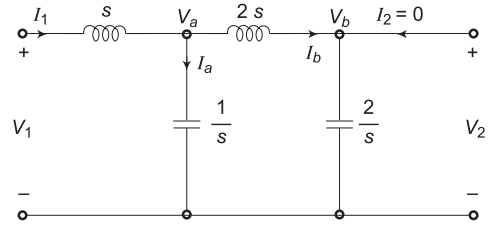


Fig. 15

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{2}{s}} = \frac{sV_2}{2}$$

$$V_a = 2sI_b + V_b = s^2V_2 + V_2 = (s^2 + 1)V_2$$

$$I_a = \frac{V_a}{\frac{1}{s}} = sV_a$$

$$I_1 = I_a + I_b = sV_a + \frac{sV_2}{2} = sV_2(s^2 + 1) + \frac{sV_2}{2} = \left[s^3 + s + \frac{s}{2}\right]V_2 = \left(s^3 + \frac{3s}{2}\right)V_2$$

$$V_1 = sI_1 + V_a$$

$$= s\left(s^3 + \frac{3}{2}s\right)V_2 + (s^2 + 1)V_2$$

$$= \left[s^4 + \frac{3}{2}s^2 + 1 + s^2\right]V_2$$

$$= \left[s^4 + \frac{5}{2}s^2 + 1\right]V_2$$

Hence,

$$\frac{V_1}{I_1} = \frac{s^4 + \frac{5}{2}s^2 + 1}{s^3 + \frac{3}{2}s}$$

$$\frac{V_2}{I_1} = \frac{1}{s^3 + \frac{3}{2}s}$$

(c) Find out Thevenin's equivalent network. **(6)**

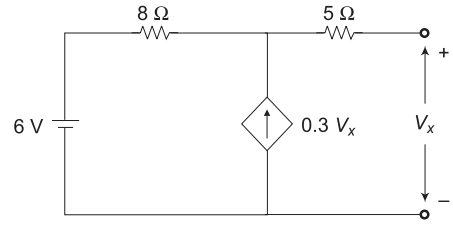


Fig. 16

Solution: Step 1 Calculation of V_{Th}

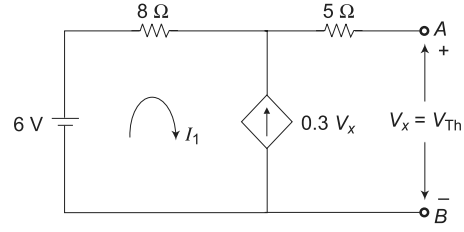


Fig. 17

From the circuit shown in Fig. 17,

$$V_x = V_{Th}$$

and $$I_1 = -0.3\ V_x$$

$$= -0.3\ V_{Th}$$

Writing the KVL equation for V_{Th},

$$6 - 8I_1 - V_{Th} = 0$$

$$6 - 8(-0.3\ V_{Th}) - V_{Th} = 0$$

$$6 + 2.4\,V_{Th} - V_{Th} = 0$$

$$6 + 1.4\ V_{Th} = 0$$

$$V_{Th} = -\frac{6}{1.4} = -4.28\ \text{V}$$

Step II Calculation of I_N

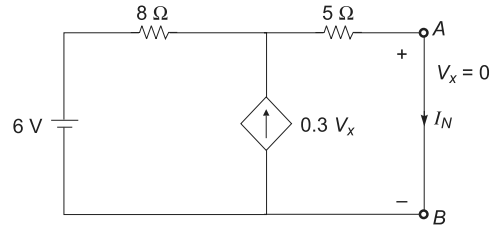


Fig. 18

From Fig. 18,

$$V_x = 0$$

The dependent source 0.3 V_x depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e., 0.3 $V_x = 0$ as shown in Fig. 19.

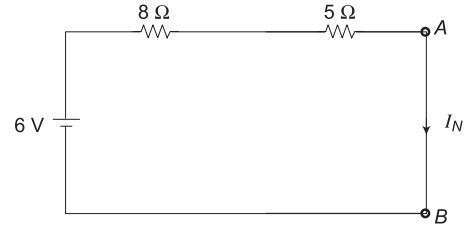


Fig. 19

$$I_N = \frac{6}{13} = 0.46 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_{\text{N}}} = -\frac{4.28}{0.46} = -9.30\ \Omega$$

Step IV Thevenin's equivalent network

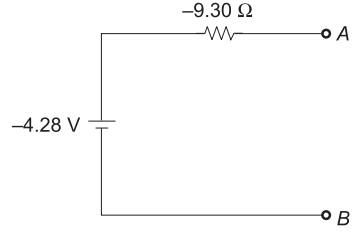


Fig. 20

Solved Question Paper May 2016

Circuit Theory
EXC 304

Time: 3 Hours **Total Marks: 80**

N.B.: (1) Question no. 1 is **compulsory**.
(2) Attempt any **three** questions out of remaining questions.
(3) Use Smith chart wherever required.
(4) Assume suitable data if required.

1. (a) Test for the following polynomial using continued fraction expansion only. **(20)**

$$P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$$

Solution: Even part of $P(s) = m(s) = 12s^4 + 60s^2 + 48$

Odd part of $P(s) = n(s) = s^5 + 45s^3 + 44s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$\begin{array}{r} \\ 12s^4 + 60s^2 + 48 \;\big)\; s^5 + 45s^3 + 44s \;\big(\; \frac{1}{12}s \\ \underline{s^5 + 5s^3 + 4s} \\ 40s^3 + 40s \;\big)\; 12s^4 + 60s^2 + 48 \;\big(\; \frac{3}{10}s \\ \underline{12s^4 + 12s^2} \\ 48s^2 + 48 \;\big)\; 40s^3 + 40s \;\big(\; \frac{5}{6}s \\ \underline{40s^3 + 40s} \\ 0 \end{array}$$

The division has terminated abruptly, i.e. the number of partial quotients, i.e., 3 is not equal to the order of polynomial, i.e., 5 with common factor $(48s^2 + 48)$.

$$\begin{aligned} P(s) &= s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48 \\ &= (48s^2 + 48)\left(\frac{1}{48}s^3 + \frac{1}{4}s^2 + \frac{44}{48}s + 1\right) \end{aligned}$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $P_1(s) = 48s^2 + 48$

Since it contains only even function, we have to find the continued fraction expansion of $\frac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 96s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{48s^2 + 48}{96s} = \frac{s}{2} + \frac{1}{2s}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let $P_2(s) = \frac{1}{48}s^3 + \frac{1}{4}s^2 + \frac{44}{48}s + 1$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.

Hence, $P(s)$ is Hurwitz.

(b) Obtain transmission parameters (ABCD) in terms of Z-parameters.

Solution: Refer Section 10.8.3 on page 10.42.

(c) List the types of damping in a series R-L-C circuit and mention the conditions for each damping.

Solution: Refer Section 6.5 on page 6.66.

(d) Obtain s-domain (Laplace Transform) equivalent circuit diagram of an inductor and capacitor with initial condition.

Solution: Refer Section 7.6 on page 7.12.

2. (a) Calculate voltage across the 6 Ω resister using source shifting technique in Fig. 1. **(10)**

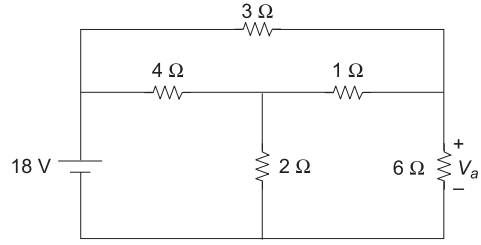


Fig. 1

Solution: Refer Example 1.8 on page 1.20.

(b) Compare and obtain Foster-I and Foster-II forms using example of RC circuit. **(10)**

$$Z(s) = \frac{(s+1)(s+6)}{s(s+4)(s+8)}$$

Solution: Foster-I form The Foster-I form is obtained by partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+4} + \frac{K_2}{s+8}$$

where $$K_0 = s\,Z(s)\big|_{s=0} = \frac{(1)(6)}{(4)(8)} = \frac{3}{16}$$

$$K_1 = (s+4)\,Z(s)\big|_{s=-4} = \frac{(-4+1)(-4+6)}{(-4)(-4+8)} = \frac{(-3)(2)}{(-4)(4)} = \frac{3}{8}$$

$$K_2 = (s+8)\,Z(s)\big|_{s=-8} = \frac{(-8+1)(-8+6)}{(-8)(-8+4)} = \frac{(-7)(-2)}{(-8)(-4)} = \frac{7}{16}$$

$$Z(s) = \frac{\frac{3}{16}}{s} + \frac{\frac{3}{8}}{s+4} + \frac{\frac{7}{16}}{s+8}$$

The first term represents the impedance of a capacitor of $\frac{16}{3}$F. The remaining terms represent the impedance of a parallel RC circuit. For a parallel RC circuit,

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{3}{32}\Omega,\ C_1 = \frac{8}{3}\text{F}$$

$$R_2 = \frac{7}{128}\Omega,\ C_2 = \frac{16}{7}\text{F}$$

The network is shown in Fig. 2.

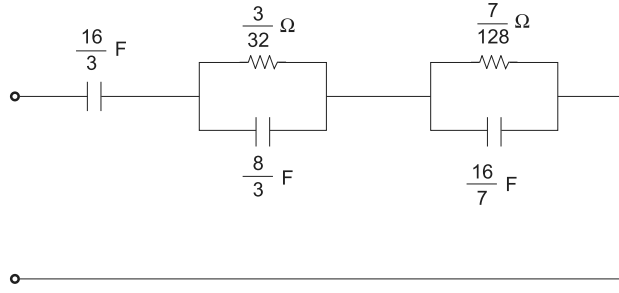


Fig. 2

Foster-II form The Foster-II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$$

$$\frac{Y(s)}{s} = \frac{(s+4)(s+8)}{(s+1)(s+6)} = \frac{s^2+12s+48}{s^2+7s+6}$$

Since the degree of numerator is equal to the degree of denominator, division is carried out first.

$$s^2+7s+6 \;\big)\; s^2+12s+48 \;\big(\; 1$$
$$\underline{s^2+\;\;7s\;+\;6}$$
$$5s\;+42$$

$$\frac{Y(s)}{s} = 1 + \frac{5s+42}{s^2+7s+6} = 1 + \frac{5s+42}{(s+1)(s+6)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = 1 + \frac{K_1}{s+1} + \frac{K_2}{s+6}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+4)(-1+8)}{(-1+6)} = \frac{(3)(7)}{5} = \frac{21}{5}$$

$$K_2 = (s+6)\frac{Y(s)}{s}\bigg|_{s=-6} = \frac{(-6+4)(-6+8)}{(-6+1)} = \frac{(-2)(2)}{-5} = \frac{4}{5}$$

$$\frac{Y(s)}{s} = 1 + \frac{\frac{21}{5}}{s+1} + \frac{\frac{4}{5}}{s+6}$$

$$Y(s) = s + \frac{\frac{21}{5}s}{s+1} + \frac{\frac{4}{5}s}{s+6}$$

The first term represents the admittance of a capacitor of 1 F. The other two terms represent the admittance of a series RC circuit. For a series RC circuit,

$$Y_{RC}(s) = \frac{\frac{1}{R_i}s}{s + \frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1 = \frac{5}{21}\Omega,\;\; C_1 = \frac{21}{5}\text{F}$$

$$R_2 = \frac{5}{4}\Omega,\;\; C_2 = \frac{2}{15}\text{F}$$

The network is shown in Fig. 3.

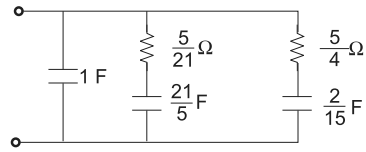


Fig. 3

3. (a) Design a short circuit shunt stub match for $\mathbf{Z}_L = 450 - 600j(\Omega)$, a line of $\mathbf{Z}_o = 300\ \Omega$ and $f = 20$ MHz. Use Smith chart. **(10)**

Solution: $\mathbf{Z}_L = 450 - j600\,\Omega,\ \mathbf{Z}_o = 300\,\Omega$

1. The normalised load impedance is given by

$$\bar{\mathbf{Z}}_L = \frac{450 - j600}{300} = 1.5 - j2$$

   Mark the normalised load impedance point A at the intersection of $r = 1.5$ and $x = -2$ circles.
2. Draw the VSWR circle with centre O and radius equal to OA.
3. From the load impedance point, move to the diametrically opposite point B and locate the corresponding normalized load admittance. It is point $0.24 + j\ 0.32$ on the Smith chart.
4. Locate the point on the VSWR circle where the real part of the admittance is unity. There are two such points C and D with normalized admittance part $1 \pm j1.7$ respectively. This can also be viewed as intersection of VSWR circle and $g = 1$ circle.
5. Distance of point C from point B can be determined as 0.13λ (i.e., $0.182 - 0.052)\lambda$.
6. At point C the susceptance is $+j1.7$. To cancel this the stub should provide $-j1.7$ susceptance. The length of the stub can be obtained by calculating distance starting from the short circuited end towards the generator in a clockwise direction. The length of the stub is $l_1 = (0.335 - 0.25)\lambda = 0.085\lambda$.

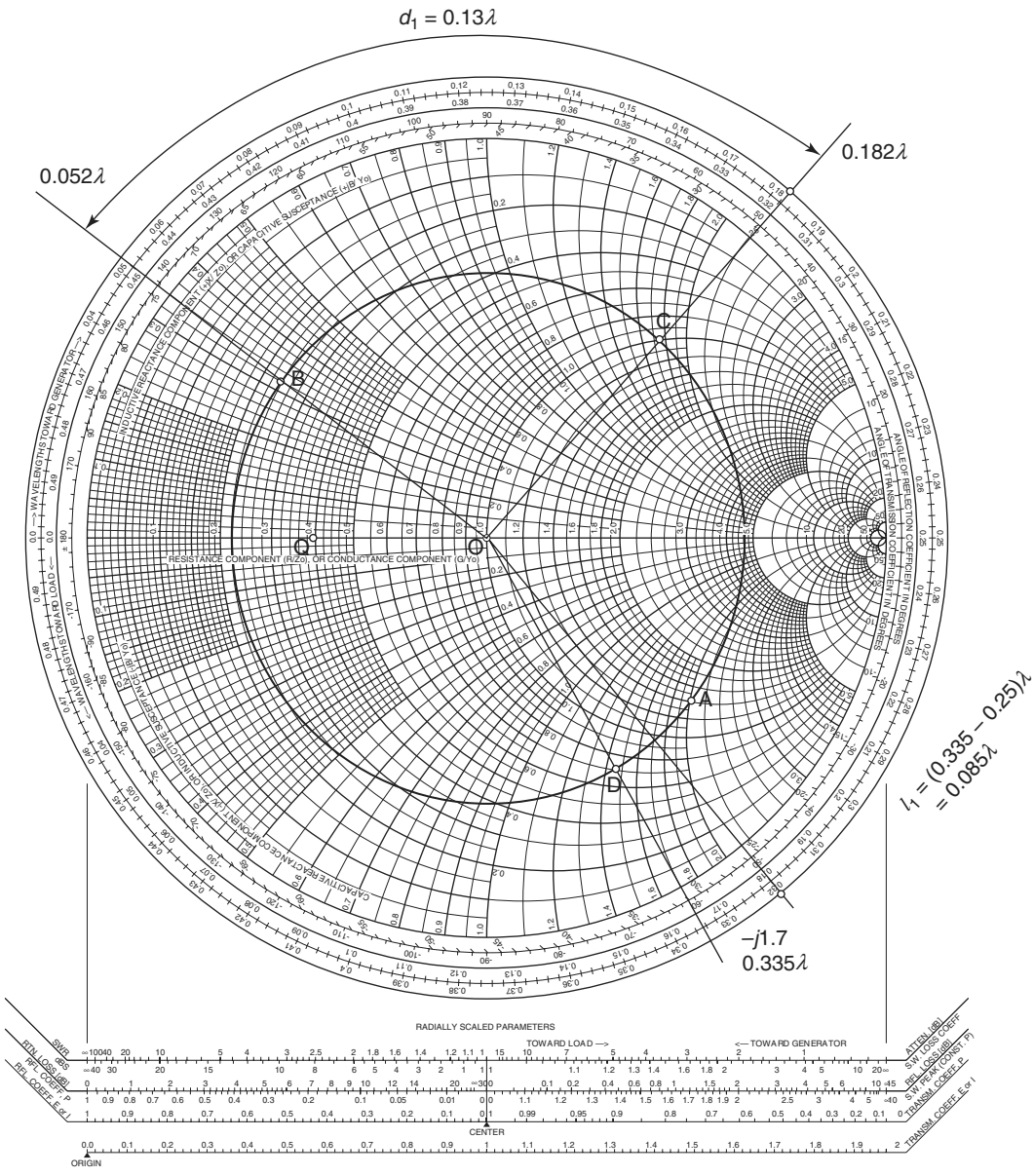

$d_1 = 0.13\lambda$
0.182λ
0.052λ
A
B
C
D
O
$l_1 = (0.335 - 0.25)\lambda$
$= 0.085\lambda$
$-j1.7$
0.335λ
RESISTANCE COMPONENT (R/Z0) OR CONDUCTANCE COMPONENT (G/Y0)
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER
ORIGIN

Fig. 4

(b) In the circuit shown in Fig. 5, determine currents i_1 and i_2 when switch is closed at $t = 0$. **(10)**

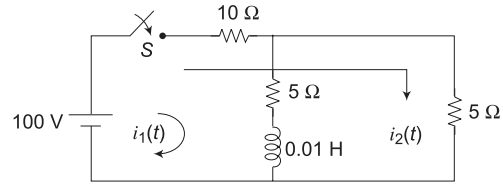


Fig. 5

Solution: Refer Section 6.3.4 on page 6.42.

4. **(a)** Determine *Y*-parameter using interconnection of two port networks in Fig. 6. **(10)**

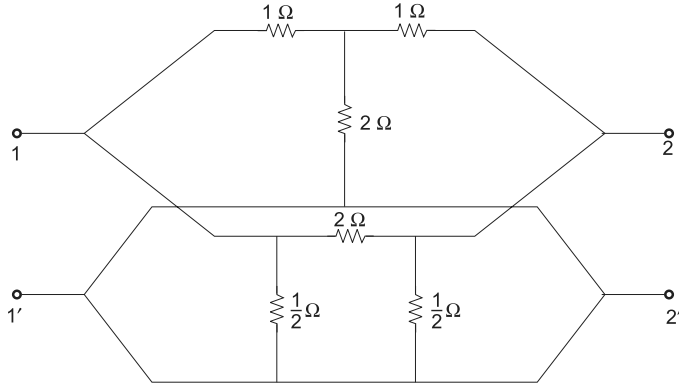


Fig. 6

Solution: The above network (Fig. 6) can be considered as a parallel combination of two networks N_1 and N_2.

The network N_1 is shown in Fig. 7.

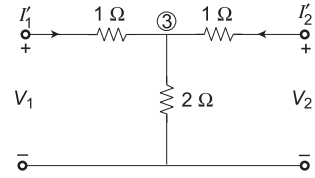


Fig. 7

Applying KCL at Node 3,

$$I_1' + I_2' = \frac{V_3}{2} \qquad \text{...(i)}$$

where

$$I_1' = \frac{V_1 - V_3}{1} = V_1 - V_3$$

$$I_2' = \frac{V_2 - V_3}{1} = V_2 - V_3$$

Substituting I_1' and I_2' in Eq. (i),

$$V_1 - V_3 + V_2 - V_3 = \frac{V_3}{2}$$

$$\frac{5}{2}V_3 = V_1 + V_2$$

$$V_3 = \frac{2}{5}V_1 + \frac{2}{5}V_2 \quad \text{...(ii)}$$

$$I_1' = V_1 - \left(\frac{2}{5}V_1 + \frac{2}{5}V_2\right) = \frac{3}{5}V_1 - \frac{2}{5}V_2 \quad \text{...(iii)}$$

$$I_2' = V_2 - \left(\frac{2}{5}V_1 + \frac{2}{5}V_2\right) = -\frac{2}{5}V_1 + \frac{3}{5}V_2 \quad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{21}' \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

The network N_2 is shown in Fig. 8.

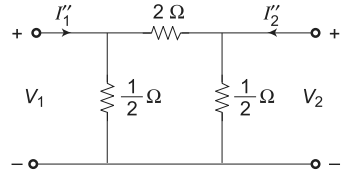


Fig. 8

Applying KCL at Node 1,

$$I_1'' = \frac{V_1}{\frac{1}{2}} + \frac{V_1 - V_2}{2} \quad \text{...(v)}$$

$$= \frac{5}{2}V_1 - \frac{1}{2}V_2$$

Applying KCL at Node 2,

$$I_2'' = \frac{V_2}{\frac{1}{2}} + \frac{V_2 - V_1}{2}$$

$$= -\frac{1}{2}V_1 + \frac{5}{2}V_2 \quad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with *Y*-parameter equations,

$$\begin{bmatrix} Y'_{11} & Y''_{12} \\ Y''_{21} & Y''_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

Hence, overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y'_{11}+Y''_{11} & Y'_{12}+Y''_{12} \\ Y'_{21}+Y''_{21} & Y'_{22}+Y''_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{5}+\frac{5}{2} & -\frac{2}{5}-\frac{1}{2} \\ -\frac{2}{5}-\frac{1}{2} & \frac{3}{5}+\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 3.1 & -0.9 \\ -0.9 & 3.1 \end{bmatrix}$$

(b) Check for positive real function test. **(5)**

$$F(s) = \frac{2s^2+2s+1}{s^3+2s^2+s+2}$$

Solution:

(i) $$F(s) = \frac{N(s)}{D(s)} = \frac{2s^2+2s+1}{s^3+2s^2+s+2} = \frac{(s+0.5-j0.5)(s+0.5+j0.5)}{(s+2)(s+j1)(s-j1)}$$

The function $F(s)$ has pole at $s_{1,2} = \pm j1, s_3 = -2.$

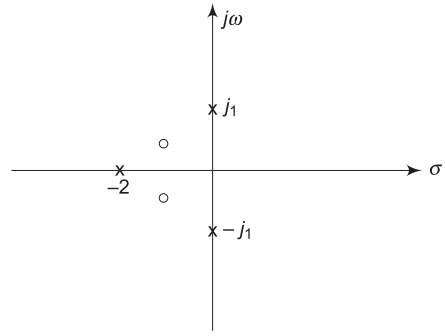


Fig. 9

Thus, pole and zero are in the left half of the *s*-plane.

(ii) Since there is pole on the $j\omega$ axis, the residue test is carried out.
By partial-fraction expansion,

$$F(s) = \frac{K_1}{(s+2)} + \frac{K_2}{(s+j1)} + \frac{K_2^*}{(s-j1)}$$

The constants K_1, K_2 and K_2^* are called residues.

$$K_1 = (s+2)\,F(s)\big|_{s=-2} = \frac{2(-2)^2+2(-2)+1}{(-2)^2+1} = 1$$

$$K_2 = (s+j1)F(s)\Big|_{s=-j1} = \frac{-1-2j}{-2-4j} = \frac{1}{2}$$

$$K_2^* = K_2 = \frac{1}{2}$$

Thus, residues are real and positive.

(iii) Even part of $N(s) = m_1 = 2s^2 + 1$
Odd part of $N(s) = n_1 = 2s$
Even part of $D(s) = m_2 = 2s^2 + 2$
Odd part of $D(s) = n_2 = s^3 + s$

$$\begin{aligned} A(\omega^2) &= m_1 m_2 - n_1 n_2\Big|_{s=j\omega} \\ &= (2s^2+1)(2s^2+2) - (2s)(s^3+s)\Big|_{s=j\omega} \\ &= 4s^4 + 4s^2 + 2s^2 + 2 - 2s^4 - 2s^2\Big|_{s=j\omega} \\ &= 2s^4 + 4s^2 + 2\Big|_{s=j\omega} \\ &= 2\omega^4 - 2\omega^2 + 2 \end{aligned}$$

$A(\omega)^2$ is positive for all $\omega \geq 0$.
Since all the three conditions are satisfied, the function is positive and real.

(c) Compare Cauer-I and Cauer-II form of LC network. **(5)**

$$Z(s) = \frac{2(s^2+1)(s^2+4)}{s(s^2+2)}$$

Solution: Cauer-I form The Cauer-I form is obtained from continued fraction expansion about pole at infinity.

(i) $$Z(s) = \frac{2(s^2+1)(s^2+4)}{s(s^2+2)} = \frac{2(s^4+4s^2+s^2+4)}{s^3+2s} = \frac{2s^4+10s^2+8}{s^3+2s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity. By continued fraction expansion,

$$\begin{array}{l} s^3+2s \;\big)\; 2s^4+10s^2+8 \;\big(\; 2s \leftarrow Z \\ \quad\; \underline{5s^4+\;4s^2} \\ \qquad\; 6s^2\;+8 \;\big)\; s^3+2s \;\big(\; \frac{1}{6}s \leftarrow Y \\ \qquad\quad\; \underline{s^3+\frac{4}{3}s} \\ \qquad\qquad \frac{2}{3}s \;\big)\; 6s^2\;+8 \;\big(\; 9s \leftarrow Z \\ \qquad\qquad\quad\; \underline{6s^2} \\ \qquad\qquad\qquad 8 \;\big)\; \frac{2}{3}s \;\big(\; \frac{2}{24}s \leftarrow Y \\ \qquad\qquad\qquad\quad\; \underline{\frac{2}{3}s} \\ \qquad\qquad\qquad\qquad 0 \end{array}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer realisation. The network is shown in Fig. 10.

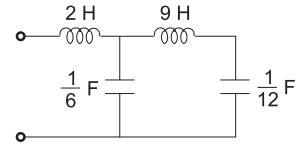


Fig. 10

(ii) **Cauer-II form** The Cauer-II form is obtained from continued fraction expansion about pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s) = \frac{8 + 10s^2 + 2s^4}{2s + s^3}$$

By continued fraction expansion,

$$\begin{array}{r} \\ 2s + s^3 \big) \; 8 + 10s^2 + 2s^4 \; \big(\frac{4}{s} \leftarrow Z \\ \underline{8 + 4s^2} \\ 6s^2 + 2s^4 \big) \; 2s + s^3 \; \big(\frac{1}{3s} \leftarrow Y \\ \underline{2s + \frac{2}{3}s^3} \\ \frac{1}{3}s^3 \big) \; 6s^2 + 2s^4 \big(\frac{18}{s} \leftarrow Z \\ \underline{6s^2} \\ 2s^4 \big) \; \frac{1}{3}s^3 \big(\frac{1}{6}s \leftarrow Y \\ \underline{\frac{1}{3}s^3} \\ 0 \end{array}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 11.

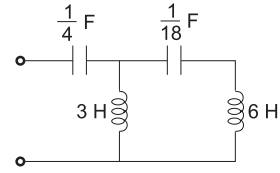


Fig. 11

5. (a) Obtain $i(t)$ for $t > 0$, where $r(t)$ is a ramp signal (Fig. 12). **(10)**

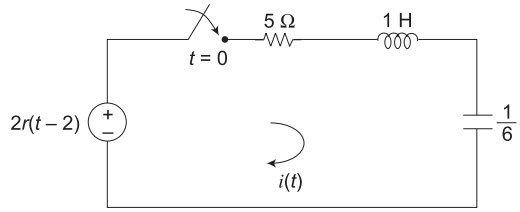


Fig. 12

Solution: For $t > 0$, the transformed network is shown in Fig. 13.

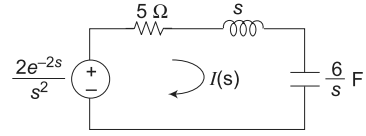


Fig. 13

Applying KVL to the Mesh for $t > 0$,

$$\frac{2e^{-2s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s(s^2+5s+6)} = \frac{2e^{-2s}}{s(s+3)(s+2)}$$

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 2e^{-2s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right]$$

$$= \frac{1}{3}\frac{e^{-2s}}{s} + \frac{2}{3}\cdot\frac{e^{-2s}}{s+3} - \frac{e^{-2s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{3}u(t-2) + \frac{2}{3}e^{-3(t-2)}u(t-2) - e^{-2(t-2)}u(t-2) \text{ for } t > 0$$

(b) Derive an expression for characteristic equation of a transmission line. Also obtain 'α', 'β' and 'γ' of the line. **(10)**

Solution: Refer Section 12.5 on page 12.3.

6. **(a)** Find Thevenin's equivalent of the network shown in Fig. 14. **(8)**

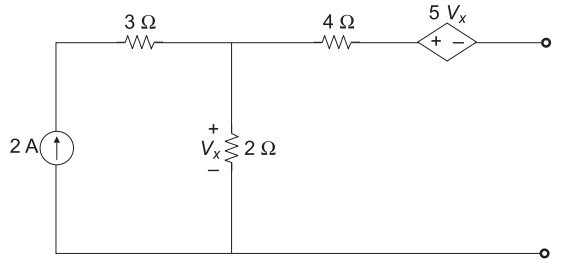


Fig. 14

Solution: Refer Example 3.52 on page 3.58.

(b) Explain various types of filters. **(4)**

Solution: Refer Section 11.2 on page 11.1.

(c) Obtain $Z_{11}(s)$, $Z_{12}(s)$, $G_{12}(s)$ for the ladder network shown in Fig. 15. **(8)**

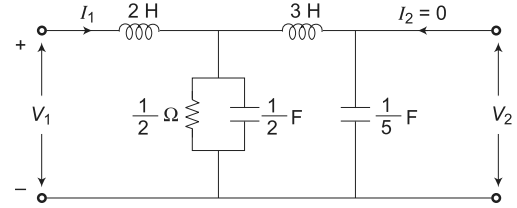


Fig. 15

Solution: The transformed network is shown in Fig. 16.

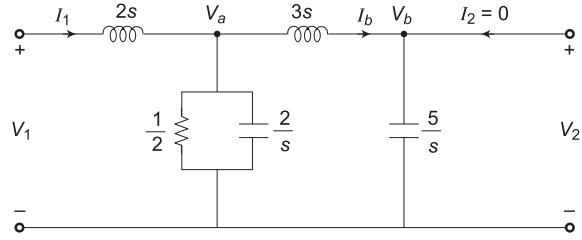


Fig. 16

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{5}{s}} = \frac{s}{5}V_2$$

$$V_a = 3sI_b + V_b = 3s\left(\frac{s}{5}V_2\right) + V_2 = \left(\frac{3s^2}{5} + 1\right)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{2}} + \frac{V_a}{\frac{2}{s}} + I_b$$

$$= \left(\frac{s}{2} + 2\right)\left(\frac{3s^2}{5} + 1\right)V_2 + \frac{s}{5}V_2$$

$$= \left[\left(\frac{s+4}{2}\right)\left(\frac{3s^2+5}{5}\right) + \frac{s}{5}\right]V_2$$

$$= \left[\frac{1}{10}(3s^3 + 12s^2 + 5s + 20) + \frac{s}{5}\right]V_2$$

$$= \frac{1}{10}(3s^3 + 12s^2 + 7s + 20)\,V_2$$

$$V_1 = 2sI_1 + V_a$$

$$= 2s\left[\frac{1}{10}(3s^3 + 12s^2 + 7s + 20)\,V_2\right] + \left(\frac{3s^2}{5} + 1\right)V_2$$

$$= \frac{1}{5}(3s^4 + 12s^3 + 10s^2 + 20s + 5)\,V_2$$

Hence,

$$Z_{11}(s) = \frac{V_1}{I_1} = \frac{2(3s^4 + 12s^3 + 10s^2 + 20s + 5)}{3s^3 + 12s^2 + 7s + 20}$$

$$Z_{12}(s) = \frac{V_2}{I_1} = \frac{10}{3s^3 + 12s^2 + 7s + 20}$$

$$G_{12}(s) = \frac{V_2}{V_1} = \frac{5}{3s^4 + 12s^3 + 10s^2 + 20s + 5}$$

Index

A

Active and Passive Elements 1.7
Active and Passive Networks 1.7
Analysis of Ladder Networks 8.5
Analysis of Non-Ladder Networks 8.15
Applications of Smith Chart 12.26
Attenuation Band 11.1
Attenuator 11.40

B

Band-Pass Filter 11.1, 11.18
Band-Stop Filter 11.1, 11.22
Bandwidth 5.3
Branch 2.1

C

Capacitance 1.3
Cascade Connection 10.63
Cauer I Form 9.33
Cauer II Form 9.34
Cauer Realisation or Ladder Realisation 9.33
Characteristic Impedance 11.2, 12.3
Characteristic of Filters 11.6
Coefficient of Coupling (k) 4.2
Composite Filter 11.37
Condition for Reciprocity 10.2
Condition for Symmetry 10.3
Conductively Coupled Equivalent Circuits 4.37
Constant Reactance Circles 12.22
Constant Resistance Circles 12.21
Constant VSWR Circles 12.25
Constant-k Half Sections 11.34
Constant-k High-Pass Filter 11.14
Constant-k Low Pass Filter 11.7
Cosine Function 7.3
Coupled Circuits 4.15
Critical Value of Mutual Inductance 4.42
Critically Damped Response 6.67
Cumulative Coupling 4.3
Current 1.1
Current Transfer Function 8.2
Current-Controlled Current Source (CCCS) 1.6
Current-Controlled Voltage Source (CCVS) 1.6
Cut-off Frequency 11.1

D

Delayed or Shifted Unit-Step Function 7.2
Delayed Unit-Ramp Function 7.3
Dependent Sources 1.5
Differential Coupling 4.4
Dot Convention 4.9
Double-Tuned Circuit 4.43
Driving Point 8.1
Driving-Point Admittance Function 8.2
Driving-Point Functions 8.1
Driving-Point Impedance at Input Port 10.87
Driving-Point Impedance at Output Port 10.89
Driving-Point Impedance Function 8.2
Dynamic Impedance of a Parallel Circuit 5.19

E

Elementary Synthesis Concepts 9.24
Energy 1.1
Equivalent Circuit Representation of Transmission Line 12.2
Exponential Function (e^{at}) 7.3
Exponentially Damped Function 7.4

F

Farad 1.3
Final Value Theorem 7.7
Forced Response 6.28
Foster I Form 9.31

Foster II Form 9.32
Foster Realisation 9.31

G

General Solution of a Transmission Line 12.11
Graphical Method for Determination of Residue 8.42

H

Henry 1.2
High-Pass Filter 11.1
Hurwitz Polynomials 9.1
Hybrid Parameters (h Parameters) 10.28
Hyperbolic Cosine Function 7.4
Hyperbolic Sine Function 7.4

I

Impedance Matching Using Smith Chart 12.36
Impedance Smith Chart 12.19
Independent Current Source 1.5
Independent Sources 1.5
Independent Voltage Source 1.5
Inductance 1.2
Inductances in Parallel 4.4
Inductances in Series 4.3
Infinite Line 12.10
Initial Conditions 6.1
Initial Conditions for the Capacitor 6.2
Initial Conditions for the Inductor 6.2
Initial Conditions for the Resistor 6.1
Initial Value Theorem 7.7
Input Port 8.1
Interconnection of Two-Port Networks 10.63
Inter-Relationships Between the Parameters 10.37
Inverse Hybrid Parameters (g Parameters) 10.33
Inverse Laplace Transform 7.7
Inverse Transmission Parameters (A′B′C′D′ Parameters) 10.24

K

Kirchhoff's Current Law (KCL) 2.1
Kirchhoff's Laws 2.1
Kirchhoff's Voltage Law (KVL) 2.1

L

Ladder-Type Attenuator 11.47
Laplace Transformation 7.1
Laplace Transforms of Some Important Functions 7.2
Lattice Attenuator 11.41
Lattice Networks 10.84
Linear and Non-linear Elements 1.6
Loop 2.1
Low-Pass Filter 11.1
Lumped and Distributed Elements 1.7

M

Maximum Power Transfer Theorem 3.91
m-Derived Filters 11.25
m-Derived Half Sections 11.35
m-Derived High-Pass Filter 11.31
m-Derived Low-Pass Filter 11.28
Mesh 2.1
Mesh Analysis 2.2
Millman's Theorem 3.112
Mutual Inductance 4.2
Mutually Induced emf 4.2

N

Natural Response 6.28
Nepers 11.40
Network and Circuit 1.6
Node 2.1
Node Analysis 2.26
Norton's Theorem 3.64

O

ohm 1.1
One-Port Network 8.1
Open-Circuit Impedance Parameters (Z Parameters) 10.2
Output Port 8.1
Overdamped Response 6.67

P

π Network 11.4
π-Type Attenuator 11.45
Parallel Connection 10.67
Parallel Resonance 5.18
Parallel–Series Connection 10.77
Partial Fraction Expansion 7.8
Pass Band 11.1
Phasor Diagram 5.2
PI (π)-Network 10.79
Poles and Zeros of Network Functions 8.20
Positive Real Functions 9.16
Potential Difference 1.1
Power Factor 5.2
Port 8.1
Primary Constants of a Transmission Line 12.2
Propagation Constant 11.2, 12.4
Properties of Hurwitz Polynomials 9.1
Properties of Laplace Transform 7.4
Properties of Positive Real Functions 9.16

Q

Quality Factor 5.5

R

Realisation of LC Functions 9.30

Realisation of RC Functions 9.47
Realisation of RL Functions 9.63
Reflection 12.15
Reflection Coefficient 12.15
Relation between Decibel and Neper 11.41
Resistance 1.1
Resistor–Capacitor Circuit 6.49, 7.19
Resistor-Inductor Circuit 6.27, 7.13
Resistor–Inductor–Capacitor Circuit 6.66, 7.25
Response 6.28
Response of RC Circuit to Various Functions 7.39
Response of RL Circuit to Various Functions 7.31

S

Scattering Parameters or S-Parameters 12.18
Secondary Constants of a Transmission Line 12.3
Self-Induced emf 4.1
Self-Inductance 4.1
Series and Parallel Combination of Capacitors 1.10
Series and Parallel Combination of Inductors 1.9
Series and Parallel Combinations of Resistors 1.7
Series Connection 10.73
Series Resonance 5.1
Series-Parallel Connection 10.75
Short-Circuit Admittance Parameters (Y Parameters) 10.8
Sine Function 7.3
Single-Tuned Circuit 4.41
Source Shifting 1.19
Source Transformation 1.13
Sources 1.4
Stability of the Network 8.42
Standing Wave Ratio 12.16
Standing Waves 12.16
Star-Delta Transformation 1.10
Steady-State Response 6.1, 6.28
Stop Band 11.1
Supermesh Analysis 2.17
Supernode Analysis 2.46
Superposition Theorem 3.1

T

Terminal Pair 8.1
Terminated Transmission Lines 12.11
Terminated Two-Port Networks 10.87
Terminating Half Sections 11.34
The Smith Chart 12.23
The Transformed Circuit 7.12
Thevenin's Theorem 3.30
Time-Domain Behaviour from the Pole-Zero Plot 8.39
Time-Invariant and Time-Variant Networks 1.7
T-Network 10.79, 11.1
Transfer Function 8.1, 8.2
Transfer Admittance Function 8.3
Transfer Impedance Function 8.3
Transient Period 6.1
Transient Response 6.1, 6.28
Transients 6.1
Transmission Lines 12.1
Transmission Parameters (ABCD Parameters) 10.18
T-type Attenuator 11.42
Tuned circuits 4.40
Types of Transmission Lines 12.1

U

Underdamped Response 6.67
Unilateral and Bilateral Elements 1.7
Unit-Impulse Function 7.3
Unit-Ramp Function 7.3
Unit-Step Function 7.2

V

Voltage 1.1
Voltage Amplification 4.42
Voltage Transfer Function 8.2
Voltage-Controlled Current Source (VCCS) 1.5
Voltage-Controlled Voltage Source (VCVS) 1.5

Z

Zero Input Response 6.28
Zero State Response 6.28