Complex Variables and Numerical Methods

Gujarat Technological University 2017

Second Edition

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Dedicated
To Our Parents

Late Shri Ramsagar Singh
and
Late Shrimati Premsheela Singh

Ravish R Singh

Late Shri Ved Prakash Sharma
and
Late Shrimati Vidyavati Hemdan

Mukul Bhatt

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Preface

Mathematics is a key area of study in any engineering course. A sound knowledge of this subject will help engineering students develop analytical skills, and thus enable them to solve numerical problems encountered in real life, as well as apply mathematical principles to physical problems, particularly in the field of engineering.

Users

This book is designed for the 4th semester GTU Mechanical Engineering students pursuing the course *Complex Variables and Numerical Methods (CODE 2141905)*. It covers the complete GTU syllabus for the course on Complex Variables and Numerical Methods for the mechanical engineering branch.

Objective

The crisp and complete explanation of topics will help students easily understand the basic concepts. The tutorial approach (i.e., teach by example) followed in the text will enable students develop a logical perspective to solving problems.

Features

Each topic has been explained from the examination point of view, wherein the theory is presented in an easy-to-understand student-friendly style. Full coverage of concepts is supported by numerous solved examples with varied complexity levels, which is aligned to the latest GTU syllabus. Fundamental and sequential explanation of topics is well aided by examples and exercises. The solutions of examples are set following a 'tutorial' approach, which will make it easy for students from any background to easily grasp the concepts. Exercises with answers immediately follow the solved examples enforcing a practice-based approach. We hope that the students will gain logical understanding from solved problems and then reiterate it through solving similar exercise problems themselves. The unique blend of theory and application caters to the requirements of both the students and the faculty. Solutions of GTU examination questions are incorporated within the text appropriately.

Highlights

- Crisp content strictly as per the latest GTU syllabus of *Complex Variables and Numerical Methods* (Regulation 2014)
- Comprehensive coverage with lucid presentation style
- Each section concludes with an exercise to test understanding of topics
- Solutions of GTU examination papers from 2010 to 2015 present appropriately within the chapters
- Solution of 2016 GTU examination paper can be accessible through weblink.
- Rich exam-oriented pedagogy:
  - ➢ Solved Examples within chapters: 473
  - ➢ Solved GTU questions tagged within chapters: 150
  - ➢ Unsolved Exercises: 474

Chapter Organization

The content spans the following twelve chapters which wholly and sequentially cover each module of the syllabus.

❏ **Chapter 1** introduces Complex Numbers.
❏ **Chapter 2** discusses Analytic Functions.
❏ **Chapter 3** presents Complex Integration.
❏ **Chapter 4** covers Power Series.
❏ **Chapter 5** deals with Applications of Contour Integration.
❏ **Chapter 6** presents Conformal Mapping and its Applications.
❏ **Chapter 7** explains Interpolation.
❏ **Chapter 8** introduces Numerical Integration.
❏ **Chapter 9** discusses Solutions of a System of Linear Equations.
❏ **Chapter 10** deals with Roots of Algebraic and Transcendental Equations.
❏ **Chapter 11** covers Eigenvalues by Power and Jacobi Methods.
❏ **Chapter 12** explains Numerical Solution of Ordinary Differential Equations.

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Ravish R Singh
Mukul Bhatt

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ROADMAP TO THE SYLLABUS

This text is useful for

Complex Variables and Numerical Methods (CODE 2141905)

Module 1: Complex Numbers and Functions

Exponential, Trigonometric, De Moivre's Theorem, Roots of a Complex Number, Hyperbolic Functions and their Properties, Multi-valued Function and its Branches: Logarithmic Function and Complex Exponent Function Limit, Continuity and Differentiability of Complex Function, Analytic Functions, Cauchy-Riemann Equations, Necessary and Sufficient Condition for Analyticity, Properties of Analytic Functions, Laplace Equation, Harmonic Functions, Harmonic Conjugate Functions and their Engineering Applications.

GO TO

CHAPTER 1: Complex Numbers

CHAPTER 2: Analytic Functions

Module 2: Complex Integration:

Curves, Line Integral (Contour Integral) and its Properties, Cauchy-Goursat Theorem, Cauchy Integral Formula, Liouville Theorem (without proof), Maximum Modulus Theorems (without proof)

GO TO

CHAPTER 3: Complex Integration

Module 3: Power Series

Convergence (Ordinary, Uniform, Absolute) of Power Series, Taylor and Laurent Theorems (without proof), Laurent Series Expansions, Zeros of Analytic Functions , Singularities of Analytic Functions and their Classification.
Residues: Residue Theorem, Rouche's Theorem (without proof)

GO TO

CHAPTER 4: Power Series

Module 4: Applications of Contour Integration

Evaluation of Various Types of Definite Real Integrals using Contour Integration Method.

GO TO
CHAPTER 5: Applications of Contour Integration

Module 5: Conformal Mapping and its Applications

Conformal and Isogonal Mappings, Translation, Rotation and Magnification, Inversion, Mobius (Bilinear), Schwarz-Christoffel Transformations

GO TO
CHAPTER 6: Conformal Mapping and its Applications

Module 6: Interpolation

Interpolation: Finite Differences, Forward, Backward and Central Operators, Interpolation by Polynomials: Newton's forward, Backward Interpolation Formulae, Newton's Divided Formula, Gauss and Stirling's Central Difference Formulae and Lagrange's Interpolation Formulae for Unequal Intervals

GO TO
CHAPTER 7: Interpolation

Module 7: Numerical Integration

Newton-Cotes Formula, Trapezoidal and Simpson's Formulae, Error Formulae, Gaussian Quadrature Formulae

GO TO
CHAPTER 8: Numerical Integration

Module 8: Solutions of a System of Linear Equations

Gauss Elimination, Partial Pivoting, Gauss-Jacobi Method and Gauss-Seidel Method

GO TO
CHAPTER 9: Solutions of a System of Linear Equations

Module 9: Roots of Algebraic and Transcendental Equations

Bisection, False Position, Secant and Newton-Raphson Methods, Rate of Convergence

GO TO
CHAPTER 10: Roots of Algebraic and Transcendental Equations

Module 10:

Eigenvalues by Power and Jacobi Methods

GO TO
CHAPTER 11: Eigenvalues by Power and Jacobi Methods

Module 11: Numerical Solution of Ordinary Differential Equations

Euler and Runge-Kutta Methods

GO TO
CHAPTER 12: Numerical Solution of Ordinary Differential Equations

CHAPTER 1

Complex Numbers

Chapter Outline

1.1 Introduction
1.2 Complex Numbers
1.3 Geometrical Representation of Complex Numbers (Argand's Diagram)
1.4 Algebra of Complex Numbers
1.5 Different Forms of Complex Numbers
1.6 Modulus and Argument (or Amplitude) of Complex Numbers
1.7 Properties of Complex Numbers
1.8 De Moivre's Theorem
1.9 Applications of De Moivre's Theorem
1.10 Circular and Hyperbolic Functions
1.11 Inverse Hyperbolic Functions
1.12 Separation into Real and Imaginary Parts
1.13 Logarithm of a Complex Number

1.1 INTRODUCTION

The complex numbers are an extension of the real numbers obtained by introducing an imaginary unit i, where $i = \sqrt{-1}$. The operations of addition, subtraction, multiplication, and division are applicable on complex numbers. A negative real number can be obtained by squaring a complex number. With a complex number, it is always possible to find solutions to polynomial equations of degree more than one. Complex numbers are used in many applications, such as control theory, signal analysis, quantum mechanics, relativity, etc.

1.2 COMPLEX NUMBERS

A complex number z is an ordered pair (x, y) of real numbers x and y. It is written as $z = (x, y)$ or $z = x + iy$, where $i = \sqrt{-1}$ is known as the imaginary unit. Here, x is called the real part of z and is written as "**Re** (z)", and y is called the imaginary part of z and is written as "**Im** (z)".

If $x = 0$ and $y \neq 0$ then $z = 0 + iy = iy$ which is purely imaginary.

If $x \neq 0$ and $y = 0$ then $z = x + i\,0 = x$ which is real.

Hence, z is **purely imaginary**, if its real part is zero, and is **real** if its imaginary part is zero.

This shows that every real number can be written in the form of a complex number by taking its imaginary part as zero. Hence, the set of real numbers is contained in the set of complex numbers.

Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

If $z = x + iy$ is a complex number then its conjugate or complex conjugate is defined as $\overline{z} = x - iy$.

The even power of i is either 1 or –1 and odd power of i is either i or $-i$.

$$i^2 = i.i = -1$$
$$i^3 = i^2.i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$
$$i^5 = i.i^4 = i, \text{ etc.}$$

1.3 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS (ARGAND'S DIAGRAM)

Any complex number $z = x + iy$ can be represented as a point $P(x, y)$ in the xy-plane with reference to the rectangular x and y axes.

The plot of a given complex number $z = x + iy$, as the point $P(x, y)$ in the xy-plane is known as **Argand's diagram** (Fig. 1.1). The x-axis is called the real axis, y-axis is called the imaginary axis, and the xy-plane is called the complex plane.

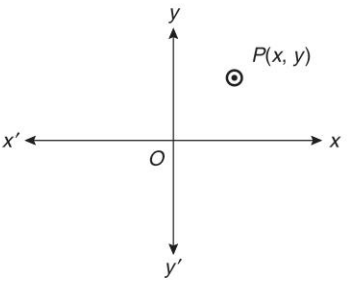


Fig. 1.1

1.4 ALGEBRA OF COMPLEX NUMBERS

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

(a) *Addition:*
$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i\,(y_1 + y_2)$$

(b) *Subtraction:* $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$= (x_1 - x_2) + i\,(y_1 - y_2)$

(c) *Multiplication:* $z_1 z_2 = (x_1 + iy_1)\,(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i\,(x_2 y_1 + y_2 x_1)$ $[\because\ i^2 = -1]$

(d) *Division:*

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \cdot \frac{(x_2 - iy_2)}{(x_2 - iy_2)}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right)$$

1.5 DIFFERENT FORMS OF COMPLEX NUMBERS

1.5.1 Cartesian or Rectangular Form

If x and y are real numbers then $z = x + iy$ is called the Cartesian form of the complex number.

1.5.2 Polar Form

The complex number $z = x + iy$ can be represented by the point P whose cartesian co-ordinates are (x, y) (Fig. 1.2). If polar coordinates of the point P are (r, θ) then $x = r\cos\theta$ and $y = r\sin\theta$.

Hence, polar form of z is

$$z = r\cos\theta + ir\sin\theta$$
$$= r\,(\cos\theta + i\sin\theta)$$

Polar form can also be written as $r \angle \theta$.

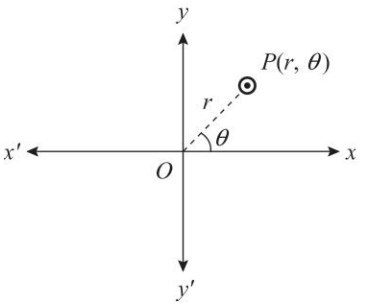


Fig. 1.2

1.5.3 Exponential Form

Polar form of z is

$$z = r\,(\cos\theta + i\sin\theta)$$
$$\cos\theta + i\sin\theta = e^{i\theta} \qquad \text{[Using Expansion]}$$

Hence, exponential form of z is

$$z = re^{i\theta}$$

Note $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Hence, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

1.6 MODULUS AND ARGUMENT (OR AMPLITUDE) OF COMPLEX NUMBERS

Let $$z = x + iy$$
$$= r(\cos\theta + i\sin\theta)$$

Here 'r' is called the modulus or absolute value of z and is denoted by $|z|$ or mod (z) and θ is called argument or amplitude of z and is denoted by arg (z) or amp (z).

Hence, $$|z| = r = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note The value of θ lying between $-\pi$ and π is called the **principal value** of argument. The argument of z is the value of θ which lies in the quadrant of point (x, y).

1.7 PROPERTIES OF COMPLEX NUMBERS

Let $z = x + iy$ and $\bar{z} = x - iy$.

(a) $\text{Re}(z) = x = \frac{1}{2}(z + \bar{z})$

(b) $\text{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$

(c) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(d) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

(e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(f) $z\bar{z} = |z|^2 = |\bar{z}|^2$ $\quad \left[\because |z| = |\bar{z}| = \sqrt{x^2 + y^2}\right]$

(g) $|z_1 z_2| = |z_1|\,|z_2|$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(h) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Example 1

Arg $(z_1, z_2) = Arg\ (z_1) + Arg\ (z_2)$? *Justify.* **[Summer 2015]**

Solution

Yes, $Arg\ (z_1\ z_2) = Arg\ (z_1) + Arg\ (z_2)$

Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = (r_1\ e^{i\theta_1})\ (r_2\ e^{i\theta_2})$$

$$= r_1\ r_2\ e^{i(\theta_1+\theta_2)}$$

$\therefore$ $Arg\,(z_1\ z_2) = \theta_1 + \theta_2$

$$= Arg\,(z_1) + Arg(z_2)$$

Example 2

Find the principal argument Arg z when $z = \dfrac{-2}{1+\sqrt{3}\,i}$. **[Winter 2014]**

Solution

$$z = \frac{-2}{1+\sqrt{3}\,i}$$

$$= \frac{-2}{1+\sqrt{3}i} \cdot \frac{\left(1-\sqrt{3}\,i\right)}{\left(1-\sqrt{3}\,i\right)}$$

$$= \frac{-2+2\sqrt{3}\,i}{1+3}$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$Arg\ z = \tan^{-1} \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)}$$

$$= \tan^{-1}\left(-\sqrt{3}\right)$$

$$= \frac{2\pi}{3}$$

Example 3

Find the modulus and principal value of the arguments of the following complex numbers:

(i) $(4+2i)\left(-3+\sqrt{2}i\right)$

(ii) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

Solution

(i)

$$\begin{aligned} z &= (4+2i)\left(-3+\sqrt{2}i\right) \\ &= \left(-12+4\sqrt{2}i-6i-2\sqrt{2}\right) \\ &= -\left(12+2\sqrt{2}\right)+i\left(4\sqrt{2}-6\right) \\ |z| &= \sqrt{\left(12+2\sqrt{2}\right)^2+\left(4\sqrt{2}-6\right)^2} \\ &= 2\sqrt{36+2+12\sqrt{2}+8+9-12\sqrt{2}} \\ &= 2\sqrt{55} \\ \arg(z) &= \tan^{-1}\left[\frac{4\sqrt{2}-6}{-\left(12+2\sqrt{2}\right)}\right] \\ &= \tan^{-1}\left(\frac{3-2\sqrt{2}}{6+\sqrt{2}}\right) \end{aligned}$$

Aliter

$$\begin{aligned} z &= (4+2i)\left(-3+\sqrt{2}i\right) = z_1 z_2, \text{ say} \\ r &= |z| \\ &= |z_1 z_2| \\ &= |z_1||z_2| \\ &= |4+2i|\left|-3+\sqrt{2}i\right| \\ &= \left(\sqrt{16+4}\right)\left(\sqrt{9+2}\right) \\ &= \sqrt{220} \\ &= 2\sqrt{55} \\ \theta &= \arg(z) \\ &= \arg(z_1 z_2) \\ &= \arg(z_1)+\arg(z_2) \\ &= \arg(4+2i)+\arg\left(-3+\sqrt{2}i\right) \end{aligned}$$

$$= \tan^{-1}\left(\frac{2}{4}\right) + \tan^{-1}\left(\frac{\sqrt{2}}{-3}\right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2} - \frac{\sqrt{2}}{3}}{1 + \frac{1}{2}\cdot\frac{\sqrt{2}}{3}}\right)$$

$$= \tan^{-1}\left(\frac{3 - 2\sqrt{2}}{6 + \sqrt{2}}\right)$$

(ii)
$$z = \frac{2 + 6\sqrt{3}i}{5 + \sqrt{3}i}$$

$$= \left(\frac{2 + 6\sqrt{3}i}{5 + \sqrt{3}i}\right)\left(\frac{5 - \sqrt{3}i}{5 - \sqrt{3}i}\right)$$

$$= \frac{28 + 28\sqrt{3}i}{28}$$

$$= 1 + \sqrt{3}i$$

$$|z| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

Example 4

Evaluate $\left[(1+i)^{100} + (1-i)^{100}\right]$.

Solution

Let $1 + i = re^{i\theta}$

$$r = |1 + i| = \sqrt{1 + 1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

$\therefore$ $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

and $1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$

$$(1+i)^{100}+(1-i)^{100}=\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{100}+\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{100}$$
$$=2^{50}(e^{25\pi i}+e^{-25\pi i})$$
$$=2^{50}(2\cos 25\pi)$$
$$=2^{51}(-1)$$
$$=-2^{51}$$

Example 5

Express in polar form:

(i) $\left(\frac{2+i}{3-i}\right)^2$ *(ii)* $1+\sin\alpha+i\cos\alpha$

Solution

(i)
$$\left(\frac{2+i}{3-i}\right)^2=\frac{4+i^2+4i}{9+i^2-6i}$$
$$=\frac{3+4i}{8-6i}$$
$$=\frac{3+4i}{8-6i}\cdot\frac{8+6i}{8+6i}$$
$$=\frac{i}{2}$$
$$=0+i\cdot\frac{1}{2}$$

Let
$$0+i\cdot\frac{1}{2}=r(\cos\theta+i\sin\theta)$$
$$r=\sqrt{0^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}$$

and
$$\theta=\tan^{-1}\left(\frac{\frac{1}{2}}{0}\right)=\tan^{-1}\infty=\frac{\pi}{2}$$

Hence, the polar form is

$$\left(\frac{2+i}{3-i}\right)^2=\frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

(ii) $$1+\sin\alpha+i\cos\alpha = 1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)$$

$$= 2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$\left[\because\ 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= 2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]$$

Comparing with the polar form $r(\cos\theta + i\sin\theta)$,

$$r = 2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$\theta = \frac{\pi}{4}-\frac{\alpha}{2}$$

Hence, the polar form is

$$1+\sin\alpha+i\cos\alpha = 2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]$$

Example 6

Find the value of $\sqrt{-5+12i}$.

Solution

Let $$\sqrt{-5+12i} = x+iy$$

$$-5+12i = (x+iy)^2$$

$$-5+12i = (x^2-y^2)+i\,(2xy)$$

Comparing real and imaginary parts,

$$x^2-y^2 = -5 \qquad \text{... (1)}$$

and $$2xy = 12$$

$$xy = 6$$

Putting $y = \frac{6}{x}$ in Eq. (1),

$$x^2-\frac{36}{x^2} = -5$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$$x^2 = -9,\ x^2 = 4$$

Since x is real, $x = \pm 2$

and $y = \pm 3$

Hence, $\sqrt{-5+12i} = \pm 2 \pm 3i$

Example 7

If x and y are real, solve the equation $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$.

Solution

$$\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$$

$$\frac{iy(3x+y) - (3y+4i)(ix+1)}{(ix+1)(3x+y)} = 0$$

$$(-3y+4x) + i(3xy + y^2 - 3xy - 4) = 0 + i0$$

$$(-3y+4x) + i(y^2 - 4) = 0 + i0$$

Comparing real and imaginary parts,

$$-3y + 4x = 0$$

and $$y^2 - 4 = 0$$

$$y = \pm 2$$

$\therefore$ $$x = \pm\frac{3}{2}$$

Hence, the solution is

$$x = \pm\frac{3}{2}, \qquad y = \pm 2$$

Example 8

Solve the equation $z^2 + (2i - 3)\,z + 5 - i = 0$. **[Summer 2015]**

Solution

$$z^2 + (2i - 3)\,z + 5 - i = 0$$

The equation is quadratic in z.

$$z = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4(5-i)}}{2}$$
$$= \frac{-(2i-3) \pm \sqrt{4i^2 + 9 - 12i - 20 + 4i}}{2}$$
$$= \frac{-(2i-3) \pm \sqrt{-15 - 8i}}{2}$$
$$= \frac{-(2i-3) \pm \sqrt{16i^2 + 1 - 8i}}{2}$$
$$= \frac{-(2i-3) \pm \sqrt{(1-4i)^2}}{2}$$
$$= \frac{-(2i-3) \pm (1-4i)}{2}$$
$$= \frac{-2i+3+1-4i}{2} \quad \text{and} \quad \frac{-2i+3-1+4i}{2}$$
$$= 2 - 3i \text{ and } 1 + i$$

Hence, the solution is

$$z = 2 - 3i \quad \text{and} \quad z = 1 + i$$

Example 9

Prove that Re $(z) > 0$ *and* $|z - 1| < |z + 1|$ *are equivalent, where* $z = x + iy$.

Solution

$$z = x + iy$$
$$\text{Re}\,(z) > 0$$
$$x > 0 \qquad \text{... (1)}$$

Now,
$$|z - 1| < |z + 1|$$
$$|x + iy - 1| < |x + iy + 1|$$
$$\sqrt{(x-1)^2 + y^2} < \sqrt{(x+1)^2 + y^2}$$
$$x^2 + 1 - 2x + y^2 < x^2 + 1 + 2x + y^2$$
$$-2x < 2x$$
$$0 < 4x$$
$$0 < x \quad \text{or} \quad x > 0 \qquad \text{... (2)}$$

From Eqs (1) and (2),

$$\text{Re}\,(z) > 0 \quad \text{and} \quad |z-1| < |z+1| \text{ are equivalent.}$$

Example 10

If $|i+z| = |i-z|$, *show that z is completely real.*

Solution

Let $z = x + iy$

$$|i+z| = |i-z|$$
$$|i+x+iy| = |i-x-iy|$$
$$|x+i(y+1)| = |-x+i(1-y)|$$
$$|x+i(y+1)|^2 = |-x+i(1-y)|^2$$
$$x^2+(y+1)^2 = (-x)^2+(1-y)^2$$
$$x^2+y^2+2y+1 = x^2+1-2y+y^2$$
$$4y = 0$$
$$y = 0$$

Hence, $z = x$ is completely real.

Example 11

Find the locus of z if $|z-5-6i| = 4$.

Solution

Let $z = x + iy$

$$|z-5-6i| = 4$$
$$|x+iy-5-6i| = 4$$
$$|(x-5)+i(y-6)| = 4$$
$$\sqrt{(x-5)^2+(y-6)^2} = 4$$

Squaring both the sides,

$$(x-5)^2 + (y-6)^2 = 16$$

which represents a circle with the centre at (5, 6) and a radius of 4 units.
Hence, the locus of z is the circle.

Example 12

If $\dfrac{z-2i}{2z-1}$ *is purely imaginary, prove that the locus of z in the Argand diagram is a circle. Find the centre and radius.*

Solution

Let $u = \dfrac{z-2i}{2z-1}$ and $z = x + iy$

$$u = \frac{(x+iy)-2i}{2(x+iy)-1}$$

$$= \frac{x+i(y-2)}{(2x-1)+2iy}$$

$$= \left[\frac{x+i(y-2)}{(2x-1)+2iy}\right]\left[\frac{(2x-1)-2iy}{(2x-1)-2iy}\right]$$

$$= \frac{[x(2x-1)+2y(y-2)]+i[(y-2)(2x-1)-2xy]}{(2x-1)^2+4y^2}$$

If u is purely imaginary,

$$\mathrm{Re}(u) = 0$$

$$\frac{x(2x-1)+2y(y-2)}{(2x-1)^2+4y^2} = 0$$

$$2x^2 - x + 2y^2 - 4y = 0$$

$$x^2 + y^2 - \frac{x}{2} - 2y = 0$$

which represents a circle with the centre at $\left(\dfrac{1}{4}, 1\right)$ and a radius of $\dfrac{\sqrt{17}}{4}$ units.

Example 13

If $u = \dfrac{z+i}{z+2}$ *and* $z = x + iy$ *then show that*

(i) locus of (x, y) *is a straight line, if u is real*
(ii) locus of (x, y) *is a circle, if u is purely imaginary*

Find the centre and radius of the circle.

Solution

$$u = \frac{z+i}{z+2} \quad \text{and} \quad z = x + iy$$

$$u = \frac{x+iy+i}{x+iy+2}$$

$$= \left[\frac{x+i(y+1)}{(x+2)+iy}\right]\left[\frac{(x+2-iy)}{(x+2)-iy}\right]$$

$$= \frac{[x(x+2)+y(y+1)]+i[(y+1)(x+2)-xy]}{(x+2)^2+y^2}$$

$$\text{Re}(u) = \frac{x(x+2)+y(y+1)}{(x+2)^2+y^2}$$

$$\text{Im}(u) = \frac{(y+1)(x+2)-xy}{(x+2)^2+y^2}$$

$$= \frac{x+2y+2}{(x+2)^2+y^2}$$

(i) If u is real then

$$\text{Im}(u) = 0$$

$$\frac{x+2y+2}{(x+2)^2+y^2} = 0$$

$$x + 2y + 2 = 0$$

which represents a straight line.

Hence, the locus of (x, y) is a straight line.

(ii) If u is purely imaginary then

$$\text{Re}(u) = 0$$

$$\frac{x(x+2)+y(y+1)}{(x+2)^2+y^2} = 0$$

$$x^2 + y^2 + 2x + y = 0$$

which represents a circle with the centre at $\left(-1, -\frac{1}{2}\right)$ and a radius of $\frac{\sqrt{5}}{2}$ units.

Hence, the locus of (x, y) is a circle.

Example 14

Find z if $\arg\ (z+1)=\dfrac{\pi}{6}$ *and* $\arg\ (z-1)=\dfrac{2\pi}{3}$.

Solution

Let $z = x+iy$

$$\arg\ (z+1) = \frac{\pi}{6}$$

$$\arg\ (x+iy+1) = \frac{\pi}{6}$$

$$\arg[(x+1)+iy] = \frac{\pi}{6}$$

$$\tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{6}$$

$$\frac{y}{x+1} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$x - y\sqrt{3} = -1 \qquad \text{...(1)}$$

Also,

$$\arg\ (z-1) = \frac{2\pi}{3}$$

$$\arg\ (x+iy-1) = \frac{2\pi}{3}$$

$$\arg\ [(x-1)+iy] = \frac{2\pi}{3}$$

$$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3} \qquad \text{...(2)}$$

$$\frac{y}{x-1} = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$x\sqrt{3} + y = \sqrt{3}$$

Solving Eqs (1) and (2),

$$x = \frac{1}{2}, \quad y = \frac{\sqrt{3}}{2}$$

Hence,

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Example 15

If the sum and product of two numbers are real, show that the two numbers must be either real or conjugate.

Solution

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

Let $z_1 + z_2 = a$, where a is real.

$$(x_1 + iy_1) + (x_2 + iy_2) = a + i \cdot 0$$

$$(x_1 + x_2) + i\,(y_1 + y_2) = a + i \cdot 0$$

Comparing real and imaginary parts,

$$x_1 + x_2 = a \qquad \ldots (1)$$

$$y_1 + y_2 = 0 \qquad \ldots (2)$$

Let $z_1\, z_2 = b$, where b is real.

$$(x_1 + iy_1)\,(x_2 + iy_2) = b + i \cdot 0$$

$$(x_1x_2 - y_1y_2) + i\,(x_2\, y_1 + x_1\, y_2) = b + i \cdot 0$$

Comparing real and imaginary parts,

$$x_1x_2 - y_1\, y_2 = b \qquad \ldots (3)$$

$$x_2\, y_1 + x_1\, y_2 = 0 \qquad \ldots (4)$$

Substituting $y_2 = -\, y_1$ from Eq. (2) in Eq. (4),

$$x_2\, y_1 - x_1\, y_1 = 0$$

$$y_1(x_2 - x_1) = 0$$

$$y_1 = 0 \quad \text{or} \quad x_2 - x_1 = 0,\ x_1 = x_2$$

If $y_1 = 0$ then $y_2 = 0$

$$z_1 = x_1 \text{ and } z_2 = x_2$$

If $x_1 = x_2$ then $z_1 = x_1 + iy_1$ and $z_2 = x_1 - iy_1$

Hence, z_1 and z_2 both are either real or conjugate.

Example 16

Prove that $\left|\exp(-2z)\right| < 1$ *if and only if* $Re\, z > 0$. **[Winter 2012]**

Solution

$$z = x + iy$$

$$\mathrm{Re}(z) = x$$

$$e^{-2z} = e^{-2(x+iy)}$$

$$= e^{-2x} e^{-2iy}$$

$$= e^{-2x}(\cos 2y - i \sin 2y)$$

$$\left| e^{-2z} \right| = \sqrt{e^{-4x} \cos^2 2y + e^{-4x} \sin^2 2y}$$
$$= \sqrt{e^{-4x}}$$
$$= e^{-2x} \qquad \ldots (1)$$

If Re $z > 0$, i.e., $x > 0$,

$$e^{2x} > e^0$$
$$e^{2x} > 1$$
$$e^{-2x} < 1$$
$$\left| e^{-2z} \right| < 1 \qquad \text{[From Eq. (1)]}$$

Conversely, if

$$\left|\exp(-2z)\right| < 1$$
$$\left|e^{-2(x+iy)}\right| < 1$$
$$\left|e^{-2x} e^{-2iy}\right| < 1$$
$$e^{-2x} < 1 \qquad \left[\because \left|re^{i\theta}\right| = r\right]$$
$$\frac{1}{e^{2x}} < 1$$
$$e^{2x} > 1$$
$$2x \log e > \log 1$$
$$2x > 0$$
$$x > 0$$

Hence, $\left|\exp(-2z)\right| < 1$ if and only if Re $z > 0$.

Example 17

Show that $\left|\frac{z}{|z|} - 1\right| \le \left|\arg(z)\right|$.

Solution

Let $z = re^{i\theta}$, where $|z| = r$ and arg $(z) = \theta$

$$\left|\frac{z}{|z|} - 1\right| = \left|\frac{re^{i\theta}}{r} - 1\right|$$
$$= \left|e^{i\theta} - 1\right|$$
$$= \left|\cos\theta + i \sin\theta - 1\right|$$
$$= \left|\cos\theta - 1 + i \sin\theta\right|$$

$$
\begin{aligned}
&= \left| -2\sin^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right| \\
&= \left| 2\sin\frac{\theta}{2} \right| \left| -\sin\frac{\theta}{2} + i\cos\frac{\theta}{2} \right| \\
&= \left| 2\sin\frac{\theta}{2} \right| \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}} \\
&= 2\left|\sin\frac{\theta}{2}\right| \\
&\le 2\left|\frac{\theta}{2}\right| \qquad \left[\because \frac{\sin\theta}{\theta} \le 1\right] \\
&\le |\theta| \\
&\le |\arg(z)|
\end{aligned}
$$

Example 18

If $\sin\alpha = i\tan\theta$, *prove that* $\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$.

Solution

$$
\begin{aligned}
i\tan\theta &= \sin\alpha \\
\frac{i\sin\theta}{\cos\theta} &= \frac{\sin\alpha}{1}
\end{aligned}
$$

Applying componendo and dividendo,

$$
\begin{aligned}
\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} &= \frac{1+\sin\alpha}{1-\sin\alpha} \\
\frac{e^{i\theta}}{e^{-i\theta}} &= \frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)}{1-\cos\left(\frac{\pi}{2}-\alpha\right)} \\
e^{2i\theta} &= \frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} \\
(e^{i\theta})^2 &= \left[\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= \cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) \\
&= \tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right] \\
&= \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \\
\cos\theta + i\sin\theta &= \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)
\end{aligned}
$$

Example 19

Prove that $\tan^{-1} z = \dfrac{i}{2}\log\left(\dfrac{i+z}{i-z}\right)$. **[Winter 2013]**

Solution

Let

$$
\begin{aligned}
\tan^{-1} z &= \theta \\
z &= \tan\theta \\
&= \frac{1}{i}\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right) \\
&= -i\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right) \\
\frac{z}{i} &= \frac{-e^{i\theta}+e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}
\end{aligned}
$$

Applying componendo and dividendo,

$$
\begin{aligned}
\frac{i+z}{i-z} &= \frac{2e^{-i\theta}}{2e^{i\theta}} \\
e^{-2i\theta} &= \frac{i+z}{i-z} \\
-2i\theta &= \log\left(\frac{i+z}{i-z}\right) \\
\theta &= -\frac{1}{2i}\log\left(\frac{i+z}{i-z}\right) \\
\tan^{-1} z &= \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)
\end{aligned}
$$

Example 20

Prove that $(1-e^{i\theta})^{-\frac{1}{2}}+(1-e^{-i\theta})^{-\frac{1}{2}}=\left(1+\operatorname{cosec}\frac{\theta}{2}\right)^{\frac{1}{2}}$.

Solution

$$(1-e^{i\theta})^{-\frac{1}{2}}+(1-e^{-i\theta})^{-\frac{1}{2}}=(1-\cos\theta-i\sin\theta)^{-\frac{1}{2}}+(1-\cos\theta+i\sin\theta)^{-\frac{1}{2}}$$

$$=\left(2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}+\left(2\sin^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}+\left(\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\right]$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left\{\cos\left(\frac{\theta}{2}-\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}-\frac{\theta}{2}\right)\right\}^{-\frac{1}{2}}+\left\{\cos\left(\frac{\theta}{2}-\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}-\frac{\theta}{2}\right)\right\}^{-\frac{1}{2}}\right]$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[\left\{e^{-i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right\}^{-\frac{1}{2}}+\left\{e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right\}^{-\frac{1}{2}}\right]$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[e^{i\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}+e^{-i\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}\right]$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[2\cos\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\right]$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[4\cos^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\right]^{\frac{1}{2}}$$

$$=\left(2\sin\frac{\theta}{2}\right)^{-\frac{1}{2}}\left[2\left\{1+\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]^{\frac{1}{2}} \qquad \left[\because\quad 1+\cos\theta=2\cos^2\frac{\theta}{2}\right]$$

$$=\left[\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right]^{\frac{1}{2}}$$

$$=\left[\operatorname{cosec}\frac{\theta}{2}+1\right]^{\frac{1}{2}}$$

Example 21

If $\alpha = i + 1,\ \beta = 1 - i$ *and* $\tan\phi = \dfrac{1}{x+1}$ *then prove that*

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \sin n\phi \operatorname{cosec}^n \phi.$$

Solution

$$\alpha = i + 1, \quad \beta = 1 - i, \quad \tan\phi = \frac{1}{x+1}$$

$$\cot\phi = x + 1,\ x = \cot\phi - 1$$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{(\cot\phi - 1 + i + 1)^n - (\cot\phi - 1 + 1 - i)^n}{i + 1 - 1 + i}$$

$$= \frac{\left(\dfrac{\cos\phi}{\sin\phi} + i\right)^n - \left(\dfrac{\cos\phi}{\sin\phi} - i\right)^n}{2i}$$

$$= \frac{(\cos\phi + i\sin\phi)^n - (\cos\phi - i\sin\phi)^n}{2i\sin^n\phi}$$

$$= \frac{(e^{i\phi})^n - (e^{-i\phi})^n}{2i\sin^n\phi}$$

$$= \frac{e^{in\phi} - e^{-in\phi}}{2i\sin^n\phi}$$

$$= \frac{2i\sin n\phi}{2i\sin^n\phi}$$

$$= \sin n\phi \ \operatorname{cosec}^n\phi$$

Example 22

If $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$, *prove that*

(i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\cdots(a_n^2 + b_n^2) = A^2 + B^2$

(ii) $\tan^{-1}\left(\dfrac{b_1}{a_1}\right) + \tan^{-1}\left(\dfrac{b_2}{a_2}\right) + \cdots + \tan^{-1}\left(\dfrac{b_n}{a_n}\right) = \tan^{-1}\left(\dfrac{B}{A}\right).$

Solution

$$(a_1 + ib_1)\,(a_2 + ib_2)\,\ldots\,(a_n + ib_n) = A + iB \qquad \ldots (1)$$

(i) Taking the modulus of Eq. (1),

$$\left|(a_1 + ib_1)(a_2 + ib_2)\cdots(a_n + ib_n)\right| = \left|A + iB\right|$$

$$\left|a_1 + ib_1\right|\left|a_2 + ib_2\right|\cdots\left|a_n + ib_n\right| = \left|A + iB\right|$$

$$\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}\cdots\sqrt{a_n^2 + b_n^2} = \sqrt{A^2 + B^2}$$

Squaring both the sides,

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)\cdots(a_n^2 + b_n^2) = A^2 + B^2$$

(ii) Taking the argument of Eq. (1),

$$\arg\,[(a_1 + ib_1)\,(a_2 + ib_2)\cdots(a_n + ib_n)] = \arg\,(A + iB)$$

$$\arg\,(a_1 + ib_1) + \arg\,(a_2 + ib_2) + \cdots + \arg\,(a_n + ib_n) = \arg\,(A + iB)$$

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \cdots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

Example 23

If $x + iy = \sqrt[3]{a + ib}$, *prove that* $\dfrac{a}{x} + \dfrac{b}{y} = 4(x^2 - y^2)$.

Solution

$$x + iy = \sqrt[3]{a + ib}$$

$$(a + ib)^{\frac{1}{3}} = x + iy$$

$$a + ib = (x + iy)^3$$

$$= x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3) \qquad [\because\ i^3 = -i]$$

Comparing real and imaginary parts,

$$a = x^3 - 3xy^2 \quad \text{and} \quad b = 3x^2y - y^3$$

$$\frac{a}{x} = x^2 - 3y^2 \quad \text{and} \quad \frac{b}{y} = 3x^2 - y^2$$

Hence, $\dfrac{a}{x} + \dfrac{b}{y} = 4(x^2 - y^2)$

Example 24

If $(1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta) = u + iv$, *prove that*

(i) $u^2 + v^2 = 16\cos^2\dfrac{\theta}{2}\cos^2\theta$ *(ii)* $\dfrac{v}{u} = \tan\dfrac{3\theta}{2}$.

Solution

$$\begin{aligned} u + iv &= (1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta) \\ &= \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)(2\cos^2\theta + i2\sin\theta\cos\theta) \\ &= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)2\cos\theta(\cos\theta + i\sin\theta) \\ &= 4\cos\frac{\theta}{2}\cos\theta\cdot e^{\frac{i\theta}{2}}\cdot e^{i\theta} \\ &= 4\cos\frac{\theta}{2}\cos\theta\cdot e^{\frac{i3\theta}{2}} \\ &= re^{i\theta}, \text{ say} \end{aligned}$$

where $r = |u + iv| = 4\cos\dfrac{\theta}{2}\cos\theta$

$$\sqrt{u^2 + v^2} = 4\cos\frac{\theta}{2}\cos\theta$$

$$u^2 + v^2 = 16\cos^2\frac{\theta}{2}\cos^2\theta$$

and $\phi = \arg(u + iv) = \dfrac{3\theta}{2}$

$$\tan^{-1}\left(\frac{v}{u}\right) = \frac{3\theta}{2}$$

$$\frac{v}{u} = \tan\left(\frac{3\theta}{2}\right)$$

Example 25

If $x_r = \cos\left(\dfrac{\pi}{2^r}\right) + i\sin\left(\dfrac{\pi}{2^r}\right)$, *show that* $\lim\limits_{n\to\infty} x_1\, x_2\, x_3 \cdots x_n = -1$.

Solution

$$x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$$

$$= e^{\frac{i\pi}{2^r}}$$

$$\lim_{n\to\infty} x_1\, x_2\, x_3 \cdots x_n = \lim_{n\to\infty} e^{\frac{i\pi}{2}}\, e^{\frac{i\pi}{2^2}}\, e^{\frac{i\pi}{2^3}} \cdots e^{\frac{i\pi}{2^n}}$$

$$= \lim_{n\to\infty} e^{i\pi\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}\right)}$$

$$\lim_{n\to\infty} x_1\, x_2\, x_3 \cdots x_n = \lim_{n\to\infty} e^{i\pi\left[1-\left(\frac{1}{2}\right)^n\right]} \qquad \text{[Using the sum of GP]}$$

$$= \lim_{n\to\infty} e^{i\pi} e^{-i\pi\left(\frac{1}{2^n}\right)}$$

$$= e^{i\pi} e^{-\frac{i\pi}{2^\infty}}$$

$$= (\cos\pi + i\sin\pi)\, e^{-\frac{i\pi}{\infty}}$$

$$= (-1 + i\cdot 0)\, e^0$$

Hence, $\lim_{n\to\infty} x_1\, x_2\, x_3 \ldots\, x_n = -1$

EXERCISE 1.1

1. Find the modulus and principal value of the argument:

(i) $-\sqrt{3} - i$

(ii) $\dfrac{1+2i}{1-3i}$

(iii) $\dfrac{1+2i}{1-(1-i)^2}$

(iv) $\sqrt{\dfrac{1+i}{1-i}}$

(v) $\tan\alpha - i$

$$\left[\text{Ans.: (i) } 2,\ \frac{-5\pi}{6}\quad \text{(ii) } \frac{1}{\sqrt{2}},\ \frac{3\pi}{4}\quad \text{(iii) } 1,\ 0\qquad \text{(iv) } 1,\ \frac{\pi}{4}\right.$$

$$\left.\text{(v) } \sec\alpha,\ \alpha - \frac{\pi}{2}\right]$$

2. Express in polar form:

(i) $\sqrt{3}-i$

(ii) $\dfrac{1+i}{1-i}$

(iii) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

$\left[\textbf{Ans.:}\ \text{(i)}\ 2\left(\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}\right)\ \text{(ii)}\ \cos\dfrac{\pi}{2}-i\sin\dfrac{\pi}{2}\ \text{(iii)}\ 2\left(\cos\dfrac{\pi}{3}-i\sin\dfrac{\pi}{3}\right)\right]$

3. Find the value of $\sqrt{3-4i}$.

[Ans. : $2 - i$ or $-2 + i$**]**

4. Find the locus of z, if $\dfrac{z-1}{z+i}$ is purely imaginary.

[Ans. : circle $x^2 + y^2 - x - y = 0$**]**

5. Find the locus of z if

(i) $\left|\dfrac{z-1}{z+1}\right|=1$ (ii) $\arg\left(\dfrac{z-1}{z+1}\right)=\dfrac{\pi}{4}$

$\left[\textbf{Ans.:}\ \text{(i)}\ x=0\ \text{(ii) circle}\ x^2+y^2-2y-1=0\right]$

6. Find two numbers whose sum is 4 and product is 8.

[Ans. : $2 \pm 2i$**]**

7. If $x + iy = \sqrt{a+ib}$, prove that $(x^2 + y^2)^2 = a^2 + b^2$.

8. If $|z| = 1$, $z \neq 1$, prove that $\dfrac{z-1}{z+1}$ is purely imaginary.

9. If $a = e^{i2\alpha}$, $b = e^{i2\beta}$, $c = e^{i2\gamma}$ then prove that $\sqrt{\dfrac{ab}{c}}+\sqrt{\dfrac{c}{ab}}=2\cos(\alpha+\beta-\gamma)$.

10. If $(\alpha+i\beta)=\dfrac{1}{a+ib}$, prove that $(\alpha^2 + \beta^2)\ (a^2 + b^2) = 1$.

11. Find the value of $x^2 - 6x + 13$, when $x = 3 + 2i$

1.8 DE MOIVRE'S THEOREM

Statement For any real number n, one of the values of $(\cos\ \theta + i\ \sin\ \theta)^n$ is $\cos n\theta + i \sin n\theta$.

Hence, $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

Proof

Case I: n is a positive integer

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, …, $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Similarly,

$$z_1 z_2 \ldots z_n = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)\ldots r_n(\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 \ldots r_n)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 \ldots r_n)[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)] \quad \ldots (1.1)$$

If $z_1 = z_2 = \ldots = z_n = z = r(\cos\theta + i\sin\theta)$ then Eq. (1.1) reduces to

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta), \text{ where } n \text{ is a positive integer.}$$

Case II: n is a negative integer

Let $n = -m$, where m is a positive integer.

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \qquad \text{[Using Case I]}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta \quad [\because\ \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$$

$$= \cos n\theta + i\sin n\theta, \quad \text{where } n \text{ is a negetive integer.}$$

Case III: n is a rational number

Let $n = \dfrac{p}{q}$, where p and q are integers and $q \neq 0$.

$$\text{Consider}\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q\cdot\frac{\theta}{q}\right) + i\sin\left(q\cdot\frac{\theta}{q}\right) \qquad \text{[Using Case I and Case II]}$$

$$= \cos\theta + i\sin\theta$$

Hence,
$$\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}=(\cos\theta+i\sin\theta)^{\frac{1}{q}}$$
$$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^{p}=(\cos\theta+i\sin\theta)^{\frac{p}{q}}$$
$$\cos\left(p\cdot\frac{\theta}{q}\right)+i\sin\left(p\cdot\frac{\theta}{q}\right)=(\cos\theta+i\sin\theta)^{\frac{p}{q}} \qquad \text{[Using Case I and Case II]}$$
$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^{n}$$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, where n is a rational number.

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for any real number n.

Example 1

Prove that $\left(\dfrac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^{n}=\cos\left(\dfrac{n\pi}{2}-n\alpha\right)+i\sin\left(\dfrac{n\pi}{2}-n\alpha\right).$

Solution

$$\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^{n}=\left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)-i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^{n}$$
$$=\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^{n}$$
$$=\left[\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^{n}$$
$$=\left[\frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}\right]^{n}$$
$$=\left[e^{2i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^{n}$$

$$= e^{i\left(\frac{n\pi}{2} - n\alpha\right)}$$
$$= \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$$

Example 2

If $x + \dfrac{1}{x} = 2\cos\theta$, *prove that*

(i) $x^n + \dfrac{1}{x^n} = 2\cos n\theta$ *(ii)* $\dfrac{x^{2n}+1}{x^{2n-1}+x} = \dfrac{\cos n\theta}{\cos(n-1)\theta}$

[Summer 2015]

Solution

$$x + x^{-1} = 2\cos\theta$$
$$= e^{i\theta} + e^{-i\theta} \qquad \text{[Using Euler's formula]}$$
$$\therefore \quad x = e^{i\theta}$$

(i) $$x^n + \frac{1}{x^n} = \left(e^{i\theta}\right)^n + \frac{1}{\left(e^{i\theta}\right)^n}$$
$$= e^{in\theta} + e^{-in\theta}$$
$$= 2\cos n\theta$$

(ii) $$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\left(e^{i\theta}\right)^{2n}+1}{\left(e^{i\theta}\right)^{2n-1}+e^{i\theta}}$$
$$= \left(\frac{e^{i2n\theta}+1}{e^{i(2n-1)\theta}+e^{i\theta}}\right)\left(\frac{e^{-in\theta}}{e^{-in\theta}}\right)$$
$$= \frac{e^{in\theta}+e^{-in\theta}}{e^{i(n-1)\theta}+e^{-i(n-1)\theta}}$$
$$= \frac{2\cos n\theta}{2\cos(n-1)\theta}$$
$$= \frac{\cos n\theta}{\cos(n-1)\theta}$$

Example 3

If $x+\dfrac{1}{x}=2\cos\theta,\ y+\dfrac{1}{y}=2\cos\phi,\ z+\dfrac{1}{z}=2\cos\psi$ *then prove that*

$(i)\quad xyz+\dfrac{1}{xyz}=2\cos(\theta+\phi+\psi)$

$(ii)\quad \dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}=2\cos(m\theta-n\phi)$

$(iii)\quad x^m y^n+\dfrac{1}{x^m y^n}=2\cos(m\theta+n\phi)$

Solution

$$x+\frac{1}{x}=2\cos\theta$$

$$x+x^{-1}=e^{i\theta}+e^{-i\theta} \qquad \text{[Using Euler's formula]}$$

$$\therefore x=e^{i\theta}$$

(i) $$xyz+\frac{1}{xyz}=e^{i\theta}e^{i\phi}e^{i\psi}+\frac{1}{e^{i\theta}e^{i\phi}e^{i\psi}}$$

$$=e^{i(\theta+\phi+\psi)}+e^{-i(\theta+\phi+\psi)}$$

$$=2\cos(\theta+\phi+\psi)$$

(ii) $$\frac{x^m}{y^n}+\frac{y^n}{x^m}=\frac{(e^{i\theta})^m}{(e^{i\phi})^n}+\frac{(e^{i\phi})^n}{(e^{i\theta})^m}$$

$$=e^{i(m\theta-n\phi)}+e^{-i(m\theta-n\phi)}$$

$$=2\cos(m\theta-n\phi)$$

(iii) $$x^m y^n+\frac{1}{x^m y^n}=(e^{i\theta})^m(e^{i\phi})^n+\frac{1}{(e^{i\theta})^m(e^{i\phi})^n}$$

$$=e^{i(m\theta+n\phi)}+e^{-i(m\theta+n\phi)}$$

$$=2\cos(m\theta+n\phi)$$

Example 4

If $x-\frac{1}{x}=2i\sin\theta, y-\frac{1}{y}=2i\sin\phi, z-\frac{1}{z}=2i\sin\psi$, *prove that*

(i) $xyz+\frac{1}{xyz}=2\cos(\pi+\theta+\psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}}+\frac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$.

Solution

$$x-\frac{1}{x}=2i\sin\theta,\ y-\frac{1}{y}=2i\sin\phi,\ z-\frac{1}{z}=2i\sin\psi$$

$$x-\frac{1}{x}=2i\sin\theta$$

$$x-x^{-1}=e^{i\theta}-e^{-i\theta} \qquad \text{[Using Euler's formula]}$$

$$\therefore\quad x=e^{i\theta}$$

Similarly, $y=e^{i\phi}, z=e^{i\psi}$

(i) $$xzy+\frac{1}{xyz}=e^{i\theta}e^{i\phi}e^{i\psi}+\frac{1}{e^{i\theta}e^{i\phi}e^{i\psi}}$$

$$=e^{i(\theta+\phi+\psi)}+e^{-i(\theta+\phi+\psi)}$$

$$=2\cos(\theta+\phi+\psi)$$

(ii) $$\frac{\sqrt[m]{x}}{\sqrt[n]{y}}+\frac{\sqrt[n]{y}}{\sqrt[m]{x}}=\frac{(e^{i\theta})^{\frac{1}{m}}}{(e^{i\phi})^{\frac{1}{n}}}+\frac{(e^{i\phi})^{\frac{1}{n}}}{(e^{i\theta})^{\frac{1}{m}}}$$

$$=e^{i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)}+e^{-i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)}$$

$$=2\cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$$

Example 5

Prove that $(4n)^{th}$ *power of* $\dfrac{1+7i}{(2-i)^2}$ *is equal to* $(-4)^n$*, where n is a positive integer.*

Solution

$$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i}$$

$$= \frac{1+7i}{4-1-4i}$$

$$= \left(\frac{1+7i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right)$$

$$= \frac{3+4i+21i+28i^2}{9+16}$$

$$= \frac{3+25i-28}{25}$$

$$= -1+i$$

Let $-1 + i = r(\cos\theta + i\sin\theta)$

where $r = |-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

[$\because$ the point (–1,1) lies in the second quadrant]

$\therefore$ $$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Now, $$\left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-1+i)^{4n}$$

$$= \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^{4n}$$

$$= \left(\sqrt{2}\right)^{4n}\left[\cos\left(4n\frac{3\pi}{4}\right) + i\sin\left(4n\frac{3\pi}{4}\right)\right]$$

$$= (2)^{2n}(\cos 3n\pi + i\sin 3n\pi)$$

$$= (4)^n\left[(-1)^{3n} + 0\right]$$

$$= \left[4(-1)^3\right]^n$$

$$= (-4)^n$$

Example 6

If α and β are the roots of the equation $x^2 - 2x + 2 = 0$, find the value of $\alpha^n + \beta^n$. Hence, deduce that $\alpha^8 + \beta^8 = 32$. **[Summer 2015]**

Solution

$$x^2 - 2x + 2 = 0$$

$$x = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm \sqrt{-1}$$

$$= 1 \pm i$$

Since α and β are the roots of the equation,

$$\alpha = 1 + i, \quad \beta = 1 - i$$

Let $1 + i = r(\cos\theta + i \sin\theta)$

$$r = |1 + i| = \sqrt{1+1} = \sqrt{2}$$

and $$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$\therefore$ $$\alpha = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{\frac{i\pi}{4}}$$

and $$\beta = 1 - i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{-\frac{i\pi}{4}}$$

$$\alpha^n + \beta^n = \left(\sqrt{2}\, e^{\frac{i\pi}{4}}\right)^n + \left(\sqrt{2}\, e^{-\frac{i\pi}{4}}\right)^n$$

$$= \left(\sqrt{2}\right)^n \left(e^{\frac{in\pi}{4}} + e^{-\frac{in\pi}{4}}\right)$$

$$= 2^{\frac{n}{2}} \left(2\cos\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}+1} \cos\left(\frac{n\pi}{4}\right)$$

Putting $n = 8$,

$$\alpha^8 + \beta^8 = 2^5 \cos\frac{8\pi}{4}$$

$$= 2^5 \cos 2\pi$$

$$= 2^5$$

$$= 32$$

Example 7

If α and β are the roots of the equation $x^2 - 2x + 4 = 0$ then show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$ and, hence, find the value of $\alpha^{15} + \beta^{15}$.

Solution

$$x^2 - 2x + 4 = 0$$

$$x = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= 1 \pm \sqrt{-3}$$

$$= 1 \pm i\sqrt{3}$$

α and β are the roots of the equation.

$\therefore$ $\alpha = 1 + i\sqrt{3}$, $\beta = 1 - i\sqrt{3}$

Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

where $r = \left|1 + i\sqrt{3}\right| = \sqrt{1+3} = \sqrt{4} = 2$

and $\theta = \arg\left(1 + i\sqrt{3}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

$\therefore$ $\alpha = 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{\frac{i\alpha}{3}}$

and $\beta = 1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2e^{-\frac{i\pi}{3}}$

$$\alpha^n + \beta^n = \left(2e^{\frac{i\pi}{3}}\right)^n + \left(2e^{-\frac{i\pi}{3}}\right)^n$$

$$= 2^n\left(e^{\frac{in\pi}{3}} + e^{-\frac{in\pi}{3}}\right)$$

$$= 2^n \cdot 2\cos\left(\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

Putting $n = 15$,

$$\alpha^{15} + \beta^{15} = 2^{16}\cos\left(\frac{15\pi}{3}\right)$$

$$= 2^{16}\cos 5\pi$$

$$= 2^{16}(-1)$$

$$= -2^{16}$$

Example 8

If α *and* β *are roots of* $z^2 \sin^2 \theta - z \sin 2\theta + 1 = 0$, *prove that* $\alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$, *where n is a positive integer.*

Solution

$$z^2 \sin^2 \theta - z \sin 2\theta + 1 = 0$$

$$z = \frac{\sin 2\theta \pm \sqrt{\sin^2 2\theta - 4\sin^2 \theta}}{2\sin^2 \theta}$$

$$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{\sin\theta}$$

$$= \frac{\cos\theta \pm i\sin\theta}{\sin\theta}$$

$$= e^{\pm i\theta} \operatorname{cosec}\theta$$

α and β are the roots of the equation.

$\therefore$ $\alpha = e^{i\theta} \operatorname{cosec}\theta, \qquad \beta = e^{-i\theta} \operatorname{cosec}\theta$

$$\alpha^n + \beta^n = (e^{i\theta} \operatorname{cosec}\theta)^n + (e^{-i\theta} \operatorname{cosec}\theta)^n$$

$$= (\operatorname{cosec}\theta)^n (e^{in\theta} + e^{-in\theta})$$

$$= (\operatorname{cosec}\theta)^n (2\cos n\theta)$$

$$= 2\cos n\theta \operatorname{cosec}^n \theta$$

Example 9

If $z = -1 + i\sqrt{3}$ *and n is an integer then prove that* $z^{2n} + 2^n z^n + 2^{2n} = 0$, *if n is not a multiple of* 3.

Solution

$$z = -1 + i\sqrt{3}$$

Let $-1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

where $r = \left|-1 + i\sqrt{3}\right| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

and $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}$

[$\because$ the point $(-1, \sqrt{3})$ lies in the second quadrant]

$$\therefore \qquad z = -1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{\frac{2i\pi}{3}}$$

Consider $$\frac{z^n}{2^n} + \frac{2^n}{z^n} = \frac{\left(2e^{\frac{2i\pi}{3}}\right)^n}{2^n} + \frac{2^n}{\left(2e^{\frac{2i\pi}{3}}\right)^n}$$

$$= e^{\frac{2in\pi}{3}} + e^{-\frac{2in\pi}{3}}$$

$$= 2\cos\frac{2n\pi}{3}$$

Let $n = 3k + 1$ where k is an integer then

$$\frac{z^n}{2^n} + \frac{2^n}{z^n} = 2\cos\frac{2\pi}{3}(3k+1)$$

$$= 2\cos\left(2k\pi + \frac{2\pi}{3}\right)$$

$$= 2\cos\frac{2\pi}{3} \qquad [\because \cos(2k\pi + \theta) = \cos\theta]$$

$$= 2\left(-\frac{1}{2}\right)$$

$$= -1$$

$$\frac{z^n}{2^n} + \frac{2^n}{z^n} = -1$$

$z^{2n} + 2^n z^n + 2^{2n} = 0$, if n is not a multiple of 3.

EXERCISE 1.2

1. If $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ then by using De Moivre's theorem, simplify $(z)^{10} + (\bar{z})^{10}$, where $\bar{z}$ is the complex conjugate of z.

[Ans.: 0]

2. If n is a positive integer, show that $(a + ib)^n + (a - ib)^n = 2\, r^n \cos n\theta$, where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Hence, or otherwise, deduce that $\left(1+i\sqrt{3}\right)^8 + \left(1-i\sqrt{3}\right)^8 = -2^8$.

3. If $x+\frac{1}{x}=2\cos\theta, y+\frac{1}{y}=2\cos\phi,$ prove that $x^r y^r+\frac{1}{x^r y^r}=2\cos(r\theta+r\phi)$.

4. If $2\cos\theta = x+\frac{1}{x}$, prove that $\frac{x^{2n}+1}{x^{2n-1}+x}=\frac{\cos n\theta}{\cos(n-1)\theta}$.

5. If $\sin\alpha+\sin\beta=0=\cos\alpha+\cos\beta$, prove that

(i) $\cos 2\alpha+\cos 2\beta=2\cos(\pi+\alpha+\beta)$

(ii) $\sin 2\alpha+\sin 2\beta=2\sin(\pi+\alpha+\beta)$

6. If α and β are the roots of the equation $x^2-2\sqrt{3}x+4=0$ then prove that $\alpha^3+\beta^3=0$.

7. If α, β are the roots of the equation $x^2-2x+2=0$, prove that $\alpha^n+\beta^n = 2\cdot 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$.

Hence, show that $\alpha^8+\beta^8=32$.

8. If α, β are the roots of the equation $x^2-\sqrt{3}x+1=0$, prove that $\alpha^n+\beta^n = 2\cos\frac{n\pi}{6}$. Hence, show that $\alpha^{12}+\beta^{12}=2$.

1.9 APPLICATIONS OF DE MOIVRE'S THEOREM

1.9.1 Roots of an Algebraic Equations

De Moivre's theorem can be used to find the roots of an algebraic equation.

Let the equation be

$$z^n=\cos\theta+i\sin\theta$$

$$z=(\cos\theta+i\sin\theta)^{\frac{1}{n}}$$

$$=\left[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)\right]^{\frac{1}{n}}$$

$$=\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Putting $k = 0, 1, 2, \ldots n-1$, all n roots of the equation are obtained.

Notes

(i) Complex roots always occur in conjugate pairs if all the coefficients of the equation including constant terms are real.

(ii) The product of all the roots of the equation is known as the *continued product*.

Example 1

Find all the values of the following:

$$(i)\ (-1)^{\frac{1}{5}} \qquad (ii)\ (1-i)^{\frac{2}{3}} \qquad (iii)\ \sqrt[3]{\frac{1+i}{\sqrt{2}}} + \sqrt[3]{\frac{1-i}{\sqrt{2}}}.$$

Solution

(i) Let $-1 = r(\cos\theta + i\sin\theta)$

where $r = |-1| = 1$

and $\theta = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1} 0 = \pi$ [$\because$ the point (–1,0) lies in the second quadrant]

$$-1 = \cos\pi + i\sin\pi$$
$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$(-1)^{\frac{1}{5}} = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{\frac{1}{5}}$$
$$= \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, all five values of $(-1)^{\frac{1}{5}}$ are obtained.

(ii) Let $1 - i = r(\cos\theta + i\sin\theta)$

where $r = |1-i| = \sqrt{1+(-1)^2} = \sqrt{2}$

and $\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$ [$\because$ the point (1, –1) lies in the fourth quadrant]

$$1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1-i)^{\frac{2}{3}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^{\frac{2}{3}}$$
$$= \left[\left(\sqrt{2}\right)^2\left(\cos\frac{2\pi}{4} - i\sin\frac{2\pi}{4}\right)\right]^{\frac{1}{3}}$$
$$= \left[2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$= \left[2\left\{ \cos\left(2k\pi + \frac{\pi}{2} \right) - i \sin\left(2k\pi + \frac{\pi}{2} \right) \right\} \right]^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}} \left[\cos(4k+1)\frac{\pi}{6} - i \sin(4k+1)\frac{\pi}{6} \right] \text{ [Using De Moivre's theorem]}$$

Putting $k = 0, 1, 2$, all three values of $(1-i)^{\frac{2}{3}}$ are obtained.

(iii) Let $\dfrac{1+i}{\sqrt{2}} = r(\cos\theta + i \sin\theta)$

$$r = \left| \frac{1+i}{\sqrt{2}} \right| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

and $\theta = \tan^{-1}(1) = \dfrac{\pi}{4}$

$$\frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}$$

and $$\frac{1-i}{\sqrt{2}} = \cos\frac{\pi}{4} - i \sin\frac{\pi}{4}$$

$$\sqrt[3]{\frac{1+i}{\sqrt{2}}} + \sqrt[3]{\frac{1-i}{\sqrt{2}}} = \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)^{\frac{1}{3}} + \left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4} \right)^{\frac{1}{3}}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4} \right) + i \sin\left(2k\pi + \frac{\pi}{4} \right) \right]^{\frac{1}{3}}$$

$$+ \left[\cos\left(2k\pi + \frac{\pi}{4} \right) - i \sin\left(2k\pi + \frac{\pi}{4} \right) \right]^{\frac{1}{3}}$$

$$= \left[\cos(8k+1)\frac{\pi}{12} + i \sin(8k+1)\frac{\pi}{12} \right]$$

$$+ \left[\cos(8k+1)\frac{\pi}{12} - i \sin(8k+1)\frac{\pi}{12} \right] \quad \text{[Using De Moivre's theorem]}$$

$$= 2\cos(8k+1)\frac{\pi}{12}$$

Putting $k = 0, 1, 2$, all three values of $\sqrt[3]{\dfrac{1+i}{\sqrt{2}}} + \sqrt[3]{\dfrac{1-i}{\sqrt{2}}}$ are obtained.

Example 2

Find the continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$.

Solution

Let
$$z=\frac{1}{2}+i\frac{\sqrt{3}}{2}$$
$$=r(\cos\theta+i\sin\theta)$$

where
$$r=\left|\frac{1}{2}+i\frac{\sqrt{3}}{2}\right|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

and
$$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$$

$$\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$$

$$=\left(\cos 3\cdot\frac{\pi}{3}+i\sin 3\cdot\frac{\pi}{3}\right)^{\frac{1}{4}}$$

$$=(\cos\pi+i\sin\pi)^{\frac{1}{4}}$$

$$=\left[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\right]^{\frac{1}{4}}$$

$$=\left[\cos(2k+1)\frac{\pi}{4}+i\sin(2k+1)\frac{\pi}{4}\right]$$

Putting $k=0, 1, 2, 3,$

$$z_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=e^{i\frac{\pi}{4}}$$

$$z_1=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=e^{i\frac{3\pi}{4}}$$

$$z_2=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=e^{i\frac{5\pi}{4}}$$

$$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = e^{i\frac{7\pi}{4}}$$

The continued product is

$$\begin{aligned} z_0 z_1 z_2 z_3 &= e^{\frac{i\pi}{4}} \cdot e^{i\frac{3\pi}{4}} \cdot e^{i\frac{5\pi}{4}} \cdot e^{i\frac{7\pi}{4}} \\ &= e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)} \\ &= e^{i\frac{16\pi}{4}} \\ &= e^{i4\pi} \\ &= \cos 4\pi + i\sin 4\pi \\ &= 1 + i\cdot 0 \end{aligned}$$

Hence, the continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ is 1.

Example 3

Find all roots of the equation $\log z = \frac{i\pi}{2}$. **[Winter 2012]**

Solution

$$\begin{aligned} \log z &= \frac{i\pi}{2} \\ z &= e^{\frac{i\pi}{2}} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right) \end{aligned}$$

where $k = 0, 1, 2, 3, \ldots$

Example 4

Show that the n^{th} *roots of unity form a geometric progression with the common ratio* $\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$ *and show that the continued product of all* n^{th} *roots is* $(-1)^{n+1}$. **[Summer 2015, 2014]**

Solution

To find n^{th} roots of unity, consider the equation

$$x^n = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi$$

$$x = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}$$

Putting $k = 0, 1, 2, \ldots n - 1$, all n roots are

$$x_0 = \cos 0 + i \sin 0 = 1$$

$$x_1 = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n} = e^{\frac{2i\pi}{n}} = \omega, \text{say}$$

$$x_2 = \cos\frac{4\pi}{n} + i \sin\frac{4\pi}{n} = e^{\frac{4i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^2 = \omega^2$$

$$x_3 = \cos\frac{6\pi}{n} + i \sin\frac{6\pi}{n} = e^{\frac{6i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^3 = \omega^3$$

.....

.....

$$x_{n-1} = \cos\frac{2(n-1)\pi}{n} + i \sin\frac{2(n-1)\pi}{n} = e^{\frac{2(n-1)i\pi}{n}} = \left(e^{\frac{2i\pi}{n}}\right)^{n-1} = \omega^{n-1}$$

Hence, 1, ω, ω^2, $\omega^3 \ldots$, ω^{n-1} represent n roots of unity which are in geometric progression with the common ratio $\omega = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

The continued product of roots is

$$\begin{aligned}
x_0 \cdot x_1 \cdot x_2 \cdots x_{n-1} &= 1 \cdot e^{\frac{2i\pi}{n}} \cdot e^{\frac{4i\pi}{n}} \cdot e^{\frac{6i\pi}{n}} \cdots e^{\frac{2(n-1)i\pi}{n}} \\
&= e^{\frac{2i\pi}{n}\{1+2+3+\cdots+(n-1)\}} \\
&= e^{\frac{2i\pi}{n} \cdot \frac{n(n-1)}{2}} \qquad \text{[Using the sum of AP]} \\
&= e^{i\pi(n-1)} \\
&= e^{in\pi} \cdot e^{-i\pi} \\
&= (\cos n\pi + i \sin n\pi)(\cos \pi - i \sin \pi) \\
&= (-1)^n (-1) \\
&= (-1)^{n+1}
\end{aligned}$$

Hence, the continued product of all n^{th} roots of unity is $(-1)^{n+1}$.

Example 5

Find and graph all sixth roots of unity in the complex plane.

[Winter 2013]

Solution

Let $x^6 = 1$

$$= \cos 0 + i \sin 0$$

$$= \cos 2k\pi + i \sin 2k\pi$$

$$x = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{6}}$$

$$= \cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6}$$

$$= \cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3}$$

Putting $k = 0, 1, 2, 3, 4, 5$, all the root are obtained (Fig. 1.3).

$$x_0 = \cos 0 + i \sin 0 = 1$$

$$x_1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = e^{\frac{i\pi}{3}} = \omega$$

$$x_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = e^{\frac{2i\pi}{3}} = \omega^2$$

$$x_3 = \cos \pi + i \sin \pi = -1$$

$$x_4 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = e^{\frac{4i\pi}{3}} = \omega^4$$

$$x_5 = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = e^{\frac{5i\pi}{3}} = \omega^5$$

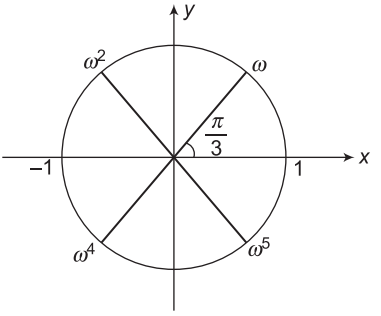


Fig. 1.3

Example 6

If ω is a 7th root of unity then prove that $S = 1 + \omega + \omega^{2m} + \omega^{3m} + \omega^{4m} + \omega^{5m} + \omega^{6m} = 7$ if m is a multiple of 7 and is 0 otherwise.

Solution

Putting $n = 7$ in Example 4, the 7th roots of unity are obtained as $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$.

where $\omega = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} = e^{i\frac{2\pi}{7}}$

$$S = 1 + \omega^m + \omega^{2m} + \omega^{3m} + \cdots + \omega^{6m}$$
$$= \frac{1[1-(\omega^m)^7]}{1-\omega^m}$$
$$= \frac{1-\omega^{7m}}{1-\omega^m} \qquad \text{[Using the Sum of GP]}$$
$$= \frac{1-(e^{2i\pi})^m}{1-\left(e^{\frac{2i\pi}{7}}\right)^m}$$

Putting $m = 7k$, where k is an integer

$$S = 1 + \omega^{7k} + (\omega^2)^{7k} + (\omega^3)^{7k} + \cdots + (\omega^6)^{7k}$$
$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + \cdots + (\omega^7)^{6k}$$
$$= 1 + (1)^k + (1)^{2k} + (1)^{3k} + \cdots + (1)^{6k} \qquad [\because \quad \omega^7 = \cos 2\pi + i \sin 2\pi = 1]$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1$$
$$= 7$$

Putting $m = 7k + 1$, where k is an integer

$$S = \frac{1-\left(e^{2i\pi}\right)^{7k+1}}{1-e^{i\left(2k\pi+\frac{2\pi}{7}\right)}}$$
$$= \frac{1-(\cos 2\pi + i\sin 2\pi)^{7k+1}}{1-\left[\cos\left(2k\pi+\frac{2\pi}{7}\right)+i\sin\left(2k\pi+\frac{2\pi}{7}\right)\right]}$$
$$= \frac{1-(1+i.0)^{7k+1}}{1-\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)}$$
$$= \frac{1-1}{1-\omega}$$
$$= 0$$

Example 7

If ω is a complex cube root of unity, prove that $(1 - \omega)^6 = -27$.

Solution

ω is the root of the equation $x^3 = 1$.

$\therefore \qquad \omega^3 = 1$

$$1 - \omega^3 = 0 \qquad \ldots (1)$$

$$(1- \omega)(1 + \omega + \omega^2) = 0$$

Since ω is a complex cube root of unity,

$$\omega \neq 1$$

$$\therefore \quad 1 + \omega + \omega^2 = 0 \qquad \ldots (2)$$

Now,
$$(1 - \omega)^6 = [(1 - \omega)^2]^3$$
$$= [1 + \omega^2 - 2\omega]^3$$
$$= (-\omega - 2\omega)^3 \qquad \text{[Using Eq. (2)]}$$
$$= (-3\omega)^3$$
$$= -27\ \omega^3$$
$$= -27 \qquad \text{[Using Eq. (1)]}$$

Example 8

Solve the following equations:

(i) $x^6 - i = 0$

(ii) $x^9 + x^5 - x^4 - 1 = 0$

(iii) $x^4 - x^3 + x^2 - x + 1 = 0$

Solution

(i) $x^6 = i$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$$

$$x = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{\frac{1}{6}}$$

$$= \cos(4k+1)\frac{\pi}{12} + i\sin(4k+1)\frac{\pi}{12} \qquad \text{[Using De Moivre's theorem]}$$

Putting $k = 0, 1, 2, 3, 4, 5$, all the 6 roots of the given equation are obtained.

(ii) $x^9 + x^5 - x^4 - 1 = 0$

$$x^5(x^4 + 1) - 1(x^4 + 1) = 0$$

$$(x^4 + 1)(x^5 - 1) = 0$$

The roots of $x^9 + x^5 - x^4 - 1 = 0$ are the roots of $x^4 + 1 = 0$ and $x^5 - 1 = 0$.

$$x^4 + 1 = 0$$
$$x^4 = -1$$
$$= \cos\pi + i\sin\pi$$
$$= \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$$
$$x = [\cos(2k_1 + 1)\pi + i\sin(2k_1 + 1)\pi]^{\frac{1}{4}}$$
$$= \cos(2k_1 + 1)\frac{\pi}{4} + i\sin(2k_1 + 1)\frac{\pi}{4}$$
$$= e^{i(2k_1+1)\frac{\pi}{4}}$$

Putting $k_1 = 0, 1, 2, 3$, all the roots of $x^4 + 1 = 0$ are obtained.

Also, $x^5 - 1 = 0$

$$x^5 = 1$$
$$= \cos 0 + i\sin 0$$
$$= \cos 2k_2\pi + i\sin 2k_2\pi$$
$$x = (\cos 2k_2\pi + i\sin 2k_2\pi)^{\frac{1}{5}}$$
$$= \cos\frac{2k_2\pi}{5} + i\sin\frac{2k_2\pi}{5}$$
$$= e^{i\frac{2k_2\pi}{5}}$$

Putting $k_2 = 0, 1, 2, 3, 4$, all the roots of $x^5 - 1 = 0$ are obtained.

Hence, all the roots of the given equations are given by $e^{i(2k_1+1)\frac{\pi}{4}}$ and $e^{i\frac{2k_2\pi}{5}}$, where $k_1 = 0, 1, 2, 3$ and $k_2 = 0, 1, 2, 3, 4$.

(iii) $x^4 - x^3 + x^2 - x + 1 = 0$

$$1 - x + x^2 - x^3 + x^4 = \frac{1\left[1 - (-x)^5\right]}{1 - (-x)} \qquad \text{[Using the sum of GP]}$$
$$= \frac{1 + x^5}{1 + x}$$
$$(x^5 + 1) = (x + 1)(1 - x + x^2 - x^3 + x^4)$$

Thus, all the roots of $x^5 + 1 = 0$ are the roots of $x^4 - x^3 + x^2 - x + 1 = 0$ except $x = -1$, which corresponds to $x + 1 = 0$.

Consider $x^5 + 1 = 0$

$$x^5 = -1$$
$$= \cos\pi + i\sin\pi$$
$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$
$$x = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{\frac{1}{5}}$$
$$= \cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, all the 5 roots of $x^5 + 1 = 0$ are

$$x_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$
$$x_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$
$$x_2 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5} = \cos\pi + i\sin\pi = -1$$
$$x_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$
$$x_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

Hence, x_0, x_1, x_3, x_4 are the roots of the given equation.

Example 9

Prove that the roots of the equation $x^2 - 2ax\cos\theta + a^2 = 0$ are also the roots of the equation $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$.

Solution

$$x^2 - 2ax\cos\theta + a^2 = 0 \qquad \ldots (1)$$
$$x = \frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta - 4a^2}}{2}$$
$$= a(\cos\theta \pm i\sin\theta)$$
$$= ae^{\pm i\theta}$$

The roots of the equation are $ae^{\pm i\theta}$.

Putting $x = ae^{\pm i\theta}$ in $x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$,

$$x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = (ae^{\pm i\theta})^{2n} - 2a^n(ae^{\pm i\theta})^n \cos n\theta + a^{2n}$$
$$= a^{2n}e^{\pm in\theta}(e^{\pm in\theta} - 2\cos n\theta + e^{\mp in\theta})$$
$$= a^{2n}e^{\pm in\theta}(e^{\pm in\theta} + e^{\mp in\theta} - 2\cos n\theta)$$

$$= a^{2n} e^{\pm in\theta} (2\cos n\theta - 2\cos n\theta)$$

$$= 0$$

Hence, the roots of the equation $x^2 - 2ax\cos\theta + a^2 = 0$ are also the roots of the equation $x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = 0$.

Example 10

If $\alpha, \alpha^2, \alpha^3, \alpha^4$ *are roots of* $x^5 - 1 = 0$ *then show that* $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Solution

One root of $x^5 - 1 = 0$ is obviously 1, the remaining roots are given as $\alpha, \alpha^2, \alpha^3, \alpha^4$.

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$[\because \quad x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1)]$$

$$x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

Putting $x = 1$ on both the sides,

$$1 + 1 + 1 + 1 + 1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

Example 11

If $\alpha, \beta, \gamma, \delta$ *are the roots of* $x^4 + x^3 + x^2 + x + 1 = 0$, *find their values and show that* $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$.

Solution

$$x^4 + x^3 + x^2 + x + 1 = 0$$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$x^5 - 1 = 0 \qquad \left[\because \; x^{n-1} = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1)\right]$$

This shows that all the roots of $x^5 - 1 = 0$, except $x = 1$, are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$.

As solved in Example 8, all roots of $x^5 - 1 = 0$ are

$$x_0 = 1 \qquad x_1 = e^{\frac{2i\pi}{5}} \qquad x_2 = e^{\frac{4i\pi}{5}} \qquad x_3 = e^{\frac{6i\pi}{5}} \qquad x_4 = e^{\frac{8i\pi}{5}}$$

Except $x_0 = 1$ remaining roots x_1, x_2, x_3, x_4 are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$.

$$\alpha = e^{\frac{2i\pi}{5}},\ \beta = e^{\frac{4i\pi}{5}},\ \gamma = e^{\frac{6i\pi}{5}},\ \delta = e^{\frac{8i\pi}{5}}$$

Since $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$,

$$x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

Putting $x = 1$ on both the sides,

$$1+1+1+1+1 = (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$$
$$5 = (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$$
$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$$

Example 12

Find the cube roots of $1 - \cos\theta - i\sin\theta$.

Solution

$$\begin{aligned}
x^3 &= 1-\cos\theta - i\sin\theta \\
&= 2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
&= 2\sin\frac{\theta}{2}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right] \\
&= 2\sin\frac{\theta}{2}\left[\cos\left\{2n\pi+\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} - i\sin\left\{2n\pi+\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]
\end{aligned}$$

$$x = \left[2\sin\frac{\theta}{2}\left\{\cos\left(2n\pi+\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(2n\pi+\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]^{\frac{1}{3}}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{\frac{1}{3}}\left[\cos\frac{1}{3}\left(2n\pi+\frac{\pi-\theta}{2}\right) - i\sin\frac{1}{3}\left(2n\pi+\frac{\pi-\theta}{2}\right)\right]$$

$$x = \left(2\sin\frac{\theta}{2}\right)^{\frac{1}{3}}\left[\cos\left\{\frac{(4n+1)\pi-\theta}{6}\right\} - i\sin\left\{\frac{(4n+1)\pi-\theta}{6}\right\}\right]$$

Putting $n = 0, 1, 2$, cube roots of $1 - \cos\theta - i\sin\theta$ are obtained.

Example 13

Show that the points representing the roots of the equation $z^n = i(z-1)^n$ *on the Argand's diagram are collinear.*

Solution

$$z^n = i\,(z-1)^n$$

$$\left(\frac{z}{z-1}\right)^n = i$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$$

$$\frac{z}{z-1} = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{\frac{1}{n}}$$

$$= \cos\,(4k+1)\frac{\pi}{2n} + i\sin(4k+1)\frac{\pi}{2n}$$

$$= e^{i(4k+1)\frac{\pi}{2n}}$$

$$= e^{i\theta}, \qquad \text{where}\,\theta = i(4k+1)\frac{\pi}{2n}$$

$$\frac{z}{z-1} = e^{i\theta}$$

$$z = ze^{i\theta} - e^{i\theta}$$

$$z(1-e^{i\theta}) = -e^{i\theta}$$

$$z = -\frac{e^{i\theta}}{1-e^{i\theta}} \cdot \frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}}$$

$$= -\frac{e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}$$

$$= -\frac{e^{\frac{i\theta}{2}}}{-2i\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}}$$

$$= \frac{1}{2i}\cot\frac{\theta}{2} + \frac{1}{2}$$

$$= -\frac{i}{2}\cot\frac{\theta}{2} + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}, \quad \text{where } \theta = i(4k+1)\frac{\pi}{2n}$$

Putting $k = 0, 1, 2, \ldots, n-1$, all roots of $z^n = i\,(z-1)^n$ are obtained.

The real parts of all the roots remain same as $\frac{1}{2}$, which means the x-coordinate of all the points represented by the roots on the Argand's diagram is $\frac{1}{2}$, i.e., constant. Therefore, all the points lie on the line $x = \frac{1}{2}$ and, hence, are collinear.

EXERCISE 1.3

1. Find all the values of $(1-i)^{\frac{2}{3}}$.

$$\left[\textbf{Ans.}: 2^{\frac{1}{3}} e^{-i(4k+1)\frac{\pi}{6}},\ k = 0, 1, 2\right]$$

2. Find the continued product of all the values of $(1+i)^{\frac{1}{8}}$.

[Ans.: $-(1 + i)$**]**

3. Solve the equations:

(i) $x^{12} - 1 = 0$ (ii) $x^5 + \sqrt{3} = i$ (iii) $x^7 + x^4 + x^3 + 1 = 0$

(iv) $(1 + x)^6 + x^6 = 0$ (v) $x^{14} + 127x^7 - 128 = 0$.

$$\left[\begin{array}{l}\textbf{Ans.}: \\ \text{(i)} \pm 1, \pm i, \pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right), \pm\left(\frac{\sqrt{3}}{2} \pm \frac{i}{2}\right) \quad \text{(ii)}\ 2^{\frac{1}{5}} e^{i\left(\frac{12k+5}{30}\right)\pi},\ k = 0, 1, 2, 3, 4 \\ \text{(iii)} -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \pm\left(\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right) \qquad \text{(iv)} -\frac{1}{2} - i\cot(2k+1)\frac{\pi}{12},\ k = 0, 1, 2, 3, 4, 5 \\ \text{(v)}\ e^{i\frac{2k_1\pi}{7}},\ e^{i(2k_2+1)\frac{\pi}{7}}, k_1 = k_2 = 0, 1, 2, 3, 4, 5, 6\end{array}\right]$$

4. Prove that $1 + \omega + \omega^2 = 0$.

5. Prove that

(i) $\alpha^{3n} + \beta^{3n} = 2$, n is an integer (ii) $\alpha e^{\alpha x} + \beta e^{\beta x}$

$= -e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right]$, where α, β are complex cube roots of unity.

6. Prove that the points representing the roots of the equation $z^3 = i\,(z-1)^3$ on the Argand's diagram are collinear.

7. If $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the roots of $x^7 - 1 = 0$, prove that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)(1-\alpha^5)(1-\alpha^6) = 7$.

1.9.2 Expansion of Trigonometric Functions

Type I Expansion of $\sin^n\theta$, $\cos^n\theta$ in terms of $\sin n\theta$, $\cos n\theta$, where n is a positive integer.

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

$$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

Hence, $$x + \frac{1}{x} = 2\cos\theta \quad \text{and} \quad x - \frac{1}{x} = 2i\sin\theta$$

Again, $$x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$$

$$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

To expand $\cos^n\theta$ and $\sin^n\theta$, write $\cos^n\theta = \frac{1}{2^n}\left(x + \frac{1}{x}\right)^n$ and $\sin^n\theta = \frac{1}{(2i)^n}\left(x - \frac{1}{x}\right)^n$ and expand RHS using binomial expansion

$$(x+a)^n = x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + \ldots + a^n$$

Example 1

Prove that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Solution

Let $x = \cos\theta + i\sin\theta$,

then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\sin^5\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^5$$

$$= \frac{1}{(2i)^5}\left[x^5 + 5\cdot x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5\right]$$

[Using binomial expansion]

$$= \frac{1}{32i^5}\left(x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}\right)$$

$$= \frac{1}{32i}\left[\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right] \qquad [\because i^5 = i^4 \cdot i = i]$$

$$= \frac{1}{32i}(2i \sin 5\theta - 5 \cdot 2i \sin 3\theta + 10 \cdot 2i \sin\theta) \qquad \left[\because x^n - \frac{1}{x^n} = 2i \sin n\theta\right]$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin\theta)$$

Example 2

Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3 \cos 4\theta + 5)$.

Solution

Let $x = \cos\theta + i \sin\theta$; then $\frac{1}{x} = \cos\theta - i \sin\theta$

$$\cos^6\theta = \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^6 = \frac{1}{2^6}\left(x + \frac{1}{x}\right)^6$$

$$= \frac{1}{2^6}\left(x^6 + 6x^5 \cdot \frac{1}{x} + 15x^4 \frac{1}{x^2} + 20x^3 \cdot \frac{1}{x^3} + 15x^2 \cdot \frac{1}{x^4} + 6x \cdot \frac{1}{x^5} + \frac{1}{x^6}\right)$$

$$= \frac{1}{2^6}\left[\left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20\right]$$

$$= \frac{1}{2^6}(2 \cos 6\theta + 6 \cdot 2\cos 4\theta + 15 \cdot 2\cos 2\theta + 20) \qquad \left[\because x^n + \frac{1}{x^n} = 2\cos n\theta\right]$$

$$= \frac{1}{2^5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10) \qquad \ldots(1)$$

$$\sin^6\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^6$$

$$= \frac{1}{(2i)^6}\left[x^6 + 6x^5\left(-\frac{1}{x}\right) + 15x^4\left(-\frac{1}{x}\right)^2 + 20x^3\left(-\frac{1}{x}\right)^3 + 15x^2\left(-\frac{1}{x}\right)^4 + 6x\left(-\frac{1}{x}\right)^5 + \left(-\frac{1}{x}\right)^6\right]$$

$$= -\frac{1}{2^6}\left[\left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20\right] \qquad [\because i^6 = (i^2)^3 = -1]$$

$$= -\frac{1}{2^6}(2\cos 6\theta - 6 \cdot 2\cos 4\theta + 15 \cdot 2\cos 2\theta - 20)$$

$$= -\frac{1}{2^5}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10) \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$\cos^6\theta + \sin^6\theta = \frac{1}{2^5}(12\cos 4\theta + 20) = \frac{1}{8}(3\cos 4\theta + 5)$$

Example 3

Expand $\sin^5\theta\cos^3\theta$ *in a series of sines of multiples of* θ.

Solution

Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\sin^5\theta\cos^3\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^5\left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^3$$

$$= \frac{1}{(2i)^5}\left(x - \frac{1}{x}\right)^2\left(x - \frac{1}{x}\right)^3 \cdot \frac{1}{(2)^3}\left(x + \frac{1}{x}\right)^3$$

$$= \frac{1}{(2^8 i)}\left(x - \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^3 \qquad [\because i^5 = i^4 \cdot i = i]$$

$$= \frac{1}{(2^8 i)}\left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= \frac{i}{(2^8 i)}\left(x^8 - 3x^4 + 3 - \frac{1}{x^4} - 2x^6 + 6x^2 - \frac{6}{x^2} + \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}\right)$$

$$= \frac{1}{(2^8 i)}\left[\left(x^8 - \frac{1}{x^8}\right) - 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) + 6\left(x^2 - \frac{1}{x^2}\right)\right]$$

$$= \frac{1}{(2^8 i)}(2i\sin 8\theta - 2 \cdot 2i\sin 6\theta - 2 \cdot 2i\sin 4\theta + 6 \cdot 2i\sin 2\theta)$$

$$= \frac{1}{2^7}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)$$

$$= \frac{1}{128}(\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta)$$

Type II Expansion of $\sin n\theta$, $\cos n\theta$ in powers of $\sin\theta$, $\cos\theta$

By De Moivre's theorem,

$$\begin{aligned}\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n \\ &= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\cdot i\sin\theta + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2 \\ &\qquad + {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \cdots \\ &= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \cdots) \\ &\qquad + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots)\end{aligned}$$

Comparing real and imaginary parts,

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \cdots$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\sin^3\theta + \cdots$$

Example 1

Prove that $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$ *and state the result which you have used.* **[Summer 2015]**

Solution

$$\begin{aligned}\cos 7\theta + i\sin 7\theta &= (\cos\theta + i\sin\theta)^7 \\ &= \cos^7\theta + 7\cos^6\theta\cdot i\sin\theta + 21\cos^5\theta(i\sin\theta)^2 \\ &\quad + 35\cos^4\theta(i\sin\theta)^3 + 35\cos^3\theta(i\sin\theta)^4 \\ &\quad + 21\cos^2\theta(i\sin\theta)^5 + 7\cos\theta(i\sin\theta)^6 + (i\sin\theta)^7 \\ &= (\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta) \\ &\quad + i(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta)\end{aligned}$$

Comparing imaginary parts,

$$\begin{aligned}\sin 7\theta &= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \\ \frac{\sin 7\theta}{\sin\theta} &= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta \\ &= 7(1-\sin^6\theta + 3\sin^4\theta - 3\sin^2\theta) - 35(1-2\sin^2\theta + \sin^4\theta)\sin^2\theta \\ &\qquad + 21(\sin^4\theta - \sin^6\theta) - \sin^6\theta \\ &= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta\end{aligned}$$

Example 2

Expand $\dfrac{\sin 5\theta}{\sin\theta}$ *in powers of* $\cos\theta$ *only.*

Solution

$$(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5 \qquad \text{[Using De Moivre's theorem]}$$

$$= \cos^5\theta + 5\cos^4\theta \cdot i\sin\theta + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i\,(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Comparing imaginary parts,

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= \sin\theta\,(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)$$

$$\frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta\,(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 16\cos^4\theta - 12\cos^2\theta + 1$$

Example 3

Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ *and, hence, deduce that*

$$5\tan^4\frac{\pi}{10} - 10\tan^2\frac{\pi}{10} + 1 = 0.$$

Solution

Comparing real and imaginary parts in Example 2,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing the numerator and denominator by $\cos^5\theta$,

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - \tan^2\theta + 5\tan^4\theta}$$

Putting $\theta = \frac{\pi}{10}$,

$$\tan 5\cdot\frac{\pi}{10} = \frac{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}{1 - \tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}} \quad \ldots (1)$$

$$\frac{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}{1 - \tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}} = \tan\frac{\pi}{2} = \infty = \frac{1}{0}$$

Hence, $1 - \tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10} = 0$

Example 4

If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, *find values of a, b, c.*

Solution

$$\begin{aligned}(\cos 6\theta + i\sin 6\theta) &= (\cos\theta + i\sin\theta)^6 \\ &= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 \\ &\quad + 20\cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4 \\ &\quad + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6 \\ &= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta) \\ &\quad + i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)\end{aligned}$$

Comparing imaginary parts,

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \quad \ldots(1)$$

Given $$\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta \quad \ldots(2)$$

Comparing Eqs (1) and (2),

$$a = 6,\ b = -20,\ c = 6$$

Example 5

Prove that $\dfrac{1+\cos 6\theta}{1+\cos 2\theta} = 16\ cos^4\ \theta - 24\ cos^2\theta\ + 9.$

Solution

Comparing real parts in Example 4,

$$\begin{aligned}\cos 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta\\ &= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3\\ &= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1-2\cos^2\theta+\cos^4\theta)\\ &\qquad - (1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)\\ &= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1\end{aligned}$$

$$\begin{aligned}\frac{1+\cos 6\theta}{1+\cos 2\theta} &= \frac{32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta}{2\cos^2\theta}\\ &= 16\cos^4\theta - 24\cos^2\theta + 9\end{aligned}$$

Example 6

Using De Moivre's theorem, prove that $2\ (1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$, *where* $x = 2\cos\theta$.

Solution

$$2\ (1 + \cos 8\theta) = 2\ .\ 2\cos^2 4\theta = (2\cos 4\theta)^2 \qquad \ldots (1)$$

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4\\ &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2\\ &\quad + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4\\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta)\\ &\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\end{aligned}$$

Comparing real parts,

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2\\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta\\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned}2\ (1 + \cos 8\theta) &= (16\cos^4\theta - 16\cos^2\theta + 2)^2\\ &= (x^4 - 4x^2 + 2)^2 \qquad \text{where } x = 2\cos\theta\end{aligned}$$

EXERCISE 1.4

1. Expand $\cos^6\theta - \sin^6\theta$ in terms of cosine multiples of θ.

$$\left[\textbf{Ans.}: \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)\right]$$

2. Expand $\cos^8\theta$ in terms of cosine multiples of θ.

$$\left[\textbf{Ans.}\ \frac{1}{2^7}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)\right]$$

3. Prove that $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$.

4. Prove that $\cos^4\theta \sin^3\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - \sin\theta)$.

5. Prove that $-2^{12}\cos^6\theta \sin^7\theta = (\sin 13\theta - \sin 11\theta - 6\sin 9\theta + 6\sin 7\theta + 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta)$.

6. Prove that $-256\sin^7\theta\cos^2\theta = \cos 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta$.

7. Expand $\frac{\sin 7\theta}{\sin\theta}$ in powers of $\sin\theta$ only.

$$\left[\textbf{Ans.}: 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta\right]$$

8. Prove that $\frac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta$.

9. Prove that $\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$

and, hence, deduce $1 - 21\tan^2\frac{\pi}{14} + 35\tan^4\frac{\pi}{14} - 7\tan^6\frac{\pi}{14} = 0$.

10. Prove that $1 - \cos 10\theta = 2(16\sin^5\theta - 20\sin^3\theta + 5\sin\theta)^2$.

11. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$.

1.10 CIRCULAR AND HYPERBOLIC FUNCTIONS

1.10.1 Circular Function

From Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

If $z = x + iy$ is a complex number then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

These are called circular functions of complex numbers.

1.10.2 Hyperbolic Function

If z is a complex number then the sine hyperbolic of z is denoted by $\sinh z$ and is given as

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

and the cosine hyperbolic of z is denoted by $\cosh z$ and is given as

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

From these expressions, other hyperbolic functions can also be obtained using

$$\tanh z = \frac{\sinh z}{\cosh z}, \text{ cosech } z = \frac{1}{\sinh z}, \text{ sech } z = \frac{1}{\cosh z} \text{ and } \coth z = \frac{1}{\tanh z}.$$

From the above definitions of $\sinh z$, $\cosh z$, $\tanh z$, the following range of hyperbolic functions is obtained:

z	$-\infty$	0	∞
$\sinh z$	$-\infty$	0	∞
$\cosh z$	∞	1	∞
$\tanh z$	–1	0	1

Note $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$

1.10.3 Relation between Circular and Hyperbolic Functions

(i) $\sin iz = i \sinh z$ and $\sinh z = -i \sin iz$

Proof By Euler's formula,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Replacing z by iz,

$$\sin iz = \frac{e^{i^2 z} - e^{-i^2 z}}{2i}$$

$$= -i\frac{(e^{-z} - e^{z})}{2} \qquad \left[\because \ \frac{1}{i} = -i\right]$$

$$= i\frac{(e^z - e^{-z})}{2}$$
$$= i \sinh z$$

and
$$\sinh z = \frac{1}{i}\sin iz$$
$$= -i \sin iz$$

(ii) $\cos iz = \cosh z$

Proof By Euler's formula,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Replacing z by iz,

$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$
$$= \frac{e^{-z} + e^{z}}{2}$$
$$= \cosh z$$

(iii) $\tan iz = i \tanh z$ and $\tanh z = -i \tan iz$

Proof
$$\tan iz = \frac{\sin iz}{\cos iz}$$
$$= \frac{i \sinh z}{\cosh z}$$
$$= i \tanh z$$

$$\tanh z = \frac{1}{i}\tan iz$$
$$= -i \tan iz$$

1.10.4 Formulae on Hyperbolic Functions

A. (i) $\cosh^2 z - \sinh^2 z = 1$
(ii) $\coth^2 z - \operatorname{cosech}^2 z = 1$
(iii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

B. (iv) $\sinh 2z = 2 \sinh z \cosh z$
(v) $\cosh 2z = \cosh^2 z + \sinh^2 z$
$= 2 \cosh^2 z - 1$
$= 1 + 2 \sinh^2 z$

(vi) $\tanh 2z = \dfrac{2\tanh z}{1+\tanh^2 z}$

C. (vii) $\sinh 3z = 3\sinh z + 4\sinh^3 z$

(viii) $\cosh 3z = 4\cosh^3 z - 3\cosh z$

(ix) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1+3\tanh^2 z}$

D. (x) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

E. (xiii) $\sinh z_1 + \sinh z_2 = 2\sinh \dfrac{z_1+z_2}{2} \cosh \dfrac{z_1-z_2}{2}$

(xiv) $\sinh z_1 - \sinh z_2 = 2\cosh \dfrac{z_1+z_2}{2} \sinh \dfrac{z_1-z_2}{2}$

(xv) $\cosh z_1 + \cosh z_2 = 2\cosh \dfrac{z_1+z_2}{2} \cosh \dfrac{z_1-z_2}{2}$

(xvi) $\cosh z_1 - \cosh z_2 = 2\sinh \dfrac{z_1+z_2}{2} \sinh \dfrac{z_1-z_2}{2}$

F. (xvii) $2\sinh z_1 \cosh z_2 = \sinh(z_1+z_2) + \sinh(z_1-z_2)$

(xviii) $2\cosh z_1 \sinh z_2 = \sinh(z_1+z_2) - \sinh(z_1-z_2)$

(xix) $2\cosh z_1 \cosh z_2 = \cosh(z_1+z_2) + \cosh(z_1-z_2)$

(xx) $2\sinh z_1 \sinh z_2 = \cosh(z_1+z_2) - \cosh(z_1-z_2)$

1.11 INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called the sine hyperbolic inverse of x, where x is real.

Similarly, $\cosh^{-1} x$, $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$ and $\operatorname{cosech}^{-1} x$ are defined.

The inverse hyperbolic functions are many-valued functions but their principal values are only considered.

If x is real,

(i) $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

(ii) $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$ **[Summer 2014]**

(iii) $\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

Proof (i) Let $\sinh^{-1} x = y$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y} - 1}{e^y}$$

$$e^{2y} - 2x\, e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$y = \log\left(x \pm \sqrt{x^2 + 1}\right)$$

But $x - \sqrt{x^2 + 1} < 0$ and log of a negative number is not defined.

$$y = \log\left(x + \sqrt{x^2 + 1}\right)$$

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$$

(ii) Let $\cosh^{-1} x = y$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y} + 1}{e^y}$$

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$y = \log\left(x \pm \sqrt{x^2 - 1}\right) \quad \text{... (1.2)}$$

Consider $$y = \log\left(x - \sqrt{x^2 - 1}\right) \quad \text{... (1.3)}$$

$$e^y = x - \sqrt{x^2 - 1}$$

$$e^{-y} = \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}$$

$$= \frac{x+\sqrt{x^2-1}}{x^2-x^2+1}$$

$$= x+\sqrt{x^2-1}$$

$$-y = \log\left(x+\sqrt{x^2-1}\right)$$

$$y = -\log\left(x+\sqrt{x^2-1}\right) \quad \text{... (1.4)}$$

Equating Eqs (1.3) and (1.4),

$$\log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right) \quad \text{... (1.5)}$$

From Eqs (1.2) and (1.5),

$$y = \pm \log\left(x+\sqrt{x^2-1}\right) \quad \text{... (1.6)}$$

$$\cosh^{-1} x = \pm \log\left(x+\sqrt{x^2+1}\right)$$

$$x = \cosh\left[\pm\log\left(x+\sqrt{x^2+1}\right)\right]$$

$$= \cosh\left[\log\left(x+\sqrt{x^2+1}\right)\right] \quad [\because \cosh(-z) = \cosh z]$$

$$\cosh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$$

(iii) Let $\tanh^{-1} x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo and dividendo,

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$= \frac{2e^y}{2e^{-y}}$$

$$= e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Example 1

Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

Solution

$$\begin{aligned}(\cosh x+\sinh x)^n &= (\cos ix - i\sin ix)^n \\ &= \cos(nix) - i\sin(nix) \quad \text{[Using De Moivre's theorem]} \\ &= \cosh nx - i\cdot i\sinh nx \\ &= \cosh nx + \sinh nx\end{aligned}$$

Example 2

Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x$.

Solution

$$\begin{aligned}\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 &= \left(\frac{1+\dfrac{\sinh x}{\cosh x}}{1-\dfrac{\sinh x}{\cosh x}}\right)^3 \\ &= \left(\frac{\cosh x+\sinh x}{\cosh x-\sinh x}\right)^3 \\ &= \left(\frac{\cos ix - i\sin ix}{\cos ix + i\sin ix}\right)^3 \\ &= \left[\left(\frac{\cos ix - i\sin ix}{\cos ix + i\sin ix}\right)\left(\frac{\cos ix - i\sin ix}{\cos ix - i\sin ix}\right)\right]^3 \\ &= \left[\frac{(\cos ix - i\sin ix)^2}{\cos^2 ix + \sin^2 ix}\right]^3 \\ &= (\cos ix - i\sin ix)^6 \\ &= \cos 6ix - i\sin 6ix \quad \text{[Using De Moivre's theorem]} \\ &= \cosh 6x - i\cdot i\sinh 6x \\ &= \cosh 6x + \sinh 6x\end{aligned}$$

Example 3

Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \cosh^2 x.$

Solution

$$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \frac{1}{1-\dfrac{1}{1-\dfrac{1}{-\sinh^2 x}}} \qquad [\because \cosh^2 x - \sinh^2 x = 1]$$

$$= \frac{1}{1-\dfrac{1}{1+\operatorname{cosech}^2 x}}$$

$$= \frac{1}{1-\dfrac{1}{\coth^2 x}} \qquad [\because 1-\coth^2 x = -\operatorname{cosech}^2 x]$$

$$= \frac{1}{1-\tanh^2 x}$$

$$= \frac{1}{\operatorname{sech}^2 x} \qquad [\because 1-\tanh^2 x = \operatorname{sech}^2 x]$$

$$= \cosh^2 x$$

Example 4

Prove that $\tanh(\log\sqrt{3}) = 0.5.$

Solution

$$\tanh(\log\sqrt{3}) = \frac{e^{\log\sqrt{3}} - e^{-\log\sqrt{3}}}{e^{\log\sqrt{3}} + e^{-\log\sqrt{3}}}$$

$$= \frac{\sqrt{3} - \dfrac{1}{\sqrt{3}}}{\sqrt{3} + \dfrac{1}{\sqrt{3}}}$$

$$= \frac{3-1}{3+1}$$

$$= 0.5$$

Example 5

Solve the equation $17\cosh x + 18\sinh x = 1$ *for real values of x.*

Solution

$$17\cosh x + 18\sinh x = 1$$

$$17\left(\frac{e^x + e^{-x}}{2}\right) + 18\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$35e^x - e^{-x} = 2$$

$$35e^{2x} - 1 = 2e^x$$

$$35e^{2x} - 2e^x - 1 = 0$$

This equation is quadratic in e^x.

$$e^x = \frac{2 \pm \sqrt{4 + 140}}{70} = \frac{2 \pm 12}{70} = \frac{14}{70}, \frac{-10}{70}$$

For real values of x, e^x should be positive.

$$\therefore \quad e^x = \frac{14}{70} = \frac{1}{5}$$

$$x = \log\frac{1}{5} = -\log 5$$

Example 6

Solve the equation $7\cosh x + 8\sinh x = 1$ *for real values of x.*

Solution

$$7\cosh x + 8\sinh x = 1$$

$$7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$15e^x - \frac{1}{e^x} = 2$$

$$15e^{2x} - 2e^x - 1 = 0$$

This equation is quadratic in e^x.

$$e^x = \frac{2 \pm \sqrt{4 + 60}}{30} = \frac{2 \pm 8}{30} = \frac{1}{3}, -\frac{1}{5}$$

For real values of x, e^x should be positive.

$$e^x = \frac{1}{3}$$

$$x = \log\frac{1}{3} = -\log 3$$

Example 7

If $\tanh x = \frac{1}{2}$, *prove that* $\sinh 2x = \frac{4}{3}$

Solution

$$\begin{aligned}\sinh 2x &= \frac{2\tanh x}{1-\tanh^2 x}\\ &= \frac{1}{1-\frac{1}{4}}\\ &= \frac{4}{3}\end{aligned}$$

Example 8

Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

Solution

$$\begin{aligned}\sinh^{-1}(\tan\theta) &= \log\left(\tan\theta+\sqrt{\tan^2\theta+1}\right)\\ &= \log(\tan\theta+\sec\theta)\\ &= \log\left(\frac{\sin\theta+1}{\cos\theta}\right)\\ &= \log\left[\frac{\cos\left(\frac{\pi}{2}-\theta\right)+1}{\sin\left(\frac{\pi}{2}-\theta\right)}\right]\\ &= \log\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]\end{aligned}$$

$$= \log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$
$$= \log\left[\tan\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$
$$= \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

Example 9

Prove that

(i) $\tanh^{-1} x = \sinh^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

(ii) $\sinh^{-1} x = \cosh^{-1}\left(\sqrt{1+x^2}\right)$.

Solution

(i) $\sinh^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \log\left(\dfrac{x}{\sqrt{1-x^2}} + \sqrt{\dfrac{x^2}{1-x^2}+1}\right)$ $\left[\because \sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)\right]$

$$= \log\left(\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)$$
$$= \log\left(\frac{x+1}{\sqrt{1-x^2}}\right)$$
$$= \log\left(\frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}}\right)$$
$$= \log\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)$$
$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$
$$= \tanh^{-1} x$$

(ii) $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \log\left(\sqrt{1+x^2}+\sqrt{1+x^2-1}\right)$ $\left[\because \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)\right]$

$$= \log\left(\sqrt{1+x^2}+x\right)$$
$$= \sinh^{-1} x$$

Example 10

Prove that $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$.

Solution

$$\begin{aligned}
\tanh^{-1}(\sin\theta) &= \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \\
&= \frac{1}{2}\log\left[\frac{(1+\sin\theta)\,(1+\sin\theta)}{(1-\sin\theta)\,(1+\sin\theta)}\right] \\
&= \frac{1}{2}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 \\
&= \frac{1}{2}\cdot 2\log(\sec\theta+\tan\theta) \\
&= \log\left(\sec\theta+\sqrt{\sec^2\theta-1}\right) \\
&= \cosh^{-1}(\sec\theta) \qquad \left[\because\ \cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right)\right]
\end{aligned}$$

Example 11

Prove that $\operatorname{cosech}^{-1} z = \log\left[\dfrac{1+\sqrt{1+z^2}}{z}\right]$. *Is it defined for all values of z?*

Solution

Let $y = \operatorname{cosech}^{-1} z$

$$\begin{aligned}
\operatorname{cosech} y &= z \\
\frac{2}{e^y - e^{-y}} &= z \\
e^y - \frac{1}{e^y} &= \frac{2}{z} \\
e^{2y} - \frac{2}{z}e^y - 1 &= 0
\end{aligned}$$

This equation is quadratic in e^y.

$$e^y = \frac{\frac{2}{z} \pm \sqrt{\frac{4}{z^2}+4}}{2}$$

$$= \frac{1+\sqrt{1+z^2}}{z} \qquad \left[\because 1-\sqrt{1+z^2} < 0 \text{ and } e^y \text{ cannot be negative}\right]$$

$$y = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$$

It is not defined for $z < 0$.

Example 12

Prove that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$.

Solution

Let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$$i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$$

$$\frac{x-a}{x+a} = \frac{-e^{i\theta}+e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$$

Applying componendo – dividendo,

$$\frac{x+a+x-a}{x+a-x+a} = \frac{e^{i\theta}+e^{-i\theta}-e^{i\theta}+e^{-i\theta}}{e^{i\theta}+e^{-i\theta}+e^{i\theta}-e^{-i\theta}}$$

$$\frac{x}{a} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

$$\frac{a}{x} = e^{2i\theta}$$

$$2i\theta = \log\left(\frac{a}{x}\right)$$

$$\theta = \frac{1}{2i}\log\left(\frac{a}{x}\right) = -\frac{i}{2}\log\left(\frac{a}{x}\right)$$

$$\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$$

EXERCISE 1.5

1. Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^n = \cosh 2nx + \sinh 2nx$.

2. Prove that cosec x + coth x = coth $\frac{x}{2}$.

3. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1+\sinh^2 x}}} = -\sinh^2 x$.

4. If 6 sinh x + 2 cosh x + 7 = 0, find tanh x.

$\left[\text{Ans.}: \frac{3}{5}, -\frac{15}{17}\right]$

5. Find the value of tanh log $\sqrt{5}$.

$\left[\text{Ans.}: \frac{2}{3}\right]$

6. If tanh $x = \frac{1}{2}$, find cosh 2x.

$\left[\text{Ans.}: \frac{4}{3}, \frac{5}{3}\right]$

7. Prove that $\sin^{-1} x = \frac{1}{i}\log(ix + \sqrt{1-x^2})$. **[Summer 2013]**

8. Prove that $\cos^{-1}x = -i\log(x \pm \sqrt{1-x^2})$.

9. Prove that $\sin^{-1}ix = 2n\pi + i\log(x + \sqrt{1+x^2})$.

10. Prove that $\sinh^{-1}(\tan x) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

11. Prove that

(i) $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\text{cosec}\,\theta)$. (ii) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$.

12. Prove that $\cosh^{-1}\left(\frac{3i}{4}\right) = \log 2 + \frac{i\pi}{2}$.

13. If $\sinh^{-1}(x + iy) + \sinh^{-1}(x - iy) = \sinh^{-1}a$, prove that $2(x^2 + y^2)\sqrt{a^2+1} = a^2 - 2x^2 + 2y^2$.

1.12 SEPARATION INTO REAL AND IMAGINARY PARTS

To separate real and imaginary parts of a complex number, the following results are used:

(i) $\sin(x \pm iy) = \sin x \cos iy \pm \cos x \sin iy = \sin x \cosh y \pm i\cos x \sinh y$

(ii) $\cos(x \pm iy) = \cos x \cos iy \mp \sin x \sin iy = \cos x \cosh y \mp i \sin x \sinh y$

(iii) $$\tan(x \pm iy) = \frac{2\sin(x \pm iy)}{2\cos(x \pm iy)} \cdot \frac{\cos(x \mp iy)}{\cos(x \mp iy)}$$
$$= \frac{\sin 2x \pm \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x \pm i \sinh 2y}{\cos 2x + \cosh 2y}$$

(iv) $$\sinh(x \pm iy) = \sinh x \cosh iy \pm \cosh x \sinh iy$$
$$= \sinh x \cos iiy \pm \cosh x (-i \sin iiy)$$
$$= \sinh x \cos(-y) \pm \cosh x [-i \sin(-y)]$$
$$= \sinh x \cos y \pm i \cosh x \sin y$$

(v) $\cosh(x \pm iy) = \cosh x \cosh iy \pm \sinh x \sinh iy = \cosh x \cos y \pm i \sinh x \sin y$

(vi) $$\tanh(x \pm iy) = \frac{2\sinh(x \pm iy)}{2\cosh(x \pm iy)} \cdot \frac{\cosh(x \mp iy)}{\cosh(x \mp iy)}$$
$$= \frac{\sinh 2x \pm \sinh 2iy}{\cosh 2x + \cosh 2iy} = \frac{\sinh 2x \pm i \sin 2y}{\cosh 2x + \cos 2y}.$$

Example 1

Separate real and imaginary parts of

(i) $\sinh z$ **[Summer 2014]**

(ii) $\cos^{-1}\left(\frac{3i}{4}\right)$ *(iii)* $\cos^{-1}(ix)$

(iv) $\sin^{-1}\left(e^{i\theta}\right)$ *(v)* $\sin^{-1}(\operatorname{cosec}\theta)$

Solution

(i) $$\sinh z = -i \sin iz$$
$$= -i \sin i(x + iy)$$
$$= -i \sin(ix - y)$$
$$= -i(\sin ix \cos y - \cos ix \sin y)$$
$$= -i(i \sinh x \cos y - \cosh x \sin y)$$
$$= -i^2 \sinh x \cos y + i \cosh x \sin y$$
$$= \sinh x \cos y + i \cosh x \sin y$$

(ii) Let $\cos^{-1}\frac{3i}{4} = x + iy$

$$\frac{3i}{4} = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

Comparing real and imaginary parts,

$$\cos x \cosh y = 0 \qquad \ldots(1)$$

$$\sin x \sinh y = -\frac{3}{4} \qquad \ldots(2)$$

From Eq. (1), $\cos x = 0 \qquad [\because \cosh y \neq 0]$

$$x = \frac{\pi}{2}$$

Putting $x = \frac{\pi}{2}$ in Eq. (2),

$$\sin\frac{\pi}{2}\sinh y = \frac{3}{4}$$

$$\sinh y = \frac{3}{4}$$

$$y = \sinh^{-1}\left(\frac{3}{4}\right)$$

$$= \log\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right)$$

$$= \log\left(\frac{3}{4} + \frac{5}{4}\right)$$

$$= \log 2$$

Hence, $\cos^{-1}\frac{3i}{4} = \frac{\pi}{2} + i\log 2$

(iii) Let $\cos^{-1}(ix) = \alpha + i\beta$

$$\cos(\alpha + i\beta) = ix$$

$$\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = ix$$

$$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = 0 + ix$$

Comparing real and imaginary parts,

$$\cos\alpha\cosh\beta = 0, \qquad \ldots(1)$$

$$\sin\alpha\sinh\beta = -x \qquad \ldots(2)$$

From Eq. (1),

$$\cos\alpha = 0 \qquad [\because \cosh\beta \neq 0]$$

$$\alpha = \frac{\pi}{2}$$

Putting $\alpha = \frac{\pi}{2}$ in Eq. (2),

$$\sin \frac{\pi}{2} \sinh \beta = -x$$

$$\sinh \beta = -x$$

$$\beta = \sinh^{-1}(-x)$$

$$= -\sinh^{-1}(x)$$

$$= -\log\left(x + \sqrt{x^2+1}\right)$$

Hence, $$\cos^{-1}(ix) = \frac{\pi}{2} - i \log\left(x + \sqrt{x^2+1}\right)$$

(iv) Let $\sin^{-1}\left(e^{i\theta}\right) = x + iy$

$$e^{i\theta} = \sin(x + iy)$$

$$\cos\theta + i\sin\theta = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$\cos\theta = \sin x \cosh y \quad \ldots (1)$$

$$\sin\theta = \cos x \sinh y \quad \ldots (2)$$

Eliminating y from Eq. (1) and Eq. (2),

$$\cosh^2 y - \sinh^2 y = \frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x}$$

$$1 = \frac{\cos^2\theta\cos^2 x - \sin^2\theta\sin^2 x}{\sin^2 x\cos^2 x}$$

$$\sin^2 x\cos^2 x = \cos^2\theta\cos^2 x - (\sin^2\theta)(1-\cos^2 x)$$

$$(1-\cos^2 x)\cos^2 x = (1-\sin^2\theta)\cos^2 x - \sin^2\theta + \sin^2\theta\cos^2 x$$

$$\cos^4 x = \sin^2\theta$$

$$\cos^2 x = \sin\theta$$

$$\cos x = \pm\sqrt{\sin\theta}$$

$$x = \cos^{-1}\left(\pm\sqrt{\sin\theta}\right)$$

From Eq. (2),

$$\sin^2\theta = \cos^2 x \sinh^2 y$$

Putting $\cos^2 x = \sin\theta$,

$$\sin^2\theta = \sin\theta \sinh^2 y$$

$$\sinh^2 y = \sin\theta$$

$$\sinh y = \pm\sqrt{\sin\theta}$$

$$y = \sinh^{-1}\left(\pm\sqrt{\sin\theta}\right)$$

$$= \log\left(\pm\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

Hence, $$\sin^{-1}\left(e^{i\theta}\right) = \cos^{-1}\left(\pm\sqrt{\sin\theta}\right) + i\log\left(\pm\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

(v) Let $\sin^{-1}(\text{cosec}\ \theta) = x + iy$

$$\text{cosec}\,\theta = \sin(x + iy)$$

$$\text{cosec}\,\theta = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$\text{cosec}\ \theta = \sin x \cosh y \quad \ldots (1)$$

$$0 = \cos x \sinh y \quad \ldots (2)$$

From Eq. (2),

$$\cos x = 0,\ x = \frac{\pi}{2}$$

Putting $x = \dfrac{\pi}{2}$ in Eq. (1),

$$\text{cosec}\ \theta = \sin\frac{\pi}{2}\cosh y$$

$$= \cosh y$$

$$y = \cosh^{-1}(\text{cosec}\ \theta)$$

$$= \log\left(\text{cosec}\,\theta + \sqrt{\text{cosec}^2\theta - 1}\right)$$

$$= \log(\text{cosec}\,\theta + \cot\theta)$$

$$= \log\left(\frac{1 + \cos\theta}{\sin\theta}\right)$$

$$= \log\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \log\left(\cot\frac{\theta}{2}\right)$$

Hence, $$\sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$$

Example 2

Expand $\cosh(z_1 + z_2)$. **[Winter 2012]**

Solution

$$\begin{aligned}\cosh(z_1+z_2) &= \cos i(z_1+z_2)\\ &= \cos(iz_1+iz_2)\\ &= \cos iz_1\cos iz_2 - \sin iz_1 \sin iz_2\\ &= \cosh z_1 \cosh z_2 - (i\sinh z_1)(i\sinh z_2)\\ &= \cosh z_1\cosh z_2 + \sinh z_1 \sinh z_2\end{aligned}$$

Example 3

If $\cos(u+iv) = x + iy$, *show that*

(i) $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$

(ii) $(1-x)^2 + y^2 = (\cosh v - \cos u)^2$

Solution

$$\cos(u+iv) = x + iy$$

$$\cos u \cos iv - \sin u \sin iv = x + iy$$

$$\cos u \cosh v - i \sin u \sinh v = x + iy$$

(i) Consider $1 + x + iy = 1 + \cos u \cosh v - i \sin u \sinh v$

Taking modulus on both the sides and squaring,

$$\begin{aligned}(1+x)^2 + y^2 &= (1+\cos u\cosh v)^2 + \sin^2 u \sinh^2 v\\ &= 1 + 2\cos u\cosh v + \cos^2 u\cosh^2 v + (1-\cos^2 u)(\cosh^2 v - 1)\\ &= 1 + 2\cos u\cosh v + \cos^2 u\cosh^2 v\\ &\qquad + \cosh^2 v - 1 - \cos^2 u\cosh^2 v + \cos^2 u\\ &= (\cosh v + \cos u)^2.\end{aligned}$$

(ii) Consider $1 - x - iy = 1 - \cos u \cosh v + i \sin u \sinh v$

Taking modulus on both the sides and squaring,

$$\begin{aligned}(1-x)^2 + y^2 &= (1-\cos u\cosh v)^2 + \sin^2 u\sinh^2 v\\ &= 1 - 2\cos u\cosh v + \cos^2 u\cosh^2 v + (1-\cos^2 u)(\cosh^2 v - 1)\\ &= (\cosh v - \cos u)^2\end{aligned}$$

Example 4

If $\sinh(\theta + i\phi) = e^{i\alpha}$, *prove that*

(i) $\sinh^4 \theta = \cos^2 \alpha$

(ii) $\cos^2 \phi = \cos^2 \alpha$

Solution

$$\sinh(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\begin{aligned}\cos\alpha + i\sin\alpha &= \sinh\theta\cosh i\phi + \cosh\theta\sinh i\phi \\ &= \sinh\theta\cos i(i\phi) + \cosh\theta\,[-i\sin i\,(i\phi)] \\ &= \sinh\theta\cos(-\phi) + \cosh\theta\,[-i\sin(-\phi)] \\ &= \sinh\theta\cos\phi + i\cosh\theta\sin\phi\end{aligned}$$

Comparing real and imaginary parts,

$$\cos\alpha = \sinh\theta\cos\phi \quad \ldots (1)$$

$$\sin\alpha = \cosh\theta\sin\phi \quad \ldots (2)$$

(i) Eliminating ϕ from Eqs (1) and (2),

$$\cos^2\phi + \sin^2\phi = \frac{\cos^2\alpha}{\sinh^2\theta} + \frac{\sin^2\alpha}{\cosh^2\theta}$$

$$1 = \frac{\cos^2\alpha}{\sinh^2\theta} + \frac{\sin^2\alpha}{\cosh^2\theta}$$

$$\sinh^2\theta\cosh^2\theta = \cos^2\alpha\cosh^2\theta + \sin^2\alpha\sinh^2\theta$$

$$\sinh^2\theta(1 + \sinh^2\theta) = \cos^2\alpha(1 + \sinh^2\theta) + (1 - \cos^2\alpha)\sinh^2\theta$$

$$\sinh^2\theta + \sinh^4\theta = \cos^2\alpha + \cos^2\alpha\sinh^2\theta + \sinh^2\theta - \cos^2\alpha\sinh^2\theta$$

$$\sinh^4\theta = \cos^2\alpha \quad \ldots (3)$$

(ii) From Eq. (1),

$$\cos^2\alpha = \sinh^2\theta\cos^2\phi$$

$$\cos^2\phi = \frac{\cos^2\alpha}{\sinh^2\theta} = \frac{\cos^2\alpha}{\cos\alpha} \quad \text{[Using (3)]}$$

$$\cos^2\phi = \cos\alpha$$

Example 5

If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$ *then prove that* $\cos 2\theta + \cosh 2\phi = 2$.

Solution

$$\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\begin{aligned}\cos\alpha + i\sin\alpha &= \cos\theta\cos i\phi - \sin\theta\sin i\phi \\ &= \cos\theta\cosh\phi - i\sin\theta\sinh\phi\end{aligned}$$

Comparing real and imaginary parts,

$$\cos\alpha = \cos\theta\cosh\phi \quad \ldots(1)$$

$$\sin\alpha = -\sin\theta\sinh\phi \quad \ldots(2)$$

Squaring and adding Eqs (1) and(2),

$$\cos^2\alpha+\sin^2\alpha=\cos^2\theta\cosh^2\phi+\sin^2\theta\sinh^2\phi$$

$$1=\frac{(1+\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2}+\frac{(1-\cos 2\theta)}{2}\frac{(\cosh 2\phi-1)}{2}$$

$$4=1+\cosh 2\phi+\cos 2\theta+\cos 2\theta\cosh 2\phi$$
$$+\cosh 2\phi-1-\cos 2\theta\cosh 2\phi+\cos 2\theta$$
$$=2(\cosh 2\phi+\cos 2\theta)$$

$$\therefore\ \cosh 2\phi+\cos 2\theta=2$$

Example 6

If $\cos(\theta+i\phi)=r(\cos\alpha+i\sin\alpha)$, *prove that*

$$\phi=\frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$$

Solution

$$\cos(\theta+i\phi)=r(\cos\alpha+i\sin\alpha)$$
$$r(\cos\alpha+i\sin\alpha)=\cos\theta\cos i\phi-\sin\theta\sin i\phi$$
$$=\cos\theta\cosh\phi-i\sin\theta\sinh\phi$$

Comparing real and imaginary parts,

$$r\cos\alpha=\cos\theta\cosh\phi \quad \text{... (1)}$$
$$r\sin\alpha=-\sin\theta\sinh\phi \quad \text{... (2)}$$

Dividing Eq. (2) by Eq. (1),

$$\frac{r\sin\alpha}{r\cos\alpha}=\frac{-\sin\theta\sinh\phi}{\cos\theta\cosh\phi}$$
$$\tan\alpha=-\tan\theta\tanh\phi$$
$$\tanh\phi=-\tan\alpha\cot\theta$$

$$\phi=\tanh^{-1}(-\tan\alpha\cot\theta)$$
$$=\frac{1}{2}\log\left(\frac{1-\tan\alpha\cot\theta}{1+\tan\alpha\cot\theta}\right)$$
$$=\frac{1}{2}\log\left(\frac{\cos\alpha\sin\theta-\sin\alpha\cos\theta}{\cos\alpha\sin\theta+\sin\alpha\cos\theta}\right)$$
$$=\frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$$

Example 7

If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, *show that* $\cos 2\theta \cosh 2\phi = 3$.

Solution

$$\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$$
$$\tan\alpha + i\sec\alpha = \sin\theta\cos i\phi + \cos\theta\sin i\phi$$
$$= \sin\theta\cosh\phi + i\cos\theta\sinh\phi$$

Comparing real and imaginary parts,

$$\tan\alpha = \sin\theta\cosh\phi \quad \ldots (1)$$
$$\sec\alpha = \cos\theta\sinh\phi \quad \ldots (2)$$

Eliminating α from Eqs (1) and (2),

$$\sec^2\alpha - \tan^2\alpha = \cos^2\theta\sinh^2\phi - \sin^2\theta\cosh^2\phi$$

$$1 = \frac{(1+\cos 2\theta)}{2}\frac{(\cosh 2\phi - 1)}{2} - \frac{(1-\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2}$$

$$4 = \cosh 2\phi - 1 + \cos 2\theta\cosh 2\phi - \cos 2\theta - 1 - \cosh 2\phi + \cos 2\theta + \cos 2\theta\cosh 2\phi$$
$$= -2 + 2\cos 2\theta\cosh 2\phi$$
$$\therefore \cos 2\theta\cosh 2\phi = 3$$

Example 8

If $\sin(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, *prove that*

(i) $R^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$

(ii) $\tan\alpha = \tanh\phi\, cot\, \theta$.

Solution

$$\sin(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$$
$$R(\cos\alpha + i\sin\alpha) = \sin\theta\cosh\phi + i\cos\theta\sinh\phi$$

Comparing real and imaginary parts,

$$R\cos\alpha = \sin\theta\cosh\phi \quad \ldots (1)$$
$$R\sin\alpha = \cos\theta\sinh\phi \quad \ldots (2)$$

Eliminating α from Eqs (1) and (2),

$$R^2(\cos^2\alpha + \sin^2\alpha) = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$

$$R^2 = \frac{(1-\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2} + \frac{(1+\cos 2\theta)}{2}\frac{(\cosh 2\phi - 1)}{2}$$
$$= \frac{2\cosh 2\phi - 2\cos 2\theta}{4}$$
$$= \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$$

Dividing Eq. (2) by Eq. (1),

$$\tan\alpha = \frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi}$$
$$= \cot\theta\tanh\phi$$

Example 9

If $u + iv = \cosh\left(\alpha + i\frac{\pi}{4}\right)$, *find* $(u^2 - v^2)$.

Solution

$$u+iv = \cosh\left(\alpha + i\frac{\pi}{4}\right)$$
$$= \cos\left(i\alpha - \frac{\pi}{4}\right)$$
$$= \cos i\alpha\cos\frac{\pi}{4} + \sin i\alpha\sin\frac{\pi}{4}$$
$$= \cosh\alpha\cos\frac{\pi}{4} + i\sinh\alpha\sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}\cosh\alpha + i\frac{1}{\sqrt{2}}\sinh\alpha$$

Comparing real and imaginary parts,

$$u = \frac{\cosh\alpha}{\sqrt{2}}, \qquad v = \frac{\sinh\alpha}{\sqrt{2}}$$
$$u^2 - v^2 = \frac{1}{2}(\cosh^2\alpha - \sinh^2\alpha) = \frac{1}{2}$$

Example 10

Prove that $2e^{2x} = \cosh 2v - \cos 2u$, *where* $e^z = \sin(u + iv)$ *and* $z = x + iy$.

Solution

$$e^z = \sin u\cos iv + \cos u\sin iv$$
$$e^{x+iy} = \sin u\cosh v + \cos u\,(i\sinh v)$$
$$e^x e^{iy} = \sin u\cosh v + i\cos u\sinh v$$
$$e^x(\cos y + i\sin y) = \sin u\cosh v + i\cos u\sinh v$$

Comparing real and imaginary parts,

$$e^x \cos y = \sin u \cosh v \quad \ldots (1)$$

$$e^x \sin y = \cos u \sinh v \quad \ldots (2)$$

Squaring and adding Eqs (1) and (2),

$$e^{2x}(\cos^2 y + \sin^2 y) = \sin^2 u \cosh^2 v + \cos^2 u \sinh^2 v$$

$$e^{2x} = (1 - \cos^2 u)\cosh^2 v + \cos^2 u (\cosh^2 v - 1)$$

$$= \cosh^2 v - \cos^2 u$$

$$= \frac{1+\cosh 2v}{2} - \frac{1+\cos 2u}{2}$$

$$2e^{2x} = \cosh 2v - \cos 2u$$

Example 11

If $\sin^{-1}(\alpha + i\beta) = x + iy$, *prove that* $\sin^2 x$ *and* $\cosh^2 y$ *are the roots of the equation* $\lambda^2 - (\alpha^2 + \beta^2 + 1)\lambda + \alpha^2 = 0$.

Solution

$$\sin^{-1}(\alpha + i\beta) = x + iy$$

$$\alpha + i\beta = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$\alpha = \sin x \cosh y$$

$$\beta = \cos x \sinh y$$

Consider $\alpha^2 + \beta^2 + 1 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y + 1$

$$= \sin^2 x \cosh^2 y + (1 - \sin^2 x)(\cosh^2 y - 1) + 1$$

$$= \sin^2 x \cosh^2 y + \cosh^2 y - 1 - \sin^2 x \cosh^2 y + \sin^2 x + 1$$

$$= \cosh^2 y + \sin^2 x$$

Also, $\alpha^2 = \cosh^2 y \sin^2 x$

Substituting $(1 + \alpha^2 + \beta^2)$ and α^2 in the given equation,

$$\lambda^2 - (\cosh^2 y + \sin^2 x)\lambda + \cosh^2 y \sin^2 x = 0$$

Comparing with x^2 – (sum of roots) x + product of roots = 0,
$\cosh^2 y$ and $\sin^2 x$ are the roots of the given equation.

Example 12

Separate into real and imaginary parts:

(i) $\tan(x + iy)$ *(ii)* $\tan^{-1}(x + iy)$ *(iii)* $\tan^{-1}(e^{i\theta})$.

Solution

(i)
$$\tan(x+iy)=\frac{\sin(x+iy)}{\cos(x+iy)}$$
$$=\frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}$$
$$=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$
$$\text{Real part}=\frac{\sin 2x}{\cos 2x+\cosh 2y}$$
$$\text{Imaginary part}=\frac{\sinh 2y}{\cos 2x+\cosh 2y}$$

(ii) Let
$$\tan^{-1}(x+iy)=a+ib \quad ...(1)$$
$$\tan^{-1}(x-iy)=a-ib \quad ...(2)$$

Adding Eqs (1) and (2),
$$2a=\tan^{-1}(x+iy)+\tan^{-1}(x-iy)$$
$$=\tan^{-1}\left[\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}\right]$$
$$a=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

Subtracting Eq. (2) from Eq. (1),
$$2ib=\tan^{-1}(x+iy)-\tan^{-1}(x-iy)$$
$$=\tan^{-1}\left[\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}\right]$$
$$\tan 2ib=\frac{2iy}{1+x^2+y^2}$$
$$i\tanh 2b=\frac{2iy}{1+x^2+y^2}$$
$$2b=\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$
$$=\frac{1}{2}\log\left[\frac{1+\dfrac{2y}{1+x^2+y^2}}{1-\dfrac{2y}{1+x^2+y^2}}\right]$$

$$= \frac{1}{2}\log\left(\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y}\right)$$

$$b = \frac{1}{4}\log\left[\frac{(1+y)^2+x^2}{(1-y)^2+x^2}\right]$$

Hence, $$\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{4}\log\left[\frac{(1+y)^2+x^2}{(1-y)^2+x^2}\right]$$

(iii) Let $$\tan^{-1}\left(e^{i\theta}\right) = x + iy \quad \ldots (1)$$

$$\tan^{-1}\left(e^{-i\theta}\right) = x - iy \quad \ldots (2)$$

Adding Eqs (1) and (2),

$$2x = \tan^{-1}(e^{i\theta}) + \tan^{-1}(e^{-i\theta}) = \tan^{-1}\left(\frac{e^{i\theta}+e^{-i\theta}}{1-e^{i\theta}\cdot e^{-i\theta}}\right) = \tan^{-1}\infty = n\pi + \frac{\pi}{2}$$

$$x = \frac{n\pi}{2} + \frac{\pi}{4}$$

Subtracting Eq. (2) from Eq. (1),

$$2iy = \tan^{-1}(e^{i\theta}) - \tan^{-1}(e^{-i\theta}) = \tan^{-1}\left(\frac{e^{i\theta}-e^{-i\theta}}{1+e^{i\theta}\cdot e^{-i\theta}}\right)$$

$$\tan 2iy = \frac{2i\sin\theta}{2}$$

$$i\tanh 2y = i\sin\theta$$

$$2y = \tanh^{-1}(\sin\theta)$$

$$= \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

$$= \frac{1}{2}\log\left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)}\right]$$

$$= \frac{1}{2}\log\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$= \frac{1}{2}\log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^2$$

$$= \log\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= -\log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$y = -\frac{1}{2}\log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

Hence, $\tan^{-1}(e^{i\theta}) = \left(n + \frac{1}{2}\right)\frac{\pi}{2} - \frac{i}{2}\log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

Example 13

If $\tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$, *prove that* $x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1$.

Solution

$$\tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$$

$$\left(\frac{\pi}{6} + i\alpha\right) = \tan^{-1}(x + iy) \quad \text{... (1)}$$

$$\left(\frac{\pi}{6} - i\alpha\right) = \tan^{-1}(x - iy) \quad \text{... (2)}$$

Adding Eqs (1) and (2),

$$\frac{2\pi}{6} = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)$$

$$= \tan^{-1}\left[\frac{x + iy + x - iy}{1 - (x + iy)(x - iy)}\right]$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{2x}{1 - x^2 - y^2}$$

$$\sqrt{3} = \frac{2x}{1 - x^2 - y^2}$$

$$1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$$

$$x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1$$

Example 14

If $\tan(x + iy) = i$, *where x and y are real then show that x is indeterminate and y is infinite.*

Solution

$$\tan(x + iy) = i$$

$$x + iy = \tan^{-1}(i) \quad \ldots (1)$$

$$x - iy = \tan^{-1}(-i) \quad \ldots (2)$$

Adding Eqs (1) and (2),

$$2x = \tan^{-1} i + \tan^{-1}(-i)$$

$$= \tan^{-1}\left[\frac{i+(-i)}{1-i(-i)}\right]$$

$$= \tan^{-1}\left(\frac{i-i}{1+i^2}\right)$$

$$= \tan^{-1}\frac{0}{0}$$

$$= \text{indeterminate}$$

Subtracting Eq. (2) from Eq. (1),

$$2iy = \tan^{-1} i - \tan^{-1}(-i)$$

$$= \tan^{-1}\left[\frac{i-(-i)}{1+i(-i)}\right]$$

$$= \tan^{-1} i$$

$$\tan 2iy = i$$

$$i \tanh 2y = i$$

$$\tanh 2y = 1$$

$$2y = \tanh^{-1}(1) = \frac{1}{2}\log\left(\frac{1+1}{1-1}\right)$$

$$y = \frac{1}{4}\log\infty = \infty$$

Example 15

If $\cot\left(\frac{\pi}{3} + i\alpha\right) = x + iy$, *prove that* $x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1$.

Solution

$$\cot\left(\frac{\pi}{3} + i\alpha\right) = x + iy$$

$$\tan\left(\frac{\pi}{3} + i\alpha\right) = \frac{1}{x+iy}$$

$$\frac{\pi}{3}+i\alpha=\tan^{-1}\left(\frac{1}{x+iy}\right)$$

$$\frac{\pi}{3}-i\alpha=\tan^{-1}\left(\frac{1}{x-iy}\right)$$

$$\left(\frac{\pi}{3}+i\alpha\right)+\left(\frac{\pi}{3}-i\alpha\right)=\tan^{-1}\left(\frac{1}{x+iy}\right)+\tan^{-1}\left(\frac{1}{x-iy}\right)$$

$$\frac{2\pi}{3}=\tan^{-1}\left[\frac{\frac{1}{x+iy}+\frac{1}{x-iy}}{1-\frac{1}{x+iy}.\frac{1}{x-iy}}\right]$$

$$=\tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right)$$

$$\tan\left(\frac{2\pi}{3}\right)=\frac{2x}{x^2+y^2-1}$$

$$-\sqrt{3}=\frac{2x}{x^2+y^2-1}$$

$$x^2+y^2+\frac{2x}{\sqrt{3}}=1$$

EXERCISE 1.6

1. Separate into real and imaginary parts:

(i) cot $(x + iy)$ (ii) sec $(x + iy)$ (iii) cosec $(x + iy)$

(iv) $(\sin\theta + i\cos\theta)^{i.}$

Ans. : (i) $\dfrac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$ (ii) $\dfrac{2(\cos x\cosh y - i\sin x\sinh y)}{\cos 2x + \cosh 2y}$

(iii) $\dfrac{2(\sin x\cosh y - i\cos x\sinh y)}{\cosh 2y - \cos 2x}$ (iv) $e^{\theta-\frac{\pi}{2}}$

2. If $\sin(\alpha + i\beta) = x + iy$, prove that

(i) $x^2\,\text{sech}^2\,\beta + y^2\,\text{cosech}^2\,\beta = 1$ (ii) $x^2\,\text{cosec}^2\,\alpha - y^2\sec^2\alpha = 1.$

3. If $\sin(\theta + i\phi) = p(\cos\alpha + i\sin\alpha)$, prove that $p^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$, $\tan\alpha = \tanh\phi\cot\theta$.

4. If $\cosh(\theta + i\phi) = e^{i\alpha}$, prove that $\sin^2\alpha = \sin^4\phi = \sinh^4\theta$.

5. If $\cos(x + iy) \cos(u + iv) = 1$, where x, y, u, v are real then prove that $\tanh^2 v \cosh^2 y = \sin^2 x$.

6. Prove that $\tan\left(\frac{u+iv}{2}\right) = \frac{\sin u + i \sinh v}{\cos u + \cosh v}$.

7. If $\tan(x + iy) = e^{i\theta}$, prove that

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$ (ii) $y = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.

8. If $\tan(\theta + i\phi) = \tan\alpha + i \sec\alpha$ then prove that

(i) $e^{2\phi} = \cot\frac{\alpha}{2}$ (ii) $2\theta = n\pi + \frac{\pi}{2} + \alpha$.

9. Prove that all solutions of the equation $\sin z = 2i \cos z$ are given by

$z = \frac{n\pi}{2} + \frac{i}{2}\log 3$.

10. If $\cot\left(\frac{\pi}{6} + i\alpha\right) = x + iy$, prove that $x^2 + y^2 - \frac{2x}{\sqrt{3}} = 1$.

11. If $\tanh\left(\alpha + \frac{i\pi}{6}\right) = x + iy$, prove that $x^2 + y^2 + \frac{2y}{\sqrt{3}} = 1$.

12. Separate real and imaginary parts of $\cos^{-1}(e^{i\theta})$.

$\left[\textbf{Ans.:}\ \sin^{-1}\sqrt{\sin\theta} + i\log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)\right]$

13. Prove that $\sin^{-1}(ix) = i\log\left(x + \sqrt{x^2+1}\right) + 2n\pi$.

14. Separate into real and imaginary parts:

(i) $\cos^{-1}(i)$ (ii) $\cos^{-1}\left(\frac{5i}{12}\right)$ (iii) $\sin^{-1}\left(\frac{3i}{4}\right)$ (iv) $\sinh^{-1}(ix)$

(v) $\tanh^{-1}(i)$.

$\left[\textbf{Ans.}: \text{(i)}\ \frac{\pi}{2} + i\log\left(\sqrt{2}-1\right)\ \text{(ii)}\ \frac{\pi}{2} + i\log\frac{2}{3}\right.$

$\left.\text{(iii)}\ i\log 2\ \ \text{(iv)}\ \cosh^{-1}x + \frac{i\pi}{2}\ \ \text{(v)}\ \frac{i\pi}{4}\right]$

15. Prove that $\sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$.

16. If $\log\sin(x + iy) = \alpha + i\beta$ then prove that $\alpha = \frac{1}{2}\log\left(\frac{\cosh 2y - \cos 2x}{2}\right)$,

$\tan\beta = \cot x \tanh y$.

1.13 LOGARITHM OF A COMPLEX NUMBER

If z and w are two complex numbers and $z = e^w$ then $w = \log z$ is called the logarithm of the complex number z.

Let $z = x + iy$

$= re^{i\theta}$

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$$\log z = \log\left(r\, e^{i\theta}\right) = \log r + \log e^{i\theta}$$

$$= \log r + i\theta \log e = \log r + i\theta$$

$$= \log\sqrt{x^2 + y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

Hence,
$$\log(x+iy) = \frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(\frac{y}{x}\right)$$

This is called the principal value of $\log(x + iy)$.

The general value of $\log(x + iy)$ is given as

$$\text{Log}(x + iy) = \log r + i(2n\pi + \theta)$$

$$= \frac{1}{2}\log\left(x^2 + y^2\right) + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= 2n\pi i + \log(x+iy)$$

The general value of $\log(x + iy)$ is denoted by $\text{Log}(x + iy)$, beginning with a capital L to distinguish it from its principal value which is denoted by $\log(x + iy)$.

Example 1

Find the value of

(i) $\log_{-3}(-2)$ *(ii)* $\log(1 + i)$.

Solution

$$\text{(i)}\ \log_{-3}(-2) = \frac{\log_e(-2)}{\log_e(-3)} = \frac{\frac{1}{2}\log 4 + i\tan^{-1}\left(\frac{0}{-2}\right)}{\frac{1}{2}\log 9 + i\tan^{-1}\left(\frac{0}{-3}\right)} = \frac{\log 2 + i\pi}{\log 3 + i\pi}$$

$$\text{(ii)}\ \log(1+i) = \frac{1}{2}\log 2 + i\tan^{-1}\left(\frac{1}{1}\right) = \frac{1}{2}\log 2 + \frac{i\pi}{4}$$

Example 2

Prove that $\log_i i = \dfrac{4n+1}{4m+1}$, *where n, m are integers.*

Solution

$$\begin{aligned}\log_i i &= \frac{\log i}{\log i}\\ &= \frac{\frac{1}{2}\log 1 + i\left(2n\pi + \frac{\pi}{2}\right)}{\frac{1}{2}\log 1 + i\left(2m\pi + \frac{\pi}{2}\right)}\\ &= \frac{i(4n+1)\pi}{i(4m+1)\pi}\\ &= \frac{4n+1}{4m+1}\end{aligned}$$

Example 3

Show that *(i)* $\sin\left(\log i^i\right) = -1$ *(ii)* $\cos\left(\log i^i\right) = 0$

Solution

(i) $$\begin{aligned}\sin\log i^i &= \sin(i\log i)\\ &= \sin\left(i\frac{i\pi}{2}\right)\\ &= \sin\left(\frac{-\pi}{2}\right)\\ &= -1\end{aligned}$$

(ii) $$\begin{aligned}\cos\log i^i &= \cos(i\log i)\\ &= \cos\left(i\frac{i\pi}{2}\right)\\ &= \cos\left(\frac{-\pi}{2}\right)\\ &= 0\end{aligned}$$

Example 4

Show that $\cos(i\overline{z}) = \overline{\cos(iz)}$ *for all z.* **[Winter 2012]**

Solution

$$\begin{aligned}\cos(i\overline{z}) &= \cosh\overline{z}\\ &= \frac{e^{\overline{z}} + e^{-\overline{z}}}{2}\\ &= \overline{\frac{e^{z} + e^{-z}}{2}}\\ &= \overline{\cosh z}\\ &= \overline{\cos(iz)}\end{aligned}$$

Example 5

Show that $\overline{\sin(iz)} = \sin(i\overline{z})$ *if and only if* $z = n\pi i\,(n \in \mathbb{Z})$.

[Winter 2014]

Solution

Let $\overline{\sin(iz)} = \sin(i\overline{z})$

$$\begin{aligned}&\Leftrightarrow \overline{i\sinh z} = i\sinh\overline{z}\\ &\Leftrightarrow \overline{i}\left(\overline{\frac{e^{z} - e^{-z}}{2}}\right) = i\left(\frac{e^{\overline{z}} - e^{-\overline{z}}}{2}\right)\\ &\Leftrightarrow -i\left(\frac{e^{\overline{z}} - e^{-\overline{z}}}{2}\right) = i\left(\frac{e^{\overline{z}} - e^{-\overline{z}}}{2}\right)\\ &\Leftrightarrow -e^{\overline{z}} + e^{-\overline{z}} = e^{\overline{z}} - e^{-\overline{z}}\\ &\Leftrightarrow 2e^{\overline{z}} = 2e^{-\overline{z}}\\ &\Leftrightarrow e^{2\overline{z}} = 1\\ &\Leftrightarrow 2\overline{z}\,\mathrm{Log}\,e = \mathrm{Log}\,1\\ &\Leftrightarrow 2\overline{z} = \log\sqrt{1+0} + i\left[2n\pi + \tan^{-1}\left(\frac{0}{1}\right)\right]\\ &\Leftrightarrow 2\overline{z} = \log 1 + 2n\pi i\\ &\Leftrightarrow 2\overline{z} = 2n\pi i\\ &\Leftrightarrow \overline{z} = n\pi i,\ n \in \mathbb{Z}\end{aligned}$$

Example 6

Prove that $\log(1 + \cos 2\theta + i \sin 2\theta) = \log(2 \cos \theta) + i\theta$.

Solution

$$\log(1+\cos 2\theta + i\sin 2\theta) = \frac{1}{2}\log\left[(1+\cos 2\theta)^2 + \sin^2 2\theta\right] + i\tan^{-1}\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right)$$

$$= \frac{1}{2}\log(4\cos^4\theta + 4\sin^2\theta\cos^2\theta) + i\tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$$

$$= \frac{1}{2}\log\left[4\cos^2\theta\,(\cos^2\theta + \sin^2\theta)\right] + i\tan^{-1}(\tan\theta)$$

$$= \frac{1}{2}\log(2\cos\theta)^2 + i\theta$$

$$= \log(2\cos\theta) + i\theta$$

Example 7

Simplify $\log\left(e^{i\alpha} + e^{i\beta}\right)$.

Solution

$$\log\left(e^{i\alpha} + e^{i\beta}\right) = \log\left[\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)\right]$$

$$= \log\left[(\cos\alpha + \cos\beta) + i\,(\sin\alpha + \sin\beta)\right]$$

$$= \log\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2i\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\left\{\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right\}\right]$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + \log\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + \log e^{i\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + i\left(\frac{\alpha+\beta}{2}\right) \quad \left[\because \log e = 1\right]$$

Example 8

Prove that $i \log\left(\dfrac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x.$

Solution

$$
\begin{aligned}
i\log\left(\frac{x-i}{x+i}\right) &= i\left[\log(x-i)-\log(x+i)\right] \\
&= i\left[\frac{1}{2}\log(x^2+1)+i\tan^{-1}\left(-\frac{1}{x}\right)-\frac{1}{2}\log(x^2+1)-\tan^{-1}\left(\frac{1}{x}\right)\right] \\
&= i\left[i(-\cot^{-1}x-\cot^{-1}x)\right] \\
&= 2\cot^{-1}x \\
&= 2\left(\frac{\pi}{2}-\tan^{-1}x\right) \\
&= \pi - 2\tan^{-1}x
\end{aligned}
$$

Example 9

Express $\log[\sin(x+iy)]$ *in the form of* $a + ib.$

Solution

$$
\begin{aligned}
\log[\sin(x+iy)] &= \log(\sin x\cos iy+\cos x\sin iy) \\
&= \log(\sin x\cosh y+i\cos x\sinh y) \\
&= \frac{1}{2}\log(\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y)+i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right) \\
&= \frac{1}{2}\log[\sin^2 x\cosh^2 y+(1-\sin^2 x)(\cosh^2 y-1)]+i\tan^{-1}(\cot x\cdot\tanh y) \\
&= \frac{1}{2}\log(\sin^2 x\cosh^2 y+\cosh^2 y-1-\sin^2 x\cosh^2 y+\sin^2 x)+i\tan^{-1}(\cot x\tanh y) \\
&= \frac{1}{2}\log(\sinh^2 y+\sin^2 x)+i\tan^{-1}(\cot x\tanh y)
\end{aligned}
$$

Example 10

Prove that $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x\tanh y).$

Solution

$$\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x+iy)] - \log[\sin(x-iy)]$$

$$= \log(\sin x\cosh y + i\cos x\sinh y) - \log(\sin x\cosh y - i\cos x\sinh y)$$

$$= \frac{1}{2}\log(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y) + i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

$$-\frac{1}{2}\log(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y) - i\tan^{-1}\left(\frac{-\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= i\tan^{-1}(\cot x\tanh y) + i\tan^{-1}(\cot x\tanh y)$$

$$= 2i\tan^{-1}(\cot x\tanh y)$$

Example 11

Prove that $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$.

Solution

$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i\log(a+ib) - i\log(a-ib)]$$

$$= \cos i\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a} - \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(-\frac{b}{a}\right)\right]$$

$$= \cos i\left(i\tan^{-1}\frac{b}{a} + i\tan^{-1}\frac{b}{a}\right)$$

$$= \cos\left(-2\tan^{-1}\frac{b}{a}\right)$$

$$= \cos\left(2\tan^{-1}\frac{b}{a}\right)$$

$$= \cos 2\theta, \quad \text{where} \quad \tan\theta = \frac{b}{a}$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}$$

$$= \frac{a^2 - b^2}{a^2 + b^2}$$

Example 12

Find the value of $(-i)^i$. **[Winter 2013]**

Solution

Let $x + iy = (-i)^i$

Taking logarithm on both the sides,

$$\log(x + iy) = i \log(-i)$$

$$= i\left(\frac{1}{2}\log 1 - \frac{i\pi}{2}\right)$$

$$= -\frac{i^2\pi}{2}$$

$$= \frac{\pi}{2}$$

$$x + iy = e^{\frac{\pi}{2}}$$

Hence, $(-i)^i = e^{\frac{\pi}{2}}$

Example 13

Find the principal value of $i^{(1-i)}$.

Solution

Let $x + iy = i^{(1-i)}$

Taking logarithm on both the sides,

$$\log(x + iy) = \log i^{(1-i)}$$

$$= (1-i)\log i$$

$$= (1-i)\left[\frac{1}{2}\log 1 + i\tan^{-1}\left(\frac{1}{0}\right)\right]$$

$$= (1-i)\left[0 + i\tan^{-1}(\infty)\right]$$

$$= (1-i)\left(\frac{i\pi}{2}\right)$$

$$= \frac{i\pi}{2} - \frac{i^2\pi}{2}$$

$$= \frac{i\pi}{2} + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{i\pi}{2}$$

$$x + iy = e^{\left(\frac{\pi}{2} + \frac{i\pi}{2}\right)}$$

$$= e^{\frac{\pi}{2}} e^{\frac{i\pi}{2}}$$

$$= e^{\frac{\pi}{2}} \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= e^{\frac{\pi}{2}} (0 + i)$$

$$= ie^{\frac{\pi}{2}}$$

Hence, the principal value of $i^{(1-i)}$ is $ie^{\frac{\pi}{2}}$.

Example 14

For the principal branch, show that Log $(i^3) \neq 3$ Log i. **[Summer 2013]**

Solution

For the principal branch,

$$\text{Log } i^3 = \log i^3$$

$$= \log(-i)$$

$$= \frac{1}{2}\log(1+0) + i\tan^{-1}\left(\frac{-1}{0}\right)$$

$$= -\frac{i\pi}{2} \qquad \ldots(1)$$

$$3 \text{ Log } i = 3\log i \qquad \text{[For the principal branch]}$$

$$= 3\left[\frac{1}{2}\log(1+0) + i\tan^{-1}\left(\frac{1}{0}\right)\right]$$

$$= 3\left(\frac{i\pi}{2}\right) \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\text{Log}(i^3) \neq 3 \text{ Log } i$$

Example 15

Separate the real and imaginary parts of

(i) $\log_{1-i}(1+i)$ *(ii)* $(1+i)^i$ *(iii)* $(-i)^{-(1-i)}$

Solution

(i) Let $x + iy = \log_{1-i}(1+i)$ $= \dfrac{\log(1+i)}{\log(1-i)}$

$$= \frac{\frac{1}{2}\log 2 + i\tan^{-1}1}{\frac{1}{2}\log 2 + i\tan^{-1}(-1)}$$

$$= \frac{\frac{1}{2}\log 2 + \frac{i\pi}{4}}{\frac{1}{2}\log 2 - \frac{i\pi}{4}}$$

$$= \frac{\left(\log 2 + \frac{i\pi}{2}\right)\left(\log 2 + \frac{i\pi}{2}\right)}{\left(\log 2 - \frac{i\pi}{2}\right)\left(\log 2 + \frac{i\pi}{2}\right)}$$

$$= \frac{(\log 2)^2 - \frac{\pi^2}{4} + i\pi\log 2}{(\log 2)^2 + \frac{\pi^2}{4}}$$

Comparing real and imaginary parts,

$$x = \frac{(\log 2)^2 - \frac{\pi^2}{4}}{(\log 2)^2 + \frac{\pi^2}{4}}, \qquad y = \frac{\pi\log 2}{(\log 2)^2 + \frac{\pi^2}{4}}$$

(ii) Let $x + iy = (1+i)^i$

Taking logarithm on both the sides,

$$\log(x+iy) = i\log(1+i)$$

$$= i\left(\frac{1}{2}\log 2 + i\tan^{-1}1\right)$$

$$= \frac{i}{2}\log 2 - \frac{\pi}{4}$$

$$x+iy = e^{\frac{i}{2}\log 2}\, e^{-\frac{\pi}{4}}$$

$$= e^{i\log\sqrt{2}}\, e^{-\frac{\pi}{4}}$$

$$= e^{-\frac{\pi}{4}}\left[\cos(\log\sqrt{2}) + i\sin(\log\sqrt{2})\right]$$

Comparing real and imaginary parts,

$$x = e^{-\frac{\pi}{4}}\cos(\log\sqrt{2})$$

$$y = e^{-\frac{\pi}{4}}\sin(\log\sqrt{2})$$

(iii) Let $x+iy = (-i)^{-(1-i)}$

$$\log(x+iy) = -(1-i)\log(-i)$$

$$= -(1-i)\left(\frac{1}{2}\log 1 - \frac{i\pi}{2}\right)$$

$$= -(1-i)\left(-\frac{i\pi}{2}\right)$$

$$= \frac{i\pi}{2} - i^2\frac{\pi}{2}$$

$$= i\frac{\pi}{2} + \frac{\pi}{2}$$

$$x+iy = e^{\frac{\pi}{2}+i\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}}\, e^{i\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= ie^{\frac{\pi}{2}}$$

Comparing real and imaginary parts,

$$x = 0$$

$$y = e^{\frac{\pi}{2}}$$

Example 16

Find the real and imaginary parts of $\tanh^{-1}(x + iy)$.

Solution

$$\tanh^{-1}(x+iy) = \frac{1}{2}\log\left[\frac{1+(x+iy)}{1-(x+iy)}\right]$$

$$= \frac{1}{2}[\log\{(1+x)+iy\} - \log\{(1-x)-iy\}]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\{(1+x)^2+y^2\} + i\tan^{-1}\left(\frac{y}{1+x}\right) - \frac{1}{2}\log\{(1-x)^2+y^2\} - i\tan^{-1}\left(-\frac{y}{1-x}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\}\right] + i\left\{\tan^{-1}\left(\frac{y}{1+x}\right) + \tan^{-1}\left(\frac{y}{1-x}\right)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\} + i\tan^{-1}\left\{\frac{\frac{y}{1+x}+\frac{y}{1-x}}{1-\left(\frac{y}{1+x}\right)\left(\frac{y}{1-x}\right)}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\} + i\tan^{-1}\left(\frac{y-xy+y+xy}{1-x^2-y^2}\right)\right]$$

$$= \frac{1}{4}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\} + i\frac{1}{2}\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)$$

Hence, $\mathrm{Re}\{\tanh^{-1}(x+iy)\} = \frac{1}{4}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\}$

and $\mathrm{Im}\{\tanh^{-1}(x+iy)\} = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)$

Example 17

If $i^{\alpha+i\beta} = \alpha + i\beta$, *prove that* $\alpha^2 + \beta^2 = e^{-(4k+1)\pi\beta}$.

Solution

$$i^{\alpha+i\beta} = \alpha + i\beta$$

Taking logarithm on both the sides,

$$(\alpha + i\beta)\log i = \log(\alpha + i\beta)$$

or $$\text{Log}(\alpha+i\beta) = (\alpha+i\beta)\left(2k\pi\, i + \log i\right)$$

$$= (\alpha+i\beta)\left(2k\pi i + \frac{i\pi}{2}\right)$$

$$= i(4k+1)\frac{\pi\alpha}{2} - (4k+1)\frac{\pi\beta}{2}$$

$$\alpha+i\beta = e^{i(4k+1)\frac{\pi\alpha}{2}} e^{-(4k+1)\frac{\pi\beta}{2}} = e^{-(4k+1)\frac{\pi\beta}{2}} e^{i(4k+1)\frac{\pi\alpha}{2}} = re^{i\theta}, \text{ say}$$

where $$r = |\alpha+i\beta| = e^{-(4k+1)\frac{\pi\beta}{2}}$$

$$\sqrt{\alpha^2+\beta^2} = e^{-(4k+1)\frac{\pi\beta}{2}}$$

$$\alpha^2+\beta^2 = e^{-(4k+1)\pi\beta}$$

Example 18

By considering only the principle value, express $(1+i\sqrt{3})^{1+i\sqrt{3}}$ *in the form of* $(a + ib)$.

Solution

Let $a + ib = (1+i\sqrt{3})^{1+i\sqrt{3}}$

Taking logarithm on both the sides,

$$\log(a+ib) = (1+i\sqrt{3})\log(1+i\sqrt{3})$$

$$\log(a+ib) = (1+i\sqrt{3})\left[\frac{1}{2}\log(1+3) + i\tan^{-1}\sqrt{3}\right]$$

$$= (1+i\sqrt{3})\left(\frac{1}{2}\log 4 + \frac{i\pi}{3}\right)$$

$$= (1+i\sqrt{3})\left(\log 2 + \frac{i\pi}{3}\right)$$

$$= \log 2 - \frac{\pi\sqrt{3}}{3} + i\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

$$a+ib = e^{\left(\log 2 - \frac{\pi\sqrt{3}}{3}\right)} e^{i\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)}$$

$$= e^{\left(\log 2 - \frac{\pi\sqrt{3}}{3}\right)}\left[\cos\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right) + i\sin\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)\right]$$

Comparing real and imaginary parts,

$$a = e^{\left(\log 2 - \frac{\pi\sqrt{3}}{3}\right)} \cos\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

$$b = e^{\left(\log 2 - \frac{\pi\sqrt{3}}{3}\right)} \sin\left(\sqrt{3}\log 2 + \frac{\pi}{3}\right)$$

Example 19

Separate the real and imaginary parts of $(\sqrt{i})^{\sqrt{i}}$.

Solution

Let $a + ib = \sqrt{i}$

Taking logarithm on both the sides,

$$\log(a+ib) = \log\sqrt{i}$$

$$= \frac{1}{2}\log i$$

$$= \frac{1}{2}\cdot\frac{i\pi}{2}$$

$$a + ib = e^{\frac{i\pi}{4}}$$

Hence, $\sqrt{i} = e^{\frac{i\pi}{4}}$

Let $\left(\sqrt{i}\right)^{\sqrt{i}} = x + iy$

Taking logarithm on both the sides,

$$\sqrt{i}\log\sqrt{i} = \log(x+iy)$$

$$e^{\frac{i\pi}{4}}\log e^{\frac{i\pi}{4}} = \log(x+iy)$$

$$\log(x+iy) = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\frac{i\pi}{4}$$

$$= \frac{i\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$$

$$x+iy=e^{\frac{-\pi}{4\sqrt{2}}}e^{\frac{i\pi}{4\sqrt{2}}}=e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}}+i\sin\frac{\pi}{4\sqrt{2}}\right)$$

Comparing real and imaginary parts,

$$x=e^{-\frac{\pi}{4\sqrt{2}}}\cos\frac{\pi}{4\sqrt{2}}$$

$$y=e^{-\frac{\pi}{4\sqrt{2}}}\sin\frac{\pi}{4\sqrt{2}}$$

Example 20

Prove that $i^{i^i}=\cos\theta+i\sin\theta$, *where* $\theta=(4n+1)\frac{\pi}{2}e^{-\left(2m+\frac{1}{2}\right)\pi}$.

Solution

Let $$i^i=x+iy$$

Taking logarithm on both the sides,

$$i\,\text{Log}\,i=\text{Log}\,(x+iy)$$

$$i\left(2m\pi i+\log i\right)=\text{Log}\,(x+iy)$$

$$\text{Log}\,(x+iy)=i\left(2m\pi i+\frac{i\pi}{2}\right)$$

$$(x+iy)=e^{-\left(2m+\frac{1}{2}\right)\pi}$$

$$i^i=e^{-\left(2m+\frac{1}{2}\right)\pi}$$

Now,

$$i^{i^i}=\cos\theta+i\sin\theta$$

$$=i^{e^{-\left(2m+\frac{1}{2}\right)\pi}}$$

Taking logarithm on both the sides,

$$\text{Log}\,(\cos\theta+i\sin\theta)=e^{-\left(2m+\frac{1}{2}\right)\pi}\text{Log}\,i$$

$$=e^{-\left(2m+\frac{1}{2}\right)\pi}\left(2n\pi i+\frac{i\pi}{2}\right)$$

$$=i(4n+1)\frac{\pi}{2}e^{-\left(2m+\frac{1}{2}\right)\pi}=i\phi,\ \text{say}$$

$$\therefore \quad \cos\theta + i\sin\theta = e^{i\phi}$$
$$e^{i\theta} = e^{i\phi}$$

Comparing both the sides,

$$\theta = \phi = (4n+1)\frac{\pi}{2}.e^{-\left(2m+\frac{1}{2}\right)\pi}$$

Example 21

Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a geometric progression. **[Summer 2014]**

Solution

Let $x + iy = i^i$

$$\log(x+iy) = \log i^i$$
$$\log(x+iy) = i\log i$$
$$= i\left(\frac{1}{2}\log 1 + i\tan^{-1}\frac{1}{0}\right)$$
$$= i(0 + i\tan^{-1}\infty)$$
$$= i\left(\frac{i\pi}{2}\right)$$
$$= -\frac{\pi}{2}$$
$$x + iy = e^{-\frac{\pi}{2}}$$

Hence, i^i is wholly real and its principal value is $e^{-\frac{\pi}{2}}$.

Let $z = x + iy = i^i$

$$\text{Log}(x+iy) = \text{Log}\, i^i$$
$$= i\,\text{Log}\, i$$
$$= i(2n\pi i + \log i)$$
$$= i\left(2n\pi i + \frac{i\pi}{2}\right)$$
$$= -2n\pi - \frac{\pi}{2}$$

$$x + iy = e^{-\frac{\pi}{2}} e^{-2n\pi}$$

$$= e^{-\frac{\pi}{2}} (e^{-2\pi})^n$$

$$\therefore \quad z = k\alpha^n, \text{ where } k = e^{-\frac{\pi}{2}}, \alpha = e^{-2\pi}$$

Putting $n = 0, 1, 2, 3, 4. \ldots$

$$z_0 = k, z_1 = k\alpha, z_2 = k\alpha^2, z_3 = k\alpha^3, \ldots$$

Hence, the values of i^i form a geometric progression with a common ratio $\alpha = e^{-2\pi}$.

EXERCISE 1.7

1. Find the general value of

(i) log $(-i)$ (ii) log $(\sqrt{3} - i)$ (iii) $\log_2 5$ (iv) sin (log i^i)
(v) cos (log i^i)

$$\left[\begin{array}{ll} \textbf{Ans.:} \text{(i) } i\left(2\pi n - \frac{\pi}{2}\right) & \text{(ii) } \log 2 + i\left(2\pi n - \frac{\pi}{6}\right) \\ \text{(iii) } [(\log 5 \log 2 + 4\pi^2 mn) + i(n\log 2 - m\log 5)2\pi] / [(\log 2)^2 + 4\pi^2 m^2] & \\ \text{(iv) } -1 & \text{(v) } 0 \end{array}\right]$$

2. Find the general value of log $(1 + i)$ + log $(1 - i)$.

[Ans.: log 2**]**

3. Find the general value of $\log(1+i\sqrt{3}) + \log(1-i\sqrt{3})$.

[Ans.: 2log 2**]**

4. Prove that sin $\log_e (i^{-i}) = 1$.

5. Show that log $(-\log i) = \log \frac{\pi}{2} - \frac{i\pi}{2}$.

6. Prove that log $(1 + i \tan a) =$ log sec $a + ia$.

7. Prove that $\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - \frac{i\theta}{2}$.

8. Separate $i^{(1-i)}$ into real and imaginary parts.

$$\left[\textbf{Ans.: } ie^{\left(2n\pi + \frac{\pi}{2}\right)}\right]$$

9. Find the principal value of $(x + iy)^i$ and show that it is entirely real if $\frac{1}{2}\log(x^2 + y^2)$ is a multiple of π.

$$\left[\textbf{Hint : put } \frac{1}{2}\log(x^2 + y^2) = n\pi\right]$$

$$\left[\textbf{Ans.: } e^{\tan^{-1}\left(\frac{y}{x}\right)}[\cos\log(x^2 + y^2) + i\sin\log(x^2 + y^2)\right]$$

10. If $i^{a+ib} = a + ib$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

Points to Remember

Algebra of Complex Numbers

(i) Addition: $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

(ii) Subtraction: $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

(iii) Multiplication: $z_1\,z_2 = (x_1\,x_2 - y_1y_2) + i\,(x_2y_1 + y_2x_1)$

(iv) Division: $$\frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Different Forms of Complex Numbers

(i) Cartesian or Rectangular Form: $z = x + iy$

(ii) Polar Form: $z = r\,(\cos\theta + i\sin\theta)$ which is also denoted as $r \angle \theta$

(iii) Exponential Form: $z = re^{i\theta}$

Modulus and Argument (or Amplitude) of Complex Numbers

Modulus: $|z| = r = \sqrt{x^2 + y^2}$

Argument (or Amplitude): $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Properties of Complex Numbers

(i) $\text{Re}\,(z) = x = \frac{1}{2}(z + \bar{z})$, $\text{Im}\,(z) = y = \frac{1}{2i}(z - \bar{z})$

(ii) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{(z_1\,z_2)} = \bar{z}_1\,\bar{z}_2$

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$

(v) $z\overline{z} = |z|^2 = |\overline{z}|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

(vii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(viii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(ix) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

De Moivre's Theorem

$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$, where n is any real number.

Circular and Hyperbolic Functions

(i) $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

(ii) $\sinh z = \frac{e^{z} - e^{-z}}{2}$, $\cosh z = \frac{e^{z} + e^{-z}}{2}$

Relation between Circular and Hyperbolic Functions

(i) $\sin iz = i \sinh z$, $\sinh z = -i \sin iz$

(ii) $\cos iz = \cosh z$

(iii) $\tan iz = i \tanh z$, $\tanh z = -i \tan iz$

Formulae on Hyperbolic Functions

(i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\coth^2 z - \operatorname{cosech}^2 z = 1$

(iii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

(iv) $\sinh 2z = 2 \sinh z \cosh z$

(v) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

(vi) $\tanh 2z = \frac{2\tanh z}{1 + \tanh^2 z}$

(vii) $\sinh 3z = 3 \sinh z + 4 \sinh^3 z$

(viii) $\cosh 3z = 4 \cosh^3 z - 3 \cosh z$

(ix) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

(x) $\sinh (z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh (z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(xii) $\tanh (z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(xiii) $\sinh z_1 + \sinh z_2 = 2\sinh \dfrac{z_1 + z_2}{2} \cosh \dfrac{z_1 - z_2}{2}$

(xiv) $\sinh z_1 - \sinh z_2 = 2\cosh \dfrac{z_1 + z_2}{2} \sinh \dfrac{z_1 - z_2}{2}$

(xv) $\cosh z_1 + \cosh z_2 = 2\cosh \dfrac{z_1 + z_2}{2} \cosh \dfrac{z_1 - z_2}{2}$

(xvi) $\cosh z_1 - \cosh z_2 = 2\sinh \dfrac{z_1 + z_2}{2} \sinh \dfrac{z_1 - z_2}{2}$

(xvii) $2 \sinh z_1 \cosh z_2 = \sinh (z_1 + z_2) + \sinh (z_1 - z_2)$

(xviii) $2 \cosh z_1 \sinh z_2 = \sinh (z_1 + z_2) - \sinh (z_1 - z_2)$

(xix) $2 \cosh z_1 \cosh z_2 = \cosh (z_1 + z_2) + \cosh (z_1 - z_2)$

(xx) $2 \sinh z_1 \sinh z_2 = \cosh (z_1 + z_2) - \cosh (z_1 - z_2)$

Inverse Hyperbolic Functions

(i) $\sinh^{-1} x = \log (x + \sqrt{x^2 + 1})$

(ii) $\cosh^{-1} x = \log (x + \sqrt{x^2 - 1})$

(iii) $\tanh^{-1} x = \dfrac{1}{2} \log \left(\dfrac{1 + x}{1 - x} \right)$

Separation into Real and Imaginary Parts

(i) $\sin (x \pm iy) = \sin x \cosh y \pm i \cos x \sinh y$

(ii) $\cos (x \pm iy) = \cos x \cosh y \pm i \sin x \sinh y$

(iii) $\tan (x \pm iy) = \dfrac{\sin 2x \pm i \sinh 2y}{\cos 2x + \cosh 2y}$

(iv) $\sinh (x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$

(v) $\cosh (x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$

(vi) $\tanh(x \pm iy) = \dfrac{\sinh 2x \pm i \sin 2y}{\cosh 2x + \cos 2y}$

Logarithm of a Complex Number

Principal Value: $\log(x + iy) = \dfrac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\left(\dfrac{y}{x}\right)$

$= \log r + i\theta$

General Value: $\text{Log}(x + iy) = \dfrac{1}{2}\log(x^2 + y^2) + i\left[2n\pi + \tan^{-1}\left(\dfrac{y}{x}\right)\right]$

$= \log r + i(2n\pi + \theta)$

CHAPTER 2
Analytic Functions

Chapter Outline

2.1 Introduction
2.2 Complex Variable
2.3 Basic Definitions
2.4 Limits
2.5 Continuity
2.6 Differentiability
2.7 Analytic Functions
2.8 Cauchy–Riemann Equations in Polar Form
2.9 Harmonic Functions
2.10 Properties of Analytic Functions
2.11 Construction of Analytic Functions: Milne–Thomson Method

2.1 INTRODUCTION

The theory of functions of a complex variable plays a very important role in solving a large number of problems in the field of engineering and science. The functions of a complex variable are particularly concerned with the analytic function of a complex variable. Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in Physics.

2.2 COMPLEX VARIABLE

Any variable of the form $z = x + iy$ is known as a *complex variable*, where x and y are real and $i = \sqrt{-1}$.

Functions of a Complex Variable

If $z = x + iy$ be a complex variable then $w = f(z) = u + iv$ is known as a function of the complex variable z and is denoted by $w = f(z)$.

2.3 BASIC DEFINITIONS

Distance: $|z - z_0|$ represents the distance between two points z and z_0.

Circle: $|z - z_0| = r$ represents a circle with centre at the point z_0 and radius r.

Interior of a circle: $|z - z_0| < r$ represents the interior of the circle.

Exterior of a circle: $|z - z_0| > r$ represents the exterior of the circle.

Annulus: The region between two concentric circles of radii r_1 and r_2 ($r_2 > r_1$) and centre at z_0 is known as the annulus region and is represented as

$$r_1 < |z - z_0| < r_2$$

Neighbourhood: The set of all points for which $|z - z_0| < r$ is known as the neighbourhood of z_0.

Boundary point: A point which does not lie in the interior or exterior of a region is known as a boundary point.

Open set: A set that does not contain its boundary points is known as an open set.

Closed set: A set that contains all its boundary points is known as a closed set.

Connected set: If any two points of the set can be joined by a polygonal line such that all the points of the line also belong to the set then the set is known as a connected set.

Domain: A set which is open and connected is known as a domain.

Bounded region: A region which can be enclosed in a circle of finite radius is known as a bounded region.

Compact region: A region that is closed and bounded is known as a compact region.

Example 1

Classify and sketch the following regions:

(i) $|z - 4| > 3$ *(ii)* $1 < |2z - 6| < 2$ *(iii)* $|z - 1| + |z + 1| \le 2$

(iv) $-\pi < \text{Re } z < \pi$ *(v)* $|z| < 1$ and $|z| > 2$

(vi) $\text{Re}(iz + 2) > 0$

Solution

(i) $$|z - 4| > 3$$

$$|(x - 4) + iy| > 3$$

$$|(x - 4) + iy|^2 > 9$$

$$(x - 4)^2 + y^2 > 9$$

which represents the exterior of the circle with its centre at (4, 0) and a radius of 3 (Fig. 2.1).

Hence, the region is open and unbounded.

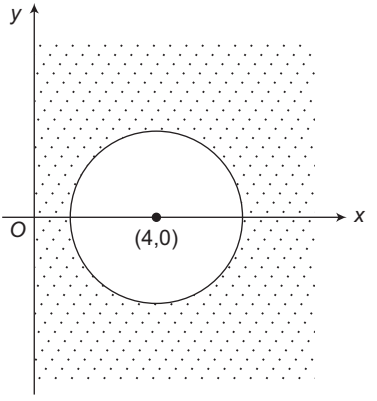


Fig. 2.1

(ii) $1 < |2z-6| < 2$

$$\frac{1}{2} < |z-3| < 1$$

$$\frac{1}{2} < |(x-3)+iy| < 1$$

$$\frac{1}{4} < |(x-3)+iy|^2 < 1$$

$$\frac{1}{4} < (x-3)^2 + y^2 < 1$$

which represents the annular region between two concentric circles with its centre at (3, 0) and radii of $\frac{1}{2}$ and 1 (Fig. 2.2).

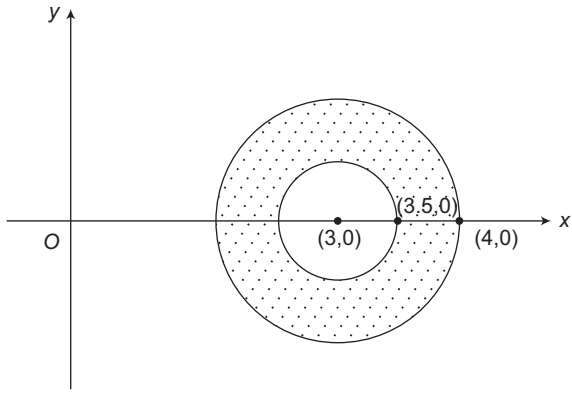


Fig. 2.2

Hence, the region is open and bounded.

(iii) Let $z-1 = r_1 e^{i\theta_1}, \quad z+1 = r_2 e^{i\theta_2}$

i.e., $(x-1)+iy = r_1 e^{i\theta_1}, \quad (x+1)+iy = r_2 e^{i\theta_2}$

where $r_1 = \sqrt{(x-1)^2 + y^2}, \qquad r_2 = \sqrt{(x+1)^2 + y^2}$

$$|z-1| + |z+1| \le 2 \qquad \ldots(1)$$

$$r_1 + r_2 \le 2$$

$$r_1 \le 2 - r_2$$

$$r_1^2 \le 4 + r_2^2 - 4r_2$$

$$(x-1)^2 + y^2 \le 4 + (x+1)^2 + y^2 - 4r_2$$

$$x^2 - 2x + 1 + y^2 \le 4 + x^2 + 2x + 1 + y^2 - 4r_2$$

$$4r_2 \le 4x + 4$$

$$r_2 \le x + 1$$

$$r_2^2 \le (x+1)^2$$

$$(x+1)^2 + y^2 \le (x+1)^2$$

$$y^2 \le 0$$

$$y = 0 \qquad \left[\because \quad y^2 \not< 0\right]$$

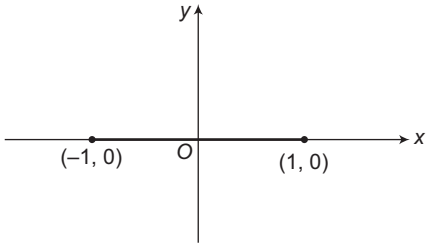


Fig. 2.3

$$\because \quad |z_1 + z_2| \le |z_1| + |z_2|$$

$$|(z-1)+(z+1)| \le |z-1| + |z+1| \le 2 \qquad [\text{From Eq. (1)}]$$

$$|2z| \le 2$$

$$|z| \le 1$$

$$|x| \le 1 \qquad [\because \ y = 0]$$

$$-1 \le x \le 1 \quad \text{and} \quad y = 0$$

which represents the closed interval [–1, 1] (Fig. 2.3). Hence, the region is closed and bounded.

(iv) $-\pi < \text{Re}\,(z) < \pi$

$-\pi < x < \pi$

which represents a vertical strip between $x = -\pi$ to $x = \pi$ (Fig. 2.4).

Hence, the region is closed and unbounded.

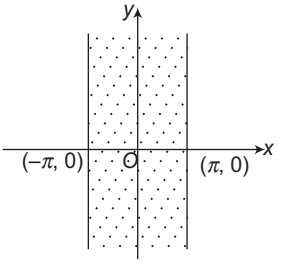


Fig. 2.4

(v) $|z| < 1$ and $|z| > 2$

$|z|^2 < 1$ and $|z|^2 > 4$

$x^2 + y^2 < 1$ and $x^2 + y^2 > 4$

which represents the interior of the circle with its centre at (0, 0) and a radius of 1, and the exterior of the circle with its centre at (0, 0) and a radius of 2 (Fig. 2.5). Hence, the region is not connected.

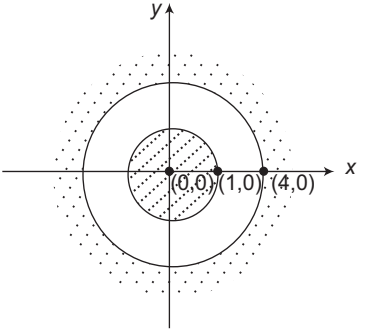


Fig. 2.5

(vi) $\text{Re}\,(iz + 2) > 0$

$\text{Re}\,(ix - y + 2) > 0$

$(2 - y) > 0$

$2 > y$

$y < 2$

which represents the half-plane below the line $y = 2$ (Fig. 2.6). Hence, the region is open and unbounded.

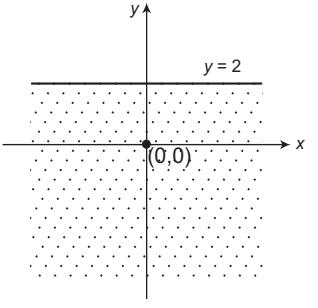


Fig. 2.6

Example 2

Sketch the following sets and determine which are domains?

(i) $\text{Im } z > 1$

(ii) $0 \le \arg z \le \dfrac{\pi}{4}$ **[Winter 2014]**

Solution

Let $z = x + iy$

(i) $\text{Im } z > 1$

$y > 1$

Yes, it is a domain (Fig. 2.7).

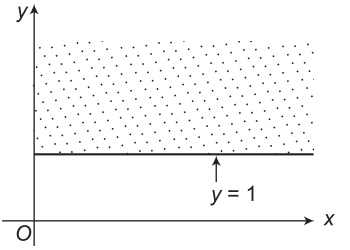


Fig. 2.7

(ii) $0 \le \arg z \le \dfrac{\pi}{4}$

$0 \le \tan^{-1}\left(\dfrac{y}{x}\right) \le \dfrac{\pi}{4}$

$\tan 0 \le \dfrac{y}{x} \le \tan\dfrac{\pi}{4}$

$0 \le \dfrac{y}{x} \le 1$

Yes, it is a domain (Fig. 2.8).

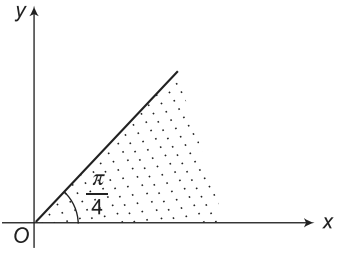


Fig. 2.8

Example 3

Sketch the set $S = (z: -1 < \text{Im } (z) < 2\}$. *Is it connected?*

Solution

Let $z = x + iy$

$-1 < \text{Im}(z) < 2$

$-1 < y < 2$

Since the lines $y = 2$ and $y = -1$ bounding the points of the set are not included in the set, the set is open (Fig. 2.9). Also, each pair of points within the set can be joined by a polygonal line.

Hence, the set S is connected.

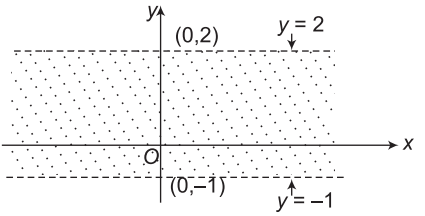


Fig. 2.9

Example 4

Sketch the region $|z| \le 1$. *Is it a domain?* **[Summer 2013]**

Solution

Let $z = x + iy$

$$|z| \le 1$$

$$|x+iy| \le 1$$

$$|x+iy|^2 \le 1$$

$$x^2 + y^2 \le 1$$

which represents the interior of the circle with its centre at (0, 0) and a radius of 1 (Fig. 2.10).

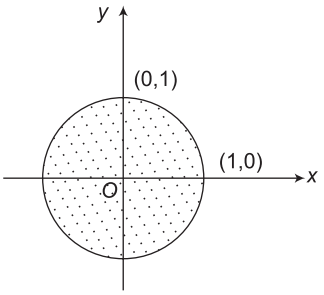


Fig. 2.10

Since, the region $|z| \le 1$ represents the boundary and the interior of the circle, it is closed.

Hence, the given region is not a domain.

Example 5

Sketch the region $|z - 2 + i| \le 1$. *Is it a domain?* **[Winter 2013]**

Solution

Let $z = x + iy$

$$|z-2+i| \le 1$$

$$|(x+iy)-2+i| \le 1$$

$$|(x-2)+i(y+1)| \le 1$$

$$|(x-2)+i(y+1)|^2 \le 1$$

$$(x-2)^2 + (y+1)^2 \le 1$$

which represents the interior of the circle with its centre at (2, –1) and a radius of 1 (Fig. 2.11).

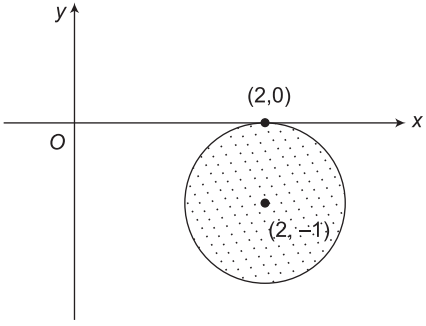


Fig. 2.11

Since the region $|z - 2 + i| \le 1$ represents the boundary and the interior of the circle, it is closed. Hence, the given region is not a domain.

Example 6

Is the set $|z - 1 + 2i| \le 2$ *a domain?* **[Summer 2014]**

Solution

Let $z = x + iy$

$$|z-1+2i| \le 2$$
$$|x+iy-1+2i| \le 2$$
$$|(x-1)+i(y+2)| \le 2$$
$$|(x-1)+i(y+2)|^2 \le 4$$
$$(x-1)^2+(y+2)^2 \le 4$$

which represents a circle with its center at (1, –2) and a radius of 2 (Fig. 2.12).

Since the set $|z - 1 + 2i| \le 2$ represents the boundary and the interior of the circle, it is closed.

Hence, the given region is not a domain.

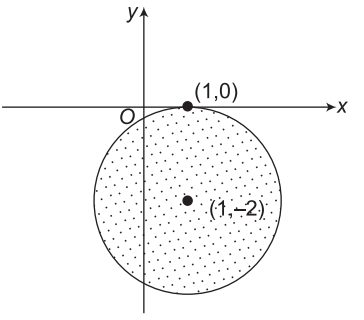


Fig. 2.12

EXERCISE 2.1

I. Classify and sketch the following regions:

1. Re $z \ge 4$

[**Ans.:** Closed, unbounded]

2. $0 < |2z - 1| \le 2$

[**Ans.:** Bounded, neither open nor closed]

3. $|z| \le |z + 1|$

[**Ans.:** Closed, unbounded]

4. $|z - 1 + 3i| \le 1$

[**Ans.:** Closed, unbounded]

II. Sketch the following sets and determine which are domains:

1. $|z - 4| \ge z$

[**Ans.:** No]

2. $|2z + 3| > 4$

[**Ans.:** Yes]

3. $|z - 2 + i| \le 1$

[**Ans.:** No]

4. Im $z = 1$

[**Ans.:** No]

2.4 LIMITS

Let $w = f(z)$ be a single-valued function. The limit of $f(z)$ is $w_0 = f(z_0)$ as z approaches z_0 if for each positive number $\in$ there exists a positive number δ such that

$$|f(z) - w_0| < \in \text{ whenever } |z - z_0| < \delta$$

Symbolically, it is represented as

$$\lim_{z \to z_0} f(z) = w_0$$

Example 1

Using the definition of limit, show that if $f(z) = iz$ in the open disk $|z| < 1$ then

$$\lim_{z \to 1} f(z) = i$$

[Winter 2014]

Solution

$$f(z) = iz, \quad z_0 = 1, \quad w_0 = i$$

Let $|f(z) - w_0| < \in, \quad \text{where } \in > 0$

$$|iz - i| < \in$$

$$|i|\,|z - 1| < \in$$

$$|z - 1| < \in \qquad [\because \ |i| = 1]$$

This condition is satisfied by the points in the open disk $|z| < 1$.

Assuming $\in = \delta$, the following conclusion can be made:

For every $\in > 0$, there exists $\delta > 0$ such that $|f(z) - i| < \in$ whenever $|z - 1| < \delta$

Hence, $\lim_{z \to 1} f(z) = i$.

Example 2

Using the definition of limit, show that $\lim_{z \to 1} \frac{i\overline{z}}{3} = \frac{i}{3}$.

Solution

$$f(z) = \frac{i\overline{z}}{3}, \quad z_0 = 1, \quad w_0 = \frac{i}{3}$$

Let $|f(z) - w_0| < \in, \qquad \text{where } \in > 0$

$$\left|\frac{i\overline{z}}{3} - \frac{i}{3}\right| < \in$$

$$\frac{|i|\,|\overline{z}-1|}{3}<\in$$

$$|\overline{z}-1|<3\in \qquad \left[\because\ |i|=1\right]$$

$$|z-1|<3\in \qquad \left[\because\ |\overline{z}-1|=|(x-1)-iy|=|(x-1)+iy|=|z-1|\right]$$

Assuming $3\in=\delta$, it is concluded that for every $\in>0$, there exists $\delta>0$ such that

$$\left|\frac{i\overline{z}}{3}-\frac{i}{3}\right|<\in \qquad \text{whenever} \quad |z-1|<\delta$$

Hence, $\lim\limits_{z\to 1}\dfrac{i\overline{z}}{3}=\dfrac{i}{3}$.

Example 3

Use the $\in$–δ definition of limit to show that

$$\lim_{z\to 3i}(3x+iy^2)=9i, \qquad \text{where } z=x+iy.$$

Solution

$$f(z)=3x+iy^2,\ z_0=3i,\ w_0=9i$$

$$|z-z_0|=|x+iy-3i|$$

$$=|x+i(y-3)|$$

$$\le |x|+|y-3|$$

Assuming $\delta_1>0$, $\delta_2>0$ such that

$$|x|<\delta_1, |y-3|<\delta_2$$

$$|z-z_0|<\delta_1+\delta_2=\delta$$

$$\therefore\ |z-3i|<\delta,\ \text{where}\ \delta=\delta_1+\delta_2$$

Now,

$$f(z)=3x+iy^2$$

$$=3x+i\left[(y-3)+3\right]^2$$

$$=3x+i\left[(y-3)^2+9+6(y-3)\right]$$

$$|f(z)-w_0|=\left|\left[3x+i(y-3)^2+9i+6i(y-3)\right]-9i\right|$$

$$=|3x+i(y-3)^2+6i(y-3)|$$

$$\le |3x|+|y-3|^2+6|y-3|$$

$$<3\delta_1+\delta_2^2+6\delta_2$$

Assuming $3\delta_1 + \delta_2^2 + 6\delta_2 = \in$,

$$|f(z) - w_0| < \in$$
$$|f(z) - 9i| < \in$$

Thus, for every $\in > 0$, there exists $\delta > 0$ such that

$$|f(z) - 9i| < \in \text{ whenever } |z - 3i| < \delta$$

Hence, $\lim_{z \to 3i} f(z) = 9i$

or $\lim_{z \to 3i} (3x + i\,y^2) = 9i$

Example 4

Show that the limit of the function does not exist.

$$f(z) = z^2 \quad \text{for } z \neq i$$
$$= 0 \quad \text{for } z = i$$

Solution

$$\lim_{z \to i} f(z) = \lim_{z \to i} z^2$$
$$= i^2$$
$$= -1$$

Given $f(i) = 0$

Since $\lim_{z \to i} f(z) \neq f(i)$, the limit does not exist.

Example 5

Show that $\lim_{z \to 0} \frac{z}{\overline{z}}$ *does not exist.*

Solution

Let $f(z) = \frac{z}{\overline{z}}$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\overline{z}}$$
$$= \lim_{z \to 0} \frac{x + iy}{x - iy}$$

The necessary condition for $\lim_{z\to 0} f(z)$ to exist is that $f(z)$ should approach the same value along all the paths leading to $z = 0$.

Let the path be $y = mx$.

$$\lim_{z\to 0} \frac{x+iy}{x-iy} = \lim_{x\to 0} \frac{x+imx}{x-imx}$$
$$= \lim_{x\to 0} \frac{1+im}{1-im}$$
$$= \frac{1+im}{1-im}$$

Since the limit value depends on m, it takes different values along different paths. Hence, the limit does not exist.

Example 6

Show that $\lim_{z\to 0} \frac{3x^2+y^2}{(x+y)^2}$ *does not exist.*

Solution

The necessary condition for $\lim_{z\to 0} f(z)$ to exist is that $f(z)$ should approach the same value along all the paths leading to $z = 0$.

Let the path be $y = mx$.

$$\lim_{z\to 0} \frac{3x^2+y^2}{(x+y)^2} = \lim_{x\to 0} \frac{3x^2+m^2x^2}{(x+mx)^2}$$
$$= \lim_{x\to 0} \frac{3+m^2}{(1+m)^2}$$
$$= \frac{3+m^2}{(1+m)^2}$$

Since the limit value depends on m, it takes different values along different paths. Hence, the limit does not exist.

2.5 CONTINUITY

$f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z)$ exists.

Example 1

Determine whether the function $f(z)$ is continuous at the origin.

$$f(z) = \frac{(x+y)^2}{x^2+y^2}, \quad z \neq 0$$
$$= 0 \qquad , \quad z = 0$$

Solution

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{(x+y)^2}{x^2+y^2}$$

Let $z \to 0$ along the line $y = mx$.

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{(x+mx)^2}{x^2+m^2x^2}$$
$$= \lim_{x \to 0} \frac{(1+m)^2}{1+m^2}$$
$$= \frac{(1+m)^2}{1+m^2}$$

Since the limit depends on m, it takes different values along different paths. Thus, the limit does not exist. Hence, $f(z)$ is not continuous at the origin.

Example 2

Discuss the continuity of $f(z)$ at the origin.

$$f(z) = \frac{\overline{z}}{z}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

Solution

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\overline{z}}{z}$$
$$= \lim_{z \to 0} \frac{x - iy}{x + iy}$$

Let $z \to 0$ along the line $y = mx$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x - imx}{x + imx}$$
$$= \lim_{x\to 0} \frac{1 - im}{1 + im}$$
$$= \frac{1 - im}{1 + im}$$

Since the limit depends on m, it takes different values along different paths. Thus, the limit does not exist. Hence, $f(z)$ is not continuous at the origin.

Example 3

Show that

$$f(z) = \frac{\mathrm{Re}(z^2)}{|z|}, \quad z \neq 0$$
$$= 0, \qquad z = 0$$

is continuous at $z = 0$.

Solution

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\mathrm{Re}(z^2)}{|z|}$$
$$= \lim_{z\to 0} \frac{\mathrm{Re}\left[(x+iy)^2\right]}{|x+iy|}$$
$$= \lim_{z\to 0} \frac{\mathrm{Re}\left[(x^2 - y^2) + i2xy\right]}{\sqrt{x^2 + y^2}}$$
$$= \lim_{z\to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$

Let $z \to 0$ along the line $y = mx$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{\sqrt{x^2 + m^2x^2}}$$
$$= \lim_{x\to 0} \frac{x^2(1 - m^2)}{x\sqrt{1 + m^2}}$$
$$= \lim_{x\to 0} \frac{x(1 - m^2)}{\sqrt{1 + m^2}}$$
$$= 0$$

Since, the limit does not depend on m, it is independent of path.

Hence, the limit exists.

Also, $\lim_{z\to 0} f(z) = f(0) = 0$

Hence, $f(z)$ is continuous at $z = 0$.

Example 4

Determine whether $f(z)$ is continuous at $z = i$. Redefine if necessary to make it continuous.

$$f(z) = z^2 + iz + 2, \quad z \neq i$$
$$= i, \quad z = i$$

Solution

$$\lim_{z\to i} f(z) = \lim_{z\to i} \left(z^2 + iz + 2\right)$$
$$= i^2 + i^2 + 2$$
$$= 2i^2 + 2$$
$$= 0$$

But $f(i) = i$ [Given]

$\therefore \quad \lim_{z\to i} f(z) \neq f(i)$

Hence, $f(z)$ is not continuous at $z = i$.

To make it continuous, it can be redefined as

$$f(z) = z^2 + iz + 2, \quad z \neq i$$
$$= 0, \quad z = i$$

Example 5

Determine whether the function $f(z)$ is continuous at $z = -i$.

$$f(z) = \frac{z^2 + 3iz - 2}{z + i}, \quad z \neq -i$$
$$= 5 \quad , \quad z = -i$$

Can the function be redefined to make it continuous at $z = -i$?

Solution

$$\lim_{z\to -i} f(z) = \lim_{z\to -i} \frac{z^2 + 3iz - 2}{z + i} \qquad \left[\frac{0}{0} \text{ form}\right]$$
$$= \lim_{z\to -i} \frac{2z + 3i}{1} \qquad \text{[Using L' Hospital's rule]}$$
$$= -2i + 3i$$
$$= i$$

But $f(-i) = 5$ [Given]

$\therefore \quad \lim_{z \to -i} f(z) \neq f(-i)$

Hence, $f(z)$ is not continuous at $z = -i$.

To make it continuous, it can be redefined as

$$f(z) = \frac{z^2 + 3iz - 2}{z + i}, \quad z \neq -i$$
$$= i \qquad\qquad , \quad z = -i$$

2.6 DIFFERENTIABILITY

A single-valued function $f(z)$ is said to be differentiable at a point z_0 if

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

Note Derivative of $f(z)$ at a point z_0 can also be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example 1

Find the derivative of $f(z) = \dfrac{1+z}{1-z}$ *by using the definition at* $z = 2$.

Solution

$$f(z) = \frac{1+z}{1-z}$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

At $z = 2$,

$$f'(2) = \lim_{\Delta z \to 0} \frac{f(2 + \Delta z) - f(2)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{1 + (2 + \Delta z)}{1 - (2 + \Delta z)} - \dfrac{1+2}{1-2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{3 + \Delta z}{-1 - \Delta z} + 3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3 + \Delta z - 3 - 3\Delta z}{-(1 + \Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-2\Delta z}{-(1+\Delta z)\,\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{1+\Delta z}$$

$$= 2$$

Example 2

Show that $f(z) = i\,z^3$ is differentiable for all z.

Solution

$$f(z) = iz^3$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{i(z+\Delta z)^3 - iz^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{i\left[z^3 + (\Delta z)^3 + 3z^2\Delta z + 3z(\Delta z)^2\right] - iz^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} i\left[(\Delta z)^2 + 3z^2 + 3z\Delta z\right]$$

$$= 3i\,z^2$$

$f'(z)$ exists for all values of z.
Hence, $f(z)$ is differentiable.

Example 3

Show that $f(z) = z$ Im (z) is differentiable only at $z = 0$ and $f'(0) = 0$.

Solution

$$f(z) = z\ \text{Im}\ (z)$$

$$= (x+iy)\ \text{Im}\ (x+iy)$$

$$= (x+iy)\ y$$

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\ \text{Im}\,(z+\Delta z) - z\ \text{Im}\,(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy+\Delta x+i\Delta y)(y+\Delta y) - (x+iy)\,y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy)y + (x+iy)\,\Delta y + (\Delta x+i\Delta y)(y+\Delta y) - (x+iy)y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{(x+iy)\Delta y + (\Delta x + i\Delta y)(y+\Delta y)}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, Δz may approach zero along any path.

(i) Let the path be parallel to the x-axis, i.e., $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{y\Delta x}{\Delta x}$$
$$= y$$

(ii) Let the path be parallel to the y-axis, i.e., $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{(x+iy)\Delta y + i\Delta y(y+\Delta y)}{i\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{x+iy+i(y+\Delta y)}{i}$$
$$= \frac{x+2iy}{i}$$
$$= -ix+2y$$

Since $f'(z)$ is different along different paths, $f(z)$ is not differentiable.

At $z = 0$, $x = 0$, $y = 0$

$\therefore$ $f'(z) = 0$ at $z = 0$ along all paths.

Hence, $f(z)$ is differentiable only at $z = 0$.

Example 4

Show that $f(z) = |z|^2$ is continuous but not differentiable, at each point in the plane. **[Winter 2014]**

Solution

Let $z = x + iy$

$f(z) = |z|^2 = x^2 + y^2$

$|z|^2$, being a polynomial function, is continuous everywhere in the z-plane.

For any z,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{\Delta x + i\Delta y}$$
$$= \lim_{\Delta z \to 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x + y^2 + (\Delta y)^2 + 2y\Delta y - x^2 - y^2}{\Delta x + i\Delta y}$$
$$= \lim_{\Delta z \to 0} \frac{(\Delta x)^2 + 2x\Delta x + (\Delta y)^2 + 2y\Delta y}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, Δz may approach zero along any path.

(i) Let the path be parallel to the x-axis, i.e., $\Delta y = 0$,

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} (\Delta x + 2x) \\ &= 2x \end{aligned}$$

(ii) Let the path be parallel to the y-axis, i.e., $\Delta x = 0$,

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{(\Delta y)^2 + 2y\Delta y}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\Delta y + 2y}{i} \\ &= \frac{2y}{i} \\ &= -2iy \end{aligned}$$

Since $f'(z)$ is different along different paths, $f(z)$ is not differentiable.

EXERCISE 2.2

Limits

1. Use the definition of limit to prove that

(i) $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$ (ii) $\lim_{z \to i} \frac{z - i}{z^2 + 1} = \frac{1}{2i}$

2. Show that the limit of the function $f(z)$ as z tends to zero does not exist.

(i) $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ (ii) $f(z) = (\bar{z})^2, \quad z \neq 0$
$= 0, \quad z = 0$

3. Prove that $\lim_{z \to i} \left(z + \frac{1}{z}\right) = 0.$

Continuity

1. Show that the function $f(z)$ is continuous of $z = i$.

$$\begin{aligned} f(z) &= \frac{z^2 + 1}{z + i}, \quad z \neq i \\ &= 0, \quad z = i \end{aligned}$$

2. Discuss the continuity of the function $f(z) = (x + y^2) + i\,xy$ in the complex plane.

[**Ans.:** Continuous everwhere]

3. Show that the function $f(z)$ is not continuous at $z = i$.

$$f(z) = z^2, \quad z \neq i$$
$$= 0, \quad z = i$$

4. Determine whether the function $f(z)$ is continuous at the origin. If not then redefine it to make it continuous.

$$f(z) = \frac{z\,\mathrm{Re}(z)}{|z|}, \quad z \neq 0$$
$$= 2, \quad z = 0$$

[**Ans.:** Not continuous, define $f(0) = 0$]

Differentiability

1. Show that the functions $\bar{z}$, Re(z) and Im(z) are not differentiable in the complex plane.

2. Find the derivative of $f(z) = \dfrac{2z - i}{z + zi}$ at $z = -i$ using definition.

$$\left[\textbf{Ans.:}\ \frac{1+i}{2}\right]$$

3. Show that the function $f(z) = z^2\,\bar{z}$ is not differentiable in the complex plane.

4. Determine whether the function $f(z)$ is differentiable at the origin.

$$f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

[**Ans.:** No]

2.7 ANALYTIC FUNCTIONS

A function $f(z)$ is said to be an analytic function if it is defined and differentiable at each point of a region R. This function is also known as a *regular* or *holomorphic function.*

Entire Function A function $f(z)$ is said to be an entire function if it is analytic everywhere in the finite plane (complex plane).

An entire function is not analytic at $z = \infty$, e.g., $\cos z$, $\sin z$, e^z and polynomial functions, etc.

2.7.1 Necessary Conditions for $f(z)$ to be Analytic (Cauchy—Riemann Equations)

The necessary conditions for a function $f(z) = u + iv$ to be an analytic function at all the points in a region R are

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Proof Let $f(z) = u + iv$ be an analytic function and Δu, Δv be the increments in u and v corresponding to increments Δx, Δy in x and y respectively.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[(u+\Delta u) - i(v+\Delta v)] - (u+iv)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta z} \qquad \ldots(2.1)$$

Since $f(z)$ is differentiable at each point of the region R, Δz may approach zero along any path. Consider two paths as follows:

(i) Parallel to *x*-axis $\Delta y = 0$

$$\Delta z = \Delta x + i\Delta y$$

$$= \Delta x$$

Substituting in Eq. (2.1),

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(2.2)$$

(ii) Parallel to *y*-axis $\Delta x = 0$

$$\Delta z = \Delta x + i\Delta y$$

$$= i\Delta y$$

Substituting in Eq. (2.1),

$$f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta u}{i\Delta y} + \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots(2.3)$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.
Equating Eqs (2.2) and (2.3),

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

These equations are known as *Cauchy–Riemann* (*C–R*) equations.

2.7.2 Sufficient Conditions for *f*(*z*) to be Analytic

For a function $f(z)=u+iv$, if the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in the region R and $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ then the function of $f(z)$ is analytic.

2.8 CAUCHY–RIEMANN EQUATIONS IN POLAR FORM

If $f(z) = u + iv$ is an analytic function where u and v are functions of r,θ and $z=re^{i\theta}$ then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Proof $f(z)=u+iv, z=re^{i\theta}$

$$u+iv=f(re^{i\theta}) \qquad ...(2.4)$$

Differentiating Eq. (2.4) partially w.r.t. r,

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta})\cdot e^{i\theta}$$

$$=f'(z)\cdot e^{i\theta}$$

$$f'(z)=\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \qquad ...(2.5)$$

Differentiating Eq. (2.4) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\cdot ire^{i\theta}$$

$$=f'(z)\cdot ire^{i\theta}$$

$$f'(z)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \boldsymbol{\theta}}\right) \qquad ...(2.6)$$

Since $f(z)$ is differentiable, $f'(z)$ must be unique.

Equating Eqs (2.5) and (2.6),

$$\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)=\frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=-\frac{i}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}$$

Comparing real and imaginary parts,

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Example 1

Find the real and imaginary part of $f(z) = z^2 + 3z$. **[Summer 2013]**

Solution

$$\begin{aligned} f(z) &= z^2+3z \\ &= (x+iy)^2+3(x+iy) \\ &= (x^2+i^2y^2+2ixy)+3(x+iy) \\ &= (x^2-y^2+3x)+i(2xy+3y) \end{aligned}$$

Real part $= x^2 - y^2 + 3x$

Imaginary part $= 2xy + 3y$

Example 2

Find the real and imaginary parts of $f(z)=2z^3-3z$.

Solution

$$\begin{aligned} f(z) &= 2z^3-3z \\ &= 2(x+iy)^3-3(x+iy) \\ &= 2[x^3+i^3y^3+3x^2\cdot iy+3x(iy)^2]-3(x+iy) \\ &= 2[x^3-iy^3+i\cdot 3x^2y-3xy^2]-3(x+iy) \\ &= (2x^3-6xy^2-3x)+i(6x^2y-2y^3-3y) \end{aligned}$$

Real part $= 2x^3-6xy^2-3x$

Imaginary part $= 6x^2y-2y^3-3y$

Example 3

Write the function $f(z) = z + \frac{1}{z}$ *in* $f(z) = u(r, \theta) + iv(r, \theta)$ *form.*

[Winter 2013]

Solution

$$f(z) = z + \frac{1}{z}$$

Let $z = re^{i\theta}$

$$f(z) = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Example 4

Determine whether the function $2xy + i(x^2 - y^2)$ *is analytic or not.*

Solution

Let $f(z) = 2xy + i(x^2 - y^2)$

$$u + iv = 2xy + i(x^2 - y^2)$$

Comparing real and imaginary parts,

$$u = 2xy, \qquad v = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2y, \qquad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2x, \qquad \frac{\partial v}{\partial y} = -2y$$

$$\therefore \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.
Hence, $f(z)$ is not analytic.

Example 5

Show that $f(z) = z^3$ is analytic everywhere. **[Summer 2014]**

Solution

$$f(z) = z^3$$

$$u + iv = (x + iy)^3$$

$$= x^3 + (iy)^3 + 3x^2 iy + 3x(iy)^2$$

$$= x^3 - 3xy^2 + i(3x^2 y - y^3)$$

Comparing real and imaginary parts,

$$u = x^3 - 3xy^2, \qquad v = 3x^2 y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C–R equations are satisfied.
Hence, z^3 is analytic everywhere.

Example 6

Test the analyticity of the function $w = \sin z$.

Solution

$$w = \sin z$$

$$u + iv = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y, \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C–R equations are satisfied.
Hence, $\sin z$ is analytic.

Example 7

Are $|z|$, Re (z), *Im*(z) *analytic? Give reasons.*

Solution

Let $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

$$\text{Re}(z) = x$$

$$\text{Im}(z) = y$$

Since all the above functions are real, $v = 0$.

$$\therefore \quad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0 \text{ for all functions.}$$

But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero. Thus, C–R equations are not satisfied for all the functions.

Hence, $|z|$, Re(z), *Im*(z) are not analytic.

Example 8

State the basic difference between the limit of a function of a real variable and that of a complex variable.

Solution

In real calculus, x approaches x_0 only along the line called real line; whereas in case of the complex variable z, z approaches z_0 from any direction in the z-plane.

Example 9

Show that the function $f(z) = \overline{z}$ *is nowhere differentiable.*

[Summer 2014]

Solution

$$f(z) = \overline{z}$$

$$u + iv = x - iy$$

Comparing real and imaginary parts,

$$u = x, \qquad v = -y$$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = -1$$

$$\therefore \quad \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial y}$$

C–R equations are not satisfied.
Hence, $f(z)$ is nowhere differentiable.

Example 10

Discuss the differentiability of $f(z) = x^2 + iy^2$. **[Summer 2015]**

Solution

$$f(z) = x^2 + iy^2$$
$$u + iv = x^2 + iy^2$$

Comparing real and imaginary parts,

$$u = x^2, \qquad v = y^2$$
$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = 2y$$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ except at the origin, i.e., at $x = 0$, $y = 0$.

Since C–R equations are satisfied only at the origin, $f(z)$ is differentiable only at $z = 0$.

Example 11

Show that $f(z) = z|z|$ *is not analytic anywhere.*

Solution

$$f(z) = z\,|z|$$
$$u + iv = (x + iy)\sqrt{x^2 + y^2}$$
$$= x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$$

Comparing real and imaginary parts,

$$u = x\sqrt{x^2 + y^2}, \qquad v = y\sqrt{x^2 + y^2}$$
$$\frac{\partial u}{\partial x} = \sqrt{x^2 + y^2} + x \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x, \qquad \frac{\partial v}{\partial x} = y \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x$$
$$= \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}} \qquad = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y, \qquad \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + y \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$$

$$= \frac{xy}{\sqrt{x^2+y^2}} \qquad = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.

Hence, $f(z)$ is not analytic anywhere.

Example 12

Find the analytic region of $f(z) = (x-y)^2 + 2i(x+y)$.

Solution

$$f(z) = (x-y)^2 + 2i(x+y)$$
$$u + iv = (x-y)^2 + 2i(x+y)$$

Comparing real and imaginary parts,

$$u = (x-y)^2, \qquad v = 2(x+y)$$
$$\frac{\partial u}{\partial x} = 2(x-y), \qquad \frac{\partial v}{\partial x} = 2$$
$$\frac{\partial u}{\partial y} = -2(x-y), \qquad \frac{\partial v}{\partial y} = 2$$

All the partial derivatives are continuous.

For $f(z)$ to be analytic, u and v should satisfy C–R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$2(x-y) = 2, \qquad -2(x-y) = -2$$
$$x - y = 1, \qquad x - y = 1$$

$f(z)$ is analytic at all points satisfying $x - y = 1$.

Hence, the analytic region is the line $x - y = 1$.

Example 13

Examine the analyticity of $\sinh z$. **[Winter 2014]**

Solution

Let
$$\begin{aligned} w &= \sinh z \\ u + iv &= -i \sin i z \\ &= -i \sin i (x + iy) \\ &= -i \sin (ix - y) \\ &= -i(\sin ix \cos y - \cos ix \sin y) \\ &= -i (i \sinh x \cos y - \cosh x \sin y) \\ &= \sinh x \cos y + i \cosh x \sin y \end{aligned}$$

Comparing real and imaginary parts,

$$\begin{aligned} u &= \sinh x \cos y, & v &= \cosh x \sin y \\ \frac{\partial u}{\partial x} &= \cosh x \cos y, & \frac{\partial v}{\partial x} &= \sinh x \sin y \\ \frac{\partial u}{\partial y} &= -\sinh x \sin y, & \frac{\partial v}{\partial y} &= \cosh x \cos y \\ \therefore \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

C–R equations are satisfied.
Hence, $\sinh z$ is analytic.

Example 14

Show that $w = e^z$ *is analytic everywhere in the complex plane. Hence, find* $\dfrac{\mathrm{d}w}{\mathrm{d}z}$.

Solution

$$\begin{aligned} w &= e^z \\ u + iv &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$
$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C–R equations are satisfied. Also e^x, cos y and sin y are continuous for all values of x and y.
Hence, e^z is analytic everywhere in the complex plane.
Since $w = u + iv$ is analytic everywhere,

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= e^x \cos y + ie^x \sin y \\ &= e^x (\cos y + i \sin y) \\ &= e^x e^{iy} \\ &= e^{x+iy} \\ &= e^z\end{aligned}$$

Example 15

Prove that cosh z *is an analytic function and, hence, find its derivative.*

Solution

$$\begin{aligned}f(z) &= \cosh z \\ &= \cos iz \\ u + iv &= \cos i(x + iy) \\ &= \cos(ix - y) \\ &= \cos ix \cos y + \sin ix \sin y \\ &= \cosh x \cos y + i \sinh x \sin y\end{aligned}$$

Comparing real and imaginary parts,

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y$$
$$\frac{\partial u}{\partial x} = \sinh x \cos y, \qquad \frac{\partial v}{\partial x} = \cosh x \sin y$$
$$\frac{\partial u}{\partial y} = -\cosh x \sin y, \qquad \frac{\partial v}{\partial y} = \sinh x \cos y$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C–R equations are satisfied.
Hence, cosh z is an analytic function.

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \sinh x \cos y + i \cosh x \sin y \\ &= -i \sin ix \cos y + i \cos ix \sin y \\ &= -i(\sin ix \cos y - \cos ix \sin y) \\ &= -i \sin(ix - y) \\ &= -i \sin(ix + i^2 y) \\ &= -i \sin i(x + iy) \\ &= -i^2 \sinh(x + iy) \\ &= \sinh z \end{aligned}$$

Example 16

Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ *is analytic.*

Solution

$$\begin{aligned} f(z) &= (x + ay) + i(bx + cy) \\ u + iv &= (x + ay) + i(bx + cy) \end{aligned}$$

Comparing real and imaginary parts,

$$\begin{aligned} u &= x + ay, & v &= bx + cy \\ \frac{\partial u}{\partial x} &= 1, & \frac{\partial v}{\partial x} &= b \\ \frac{\partial u}{\partial y} &= a, & \frac{\partial v}{\partial y} &= c \end{aligned}$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ 1 &= c, & a &= -b \\ c &= 1 \end{aligned}$$

Example 17

Find the constants a, b, c, d so that the function
$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ *is analytic.*

Solution

$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$
$$u + iv = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$

Comparing real and imaginary parts,

$$u = x^2 + axy + by^2, \qquad v = cx^2 + dxy + y^2$$
$$\frac{\partial u}{\partial x} = 2x + ay, \qquad \frac{\partial v}{\partial x} = 2cx + dy$$
$$\frac{\partial u}{\partial y} = ax + 2by, \qquad \frac{\partial v}{\partial y} = dx + 2y$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$2x + ay = dx + 2y \quad \ldots(1), \quad ax + 2by = -(2cx + dy) \quad \ldots(2)$$

Comparing coefficients of x and y in Eqs (1) and (2),

$$d = 2, \qquad a = 2$$
$$a = -2c, \qquad 2b = -d$$
$$c = -1, \qquad b = -1$$

Example 18

Find the values of a and b such that the function $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ *is analytic. Also, find* $f'(z)$.

Solution

$$f(z) = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$$
$$u + iv = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$$

Comparing real and imaginary parts,

$$u = x^2 + ay^2 - 2xy, \qquad v = bx^2 - y^2 + 2xy$$
$$\frac{\partial u}{\partial x} = 2x - 2y, \qquad \frac{\partial v}{\partial x} = 2bx + 2y$$
$$\frac{\partial u}{\partial y} = 2ay - 2x, \qquad \frac{\partial v}{\partial y} = -2y + 2x$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$2x - 2y = -2y + 2x \quad \ldots(1), \quad 2ay - 2x = -2bx - 2y \quad \ldots(2)$$

Comparing coefficients of x and y in Eqs (1) and (2),

$$2a = -2, \qquad -2 = -2b$$
$$a = -1, \qquad b = 1$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= (2x - 2y) + i(2bx + 2y)$$
$$= (2x - 2y) + i(2x + 2y) \qquad [\because b = 1]$$
$$= 2(x + i^2 y + ix + iy) \qquad [\because i^2 = -1]$$
$$= 2[x(1+i) + iy(1+i)]$$
$$= 2(1+i)(x+iy)$$
$$= 2(1+i)z$$

Example 19

If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.

Solution

Since $u + iv$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots(2)$$

(i) C–R equations for $v - iu$ are

$$\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x} = \frac{\partial u}{\partial x}$$

Equations (1) and (2) are satisfied.
Hence, $v - iu$ is analytic.

(ii) C–R equations for $-v + iu$ are

$$\frac{\partial(-v)}{\partial x} = \frac{\partial u}{\partial y}, \qquad \frac{\partial(-v)}{\partial y} = -\frac{\partial u}{\partial x}$$
$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Equations (1) and (2) are satisfied.
Hence, $-v + iu$ is also analytic.

Example 20

Show that an analytic function with constant imaginary part is constant.

Solution

$f(z) = u + iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Imaginary part is constant.

$$v = c$$

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0$$

But $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 0 + i \cdot 0$$

$$= 0$$

Hence, $f(z) =$ constant

Example 21

If $f(z)$ and $\overline{f(z)}$ are both analytic then show that $f(z)$ is constant.

Solution

Let $$f(z) = u + iv$$

$$\overline{f(z)} = u - iv$$

$f(z)$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots(2)$$

$\overline{f(z)}$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y}$$

$$= -\frac{\partial v}{\partial y} \qquad \ldots(3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x}$$

$$= \frac{\partial v}{\partial x} \qquad \ldots(4)$$

Adding Eqs (1) and (3),

$$2\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

Subtracting Eq. (2) from Eq. (4),

$$0 = 2\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 0 + i \cdot 0$$

$$= 0$$

Hence, $f(z) = \text{constant}$

Example 22

If $f(z)$ is an analytic function with constant modulus, prove that $f(z)$ is constant.

Solution

$$|f(z)| = \text{constant}$$

$$\sqrt{u^2 + v^2} = \text{constant}$$

$$u^2 + v^2 = \text{constant} \quad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t x,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \ldots(2)$$

Differentiating Eq. (1) partially w.r.t y,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \ldots(3)$$

Since $f(z) = u + iv$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Substituting in Eq. (3),

$$u\left(-\frac{\partial v}{\partial x}\right)+v\frac{\partial u}{\partial x}=0$$

$$v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}=0 \qquad \ldots(4)$$

Multiplying Eq. (2) by u and Eq. (4) by v and adding,

$$(u^2+v^2)\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial x}=0$$

Multiplying Eq. (2) by v and Eq. (4) by u and subtracting,

$$(v^2+u^2)\frac{\partial v}{\partial x}=0$$

$$\frac{\partial v}{\partial x}=0$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$=0+i\cdot 0$$

$$=0$$

Hence, $f(z)$ = constant

Example 23

Show that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\overline{z}$.

Solution

$$z=x+iy, \qquad \overline{z}=x-iy$$

$$x=\frac{1}{2}(z+\overline{z}), \qquad y=\frac{1}{2i}(z-\overline{z})$$

$\therefore$ u and v can be considered as functions of z and $\overline{z}$ (Fig. 2.13). Hence, w also depends on z and $\overline{z}$.

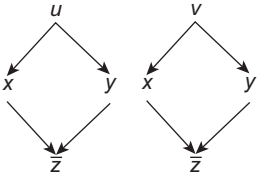


Fig. 2.13

$$w=u+iv$$

$$\frac{\partial w}{\partial \overline{z}}=\frac{\partial u}{\partial \overline{z}}+i\frac{\partial v}{\partial \overline{z}}$$

$$=\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\right)+i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\right)$$

$$=\left[\frac{\partial u}{\partial x}\cdot\frac{1}{2}+\frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right]+i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2}+\frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$= \left[\frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}\right] \quad \text{[Using C–R equations]}$$

$$= 0$$

$\therefore$ w does not depend on $\bar{z}$.

Hence, w can be expressed as a function of z alone, not as a function of $\bar{z}$.

Example 24

If $w = f(z)$ is analytic, prove that $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$, where $z = x + iy$.

Solution

Let $w = u + iv$

Since w is an analytic function,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$w = f(z) = u + iv$$

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) \quad \text{[Using C–R equations]}$$

$$= i\left(\frac{1}{i}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right)$$

$$= i\left(-i\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right)$$

$$= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial}{\partial y}(u + iv)$$

$$= -i\frac{\partial w}{\partial y} \quad \text{... (1)}$$

Again, $$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial x}(u + iv)$$

$$= \frac{\partial w}{\partial x} \quad \text{... (2)}$$

From Eqs (1) and (2),

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

Example 25

Show that $f(z) = \sqrt{|xy|}$ *is not differentiable at origin although C–R equations are satisfied.* **[Summer 2015]**

Solution

$$f(z) = \sqrt{|xy|}$$

$$u + iv = \sqrt{|xy|}$$

Comparing real and imaginary parts,

$$u = \sqrt{|xy|}, \qquad v = 0$$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x}$$

$$= 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - \boldsymbol{u}(x, y)}{\Delta y}$$

At the origin,

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

$$= 0$$

$\because \quad v = 0$

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C–R equations are satisfied at the origin.

Now,
$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$
$$= \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$$

Let $z \to 0$ along the line $y = mx$.

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x.mx|}}{x+imx}$$
$$= \lim_{x\to 0} \frac{x\sqrt{|m|}}{x(1+im)}$$
$$= \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

Since limit depends on m (i.e., on the path), it is not unique. Therefore, $f'(0)$ does not exist.

Hence, $f(z)$ is not analytic at the origin.

Example 26

For what values of z does the function w defined by $z = e^{-v}\left(\cos u + i \sin u\right)$ *cease to be analytic?*

Solution

$$z = e^{-v}(\cos u + i \sin u)$$
$$= e^{-v} e^{iu}$$
$$= e^{i^2 v} e^{iu}$$
$$= e^{i(iv+u)}$$
$$= e^{iw}$$
$$\log z = \log e^{iw}$$
$$= iw$$
$$w = \frac{1}{i}\log z$$
$$= -i \log z$$
$$\frac{dw}{dz} = -\frac{i}{z}$$

w ceases to be analytic, where $\frac{dw}{dz}$ does not exist. i.e., $\frac{dw}{dz} \to \infty$.

At $z = 0$ $\dfrac{dw}{dz} \to \infty$.

Hence, w is not analytic at $z = 0$.

Example 27

Prove that $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \; z \neq 0$

$= 0, \qquad z = 0$

satisfy C–R equations at the origin but $f'(0)$ *does not exist.*

Solution

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \; z \neq 0$$

and $f(0) = 0$

$$u + iv = \left(\frac{x^3 - y^3}{x^2 + y^2}\right) + i\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$$

Comparing real and imaginary parts,

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad v = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

At the origin, $\dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{u(\Delta x, 0) - u(0,0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x}$$

$$= 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

At the origin, $\dfrac{\partial u}{\partial y} = \lim\limits_{\Delta y \to 0} \dfrac{u(0, \Delta y) - u(0,0)}{\Delta y}$

$$= \lim_{\Delta y \to 0} \frac{-(\Delta y) - 0}{\Delta y}$$

$$= -1$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

At the origin, $$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x}$$

$$= 1$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

At the origin, $$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0,0)}{\Delta y}$$

$$= \lim_{x \to 0} \frac{\Delta y}{\Delta y}$$

$$= 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C–R equations are satisfied at the origin.

Now, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\dfrac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)} - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$.

$$f'(0) = \lim_{z \to 0} \frac{(x^3 - m^3 x^3) + i(x^3 + m^3 x^3)}{(x^2 + m^2 x^2)(x + imx)}$$

$$= \lim_{x \to 0} \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

Since limit depends on m (i.e., on the path), it is not unique.

Therefore, $f'(0)$ does not exist.

Hence, $f(z)$ is not analytic at the origin.

Example 28

Show that the function $$f(z) = \frac{\bar{z}^2}{z}, \quad z \neq 0$$

$$= 0, \qquad z = 0$$

satisfies the Cauchy–Riemann equations at the origin, but $f'(0)$ fails to exist. **[Winter 2012]**

Solution

$$f(z) = \frac{\bar{z}^2}{z}, \quad z \neq 0$$

$$u + iv = \frac{(x - iy)^2}{(x + iy)} \times \frac{(x - iy)}{(x - iy)}$$

$$= \frac{(x^2 - y^2 - 2ixy)(x - iy)}{x^2 + y^2}$$

$$= \frac{(x^3 - xy^2 - 2xy^2) - i(x^2 y - y^3 + 2x^2 y)}{x^2 + y^2}$$

$$= \frac{(x^3 - 3xy^2)}{(x^2 + y^2)} + i\left(\frac{y^3 - 3x^2 y}{x^2 + y^2}\right)$$

Comparing real and imaginary parts,

$$u = \frac{x^3 - 3xy^2}{x^2 + y^2}, \quad v = \frac{y^3 - 3x^2 y}{x^2 + y^2}$$

Also, $f(z) = 0$ when $z = 0$, i.e., $u = 0$, $v = 0$ at the origin.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x}$$

$$= 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

At the origin,

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

$$= 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

At the origin, $$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x}$$

$$= 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

At the origin, $$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y}$$

$$= 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

Hence, C–R equations are satisfied at the origin.

Now, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\dfrac{(x^3 - 3xy^2) + i(y^3 - 3x^2 y)}{x^2 + y^2}}{x + iy}$$

Let $z \to 0$ along the line $y = mx$.

$$f'(0) = \lim_{x \to 0} \frac{(x^3 - 3m^2 x^3) + i(m^3 x^3 - 3mx^3)}{(x^2 + m^2 x^2)(x + imx)}$$

$$= \lim_{x \to 0} \frac{(1 - 3m^2) + i(m^3 - 3m)}{(1 + m^2)(1 + im)}$$

$$= \frac{(1 - 3m^2) + i(m^3 - 3m)}{(1 + m^2)(1 + im)}$$

Since the limit depends on m (i.e., on the path), it is not unique.
Hence, $f'(0)$ does not exist.

Example 29

Find p such that the function $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ *is analytic.*

Solution

$$f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$$

$$u + iv = r^2 \cos 2\theta + ir^2 \sin p\theta$$

Comparing real and imaginary parts,

$$u = r^2 \cos 2\theta, \qquad v = r^2 \sin p\theta$$

Since $f(z)$ is analytic, C–R equations are satisfied.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

From the second equation,

$$r^2(-2\sin 2\theta) = -r(2r\sin p\theta)$$

$$\sin 2\theta = \sin p\theta$$

On comparing,

$$p = 2$$

Example 30

Show that $f(z) = \dfrac{1}{z}$ *is analytic everywhere except at* $z = 0$ *and find* $f'(z)$.

Solution

Let $z = re^{i\theta}$

$$f(z) = \frac{1}{z}$$

$$u + iv = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{r}e^{-i\theta}$$

$$= \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u = \frac{1}{r}\cos\theta, \qquad v = -\frac{1}{r}\sin\theta$$

$$\frac{\partial u}{\partial r} = -\frac{1}{r^2}\cos\theta, \quad \frac{\partial v}{\partial r} = \frac{1}{r^2}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = -\frac{1}{r}\sin\theta, \quad \frac{\partial v}{\partial \theta} = -\frac{1}{r}\cos\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

C–R equations are satisfied.

All the derivatives are continuous everywhere except at $r = 0$ (i.e., $z = 0$).

Hence, $f(z)$ is analytic everywhere except at $z = 0$.

Now, $f(z) = u + iv$

Differentiating w.r.t. *r,*

$$f'(z)\cdot\frac{\partial z}{\partial r}=\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}$$

$$f'(z)\cdot e^{i\theta}=-\frac{1}{r^2}\cos\theta+i\frac{1}{r^2}\sin\theta$$

$$=-\frac{(\cos\theta-i\sin\theta)}{r^2}$$

$$f'(z)=\frac{-e^{i\theta}}{r^2e^{i\theta}}$$

$$=-\frac{1}{r^2e^{2i\theta}}$$

$$=-\frac{1}{(re^{i\theta})^2}$$

$$=-\frac{1}{z^2}$$

Example 31

Show that $f(z) = \log z$ is analytic everywhere except at the origin and find its derivative.

Solution

Let $z=re^{i\theta}$

$$f(z)=\log z$$

$$u+iv=\log(re^{i\theta})$$

$$=\log r+\log e^{i\theta}$$

$$=\log r+i\theta$$

Comparing real and imaginary parts,

$$u=\log r,\qquad v=\theta$$

$$\frac{\partial u}{\partial r}=\frac{1}{r},\qquad \frac{\partial v}{\partial r}=0$$

$$\frac{\partial u}{\partial\theta}=0,\qquad \frac{\partial v}{\partial\theta}=1$$

$$\therefore\quad \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\qquad \frac{\partial u}{\partial\theta}=-r\frac{\partial v}{\partial r}$$

C–R equations are satisfied.

But $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}$ is not continuous at $r = 0$ (i.e., $z = 0$).

Hence, log z is analytic everywhere except at the origin ($z = 0$).

Now, $$f(z) = u + iv$$

Differentiating w.r.t. r,

$$f'(z) \cdot \frac{\partial z}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$f'(z) \cdot e^{i\theta} = \frac{1}{r} + i(0)$$

$$f'(z) = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{z}$$

EXERCISE 2.3

1. Show that an analytic function with a constant real part is constant.

2. Determine which of the following functions are analytic.

(i) $xy + iy$ (ii) $e^x(\cos y - i \sin y)$ (iii) $z^2 + z$ (iv) z^3 (v) $z + 2\bar{z}$

[Ans.: (i) not analytic (ii) not analytic (iii) analytic (iv) analytic (v) not analytic]

3. Find the constants a, b, c, d, and e if the function $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic.

[Ans.: $a = 1, b = -6, c = 1, d = 2, e = 4$]

4. Determine p such that the function $f(z) = \dfrac{1}{2}\log(x^2 + y^2) + \tan^{-1}\left(\dfrac{px}{y}\right)$ is analytic.

[Ans.: $p = -1$]

5. For what values of z does the function cease to be analytic?

(i) $\dfrac{z^2 - 4}{z^2 + 1}$ (ii) $\dfrac{z}{z^2 - 1}$ (iii) $\tan^2 z$

$$\left[\textbf{Ans.:}\ \text{(i) } z = \pm i \ \text{ (ii) } z = \pm 1 \ \text{ (iii) } z = \frac{(2n-1)\pi}{2},\ n = 1, 2, \ldots\right]$$

6. Show that $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}, z \neq 0$

$= 0, \qquad z = 0$

is not analytic at the origin although C–R equations are satisfied at the origin.

2.9 HARMONIC FUNCTIONS

A real function ϕ of two variables x and y is said to be a harmonic function in a region R if it has continuous second-order partial derivatives and satisfies the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\nabla^2 \phi = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

2.10 PROPERTIES OF ANALYTIC FUNCTIONS

(1) If $f(z) = u + iv$ is an analytic function, u and v are harmonic functions.

Proof

$f(z) = u + iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \cdots(2.7)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \cdots(2.8)$$

Differentiating Eq. (2.7) w.r.t. x and Eq. (2.8) w.r.t. y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \cdots(2.9)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \cdots(2.10)$$

Adding Eqs (2.9) and (2.10),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= 0 \qquad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \right]$$

Hence, u is a harmonic function.

Differentiating Eq. (2.7) w.r.t. y and Eq. (2.8) w.r.t. x,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \cdots(2.11)$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \cdots(2.12)$$

Subtracting Eq. (2.12) from Eq. (2.11),

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

Hence, v is a harmonic function.

Note u and v of an analytic function $u + iv$ are called *conjugate harmonic functions* of each other.

(2) If $f(z) = u + iv$ is an analytic function then the family of curves $u(x, y) = c_1$ and the family of curves $v(x, y) = c_2$ cut orthogonally.

Proof

$f(z) = u + iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots(2.13)$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots(2.14)$$

$u(x, y) = c_1$ (Fig. 2.14)

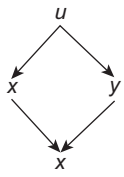


Fig. 2.14

$$\frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1, \text{ say}$$

which represents the slope of the family of curves $u(x, y) = c_1$

$v(x, y) = c_2$

$$\frac{\partial v}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}=m_2, \text{ say}$$

which represents slope of family of curves $v(x, y) = c_2$

$$m_1m_2=\left[-\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[-\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}\right]$$

$$=\left[\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right]\left[\frac{-\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}\right] \qquad \text{[Using Eqs (2.7) and (2.8)]}$$

$$=-1$$

Hence, both the families of curves cut orthogonally.

Note In polar form, the condition of orthogonality for the families of curves $u(r, \theta) = c_1$ and $v(r, \theta) = c_2$ is

$$\left(r\frac{d\theta}{dr}\right)_{u=c_1}\cdot\left(r\frac{d\theta}{dr}\right)_{v=c_2}=-1$$

Example 1

Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

Solution

$$u=2x-x^3+3xy^2$$

$$\frac{\partial u}{\partial x}=2-3x^2+3y^2, \quad \frac{\partial u}{\partial y}=6xy$$

$$\frac{\partial^2 u}{\partial x^2}=-6x \qquad \frac{\partial^2 u}{\partial y^2}=6x$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-6x+6x$$

$$=0$$

Hence, u is harmonic.

Example 2

Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ *is harmonic.*

Solution

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6, \qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x + 6) + (-6x - 6)$$

$$= 0$$

Hence, u is harmonic.

Example 3

If $f(z) = e^z$ *then show that u and v are harmonic functions.*

Solution

$$f(z) = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \qquad \frac{\partial^2 v}{\partial x^2} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y, \qquad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y + (-e^x \cos y)$$

$$= 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \sin y + (-e^x \sin y)$$
$$= 0$$

Hence, *u* and *v* are harmonic functions.

Example 4

Prove that $u = e^{-y} \cos x$ and $v = e^{-x} \sin y$ *the satisfy the Laplace equation, but u + iv is not an analytic function of z.*

Solution

$$u = e^{-y} \cos x, \qquad v = e^{-x} \sin y$$
$$\frac{\partial u}{\partial x} = -e^{-y} \sin x, \qquad \frac{\partial v}{\partial x} = -e^{-x} \sin y$$
$$\frac{\partial^2 u}{\partial x^2} = -e^{-y} \cos x, \qquad \frac{\partial^2 v}{\partial x^2} = e^{-x} \sin y$$
$$\frac{\partial u}{\partial y} = -e^{-y} \cos x, \qquad \frac{\partial v}{\partial y} = e^{-x} \cos y$$
$$\frac{\partial^2 u}{\partial y^2} = e^{-y} \cos x, \qquad \frac{\partial^2 v}{\partial y^2} = -e^{-x} \sin y$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{-y} \cos x + e^{-y} \cos x$$
$$= 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-x} \sin y - e^{-x} \sin y$$
$$= 0$$

Hence, *u* and *v* satisfy the Laplace equation.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.
Hence, *u + iv* is not an analytic function.

Example 5

Prove that $u = x^2 - y^2$ *and* $v = -\dfrac{y}{x^2 + y^2}$ *are harmonic but u + iv is not regular.*

Solution

$$u = x^2 - y^2, \qquad v = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = \frac{y}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 v}{\partial x^2} = 2y\left[\frac{(x^2+y^2)^2 - x\cdot 2(x^2+y^2)^2\cdot 2x}{(x^2+y^2)^4}\right]$$

$$\frac{\partial u}{\partial y} = -2y \qquad = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right]$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2$$

$$= 0$$

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$= 0$$

Hence, u and v are harmonic.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C–R equations are not satisfied.

Hence, $u + iv$ is not regular.

Example 6

If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ *is an analytic function of $z = x + iy$.*

Solution

Let $$U+iV=\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Comparing real and imaginary parts,

$$U=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}, \qquad V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2}, \qquad \frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}$$

$$\frac{\partial U}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y\,\partial x}, \qquad \frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\,\partial y}, \qquad =\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y^2}$$

Since u and v are harmonic functions,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 v}{\partial y^2}$$

$$\therefore \frac{\partial U}{\partial y}=-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 v}{\partial x\,\partial y}$$

and

$$\frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$$

Hence, $U + iV$, i.e., the given function, is analytic.

Example 7

Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $u + iv = z^3$.

Solution

$$\begin{aligned} u + iv &= z^3 \\ &= (x+iy)^3 \\ &= x^3 + i^3y^3 + 3x^2 \cdot iy + 3x \cdot i^2y^2 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \end{aligned}$$

Comparing real and imaginary parts,

$$u = x^3 - 3xy^2, \qquad v = 3x^2y - y^3$$

Slope of families of curves $u = c_1$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}$$

$$= -\frac{3x^2 - 3y^2}{-6xy}$$

$$= \frac{x^2 - y^2}{2xy} = m_1, \text{ say}$$

Slope of families of curves $v = c_2$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= -\frac{6xy}{3x^2 - 3y^2}$$

$$= -\frac{2xy}{x^2 - y^2} = m_2, \text{say}$$

$$m_1m_2 = \left(\frac{x^2 - y^2}{2xy}\right)\left(-\frac{2xy}{x^2 - y^2}\right)$$

$$= -1$$

Hence, families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 8

Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = e^{z^2}$.

Solution

$$w = e^{z^2}$$

$$u + iv = e^{(x+iy)^2}$$

$$= e^{x^2 - y^2 + 2ixy}$$

$$= e^{x^2 - y^2} e^{i2xy}$$

$$= e^{x^2 - y^2} (\cos 2xy + i \sin 2xy)$$

Comparing real and imaginary parts,

$$u = e^{x^2 - y^2} \cos 2xy, \qquad v = e^{x^2 - y^2} \sin 2xy$$

Slope of families of curves $u = c_1$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}$$

$$= -\frac{e^{x^2 - y^2} \cdot 2x \cos 2xy + e^{x^2 - y^2} (-2y \sin 2xy)}{e^{x^2 - y^2} (-2y) \cos 2xy + e^{x^2 - y^2} (-2x \sin 2xy)}$$

$$= \frac{x \cos 2xy - y \sin 2xy}{y \cos 2xy + x \sin 2xy}$$

$$= m_1, \quad \text{say}$$

Slope of families of curves $v = c_2$ is

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$$

$$= -\frac{e^{x^2 - y^2} \cdot 2x \sin 2xy + e^{x^2 - y^2} (2y \cos 2xy)}{e^{x^2 - y^2} (-2y) \sin 2xy + e^{x^2 - y^2} (2x \cos 2xy)}$$

$$= -\frac{x \sin 2xy + y \cos 2xy}{-y \sin 2xy + x \cos 2xy}$$

$$= m_2, \text{say}$$

$$m_1 m_2 = \left(\frac{x \cos 2xy - y \sin 2xy}{x \sin 2xy + y \cos 2xy}\right)\left(-\frac{x \sin 2xy + y \cos 2xy}{x \cos 2xy - y \sin 2xy}\right)$$

$$= -1$$

Hence, families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 9

Show that the families of the curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ cut orthogonally.

Solution

Let $u = r^n - a \sec n\theta = 0$ $\cdots(1)$

and $v = r^n - b \operatorname{cosec} n\theta = 0$ $\cdots(2)$

Differentiating Eq. (1) w.r.t. r,

$$nr^{n-1} - a(\sec n\theta \tan n\theta) \cdot n \frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{r^{n-1}}{a \sec n\theta \tan n\theta}$$

$$r \frac{d\theta}{dr} = \frac{r^n}{a \sec n\theta \tan n\theta}$$

$$= \frac{a \sec n\theta}{a \sec n\theta \tan n\theta}$$

$$= \cot n\theta$$

$$= m_1, \text{ say}$$

Differentiating Eq. (2) w.r.t. r,

$$nr^{n-1} - b(-\operatorname{cosec} n\theta \cot n\theta) n \frac{d\theta}{dr} = 0$$

$$\frac{d\theta}{dr} = \frac{-r^{n-1}}{b \operatorname{cosec} n\theta \cot n\theta}$$

$$r \frac{d\theta}{dr} = -\frac{r^n}{b \operatorname{cosec} n\theta \cot n\theta}$$

$$= -\frac{r^n}{r^n \cot n\theta}$$

$$= -\frac{1}{\cot n\theta}$$

$$= m_2, \text{say}$$

$$m_1 m_2 = \cot n\theta \left(-\frac{1}{\cot n\theta} \right)$$

$$= -1$$

Hence, families of the given curves cut orthogonally.

Example 10

Show that $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}=4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$.

Solution

Let $f(z)$ be a function of x and y (Fig. 2.15).

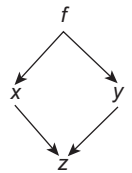


Fig. 2.15

where $z = x+iy$

$$\bar{z} = x-iy$$

$$\therefore \quad x=\frac{1}{2}(z+\bar{z})$$

$$y=\frac{1}{2i}(z-\bar{z})$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$=\frac{\partial f}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial f}{\partial y}\left(\frac{1}{2i}\right)$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}+\frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y} \qquad \cdots(1)$$

$$\frac{\partial f}{\partial \bar{z}}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$=\frac{\partial f}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}-\frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$2\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y} \qquad \cdots(2)$$

From Eqs (1) and (2),

$$2\frac{\partial}{\partial z}\left(2\frac{\partial}{\partial \bar{z}}\right)=\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$$

$$4\frac{\partial^2}{\partial z\,\partial \bar{z}}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

This is also known as the complex form of the Laplace equation.

Example 11

If $f(z) = u + iv$ is analytic in the domain D then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|\text{Re } f(z)\right|^2 = 2\left|f'(z)\right|^2. \qquad \textbf{[Summer 2015]}$$

Solution

Let $f(z) = u + iv$

Since $f(z)$ is analytic, u and v are harmonic functions.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(2)$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\left|f'(z)\right| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|\text{Re } f(z)\right|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2$$

$$= \frac{\partial^2}{\partial x^2}u^2+\frac{\partial^2}{\partial y^2}u^2$$

$$= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$= 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \qquad \text{[Using Eq. (1)]}$$

$$= 2\left|f'(z)\right|^2$$

Example 12

If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

Solution

Let $f(z) = u + iv$

Since $f(z)$ is analytic, u and v are harmonic functions.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \cdots(1)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad \cdots(2)$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$|f'(z)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \qquad \cdots(3)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$=\frac{\partial^2}{\partial x^2}(u^2+v^2)+\frac{\partial^2}{\partial y^2}\left(u^2+v^2\right)$$

$$=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right)$$

$$=2\left[\left\{u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right\}+\left\{v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right\}+\left\{u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right\}\right.$$
$$\left.+\left\{v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right\}\right]$$

$$=2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right.$$
$$\left.+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] \quad \left[\begin{array}{l}\text{Using equations (1) and (2)}\\ \text{and C–R equations}\end{array}\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4\left|f'(z)\right|^2$$

Aliter

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\cdot\overline{f(z)}\right]$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\cdot f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left\{f(z)\cdot f(\bar{z})\right\}\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\frac{\partial}{\partial \bar{z}}f(\bar{z})\right] \qquad \left[\because f(z) \text{ is a function of } z \text{ alone}\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\cdot f'(\bar{z})\right]$$

$$= 4f'(\bar{z})\frac{\partial}{\partial z}f(z) \qquad \left[\because f'(\bar{z}) \text{ is a function of } \bar{z} \text{ alone}\right]$$

$$= 4f'(\bar{z})\,f'(z)$$

$$= 4\overline{f'(z)}\,f'(z)$$

$$= 4\left|f'(z)\right|^2$$

Example 13

If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log\left|f(z)\right| = 0.$

Solution

Let $f(z) = u + iv$

$$\left|f(z)\right| = \sqrt{u^2 + v^2}$$

$$\left|f(z)\right|^2 = u^2 + v^2$$

Since $f(z)$ is an analytic function, u and v are harmonic functions.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \cdots(1)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad \cdots(2)$$

$$\log|f(z)| = \log\left(|f(z)|^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log|f(z)|^2$$

$$= \frac{1}{2}\log(u^2+v^2)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\frac{1}{2}\log(u^2+v^2)\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\log(u^2+v^2)+\frac{\partial^2}{\partial y^2}\log(u^2+v^2)\right]$$

$$\frac{\partial^2}{\partial x^2}\log(u^2+v^2) = \frac{\partial}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)\right]$$

$$= 2\frac{\partial}{\partial x}\left[\frac{u u_x + v v_x}{u^2+v^2}\right]$$

$$= 2\left[\frac{\left\{\frac{\partial}{\partial x}\left(u u_x + v v_x\right)\right\}(u^2+v^2)-(u u_x + v v_x)\frac{\partial}{\partial x}(u^2+v^2)}{(u^2+v^2)^2}\right]$$

$$= \frac{2}{(u^2+v^2)^2}\Big[(uu_{xx}+vv_{xx}+u_x^2+v_x^2)(u^2+v^2)$$

$$-(u u_x + v v_x)(2u u_x + 2v v_x)\Big]$$

$$= \frac{2}{(u^2+v^2)^2}\Big[(u^3u_{xx}+u^2v v_{xx}+u^2u_x^2+u^2v_x^2+uv^2u_{xx}+v^3v_{xx}$$

$$+v^2u_x^2+v^2v_x^2)-(2u^2u_x^2+2uvu_xv_x+2uvu_xv_x+2v^2{v_x}^2)\Big]$$

$$= \frac{2}{(u^2+v^2)^2}\Big[u^3u_{xx}+v^3\,v_{xx}+uv^2u_{xx}+u^2v\,v_{xx}-u^2u_x^2+u^2v_x^2$$

$$+v^2u_x^2-v^2v_x^2-4uvu_xv_x\Big] \qquad \cdots(3)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log(u^2+v^2)=\frac{2}{(u^2+v^2)^2}\Big[u^3u_{yy}+v^3v_{yy}+uv^2u_{yy}+u^2vv_{yy}-u^2u_y^2+u^2v_y^2 + v^2u_y^2-v^2v_y^2-4uvu_yv_y\Big] \quad \cdots(4)$$

Adding Eqs (3) and (4),

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log(u^2+v^2)=\frac{2}{(u^2+v^2)^2}\Big[u^3(u_{xx}+u_{yy})+v^3(v_{xx}+v_{yy})+uv^2(u_{xx}+u_{yy}) + u^2v(v_{xx}+v_{yy})-u^2(u_x^2+u_y^2)+u^2(v_x^2+v_y^2) + v^2(u_x^2+u_y^2)-v^2(v_x^2+v_y^2)-4uv(u_xv_x+u_yv_y)\Big]$$

$$=\frac{2}{(u^2+v^2)^2}\Big[u^3(0)+v^3(0)+uv^2(0)+u^2v(0)-u^2(u_x^2+u_y^2) + u^2(u_y^2+u_x^2)+v^2(u_x^2+u_y^2) - v^2(u_x^2+u_y^2)-4uv(-u_xu_y+u_yv_x)\Big]$$

$$\left[\begin{array}{l}\text{Using equations (1)}\\ \text{and (2) and C–R equations}\end{array}\right]$$

$$=0$$

Hence, $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\log|f(z)|=0$

Aliter

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|=4\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{1}{2}\log|f(z)|^2\right]$$

$$=2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\log\left\{f(z)\cdot\overline{f(z)}\right\}\right]$$

$$=2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\log f\left(z\right)+\log f\left(\bar{z}\right)\right]$$

$$=2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\log f(z)+\frac{\partial}{\partial\bar{z}}\log f(\bar{z})\right]$$

$$=2\frac{\partial}{\partial z}\left[0+\frac{1}{f(\bar{z})}f'(\bar{z})\right] \qquad \left[\because \log f(z) \text{ is a function of } z \text{ alone}\right]$$

$$=0 \qquad \left[\because f(\bar{z}) \text{ and } f'(\bar{z}) \text{ are functions of } \bar{z} \text{ alone}\right]$$

Example 14

If $f(z)$ is an analytic function of z in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f'(z)|^2\,|f(z)|^{p-2}.$$

Solution

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[|f(z)|^2\right]^{\frac{p}{2}}$$

$$= 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[f(z)\cdot\overline{f(z)}\right]^{\frac{p}{2}}$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\left\{f(z)\cdot f(\bar{z})\right\}^{\frac{p}{2}}\right] \qquad \left[\because \overline{f(z)} = f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[\left\{f(z)\right\}^{\frac{p}{2}}\cdot\frac{p}{2}\left\{f(\bar{z})\right\}^{\frac{p}{2}-1}f'(\bar{z})\right] \left[\because f(z) \text{ is a function of } z \text{ alone}\right]$$

$$= 4\cdot\frac{p}{2}\left[f(\bar{z})\right]^{\frac{p}{2}-1}f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z)\right]^{\frac{p}{2}} \quad \left[\because f(\bar{z}) \text{ is a function of } \bar{z} \text{ alone}\right]$$

$$= 4\cdot\frac{p}{2}\left[f(\bar{z})\right]^{\frac{p}{2}-1}f'(\bar{z})\cdot\frac{p}{2}\left[f(z)\right]^{\frac{p}{2}-1}f'(z)$$

$$= p^2\left[f(\bar{z})\cdot f(z)\right]^{\frac{p}{2}-1}\left[f'(\bar{z})\cdot f'(z)\right]$$

$$= p^2\left[\overline{f(z)}\cdot f(z)\right]^{\frac{p-2}{2}}\cdot\left[\overline{f'(z)}\cdot f'(z)\right]$$

$$= p^2\left[|f(z)|^2\right]^{\frac{p-2}{2}}\left[|f'(z)|^2\right]$$

$$= p^2\,|f'(z)|^2\,|f(z)|^{p-2}$$

EXERCISE 2.4

1. Prove that $\tan^{-1}\left(\frac{y}{x}\right)$ is harmonic.

2. Prove that $\log\sqrt{x^2+y^2}$ is harmonic.

3. If $f(z)$ is analytic, prove that

(i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re}\, f(z)|^2 = 2|f'(z)|^2$

(ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Im}\, f(z)|^2 = 2|f'(z)|^2$

(iii) $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

4. Prove that $\log\{(x-1)^2+(y-2)^2\}$ is harmonic in every region which does not include the point (1, 2).

2.11 CONSTRUCTION OF ANALYTIC FUNCTIONS: MILNE—THOMSON METHOD

Case (i) If the real part u of $f(z)$ is given

Step 1: Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Step 2: Find $f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$ [Using C–R equations]

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z)$ to obtain $f(z)$.

Case (ii) If the imaginary part v of $f(z)$ is given

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Step 2: Find $f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$ [Using C–R equations]

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z)$ to obtain $f(z)$.

Example 1

Find the analytic function $f(z) = u + iv$ if $u = x^3 - 3xy$.

[Summer 2014]

Solution

$$u = x^3 - 3xy$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial u}{\partial y} = -3x$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (3x^2 - 3y) - i(-3x)$$

$$= 3x^2 - 3y + 3ix$$

Putting $x = z$, $y = 0$,

$$f'(z) = 3z^2 + 3iz$$

Integrating w.r.t. z,

$$f(z) = \int (3z^2 + 3iz)\,\mathrm{d}z$$

$$= z^3 + i\frac{3z^2}{2} + c$$

Example 2

Construct the analytic function whose real part is
$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$

Solution

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x, \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (3x^2 - 3y^2 + 6x) - i(-6xy - 6y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = (3z^2 + 6z) + i(0)$$

Integrating w.r.t. z,

$$\begin{aligned} f(z) &= \int (3z^2 + 6z)\mathrm{d}z \\ &= 3\left(\frac{z^3}{3}\right) + 6\left(\frac{z^2}{2}\right) + c \\ &= z^3 + 3z^2 + c \end{aligned}$$

Example 3

Find the analytic function $w = u + iv$ whose imaginary part is given by $v = e^x (x \sin y + y \cos y)$.

Solution

$$\begin{aligned} v &= e^x (x\sin y + y\cos y) \\ \frac{\partial v}{\partial x} &= e^x (x\sin y + y\cos y) + e^x (\sin y) \\ \frac{\partial v}{\partial y} &= e^x (x\cos y + \cos y - y\sin y) \\ f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]} \\ &= e^x (x\cos y + \cos y - y\sin y) + i e^x (x\sin y + y\cos y + \sin y) \end{aligned}$$

Putting $x = z$, $y = 0$,

$$\begin{aligned} f'(z) &= e^z (z\cos 0 + \cos 0 - 0) + i e^z (z\sin 0 + 0 + \sin 0) \\ &= e^z (z + 1) \end{aligned}$$

Integrating w.r.t. z,

$$\begin{aligned} f(z) &= \int (z+1)e^z \, \mathrm{d}z \\ &= (z+1)e^z - (1)e^z + c \\ &= ze^z + c \end{aligned}$$

Example 4

Find the analytic function $f(z) = u + iv$, *whose real part* u *is* $\dfrac{x}{x^2+y^2}$.

Solution

$$u = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(1)(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{(x^2+y^2)^2}\cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad [\text{Using C–R equations}]$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} - i\left[-\frac{2xy}{(x^2+y^2)^2}\right]$$

Putting $x = z$, $y = 0$,

$$f'(z) = \frac{-z^2}{z^4} + i(0)$$

$$= -\frac{1}{z^2}$$

Integrating w.r.t. z,

$$f(z) = \int -\frac{1}{z^2}\,dz$$

$$= -\left(\frac{z^{-1}}{-1}\right) + c$$

$$= \frac{1}{z} + c$$

Example 5

Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \sin 2x\left[-\frac{1}{(\cosh 2y - \cos 2x)^2} \cdot (2\sinh 2y)\right]$$

$$= -\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Putting $x = z$, $y = 0$,

$$f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} \qquad [\because \cosh 0 = 1, \sinh 0 = 0]$$

$$= -\frac{2}{1 - \cos 2z}$$

$$= -\frac{2}{2\sin^2 z}$$

$$= -\text{cosec}^2 z$$

Integrating w.r.t. z,

$$f(z) = \int -\text{cosec}^2 z \, dz$$

$$= \cot z + c$$

Example 6

Find the analytic function $f(z) = u + iv$, *where* $v = \dfrac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$.

Solution

$$v = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\frac{\partial v}{\partial x} = 2\sinh y \left[\frac{\cos x(\cos 2x + \cosh 2y) - \sin x(-2\sin 2x)}{(\cos 2x + \cosh 2y)^2}\right]$$

$$\frac{\partial v}{\partial y} = 2\sin x \left[\frac{\cosh y(\cos 2x + \cosh 2y) - \sinh y(2\sinh 2y)}{(\cos 2x + \cosh 2y)^2}\right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$

$$= 2\sin x \left[\frac{\cosh y(\cos 2x + \cosh 2y) - 2\sinh y \sinh 2y}{(\cos 2x + \cosh 2y)^2}\right]$$

$$+ i2\sinh y \left[\frac{\cos x(\cos 2x + \cosh 2y) + 2\sin x \sin 2x}{(\cos 2x + \cosh 2y)^2}\right]$$

Putting $x = z$, $y = 0$,

$$f'(z) = 2\sin z \left[\frac{\cosh 0(\cos 2z + \cosh 0) - 0}{(\cos 2z + \cosh 0)^2}\right] + i(0) \quad [\because \sinh 0 = 0]$$

$$= \frac{2\sin z(\cos 2z + 1)}{(\cos 2z + 1)^2}$$

$$= \frac{2\sin z}{\cos 2z + 1}$$

$$= \frac{2\sin z}{2\cos^2 z}$$

$$= \tan z \sec z$$

Integrating w.r.t. z,

$$f(z) = \int \tan z \sec z \, dz + c$$

$$= \sec z + c$$

Example 7

Find the analytic function u + iv, if $u = (x-y)(x^2+4xy+y^2)$. *Also, find the conjugate harmonic function v.*

Solution

$$\begin{aligned}
u &= (x-y)(x^2+4xy+y^2)\\
&= x^3+4x^2y+xy^2-x^2y-4xy^2-y^3\\
&= x^3+3x^2y-3xy^2-y^3\\
\frac{\partial u}{\partial x} &= 3x^2+6xy-3y^2\\
\frac{\partial u}{\partial y} &= 3x^2-6xy-3y^2\\
f'(z) &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\\
&= \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}\\
&= (3x^2+6xy-3y^2)-i(3x^2-6xy-3y)
\end{aligned}$$

Putting $x = z$, $y = 0$,

$$\begin{aligned}
f'(z) &= 3z^2-i(3z^2)\\
&= 3(1-i)z^2
\end{aligned}$$

Integrating w.r.t. z,

$$\begin{aligned}
f(z) &= \int 3(1-i)z^2\,dz\\
&= (1-i)z^3+c\\
u+iv &= (1-i)(x+iy)^3+c\\
&= (1-i)\left[x^3+i^3y^3+3x^2(iy)+3x(iy)^2\right]+c\\
&= (1-i)\left[x^3-iy^3+3ix^2y-3xy^2\right]+c\\
&= x^3-iy^3+3ix^2y-3xy^2-ix^3+i^2y^3-3i^2x^2y+3ixy^2+c\\
&= (x^3-y^3+3x^2y-3xy^2)+i(-y^3+3x^2y-x^3+3xy^2)+c
\end{aligned}$$

Comparing imaginary parts,

$$v = -x^3-y^3+3x^2y+3xy^2+c$$

Example 8

Show that $u = \frac{1}{2}\log(x^2 + y^2)$ *is harmonic. Determine its analytic function. Find also its conjugate.*

Solution

$$u = \frac{1}{2}\log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{(x^2 + y^2)} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= 0$$

Hence, u is a harmonic function.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Putting $x = z$, $y = 0$,

$$f'(z) = \frac{z}{z^2} - i(0)$$

$$= \frac{1}{z}$$

Integrating w.r.t. z,

$$f(z) = \int \frac{1}{z}\,dz$$

$$= \log z + c$$

$$u + iv = \log(x + iy) + c$$
$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + c$$

Comparing imaginary parts,

$$v = i\tan^{-1}\frac{y}{x} + c$$

Example 9

If $w = u + iv$ is an analytic function and $v = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, find u.

Solution

$$v = x^2 - y^2 + \frac{x}{x^2 + y^2}$$
$$\frac{\partial v}{\partial x} = 2x + \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$\frac{\partial v}{\partial y} = -2y - \frac{x}{(x^2 + y^2)^2} \cdot 2y = -2y - \frac{2xy}{(x^2 + y^2)^2}$$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[Using C–R equations]}$$
$$= -\left[2y + \frac{2xy}{(x^2 + y^2)^2}\right] + i\left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

Putting $x = z$, $y = 0$,

$$f'(z) = 0 + i\left(2z + \frac{0 - z^2}{z^4}\right)$$
$$= i(2z - z^{-2})$$

Integrating w.r.t. z,

$$f(z) = \int i\left(2z - z^{-2}\right)\mathrm{d}z$$
$$= i\left(z^2 - \frac{z^{-1}}{-1}\right) + c$$
$$= i\left(z^2 + \frac{1}{z}\right) + c$$

$$u+iv = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c$$

$$= i\left[x^2 + i^2y^2 + 2ixy + \frac{x-iy}{x^2+y^2}\right] + c$$

$$= ix^2 - iy^2 + 2i^2xy + \frac{ix}{x^2+y^2} - \frac{i^2y}{x^2+y^2} + c$$

$$= i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) - 2xy + \frac{y}{x^2+y^2} + c$$

$$= \left(\frac{y}{x^2+y^2} - 2xy\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$$

Comparing real parts,

$$u = \frac{y}{x^2+y^2} - 2xy + c$$

Example 10

Find the analytic function $w = u + iv$ when $v = e^{-2y}(y\cos 2x + x\sin 2x)$ and find u.

Solution

$$v = e^{-2y}(y\cos 2x + x\sin 2x)$$

$$\frac{\partial v}{\partial x} = e^{-2y}(-2y\sin 2x + \sin 2x + 2x\cos 2x)$$

$$\frac{\partial v}{\partial y} = -2e^{-2y}(y\cos 2x + x\sin 2x) + e^{-2y}(\cos 2x)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad [\text{Using C–R equations}]$$

$$= -2e^{-2y}(y\cos 2x + x\sin 2x) + e^{-2y}\cos 2x + ie^{-2y}(-2y\sin 2x + \sin 2x + 2x\cos 2x)$$

Putting $x = z$, $y = 0$,

$$f'(z) = -2e^0(0 + z\sin 2z) + e^0\cos 2z + ie^0(-0 + \sin 2z + 2z\cos 2z)$$

$$= -2z\sin 2z + \cos 2z + i(\sin 2z + 2z\cos 2z)$$

Integrating w.r.t z,

$$f(z) = \int (\cos 2z - 2z\sin 2z)dz + i\int (\sin 2z + 2z\cos 2z)dz$$

$$= \frac{\sin 2z}{2} - 2\left[z\left(-\frac{\cos 2z}{2}\right) - (1)\left(-\frac{\sin 2z}{4}\right)\right]$$

$$+ i\left[-\frac{\cos 2z}{2} + 2\left\{z\frac{\sin 2z}{2} - (1)\left(-\frac{\cos 2z}{4}\right)\right\}\right] + c$$

$$= \frac{\sin 2z}{2} + z\cos 2z - \frac{\sin 2z}{2} + i\left[-\frac{\cos 2z}{2} + z\sin 2z + \frac{\cos 2z}{2}\right] + c$$

$$= z\cos 2z + iz\sin 2z + c$$

$$= z(\cos 2z + i\sin 2z) + c$$

$$= ze^{i2z} + c$$

$$u + iv = (x + iy)e^{i2(x+iy)} + c$$

$$= (x + iy)e^{2ix-2y} + c$$

$$= (x + iy)e^{-2y}e^{2ix} + c$$

$$= e^{-2y}(x + iy)(\cos 2x + i\sin 2x) + c$$

$$= e^{-2y}\left[(x\cos 2x - y\sin 2x) + i(y\cos 2x + x\sin 2x)\right] + c$$

Comparing real parts,

$$u = e^{-2y}(x\cos 2x - y\sin 2x)$$

Example 11

Show that $u(x, y) = x^2 - y^2$ is harmonic. Find the corresponding analytic function $f(z) = u + iv$. **[Summer 2013]**

Solution

(i) $u(x, y) = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2$$

$$= 0$$

Hence, u is harmonic.

(ii) $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= 2x - i(-2y)$$

Putting $x = z$, $y = 0$,

$f'(z) = 2z$

Integrating w.r.t. z

$$\int f'(z)\,dz = \int 2z\,dz$$

$$f(z) = z^2 + c$$

Example 12

Show that $u(x, y) = x^2 - y^2 + x$ is harmonic. Find the corresponding analytic function $f(z) = u + iv$. **[Winter 2013]**

Solution

(i) $u(x, y) = x^2 - y^2 + x$

$$\frac{\partial u}{\partial x} = 2x + 1, \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2$$

$$= 0$$

Hence, u is harmonic.

(ii) $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= (2x + 1) - i(-2y)$$

Putting $x = z$, $y = 0$,

$f'(z) = 2z + 1$

Integrating w.r.t. z,

$$\int f'(z)\,dz = \int (2z+1)\,dz$$
$$= z^2 + z + c$$

Example 13

Prove that $u = e^x (x \cos y - y \sin y)$ *is harmonic and, hence, find the analytic function* $f(z) = u + iv$.

Solution

$$u = e^x (x\cos y - y\sin y)$$
$$\frac{\partial u}{\partial x} = e^x (x\cos y - y\sin y) + e^x (\cos y)$$
$$\frac{\partial^2 u}{\partial x^2} = e^x (x\cos y - y\sin y + \cos y) + e^x (\cos y)$$
$$= e^x (x\cos y - y\sin y + 2\cos y)$$
$$\frac{\partial u}{\partial y} = e^x (-x\sin y - \sin y - y\cos y)$$
$$\frac{\partial^2 u}{\partial y^2} = e^x (-x\cos y - \cos y - \cos y + y\sin y)$$
$$= e^x (-x\cos y - 2\cos y + y\sin y)$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x (x\cos y - y\sin y + 2\cos y) + e^x (-x\cos y - 2\cos y + y\sin y)$$
$$= 0$$

Hence, u is a harmonic function.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$
$$= e^x (x\cos y - y\sin y + \cos y) - ie^x (-x\sin y - \sin y - y\cos y)$$

Putting $x = z, y = 0,$

$$f'(z) = e^z (z\cos 0 - 0 + \cos 0) - ie^z (0 - 0 - 0)$$
$$= e^z (z+1)$$

Integrating w.r.t. z,

$$f(z) = \int (z+1)e^z \mathrm{d}z$$
$$= (z+1)e^z - (1)e^z + c$$
$$= ze^z + c$$

Example 14

Prove that $x^2 - y^2 + e^{-2x} \cos 2y$ *is harmonic and find its harmonic conjugate.*

Solution

Let $u = x^2 - y^2 + e^{-2x}\cos 2y$

$$\frac{\partial u}{\partial x} = 2x - 2e^{-2x}\cos 2y, \qquad \frac{\partial u}{\partial y} = -2y - 2e^{-2x}\sin 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 4e^{-2x}\cos 2y, \qquad \frac{\partial^2 u}{\partial y^2} = -2 - 4e^{-2x}\cos 2y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (2 + 4e^{-2x}\cos 2y) + (-2 - 4e^{-2x}\cos 2y)$$
$$= 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$
$$= (2x - 2e^{-2x}\cos 2y) - i(-2y - 2e^{-2x}\sin 2y)$$

Putting $x = z$, $y = 0$,

$$f'(z) = 2(z - e^{-2z}\cos 0) - i(0)$$
$$= 2(z - e^{-2z})$$

Integrating w.r.t. z,

$$f(z) = \int 2(z - e^{-2z})\,\mathrm{d}z$$
$$= 2\left(\frac{z^2}{2} - \frac{e^{-2z}}{-2}\right) + c$$
$$= z^2 + e^{-2z} + c$$

$$u + iv = (x + iy)^2 + e^{-2(x+iy)} + c$$
$$= x^2 + i^2y^2 + 2ixy + e^{-2x}\,e^{-2iy} + c$$

$$= x^2 - y^2 + 2ixy + e^{-2x}(\cos 2y - i \sin 2y) + c$$
$$= x^2 - y^2 + e^{-2x} \cos 2y + i(2xy - e^{-2x} \sin 2y) + c$$

Comparing imaginary parts,

$$v = 2xy - e^{-2x} \sin 2y + c$$

Example 15

Find the analytic function $u + iv$, if $u - v = (x - y)(x^2 + 4xy + y^2)$. **[Winter 2012]**

Solution

$$f(z) = u + iv \quad \ldots(1)$$
$$i f(z) = iu + i^2 v$$
$$= iu - v \quad \ldots(2)$$

Adding Eqs (1) and (2),

$$f(z) + i f(z) = (u + iv) + (iu - v)$$
$$(1 + i) f(z) = (u - v) + i(u + v)$$
$$F(z) = U + iV$$

where $F(z) = (1 + i) f(z)$, $U = u - v$, $V = u + v$

Since, $f(z)$ is analytic, $F(z)$ is analytic.

$$U = u - v$$
$$= (x - y)(x^2 + 4xy + y^2)$$
$$= x^3 + 4x^2 y + xy^2 - x^2 y - 4xy^2 - y^3$$
$$= x^3 + 3x^2 y - 3xy^2 - y^3$$
$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$$
$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

$$F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x}$$
$$= \frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y} \qquad \text{[Using C–R equations]}$$
$$= (3x^2 + 6xy - 3y^2) - i(3x^2 - 6xy - 3y^2)$$

Putting $x = z$, $y = 0$,

$$F'(z) = 3z^2 - i3z^2$$
$$= 3(1 - i) z^2$$

Integrating w.r.t. z,

$$F(z) = \int 3(1-i)z^2 \, dz$$

$$= 3(1-i)\frac{z^3}{3} + c$$

$$(1+i)f(z) = (1-i)z^3 + c \qquad \text{[Resubstituting } F(z)\text{]}$$

$$f(z) = \frac{(1-i)}{1+i}z^3 + \frac{c}{1+i}$$

$$= \frac{(1-i)^2}{(1+i)(1-i)}z^3 + \frac{c}{1+i}$$

$$= \frac{1+i^2-2i}{2}z^3 + \frac{c}{1+i}$$

$$= -i\,z^3 + \frac{c}{1+i}$$

$$= -i\,z^3 + c'$$

where $c' = \dfrac{c}{1+i}$

Example 16

Determine the analytic function $f(z) = u + iv$ such that $u - v = e^x(\cos y - \sin y)$. **[Summer 2015]**

Solution

$$f(z) = u + iv \qquad \cdots(1)$$

$$i\,f(z) = iu + i^2 v$$

$$= iu - v \qquad \cdots(2)$$

Adding Eqs (1) and (2),

$$f(z) + i\,f(z) = (u+iv) + (iu - v)$$

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

where $F(z) = (1+i)f(z), U = u - v, V = u + v$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$U = u - v = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y), \qquad \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= e^x(\cos y - \sin y) - ie^x(-\sin y - \cos y)$$

Putting $x = z,\ y = 0$,

$$F'(z) = e^z(\cos 0 - \sin 0) + ie^z(\sin 0 + \cos 0)$$
$$= e^z(1+i)$$

Integrating w.r.t. z,

$$F(z) = \int (1+i)e^z \, dz$$
$$= (1+i)e^z + c$$
$$(1+i)f(z) = (1+i)e^z + c$$
$$f(z) = e^z + \frac{c}{1+i}$$
$$= e^z + c'$$

where $c' = \dfrac{c}{1+i}$

Example 17

If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, *find the analytic function* $f(z) = u + iv$.

Solution

$$f(z) = u + v \quad \ldots(1)$$
$$if(z) = iu - v \quad \ldots(2)$$

Adding Eqs (1) and (2),

$$(1+i)f(z) = (u - v) + i(u + v)$$
$$F(z) = U + iV$$

where $F(z) = (1+i)f(z), U = u - v, V = u + v$

$$V = u + v$$
$$= \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$
$$= \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$$
$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$
$$\frac{\partial V}{\partial x} = \frac{(2\cos 2x)(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$
$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial V}{\partial y} = -\frac{\sin 2x}{(\cosh 2y - \cos 2x)^2}(2\sinh 2y)$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \qquad \text{[Using C–R equations]}$$

$$= -\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} + i\frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2}$$

Putting $x = z$, $y = 0$,

$$F'(z) = -0 + i\frac{2(\cos 2z \cdot \cosh 0 - 1)}{(\cosh 0 - \cos 2z)^2}$$

$$= i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= -i\frac{2}{1 - \cos 2z}$$

$$= -i\frac{2}{2\sin^2 z}$$

$$= -\text{cosec}^2 z$$

Integrating w.r.t. z,

$$F(z) = \int -\text{cosec}^2 z \, dz$$

$$= \cot z + c$$

$$(1+i)f(z) = \cot z + c$$

$$f(z) = \frac{\cot z}{1+i} + \frac{c}{1+i}$$

$$= \frac{(1-i)}{2}\cot z + c'$$

where $c' = \dfrac{c}{1+i}$

Example 18

Determine the analytic function $f(z) = u + iv$ *if* $3u + 2v = y^2 - x^2 + 16xy$.

Solution

$$f(z) = u + iv \qquad \text{....(1)}$$

$$i f(z) = iu - v \qquad \text{...(2)}$$

Multiplying Eq. (1) by 2 and Eq. (2) by 3 and adding,

$$2f(z)+3i\,f(z)=2(u+iv)+3(iu-v)$$
$$(2+3i)f(z)=(2u-3v)+i(3u+2v)$$
$$F(z)=U+iV$$
$$\text{where } F(z)=(2+3i)f(z), U=2u-3v, V=3u+2v$$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$V=3u+2v=y^2-x^2+16xy$$
$$\frac{\partial V}{\partial x}=-2x+16y, \quad \frac{\partial V}{\partial y}=2y+16x$$
$$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}$$
$$=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x} \qquad [\text{Using C–R equations}]$$
$$=(2y+16x)+i(-2x+16y)$$

Putting $x = z$, $y = 0$,

$$F'(z)=16z+i(-2z)$$
$$=(16-2i)z$$

Integrating, w.r.t. z,

$$F(z)=\int(16-2i)z\,dz$$
$$=(16-2i)\frac{z^2}{2}+c$$
$$=(8-i)z^2+c$$
$$(2+3i)f(z)=(8-i)z^2+c$$
$$f(z)=\frac{(8-i)}{2+3i}z^2+\frac{c}{2+3i}$$
$$=\frac{(8-i)(2-3i)}{4+9}z^2+c', \text{ where } c'=\frac{c}{2+3i}$$
$$=\frac{13-26i}{13}z^2+c'$$
$$=(1-2i)z^2+c'$$

Example 19

Find the analytic function $f(z)=u+iv$ *if* $u+v=\dfrac{x}{x^2+y^2}$ *and* $f(1)=1$.

Solution

$$f(z) = u + iv \qquad \cdots(1)$$

$$i f(z) = iu + i^2 v$$

$$= iu - v \qquad \cdots(2)$$

Adding Eqs (1) and (2),

$$f(z) + i f(z) = (u + iv) + (iu - v)$$

$$(1+i) f(z) = (u - v) + i(u + v)$$

$$F(z) = U + iV$$

where $F(z) = (1+i) f(z), U = u - v, V = u + v$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$V = u + v$$

$$= \frac{x}{x^2 + y^2}$$

$$\frac{\partial V}{\partial x} = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial V}{\partial y} = -\frac{x}{(x^2 + y^2)^2} \cdot 2y$$

$$= \frac{-2xy}{(x^2 + y^2)^2}$$

$$F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial y} + i \frac{\partial V}{\partial x}$$

$$= \frac{-2xy}{(x^2 + y^2)^2} + i \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$= \frac{-2xy + i(y^2 - x^2)}{(x^2 + y^2)^2}$$

Putting $x = z$, $y = 0$,

$$F'(z) = -\frac{iz^2}{z^4}$$

$$= -\frac{i}{z^2}$$

Integrating w.r.t. z,

$$F(z) = \int -\frac{i}{z^2}\,dz$$

$$= \frac{i}{z} + c$$

$$(1+i)f(z) = \frac{i}{z} + c$$

Putting $z = 1$,

$$(1+i)(1) = i + c \qquad [\because \quad f(1) = 1]$$

$$c = 1$$

$$\therefore\ (1+i)f(z) = \frac{i}{z} + 1$$

$$f(z) = \frac{i}{z(1+i)} + \frac{1}{1+i}$$

$$= \frac{1+i}{2z} + \frac{1-i}{2}$$

Example 20

Determine the analytic function $f(z) = u + iv$ such that

$u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ *where* $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$.

Solution

$$f(z) = u + iv \qquad \cdots(1)$$

$$i\,f(z) = iu + i^2 v$$

$$= iu - v \qquad \cdots(2)$$

Adding Eqs (1) and (2),

$$f(z) + i\,f(z) = (u + iv) + (iu - v)$$

$$(1+i)f(z) = (u - v) + i(u + v)$$

$$F(z) = U + iV$$

where $F(z) = (1+i)f(z), U = u - v, V = u + v$

Since $f(z)$ is analytic, $F(z)$ is also analytic.

$$U = u - v$$

$$= \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$$\frac{\partial U}{\partial x} = \frac{(\sin x + \cos x)(\cosh y - \cos x) - (e^y - \cos x + \sin x)(\sin x)}{(\cosh y - \cos x)^2}$$

$$\frac{\partial U}{\partial y} = \frac{e^y(\cosh y - \cos x) - (e^y - \cos x + \sin x)\sinh y}{(\cosh y - \cos x)^2}$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= \frac{(\sin x + \cos x)(\cosh y - \cos x) - (e^y - \cos x + \sin x)\sin x}{(\cosh y - \cos x)^2}$$

$$- i\frac{e^y(\cosh y - \cos x) - (e^y - \cos x + \sin x)\sinh y}{(\cosh y - \cos x)^2}$$

Putting $x = z$, $y = 0$,

$$F'(z) = \frac{(\sin z + \cos z)(1 - \cos z) - (1 - \cos z + \sin z)\sin z}{(1 - \cos z)^2}$$

$$- i\frac{(1 - \cos z) - (1 - \cos z + \sin z)\cdot 0}{(1 - \cos z)^2}$$

$$= \frac{\sin z - \sin z\cos z + \cos z - \cos^2 z - \sin z + \sin z\cos z - \sin^2 z - i(1 - \cos z)}{(1 - \cos z)^2}$$

$$= \frac{\cos z - 1 - i(1 - \cos z)}{(1 - \cos z)^2}$$

$$= \frac{(-1 - i)(1 - \cos z)}{(1 - \cos z)^2}$$

$$= \frac{-(i + 1)}{1 - \cos z}$$

$$= \frac{-(i + 1)}{2\sin^2 \frac{z}{2}}$$

Integrating w.r.t. z,

$$F(z) = \int \frac{-(i + 1)}{2\sin^2 \frac{z}{2}} \mathrm{d}z$$

$$= -\frac{(i + 1)}{2} \int \mathrm{cosec}^2 \frac{z}{2} \mathrm{d}z$$

$$= -\frac{(i + 1)}{2}\left(-2\cot \frac{z}{2}\right) + c$$

$$= (i+1)\cot\frac{z}{2} + c$$

$$(1+i)f(z) = (i+1)\cot\frac{z}{2} + c$$

Putting $z = \frac{\pi}{2}$,

$$(1+i)\left(\frac{3-i}{2}\right) = (i+1)\cot\frac{\pi}{4} + c$$

$$c = 2 + i - i - 1$$

$$= 1$$

$$\therefore \quad f(z) = \cot\frac{z}{2} + \frac{1}{i+1}$$

$$= \cot\frac{z}{2} + \frac{1-i}{2}$$

Example 21

Show that $2x\,(1-y)$ *can be the imaginary part of an analytic function.*

Solution

Since real and imaginary parts of an analytic function are harmonic functions, they satisfy the Laplace equation.

Let $v = 2x\,(1-y)$.

$$\frac{\partial v}{\partial x} = 2(1-y), \qquad \frac{\partial v}{\partial y} = -2x$$

$$\frac{\partial^2 v}{\partial x^2} = 0, \qquad \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence, v can be the imaginary part of an analytic function.

Example 22

Find the orthogonal trajectory of the family of the curves $3x^2y - y^3 = a$.

Solution

Let $u = 3x^2y - y^3$ and $f(z) = u + iv$ be an analytic function.

Then $v = b$ will be the orthogonal trajectory to $u = a$.

$$u = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using C–R equations]}$$

$$= 6xy - i(3x^2 - 3y^2)$$

Putting $x = z$, $y = 0$,

$$f'(z) = -i(3z^2)$$

Integrating w.r.t. z,

$$f(z) = -i\int 3z^2 \, dz$$

$$= -i\,z^3 + c$$

$$u + iv = -i(x + iy)^3 + c$$

$$= -i(x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2) + c$$

$$= -ix^3 + i^2y^3 - 3x^2i^2y + 3ixy^2 + c$$

$$= (-y^3 + 3x^2y) + i(-x^3 + 3xy^2) + c$$

Comparing imaginary parts,

$$v = 3xy^2 - x^2 + c$$

Hence, the orthogonal trajectory is

$$3xy^2 - x^3 + c = b$$

$$3xy^2 - x^3 = c'$$

where $c' = b - c$

EXERCISE 2.5

1. Prove that the function $v = e^{-x}(x\cos y + y\sin y)$ is harmonic and determine the corresponding analytic function $f(z) = u + iv$.
2. Construct the analytic function whose real part is

(i) $\dfrac{y}{x^2 + y^2}$ (ii) $e^{2x}(x\cos 2y - y\sin 2y)$

(iii) $\cos x \cosh y$ (iv) $e^x\left[(x^2 - y^2)\cos y - 2xy\sin y\right]$

[Ans.: (i) $\log z + c$ (ii) $ze^{2z} + c$ (iii) $\cos z + c$ (iv) $x^2e^z + c$**]**

3. Construct the analytic function whose imaginary part is

(i) $\dfrac{(x-y)}{(x^2+y^2)}$ (ii) $-\sin x \sinh y$

(ii) $e^{-x}(x\sin y - y\cos y)$ (iv) $\dfrac{2\sin x \sin y}{\cos 2x + \cosh 2y}$

$$\left[\textbf{Ans.:}(\text{i})\frac{1+i}{z}+c\ (\text{ii})\cos z + c\ \ (\text{iii})\ \bar{z}\,\bar{e}^{\bar{z}}+c\ \ (\text{iv})\frac{2\cos x\cosh y}{\cos 2x+\cosh 2y}+c\right]$$

4. Show that the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic and find the analytic function $f(z) = u + iv$.

$$\left[\textbf{Ans.:}\,(1-2i)(\sin z + z^2)+c\right]$$

5. Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$.

$$\left[\textbf{Ans.:}\,\frac{(-1+5i)e^z}{13}+c\right]$$

6. Find the analytic function $f(\underline{z}) = u + iv$, given that $2u + v = e^{2x}\left\{(2x+y)\cos 2y + (x-2y)\sin 2y\right\}$.

$$\left[\textbf{Ans.:}\,f(z) = ze^{2z}+c\right]$$

7. Find the analytic function $z = u + iv$ if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$.

$$\left[\textbf{Ans.:}\,\frac{1}{2} - \cot\frac{z}{2}\right]$$

8. Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z.

$$\left[\textbf{Ans.:}\,v = -e^{-2xy}\cos(x^2-y^2)+c, f(z) = -ie^{iz^2}+c\right]$$

9. Find the orthogonal trajectories of the family of curves

(i) $x^3y - xy^3 = c$ (ii) $e^x - \cos y - xy = c$

$$\left[\textbf{Ans.:}(\text{i})\,x^4 - 6x^2y^2 + y^4 = c\ \ (\text{ii})\,x^2 - y^2 + 2e^x - \sin y = c\right]$$

Points to Remember

Analytic Functions

A function $f(z)$ is said to be an analytic function if it is defined and differentiable at each point of a region R.

Entire Function A function $f(z)$ is said to be an entire function if it is analytic everywhere in the finite plane (complex plane).

1. Necessary Conditions for $f(z)$ to be Analytic (Cauchy—Riemann Equations)
The necessary conditions for a function $f(z) = u + iv$ to be an analytic function at all the points in a region R are

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

2. Sufficient Conditions for $f(z)$ to be Analytic

For a function $f(z) = u + iv$, if the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are continuous in the region R and $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ then the function of $f(z)$ is analytic.

Cauchy–Riemann Equations in Polar Form

If $f(z) = u + iv$ is an analytic function where u and v are functions of r, θ and $z = re^{i\theta}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Harmonic Functions

A real function ϕ of two variables x and y is said to be a harmonic function in a region R if it has continuous second-order partial derivatives and satisfies the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\nabla^2 \phi = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Properties of Analytic Functions

(i) If $f(z) = u + iv$ is an analytic function, u and v are harmonic functions.

(ii) If $f(z) = u + iv$ is an analytic function then the family of curves $u(x, y) = c_1$ and the family of curves $v(x, y) = c_2$ cut orthogonally.

CHAPTER 3
Complex Integration

Chapter Outline

3.1 Introduction
3.2 Some Basic Definitions
3.3 Line Integral
3.4 Simply connected and Multiply connected Regions
3.5 Cauchy's Integral Theorem
3.6 Cauchy's Integral Formula
3.7 Liouville's Theorem
3.8 Maximum Modulus Theorem

3.1 INTRODUCTION

The concept of a real line integral is extended to that of a complex line integral. Complex integration plays an important role in the evaluation of complicated real integrals. It is a powerful tool in evaluating certain integrals. In case of real integration, $\int_a^b f(x)\,dx$, the path of integration is always along the x-axis from $x = a$ to $x = b$. In case of complex integration, $\int_{z_1}^{z_2} f(z)\,dz$, the path of integration can be along any curve from $z = z_1$ to $z = z_2$ but the value of the integral does not depend upon the path if $f(z)$ is analytic.

3.2 SOME BASIC DEFINITIONS

1. Continuous Arc The set of points (x, y) defined by $x = \phi(t)$, $y = \psi(t)$, where t is a parameter in the interval (a, b), represents a continuous arc (or curve) provided ϕ and ψ are continuous functions.

2. Smooth Arc If ϕ and ψ are differentiable in the closed interval $a \le t \le b$ and $z'(t) \neq 0$, the arc is said to be smooth.

3. Simple Curve The curve which does not intersect itself is said to be a simple curve.

4. Simple Closed Curve The curve which does not intersect itself and has coincident end points $[z(b) = z(a)]$ is said to be a simple closed curve.

5. Contour A contour is a continuous chain of a finite number of smooth arcs.

6. Closed Contour A closed contour is a piecewise smooth closed curve in which only the initial and final values of $z(t)$ are same.

3.3 LINE INTEGRAL

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve C whose end points are A and B (Fig. 3.1). Divide the curve C into n parts at the points.

$A = P_0(z_0), P_1(z_1), P_2(z_2), \cdots P_i(z_i) \cdots P_n(z_n) = B$

Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1}\, P_i$. The limit of the sum $\sum_{i=1}^{n} f(\xi_i)\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the line integral of $f(z)$ along the path C and is denoted by $\int_C f(z)\,dz$. If C is a closed curve, i.e., if P_o and P_n coincide, the integral is called the contour integral and is denoted by $\oint_C f(z)\,dz$.

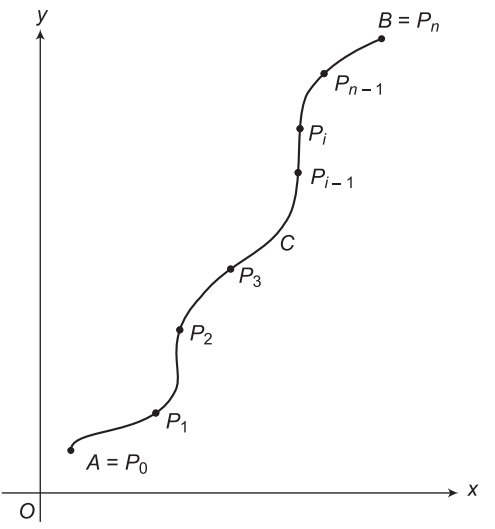


Fig. 3.1

3.3.1 Properties of Line Integrals

If $f(z)$ and $g(z)$ are integrable along a curve C then the following properties hold.

1. Linearity

$$\int_C \left[k_1 f(z) + k_2 g(z)\right] dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)\,dz$$

2. Sense Reversal

$$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

3. Partitioning of Path

If the curve C consists of the curves C_1 and C_2 then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

4. Integral Inequality

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)||dz|$$

5. ML Inequality

If $f(z)$ is continuous on the curve C of length L and $|f(z)| \le M$ then

$$\left|\int_C f(z)\,dz\right| \le ML$$

Example 1

Find an upper bound for the absolute value of the integral $\int_C z^2\,dz$, *where* C *is the straight-line segment from* 0 *to* $1 + i$. **[Summer 2014]**

Solution

The length of the line segment OA from 0 to $1 + i$ (Fig. 3.2) is

$$L = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$$

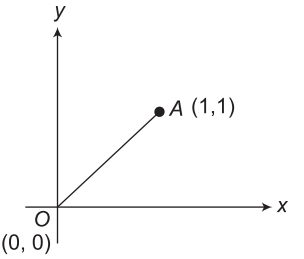


Fig. 3.2

Let $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

Along the line OA, $y = x$

$$f(z) = x^2 - x^2 + 2ix^2$$
$$= 2ix^2$$
$$|f(z)| = |2ix^2|$$
$$= 2x^2$$
$$\le 2 \qquad [\because \quad 0 \le x \le 1]$$
$$M = 2$$

By *ML* inequality,

$$\left|\int_C f(z)\,dz\right| \le ML$$

$$\left|\int_C z^2\,dz\right| \le 2\sqrt{2}$$

Example 2

Find an upper bound for the absolute value of the integral $\oint_C \frac{e^z}{z+1}dz$, *where C is* $|z| = 4$. **[Summer 2013]**

Solution

The length of the circular arc $|z| = 4$ (Fig. 3.3) is

$$L = 2\pi r$$
$$= 2\pi(4)$$
$$= 8\pi$$

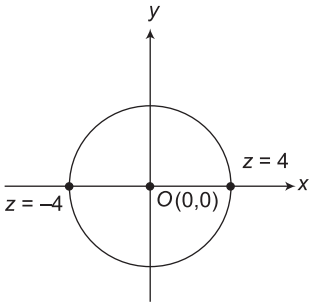


Fig. 3.3

Let $f(z) = \frac{e^z}{z+1}$

$$|f(z)| = \left|\frac{e^z}{z+1}\right|$$

$$\le \frac{|e^z|}{|z+1|}$$

$$\le \frac{e^4}{3} \qquad \left[\because \max(z) = 4,\quad e^z \le e^4,\quad \min(z) = -4, |z+1| \ge 3, \frac{1}{|z+1|} \le \frac{1}{3}\right]$$

$$M = \frac{e^4}{3}$$

By *ML* inequality,

$$\left|\int_C f(z)\,dz\right| \le ML$$

$$\left|\oint_C \frac{e^z}{z+1}dz\right| \le \frac{8\pi e^4}{3}$$

Example 3

Find an upper bound for the integral $\int_C \frac{e^z}{z^2}\,dz$, *where C is the line segment from* $z = i$ *to* $z = 2$.

Solution

Let $z = i$ is the point $A(0, 1)$ and $z = 2$ is the point $B(2, 0)$.

The length of the line segment AB (Fig. 3.4) is

$$L = \sqrt{(0-2)^2 + (1-0)^2}$$
$$= \sqrt{5}$$

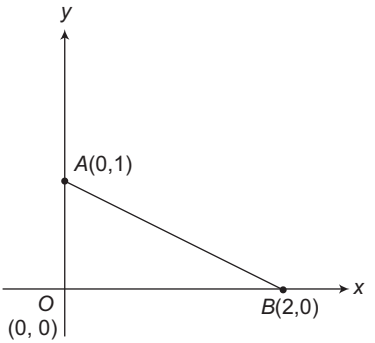


Fig. 3.4

The equation of the line AB is

$$y - 0 = \frac{1-0}{0-2}(x-2)$$
$$-2y = x - 2$$
$$x + 2y = 2$$

Let $f(z) = \frac{e^z}{z^2}$

$$= \frac{e^{x+iy}}{(x+iy)^2}$$
$$= \frac{e^x e^{iy}}{(x+iy)^2}$$
$$|f(z)| = \left|\frac{e^x e^{iy}}{(x+iy)^2}\right|$$
$$= \frac{|e^x||e^{iy}|}{|x+iy|^2}$$
$$= \frac{e^x}{x^2+y^2} \qquad [\because\ e^x \geq 0 \text{ for all values of } x]$$

Along the line AB, $y = \frac{2-x}{2}$

$$|f(z)| = \frac{e^x}{x^2 + \left(\frac{2-x}{2}\right)^2}$$
$$= \frac{4e^x}{4x^2 + 4 + x^2 - 4x}$$

$$= \frac{4e^x}{5x^2 - 4x + 4}$$

For maximum value,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left|f(z)\right| = 0$$

$$\frac{4e^x(5x^2 - 4x + 4) - 4e^x(10x - 4)}{(5x^2 - 4x + 4)^2} = 0$$

$$5x^2 - 14x + 8 = 0$$

$$x = 2, \frac{4}{5}$$

But $|f(z)|$ is maximum at $x = \frac{4}{5}$.

$$\left|f(z)\right| = \frac{4e^x}{5x^2 - 4x + 4}$$

$$\leq \frac{4e^{\frac{4}{5}}}{5\left(\frac{4}{5}\right)^2 - 4\left(\frac{4}{5}\right) + 4}$$

$$\left|\frac{e^z}{z^2}\right| \leq e^{\frac{4}{5}}$$

$$M = e^{\frac{4}{5}}$$

By *ML* inequality,

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq ML$$

$$\left|\int_C \frac{e^z}{z^2}\mathrm{d}z\right| \leq \sqrt{5}\, e^{\frac{4}{5}}$$

3.3.2 Evaluation of Line Integrals

Let $f(z) = u + iv$

$z = x + iy, \qquad \mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$

$$\int_C f(z)\,\mathrm{d}z = \int_C (u + iv)\,(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y)$$

Here, u and v are functions of x and y. To evaluate the line integral, it is converted to a single variable x or y using the equation of the curve C.

Note If $f(z)$ is analytic in a simply connected domain D then the line integral is independent of the path, i.e.,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where $F'(z) = f(z)$

Example 1

Evaluate $\int_0^{2+i} z^2\,dz$ *along the line* $y = \frac{x}{2}$. **[Summer 2013]**

Solution

$$f(z) = z^2$$
$$f'(z) = 2z$$

Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.

Hence, $\int_0^{2+i} z^2\,dz$ will be independent of the path.

$$\int_0^{2+i} z^2\,dz = \left|\frac{z^3}{3}\right|_0^{2+i}$$
$$= \frac{1}{3}(2+i)^3$$
$$= \frac{1}{3}(8 + i^3 + 6i^2 + 12i)$$
$$= \frac{1}{3}(8 - i - 6 + 12i)$$
$$= \frac{1}{3}(2 + 11i)$$

Example 2

Evaluate $\int_0^{1+i} (x - y + ix^2)\,dz$ *along the line from* $z = 0$ *to* $z = 1 + i$.

Solution

Let $z = x + iy$

$dz = dx + i\,dy$

The equation of the line OA (Fig. 3.5) is

$$y-0=\frac{1-0}{1-0}(x-0)$$

$$y=x$$

$$\int_0^{1+i}(x-y+ix^2)\,dz=\int_0^{1+i}(x-y+ix^2)\,(dx+i dy)$$

$$=\int_{0A}(x-y+ix^2)\,(dx+i dy)$$

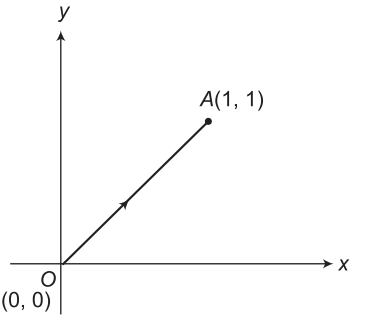


Fig. 3.5

Along OA, $y = x$, $dx = dy$, x varies from 0 to 1.

$$\int_0^{1+i}(x-y+ix^2)dz=\int_0^1 ix^2(dx+i dx)$$

$$=\int_0^1 ix^2(1+i)\,dx$$

$$=(-1+i)\int_0^1 x^2\,dx$$

$$=(-1+i)\left|\frac{x^3}{3}\right|_0^1$$

$$=\frac{-1+i}{3}$$

$$=-\frac{1}{3}+\frac{1}{3}i$$

Example 3

Evaluate $\int_C \mathrm{Re}\, z\, dz$, *where* C *is the shortest path from* $1 + i$ *to* $3 + 2i$. **[Summer 2014]**

Solution

Let $z = x + iy$

$\mathrm{Re}\, z = x$

$dz = dx + i dy$

The shortest distance from $A(1 + i)$ to $B(3 + 2i)$ is the line joining these points (Fig. 3.6).
The equation of the line AB is

$$y-1=\frac{2-1}{3-1}(x-1)$$

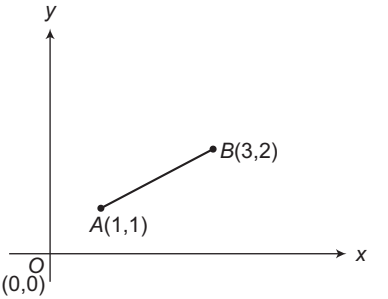


Fig. 3.6

$$2(y-1) = x-1$$
$$x-2y = -1$$
$$\int_C \text{Re } z\, dz = \int_C x(dx + i\, dy)$$

Along AB, $y = \frac{x+1}{2}$, $dy = \frac{1}{2}dx$, x varies from 1 to 3.

$$\int_C \text{Re } z\, dz = \int_1^3 x\left(dx + i\frac{1}{2}dx\right)$$
$$= \left(1+\frac{i}{2}\right)\int_1^3 x\, dx$$
$$= \left(1+\frac{i}{2}\right)\left|\frac{x^2}{2}\right|_1^3$$
$$= \left(1+\frac{i}{2}\right)\left(\frac{9}{2}-\frac{1}{2}\right)$$
$$= \left(1+\frac{i}{2}\right)4$$
$$= 4+2i$$

Example 4

Evaluate $\int_C z^2\, dz$, *where C is the line joining the points* (0, 0) *and* (4, 2). **[Winter 2013]**

Solution

Let $z = x + iy$

$dz = dx + i dy$

The equation of the line joining $O(0, 0)$ and $A(4, 2)$ (Fig. 3.7) is

$$y - 0 = \frac{2-0}{4-0}(x-0)$$
$$2y = x$$

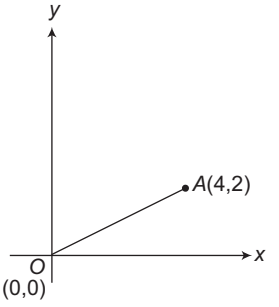


Fig. 3.7

Along OA, $x = 2y$, $dx = 2dy$, y varies from 0 to 2.

$$\int_0^{4+2i} z^2\, dz = \int_C (x+iy)^2\,(dx + i\, dy)$$
$$= \int_C (x^2 - y^2 + 2ixy)\,(dx + i dy)$$

$$= \int_0^2 (4y^2 - y^2 + 4iy^2)\,(2\mathrm{d}y + i\mathrm{d}y)$$

$$= \int_0^2 (3+4i)\,y^2 (2+i)\,\mathrm{d}y$$

$$= (6+3i+8i+4i^2)\left|\frac{y^3}{3}\right|_0^2$$

$$= (2+11i)\left(\frac{2^3}{3}\right)$$

$$= \frac{8}{3}(2+11i)$$

Aliter:

Let $f(z) = z^2$

$f'(z) = 2z$

Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.

Hence, $\int_0^{4+2i} z^2\,\mathrm{d}z$ will be independent of the path.

$$\int_0^{4+2i} z^2\,\mathrm{d}z = \left|\frac{z^3}{3}\right|_0^{4+2i}$$

$$= \frac{1}{3}(4+2i)^3$$

$$= \frac{2^3}{3}(2+i)^3$$

$$= \frac{8}{3}(8+i^3+6i^2+12i)$$

$$= \frac{8}{3}(8-i-6+12i)$$

$$= \frac{8}{3}(2+11i)$$

Example 5

Evaluate $\int_C \overline{z}\,\mathrm{d}z$, *where C is along the sides of the triangle having vertices* $z = 0, 1, i$. **[Summer 2015]**

Solution

Let $z = x + iy,\ \bar{z} = x - iy$

$$dz = dx + i\,dy$$

$$\int_C \bar{z}\,dz = \int_C (x - iy)(dx + i\,dy)$$

The equation of the line joining $A(1, 0)$ and $B(0, 1)$ (Fig. 3.8) is

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$y = -x + 1$$

$$x + y = 1$$

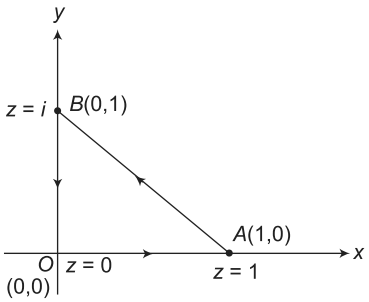


Fig. 3.8

Along OA, $y = 0$, $dy = 0$, x varies from 0 to 1.

$$\int_C \bar{z}\,dz = \int_0^1 x\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2} \qquad \ldots(1)$$

Along AB, $y = -x + 1$, $dy = -dx$, x varies from 1 to 0.

$$\int_C \bar{z}\,dz = \int_1^0 \left[x - i(-x+1)\right](dx - i\,dx)$$

$$= \int_1^0 \left[(1+i)x - i\right](1-i)\,dx$$

$$= (1-i)\left|(1+i)\frac{x^2}{2} - ix\right|_1^0$$

$$= (1-i)\left[-\frac{(1+i)}{2} + i\right]$$

$$= (1-i)\left[\frac{-1-i+2i}{2}\right]$$

$$= \frac{(1-i)(i-1)}{2}$$

$$= i \qquad \ldots(2)$$

Along BO, $x = 0$, y varies from 1 to 0.

$$\int_C \bar{z}\,dz = \int_1^0 (-iy)\,i\,dy$$

$$= \int_1^0 y\,dy$$

$$= \left|\frac{y^2}{2}\right|_1^0$$

$$= -\frac{1}{2} \qquad \ldots(3)$$

Adding Eqs (1), (2), and (3),

$$\int_C \bar{z}\,dz = \frac{1}{2} + i - \frac{1}{2}$$

$$= i$$

Example 6

Find the value of the integral $\int_C \bar{z}\,dz$, *where C is the right-hand half* $z = 2e^{i\theta}\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ *of the circle* $|z| = 2$, *from* $z = -2i$ *to* $z = 2i$.

[Winter 2014]

Solution

$$z = 2e^{i\theta}$$

$$dz = 2ie^{i\theta}\,d\theta$$

$$\bar{z} = 2e^{-i\theta}$$

$$\int_C \bar{z}\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} \cdot 2ie^{i\theta}\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4i\,d\theta$$

$$= 4i\,|\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

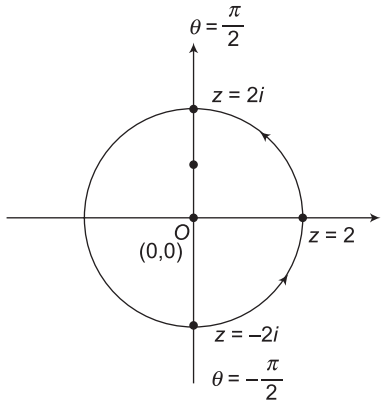


Fig. 3.9

$$= 4i\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$= 4i\pi$$

Example 7

Show that $\int_C \frac{dz}{z} = \pi i$, *where C is the right half of the circle* $|z| = 2$.

Solution

Let $z = re^{i\theta}$

For the circle $|z| = 2$, $r = 2$

$$z = 2e^{i\theta}$$
$$dz = 2ie^{i\theta}d\theta$$

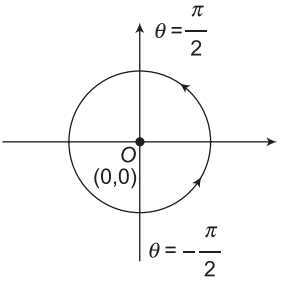


Fig. 3.10

For the right half of the circle $|z| = 2$, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (Fig. 3.10).

$$\int_C \frac{dz}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2ie^{i\theta}\,d\theta}{2e^{i\theta}}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\,d\theta$$

$$= i\left|\theta\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= i\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \pi i$$

Example 8

Evaluate $\int_C |z|\,dz$, *where C is the left half of the unit circle* $|z| = 1$ *from* $z = -i$ *to* $z = i$.

Solution

Let $z = re^{i\theta} = e^{i\theta} \quad [\because\ r = 1 \text{ for } |z| = 1]$

$dz = ie^{i\theta} d\theta$

In the left half of the unit circle, θ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$ (Fig. 3.11).

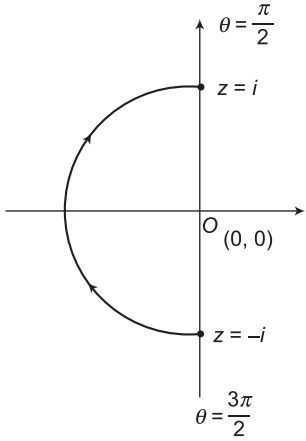


Fig. 3.11

$$\int_C |z|\, dz = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} 1 \cdot ie^{i\theta}\, d\theta$$

$$= i\left|\frac{e^{i\theta}}{i}\right|_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$= e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}$$

$$= 0 + i - 0 - i(-1)$$

$$= 2i$$

Example 9

Evaluate $\int_C (3z^2 + 2z + 1)\, dz$, *where C is the arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, *between* $\theta = 0$ *to* $= 2\pi$.

Solution

Let $f(z) = 3z^2 + 2z + 1$

Since $f(z)$ is a polynomial, it is analytic everywhere. Hence, the integral is independent of the path.

$$z = x + iy$$

$$= a(\theta + \sin\theta) + ia(1 - \cos\theta)$$

At $\theta = 0$, $z = 0$

At $\theta = 2\pi$, $z = 2a\pi$

$$\int_C f(z)\, dz = \int_0^{2a\pi} (3z^2 + 2z + 1)\, dz$$

$$= \left| z^3 + z^2 + z \right|_0^{2a\pi}$$
$$= 2a\pi\,(4a^2\pi^2 + 2\pi a + 1)$$

Example 10

Evaluate $\int_C (x^2 - iy^2)\,dz$ *along the parabola* $y = 2x^2$ *from* (1, 2) *to* (2, 8).

Solution

Along $y = 2x^2$, $dy = 4x\,dx$,

x varies from 1 to 2 (Fig. 3.12).

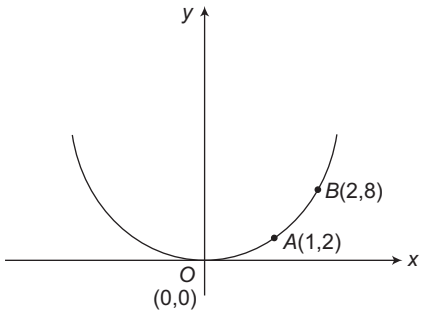


Fig. 3.12

$$dz = dx + i\,dy$$
$$= dx + i \cdot 4x\,dx$$
$$= (1 + 4ix)\,dx$$

$$\int_C (x^2 - iy^2)\,dz = \int_1^2 (x^2 - i \cdot 4x^4)\,(1 + 4ix)\,dx$$
$$= \int_1^2 (x^2 + 4ix^3 - 4ix^4 - 16i^2x^5)\,dx$$
$$= \int_1^2 (x^2 + 4ix^3 - 4ix^4 + 16x^5)\,dx$$
$$= \left| \frac{x^3}{3} + 4i\frac{x^4}{4} - 4i\frac{x^5}{5} + 16\frac{x^6}{6} \right|_1^2$$
$$= \frac{1}{3}(8-1) + i(16-1) - \frac{4i}{5}(32-1) + \frac{8}{3}(64-1)$$
$$= \frac{7}{3} + \frac{504}{3} + i\left(15 - \frac{124}{5}\right)$$
$$= \frac{511}{3} - i\frac{49}{5}$$

Example 11

Evaluate $\int_0^{1+i} (x^2 + iy)\,dz$ *along the path (i)* $y = x$, *(ii)* $y = x^2$. *Is the line integral independent of the path?*

Solution

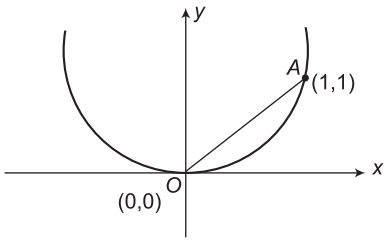


Fig. 3.13

(i) Along the path $y = x$

$y = x$, $dy = dx$,

x varies from 0 to 1 (Fig. 3.13).

$$\int_0^{1+i} (x^2 + iy)\,dz = \int_0^1 (x^2 + iy)\,(dx + i dy)$$

$$= \int_0^1 (x^2 + ix)(1 + i)\,dx$$

$$= (1 + i)\left|\frac{x^3}{3} + i\frac{x^2}{2}\right|_0^1$$

$$= (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

$$= (1 + i)\frac{(2 + 3i)}{6}$$

$$= \frac{2 + 2i + 3i - 3}{6}$$

$$= \frac{-1 + 5i}{6}$$

(ii) Along the path $y = x^2$, $y = x^2$, $dy = 2x dx$, x varies form 0 to 1.

$$dz = dx + i2x\,dx$$

$$\int_0^{1+i} (x^2 + iy)\,dz = \int_0^1 (x^2 + iy)(dx + 2ix\,dx)$$

$$= \int_0^1 (x^2 + ix^2)\,(1 + 2ix)\,dx$$

$$= (1 + i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$= (1 + i)\left|\frac{x^3}{3} + 2i\frac{x^4}{4}\right|_0^1$$

$$= (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

$$= \frac{-1 + 5i}{6}$$

Both the line integrals are equal.

$$f(z) = x^2 + iy$$

$$u + iv = x^2 + iy$$

Comparing real and imaginary parts,

$$u = x^2, \qquad v = y$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = 1$$

Since *C–R* equations are not satisfied, $f(z)$ is not analytic.
Hence, the line integral is not independent of the path.

Example 12

Evaluate $\int_0^{4+2i} \overline{z}\, \mathrm{d}z$ *along the curve* $z = t^2 + it$.

Solution

$$f(z) = \overline{z}$$

Along the curve $z = t^2 + it$,

$$f(z) = \overline{z} = t^2 - it$$

$$\mathrm{d}z = 2t\, \mathrm{d}t + i\mathrm{d}t$$

$$= (2t + i)\mathrm{d}t$$

When $z = 0, \quad t^2 + it = 0, t = 0$

When $z = 4 + 2i, t^2 + it = 4 + 2i, t = 2$

$\therefore$ t varies from 0 to 2.

$$\int_0^{4+2i} \overline{z}\, \mathrm{d}z = \int_0^2 (t^2 - it)\,(2t + i)\, \mathrm{d}t$$

$$= \int_0^2 (2t^3 + it^2 - i2t^2 - i^2 t)\, \mathrm{d}t$$

$$= \left| \frac{2t^4}{4} - i\frac{t^3}{3} + \frac{t^2}{2} \right|_0^2$$

$$= \left(8 - i\frac{8}{3} + 2 \right)$$

$$= 10 - i\frac{8}{3}$$

Example 13

Integrate $f(z) = x^2 + ixy$ *from* (1, 1) *to* (2, 4) *along the curve* $x = t$, $y = t^2$.

Solution

$$f(z) = x^2 + ixy$$

Along the curve $x = t, y = t^2$

$$f(z) = t^2 + i(t)\,(t^2)$$
$$= t^2 + it^3$$

$$dz = dx + idy$$
$$= dt + i2t\,dt$$
$$= (1 + 2it)\,dt$$

When $x = 1$, $t = 1$
When $x = 2$, $t = 2$
$\therefore$ t varies from 1 to 2.

$$\int_C f(z)\,dz = \int_1^2 (t^2 + it^3)\,(1 + 2it)\,dt$$
$$= \int_1^2 (t^2 + 2it^3 + it^3 + 2i^2t^4)\,dt$$
$$= \int_1^2 \left[t^2 - 2t^4 + 3it^3\right] dt$$
$$= \left|\frac{t^3}{3} - 2\frac{t^5}{5} + 3i\frac{t^4}{4}\right|_1^2$$
$$= \left(\frac{8}{3} - \frac{64}{5} + 3i\frac{16}{4}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{3i}{4}\right)$$
$$= -\frac{151}{15} + i\frac{45}{4}$$

EXERCISE 3.1

1. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$ along (i) the straight line joining $(1 - i)$ to $(2 + i)$, and (ii) $x = t + 1$, $y = 2t^2 - 1$.

$$\left[\textbf{Ans.: (i) } 4(1 + 2i),\ \text{(ii) } 4 + \frac{25}{3}i\right]$$

2. Evaluate $\int_C \frac{2z+3}{z} dz$, where C is (i) the upper half of the circle $|z| = 2$, (ii) the lower half of the circle $|z| = 2$, and (iii) the whole circle in the anti-clockwise direction.

[Ans.: (i) $2(3\pi i - 4)$, (ii) $2(4 - 3\pi i)$, (iii) 12**]**

3. Show that $\int_C \log z dz = 2\pi i$, where C is the unit circle in the Z-plane.

4. Evaluate $\int_C (z - z^2) dz$, where C is the upper half of the circle $|z| = 1$.

$$\left[\textbf{Ans.:}\ \frac{2}{3}\right]$$

5. Evaluate $\int_C |z|^2 dz$, where C is the boundary of the square C with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

[Ans.: $-1 + i$**]**

6. Evaluate $\int_C (2z^3 + 8z + 2) dz$, where C is the arc of the cycloid $x = a(\theta - \sin\theta)$ $y = a(1 - \cos\theta)$ between the points $(0, 0)$ and $(2\pi a, 0)$.

[Ans.: $4\pi a(2\pi^3 a^3 + 4\pi a + 1)$**]**

7. Evaluate $\int_C \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$, where C is the curve given by $z = t^2 + it$.

$$\left[\textbf{Ans.:}\ 10 - \frac{8}{3}i\right]$$

8. Prove that $\int_{-2}^{-2+i} (2 + z)^2 = dz = -\frac{i}{3}$.

9. Evaluate $\int_C (z^2 - 2\bar{z} + 1) dz$, where C is the circle $x^2 + y^2 = 2$.

[Ans.: $-8\pi i$**]**

10. Evaluate $\int_C z^2 dz$ from $P(1, 1)$ to $\theta(2, 4)$, where C is the curve $x = t$, $y = t^2$.

$$\left[\textbf{Ans.:}\ -\frac{86}{3} - 6i\right]$$

3.4 SIMPLY CONNECTED AND MULTIPLY CONNECTED REGIONS

3.4.1 Simply Connected Region

A simply connected region R is the region enclosed by a simple curve, e.g., interior of a circle, rectangle, triangle, ellipse, etc., (Fig. 3.14).

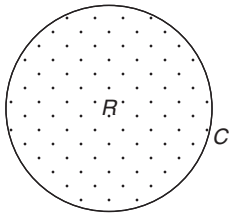


Fig. 3.14

3.4.2 Multiply Connected Region

A multiply connected region is the region enclosed by more than one simple curve, e.g., annulus region, regions with holes, etc., (Fig. 3.15a).

A multiply connected region can be converted into a simply connected region by introducing cross-cuts (Fig. 3.15b).

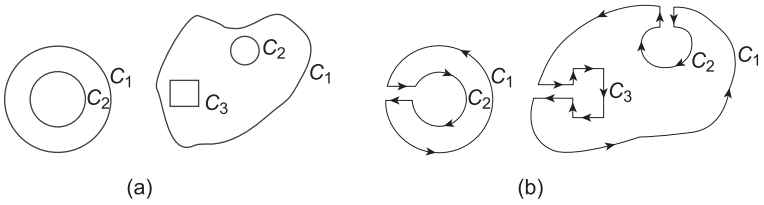


Fig. 3.15

3.4.3 Independence of Path

If $f(z)$ is analytic in a simply connected region then the line integral is independent of the path.

3.5 CAUCHY'S INTEGRAL THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point inside and on a closed curve C then

$$\oint_C f(z)\,dz = 0$$

Proof

Let $f(z) = u + iv, \quad z = x + iy$

$$dz = dx + i\,dy$$

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy \quad [\text{Using Green's theorem}]$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dx\, dy \quad [\text{Using C-R equations}]$$

$$= 0$$

3.5.1 Cauchy–Goursat Theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C contained in a simply connected domain D then

$$\oint_C f(z)\, dz = 0$$

3.5.2 Extension of Cauchy's Integral Theorem to a Multiply Connected Region

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz$$

Proof The multiply connected region is made simply connected region by introducing a cross-cut AB (Fig. 3.16).

The path of integration along AB and C_2 is in the clockwise direction, and along BA and C_1 is in the anticlockwise direction.

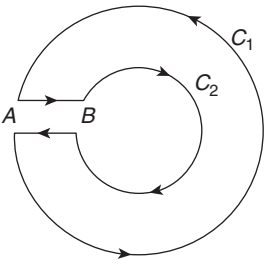


Fig. 3.16

By Cauchy's integral theorem, in a simply connected region C_1ABC_2,

$$\oint f(z)\, dz = 0$$

$$\int_{AB} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{BA} f(z)\, dz + \int_{C_1} f(z)\, dz = 0$$

$$\int_{C_2} f(z)\, dz + \int_{C_1} f(z)\, dz = 0 \quad \left[\because \int_{AB} f(z)\, dz = -\int_{BA} f(z)\, dz \right]$$

Reversing the direction of the integral around C_2,

$$-\int_{C_2} f(z)\, dz + \int_{C_1} f(z)\, dz = 0$$

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

Note If $C_1, C_2, C_3, \ldots, C_n$ be n closed curves within C (Fig. 3.17) then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

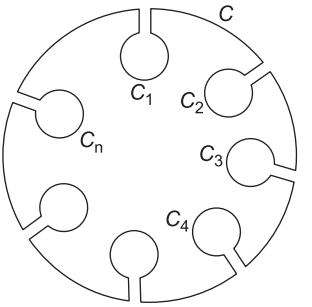


Fig. 3.17

Example 1

Evaluate $\int_C e^z\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = e^z$

(ii) Since $f(z)$ is differentiable, it is analytic inside and on C.
$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C e^z\,dz = 0$$

Example 2

Evaluate $\oint_C e^{z^2}\,dz$, *where C is any closed contour. Justify your answer.*

[Winter 2013]

Solution

(i) Let $f(z) = e^{z^2}$

$$f'(z) = e^{z^2} \cdot 2z$$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.
$f'(z)$ is continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\oint_C e^{z^2}\,dz = 0$$

Example 3

If C is any simple closed contour, in either direction then show that $\int_C \exp(z^3)\,dz = 0$. **[Winter 2014]**

Solution

(i) Let $f(z) = \exp(z^3) = e^{z^3}$

$f'(z) = e^{z^3}(3z^2)$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane. $f'(z)$ is also continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\int_C \exp(z^3)\,dz = 0$$

Example 4

Evaluate $\int_C e^{\sin z^2}\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = e^{\sin z^2}$

(ii) Since $f(z)$ is differentiable, it is analytic inside and on C.

$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C e^{\sin z^2}\,dz = 0$$

Example 5

Evaluate $\oint_C (z^2 + 3)\,dz$, *where C is any closed contour. Justify your answer.* **[Summer 2013]**

Solution

(i) Let $f(z) = z^2 + 3$

$f'(z) = 2z$

(ii) Since $f(z)$ is differentiable for all finite values of z, it is analytic in the z-plane.
$f'(z)$ is also continuous in the z-plane.

(iii) By Cauchy's integral theorem,

$$\oint_C (z^2 + 3)\,dz = 0$$

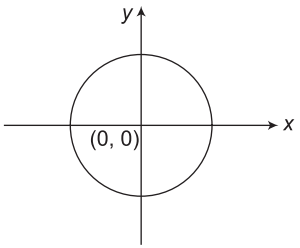


Fig. 3.18

Example 6

Evaluate $\int_C (z^2 + 2z)\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = z^2 + 2z$

(ii) Since $f(z)$ is a polynomial, it is analytic inside and on C.
$f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C (z^2 + 2z)\,dz = 0$$

Example 7

Evaluate $\int_C \frac{dz}{z+4}$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{1}{z+4}$

$f(z)$ is not analytic at $z = -4$.

(ii) C is the circle $|z| = 2$ with the centre at $(0,0)$ and a radius of 2 (Fig. 3.19).

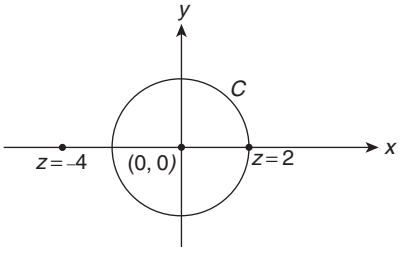


Fig. 3.19

(iii) For $z = -4, |z| = |-4| = 4 > 2$

Hence, $z = -4$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{dz}{z+4} = 0$$

Example 8

Evaluate $\int_C \frac{3z^2+7z+1}{z+1} dz$, *where C is* $|z| = \frac{1}{2}$.

Solution

(i) Let $f(z) = \frac{3z^2+7z+1}{z+1}$

$f(z)$ is not analytic at $z = -1$.

(ii) C is the circle $|z| = \frac{1}{2}$ with the centre at $(0,0)$ and a radius of $\frac{1}{2}$ (Fig. 3.20).

(iii) For $z = -1, |z| = |-1| = 1 > \frac{1}{2}$

Hence, $z = -1$ lies outside C

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C

(v) By Cauchy's integral theorem,

$$\int_C \frac{3z^2+7z+1}{z+1} dz = 0$$

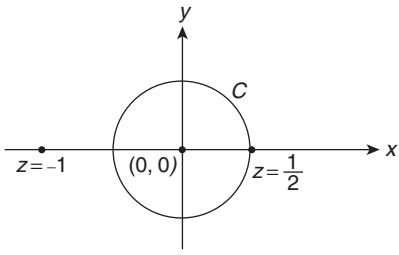


Fig. 3.20

Example 9

Evaluate $\int_C \frac{dz}{(z-3)^2}$, *where C is the circle* $|z| = 1$.

Solution

(i) Let $f(z) = \dfrac{1}{(z-3)^2}$

$f(z)$ is not analytic at $z = 3$.

(ii) C is the circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 3.21).

(iii) For $z = 3$, $|z| = |3| = 3 > 1$.

Hence, $z = 3$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{dz}{(z-3)^2} = 0$$

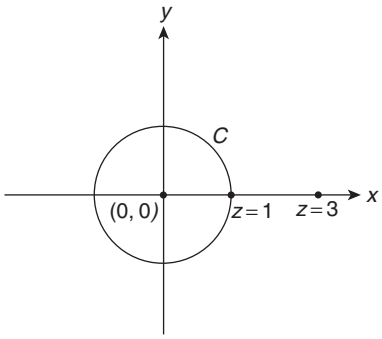


Fig. 3.21

Example 10

Evaluate $\displaystyle\int_C \frac{z\,dz}{(z-1)(z-2)}$, *where C is the circle* $|z| = \dfrac{1}{2}$.

Solution

(i) Let $f(z) = \dfrac{z}{(z-1)(z-2)}$

$f(z)$ is not analytic at $z = 1$ and $z = 2$.

(ii) C is the circle $|z| = \dfrac{1}{2}$ with the centre at (0,0) and a radius of $\dfrac{1}{2}$ (Fig. 3.22).

(iii) For $z = 1$, $|z| = |1| = 1 > \dfrac{1}{2}$.

Hence, $z = 1$ lies outside C.

For $z = 2$, $|z| = |2| = 2 > \dfrac{1}{2}$

Hence, $z = 2$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

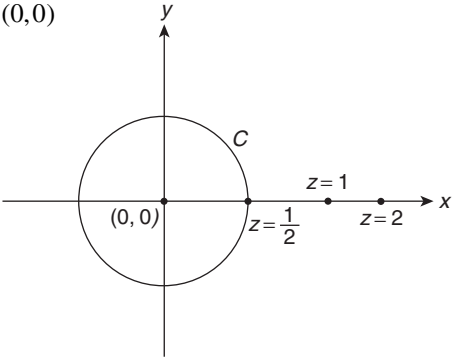


Fig. 3.22

(v) By Cauchy's integral theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)} = 0$$

Example 11

Evaluate $\oint_C \frac{e^z}{z+i} dz$, *where* $C : |z-1| = 1$. **[Summer 2015]**

Solution

(i) Let $f(z) = \frac{e^z}{z+i}$

(ii) C is a circle $|z-1| = 1$ with the centre at (1, 0) and a radius of 1 (Fig. 3.23).

(iii) For $z = -i, |z-1| = |-i-1| = \sqrt{2} > 1$
Hence, $z = -i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.
$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\oint_C \frac{e^z}{z+i} dz = 0$$

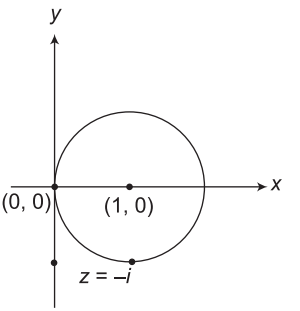


Fig. 3.23

Example 12

Evaluate $\int_C \frac{e^{2z}}{z^2+1} dz$, *where* C *is* $|z| = \frac{1}{2}$.

Solution

(i) Let $f(z) = \frac{e^{2z}}{z^2+1} = \frac{e^{2z}}{(z+i)(z-i)}$

$f(z)$ is not analytic at $z = \pm 1$.

(ii) C is the circle $|z| = \frac{1}{2}$ with the centre at (0, 0) and a radius of $\frac{1}{2}$ (Fig. 3.24).

(iii) For $z = -i, |z| = |-i| = 1 > \frac{1}{2}$
Hence, $z = -i$ lies outside C.

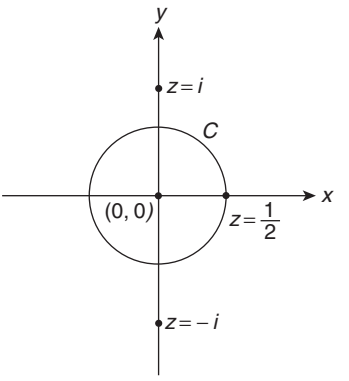


Fig. 3.24

For $z = i, |z| = |i| = 1 > \frac{1}{2}$

Hence, $z = i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{e^{2z}}{z^2+1} dz = 0$$

Example 13

Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2} dz$, *where C is* $|z-1+i| = 2$.

Solution

(i) Let $f(z) = \frac{z+1}{(z^2+2z+4)^2}$

$$= \frac{z+1}{\left[(z+1-\sqrt{3}i)(z+1+\sqrt{3}i)\right]^2}$$

$f(z)$ is not analytic at $z = -1+\sqrt{3}i$ and $z = -1-\sqrt{3}i$.

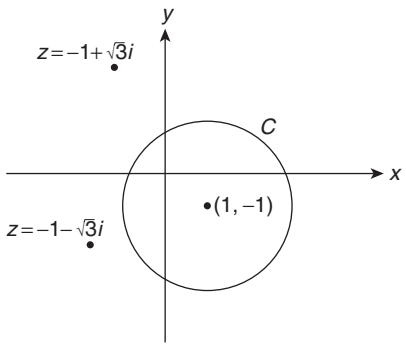


Fig. 3.25

(ii) C is the circle $|z-1+i| = 2$ with the centre at $(1, -1)$ and a radius of 2 (Fig. 3.25).

(iii) For $z = -1+\sqrt{3}i, |z-1+i| = |-1+\sqrt{3}i-1+i|$

$= 3.39 > 2$

Hence, $z = -1+\sqrt{3}i$ lies outside C.

For $z = -1-\sqrt{3}i, |z-1+i| = |-1-\sqrt{3}i-1+i| = 2.13 > 2$

Hence, $z = -1-\sqrt{3}i$ lies outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{z+1}{(z^2+2z+4)^2} dz = 0$$

Example 14

Evaluate $\int_C \sec z\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \sec z = \dfrac{1}{\cos z}$

$f(z)$ is not analytic at $\cos z = 0$,

i.e., $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \cdots$

(ii) C is the circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 3.26).

(iii) For $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots\ldots$

$|z| > 1$

Hence, all these points lie outside C.

(iv) $f(z)$ is analytic inside and on C.

$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \sec z\,dz = 0$$

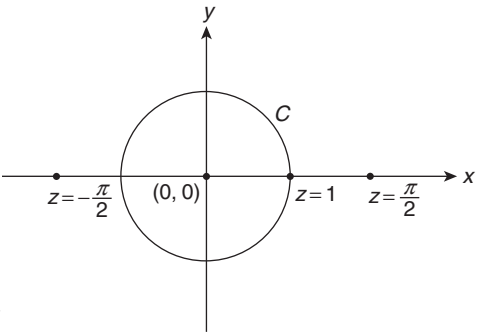


Fig. 3.26

Example 15

Evaluate $\int_C \tan z\,dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

$f(z)$ is not analytic at $\cos z = 0$, i.e., $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \cdots$

(ii) C is the circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 3.27).

(iii) For $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

$|z| > 1$

Hence, all these points lie outside C.

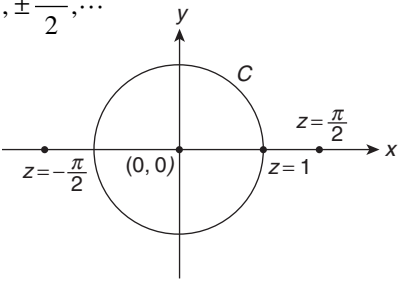


Fig. 3.27

(iv) $f(z)$ is analytic inside and on C.
$f'(z)$ is continuous inside C.

(v) By Cauchy's integral theorem,

$$\int_C \tan z \, dz = 0$$

EXERCISE 3.2

Evaluate the following integrals using Cauchy's integral theorem:

1. $\int_C \frac{z+3}{z^2-2z+5}$, where C is $|z-1|=1$

[Ans.: 0]

2. $\int_C \frac{z}{z-2} dz$, where C is $|z|=1$

[Ans.: 0]

3. $\int_C \frac{1}{2z-3} dz$, where C is $|z|=1$

[Ans.: 0]

4. $\int_C \frac{e^{2z}}{z^2+1} dz$, where C is $|z|=\frac{1}{2}$

[Ans.: 0]

5. $\int_C \frac{3z-1}{z^3-z} dz$, where C is $|z|=2$

[Ans.: 0]

6. $\int_C (x^2-y^2+2ixy)\, dz$, where C is $|z|=2$

[Ans.: 0]

7. $\int_C \frac{e^{2z}}{z-1} dz$, where C is $|z|=\frac{1}{2}$

[Ans.: 0]

8. $\int_C \cot z \, dz$, where C is $\left|z+\frac{1}{2}\right|=\frac{1}{3}$

[Ans.: 0]

3.6 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic inside and on a closed curve C and if a is any point inside C then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} \mathrm{d}z$$

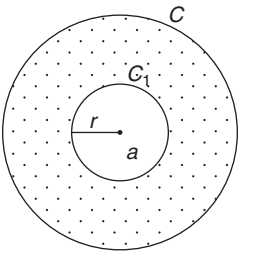


Fig. 3.28

Proof Since $f(z)$ is analytic inside and on C, $\dfrac{f(z)}{z-a}$ is also analytic inside and on C except at $z = a$.

Draw a small circle C_1 with the centre at $z = a$ and the radius r lying entirely inside C (Fig. 3.28).

Now, $\dfrac{f(z)}{z-a}$ is analytic in the region between C and C_1.

By Cauchy's integral theorem for a multiply connected region,

$$\oint_C \frac{f(z)}{z-a} \mathrm{d}z = \oint_{C_1} \frac{f(z)}{z-a} \mathrm{d}z$$

For the circle C_1, let $z - a = re^{i\theta}, \mathrm{d}z = rie^{i\theta}\, \mathrm{d}\theta$

$$\oint_C \frac{f(z)}{z-a} \mathrm{d}z = \oint_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} \mathrm{d}\theta$$

$$= i \oint_0^{2\pi} f(a+re^{i\theta})\, d\theta \qquad \ldots(3.1)$$

In the limiting position, as $r \to 0$, the circle C_1 shrinks to the point a.
Hence, Eq. (3.1) reduces to

$$\oint_0 \frac{f(z)}{z-a} \mathrm{d}z = i \oint_0^{2\pi} f(a)\, d\theta$$

$$= i f(a) \left|\theta\right|_0^{2\pi}$$

$$= i f(a)(2\pi - 0)$$

$$= 2\pi i f(a)$$

$$\therefore \qquad f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} \mathrm{d}z$$

3.6.1 Cauchy's Integral Formula for the Derivative of an Analytic Function

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i} \oint_0 \frac{f(z)}{(z-a)^2} \mathrm{d}z$$

where C is any closed curve in R surrounding the point $z = a$.

Proof By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} \mathrm{d}z$$

Differentiating w.r.t. a using Differentiation Under the Integral Sign (DUIS),

$$\begin{aligned} f'(a) &= \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right] \mathrm{d}z \\ &= \frac{1}{2\pi i} \oint_C f(z)\left[-\frac{1}{(z-a)^2}(-1)\right] \mathrm{d}z \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} \mathrm{d}z \end{aligned}$$

Differentiating again w.r.t. a,

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$$

Similarly,

$$f'''(a) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^4} \mathrm{d}z$$

In general,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

3.7 LIOUVILLE THEOREM

If $f(z)$ is analytic and bounded in the entire z-plane for all z then $f(z)$ is constant.

3.8 MAXIMUM MODULUS THEOREM

If $f(z)$ is nonconstant and analytic inside and on a simple closed contour C then the maximum value of $|f(z)|$ occurs on C.

Example 1

Prove that $\int_C \frac{dz}{z-a} = 2\pi i$ *and* $\int_C (z-a)^n \, dz = 0$ *(n is an integer and* $n \neq -1$*) where C is the circle* $|z-a| = r$. **[Summer 2014]**

Solution

(a) (i) Let $I = \int_C \frac{dz}{z-a}$

(ii) C is the circle $|z-a| = r$ with the centre at $(a, 0)$ and a radius of r (Fig. 3.29).

(iii) For $z = a$, $|z-a| = |a-a| = 0 < r$

Hence, $z = a$ lies inside C.

(iv) Let $f(z) = 1$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

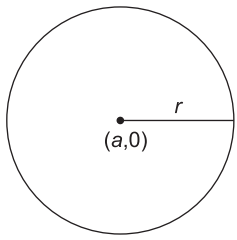


Fig. 3.29

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{dz}{z-a} = 2\pi i \, (1)$$

$$= 2\pi i$$

(b) (i) Let $f(z) = (z-a)^n$

(ii) Since $f(z)$ is a polynomial, it is analytic inside and on C. $f'(z)$ is continuous inside C.

(iii) By Cauchy's integral theorem,

$$\int_C (z-a)^n dz = 0$$

Example 2

Evaluate $\int_C \frac{z}{z-2} dz$, *where C is the circle* $|z-2| = \frac{3}{2}$.

Solution

(i) Let $I = \int_C \frac{z}{z-2} dz$

(ii) C is the circle $|z-2| = \frac{3}{2}$ with the centre at (2, 0) and a radius of $\frac{3}{2}$ (Fig. 3.30).

(iii) For $z = 2$, $|z-2| = |2-2| = 0 < \frac{3}{2}$

Hence, $z = 2$ lies inside C.

(iv) Let $f(z) = z$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z}{z-2}\,dz = 2\pi i[z]_{z=2}$$

$$= 2\pi i(2)$$

$$= 4\pi i$$

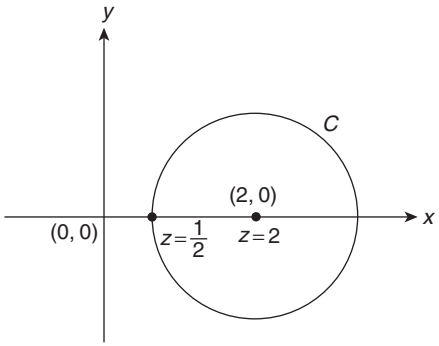


Fig. 3.30

Example 3

Evaluate $\int_C \frac{e^z}{z-1}\,dz$ *if* C *is* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{e^z}{z-1}\,dz$

(ii) C is the circle $|z| = 2$ with the centre at $(0,0)$ and a radius of 2 (Fig. 3.31).

(iii) For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

(iv) Let $f(z) = e^z$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

$$\int_C \frac{e^z}{z-1}\,dz = 2\pi i\left[e^z\right]_{z=1}$$

$$= 2\pi ie$$

$$= 2\pi ei$$

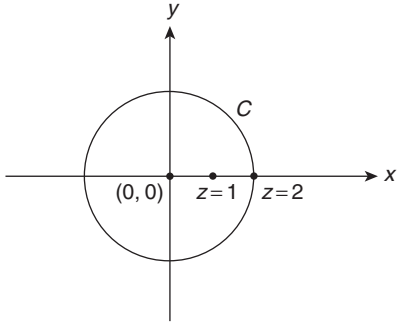


Fig. 3.31

Example 4

Evaluate $\int_C \frac{\cos \pi z}{z-1} dz$ *if C is* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{\cos \pi z}{z-1} dz$

(ii) C is the circle $|z| = 2$ with the centre at $(0,0)$ and a radius of 2 (Fig. 3.32).

(iii) For $z = 1, |z| = |1| = 1 < 2$.

Hence, $z = 1$ lies inside C.

(iv) Let $f(z) = \cos \pi z$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

$$\int_C \frac{\cos \pi z}{z-1} dz = 2\pi i \left[\cos \pi z\right]_{z=1}$$

$$= 2\pi i \left[\cos \pi\right]$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

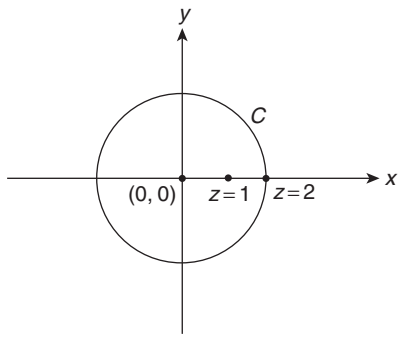


Fig. 3.32

Example 5

Evaluate $\oint_C \frac{z^2 - 4z + 4}{z+i} dz$, *where C is* $|z| = 2$.

Solution

(i) Let $I = \oint_C \frac{z^2 - 4z + 4}{z+i} dz$

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 3.33).

(iii) For $z = -i$, $|z| = |-i| = 1 < 2$

Hence, $z = -i$ lies inside C.

(iv) Let $f(z) = z^2 - 4z + 4$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

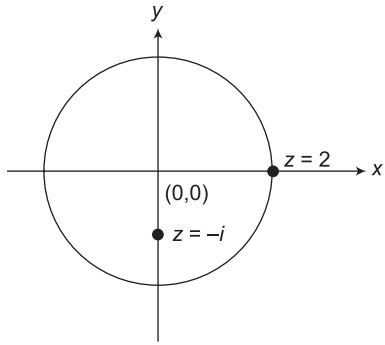


Fig. 3.33

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\oint \frac{z^2-4z+4}{z+i}\,dz = 2\pi i[z^2-4z+4]_{z=-i}$$

$$= 2\pi i[i^2+4i+4]$$

$$= 2\pi i(3+4i)$$

$$= 2\pi(3i-4)$$

Example 6

Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$, *where C is* $|z| = \frac{3}{2}$.

Solution

(i) Let $I = \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$

(ii) C is the circle $|z| = \frac{3}{2}$ with the centre at (0, 0) and a radius of $\frac{3}{2}$ (Fig. 3.34).

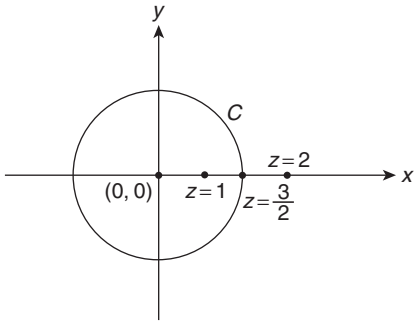


Fig. 3.34

(iii) For $z = 1$, $|z| = |1| = 1 < \frac{3}{2}$

Hence, $z = 1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 > \frac{3}{2}$

Hence, $z = 2$ lies outside C.

(iv) Let $f(z) = \frac{\cos \pi z^2}{z-2}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} = \int_C \frac{\frac{\cos \pi z^2}{z-2}}{z-1}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

$$\int_C \frac{\frac{\cos \pi z^2}{z-2}}{z-1} dz = 2\pi i \left[\frac{\cos \pi z^2}{z-2} \right]_{z=1}$$

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \left[\frac{\cos \pi}{1-2} \right]$$

$$= 2\pi i (1)$$

$$= 2\pi i$$

Example 7

Evaluate $\int_C \frac{dz}{z^2 - 7z + 12}$, *where C is the circle* $|z| = 3.5$.

Solution

(i) Let $I = \int_C \frac{dz}{z^2 - 7z + 12}$

$$= \int_C \frac{dz}{(z-4)(z-3)}$$

(ii) C is the circle $|z| = 3.5$ with the centre at (0, 0) and a radius of 3.5 (Fig. 3.35).

(iii) For $z = 4$, $|z| = |4| = 4 > 3.5$

Hence, $z = 4$ lies outside C.

For $z = 3$, $|z| = |3| = 3 < 3.5$

Hence, $z = 3$ lies inside C.

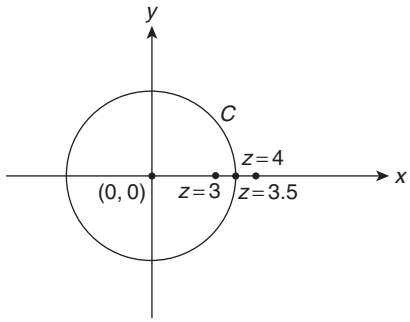


Fig. 3.35

(iv) Let $f(z) = \frac{1}{z-4}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{dz}{z^2 - 7z + 12} = \int_C \frac{\frac{1}{z-4}}{z-3} dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$$

$$\int_C \frac{\frac{1}{z-4}}{z-3}\,dz = 2\pi i\left[\frac{1}{z-4}\right]_{z=3}$$

$$\int_C \frac{dz}{z^2-7z+12} = 2\pi i\left[\frac{1}{3-4}\right]$$

$$= -2\pi i$$

Example 8

Evaluate $\int_C \frac{1}{z^2-1}\,dz$, *where C is the circle with the centre at* (1, 0) *and the radius* = 1.

Solution

(i) Let $I = \int_C \frac{1}{z^2-1}\,dz$

$$= \int_C \frac{1}{(z+1)(z-1)}\,dz$$

(ii) C is the circle with the centre at (1, 0) and a radius of 1 (Fig. 3.36).

$\therefore\ |z-1| = 1$

(iii) For $z = -1$, $|z-1| = |-1-1| = 2 > 1$

Hence, $z = -1$ lies outside C.

For $z = 1$, $|z-1| = |1-1| = 0 < 1$.

Hence, $z = 1$ lies inside C.

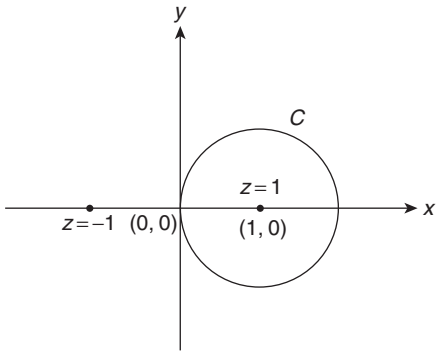


Fig. 3.36

(iv) Let $f(z) = \frac{1}{z+1}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{1}{z^2-1}\,dz = \int_C \frac{\frac{1}{z+1}}{z-1}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\frac{1}{z+1}}{z-1}\,dz = 2\pi i\left[\frac{1}{z+1}\right]_{z=1}$$

$$\int_C \frac{1}{z^2-1}\,dz = 2\pi i\left[\frac{1}{1+1}\right]$$

$$= \pi i$$

Example 9

Evaluate $\int_C \frac{z^2+1}{z^2-1}\,dz$, *where C is* $|z-1|=1$.

Solution

(i) Let $I = \int_C \frac{z^2+1}{z^2-1}\,dz$

$$= \int_C \frac{z^2+1}{(z+1)(z-1)}\,dz$$

(ii) C is the circle $|z-1|=1$ with the centre at $(1,0)$ and a radius of 1 (Fig. 3.37).

(iii) For $z=-1$, $|z-1| = |-1-1| = 2 > 1$.

Hence, $z=-1$ lies outside C.

For $z=1$, $|z-1| = |1-1| = 0 < 1$

Hence, $z=1$ lies inside C.

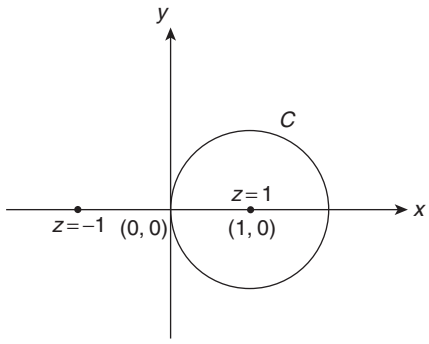


Fig. 3.37

(iv) Let $f(z) = \frac{z^2+1}{z+1}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

$$\int_C \frac{\frac{z^2+1}{z+1}}{z-1} dz = 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1}$$

$$\int_C \frac{z^2+1}{z^2-1} dz = 2\pi i \left[\frac{1+1}{1+1}\right]$$

$$= 2\pi i$$

Example 10

Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, *where C is the circle* $|z+1+i| = 2$.

Solution

(i) Let $I = \int_C \frac{z+4}{z^2+2z+5} dz$

$$= \int_C \frac{z+4}{(z+1+2i)(z+1-2i)} dz$$

(ii) C is the circle $|z+1+i| = 2$ with the centre at $(-1,-1)$ and a radius of 2 (Fig. 3.38).

(iii) For $z = -1-2i$,

$$|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$$

Hence, $z = -1-2i$ lies inside C.

For $z = -1+2i$,

$$|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$$

Hence, $z = -1+2i$ lies outside C.

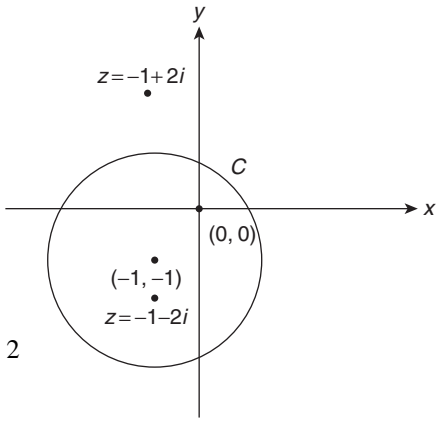


Fig. 3.38

(iv) Let $f(z) = \dfrac{z+4}{z+1-2i}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{z+4}{z^2+2z+5} = \int_C \frac{\dfrac{z+4}{z+1-2i}}{(z+1+2i)} dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\ f(a)$$

$$\int_C \frac{\dfrac{z+4}{z+1-2i}}{z+1+2i} dz = 2\pi i \left[\frac{z+4}{z+1-2i}\right]_{z=-1-2i}$$

$$\int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i \left(\frac{-1-2i+4}{-1-2i+1-2i}\right)$$

$$= 2\pi i \left(\frac{3-2i}{-4i}\right)$$

$$= \frac{\pi}{2}(2i-3)$$

Example 11

Evaluate $\int_C \dfrac{z}{z^2+1} dz$, *where C is the circle* $|z+i| = 1$.

Solution

(i) Let $I = \int_C \dfrac{z}{z^2+1} dz$

$$= \int_C \frac{z}{(z+i)(z-i)} dz$$

(ii) C is the circle $|z+i| = 1$ with the centre at $(0,-1)$ and a radius of 1 (Fig. 3.39).

(iii) For $z = -i, |z+i| = |-i+i| = 0 < 1$

Hence, $z = -i$ lies inside C.

For $z = i, |z+i| = |i+i| = |2i| = 2 > 1$

Hence, $z = i$ lies outside C.

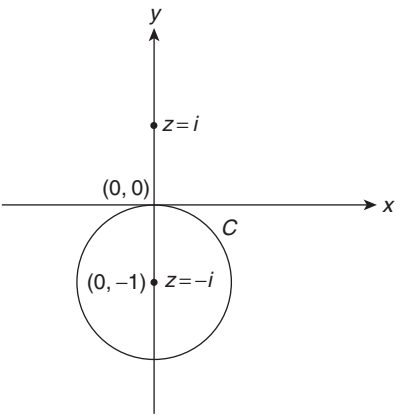


Fig. 3.39

(iv) Let $f(z) = \dfrac{z}{z-i}$

$f(z)$ is analytic inside and on C.

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{\dfrac{z}{z-i}}{z+i}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\dfrac{z}{z-i}}{z+i}\,dz = 2\pi i\left[\frac{z}{z-i}\right]_{z=-i}$$

$$\int_C \frac{z}{z^2+1}\,dz = 2\pi i\left[\frac{-i}{-i-i}\right]$$

$$= 2\pi i\left(\frac{1}{2}\right)$$

$$= \pi i$$

Example 12

Evaluate $\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, *where C is the circle* $|z| = \dfrac{3}{2}$.

Solution

(i) Let $I = \displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$

(ii) C is the circle $|z| = \dfrac{3}{2}$ with the centre at $(0, 0)$ and a radius of $\dfrac{3}{2}$ (Fig. 3.40).

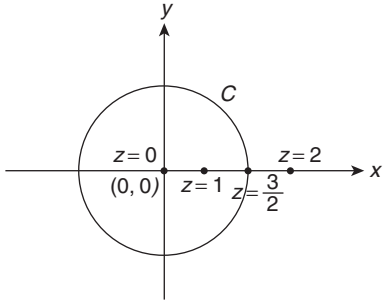


Fig. 3.40

(iii) For $z = 0, |z| = 0 < \dfrac{3}{2}$

Hence, $z = 0$ lies inside C.

For $z = 1, |z| = |1| = 1 < \dfrac{3}{2}$

Hence, $z = 1$ lies inside C.

For $z = 2, |z| = |2| = 2 > \frac{3}{2}$

Hence, $z = 2$ lies outside C.

(iv) Let $f(z) = \frac{4-3z}{z-2}$

$f(z)$ is analytic inside and on C.

$$\frac{4-3z}{z(z-1)(z-2)} = \frac{\left(\frac{4-3z}{z-2}\right)}{z(z-1)}$$

$$= \left(\frac{4-3z}{z-2}\right)\left[\frac{z-(z-1)}{z(z-1)}\right]$$

$$= \frac{4-3z}{z-2}\left[\frac{1}{z-1} - \frac{1}{z}\right]$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \left(\frac{\frac{4-3z}{z-2}}{z-1} - \frac{\frac{4-3z}{z-2}}{z}\right) dz$$

$$= \int_C \frac{\frac{4-3z}{z-2}}{z-1}\,dz - \int_C \frac{\frac{4-3z}{z-2}}{z}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\frac{4-3z}{z-2}}{z-1}\,dz - \int_C \frac{\frac{4-3z}{z-2}}{2}\,dz = 2\pi i\, f(1) - 2\pi i\, f(0)$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i\left(\frac{4-3}{1-2}\right) - 2\pi i\left(\frac{4-0}{0-2}\right)$$

$$= -2\pi i + 4\pi i$$

$$= 2\pi i$$

Example 13

If C is the circle $|z| = 3$ *and if* $g(z_0) = \int_C \frac{2z^2 - z - 2}{z - z_0}\,dz$ *then find* $g(2)$.

Solution

(i) $g(z_0) = \int_C \frac{2z^2 - z - 2}{z - z_0} dz$

(ii) C is the circle $|z| = 3$ with the centre at $(0, 0)$ and a radius of 3 (Fig. 3.41).

(iii) $z_0 = 2$ lies inside C.

(iv) Let $f(z) = 2z^2 - z - 2$

$f(z)$ is analytic inside and on C.

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z - a} dz = 2\pi i\, f(a)$$

$$\int_C \frac{2z^2 - z - 2}{z - 2} dz = 2\pi i \left[2z^2 - z - 2\right]_{z=2}$$

$$= 2\pi i \left[2(2)^2 - 2 - 2\right]$$

$$= 2\pi i(4)$$

$$= 8\pi i$$

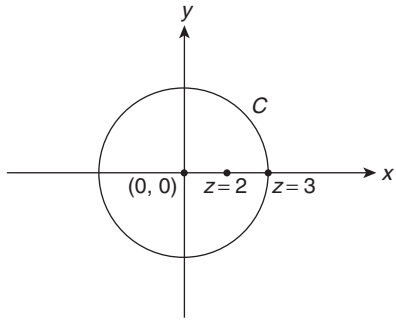


Fig. 3.41

Example 14

Evaluate $\int_C \frac{z - 2}{z(z - 1)} dz$, *where C is the circle* $|z| = 3$.

Solution

(i) Let $I = \int_C \frac{z - 2}{z(z - 1)} dz$

(ii) C is the circle $|z| = 3$ with the centre at $(0, 0)$ and a radius of 3 (Fig. 3.42).

(iii) For $z = 0, |z| = 0 < 3$

Hence, $z = 0$ lies inside C.

For $z = 1, |z| = 1 < 3$

Hence, $z = 1$ lies inside C.

(iv) Let $f(z) = z - 2$

$f(z)$ is analytic inside and on C.

$$\frac{1}{z(z - 1)} = \frac{z - (z - 1)}{z(z - 1)}$$

$$= \frac{1}{z - 1} - \frac{1}{z}$$

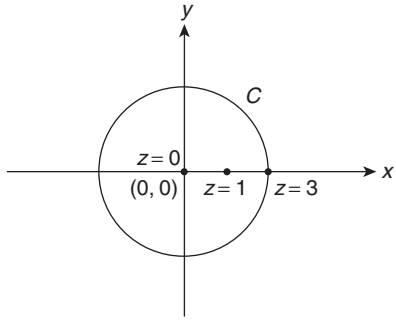


Fig. 3.42

$$\int_C \frac{z-2}{z(z-1)}\,dz = \int_C \left(\frac{1}{z-1} - \frac{1}{z}\right) dz$$

$$= \int_C \frac{1}{z-1}\,dz - \int_C \frac{1}{z}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z-2}{z(z-1)}\,dz = 2\pi i\, f(1) - 2\pi i\, f(0)$$

$$= 2\pi i(1-2) - 2\pi i(0-2)$$

$$= -2\pi i + 4\pi i$$

$$= 2\pi i$$

Example 15

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, *where C is* $|z| = 3$.

Solution

(i) Let $I = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$

(ii) C is the circle $|z| = 3$ with the centre at $(0,0)$ and a radius of 3 (Fig. 3.43).

(iii) For $z = 1, |z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2, |z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

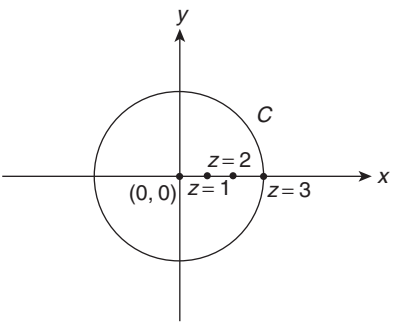


Fig. 3.43

(iv) Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

$f(z)$ is analytic inside and on C.

$$\frac{1}{(z-1)(z-2)} = \frac{(z-1)-(z-2)}{(z-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \int_C \left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz \right)$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i(\sin \pi + \cos \pi)$$

$$= 2\pi i(0+1) - 2\pi i(0-1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

Example 16

Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$, *where C is* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{z}{(z-1)^3}\,dz$

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 3.44).

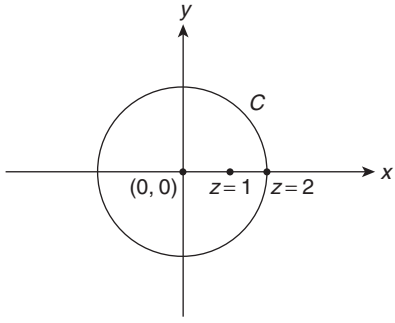


Fig. 3.44

(iii) For $z = 1$, $|z| = 1 < 2$

Hence, $z = 1$ lies inside C.

(iv) Let $f(z) = z$

$f(z)$ is analytic inside and on C.

$$f'(z) = 1$$

$$f''(z) = 0$$

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$

$$\int_C \frac{z}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1)$$

$$= \pi i(0)$$

$$= 0$$

Example 17

Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, *where* C *is* $|z|=1$.

Solution

(i) Let $I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$

(ii) C is the circle $|z|=1$ with the centre at (0, 0) and a radius of 1 (Fig. 3.45).

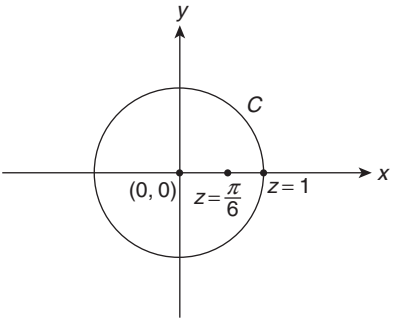


Fig. 3.45

(iii) For $z=\frac{\pi}{6}, |z| = \left|\frac{\pi}{6}\right| = 0.52 < 1$

Hence, $z=\frac{\pi}{6}$ lies inside C.

(iv) Let $f(z) = \sin^6 z$

$f(z)$ is analytic inside and on C.

$f'(z) = 6\sin^5 z \cos z$

$f''(z) = 6(5\sin^4 z\cos^2 z - \sin^6 z)$

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

$$\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!}\left[6(5\sin^4 z\cos^2 z - \sin^6 z)\right]_{z=\frac{\pi}{6}}$$

$$= 6\pi i\left(5\sin^4\frac{\pi}{6}\cos^2\frac{\pi}{6} - \sin^6\frac{\pi}{6}\right)$$

Example 18

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $I = \int_C \frac{e^{2z}}{(z+1)^4}\,dz$

(ii) C is the circle $|z| = 2$ with the centre at $(0, 0)$ and a radius of 2 (Fig. 3.46).

(iii) For $z = -1, |z| = |-1| = 1 < 2$

Hence, $z = -1$ lies inside C.

(iv) Let $f(z) = e^{2z}$

$f(z)$ is analytic inside and on C.

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

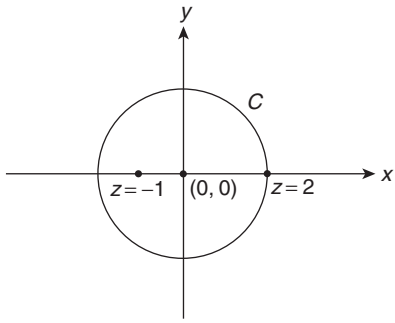


Fig. 3.46

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^4}\,dz = \frac{2\pi i}{3!} f'''(a)$$

$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{3!}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{6}\left[8e^{-2}\right]$$

$$= \frac{8\pi i}{3e^2}$$

Example 19

Evaluate $\int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz$, *where C :* $|z| = 2$. **[Summer 2015]**

Solution

(i) Let $I = \int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz$

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 3.47).

(iii) For $z = i$, $|z| = |i| = 1 < 2$.
Hence, $z = i$ lies inside C.

(iv) Let $f_1(z) = 3, f_2(z) = 6$
$f_1(z)$ and $f_2(z)$ are analytic inside and on C.

(v) By Cauchy's integral formula,

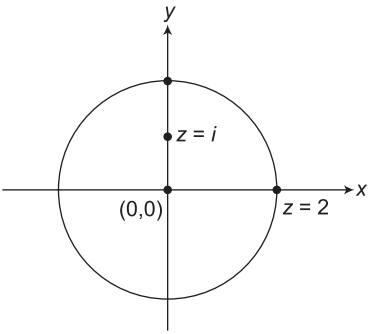


Fig. 3.47

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a) \quad \text{and} \quad \int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\int_C \left[\frac{3}{z-i} - \frac{6}{(z-i)^2}\right] dz = 2\pi i \left[f_1(z)\right]_{z=i} - 2\pi i \left[f_2'(z)\right]_{z=i}$$

$$= 2\pi i\,(3) - 2\pi i\,(0)$$

$$= 6\pi i$$

Example 20

Evaluate $\int_C \frac{dz}{z^4 e^z}$, *where C is* $|z| = 1$.

Solution

(i) Let $I = \int_C \frac{dz}{z^4 e^z}$

$$= \int_C \frac{e^{-z}}{z^4} dz$$

(ii) C is the circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 3.48).

(iii) For $z = 0, |z| = 0 < 1$
Hence, $z = 0$ lies inside C.

(iv) Let $f(z) = e^{-z}$
$f(z)$ is analytic inside and on C.

$f'(z) = -e^{-z}$

$f''(z) = e^{-z}$

$f'''(z) = -e^{-z}$

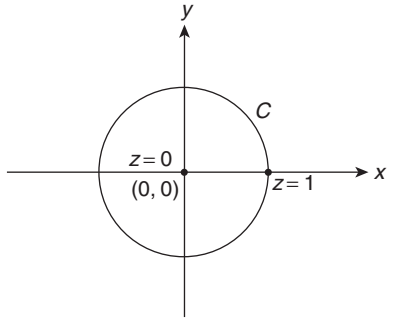


Fig. 3.48

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^4}\,dz = \frac{2\pi i}{3!} f'''(a)$$

$$\int_C \frac{dz}{z^4 e^4} = \frac{2\pi i}{3!}\left[-e^{-z}\right]_{z=0}$$

$$= \frac{2\pi i}{6}(-1)$$

$$= -\frac{\pi i}{3}$$

Example 21

Evaluate $\int_C \frac{dz}{(z+1)^2(z-2)}$, *where C is the circle* $|z| = \frac{3}{2}$.

Solution

(i) Let $I = \int_C \frac{dz}{(z+1)^2(z-2)}$

(ii) C is the circle $|z| = \frac{3}{2}$ with the centre at (0, 0) and a radius of $\frac{3}{2}$ (Fig. 3.49).

(iii) For $z = -1$, $|z| = |-1| = 1 < \frac{3}{2}$

Hence, $z = -1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 > \frac{3}{2}$

Hence, $z = 2$ lies outside C.

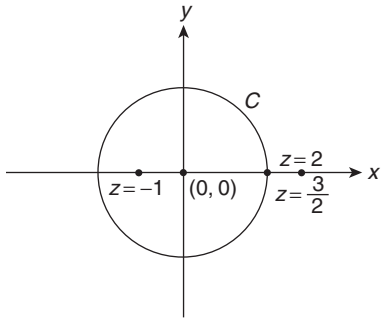


Fig. 3.49

(iv) Let $f(z) = \frac{1}{z-2}$

$f(z)$ is analytic inside and on C.

$$f'(z) = -\frac{1}{(z-2)^2}$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{(z+1)^2} dz$$

(v) By Cauchy's integral formula for derivative,

$$\int_C \frac{f(z)}{(z-a)^2} = 2\pi i\, f'(a)$$

$$\int_C \frac{\frac{1}{z-2}}{(z+1)^2}\,dz = 2\pi i\left[-\frac{1}{(z-2)^2}\right]_{z=-1}$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = 2\pi i\left[-\frac{1}{(-1-2)^2}\right]$$

$$= 2\pi i\left(-\frac{1}{9}\right)$$

$$= -\frac{2\pi i}{9}$$

Example 22

Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}\,dz$, *where C is* $|z-2| = \frac{1}{2}$.

Solution

(i) Let $I = \int_C \frac{z}{(z-1)(z-2)^2}\,dz$

(ii) C is the circle $|z-2| = \frac{1}{2}$ with the centre at $(2,0)$ and a radius of $\frac{1}{2}$ (Fig. 3.50).

(iii) For $z = 1, |z-2| = |1-2| = 1 > \frac{1}{2}$

Hence, $z = 1$ lies outside C.

For $z = 2, |z-2| = |2-2| = 0 < \frac{1}{2}$

Hence, $z = 2$ lies inside C.

(iv) Let $f(z) = \frac{z}{z-1}$

$f(z)$ is analytic inside and on C.

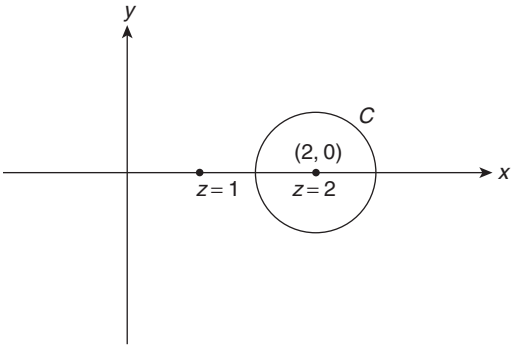


Fig. 3.50

$$f'(z) = \frac{(z-1)(1) - z(1)}{(z-1)^2}$$

$$= \frac{z-1-z}{(z-1)^2}$$

$$= -\frac{1}{(z-1)^2}$$

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = \int_C \frac{\overline{\frac{z}{z-1}}}{(z-2)^2}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

$$\int_C \frac{\overline{\frac{z}{z-1}}}{(z-2)^2}\,dz = 2\pi i - \left[\frac{1}{(z-1)^2}\right]_{z=2}$$

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = 2\pi i - \left[\frac{1}{(2-1)^2}\right]$$

$$= -2\pi i$$

Example 23

Evaluate $\int_C \frac{z+1}{z^4 - 4z^3 + 4z^2}\,dz$, *where C is the circle* $|z-2-i| = 2$.

Solution

(i) Let $I = \int_C \frac{z+1}{z^4 - 4z^3 + 4z^2}\,dz$

$$= \int_C \frac{z+1}{z^2(z^2 - 4z + 4)}\,dz$$

$$= \int_C \frac{z+1}{z^2(z-2)^2}\,dz$$

(ii) C is the circle $|z-2-i| = 2$ with the centre at (2, 1) and a radius of 2 (Fig. 3.51).

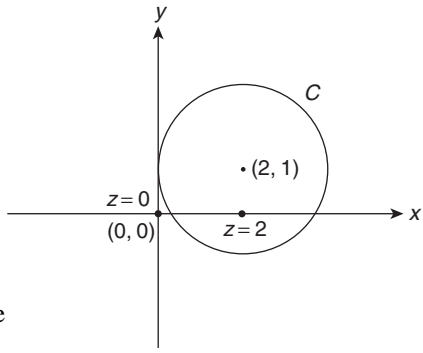


Fig. 3.51

(iii) For $z=0, |z-2-i| = |0-2-i| = \sqrt{5} > 2$

Hence, $z=0$ lies outside C.

For $z=2, |z-2-i| = |2-2-i| = 1 < 2$

Hence, $z=2$ lies inside C.

(iv) Let $f(z) = \dfrac{z+1}{z^2}$

$f(z)$ is analytic inside and on C.

$$f'(z) = \frac{z^2(1)-(z+1)2z}{z^4}$$

$$= \frac{-z^2-2z}{z^4}$$

$$= -\frac{z+2}{z^3}$$

$$\int_C \frac{z+1}{z^4-4z^3+4z^2}\,dz = \int_C \frac{z+1}{z^2(z-2)^2}$$

$$= \int_C \frac{\dfrac{z+1}{z^2}}{(z-2)^2}\,dz$$

(v) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

$$\int_C \frac{\dfrac{z+1}{z^2}}{(z-2)^2}\,dz = 2\pi i\left[-\frac{z+2}{z^3}\right]_{z=2}$$

$$\int_C \frac{z+1}{z^4-4z^3+4z^2}\,dz = 2\pi i\left[-\frac{2+2}{2^3}\right]$$

$$= 2\pi i\left(-\frac{1}{2}\right)$$

$$= -\pi i$$

Example 24

Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z^2+5}{z-3}\,dz$, *where C is* $|z|=4$.

Solution

(i) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz = f(a) \qquad \cdots(1)$$

Given $$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz \qquad \cdots(2)$$

Comparing integrand of Eqs (1) and (2),

$$f(z) = z^2 + 5$$

(ii) C is the circle $|z| = 4$ with the centre at (0, 0) and a radius of 4 (Fig. 3.52).

(iii) For $z = 3, |z| = |3| = 3 < 4$

Hence, $z = 3$ lies inside C.

(iv) $f(z)$ is analytic inside and on C.

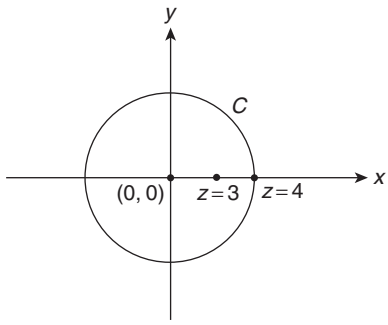


Fig. 3.52

(v) $$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = f(3)$$

$$= [z^2+5]_{z=3}$$

$$= 3^2 + 5$$

$$= 14$$

Example 25

If $f(a) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$, *where C is the circle* $|z| = 2$, *find the values of* $f(3), f'(1+i)$ *and* $f''(1+i)$.

Solution

(i) By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz \qquad \cdots(1)$$

Given $$f(a)=\int_C \frac{3z^2+7z+1}{z-a}\,dz \qquad \cdots(2)$$

Comparing the integrands of Eqs (1) and (2),

$$\frac{f(z)}{2\pi i}=3z^2+7z+1$$

$$f(z)=2\pi i(3z^2+7z+1)$$

(ii) C is the circle $|z|=2$ with the centre at (0, 0) and a radius of 2 (Fig. 3.53).

(iii) For $z=3, |z|=|3|=3>2$

Hence, $z=3$ lies outside C.

(iv) $\frac{f(z)}{z-3}$ is analytic inside and on C.

(v) By Cauchy's integral theorem,

$$\int_C \frac{f(z)}{z-3}\,dz=0$$

$$\int_C \frac{3z^2+7z+1}{z-3}\,dz=0$$

$$f(3)=0 \qquad [\because \ a=3]$$

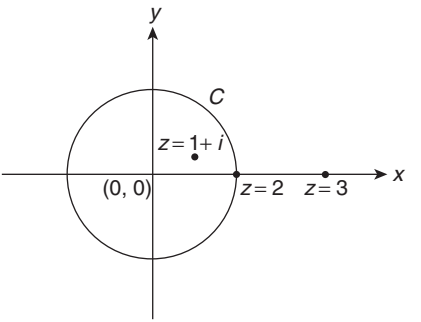


Fig. 3.53

(vi) $f'(z)=2\pi i(6z+7)$

$f''(z)=2\pi i(6)=12\pi i$

For $z=1+i, |z|=|1+i|=\sqrt{2}<2$

Hence, $z=1+i$ lies inside C.

$$f'(1+i)=2\pi i[6(1+i)+7]$$

$$=2\pi i(13+6i)$$

$$=2\pi(-6+13i)$$

$$f''(1+i)=12\,\pi i$$

EXERCISE 3.3

Evaluate the following integrals using Cauchy's integral formula:

1. $\int_C \frac{z\,dz}{(z-1)(z-2)}$, where C is $|z-2|=\frac{1}{2}$

[Ans.: $4\pi i$**]**

2. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$, where C is $|z| = 4$

[Ans.: $-4\pi i$**]**

3. $\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z-i| = 2$

$$\left[\textbf{Ans.:} \frac{\pi}{16}\right]$$

4. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$, where C is $|z| = 3$

[Ans.: $4\pi(\pi+1)i$ **]**

5. $\int_C \frac{e^z}{(z+2)(z+1)^2} dz$, where C is $|z| = 3$

$$\left[\textbf{Ans.:} \frac{2\pi i}{e^2}\right]$$

6. $\int_C \frac{z+1}{z^2+2z+4} dz$, where C is $|z+1-i| = 2$

[Ans.: πi**]**

7. $\int_C \frac{z^3 - z}{(z-2)^3} dz$, where C is $|z| = 3$

[Ans.: 12 πi**]**

8. $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i| = 2$

$$\left[\textbf{Ans.:} \frac{\pi}{2}(3+2i)\right]$$

9. $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is $|z| = 3$

[Ans.: $4\pi i$**]**

10. $\int_C \frac{dz}{z^3(z+4)}$, where C is $|z| = 2$

$$\left[\textbf{Ans.:} \frac{2\pi i}{27}\right]$$

11. $\int_C \frac{e^z}{(z-1)(z-4)} dz$, where C is $|z| = 2$

$\left[\text{Ans.: } -\frac{2\pi i e}{3}\right]$

12. $\int_C \frac{z^2+4}{(z-2)(z+3i)} dz$, where C is $|z-2| = 2$

$\left[\text{Ans.: } \frac{16\pi i}{2+3i}\right]$

13. $\int_C \frac{z+1}{z^3-2z} dz$, where C is $|z| = 1$

$\left[\text{Ans.: } -\frac{3\pi i}{2}\right]$

14. $\int_C \frac{ze^{2z}}{(z-1)^3} dz$, where C is $|z+i| = 2$

[Ans.: $8\pi i e^2$]

15. $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is $|z-i| = 2$

$\left[\text{Ans.: } -\frac{2\pi i}{9}\right]$

Points to Remember

Simply Connected Region

A simply connected region R is the region enclosed by a simple curve.

Multiply Connected Region

A multiply connected region is the region enclosed by more than one simple curve. A multiply connected region can be converted into a simply connected region by introducing cross-cuts.

Independence of Path

If $f(z)$ is analytic in a simply connected region then the line integral is independent of the path.

Cauchy's Integral Theorem

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point inside and on a closed curve C then

$$\oint_C f(z)\,dz = 0$$

1. Cauchy–Goursat Theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C contained in a simply connected domain D then

$$\oint_C f(z)\,dz = 0$$

2. Extension of Cauchy's Integral Theorem to a Multiply Connected Region

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2 then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a closed curve C and if a is any point inside C then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

1. Cauchy's Integral Formula for the Derivative of an Analytic Function

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i}\oint_0 \frac{f(z)}{(z-a)^2}\,dz$$

where C is any closed curve in R surrounding the point $z = a$.

Liouville Theorem

If $f(z)$ is analytic and bounded in the entire z-plane for all z then $f(z)$ is constant.

Maximum Modulus Theorem

If $f(z)$ is non-constant and analytic inside and on a simple closed contour C then the maximum value of $|f(z)|$ occurs on C.

CHAPTER 4
Power Series

Chapter Outline

4.1 Introduction
4.2 Sequences and Series
4.3 Power Series
4.4 Convergence of a Power Series
4.5 Taylor's Series
4.6 Laurent's Series
4.7 Singular Points
4.8 Residues
4.9 Cauchy's Residue Theorem
4.10 Argument Theorem
4.11 Rouche's Theorem

4.1 INTRODUCTION

Series expansions are used to approximate a large class of arbitrary functions, which are often difficult to analyze, with simpler, well-understood functions such as Taylor series. Power series are natural generalizations of Taylor polynomials. A power series may represent a function, in the sense that wherever the series converges, it converges as well. Any polynomial can be easily expressed as a power series around any centre.

In this chapter, we will study Taylor's series, Laurent's series and Cauchy's residue theorem. The Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions where a Taylor series expansion cannot be applied. Residues obtained in Laurent's series are used to evaluate complex integrals by summing the residues at the isolated singularities of a function within the closed contour by Cauchy's residue theorem.

4.2 SEQUENCES AND SERIES

A sequence of complex numbers $z_1, z_2, z_3, \ldots z_n$ is a function which assigns a complex number z_n to each positive integer n. A sequence is denoted by $\{z_n\}$.

A sequence $\{z_n\}$ is said to be convergent if

$$\lim_{n\to\infty} z_n = l$$

where l is a finite quantity.

An infinite series of complex numbers z_1, z_2, z_3, ..., z_n, ... is the sum of the terms of a sequence $\{z_n\}$ given as

$$z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

It is denoted as $\sum_{n=1}^{\infty} z_n$.

The n^{th} partial sum of a series is the sum of the first n terms of the series and is denoted as

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n = \sum_{m=1}^{n} z_m$$

A series is said to be convergent if

$$\lim_{n\to\infty} S_n = S$$

otherwise, the series is divergent.

A necessary condition for the convergence of a complex series is

$$\lim_{n\to\infty} z_n = 0$$

where z_n is the n^{th} term of the series.

4.3 POWER SERIES

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots + a_n(z - z_0)^n + \cdots$$

where a_0, a_1, a_2, ..., a_n,... are complex or real constants, z_0 is a constant known as the centre of the series.

This series is also known as the power series about the point $z = z_0$.

The power series about $z = 0$, i.e., the origin is given as

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

Circle and Radius of Convergence of a Power Series

A power series may converge for all z or only at the point $z = z_0$ or everywhere inside a circular disk $|z - z_0| = R$ (i.e., in the region $|z - z_0| < R$) and diverges everywhere outside

circular disk (i.e., in the region $|z - z_0| > R$). The circular disk $|z - z_0 = R$ is known as the circle of convergence and R is known as the radius of convergence. The set of all points for which the series converges is known as the *region of convergence*.

A power series is uniformly convergent in the region which lies entirely within its circle of convergence.

4.4 CONVERGENCE OF A POWER SERIES

If the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at a point $z = a$ then it converges absolutely in the open circular disk $|z - z_0| < |a - z_0|$. A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with a radius of convergence $R > 0$ converges uniformly inside and on every circular disc $|z - z_0| = r$, where $r < R$.

Theorem 1 A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and the k times differentiated series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (z - z_0)^{n-k}$$

have the same radius of convergence.

Theorem 2 Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence R, then

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Absolute Convergence The series $\sum_{n=1}^{\infty} u_n(z)$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n(z)|$ is also convergent.

Uniform Convergence The series $\sum_{n=0}^{\infty} u_n(z)$ is said to be uniformly convergent in a region R if there exists a convergent series of positive real constants $\sum_{n=0}^{\infty} M_n$ such that

$$|u_n(z)| \le M_n \text{ for all } z \text{ in } R$$

Example 1

Find the radii of convergence and region of convergence of the following:

(i) $\sum_{n=1}^{\infty} \frac{z^n}{2^n+1}$ *(ii)* $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (z-2i)^n$ *(iv)* $\sum_{n=1}^{\infty} \left(\frac{6n+1}{2n+5}\right)^2 (z-2i)^n$

[Summer 2015]

Solution

(i) $a_n = \frac{1}{2^n+1}$

Radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{2^{n+1}+1}{2^n+1} \right| \\ &= \lim_{n\to\infty} \left| \frac{2+\frac{1}{2^n}}{1+\frac{1}{2^n}} \right| \\ &= \frac{2+0}{1+0} \\ &= 2 \end{aligned}$$

Region of convergence is $|z| < 2$.

(ii) $a_n = \frac{n!}{n^n}$

Radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n (n+1)} \right| \end{aligned}$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right)^n \right|$$

$$= e$$

Region of convergence is $|z| < e$.

(iii) $a_n = \dfrac{(-1)^n}{n}$

Radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \cdot \frac{(n+1)}{(-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| -\left(1+\frac{1}{n}\right) \right|$$

$$= |-1|$$

$$= 1$$

Region of convergence is $|z - 2i| < 1$.

(iv) $a_n = \left(\dfrac{6n+1}{2n+5}\right)^2$

Radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{6n+1}{2n+5}\right)^2 \cdot \left(\frac{2n+2+5}{6n+6+1}\right)^2 \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{6+\frac{1}{n}}{2+\frac{5}{n}}\right)^2 \cdot \left(\frac{2+\frac{7}{n}}{6+\frac{7}{n}}\right)^2 \right|$$

$$= \left(\frac{6+0}{2+0}\right)^2 \left(\frac{2+0}{6+0}\right)^2$$

$$= 1$$

Region of convergence is $|z - 2i| < 1$.

Example 2

Discuss the convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ *and also find the radius and circle of convergence.*

Solution

$$a_n = \frac{(2n)!}{(n!)^2}$$

Radius of convergence is

$$\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n)!}{(n!)^2} \cdot \frac{[(n+1)!]^2}{[2(n+1)]!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n)!}{(n!)^2} \cdot \frac{(n+1)^2\ (n!)^2}{(2n+2)\ (2n+1)\ (2n)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} \right| \\
&= \frac{1+0}{(2+0)\ (2+0)} \\
&= \frac{1}{4}
\end{aligned}$$

The series is convergent in the region $|z-3i| < \frac{1}{4}$ and is divergent in the region $|z-3i| > \frac{1}{4}$.

Hence, the circle of convergence is $|z-3i| = \frac{1}{4}$.

Example 3

Discuss the convergence of the $\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$.

Solution

$$a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$

Radius of convergence is

$$\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \left|\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{n+1}{2n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right| \\
&= \left|\frac{1+0}{2+0}\right| \\
&= \frac{1}{2}
\end{aligned}$$

The series is convergent in the region

$$\left|\frac{1-z}{z}\right| < \frac{1}{2}$$

$$\frac{|1-z|}{|z|} < \frac{1}{2}$$

$$|1-z|^2 < \left(\frac{1}{2}\right)^2 |z|^2$$

$$|1-x-iy|^2 < \frac{1}{4}|x+iy|^2$$

$$(1-x)^2 + y^2 < \frac{1}{4}(x^2+y^2)$$

$$1+x^2-2x+y^2 < \frac{1}{4}(x^2+y^2)$$

$$4+4x^2-8x+4y^2 < x^2+y^2$$

$$3x^2+3y^2-8x+4<0$$

$$x^2+y^2-\frac{8}{3}x+\frac{4}{3}<0$$

$$\left(x-\frac{4}{3}\right)^2+y^2-\frac{16}{9}+\frac{4}{3}<0$$

$$\left(x-\frac{4}{3}\right)^2+y^2<\frac{4}{9}$$

Hence, the series is convergent inside the circle with its centre at $\left(\frac{4}{3}, 0\right)$ and a radius of $\frac{2}{3}$, i.e., in the region $\left|z-\frac{4}{3}\right|<\frac{2}{3}$ and is divergent in the region $\left|z-\frac{4}{3}\right|>\frac{2}{3}$.

4.5 TAYLOR'S SERIES

If $f(z)$ is analytic inside a circle C with centre at $z = a$ then for each z inside C, $f(z)$ can be expanded as a power series about $z = a$ as

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots+\frac{(z-a)^n}{n!}f^{(n)}(a)+\cdots$$

Proof Let z be any point inside the circle C. Draw a circle C_1 inside C with the centre at $z = a$, enclosing the point z (Fig. 4.1).
Let w be any point on C_1.

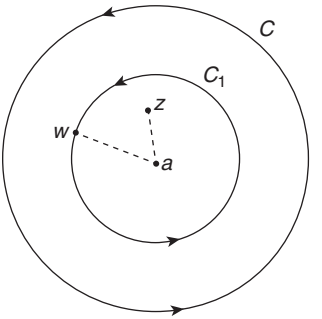


Fig. 4.1

$$|z-a|<|w-a|$$

$$\frac{|z-a|}{|w-a|}<1$$

$$\left|\frac{z-a}{w-a}\right|<1$$

By Cauchy's integral formula,

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw \qquad ...(4.1)$$

$$\text{Consider } \frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}$$

$$=\frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$=\frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$=\frac{1}{w-a}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$\left[\because \left|\frac{z-a}{w-a}\right|<1 \quad \text{Using binomial expansion}\right]$$

$$= \frac{1}{(w-a)} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (4.1),

$$f(z) = \frac{1}{2\pi i} \int_{C_1} f(w) \left[\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \right] \mathrm{d}w$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w \right]$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[2\pi i \frac{f^{(n)}(a)}{n!} \right] \left[\begin{matrix} \text{Using Cauchy's integral formula} \\ \text{for derivatives} \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Example 1

Expand $f(z) = e^z$ *in a Taylor series about* $z = 0$.

Solution

$$f(z) = e^z, \quad a = 0$$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Putting $a = 0$,

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots \quad \text{...(1)}$$

$$f(z) = e^z \qquad f(0) = e^0 = 1$$

$$f'(z) = e^z \qquad f'(0) = e^0 = 1$$

$$f''(z) = e^z \qquad f''(0) = e^0 = 1$$

$$f'''(z) = e^z \qquad f'''(0) = e^0 = 1$$

Substituting in Eq. (1),

$$f(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots$$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots$$

Example 2

Expand $\frac{1}{z+2}$ at $z = 1$ *in Taylor's series.*

Solution

Let $f(z) = \frac{1}{z+2}$, $a = 1$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Putting $a = 1$,

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots \quad \ldots(1)$$

$$f(z) = \frac{1}{z+2} \qquad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

$$f'(z) = -\frac{1}{(z+2)^2} \qquad f'(1) = -\frac{1}{(1+2)^2} = -\frac{1}{9}$$

$$f''(z) = \frac{2}{(z+2)^3} \qquad f''(1) = \frac{2}{(1+2)^3} = \frac{2}{27}$$

$$f'''(z) = -\frac{6}{(z+2)^4} \qquad f'''(1) = -\frac{6}{(1+2)^4} = -\frac{6}{81} = -\frac{2}{27}$$

Substituting in Eq. (1),

$$f(z) = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{1}{2!}\frac{2}{27}(z-1)^2 + \frac{1}{3!}\left(-\frac{2}{27}\right)(z-1)^3 + \cdots$$

$$\frac{1}{z+2} = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{1}{27}(z-1)^2 - \frac{1}{81}(z-1)^3 + \cdots$$

Aliter:

$$f(z) = \frac{1}{z+2}$$

Let $z - 1 = t$ then $z = t + 1$.

$$f(z)=\frac{1}{t+1+2}$$

$$\frac{1}{z+2}=\frac{1}{3+t}$$

$$=\frac{1}{3\left(1+\frac{t}{3}\right)}$$

$$=\frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}$$

$$=\frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{27}+\cdots\right)$$

$$=\frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\cdots$$

$$=\frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\ldots$$

Example 3

Obtain the Taylor series of $f(z)=\dfrac{1-z}{z^2}$ *in powers of* $z-1$.

Solution

$$f(z)=\frac{1-z}{z^2}=\frac{1}{z^2}-\frac{1}{z},\quad a=1$$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ldots$$

Putting $a=1$,

$$f(z)=f(1)+f'(1)(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(z-1)^3+\cdots \qquad \ldots(1)$$

$$f(z)=\frac{1}{z^2}-\frac{1}{z} \qquad f(1)=\frac{1}{1}-\frac{1}{1}=0$$

$$f'(z)=-\frac{2}{z^3}+\frac{1}{z^2} \qquad f'(1)=-2+1=-1$$

$$f''(z)=\frac{6}{z^4}-\frac{2}{z^3} \qquad f''(1)=6-2=4$$

$$f'''(z)=-\frac{24}{z^5}+\frac{6}{z^4} \qquad f'''(1)=-24+6=-18$$

Substituting in Eq. (1),

$$f(z) = 0 - 1(z-1) + \frac{4}{2!}(z-1)^2 - \frac{18}{3!}(z-1)^3 + \cdots$$

$$\frac{1-z}{z^2} = -(z-1) + 2(z-1)^2 - 3(z-1)^3 + \cdots$$

Example 4

Expand $\dfrac{z-1}{z+1}$ *in a Taylor series about the point* $z = 1$.

Solution

Let $\quad f(z) = \dfrac{z-1}{z+1}, \quad a = 1$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Putting $a = 1$,

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots \qquad \ldots(1)$$

$$f(z) = \frac{z-1}{z+1} \qquad f(1) = 0$$

$$f'(z) = \frac{(z+1)(1) - (z-1)(1)}{(z+1)^2} \qquad f'(1) = \frac{1}{2}$$

$$= \frac{2}{(z+1)^2}$$

$$f''(z) = \frac{0 - 2\{2(z+1)\}}{(z+1)^4} \qquad f''(1) = -\frac{4}{2^3} = -\frac{1}{2}$$

$$= -\frac{4}{(z+1)^3}$$

Substituting in Eq. (1),

$$f(z) = 0 + \frac{1}{2}(z-1) - \frac{1}{2!}\frac{1}{2}(z-1)^2 + \cdots$$

$$\frac{z-1}{z+1} = \frac{1}{2}(z-1) - \frac{1}{4}(z-1)^2 + \cdots$$

Example 5

Expand $\log(1 + z)$ *as a Taylor series about* $z = 0$ *when* $|z| < 1$.

Solution

Let $f(z) = \log(1 + z),\ a = 0$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Putting $a = 0$,

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \ldots \quad \ldots(1)$$

$$f(z) = \log(1+z) \qquad f(0) = \log 1 = 0$$

$$f'(z) = \frac{1}{1+z} \qquad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = -\frac{1}{(1+z)^2} \qquad f''(0) = -\frac{1}{1} = -1$$

$$f'''(z) = \frac{2}{(1+z)^3} \qquad f'''(0) = 2$$

Substituting in Eq. (1),

$$f(z) = 0 + z - \frac{z^2}{2!} + \frac{2}{3!}z^3 + \ldots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

Example 6

Expand $f(z) = \sin z$ *about* $z = \dfrac{\pi}{4}$.

Solution

$$f(z) = \sin z, \qquad a = \frac{\pi}{4}$$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Putting $a = \frac{\pi}{4}$,

$$f(z) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots \quad ...(1)$$

$$f(z) = \sin z \qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z \qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z \qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z \qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Substituting in Eq. (1),

$$f(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right)^3 + \cdots$$

$$\sin z = \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(z - \frac{\pi}{4}\right)^3 + \cdots\right]$$

Example 7

Expand $f(z) = \dfrac{z}{(z+1)(z-3)}$ *as a Taylor's series about* $z = 0$.

Solution

$$f(z) = \frac{z}{(z+1)(z-3)}$$

$$= \frac{A}{z+1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z+1)$$

Putting $z = -1$,

$$-1 = -4A$$

$$A = \frac{1}{4}$$

Putting $z = 3$,

$$3 = 4B$$
$$B = \frac{3}{4}$$
$$\therefore f(z) = \frac{1}{4}\frac{1}{z+1} + \frac{3}{4}\frac{1}{z-3}$$
$$= \frac{1}{4}\frac{1}{1+z} + \frac{3}{4}\frac{1}{-3\left(1-\frac{z}{3}\right)}$$
$$= \frac{1}{4}(1+z)^{-1} - \frac{1}{4}\left(1-\frac{z}{3}\right)^{-1}$$
$$= \frac{1}{4}(1-z+z^2-\ldots) - \frac{1}{4}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\right]$$
$$= -\frac{1}{3}z + \frac{2}{9}z^2 - \ldots$$

Example 8

Expand $f(z) = \dfrac{z+1}{(z-3)(z-4)}$ *as a Taylor's series about* $z = 2$.

Solution

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$
$$= \frac{A}{z-3} + \frac{B}{z-4}$$
$$z+1 = A(z-4) + B(z-3)$$

Putting $z = 3$,

$$4 = -A$$
$$A = -4$$

Putting $z = 4$,

$$5 = B(1)$$
$$B = 5$$
$$\therefore f(z) = -\frac{4}{z-3} + \frac{5}{z-4}$$

Let $z-2=t$ then $t=z+2$.

$$f(z)=-\frac{4}{z-2-1}+\frac{5}{z-2-2}$$

$$=-\frac{4}{t-1}+\frac{5}{t-2}$$

$$=\frac{4}{1-t}-\frac{5}{2\left(1-\frac{t}{2}\right)}$$

$$=4(1-t)^{-1}-\frac{5}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$=4(1-t+t^2-\cdots)-\frac{5}{2}\left[1-\frac{t}{2}+\left(\frac{t}{2}\right)^2-\cdots\right]$$

$$=\frac{3}{2}-\frac{11}{4}t+\frac{27}{8}t^2+\cdots$$

$$=\frac{3}{2}-\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+\cdots$$

Example 9

Find the Taylor series to represent the function $\dfrac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

Solution

Let $$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of numerator,

$$f(z)=1+\frac{-5z-7}{(z+2)(z+3)}$$

Let $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Putting $z=-2$,

$$-5(-2)-7=A(-2+3)$$

$$A=3$$

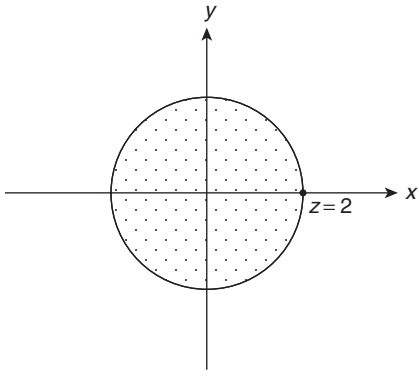


Fig. 4.2

Putting $z = -3$,

$$-5(-3) - 7 = B(-3+2)$$

$$B = -8$$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ is not analytic at $z = -2$ and $z = -3$. But $f(z)$ is analytic inside the region $|z| < 2$ (Fig. 4.2).

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1, \quad \left|\frac{z}{3}\right| < 1$$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$= -\frac{1}{6} + \frac{5}{36}z + \frac{17}{216}z^2 + \cdots$$

EXERCISE 4.1

Find the Taylor's series for the following functions about the indicated points:

1. $f(z) = \cos z$ about $z = \dfrac{\pi}{4}$

$$\left[\textbf{Ans.:}\ \frac{1}{\sqrt{2}}\left[1 - \left(z - \frac{\pi}{4}\right) + \frac{1}{2}\left(z - \frac{\pi}{4}\right)^2 - \cdots\right]\right]$$

2. $f(z) = \cos z$ about $z = 0$

$$\left[\textbf{Ans.:}\ 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right]$$

3. $f(z) = e^{2z}$ about $z = 2i$

$$\left[\textbf{Ans.:}\ e^{4i}\left\{1 + 2(z-2i) - (z-2i)^2 + \cdots\right\}\right]$$

4. $f(z) = \tanh z$ about $z = 0$

$$\left[\textbf{Ans.:}\ z - \frac{z^3}{3} + \cdots\right]$$

5. $f(z) = \dfrac{z-1}{z+1}$ in the region $|z| < 1$

$$[\textbf{Ans.:}\ 1 - 2(1 + z + z^2 + z^3 + \cdots)]$$

6. $f(z) = \dfrac{1}{z^2+4}$ about $z = -i$

$$\left[\textbf{Ans.:}\ \frac{1}{3} + \frac{1}{18}(z+i).i + \frac{7}{27}(z+1)^2 - \cdots\right]$$

7. $f(z) = \dfrac{z-1}{z^2}$ about $z = 1$

$$[\textbf{Ans.:}\ (z-1) - 2(z-1)^2 + 3(z-1)^3 - \cdots]$$

8. $f(z) = \dfrac{1}{z^2 - 4z + 3}$ about $z = 4$

$$\left[\textbf{Ans.:}\ \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 - \cdots\right]$$

4.6 LAURENT'S SERIES

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with the centre at $z = a$ and radii r_1, $r_2 (r_2 < r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \, dw$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \, dw$$

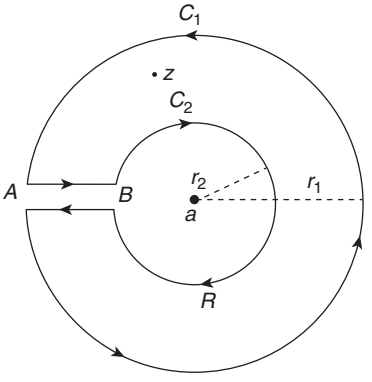


Fig. 4.3

Proof

Let C_1, C_2 be two concentric circles with centre at $z = a$ and radii r_1 and $r_2 (r_2 < r_1)$ respectively (Fig. 4.3).

The annular region R between C_1 and C_2 is a multiply connected region. This multiply connected region is converted to a simply connected region by introducing a cross-cut AB.

$f(z)$ is analytic in this simply connected region.

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw$$

$$\left[\because \text{integrals around } AB \text{ and } BA \text{ cancel each other}\right]$$

$$= f_1(z) - f_2(z)$$

(i) $$f_1(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw \qquad \ldots(4.2)$$

z is any point in R and for this integral, w lies on C_1.

$$\therefore \quad |z-a| < |w-a|$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

Consider $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left(1-\dfrac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$= \frac{1}{(w-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (4.2),

$$f_1(z) = \frac{1}{2\pi i}\int_{C_1} f(w)\left[\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right]\mathrm{d}w$$

$$= \sum_{n=0}^{\infty}\left[(z-a)^n\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\mathrm{d}w\right\}\right]$$

$$= \sum_{n=0}^{\infty}(z-a)^n a_n \qquad \cdots(4.3)$$

where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\mathrm{d}w$

(ii) $$f_2(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\mathrm{d}w \qquad \cdots(4.4)$$

z is any point in R and for this integral, w lies on C_2.

$\therefore \quad |z-a| > |w-a|$

$$1 > \left|\frac{w-a}{z-a}\right|$$

$$\left|\frac{w-a}{z-a}\right| < 1$$

Consider $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(z-a)\left(\frac{w-a}{z-a}-1\right)}$$

$$= -\frac{1}{(z-a)}\cdot\frac{1}{\left(1-\frac{w-a}{z-a}\right)}$$

$$= -\frac{1}{(z-a)}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{(z-a)}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n-1}+\cdots\right]$$

$$= -\frac{1}{(z-a)}\sum_{n=1}^{\infty}\left(\frac{w-a}{z-a}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty}-\frac{(w-a)^{n-1}}{(z-a)^n}$$

Substituting $\frac{1}{w-z}$ in Eq. (4.4),

$$f_2(z) = \frac{1}{2\pi i}\int_{C_2} f(w)\left[\sum_{n=1}^{\infty} -\frac{(w-a)^{n-1}}{(z-a)^n}\right]\mathrm{d}w$$

$$= \sum_{n=1}^{\infty}\left[\frac{-1}{(z-a)^n}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\mathrm{d}w\right\}\right]$$

$$= \sum_{n=1}^{\infty} -\frac{1}{(z-a)^n} b_n$$

where $$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\mathrm{d}w$$

$$\therefore f(z) = f_1(z) - f_2(z)$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $$a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\mathrm{d}w$$

$$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\mathrm{d}w$$

The second term $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ is known as the principal part of Laurent's series.

Example 1

Find the Laurent's series of $f(z) = z^2 e^{\frac{1}{z}}$ *about* $z = 0$.

Solution

$$f(z) = z^2 e^{\frac{1}{z}}$$

$f(z)$ is analytic for $|z| > 0$.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \cdots$$

Example 2

Find the Laurent's series of $f(z)=\dfrac{e^{2z}}{(z-1)^3}$ *about* $z = 1$.

Solution

$$f(z)=\frac{e^{2z}}{(z-1)^3}$$

$f(z)$ is not analytic at $z = 1$. But it is analytic in the region $|z-1|>0$ about $z = 1$.
Let $z - 1 = t$ then $z = t + 1$.

$$\begin{aligned}
f(z)&=\frac{e^{2(t+1)}}{t^3}\\
&=\frac{e^2}{t^3}e^{2t}\\
&=\frac{e^2}{t^3}\left[1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots\right]\\
&=e^2\left[\frac{1}{t^3}+\frac{2}{t^2}+\frac{2}{t}+\frac{4}{3}+\cdots\right]\\
&=e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)}+\frac{4}{3}+\cdots\right]
\end{aligned}$$

Example 3

Find Laurent's series expansion in powers of z *that represent* $f(z)=\dfrac{1}{z^2(1-z)}$ *for the following domains: (i)* $|z|<1$ *(ii)* $|z|>1$.

[Summer 2013]

Solution

$$\begin{aligned}
f(z)&=\frac{1}{z^2(1-z)}\\
&=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{1-z}\\
1&=Az(1-z)+B(1-z)+Cz^2
\end{aligned}$$

Putting $z = 0$,

$$1 = B$$

Putting $z = 1$,

$$1 = C$$

Putting $z = -1$,

$$1 = -2A + 2B + C$$
$$= -2A + 2 + 1$$
$$A = 1$$

$$\therefore \quad f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.4).

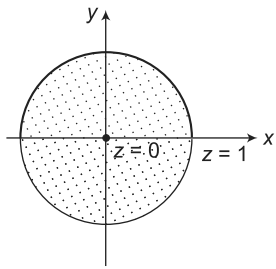


Fig. 4.4

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + (1-z)^{-1}$$
$$= \frac{1}{z} + \frac{1}{z^2} + (1 + z + z^2 + \cdots)$$

(ii) $|z| > 1$

$f(z)$ is analytic in the region $|z| > 1$ about $z = 0$ (Fig. 4.5).

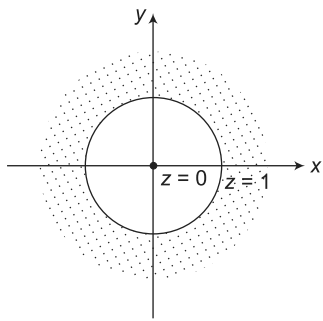


Fig. 4.5

$\because \quad |z| > 1, \qquad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{-z\left(-\frac{1}{z} + 1\right)}$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

$$= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

Example 4

Expand $f(z) = \dfrac{1}{(z+1)(z-2)}$ *in Laurent's series in the following regions:*

(i) $|z| < 1$ *(ii)* $1 < |z| < 2$ *(iii)* $|z| > 2$. **[Summer 2015]**

Solution

$$f(z) = \frac{1}{(z+1)(z+2)}$$
$$= \frac{1}{3}\frac{(z+1)-(z-2)}{(z+1)(z-2)}$$
$$= \frac{1}{3}\left[\frac{1}{z-2} - \frac{1}{z+1}\right]$$

$f(z)$ is not analytic at $z = 2$ and $z = -1$.

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.6).

$\because \quad |z| < 1 < 2, \qquad \left|\frac{z}{2}\right| < 1$

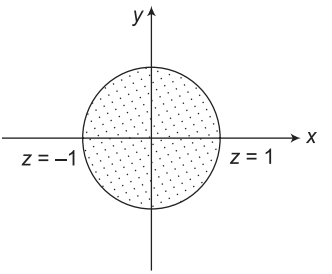


Fig. 4.6

$$f(z) = \frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z+1}\right]$$
$$= -\frac{1}{3}\left[\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1+z)^{-1}\right]$$
$$= -\frac{1}{3}\left[\frac{1}{2}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right\} + (1-z+z^2-\cdots)\right]$$
$$= -\frac{1}{3}\left(\frac{3}{2} - \frac{3}{4}z + \frac{9}{8}z^2 - \cdots\right)$$
$$= -\frac{1}{2} + \frac{1}{4}z - \frac{3}{8}z^2 + \cdots$$

(ii) $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.7).

$1 < |z|, \left|\frac{1}{z}\right| < 1$

$|z| < 2, \left|\frac{z}{2}\right| < 1$

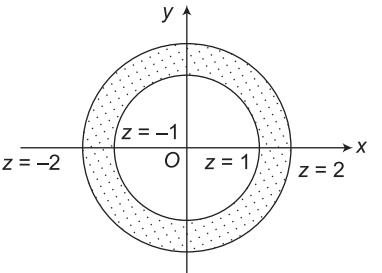


Fig. 4.7

$$f(z) = \frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$
$$= \frac{1}{3}\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$
$$= \frac{1}{3}\left[-\frac{1}{2}\left\{1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right\} - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)\right]$$

$$= -\frac{1}{3}\left[\left(\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots\right)+\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.8).

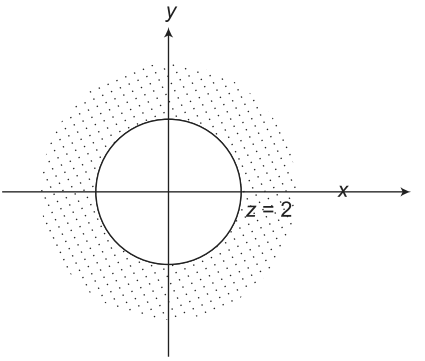


Fig. 4.8

$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{3}\left[\frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$

$$= \frac{1}{3}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{3}\left[\frac{1}{z}\left\{1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right\} - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right)-\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

$$= \frac{1}{3}\left(\frac{3}{z^2}+\frac{3}{z^3}+\cdots\right)$$

$$= \frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

Example 5

Find all possible Laurent's expansion of $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ *about* $z = 0$ *Indicate the region of convergence in each case.*

Solution

$$f(z) = \frac{4-3z}{z(1-z)(2-z)}$$

$$= \frac{A}{z}+\frac{B}{1-z}+\frac{C}{2-z}$$

$$4-3z = A(1-z)(2-z)+Bz(2-z)+Cz(1-z)$$

Putting $z = 0$,

$$4 = 2A$$

$$A = 2$$

Putting $z = 1$,

$$B = 1$$

Putting $z = 2$,

$$4 - 3(2) = -2C$$

$$C = 1$$

$$\therefore \quad f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = 2$.

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.9).

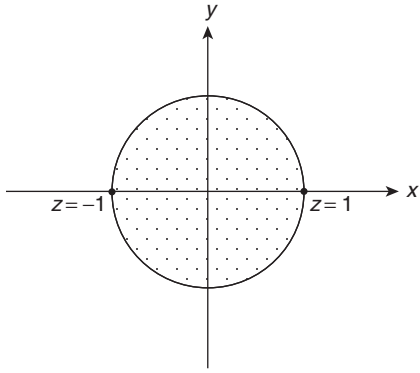


Fig. 4.9

$$\because \quad |z| < 1 < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} + \left(1 + z + z^2 + \cdots\right) + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots\right) + \frac{2}{z}$$

(ii) $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.10).

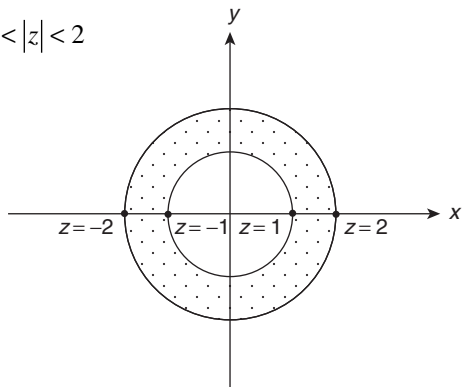


Fig. 4.10

$$1 < |z|, \quad \left|\frac{1}{z}\right| < 1$$

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) + \left(\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right)$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.11).

$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$-\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$= -\frac{3}{z^2} - \frac{5}{z^3} - \cdots$$

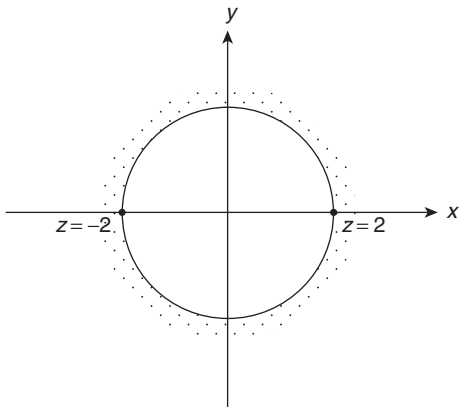


Fig. 4.11

Example 6

Find the Laurent's series expansion of

$$f(z) = \frac{4z+1}{z(z^2+z-2)} \text{ in } |z| > 2.$$

Solution

$$f(z) = \frac{4z+1}{z(z^2+z-2)}$$

$$= \frac{4z+1}{z(z-1)(z+2)}$$

$$= \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$4z+1 = A(z+2)(z-1) + Bz(z+2) + Cz(z-1)$$

Putting $z = 0$,

$$1 = (2)(-1)A$$

$$A = -\frac{1}{2}$$

Putting $z = 1$,

$$5 = 3B$$

$$B = \frac{5}{3}$$

Putting $z = -2$,

$$-7 = (-2)(-3)C$$

$$C = -\frac{7}{6}$$

$$\therefore \quad f(z) = -\frac{1}{2}\frac{1}{z} + \frac{5}{3}\frac{1}{z-1} - \frac{7}{6}\frac{1}{z+2}$$

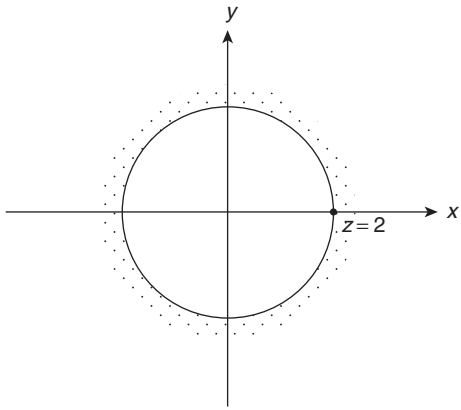


Fig. 4.12

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.12).

$$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = -\frac{1}{2z} + \frac{5}{3z\left(1-\frac{1}{z}\right)} - \frac{7}{6z\left(1+\frac{2}{z}\right)}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{7}{6z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{7}{6z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\cdots\right]$$

$$= -\frac{1}{2z} + \left(\frac{5}{3z}+\frac{5}{3z^2}+\frac{5}{3z^3}+\cdots\right) - \left(\frac{7}{6z}-\frac{7}{3z^2}+\frac{14}{3z^3}-\cdots\right)$$

$$= \frac{4}{z^2} - \frac{3}{z^3} + \cdots$$

Example 7

Evaluate $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in the Laurent series valid in the regions* $|z| > 3$ and $1 < |z| < 3$.

Solution

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

Putting $z = -1$,

$$1 = A(-1+3)$$

$$A = \frac{1}{2}$$

Putting $z = -3$,

$$1 = B(-3+1)$$

$$B = -\frac{1}{2}$$

$$\therefore f(z) = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}$$

$f(z)$ is not analytic at $z = -1$ and $z = -3$.

(i) $|z| > 3$

$f(z)$ is analytic in the region $|z| > 3$ about $z = 0$ (Fig. 4.13).

$$\because \quad |z| > 3 > 1, \left|\frac{3}{z}\right| < 1, \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \cdots\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \cdots\right) - \left(\frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \cdots\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} - \cdots$$

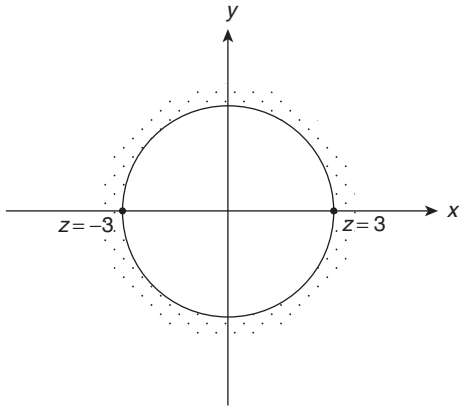


Fig. 4.13

(ii) $1 < |z| < 3$

$f(z)$ is analytic in the annular region $1 < |z| < 3$ about $z = 0$ (Fig. 4.14).

$$1 < |z|, \quad \left|\frac{1}{z}\right| < 1$$

$$|z| < 3, \quad \left|\frac{z}{3}\right| < 1$$

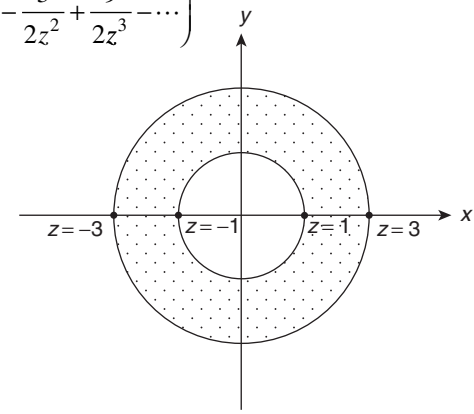


Fig. 4.14

$$f(z)=\frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2(3)}\frac{1}{\left(1+\frac{z}{3}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\cdots\right]$$

$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right)-\left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\cdots\right)$$

$$=\left(-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\cdots\right)+\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right)$$

Example 8

Expand $f(z)=\dfrac{1}{(z+2)(z+4)}$ *valid for the following regions:*

(i) $|z|<2$ *(ii)* $2<|z|<4$ *(iii)* $|z|>4$.

Solution

$$f(z)=\frac{1}{(z+2)(z+4)}$$

$$=\frac{(z+4)-(z+2)}{2(z+2)(z+4)}$$

$$=\frac{1}{2}\left[\frac{1}{z+2}-\frac{1}{z+4}\right]$$

(i) $|z|<2$

$f(z)$ is analytic in the region $|z|<2$ about $z=0$ (Fig. 4.15).

$$\because\quad |z|<2<4,\quad \left|\frac{z}{2}\right|<1,\quad \left|\frac{z}{4}\right|<1$$

$$f(z)=\frac{1}{2}\frac{1}{2\left(\frac{z}{2}+1\right)}-\frac{1}{2}\frac{1}{4\left(\frac{z}{4}+1\right)}$$

$$=\frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}-\frac{1}{8}\left(1+\frac{z}{4}\right)^{-1}$$

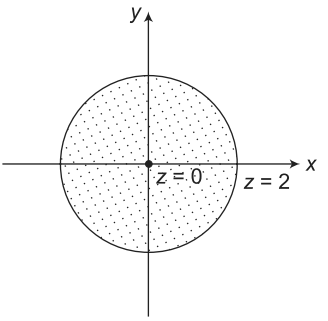


Fig. 4.15

$$= \frac{1}{4}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right] - \frac{1}{8}\left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{4} - \frac{z}{8} + \frac{z^2}{16} - \cdots\right) - \left(\frac{1}{8} - \frac{z}{32} + \frac{z^2}{128} - \cdots\right)$$

$$= \frac{1}{8} - \frac{3z}{32} + \frac{7z^2}{128} - \cdots$$

(ii) $2 < |z| < 4$

$f(z)$ is analytic in the annular region $2 < |z| < 4$ about $z = 0$ (Fig. 4.16).

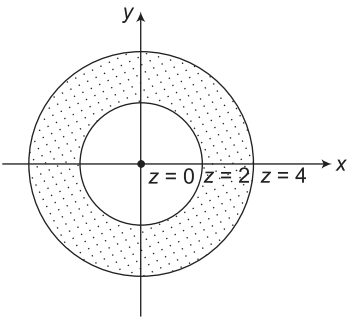


Fig. 4.16

$$2 < |z|, \quad \left|\frac{2}{z}\right| < 1$$

$$|z| < 4, \quad \left|\frac{z}{4}\right| < 1$$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1 + \frac{2}{z}\right)} - \frac{1}{2}\frac{1}{4\left(\frac{z}{4} + 1\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{8}\left(1 + \frac{z}{4}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{1}{8}\left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{z^2} + \frac{2}{z^3} - \cdots\right) - \frac{1}{8}\left(1 - \frac{z}{4} + \frac{z^2}{16} - \cdots\right)$$

(iii) $|z| > 4$

$f(z)$ is analytic in the region $|z| > 4$ about $z = 0$ (Fig. 4.17).

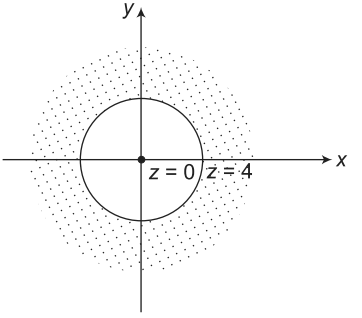


Fig. 4.17

$\because \quad |z| > 4 > 2, \quad \left|\frac{4}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1 + \frac{2}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1 + \frac{4}{z}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{2z}\left(1 + \frac{4}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{1}{2z}\left[1 - \frac{4}{z} + \left(\frac{4}{z}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{z^2} + \frac{2}{z^3} - \cdots\right) - \left(\frac{1}{2z} - \frac{2}{z^2} + \frac{8}{z^3} - \cdots\right)$$

$$= \frac{1}{z^2} - \frac{6}{z^3} + \cdots$$

Example 9

Expand $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ *as a Laurent's series if* (*i*) $2 < |z| < 3$, *and* (*ii*) $|z| > 3$.

Solution

Let $$f(z) = \frac{z^2 - 1}{(z+2)(z+3)} = \frac{z^2 - 1}{z^2 + 5z + 6}$$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of the numerator,

$$f(z) = 1 + \frac{-5z - 7}{(z+2)(z+3)}$$

Let $$\frac{-5z - 7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$-5z - 7 = A(z+3) + B(z+2)$$

Putting $z = -2$,

$$-5(-2) - 7 = A(-2+3)$$

$$A = 3$$

Putting $z = -3$,

$$-5(-3) - 7 = B(-3+2)$$

$$B = -8$$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ is not analytic at $z = -2$ and $z = -3$.

(i) $2 < |z| < 3$

$f(z)$ is analytic in the annular region $2 < |z| < 3$ about $z = 0$ (Fig. 4.18).

$$2 < |z|, \quad \left|\frac{2}{z}\right| < 1$$

$$|z| < 3, \quad \left|\frac{z}{3}\right| < 1$$

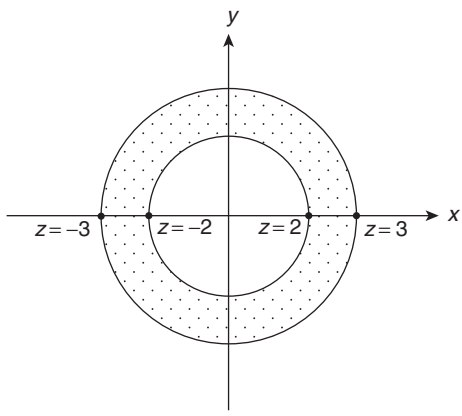


Fig. 4.18

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$=1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

(ii) $|z|>3$

$f(z)$ is analytic in the region $|z|>3$ about $z=0$ (Fig. 4.19).

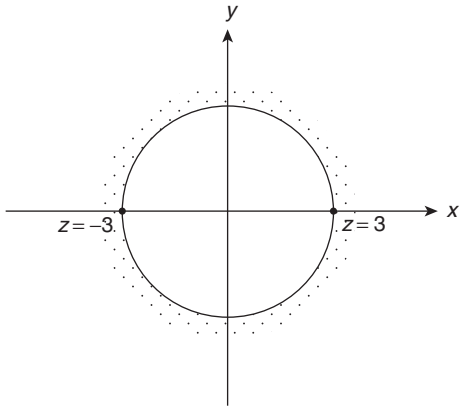


Fig. 4.19

$\because \quad |z|>3>1, \quad \left|\frac{3}{z}\right|<1, \quad \left|\frac{1}{z}\right|<1$

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$=1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right)-8\left(\frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\frac{27}{z^4}+\cdots\right)$$

$$=1-\frac{5}{z}+\frac{18}{z^2}-\frac{60}{z^3}+\frac{192}{z^4}-\cdots$$

Example 10

Find the Laurent's series of $f(z)=\dfrac{z}{(z^2+1)(z^2+4)}$ in $1<|z|<2$.

Solution

$$f(z) = \frac{z}{(z^2+1)(z^2+4)}$$

$$= \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+4}$$

$$z = (Az+B)(z^2+4) + (Cz+D)(z^2+1)$$

$$= Az^3 + Bz^2 + 4Az + 4B + Cz^3 + Dz^2 + Cz + D$$

$$= (A+C)z^3 + (B+D)z^2 + (4A+C)z + 4B + D$$

Equating coefficients of z^3,

$$0 = A + C$$

$$A = -C$$

Equating coefficients of z^2,

$$0 = B + D$$

$$B = -D$$

Equating coefficients of z,

$$1 = 4A + C$$

$$1 = 4A - A$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$C = -\frac{1}{3}$$

Equating constant term,

$$0 = 4B + D$$

$$0 = -4D + D$$

$$D = 0$$

$$B = 0$$

$$\therefore\ f(z) = \frac{1}{3}\frac{z}{z^2+1} - \frac{1}{3}\frac{z}{z^2+4}$$

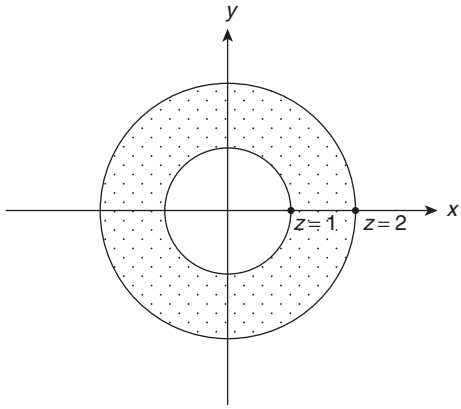


Fig. 4.20

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.20).

$$1 < |z|,\ \left|\frac{1}{z}\right| < 1,\ \left|\frac{1}{z^2}\right| < 1$$

$$|z| < 2,\ |z^2| < 4,\ \left|\frac{z^2}{4}\right| < 1 \quad \left[\because\ |z|^2 = |z^2|\right]$$

$$f(z)=\frac{z}{3}\frac{1}{z^2\left(1+\frac{1}{z^2}\right)}-\frac{z}{3}\frac{1}{4\left(1+\frac{z^2}{4}\right)}$$

$$=\frac{1}{3z}\left(1+\frac{1}{z^2}\right)^{-1}-\frac{z}{12}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$=\frac{1}{3z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right)-\frac{z}{12}\left[1-\frac{z^2}{4}+\left(\frac{z^2}{4}\right)^2-\cdots\right]$$

$$=\left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\cdots\right)-\left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\cdots\right)$$

$$=-\left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\cdots\right)+\left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\cdots\right)$$

Example 11

Find the series of $f(z)=\dfrac{z}{(z-1)(z-4)}$ *in terms of* $(z+3)$ *valid for* $|z+3|<4$.

Solution

$$f(z)=\frac{z}{(z-1)(z-4)}$$

$$=\frac{A}{z-1}+\frac{B}{z-4}$$

$$z=A(z-4)+B(z-1)$$

Putting $z=1$,

$$1=-3A$$

$$A=-\frac{1}{3}$$

Putting $z=4$,

$$4=3B$$

$$B=\frac{4}{3}$$

$$\therefore\quad f(z)=-\frac{1}{3}\frac{1}{z-1}+\frac{4}{3}\frac{1}{z-4}$$

$f(z)$ is analytic in the region $|z+3|<4$ about $z=-3$ (Fig. 4.21).

Let $z+3=t$ then $z=t-3$

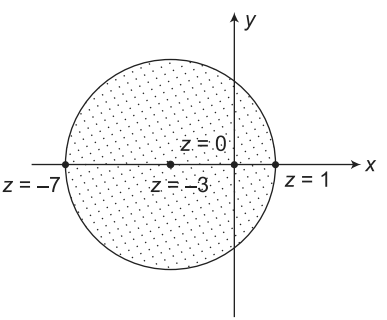


Fig. 4.21

$$f(z) = -\frac{1}{3}\frac{1}{t-4} + \frac{4}{3}\frac{1}{t-7}$$

$\because \quad |z+3| < 4$, i.e., $|t| < 4 < 7$, $\left|\frac{t}{4}\right| < 1$, $\left|\frac{t}{7}\right| < 1$

$$f(z) = -\frac{1}{3}\left[\frac{1}{-4\left(1-\frac{t}{4}\right)}\right] + \frac{4}{3}\left[\frac{1}{-7\left(1-\frac{t}{7}\right)}\right]$$

$$= \frac{1}{12}\left(1-\frac{t}{4}\right)^{-1} - \frac{4}{21}\left(1-\frac{t}{7}\right)^{-1}$$

$$= \frac{1}{12}\left[1+\frac{t}{4}+\left(\frac{t}{4}\right)^2+\cdots\right] - \frac{4}{21}\left[1+\frac{t}{7}+\left(\frac{t}{7}\right)^2+\cdots\right]$$

$$= \left(\frac{1}{12}+\frac{t}{48}+\frac{t^2}{192}+\cdots\right) - \left(\frac{4}{21}+\frac{4t}{147}+\frac{4t^2}{1029}+\cdots\right)$$

$$= -\frac{3}{28} - \frac{5t}{784} + \frac{29t^2}{21952} + \cdots$$

$$= -\frac{3}{28} - \frac{5}{784}(z+3) + \frac{29}{21952}(z+3)^2 + \cdots$$

Example 12

Find Laurent's series expansion in powers of z that represent $f(z) = \frac{1}{z(z-1)}$ for the following domains: (i) $0 < |z| < 1$ (ii) $0 < |z-1| < 1$. **[Winter 2013]**

Solution

$$f(z) = \frac{1}{z(z-1)}$$

$$= \frac{z-(z-1)}{z(z-1)}$$

$$= \frac{1}{z-1} - \frac{1}{z}$$

$f(z)$ is not analytic at $z = 1$ and $z = 0$.

(i) $0 < |z| < 1$

$f(z)$ is analytic in the region $0 < |z| < 1$ (Fig. 4.22).

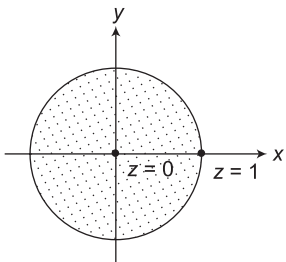


Fig. 4.22

$$f(z) = -\frac{1}{(1-z)} - \frac{1}{z}$$

$$= -(1-z)^{-1} - \frac{1}{z}$$

$$= -(1+z+z^2+z^3+\cdots) - \frac{1}{z}$$

(ii) $0 < |z-1| < 1$

$f(z)$ is analytic in the region $0 < |z-1| < 1$ (Fig. 4.23).

Let $z - 1 = t$ then $z = t + 1$

$0 < |z-1| < 1,\ 0 < |t| < 1$

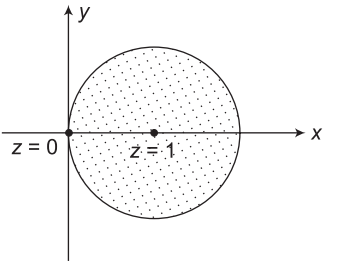


Fig. 4.23

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{1}{t} - \frac{1}{1+t}$$

$$= \frac{1}{t} - (1+t)^{-1}$$

$$= \frac{1}{t} - (1 - t + t^2 - t^3 + \cdots)$$

$$= \frac{1}{z-1} - \left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right]$$

Example 13

Find the Laurent's series expansion of $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ *in* $1 < |z+1| < 3$.

Solution

$$f(z) = \frac{7z-2}{z(z-2)(z+1)}$$

By partial fraction expansion,

$$f(z) = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Putting $z = 0$,

$$-2 = A(-2)(1)$$

$$A = 1$$

Putting $z = 2$,

$$14 - 2 = B(2)(2+1)$$

$$B = 2$$

Purring $z = -1$,

$$-7 - 2 = C(-1)(-1-2)$$

$$C = -3$$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$f(z)$ is not analytic at $z = 0$, $z = 2$ and $z = -1$.
$f(z)$ is analytic in the annular region $1 < |z + 1| < 3$ about $z = -1$ (Fig. 4.24).
Let $z + 1 = t$ then $z = t - 1$.

$$1 < |z+1| < 3, \text{ i.e., } 1 < |t| < 3$$

$$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$$

$$|t| < 3, \quad \left|\frac{t}{3}\right| < 1$$

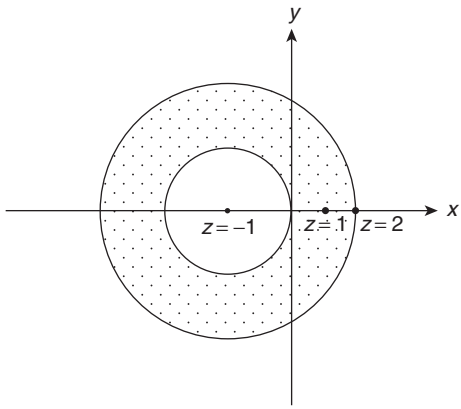


Fig. 4.24

$$f(z) = \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{-3\left(1-\frac{t}{3}\right)} - \frac{3}{t}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t}$$

$$= \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\cdots\right) - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\cdots\right] - \frac{3}{t}$$

$$= \left(-\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\right) - \frac{2}{3}\left[1+\frac{t}{3}+\frac{t^2}{9}+\cdots\right]$$

$$= \left(-\frac{2}{3}-\frac{2}{9}t-\frac{2}{27}t^2-\cdots\right) + \left[-\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\right]$$

$$= \left[-\frac{2}{3}-\frac{2}{9}(z+1)-\frac{2}{27}(z+1)^2-\cdots\right] + \left[-\frac{2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right]$$

Example 14

Find the Laurent's expansion of $f(z) = \dfrac{1}{z - z^3}$ *in the region* $1 < |z - 1| < 2$.

Solution

$$f(z) = \frac{1}{z-z^3}$$

$$= \frac{1}{z(1-z^2)}$$

$$= \frac{1}{z(1-z)(1+z)}$$

$$= \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z}$$

$$1 = A(1-z)(1+z) + Bz(1+z) + Cz(1-z)$$

Putting $z = 0$,

$$A = 1$$

Putting $z = 1$,

$$1 = 2B$$

$$B = \frac{1}{2}$$

Putting $z = -1$,

$$1 = -2C$$

$$C = -\frac{1}{2}$$

$$\therefore \quad f(z) = \frac{1}{z} + \frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1+z}$$

$$= \frac{1}{z} - \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$$

$f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = -1$.

(i) $1 < |z-1| < 2$

$f(z)$ is analytic in the annular region $1 < |z-1| < 2$ about $z = 1$ (Fig. 4.25).

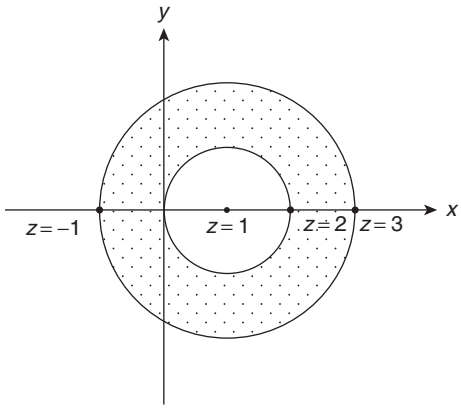


Fig. 4.25

Let $z - 1 = t$ then $z = t + 1$.

$$1 < |t| < 2$$

$$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$$

$$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$$

$$f(z) = \frac{1}{t+1} - \frac{1}{2}\frac{1}{t} - \frac{1}{2}\frac{1}{t+2}$$

$$= \frac{1}{t}\frac{1}{\left(1+\frac{1}{t}\right)} - \frac{1}{2}\frac{1}{t} - \frac{1}{4}\frac{1}{\left(1+\frac{t}{2}\right)}$$

$$= \frac{1}{t}\left(1+\frac{1}{t}\right)^{-1} - \frac{1}{2t} - \frac{1}{4}\left(1+\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{t}\left(1 - \frac{1}{t} + \frac{1}{t^2} - \cdots\right) - \frac{1}{2t} - \frac{1}{4}\left[1 - \frac{t}{2} + \left(\frac{t}{2}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3} - \cdots\right) - \frac{1}{2t} + \left(-\frac{1}{4} + \frac{t}{8} - \frac{t^2}{16} + \cdots\right)$$

$$= \left(-\frac{1}{4} + \frac{t}{8} - \frac{t^2}{16} + \cdots\right) + \left(\frac{1}{2t} - \frac{1}{t^2} + \frac{1}{t^3} - \cdots\right)$$

$$= \left[-\frac{1}{4} + \frac{z-1}{8} - \frac{(z-1)^2}{16} + \cdots\right] + \left[\frac{1}{2(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \cdots\right]$$

Example 15

Find the Laurent's series expansion of $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$ *in the region* $|z-1| > 4$.

Solution

$$f(z) = \frac{z+4}{(z+3)(z-1)^2}$$

$$= \frac{A}{z+3} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z + 4 = A(z-1)^2 + B(z+3)(z-1) + C(z+3)$$

Putting $z = -3$,

$$1 = 16A$$

$$A = \frac{1}{16}$$

Putting $z = 1$,

$$5 = 4C$$

$$C = \frac{5}{4}$$

Putting $z = 0$,

$$4 = A - 3B + 3C$$

$$4 = \frac{1}{16} - 3B + \frac{15}{4}$$

$$B = -\frac{1}{16}$$

$$\therefore f(z) = \frac{1}{16}\frac{1}{z+3} - \frac{1}{16}\frac{1}{z-1} + \frac{5}{4}\frac{1}{(z-1)^2}$$

$f(z)$ is not analytic at $z = 1$ and $z = -3$.

$f(z)$ is analytic in the region $|z-1|>4$ about $z=1$ (Fig. 4.26).
Let $z-1=t$ then $z=t+1$.

$$|z-1|>4, \text{ i.e., } |t|>4, \quad \left|\frac{1}{t}\right|<\frac{1}{4}<1$$

$$f(z)=\frac{1}{16}\frac{1}{t+4}-\frac{1}{16}\frac{1}{t}+\frac{5}{4}\frac{1}{t^2}$$

$$=\frac{1}{16}\frac{1}{t\left(1+\frac{4}{t}\right)}-\frac{1}{16}\frac{1}{t}+\frac{5}{4}\frac{1}{t^2}$$

$$=\frac{1}{16t}\left(1+\frac{4}{t}\right)^{-1}-\frac{1}{16t}+\frac{5}{4t^2}$$

$$=\frac{1}{16t}\left[1-\frac{4}{t}+\left(\frac{4}{t}\right)^2-\cdots\right]-\frac{1}{16t}+\frac{5}{4t^2}$$

$$=\frac{1}{16t}\left(-\frac{4}{t}+\frac{16}{t^2}-\cdots\right)+\frac{5}{4t^2}$$

$$=\frac{1}{16(z-1)}\left[-\frac{4}{(z-1)}+\frac{16}{(z-1)^2}-\cdots\right]+\frac{5}{4(z-1)^2}$$

$$=\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}-\cdots$$

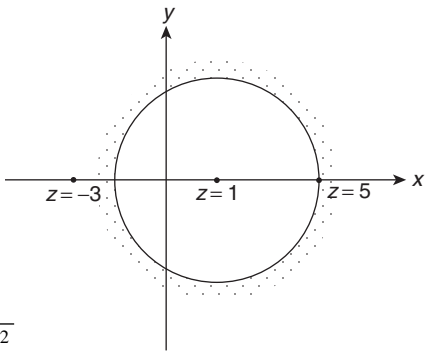


Fig. 4.26

Example 16

Find the Laurent's series of $f(z)=\dfrac{1}{z(1-z)}$ *valid in the following region*

(i) $|z+1|<1$ *(ii)* $|z+1|>2$ *(iii)* $1<|z+1|<2$.

Solution

$$f(z)=\frac{1}{z(1-z)}$$

$$=\frac{1-z+z}{z(1-z)}$$

$$=\frac{1}{z}+\frac{1}{1-z}$$

$f(z)$ is not analytic at $z=0$ and $z=1$.

(i) $|z+1|<1$

$f(z)$ is analytic in the region $|z+1|<1$ about $z=-1$ (Fig. 4.27).

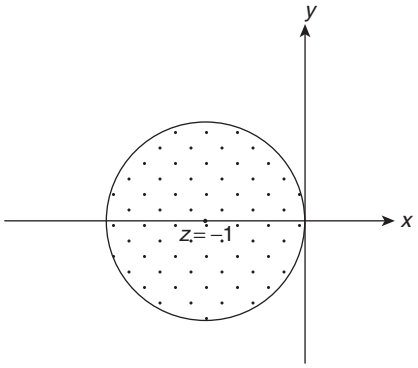


Fig. 4.27

Let $z + 1 = t$ then $z = t - 1$.

$$|z+1|<1, \text{ i.e., } |t|<1<2,$$

$$\left|\frac{t}{2}\right|<1$$

$$f(z)=\frac{1}{t-1}+\frac{1}{1-(t-1)}$$

$$=\frac{1}{t-1}+\frac{1}{2-t}$$

$$=-\frac{1}{1-t}+\frac{1}{2\left(1-\frac{t}{2}\right)}$$

$$=-(1-t)^{-1}+\frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$=-(1+t+t^2+\cdots)+\frac{1}{2}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$=-\frac{1}{2}\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\cdots\right)$$

$$=-\frac{1}{2}\left[1+\frac{3}{2}(z+1)+\frac{7}{4}(z+1)^2+\cdots\right]$$

(ii) $|z + 1| > 2$

$f(z)$ is analytic in the region $|z + 1| > 2$ about $z = -1$ (Fig. 4.28).

Let $z+1=t$ then $z=t-1$.

$$|z+1|>2, \text{ i.e., } |t|>2, \quad \left|\frac{1}{t}\right|<\frac{1}{2}<1$$

$$f(z)=\frac{1}{t-1}+\frac{1}{2-t}$$

$$=\frac{1}{t\left(1-\frac{1}{t}\right)}-\frac{1}{t\left(1-\frac{2}{t}\right)}$$

$$=\frac{1}{t}\left(1-\frac{1}{t}\right)^{-1}-\frac{1}{t}\left(1-\frac{2}{t}\right)^{-1}$$

$$=\frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\cdots\right)$$

$$-\frac{1}{t}\left[1+\frac{2}{t}+\left(\frac{2}{t}\right)^2+\cdots\right]$$

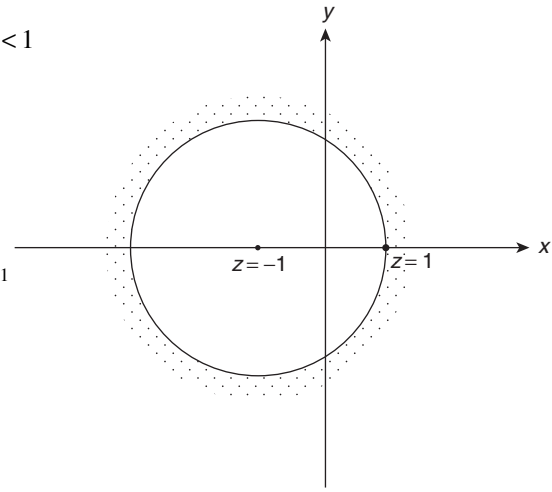


Fig. 4.28

$$= \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots - \left(\frac{1}{t} + \frac{2}{t^2} + \frac{4}{t^3} + \cdots \right)$$

$$= -\frac{1}{t^2} - \frac{3}{t^3} - \cdots$$

$$= -\frac{1}{(z+1)^2} - \frac{3}{(z+1)^3} - \cdots$$

(iii) $1 < |z+1| < 2$

$f(z)$ is analytic in the annular region $1 < |z+1| < 2$ about $z = -1$ (Fig. 4.29).

Let $z+1 = t$ then $z = t-1$.

$1 < |z+1| < 2$ i.e., $1 < |t| < 2$

$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$

$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$

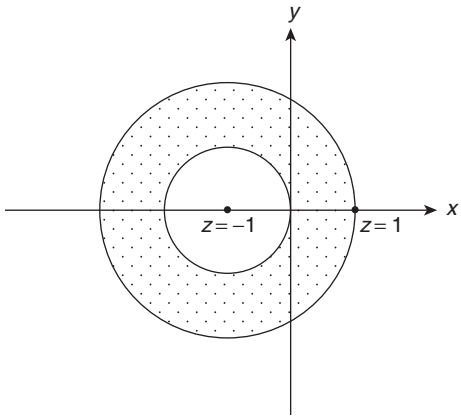


Fig. 4.29

$$f(z) = \frac{1}{t-1} + \frac{1}{2-t}$$

$$= \frac{1}{t\left(1 - \frac{1}{t}\right)} + \frac{1}{2\left(1 - \frac{t}{2}\right)}$$

$$= \frac{1}{t}\left(1 - \frac{1}{t}\right)^{-1} + \frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1}$$

$$= \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \cdots\right) + \frac{1}{2}\left[1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots\right) + \left(\frac{1}{2} + \frac{t}{4} + \frac{t^2}{8} + \cdots\right)$$

$$= \left[\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right] + \left[\frac{1}{2} + \frac{z+1}{4} + \frac{(z+1)^2}{8} + \cdots\right]$$

$$= \left[\frac{1}{2} + \frac{z+1}{4} + \frac{(z+1)^2}{8} + \cdots\right] + \left[\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right]$$

Example 17

Expand $-\dfrac{1}{(z-1)(z-2)}$ *in Laurent's series valid for*

(i) $|z-1|<1$ *(ii)* $1<|z|<2$ *(iii)* $|z|>2$

Solution

Let $$f(z)=-\frac{1}{(z-1)(z-2)}$$

$$=-\frac{(z-1)-(z-2)}{(z-1)(z-2)}$$

$$=-\frac{1}{z-2}+\frac{1}{z-1}$$

$f(z)$ is not analytic at $z=2$ and $z=1$.

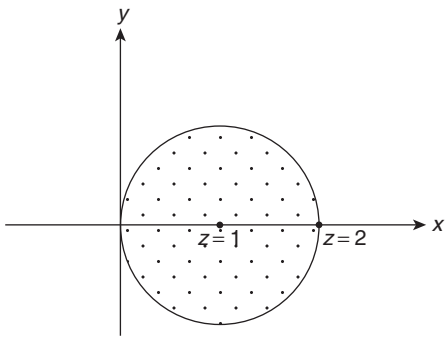


Fig. 4.30

(i) $|z-1|<1$

$f(z)$ is analytic in the region $|z-1|<1$ about $z=1$ (Fig. 4.30).

Let $z-1=t$ then $z=t+1$.

$|z-1|<$ i.e., $|t|<1$

$$f(z)=-\frac{1}{t+1-2}+\frac{1}{t}$$

$$=-\frac{1}{t-1}+\frac{1}{t}$$

$$=\frac{1}{1-t}+\frac{1}{t}$$

$$=(1-t)^{-1}+\frac{1}{t}$$

$$=(1+t+t^2+\cdots)+\frac{1}{t}$$

$$=\left[1+(z-1)+(z-1)^2+\cdots\right]+\frac{1}{z-1}$$

(ii) $1<|z|<2$

$f(z)$ is analytic in the annular region $1<|z|<2$ about $z=0$ (Fig. 4.31).

$$1<|z|,\quad \left|\frac{1}{z}\right|<1$$

$$|z|<2,\quad \left|\frac{z}{2}\right|<1$$

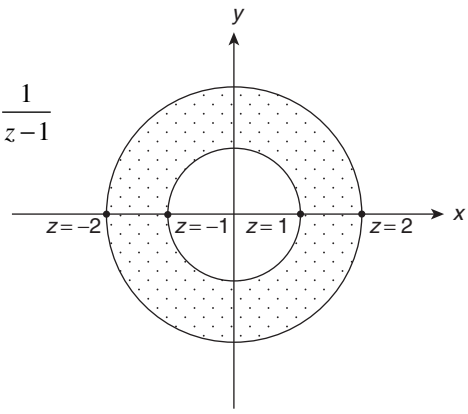


Fig. 4.31

$$f(z) = -\frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right] + \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$= \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)+\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.32).

$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$

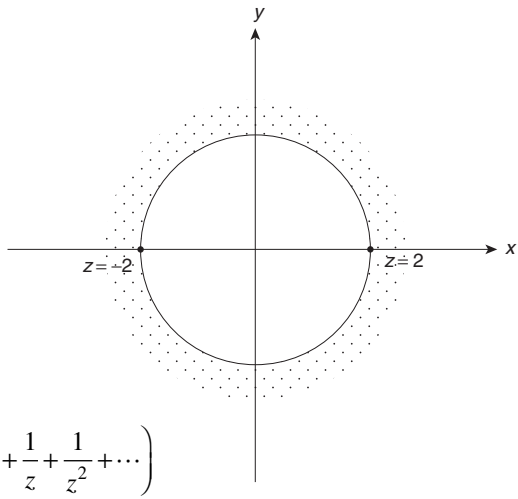


Fig. 4.32

$$f(z) = -\frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right] + \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= -\left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right)+\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= -\frac{1}{z^2}+\frac{3}{z^3}+\cdots$$

Example 18

Expand $f(z) = \dfrac{z}{(z-1)(z-3)}$ *as Laurent's series valid in the following regions:*

(i) $1 < |z| < 3$ *(ii)* $0 < |z-1| < 2$ *(iii)* $|z| > 3$. **[Winter 2012]**

Solution

$$f(z)=\frac{z}{(z-1)(z-3)}$$

$$=\frac{A}{z-1}+\frac{B}{z-3}$$

$$z=A(z-3)+B(z-1)$$

Putting $z=1$,

$$1=A(1-3)$$

$$A=-\frac{1}{2}$$

Putting $z=3$,

$$3=B(3-1)$$

$$B=\frac{3}{2}$$

$$\therefore\quad f(z)=-\frac{1}{2}\frac{1}{z-1}+\frac{3}{2}\frac{1}{z-3}$$

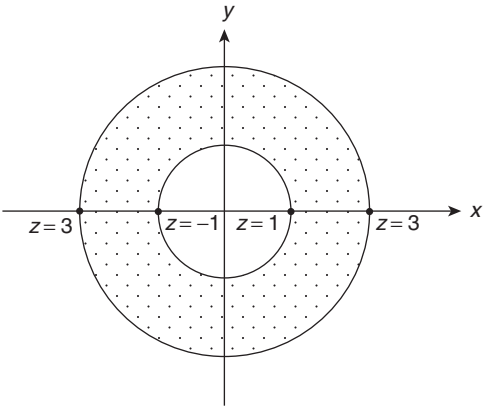


Fig. 4.33

$f(z)$ is not analytic at $z = 1$ and $z = 3$.

(i) $1<|z|<3$

$f(z)$ is analytic in the annular region $1<|z|<3$ about $z = 0$ (Fig. 4.33).

$$1<|z|,\qquad \left|\frac{1}{z}\right|<1$$

$$|z|<3,\qquad \left|\frac{z}{3}\right|<1$$

$$f(z)=-\frac{1}{2}\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{3}{2}\frac{1}{(-3)\left(1-\frac{z}{3}\right)}$$

$$=-\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{2}\left(1-\frac{z}{3}\right)^{-1}$$

$$=-\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)-\frac{1}{2}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

$$=-\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)$$

$$=-\frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)-\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

(ii) $0 < |z-1| < 2$

$f(z)$ is analytic in the annular region $0 < |z-1| < 2$ about $z = 1$ (Fig. 4.34).

Let $z-1 = t$ then $z = t+1$.

$0 < |z-1| < 2$, i.e., $0 < |t| < 2$

$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$

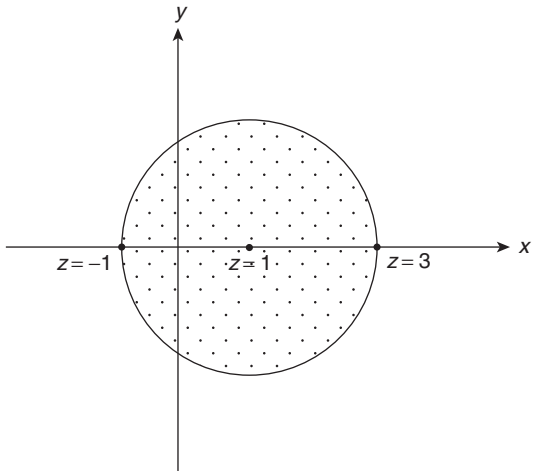


Fig. 4.34

$$f(z) = -\frac{1}{2}\frac{1}{t} + \frac{3}{2}\frac{1}{t+1-3}$$

$$= -\frac{1}{2t} + \frac{3}{2}\frac{1}{t-2}$$

$$= -\frac{1}{2t} + \frac{3}{2}\frac{1}{(-2)\left(1-\frac{t}{2}\right)}$$

$$= -\frac{1}{2t} - \frac{3}{4}\left(1-\frac{t}{2}\right)^{-1}$$

$$= -\frac{1}{2t} - \frac{3}{4}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right]$$

$$= -\frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right] - \frac{1}{2(z-1)}$$

(iii) $|z| > 3$

$f(z)$ is analytic in the region $|z| > 3$ about $z = 0$ (Fig. 4.35).

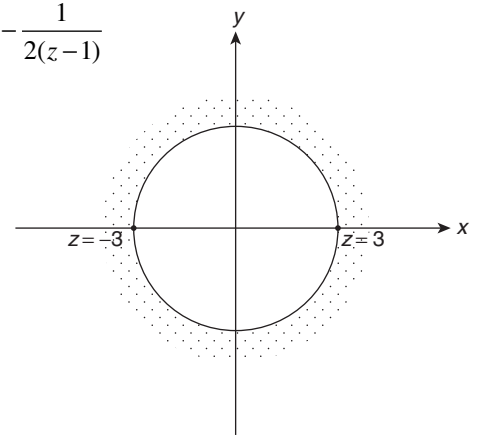


Fig. 4.35

$$|z| > 3, \left|\frac{1}{z}\right| < \frac{1}{3} < 1$$

$$f(z) = -\frac{1}{2z\left(1-\frac{1}{z}\right)} + \frac{3}{2}\frac{1}{z\left(1-\frac{3}{z}\right)}$$

$$= -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} + \frac{3}{2z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{3}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\cdots\right]$$

$$= -\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)+\frac{3}{2}\left(\frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\cdots\right)$$
$$= \frac{1}{z}+\frac{4}{z^2}+\frac{13}{z^3}+\cdots$$

EXERCISE 4.2

Expand the following functions in Laurent's series:

1. $f(z)=\dfrac{z-1}{(z+2)(z+3)}, \qquad 2<|z|<3$

$$\left[\textbf{Ans.:}\ -\frac{3}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}\cdots\right]+\frac{4}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}\cdots\right]\right]$$

2. $f(z)=\dfrac{1}{z(z-1)}$ *f* or $0<|z|<1$ and $0<|z-1|<1$ **[Winter 2013]**

$$\left[\textbf{Ans.:}\ -\frac{1}{z}(1+z+z^2+\cdots) \text{ and } \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-\cdots\right]\right]$$

3. $f(z)=\dfrac{z-1}{z^2}$ in $|z-1|>1$

$$\left[\textbf{Ans.}:\ \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots\right]$$

4. $f(z)=\dfrac{z+3}{z(z^2-z-2)}$ in $1<|z|<2$

$$\left[\textbf{Ans.:}\ -\frac{5}{12}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)-\frac{3}{2z}+\frac{2}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\cdots\right)\right]$$

5. $f(z)=\dfrac{1}{z(1-z)^2}$ in the region $0<|z|<1$ and $0<|z-1|<1$

$$\left[\textbf{Ans.:}\ (\text{i})\frac{1}{z}+2+3z+4z^2+\cdots\ (\text{ii})\ \frac{1}{(z-1)^2}-\frac{1}{z-1}+1-(z-1)+(z-1)^2-\cdots\right]$$

6. $f(z)=\dfrac{1}{z(1-z)^2}$ in the region $|z|>1$

$$\left[\textbf{Ans.:}\ \frac{1}{z^3}+\frac{2}{z^4}+\frac{3}{z^5}+\cdots\right]$$

7. $f(z)=\dfrac{1}{z^3-3z^2+2z}$ about $z=0$ for (i) $1<|z|<2$ (ii) $|z|>2$

$$\left[\textbf{Ans.:}\ (i)\ -\frac{1}{4}-\frac{1}{8}z-\frac{1}{16}z^2-\cdots-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots\ (ii)\ \frac{1}{z^3}+\frac{3}{z^4}+\frac{7}{z^5}+\cdots\right]$$

8. $f(z)=\dfrac{z-1}{z^2-2z-3}f$ or (i) $1<|z|<3$ (ii) $|z|>3$

$$\left[\textbf{Ans.:}\ (i)\ \frac{1}{2}\left(-\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\cdots-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\ (ii)\ \frac{1}{z}+\frac{1}{z^2}+\frac{5}{z^3}+\cdots\right]$$

9. $f(z)=\dfrac{1}{z^2(z-2)};0<|z|<2$

$$\left[\textbf{Ans.:}-\frac{1}{2z^2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)\right]$$

10. $f(z)=\dfrac{1+2z}{z+z^2};\ 0<|z|<1$

$$\left[\textbf{Ans.:}\ \frac{1}{z}(1+z+z^2+\cdots)\right]$$

11. $f(z)=\dfrac{z}{(z+1)(z+2)};\ 0<|z+2|<1$

$$\left[\textbf{Ans.:}\ \frac{2}{z+2}+1+(z+2)+(z+2)^2+\cdots\right]$$

12. $f(z)=\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in (i) $1<|z|<4$ (ii) $|z|>4$

$$\left[\textbf{Ans.:}\ (i)\ 1-\left(\frac{1}{z}-\frac{1}{z^2}+\cdots\right)-\left(1-\frac{z}{4}+\frac{z^2}{16}-\cdots\right)\right.$$
$$\left.(ii)\ 1-\left(\frac{1}{z}-\frac{1}{z^2}+\cdots\right)-4\left(\frac{1}{z}-\frac{4}{z^2}+\cdots\right)\right]$$

4.7 SINGULAR POINTS

A point at which the function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Types of Singularities

1. Isolated Singularity

A singular point $z = a$ is known as an isolated singularity if there is no other singularity within a small circle surrounding the point $z = a$; otherwise it is called a non-isolated singularity.

For example, $f(z) = \dfrac{z^2}{z-2}$ has isolated singularity at $z = 2$.

2. Removable Singularity

An isolated singular point $z = a$ is known as a removable singularity if $\lim\limits_{z\to a} f(z)$ exists or there is no negative power term of $(z - a)$ in Laurent's series of $f(z)$, i.e., $b_n = 0$.

For example, $f(z) = \dfrac{\sin z}{z}$ **[Summer 2013]**

$z = 0$ is a singularity of $f(z)$.

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

$\therefore$ $f(z)$ has a removable singularity at $z = 0$.

3. Essential Singularity

A singular point $z = a$ is known as an essential singularity if the number of negative power terms of $(z - a)$ in Laurent's series is infinite.

For example, $f(z) = e^{\frac{1}{z-2}}$

$$= 1 + \frac{1}{z-2} + \frac{1}{2!}\frac{1}{(z-2)^2} + \cdots$$

Since the number of negative power terms of $(z - 2)$ is infinite, $z = 2$ is an essential singularity.

4. Pole of Order *m*

An isolated singularity $z = a$ is known as a pole of order m if in Laurent's series, all the negative powers of $(z - a)$ after the m^{th} power are zero, i.e., highest power of $\dfrac{1}{z-a}$ is m.

For example, $$f(z)=\frac{1}{(z-1)(z-3)^2}$$

$z = 1$ is a pole of order 1 and $z = 3$ is a pole of order 2.

Note A pole of order one is also known as a simple pole.

Meromorphic Function A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example 1

Define the singular point of the function and state whether the function $f(z)=\frac{z^2+1}{(z+2)(z^2+2z+2)}$ *is analytic?* **[Winter 2014]**

Solution

A point at which the function $f(z)$ is not analytic is known as a singular point or a singularity of the function.

$f(z)$ is analytic everywhere except at the points given by

$$(z+2)(z^2+2z+2)=0$$

$$z+2=0, \quad z^2+2z+2=0$$

$$z=-2, \quad z=\frac{-2\pm\sqrt{4-8}}{2}$$

$$=\frac{-2\pm 2i}{2}$$

$$=-1\pm i$$

The singular points are $z = –2$ and $z = –1 \pm i$.

Hence, z is analytic everywhere except at $z = –2$ and $z = –1 \pm i$.

Example 2

Identify the type of singularities of the function $f(z)=e^{\frac{1}{z-1}}$.

Solution

$$f(z)=e^{\frac{1}{z-1}}$$

$$=1+\frac{1}{(z-1)}+\frac{1}{2!(z-1)^2}+\cdots\infty$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z=1$ is an essential singularity.

Example 3

Classify the poles of $f(z)=\dfrac{1}{z^2-z^6}$.

Solution

$$f(z)=\frac{1}{z^2(1-z^4)}$$

The poles are given by

$$z^2(1-z^4)=0$$
$$z^2(1+z^2)(1-z^2)=0$$
$$z=0,0,\pm 1,\pm i$$

$z=0$ is a pole of order 2 and $z=\pm 1,\pm i$ are poles of order one or simple poles.

Example 4

Find the residue at $z=0$ *of* $f(z)=z\cos\dfrac{1}{z}$.

Solution

$$f(z)=z\cos\frac{1}{z}$$
$$=z\left[1-\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^4}{4!}-\cdots\right]$$
$$=z-\frac{1}{2z}+\frac{1}{24z^3}-\cdots$$

Res $[f(z); z=0]$ = coefficient of $\dfrac{1}{z}$

$$=-\frac{1}{2}$$

Example 5

Find the pole and its order of the function $f(z)=\dfrac{\sin z}{z^4}$. **[Winter 2013]**

Solution

$$\begin{aligned} f(z) &= \frac{\sin z}{z^4} \\ &= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} - \frac{z^3}{5040} + \cdots \end{aligned}$$

Since the highest power of $\frac{1}{z}$ is 3, $z = 0$ is a pole of order 3.

Example 6

Expand $f(z) = \dfrac{z - \sin z}{z^2}$ *at* $z = 0$, *classify the singular point* $z = 0$.

[Summer 2015]

Solution

$$\begin{aligned} f(z) &= \frac{z - \sin z}{z^2} \\ &= \frac{1}{z^2}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right] \\ &= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} + \cdots \\ \lim_{z\to 0} f(z) &= \lim_{z\to 0}\left(\frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots\right) \\ &= 0 \end{aligned}$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 7

What is the nature of the singularity at $z = 0$ *of the function* $f(z) = \dfrac{\sin z - z}{z^3}$?

Solution

$$f(z) = \frac{\sin z - z}{z^3}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to 0} \frac{\cos z - 1}{3z^2} \qquad [\text{Using L'Hospital's rule}]$$

$$= \lim_{z\to 0} \frac{-\sin z}{6z} \qquad [\text{Using L'Hospital's rule}]$$

$$= -\frac{1}{6} \qquad \left[\because \lim_{z\to 0} \frac{\sin z}{z} = 1\right]$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 8

Identify the type of singularity of $f(z) = \dfrac{2 - e^z}{z^3}$.

Solution

$$f(z) = \frac{2 - e^z}{z^3}$$

$$= \frac{1}{z^3}\left[2 - \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots\right)\right]$$

$$= \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{6} - \frac{z}{24} - \frac{z^2}{120} - \cdots$$

Since the highest power of $\dfrac{1}{z}$ is 3, $z = 0$ is a pole of order 3.

Example 9

Write the singularity of the function $\dfrac{1}{1 - e^z}$.

Solution

Let $$f(z) = \frac{1}{1 - e^z}$$

$$= \frac{1}{1 - \left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots\right)}$$

$$= \frac{1}{-\left(z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right)}$$

$$= \frac{1}{-z\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \ldots\right)}$$

$$= -\frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \cdots\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \cdots\right)\right]^{-1}$$

$$= -\frac{1}{z}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \cdots\right) + \frac{\left(\frac{z}{2} + \frac{z^2}{6} + \cdots\right)^2}{2!} - \cdots\right]$$

$$= -\frac{1}{z} + \frac{1}{2} + \frac{z}{24} - \frac{z^2}{24} + \cdots$$

Since the highest power of $\frac{1}{z}$ is 1, $z = 0$ is a pole of order 1.

Example 10

Identify the type of singularity of $f(z) = e^{e^{\frac{1}{z^2}}}$.

Solution

$$f(z) = e^{e^{\frac{1}{z^2}}}$$

$$= e^{\left(1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \cdots\right)}$$

$$= e^1 e^{\left(\frac{1}{z^2} + \frac{1}{2!z^4} + \cdots\right)}$$

$$= e\left[1 + \left(\frac{1}{z^2} + \frac{1}{2z^4} + \cdots\right) + \frac{\left(\frac{1}{z^2} + \frac{1}{2z^4} + \cdots\right)^2}{2!} + \cdots\right]$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

Example 11

Find the singularities of the function $f(z) = \dfrac{\tan z}{z}$.

Solution

$$f(z) = \frac{\tan z}{z}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\tan z}{z}$$

$$= \lim_{z\to 0} \frac{z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots}{z}$$

$$= \lim_{z\to 0} \left(1 + \frac{z^2}{3} + \frac{2z^4}{15} + \cdots\right)$$

$$= 1$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 12

Find the singularities of the function $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$.

Solution

$$f(z) = \frac{\cot \pi z}{(z-a)^3}$$

$$= \frac{\cos \pi z}{(z-a)^3 \sin \pi z}$$

Singularities are at $(z - a)^3 \sin \pi z = 0$.

$$z - a = 0, \qquad \sin \pi z = 0$$

$$z = a, \qquad \pi z = n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$z = n$$

$$= 0, \pm 1, \pm 2, \pm 3, \ldots$$

Since in the neighbourhood of $z = a$, there are infinite number of singularities, $z = a$, is a non-isolated singularity.

Example 13

Identify the type of singularity of $f(z)=\dfrac{z-2}{4}\sin\left(\dfrac{1}{z-1}\right)$.

Solution

$$f(z)=\frac{z-2}{4}\sin\left(\frac{1}{z-1}\right)$$

Let $z-1=t$ then $z=t+1$.

$$f(z)=\frac{t+1-2}{4}\sin\left(\frac{1}{t}\right)$$

$$=\frac{t-1}{4}\left[\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\cdots\right]$$

$$=\frac{1}{4}\left[\left(1-\frac{1}{3!t^2}+\frac{1}{5!t^4}-\cdots\right)-\left(\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\cdots\right)\right]$$

$$=\frac{1}{4}\left[1-\frac{1}{t}-\frac{1}{3!t^2}+\frac{1}{3!t^3}+\cdots\right]$$

$$=\frac{1}{4}\left[1-\frac{1}{z-1}-\frac{1}{3!(z-1)^2}+\frac{1}{3!(z-1)^3}+\cdots\right]$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z=1$ is an essential singularity.

Example 14

Find the singular points of $f(z)=\dfrac{1}{\sin\dfrac{1}{z-a}}$. *State their nature.*

Solution

$$f(z)=\frac{1}{\sin\dfrac{1}{z-a}}$$

Singularity is at $\sin\dfrac{1}{z-a}=0$.

$$\frac{1}{z-a} = n\pi, \quad n = \pm 1, \pm 2, \ldots$$

$$z - a = \frac{1}{n\pi}$$

$$z = a + \frac{1}{n\pi}$$

Since in the neighbourhood of $z = a$, there are infinite number of singularities, $z = a$ is non-isolated singularity.

Example 15

Find the singularities of the function $f(z) = \dfrac{z^2 + 1}{e^z}$.

Solution

$$f(z) = \frac{z^2 + 1}{e^z}$$

$$= (z^2 + 1)e^{-z}$$

$$= (z^2 + 1)\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)$$

$f(z)$ is analytic everywhere.
Hence, there is no singularity.

EXERCISE 4.3

Determine the nature of singularities of the following functions:

1. $z^2 e^{\frac{1}{z}}$

[**Ans.:** $z = 0$ is an essential singularity.]

2. $\dfrac{\sin 3z}{z}$

[**Ans.:** $z = 0$ is a removable singularity.]

3. $e^{-\frac{1}{z^2}}$

[**Ans.:** No singularity]

4. $\dfrac{2 - e^z}{z^3}$

[**Ans.:** $z = 0$ is a pole of order 3.]

5. $z^3 e^{\frac{1}{z-1}}$

[**Ans.:** $z = 1$ is an essential singularity.]

6. $\dfrac{1}{(z-5)^2(z^2-4)}$

[**Ans.:** $z = 2, -2$ are simple poles and $z = 5$ is a pole of order 2.]

7. $\sin\dfrac{1}{z-2}$

[**Ans.:** $z = 2$ is an essential singularity.]

8. $ze^{\frac{1}{z^2}}$

[**Ans.:** $z = 0$ is an essential singularity.]

4.8 RESIDUES

The coefficient of $\dfrac{1}{z-a}$ in the Laurent series expansion of $f(z)$ is known as the residue of $f(z)$ at $z = a$.

The Laurent series expansion about $z = a$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$\text{Residue (at } z = a) = \text{Coefficient of } \frac{1}{z-a}$$

$$= b_1$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-1+1}}\,dz \quad \left[\because \quad b_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{-n+1}}\,dz\right]$$

$$= \frac{1}{2\pi i}\int_C f(z)\,dz$$

Evaluation of Residues

1. Residue at a Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$ then

$$\text{Res}\left[f(z); z = a\right] = \lim_{z \to a} (z-a) f(z)$$

(ii) If $f(z)$ is of the form $f(z)=\dfrac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic at $z=a$.

If $h(a)=0$ but $h'(a)\neq 0$ then $z=a$ is a simple pole.

If $h(a)=0$ but $g(a)\neq 0$ then

$$\text{Res}\left[f(z);\, z=a\right]=\lim_{z\to a}\frac{g(z)}{h'(z)}$$

2. Residue at a Pole of Order *m*

If $f(z)$ has a pole of order m at $z=a$ then

$$\text{Res}\left[f(z);z=a\right]=\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m\, f(z)\right]$$

Example 1

If $f(z)=\dfrac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\cdots\right]$, *find the residue of* $f(z)$ *at* $z=1$.

Solution

$$\text{Res}\left[f(z);z=1\right]=\text{Coefficient of }\frac{1}{z-1}\text{ in the Laurent's series of }f(z)\text{ about }z=1$$

$$=-1$$

Example 2

Find the residue of $f(z)=z^2\sin\left(\dfrac{1}{z}\right)$ at $z=0$.

Solution

$$f(z)=z^2\sin\left(\frac{1}{z}\right)$$

$$=z^2\left[\frac{1}{z}-\frac{\left(\frac{1}{z}\right)^3}{3!}+\cdots\right]$$

$$=z-\frac{1}{6z}+\cdots$$

$$\text{Res}\left[f(z);\, z=0\right] = \text{Coefficient of } \frac{1}{z}$$

$$= -\frac{1}{6}$$

Example 3

Obtain the residue of $f(z) = \dfrac{z-3}{(z+1)(z+2)}$ *at its poles.*

Solution

$$f(z) = \frac{z-3}{(z+1)(z+2)}$$

The poles are given by

$$(z+1)(z+2) = 0$$

$$z = -1, -2$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z=-1\right] = \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to -1}\frac{z-3}{z+2}$$

$$= \frac{-1-3}{-1+2}$$

$$= -4$$

$z = -2$ is a simple pole.

$$\text{Res}\left[f(z); z=-2\right] = \lim_{z\to -2}(z+2)f(z)$$

$$= \lim_{z\to -2}\frac{z-3}{z+1}$$

$$= \frac{-2-3}{-2+1}$$

$$= 5$$

Example 4

Test for singularity of $\dfrac{1}{z^2+1}$ *and, hence, find the corresponding residues.*

Solution

Let $$f(z) = \frac{1}{z^2+1}$$

$$= \frac{1}{(z+i)(z-i)}$$

$z = -i$ and $z = i$ are simple poles.

$$\text{Res}\left[f(z); z=-i\right] = \lim_{z\to -i}(z+i)f(z)$$

$$= \lim_{z\to -i}\frac{1}{z-i}$$

$$= \frac{1}{-i-i}$$

$$= -\frac{1}{2i}$$

$$\text{Res}\left[f(z); z=i\right] = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}\frac{1}{z+i}$$

$$= \frac{1}{i+i}$$

$$= \frac{1}{2i}$$

Example 5

Find the residue of $f(z) = \dfrac{z}{z^2+1}$ about $z = i$.

Solution

$$f(z) = \frac{z}{z^2+1}$$

The poles are given by

$$z^2 + 1 = 0$$

$$z^2 = -1$$

$$z = \pm i$$

$$\therefore \quad f(z) = \frac{z}{(z+i)(z-i)}$$

$z = i$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z = i\right] &= \lim_{z \to i} (z - i) f(z) \\ &= \lim_{z \to i} \frac{z}{z + i} \\ &= \frac{i}{i + i} \\ &= \frac{1}{2}\end{aligned}$$

Example 6

Find the residue of the function $f(z) = \dfrac{4}{z^3(z-2)}$ *at a simple pole.*

Solution

$$f(z) = \frac{4}{z^3(z-2)}$$

The poles are given by

$$\begin{aligned}z^3(z-2) &= 0 \\ z &= 0, 0, 0, 2\end{aligned}$$

$z = 2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z = 2\right] &= \lim_{z \to 2} (z - 2) f(z) \\ &= \lim_{z \to 2} \frac{4}{z^3} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

Example 7

Find the residue of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *at* $z = -2.$

Solution

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

$z = -2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=-2\right] &= \lim_{z\to -2}(z+2)f(z)\\ &= \lim_{z\to -2}\frac{z^2}{(z-1)^2}\\ &= \frac{(-2)^2}{(-2-1)^2}\\ &= \frac{4}{9}\end{aligned}$$

Example 8

Find the residue of $\tan z$ at $z = \dfrac{\pi}{2}$.

Solution

$$f(z) = \tan z$$

$z = \dfrac{\pi}{2}$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=\frac{\pi}{2}\right] &= \lim_{z\to \frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\tan z\\ &= \lim_{z\to \frac{\pi}{2}}\frac{z-\frac{\pi}{2}}{\cot z} \qquad \left[\frac{0}{0}\text{form}\right]\\ &= \lim_{z\to \frac{\pi}{2}}\frac{1}{-\text{cosec}^2 z} \qquad \left[\text{Using L'Hospital's rule}\right]\\ &= -\frac{1}{\text{cosec}^2\frac{\pi}{2}}\\ &= -1\end{aligned}$$

Example 9

Find the residue of $f(z)=\dfrac{z}{(z-1)^2}$ *at its pole.*

Solution

$$f(z)=\frac{z}{(z-1)^2}$$

$z = 1$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);\, z=1\right] &= \frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]\\ &= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}(z)\\ &= \lim_{z\to 1}(1)\\ &= 1\end{aligned}$$

Example 10

Find the residue at the pole of the function $f(z)=\dfrac{2z+3}{(z+2)^2}$.

Solution

$$f(z)=\frac{2z+3}{(z+2)^2}$$

$z = 2$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);\, z=-2\right] &= \frac{1}{(2-1)!}\lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+2)^2 f(z)\right]\\ &= \lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}(2z+3)\\ &= \lim_{z\to -2}(2)\\ &= 2\end{aligned}$$

Example 11

Calculate the residue of $f(z)=\dfrac{e^{2z}}{(z+1)^2}$ *at its pole.*

Solution

$$f(z)=\frac{e^{2z}}{(z+1)^2}$$

The poles are given by

$$(z+1)^2=0$$
$$z+1=0$$
$$z=-1$$

$z=-1$ is a pole of order 2.

$$\text{Res}\left[f(z);z=-1\right]=\frac{1}{(2-1)!}\lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]$$
$$=\frac{1}{1!}\lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[e^{2z}\right]$$
$$=\lim_{z\to-1}2e^{2z}$$
$$=2e^{-2}$$

Example 12

Find the residue of $f(z)=\dfrac{1-e^z}{z^3}$ *at* $z=0$.

Solution

$$f(z)=\frac{1-e^z}{z^3}$$

$z=0$ is a pole of order 3.

$$\text{Res}\left[f(z);z=0\right]=\frac{1}{(3-1)!}\lim_{z\to0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(z-0)^3 f(z)\right]$$
$$=\frac{1}{2!}\lim_{z\to0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(1-e^z)$$
$$=\frac{1}{2}\lim_{z\to0}\frac{\mathrm{d}}{\mathrm{d}z}(-e^z)$$
$$=\frac{1}{2}\lim_{z\to0}(-e^z)$$
$$=\frac{1}{2}(-e^0)$$
$$=-\frac{1}{2}$$

Example 13

Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at $z=0$.

Solution

$$f(z)=\frac{1-e^{2z}}{z^4}$$

$z = 0$ is a pole of order 4.

$$\begin{aligned}\operatorname{Res}\left[f(z);z=0\right]&=\frac{1}{(4-1)!}\lim_{z\to 0}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[(z-0)^4 f(z)\right]\\&=\frac{1}{3!}\lim_{z\to 0}\frac{\mathrm{d}^3}{\mathrm{d}z^3}(1-e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(-2e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}(-4e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}(-8e^{2z})\\&=\frac{1}{6}(-8e^{0})\\&=-\frac{4}{3}\end{aligned}$$

Example 14

Find the residue of $f(z)=\dfrac{50z}{(z+4)(z-1)^2}$ *at* $z=1$.

Solution

$$f(z)=\frac{50z}{(z+4)(z-1)^2}$$

$z = 1$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);z=1\right]&=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]\\&=\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{50z}{z+4}\right]\end{aligned}$$

$$= \lim_{z \to 1} \left[\frac{(z+4)50 - 50z}{(z+4)^2} \right]$$
$$= \frac{5(50) - 50}{(5)^2}$$
$$= 8$$

Example 15

Find the residues of the function $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ *at each of its poles.*

[Winter 2014]

Solution

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

The poles are given by

$$(z+1)^2(z^2+4) = 0$$
$$z = -1, -1, \pm 2i$$
$$\therefore \quad f(z) = \frac{z^2 - 2z}{(z+1)^2(z+2i)(z-2i)}$$

$z = -1$ is a pole of order 2.

$$\text{Res}\left[f(z); z = -1\right] = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]$$
$$= \lim_{z \to -1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2 - 2z}{z^2+4}\right]$$
$$= \lim_{z \to -1} \left[\frac{(2z-2)(z^2+4) - (z^2-2z)\cdot 2z}{(z^2+4)^2}\right]$$
$$= \lim_{z \to -1} \left[\frac{2z^3 + 8z - 2z^2 - 8 - 2z^3 + 4z^2}{(z^2+4)^2}\right]$$
$$= \lim_{z \to -1} \left[\frac{2z^2 + 8z - 8}{(z^2+4)^2}\right]$$
$$= \frac{2 - 8 - 8}{25}$$
$$= -\frac{14}{25}$$

$z = 2i$ is a simple pole,

$$\text{Res}\left[f(z); z = 2i\right] = \lim_{z \to 2i}\left[(z-2i)\, f(z)\right]$$

$$= \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2 (z+2i)}$$

$$= \frac{(2i)^2 - 2(2i)}{(2i+1)^2 4i}$$

$$= \frac{-4-4i}{(-4+1+4i)\, 4i}$$

$$= \frac{-(1+i)}{i(4i-3)}$$

$$= \frac{(i-1)}{(4i-3)} \times \frac{(4i+3)}{(4i+3)} \qquad \left[\because \quad \frac{1}{i} = -i\right]$$

$$= \frac{(4i^2 + 3i - 4i - 3)}{-25}$$

$$= \frac{(7+i)}{25}$$

$z = -2i$ is a simple pole.

$$\text{Res}\ \left[f(z); z = -2i\right] = \lim_{z \to -2i}\left[(z+2i) f(z)\right]$$

$$= \lim_{z \to -2i} \frac{z^2 - 2z}{(z+1)^2 (z-2i)}$$

$$= \frac{(-2i)^2 + 4i}{(-2i+1)^2 (-4i)}$$

$$= \frac{-4+4i}{(-4+1-4i)(-4i)}$$

$$= \frac{-(1+i)}{-(3+4i)} \times \frac{(3-4i)}{(3-4i)}$$

$$= \frac{3-4i+3i-4i^2}{25}$$

$$= \frac{7-i}{25}$$

Example 16

Determine poles and their orders for the function $\dfrac{z+2}{(z+1)^2(z-2)}$. *Find the residues at the poles.*

Solution

$$f(z)=\frac{z+2}{(z+1)^2(z-2)}$$

The poles are given by

$$(z+1)^2(z-2)=0$$
$$z=-1,\,-1,\,2$$

$z=-1$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);\,z=-1\right]&=\frac{1}{(2-1)!}\lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]\\&=\lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z+2}{z-2}\right]\\&=\lim_{z\to-1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right]\\&=\lim_{z\to-1}\left[\frac{-4}{(z-2)^2}\right]\\&=\frac{-4}{(-1-2)^2}\\&=-\frac{4}{9}\end{aligned}$$

$z=2$ is a simple pole.

$$\begin{aligned}\operatorname{Res}\left[f(z);\,z=2\right]&=\lim_{z\to2}(z-2)f(z)\\&=\lim_{z\to2}\frac{z+2}{(z+1)^2}\\&=\frac{2+2}{(2+1)^2}\\&=\frac{4}{9}\end{aligned}$$

Example 17

Obtain the Laurent's expansion of the function $\dfrac{e^z}{(z-1)^2}$ *in the neighbourhood of its singular point. Hence, find the residue at that point.*

Solution

Let $$f(z)=\frac{e^z}{(z-1)^2}$$

$f(z)$ is not analytic at $z = 1$. But it is analytic in the region $|z-1|>0$ about $z = 1$.
Let $z - 1 = t$, then $z = t + 1$.

$$\begin{aligned}
f(z)&=\frac{e^z}{(z-1)^2}\\
&=\frac{e^{t+1}}{t^2}\\
&=\frac{e}{t^2}(e^t)\\
&=\frac{e}{t^2}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)\\
&=\frac{e}{t^2}+\frac{e}{t}+\frac{e}{2}+e\frac{t}{6}+\cdots\\
&=\frac{e}{(z-1)^2}+\frac{e}{z-1}+\frac{e}{2}+e\frac{(z-1)}{6}+\cdots
\end{aligned}$$

$$\begin{aligned}
\text{Res}\left[f(z);z=1\right]&=\text{Coefficient of }\frac{1}{z-1}\\
&=e
\end{aligned}$$

Example 18

Find the residues of $f(z)=\dfrac{z^2}{(z-1)^2(z+2)^2}$ *at its isolated singularities using Laurent's series expansion. Also, state the valid region.*

Solution

$$f(z) = \frac{z^2}{(z-1)^2(z+2)^2}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{z+2} + \frac{D}{(z+2)^2}$$

$$z^2 = A(z-1)(z+2)^2 + B(z+2)^2 + C(z+2)(z-1)^2 + D(z-1)^2$$

Putting $z = 1$,

$$1 = 9B$$

$$B = \frac{1}{9}$$

Putting $z = -2$,

$$4 = 9D$$

$$D = \frac{4}{9}$$

Equating the coefficient of z^3,

$$0 = A + C$$

$$A = -C$$

Putting $z = 0$,

$$0 = -4A + 4B + 2C + D$$

$$0 = -4A + \frac{4}{9} + 2C + \frac{4}{9}$$

$$4A - 2C = \frac{8}{9}$$

$$-4C - 2C = \frac{8}{9}$$

$$-6C = \frac{8}{9}$$

$$C = -\frac{4}{27}$$

$$A = \frac{4}{27}$$

$$\therefore \quad f(z) = \frac{4}{27}\frac{1}{z-1} + \frac{1}{9}\frac{1}{(z-1)^2} - \frac{4}{27}\frac{1}{z+2} + \frac{4}{9}\frac{1}{(z+2)^2}$$

$z = 1$ and $z = -2$ are poles of order 2. Since poles are isolated singularities, $z = 1$ and $z = -2$ are isolated singularities.

(i) For Laurent's series about $z = 1$,

Let $z - 1 = t$ then $z = t + 1$

$$f(z) = \frac{4}{27}\frac{1}{t} + \frac{1}{9}\frac{1}{t^2} - \frac{4}{27}\frac{1}{(t+1+2)} + \frac{4}{9}\frac{1}{(t+1+2)^2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{27}\frac{1}{3\left(1+\frac{t}{3}\right)} + \frac{4}{9}\frac{1}{3^2\left(1+\frac{t}{3}\right)^2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{81}\left(1+\frac{t}{3}\right)^{-1} + \frac{4}{81}\left(1+\frac{t}{3}\right)^{-2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{81}\left(1 - \frac{t}{3} + \frac{t^2}{9} - \cdots\right) + \frac{4}{81}\left(1 - \frac{2t}{3} + \frac{3t^2}{9} - \cdots\right)$$

$$= \frac{4}{27(z-1)} + \frac{1}{9(z-1)^2} - \frac{4}{81}\left[1 - \frac{z-1}{3} + \frac{(z-1)^2}{9} - \cdots\right]$$

$$+ \frac{4}{81}\left[1 - \frac{2(z-1)}{3} + \frac{3(z-1)^2}{9} - \cdots\right]$$

$$\text{Res}\,[f(z); z = 1] = \text{Coefficient of } \frac{1}{z-1}$$

$$= \frac{4}{27}$$

(ii) For Laurent's series about $z = -2$,

Let $z + 2 = t$ then $z = t - 2$.

$$f(z) = \frac{4}{27}\frac{1}{(t-2-1)} + \frac{1}{9}\frac{1}{(t-2-1)^2} - \frac{4}{27}\frac{1}{t} + \frac{4}{9}\frac{1}{t^2}$$

$$= \frac{4}{27}\left[\frac{1}{-3\left(1-\frac{t}{3}\right)}\right] + \frac{1}{9}\left[\frac{1}{(-3)^2\left(1-\frac{t}{3}\right)^2}\right] - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left(1-\frac{t}{3}\right)^{-1} + \frac{1}{81}\left(1-\frac{t}{3}\right)^{-2} - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left(1 + \frac{t}{3} + \frac{t^2}{9} + \cdots\right) + \frac{1}{81}\left(1 + \frac{2t}{3} + \frac{3t^2}{9} + \cdots\right) - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left[1+\frac{z+2}{3}+\frac{(z+2)^2}{9}+\cdots\right]+\frac{1}{81}\left[1+\frac{2(z+2)}{3}+\frac{(z+2)^2}{3}+\cdots\right]$$
$$-\frac{4}{27(z+2)}+\frac{4}{9(z+2)^2}$$

$$\text{Res}\,[f(z);\, z=-2] = \text{Coefficient of } \frac{1}{z+2}$$
$$= -\frac{4}{27}$$

EXERCISE 4.4

Find the residues of $f(z)$ at the singular points:

1. $\dfrac{z}{(z-1)^2}$

$\left[\text{Ans.: } -\dfrac{1}{2i}\right]$

2. $\dfrac{z+2}{(z+1)^2}$

[Ans.: 1]

3. $\dfrac{z^2-z}{(z+1)^2(z^2+4)}$

$\left[\text{Ans.: } -\dfrac{11}{25}, \dfrac{11\pm 2i}{50}\right]$

4. $\dfrac{e^z}{(z-1)^3}$

$\left[\text{Ans.: } \dfrac{e}{2}\right]$

5. $\dfrac{\sin^2 z}{z^3}$

$\left[\text{Ans.: } \dfrac{1}{4}\right]$

6. $\dfrac{z+3}{z(z-1)(z+2)}$

$\left[\text{Ans. : } -\dfrac{3}{2}, \dfrac{4}{3}, \dfrac{1}{6}\right]$

7. $\dfrac{e^{2z}}{z^2+\pi^2}$

$\left[\textbf{Ans.:}\ \dfrac{1}{2\pi i}, -\dfrac{1}{2\pi i}\right]$

8. $\dfrac{z^3}{(z+a)^2}$

[Ans.: $3a^2$**]**

9. $\dfrac{z^2+1}{z(z-2)}$

$\left[\textbf{Ans.:}\ -\dfrac{1}{2}, \dfrac{5}{2}\right]$

10. $z\cos\dfrac{1}{z}$

[Ans.: -4**]**

4.9 CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points inside C then

$$\oint_C f(z)\,dz = 2\pi i \ \text{(sum of residues)}$$

where the integral is taken in the anticlockwise direction around C.

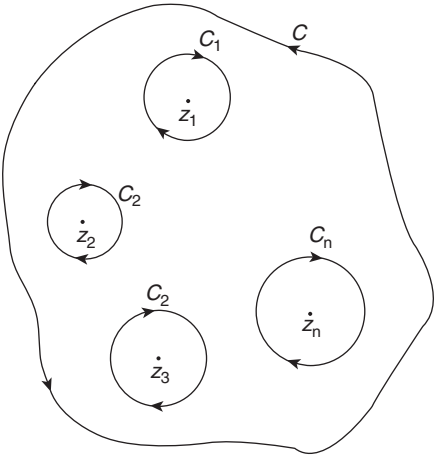


Fig. 4.36

Proof

Let $z_1, z_2, \ldots, z_n$ be finite numbers of singular points inside C (Fig. 4.36). Enclose each of the singular point in a small circle such that no other singular point lies inside this circle. The curve C along with these small circles C_1, C_2, C_3, …, C_n form a multiply connected region and $f(z)$ is analytic in this region.

By Cauchy's integral theorem for a multiply connected region,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz \\ &= 2\pi i\left[\text{Res. at } z_1\right] + 2\pi i\left[\text{Res. at } z_2\right] + \cdots + 2\pi i\left[\text{Res. at } z_n\right] \\ &= 2\pi i \ \text{(sum of residues)}\end{aligned}$$

Example 1

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$ *where C is* $|z| = 2$. **[Winter 2013]**

Solution

(i) Let $f(z) = \frac{z^2-4z+4}{z+i}$

The poles are given by $z = -i$.

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.37).

(iii) For $z = -i$, $|z| = |-i| = 1 < 2$

Hence, $z = -i$ lies inside C.

(iv) $z = -i$ is a simple pole.

$$\text{Res}\left[f(z); z=-i\right] = \lim_{z\to -i}(z+i)f(z)$$

$$= \lim_{z\to -i}(z^2-4z+4)$$

$$= i^2+4i+4$$

$$= 3+4i$$

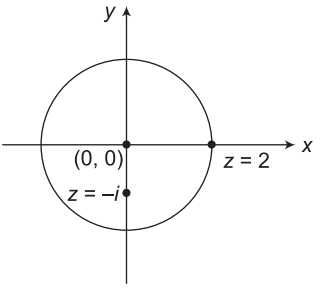


Fig. 4.37

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2-4z+4}{z+i} = 2\pi i\,(3+4i)$$

$$= 2\pi(3\text{i}-4)$$

Example 2

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$ *where C is* $|z| = 2$.

Solution

(i) $f(z) = \frac{1}{(z-1)^2(z-3)}$

The poles are given by $z = 1$, $z = 3$.

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.38).

(iii) For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3$, $|z| = |3| = 3 > 2$

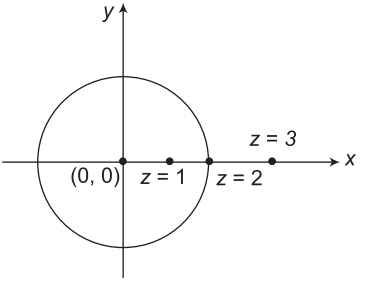


Fig. 4.38

Hence, $z = 3$ lies outside C.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}\left[f(z); z=1\right] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2\ f(z)\right]$$

$$= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{z-3}\right]$$

$$= \lim_{z\to 1}\left[-\frac{1}{(z-3)^2}\right]$$

$$= -\frac{1}{4}$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,\mathrm{d}z = 2\pi i\left(-\frac{1}{4}\right)$$

$$= -\frac{\pi i}{2}$$

Example 3

Evaluate $\int_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,\mathrm{d}z$, *where C is* $|z| = 3$.

Solution

(i) Let $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are given by $z = 1$, $z = 2$.

(ii) C is a circle $|z| = 3$ with the centre at (0, 0) and a radius of 3 (Fig. 4.39).

(iii) For $z = 1$, $|z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

(iv) $z = 1$ is a simple pole.

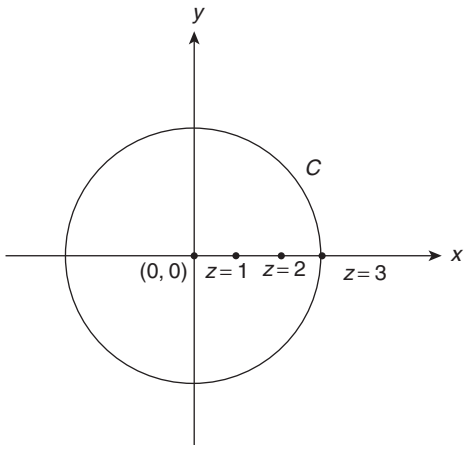


Fig. 4.39

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1}\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$= \frac{\sin \pi + \cos \pi}{-1}$$

$$= 1$$

$z = 2$ is a simple pole.

$$\text{Res}\left[f(z); z=2\right] = \lim_{z\to 2}(z-2)\, f(z)$$

$$= \lim_{z\to 2}\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1}$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i(1+1)$$

$$= 4\pi i$$

Example 4

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\, dz$, *where C is* $|z-2| = 2$. **[Summer 2013]**

Solution

(i) Let $f(z) = \frac{5z+7}{z^2+2z-3}$

$$= \frac{5z+7}{(z+3)(z-1)}$$

The poles are given by $z = -3$, $z = 1$.

(ii) C is a circle $|z-2| = 2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.40).

(iii) For $z = -3$, $|z-2| = |-3-2| = 5 > 2$
Hence, $z = -3$ lies outside C.
For $z = 1$, $|z-2| = |1-2| = 1 < 2$
Hence, $z = 1$ lies inside C.

(iv) $z = 1$ is a simple pole.

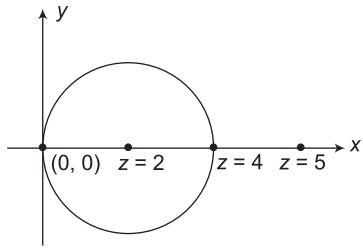


Fig. 4.40

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\,f(z)$$
$$= \lim_{z\to 1}\frac{5z+7}{z+3}$$
$$= \frac{5+7}{1+3}$$
$$= 3$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 2\pi i\,(3)$$
$$= 6\pi \text{ i}$$

Example 5

Evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\,dz$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)} = \frac{3z^2+z-1}{(z+1)(z-1)(z-3)}$

The poles are given by $z = -1$, $z = 1$, $z = 3$.

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.41).

(iii) For $z = -1$, $|z| = |-1| = 1 < 2$

Hence, $z = -1$ lies inside C.

For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3$, $|z| = |3| = 3 > 2$

Hence, $z = 3$ lies outside C.

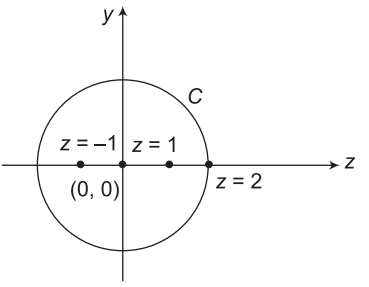


Fig. 4.41

(iv) $z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z=-1\right] = \lim_{z\to -1}(z+1)\,f(z)$$
$$= \lim_{z\to -1}\frac{3z^2+z-1}{(z-1)(z-3)}$$
$$= \frac{3(-1)^2+(-1)-1}{(-1-1)(-1-3)}$$
$$= \frac{1}{8}$$

$z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1}\frac{3z^2+z-1}{(z+1)(z-3)}$$

$$= \frac{3+1-1}{(1+1)\,(1-3)}$$

$$= -\frac{3}{4}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\mathrm{d}z = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right)$$

$$= 2\pi i\left(-\frac{5}{8}\right)$$

$$= -\frac{5\pi i}{4}$$

Example 6

Using the residue theorem, evaluate $\int_C \frac{e^z+z}{z^3-z}\mathrm{d}z$, *where* $C : |z| = \frac{\pi}{2}$.

[Summer 2015]

Solution

(i) Let $f(z) = \frac{e^z+z}{z^3-z} = \frac{e^z+z}{z(z+1)(z-1)}$

The poles are given by $z = 0$, $z = -1$, $z = 1$.

(ii) C is the circle $|z| = \frac{\pi}{2} = 1.57$ with the centre (0, 0) and a radius of $\frac{\pi}{2}$ (Fig. 4.42).

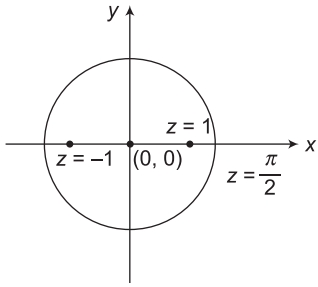


Fig. 4.42

(iii) For $z = 0$, $|z| = 0 < \frac{\pi}{2}$

Hence, $z = 0$ lies inside C.

For $z = -1$, $|z| = |-1| = 1 < \frac{\pi}{2}$

Hence, $z = -1$ lies inside C.

For $z = 1$, $|z| = |1| = 1 < \dfrac{\pi}{2}$

Hence, $z = 1$ lies inside C.

(iv) $z = 0$ is a simple pole.

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z f(z)$$

$$= \lim_{z \to 0} \frac{e^z + z}{(z+1)(z-1)}$$

$$= \frac{e^0}{-1}$$

$$= -1$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z = -1\right] = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} \frac{e^z + z}{z(z-1)}$$

$$= \frac{e^{-1} - 1}{-1(-2)}$$

$$= \frac{1 - e}{2e}$$

$z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z = 1\right] = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{e^z + z}{z(z+1)}$$

$$= \frac{e+1}{2}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^z + z}{z^3 - z}\,\mathrm{d}z = 2\pi i \left(-1 + \frac{1-e}{2e} + \frac{e+1}{2}\right)$$

$$= 2\pi i \left(\frac{-2e + 1 - e + e^2 + e}{2e}\right)$$

$$= \frac{\pi i}{e}(e^2 - 2e + 1)$$

$$= \frac{\pi i}{e}(e-1)^2$$

Example 7

Use residues to evaluate the integrals of the function $\dfrac{\exp(-z)}{z^2}$ *around the circle* $|z| = 3$ *in the positive sense.*

[Winter 2012]

Solution

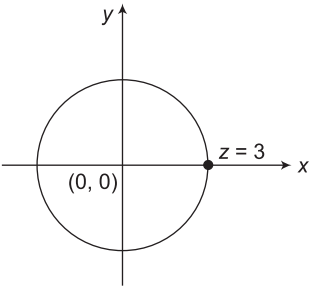


Fig. 4.43

(i) Let $f(z) = \dfrac{\exp(-z)}{z^2} = \dfrac{e^{-z}}{z^2}$

The poles are given by $z = 0$.

(ii) C is the circle $|z| = 3$ with the centre at (0, 0) and a radius of 3 (Fig. 4.43).

(iii) For $z = 0$, $|z| = 0 < 3$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}\left[f(z); z = 0\right] = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 f(z)\right]$$

$$= \lim_{z\to 0} e^{-z}$$

$$= e^0$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^{-z}}{z^2}\,\mathrm{d}z = 2\pi i\,(1)$$

$$= 2\pi i$$

Example 8

Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z-\pi i)^3}\,\mathrm{d}z$, *where C is* $|z - 2i| = 2$.

Solution

(i) Let $f(z) = \dfrac{e^{2z}}{(z-\pi i)^3}$

The poles are given by $z = \pi i$.

(ii) C is a circle $|z - 2i| = 2$ with the centre at (0, 2) and a radius of 2 (Fig. 4.44).

(iii) For $z = 2\pi$,

$$|z - 2i| = |\pi i - 2i| = \pi - 2 = 1.14 < 2$$

Hence, $z = \pi i$ lies inside C.

(iv) $z = \pi i$ is a pole of order 3.

$$\text{Res}\left[f(z); z = \pi i\right] = \frac{1}{(3-1)!}\lim_{z\to\pi i}\frac{d^2}{dz^2}\left[(z-\pi i)^3 f(z)\right]$$

$$= \frac{1}{2!}\lim_{z\to\pi i}\frac{d^2}{dz^2}(e^{2z})$$

$$= \frac{1}{2}\lim_{z\to\pi i}\frac{d}{dz}(2e^{2z})$$

$$= \frac{1}{2}\lim_{z\to\pi i} 4e^{2z}$$

$$= 2e^{2\pi i}$$

$$= 2\left(\cos 2\pi + i\sin 2\pi\right)$$

$$= 2$$

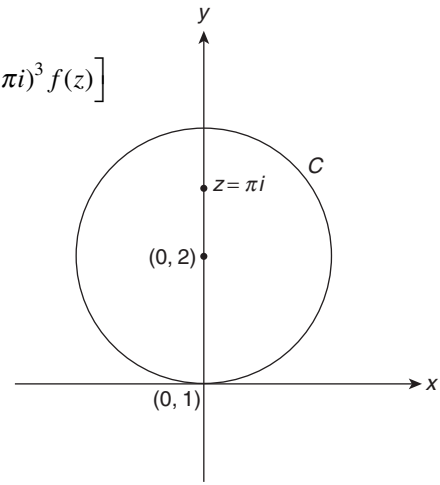


Fig. 4.44

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^{2z}}{(z-\pi i)^3}\,dz = 2\pi i(2)$$

$$= 4\pi i$$

Example 9

Evaluate $\displaystyle\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$, *where C is the circle* $|z| = 4$.

Solution

(i) Let $f(z) = \dfrac{e^z}{(z^2+\pi^2)^2} = \dfrac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

The poles are given by $z = -\pi i$, $z = \pi i$.

(ii) C is a circle $|z| = 4$ with the centre at (0, 0) and a radius of 4 (Fig. 4.45).

(iii) For $z = -\pi i, |z| = |-\pi i| = \pi < 4$

Hence, $z = -\pi i$ lies inside C.

For $z = \pi i, |z| = |\pi i| = \pi < 4$

Hence, $z = \pi i$ lies inside C.

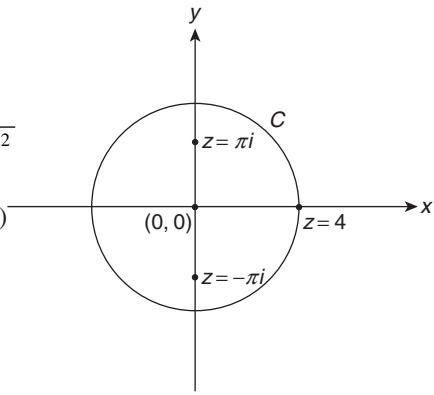


Fig. 4.45

(iv) $z = -\pi i$ is a pole of order 2.

$$\text{Res}\left[f(z); z = -\pi i\right] = \frac{1}{(2-1)!}\lim_{z\to -\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+\pi i)^2 f(z)\right]$$

$$= \lim_{z\to -\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{(z-\pi i)^2 e^z - e^z\cdot 2(z-\pi i)(1)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{e^z(z-\pi i)(z-\pi i-2)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{e^z\left(z-\pi i-2\right)}{\left(z-\pi i\right)^3}\right]$$

$$= \frac{e^{-\pi i}\left(-\pi i-\pi i-2\right)}{\left(-\pi i-\pi i\right)^3}$$

$$= \frac{-\left(\cos\pi - i\sin\pi\right)\left(2\pi i+2\right)}{-8\pi^3 i^3}$$

$$= \frac{(\pi i+1)}{4\pi^3 i}$$

$$= \frac{-i(\pi i+1)}{4\pi^3}$$

$$= \frac{\pi - i}{4\pi^3}$$

Similarly, replacing i by $-i$ in $\text{Res}\left[f(z); z = -\pi i\right]$,

$$\text{Res}\left[f(z); z = \pi i\right] = \frac{\pi + i}{4\pi^3}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}\,\mathrm{d}z = 2\pi i\left(\frac{\pi - i}{4\pi^3} + \frac{\pi + i}{4\pi^3}\right)$$

$$= \frac{2\pi i(2\pi)}{4\pi^3}$$

$$= \frac{i}{\pi}$$

Example 10

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$ *where C is* $|z-i|=2$. **[Summer 2013]**

Solution

(i) Let $f(z)=\frac{2z+6}{z^2+4}$

$$=\frac{2z+6}{(z+2i)(z-2i)}$$

The poles are given by $z=-2i$, $z=2i$.

(ii) C is a circle $|z-i|=2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.46).

(iii) For $z=2i$, $|z-i|=|2i-i|=|i|=1<2$

Hence, $z=2i$ lies inside C.

For $z=-2i$, $|z-i|=|-2i-i|=|-3i|=3>2$

Hence, $z=-2i$ lies outside C.

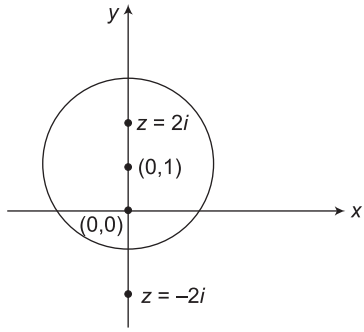


Fig. 4.46

(iv) $z=2i$ is a simple pole.

$$\text{Res}\left[f(z); z=2i\right]=\lim_{z\to 2i}\left[(z-2i)\,f(z)\right]$$

$$=\lim_{z\to 2i}\frac{2z+6}{z+2i}$$

$$=\frac{2(2i)+6}{2i+2i}$$

$$=\frac{2i+3}{2i}$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\left(\frac{2i+3}{2i}\right)$$

$$=\pi(2i+3)$$

Example 11

Find the value of the integral $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$ *taken counterclockwise around the circle C :* $|z-2|=2$. **[Winter 2012]**

Solution

(i) Let $f(z)=\frac{3z^2+2}{(z-1)(z^2+9)}=\frac{3z^2+2}{(z-1)(z+3i)(z-3i)}$

The poles are given by $z = 1$, $z = -3i$, $z = 3i$.

(ii) C is the circle $|z - 2| = 2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.47).

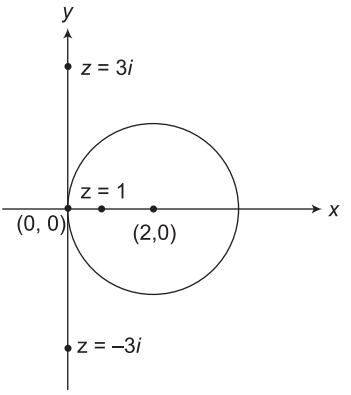


Fig. 4.47

(iii) For $z = 1$, $|z-2| = |1-2| = |-1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3i$,

$$|z-2| = |3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = 3i$ lies outside C.

For $z = -3i$,

$$|z-2| = |-3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = -3i$ lies outside C.

(iv) $z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\,f(z)$$

$$= \lim_{z\to 1}\frac{3z^2+2}{z^2+9}$$

$$= \frac{3+2}{1+9}$$

$$= \frac{1}{2}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2}\right)$$

$$= \pi i$$

Example 12

Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$, *where C is the circle* $|z| = 2.5$.

[Summer 2014]

Solution

(i) Let $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

The poles are given by $z = 1$, $z = -2$.

(ii) C is a circle $|z| = 2.5$ with the centre at (0, 0) and a radius of 2.5 (Fig. 4.48).

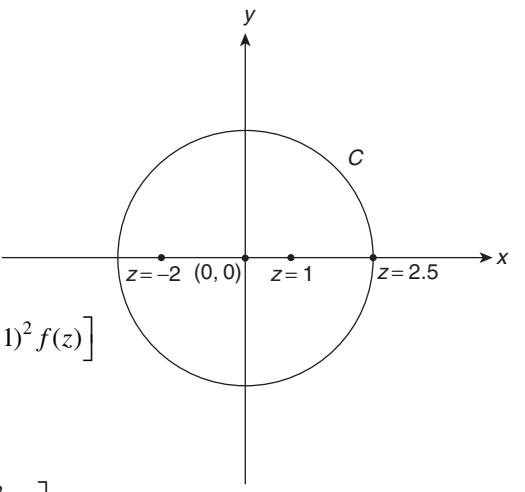


Fig. 4.48

(iii) For $z = 1, |z| = |1| = 1 < 2.5$

Hence, $z = 1$ lies inside C.

For $z = -2, |z| = |-2| = 2 < 2.5$

Hence, $z = -2$ lies inside C.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}\left[f(z); z=1\right] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)(2z)-z^2(1)}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\frac{2z^2+4z-z^2}{(z+2)^2}$$

$$= \frac{2+4-1}{9}$$

$$= \frac{5}{9}$$

$z = -2$ is a simple pole.

$$\text{Res}\,[f(z); z=-2] = \lim_{z\to -2}(z+2)\,f(z)$$

$$= \lim_{z\to -2}\frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2}$$

$$= \frac{4}{9}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}\,\mathrm{d}z = 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right)$$

$$= 2\pi i(1)$$

$$= 2\pi i$$

Example 13

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$, *where C is* $|z-2|=2$.

Solution

(i) Let $f(z)=\frac{5z+7}{z^2+2z-3}=\frac{5z+7}{(z+3)(z-1)}$

The poles are given by $z=1,\quad z=-3$.

(ii) C is the circle $|z-2|=2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.47).

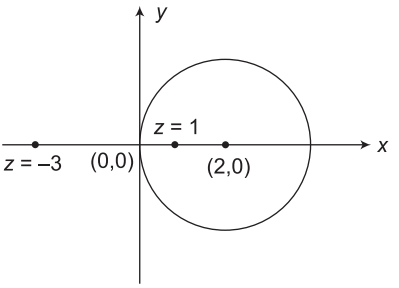


Fig. 4.49

(iii) For $z=1$, $|z-2|=|-1|=1<2$

Hence, $z=1$ lies inside C.

For $z=-3$, $|z-2|=|-3-2|=|-5|=5>2$

Hence, $z=-3$ lies outside C.

(iv) $z=1$ is a simple pole

$$\text{Res}\left[f(z);z=1\right]=\lim_{z\to 1}(z-1)f(z)$$
$$=\lim_{z\to 1}\frac{5z+7}{z+3}$$
$$=\frac{12}{4}$$
$$=3$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz=2\pi i\ (\text{sum of residues})$$
$$=2\pi i(3)$$
$$=6\pi i$$

Example 14

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, *where C is* $|z-i|=2$.

Solution

(i) Let $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$

The poles are given by $z = -1$, $z = 2$.

(ii) C is a circle $|z - i| = 2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.50).

(iii) For $z = -1, |z-i| = |-1-i| = \sqrt{2} < 2$

Hence, $z = -1$ lies inside C.

For $z = 2, |z-i| = |2-i| = \sqrt{5} > 2$

Hence, $z = 2$ lies outside C.

(iv) $z = -1$ is a pole of order 2.

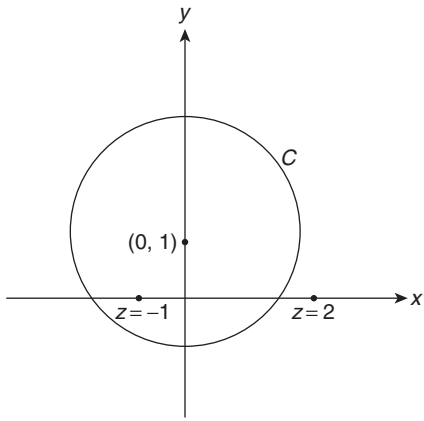


Fig. 4.50

$$\text{Res}\left[f(z); z=-1\right] = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z=-1}\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}$$

$$= \lim_{z=-1}\frac{-1}{(z-2)^2}$$

$$= -\frac{1}{(-1-2)^2}$$

$$= -\frac{1}{9}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,\mathrm{d}z = 2\pi i\left(-\frac{1}{9}\right)$$

$$= -\frac{2\pi i}{9}$$

Example 15

Evaluate $\displaystyle\int_C \frac{\mathrm{d}z}{(z^2+4)^2}$, *where C is the circle* $|z - i| = 2$.

Solution

(i) Let $f(z) = \dfrac{1}{(z^2+4)^2} = \dfrac{1}{(z+2i)^2(z-2i)^2}$

The poles are given by $z = -2i$, $z = 2i$.

(ii) C is a circle $|z - i| = 2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.51).

(iii) For $z = -2i, |z-i| = |-2i-i| = |-3i| = 3 > 2$

Hence, $z = -2i$ lies outside C.

For $z = 2i, |z-i| = |2i-i| = |i| = 1 < 2$

Hence, $z = 2i$ lies inside C.

(iv) $z = 2i$ is a pole of order 2.

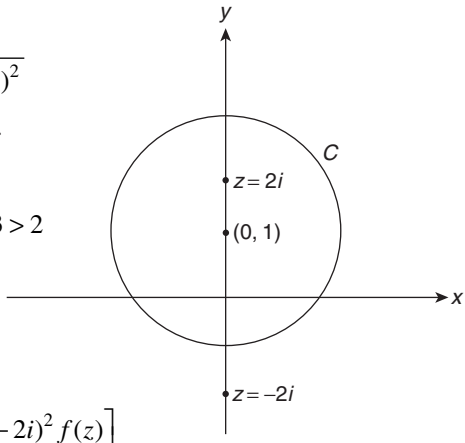


Fig. 4.51

$$\text{Res}\left[f(z); z = 2i\right] = \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-2i)^2 f(z)\right]$$

$$= \lim_{z\to 2i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z+2i)^2}\right]$$

$$= \lim_{z\to 2i}\left[-\frac{2}{(z+2i)^3}\right]$$

$$= -\frac{2}{(4i)^3}$$

$$= -\frac{2}{-64i}$$

$$= \frac{1}{32i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\mathrm{d}z}{(z^2+4)^2} = 2\pi i\left(\frac{1}{32i}\right)$$

$$= \frac{\pi}{16}$$

Example 16

Evaluate $\int \dfrac{z^4}{(z+1)(z-i)^2}\mathrm{d}z$, *where C:* $9x^2 + 4y^2 = 36$ *by using the residue theorem.* **[Summer 2015]**

Solution

(i) Let $f(z) = \dfrac{z^4}{(z+1)(z-i)^2}$

The poles are given by $z = -1$, $z = i$.

(ii) C is an ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ (Fig. 4.52)

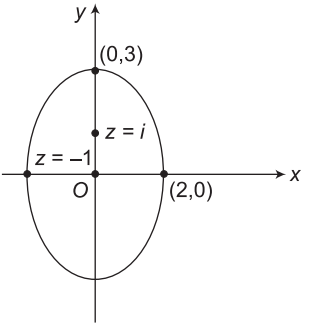


Fig. 4.52

(iii) For $z = -1$, $x = -1$, $y = 0$

$$\therefore \quad \frac{x^2}{4} + \frac{y^2}{9} = \frac{(-1)^2}{4} + 0 = \frac{1}{4} < 1$$

Hence, $z = -1$ lies inside C.

For $z = i$, $x = 0$, $y = 1$

$$\therefore \quad \frac{x^2}{4} + \frac{y^2}{9} = 0 + \frac{1}{9} < 1$$

Hence, $z = i$ lies inside C.

(iv) $z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z = -1\right] = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} \frac{z^4}{(z-i)^2}$$

$$= \frac{(-1)^4}{(-1-i)^2}$$

$$= \frac{1}{2i}$$

$z = i$ is a pole of order 2.

$$\text{Res}\left[f(z); z = i\right] = \frac{1}{(2-1)!} \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-i)^2 \, f(z)\right]$$

$$= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^4}{z+1}\right)$$

$$= \lim_{z \to i} \frac{4z^3(z+1) - z^4}{(z+1)^2}$$

$$= \lim_{z \to i} \frac{3z^4 + 4z^3}{(z+1)^2}$$

$$= \frac{3i^4 + 4i^3}{(i+1)^2}$$

$$= \frac{3 - 4i}{2i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^4}{(z+1)(z-i)^2}\,dz = 2\pi i\left(\frac{1}{2i}+\frac{3-4i}{2i}\right)$$

$$= 4\pi(1-i)$$

Example 17

Evaluate $\oint_C e^{\frac{3}{z}}\,dz$ *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = e^{\frac{3}{z}}$

$$= 1+\frac{3}{z}+\frac{1}{2!}\left(\frac{3}{z}\right)^2+\cdots$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

$$\text{Res}\left[f(z); z=0\right] = \text{coefficient of } \frac{1}{z}$$

$$= 3$$

By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C e^{\frac{3}{z}}\,dz = 2\pi i\,(3)$$

$$= 6\pi i$$

Example 18

Evaluate $\int_C z^2 e^{\frac{1}{z}}\,dz$, *where C is* $|z| = 1$.

Solution

Let $$f(z) = z^2 e^{\frac{1}{z}}$$

$$= z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}z^{-1} + \frac{1}{24}z^{-2} + \cdots$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

$$\text{Res}\,[f(z); z = 0] = \text{Coefficient of } \frac{1}{z}$$

$$= \frac{1}{6}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C z^2 e^{\frac{1}{z}}\,dz = 2\pi i\left(\frac{1}{6}\right)$$

$$= \frac{\pi i}{3}$$

Example 19

Evaluate $\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz$, *where C is* $|z| = 1$.

Solution

Let $$f(z) = e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right)$$

$$= \left(1 - \frac{1}{z} + \frac{1}{2z^2} - \cdots\right)\left(\frac{1}{z} - \frac{1}{6z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \cdots$$

$z = 0$ is an isolated essential singularity.

$$\text{Res}\left[f(z); z = 0\right] = \text{Coefficient of } \frac{1}{z}$$
$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz = 2\pi i(1)$$
$$= 2\pi i$$

Example 20

Evaluate $\int_C \frac{dz}{\cos z}$ *where C is* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{1}{\cos z}$

The poles are given by

$\cos z = 0$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.53).

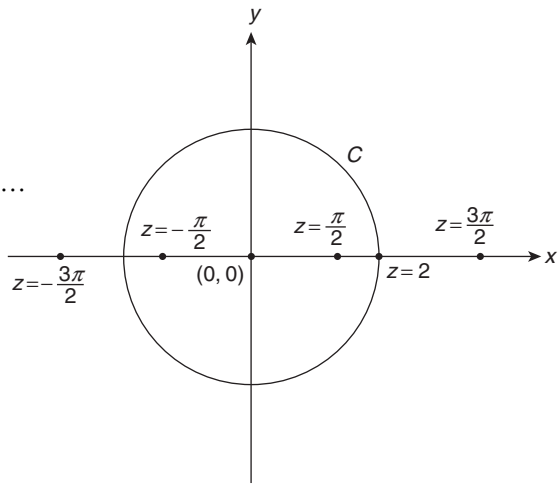


Fig. 4.53

(iii) For $z = \pm\frac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.52 < 2$$

Hence, $z = \pm\frac{\pi}{2}$ lies inside C.

Other poles, i.e., $z = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ lies outside C.

(iv) $z = \frac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{1}{\cos z}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)}{\cos z} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{1}{-\sin z} \qquad \left[\text{Using L'Hospital's rule}\right]$$

$$= -\frac{1}{\sin\frac{\pi}{2}}$$

$$= -1$$

Similarly,

$$\text{Res}\left[f(z); z=-\frac{\pi}{2}\right] = \lim_{z\to-\frac{\pi}{2}}\left(z+\frac{\pi}{2}\right)f(z)$$

$$= \lim_{z\to-\frac{\pi}{2}}\frac{\left(z+\frac{\pi}{2}\right)}{\cos z} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z\to-\frac{\pi}{2}}\frac{1}{-\sin z} \qquad \left[\text{Using L'Hospital's rule}\right]$$

$$= \frac{1}{-\sin\left(-\frac{\pi}{2}\right)}$$

$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C \frac{dz}{\cos z} = 2\pi i\,(-1+1)$$

$$= 0$$

Example 21

Evaluate $\int_C \text{cosec}\, z\, dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \operatorname{cosec} z$

$$= \frac{1}{\sin z}$$

The poles are given by $\sin z = 0$

$$z = 0$$

$$f(z) = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$= \frac{1}{z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\right)}$$

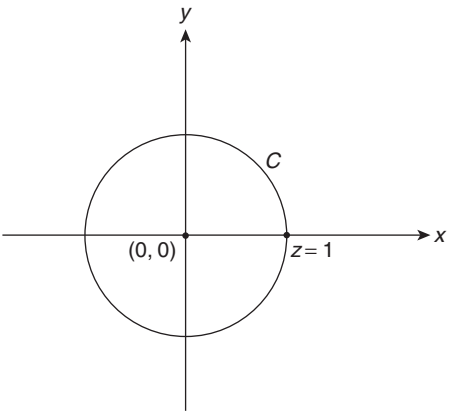


Fig. 4.54

(ii) C is a circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 4.54).

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a simple pole.

$$\operatorname{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z\, f(z)$$

$$= \lim_{z \to 0} \frac{1}{1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots}$$

$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \operatorname{cosec} z\, dz = 2\pi i (1)$$

$$= 2\pi i$$

Example 22

Evaluate $\int_C \frac{dz}{\sinh 2z}$, *where C is* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{1}{\sinh 2z}$

The poles are given by

$$\sinh 2z = 0$$

$$z = 0$$

$$f(z) = \frac{1}{2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots}$$

$$= \frac{1}{2z\left[1 + \frac{(2z)^2}{3!} + \frac{(2z)^4}{5!} + \cdots\right]}$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.55).

(iii) For $z = 0$, $|z| = 0 < 2$
Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a simple pole.

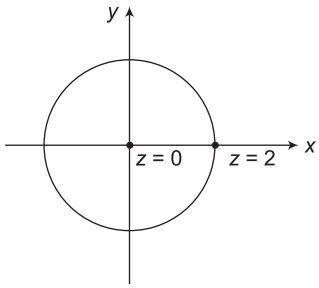


Fig. 4.55

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z\, f(z)$$

$$= \lim_{z \to 0} \frac{1}{2\left[1 + \frac{(2z)^2}{3!} + \frac{(2z)^4}{5!} + \cdots\right]}$$

$$= \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \,(\text{sum of residues})$$

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i\left(\frac{1}{2}\right)$$

$$= \pi i$$

Example 23

Evaluate $\int_C \tan z\, dz$, *where C is the circle* $|z| = 2$. **[Summer 2013]**

Solution

(i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

The poles are given by

$$\cos z = 0$$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} + \cdots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.56).

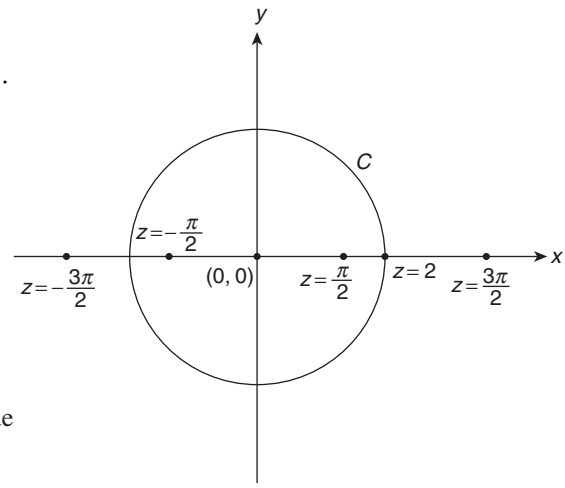


Fig. 4.56

(iii) For $z = \pm\dfrac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.57 < 2$$

Hence, $z = \pm\dfrac{\pi}{2}$, lies inside C.

(iv) $z = \dfrac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)\tan z$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)}{\cot z} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{1}{-\text{cosec}^2 z} \qquad [\text{Using L'Hospital's rule}]$$

$$= -\frac{1}{\text{cosec}^2\frac{\pi}{2}}$$

$$= -1$$

$z = -\dfrac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = -\frac{\pi}{2}\right] = \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right)\tan z$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{z + \frac{\pi}{2}}{\cot z} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{1}{-\text{cosec}^2 z} \quad \left[\text{Using L'Hospital's rule}\right]$$

$$= -\frac{1}{\text{cosec}^2\left(-\frac{\pi}{2}\right)}$$

$$= -1$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \tan z\,dz = 2\pi i(-1-1)$$

$$= -4\pi i$$

Example 24

Evaluate $\int_C \frac{dz}{z \sin z}$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \frac{1}{z \sin z}$

The poles are given by $z \sin z = 0$

$z = 0, \quad \sin z = 0$

$z = 0, \pm\pi, \pm 2\pi, \ldots$

(ii) C is a circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 4.57).

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

Other poles $z = \pm\pi, \pm 2\pi, \ldots$ lies outside C.

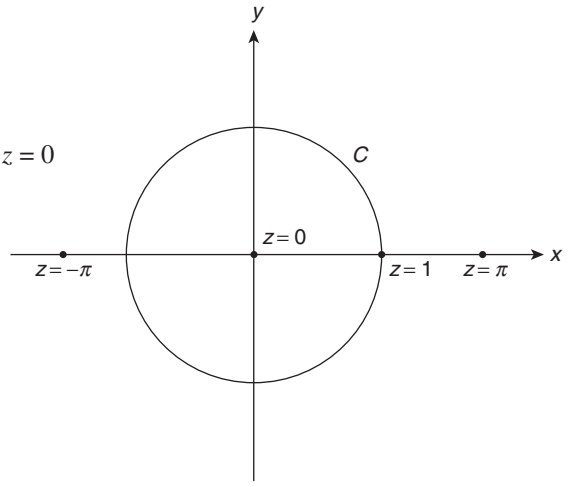


Fig. 4.57

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}\left[f(z); z=0\right] = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2\ f(z)\right]$$

$$= \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{\sin z}\right)$$

$$= \lim_{z\to 0}\frac{(1)\sin z - z\,(\cos z)}{(\sin z)^2} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z\to 0}\frac{\cos z - (\cos z - z\sin z)}{2\sin z\cos z} \qquad \left[\text{Using L'Hospital's rule}\right]$$

$$= \lim_{z\to 0}\frac{z\sin z}{2\sin z\cos z}$$

$$= \lim_{z\to 0}\frac{z}{2\cos z}$$

$$= 0$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\mathrm{d}z}{z\sin z} = 2\pi i\,(0)$$

$$= 0$$

Example 25

Evaluate $\int \dfrac{z\sec z}{1-z^2}\,\mathrm{d}z$, *where C is circle* $|z| = 2$.

Solution

(i) Let $f(z) = \dfrac{z\sec z}{1-z^2} = \dfrac{z}{(1-z^2)\cos z}$

$$= \frac{z}{(1+z)(1-z)\cos z}$$

The poles are given by

$(1 - z^2)\cos z = 0$

$(1 - z^2) = 0,\ \cos z = 0$

$z = \pm 1,\ z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.58).

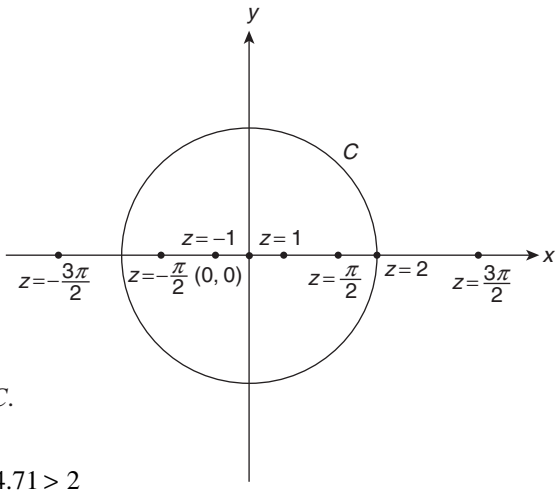


Fig. 4.58

(iii) For $z = \pm 1, |z| = |\pm 1| = 1 < 2$

Hence, $z = \pm 1$ lies inside C.

For $z = \pm\dfrac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.57 < 2$$

Hence, $z = \pm\dfrac{\pi}{2}$ lies inside C.

For $z = \pm\dfrac{3\pi}{2}, |z| = \left|\pm\dfrac{3\pi}{2}\right| = 4.71 > 2$

Hence, $z = \pm\dfrac{3\pi}{2}$ lies outside C.

Only poles at $z = \pm 1, \pm\dfrac{\pi}{2}$ lie inside C.

(iv) $z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z = 1\right] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1}\left[-\frac{z}{(1+z)\cos z}\right]$$

$$= -\frac{1}{2\cos 1}$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z = -1\right] = \lim_{z\to -1}(z+1)\, f(z)$$

$$= \lim_{z\to -1}\frac{z}{(1-z)\cos z}$$

$$= -\frac{1}{2\cos 1}$$

$z = \dfrac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right) f(z)$$

$$= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{z}{\left(1-z^2\right)\cos z}$$

$$= \lim_{z\to\frac{\pi}{2}}\left(\frac{z}{1-z^2}\right)\cdot \lim_{z\to\frac{\pi}{2}}\frac{z-\frac{\pi}{2}}{\cos z} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \frac{\frac{\pi}{2}}{1-\frac{\pi^2}{4}}\cdot \lim_{z\to\frac{\pi}{2}}\frac{1}{(-\sin z)} \qquad \left[\text{Using L' Hospital's rule}\right]$$

$$= \frac{2\pi}{4-\pi^2}\cdot\frac{1}{\left(-\sin\frac{\pi}{2}\right)}$$

$$= -\frac{2\pi}{4-\pi^2}$$

$$= \frac{2\pi}{\pi^2-4}$$

$z=-\frac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z);z=-\frac{\pi}{2}\right] = \lim_{z\to-\frac{\pi}{2}}\left(z+\frac{\pi}{2}\right)f(z)$$

$$= \lim_{z\to-\frac{\pi}{2}}\left(z+\frac{\pi}{2}\right)\frac{z}{(1-z^2)\cos z}$$

$$= \lim_{z\to-\frac{\pi}{2}}\left(\frac{z}{1-z^2}\right)\cdot \lim_{z\to-\frac{\pi}{2}}\frac{z+\frac{\pi}{2}}{\cos z} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \frac{-\frac{\pi}{2}}{1-\frac{\pi^2}{4}}\lim_{z\to-\frac{\pi}{2}}\frac{1}{(-\sin z)} \qquad \left[\text{Using L' Hospital's rule}\right]$$

$$= -\frac{2\pi}{4-\pi^2}\cdot\frac{1}{\left[-\sin\left(-\frac{\pi}{2}\right)\right]}$$

$$= \frac{2\pi}{\pi^2-4}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left[-\frac{1}{2\cos 1}-\frac{1}{2\cos 1}+\frac{2\pi}{(\pi^2-4)}+\frac{2\pi}{(\pi^2-4)}\right]$$

$$= 2\pi i\left[-\frac{1}{\cos 1}+\frac{4\pi}{(\pi^2-4)}\right]$$

$$= -\frac{2\pi i}{\cos 1}+\frac{8\pi^2 i}{\pi^2-4}$$

Example 26

Evaluate $\int \frac{\cos \pi z}{z^2-1}\,dz$, *where C is the rectangle whose vertices are* $2 \pm i, -2 \pm i$.

Solution

(i) Let $f(z) = \frac{\cos \pi z}{z^2-1} = \frac{\cos \pi z}{(z+1)(z-1)}$

The poles are given by $z = -1, z = 1$.

(ii) C is a rectangle with vertices $2 \pm i$ and $-2 \pm i$. (Fig. 4.59).

(iii) The poles $z = \pm 1$ lie inside C.

(iv) $z = 1$ is a simple pole.

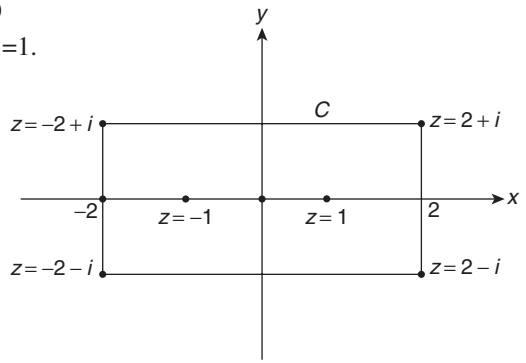


Fig. 4.59

$$\text{Res}[f(z); z=1] = \lim_{z\to 1}(z-1)\,f(z)$$

$$= \lim_{z\to 1}\frac{\cos \pi z}{z+1}$$

$$= -\frac{1}{2}$$

$z = -1$ is a simple pole.

$$\text{Res}[f(z); z=-1] = \lim_{z\to -1}(z+1)\,f(z)$$

$$= \lim_{z\to -1}\frac{\cos \pi z}{z-1}$$

$$= \frac{-1}{-2}$$

$$= \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \left(\text{sum of residues}\right)$$

$$\int_C \frac{\cos \pi z}{z^2 - 1}\,dz = 2\pi i\left(-\frac{1}{2} + \frac{1}{2}\right)$$

$$= 0$$

EXERCISE 4.5

Evaluate the following integrals using Cauchy's residue theorem:

1. $\int_C \frac{\sin z}{z^6}\,dz$, where C is $|z| = 1$

[**Ans.:** $\frac{\pi i}{60}$]

2. $\int_C z\, e^{\frac{1}{z}}\,dz$, where C is $|z| = 1$

[**Ans. :** πi]

3. $\int_C \frac{z}{(z-1)^2(z+1)}\,dz$, where C is $|z| = \frac{3}{4}$

[**Ans. :** 0]

4. $\int_C \frac{z}{(z+1)^2(z-2)}\,dz$, where C is $|z - i| = 2$

[**Ans. :** $-\frac{4\pi i}{9}$]

5. $\int_C \frac{dz}{(z^2+1)^2}$, where C is $|z - i| = 1$

[**Ans. :** $\frac{\pi}{2}$]

6. $\int_C \frac{(z+4)^2}{z^4 + 5z^3 + 6z^2}$, where C is $|z| = 1$

[**Ans. :** $-\frac{16\pi i}{9}$]

7. $\int_C \frac{3z^2 + 2z - 4}{z^3 - 4z}$, where C is $|z - i| = 3$

[**Ans. :** $6\pi i$]

8. $\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is $|z+1-i|=2$

[Ans. : $\pi(2+i)$**]**

9. $\int_C \frac{z^2+3}{z^2-1}$, where C is $|z-1|=1$

[Ans. : $4\pi i$**]**

10. $\int_C \frac{15z+9}{z^3-9z}$, where C is $|z-1|=3$

[Ans. : 7**]**

4.10 ARGUMENT THEOREM

Let $f(z)$ be analytic on and within a simple closed curve C except at a pole $z=a$ of order P inside C. Also, suppose that $f(z)$ has only one zero $z=b$ of order N inside C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N-P$$

Proof Enclose $z=a$ and $z=b$ by drawing non-overlapping circle C_1 and C_2 around them with centres at $z=a$ and $z=b$ respectively (Fig. 4.60). Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f'(z)}{f(z)}dz \qquad ...(4.5)$$

Since $f(z)$ has a pole of order P at $z=a$,

$$f(z)=\frac{F(z)}{(z-a)^P}$$

where $F(z)$ is analytic and nonzero on and within C_1.

Taking logarithm on both the sides,

$$\log f(z) = \log F(z) - P\log(z-a)$$

Differentiating w.r.t. z,

$$\frac{1}{f(z)}f'(z) = \frac{1}{F(z)}F'(z) - \frac{P}{z-a}$$

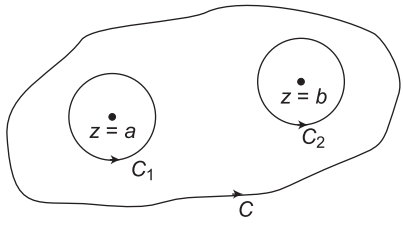


Fig. 4.60

$$\therefore \quad \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}dz - \frac{1}{2\pi i}\oint_{C_1} \frac{P}{z-a}dz \qquad ...(4.6)$$

Since $F(z)$ is analytic in C_1, $\frac{F'(z)}{F(z)}$ is analytic in C_1.

By Cauchy's integral theorem,

$$\oint_{C_1} \frac{F'(z)}{F(z)} dz = 0$$

Also, $\frac{P}{z-a}$ is not analytic at $z = a$.

By Cauchy's integral formula,

$$\oint_{C_1} \frac{P}{z-a} dz = 2\pi i\,(P)$$

Substituting these integral values in Eq. (4.6),

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f'(z)}{f(z)} dz = 0 - \frac{1}{2\pi i}(2\pi i P)$$

$$= -P \qquad \text{...(4.7)}$$

Since $f(z)$ has a zero of order N at $z = b$,

$$f(z) = (z - b)^N G(z)$$

where $G(z)$ is analytic and non-zero on and within C_2.

Taking logarithm on both the sides,

$$\log f(z) = \log G(z) + N \log (z - b)$$

Differentiating w.r.t. z,

$$\frac{1}{f(z)} f'(z) = \frac{1}{G(z)} G'(z) + \frac{N}{z-b}$$

$$\frac{1}{2\pi i} \oint_{C_2} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{C_2} \frac{G'(z)}{G(z)} dz + \frac{1}{2\pi i} \oint_{C_2} \frac{N}{z-b} dz$$

$$= 0 + \frac{1}{2\pi i}(2\pi i N) \qquad \left[\begin{array}{l}\text{Using Cauchy's integral theorem} \\ \text{and Cauchy's integral formula}\end{array}\right]$$

$$= N \qquad \text{...(4.8)}$$

Substituting Eqs (4.7) and (4.8) in Eq. (4.5)

$$\frac{1}{2\pi i} \oint_{C} \frac{f'(z)}{f(z)} dz = -P + N = N - P$$

Corollary If $f(z)$ is analytic everywhere, i.e., $P = 0$ then

$$\frac{1}{2\pi i} \oint_{C} \frac{f'(z)}{f(z)} dz = N$$

4.11 ROUCHE'S THEOREM

Let $f(z)$ and $g(z)$ be analytic in a domain D and have finite number of zeros in D. Suppose C is a simple closed contour in D such that no zeros of $f(z)$ and $g(z)$ lie on C and $|g(z)| < |f(z)|$ on C. Then the total number of zeros of $f(z) + g(z)$ is equal to the total number of zeros of $f(z)$ inside C.

i.e., $N_{f+g} = N_f$

Proof $|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0 \qquad \left[\because \ |f(z)| > |g(z)|\right]$

Thus, $f(z) + g(z)$ and $f(z)$ are nonzero on C. By the argument-principle corollary,

$$N_f = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

$$N_{f+g} = \frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz$$

$$N_f - N_{f+g} = \frac{1}{2\pi i}\int_C \left[\frac{f'(z)}{f(z)} - \frac{f'(z)+g'(z)}{f(z)+g(z)}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f'(z)\left[f(z)+g(z)\right] - f(z)\left[f'(z)+g'(z)\right]}{f(z)\left[f(z)+g(z)\right]}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\left[f'(z)g(z) - f(z)g'(z)\right]}{f(z)\left[f(z)+g(z)\right]}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\left[f'(z)g(z) - f(z)g'(z)\right]}{\left[f(z)\right]^2\left[\dfrac{(f(z)+g(z)}{f(z)}\right]}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\left[\dfrac{f'(z)g(z) - f(z)g'(z)}{\left\{f(z)\right\}^2}\right]}{\left[1+\dfrac{g(z)}{f(z)}\right]}\,dz$$

$$= -\frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}\,dz \qquad \text{...(4.9)}$$

where $F(z) = 1 + \dfrac{g(z)}{f(z)}$

Let $w = F(z) = 1 + \dfrac{g(z)}{f(z)}$

$$w-1=\frac{g(z)}{f(z)}$$

$$|w-1|=\left|\frac{g(z)}{f(z)}\right|<1 \qquad [\because \quad g(z)<f(z)]$$

Thus, the function $F(z)$ maps C into $|w-1|<1$.

As z traverses C, the point $w = F(z)$ traverses a closed contour C^* lying completely inside the domain $|w-1|<1$ so that $w = 0$ lies outside C^* (Fig. 4.61).

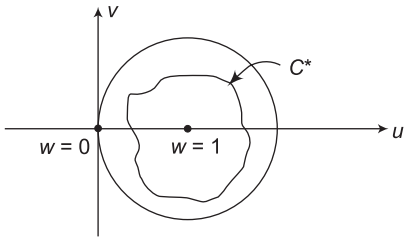


Fig. 4.61

By Cauchy integral formula,

$$\int_C \frac{F'(z)}{F(z)}dz=\int_{C^*}\frac{dw}{w}=0$$

Hence, from Eq. (4.9),

$$N_f - N_{f+g}=0$$

$$N_{f+g}=N_f$$

Example 1

Use Rouche's theorem to determine the number of zeros of the polynomial $2z^5 + 8z - 1$ *inside the circle* $|z| = 2$.

Solution

Let $F(z) = 2z^5 + 8z - 1$

Let $f(z) = 2z^5$, $g(z) = 8z - 1$, C: $|z| = 2$

On circle C: $|z| = 2$,

$$|f(z)|=|2z^5|$$

$$=2|2^5|$$

$$=64$$

and

$$|g(z)|=|8z-1|$$

$$\le |8z|+|1|$$

$$=8|2|+1$$

$$=17$$

$\therefore \quad |g(z)|\le 17$

and $|f(z)|=64>17\ge|g(z)|$

$\therefore \quad |f(z)|>|g(z)|$

Since $f(z)$ and $g(z)$ are analytic within and on C, also $f(z)$ and $f(z) + g(z)$ are nonzero on C, by Rouche's theorem,

$$N_{f+g} = N_f$$

$f(z) = 2z^5$ has a zero of order 5 inside C.

$$N_f = 5$$

$$\therefore \quad N_{f+g} = 5$$

Hence, $F(z)$ has five zeros inside C.

Example 2

Use Rouche's theorem to determine the number of zeros of the polynomial $z^6 - 5z^4 + z^3 - 2z$ *inside the circle* $|z| = 1$. **[Summer 2012]**

Solution

Let $F(z) = z^6 - 5z + z^3 - 2z$

Let $f(z) = -5z^4,\ g(z) = z^6 + z^3 - 2z,\ C: |z| = 1$

On circle C: $|z| = 1$,

$$|f(z)| = |-5z^4| = 5$$

and

$$|g(z)| = |z^6 + z^3 - 2z|$$

$$\leq |z^6| + |z^3| + 2|z|$$

$$= 4$$

$$\therefore \quad |g(z)| \leq 4$$

$$\therefore \quad |f(z)| = 5 > 4 \geq |g(z)|$$

$$\therefore \quad |f(z)| > |g(z)|$$

Since $f(z)$ and $g(z)$ are analytic within and on C, also $f(z)$ and $f(z) + g(z)$ are nonzero on C, by Rouche's theorem,

$$N_{f+g} = N_f$$

$f(z) = -5z^4$ has a zero of order 4 inside C.

$$N_f = 4$$

$$\therefore \quad N_{f+g} = 4$$

Hence, $F(z)$ has four zeros inside C.

Example 3

Use Rouche's theorem to determine the number of zeros of the polynomial $z^4 - 8z + 10$ *in the annulus region* $1 < |z| < 3$.

Solution

Let $F(z) = z^4 - 8z + 10$

(i) Let $f(z) = 10, \quad g(z) = z^4 - 8z, \quad C_1: |z| = 1$

On circle $C_1 : |z| = 1,$

$$|f(z)| = 10$$

and

$$\begin{aligned} |g(z)| &= |z^4 - 8z| \\ &\le |z^4| + |8|\,|z| \\ &= 1 + 8 \\ &= 9 \end{aligned}$$

$$|g(z)| \le 9$$

$$|f(z)| = 10 > 9 \ge |g(z)|$$

$$\therefore \quad |f(z)| > |g(z)|$$

Since $f(z)$ and $g(z)$ are analytic within and on C_1, also $f(z)$ and $f(z) + g(z)$ are nonzero on C_1, by Rouche's theorem,

$$N_{f+g} = N_f$$

$f(z) = 10$ has no zero inside C_1.

$$N_f = 0$$

$$\therefore \quad N_{f+g} = 0$$

Hence, $F(z)$ has no zero inside C_1.

(ii) Let $f(z) = z^4$, $g(z) = -8z + 10$, $C_2: |z| = 3$

On circle C_2: $|z| = 3$,

$$\begin{aligned} |f(z)| &= |z^4| \\ &= |3^4| \\ &= 81 \end{aligned}$$

and

$$\begin{aligned} |g(z)| &= |-8z + 10| \\ &\le 8|z| + |10| \\ &= 24 + 10 \\ &= 34 \end{aligned}$$

$$\therefore \quad |g(z)| \le 34$$

$$|f(z)| = 81 > 34 \ge |g(z)|$$

$$\therefore \quad |f(z)| > |g(z)|$$

Since $f(z)$ and $g(z)$ are analytic within and on C_2, also $f(z)$ and $f(z) + g(z)$ are nonzero on C_2, by Rouche's theorem,

$$N_{f+g} = N_f$$

$f(z) = z^4$ has a zero of order 4 inside C_2.

$$N_f = 4$$

$\therefore \quad N_{f+g} = 4$

Hence, $F(z)$ has four zeros inside C_2.

Total number of zeros of $F(z)$ inside $1 < |z| < 2 = [N_{f+g}]_{C_2} - [N_{f+g}]_{C_1}$

$= 4 - 0$

$= 4$

Example 4

Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ *lie between the circles* $|z| = 1$ *and* $|z| = 2$*, using Rouche's theorem.*

Solution

Let $F(z) = z^7 - 5z^3 + 12$

(i) Let $f(z) = 12$, $g(z) = z^7 - 5z^3$, C_1: $|z| = 1$

On circle C_1: $|z| = 1$,

$$|f(z)| = |12|$$
$$= 12$$

and
$$|g(z)| = |z^7 - 5z^3|$$
$$\le |z^7| + 5|z^3|$$
$$= 1 + 5$$
$$= 6$$

$\therefore \quad |g(z)| \le 6$

$$|f(z)| = 12 > 6 \ge |g(z)|$$

$\therefore \quad |f(z)| > |g(z)|$

Since $f(z)$ and $g(z)$ are analytic within and on C_1, and also $f(z)$ and $f(z) + g(z)$ are nonzero on C_1, by Rouche's theorem,

$$N_{f+g} = N_f$$
$$f(z) = 12 \text{ has no zero inside } C_1$$
$$N_f = 0$$
$$\therefore \quad N_{f+g} = 0$$

Hence, $F(z)$ has no zero inside C_1.

(ii) Let $f(z) = z^7$, $g(z) = 12 - 5z^3$, C_2 :| z | = 2

On circle C_2: $|z| = 2$

$$|f(z)| = |z^7|$$
$$= |2^7|$$
$$= 128$$

and $$|g(z)| = 12 - 5z^3$$
$$\le |12| + 5|z^3|$$
$$= 12 + 5|2^3|$$
$$= 52$$

$\therefore \quad |g(z)| > 52$

$$|f(z)| = 128 > 52 \ge |g(z)|$$

$\therefore \quad |f(z)| > |g(z)|$

Since $f(z)$ and $g(z)$ are analytic within and on C_2, and also $f(z)$ and $f(z) + g(z)$ are non-zero on C_2, by Rouches' theorem,

$N_{f+g} = N_f$

$f(z) = z^7$ has a zero of order 7 inside C_2.

$N_f = 7$

$\therefore \quad N_{f+g} = 7$

Hence, $F(z)$ has four zeros inside C_2.

Total number of zeros of $F(z)$ inside $1 < |z| < 2 = \left(N_{f+g}\right)_{C_2} - \left(N_{f+g}\right)_{C_1}$

$$= 7 - 0$$
$$= 7$$

Hence, all the roots of $F(z) = z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Example 5

Use Rouche's theorem to determine the number of zeros of the polynomial $z^5 + 15z + 1$ *inside the annulus region* $\frac{3}{2} < |z| < 2$.

Solution

Let $F(z) = z^5 + 15z + 1$

(i) Let $f(z) = 15z, \; g(z) = z^5 + 1, \; C_1 : |z| = \frac{3}{2}$

On circle $C_1 : |z| = \frac{3}{2}$,

$$|f(z)| = |15z|$$
$$= 15|z|$$
$$= 15\left(\frac{3}{2}\right)$$
$$= \frac{45}{2}$$

and $$|g(z)| = |z^5 + 1|$$
$$\leq |z^5| + |1|$$
$$= \left(\frac{3}{2}\right)^5 + 1$$
$$= \frac{275}{32}$$

$\therefore \quad |g(z)| \leq \frac{275}{32}$

$$|f(z)| = \frac{45}{2} > \frac{275}{32} \geq |g(z)|$$

$\therefore \quad |f(z)| > |g(z)|$

Since $f(z)$ and $g(z)$ are analytic within and on C_1, also $f(z)$ and $f(z) + g(z)$ are nonzero on C_1, by Rouche's theorem,

$N_{f+g} = N_f$

$f(z) = 15z$ has a zero of order 1 inside C_1.

$N_f = 1$

$\therefore \quad N_{f+g} = 1$

Hence, $F(z)$ has one zero inside C_1.

(ii) Let $f(z) = z^5$, $g(z) = 15z + 1$, $C_2 : |z| = 2$ on $C_2 : |z| = 2$,

On circle C_2: $|z| = 2$,

$$|f(z)| = |z^5|$$
$$= |2^5|$$
$$= 32$$

and $$|g(z)| = |15z + 1|$$
$$\leq 15|z| + |1|$$
$$= 15|2| + 1$$
$$= 30 + 1$$
$$= 31$$

$\therefore \quad |g(z)| \leq |3|$

$$|f(z)| = 32 > 31 > |g(z)|$$

Since $f(z)$ and $g(z)$ are analytic within and on C_2, also $f(z)$ and $f(z) + g(z)$ are nonzero on C_2, by Rouche's theorem,

$N_{f+g} = N_f$

$f(z) = z^5$ has a zero of order 5 inside C_2

$N_f = 5$

$N_{f+g} = 5$

Hence, $F(z)$ has five zeros inside C_2.

Total number of zeros of $F(z)$ inside $\frac{3}{2} < |z| < 2 = \left(N_{f+g}\right)_{C_2} - \left(N_{f+g}\right)_{C_1}$

$= 5 - 1$

$= 4$

EXERCISE 4.6

Use Rouche's theorem to determine the number of zeros of *F*(z) within the given region:

1. $F(z) = z^4 - 7z - 1, |z| = 1$

[Ans.: 1]

2. $F(z) = z^7 - 5z^4 + z^2 - 2, |z| = 1$

[Ans.: 4]

3. $F(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9, |z| = 1$

[Ans.: 0]

4. $F(z) = 2z^5 - z^3 + 3z^2 - z + 8, |z| = 1$

[Ans.: 0]

5. $F(z) = z^7 - 4z^3 + z - 1, |z| = 1$

[Ans.: 3]

6. $F(z) = z^4 - 5z + 1, 1 < |z| < 2$

[Ans.: 3]

7. $F(z) = z^7 - z^3 - 12, 1 < |z| < 2$

[Ans.: 7]

8. $F(z) = 2z^5 - 6z^2 + z + 1, 1 < |z| < 2$

[Ans.: 3]

Points to Remember

Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots + a_n(z-z_0)^n + \cdots$$

where $a_0, a_1, a_2, ..., a_n, ...$ are complex or real constants, z_0 is a constant known as the centre of the series.

The power series about $z = 0$, i.e., the origin is given as

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at $z = a$ then for each z inside C, $f(z)$ can be expanded as a power series about $z = a$ as

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Laurent's Series

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with the centre at $z = a$ and radii r_1, $r_2 (r_2 < r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \, dw$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \, dw$$

Singular Points

A point at which the function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Types of Singularities

1. Isolated Singularity: A singular point $z = a$ is known as an isolated singularity if there is no other singularity within a small circle surrounding the point $z = a$; otherwise it is called a non-isolated singularity.

2. Removable Singularity: An isolated singular point $z = a$ is known as a removable singularity if $\lim_{z\to a} f(z)$ exists or there is no negative power term of $(z - a)$ in Laurent's series of $f(z)$, i.e., $b_n = 0$.
3. Essential Singularity: A singular point $z = a$ is known as an essential singularity if the number of negative power terms of $(z - a)$ in Laurent's series is infinite.
4. Pole of Order m: An isolated singularity $z = a$ is known as a pole of order m if in Laurent's series, all the negative powers of $(z - a)$ after the m^{th} power are zero, i.e., highest power of $\dfrac{1}{z-a}$ is m.

Meromorphic Function: A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Residues

The coefficient of $\dfrac{1}{z-a}$ in the Laurent series expansion of $f(z)$ is known as the residue of $f(z)$ at $z = a$.

Evaluation of Residues

1. Residue at a Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$ then

$$\text{Res}\big[f(z); z = a\big] = \lim_{z\to a} (z-a) f(z)$$

(ii) If $f(z)$ is of the form $f(z) = \dfrac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic at $z = a$.

If $h(a) = 0$ but $h'(a) \neq 0$ then $z = a$ is a simple pole.

If $h(a) = 0$ but $g(a) \neq 0$ then

$$\text{Res}\big[f(z); z = a\big] = \lim_{z\to a} \frac{g(z)}{h'(z)}$$

2. Residue at a Pole of Order **m**

If $f(z)$ has a pole of order m at $z = a$ then

$$\text{Res}\big[f(z); z = a\big] = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m \, f(z) \right]$$

Cauchy's Residue Theorem

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points inside C then

$$\oint_C f(z)\, dz = 2\pi i \ \text{(sum of residues)}$$

where the integral is taken in the anticlockwise direction around C.

Argument Theorem

Let $f(z)$ be analytic on and within a simple closed curve C except at a pole $z = a$ of order P inside C. Also, suppose that $f(z)$ has only one zero $z = b$ of order N inside C.

Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$

Rouche's Theorem

Let $f(z)$ and $g(z)$ be analytic in a domain D and have finite number of zeros in D. Suppose C is a simple closed contour in D such that no zeros of $f(z)$ and $g(z)$ lie on C and $|g(z)| < |f(z)|$ on C. Then the total number of zeros of $f(z) + g(z)$ is equal to the total number of zeros of $f(z)$ inside C.

i.e., $N_{f+g} = N_f$

CHAPTER 5
Applications of Contour Integration

Chapter Outline

5.1 Introduction
5.2 Evaluation of a Real Definite Integral of a Rational Function of cos θ and sin θ
5.3 Evaluation of Improper Real Integral of a Rational Function
5.4 Evaluation of Improper Real Integral of a Rational Function Including Trigonometric Functions
5.5 Evaluation of Improper Integral When Simple Poles Lie on the Real Axis

5.1 INTRODUCTION

The Cauchy's residue theorem is very important in the development and applications of the theory of functions of a complex variable. It is a powerful tool to evaluate line integrals of analytic functions over closed curves. The residue theorem can be applied to real definite integrals. These real integrals are evaluated by expressing them in terms of complex functions over a suitable contour. The process of evaluation of these integrals is called *contour integration*.

5.2 EVALUATION OF A REAL DEFINITE INTEGRAL OF A RATIONAL FUNCTION OF cos θ AND sin θ

Let $$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$$

where $f(\sin\theta, \cos\theta)$ is a rational function of sin θ and cos θ.

To evaluate the integral, putting $z = e^{i\theta}$,

$$dz = i\, e^{i\theta}\, d\theta = i\, z\, d\theta$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 - 1}{2iz}$$

$$|z| = |e^{i\theta}| = 1$$

Substituting these values in the integral,

$$I = \oint_C f(z)\,dz$$

where C is a unit circle $|z| = 1$

By Cauchy's residue theorem,

$I = 2\pi i$ (sum of residues at all poles lying inside the circle $|z| = 1$)

Example 1

Evaluate $\int_0^{2\pi} \frac{d\theta}{5-\sin\theta}$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{5-\sin\theta}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.1).

Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

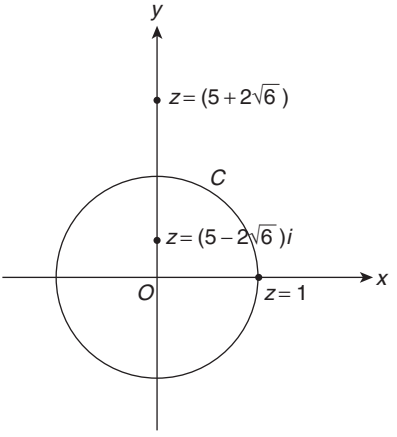


Fig. 5.1

(ii)
$$I = \int_C \frac{\frac{dz}{iz}}{5-\left(\frac{z^2-1}{2iz}\right)}$$

$$= \int_C \frac{2}{-z^2+10iz+1}\,dz$$

$$= -2\int_C \frac{1}{z^2-10iz-1}\,dz \qquad ...(1)$$

(iii) Let $f(z) = \frac{1}{z^2-10iz-1}$

The poles are given by

$$z^2 - 10iz - 1 = 0$$

$$z = \frac{10i \pm \sqrt{(-10i)^2+4}}{2}$$

$$= \frac{10i \pm \sqrt{(-96)}}{2}$$

$$= 5i \pm 2\sqrt{6i}$$

$$= (5 \pm 2\sqrt{6})\, i$$

$$\therefore \quad f(z) = \frac{1}{\left[z-(5+2\sqrt{6})i\right]\left[z-(5-2\sqrt{6})i\right]}$$

(iv) For $z = (5+2\sqrt{6})i, |z| = \left|(5+2\sqrt{6})\, i\right| = 9.89 > 1$

Hence, $z = (5+2\sqrt{6})\, i$ lies outside C.

For $z = (5-2\sqrt{6})i, |z| = \left|(5-2\sqrt{6})\, i\right| = 0.10 < 1$

Hence, $z = (5-2\sqrt{6})\, i$ lies inside C.

(v) Res $[f(z); z = (5-2\sqrt{6})i] = \lim\limits_{z \to (5-2\sqrt{6})i} [z-(5-2\sqrt{6})i] f(z)$

$$= \lim_{z \to (5-2\sqrt{6})i} \frac{1}{z-(5+2\sqrt{6})i}$$

$$= \frac{1}{(5-2\sqrt{6i})-(5+2\sqrt{6})i}$$

$$= -\frac{1}{4\sqrt{6i}}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, \mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\mathrm{d}z}{z^2 - 10iz - 1} = 2\pi i \left(-\frac{1}{4\sqrt{6i}}\right)$$

$$= -\frac{\pi}{2\sqrt{6}}$$

Substituting in Eq. (1),

$$I = -2\left(-\frac{\pi}{2\sqrt{6}}\right)$$

$$= \frac{\pi}{\sqrt{6}}$$

Example 2

Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.2).

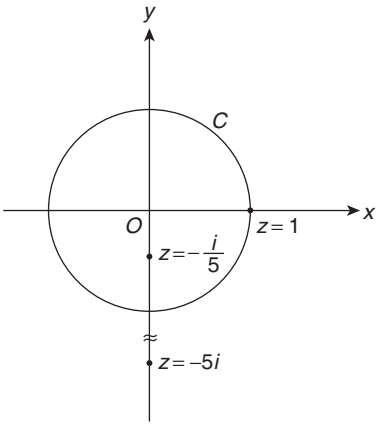


Fig. 5.2

$$z = e^{i\theta},\ d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z^2-1}{2iz}$$

(ii) $$I = \int_C \frac{\frac{dz}{iz}}{13+5\left(\frac{z^2-1}{2iz}\right)}$$

$$= 2\int_C \frac{dz}{5z^2+26iz-5} \qquad \ldots(1)$$

(iii) Let $f(z) = \frac{1}{5z^2+26iz-5}$

The poles are given by

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4(5)(-5)}}{10}$$

$$= \frac{-26i \pm \sqrt{-576}}{10}$$

$$= \frac{-26i \pm 24i}{10}$$

$$= -\frac{i}{5}, -5i$$

$$\therefore\ f(z) = \frac{1}{5\left(z+\frac{i}{5}\right)(z+5i)} \qquad \left[\because\ ax^2+bx+c = a(x-\alpha)(x-\beta)\right]$$

(iv) For $z = -\frac{i}{5}$, $|z| = \left|-\frac{i}{5}\right| = \frac{1}{5} < 1$

Hence, $z = -\frac{i}{5}$ lies inside C.

For $z = -5i, |z| = |-5i| = 5 > 1$

Hence, $z = -5i$ lies outside C.

(v) $$\text{Res}\left[f(z); z = -\frac{i}{5}\right] = \lim_{z \to -\frac{i}{5}} \left[\left(z + \frac{i}{5}\right) f(z)\right]$$

$$= \lim_{z \to -\frac{i}{5}} \frac{1}{5(z + 5i)}$$

$$= \frac{1}{5\left(-\frac{i}{5} + 5i\right)}$$

$$= \frac{1}{24i}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{5z^2 + 26iz - 5} = 2\pi i \left(\frac{1}{24i}\right)$$

$$= \frac{\pi}{12}$$

Substituting in Eq. (1),

$$I = 2\left(\frac{\pi}{12}\right)$$

$$= \frac{\pi}{6}$$

Example 3

Evaluate $\int_0^{\pi} \frac{1}{17 - 8\cos\theta} d\theta$, *by integrating around a unit circle.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{1}{17 - 8\cos\theta} d\theta$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.3).

Let $z = e^{i\theta}$, $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{z^2+1}{2z}$

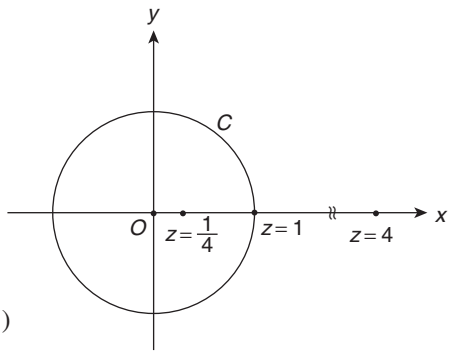


Fig. 5.3

(ii) $$I = \int_C \frac{1}{17 - 8\left(\dfrac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{17z - 4z^2 - 4}\, dz$$

$$= -\frac{1}{i}\int_C \frac{1}{4z^2 - 17z + 4} dz \quad \ldots(1)$$

(iii) Let $f(z) = \dfrac{1}{4z^2 - 17z + 4}$

The poles are given by

$$4z^2 - 17z + 4 = 0$$

$$z = \frac{17 \pm \sqrt{289 - 64}}{8}$$

$$= \frac{17 \pm \sqrt{225}}{8}$$

$$= \frac{17 \pm 15}{8}$$

$$= 4, \frac{1}{4}$$

$$\therefore \quad f(z) = \frac{1}{4(z-4)\left(z - \dfrac{1}{4}\right)} \quad \left[\because ax^2 + bx = c = a(x-\alpha)(x-\beta)\right]$$

(iv) For $z = 4$, $|z| = |4| = 4 > 1$

Hence, $z = 4$ lies outside C.

For $z = \dfrac{1}{4}$, $|z| = \left|\dfrac{1}{4}\right| = \dfrac{1}{4} < 1$

Hence, $z = \dfrac{1}{4}$ lies inside C.

(v) $$\text{Res}\left[f(z); z = \frac{1}{4}\right] = \lim_{z \to \frac{1}{4}} \left[\left(z - \frac{1}{4}\right) f(z)\right]$$

$$= \lim_{z \to \frac{1}{4}} \frac{1}{4(z-4)}$$

$$= \frac{1}{4\left(\frac{1}{4} - 4\right)}$$

$$= -\frac{1}{15}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{1}{4z^2 - 17z + 4}\,dz = 2\pi i\left(-\frac{1}{15}\right)$$

$$= -\frac{2\pi i}{15}$$

Substituting in Eq. (1),

$$I = -\frac{1}{i}\left(-\frac{2\pi i}{15}\right)$$

$$= \frac{2\pi}{15}$$

$$\int_0^{2\pi} \frac{1}{17 - 8\cos\theta}\,d\theta = \frac{2\pi}{15}$$

$$2\int_0^{\pi} \frac{1}{17 - 8\cos\theta}\,d\theta = \frac{2\pi}{15} \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a - x) = f(x)\right]$$

$$\int_0^{\pi} \frac{1}{17 - 8\cos\theta}\,d\theta = \frac{\pi}{15}$$

Example 4

Evaluate $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.4).

Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2 + 1}{2z}$

(ii) $I = \int_C \frac{\frac{dz}{iz}}{2 + \frac{z^2+1}{2z}}$

$$= \frac{2}{i}\int_C \frac{dz}{4z + z^2 + 1}$$

$$= \frac{2}{i}\int_C \frac{dz}{z^2 + 4z + 1} \quad \ldots(1)$$

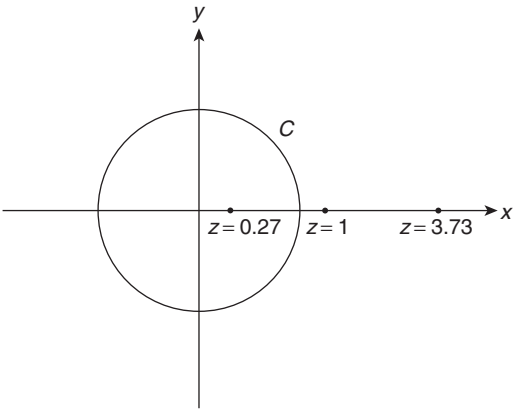


Fig. 5.4

(iii) Let $f(z) = \frac{1}{z^2 + 4z + 1}$

The poles are given by

$$z^2 - 4z + 1 = 0$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

$$\therefore f(z) = \frac{1}{\left[z - (-2+\sqrt{3})\right]\left[z - (-2-\sqrt{3})\right]}$$

(iv) For $z = -2+\sqrt{3}, |z| = \left|-2+\sqrt{3}\right| = 0.27 < 1$

Hence, $z = -2+\sqrt{3}$ lies inside C.

For $z = -2-\sqrt{3}, |z| = \left|-2-\sqrt{3}\right| = 3.73 > 1$

Hence, $z = -2-\sqrt{3}$ lies outside C.

(v) Res $\left[f(z); z = -2+\sqrt{3}\right] = \lim_{z \to -2+\sqrt{3}} \left[(z+2-\sqrt{3}) f(z)\right]$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}}$$

$$= \frac{1}{-2+\sqrt{3}+2+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{z^2+4z+1} = 2\pi i\left(\frac{1}{2\sqrt{3}}\right)$$

$$= \frac{\pi i}{\sqrt{3}}$$

Substituting in Eq. (1),

$$I = \frac{2}{i}\left(\frac{\pi i}{\sqrt{3}}\right)$$

$$= \frac{2\pi}{\sqrt{3}}$$

Example 5

Evaluate a real integral $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta$ *using residue.* **[Summer 2013]**

Solution

(i) Let $I = \int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.5).

Let $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \cos\theta = \frac{z^2+1}{2z}$

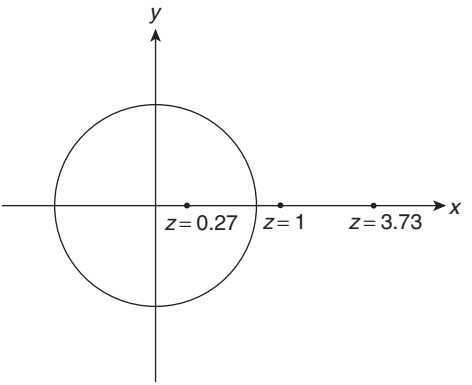


Fig. 5.5

(ii) $I = \int_C \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}$

$$= \frac{4}{i}\int_C \frac{z}{(4z+z^2+1)^2}\,dz$$

$$= \frac{4}{i}\int_C \frac{z}{(z^2+4z+1)^2}\,dz \quad \ldots(1)$$

(iii) Let $f(z) = \frac{z}{(z^2+4z+1)^2}$

The poles are given by

$$(z^2+4z+1)^2=0$$

$$z=\frac{-4\pm\sqrt{16-4}}{2}$$

$$=-2\pm\sqrt{3}$$

$$\therefore\quad f(z)=\frac{z}{\left[z-\left(-2+\sqrt{3}\right)\right]^2\left[z-\left(-2-\sqrt{3}\right)\right]^2}$$

$$=\frac{z}{\left(z+2-\sqrt{3}\right)^2\left(z+2+\sqrt{3}\right)^2}$$

(iv) For $z=-2+\sqrt{3}, |z|=\left|-2+\sqrt{3}\right|=0.27<1$

Hence, $z=-2+\sqrt{3}$ lies inside C.

For $z=-2-\sqrt{3}, |z|=\left|-2-\sqrt{3}\right|=3.73>1$

Hence, $z=-2-\sqrt{3}$ lies outside C.

(v) $z=-2+\sqrt{3}$ is a pole of order 2.

$$\text{Res}\left[f(z); z=-2+\sqrt{3}\right]=\frac{1}{(2-1)!}\lim_{z\to\left(-2+\sqrt{3}\right)}\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(z+2-\sqrt{3}\right)^2 f(z)\right]$$

$$=\lim_{z\to\left(-2+\sqrt{3}\right)}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{\left(z+2+\sqrt{3}\right)^2}$$

$$=\lim_{z\to\left(-2+\sqrt{3}\right)}\left[-\frac{2z}{\left(z+2+\sqrt{3}\right)^3}+\frac{1}{\left(z+2+\sqrt{3}\right)^2}\right]$$

$$=-\frac{2\left(-2+\sqrt{3}\right)}{\left(-2+\sqrt{3}+2+\sqrt{3}\right)^3}+\frac{1}{\left(-2+\sqrt{3}+2+\sqrt{3}\right)^2}$$

$$=-\frac{\left(-4+2\sqrt{3}\right)}{\left(2\sqrt{3}\right)^3}+\frac{1}{\left(2\sqrt{3}\right)^2}$$

$$=-\frac{\left(-2+\sqrt{3}\right)}{12\sqrt{3}}+\frac{1}{12}$$

$$=\frac{2-\sqrt{3}+\sqrt{3}}{12\sqrt{3}}$$

$$=\frac{2}{12\sqrt{3}}$$

$$=\frac{1}{6\sqrt{3}}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z}{(z^2+4z+1)^2}\,dz = 2\pi i\left(\frac{1}{6\sqrt{3}}\right)$$

$$= \frac{\pi i}{3\sqrt{3}}$$

Substituting in Eq. (1),

$$I = \frac{4}{i}\left(\frac{\pi i}{3\sqrt{3}}\right)$$

$$= \frac{4\pi}{3\sqrt{3}}$$

Example 6

Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.6).

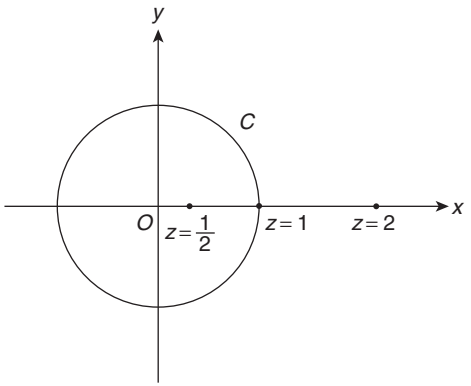


Fig. 5.6

$$z = e^{i\theta},\ d\theta = \frac{dz}{iz},\ \cos\theta = \frac{z^2+1}{2z}$$

$$z^3 = e^{3i\theta} = \cos 3\theta + i\cos 3\theta$$

$$\cos 3\theta = \text{RP}\ z^3$$

(ii) $$I = \text{RP}\int_C \frac{z^3}{5-4\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz}$$

$$= \text{RP}\,\frac{1}{i}\int_C \frac{z^3}{5z-2z^2-2}\,dz$$

$$= \text{RP}\left(-\frac{1}{i}\right)\int_C \frac{z^3}{2z^2-5z+2}\,dz \qquad \ldots(1)$$

(iii) Let $f(z) = \dfrac{z^3}{2z^2 - 5z + 2}$

The poles are given by

$$2z^2 - 5z + 2 = 0$$

$$2z^2 - 4z - z + 2 = 0$$

$$2z(z-2) - 1(z-2) = 0$$

$$(z-2)(2z-1) = 0$$

$$z = 2, \ \frac{1}{2}$$

$$\therefore \quad f(z) = \frac{z^3}{2(z-2)\left(z - \frac{1}{2}\right)} \qquad [\therefore ax^2 + bx + c = a(x-\alpha)(x-\beta)]$$

(iv) For $z = 2, |z| = |2| = 2 > 1$

Hence, $z = 2$ lies outside C.

For $z = \dfrac{1}{2}, |z| = \left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$

Hence, $z = \dfrac{1}{2}$ lies inside C.

(v) Res $\left[f(z); z = \dfrac{1}{2}\right] = \lim\limits_{z \to \frac{1}{2}} \left[\left(z - \dfrac{1}{2}\right) f(z)\right]$

$$= \lim_{z \to \frac{1}{2}} \frac{z^3}{2(z-2)}$$

$$= \frac{\frac{1}{8}}{2\left(\frac{1}{2} - 2\right)}$$

$$= -\frac{1}{24}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i (\text{sum of residues})$$

$$\int_C \frac{z^3 dz}{2z^2 - 5z + 2} = 2\pi i \left(-\frac{1}{24}\right)$$

$$= -\frac{\pi i}{12}$$

Substituting in Eq. (1),

$$I = \text{RP}\left(-\frac{1}{i}\right)\left(\frac{-\pi i}{12}\right)$$

$$= \frac{\pi}{12}$$

Example 7

Evaluate $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.7).

Let $z = e^{i\theta}$, $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$,

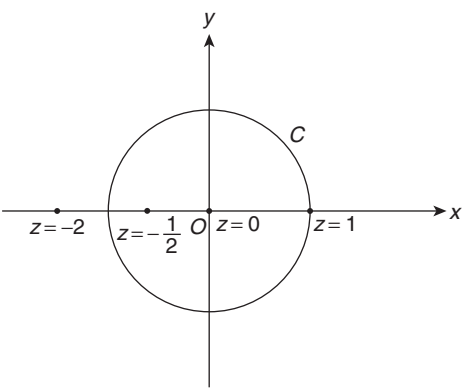


Fig. 5.7

(ii) $$I = \int_C \frac{1+2\left(\frac{z^2+1}{2z}\right)}{5+4\left(\frac{z^2+1}{2z}\right)}\frac{\mathrm{d}z}{iz}$$

$$= \frac{1}{i}\int_C \frac{z+z^2+1}{z(5z+2z^2+2)}\,\mathrm{d}z$$

$$= \frac{1}{i}\int_C \frac{z^2+z+1}{z(2z^2+5z+2)}\,\mathrm{d}z \qquad \ldots(1)$$

(iii) Let $f(z) = \frac{z^2+z+1}{z(2z^2+5z+2)}$

The poles are given by

$$z(2z^2+5z+2) = 0$$

$$z(z+2)(2z+1) = 0$$

$$z = 0, -2, -\frac{1}{2}$$

$$\therefore f(z) = \frac{z^2+z+1}{2z(z+2)\left(z+\frac{1}{2}\right)} \qquad [\because ax^2 + bx + c = a(x-\alpha)(x-\beta)]$$

(iv) For $z = 0, |z| = 0 < 1$

Hence, $z = 1$ lies inside C.

For, $z = -2,\ |z| = |-2| = 2 > 1$

Hence, $z = -2$ lies outside C.

For, $z = -\frac{1}{2}, |z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

Hence, $z = -\frac{1}{2}$ lies inside C.

(v) $\text{Res}\ \left[f(z); z = 0\right] = \lim\limits_{z \to 0} zf(z)$

$$= \lim_{z \to 0} \frac{z^2 + z + 1}{2(z+2)\left(z + \frac{1}{2}\right)}$$

$$= \frac{0 + 0 + 1}{2(0+2)\left(0 + \frac{1}{2}\right)}$$

$$= \frac{1}{2}$$

$$\text{Res}\left[f(z); z = -\frac{1}{2}\right] = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{1}{2}} \frac{z^2 + z + 1}{2z(z+2)}$$

$$= \frac{\frac{1}{4} - \frac{1}{2} + 1}{2\left(-\frac{1}{2}\right)\left(-\frac{1}{2} + 2\right)}$$

$$= -\frac{1}{2}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2 + z + 1}{z(2z^2 + 5z + 2)} dz = 2\pi i \left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= 0$$

Substituting in Eq. (1),

$$I = \frac{1}{i}(0) = 0$$

Example 8

Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.8).

$z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $\sin\theta = \frac{z^2-1}{2iz}$,

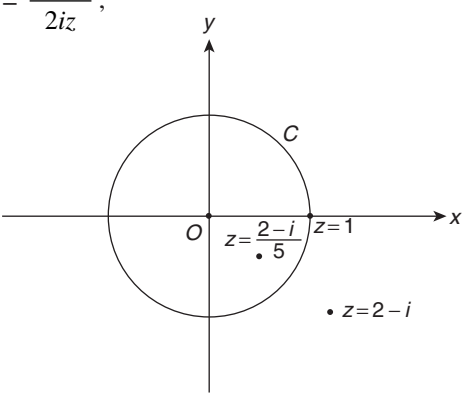


Fig. 5.8

(ii) $$I = \int_C \frac{\frac{dz}{iz}}{3-2\left(\frac{z^2+1}{2z}\right)+\left(\frac{z^2-1}{2iz}\right)}$$

$$= \int_C \frac{\frac{dz}{iz}}{3-\frac{2z^2+2}{2z}+\frac{z^2-1}{2iz}}$$

$$= \int_C \frac{2dz}{6iz-(2z^2+2)i+(z^2-1)}$$

$$= 2\int \frac{dz}{(1-2i)z^2+6iz-(2i+1)} \quad \ldots(1)$$

(iii) Let $f(z) = \frac{1}{(1-2i)z^2+6iz-(2i+1)}$

The poles are given by

$$(1-2i)z^2+6iz-(2i+1) = 0$$

$$z = \frac{-6i \pm \sqrt{(6i)^2+4(1-2i)(2i+1)}}{2(1-2i)}$$

$$= \frac{-6i \pm \sqrt{-36+4(2i+1+4-2i)}}{2(1-2i)}$$

$$= \frac{-6i \pm \sqrt{-36+20}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)}$$

$$= \frac{i}{1-2i}, \frac{-5i}{1-2i}$$

$$= \frac{2-i}{5},\ 2-i$$

$$\therefore \quad f(z) = \frac{1}{(1-2i)\left[z-(2-i)\right]\left[z-\left(\frac{2-i}{5}\right)\right]} \qquad [\because\ ax^2+bx+c = a(x-\alpha)\,(x-\beta)]$$

(iv) For $z = 2-i, |z| = |2-i| = \sqrt{5} > 1$

Hence, $z = 2 - i$ lies outside C.

For $z = \frac{2-i}{5}, |z| = \left|\frac{2-i}{5}\right| = \frac{\sqrt{5}}{5} = 0.45 < 1$

Hence, $z = \frac{2-i}{5}$ lies inside C.

(v) $\text{Res}\left[f(z); z = \frac{2-i}{5}\right] = \lim\limits_{z \to \frac{2-i}{5}} \left[\left(z - \frac{2-i}{5}\right) f(z)\right]$

$$= \lim_{z \to \frac{2-i}{5}} \frac{1}{(1-2i)(z-2+i)}$$

$$= \frac{1}{(1-2i)\left(\frac{2-i}{5}-2+i\right)}$$

$$= \frac{1+2i}{5\left(\frac{-8+4i}{5}\right)}$$

$$= \frac{1+2i}{4i(2i+1)}$$

$$= \frac{1}{4i}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{(1-2i)z^2 + 6iz - (2i+1)} = 2\pi i\left(\frac{1}{4i}\right)$$

$$= \frac{\pi}{2}$$

Substituting in Eq. (1),

$$I = 2\left(\frac{\pi}{2}\right) = \pi$$

Example 9

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $(0<a<1)$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.9).

Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$,

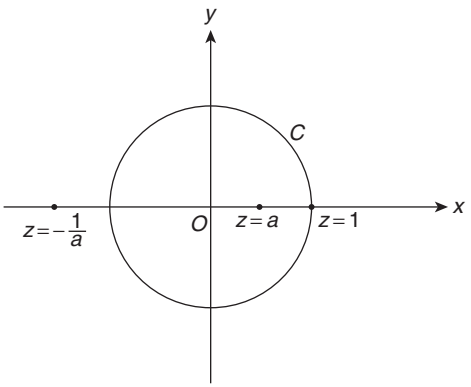


Fig. 5.9

(ii) $$I = \int_C \frac{\frac{dz}{iz}}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2}$$

$$= \frac{1}{i}\int_C \frac{dz}{z-az^2-a+a^2z}$$

$$= \frac{1}{i}\int_C \frac{dz}{-az^2+(a^2+1)z-a}$$

$$= -\frac{1}{i}\int_C \frac{dz}{az^2-(a^2+1)z+a} \quad \ldots(1)$$

(iii) Let $f(z) = \frac{1}{az^2-(a^2+1)z+a}$

The poles are given by

$$z^2 - \left(\frac{a^2+1}{a}\right)z + 1 = 0$$

$$z = \frac{a^2+1\pm\sqrt{(a^2+1)^2-4a^2}}{2a}$$

$$= \frac{a^2+1\pm\sqrt{a^2+2a^2+1-4a^2}}{2a}$$

$$= \frac{a^2 + 1 \pm \sqrt{(a^2-1)^2}}{2a}$$

$$= \frac{a^2 + 1 \pm (a^2 - 1)}{2a}$$

$$= a, \frac{1}{a}$$

$$\therefore\ f(z) = \frac{1}{a(z-a)\left(z - \frac{1}{a}\right)} \qquad [\because\ ax^2 + bx + c = a(x-\alpha)\,(x-\beta)]$$

(iv) For $z = a, |z| = |a| = a < 1$

Hence, $z = a$ lies inside C.

For $z = \frac{1}{a}, |z| = \left|\frac{1}{a}\right| = \frac{1}{a} > 1$

Hence, $z = \frac{1}{a}$ lies outside C.

(v) $\text{Res}\left[f(z); z = a\right] = \lim\limits_{z \to a} \left[(z-a) f(z)\right]$

$$= \lim_{z \to a} \frac{1}{a\left(z - \frac{1}{a}\right)}$$

$$= \frac{1}{a\left(a - \frac{1}{a}\right)}$$

$$= \frac{1}{a^2 - 1}$$

(vi) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{dz}{az^2 - (a^2+1)z + a} = 2\pi i \left(\frac{1}{a^2 - 1}\right)$$

Substituting in Eq. (1),

$$I = -\frac{1}{i} 2\pi i \left(\frac{1}{a^2 - 1}\right)$$

$$= -\frac{2\pi}{a^2 - 1}$$

$$= \frac{2\pi}{1 - a^2}, 0 < a < 1$$

Example 10

Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,\mathrm{d}\theta,\ a > b > 0$ *using contour integration.*

Solution

(i) Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,\mathrm{d}\theta$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)}\,\mathrm{d}\theta$$

Consider the contour C as the unit circle $|z| = 1$ (Fig. 5.10).

$z = e^{i\theta}$, $\mathrm{d}\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$z^2 = e^{2i\theta} = \cos 2\theta + i \sin 2\theta$

$\cos 2\theta = \mathrm{RP}\ z^2$

$1 - \cos 2\theta = \mathrm{RP}\ (1 - z^2)$

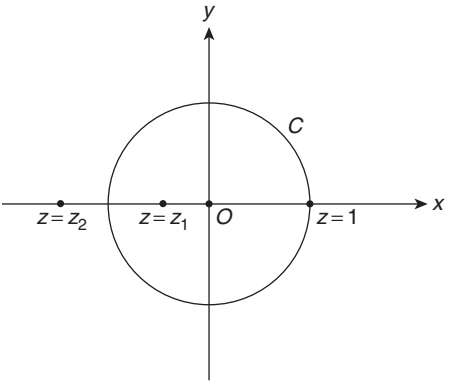


Fig. 5.10

(ii) $I = \mathrm{RP} \int_C \frac{(1-z)^2 \frac{\mathrm{d}z}{iz}}{2\left[a+b\left(\frac{z^2+1}{2z}\right)\right]}$

$$= \mathrm{RP}\frac{1}{i}\int_C \frac{(1-z^2)\mathrm{d}z}{2az+b(z^2+1)}$$

$$= \mathrm{RP}\frac{1}{i}\int_C \frac{(1-z^2)\mathrm{d}z}{bz^2+2az+b} \qquad \ldots(1)$$

(iii) Let $f(z) = \frac{1-z^2}{bz^2-2az+b}$

The poles are given by

$$bz^2 + 2az + b = 0$$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$= \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Let $z_1 = \dfrac{-a+\sqrt{a^2-b^2}}{b}$, $z_2 = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

$\therefore \quad f(z) = \dfrac{1-z^2}{b(z-z_1)(z-z_2)}$

(iv) For $z = z_1 = \dfrac{-a+\sqrt{a^2-b^2}}{b}$

$$|z| = |z_1| = \left|\frac{-a+\sqrt{a^2-b^2}}{b}\right| = \frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1} < 1 \qquad \left[\because \quad a > b, \frac{a}{b} > 1\right]$$

Hence, $z = z_1$ lies inside C.

For $z = z_2 = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

$$|z| = |z_2| = \left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| = \frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1} > 1 \qquad \left[\because \quad a > b, \frac{a}{b} > 1\right]$$

Hence, $z = z_2$ lies outside C.

(v) $\text{Res}\left[f(z); z = z_1\right] = \lim\limits_{z \to z_1} \left[(z-z_1)f(z)\right]$

$$= \lim_{z \to z_1} \frac{(1-z^2)}{b(z-z_2)}$$

$$= \frac{1-z_1^{\,2}}{b(z_1-z_2)}$$

$$= \frac{\left[1-\left(\dfrac{-a+\sqrt{a^2-b^2}}{b}\right)^2\right]}{b\left(\dfrac{2\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{b^2-\left[a^2+(a^2-b^2)-2a\sqrt{a^2-b^2}\right]}{2b^2\sqrt{a^2-b^2}}$$

$$= \frac{2(b^2-a^2)-2a\sqrt{a^2-b^2}}{2b^2\sqrt{a^2-b^2}}$$

$$= \frac{-\sqrt{a^2-b^2}-a}{b^2}$$

$$= -\left(\frac{a+\sqrt{a^2-b^2}}{b^2}\right)$$

(vi) $\int_C f(z)\,dz = 2\pi i$ (sum of residues)

$$\int_C \frac{(1-z^2)}{bz^2-2az+b}dz = -2\pi i\left(\frac{a+\sqrt{a^2-b^2}}{b^2}\right)$$

Substituting in Eq. (1),

$$I = \text{RP}\,\frac{1}{i}\left[-2\pi i\left(\frac{a+\sqrt{a^2-b^2}}{b^2}\right)\right]$$

$$= \frac{-2\pi\left(a+\sqrt{a^2-b^2}\right)}{b^2}$$

EXERCISE 5.1

Evaluate the following integrals using contour integration:

1. $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ [Ans.: $\frac{2\pi}{3}$]

2. $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ [Ans.: $\frac{2\pi}{3}$]

3. $\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}d\theta$ [Ans.: 0]

4. $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}$ [Ans.: $\frac{\pi}{6}$]

5. $\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}d\theta$ [Ans.: $\frac{2\pi}{3}$]

6. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, $a > b > 0$ [Ans.: $\frac{2\pi}{\sqrt{a^2-b^2}}$]

7. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ $\left[\text{Ans.: } \frac{2\pi}{\sqrt{3}}\right]$

8. $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$ $\left[\text{Ans.: } \frac{\pi}{2}\right]$

9. $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a > b > 0$ $\left[\text{Ans.: } \frac{2\pi}{\sqrt{a^2-b^2}}\right]$

10. $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}, |a| < 1$ $\left[\text{Ans.: } \frac{2\pi}{\sqrt{1-a^2}}\right]$

11. $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, a > b > 0$ $\left[\text{Ans.: } \frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}\right]$

12. $\int_0^{2\pi} \frac{d\theta}{(5-4\cos\theta)^2}$ $\left[\text{Ans.: } \frac{10\pi}{27}\right]$

5.3 EVALUATION OF IMPROPER REAL INTEGRAL OF A RATIONAL FUNCTION

Let $I = \int_{-\infty}^{\infty} f(x)\,dx$

where $f(x) = \frac{P(x)}{Q(x)}$ and the degree of $Q(x)$ is greater than the degree of $P(x)$ by at least 2 and $Q(x)$ has no real roots.

To evaluate the integral, consider $\int_C f(z)dz$, where C is the contour consisting of the upper semicircle C_1 of radius R with centre at the origin and the part of real axis from $-R$ to R (Fig. 5.11).

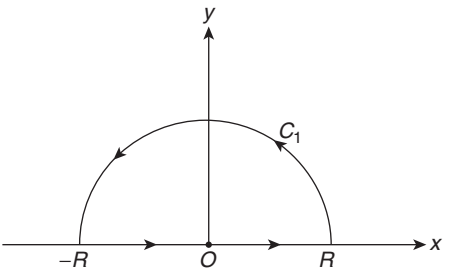


Fig. 5.11

$$\int_C f(z)\,dz = \int_{C_1} f(z)dz + \int_{-R}^{R} f(x)\,dx$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles in the upper half of the } z\text{-plane)}$$

$$\int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \text{ (sum of resides)}$$

Taking limit $R \to \infty$, $\lim_{R\to\infty} \int_{C_1} f(z)\,dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i$ (sum of residues)

By Cauchy's lemma, $\lim_{R\to\infty} \int_{C_1} f(z)dz = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

Example 1

Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$ *using contour integration.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{z^2\,dz}{(z^2+1)(z^2+4)}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and part of the axis from $-R$ to R (Fig. 5.12).

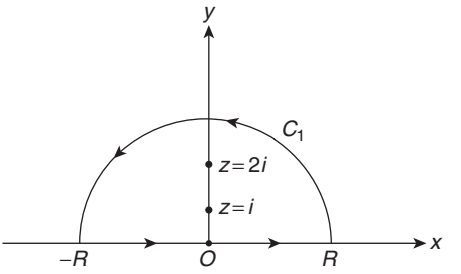


Fig. 5.12

(ii) Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$

The poles are given by

$(z^2+1)(z^2+4) = 0$

$z = \pm i, \pm 2i$

$$\therefore \quad f(z) = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$$

(iii) The poles $z = i$ and $z = 2i$ lie inside C.

(iv) $\text{Res}\left[f(z); z = i\right] = \lim_{z\to i} \left[(z-i) f(z)\right]$

$$= \lim_{z \to i} \frac{z^2}{(z+i)(z+2i)(z-2i)}$$

$$= \frac{i^2}{(2i)(3i)(-i)}$$

$$= -\frac{1}{6i}$$

$$\text{Res}\left[f(z); z = 2i\right] = \lim_{z \to 2i} \left[\left(z-2i\right) f(z)\right]$$

$$= \lim_{z \to 2i} \frac{z^2}{(z+i)(z-i)(z+2i)}$$

$$= \frac{4i^2}{(3i)(2i)(4i)}$$

$$= \frac{-4}{-12i}$$

$$= \frac{1}{3i}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)} = 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right)$$

$$= 2\pi\left(\frac{1}{6}\right)$$

$$= \frac{\pi}{3}$$

Example 2

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,\mathrm{d}x$ *using contour integration.*

Solution

(i) Let $\displaystyle\int_C f(z)\mathrm{d}z = \int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,\mathrm{d}z$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.13).

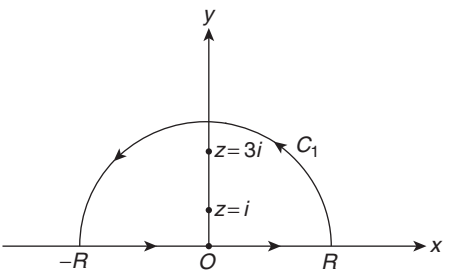


Fig. 5.13

(ii) Let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

The poles are given by

$$z^4 + 10z^2 + 9 = 0$$

$$(z^2 + 1)(z^2 + 9) = 0$$

$$z = \pm i, \pm 3i$$

$$\therefore \quad f(z) = \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)(z-3i)}$$

(iii) The poles $z = i$ and $z = 3i$ lie inside C.

(iv) $$\text{Res}\left[f(z); z = i\right] = \lim_{z \to i} (z - i) f(z)$$

$$= \lim_{z \to i} \frac{z^2 - z + 2}{(z+i)(z+3i)(z-3i)}$$

$$= \frac{i^2 - i + 2}{(2i)(4i)(-2i)}$$

$$= \frac{-1 - i + 2}{16i}$$

$$= \frac{1 - i}{16i}$$

$$\text{Res}\left[f(z); z = 3i\right] = \lim_{z \to 3i} (z - 3i) f(z)$$

$$= \lim_{z \to 3i} \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)}$$

$$= \frac{(3i)^2 - 3i + 2}{(4i)(2i)(6i)}$$

$$= \frac{-9 - 3i + 2}{-48i}$$

$$= \frac{7 + 3i}{48i}$$

(v) $$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = 2\pi i \left(\frac{1 - i}{16i} + \frac{7 + 3i}{48i} \right)$$

$$= 2\pi i \left[\frac{3(1 - i) + 7 + 3i}{48i} \right]$$

$$= \frac{10\pi}{24}$$

$$= \frac{5\pi}{12}$$

Example 3

Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$ *using contour integration.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{z^2}{(z^2+1)^2}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.14).

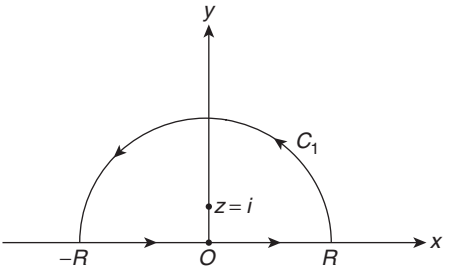


Fig. 5.14

(ii) Let $f(z) = \frac{z^2}{(z^2+1)^2}$

The poles are given by

$$(z^2+1)^2 = 0$$

$$z = \pm i, \pm i$$

$$\therefore \quad f(z) = \frac{z^2}{(z+i)^2(z-i)^2}$$

(iii) The poles $z = i$ of order 2 lies inside C.

(iv) $\text{Res}\,[f(z); z = i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)]$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]$$

$$= \lim_{z\to i}\frac{(z+i)^2(2z) - z^2 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{(z+i)[(z+i)(2z) - 2z^2]}{(z+i)^4}$$

$$= \lim_{z\to i}\frac{2iz}{(z+i)^3}$$

$$= \frac{2i \cdot i}{(i+i)^3}$$

$$= \frac{-2}{-8i}$$

$$= \frac{1}{4i}$$

(v) $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i(\text{sum of residues})$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = 2\pi i\left(\frac{1}{4i}\right)$$

$$= \frac{\pi}{2}$$

Example 4

Evaluate $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ *using contour integration.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{1}{z^2+1}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.15).

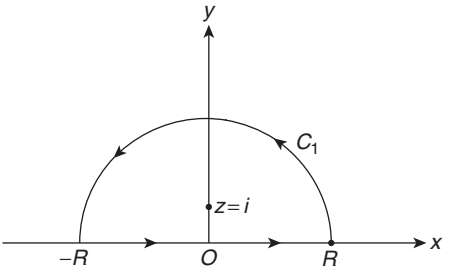


Fig. 5.15

(ii) Let $f(z) = \frac{1}{z^2+1}$

The poles are given by

$$z^2 + 1 = 0$$

$$z = \pm i$$

$$\therefore \quad f(z) = \frac{1}{(z+i)(z-i)}$$

(iii) The pole $z = i$ lies inside C.

(iv) $\text{Res}[f(z); z = i] = \lim_{z \to i}(z-i)f(z)$

$$= \lim_{z \to i} \frac{1}{z+i}$$

$$= \frac{1}{2i}$$

(v) $$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi i(\text{sum of residues})$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = 2\pi i\left(\frac{1}{2i}\right)$$

$$= \pi$$

$$2\int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \pi \quad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2}$$

Example 5

Evaluate $\int_{0}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^2}$ *using contour integration.*

Solution

(i) Let $$\int_C f(z)\mathrm{d}z = \int_C \frac{\mathrm{d}z}{(1+z^2)^2}$$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.16).

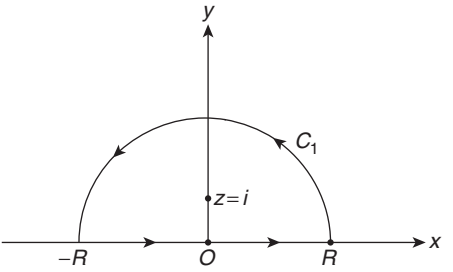


Fig. 5.16

(ii) Let $f(z) = \frac{1}{(1+z^2)^2}$

The poles are given by

$$(1+z^2)^2 = 0$$

$$1+z^2 = 0$$

$$z^2 = -1$$

$$z = \pm i$$

$$\therefore \quad f(z) = \frac{1}{(z+i)^2(z-i)^2}$$

(iii) The pole $z = i$ of order 2 lies inside C.

(iv) $$\text{Res}[f(z); z = i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}[(z-i)^2 f(z)]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= \lim_{z \to i}\left[-\frac{2}{(z+i)^3}\right]$$

$$= -\frac{2}{(i+i)^3}$$

$$= \frac{2}{8i}$$

$$= \frac{1}{4i}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = 2\pi i\left(\frac{1}{4i}\right)$$

$$= \frac{\pi}{2}$$

$$2\int_{0}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{2} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{4}$$

Example 6

Evaluate $\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$, $a > 0$, $b > 0$

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.17).

(ii) Let $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

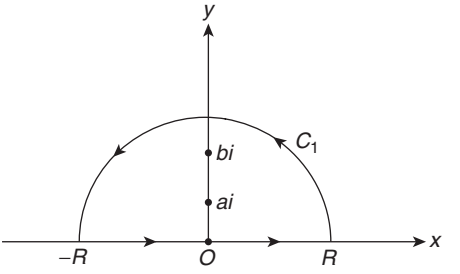


Fig. 5.17

The poles are given by

$$(z^2 + a^2)(z^2 + b^2) = 0$$

$$z = \pm ai, \pm bi$$

$$\therefore f(z) = \frac{z^2}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

(iii) The poles $z = ai$ and $z = bi$ lie inside C as $a > 0$, $b > 0$.

(iv)
$$\text{Res}[f(z); z = ai] = \lim_{z \to ai}(z - ai)f(z)$$

$$= \lim_{z \to ai} \frac{z^2}{(z+ai)(z+bi)(z-bi)}$$

$$= \frac{a^2 i^2}{(2ai)(ai+bi)(ai-bi)}$$

$$= -\frac{a}{2i(b^2 - a^2)}$$

$$\text{Res}[f(z); z = bi] = \lim_{z \to bi}(z - bi)f(z)$$

$$= \lim_{z \to bi} \frac{z^2}{(z+ai)(z-ai)(z+bi)}$$

$$= \frac{b^2 i^2}{(bi+ai)(bi-ai)(2bi)}$$

$$= -\frac{b}{2i(a^2 - b^2)}$$

$$= \frac{b}{2i(b^2 - a^2)}$$

(v)
$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi i(\text{sum of residues})$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x = 2\pi i\left[-\frac{a}{2i(b^2 - a^2)} + \frac{b}{2i(b^2 - a^2)}\right]$$

$$= \pi\left[\frac{b-a}{b^2 - a^2}\right]$$

$$= \frac{\pi}{b+a}$$

$$2\int_{0}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x = \frac{\pi}{b+a} \quad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\mathrm{d}x = \frac{\pi}{2(b+a)}$$

Example 7

Show that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{dz}{1+z^4}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.18).

(ii) Let $f(z) = \frac{1}{1+z^4}$

The poles are given by

$$1+z^4 = 0$$

$$z^4 = -1$$

$$z = (-1)^{\frac{1}{4}}$$

$$= e^{\frac{i\pi}{4}}, e^{-\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{-\frac{i3\pi}{4}}$$

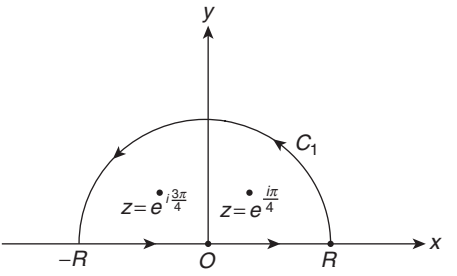


Fig. 5.18

(iii) The poles $z = e^{\frac{i\pi}{4}}$ and $z = e^{\frac{i3\pi}{4}}$ lie inside C.

(iv) $\text{Res}\left[f(z); z = e^{i\frac{\pi}{4}}\right] = \lim_{z \to e^{i\frac{\pi}{4}}} (z - e^{i\frac{\pi}{4}}) f(z)$

$$= \lim_{z \to e^{i\frac{\pi}{4}}} \frac{z - e^{i\frac{\pi}{4}}}{1+z^4} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z \to e^{i\frac{\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4e^{3i\frac{\pi}{4}}}$$

$$= \frac{1}{4} e^{-i\frac{3\pi}{4}}$$

$$= \frac{1}{4}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i - \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{(1+i)}{4\sqrt{2}}$$

$$\text{Res}\,[f(z); z = e^{i\frac{3\pi}{4}}] = \lim_{z \to e^{i\frac{3\pi}{4}}} (z - e^{i\frac{3\pi}{4}}) f(z)$$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{z - e^{i\frac{3\pi}{4}}}{1 + z^4} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{4z^3} \qquad [\text{Applying L'Hospital rule}]$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$= \frac{1}{4} e^{-i\frac{9\pi}{4}}$$

$$= \frac{1}{4}\left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right)$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1-i}{4\sqrt{2}}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x = 2\pi i\left[-\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right]$$

$$= 2\pi i\left(-\frac{2i}{4\sqrt{2}}\right)$$

$$= \frac{\pi}{\sqrt{2}}$$

$$2\int_{0}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}} \qquad \left[\because \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}$$

Example 8

Using the theory of residues, evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$. **[Summer 2015]**

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z^2+1}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.19).

(ii) Let $f(z) = \frac{e^{iz}}{z^2+1}$

The poles are given by

$$z^2+1=0$$
$$z=\pm i$$
$$\therefore \quad f(z)=\frac{e^{iz}}{(z+i)(z-i)}$$

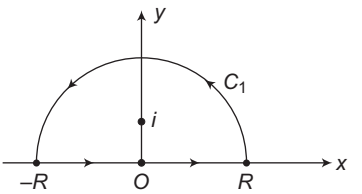


Fig. 5.19

(iii) The pole $z = i$ lies inside C.

(iv) $$\text{Res}\,[f(z); z=i] = \lim_{z\to i}(z-i)\,f(z)$$
$$= \lim_{z\to i}\frac{e^{iz}}{(z+i)}$$
$$= \frac{e^{i^2}}{2i}$$
$$= \frac{e^{-1}}{2i}$$
$$= \frac{1}{2ie}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \ \text{(sum of residues)}$$
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} = 2\pi i\left(\frac{1}{2ie}\right)$$
$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+1}\,dx = \frac{\pi}{e}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$$

EXERCISE 5.2

Evaluate the following integrals using contour integration:

1. $\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)}$ [Ans.: $\frac{\pi}{10}$]

2. $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a > b > 0$ [Ans.: $\frac{\pi}{2ab(a+b)}$]

3. $\int_0^{\infty} \frac{dx}{x^4+10x^2+9}$ [Ans.: $\frac{\pi}{24}$]

4. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ [Ans.: $\frac{3\pi}{8}$]

5. $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$, $a > 0$ [Ans.: $\frac{\pi}{4a^3}$]

6. $\int_{-\infty}^{\infty} \frac{x^2+x+3}{x^4+5x^2+4}\,dx$ [Ans.: $\frac{5\pi}{6}$]

7. $\int_0^{\infty} \frac{x^2}{(x^2+1)^3}\,dx$ [Ans.: $\frac{\pi}{6}$]

8. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx,\ a>0$

[**Ans.:** $\frac{3\pi}{2a}$]

9. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+2x+2)}\,dx$

[**Ans.:** $\frac{3\pi}{5}$]

10. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+25)}$

[**Ans.:** $\frac{\pi}{8}$]

5.4 EVALUATION OF IMPROPER REAL INTEGRAL OF A RATIONAL FUNCTION INCLUDING TRIGONOMETRIC FUNCTIONS

Let $I = \int_{-\infty}^{\infty} \frac{\cos mx}{Q(x)}\,dx$ or $\int_{-\infty}^{\infty} \frac{\sin mx}{Q(x)}\,dx$

where $Q(x)$ is a polynomial in x.

To evaluate the integral,

consider $\int_C f(z)\,dz = \int e^{imz} g(z)\,dz$

where $g(z) = \frac{1}{Q(z)}$ and C is the contour consisting of the upper semicircle C_1 of radius R with centre at the origin and the part of the real axis from $-R$ to R (Fig. 5.20).

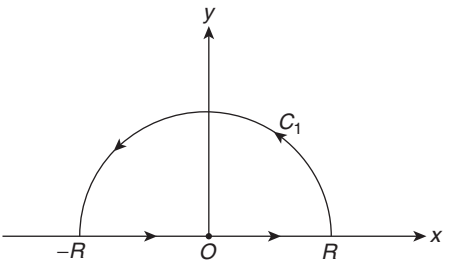


Fig. 5.20

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles in the upper half of } z\text{-plane)}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_1} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

Taking $R \to \infty$,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_1} f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \int_{C_1} e^{imz} g(z)\,dz = 2\pi i \text{(sum of residues)}$$

By Jordan's lemma, $\lim_{R\to\infty} \int_{C_1} e^{imz} g(z)\,dz = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

Example 1

Evaluate $\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\,dx, a > 0, m > 0$ *using contour integration.*

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{imz}}{z^2 + a^2}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.21).

(ii) Let $f(z) = \frac{e^{imz}}{z^2 + a^2}$

The poles are given by

$$z^2 + a^2 = 0$$

$$z = \pm ai$$

$$\therefore f(z) = \frac{e^{imz}}{(z + ai)(z - ai)}$$

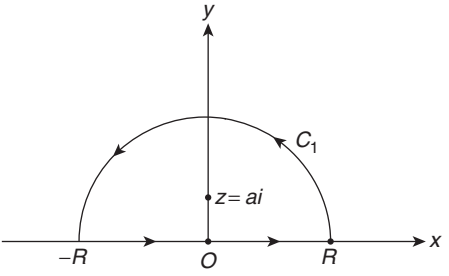


Fig. 5.21

(iii) The pole $z = ai$ lies inside C.

(iv) $\text{Res}[f(z); z = ai] = \lim_{z\to ai} (z - ai) f(z)$

$$= \lim_{z\to ai} \frac{e^{imz}}{(z + ai)}$$

$$= \frac{e^{im(ai)}}{(ai + ai)}$$

$$= \frac{e^{-ma}}{2ai}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = 2\pi i\left(\frac{e^{-ma}}{2ai}\right)$$

$$= \frac{\pi e^{-ma}}{a}$$

$$\int_{-\infty}^{\infty} \frac{(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a} \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2a}$$

Example 2

Evaluate a real integral $\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx$ *using residue.* **[Winter 2013]**

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{ze^{iz}}{z^2+9}\,dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.22).

(ii) Let $f(z) = \frac{ze^{iz}}{z^2+9}$

The poles are given by

$$z^2 + 9 = 0$$

$$z = \pm 3i$$

$$\therefore \quad f(z) = \frac{ze^{iz}}{(z+3i)(z-3i)}$$

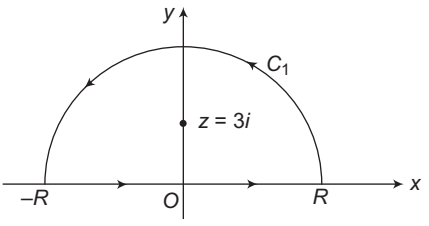


Fig. 5.22

(iii) The pole $z = 3i$ lies inside C.

(iv) $\text{Res}\,[f(z); z = 3i] = \lim_{z \to 3i} (z - 3i) f(z)$

$$= \lim_{z \to 3i} \frac{ze^{iz}}{z + 3i}$$

$$= \frac{3ie^{3i^2}}{3i + 3i}$$

$$= \frac{3ie^{-3}}{6i}$$

$$= \frac{e^{-3}}{2}$$

$$= \frac{1}{2e^3}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 9}\,dx = 2\pi i\left(\frac{1}{2e^3}\right)$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{x^2 + 9}\,dx = \frac{\pi i}{e^3}$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{e^3}$$

$$2\int_{0}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{e^3} \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + 9}\,dx = \frac{\pi}{2e^3}$$

Example 3

Evaluate $\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2}\,dx, a > 0, m > 0.$

Solution

(i) Let $\int_C f(z)dz = \frac{ze^{imz}}{z^2 + a^2}\,dz$

Consider the contour C consisting of upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.23).

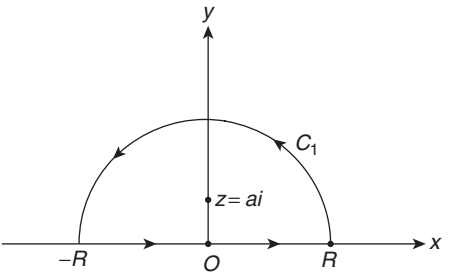


Fig. 5.23

(ii) Let $f(z) = \dfrac{ze^{imz}}{z^2 + a^2}$

The poles are given by

$$z^2 + a^2 = 0$$

$$z = \pm ai$$

$$\therefore f(z) = \frac{ze^{imz}}{(z + ai)(z - ai)}$$

(iii) The pole $z = ai$ lies inside C.

(iv) $\text{Res}\left[f(z); z = ai\right] = \lim\limits_{z \to ai} (z - ai) f(z)$

$$= \lim_{z \to ai} \frac{ze^{imz}}{z + ai}$$

$$= \frac{aie^{i^2 ma}}{ai + ai}$$

$$= \frac{aie^{-ma}}{2ai}$$

$$= \frac{e^{-ma}}{2}$$

(v) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \,(\text{sum of residues})$

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2 + a^2} dx = 2\pi i \left(\frac{e^{-ma}}{2} \right)$$

$$= \pi i e^{-ma}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos mx + i \sin mx)}{x^2 + a^2} dx = \pi i e^{-ma}$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \pi e^{-ma}$$

$$2\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \pi e^{-ma} \quad \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x) \text{ if } f(-x) = f(x) \right]$$

$$\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi e^{-ma}}{2}$$

Example 4

Evaluate $\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx, m>0$ *using contour integration.*

Solution

(i) Let $f(z)dz = \int_C \frac{e^{imz}}{(1+z^2)^2}\,dz$

Consider the contour C consisting of the upper semicircle of radius R and the part of the real axis from $-R$ to R (Fig. 5.24).

(ii) Let $f(z) = \frac{e^{imz}}{(1+z^2)^2}$

The poles are given by

$$(1+z^2)^2 = 0$$
$$1+z^2 = 0$$
$$z = \pm i$$
$$\therefore f(z) = \frac{e^{imz}}{(z+i)^2(z-i)^2}$$

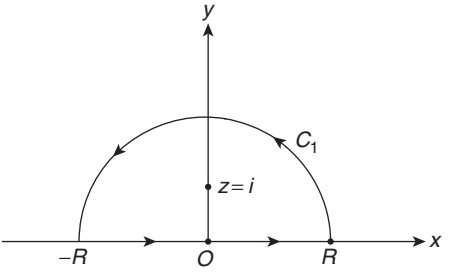


Fig. 5.24

(iii) The pole $z = i$ of order 2 lies inside C.

(iv) $\text{Res}[f(z); z=i] = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)]$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{imz}}{(z+i)^2}\right]$$
$$= \lim_{z\to i}\frac{(z+i)^2 e^{imz}\cdot im - e^{imz}2(z+i)}{(z+i)^4}$$
$$= \lim_{z\to i}\frac{[im(z+i)-2]e^{imz}}{(z+i)^3}$$
$$= \frac{[(i+i)im-2]e^{i^2 m}}{(i+i)^3}$$
$$= \frac{(-2m-2)e^{-m}}{-8i}$$
$$= \frac{(m+1)e^{-m}}{4i}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i(\text{sum of residues})$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(1+x^2)^2}\,dx = 2\pi i\left[\frac{(m+1)e^{-m}}{4i}\right]$$

$$= \frac{\pi(m+1)e^{-m}}{2}$$

$$\int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{2} \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi(m+1)e^{-m}}{4}$$

Example 5

Evaluate $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}, a>0, b>0$ *using contour integration.*

Solution

(i) Let $\int_C f(z)dz = \int_C \frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)}$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.25).

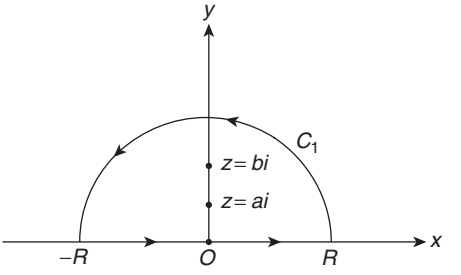


Fig. 5.25

(ii) Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

The poles are given by

$(z^2+a^2)(z^2+b^2) = 0$

$z = \pm ai, \pm bi$

$$\therefore f(z) = \frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

(iii) The poles $z = ai$ and $z = bi$ lie inside C.

(iv) $\text{Res}\,[f(z); z = ai] = \lim_{z \to ai} (z - ai) f(z)$

$$= \lim_{z \to ai} \frac{e^{iz}}{(z+ai)(z+bi)(z-bi)}$$

$$= \frac{e^{i(ai)}}{(2ai)(ai+bi)(ai-bi)}$$

$$= \frac{e^{-a}}{-2ai(a^2 - b^2)}$$

$$\text{Res}[f(z); z = bi] = \lim_{z \to bi} (z - bi) f(z)$$

$$= \lim_{z \to bi} \frac{e^{iz}}{(z+ai)(z-ai)(z+bi)}$$

$$= \frac{e^{i(bi)}}{(bi+ai)(bi-ai)(2bi)}$$

$$= \frac{e^{-b}}{2bi(a^2 - b^2)}$$

(v) $$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = 2\pi i \left[\frac{e^{-a}}{-2ai(a^2-b^2)} + \frac{e^{-b}}{2bi(a^2-b^2)} \right]$$

$$= \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{\cos x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Example 6

Evaluate $\int_0^\infty \frac{x^2 \cos mx}{(x^2+a^2)(x^2+b^2)}\,dx,\ m>0,\ a>b>0.$

Solution

(i) Let $\int_C f(z)dz = \int_C \frac{z^2 e^{imz}}{(z^2+a^2)(z^2+b^2)}dz$

Consider the contour C consisting of the upper semicircle C_1 of radius R and the part of the real axis from $-R$ to R (Fig. 5.26).

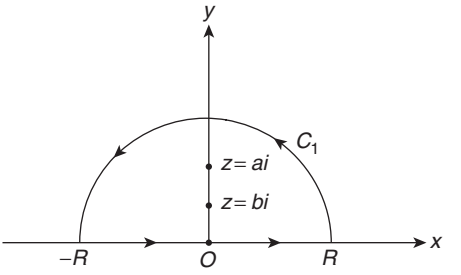


Fig. 5.26

(ii) Let $f(z) = \frac{z^2 e^{imz}}{(z^2+a^2)(z^2+b^2)}$

The poles are given by

$(z^2+a^2)(z^2+b^2)=0$

$z = \pm ai, \pm bi$

$\therefore f(z) = \frac{z^2 e^{imz}}{(z+ai)(z-ai)(z+bi)(z-bi)}$

(iii) The poles $z = ai$ and $z = bi$ lie inside C.

(iv) $\text{Res}[f(z); z=ai] = \lim_{z\to ai}(z-ai)f(z)$

$= \lim_{z\to ai} \frac{z^2 e^{imz}}{(z+ai)(z+bi)(z-bi)}$

$= \frac{(ai)^2 e^{i^2 ma}}{(2ai)(ai+bi)(ai-bi)}$

$= \frac{-a^2 e^{-ma}}{2ai(b^2-a^2)}$

$= \frac{-ae^{-ma}}{2i(b^2-a^2)}$

$\text{Res}[f(z); z=bi] = \lim_{z\to bi}(z-bi)f(z)$

$= \lim_{z\to bi} \frac{z^2 e^{imz}}{(z+ai)(z-ai)(z+bi)}$

$$= \frac{(bi)^2 e^{i^2 mb}}{(bi+ai)(bi-ai)(2bi)}$$

$$= \frac{-b^2 e^{-mb}}{(a^2-b^2)(2bi)}$$

$$= \frac{be^{-mb}}{2i(b^2-a^2)}$$

(v) $$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi i \text{ (sum of residues)}$$

$$\int_{-\infty}^{\infty} \frac{x^2 e^{imx}}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x = 2\pi i\left[-\frac{ae^{-ma}}{2i(b^2-a^2)} + \frac{be^{-mb}}{2i(b^2-a^2)}\right]$$

$$= \pi\left[\frac{be^{-mb}-ae^{-ma}}{b^2-a^2}\right]$$

$$\int_{-\infty}^{\infty} \frac{x^2(\cos mx + i\sin mx)}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x = \frac{\pi(be^{-mb}-ae^{-ma})}{b^2-a^2}$$

Comparing real parts,

$$\int_{-\infty}^{\infty} \frac{x^2\cos mx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x = \frac{\pi(be^{-mb}-ae^{-ma})}{(b^2-a^2)}$$

$$2\int_{0}^{\infty} \frac{x^2\cos mx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x = \frac{\pi(be^{-mb}-ae^{-ma})}{(b^2-a^2)} \quad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x \text{ if } f(-x)=f(x)\right]$$

$$\int_{0}^{\infty} \frac{x^2\cos mx}{(x^2+a^2)(x^2+b^2)}\mathrm{d}x = \frac{\pi(be^{-mb}-ae^{-ma})}{2(b^2-a^2)}$$

EXERCISE 5.3

Evaluate the following integrals using contour integration:

1. $\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\mathrm{d}x,\ a>0, m>0$

$\left[\textbf{Ans.:}\ \frac{\pi}{4a^3}(1+ma)e^{-ma}\right]$

2. $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\mathrm{d}x$

$\left[\textbf{Ans.:}\ \frac{\pi}{e}\right]$

3. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx$

$\left[\textbf{Ans.:} \frac{\pi}{e}(\sin 1 + 1)\right]$

4. $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 5x^2 + 4} dx$

$\left[\textbf{Ans.:} \pi\left(\frac{1}{3e} - \frac{1}{6e^2}\right)\right]$

5. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx$

$\left[\textbf{Ans.:} -\frac{\pi}{e}\sin 2\right]$

6. $\int_{-\infty}^{\infty} \frac{\cos ax}{x^4 + 10x^2 + 9} dx,\ a > 0$

$\left[\textbf{Ans.:} \frac{\pi}{8}\left(e^{-a} - \frac{e^{-3a}}{3}\right)\right]$

5.5 EVALUATION OF IMPROPER INTEGRAL WHEN SIMPLE POLES LIE ON THE REAL AXIS

Let $I = \int_{-\infty}^{\infty} f(x)\,dx$

To evaluate the integral, consider $\int_C f(z)\,dz$.

If $f(z)$ has a simple pole on the real axis then this pole is deleted by indenting the contour. The contour is indented by drawing a small semi-circle C_1 of radius r having the pole as its centre (Fig. 5.27).

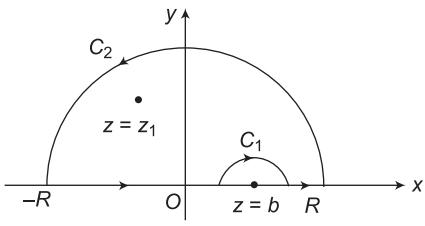


Fig. 5.27

If $f(z)$ has a simple pole at $z = b$ on the real axis then

$$\lim_{r \to 0} \int_{C_1} f(z)\,dz = \pi i \operatorname{Res}\left[f(z); z = b\right]$$

If $f(z)$ has a simple pole $z = b$ on the real axis and another simple pole $z = z_1$ in the upper half of the z-plane then

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \operatorname{Res}\left[f(z); z = b\right] + 2\pi i \operatorname{Res}\left[f(z); z = z_1\right]$$

Example 1

Find the principal value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{dz}{(z^2-3z+2)(z^2+1)}$

(ii) Let $f(z) = \frac{1}{(z^2-3z+2)(z^2+1)}$

The poles are given by

$$(z^2-3z+2)(z^2+1) = 0$$

$$(z-1)(z-2)(z+i)(z-i) = 0$$

$$z = 1, 2, \pm i$$

$$\therefore \quad f(z) = \frac{1}{(z-1)(z-2)(z+i)(z-i)}$$

(iii) The poles $z = 1$, $z = 2$ lie on the real axis and the pole $z = i$ lies in the upper half of the z-plane (Fig. 5.28).

(iv) $\text{Res}\,[f(z); z=1] = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1} \frac{1}{(z-2)(z^2+1)}$$

$$= -\frac{1}{2}$$

$\text{Res}\,[f(z); z=2] = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2} \frac{1}{(z-1)(z^2+1)}$$

$$= \frac{1}{5}$$

$\text{Res}\,[f(z); z=i] = \lim_{z\to i}(z-i)f(z)$

$$= \lim_{z\to i} \frac{1}{(z^2-3z+2)(z+i)}$$

$$= \frac{1}{(-1-3i+2)2i}$$

$$= \frac{1}{(1-3i)2i}$$

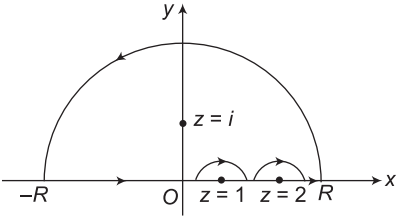


Fig. 5.28

$$= \frac{1}{2(i+3)} \times \frac{(-i+3)}{(-i+3)}$$

$$= \frac{-i+3}{2(1+9)}$$

$$= \frac{3-i}{20}$$

(v) PV of $\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = \pi i \left\{ \text{Res}\left[f(z); z=1 \right] + \left[f(z); z=2 \right] \right\}$

$$+ 2\pi i \text{ Res}\left[f(z); z=i \right]$$

$$= \pi i \left(-\frac{1}{2} + \frac{1}{5} \right) + 2\pi i \left(\frac{3-i}{20} \right)$$

$$= \pi i \left(-\frac{3}{10} \right) + \pi i \left(\frac{3-i}{10} \right)$$

$$= -\frac{3\pi i}{10} + \frac{3\pi i}{10} - \frac{\pi i^2}{10}$$

$$= \frac{\pi}{10}$$

Example 2

Show that the principal value of $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2} \right)$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{(z^2+4)(z-1)} dz$

(ii) Let $f(z) = \frac{e^{iz}}{(z^2+4)(z-1)}$

The poles are given by

$$(z^2+4)(z-1) = 0$$

$$z = \pm 2i, 1$$

$$\therefore \quad f(z) = \frac{e^{iz}}{(z+2i)(z-2i)(z-1)}$$

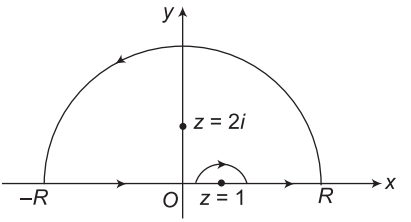


Fig. 5.29

(iii) The pole $z = 1$ lies on the real axis and the pole $z = 2i$ lies in the upper half of the z-plane (Fig. 5.29).

(iv) $\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}\frac{e^{iz}}{(z^2+4)}$$

$$= \frac{e^i}{5}$$

$$\text{Res}\left[f(z); z=2i\right] = \lim_{z\to 2i}(z-2i)\,f(z)$$

$$= \lim_{z\to 2i}\frac{e^{iz}}{(z+2i)(z-1)}$$

$$= \frac{e^{i(2i)}}{4i(2i-1)}$$

$$= \frac{e^{-2}}{-8-4i}$$

$$= \frac{e^{-2}}{-4(2+i)}\times\frac{(2-i)}{(2-i)}$$

$$= \frac{e^{-2}(2-i)}{-4(5)}$$

$$= \frac{e^{-2}(i-2)}{20}$$

$$= \frac{i-2}{20e^2}$$

(v) $\text{PV of}\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x-1)}\mathrm{d}z = \pi i\,\text{Res}\left[f(z); z=1\right] + 2\pi i\,\text{Res}\left[f(z); z=2i\right]$

$$\text{PV of}\int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{(x^2+4)(x-1)}\mathrm{d}x = \pi i\left(\frac{e^i}{5}\right) + 2\pi i\left(\frac{i-2}{20e^2}\right)$$

$$= \frac{\pi i}{5}(\cos 1 + i\sin 1) + \frac{\pi}{10e^2}(-1-2i)$$

$$= \frac{\pi}{5}(i\cos 1 - \sin 1) - \frac{\pi}{10e^2}(1+2i)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^2}\right) + i\left(\frac{\pi}{5}\cos 1 - \frac{2\pi}{10e^2}\right)$$

$$= -\frac{\pi}{5}\left(\sin 1 + \frac{1}{2e^2}\right) + i\frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$$

Comparing imaginary parts,

$$\text{PV of } \int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$$

Example 3

Using indentation along a branch cut, show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution

(i) Let $\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z} dz$

(ii) Let $f(z) = \frac{e^{iz}}{z}$

The pole is given by

$z = 0$

(iii) The pole $z = 0$ lies on the real axis (Fig. 5.30).

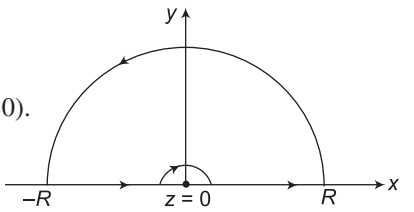


Fig. 5.30

(iv) $\text{Res}\left[f(z); z=0\right] = \lim_{z\to 0} z\, f(z)$

$= \lim_{z\to 0} e^{iz}$

$= e^0$

$= 1$

(v) $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i \text{ Res}\left[f(z); z=0\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = \pi i \, (1)$$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$2\int_0^{\infty} \frac{\sin x}{x} dx = \pi \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

EXERCISE 5.4

Evaluate the following integrals using contour integration.

1. $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$

$\left[\textbf{Ans.:}\ \frac{\pi}{2}\right]$

2. $\int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx$

$\left[\textbf{Ans.:}\ -\frac{\pi\sqrt{3}}{6}\right]$

3. $\int_0^\infty \frac{\sin mx}{x}\,dx,\ m>0$

$\left[\textbf{Ans.:}\ \frac{\pi}{2}\right]$

4. $\int_{-\infty}^{\infty} \frac{\cos mx}{x}\,dx$

$\left[\textbf{Ans.:}\ 0\right]$

5. $\int_{-\infty}^{\infty} \frac{dx}{x^2-ix}$

$\left[\textbf{Ans.:}\ \pi\right]$

6. $\int_{-\infty}^{\infty} \frac{x^2}{x^4-1}\,dx$

$\left[\textbf{Ans.:}\ \frac{\pi}{2}\right]$

7. $\int_{-\infty}^{\infty} \frac{dx}{x^4+3x^2-4}$

$\left[\textbf{Ans.:}\ -\frac{\pi}{10}\right]$

8. $\int_{-\infty}^{\infty} \frac{3x+2}{x(x-4)(x^2+9)}\,dx$

$\left[\textbf{Ans.:} -\frac{14}{75}\pi\right]$

Points to Remember

Evaluation of a Real Definite Integral of a Rational Function of $\cos\theta$ and $\sin\theta$

$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = 2\pi i$ (sum of residues at all poles lying inside the circle $|z| = 1$)

Evaluation of Improper Real Integral of a Rational Function

$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ (sum of residues at poles in the upper half of the z-plane)

where $f(x) = \frac{P(x)}{Q(x)}$ and the degree of $Q(x)$ is greater than the degree of $P(x)$ by at least 2 and $Q(x)$ has no real roots.

Evaluation of Improper Real Integral of a Rational Function Including Trigonometric Functions

$\int_{-\infty}^{\infty} \frac{\cos mx}{Q(x)}\,dx = 2\pi i$ (sum of residues at poles in the upper half of the z-plane)

$\int_{-\infty}^{\infty} \frac{\sin mx}{Q(x)}\,dx = 2\pi i$ (sum of residues at poles in the upper half of the z-plane)

where $Q(x)$ is a polynomial in x.

Evaluation of Improper Integral When Simple Poles Lie on the Real Axis

If $f(z)$ has a simple pole $z = b$ on the real axis and another simple pole $z = z_1$ in the upper half of the z-plane then

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \operatorname{Res}\left[f(z); z = b\right] + 2\pi i \operatorname{Res}\left[f(z); z = z_1\right]$$

CHAPTER 6
Conformal Mapping and Its Applications

Chapter Outline

6.1 Introduction
6.2 Conformal Mapping
6.3 Some Standard Transformations
6.4 Some Special Transformations
6.5 Schwarz—Christoffel Transformation
6.6 Bilinear Transformation

6.1 INTRODUCTION

A *conformal mapping*, also called a *conformal map* or *conformal transformation*, is a transformation that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative. Conformal mappings are useful for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable but exhibit inconvenient geometrics. By choosing an appropriate mapping, the inconvenient geometry can be transformed into a much more convenient geometry.

6.2 CONFORMAL MAPPING

Conformal mapping transforms curves from one complex plane to another with respect to size and orientation. If the point $z\ (x, y)$ describes some curve C in the z-plane, the point $w\ (u, v)$ describes a corresponding curve C' in the w-plane. Hence, for each point (x, y) in the z-plane, there corresponds a point (u, v) in the w-plane. Thus, a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the mapping $w = f(z)$.

A mapping or transformation $w = f(z)$ is said to be conformal at a point z_0 if it preserves the angle between any two curves passing through z_0 in magnitude and sense.

Isogonal Transformation A transformation which preserves the angle between any two curves passing through a point only in magnitude and not in sense is known as the isogonal transformation at that point.

Conformal Property of Analytic Function If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of the z-plane then the mapping $w = f(z)$ is conformal at all points of the region R.

Critical Points A point of the transformation $w = f(z)$ at which $f'(z) = 0$ is known as the critical point. At this point, mapping is not conformal.

Example 1

At what points is the mapping $w = z^2 + \dfrac{1}{z^2}$ *not conformal?*

Solution

The mapping is not conformal at critical points, i.e., where $f'(z) = 0$.

$$w = f(z) = z^2 + \frac{1}{z^2}$$

$$f'(z) = 2z - \frac{2}{z^3}$$

$$f'(z) = 0$$

$$2z - \frac{2}{z^3} = 0$$

$$2z^4 - 2 = 0$$

$$z^4 = 1$$

$$z^2 = \pm 1$$

$$z^2 = 1, z^2 = -1 = i^2$$

$$z = \pm 1, z = \pm i$$

Hence, the mapping is not conformal at $z = \pm 1$, $z = \pm i$.

6.3 SOME STANDARD TRANSFORMATIONS

6.3.1 Translation

The transformation $w = z + c$, where c is a complex constant, represents translation.

Let $z = x + iy, w = u + iv, c = a + ib$

$$w = z + c$$
$$u + iv = (x + iy) + (a + ib)$$
$$= (x + a) + i(y + b)$$

Comparing real and imaginary parts,

$$u = x + a, \qquad v = y + b$$

Therefore, the image of the point (x, y) in the z-plane is the point $(x + a, y + b)$ in the w-plane.

Hence, $w = z + c$ is simply a translation of the axes and preserves shape and size of the region of the z-plane.

Example 1

Find the image of $2x + y - 3 = 0$ *under the transformation* $w = z + 2i$.

Solution

$$w = z + 2i$$
$$u + iv = x + iy + 2i$$
$$= x + i(y + 2)$$

Comparing real and imaginary parts,

$$u = x, \quad v = y + 2$$
$$x = u, \quad y = v - 2$$

Given
$$2x + y - 3 = 0$$
$$2u + (v - 2) - 3 = 0$$
$$2u + v - 5 = 0$$

Hence, the line $2x + y - 3 = 0$ in the z-plane is mapped onto a line $2u + v - 5 = 0$ in the w-plane (Fig. 6.1).

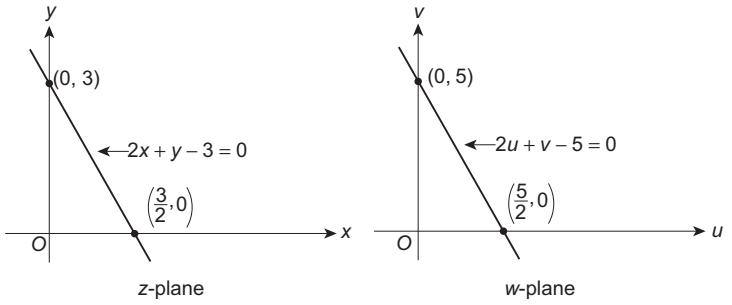


Fig. 6.1

Example 2

Determine and sketch the image of $|z| = 1$ *under the transformation* $w = z + i$.

Solution

$$w = z + i$$
$$u + iv = x + iy + i$$
$$= x + i(y + 1)$$

Comparing real and imaginary parts,

$$u = x, \quad v = y + 1$$
$$x = u, \quad y = v - 1$$

Given $|z| = 1$

$$\sqrt{x^2 + y^2} = 1$$
$$x^2 + y^2 = 1$$
$$u^2 + (v-1)^2 = 1$$

Hence, the image of the circle $|z| = 1$ in the z-plane is the circle $u^2 + (v - 1)^2 = 1$ with the centre at (0, 1) and a radius of 1 in the w-plane (Fig. 6.2).

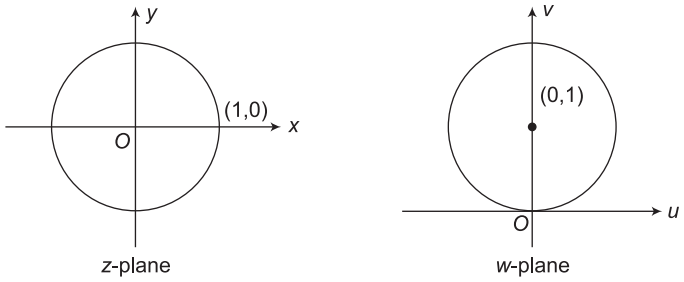


Fig. 6.2

Example 3

Find the image of $|z| = 2$ *under the mapping* $w = z + 3 + 2i$.

Solution

$$w = z + 3 + 2i$$
$$u + iv = x + iy + 3 + 2i$$
$$= (x + 3) + i(y + 2)$$

Comparing real and imaginary parts,

$$u = x + 3, \quad v = y + 2$$
$$x = u - 3, \quad y = v - 2$$

Given
$$|z| = 2$$
$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$
$$(u-3)^2 + (v-2)^2 = 4$$

Hence, the circle $|z| = 2$ in the z-plane is mapped onto a circle with the centre at (3, 2) and a radius of 2 in the w-plane (Fig. 6.3).

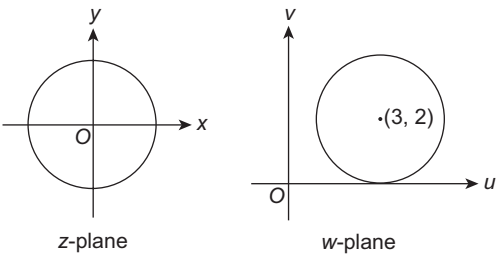


Fig. 6.3

Example 4

Find the image of the triangular region whose vertices are $i, 1+i, 1-i$ *under the transformation* $w = z + 4 - 2i$ *. Show the region graphically.*

Solution

$$w = z + 4 - 2i$$
$$u + iv = x + iy + 4 - 2i$$
$$= (x+4) + i(y-2)$$

Comparing real and imaginary parts,
$$u = x + 4, \quad v = y - 2$$

(i) For vertex i,
$$z = i$$
$$x + iy = i$$

Comparing real and imaginary parts,
$$x = 0, y = 1$$
$$u = 4, v = -1$$

Hence, the point $A(0,1)$ is mapped onto the point $A'(4,-1)$.

(ii) For vertex $1+i$,
$$z = 1 + i$$
$$x + iy = 1 + i$$

Comparing real and imaginary parts,

$$x = 1, y = 1$$

$$u = 5, v = -1$$

Hence, the point $B(1, 1)$ is mapped onto the point $B'(5, -1)$.

(iii) For vertex $1 - i$,

$$z = 1 - i$$

$$x + iy = 1 - i$$

Comparing real and imaginary parts,

$$x = 1, \quad y = -1$$

$$u = 5, \quad v = -3$$

Hence, the point $C(1, -1)$ is mapped onto the point $C'(5, -3)$.

Hence, the image of the triangular region ABC in z-plane is the triangular region $A'B'C'$ with vertices $(4 - i)$, $(5 - i)$ and $(5 - 3i)$, in the w-plane (Fig. 6.4).

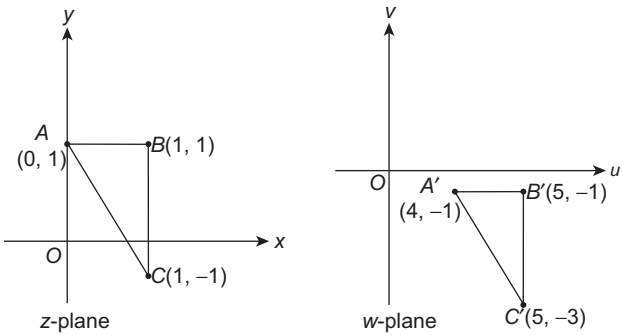


Fig. 6.4

Example 5

Find the image of the circle $|z - \alpha| = r$ *by the mapping* $w = z + c$, *where* c *is a real constant.*

Solution

$$w = z + c$$

$$u + iv = x + iy + c$$

$$= (x + c) + iy$$

Comparing real and imaginary parts,

$$u = x + c, \quad v = y$$

$$x = u - c, \quad y = v$$

Given

$$|z - \alpha| = r$$

$$|x + iy - \alpha| = r$$

$$|(x - \alpha) + iy| = r$$

$$\sqrt{(x-\alpha)^2+y^2} = r$$
$$(x-\alpha)^2+y^2 = r^2$$
$$(u-c-\alpha)^2+v^2 = r^2$$
$$\left[u-(c+\alpha)\right]^2+v^2 = r^2$$

Hence, the circle $|z-\alpha| = r$ in the z-plane is mapped onto a circle $\left[u-(c+\alpha)\right]^2+v^2 = r^2$ with the centre at $[(c+\alpha), 0]$ and a radius of r in the w-plane.

6.3.2 Magnification and Rotation

The transformation $w = cz$, where c is a complex constant represents magnification and rotation.

Let $w = \mathrm{Re}^{i\phi}, z = re^{i\theta}, c = \rho e^{i\alpha}$

$$w = cz$$
$$\mathrm{Re}^{i\phi} = (\rho e^{i\alpha})(re^{i\theta})$$
$$= \rho r e^{i(\alpha+\theta)}$$

Comparing both the sides,

$$R = \rho r,\ \phi = \theta+\alpha$$

Therefore, this transformation maps the point (r, θ) in the z-plane onto the point $(\rho r, \theta+\alpha)_n$ in the w-plane. The radius vector r is magnified by ρ and is rotated through an angle α.

Hence, geometrically it maps any figure in the z-plane into a similar figure in the w-plane. In particular, this transformation maps circles into circles.

Example 1

Find the map of the circle $|z| = 3$ *under the transformation* $w = 2z$.

Solution

$$w = 2z$$
$$u+iv = 2(x+iy)$$

Comparing real and imaginary parts,

$$u = 2x, \qquad v = 2y$$
$$x = \frac{u}{2}, \qquad y = \frac{v}{2}$$

Given $|z| = 3$

$$\sqrt{x^2+y^2} = 3$$

$$x^2 + y^2 = 9$$

$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9$$

$$u^2 + v^2 = 36$$

Hence, the map of the circle $|z| = 3$ in the z-plane is a circle $u^2 + v^2 = 36$ with the centre at (0,0) and a radius of 6 in the w-plane (Fig. 6.5).

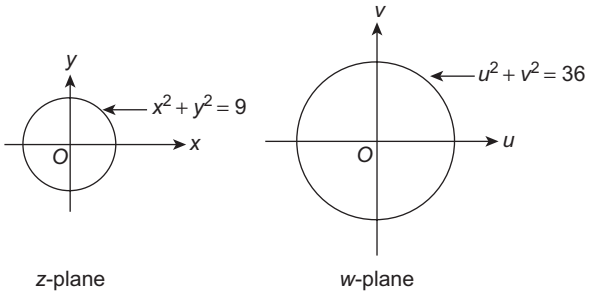


Fig. 6.5

Example 2

Find the image of $x^2 + y^2 = 4$ *under the transformation* $w = 3z$.

Solution

$$w = 3z$$

$$u + iv = 3(x + iy)$$

Comparing real and imaginary parts,

$$u = 3x, \qquad v = 3y$$

$$x = \frac{u}{3}, \qquad y = \frac{v}{3}$$

Given $x^2 + y^2 = 4$

$$\left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4$$

$$u^2 + v^2 = 36$$

Hence, the image of $x^2 + y^2 = 4$ in the z-plane is a circle $u^2 + v^2 = 36$ with the centre at (0,0) and a radius of 6 in the w-plane (Fig. 6.6).

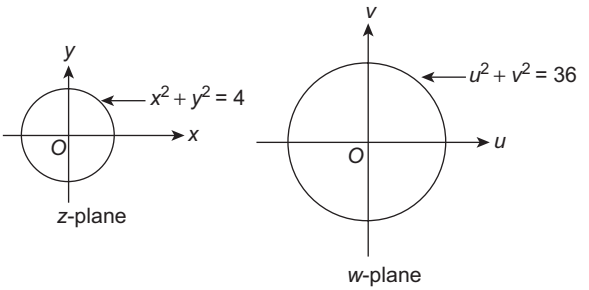


Fig. 6.6

Example 3

Find and sketch the image of the region $|z| > 1$ *under the transformation* $w = 4z$. **[Summer 2014]**

Solution

$$w = 4z$$

$$u + iv = 4(x + iy)$$

Comparing real and imaginary parts,

$$u = 4x, \qquad v = 4y$$

$$x = \frac{u}{4}, \qquad y = \frac{v}{4}$$

Given $|z| > 1$

$$\sqrt{x^2 + y^2} > 1$$

$$x^2 + y^2 > 1$$

$$\frac{u^2}{16} + \frac{v^2}{16} > 1$$

$$u^2 + v^2 > 16$$

Hence, the image of $|z| > 1$ (Exterior of the circle $|z| = 1$) in the z-plane is the exterior of the circle $u^2 + v^2 = 16$ in the w-plane (Fig. 6.7).

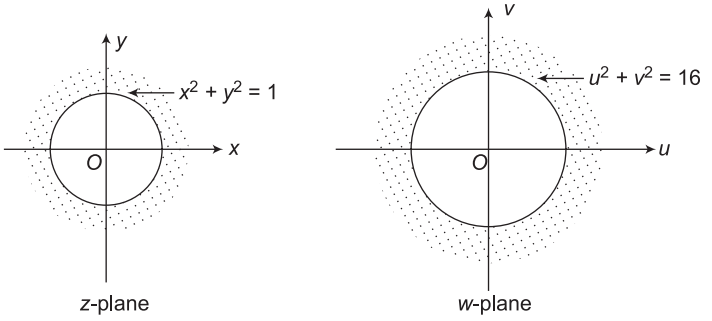


Fig. 6.7

Example 4

Determine the region in the w-plane into which the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the z-plane is mapped under the transformation $w = 4z$.

Solution

$$w = 4z$$

$$u + iv = 4(x + iy)$$

Comparing real and imaginary parts,

$$u = 4x, \qquad v = 4y$$

$$x = \frac{u}{4}, \qquad y = \frac{v}{4}$$

(i) When $x = 0, u = 0$

Hence, the line $x = 0$ is mapped onto the line $u = 0$.

(ii) When $y = 0, v = 0$

Hence, the line $y = 0$ is mapped onto the line $v = 0$.

(iii) When $x + y = 1$,

$$\frac{u}{4} + \frac{v}{4} = 1$$

$$u + v = 4$$

Hence, the line $x + y = 1$ is mapped onto the line $u + v = 4$.

Hence, the triangle OAB in the z-plane is mapped onto the triangular region $OA'B'$ in the w-plane (Fig. 6.8).

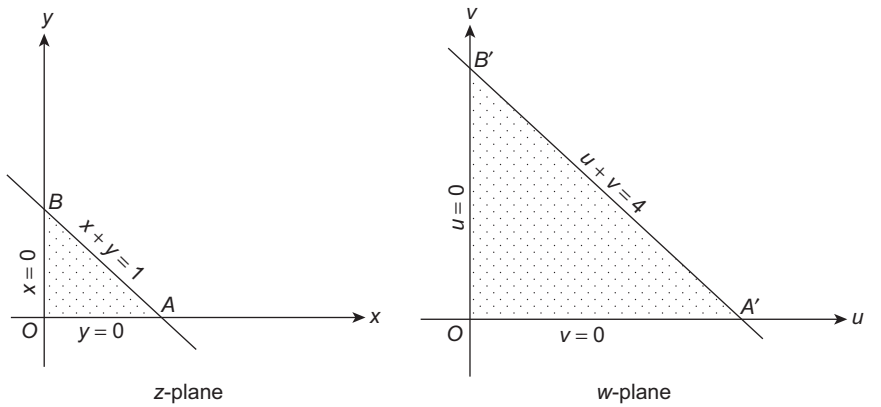


Fig. 6.8

Example 5

Draw the image of the square whose vertices are at (0, 0), (1, 0), (1, 1), (0, 1) *in the z-plane, under the transformation $w = (1 + i)z$. What has this transformation done to the original square?*

Solution

$$w = (1+i)z$$
$$u+iv = (1+i)(x+iy)$$
$$= x+iy+ix+i^2y$$
$$= (x-y)+i(x+y)$$

Comparing real and imaginary parts,

$$u = x-y, \qquad v = x+y$$

Let $ABCD$ be the square with vertices A (0, 0), B (1, 0), C (1, 1), D (0, 1) in the z-plane.

(i) At $(0, 0), x = 0, y = 0$

$$u = 0, v = 0$$

Hence, the point $A(0,0)$ is mapped onto the point $A'(0, 0)$.

(ii) At $(1, 0), x = 1, y = 0$

$$u = 1, v = 1$$

Hence, the point $B(1, 0)$ is mapped onto the point $B'(1, 1)$.

(iii) At $(1, 1), x = 1, y = 1$

$$u = 0, v = 2$$

Hence, the point $C(1, 1)$ is mapped onto the point $C'(0, 2)$.

(iv) At $(0, 1), x = 0, y = 1$

$$u = -1, v = 1$$

Hence, the point $D(0, 1)$ is mapped onto the point $D'(-1, 1)$.

Hence, the image of the square $ABCD$ in the z-plane is the square $A'B'C'D'$ in the w-plane (Fig. 6.9). This transformation has rotated the square by an angle $\frac{\pi}{4}$ but has preserved the size of the square.

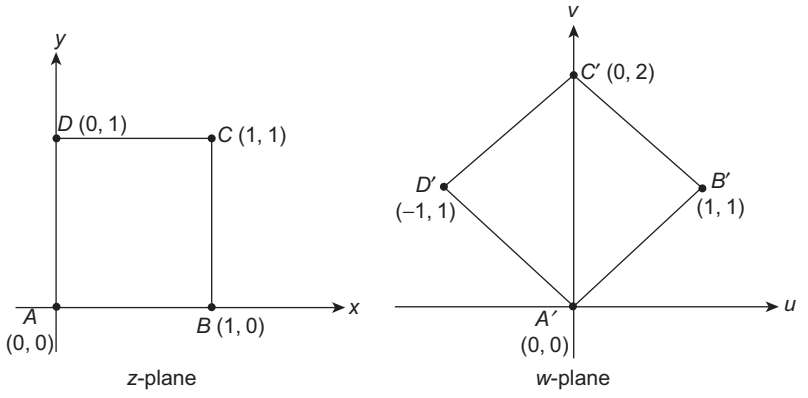


Fig. 6.9

Example 6

Determine the region in the w-plane into which the region bounded by $x = 0, y = 0, x = 1, y = 2$ *in the z-plane is mapped under the transformation* $w = (1+i)z + (2-i)$.

Solution

$$\begin{aligned} w &= (1+i)z + (2-i) \\ u + iv &= (1+i)(x+iy) + (2-i) \\ &= x + iy + ix - y + 2 - i \\ &= (x - y + 2) + i(x + y - 1) \end{aligned}$$

Comparing real and imaginary parts,

$$u = x - y + 2, \qquad v = x + y - 1$$

$$x - y = u - 2 \qquad \ldots(1)$$

$$x + y = v + 1 \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$2x = u + v - 1$$

$$x = \frac{1}{2}(u + v - 1)$$

Subtracting Eq. (1) from Eq. (2),

$$2y = (v+1) - (u-2)$$

$$y = \frac{1}{2}(v - u + 3)$$

Let *ABCD* be the given region in the *z*-plane.

(i) When $x = 0, u + v = 1$

Hence, the line $x = 0$ is mapped onto the line $u + v = 1$.

(ii) When $y = 0, u - v = 3$

Hence, the line $y = 0$ is mapped onto the line $u - v = 3$.

(iii) When $x = 1, u + v = 3$

Hence, the line $x = 1$ is mapped onto the line $u + v = 3$.

(iv) When $y = 2, u - v = -1$

Hence, the line $y = 2$ is mapped onto the line $u - v = -1$.

Hence, the region *ABCD* in the *z*-plane is mapped onto the region $A'B'C'D'$ in the *w*-plane (Fig. 6.10).

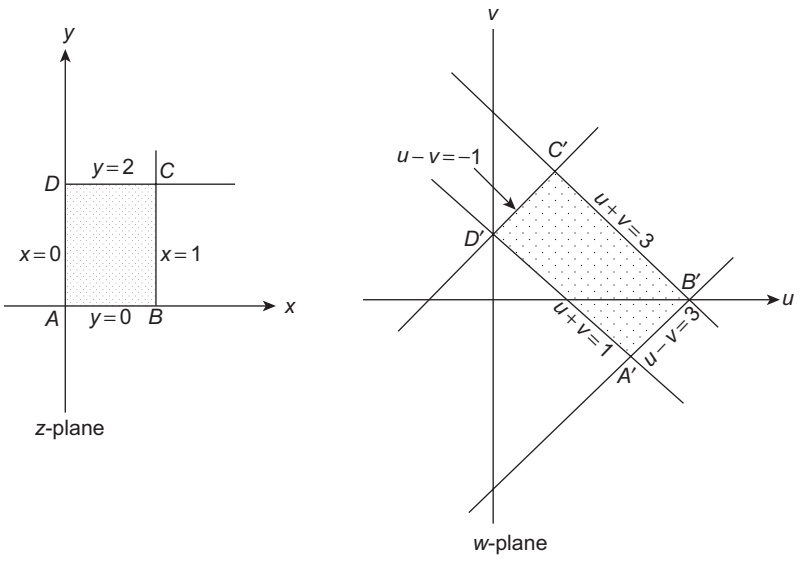


Fig. 6.10

Example 7

Find the image of the infinite strip $0 \le x \le 1$ *under the transformation* $w = iz + 1$. *Sketch the region in the w-plane.* **[Summer 2015]**

Solution

$$w = iz + 1$$
$$u + iv = i(x + iy) + 1$$
$$= ix - y + 1$$
$$= (1 - y) + ix$$

Comparing real and imaginary parts,

$$u = 1 - y, \qquad v = x$$
$$y = 1 - u, \qquad x = v$$

Given $0 \le x \le 1$

$0 \le v \le 1$

Hence, the image of the infinite strip $0 \le x \le 1$ in z-plane is the infinite strip $0 \le v \le 1$ in the w-plane (Fig. 6.11).

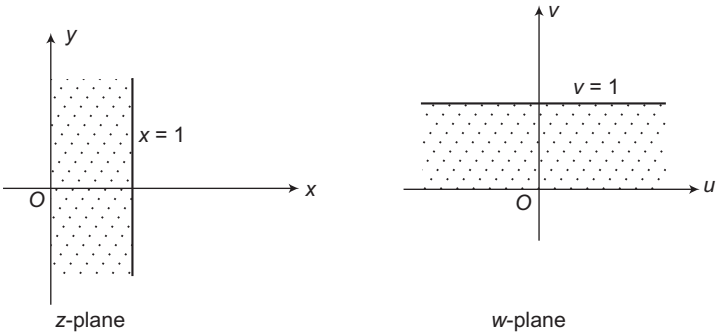


Fig. 6.11

Example 8

Find the image of the semi-infinite strip $x > 0, 0 < y < 2$ when $w = iz + 1$. Sketch the strip and its image.

Solution

$$w = iz + 1$$
$$u + iv = i(x + iy) + 1$$
$$= ix - y + 1$$
$$= (1 - y) + ix$$

Comparing real and imaginary parts,

$$u = 1 - y, \qquad v = x$$
$$y = 1 - u, \qquad x = v$$

Given $x > 0$

$v > 0$

and $0 < y < 2$

$0 < 1 - u < 2$

$-1 < -u < 1$

$1 > u > -1$

or $-1 < u < 1$

Hence, the image of the semi-infinite strip $x > 0$, $0 < y < 2$ in the z-plane is the semi-infinite strip $v > 0$, $-1 < u < 1$ in the w-plane (Fig. 6.12).

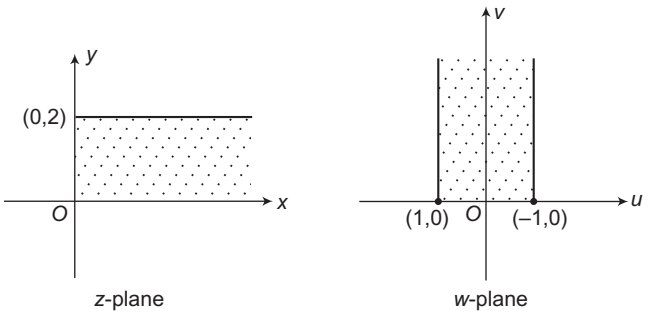


Fig. 6.12

6.3.3 Inversion

The transformation $w = \dfrac{1}{z}$ represents inversion and reflection.

This transformation is conformal for all $z \neq 0$.

Let $w = \mathrm{Re}^{i\phi}, z = re^{i\theta}$

where $R = |w|, \quad \phi = \arg(w), \quad r = |z|, \quad \theta = \arg(z)$

$$w = \frac{1}{z}$$

$$\mathrm{Re}^{i\phi} = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{r} e^{-i\theta}$$

Comparing both the sides,

$$R = \frac{1}{r}, \qquad \phi = -\theta$$

$$|w| = \frac{1}{|z|}, \qquad \arg(w) = -\arg(z)$$

Hence, this transformation maps the point (r, θ) in the z-plane onto the point $\left(\dfrac{1}{r}, -\theta\right)$ in the w-plane. Thus, this transformation represents inversion with respect to the unit circle $|z| = 1$ followed by the reflection in the real axis.

If $|z| < 1, \quad |w| > 1$

and $|z| > 1, \quad |w| < 1$

Hence, this transformation maps the interior of the circle $|z| = 1$ onto the exterior of the circle $|w| = 1$, and the exterior of the circle $|z| = 1$ onto the interior of the circle $|w| = 1$. Thus, the unit circle $|z| = 1$ in the z-plane is mapped onto the unit circle $|w| = 1$ in the w-plane.

Example 1

Show that the transformation $w = \dfrac{1}{z}$ *maps a circle in the z-plane into a circle or into a straight line in the w-plane.*

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u + iv}$$

$$x + iy = \frac{(u - iv)}{(u + iv)(u - iv)}$$

$$= \frac{u - iv}{u^2 + v^2}$$

$$= \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

The general equation of the circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Substituting x and y,

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(-\frac{v}{u^2 + v^2}\right)^2 + 2g\left(\frac{u}{u^2 + v^2}\right) + 2f\left(-\frac{v}{u^2 + v^2}\right) + c = 0$$

$$\frac{u^2 + v^2}{\left(u^2 + v^2\right)^2} + \frac{2gu - 2fv}{u^2 + v^2} + c = 0$$

$$1 + 2gu - 2fv + c\left(u^2 + v^2\right) = 0$$

$$c\left(u^2 + v^2\right) + 2gu - 2fv + 1 = 0 \qquad \cdots(1)$$

(i) If $c \neq 0$, Eq. (1) reduces to $u^2 + v^2 + \dfrac{2g}{c}u - \dfrac{2f}{c}v + \dfrac{1}{c} = 0$

which represents a circle in the w-plane.

(ii) If $c = 0$, Eq. (1) reduces to $2gu - 2fv + 1 = 0$

which represents a straight line in the w-plane.

Hence, a circle in the z-plane is mapped onto a circle or onto a straight line in the w-plane.

Example 2

Find the image of the line $x - y = 1$ *under the transformation* $w = \dfrac{1}{z}$.

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

Given $x - y = 1$

Substituting x and y,

$$\frac{u}{u^2 + v^2} - \left(-\frac{v}{u^2 + v^2}\right) = 1$$

$$u + v = u^2 + v^2$$

$$u^2 + v^2 - u - v = 0$$

which represents a circle with the centre at $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and a radius of $\dfrac{1}{\sqrt{2}}$.

Hence, the image of the line $x - y = 1$ in the z-plane is a circle $u^2 + v^2 - u - v = 0$ in the w-plane (Fig. 6.13).

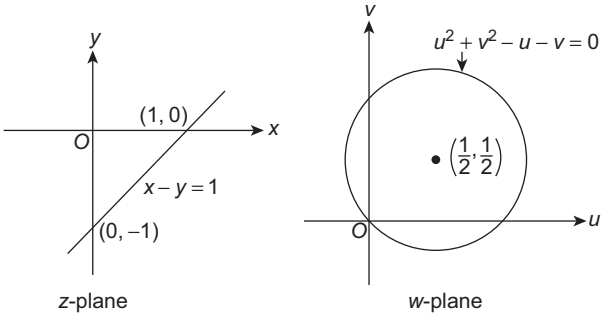


Fig. 6.13

Example 3

Find the image of $|z+1|=1$ *under the mapping* $w=\dfrac{1}{z}$.

Solution

$$w=\frac{1}{z}$$

$$z=\frac{1}{w}$$

$$=\frac{1}{u+iv}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x=\frac{u}{u^2+v^2},\qquad y=-\frac{v}{u^2+v^2}$$

Given $|z+1|=1$

$$|x+iy+1|=1$$

$$|(x+1)+iy|^2=1$$

$$(x+1)^2+y^2=1$$

$$x^2+2x+1+y^2=1$$

$$x^2+y^2+2x=0$$

Substituting x and y,

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(-\frac{v}{u^2+v^2}\right)^2+2\left(\frac{u}{u^2+v^2}\right)=0$$

$$\frac{u^2+v^2}{\left(u^2+v^2\right)^2}+\frac{2u}{u^2+v^2}=0$$

$$\frac{1+2u}{u^2+v^2}=0$$

$$1+2u=0$$

$$u=-\frac{1}{2}$$

which represents a straight line.

Hence, the image of the circle $|z+1|=1$ in the z-plane is the straight line $u=-\frac{1}{2}$ in the w-plane (Fig. 6.14).

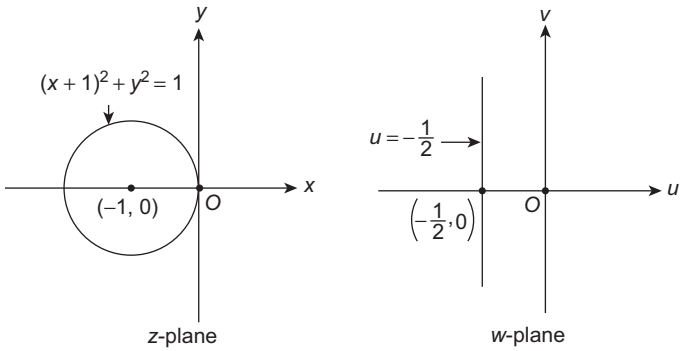


Fig. 6.14

Example 4

Find the image of $|z+2i|=2$ *under the transformation* $w=\frac{1}{z}$.

Solution

$$w=\frac{1}{z}$$

$$z=\frac{1}{w}$$

$$=\frac{1}{u+iv}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x=\frac{u}{u^2+v^2},\qquad y=-\frac{v}{u^2+v^2}$$

Given

$$|z+2i|=2$$

$$|x+iy+2i|=2$$

$$|x+i(y+2)|^2=4$$

$$x^2+(y+2)^2=4$$

$$x^2+y^2+4y=0$$

Substituting x and y,

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(-\frac{v}{u^2+v^2}\right)^2+4\left(-\frac{v}{u^2+v^2}\right)=0$$

$$\frac{u^2+v^2}{\left(u^2+v^2\right)^2}-\frac{4v}{u^2+v^2}=0$$

$$1-4v=0$$

$$v=\frac{1}{4}$$

which represents a straight line.

Hence, the image of the circle $|z+2i|=2$ in the z-plane is the straight line $v=\frac{1}{4}$ in the w-plane (Fig. 6.15).

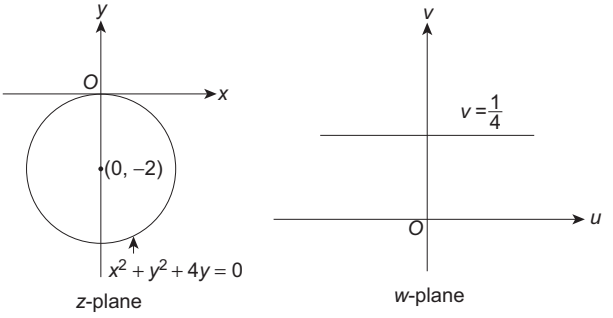


Fig. 6.15

Example 5

Find the image of the half-plane $x>c$, when $c>0$ under the transformation $w=\frac{1}{z}$. Show the regions graphically.

Solution

$$w=\frac{1}{z}$$

$$z=\frac{1}{w}$$

$$=\frac{1}{u+iv}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

Given $x > c, \text{when } c > 0$

$$\frac{u}{u^2 + v^2} > c$$

$$\frac{u}{c} > u^2 + v^2$$

$$u^2 + v^2 < \frac{u}{c}$$

$$u^2 + v^2 - \frac{u}{c} < 0$$

which represents a circle with the centre at $\left(\frac{1}{2c}, 0\right)$ and a radius of $\frac{1}{2c}$.

Hence, the image of $x > c$ in the z-plane is the interior of the circle in the w-plane (Fig. 6.16).

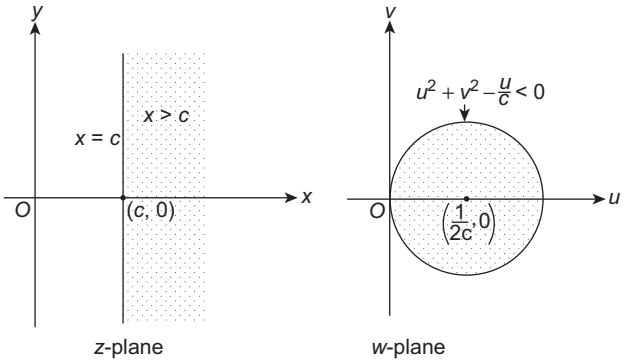


Fig. 6.16

Example 6

Find the image of the infinite strip $0 < y < \frac{1}{2}$ *under the transformation* $w = \frac{1}{z}$.

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u+iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$$

Given $$0 < y < \frac{1}{2}$$

$$0 < -\frac{v}{u^2+v^2} < \frac{1}{2}$$

$$0 < -v < \frac{1}{2}(u^2+v^2)$$

(i) $0 < -v$

$-v > 0$

$v < 0$

which represents lower half of the w-plane.

(ii) $-v < \frac{1}{2}(u^2+v^2)$

$-2v < u^2+v^2$

$u^2+v^2+2v > 0$

which represents the exterior of the circle $u^2+v^2+2v=0$ with the centre at (0, –1) and a radius of 1.

Hence, the image of the strip $0 < y < \frac{1}{2}$ in the z-plane is the lower half (below u-axis) of the w-plane outside the circle $u^2+v^2+2v=0$ in the w-plane (Fig. 6.17).

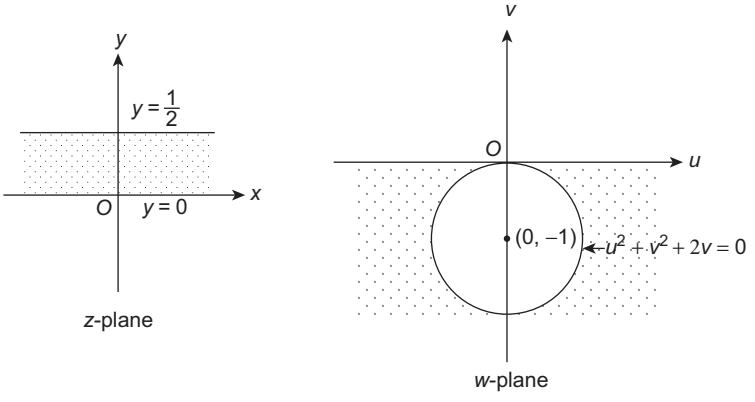


Fig. 6.17

Example 7

Find the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ *under the transformation* $w = \frac{1}{z}$. *Also, show the regions graphically.*

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

Given $\frac{1}{4} \le y \le \frac{1}{2}$

$$\frac{1}{4} \le -\frac{v}{u^2 + v^2} \le \frac{1}{2}$$

$$u^2 + v^2 \le -4v \le 2(u^2 + v^2) \qquad \left[\text{Multiplying by } 4(u^2 + v^2)\right]$$

(i) $u^2 + v^2 < -4v$

$u^2 + v^2 + 4v < 0$

which represents the interior of a circle with the centre at (0, –2) and a radius of 2.

(ii) $-4v < 2(u^2 + v^2)$

$0 < u^2 + v^2 + 2v$

$u^2 + v^2 + 2v > 0$

which represents the exterior of a circle with the centre at (0, –1) and a radius of 1.

Hence, the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ in the z-plane is the region between the circles $u^2 + v^2 + 2v = 0$ and $u^2 + v^2 + 4v = 0$ in the w-plane (Fig. 6.18).

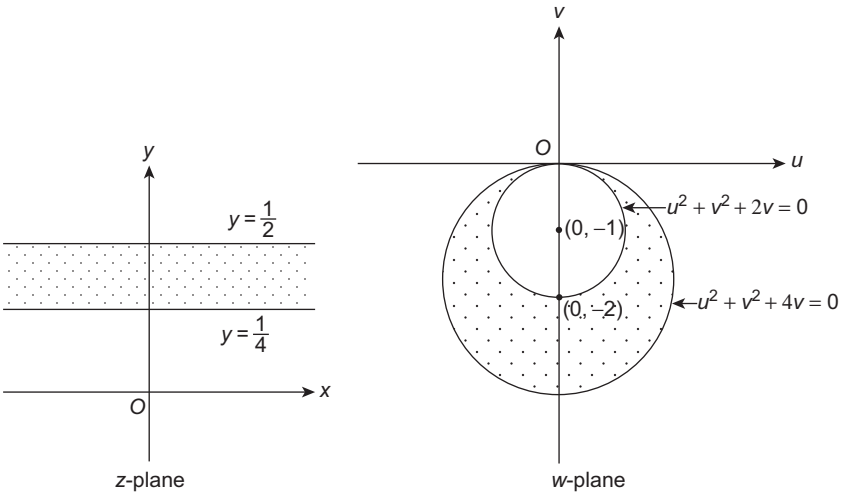


Fig. 6.18

Example 8

Find the image of the strip $1 < x < 2$ *under the transformation* $w = \dfrac{1}{z}$.

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \qquad y = -\frac{v}{u^2 + v^2}$$

Given $1 < x < 2$

$$1 < \frac{u}{u^2 + v^2} < 2$$

$$u^2 + v^2 < u < 2(u^2 + v^2)$$

(i) $u^2 + v^2 < u$

$u^2 + v^2 - u < 0$

which represents the interior of a circle with the centre at $\left(\frac{1}{2}, 0\right)$ and a radius of $\frac{1}{2}$.

(ii) $u < 2(u^2 + v^2)$

$\frac{u}{2} < u^2 + v^2$

$u^2 + v^2 - \frac{u}{2} > 0$

which represents the exterior of a circle with the centre at $\left(\frac{1}{4}, 0\right)$ and a radius of $\frac{1}{4}$.

Hence, the image of the strip $1 < x < 2$ in the z-plane is the region between the circles $u^2 + v^2 - u = 0$ and $u^2 + v^2 - \frac{u}{2} = 0$ in the w-plane (Fig. 6.19).

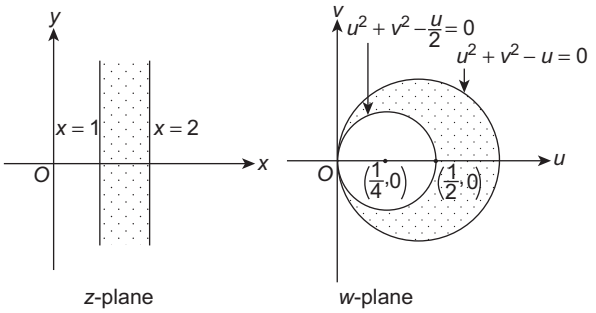


Fig. 6.19

Example 9

Find the image of the hyperbola $x^2 - y^2 = 1$ *under the transformation* $w = \frac{1}{z}$.

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u+iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \qquad y = -\frac{v}{u^2+v^2}$$

Given $x^2 - y^2 = 1$

$$\left(\frac{u}{u^2+v^2}\right)^2 - \left(-\frac{v}{u^2+v^2}\right)^2 = 1$$

$$u^2 - v^2 = (u^2+v^2)^2$$

which represents a lemniscate.

Hence, the image of the hyperbola $x^2 - y^2 = 1$ in the z-plane is the lemniscate $u^2 - v^2 = (u^2+v^2)^2$ in the w-plane (Fig. 6.20).

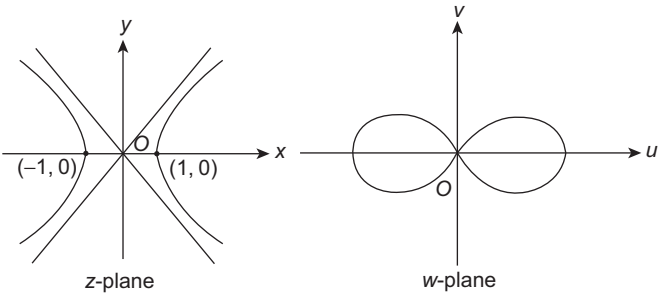


Fig. 6.20

Note Putting $u = r\cos\theta, v = r\sin\theta$

$$r^2\cos^2\theta - r^2\sin^2\theta = r^4$$

$$\cos 2\theta = r^2$$

$$r^2 = \cos 2\theta$$

which represents a lemniscate.

Example 10

Find the image of the square whose vertices are (1 + i), (3 + i), (1 + 3i), and (3 + 3i) *under the transformation* $w = \frac{1}{z}$.

Solution

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$= \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \quad y = -\frac{v}{u^2 + v^2}$$

Let $ABCD$ be the square with vertices $A(1+i)$, i.e., $(1,1)$, $B(3+i)$, i.e., $(3,1)$, $C(1+3i)$ i.e., $(1,3)$ and $D(3+3i)$, i.e., $(3, 3)$.

(i) Equation of the line AB is $y = 1$.

$$-\frac{v}{u^2 + v^2} = 1$$

$$u^2 + v^2 + v = 0$$

The image of $y = 1$ is a circle $u^2 + v^2 + v = 0$ with the centre at $\left(0, -\frac{1}{2}\right)$ and a radius of $\frac{1}{2}$.

(ii) Equation of the line BC is $x = 3$.

$$\frac{u}{u^2 + v^2} = 3$$

$$u^2 + v^2 - \frac{u}{3} = 0$$

The image of $x = 3$ is a circle with the centre at $\left(\frac{1}{6}, 0\right)$ and a radius of $\frac{1}{6}$.

(iii) Equation of the line CD is $y = 3$.

$$-\frac{v}{u^2 + v^2} = 3$$

$$u^2 + v^2 + \frac{v}{3} = 0$$

The image of the line $y = 3$ is the circle with the centre at $\left(0, -\frac{1}{6}\right)$ and a radius of $\frac{1}{6}$.

(iv) Equation of the line DA is $x = 1$.

$$\frac{u}{u^2 + v^2} = 1$$

$$u^2 + v^2 - u = 0$$

The image of the line $x = 1$ is the circle with the centre at $\left(\frac{1}{2}, 0\right)$ and a radius of $\frac{1}{2}$.

Hence, the image of the square $ABCD$ in the z-plane is the region $A'B'C'D'$ bounded by all four circles in the w-plane (Fig. 6.21).

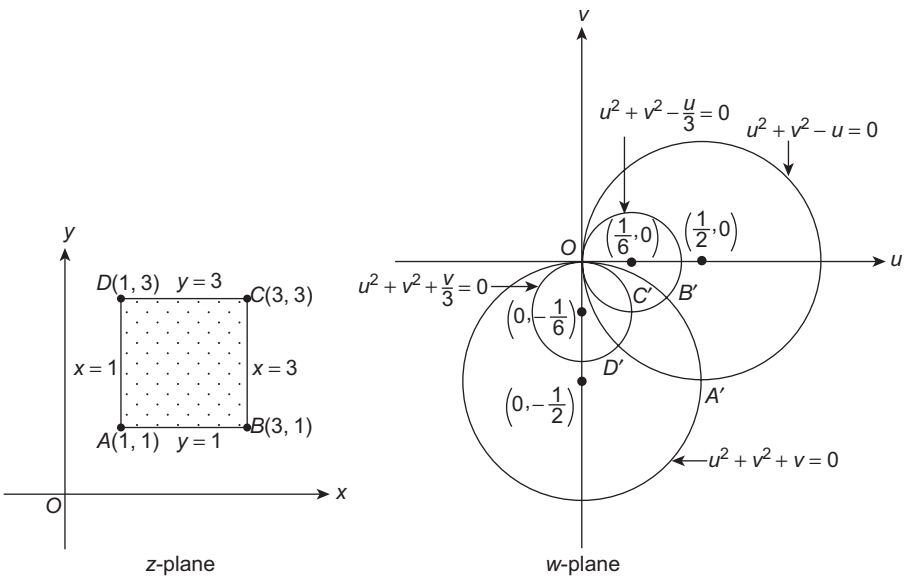


Fig. 6.21

6.4 SOME SPECIAL TRANSFORMATIONS

6.4.1 $w = z^2$

$$u + iv = (x + iy)^2$$
$$= x^2 - y^2 + 2ixy$$

Comparing real and imaginary parts,

$$u = x^2 - y^2, \qquad v = 2xy$$

(i) Let x = constant = a, say

$$u = a^2 - y^2, \quad v = 2ay$$
$$y = \frac{v}{2a}$$

Substituting y in u,

$$u = a^2 - \frac{v^2}{4a^2}$$
$$\frac{v^2}{4a^2} = a^2 - u$$
$$v^2 = -4a^2(u - a^2)$$

which represents a parabola with vertex $(a^2, 0)$.

Hence, the lines parallel to the y-axis in the z-plane are mapped into the parabolas in the w-plane.

(ii) Let y = constant = b, say

$$u = x^2 - b^2, \quad v = 2xb$$
$$x = \frac{v}{2b}$$

Substituting x in u,

$$u = \frac{v^2}{4b^2} - b^2$$
$$v^2 = 4b^2(u + b^2)$$

which represents a parabola with vertex $(-b^2, 0)$

Hence, the lines parallel to the x-axis in the z-plane are mapped into the parabolas in the w-plane.

Example 1

Find the critical points of the transformation $w = z^2$.

Solution

$$w = z^2$$

For critical points,

$$\frac{dw}{dz} = 0$$
$$2z = 0$$
$$z = 0$$

Hence, the critical point is $z = 0$ (i.e., origin).

Example 2

Find the image of the triangular region bounded by the lines $x = 1, y = 1, x + y = 1$ *under the transformation* $w = z^2$.

Solution

$$w = z^2$$
$$u + iv = (x + iy)^2$$
$$= x^2 - y^2 + 2ixy$$

Comparing real and imaginary parts,

$$u = x^2 - y^2, \qquad v = 2xy$$

(i) When $x = 1$,

$$u = 1 - y^2, \quad v = 2y$$
$$y = \frac{v}{2}$$

Substituting y in u,

$$u = 1 - \frac{v^2}{4}$$
$$4u = 4 - v^2$$
$$v^2 = -4(u - 1)$$

which represents a parabola with vertex (1, 0) and opening to the left of the vertex.

The image of the line $x = 1$ is the parabola $v^2 = -4(u - 1)$.

(ii) When $y = 1$,

$$u = x^2 - 1, \qquad v = 2x$$
$$x = \frac{v}{2}$$

Substituting x in u,

$$u = \frac{v^2}{4} - 1$$
$$4u = v^2 - 4$$
$$v^2 = 4(u + 1)$$

which represents a parabola with vertex (–1,0) and opening to the right of the vertex.

The image of the line $y = 1$ is the parabola $v^2 = 4(u+1)$.

(ii) When $x + y = 1$

$$(x+y)^2 = 1$$

$$x^2 + y^2 + 2xy = 1$$

$$(x^2 + y^2)^2 = (1 - 2xy)^2$$

$$(x^2 - y^2)^2 + 4x^2y^2 = 1 + 4x^2y^2 - 4xy$$

$$(x^2 - y^2)^2 = 1 - 4xy$$

Substituting $u = (x^2 - y^2)$ and $v = 2xy$,

$$u^2 = 1 - 2v = -2\left(v - \frac{1}{2}\right)$$

which represents a parabola with vertex $\left(0, \frac{1}{2}\right)$ opening below the vertex.

The image of the line $x + y = 1$ is the parabola $u^2 = 1 - 2v$.
Hence, the image of the triangular region ABC in the z-plane is the curvilinear triangular region $A'B'C'$ bounded by the parabolas in the w-plane (Fig. 6.22).

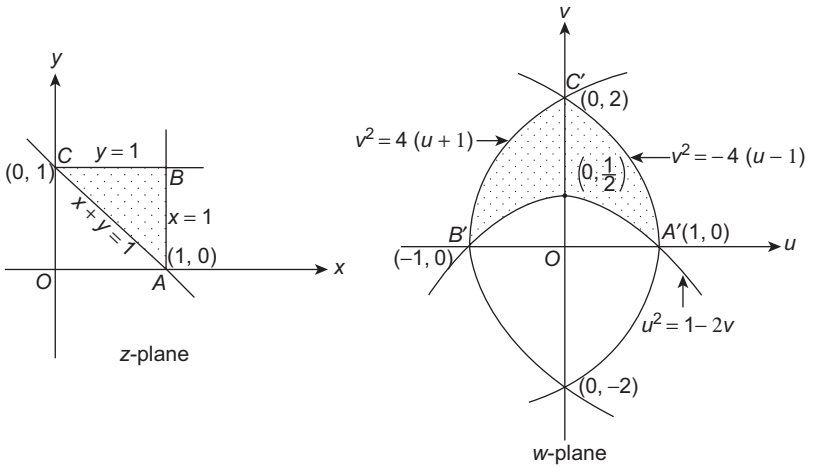


Fig. 6.22

Example 3

Find the image of the square with vertices (0, 0), (2, 0), (2, 2), (0, 2) *in the z-plane under the transformation* $w = z^2$.

Solution

$$w = z^2$$

$$u + iv = (x + iy)^2$$

$$u + iv = x^2 - y^2 + 2ixy$$

Comparing real and imaginary parts,

$$u = x^2 - y^2, \qquad v = 2xy$$

Let $ABCD$ be the square in the z-plane with vertices A (0, 0), B (2, 0), C (2, 2), D (0, 2).

(i) Equation of the line AB is $y = 0$.

Substituting in u and v,

$u = x^2, \qquad v = 0$

which represents the right part of the u-axis where $u > 0$.

The image of the line $y = 0$ (x-axis) is the right part of the u-axis ($u > 0$).

(ii) Equation of the line BC is $x = 2$.

Substituting in u and v,

$u = 4 - y^2, \qquad v = 4y$

$$y = \frac{v}{4}$$

Substituting y in u,

$$u = 4 - \frac{v^2}{16}$$

$$16u = 64 - v^2$$

$$v^2 = -16(u - 4)$$

which represents a parabola with vertex (4, 0) and opening to the left of the vertex.

The image of the line $x = 2$ is a parabola $v^2 = -16(u - 4)$.

(iii) Equation of the line CD is $y = 2$.

Substituting in u and v,

$u = x^2 - 4, \qquad v = 4x$

$$x = \frac{v}{4}$$

Substituting x in u,

$$u = \frac{v^2}{16} - 4$$

$$16u = v^2 - 64$$

$$v^2 = 16(u + 4)$$

which represents a parabola with vertex (–4, 0) and opening to the right of the vertex.

The image of the line $y = 2$ is the parabola $v^2 = 16(u + 4)$.

(iv) Equation of the line DA is $x = 0$.

Substituting in u and v,

$u = -y^2, \qquad v = 0$

which represents the left part of the u-axis, where $u < 0$.

The image of the line $x = 0$ is the left part of the u-axis ($u < 0$).

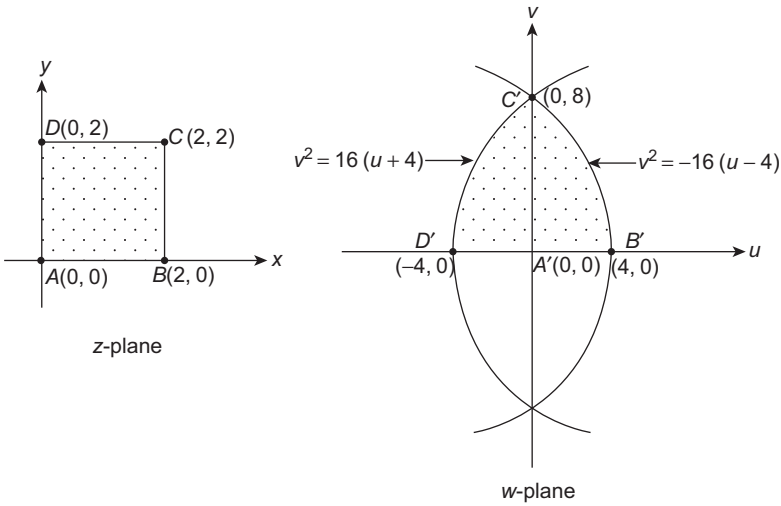


Fig. 6.23

$\because \; x > 0, y > 0$

$v = 2xy > 0$

Hence, the image of the square $ABCD$ in the z-plane is the curvilinear triangle $B'C'D'$ in the upper half of the w-plane (Fig. 6.23).

Example 4

Find the image of $|z-1| = 1$ *under the transformation* $w = z^2$.

Solution

$w = z^2$

$\text{Re}^{i\phi} = (re^{i\theta})^2 \qquad [\text{Considering polar form}]$

$= r^2 e^{2i\theta}$

$\therefore \quad R = r^2, \phi = 2\theta$

Given $|z-1| = 1$

$|re^{i\theta} - 1| = 1$

$$|r(\cos\theta + i\sin\theta) - 1| = 1$$

$$|(r\cos\theta - 1) + ir\sin\theta|^2 = 1$$

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

$$r^2 = 4\cos^2\theta$$

$$= 2(1 + \cos 2\theta) \qquad [\because \cos 2\theta = 2\cos^2\theta - 1]$$

Substituting r^2 and 2θ r^2 and 2θ, ,

$$R = 2(1+\cos\phi)$$

which represents a cardioid.

Hence, the image of the circle $|z-1| = 1$ in the z-plane is the cardioid $R = 2(1+\cos\phi)$ in the w-plane (Fig. 6.24).

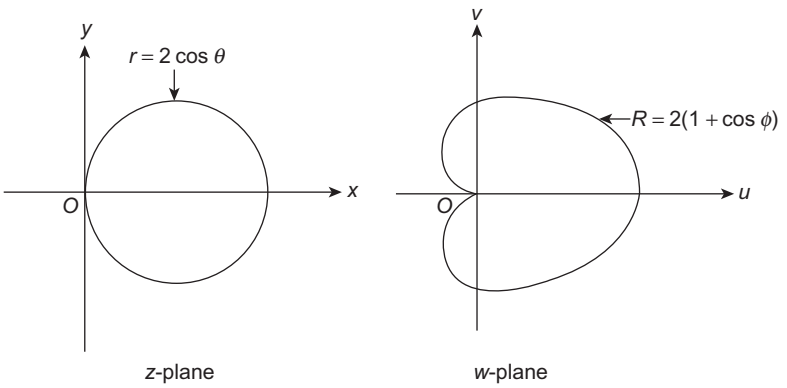


Fig. 6.24

Example 5

Find the image of the region bounded by $1 \le r \le 2$ *and* $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ *in the z-plane under the transformation* $w = z^2$. *Show the regions graphically.*

Solution

$$w = z^2$$

$$\mathrm{Re}^{i\phi} = (re^{i\theta})^2$$

$$= r^2\, e^{2i\theta}$$

Comparing both the sides,

$$R = r^2,\ \phi = 2\theta$$

Given $1 \le r \le 2$

$$1 \le r^2 \le 4$$
$$1 \le R \le 4$$

and
$$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$
$$\frac{\pi}{6} \le \frac{\phi}{2} \le \frac{\pi}{3}$$
$$\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$$

Hence, the image of the region bounded by $1 \le r \le 2$ and $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ in the z-plane is the region $1 \le R \le 4$ and $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ in the w-plane (Fig. 6.25).

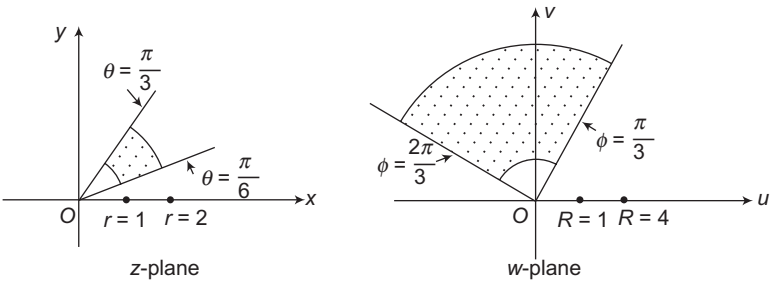


Fig. 6.25

6.4.2 $w = e^z$

$$u + iv = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

(i) Let x = constant = a, say

Substituting $x = a$ in u and v,

$$u = e^a \cos y \qquad \cdots(6.1)$$

$$v = e^a \sin y \qquad \cdots(6.2)$$

Squaring and adding Eqs (6.1) and (6.2),

$$u^2 + v^2 = e^{2a}$$

which represents a circle with the centre at (0, 0) and a radius of e^a.

For $a > 0$, $e^a > 1$, the radius of the circle is greater than 1.

For $a < 0$, $e^a < 1$, the radius of the circle is less than 1.

For $a = 0$, $e^a = 1$, the radius of the circle is 1.

Hence, the lines parallel to the y-axis in the z-plane are mapped onto the circles in the w-plane with the centre at (0, 0) and radius, $r > 1$ or $r < 1$ or $r = 1$ according to whether $a > 0$ or $a < 0$ or $a = 0$ respectively.

(ii) Let $y = \text{constant} = b$, say

Substituting $y = b$ in u and v,

$$u = e^x \cos b \qquad \cdots(6.3)$$

$$v = e^x \sin b \qquad \cdots(6.4)$$

Dividing Eq. (6.3) by Eq. (6.4),

$$\frac{v}{u} = \frac{e^x \sin b}{e^x \cos b}$$

$$= \tan b$$

$$v = u \tan b$$

which represents a radial line.

Hence, the lines parallel to the x-axis in the z-plane are mapped onto radial lines in the w-plane.

Example 1

Prove that the transformation $w = e^z$ transforms the region between the real axis and a line parallel to the real axis $y = \pi$ into the upper half of the w-plane.

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

Given region lies between real axis $(\text{i.e., } y = 0)$ and the line $y = \pi$.

In the region $0 < y < \pi$,

(i) $\sin y > 0$

$e^x \sin y > 0 \qquad \left[\because \ e^x \text{ is always positive}\right]$

$v > 0$

(ii) $\cos y > 0, 0 < y < \dfrac{\pi}{2}$

and $\cos y < 0, \dfrac{\pi}{2} < y < \pi$

$\therefore$ u may be negative or positive.

Hence, the region between the real axis and the line $y = \pi$ in the z-plane is transformed into the upper half of the w-plane (i.e., $v > 0$).

Example 2

Determine and sketch the image of the region $0 \le x \le 1, 0 \le y \le \pi$ *under the transformation* $w = e^z$. **[Summer 2015]**

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

The given region is $0 \le x \le 1$ and $0 \le y \le \pi$.

In the region $0 \le y \le \pi$,

(i) $\sin y \ge 0$

and $0 \le \sin y \le 1$ ... (1)

Also, $0 \le x \le 1$

$$e^0 \le e^x \le e$$

$$1 \le e^x \le e \quad \text{... (2)}$$

From Eqs (1) and (2),

$$0 \le e^x \sin y \le e$$

$$0 \le v \le e$$

(ii) $\cos y \ge 0, \quad 0 \le y \le \dfrac{\pi}{2}$

$$\cos y \le 0, \quad \frac{\pi}{2} \le y \le \pi$$

$$-1 \le \cos y \le 1 \quad \text{... (3)}$$

From Eqs (2) and (3),

$$-1 \le e^x \cos y \le e$$

$$-1 \le u \le e$$

Hence, the image of the region $0 \le x \le 1$, $0 \le y \le \pi$, in the z-plane is the region $-1 \le u \le e$, $0 \le v \le e$, in the w-plane (Fig. 6.26).

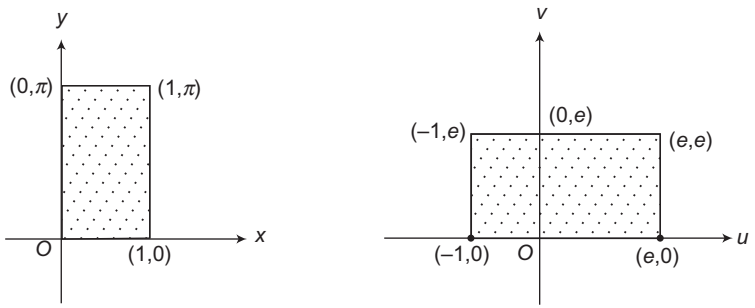


Fig. 6.26

Example 3

Prove that the image of the straight line y = mx is an equiangular spiral under the transformation $w = e^z$.

Solution

$$w = e^z$$

$$\text{Re}^{i\phi} = e^{x+iy} \qquad \left[\text{Considering the polar form of } w\right]$$

$$= e^x e^{iy}$$

$$R = e^x, \phi = y$$

$$x = \log R$$

Given $y = mx$.

Substituting x and y,

$$\phi = m \log R$$

$$\log R = \frac{\phi}{m}$$

$$R = e^{\frac{\phi}{m}}$$

which represents an equiangular spiral.

Hence, the image of the straight line $y = mx$ in the z-plane is an equiangular spiral $R = e^{\frac{\phi}{m}}$ in the w-plane.

6.4.3 $w = \sin z$

$$\begin{aligned} u + iv &= \sin(x + iy) \\ &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + \cos x (i \sinh y) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

Comparing real and imaginary parts,

$u = \sin x \cosh y, \quad v = \cos x \sinh y$

(i) Let x = constant = a, say

Substituting in u and v,

$u = \sin a \cosh y, \quad v = \cos a \sinh y$

$$\frac{u}{\sin a} = \cosh y, \quad \frac{v}{\cos a} = \sinh y$$

Squaring and subtracting,

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = \cosh^2 y - \sinh^2 y$$

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$$

which represents a hyperbola.

Hence, the lines parallel to the y-axis are mapped onto the hyperbolas.

(ii) Let y = constant = b, say

Substituting in u and v,

$u = \sin x \cosh b, \; v = \cos x \sinh b$

$$\frac{u}{\cosh b} = \sin x, \quad \frac{v}{\sinh b} = \cos x$$

Squaring and adding,

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

which represents an ellipse.

Hence, the lines parallel to the x-axis are mapped onto the ellipses.

Example 1

Show that the transformation $w = \sin z$ *maps the top half* $(y > 0)$ *of the line* $x = c_1 \left(-\frac{\pi}{2} < c_1 < 0 \right)$ *in a one-to-one manner onto the half* $(v > 0)$ *of the left-hand branch of the hyperbola* $\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1$.

[Winter 2012]

Solution

$$w = \sin z$$

$$u + iv = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Comparing real and imaginary parts,

$$u = \sin x \cosh y,\ v = \cos x \sinh y$$

For the line $x = c_1$,

$$u = \sin c_1 \cosh y \quad ...(1), \qquad v = \cos c_1 \sinh y \quad ...(2)$$

$$\frac{u}{\sin c_1} = \cosh y, \qquad \frac{v}{\cos c_1} = \sinh y$$

Squaring and subtracting,

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = \cosh^2 y - \sinh^2 y = 1$$

From Eqs (1) and (2),

$$-\frac{\pi}{2} < c_1 < 0 \quad \text{and} \quad y > 0,$$

$$u < 0 \quad \text{and} \quad v > 0 \qquad \left[\because \ \sin(-x) = -\sin x,\ \cos(-x) = \cos x\right]$$

Hence, the image of the top half ($y > 0$) of the line $x = c_1 \left(-\frac{\pi}{2} < c_1 < 0\right)$ in the z-plane (Fig. 6.27) is the top half ($v > 0$) and the left-hand branch of the hyperbola

$$\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1.$$

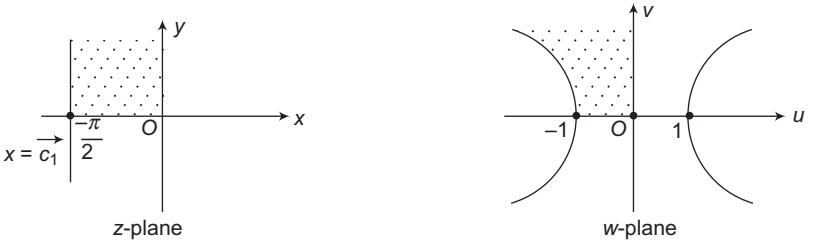


Fig. 6.27

6.4.4 Joukowski's Transformation $\left(w = z + \frac{1}{z}\right)$

$$w = z + \frac{1}{z}$$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Comparing real and imaginary parts,

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \left(r - \frac{1}{r}\right)\sin\theta$$

(i) Let r = constant = a, say

Substituting in u and v,

$$u = \left(a + \frac{1}{a}\right)\cos\theta, \qquad v = \left(a - \frac{1}{a}\right)\sin\theta$$

$$\frac{u}{\left(a + \frac{1}{a}\right)} = \cos\theta, \qquad \frac{v}{a - \frac{1}{a}} = \sin\theta$$

Squaring and adding,

$$\frac{u^2}{\left(a + \frac{1}{a}\right)^2} + \frac{v^2}{\left(a - \frac{1}{a}\right)^2} = 1, a \neq 1$$

which represents an ellipse.

Hence, the circles $r = a$ of the z-plane are mapped onto the ellipses of the w-plane.

If $a = 1$,

$u = 2\cos\theta, \qquad v = 0$

$\because \quad -1 \le \cos\theta \le 1, \qquad -2 \le u \le 2$

Hence, the unit circle $|z| = 1$ in the z-plane is mapped onto the line segment $-2 \le u \le 2$ of the real axis in the w-plane.

(ii) Let θ = constant = b, say

Substituting in u and v,

$$u = \left(r + \frac{1}{r}\right)\cos b, \qquad v = \left(r - \frac{1}{r}\right)\sin b$$

$$\frac{u}{\cos b} = r + \frac{1}{r}, \qquad \frac{v}{\sin b} = r - \frac{1}{r}$$

Squaring and subtracting,

$$\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

$$= 2 + 2$$

$$= 4$$

$$\frac{u^2}{4\cos^2 b} - \frac{v^2}{4\sin^2 b} = 1$$

which represents a hyperbola.

Hence, the radial lines θ = constant of the z-plane are mapped onto the hyperbolae of the w-plane.

Example 1

Determine the points where $w = \left(z + \frac{1}{z}\right)$ *is not conformal mapping. Also, find the image of the circle* $|z| = 2$ *under the transformation* $w = z + \frac{1}{z}$.

[Summer 2014]

Solution

$$w = f(z) = z + \frac{1}{z}$$

The mapping is not conformal at critical points, i.e., where $f'(z) = 0$.

$$f'(z) = 0$$

$$1 - \frac{1}{z^2} = 0$$

$$z^2 = 1$$

$$z = \pm 1$$

Hence, mapping is not conformal at $z = \pm 1$.

$$w = z + \frac{1}{z}$$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

For circle $|z| = 2$, $r = 2$

$$\therefore \quad u + iv = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$$

$$= 2(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta - i\sin\theta)$$

$$= \left(2 + \frac{1}{2}\right)\cos\theta + i\left(2 - \frac{1}{2}\right)\sin\theta$$

$$= \frac{5}{2}\cos\theta + i\frac{3}{2}\sin\theta$$

Comparing real and imaginary parts,

$$u = \frac{5}{2}\cos\theta, \qquad v = \frac{3}{2}\sin\theta$$

$$\frac{2u}{5} = \cos\theta, \qquad \frac{2v}{3} = \sin\theta$$

Squaring and adding,

$$\frac{4u^2}{25} + \frac{4v^2}{9} = \cos^2\theta + \sin^2\theta$$

$$\frac{4u^2}{25} + \frac{4v^2}{9} = 1$$

which represents an ellipse.

Hence, the image of the circle $|z| = 2$ in the z-plane is the ellipse $\frac{4u^2}{25} + \frac{4v^2}{9} = 1$ in the w-plane.

Example 2

Show that the transformation $w = z + \frac{1}{z}$ *maps the straight line arg* $z = \alpha$, $|\alpha| < \frac{\pi}{2}$ *into a branch of hyperbola of eccentricity sec* α.

Solution

$$w = z + \frac{1}{z}$$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Comparing real and imaginary parts,

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta$$

Given $\arg(z) = \alpha$

$$\theta = \alpha$$

Substituting in u and v,

$$u = \left(r + \frac{1}{r}\right)\cos\alpha, \qquad v = \left(r - \frac{1}{r}\right)\sin\alpha$$

$$\frac{u}{\cos\alpha} = r + \frac{1}{r}, \qquad \frac{v}{\sin\alpha} = r - \frac{1}{r}$$

Squaring and subtracting,

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

$$= 4$$

$$\frac{u^2}{4\cos^2\alpha} - \frac{v^2}{4\sin^2\alpha} = 1$$

which represents a hyperbola.

The eccentricity is given as

$$b^2 = a^2(e^2 - 1)$$

$$4\sin^2\alpha = 4\cos^2\alpha(e^2 - 1)$$

$$\tan^2\alpha = e^2 - 1$$

$$\tan^2\alpha + 1 = e^2$$

$$\sec^2\alpha = e^2$$

$$e = \sec\alpha$$

EXERCISE 6.1

1. Find the image of the circle $|z| = 1$ under the transformation $w = z + 2 + 4i$.

[Ans.: $(u-2)^2 + (v-4)^2 = 1$**]**

2. Find the image of the circle $|z| = a$ under the transformation $w = (1+i)z + 2 - i$.

[Ans.: $(u-2)^2 + (v+1)^2 = 2a^2$**]**

3. Find the image of the region bounded by the lines $x = 0$, $y = 0$, $x = 2$, $y = 1$ under the transformation $w = z + 1 - 2i$.

[Ans.: Rectangle bounded by the lines $u = 1, v = -2, u = 3, v = -1$**]**

4. Find the image of the line $2x + y - 3 = 0$ under the transformation $w = z + 2i$.

[Ans.: $2u + v - 5 = 0$**]**

5. Find the image of the circle $|z| = 2$ under the transformation $w = 3z$.

$$\left[\textbf{Ans.:}\ \frac{u^2}{9} + \frac{v^2}{9} = 4\right]$$

6. Find the image of the triangular region bounded by the lines $x = 0, y = 0, x + y = 1$ under the transformation $w = ze^{\frac{-i\pi}{4}}$.

$$\left[\textbf{Ans.:}\ \text{Triangle bounded by the lines } v = u, v = -u, v = \frac{1}{\sqrt{2}}\right]$$

7. Find the image of the strip $x > 0,\ 0 < y < 2$ under the transformation $w = iz + 1$.

$$\left[\textbf{Ans.:}\ \text{Strip} -1 < u < 1,\quad v > 0\right]$$

8. Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$.

$$\left[\textbf{Ans.:}\ u + v > 2\right]$$

9. Find the image of the circle $|z - 1| = 1$ under the transformation $w = \frac{1}{z}$.

$$\left[\textbf{Ans.:}\ 2u - 1 = 0\right]$$

10. Find the image of $|z - 2i| = 3$ under the transformation $w = \frac{1}{z}$.

$$\left[\textbf{Ans.:}\ u^2 + v^2 - \frac{4}{5}v - \frac{1}{5} = 0\right]$$

11. What will be the image of a circle passing through the origin in the xy-plane under the transformation $w = \frac{1}{z}$.

[**Ans.:** Straight line $2gu - 2fv + 1 = 0$ if centre of the circle is $(-g, -f)$]

12. Show that the transformation $w = \frac{1}{z}$ maps the circle $|z - 3| = 5$ into a circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$ in the w-plane.

6.5 SCHWARZ—CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon in the w-plane into the upper half of the z-plane and the boundary of the polygon into the real axis.

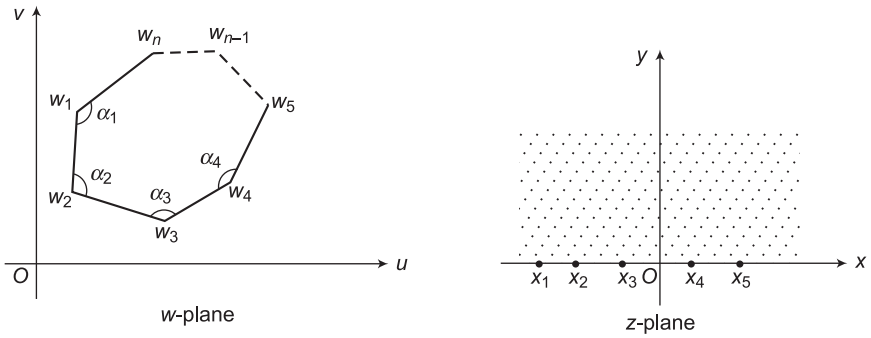


Fig. 6.28

Consider a polygon in the w-plane with vertices $w_1, w_2, w_3, ..., w_n$ and interior angles at these vertices as $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$. Let these points map onto points $x_1, x_2, x_3, ..., x_n$ on the real axis of z-plane (Fig. 6.28). Then the Schwarz–Christoffel formula is given as

$$w = A\int (z-x_1)^{\frac{\alpha_1}{\pi}-1}(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1}\,dz + B$$

where A and B are complex constants. These complex constants determine the size and position of the polygon.

The differential form of this formula is

$$\frac{dw}{dz} = A(z-x_1)^{\frac{\alpha_1}{\pi}-1}(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1} \qquad ...(6.5)$$

Justification of the Formula

From Eq. (6.5),

$$\arg\left(\frac{dw}{dz}\right) = \arg A + \left(\frac{\alpha_1}{\pi}-1\right)\arg(z-x_1) + \left(\frac{\alpha_2}{\pi}-1\right)\arg(z-x_2) + \cdots + \left(\frac{\alpha_n}{\pi}-1\right)\arg(z-x_n) \qquad ...(6.6)$$

As z moves along the real axis in the positive direction and lies between two of the x_i, all the terms in Eq. (6.6) are constant. Therefore, w lies in a straight line. But as z crosses x_i from left to right, all the terms except those involving x_i remain constant.

When $z < x_i$, $\arg(z-x_i) = \pi$

$\therefore$ $\left(\frac{\alpha_i}{\pi}-1\right)\arg(z-x_i) = \left(\frac{\alpha_i}{\pi}-1\right)\pi = \alpha_i - \pi$

When $z > x_i$, $\arg(z-x_i) = 0$

$\therefore$ $\left(\frac{\alpha_i}{\pi}-1\right)\arg(z-x_i) = 0$

Therefore, as z crosses x_i, the term $\left(\frac{\alpha_i}{\pi}-1\right)\arg(z-x_i)$ decreases by $(\alpha_i - \pi)$ or increases by $(\pi - \alpha_i)$ in the anti-clockwise direction. The direction of w rotates in the positive sense by $(\pi - \alpha_i)$. This change in direction of w generates a corner in the polygon with interior angle α_i.

Hence, as z moves along the x-axis, w traces a polygon.

Example 1

Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0$, $v = \pi$, and $u = 0$ into the upper half of the z-plane.

Solution

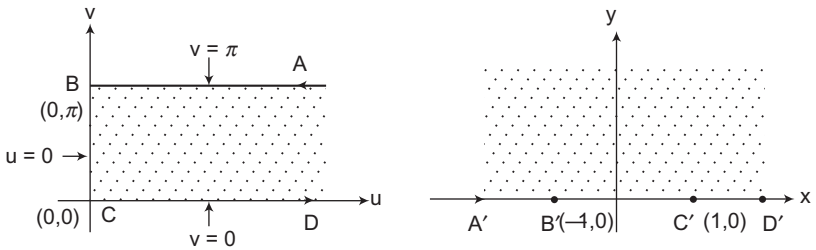


Fig. 6.29

Consider $ABCD$ as the semi-infinite strip of width π with two vertices B and C on the line $u = 0$ and the third vertex A or D at infinity.

Let the vertices B and C map into the points $B'(-1, 0)$ and $C'(1, 0)$ of the z-plane (Fig. 6.29) respectively.

$$w_1 = i\pi, \quad w_2 = 0, \quad w_3 = \infty$$
$$x_1 = -1, \quad x_2 = 1, \quad x_3 = \infty$$

The interior angles at vertices B and C are $\frac{\pi}{2}$.

$$\alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = \frac{\pi}{2}$$

By the Schwarz–Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\left(\frac{\pi}{2}\right)}{\pi}-1}(z-1)^{\frac{\left(\frac{\pi}{2}\right)}{\pi}-1}$$
$$= A(z+1)^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}$$
$$= \frac{A}{\sqrt{z^2-1}}$$

Integrating w.r.t. z,

$$w = \int \frac{A}{\sqrt{z^2 - 1}} dz + B$$

$$= A \cosh^{-1} z + B$$

When $z = 1,\ w = 0$

$$0 = A \cosh^{-1} 1 + B$$

$$= 0 + B$$

$$B = 0$$

When $z = -1,\ w = i\pi$

$$i\pi = A \cosh^{-1}(-1) + B$$

$$= A \cosh^{-1}(-1) + 0$$

$$\frac{i\pi}{A} = \cosh^{-1}(-1)$$

$$\cosh\left(\frac{i\pi}{A}\right) = -1$$

$$\cos i\left(\frac{i\pi}{A}\right) = -1 \qquad [\because \quad \cosh z = \cos iz]$$

$$\cos\left(-\frac{\pi}{A}\right) = -1$$

$$\cos\frac{\pi}{A} = -1 \qquad [\because \quad \cos(-\theta) = \cos\theta]$$

$$= \cos\pi$$

$$\frac{\pi}{A} = \pi$$

$$A = 1$$

Hence, $w = \cosh^{-1} z$

$$z = \cosh w$$

Example 2

Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0$, $v = \pi$ onto the upper half of the z-plane.

Solution

Consider $ABCD$ as the infinite strip bounded by the lines $v = 0$, $v = \pi$ with two vertices B and C on the line $u = 0$ and the third vertex A or D at infinity.

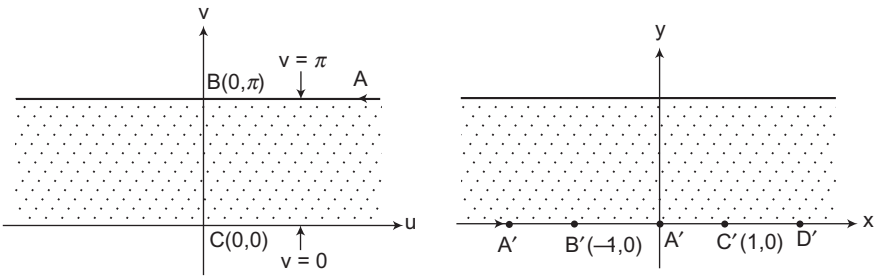


Fig. 6.30

Let the vertices B and C map into the points $B'(-1, 0)$ and $C'(1, 0)$ of the z-plane and the vertex A at infinity map into $A'(0, 0)$ (Fig. 6.30).

$$w_1 = i\pi, \qquad w_2 = 0, \qquad w_3 = \infty$$
$$x_1 = -1, \qquad x_2 = 1, \qquad x_3 = 0$$

The interior angles at vertices B and C are π and at A is 0.

$$\alpha_1 = \pi, \qquad \alpha_2 = \pi, \qquad \alpha_3 = 0$$

By the Schwarz–Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi}{\pi}-1}(z-1)^{\frac{\pi}{\pi}-1}(z-0)^{-1}$$
$$= A(z+1)^0(z-1)^0(z)^{-1}$$
$$= \frac{A}{z}$$

Integrating w.r.t. z,

$$w = \int \frac{A}{z}dz + B$$
$$= A\log z + B$$

6.6 BILINEAR TRANSFORMATION

The transformation $w = \dfrac{az+b}{cz+d}$, where *a, b, c, d* are complex or real constants such that $ad - bc \neq 0$ is known as bilinear transformation. Bilinear transformation is also known as *Mobius transformation*.

Note
$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}$$
$$= \frac{ad-bc}{(cz+d)^2}$$
$$\neq 0 \qquad [\because \ ad - bc \neq 0]$$

Thus, bilinear transformation is conformal.

Special Cases

(i) If $c = 0, a = d \neq 0$,

$$w = z + \frac{b}{d}$$

which represents translation.

(ii) If $c = 0,\ b = 0, d \neq 0$,

$$w = \frac{a}{d} z$$

which represents rotation and magnification.

(iii) If $a = 0, d = 0, b = c \neq 0$,

$$w = \frac{1}{z}$$

which represents inversion and reflection.

6.6.1 Fixed (Invariant) Points

A fixed point is a point $z = x + iy$ which maps into itself in the w-plane (i.e., $w = z$) under the bilinear transformation $w = \dfrac{az + b}{cz + d}$.

$$z = \frac{az + b}{cz + d}$$

$$cz^2 + z(d - a) - b = 0$$

This is a quadratic equation if $c \neq 0$. The roots of this equation are the fixed point of the bilinear transformation.

Notes

(i) If both the roots are equal then the bilinear transformation is said to be parabolic.

(ii) A bilinear transformation with two finite fixed points α, β can be written as $\dfrac{w - \alpha}{w - \beta} = \lambda\left(\dfrac{z - \alpha}{z - \beta}\right)$, where λ is a complex constant.

(iii) If a bilinear transformation has only one fixed point α then it can be written as $\dfrac{1}{w - \alpha} = k + \dfrac{1}{z - \alpha}$, where $k = \dfrac{c}{a - \alpha c} \neq 0$

This form is known as *normal form* or *canonical form* of a bilinear transformation.

6.6.2 Cross Ratio

The cross ratio of four points z_1, z_2, z_3, z_4 is defined as $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ and is denoted by (z_1, z_2, z_3, z_4).

Theorem

A bilinear transformation preserves the cross ratio of four points, i.e., the cross ratio of four points is invariant under a bilinear transformation.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Proof

Let $w = \dfrac{az + b}{cz + d}$ be the bilinear transformation.

Let z, z_1, z_2, z_3 are four points in the z-plane which are mapped to the points w, w_1, w_2, w_3 in the w-plane respectively.

$$w_1 = \frac{az_1 + b}{cz_1 + d}$$

$$w - w_1 = \frac{az + b}{cz + d} - \frac{az_1 + b}{cz_1 + d}$$

$$= \frac{(az + b)(cz_1 + d) - (cz + d)(az_1 + b)}{(cz + d)(cz_1 + d)}$$

$$= \frac{aczz_1 + adz + bcz_1 + bd - cazz_1 - cbz - daz_1 - db}{(cz + d)(cz_1 + d)}$$

$$= \frac{z(ad - bc) - z_1(ad - bc)}{(cz + d)(cz_1 + d)}$$

$$= \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)}$$

Similarly,

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

$$(w - w_1)(w_2 - w_3) = \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)} \cdot \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

Similarly,

$$(w - w_3)(w_2 - w_1) = \frac{(ad - bc)(z - z_3)}{(cz + d)(cz_3 + d)} \cdot \frac{(ad - bc)(z_2 - z_1)}{(cz_2 + d)(cz_1 + d)}$$

$$\therefore \quad \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Thus, bilinear transformation preserves the cross ratio.

Example 1

Find the fixed points of the transformation $w = \dfrac{6z-9}{z}$.

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{6z-9}{z}$$
$$z^2 = 6z-9$$
$$z^2 - 6z + 9 = 0$$
$$(z-3)^2 = 0$$
$$z = 3,3$$

Example 2

Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = \frac{1+z}{1-z}$$
$$z - z^2 = 1+z$$
$$z^2 = -1$$
$$z = \pm i$$

Example 3

Find the invariant points of the transformation $w = \dfrac{2z+6}{z+7}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = \frac{2z+6}{z+7}$$
$$z^2 + 7z = 2z+6$$
$$z^2 + 5z - 6 = 0$$
$$(z+6)(z-1) = 0$$
$$z = -6,1$$

Example 4

Find the fixed points of the mapping $w = \dfrac{1}{z+2i}$.

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{1}{z+2i}$$
$$z^2 + 2zi = 1$$
$$z^2 + 2zi - 1 = 0$$
$$z = \frac{-2i \pm \sqrt{-4+4}}{2}$$
$$= -i$$

Example 5

Obtain the invariant points of the transformation $w = 2 - \dfrac{2}{z}$.

Solution

The invariant points are obtained by putting $w = z$.

$$z = 2 - \frac{2}{z}$$
$$= \frac{2z-2}{z}$$
$$z^2 = 2z - 2$$
$$z^2 - 2z + 2 = 0$$
$$z = \frac{2 \pm \sqrt{4-8}}{2}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2i}{2}$$
$$= 1 \pm i$$

Example 6

Find the fixed point of $w = \dfrac{z+1}{z}$. *Verify your result.*

Solution

The fixed points are obtained by putting $w = z$.

$$z = \frac{z+1}{z}$$

$$z^2 = z+1$$

$$z^2 - z - 1 = 0$$

$$z = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Verification

$$w = \frac{z+1}{z}$$

Putting $z = \dfrac{1+\sqrt{5}}{2}$,

$$w = \frac{\frac{1+\sqrt{5}}{2}+1}{\frac{1+\sqrt{5}}{2}}$$

$$= \frac{3+\sqrt{5}}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{3-3\sqrt{5}+\sqrt{5}-5}{1-5}$$

$$= \frac{-2-2\sqrt{5}}{-4}$$

$$= \frac{1+\sqrt{5}}{2}$$

Putting $z = \dfrac{1-\sqrt{5}}{2}$,

$$w = \frac{\frac{1-\sqrt{5}}{2}+1}{\frac{1-\sqrt{5}}{2}}$$

$$= \frac{3-\sqrt{5}}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$
$$= \frac{3+3\sqrt{5}-\sqrt{5}-5}{1-5}$$
$$= \frac{-2+2\sqrt{5}}{-4}$$
$$= \frac{1-\sqrt{5}}{2}$$

This verifies that the point $z = \frac{1 \pm \sqrt{5}}{2}$ maps into itself under the given transformation.

Example 7

Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Solution

Let $z_1 = 0, z_2 = -i, z_3 = -1$ and $w_1 = i, w_2 = 1, w_3 = 0$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$
$$\frac{w-i}{w(1-i)} = \frac{z(-i+1)}{(z+1)(-i)}$$
$$\frac{w-i}{w-iw} = \frac{-iz+z}{-iz-i}$$
$$(w-i)(-iz-i) = (w-iw)(-iz+z)$$
$$-iwz-iw+i^2z+i^2 = -iwz+i^2wz+wz-iwz$$
$$-iwz-iw-z-1 = -iwz-wz+wz-iwz$$
$$-iw+iwz = z+1$$
$$iw(-1+z) = z+1$$
$$w = \frac{1}{i} \cdot \frac{z+1}{z-1} = -i\frac{z+1}{z-1}$$

Example 8

Find the bilinear transformation which maps the points –2, 0, 2 into the points w = 0, i, – i respectively.

Solution

Let $z_1 = -2, z_2 = 0, z_3 = 2$ and $w_1 = 0, w_2 = i, w_3 = -i.$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i+i)}{(w+i)(i-0)} = \frac{(z+2)(0-2)}{(z-2)(0+2)}$$

$$\frac{w(2i)}{(w+i)i} = \frac{(z+2)(-2)}{(z-2)(2)}$$

$$\frac{2w}{w+i} = -\frac{z+2}{z-2}$$

$$2w(z-2) = -(z+2)(w+i)$$

$$2w(z-2) = -w(z+2) - i(z+2)$$

$$w\left[2(z-2)+z+2\right] = -i(z+2)$$

$$w(3z-2) = -i(z+2)$$

$$w = -\frac{i(z+2)}{3z-2}$$

Example 9

Find the bilinear transformation which maps the points $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ *into the points* $w_1 = 0, w_2 = i,$ and $w_3 = 3i$ *respectively.*

Solution

Let $z_1 = -1, z_2 = 0, z_3 = 1$ and $w_1 = 0, w_2 = i, w_3 = 3i.$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$\frac{w(-2i)}{(w-3i)i} = \frac{(z+1)(-1)}{z-1}$$

$$\frac{-2w}{w-3i} = \frac{-(z+1)}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (z+1)(w-3i)$$
$$2wz - 2w = wz - i3z + w - 3i$$
$$wz - 3w = -3iz - 3i$$
$$w(z-3) = -3i(z+1)$$
$$w = -3i\left(\frac{z+1}{z-3}\right)$$

Example 10

Find the bilinear transformation which maps –1, 0, 1 *of the z-plane onto* –1, *–i,* 1 *of the w-plane.*

Solution

Let $z_1 = -1, z_2 = 0, z_3 = 1$ and $w_1 = -1, w_2 = -i, w_3 = 1.$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$
$$\frac{(w+1)(1+i)}{(w-1)(1-i)} = \frac{z+1}{z-1}$$
$$\frac{(w+1)(1+i)(1+i)}{(w-1)(1-i)(1+i)} = \frac{z+1}{z-1}$$
$$\frac{(w+1)(1+i)^2}{(w-1)(2)} = \frac{z+1}{z-1}$$
$$\frac{(w+1)(2i)}{(w-1)(2)} = \frac{z+1}{z-1}$$
$$\frac{(w+1)i}{(w-1)} = \frac{z+1}{z-1}$$
$$\frac{w+1}{w-1} = \frac{1}{i}\left(\frac{z+1}{z-1}\right)$$
$$= -i\left(\frac{z+1}{z-1}\right)$$
$$= \frac{-iz-i}{z-1}$$

Applying componendo-dividendo,

$$\frac{w+1+w-1}{w+1-w+1}=\frac{-iz-i+z-1}{-iz-i-z+1}$$

$$\frac{2w}{2}=\frac{(-i+1)z-(i+1)}{(-i-1)z+(1-i)}$$

$$w=\left[\frac{(1-i)z-(1+i)}{-(1+i)z+(1-i)}\right]\left(\frac{1-i}{1-i}\right)$$

$$=\frac{(1-i)^2z-(1+i)(1-i)}{-(1+i)(1-i)z+(1-i)^2}$$

$$=\frac{-2iz-2}{-2z-2i}$$

$$=\frac{iz+1}{z+i}$$

Example 11

Find the bilinear transformation that maps the points $z=\infty, i, 0$ on to $w=0, i, \infty$ *respectively.*

Solution

Let $z_1=\infty, z_2=i, z_3=0$ and $w_1=0, w_2=i, w_3=\infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)} \qquad \left[\because z_1=\infty, \frac{z}{z_1}=0, \frac{z_2}{z_1}=0 \quad \because w_3=\infty, \frac{w_2}{w_3}=0, \frac{w}{w_3}=0\right]$$

$$\frac{-w}{-i}=\frac{-i}{-z}$$

$$\frac{w}{i}=\frac{i}{z}$$

$$w=-\frac{1}{z}$$

Example 12

Determine the bilinear transformation which maps the points $z = 0, i, 1$ *into* $w = i, -1, \infty$. **[Summer 2015]**

Solution

Let $z_1 = 0, z_2 = i, z_3 = 1$ and $w_1 = i, w_2 = -1, w_3 = \infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0-1)}{(0-1)(-1-i)} = \frac{(z-0)(i-1)}{(z-1)(i-0)} \qquad \left[\because\ w_3 = \infty, \frac{w_2}{w_3} = 0, \frac{w}{w_3} = 0\right]$$

$$w - i = \frac{(1+i)(1-i)z}{i(z-1)}$$

$$w = \frac{2z(-i)}{z-1} + i$$

$$= \frac{-2iz + iz - i}{z-1}$$

$$= \frac{-i(z+1)}{z-1}$$

Example 13

Determine the linear fractional transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$, on to $w_1 = -1, w_2 = -i, w_3 = 1$ respectively. **[Summer 2014]**

Solution

Let $z_1 = 0, z_2 = 1, z_3 = \infty$ and $w_1 = -1, w_2 = -i, w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$\frac{w+1}{w-1} = -\frac{z(1-i)}{1+i}$$

$$= -\frac{z(1-i)(1-i)}{(1+i)\,(1-i)}$$

$$= -\frac{z(-2i)}{2}$$

$$= \frac{z}{-i}$$

Applying componendo-dividendo,

$$\frac{w+1+w-1}{w+1-w+1} = \frac{z-i}{z+i}$$

$$\frac{2w}{2} = \frac{z-i}{z+i}$$

$$w = \frac{z-i}{z+i}$$

Example 14

Determine the bilinear transformation which maps the points 0, ∞, *i into* ∞, 1, 0.

Solution

Let $z_1 = 0, z_2 = \infty, z_3 = i$ and $w_1 = \infty, w_2 = 1, w_3 = 0$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{\left(\frac{w}{w_1}-1\right)(w_2-w_3)}{(w-w_3)\left(\frac{w_2}{w_1}-1\right)} = \frac{(z-z_1)\left(1-\frac{z_3}{z_2}\right)}{(z-z_3)\left(1-\frac{z_1}{z_2}\right)}$$

$$\frac{(0-1)(1-0)}{(w-0)(0-1)} = \frac{(z-0)(1-0)}{(z-i)(1-0)}$$

$$\frac{1}{w} = \frac{z}{z-i}$$

$$w = \frac{z-i}{z}$$

$$\left[\because z_2 = \infty, \frac{z_3}{z_2} = 0, \frac{z_1}{z_2} = 0 \quad \because w_1 = \infty, \frac{w}{w_1} = 0, \frac{w_2}{w_1} = 0\right]$$

Example 15

Find the bilinear transformation which maps the points $z = 0, 1, \infty$ into $w = i, 1, -i$.

Solution

Let $z_1 = 0, z_2 = 1, z_3 = \infty$ and $w_1 = i, w_2 = 1, w_3 = -i$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{(z-0)(0-1)}{(0-1)(1-0)} \qquad \left[\because \quad z_3 = \infty, \frac{z_2}{z_3} = 0, \frac{z}{z_3} = 0\right]$$

$$\frac{(w-i)(1+i)}{(w+1)(1-i)} = z$$

$$\frac{w-i}{w+i} = z\left(\frac{1-i}{1+i}\right)$$

$$= z\left(\frac{1-i}{1+i}\right)\left(\frac{1-i}{1-i}\right)$$

$$= z\frac{(1-i)^2}{2}$$

$$= z\frac{(-2i)}{2}$$

$$= -iz$$

$$w - i = -iwz + z$$

$$w + iwz = z + i$$

$$w(1+iz) = z + i$$

$$w = \frac{z+i}{1+iz}$$

Example 16

Find the bilinear transformation that maps the points $1+i, -i, 2-i$ *of the z-plane into the points* $0, 1, i$ *of the w-plane.*

Solution

Let $z_1 = 1+i, z_2 = -i, z_3 = 2-i$ and $w_1 = 0, w_2 = 1, w_3 = i.$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{(z-1-i)(-i-2+i)}{(z-2+i)(-i-1-i)}$$

$$\frac{w(1-i)}{w-i} = \frac{(z-1-i)(-2)}{(z-2+i)(-1-2i)}$$

$$\frac{w(1-i)}{w-i} = \frac{(-2)(z-1-i)}{(1+2i)(2-z-i)}$$

$$\frac{w}{w-i} = \frac{(-2)}{(1+2i)(1-i)}\left(\frac{z-1-i}{2-z-i}\right)$$

$$= \frac{(-2)}{(1-i+2i+2)}\left(\frac{z-1-i}{2-z-i}\right)$$

$$= \frac{(-2)}{3+i}\left(\frac{z-1-i}{2-z-i}\right)$$

$$\frac{w-i}{w} = \frac{(3+i)}{(-2)}\left(\frac{2-z-i}{z-1-i}\right)$$

$$1-\frac{i}{w} = \frac{(3+i)}{(-2)}\left(\frac{2-z-i}{z-1-i}\right)$$

$$\frac{i}{w} = 1-\frac{(3+i)}{(-2)}\left(\frac{2-z-i}{z-1-i}\right)$$

$$= 1+\frac{(3+i)}{2}\left(\frac{2-z-i}{z-1-i}\right)$$

$$= \frac{2(z-1-i)+(3+i)(2-z-i)}{2(z-1-i)}$$

$$\frac{w}{i} = -\frac{2(z-1-i)}{2(z-1-i)+(3+i)(2-z-i)}$$

$$w = \frac{2i(z-1-i)}{2(z-1-i)+(3+i)(2-z-i)}$$

$$= \frac{2i(z-1-i)}{2z-2-2i+6-3z-3i+2i-iz+1}$$

$$= \frac{2i(z-1-i)}{-z+5-3i-iz}$$

$$= \frac{2iz+(2-2i)}{-z(1+i)+(5-3i)}$$

Example 17

Find the bilinear transformation that maps the points $z_1 = 1, z = i, z_3 = -1,$ onto $w_1 = -1, w_2 = 0, w_3 = 1$ *respectively. Find the image of* $|z| < 1$ *under this transformation.* **[Winter 2013]**

Solution

(i) Let $z_1 = 1, z_2 = i, z_3 = -1$ onto $w_1 = -1, w_2 = 0, w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w+1)(-1)}{(w-1)} = \frac{(z-1)}{(z+1)}\frac{(i+1)^2}{(i-1)(i+1)}$$

$$= \frac{(z-1)(2i)}{(z+1)(-2)}$$

$$\frac{w+1}{w-1} = \frac{iz-i}{z+1}$$

Applying componendo-dividendo,

$$\frac{w+1+w-1}{w+1-w+1} = \frac{iz-i+z+1}{iz-i-z-1}$$

$$\frac{2w}{2} = \left[\frac{z(i+1)-(i-1)}{z(i-1)-(i+1)}\right]\left(\frac{i+1}{i+1}\right)$$

$$w = \frac{z(i+1)^2-(i-1)(i+1)}{z(i-1)(i+1)-(i+1)^2}$$

$$= \frac{2iz+2}{-2z-2i}$$

$$= -\frac{iz+1}{z+i}$$

(ii) $$w = -\frac{iz+1}{z+i}$$

$$(z+i)w = -(iz+1)$$

$$zw + iw = -iz - 1$$

$$z(w+i) = -1 - iw = i^2 - iw = i(i-w)$$

$$z = \frac{i(i-w)}{(i+w)}$$

$$|z|<1$$
$$\left|\frac{i(i-w)}{(i+w)}\right|<1$$
$$|i|\,|i-w|<|i+w|$$
$$1\cdot|i-(u+iv)|<|i+u+iv|$$
$$|-u+i(1-v)|<|u+i(1+v)|$$
$$\sqrt{u^2+(1-v)^2}<\sqrt{u^2+(1+v)^2}$$
$$u^2+(1-v)^2<u^2+(1+v)^2$$
$$u^2+1-2v+v^2<u^2+1+2v+v^2$$
$$4v>0$$
$$v>0$$

Hence, the image of $|z|<1$ in the z-plane is the upper half of the w-plane.

Example 18

Find the bilinear transformation which maps the points 1, *i*, –1 *onto the points* 0, 1, ∞. *Also show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane.*

Solution

(i) Let $z_1=1, z_2=i, z_3=-1$ and $w_1=0, w_2=1, w_3=\infty$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-0)(0-1)}{(0-1)(1-0)}=\frac{(z-1)(i+1)}{(z+1)(i-1)}\quad\left[\because w_3=\infty,\frac{w_2}{w_3}=0,\frac{w}{w_3}=0\right]$$
$$\frac{-w}{-1}=-\frac{(z-1)(1+i)}{(z+1)(1-i)}$$
$$=-\frac{(z-1)(1+i)}{(z+1)(1-i)}$$
$$=-\frac{(z-1)(1+i)(1+i)}{(z+1)(1-i)(1+i)}$$

$$= -\frac{(z-1)(1+i)^2}{(z+1)(2)}$$

$$= -\frac{(z-1)(2i)}{(z+1)(2)}$$

$$= -i\frac{z-1}{z+1}$$

(ii) $(z+1)w = -i(z-1)$

$$wz + w = -iz + i$$

$$(w+i)z = i - w$$

$$z = \frac{i-w}{w+i}$$

$$= -\left(\frac{w-i}{w+i}\right)$$

Interior of the unit circle is given by

$$|z| < 1$$

$$\left|-\left(\frac{w-i}{w+i}\right)\right| < 1$$

$$\left|\frac{w-i}{w+i}\right| < 1$$

$$\frac{|w-i|}{|w+i|} < 1$$

$$\frac{|u+iv-i|}{|u+iv+i|} < 1$$

$$|u+iv-i| < |u+iv+i|$$

$$|u+i(v-1)| < |u+i(v+1)|$$

$$u^2 + (v-1)^2 < u^2 + (v+1)^2$$

$$(v-1)^2 < (v+1)^2$$

$$v^2 - 2v + 1 < v^2 + 2v + 1$$

$$-4v < 0$$

$$v > 0$$

Hence, the interior of unit circle of the z-plane is mapped onto the upper half of the w-plane (Fig. 6.31).

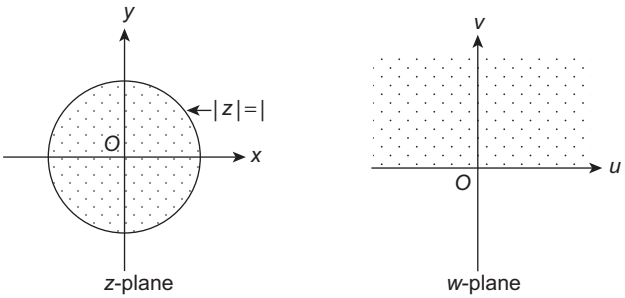


Fig. 6.31

Example 19

Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. What are the invariant points of the map? Also find the image of the interior of the unit circle of the z-plane.

Solution

(i) Let $z_1 = 1, z_2 = i, z_3 = -1$ and $w_1 = i, w_2 = 0, w_3 = -i$.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)(i)}{(w+i)(-i)} = \frac{(z-1)(1+i)}{-(z+1)(1-i)}$$

$$\frac{w-i}{w+i} = \frac{(z-1)(1+i)}{(z+1)(1-i)}$$

By componedo-dividendo,

$$\frac{w-i+w+i}{w-i-w-i} = \frac{(z-1)(1+i)+(z+1)(1-i)}{(z-1)(1+i)-(z+1)(1-i)}$$

$$\frac{2w}{-2i} = \frac{z+iz-1-i+z-iz+1-i}{(z+iz-1-i)-(z-iz+1-i)}$$

$$\frac{w}{-i} = \frac{2z-2i}{2iz-2}$$

$$\frac{w}{-i} = \frac{z-i}{iz-1}$$

$$w = \frac{-i(z-i)}{iz-1}$$

$$= \frac{-iz-1}{iz-1}$$

$$= \frac{1+iz}{1-iz}$$

(ii) The invariant points are obtained by putting $w = z$.

$$z = \frac{1+iz}{1-iz}$$

$$z(1-iz) = 1+iz$$

$$-iz^2 + (1-i)z - 1 = 0$$

Multiplying by i,

$$z^2 + (1+i)z - i = 0$$

$$z = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4i}}{2}$$

$$= \frac{-(1+i) \pm \sqrt{6i}}{2}$$

(iii) $$w = \frac{1+iz}{1-iz}$$

$$w(1-iz) = 1+iz$$

$$iz(1+w) = w-1$$

$$z = \frac{w-1}{i(w+1)}$$

$$= \frac{-i(w-1)}{w+1}$$

Given $$|z| < 1$$

$$\left|\frac{-i(w-1)}{w+1}\right| < 1$$

$$\left|\frac{w-1}{w+1}\right| < 1$$

$$\frac{|w-1|}{|w+1|} < 1$$

$$|w-1| < |w+1|$$

$$|u+iv-1| < |u+iv+1|$$

$$|(u-1)+iv|^2 < |(u+1)+iv|^2$$

$$(u-1)^2 + v^2 < (u+1)^2 + v^2$$

$$u^2 - 2u + 1 + v^2 < u^2 + 2u + 1 + v^2$$

$$-4u < 0$$
$$4u > 0$$
$$u > 0$$

Hence, the image of the interior of the unit circle of the z-plane is right half of the w-plane (Fig. 6.32).

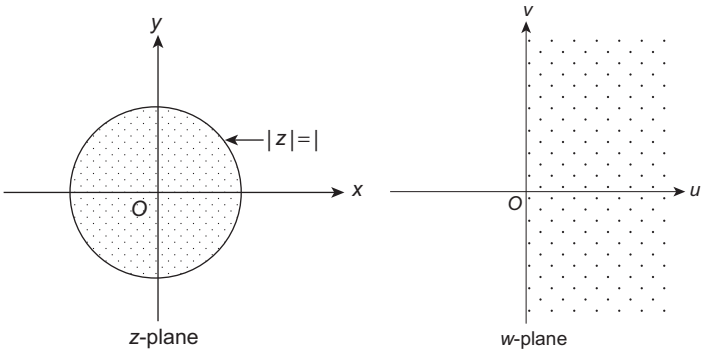


Fig. 6.32

Example 20

Prove that the transformation $w = \dfrac{z}{1-z}$ *maps the upper half of the z-plane onto the upper half of the u-plane. What is the image of* $|z| = 1$ *under this transformation?*

Solution

$$w = \frac{z}{1-z}$$
$$w(1-z) = z$$
$$w - wz = z$$
$$w = z + wz$$
$$= z(1+w)$$
$$z = \frac{w}{1+w}$$
$$x + iy = \frac{u+iv}{1+u+iv}$$
$$= \frac{(u+iv)[(1+u)-iv]}{[(1+u)+iv][(1+u)-iv]}$$
$$= \frac{[u(1+u)+v^2]+i[v(1+u)-uv]}{(1+u)^2+v^2}$$

$$= \frac{(u^2 + v^2 + u) + iv}{(1+u)^2 + v^2}$$

Comparing real and imaginary parts,

$$x = \frac{u^2 + v^2 + u}{(1+u)^2 + v^2}, \quad y = \frac{v}{(1+u)^2 + v^2}$$

(i) Upper half of the z-plane is given by

$$y > 0$$

$$\frac{v}{(1+u)^2 + v^2} > 0$$

$$v > 0$$

which represents the upper half of the w-plane.

Hence, the upper half of the z-plane is mapped onto the upper half of the w-plane (Fig. 6.33).

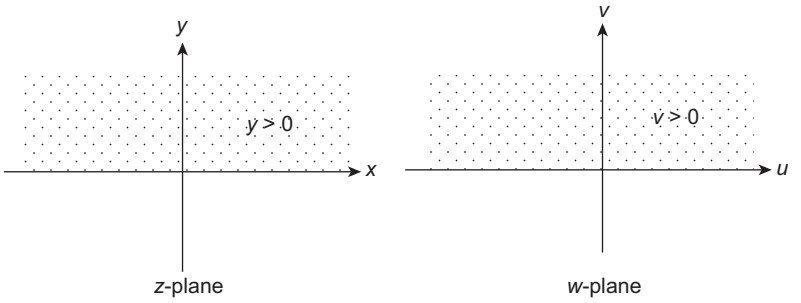


Fig. 6.33

(ii)
$$|z| = 1$$

$$\left|\frac{w}{1+w}\right| = 1$$

$$\frac{|w|}{|1+w|} = 1$$

$$\frac{|u+iv|}{|1+u+iv|} = 1$$

$$\frac{|u+iv|^2}{|(1+u)+iv|^2} = 1$$

$$\frac{u^2 + v^2}{(1+u)^2 + v^2} = 1$$

$$u^2 + v^2 = 1 + 2u + u^2 + v^2$$
$$0 = 1 + 2u$$
$$u = -\frac{1}{2}$$

which represents a straight line.

Hence, the image of $|z| = 1$ in the z-plane is the line $u = -\frac{1}{2}$ in the w-plane (Fig. 6.34).

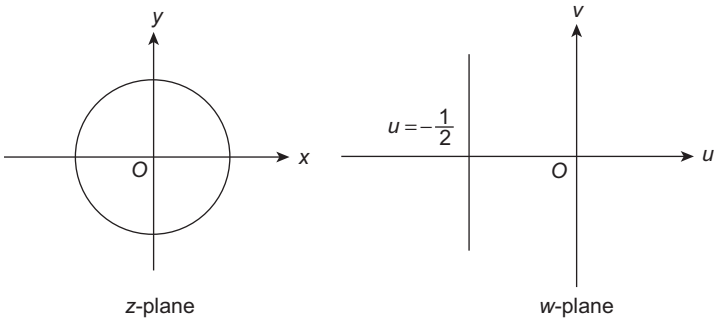


Fig. 6.34

Example 21

Find the bilinear transformation that maps the points $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ onto $w_1 = -i$, $w_2 = 1$, $w_3 = i$ *respectively. Also, find w for* $z = \infty$.

[Summer 2013]

Solution

(i) $z_1 = -1, z_2 = 0, z_3 = 1$ and $w_1 = -i, w_2 = 1, w_3 = i$

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$
$$\frac{(w + i)(1 - i)}{(w - i)(1 + i)} = \frac{(z + 1)(0 - 1)}{(z - 1)(0 + 1)}$$
$$\frac{w + i}{w - i} = -\frac{(z + 1)(1 + i)}{(z - 1)(1 - i)}\frac{(1 + i)}{(1 + i)}$$
$$= -\frac{(z + 1)(2i)}{(z - 1)(2)}$$
$$= \frac{iz + i}{-z + 1}$$

Applying componendo-dividendo,

$$\frac{w+i+w-i}{w+i-w+i} = \frac{iz+i-z+1}{iz+i+z-1}$$

$$\frac{2w}{2i} = \frac{z(i-1)+(i+1)}{z(i+1)+(i-1)}$$

$$w = \left[\frac{z(-1-i)+(-1+i)}{z(i+1)+(i-1)}\right]\left(\frac{i-1}{i-1}\right)$$

$$= \frac{-z(i+1)(i-1)+(i-1)^2}{z(i+1)(i-1)+(i-1)^2}$$

$$= \frac{2z-2i}{-2z-2i}$$

$$= -\frac{z-i}{z+i}$$

(ii) $$w = -\frac{z-i}{z+i}$$

$$= -\frac{1-\frac{i}{z}}{1+\frac{i}{z}}$$

At $z = \infty$,

$w = -1$

Example 22

Find the bilinear transformation which maps the points $1, -1, \infty$ *onto the points* $1 + i$, $1 - i$, 1 *respectively. Also, find its fixed points.*

[Winter 2012]

Solution

(i) Let $z_1 = 1, z_2 = -1, z_3 = \infty$ and $w_1 = 1+i, w_2 = 1-i, w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-1-i)(1-i-1)}{(w-1)(1-i-1-i)} = \frac{(z-1)(0-1)}{(0-1)(-1-1)} \quad \left[\because z_3 = \infty, \frac{z_2}{z_3} = 0, \frac{z}{z_3} = 0\right]$$

$$\frac{(w-1-i)(-i)}{(w-1)(-2i)} = \frac{z-1}{-2}$$

$$\frac{w-1-i}{w-1} = 1-z$$

$$w-1-i = w(1-z)-(1-z)$$

$$w - w(1-z) = 1+i-1+z$$

$$wz = z+i$$

$$w = \frac{z+i}{z}$$

(ii) The fixed points are obtained by putting $w = z$.

$$z = \frac{z+i}{z}$$

$$z^2 - z - i = 0$$

$$z = \frac{1 \pm \sqrt{1+4i}}{2}$$

EXERCISE 6.2

1. Find the invariant point of the transformation $w = \dfrac{1}{z-2i}$.

[Ans.: i**]**

2. Find the invariant points of the bilinear transformation $\dfrac{z-1}{z+1}$.

[Ans.: $\pm i$**]**

3. Find the fixed points of $w = \dfrac{3z-4}{z-1}$.

[Ans.: 2, 2**]**

4. Find the fixed points under the transformation $w = \dfrac{2z-5}{z+4}$.

[Ans.: $-1 \pm 2i$ **]**

5. Find the bilinear transformation that maps the points -1, 0, 1 in the z-plane onto the points 0, i, $3i$ in the w-plane.

$$\left[\textbf{Ans.:} -3i\frac{(z+1)}{(z-3)}\right]$$

6. Find the bilinear transformation that maps $z = 0, 1, \infty$ onto the points $w = -5, -1, 3$ respectively. What are the fixed points?

$$\left[\textbf{Ans.:}\ \frac{3z-5}{z+1}; 1 \pm 2i\right]$$

7. Find the bilinear transformation mapping the points $z = 1, i, -1$ into the points $w = 2, i, -2$.

$$\left[\textbf{Ans.:}\ -\frac{6z-2i}{iz-3}\right]$$

8. Find the bilinear transformation which maps points $i, -1, 1$ of the z-plane into the points $0, 1, \infty$ of the w-plane respectively.

$$\left[\textbf{Ans.:}\ \frac{2z-2i}{(1+i)(z-1)}\right]$$

9. Find the bilinear transformation which maps the points $\infty, i, 0$ of the z-plane into the points $0, -i, \infty$ of the w-plane.

$$\left[\textbf{Ans.:}\ \frac{1}{z}\right]$$

10. Obtain the bilinear transformation which maps $z = 0$ onto $w = -i$ and has $-1, 1$ as fixed points. Also show that the upper half plane is mapped onto the interior of the unit circle in the w-plane.

$$\left[\textbf{Ans.:}\ \frac{z-1}{1-iz}\right]$$

Points to Remember

Conformal Mapping

Conformal mapping transforms curves from one complex plane to the other with respect to size and orientation.

A mapping or transformation $w = f(z)$ is said to be conformal at a point z_0 if it preserves the angle between any two curves passing through z_0 in magnitude and sense.

Isogonal Transformation

A transformation which preserves the angle between any two curves passing through a point only in magnitude and not in sense is known as the isogonal transformation at that point.

Conformal Property of Analytic Function

If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of the z-plane then the mapping $w = f(z)$ is conformal at all points of the region R.

Critical Points

A point of the transformation $w = f(z)$ at which $f'(z) = 0$ is known as the critical point. At this point, mapping is not conformal.

Some Transformations

(i) The transformation $w = z + c$ represents a translation of axes. It preserves the shape and size of the region of the z-plane.

(ii) The transformation $w = cz$ represents magnification and rotation. It maps any figure in the z-plane into a similar figure in the w-plane.

(iii) The transformation $w = \frac{1}{z}$ represents inversion and reflection. It maps the point (r, θ) in the z-plane onto the point $\left(\frac{1}{r}, -\theta\right)$ in the z-plane.

(iv) The transformation $w = z^2$ maps the lines parallel to x-axis and y-axis in the z-plane into the parabolas in the w-plane.

(v) The transformation $w = e^z$ maps the lines parallel to the y-axis in the z-plane onto the circles in w-plane and the lines parallel to the x-axis in the z-plane onto radial lines in the w-plane.

(vi) The transformation $w = \sin z$ maps the lines parallel to the y-axis onto the hyperbolas and the lines parallel to the x-axis onto the ellipses.

Bilinear Transformation

The transformation $w = \frac{az + b}{cz + d}$, where a, b, c, d are complex or real constants such that $ad - bc \neq 0$ is known as bilinear transformation.

Fixed (Invariant) Points

Fixed point is a point $z = x + iy$ which maps into itself in the w-plane (i.e., $w = z$) under the bilinear transformation $w = \frac{az + b}{cz + d}$.

$$\therefore \quad z = \frac{az + b}{cz + d}$$

$$cz^2 + z(d - a) - b = 0$$

This is a quadratic equation if $c \neq 0$. The roots of this equation are the fixed point of the bilinear transformation.

Cross Ratio

The cross ratio of four points z_1, z_2, z_3, z_4 is defined as $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ and is denoted by (z_1, z_2, z_3, z_4).

CHAPTER 7
Interpolation

Chapter Outline

7.1 Introduction
7.2 Finite Differences
7.3 Different Operators and their Relations
7.4 Interpolation
7.5 Newton's Forward Interpolation Formula
7.6 Newton's Backward Interpolation Formula
7.7 Central Difference Interpolation
7.8 Gauss's Forward Interpolation Formula
7.9 Gauss's Backward Interpolation Formula
7.10 Stirling's Formula
7.11 Interpolation with Unequal Intervals
7.12 Lagrange's Interpolation Formula
7.13 Divided Differences
7.14 Newton's Divided Difference Formula
7.15 Inverse Interpolation

7.1 INTRODUCTION

Interpolation is the process of reading between the lines of a table. It is the process of computing intermediate values of a function from a given set of tabular values of the function. Extrapolation is used to denote the process of finding the values outside the given interval.

In the interpolation process, the given set of tabular values are used to find an expression for $f(x)$ and then using it to find its required value for a given value of x. But it is difficult to find an exact form of $f(x)$ using the limited values in the table. Hence, $f(x)$ is replaced by another function $\phi(x)$, which matches with $f(x)$ at the discrete values in the table. This function $\phi(x)$ is known as the *interpolating function*.

When the interpolating function is a polynomial function, the process is known as *polynomial interpolation*. Polynomial interpolations are preferred because of the following reasons:

(i) They are simple forms of functions which can be easily manipulated.

(ii) Polynomials are free from singularities whereas rational functions or other types have singularities.

7.2 FINITE DIFFERENCES

Let the function $y = f(x)$ be tabulated for the equally spaced values $y_0 = f(x_0)$, $y_1 = f(x_0 + h)$, $y_2 = f(x_0 + 2h)$, …, $y_n = f(x_0 + nh)$, as

x	x_0	$x_0 + h$	$x_0 + 2h$	….	$x_0 + nh$	….
$y = f(x)$	y_0	y_1	y_2	….	y_n	….

To determine the values of $f(x)$ for some intermediate values of x, the following three types of differences can be used.

7.2.1 Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the first forward differences of y and are denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively.

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
&\vdots \quad \vdots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

where Δ is called the *forward difference operator*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots, \Delta^2 y_{n-1}$. Similarly, third forward differences, fourth forward differences, etc., can be defined.

$$\begin{aligned}
\Delta^2 y_0 &= \Delta(\Delta y_0) \\
&= \Delta(y_1 - y_0) \\
&= \Delta y_1 - \Delta y_0 \\
&= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0 \\
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0
\end{aligned}$$

$$\begin{aligned}\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0 \\ &= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) \\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\end{aligned}$$

Since the coefficients occurring on the right-hand side are the binomial coefficients, the general form is

$$\Delta^n y_0 = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} - \cdots + (-1)^n y_0$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
$x_0 + h = x_1$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
$x_0 + 2h = x_2$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
$x_0 + 3h = x_3$	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
$x_0 + 4h = x_4$	y_4				

In a difference table, x is called the *argument*, and y is called the *function* or *entry*.

Note When $(n + 1)$ values are given, the highest-order difference is n, e.g., when 5 values are given, the highest-order difference is 4.

7.2.2 Backward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *first backward differences* of y and are denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$, respectively.

$$\begin{aligned}\nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \quad \vdots \quad \vdots \\ \nabla y_n &= y_n - y_{n-1}\end{aligned}$$

where ∇ is called the *backward difference operator.* Similarly, backward differences of higher order can be defined.

$$\nabla^2 y_2 = \nabla\,(\nabla y_2)$$

$$
\begin{aligned}
&= \nabla (y_2 - y_1) \\
&= \nabla y_2 - \nabla y_1 \\
&= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0 \\
\nabla^3 y_3 &= \nabla^2 y_3 - \nabla^2 y_2 \\
&= y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.}
\end{aligned}
$$

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

7.2.3 Central Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *central differences* of y and are denoted by $\delta y_{\frac{1}{2}}, \delta y_{\frac{3}{2}}, \ldots, \delta y_{n-\frac{1}{2}}$ respectively.

$$
\begin{aligned}
\delta y_{\frac{1}{2}} &= y_1 - y_0 \\
\delta y_{\frac{3}{2}} &= y_2 - y_1 \\
&\vdots \quad \vdots \quad \vdots \\
\delta y_{n-\frac{1}{2}} &= y_n - y_{n-1}
\end{aligned}
$$

where δ is called the *central difference operator*. Similarly, higher-order central differences can be defined.

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$
$$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1 \text{ etc.}$$

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{\frac{1}{2}} = y_1 - y_0$			
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$		
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$	
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$		$\delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$	
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$		
		$\delta y_{\frac{7}{2}} = y_4 - y_3$			
x_4	y_4				

From the central difference table, it is clear that the central differences on the same horizontal line have the same suffix. Also, the differences of odd orders are known only for half values of the suffix and those of even orders, for only integral values of the suffix.

Note It is clear from the three difference tables that it is only the notations which change and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$$

7.3 DIFFERENT OPERATORS AND THEIR RELATIONS

1. Forward Difference Operator The forward difference operator is denoted by Δ and is defined as

$$\Delta f(x) = f(x + h) - f(x)$$

or

$$\Delta y_r = y_{r+1} - y_r$$

where h is known as the *interval of differencing*.

2. Backward Difference Operator The backward difference operator is denoted by ∇ and is defined as

$$\nabla f(x) = f(x) - f(x - h)$$

or

$$\nabla y_r = y_r - y_{r-1}$$

3. Central Difference Operator The central difference operator is denoted by δ and is defined as

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

or

$$\delta y_r = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$$

4. Shift Operator The shift operator is denoted by E and is defined as

$$E\,[f(x)] = f(x + h)$$

or

$$E\,y_r = y_{r+1}$$

Similarly,

$$E^{-1}\,[f(x)] = f(x - h)$$

or

$$E^{-1}\,y_r = y_{r-1}$$

5. Averaging Operator The averaging operator is denoted by μ and is defined as

$$\mu f(x) = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

6. Differential Operator The differential operator is denoted by D and is defined as

$$D f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

7.3.1 Relations between Operators

1. Relation between Δ and *E*

[Summer 2015, Winter 2014, Summer 2014, Summer 2013]

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x) \\ &= Ef(x) - f(x) \qquad [\because Ef(x) = f(x+h)] \\ &= (E-1) f(x)\end{aligned}$$

$$\therefore \quad \Delta \equiv E - 1 \quad \text{or} \quad E \equiv 1 + \Delta$$

2. Relation between ∇ and *E*

[Winter 2014, Winter 2013]

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h) \\ &= f(x) - E^{-1} f(x) \qquad [\because E^{-1} f(x) = f(x-h)] \\ &= (1 - E^{-1}) f(x)\end{aligned}$$

$$\therefore \quad \nabla \equiv 1 - E^{-1}$$

3. Relation between δ and *E*

$$\begin{aligned}\delta f(x) &= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \\ &= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x) \\ &= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)\end{aligned}$$

$$\therefore \quad \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

4. Relation between μ and *E*

$$\begin{aligned}\mu f(x) &= \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2} \\ &= \frac{E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)}{2}\end{aligned}$$

$$\therefore \quad \mu \equiv \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$$

5. Relation between *D* and *E*

[Winter 2014, Summer 2014]

$$
\begin{aligned}
E f(x) &= f(x+h) \\
&= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots \qquad \text{[By Taylor's series]} \\
&= f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots \\
&= \left(1 + hD + \frac{h^2}{2!} D^2 + \cdots\right) f(x) \\
&= e^{hD} f(x)
\end{aligned}
$$

$$\therefore \quad E \equiv e^{hD}$$

Also, $$hD \equiv \log E \equiv \log(1+\Delta)$$

Corollary $$E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$$

Proof

$$
\begin{aligned}
E^{-\frac{1}{2}} f(x) &= f\left(x - \frac{h}{2}\right) \\
&= f(x) - \frac{h}{2} f'(x) + \frac{\left(\frac{h}{2}\right)^2}{2!} f''(x) - \cdots \\
&= \left[1 - \frac{h}{2} D + \frac{\left(\frac{h}{2}\right)^2}{2!} D^2 - \cdots\right] f(x) \\
&= e^{-\frac{hD}{2}} f(x)
\end{aligned}
$$

$$\therefore \quad E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$$

6. Relation between μ and δ

$$
\begin{aligned}
\mu &\equiv \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \\
\mu^2 &\equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)^2 \\
&\equiv \frac{E + 2 + E^{-1}}{4}
\end{aligned}
$$

$$\equiv \frac{4+(E-E^{-1})^2}{4}$$

$$\equiv 1+\frac{\delta^2}{4}$$

$$\mu \equiv \sqrt{1+\frac{\delta^2}{4}}$$

Example 1

Prove that $(1 + \Delta)\,(1 - \nabla) = 1$

Solution

$$E \equiv 1 + \Delta$$

$$\nabla \equiv 1 - E^{-1}$$

$$E^{-1} \equiv 1 - \nabla$$

$$(1 + \Delta)(1 - \nabla) \equiv EE^{-1} = 1$$

Example 2

Prove that $\delta \equiv 2 \sinh \dfrac{hD}{2}$

Solution

$$\delta \equiv f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$$

$$\equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

$$\equiv e^{\frac{hD}{2}}-e^{-\frac{hD}{2}}$$

$$\equiv 2\sinh\left(\frac{hD}{2}\right)$$

Example 3

Prove that $hD \equiv \sinh^{-1}(\mu\delta)$

Solution

$$\mu\delta \equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

$$\equiv \frac{1}{2}(E - E^{-1})$$

$$2\mu\delta \equiv E - E^{-1}$$

$$\equiv e^{hD} - e^{-hD}$$

$$\equiv 2\sinh(hD)$$

$$\therefore \quad hD \equiv \sinh^{-1}(\mu\delta)$$

Example 4

Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log \frac{Ef(x)}{f(x)}$$

$$= \log \frac{(1+\Delta)f(x)}{f(x)}$$

$$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Example 5

Evaluate (*a*) $\Delta(x^2 + \sin x)$, *and* (*b*) $\Delta^2 \cos 3x$, *the interval of differencing being h.*

Solution

(i) $$\begin{aligned}\Delta(x^2+\sin x) &= [(x+h)^2+\sin(x+h)]-(x^2+\sin x)\\ &= h^2+2hx+\sin(x+h)-\sin x\\ &= h^2+2hx+2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}\end{aligned}$$

(ii) $$\begin{aligned}\Delta^2\cos 3x &= \Delta(\Delta\cos 3x)\\ &= \Delta[\cos 3(x+h)-\cos 3x]\\ &= \Delta\cos 3(x+h)-\Delta\cos 3x\\ &= \cos 3[(x+h)+h]-\cos 3(x+h)-\cos 3(x+h)+\cos 3x\\ &= \cos 3(x+2h)-2\cos 3(x+h)+\cos 3x\\ &= \cos 3(x+2h)+\cos 3x-2\cos 3(x+h)\end{aligned}$$

Example 6

Prove that $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E\,e^x}{\Delta^2 e^x} = e^x$

Solution

$$\begin{aligned}\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E\,e^x}{\Delta^2 e^x} &= \frac{(E-1)^2}{E}e^x \cdot \frac{e^{x+h}}{\Delta(e^{x+h}-e^x)}\\ &= \left(\frac{E^2-2E+1}{E}\right)e^x \frac{e^{x+h}}{\left(e^{x+2h}-e^{x+h}-e^{x+h}+e^x\right)}\\ &= (E-2+E^{-1})e^x \cdot \frac{e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}\\ &= \frac{\left(e^{x+h}-2e^x+e^{x-h}\right)e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}\end{aligned}$$

$$= \frac{e^x\left(e^{x+2h} - 2e^{x+h} + e^x\right)}{e^{x+2h} - 2e^{x+h} + e^x}$$
$$= e^x$$

Example 7

Prove that $\Delta\nabla \equiv (\Delta - \nabla)$

Solution:

$$\begin{aligned}\Delta\nabla f(x) &= \Delta[f(x) - f(x-h)] \\ &= \Delta f(x) - \Delta f(x-h) \\ &= \Delta f(x) - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) \\ &= (\Delta - \nabla) f(x)\end{aligned}$$

$\therefore \quad \Delta\nabla \equiv (\Delta - \nabla)$

Example 8

Prove that $\Delta \equiv E\nabla \equiv \nabla E$

Solution

$$\Delta f(x) = f(x+h) - f(x) \qquad \text{...(1)}$$

$$\begin{aligned}E\nabla f(x) &= E\{f(x) - f(x-h)\} \\ &= Ef(x) - Ef(x-h) \\ &= f(x+h) - f(x)\end{aligned} \qquad \text{...(2)}$$

$$\begin{aligned}\nabla E f(x) &= \nabla f(x+h) \\ &= f(x+h) - f(x)\end{aligned} \qquad \text{...(3)}$$

From Eqs (1), (2), and (3),

$$\Delta \equiv E\nabla \equiv \nabla E$$

Example 9

Find the missing term in the following table:

x	1	2	3	4	5
y	7	–	13	21	37

Solution

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7				
		$y_1 - 7$			
2	y_1		$20 - 2y_1$		
		$13 - y_1$		$3y_1 - 25$	
3	13		$y_1 - 5$		$38 - 4y_1$
		8		$13 - y_1$	
4	21		8		
		16			
5	37				

Since only four entries are given, the fourth-order difference will be zero.

$$\Delta^4 y_0 = 0$$
$$38 - 4y_1 = 0$$
$$y_1 = 9.5$$

Example 10

Obtain the estimate of missing terms in the following table:

x	1	2	3	4	5	6	7	8
y	2	4	8	–	32	–	128	256

Solution

Let $f(4) = a, \quad f(6) = b$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	2						
		2					
2	4		2				
		4		$a - 14$			
3	8		$a - 12$		$66 - 4a$		
		$a - 8$		$52 - 3a$		$10a + b - 222$	
4	a		$40 - 2a$		$b + 6a - 156$		$706 - 20a - 6b$
		$32 - a$		$b + 3a - 104$		$484 - 10a - 5b$	
5	32		$b + a - 64$		$328 - 4a - 4b$		$15b + 15a - 1196$
		$b - 32$		$224 - 3b - a$		$10b + 5a - 712$	
6	b		$160 - 2b$		$6b + a - 384$		
		$128 - b$		$3b - 160$			
7	128		b				
		128					
8	256						

Since only six values are given,

$$\Delta^6 y_0 = 0$$

$$20a + 6b = 706 \quad \ldots(1)$$

$$15a + 15b = 1196 \quad \ldots(2)$$

Solving Eqs (1) and (2),

$$a = 16.26$$

$$b = 63.48$$

Example 11

The following table gives the value of y which is a polynomial of degree five. It is known that $y = f(3)$ is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1025	3126	7777

Solution

Let $f(3) = a$

Since y is a polynomial of degree 5,

$$\Delta^6 y_0 = 0$$

$$(E-1)^6 y_0 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y_0 = 0$$

$$y^6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

$$7777 - 6(3126) + 15(1025) - 20a + 15(33) - 6(2) + 1 = 0$$

$$-20a = -4880$$

$$a = 244$$

$$\text{Error} = 254 - 244 = 10$$

Example 12

If u_x is a function for which the fifth difference is constant and $u_1 + u_7 = -784$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$, find u_4.

Solution

Since the fifth difference is constant,

$$\Delta^6 u_1 = 0$$

$$(E-1)^6 u_1 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, u_1 = 0$$

$$u_7 - 6u_6 + 15u_5 - 20u_4 + 15u_3 - 6u_2 + u_1 = 0$$

$$(u_7 + u_1) - 6(u_6 + u_2) + 15(u_5 + u_3) - 20u_4 = 0$$

$$-784 - 6(686) + 15(1088) - 20u_4 = 0$$

$$20u_4 = 11420$$

$$u_4 = 571$$

7.3.2 Factorial Notation

A product of the form $x(x-1)\,(x-2)\ldots(x-n+1)$ is called a factorial polynomial or function and is denoted by $[x]^n$.

$$[x]^n = x\,(x-1)\,(x-2)\,\ldots(x-n+1)$$

If the interval of differencing is h then

$$[x]^n = x\,(x-h)\,(x-2h)\,\ldots\,\{x-(n-1)\,h\}$$

The factorial notation is of special utility in the theory of finite differences. It is useful in finding the successive differences of a polynomial directly by simple rule of differentiation.

Example 1

Write $f(x) = x^4 - 2x^3 + x^2 - 2x + 1$ *in factorial notation and find* $\Delta^4 f(x)$.

Solution

$$f(x) = x^4 - 2x^3 + x^2 - 2x + 1$$

Let $$f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$$

Using synthetic division,

1	1	–2	1	–2	$1 = E$
	0	1	–1	0	
2	1	–1	0	$-2 = D$	
	0	2	2		
3	1	1	$2 = C$		
	0	3			
	$1 = A$	$4 = B$			

$\therefore$ $$f(x) = [x]^4 + 4[x]^3 + 2[x]^2 - 2[x]^1 + 1$$

$$\Delta f(x) = 4[x]^3 + 12[x]^2 + 4[x] - 2$$

$$\Delta^2 f(x) = 12[x]^2 + 24[x] + 4$$

$$\Delta^3 f(x) = 24[x] + 24$$

$$\Delta^4 f(x) = 24$$

Example 2

Express $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ *and its successive differences in terms of factorial polynomials. Also, find the function whose first difference is* $f(x)$.

Solution

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

Let $$f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$$

Using synthetic division,

1	1	–12	42	–30	9 = *E*
	0	1	–11	31	
2	1	–11	31	1 = *D*	
	0	2	–18		
3	1	–9	13 = *C*		
	0	3			
	1 = *A*	–6 = *B*			

$\therefore \quad f(x) = [x]^4 - 6[x]^3 + 13[x]^2 + [x]^1 + 9$

$$\Delta f(x) = 4[x]^3 - 18[x]^2 + 26[x]^1 + 1$$

$$\Delta^2 f(x) = 12[x]^2 - 36[x]^1 + 26$$

$$\Delta^3 f(x) = 24[x]^1 - 36$$

$$\Delta^4 f(x) = 24$$

$$\Delta^5 f(x) = 0$$

By integrating $f(x)$, the function $\phi(x)$ whose first difference is $f(x)$, is obtained.

$$\phi(x) = \frac{[x]^5}{5} - \frac{6[x]^4}{4} + \frac{13[x]^3}{3} + \frac{[x]^2}{2} + 9[x]^1 + c$$

Example 3

Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ *in factorial polynomial and, hence, show that* $\Delta^3 f(x) = 12$.

Solution

$$f(x) = 2x^3 - 3x^2 + 3x - 10$$

Let $\quad f(x) = A[x]^3 + B[x]^2 + C[x]^1 + D$

Using synthetic division,

1	2	–3	3	–10 = *D*
	0	2	–1	
2	2	–1	2 = *C*	
	0	4		
	2 = *A*	3 = *B*		

$\therefore$
$$f(x) = 2[x]^3 + 3[x]^2 + 2[x]^1 - 10$$
$$\Delta f(x) = 6[x]^2 + 6[x]^1 + 2$$
$$\Delta^2 f(x) = 12[x]^1 + 6$$
$$\Delta^3 f(x) = 12$$

EXERCISE 7.1

1. Prove the following identities:

(i) $\Delta\nabla \equiv \nabla\Delta$

(ii) $\nabla \equiv E^{1}\,\Delta$

(iii) $E\nabla \equiv \nabla E$

(iv) $hD \equiv -\log\,(1 - \nabla)$

(v) $\Delta + \nabla \equiv \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

(vi) $\left(E^{\frac{1}{2}} - E^{\frac{-1}{2}}\right)(1-\nabla)^{\frac{1}{2}} \equiv \nabla$

2. Find

(i) $\Delta\left[\dfrac{2^x}{(x+1)!}\right]$

(ii) $\Delta\,\tan^{-1} x$

(iii) $\Delta^n\, e^{ax}$

(iv) $\Delta\,(x + \cos x)$

(v) $\Delta^4\,(ax^4 + bx^2 + cx + d)$

[**Ans.:** (i) $-\dfrac{x \cdot 2^x}{(x+2)!}$ (ii) $\tan^{-1}\left(\dfrac{h}{1+hx+x^2}\right)$ (iii) $(e^a - 1)^n \cdot e^x$ (iv) $1 - 2\sin\left(x + \dfrac{1}{2}\right)\sin x$ (v) $a^4 \cdot 4!$]

3. Evaluate $\left(\dfrac{\Delta^2}{E}\right)\sin x$, where the interval of difference is h.

[**Ans.:** $\sin\,(x + h) - 2\,\sin x + \sin\,(x - h)$]

4. Prove that

(i) $\Delta[x\,f(x)] = (x + h)\,\Delta\,f(x) + hf(x)$

(ii) $(\Delta + \nabla)^2\,(x^2 + x) = 8h^2$

5. Prove that $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

6. Find $\dfrac{\Delta^2(x^2)}{E(x^2)}$ and $\left(\dfrac{\Delta^2}{E}\right)x^2$.

$$\left[\textbf{Ans.:}\ \frac{2}{(1+x)^2}, 2\right]$$

7. If $y = a3^x + b(-2)^x$ and $h = 1$, prove that $(\Delta^2 + \Delta - 6)y = 0$.

8. Find the missing term from the following data:

x	0	1	2	3	4
y	1	3	9	–	81

[Ans.: 31]

9. From the following table, estimate y at $x = 2$.

x	4	6	8	10	12
y	6	7	13	32	77

[Ans.: 7]

10. If $u_0 = -10$, $u_1 = -6$, $u_2 = 2$, $u_3 = 12$, $u_4 = 26$, $u_5 = 42$, find u_6.

[Ans.: 46]

11. If $u_3 = 4$, $u_4 = 12$, $u_5 = 22$, $u_6 = 37$, $u_7 = 55$, find u_8.

[Ans.: 69]

12. From the following table, find $(15)^3$.

x	3	5	7	9	11
y	27	125	343	721	1331

[Ans.: 3375]

7.4 INTERPOLATION

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of x. The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus, interpolation is a technique of finding the value of a function for any intermediate value of the independent variable. The process of computing the value of the function outside the range of given values of the variable is called *extrapolation*. The study of interpolation is based on the concept of finite differences which were discussed in the preceding section.

7.5 NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$
\begin{aligned}
y_r &= f(x_0 + rh) \\
&= E^r f(x_0) \\
&= (1+\Delta)^r f(x_0) \\
&= (1+\Delta)^r y_0 \\
&= \left[1 + r\Delta + \frac{r(r-1)}{2!}\Delta^2 + \frac{r(r-1)(r-2)}{3!}\Delta^3 + \ldots.\right] y_0
\end{aligned}
$$

[Using Binomial theorem]

$$
= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \qquad \ldots(7.1)
$$

Equation (7.1) is known as *Newton's forward interpolation formula*.

Note This formula is used for evaluating the value of y near the initial tabulated value of x, i.e., near x_0.

Example 1

Compute cosh (0.56) *using Newton's forward difference formula from the following table:*

x	0.5	0.6	0.7	0.8
$f(x)$	1.127626	1.185465	1.255169	1.337435

Solution

Let $x = 0.56,\ x_0 = 0.5,\ h = 0.1$

$$
r = \frac{x - x_0}{h} = \frac{0.56 - 0.5}{0.1} = 0.6
$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.5	1.127626			
		0.057839		
0.6	1.185465		0.011865	
		0.069704		0.000697
0.7	1.255169		0.012562	
		0.082266		
0.8	1.337435			

By Newton's forward difference formula,

$$f(x_0 + rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) + \cdots$$

$$\cosh(0.56) = 1.127626 + 0.6(0.057839) + \frac{0.6(0.6-1)}{2!}(0.011865)$$

$$+ \frac{0.6(0.6-1)(0.6-2)}{3!}(0.000697)$$

$$= 1.127626 + 0.034703 - 0.001424 + 0.000039$$

$$= 1.160944$$

Example 2

Find the value of $\sin 52°$ *using Newton's forward interpolation formula from the following table:*

$\theta°$	45°	50°	55°	60°
$\sin\theta°$	0.7071	0.7660	0.8192	0.8660

Solution

Let $x = 52°, x_0 = 45°, h = 5°$

$$r = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4°$$

Difference Table

$x = \theta°$	$y = \sin\theta°$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60°	0.8660			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\sin 52° = 0.7071 + 1.4(0.0589) + \frac{1.4(1.4-1)}{2!}(-0.0057) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-0.0007)$$
$$= 0.7071 + 0.0825 - 0.0016 + 0.00004$$
$$= 0.7880$$

Example 3

Using Newton's forward interpolation formula, find the value of $f(1.6)$.

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

Solution

Let $x = 1.6, x_0 = 1, h = 0.4$

$$r = \frac{x - x_0}{h} = \frac{1.6 - 1}{0.4} = 1.5$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		−0.19	
		1.14		−0.41
1.8	5.96		−0.6	
		0.54		
2.2	6.5			

By Newton's forward interpolation formula,

$$f(x_0 + rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$
$$f(1.6) = 3.49 + 1.5(1.33) + \frac{1.5(1.5-1)}{2!}(-0.19) + \frac{1.5(1.5-1)(1.5-2)}{3!}(-0.41)$$
$$= 3.49 + 1.995 - 0.0713 + 0.0256$$
$$= 5.4393$$

Example 4

Use Newton's forward difference method to find the approximate value of $f(1.3)$ from the following data:

x	1	2	3	4
$f(x)$	1.1	4.2	9.3	16.4

Solution

Let $x = 1.3, x_0 = 1, h = 1$

$$r = \frac{x - x_0}{h} = \frac{1.3 - 1}{1} = 0.3$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1.1			
		3.1		
2	4.2		2	
		5.1		0
3	9.3		2	
		7.1		
4	16.4			

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$

$$f(1.3) = 1.1 + 0.3(3.1) + \frac{0.3(0.3-1)}{2!}(2) + 0$$

$$= 1.1 + 0.93 - 0.21$$

$$= 1.82$$

Example 5

Use Newton's forward difference method to find the approximate value of $f(2.3)$ from the following data:

x	2	4	6	8
$f(x)$	4.2	8.2	12.2	16.2

Solution

Let $x = 2.3,\ x_0 = 2,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{2.3 - 2}{2} = 0.15$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	4.2		
		4	
4	8.2		0
		4	
6	12.2		0
		4	
8	16.2		

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \cdots$$

$$\begin{aligned} f(2.3) &= 4.2 + 0.15(4) + 0 \\ &= 4.2 + 0.6 \\ &= 4.8 \end{aligned}$$

Example 6

Using Newton's forward interpolation formula, find the value of $f(218)$.

x	100	150	200	250	300	350	400
$f(x)$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[Summer 2014]

Solution

Let $x = 218,\ x_0 = 100,\ h = 50$

$$r = \frac{x - x_0}{h} = \frac{218 - 100}{50} = 2.36$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
100	10.63						
		2.4					
150	13.03		−0.39				
		2.01		0.15			
200	15.04		−0.24		−0.07		
		1.77		0.08		0.02	
250	16.81		−0.16		−0.05		0.02
		1.61		0.03		0.04	
300	18.42		−0.13		−0.01		
		1.48		0.02			
350	19.90		−0.11				
		1.37					
400	21.27						

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0)$$
$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f(x_0) + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 f(x_0)$$
$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 f(x_0) + \cdots$$

$$f(218) = 10.63 + 2.36(2.4) + \frac{2.36(2.36-1)}{2!}(-0.39) + \frac{2.36(2.36-1)(2.36-2)}{3!}(0.15)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)}{4!}(-0.07)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{5!}(0.02)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)(2.36-5)}{6!}(0.02)$$
$$= 10.63 + 5.664 - 0.6259 + 0.0289 + 0.0022 + 0.0002 - 0.00009$$
$$= 15.6993$$

Example 7

From the following table, estimate the number of students who obtained marks between 40 *and* 45:

Marks	30–40	40–50	50–60	60–70	70–80
Number of students	31	42	51	35	31

[Summer 2015]

Solution

Cumulative Frequency Table

Marks less than (x)	40	50	60	70	80
Number of students (y)	31	73	124	159	190

Since $x = 45$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 45, x_0 = 40, h = 10$

$$r = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(45) = 31 + 0.5(42) + \frac{0.5(0.5-1)}{2!}(9) + \frac{0.5(0.5-1)(0.5-2)}{3!}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$$= 31 + 21 - 1.1250 - 1.5625 - 1.4453$$

$$= 47.8672$$

$$\approx 48$$

The number of students with marks less than 45 = 48
The number of students with marks less than 40 = 31
Hence, the number of students obtaining marks between 40 and 45 = 48 – 31 = 17

Example 8

Determine the polynomial by Newton's forward difference formula from the following table:

x	0	1	2	3	4	5
y	–10	–8	–8	–4	10	40

Solution

Let $x_0 = 0, h = 1$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–10				
		2			
1	–8		–2		
		0		6	
2	–8		4		0
		4		6	
3	–4		10		0
		14		6	
4	10		16		
		30			
5	40				

By Newton's forward difference formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$= -10 + x(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(6) + 0$$

$$= -10 + 2x - x(x-1) + x(x-1)(x-2)$$

$$= -10 + 2x - x^2 + x + x(x^2 - 3x + 2)$$

$$= -10 + 2x - x^2 + x + x^3 - 3x^2 + 2x$$

$$= x^3 - 4x^2 + 5x - 10$$

Example 9

Find a polynomial of degree 2 which takes the following values:

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29

Solution

Let $x_0 = 0, h = 1$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		
6	22		1	
		7		
7	29			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + x(1) + \frac{x(x-1)}{2!}(1) + 0$$

$$= 1 + x + \frac{x^2 - x}{2}$$

$$= 1 + x + \frac{x^2}{2} - \frac{x}{2}$$

$$= \frac{1}{2}(x^2 + x + 2)$$

Example 10

Construct Newton's forward interpolation polynomial for the following data:

x	4	6	8	10
y	1	3	8	16

[Summer 2015]

Solution

Let $x_0 = 4,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-4}{2}$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2!}(3) + 0$$

$$= 1 + (x-4) + \frac{(x-4)(x-6)}{8}(3)$$

$$= x - 3 + \frac{(x^2 - 10x + 24)}{8}(3)$$

$$= x - 3 + \frac{3x^2}{8} - \frac{15x}{4} + 9$$

$$= \frac{3x^2}{8} - \frac{11x}{4} + 6$$

$$y(5) = \frac{3(25)}{8} - \frac{11(5)}{4} + 6 = \frac{13}{8}$$

7.6 NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$y_r = f(x_n + rh)$$

$$= E^r f(x_n)$$

$$= (E^{-1})^{-r} y_n$$

$$= (1 - \nabla)^{-r} y_n$$

$$= \left[1 + r\nabla + \frac{r(r+1)}{2!}\nabla^2 + \frac{r(r+1)(r+2)}{3!}\nabla^3 + \cdots\right] y_n$$

[Using Binomial theorem]

$$= y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots \quad \ldots(7.2)$$

Equation (7.2) is known as *Newton's backward interpolation formula*.

Note This formula is used for evaluating the value of y near to the end of tabulated value of x, i.e., near x_n.

Example 1

Consider the following tabular values:

x	50	100	150	200	250
y	618	724	805	906	1032

Determine $y(300)$ using Newton's backward interpolation formula.

Solution

Let $x = 300, x_n = 250, h = 50$

$$r = \frac{x - x_n}{h} = \frac{300 - 250}{50} = 1$$

Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	618				
		106			
100	724		–25		
		81		45	
150	805		20		–40
		101		5	
200	906		25		
		126			
250	1032				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(300) = 1032 + 1(126) + \frac{1(2)}{2!}(25) + \frac{1(2)(3)}{3!}(5) + \frac{1(2)(3)(4)}{4!}(-40)$$

$$= 1032 + 126 + 25 + 5 - 40$$

$$= 1148$$

Example 2

The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of a diameter of 105 *units using Newton's interpolation formula.* **[Summer 2015]**

Solution

Since $x = d = 105$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $x = 105,\ x_n = 100,\ h = 5$

$$r = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

Difference Table

$x = d$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		–2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(105) = 7854 + 1(766) + \frac{1(2)}{2!}(40) + \frac{1(2)(3)}{3!}(2) + \frac{1(2)(3)(4)}{4!}(4)$$

$$= 7854 + 766 + 40 + 2 + 4$$

$$= 8666$$

Example 3

From the following table, find P when t = 142°C and 175°C using appropriate Newton's interpolation formula.

Temperature t°C	140	150	160	170	180
Pressure P	3685	4845	6302	8076	10225

[Winter 2014]

Solution

Since $x = 142$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 142, \quad x_0 = 140, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{142 - 140}{10} = 0.2$$

Difference Table

$x = t°$	$y = P$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3685				
		1169			
150	4854		279		
		1448		47	
160	6302		326		2
		1774		49	
170	8076		375		
		2149			
180	10225				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$P(142) = y(142) = 3685 + 0.2(1169) + \frac{0.2(0.2-1)}{2!}(279) + \frac{0.2(0.2-1)(0.2-2)}{3!}(47) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}(2)$$

$$= 3685 + 233.8 - 22.32 + 2.256 - 0.0672$$

$$= 3898.6688$$

Since $x = 175$ is near to the end of the table, Newton's backward interpolation formula is used.

$$x = 175, \quad x_n = 180, \quad h = 10$$

$$r = \frac{x - x_n}{h} = \frac{175 - 180}{10} = -0.5$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$P(175) = y(175) = 10225 + (-0.5)(2149) + \frac{(-0.5)(-0.5+1)}{2!}(375) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(49) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(2)$$

$$= 10225 - 1074.5 - 46.875 - 3.0625 - 0.0781$$

$$= 9100.4844$$

Example 4

The population of a town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.

Year	x	1891	1901	1911	1921	1931
Population (in thousand)	y	46	66	81	93	101

[Summer 2015]

Solution

Since $x = 1895$ is near to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 1895, \; x_0 = 1891, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(-5) + \frac{0.4(0.4-1)(0.4-2)}{3!}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$= 54.8528 \text{ thousands}$$

Since $x = 1930$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $x = 1930,\ x_n = 1931,\ h = 10$

$$r = \frac{x - x_n}{h} = \frac{1930 - 1931}{10} = -0.1$$

By Newton's forward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(1930) = 101 + (-0.1)(8) + \frac{(-0.1(-0.1+1)}{2!}(-4) + \frac{(-0.1)(-0.1+1)(-0.1+2)}{3!}(-1)$$
$$+ \frac{(-0.1)(-0.1+1)(-0.1+2)(-0.1+3)}{4!}(-3)$$
$$= 101 - 0.8 + 0.18 + 0.0285 + 0.062$$
$$= 100.4705 \text{ thousands}$$

Example 5

In the table below, the values of y are consecutive terms of a series of which 23.6 *is the sixth term. Find the first and tenth terms of the series:*

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution

To find the first term, Newton's forward interpolation formula is used.

Let $x = 1, x_0 = 3, h = 1$

$$r = \frac{x - x_0}{h} = \frac{1-3}{1} = -2$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$y(1) = 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2!}(2.5) + \frac{(-2)(-3)(-4)}{3!}(0.5)$$

$$= 4.8 + 7.2 + 7.5 - 2$$

$$= 3.1$$

To find the tenth term, Newton's backward interpolation formula is used.

Let $x = 10, x_n = 9, h = 1$

$$r = \frac{x - x_n}{h} = \frac{10 - 9}{1} = 1$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$y(10) = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5)$$

$$= 73.9 + 21.1 + 4.5 + 0.5$$

$$= 100$$

EXERCISE 7.2

1. Find tan 67° 20′ from the table:

θ	65°	66°	67°	68°	69°
tan θ	2.1445	2.2460	2.3559	2.4751	2.6051

[Ans.: 2.3932]

2. Find $(5.5)^3$ from the following table:

x	3	5	7	9	11
$y = x^3$	27	125	343	729	1331

[Ans.: 166.375]

3. Calculate $e^{1.85}$ from the following table:

x	1.7	1.8	1.9	2	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

[Ans.: 6.36]

4. Find $\sqrt{x}$ at $x = 2.52$ and $x = 2.62$ from the table:

x	2.5	2.55	2.6	2.65	2.7	2.75
$\sqrt{x}$	1.58114	1.59687	1.61245	1.62788	1.64317	1.65831

[Ans.: 1.58745, 1.6186]

5. The values of specific heat (C_p) at constant pressure of a gas are tabulated below for various temperatures. Find the specific heat at 900°C.

Temperature °C	500	1000	1500	2000
C_p	31.23	35.01	39.18	43.75

[Ans.: 34.223]

6. P and V are connected by the following data:

V	10	20	30	40
P	1.1	2	4.4	7.9

Determine P when $V = 25$ and $V = 45$. **[Ans.:** 3.0375, 9.9375]

7. Find the number of persons getting wages less than ₹15 from the following data:

Wages in ₹	0–10	10–20	20–30	30–40
Number of persons	9	30	35	42

[Ans.: 24]

8. Find the number of students getting marks less than 70 from the following data:

Marks	0–20	20–40	40–60	60–80	80–100
Number of students	41	62	65	50	17

[Ans.: 196]

9. From the following table, estimate the profit in the year 1925.

Year	1891	1901	1911	1921	1931
Profit in lakhs	46	66	81	93	101

[Ans.: ₹ 96.8365 lakhs]

10. Find the polynomial of degree three which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.:}\ \frac{1}{12}(x^3 + 3x^2 + 32x + 24)\right]$$

11. Obtain the cubic polynomial which takes the values

x	0	1	2	3
y	1	2	1	0

and, hence, find $f(4)$. **[Ans.:** $2x^3 - 7x^2 + 6x + 1$, 41]

12. Find a polynomial of degree 4 which takes the values

x	2	4	6	8	10
y	0	0	1	0	0

$$\left[\textbf{Ans.:}\ \frac{1}{64}(x^4 - 24x^3 + 196x^2 - 624x + 640)\right]$$

13. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. **[Ans.:** 42, 49**]**

14. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$. Without forming the difference table, find $\Delta^5 y_0$. **[Ans.:** 755**]**

15. Find the polynomial of least degree passing through the points (0,–1), (1,1), (2,1), and (3,–2).

$$\left[\textbf{Ans.:}\ -\frac{1}{6}(x^3 + 3x^2 - 16x + 6)\right]$$

7.7 CENTRAL DIFFERENCE INTERPOLATION

Central difference interpolation formulae are used for interpolation near the middle of the tabulated values. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, the difference tables in the two notations are given as follows:

x	y	First Difference	Second Difference	Third Difference	Fourth Difference
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} = \delta y_{-\frac{3}{2}}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \delta^2 y_{-1}$		
		$\Delta y_{-1} = \delta y_{-\frac{1}{2}}$		$\Delta^3 y_{-2} = \delta^3 y_{-\frac{1}{2}}$	
x_0	y_0		$\Delta^2 y_{-1} = \delta^2 y_0$		$\Delta^4 y_{-2} = \delta^4 y_0$
		$\Delta y_0 = \delta y_{\frac{1}{2}}$		$\Delta^3 y_{-1} = \delta^3 y_{\frac{1}{2}}$	
$x_0 + h$	y_1		$\Delta^2 y_0 = \delta^2 y_1$		
		$\Delta y_1 = \delta y_{\frac{3}{2}}$			
$x_0 + 2h$	y_2				

7.8 GAUSS'S FORWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{...(7.3)}$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta^2 y_0 = \Delta^2 Ey_{-1} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad \text{...(7.4)}$$

$$\Delta^3 y_0 = \Delta^3 Ey_{-1} = \Delta^3 (1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad \text{... (7.5)}$$

$$\Delta^4 y_0 = \Delta^4 Ey_{-1} = \Delta^4 (1+\Delta) y_{-1} = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad \text{...(7.6)}$$

Also, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \quad \text{...(7.7)}$$

Substituting the values of $\Delta^2 y_0$ $\Delta^3 y_0$, ... in Eq. (7.3),

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

[Using Eqs (7.4), (7.5), and (7.6)]

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \left[\frac{r(r-1)}{2!}\Delta^3 y_{-1} + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_{-1}\right] + \left[\frac{r(r-1)(r-2)}{3!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_{-1} + \cdots\right]$$

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + \cdots$$

[Using Eq. (7.7)]

$$= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{...(7.8)}$$

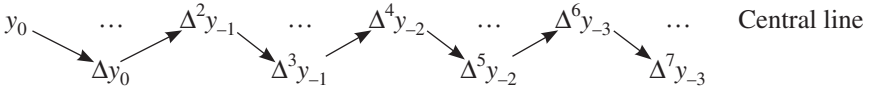
Equation (7.8) is known as *Gauss's forward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\delta y_{\frac{1}{2}} + \frac{r(r-1)}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{(r+1)r(r-1)(r-2)}{4!}\delta^4 y_0 + \cdots$$

Notes

(i) This formula involves odd differences below the central line and even differences on the central line.



(ii) This formula is used to evaluate the values of y for r between 0 and 1.

Example 1

Find $y(32)$ from the following table:

x	25	30	35	40
$y = f(x)$	0.2707	0.3027	0.3386	0.3794

Solution

Let $x = 32, x_0 = 30, h = 5$

$$r = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	–1	0.2707			
			0.0320		
30	0	0.3027		0.0039	
			0.0359		0.0010
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$y(32) = 0.3027 + 0.4(0.0359) + \frac{0.4(0.4-1)}{2!}(0.0039) + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(0.0010)$$

$$= 0.3027 + 0.0144 - 0.0005 - 0.0001$$

$$= 0.3165$$

Example 2

Use Gauss's forward interpolation formula to find y(3.3) from the following table:

x	1	2	3	4	5
y	15.3	15.1	15	14.5	14

Solution

Let $x = 3.3, x_0 = 3, h = 1$

$$r = \frac{x - x_0}{h} = \frac{3.3 - 3}{1} = 0.3$$

Central Difference Table

x	*r*	*y*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	–2	15.3				
			–0.2			
2	–1	15.1		0.1		
			–0.1		–0.5	
3	0	15		–0.4		0.9
			–0.5		0.4	
4	1	14.5		0		
			–0.5			
5	2	14				

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(3.3) = 15 + 0.3(-0.5) + \frac{0.3(0.3-1)}{2!}(-0.4) + \frac{(0.3+1)(0.3)(0.3-1)}{3!}(0.4) + \frac{(0.3+1)(0.3)(0.3-1)(0.3-2)}{4!}(0.9)$$

$$= 15 - 0.15 + 0.042 - 0.0182 + 0.0174$$

$$= 14.8912$$

Example 3

Find the polynomial which fits the data in the following table using Gauss's forward interpolation formula.

x	3	5	7	9	11
y	6	24	58	108	174

Solution

Let $x_0 = 7,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-7}{2}$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	–2	6				
			18			
5	–1	24		16		
			34		0	
7	0	58		16		0
			50		0	
9	1	108		16		
			66			
11	2	174				

By Gauss's forward interpolation formula,

$$\begin{aligned} y(x) &= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \cdots \\ &= 58 + \left(\frac{x-7}{2}\right)(50) + \frac{1}{2}\left(\frac{x-7}{2}\right)\left(\frac{x-7}{2} - 1\right)(16) \\ &= 58 + 25(x-7) + 2(x-7)(x-9) \\ &= 58 + 25x - 175 + 2x^2 - 32x + 126 \\ &= 2x^2 - 7x + 9 \end{aligned}$$

7.9 GAUSS'S BACKWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{...(7.9)}$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1} \quad \text{...(7.10)}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad \text{...(7.11)}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad \text{...(7.12)}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad \text{...(7.13)}$$

Also, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \quad \text{...(7.14)}$$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, …. in Eq. (7.9),

$$\begin{aligned} y_r = y_0 &+ r(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) \\ &+ \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots \end{aligned}$$

[Using Eqs (7.10), (7.11), and (7.12)]

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \frac{(r+1)r(r-1)(r-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots.$$

[Using Eq. (7.14)]

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \ldots(7.15)$$

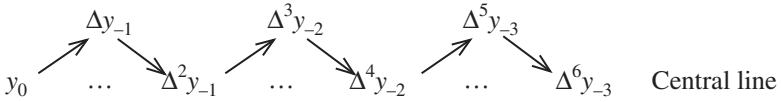
Equation (7.15) is known as *Gauss's backward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\delta y_{-\frac{1}{2}} + \frac{(r+1)r}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{-\frac{1}{2}} + \frac{(r+2)(r+1)r(r-1)}{4!}\delta^4 y_0 + \ldots$$

Notes

(i) This formula involves odd differences above the central line and even differences on the central line.



(ii) This formula is used to evaluate the values of y for r between –1 and 0.

Example 1

Using Gauss's backward interpolation formula, find the population for the year 1936 *given that*

Year (x)	1901	1911	1921	1931	1941	1951
Population in thousands (y)	12	15	20	27	39	52

Solution

Let $x = 1936,\ x_0 = 1941,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	−4	12					
			3				
1911	−3	15		2			
			5		0		
1921	−2	20		2		3	
			7		3		−10
1931	−1	27		5		−7	
			12		−4		
1941	0	39		1			
			13				
1951	1	52					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y(1936) = 39 + (-0.5)(12) + \frac{(-0.5+1)(-0.5)}{2!}(1) + \frac{(-0.5+1)(-0.5)(-0.5-1)}{3!}(-4)$$

$$= 39 - 6 - 0.1250 - 0.25$$

$$= 32.625 \text{ thousands}$$

Example 2

Find $y(2.36)$ from the following table:

x	1.6	1.8	2	2.2	2.4	2.6
y	4.95	6.05	7.39	9.03	11.02	13.46

Solution

Let $x = 2.36,\ x_0 = 2.4,\ h = 0.2$

$$r = \frac{x - x_0}{h} = \frac{2.36 - 2.4}{0.2} = -0.2$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.6	−4	4.95					
			1.1				
1.8	−3	6.05		0.24			
			1.34		0.06		
2	−2	7.39		0.3		−0.01	
			1.64		0.05		0.06
2.2	−1	9.03		0.35		0.05	
			1.99		0.1		
2.4	0	11.02		0.45			
			2.44				
2.6	1	13.46					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y(2.36) = 11.02 + (-0.2)(1.99) + \frac{(-0.2+1)(-0.2)}{2!}(0.45) + \frac{(-0.2+1)(-0.2)(-0.2-1)}{3!}(0.1)$$

$$= 11.02 - 0.398 - 0.036 + 0.0032$$

$$= 10.5892$$

Example 3

From the following table, find y when $x = 38$.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Solution

Let $x = 38, \quad x_0 = 40, \quad h = 5$

$$r = \frac{x - x_0}{h} = \frac{38 - 40}{5} = -0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	–2	15.9				
			–1			
35	–1	14.9		0.2		
			–0.8		–0.2	
40	0	14.1		0		0.2
			–0.8		0	
45	1	13.3		0		
			–0.8			
50	2	12.5				

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(38) = 14.1 + (-0.4)(-0.8) + \frac{(-0.4+1)(-0.4)}{2!}(0) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(-0.2)$$
$$+ \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(0.2)$$
$$= 14.1 + 0.32 + 0 - 0.0112 + 0.0045$$
$$= 14.4133$$

7.10 STIRLING'S FORMULA

By Gauss's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{...(7.16)}$$

By Gauss's backward interpolation formula,

$$y_r = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \quad (7.17)$$

Adding Eqs (7.16) and (7.17) and then dividing by 2,

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{...(7.18)}$$

Equation (7.18) is known as *Stirling's formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\mu\delta y_0 + \frac{r^2}{2!}\delta^2 y_0 + \frac{r(r^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{r^2(r^2-1^2)}{4!}\delta^4 y_0 + \cdots$$

$$\frac{1}{2}(\Delta y_{-1} + \Delta y_0) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-2} + \Delta^3 y_{-1}) = \frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) = \mu\delta^3 y_0, \quad \text{etc.}$$

Notes

(i) This formula involves means of the odd differences just above and below the central line and even differences on the central line.

$$y_0 \begin{pmatrix} \Delta y_{-1} \\ \Delta y_0 \end{pmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{pmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{pmatrix} \cdots \Delta^4 y_{-2} \quad \cdots \text{Central line}$$

(ii) This formula gives fairly accurate values of y for r between –0.25 and 0.2 but can be used for r between –0.5 to 0.5.

Example 1

Using Stirling's formula, estimate the value of tan 16°.

x	0°	5°	10°	15°	20°	25°	30°
$y = \tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Solution

Let $x = 16°, x_0 = 15°, h = 5°$

$$r = \frac{x - x_0}{h} = \frac{16° - 15°}{5°} = 0.2$$

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0°	–3	0						
			0.0875					
5°	–2	0.0875		0.0013				
			0.0888		0.0015			
10°	–1	0.1763		0.0028		0.0002		
			0.0916		0.0017		–0.0002	
15°	0	0.2679		0.0045		0		0.0011
			0.0961		0.0017		0.0009	
20°	1	0.3640		0.0062		0.0009		
			0.1023		0.0026			
25°	2	0.4663		0.0088				
			0.1111					
30°	3	0.5774						

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$
$$+ \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \frac{r(r^2-1)(r^2-4)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$
$$+ \frac{r^2(r^2-1)(r^2-4)}{6!}\Delta^6 y_{-3} + \cdots$$

$$y(16) = 0.2679 + 0.2\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045)$$
$$+ \frac{(0.2)(0.2^2-1)}{3!}\left(\frac{0.0017 + 0.0017}{2}\right) + 0$$
$$+ \frac{(0.2)(0.2^2-1)(0.2^2-4)}{5!}\left(\frac{-0.0002 + 0.0009}{2}\right)$$
$$+ \frac{(0.2)^2(0.2^2-1)(0.2^2-4)}{6!}(0.0011)$$
$$= 0.2679 + 0.0188 + (9\times10^{-5}) - (5.44\times10^{-5})$$
$$+ 0 + (2.2176\times10^{-6}) + (2.3232\times10^{-7})$$
$$= 0.2867$$

Example 2

Employ Stirling's formula to compute $y(35)$ *from the following table:*

x	20	30	40	50
y	512	439	346	243

Solution

Let $x = 35,\ x_0 = 30,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	512			
			–73		
30	0	439		–20	
			–93		10
40	1	346		–10	
			–103		
50	2	243			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(35) = 439 + 0.5\left(\frac{-73 - 93}{2}\right) + \frac{(0.5)^2}{2!}(-20) + \frac{0.5(0.5^2 - 1)}{3!}\left(\frac{10}{2}\right)$$

$$= 439 - 41.5 - 2.5 - 0.3125$$

$$= 394.6875$$

Example 3

Let $f(40) = 836$, $f(50) = 682$, $f(60) = 436$, $f(70) = 272$. *Use Stirling's formula to find* $f(55)$.

Solution

Let $x = 55,\quad x_0 = 50,\quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{55 - 50}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	–1	836			
			–154		
50	0	682		–92	
			–246		174
60	1	436		–82	
			–164		
70	2	272			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(55) = 682 + 0.5\left(\frac{-154 - 246}{2}\right) + \frac{(0.5)^2}{2!}(-92) + \frac{0.5(0.5^2 - 1)}{3!}\left(\frac{174}{2}\right)$$

$$= 682 - 100 - 11.5 - 5.4375$$

$$= 565.0625$$

Example 4

Using Stirling's formula, find $y(25)$ from the following table:

x	20	24	28	32
y	0.01427	0.01581	0.01772	0.01996

Solution

Let $x = 25, \quad x_0 = 24, \quad h = 4$

$$r = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	0.01427			
			0.00154		
24	0	0.01581		0.00037	
			0.00191		–0.00004
28	1	0.01772		0.00033	
			0.00224		
32	2	0.01996			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(25) = 0.01581 + 0.25\left(\frac{0.00154 + 0.00191}{2}\right) + \frac{(0.25)^2}{2!}(0.00037)$$

$$+ \frac{0.25(0.25^2 - 1)}{3!}\left(\frac{-0.00004}{2}\right)$$

$$= 0.01581 + 4.3125 \times 10^{-4} + 1.15625 \times 10^{-5} + 7.8125 \times 10^{-7}$$

$$= 0.01625$$

Example 5

Find the value of $y(1.63)$ *from the following table using Stirling's formula:*

x	1.5	1.6	1.7	1.8	1.9
$y = f(x)$	17.609	20.412	23.045	25.527	27.875

Solution

Let $x = 1.63, \quad x_0 = 1.6, \quad h = 0.1$

$$r = \frac{x - x_0}{h} = \frac{1.63 - 1.6}{0.1} = 0.3$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	–1	17.609				
			2.803			
1.6	0	20.412		–0.17		
			2.633		0.019	
1.7	1	23.045		–0.151		–0.002
			2.482		0.017	
1.8	2	25.527		–0.134		
			2.348			
1.9	3	27.875				

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(1.63) = 20.412 + 0.3\left(\frac{2.803 + 2.633}{2}\right) + \frac{(0.3)^2}{2!}(-0.17) + \frac{0.3(0.3^2 - 1)}{3!}\left(\frac{0.019}{2}\right)$$

$$= 20.412 + 0.8154 - 7.65 \times 10^{-3} - 4.3225 \times 10^{-4}$$

$$= 21.2193$$

EXERCISE 7.3

1. Use Gauss's interpolation formula to find y_{16}.

x	5	10	15	20	25
y	26.782	19.951	14.001	8.762	4.163

[Ans.: 12.901]

2. Find $e^{-1.7425}$ by Gauss's forward formula.

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

[Ans.: 0.17508]

3. Find $f(25)$ given $f(20) = 14$, $f(24) = 32$, $f(28) = 35$, and $f(32) = 40$ using Gauss's formula.

[Ans.: 33.41]

4. Apply Gauss's backward formula to find the population in 1926.

Year	x	1911	1921	1931	1941	1951
Population in lacs	y	15	20	27	39	52

[Ans. : 22.898 lacs]

5. Apply Gauss's backward interpolation formula to find sin 45°.

$x°$	20	30	40	50	60	70
sin $x°$	0.34202	0.50200	0.64279	0.76604	0.86603	0.93969

[Ans. : 0.705990]

6. Use Gauss's backward formula, find $f(5.8)$ given that $f(x)$ is a polynomial of degree four and $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682$, $\Delta^2 f(4) = 132$

[Ans.: 1163]

7. Using Stirling's formula, find $y(5)$ from the following table:

x	0	4	8	12
y	14.27	15.81	17.72	19.96

[Ans.: 16.25]

8. Find $\sqrt{1.12}$ using Stirling's formula from the following table:

x	1.0	1.05	1.10	1.15	1.20	1.25	1.30
$f(x)$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

[Ans.: 1.05830]

9. Use Stirling's formula to find tan 89° 26′ from the table:

x	89° 21′	89° 23′	89° 25′	89° 27′	89° 29′
tan x	88.14	92.91	98.22	104.17	110.90

[Ans.: 101.107]

7.11 INTERPOLATION WITH UNEQUAL INTERVALS

If the values of x are unequally spaced then interpolation formulae for equally spaced points cannot be used. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. There are two such formulae for unequally spaced values of x.

(i) Lagrange's interpolation formula
(ii) Newton's interpolation formula with divided difference

7.12 LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which take the values y_0, y_1, y_2, …, y_n corresponding to $x = x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of x and y, $f(x)$ can be represented by a polynomial in x of degree n.

$$y = f(x) = a_0 (x - x_1)(x - x_2) \ldots (x - x_n) + a_1 (x - x_0)(x - x_2) \ldots (x - x_n) + \ldots + a_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(7.19)$$

where $a_0, a_1, a_2, ..., a_n$ are constants.

Putting $x = x_0$, $y = y_0$ in Eq. (7.19),

$$y_0 = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Similarly, putting $x = x_1$, $y = y_1$ in Eq. (7.19),

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)}$$

Proceeding in the same way,

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2 \ldots, a_n$ in Eq. (7.19),

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1 + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n \quad \ldots(7.20)$$

Equation (7.20) is known as *Lagrange's interpolation formula*.

Note This formula can also be used to split the given function into partial fractions. Dividing both sides of Eq. (7.20) by $(x - x_0)(x - x_1) \ldots (x - x_n)$,

$$\frac{f(x)}{(x - x_0)(x - x_1)\ldots(x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}\left(\frac{1}{x - x_0}\right) + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}\left(\frac{1}{x - x_1}\right) + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}\left(\frac{1}{x - x_n}\right)$$

Example 1

Compute f(9.2) *by using Lagrange's interpolation method from the following data:*

x	9	9.5	11
$f(x)$	2.1972	2.2513	2.3979

[Summer 2013]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$f(9.2)=\frac{(9.2-9.5)(9.2-11)}{(9-9.5)(9-11)}(2.1972)+\frac{(9.2-9)(9.2-11)}{(9.5-9)(9.5\text{-}11)}(2.2513)+\frac{(9.2-9)(9.2-9.5)}{(11-9)(11-9.5)}(2.3979)$$

$$=1.1865+1.0806-0.048$$

$$=2.2191$$

Example 2

Find the value of y when $x = 10$ *from the following table:*

x	5	6	9	11
y	12	13	14	16

Solution

By Lagrange's interpolation formula,

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$=2-4.3333+11.6666+5.3333$$

$$=14.6666$$

Example 3

Compute $f(4)$ from the tabular values given:

x	2	3	5	7
$f(x)$	0.1506	0.3001	0.4517	0.6259

using Lagrange's interpolation formula. **[Winter 2012]**

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$f(4)=\frac{(4-3)(4-5)(4-7)}{(2-3)(2-5)(2-7)}(0.1506)+\frac{(4-2)(4-5)(4-7)}{(3-2)(3-5)(3-7)}(0.3001)$$
$$+\frac{(4-2)(4-3)(4-7)}{(5-2)(5-3)(5-7)}(0.4517)+\frac{(4-2)(4-3)(4-5)}{(7-2)(7-3)(7-5)}(0.6259)$$
$$=-0.0301+0.2251+0.2259-0.0313$$
$$=0.3896$$

Example 4

Compute $f(2)$ by using Lagrange's interpolation method from the following data:

x	–1	0	1	3
$f(x)$	2	1	0	–1

[Winter 2013, Summer 2015]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$f(2)=\frac{(2-0)(2-1)(2-3)}{(-1-0)(-1-1)(-1-3)}(2)+\frac{(2+1)(2-1)(2-3)}{(0+1)(0-1)(0-3)}(1)$$
$$+\frac{(2+1)(2-0)(2-3)}{(1+1)(1-0)(1-3)}(0)+\frac{(2+1)(2-0)(2-1)}{(3+1)(3-0)(3-1)}(-1)$$
$$=0.5-1+0-0.25$$
$$=-0.75$$

Example 5

By using Lagrange's formula, find y when x = 10.

x	5	6	9	11
y	12	13	14	16

[Summer 2015]

Solution

By Lagrange's interpolation formula,

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$
$$=2-4.3333+11.6667+5.3333$$
$$=14.6667$$

Example 6

Evaluate f(9) by using Lagrange's interpolation method from the following data:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$f(9)=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150)$$
$$+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392)$$
$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452)$$
$$+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}(2366)$$
$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202)$$
$$=-16.6667+209.0667+1290.6667-788.6667+115.6$$
$$=810$$

Example 7

Determine y(12) *by using Lagrange's interpolation method from the following data:*

x	11	13	14	18	20	23
y	25	47	68	82	102	124

[Winter 2014]

Solution

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} y_3$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} y_4$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} y_5$$

$$y(12) = \frac{(12-13)(12-14)(12-18)(12-20)(12-23)}{(11-13)(11-14)(11-18)(11-20)(11-23)}(25)$$
$$= \frac{(12-11)(12-14)(12-18)(12-20)(12-23)}{(13-11)(13-14)(13-18)(13-20)(13-23)}(47)$$
$$= \frac{(12-11)(12-13)(12-18)(12-20)(12-23)}{(14-11)(14-13)(14-18)(14-20)(14-23)}(68)$$
$$= \frac{(12-11)(12-13)(12-14)(12-20)(12-23)}{(18-11)(18-13)(18-14)(18-20)(18-23)}(82)$$
$$= \frac{(12-11)(12-13)(12-14)(12-18)(12-23)}{(20-11)(20-13)(20-14)(20-18)(20-23)}(102)$$
$$= \frac{(12-11)(12-13)(12-14)(12-18)(12-20)}{(23-11)(23-13)(23-14)(23-18)(23-20)}(124)$$
$$= 5.8201 + 70.9029 - 55.4074 + 10.3086 - 5.9365 + 0.7348$$
$$= 26.4225$$

Example 8

Find a second-degree polynomial passing through the points (0, 0), (1, 1) *and* (2, 20) *using Larange's interpolation.* **[Summer 2015]**

Solution

Let $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$

$f(x_0) = 0, \quad f(x_1) = 1, \quad f(x_2) = 20$

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(20)$$

$$= 0 - x(x-2) + 10x(x-1)$$

$$= 9x^2 - 8x$$

Example 9

Using Lagrange's interpolation formula, find the interpolating polynomial for the following table:

x	0	1	2	5
$f(x)$	2	3	12	147

Solution

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x^2-7x+10)}{-5} + \frac{3x(x^2-7x+10)}{4} + \frac{2x(x^2-6x+5)}{-1} + \frac{49x(x^2-3x+2)}{20}$$

$$= \frac{\left\{\begin{array}{l} -4(x^3-8x^2+17x-10) + (15x^3-105x^2+150x) \\ \qquad -20(2x^3-12x^2+10x) + (49x^3-147x^2+98x) \end{array}\right\}}{20}$$

$$= \frac{20x^3+20x^2-20x+40}{20}$$

$$= x^3 + x^2 - x + 2$$

Example 10

Find the Lagrange interpolating polynomial from the following data:

x	0	1	4	5
$f(x)$	1	3	24	39

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$=\frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(1)+\frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$
$$+\frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24)+\frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$
$$=-\frac{(x-1)(x-4)(x-5)}{20}+\frac{x(x-4)(x-5)}{4}$$
$$-2x(x-1)(x-5)+\frac{39x(x-1)(x-4)}{20}$$
$$=-\frac{x^3-10x^2+29x-20}{20}+\frac{x^3-9x^2+20x}{4}$$
$$-(2x^3-12x^2+10x)+\frac{39(x^3-5x^2+4x)}{20}$$
$$=\frac{1}{20}(3x^3+10x^2+27x+20)$$

Example 11

Use Lagrange's formula to fit a polynomial to the data:

x	–1	0	2	3
y	8	3	1	12

and hence, find $y(2)$.

Solution

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$
$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)$$
$$= -\frac{2x(x-2)(x-3)}{3} + \frac{(x+1)(x-2)(x-3)}{2}$$
$$- \frac{(x+1)(x)(x-3)}{6} + (x+1)(x)(x-2)$$
$$= \frac{1}{3}(2x^3 + 2x^2 - 15x + 9)$$
$$y(2) = \frac{1}{3}\left[2(8) + 2(4) - 15(2) + 9\right] = 1$$

Example 12

Express the given rational function as a sum of partial fractions:

$$y = \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$

Solution

Let $f(x) = 3x^2 + x + 1$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	5	15	31

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$
$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$
$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$\therefore \quad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$
$$= \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}$$

Example 13

Express the function $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions using Lagrange's formula.*

Solution

Let $f(x) = 3x^2 - 12x + 11$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	2	–1	2

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$
$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(2) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(-1) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(2)$$
$$= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$$

$$\therefore \quad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$
$$= \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Example 14

The following values of the function $f(x)$ *are given as* $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. *Find the value of* $f(6)$ *and also the value of* x *for which* $f(x)$ *is maximum or minimum.*

Solution

Tabular form of the data is

x	1	2	7	8
$f(x)$	4	5	5	4

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$
$$=\frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}(4)+\frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}(5)+\frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}(5)$$
$$+\frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}(4)$$
$$=-\frac{2}{21}(x^3-17x^2+86x-112)+\frac{1}{6}(x^3-16x^2+71x-56)$$
$$-\frac{1}{6}(x^3-11x^2+26x-16)+\frac{2}{21}(x^3-10x^2+23x-14)$$
$$=-\frac{1}{6}x^2+\frac{3}{2}x+\frac{8}{3}$$
$$f(6)=-\frac{1}{6}(6)^2+\frac{3}{2}(6)+\frac{8}{3}=\frac{17}{3}$$

For extreme values, $f'(x)=0$

$$-\frac{1}{3}x+\frac{3}{2}=0$$
$$x=4.5$$

Also, $f''(x)=-\frac{1}{3}<0$

Since $f''(x)$ is negative, $f(x)$ is maximum at $x = 4.5$.

Example 15

A body moving with velocity v at any time t satisfies the data

t	0	1	3	4
v	21	15	12	10

Obtain the distance travelled in 4 seconds and acceleration at the end of 4 *seconds.*

Solutions

By Lagrange's interpolation formula,

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$
$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}v_3$$
$$= \frac{(t-1)(t-3)(t-4)}{(0-1)(0-3)(0-4)}(21) + \frac{(t-0)(t-3)(t-4)}{(1-0)(1-3)(1-4)}(15)$$
$$+ \frac{(t-0)(t-1)(t-4)}{(3-0)(3-1)(3-4)}(12) + \frac{(t-0)(t-1)(t-3)}{(4-0)(4-1)(4-3)}(10)$$
$$= \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

If s is the distance travelled in time t,

$$v = \frac{ds}{dt} = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$
$$s = \int_0^4 v\,dt$$
$$= \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)$$
$$= \frac{1}{12}\left|-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right|_0^4$$
$$= \frac{1}{12}\left[-\frac{5}{4}\times 256 + \frac{38}{3}\times 64 - \frac{105}{2}\times 16 + 1008\right]$$
$$= 54.88$$

Hence, the distance travelled in 4 seconds = 54.88

$$a = \frac{dv}{dt} = \frac{1}{12}(-15t^2 + 76t - 105)$$

At $t = 4$,

$$a = \frac{1}{12}(-15\times 16 + 76\times 4 - 105) = 3.416$$

EXERCISE 7.4

1. From the table given below, find $y(x = 2)$.

x	0	1	3	4
y	5	6	50	105

[Ans. : 19]

2. Use Lagrange's formula to find the velocity of the particle $v = f(t)$ and then at $t = 3.5$ from the table

t	0	1	2	3
v	21	15	12	10

[Ans. : 8.75]

3. Find $f(27)$ from the following table:

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

[Ans. : 49.3]

4. Find $f(6)$ from the following table:

x	2	5	7	10	12
$f(x)$	18	180	448	1210	2028

[Ans. : 294]

5. Find $f(9)$ from the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Ans. : 809.997]

6. If $y_0 = 4.3315$, $y_1 = 7.4046$, $y_3 = 5.6713$, $y_5 = 7.1154$, find the curve passing through these points. Hence, find y_2 and y_4.

[Ans. : 5.1420, 6.3199]

7. If $f(1) = 3, f(2) = -5, f(-4) = 4$, find the three-point Lagrange's interpolation polynomial that takes the same values.

$$\left[\textbf{Ans.:}\ \frac{1}{20}(-39x^2 - 123x + 252)\right]$$

8. A third-degree polynomial passes through the points (0,–1), (1,1), (2,1), (3,–2). Find the polynomial.

$$\left[\textbf{Ans.:}\ \frac{1}{6}(-x^3 - 3x^2 + 16x - 6)\right]$$

9. If $y = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points

x	1	3	5	7
y	0	50	236	654

find a_0, a_1, a_2 and a_3.

[Ans.: –4, 3, –1, 2]

10. Find the polynomial of degree 3 which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.:}\ \frac{1}{12}\left(x^3 + 3x^2 + 32x + 24\right)\right]$$

11. Find the polynomial which takes the values $f(1) = 1$, $f(2) = 9$, $f(3) = 25$, $f(4) = 55$, $f(5) = 105$.

[Ans.: $x^3 - 2x^2 + 7x - 5$]

12. Find $f(x)$ from the following table:

x	0	2	3	6
$f(x)$	659	705	729	804

$$\left[\textbf{Ans.:}\ \frac{1}{72}(-x^3 + 29x^2 + 1604x + 47448)\right]$$

13. For the following table:

x	1	3	4	6
$f(x)$	–3	9	30	132

Express $f(x)$ as a third-degree polynomial in x. Also, find $f'(x)$, $f''(x)$ at $x = 1$.

[Ans.: $x^3 - 3x^2 + 5x - 6$, 2, 0]

14. Using Lagrange's formula for unequal intervals, express the function $\dfrac{x^2 + 6x - 1}{(x^2 - 1)(x - 4)(x - 6)}$ as a sum of partial fractions.

$$\left[\textbf{Ans.:}\ \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}\right]$$

7.13 DIVIDED DIFFERENCES

In Lagrange's interpolation formula, if another interpolation value is added then the interpolation coefficients are required to be recalculated. To avoid this recalculation, Newton's general interpolation formula is used.

If (x_0, y_0), (x_1, y_1), (x_2, y_2) …. be given points then the first divided difference for x_0, x_1 is defined by the relation,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$, etc.

The second divided difference for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

Notes

(i) The divided differences are symmetrical in their arguments, i.e., independent of the order of arguments:

$$[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$
$$= [x_1, x_0]$$

$$[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$
$$= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1]$$

(ii) The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$
$$= \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right)$$
$$= \frac{1}{2!h^2}\Delta^2 y_0$$

In general,

$$[x_0, x_1, x_2 ..., x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is an n^{th} degree polynomial, $\Delta^n y_0$ will be constant. Hence, the n^{th} divided differences will also be constant.

7.14 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let the function $y = f(x)$ take values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$ respectively. According to the definition of divided differences,

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$
$$y = y_0 + (x - x_0)[x, x_0] \qquad ...(7.21)$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$
$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting the value of $[x, x_0]$ in Eq. (7.21),

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x, x_0, x_1] \qquad ...(7.22)$$

Also, $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)\,[x, x_0, x_1, x_2]$$

Substituting the value of $[x, x_0, x_1]$ in Eq. (7.22),

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x, x_0, x_1, x_2]$$

Proceeding in the same manner,

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots$$
$$+ (x - x_0)\,(x - x_1)\,\ldots\,(x - x_{n-1})\,[x, x_0, x_1, \ldots, x_n] \qquad \ldots(7.23)$$

Equation (7.23) is known as *Newton's general interpolation formula* with divided differences.

Example 1

If $f(x)=\frac{1}{x}$, *find the divided difference* $[a, b]$ *and* $[a, b, c]$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference
a	$\frac{1}{a}$		
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a}=-\frac{1}{ab}$	
b	$\frac{1}{b}$		$\frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a}=\frac{1}{abc}$
		$\frac{\frac{1}{c}-\frac{1}{b}}{c-b}=-\frac{1}{bc}$	
c	$\frac{1}{c}$		

$$[a,b]=-\frac{1}{ab}$$

$$[a,b,c]=\frac{1}{abc}$$

Example 2

Find the second divided difference for the argument $x = 1, 2, 5,$ *and* 7 *for the function* $f(x) = x^2$. **[Summer 2015]**

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		$\frac{4-1}{2-1}=3$		
2	4		$\frac{7-3}{5-1}=1$	
		$\frac{25-4}{5-2}=7$		0
5	25		$\frac{12-7}{7-2}=1$	
		$\frac{49-25}{7-5}=12$		
7	49			

Example 3

Find the third divided difference with arguments $2, 4, 9, 10$ *of the function* $f(x) = x^3 - 2x$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
2	4			
		$\frac{56-4}{4-2}=26$		
4	56		$\frac{131-26}{9-2}=15$	
		$\frac{711-56}{9-4}=131$		$\frac{23-15}{10-2}=1$
9	711		$\frac{269-131}{10-4}=23$	
		$\frac{980-711}{10-9}=269$		
10	980			

Example 4

Construct the divided difference for the data given below:

x	−4	−1	0	2	5
$f(x)$	1245	33	5	9	1335

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
−4	1245				
		$\frac{33-1245}{-1+4}=-404$			
−1	33		$\frac{-28+404}{0+4}=94$		
		$\frac{5-33}{0+1}=-28$		$\frac{10-94}{2+4}=-14$	
0	5		$\frac{2+28}{2+1}=10$		$\frac{13+14}{5+1}=13$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5+1}=13$	
2	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
5	1335				

Example 5

Complete $f(9.2)$ from the following data by using Newton's divided difference interpolation formula.

x	8	9	9.5	11
$f(x)$	2.079442	2.197225	2.251292	2.397895

[Winter 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
8	2.079442			
		0.117783		
9	2.197225		−0.006433	
		0.108134		0.000411
9.5	2.251292		−0.005200	
		0.097735		
11	2.397895			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(9.2) = 2.079442 + (9.2 - 8)\,(0.117783) + (9.2 - 8)\,(9.2 - 9)\,(-0.006433) + (9.2 - 8)\,(9.2 - 9)\,(9.2 - 9.5)\,(0.00041)$$

$$= 2.079442 + 0.141340 - 0.001544 - 0.000030$$

$$= 2.219208$$

Example 6

Using Newton's divided difference formula, compute $f(10.5)$ from the following data:

x	10	11	13	17
$f(x)$	2.3026	2.3979	2.5649	2.8332

[Summer 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
10	2.3026			
		0.0953		
11	2.3979		−0.0039	
		0.0835		0.0002
13	2.5649		−0.0027	
		0.0671		
17	2.8332			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(10.5) = 2.3026 + (10.5 - 10)\,(0.0953) + (10.5 - 10)\,(10.5 - 11)\,(-0.0039) + (10.5 - 10)\,(10.5 - 11)\,(10.5 - 13)\,(0.0002)$$

$$= 2.3026 + 0.0477 + 0.00098 + 0.00013$$

$$= 2.3514$$

Example 7

Using Newton's divided difference interpolation, compute the value of $f(6)$ *from the table given below:*

x	1	2	7	8
$f(x)$	1	5	5	4

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		−1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(6) = 1 + (6 - 1)\,(4) + (6 - 1)\,(6 - 2)\left(-\frac{2}{3}\right) + (6 - 1)\,(6 - 2)\,(6 - 7)\left(\frac{1}{14}\right)$$

$$= 1 + 20 - 13.3333 - 1.4286$$

$$= 6.2381$$

Example 8

Evaluate $f(9)$ using the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)\,(x - x_2)\,(x - x_3)\,[x_0, x_1, x_2, x_3, x_4]$$

$$f(9) = 150 + (9 - 5)\,(121) + (9 - 5)\,(9 - 7)\,(24) + (9 - 5)\,(9 - 7)\,(9 - 11)\,(1) + 0$$

$$= 150 + 484 + 192 - 16$$

$$= 810$$

Example 9

Compute $f(8)$ from the following values using Newton's divided difference formula:

x	4	5	7	10	11	13
$f(x)$	48	100	244	900	1210	2028

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
4	48				
		52			
5	100		15		
		97		1	
7	244		21		0
		202		1	
10	900		27		0
		310		1	
11	1210				
		409	33		
13	2028				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(8) = 48 + (8 - 4)\,(52) + (8 - 4)\,(8 - 5)\,(15) + (8 - 4)\,(8 - 5)\,(8 - 7)\,(1) + 0$$

$$= 48 + 208 + 180 + 12$$

$$= 448$$

Example 10

From the following table, find f(x) using Newton's divided difference formula:

x	1	2	7	8
$f(x)$	1	5	5	4

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		–1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 1 + (x - 1)\,4 + (x - 1)\,(x - 2)\left(-\frac{2}{3}\right) + (x - 1)\,(x - 2)\,(x - 7)\left(\frac{1}{14}\right)$$

$$= 1 + 4x - 4 - \frac{2}{3}(x^2 - 3x + 2) + \frac{1}{14}(x^3 - 10x^2 + 23x - 14)$$

$$= \frac{1}{14}x^3 - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{3}$$

Example 11

Using Newton's divided difference formula, find a polynomial and also, find f(–1) and f(6).

x	1	2	4	7
$f(x)$	10	15	67	430

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	10			
		5		
2	15		7	
		26		2
4	67		19	
		121		
7	430			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 10 + (x-1)(5) + (x-1)(x-2)(7) + (x-1)(x-2)(x-4)2$$

$$= 10 + 5x - 5 + 7x^2 - 21x + 14 + 2x^3 - 14x^2 + 28x - 16$$

$$= 2x^3 - 7x^2 + 12x + 3$$

$$f(-1) = 2(-1)^3 - 7(-1)^2 + 12(-1) + 3 = -18$$

$$f(6) = 2(6)^3 - 7(6)^2 + 12(6) + 3 = 255$$

Example 12

Establish a cubic polynomial of the curve $y = f(x)$ passing through the points (1, –3), (3, 9), (4, 30), (6, 132). *Hence, find* $f(2)$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	–3			
		6		
3	9		5	
		21		1
4	30		10	
		51		
6	132			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= -3 + (x - 1)\,(6) + (x - 1)\,(x - 3)\,(5) + (x - 1)\,(x - 3)\,(x - 4)\,(1)$$

$$= -3 + 6x - 6 + 5x^2 - 20x + 15 + x^3 - 8x^2 + 19x - 12$$

$$= x^3 - 3x^2 + 5x - 6$$

$$f(2) = (2)^3 - 3(2)^2 + 5(2) - 6 = 0$$

Example 13

The shear stress in kilopound per square foot (ksf) for 5 specimens in a clay stratum are as follows:

Depth (*m*)	1.9	3.1	4.2	5.1	5.8
Stress (*ksf*)	0.3	0.6	0.4	0.9	0.7

Use Newton's dividend difference interpolating polynomial to compute the stress at 4.5 m depth. **[Winter 2012]**

Solution

Divided Difference Table

Depth x	Stress y	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
1.9	0.3				
		0.25			
3.1	0.6		–0.1877		
		–0.1818		0.1739	
4.2	0.4		0.3687		–0.1295
		0.5556		–0.3313	
5.1	0.9		–0.5258		
		–0.2857			
5.8	0.7				

By Newton's divided difference formula,

$$y(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + (x - x_0)\,(x - x_1)\,(x - x_2)\,(x - x_3)\,[x_0, x_1, x_2, x_3, x_4]$$

$$y(4.5) = 0.3 + (4.5 - 1.9)(0.25) + (4.5 - 1.9)(4.5 - 3.1)(-0.1877) + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(0.1739) + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(4.5 - 5.1)(-0.1295)$$
$$= 0.3 + 0.65 - 0.6832 + 0.1899 + 0.0848$$
$$= 0.5415 \text{ ksf}$$

EXERCISE 7.5

1. If $f(x) = \frac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$, and $f(a, b, c, d)$.

$$\left[\textbf{Ans.:} -\frac{(a+b)}{a^2b^2}, \frac{ab+bc+ca}{a^2b^2c^2}, -\frac{(abc+bcd+acd+abd)}{a^2b^2c^2d^2}\right]$$

2. Find the third divided difference of $f(x)$ with arguments 2, 4, 9, 10 where $f(x) = x^3 - 2x$.

[Ans.: 1**]**

3. Obtain the value of $\log_{10} 656$ given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 666 = 2.8202$.

[Ans.: 2.8169**]**

4. Find $f(5)$ from the following table:

x	0	1	3	6
$f(x)$	1	4	88	1309

[Ans.: 636**]**

5. Find $y(x = 20)$ from the table:

x	12	18	22	24	32
$y(x)$	146	836	19481	2796	9236

[Ans.: 1305.36**]**

6. Find a polynomial $f(x)$ of lowest degree which takes the values 3, 7, 9, and 19 when $x = 2, 4, 5, 10$.

[Ans.: $2x - 1$**]**

7. Using the divided difference table, find $f(x)$ which takes the values 1, 4, 40, 85 as $x = 0, 1, 3, 4$.

[Ans.: $x^3 + x^2 + x + 1$**]**

8. Find $f(x)$ as a polynomial by using Newton's formula:

x	–1	0	3	6	7
$f(x)$	3	–6	39	822	1611

[Ans.: $x^4 - 3x^3 + 5x^2 - 6$**]**

9. Find the polynomial $y = f(x)$ passing through (5, 1355), (2, 9), (0, 5), (–1, 33), and (–4, 1245).

[Ans.: $3x^4 - 5x^3 + 6x^2 + 14x + 5$**]**

10. Find the polynomial equation of degree 4 passing through the points (8, 1515), (7, 778), (5, 138), (4, 43), and (2, 3).

[Ans.: $x^4 - 10x^3 + 36x^2 - 36x - 5$**]**

11. Find the function $y(x)$ in powers of $(x - 1)$ given $y(0) = 8$, $y(1) = 11$, $y(4) = 68$, $y(5) = 123$.

[Ans.: $11 + 4(x - 1) + 2(x - 1)^2 + (x - 1)^3$**]**

12. Using the following table, find $f(x)$ as a polynomial in powers of $(x - 6)$.

x	–1	0	2	3	7	10
$f(x)$	–11	1	1	1	141	561

[Ans.: $73 + 54(x - 6) + 13(x - 6)^2 + (x - 6)^3$**]**

7.15 INVERSE INTERPOLATION

The process of evaluating the value of x for a value of y (which is not in the table) is called *inverse interpolation*. Lagrange's formula is a relation between two variables, either of which may be taken as the independent variable. On interchanging x and y in the Lagrange's interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_1)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1 + \ldots$$
$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n \qquad \ldots(7.24)$$

Equation (7.24) is used for inverse interpolation.

Example 1

From the data given, find the value of x when $y = 13.5$.

x	93	96.2	100	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

Solution

By Lagrange's formula for inverse interpolation,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y-y_4)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)}x_4$$

$$= \frac{(13.5-12.80)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)}(93)$$
$$+ \frac{(13.5-11.38)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)}(96.2)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-17.07)(13.5-19.91)}{(14.70-11.38)(14.70-12.80)(14.70-17.07)(14.70-19.91)}(100)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)}(104.2)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)}(108.7)$$
$$x = -7.8137 + 68.4669 + 43.6076 - 7.2758 + 0.7711$$
$$= 97.7561$$

Example 2

Find the root of the equation $f(x) = 0$, *given that* $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, *and* $f(42) = 18$.

Solution

Let $x_0 = 30, \quad x_1 = 34, x_2 = 38, x_3 = 42$
$y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x for $y = f(x) = 0$.
By Lagrange's formula for inverse interpolation,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$
$$= \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}(34)$$
$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$
$$= -0.782 + 6.5323 + 33.6818 - 2.2016$$
$$= 37.2305$$

Hence, the root of $f(x) = 0$ is 37.2305.

EXERCISE 7.6

1. Find x given $y = 0.3887$ from the following data:

x	21	23	25
y	0.3706	0.4068	0.4433

[Ans.: 22]

2. Find x corresponding to $y = 85$ from the following table:

x	2	5	8	14
y	94.8	87.9	81.3	68.7

[Ans.: 6.5928]

3. Find x corresponding to $y = 100$ from the following table:

x	3	5	7	9	11
y	6	24	58	108	174

[Ans.: 8.656]

4. Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^{\theta} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}$ using the table:

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

[Ans.: 22.0020°]

5. Find the age corresponding to the annuity value 13.6 from the given table:

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

[Ans.: 43]

Points to Remember

Forward Differences

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Backward Differences

$$\nabla y_n = y_n - y_{n-1}$$

Central Differences

$$\delta y_{n-\frac{1}{2}} = y_n - y_{n-1}$$

Newton's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Newton's Backward Interpolation Formula

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Gauss's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss's Backward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Stirling's Formula

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

Newton's Divided Difference Formula

$$y = y_0 + (x-x_0)\,[x_0, x_1] + (x-x_0)\,(x-x_1)\,[x_0, x_1, x_2] + (x-x_0)\,(x-x_1)\,(x-x_2)\,[x_0, x_1, x_2, x_3] + \ldots + (x-x_0)\,(x-x_1)\,\ldots\,(x-x_{n-1})\,[x, x_0, x_1, \ldots, x_n]$$

Inverse Interpolation

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_1)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1 + \ldots$$
$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n$$

CHAPTER 8
Numerical Integration

Chapter Outline

8.1 Introduction
8.2 Newton–Cotes Quadrature Formula
8.3 Trapezoidal Rule
8.4 Simpson's 1/3 Rule
8.5 Simpson's 3/8 Rule
8.6 Gaussian Quadrature Formulae

8.1 INTRODUCTION

The process of evaluating a definite integral from a set of tabulated values of $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable is known as *quadrature*. In numerical integration, $f(x)$ is represented by an interpolation formula and then it is integrated between the given limits. In this way, the quadrature formula is derived for approximate integration of a function defined by a set of numerical values only.

8.2 NEWTON–COTES QUADRATURE FORMULA

Let the function $y = f(x)$ takes values $y_0, y_1, y_2 \ldots, y_n$ for $x_0, x_1, x_2 \ldots, x_n$ respectively (Fig. 8.1).

Let $I = \int_a^b f(x)\,dx$. Dividing the interval (a, b) into n sub-intervals of width h such that

$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_n = x_0 + nh = b$

$$\therefore \qquad \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

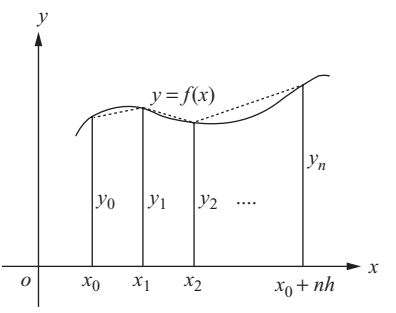


Fig. 8.1

Putting $x = x_0 + rh$, $\mathrm{d}x = h\mathrm{d}r$
When $x = x_0$, $r = 0$
When $x = x_0 + nh$, $r = n$

$$\int_a^b f(x)\,\mathrm{d}x = h\int_0^n f(x_0 + rh)\,\mathrm{d}r$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] \mathrm{d}r$$

[By Newton's forward interpolation formula]

$$= h\left| ry_0 + \frac{r^2}{2}\Delta y_0 + \left(\frac{\frac{r^3}{3} - \frac{r^2}{2}}{2} \right)\Delta^2 y_0 + \left(\frac{\frac{r^4}{4} - r^3 + r^2}{6} \right)\Delta^3 y_0 + \cdots \right|_0^n$$

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right]$$

This equation is known as the *Newton–Cotes quadrature formula*.

8.3 TRAPEZOIDAL RULE

By the Newton-Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad \ldots(8.1)$$

Putting $n = 1$ in Eq. (8.1) and ignoring the differences of order higher than one,

$$\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = h\left(y_0 + \frac{1}{2}\Delta y_0 \right)$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)\,\mathrm{d}x = \frac{h}{2}(y_1 + y_2)$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]$$

$$= \frac{h}{2}[X + 2R]$$

where X = extreme terms, R = remaining terms
This is known as the *trapezoidal rule*.

Errors in the Trapezoidal Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots \qquad \ldots(8.2)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_1} y\,\mathrm{d}x = \int_{x_0}^{x_1}\left[y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots\right]\mathrm{d}x$$

$$= \left|y_0 x + \frac{(x - x_0)^2}{2!}y_0' + \frac{(x - x_0)^3}{3!}y_0'' + \cdots\right|_{x_0}^{x_1}$$

$$= y_0(x_1 - x_0) + \frac{(x_1 - x_0)^2}{2!}y_0' + \frac{(x_1 - x_0)^3}{3!}y_0'' + \cdots$$

$$= h\,y_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \cdots \qquad \ldots(8.3)$$

where $x_1 - x_0 = h$

Also, $\int_{x_0}^{x_1} y\,\mathrm{d}x \approx \frac{h}{2}(y_0 + y_1) = \text{Area of the first trapezium} = A_1$ ...(8.4)

Putting $x = x_1$ in Eq. (8.2),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\,y_0' + \frac{(x_1 - x_0)^2}{2!}y_0'' + \cdots$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \ldots(8.5)$$

Substituting Eq. (8.5) in Eq. (8.4),

$$A_1 \approx \frac{h}{2}\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots\right]$$

$$\approx h\,y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2(2!)}y_0'' + \cdots \qquad \ldots(8.6)$$

Subtracting Eq. (8.6) from Eq. (8.3),

$$\int_{x_0}^{x_1} y\,dx - A_1 = h^3 y_0'' \left[\frac{1}{3!} - \frac{1}{2(2!)}\right] + \cdots$$

$$= -\frac{1}{12} h^3 y_0'' + \cdots$$

Hence, the error in the first interval (x_0, x_1), neglecting other terms, is $-\frac{1}{12} h^3\ y_0''$.

Similarly, the error in the interval (x_1, x_2) is $-\frac{1}{12} h^3\ y_1''$ and the error in the interval (x_{n-1}, x_n) is $-\frac{1}{12} h^3 y_{n-1}''$.

Hence, the total error is

$$E = -\frac{1}{12} h^3 (y_0'' + y_1'' + \cdots + y_{n-1}'')$$

Let $y''(\xi)$ be the largest value of $y_0'', y_1'', \ldots, y_{n-1}''$ where $x_0 < \xi < x_n$.

$$E < -\frac{1}{12} nh^3\ y''(\xi)$$

$$< -\frac{(x_n - x_0)}{12} h^2 y''(\xi) \qquad [\because\ nh = x_n - x_0]$$

Example 1

Find the area bounded by the curve and the x-axis from $x = 7.47$ *to* $x = 7.52$ *from the following table, by using the trapezoidal rule.*

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution

$$a = 7.47,\ b = 7.52,\ h = 0.01$$

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,dx$$

By the trapezoidal rule,

$$\int_{7.47}^{7.52} f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_5\right) + 2\left(y_1 + y_2 + y_3 + y_4\right)\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.0996$$

Example 2

Consider the following tabular values:

x	25.0	25.1	25.2	25.3	25.4	25.5	25.6
$f(x)$	3.205	3.217	3.232	3.245	3.256	3.268	3.280

Determine the area bounded by the given curve and the x-axis between $x = 25$ and $x = 25.6$ by the trapezoidal rule.

Solution

$$a = 25, \quad b = 25.6, \quad h = 0.1$$

By the trapezoidal rule,

$$\int_{25}^{25.6} y\,\mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= \frac{0.1}{2}\left[(3.205 + 3.280) + 2(3.217 + 3.232 + 3.245 + 3.256 + 3.268)\right]$$

$$= 1.9461$$

Example 3

Given the data below, find the isothermal work done on the gas if it is compressed from $v_1 = 22$ L to $v_2 = 2$ L.

Use $W = -\int_{v_1}^{v_2} p\,\mathrm{d}v$

v, L	2	7	12	17	22
P, atm	12.20	3.49	2.049	1.44	1.11

[Winter 2012]

Solution

$$v_1 = 22, v_2 = 2, h = 5$$

By the trapezoidal rule,

$$W = -\int_{v_1}^{v_2} p\,\mathrm{d}v$$

$$= -\int_{22}^{2} p\,\mathrm{d}v$$

$$= \int_2^{22} p\,dv$$
$$= \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$
$$= \frac{5}{2}\left[(12.20 + 1.11) + 2(3.49 + 2.049 + 1.44)\right]$$
$$= 68.17$$

Example 4

Use trapezoidal rule to evaluate $\int_0^2 \frac{x}{\sqrt{2+x^2}}\,dx$, *dividing the interval into four equal parts.*

Solution

$$a = 0,\ b = 2,\ n = 4$$
$$h = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$
$$y = f(x) = \frac{x}{\sqrt{2+x^2}}$$

x	0	0.5	1	1.5	2
$y = f(x)$	0	0.3333	0.5774	0.7276	0.8165
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_0^2 \frac{x}{\sqrt{2+x^2}}\,dx = \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$
$$= \frac{0.5}{2}\left[(0 + 0.8165) + 2(0.3333 + 0.5774 + 0.7276)\right]$$
$$= 1.0233$$

Example 5

Evaluate $\int_0^1 e^x\,dx$, *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.10
$f(x)$	1	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 e^x \, dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(1 + 2.7183) + 2(1.1052 + 1.2214 + 1.3499 + 1.4918 + 1.6487 + 1.8211 + 2.0138 + 2.2255 + 2.4596)\right]$$

$$= 1.7196$$

Example 6

Calculate $\int_0^1 2\,e^x \, dx$ *with* $n = 10$ *using the trapezoidal rule.*

[Winter 2014]

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = 2e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	2	2.2103	2.4428	2.6997	2.9836	3.2974	3.6442	4.0275	4.4511	4.9192	5.4365
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 2e^x dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(2 + 5.4365) + 2(2.2103 + 2.4428 + 2.6997 + 2.9836 + 3.2974 + 3.6442 + 4.0275 + 4.4511 + 4.9192)\right]$$

$$= 3.4394$$

Example 7

Compute the integral $\int_{-1}^{1} e^x \, dx$ *using the trapezoidal rule for* $n = 4$.

Solution

$$a = -1,\ b = 1,\ n = 4$$

$$h = \frac{x_n - x_0}{n} = \frac{1-(-1)}{4} = 0.5$$

$$y = f(x) = e^x$$

x	–1	–0.5	0	0.5	1
$f(x)$	0.3679	0.6065	1	1.6487	2.7183
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_{-1}^{1} e^x \, dx = \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$

$$= \frac{0.5}{2}\left[(0.3679 + 2.7183) + 2(0.6065 + 1 + 1.6487)\right.$$

$$= 2.39916$$

Example 8

Evaluate $\int_0^1 e^{-x^2} \, dx$ *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977	0.6126	0.5273	0.4449	0.3679
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{2}\left[\left(y_0 + y_{10}\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9\right)\right]$$

$$= \frac{0.1}{2}\left[(1 + 0.3679) + 2(0.99 + 0.9608 + 0.9139 + 0.8521 + 0.7788 + 0.6977 + 0.6126 + 0.5273 + 0.4449)\right]$$

$$= 0.7462$$

8.4 SIMPSON'S 1/3 RULE

By the Newton-Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \ldots(8.7)$$

Putting $n = 2$ in Eq. (8.7) and ignoring the differences of order higher than 2,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \left(y_1 - y_0\right) + \left(\frac{y_2 - 2y_1 + y_0}{6}\right)\right]$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[\left(y_0+y_n\right)+4\left(y_1+y_3+\cdots+y_{n-1}\right)+2\left(y_2+y_4+\cdots+y_{n-2}\right)\right]$$

$$= \frac{h}{3}\left[X+4O+2E\right]$$

where X = extreme terms, O = odd terms, E = even terms
This is known as Simpson's 1/3 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 2.

Errors in Simpson's 1/3 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{\text{iv}} + \cdots \qquad ...(8.8)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2}\left[y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{\text{iv}} + \cdots\right]dx$$

$$= \left|y_0 x + \frac{(x-x_0)^2}{2!}\,y_0' + \frac{(x-x_0)^3}{3!}\,y_0'' + \frac{(x-x_0)^4}{4!}\,y_0''' + \frac{(x-x_0)^5}{5!}\,y_0^{\text{iv}} + \cdots\right|_{x_0}^{x_2}$$

$$= y_0(x_2-x_0) + \frac{(x_2-x_0)^2}{2!}\,y_0' + \frac{(x_2-x_0)^3}{3!}\,y_0'' + \frac{(x_2-x_0)^4}{4!}\,y_0''' + \frac{(x_2-x_0)^5}{5!}\,y_0^{\text{iv}} + \cdots$$

$$= 2h\,y_0 + \frac{4h^2}{2!}\,y_0' + \frac{8h^3}{3!}\,y_0'' + \frac{16h^4}{4!}\,y_0''' + \frac{32h^5}{5!}\,y_0^{\text{iv}} + \cdots$$

$$= 2h\,y_0 + 2h^2 y_0' + \frac{4h^3}{3}\,y_0'' + \frac{2h^4}{3}\,y_0''' + \frac{4h^5}{15!}\,y_0^{\text{iv}} + \cdots \qquad ...(8.9)$$

where $x_2 - x_0 = 2h$

Also, $\int_{x_0}^{x_2} y\,dx = \frac{h}{3}\left(y_0+4y_1+y_2\right) = \text{Area in the interval } \left(x_0, x_2\right) = A_1 \qquad ...(8.10)$

Putting $x = x_1$ in Eq. (8.8),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\, y_0' + \frac{(x_1 - x_0)^2}{2!} y_0'' + \frac{(x_1 - x_0)^3}{3!} y_0''' + \frac{(x_1 - x_0)^4}{4!} y_0^{\text{iv}} + \cdots$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{\text{iv}} + \cdots \qquad ...(8.11)$$

Putting $x = x_2$ in Eq. (8.8),

$$y(x_2) = y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + \frac{16h^4}{4!} y_0^{\text{iv}} + \cdots \qquad ...(8.12)$$

Substituting Eq. (8.11) and (8.12) in Eq. (8.10),

$$A_1 = 2h\, y_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0''' + \frac{5h^5}{18} y_0^{\text{iv}} + \cdots \qquad ...(8.13)$$

Subtracting Eq. (8.13) from Eq. (8.9),

$$\int_{x_0}^{x_2} y\, \mathrm{d}x - A_1 = \left(\frac{4}{15} - \frac{5}{18} \right) h^5\, y_0^{\text{iv}} + \cdots$$

$$= -\frac{1}{90} h^5 y_0^{\text{iv}} + \cdots$$

Hence, the error in the interval (x_0, x_2), neglecting higher powers of h, is $-\frac{1}{90} h^5\, y_0^{\text{iv}} + \cdots$

Similarly, the error in the interval (x_2, x_4) is $-\frac{1}{90} h^5 y_2^{\text{iv}}$.

Hence, the total error is

$$E = -\frac{1}{90} h^5 \left(y_0^{\text{iv}} + y_2^{\text{iv}} + \cdots \right)$$

Let $y^{\text{iv}}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_2^{\text{iv}}, \ldots, y_{2n-2}^{\text{iv}}$ where $x_0 < \xi < x_{2n}$.

$$E < -\frac{1}{90} nh^5\, y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{2n} - x_0)}{180} h^4\, y^{\text{iv}}(\xi) \qquad [\because \quad 2nh = x_{2n} - x_0]$$

Example 1

Consider the following values:

x	10	11	12	13	14	15	16
y	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Find $\int_{10}^{16} y\,dx$ *by Simpson's* 1/3 *rule.*

Solution

$$a = 10, \quad b = 16, \quad h = 1$$

By Simpson's 1/3 rule,

$$\int_{10}^{16} y\,dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)\right]$$

$$= 4.7233$$

Example 2

A rocket is launched from the ground. Its acceleration is registered during the first 80 *seconds and is given as follows:*

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

By Simpson's 1/3 *rule, find the velocity at* $t = 80$ *s.*

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

By Simpson's 1/3 rule,

$$\text{Velocity} = \int_0^{80} a\,dt$$

$$= \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

Example 3

A river is 80 *metres wide. The depth 'd' in metres at a distance x metres from one bank is given by the following table. Calculate the area of cross section of the river using Simpson's* 1/3 *rule.* **[Summer 2015]**

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	7

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

$$A = \int_0^{80} y\,\mathrm{d}x$$

By Simpson's 1/3 rule,

$$\int_0^{80} y\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0 + y_8\right) + 4\left(y_1 + y_3 + y_5 + y_7\right) + 2\left(y_2 + y_4 + y_6\right)\right]$$

$$= \frac{10}{3}\left[\left(0+7\right) + 4\left(4+9+15+8\right) + 2\left(7+12+14\right)\right]$$

$$= 723.33\ \mathrm{m}^2$$

Example 4

Evaluate $\int_0^6 \frac{1}{1+x}\,\mathrm{d}x$ *taking* $h = 1$ *using Simpson's* 1/3 *rule. Hence, obtain an approximate value of* $\log 7$. **[Winter 2013]**

Solution

$$a = 0, \quad b = 6, \quad h = 1$$

$$n = \frac{b-a}{h} = \frac{6-0}{1} = 6$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.1429)+4(0.5+0.25+0.1667)+2(0.3333+0.2)\right]$$

$$= 1.9588 \qquad \ldots(1)$$

By direct integration,

$$\int_0^6 \frac{1}{1+x}\,dx = \left|\log(1+x)\right|_0^6 = \log 7 \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\log 7 = 1.9588$$

Example 5

Evaluate $\int_0^5 \frac{dx}{4x+5}$ *by using Simpson's 1/3 rule, taking* 10 *equal parts. Hence, find the approximate value of* $\log_e 5$.

Solution

$$a = 0, \quad b = 5, \quad n = 10$$

$$h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$$

$$y = f(x) = \frac{1}{4x+5}$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.2	0.1428	0.1111	0.0910	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Simpson's 1/3 rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{0.5}{3}\left[(0.2+0.04)+4(0.1428+0.0910+0.0667+0.0526+0.0435)\right.$$

$$\left.+\,2(0.1111+0.0769+0.0588+0.0476)\right]$$

$$= 0.4026 \qquad \ldots(1)$$

By the direct method,

$$\int_0^5 \frac{dx}{4x+5} = \left| \frac{\log_e(4x+5)}{4} \right|_0^5$$

$$= \frac{1}{4}(\log_e 25 - \log_e 5)$$

$$= \frac{1}{4}\log_e \frac{25}{5}$$

$$= \frac{1}{4}\log_e 5 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$\frac{1}{4}\log_e 5 = 0.4026$$

$$\log_e 5 = 1.6104$$

Example 6

Evaluate the integral $\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\, dx$ *by Simpson's* 1/3 *rule with taking* 6 *sub-intervals. Use four digits after the decimal point for calculations.* **[Winter 2012]**

Solution

$$a = -2, \quad b = 6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{6-(-2)}{6} = \frac{4}{3}$$

$$y = f(x) = (1+x^2)^{\frac{3}{2}}$$

x	-2	$-\frac{2}{3}$	$\frac{2}{3}$	2	$\frac{10}{3}$	$\frac{14}{3}$	6
$f(x)$	11.1803	1.7360	1.7360	11.1803	42.1479	108.7094	225.0622
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{4}{9}\left[(11.1803+225.0622)+4(1.7360+11.1803+108.7094)\right.$$

$$\left.+2(1.7360+42.1479)\right]$$

$$= 360.2280$$

Example 7

Using Simpson's 1/3 *rule, find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking* $n = 6$.

[Summer 2015]

Solution

$$a = 0, \quad b = 0.6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

$$y = f(x) = e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6977)+4(0.99+0.9139+0.7788)+2(0.9608+0.8521)\right]$$

$$= 0.5351$$

Example 8

Estimate $\int_0^3 \cos^2 x\,dx$ *by using Simpson's* 1/3 *rule with* 6 *intervals.*

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \cos^2 x$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	1	0.9999	0.9996	0.9993	0.9988	0.9981	0.9973
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^3 \cos^2 x \, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.5}{3}\left[(1 + 0.9973) + 4(0.9999 + 0.9993 + 0.9981) + 2(0.9996 + 0.9988)\right]$$

$$= 2.9978$$

Example 9

Compute the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx$ *for* $n = 6$ *with an accuracy to five decimal places using Simpson's* 1/3 *rule.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(x) = \sqrt{\sin x}$$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	0	0.5087	0.7071	0.8409	0.9306	0.9828	1.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 4(0.5087 + 0.8409 + 0.9828) + 2(0.7071 + 0.9306)\right]$$

$$= 1.1873$$

Example 10

The speed v metres per second, of a car, t seconds after it starts, is shown in the following table:

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	4.50	4.5	5.4	9.0

Using Simpson's 1/3 rule, find the distance travelled by the car in 2 minutes.

Solution

Let s (metres) distance be travelled in t (seconds).

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\, dt$$

$$s = \int v\, dt$$

The distance travelled in 2 minutes i.e., 120 seconds is

$$s = \int_0^{120} v\, dt$$

Also, $h = 12$ seconds

By Simpson's 1/3 rule,

$$\int_0^{120} v\, dt = \frac{h}{3}\left[\left(y_0 + y_{10}\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right) + 2\left(y_2 + y_4 + y_6 + y_8\right)\right]$$

$$= \frac{12}{3}\left[(0 + 9.0) + 4(3.60 + 18.90 + 18.54 + 4.50 + 5.4) + 2(10.08 + 21.60 + 10.26 + 4.5)\right]$$

$$= 1222.56 \text{ metres}$$

8.5 SIMPSON'S 3/8 RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \ldots(8.14)$$

Putting $n = 3$ in Eq. (8.14) and ignoring the differences of order higher than 3,

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[\left(y_0 + y_n\right) + 2\left(y_3 + y_6 + \cdots + y_{n-3}\right) + 3\left(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}\right)\right]$$

$$= \frac{3h}{8}\left[X + 2T + 3R\right]$$

where X = extreme terms, T = multiple of three terms, R = remaining terms
This is known as Simpson's 3/8 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 3.

Errors in Simpson's 3/8 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)\,y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{\text{iv}} + \cdots \quad \ldots(8.15)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_3} y\,dx = \int_{x_0}^{x_3}\left[y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{iv} + \cdots\right]$$

$$= \left|y_0 x + \frac{(x-x_0)^2}{2!}\,y_0' + \frac{(x-x_0)^3}{3!}\,y_0'' + \frac{(x-x_0)^4}{4!}\,y_0''' + \frac{(x-x_0)^5}{5!}\,y_0^{iv} + \cdots\right|_{x_0}^{x_3}$$

$$= y_0(x_3 - x_0) + \frac{(x_3-x_0)^2}{2!}\,y_0' + \frac{(x_3-x_0)^3}{3!}\,y_0'' + \frac{(x_3-x_0)^4}{4!}\,y_0''' + \frac{(x_3-x_0)^5}{5!}\,y_0^{iv} + \cdots$$

$$= 3h\,y_0 + \frac{9h^2}{2!}\,y_0' + \frac{27h^3}{3!}\,y_0'' + \frac{81h^4}{4!}\,y_0''' + \frac{243h^5}{5!}\,y_0^{iv} + \cdots \quad ...(8.16)$$

where $x_3 - x_0 = 3h$

Also, $\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3) = \text{Area in the interval } (x_0, x_3) = A_1$ ...(8.17)

Putting $x = x_1$ in Eq. (8.15),

$$y(x_1) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \frac{h^4}{4!}\,y_0^{iv} + \cdots \quad ...(8.18)$$

Putting $x = x_2$ in Eq. (8.16),

$$y(x_2) = y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}\,y_0'' + \frac{8h^3}{3!}\,y_0''' + \frac{16h^4}{4!}\,y_0^{iv} + \cdots \quad ...(8.19)$$

Putting $x = x_3$ in Eq. (8.17),

$$y(x_3) = y_3 = y_0 + 3h\,y_0' + \frac{9h^2}{2!}\,y_0'' + \frac{27h^3}{3!}\,y_0''' + \frac{81h^4}{4!}\,y_0^{iv} + \cdots \quad ...(8.20)$$

Substituting Eqs (8.18), (8.19) and (8.20) in Eq. (8.17),

$$A_1 = 3h\,y_0 + \frac{9h^2}{2!}\,y_0' + \frac{27h^3}{3!}\,y_0'' + \frac{81h^4}{4!}\,y_0''' + \frac{33h^5}{16}\,y^{iv} + \cdots \quad ...(8.21)$$

Subtracting Eq. (8.21) from Eq. (8.16),

$$\int_{x_0}^{x_3} y\,dx - A_1 = \left(\frac{81}{40} - \frac{33}{16}\right)h^5\,y_0^{iv} + \cdots$$

$$= -\frac{3}{80}h^5\,y_0^{iv} + \cdots$$

Hence, the error in the interval (x_0, x_3), neglecting higher powers of h, is $-\frac{3}{80}h^5\,y_0^{iv}$.

Similarly, the error in the interval (x_3, x_6) is $-\frac{3}{80}h^5\,y_3^{iv}$.

Hence, the total error is

$$E = -\frac{3}{80} h^5 \left(y_0^{\text{iv}} + y_3^{\text{iv}} + \cdots + y_{3n-3}^{\text{iv}} \right)$$

Let $y^{iv}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_3^{\text{iv}}, ..., y_{n-3}^{\text{iv}}$ where $x_0 < \xi < x_{3n}$.

$$E < -\frac{3}{80} nh^5 \, y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{3n} - x_0)}{80} h^4 y^{\text{iv}}(\xi) \qquad \left[\because \quad 3nh = x_{3n} - x_0 \right]$$

Example 1

Evaluate $\int_0^3 \frac{1}{1+x} \mathrm{d}x$ *with* $n = 6$ *by using Simpson's* 3/8 *rule and, hence, calculate* log 2. **[Summer 2014]**

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^3 \frac{1}{1+x} \mathrm{d}x = \frac{3h}{8} \left[\left(y_0 + y_6 \right) + 2\left(y_3 \right) + 3\left(y_1 + y_2 + y_4 + y_5 \right) \right]$$

$$= \frac{3(0.5)}{8} \left[(1 + 0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2857) \right]$$

$$= 1.3888 \qquad \text{...(1)}$$

By direct integration,

$$\int_0^3 \frac{1}{1+x} \mathrm{d}x = \left| \log(1+x) \right|_0^3$$

$$= \log 4$$

$$= \log (2)^2$$

$$= 2 \log 2 \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$2 \log 2 = 1.3888$$

$$\log 2 = 0.6944$$

Example 2

Evaluate $\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\,dx$ *by using Simpson's 3/8 rule.*

Solution

$$a = 0, \quad b = \pi$$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

$$y = f(x) = \frac{\sin^2 x}{5 + 4\cos x}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$f(x)$	0	0.02954	0.10714	0.2	0.25	0.16277	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{6}\right)\left[(0+0) + 2(0.2) + 3(0.02954 + 0.10714 + 0.25 + 0.16277)\right]$$

$$= \frac{\pi}{16}(2.04835)$$

$$= 0.40219$$

Example 3

Find $\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\,dt$ *using one of the methods of numerical integration.*

Solution

Dividing the interval $\left[0, \frac{\pi}{2}\right]$ into six equal parts and applying Simpson's 3/8 rule,

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(t) = \sqrt{1 - \frac{1}{2}\sin^2 t}$$

t	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	1	0.9831	0.9354	0.8660	0.7906	0.7304	0.7071
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\ \mathrm{d}t = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)\left[(1 + 0.7071) + 2(0.8660) + 3(0.9831 + 0.9354 + 0.7906 + 0.7304)\right]$$

$$= 1.3496$$

Example 4

Find $\int_0^{\frac{\pi}{2}} e^{\sin\theta}\mathrm{d}\theta$ *by Simpson's 3/8 rule, dividing the interval* $\left[0, \frac{\pi}{12}\right]$ *into six equal parts.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(\theta) = e^{\sin\theta}$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(\theta)$	1	1.2953	1.6487	2.0281	2.3773	2.6247	2.7182
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin\theta}\,d\theta = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)\left[(1+2.7182)+2(2.0281)\right.$$
$$\left.+3(1.2953+1.6487+2.3773+2.6247)\right]$$
$$= 3.1012$$

Example 5

By Simpson's 3/8 rule, evaluate $\int_0^1 \frac{\sin x}{x}$ *taking* $h = \frac{1}{6}$.

Solution

$$a = 0, \quad b = 1, \quad h = \frac{1}{6}$$

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$$y = f(x) = \frac{\sin x}{x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	0.9954	0.9816	0.9589	0.9276	0.8882	0.8415
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^1 \frac{\sin x}{x}\,dx = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[(1 + 0.8415) + 2(0.9589) + 3(0.9954 + 0.9816 + 0.9276 + 0.8882)\right]$$

$$= 0.9461$$

Example 6

The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in km/h.

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0

Estimate approximately the distance covered in 18 minutes by Simpson's 3/8 rule.

Solution

Let s km distance be covered in t minutes.

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\,dt$$

$$s = \int v\,dt$$

The distance covered in 18 minutes is

$$s = \int_0^{18} v\,dt$$

Since the train starts from rest, at $t = 0$, $v = 0$ $\quad \therefore\ y_0 = 0$

Time (t)	0	3	6	9	12	15	18
Velocity (v)	0	22	29	31	20	4	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Also, $h = 3$ minutes $= \frac{3}{60} = \frac{1}{20}$ hours

By Simpson's 3/8 rule,

$$\int_0^{18} v\,dt = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{20}\right)\left[(0+0)+2(31)+3(22+29+20+4)\right]$$
$$= 5.38125$$

Example 7

Find the volume of a solid of revolution formed by rotating about the x-axis the area bounded by the lines $x = 0$, $x = 1.5$, $y = 0$, and the curve passing through the following points:

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

Solution

Volume is given by

$$V = \int \pi y^2 \, dx$$

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y^2	1.00	0.9655	0.9195	0.8261	0.7081	0.5812	0.5759
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = 0.25$$

By Simpson's 3/8 rule,

$$\int y^2 \, dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$
$$= \frac{3(0.25)}{8}\left[(1.00 + 0.5759) + 2(0.8261) + 3(0.9655 + 0.9195 + 0.7081 + 0.5812)\right]$$
$$= 1.1954$$
$$\text{Volume} = \pi \int y^2 \, dx$$
$$= \pi(1.1954)$$
$$= 3.7555$$

Example 8

Evaluate $\int_4^{5.2} \log x$ *using the trapezoidal rule and Simpson's* 3/8 *rule, take* $h = 0.2$.

Solution

$$a = 4, \quad b = 5.2, \quad h = 0.2$$

$$n = \frac{b-a}{h} = \frac{5.2-4}{0.2} = 6$$

$$y = f(x) = \log x$$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By the trapezoidal rule,

$$\int_4^{5.2} \log\, x\,dx = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= \frac{0.2}{2}\left[(1.3863 + 1.6487) + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)\right]$$

$$= 1.8277$$

By Simpson's 3/8 rule,

$$\int_4^{5.2} \log dx = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3(0.2)}{8}\left[(1.3863 + 1.6487) + 2(1.5261) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094)\right]$$

$$= 1.8278$$

Example 9

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ *taking* $h = \frac{1}{6}$ *using Simpson's* 3/8 *rule and the trapezoidal rule.*

Solution

$$a = 0, \quad b = 1, \quad h = \frac{1}{6}$$

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{1}{2}\right)+2\left(\frac{4}{5}\right)+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= 0.7854$$

By the trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= 0.7842$$

Example 10

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) trapezoidal rule, (ii) Simpson's* 1/3 *rule, (iii) Simpson's* 3/8 *rule.* **[Summer 2014]**

Solution

$a = 0, b = 6$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By the trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027) + 2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= 1.4108$$

(ii) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027) + 4(0.5+0.1+0.0385) + 2(0.2+0.0588)\right]$$

$$= 1.3662$$

(iii) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left[(1+0.027) + 2(0.1) + 3(0.5+0.2+0.0588+0.0385)\right]$$

$$= 1.3571$$

EXERCISE 8.1

1. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ taking $h = 0.2$, using trapezoidal rule.

[Ans.: 0.3228]

2. Evaluate the value of $\int_0^{0.3} \sqrt{1-8x^3}\, dx$ using Simpson's 3/8 rule.

[Ans.: 0.2916]

3. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$ by Simpson's 3/8 rule.

[Ans.: 3.1044]

4. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) trapezoidal rule, (ii) Simpson's 1/3 rule, and (iii) Simpson's 3/8 rule. Take $h = 0.25$.

[Ans.: (i) 0.6970 (ii) 0.6932 (iii) 0.6932]

5. Calculate $\int_0^{\frac{\pi}{2}} \sin x\,dx$ by dividing the interval into ten equal parts, using the trapezoidal rule and Simpson's 1/3 rule. **[Ans.:** 0.9981, 1.0006]

6. Find the value of $\log 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2}{1+x^3}$ using Simpson's 1/3 rule with $h = 0.25$. **[Ans.:** 0.2311]

7. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx$ taking $h = 0.2$ and using the trapezoidal rule, and Simpson's rule. **[Ans.:** 4.0715, 4.0521]

8. Evaluate $\int_{0.5}^{0.7} \sqrt{x}\, e^{-x}\,dx$ using Simpson's 3/8 rule.

[Ans.: 0.0841]

9. Evaluate $\int_0^1 \frac{dx}{x^3 + x + 1}$ using Simpson's 1/3 rule, taking $h = 0.25$.

[Ans.: 0.6305]

10. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Obtain the area bounded by the curve, the x-axis, and the lines $x = 1$ and $x = 4$ by any method.

[Ans.: 7.7833]

8.6 GAUSSIAN QUADRATURE FORMULAE

An n-point Gaussian quadrature formula is a quadrature formula constructed to give an exact result for polynomials of degree $2n - 1$ or less by a suitable choice of the points x_i and weights w_i for $i = 1, 2, ..., n$. Gauss quadrature formula can be expressed as

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) \qquad ...(8.22)$$

8.6.1 One-point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval $[-1, 1]$ with sampling point x_1 and weight w_1. The one-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) \qquad ...(8.23)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(1) - 1 = 1$, i.e., it is exact for $f(x) = 1$ and x.

Substituting $f(x)$ in Eq. (8.23) successively,

$$\int_{-1}^{1} 1\,dx = w_1$$

$$\left|x\right|_{-1}^{1} = w_1$$

$$2 = w_1 \qquad ...(8.24)$$

$$\int_{-1}^{1} x\,dx = w_1 x_1$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1$$

$$0 = w_1 x_1 \qquad ...(8.25)$$

Solving Eqs (8.24) and (8.25),

$$w_1 = 2$$

$$x_1 = 0$$

Hence, $$\int_{-1}^{1} f(x)\,dx = 2f(0) \qquad ...(8.26)$$

Equation (8.26) is known as *one-point Gaussian quadrature formula*. This formula is exact for polynomials up to degree one.

8.6.2 Two-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2 and weights w_1, w_2 respectively. The two-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) \qquad ...(8.27)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(2) - 1 = 3$, i.e., it is exact for $f(x) = 1, x, x^2$ and x^3.

Substituting $f(x)$ in Eq. (8.27), successively,

$$\int_{-1}^{1} 1\,dx = w_1 + w_2$$

$$|x|_{-1}^{1} = w_1 + w_2$$

$$2 = w_1 + w_2 \qquad ...(8.28)$$

$$\int_{-1}^{1} x\,dx = w_1 x_1 + w_2 x_2$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1 + w_2 x_2$$

$$0 = w_1 x_1 + w_2 x_2 \qquad ...(8.29)$$

$$\int_{-1}^{1} x^2\,dx = w_1 x_1^2 + w_2 x_2^2$$

$$\left|\frac{x^3}{3}\right|_{-1}^{1} = w_1 x_1^2 + w_2 x_2^2$$

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 \qquad ...(8.30)$$

$$\int_{-1}^{1} x^3\,dx = w_1 x_1^3 + w_2 x_2^3$$

$$\left|\frac{x^4}{4}\right|_{-1}^{1} = w_1 x_1^3 + w_2 x_2^3$$

$$0 = w_1 x_1^3 + w_2 x_2^3 \qquad ...(8.31)$$

Solving Eqs (8.28), (8.29), (8.30), and (8.31),

$$w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}}$$

Hence,
$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \qquad ...(8.32)$$

Equation (8.32) is known as the two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

8.6.3 Three-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2, x_3 and weights w_1, w_2, w_3 respectively. The three-point Gaussian Quadrature formula is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \qquad ...(8.33)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2\,(3) - 1 = 5$, i.e., it is exact for $f(x) = 1, x, x^2, x^3, x^4$ and x^5.

Substituting $f(x)$ in Eq. (8.33) successively,

$$\int_{-1}^{1} 1\,dx = w_1 + w_2 + w_3$$

$$|x|_{-1}^{1} = w_1 + w_2 + w_3$$

$$0 = w_1 + w_2 + w_3 \qquad ...(8.34)$$

$$\int_{-1}^{1} x\,dx = w_1 x_1 + w_2 x_2 + w_3 x_3$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1 + w_2 x_2 + w_3 x_3$$

$$0 = w_1 x_1 + w_2 x_2 + w_3 x_3 \qquad ...(8.35)$$

$$\int_{-1}^{1} x^2\,dx = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2$$

$$\left|\frac{x^3}{3}\right|_{-1}^{1} = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2$$

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 \qquad ...(8.36)$$

$$\int_{-1}^{1} x^3 \, dx = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3$$

$$\left| \frac{x^4}{4} \right|_{-1}^{1} = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3$$

$$0 = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3 \qquad \text{...(8.37)}$$

$$\int_{-1}^{1} x^4 \, dx = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4$$

$$\left| \frac{x^5}{5} \right|_{-1}^{1} = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4$$

$$\frac{2}{5} = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4 \qquad \text{...(8.38)}$$

$$\int_{-1}^{1} x^5 \, dx = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5$$

$$\left| \frac{x^6}{6} \right|_{-1}^{1} = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5$$

$$0 = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5 \qquad \text{...(8.39)}$$

Solving Eqs (8.34), (8.35), (8.36), (8.37), (8.38), and (8.39),

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}$$

$$x_1 = -\sqrt{\frac{3}{5}}, \; x_2 = 0, \; x_3 = \sqrt{\frac{3}{5}}$$

Hence, $$\int_{-1}^{1} f(x) \, dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \qquad \text{...(8.40)}$$

Equation (8.40) is known as the three-point Gaussian quadrature formula. This formula is exact for polynomials up to degree 5.

Example 1

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ *by one-point, two-point, and three-point Gaussian formulae.*

Solution

$$f(x) = \frac{1}{1+x^2}$$

By the one-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 2f(0)$$
$$= 2\left(\frac{1}{1+0}\right)$$
$$= 2$$

By the two-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}}$$
$$= 1.5$$

By the three-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$
$$= \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right) + \frac{8}{9}\left(\frac{1}{1+0}\right) + \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right)$$
$$= 1.5833$$

Example 2

Evaluate $\int_0^1 \frac{dt}{1+t}$ *by one-point, two-point, and three-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0, \quad b = 1$

$$t = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(x+1)$$

$$dt = \frac{1}{2}\,dx$$

When $t = 0, \quad x = -1$

When $t = 1, \quad x = 1$

$$\int_0^1 \frac{dt}{1+t} = \frac{1}{2}\int_{-1}^1 \frac{dx}{1+\frac{1}{2}(x+1)}$$

$$= \int_{-1}^1 \frac{dx}{x+3}$$

$$f(x) = \frac{1}{x+3}$$

By the one-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^1 \frac{dx}{x+3}$$

$$= 2f(0)$$

$$= 2\left(\frac{1}{0+3}\right)$$

$$= 0.6667$$

By the two-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^1 \frac{dx}{x+3}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{-\sqrt{\frac{1}{3}}+3} + \frac{1}{\sqrt{\frac{1}{3}}+3}$$

$$= 0.6923$$

By the three-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^1 \frac{dx}{x+3}$$

$$= \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\left(\frac{1}{-\sqrt{\frac{3}{5}}+3}\right)+\frac{8}{9}\left(\frac{1}{0+3}\right)+\frac{5}{9}\left(\frac{1}{\sqrt{\frac{3}{5}}+3}\right)$$

$$= 0.6931$$

Example 3

Evaluate the integral $\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx$ *by the Gaussian formula for* $n = 3$.

[Winter 2012]

Solution

Let $x = \frac{b-a}{2}t+\frac{b+a}{2}$

Here, $a = -2, \quad b = 6$

$$x = 4t+2$$

$$dx = 4dt$$

When $x = -2, \quad t = -1$

When $x = 6, \quad t = 1$

$$\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx = \int_{-1}^{1}\left[1+(4t+2)^2\right]^{\frac{3}{2}}4\,dt$$

$$= 4\int_{-1}^{1}(16t^2+16t+5)^{\frac{3}{2}}\,dt$$

$$f(t) = (16\,t^2+16t+5)^{\frac{3}{2}}$$

By the three-point Gaussian formula,

$$\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx = 4\int_{-1}^{1}(16t^2+16t+5)^{\frac{3}{2}}\,dt$$

$$= 4\left[\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right)+\frac{8}{9}f(0)+\frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= 4\left[\frac{5}{9}\left\{16\left(\frac{3}{5}\right)+16\left(-\sqrt{\frac{3}{5}}\right)+5\right\}^{\frac{3}{2}}+\frac{8}{9}(5)^{\frac{3}{2}}+\frac{5}{9}\left\{16\left(\frac{3}{5}\right)+16\left(\sqrt{\frac{3}{5}}\right)+5\right\}^{\frac{3}{2}}\right]$$

$$= 358.6928$$

Example 4

Evaluate $\int_0^1 e^{-x^2}\,dx$ *by using the Gaussian quadrature formula with* $n = 3$. **[Winter 2014, Summer 2015]**

Solution

Let $$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Here, $a = 0, \quad b = 1$

$$x = \frac{1}{2}t + \frac{1}{2} = \frac{1}{2}(t+1)$$

$$dx = \frac{1}{2}dt$$

When $x = 0, \quad t = -1$

When $x = 1, \quad t = 1$

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\int_{-1}^1 e^{-\frac{1}{4}(t+1)^2}\,dt$$

$$f(x) = e^{-\frac{1}{4}(t+1)^2}$$

By the three-point Gaussian quadrature formula,

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\int_{-1}^1 e^{-\frac{1}{4}(t+1)^2}\,dt$$

$$= \frac{1}{2}\left[\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}e^{-\frac{1}{4}\left(-\sqrt{\frac{3}{5}}+1\right)^2} + \frac{8}{9}e^{-\frac{1}{4}(0+1)^2} + \frac{5}{9}e^{-\frac{1}{4}\left(\sqrt{\frac{3}{5}}+1\right)^2}\right]$$

$$= 0.746815$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} \sin t\,dt$ *by the two-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0, \quad b = \frac{\pi}{2}$

$$t = \frac{\pi}{4}x + \frac{\pi}{4} = \frac{\pi}{4}(x+1)$$

$$\mathrm{d}t = \frac{\pi}{4}\,\mathrm{d}x$$

When $t = 0, \quad x = -1$

When $t = \frac{\pi}{2}, \quad x = 1$

$$\int_0^{\frac{\pi}{2}} \sin t \, \mathrm{d}t = \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x$$

$$f(x) = \sin\frac{\pi}{4}(x+1)$$

By the two-point Gaussian formula,

$$\int_0^{\frac{\pi}{2}} \sin t \, \mathrm{d}t = \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x$$

$$= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{4}\left[\sin\frac{\pi}{4}\left(-\frac{1}{\sqrt{3}}+1\right) + \sin\frac{\pi}{4}\left(\frac{1}{\sqrt{3}}+1\right)\right]$$

$$= 0.99847$$

EXERCISE 8.2

Evaluate the following integrals by using Gaussian quadrature formulae:

1. $\int_0^1 e^x \, \mathrm{d}x$ (2 points)

[Ans.: 2.342696**]**

2. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$ (2 points)

[Ans.: 1.311028**]**

3. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\sin^2\theta}}$ (2 points)

[Ans.: 1.226**]**

4. $\int_0^{\frac{\pi}{2}} \log(1+x)\,dx$ (2 points)

[Ans.: 0.858**]**

5. $\int_0^3 x^2 \cos x\,dx$ (3 points)

[Ans.: −4.936**]**

6. $\int_1^2 e^x\,dx$ (3 points)

[Ans.: 4.67077**]**

Points to Remember

Newton–Cotes Quadrature Formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1})\right]$$

Gaussian Quadrature Formulae

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

1. One-point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

2. Two-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

3. Three-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

CHAPTER 9

Solutions of a System of Linear Equations

Chapter Outline

9.1 Introduction
9.2 Solutions of a System of Linear Equations
9.3 Elementary Transformations
9.4 Numerical Methods for Solution of a System of Linear Equations
9.5 Gauss Elimination Method
9.6 Gauss Elimination Method with Partial Pivoting
9.7 Gauss–Jordan Method
9.8 Gauss–Jacobi Method
9.9 Gauss–Siedel Method

9.1 INTRODUCTION

A system of m nonhomogenous linear equations in n variables $x_1, x_2, ..., x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \quad \vdots \qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the *coefficient matrix* of order $m \times n$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ is any vector of order } n \times 1$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ is any vector of order } m \times 1$$

9.2 SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has a unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix $$[A : B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the *augmented matrix* of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to echelon form.

9.3 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any nonzero number
(iii) The addition or subtraction of k times the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij}: Interchange of i^{th} and j^{th} row
(ii) kR_i: Multiplication of i^{th} row by a nonzero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i, and $C_i + kC_j$ respectively.

9.3.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

9.3.2 Equivalence of Matrices

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A then A is equivalent to B. Symbolically, we can write $A \sim B$.

9.3.3 Echelon Form of a Matrix

A matrix A is said to be in echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a nonzero row.

(ii) In a nonzero row the first nonzero number from the left is 1. This is called a leading 1.

(iii) For each nonzero row, the leading 1 appears to the right of any leading 1 in preceding rows.

The following matrices are in echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

9.4 NUMERICAL METHODS FOR SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

There are two methods to solve linear algebraic equations:

(i) Direct methods

(ii) Iterative methods

9.4.1 Direct Methods

Direct methods transform the original equations into equivalent equations that can be solved easily. The transformation of the original equations is carried out by applying elementary row transformations to the augmented matrix of the system of equations.

We will discuss two direct methods:

(i) Gauss elimination method

(ii) Gauss–Jordan method

9.4.2 Iterative Methods

The direct methods lead to exact solutions in many cases but are subject to errors due to roundoff and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.

We will discuss two iterative methods.

(i) Gauss–Jacobi method

(ii) Gauss–Seidel method

9.5 GAUSS ELIMINATION METHOD

This method solves a given system of equations by transforming the augmented matrix to an echelon form. The corresponding linear system of equations is then solved for the unknowns by back substitution.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Reducing the augmented matrix to echelon form by using elementary row transformations,

$$[A : B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right]$$

The corresponding system of equations is

$$c_{11}x + c_{12}y + c_{13}z = d_1$$

$$c_{22}y + c_{23}z = d_2$$
$$c_{33}z = d_3$$

The solution of the system is obtained by solving these equations by back substitution.

Working Rule

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.
(iv) Write the corresponding linear system of equations from the echelon form.
(v) Solve the corresponding linear system of equations by back substitution.

Example 1

Solve the following system of equations:

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$
$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - 2R_1, R_3 - R_1$$
$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 3y + 2z &= 5 \\ y + 5z &= 7 \\ -9z &= -9 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} z &= 1 \\ y &= 7 - 5z = 7 - 5(1) = 2 \\ x &= 5 - 3y - 2z = 5 - 3(2) - 2(1) = -3 \end{aligned}$$

Hence, the solution is

$$x = -3,\ y = 2,\ z = 1$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_{13}$$

$$[A:B]\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$$R_2-3R_1, R_3-2R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$$\left(-\frac{1}{10}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$$R_3+7R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & 0 & -\frac{1}{5} & -1 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x+4y+\ 9z &= 16 \\ y+\frac{24}{10}z &= 3 \\ -\ \frac{1}{5}z &= -1 \end{aligned}$$

Solving these equations by back substitution,

$$z = 5$$

$$y = 3 - \frac{24}{10}\ z = 3 - \frac{24}{10}(5) = -9$$

$$x = 16 - 4y - 9z = 16 - 4(-9) - 9(5) = 7$$

Hence, the solution is

$$x = 7,\ y = -9,\ z = 5$$

Example 3

Solve the following system of equations:

$$\begin{aligned} 6x - \ y - \ z &= 19 \\ 3x + 4y + \ z &= 26 \\ x + 2y + 6z &= 22 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 6 & -1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 22 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 6 & -1 & -1 & 19 \\ 3 & 4 & 1 & 26 \\ 1 & 2 & 6 & 22 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$[A : B] \sim \overset{R_{13}}{\left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 3 & 4 & 1 & 26 \\ 6 & -1 & -1 & 19 \end{array}\right]}$$

$$R_2 - 3R_1,\ R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & -2 & -17 & -40 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & 0 & \frac{147}{2} & 147 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 2y + \quad 6z &= 22 \\ y + \frac{17}{2}z &= 20 \\ \frac{147}{2}z &= 147 \end{aligned}$$

Solving these equations by back substitution,

$$z = 2$$

$$y = 20 - \frac{17}{2}z = 20 - \frac{17}{2}(2) = 3$$

$$x = 22 - 2y - 6z = 22 - 2(3) - 6(2) = 4$$

Hence, the solution is

$$x = 4,\ y = 3,\ z = 2$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 5x + 5y + 2z &= 12 \\ 2x + 4y + 5z &= 2 \\ 39x + 43y + 45z &= 74 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 5 & 5 & 2 \\ 2 & 4 & 5 \\ 39 & 43 & 45 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 74 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 5 & 2 & 12 \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\left(\frac{1}{5}\right)R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - 39R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 2 & \frac{21}{5} & -\frac{14}{5} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$$R_3 - 4R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 0 & 21 & -14 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y + \frac{2}{5}z &= \frac{12}{5} \\ y + \frac{21}{10}z &= -\frac{14}{10} \\ 21z &= -14 \end{aligned}$$

Solving these equations by back substitution,

$$z = -\frac{14}{21} = -\frac{2}{3}$$

$$y = -\frac{14}{10} - \frac{21}{10}z = -\frac{14}{10} - \frac{21}{10}\left(-\frac{2}{3}\right) = 0$$

$$x = \frac{12}{5} - y - \frac{2}{5}z = \frac{12}{5} - \frac{2}{5}\left(-\frac{2}{3}\right) = \frac{8}{3}$$

Hence, the solution is

$$x = \frac{8}{3},\ y = 0,\ z = -\frac{2}{3}$$

Example 5

Use the Gauss elimination method to solve the following equations:

$$\begin{aligned} x + 4y - z &= -5 \\ x + y - 6z &= -12 \\ 3x - y - z &= 4 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - R_1,\ R_3 - 3R_1$$

$$[A : B] \sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & \frac{71}{3} & \frac{148}{3} \end{array}\right]$$

The corresponding system of equations is

$$x + 4y - z = -5$$

$$y + \frac{5}{3}z = \frac{7}{3}$$

$$\frac{71}{3}z = \frac{148}{3}$$

Solving these equations by back substitution,

$$z = \frac{148}{71}$$

$$y = \frac{7}{3} - \frac{5}{3}z = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71}$$

$$x = -5 - 4y + z = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71}$$

Hence, the solution is

$$x = \frac{117}{71}, \quad y = -\frac{81}{71}, \quad z = \frac{148}{71}$$

Example 6

Solve the following system of linear equations:

$$8y + 2z = -7$$

$$3x + 5y + 2z = 8$$

$$6x + 2y + 8z = 26$$

[Summer 2014]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 5 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 26 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$[A : B] \overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$R_3 - 6R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$\left(\frac{1}{8}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$R_3 + 8R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & 0 & 6 & 3 \end{array}\right]$$

The corresponding system of equations is

$$x + \frac{5}{3}y + \frac{2}{3}z = \frac{8}{3}$$

$$y + \frac{1}{4}z = -\frac{7}{8}$$

$$6z = 3$$

Solving these equations by back substitution,

$$z = \frac{1}{2}$$

$$y = -\frac{7}{8} - \frac{1}{4}z = -\frac{7}{8} - \frac{1}{4}\left(\frac{1}{2}\right) = -1$$

$$x = \frac{8}{3} - \frac{5}{3}y - \frac{2}{3}z = \frac{8}{3} - \frac{5}{3}(-1) - \frac{2}{3}\left(\frac{1}{2}\right) = 4$$

Hence, the solution is

$$x = 4, \quad y = -1, \quad z = \frac{1}{2}$$

9.6 GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING

For a large system of linear equations, the Gaussian elimination method can involve a large number of arithmetic computations, each of which can produce rounding errors. This is due to the fact that every computation is dependent on previous results.

Consequently, an error in the early step will tend to propagate, i.e., it will cause errors in subsequent steps, and the final solution will become inaccurate. The rounding error can be reduced by the Gaussian elimination method with partial pivoting.

Consider the system of equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

For the partial pivoting process, the left column is searched for the largest absolute-value entry. This entry is called the *pivot*. The row interchange is performed, if necessary, to bring the pivot in the first row. The first row is divided by the pivot and elementary row operations are used to reduce the remaining entries in the first column to zero. The completion of these steps is called a *pass*. After performing the first pass, the first row and first column are ignored and the process is repeated on the remaining submatrix. This process is continued until the matrix is in the row echelon form.

The term *partial* in partial pivoting refers to the fact that in each pivot search, only entries in the left column of the matrix or submatrix are considered. This search can be extended to include every entry in the coefficient matrix or submatrix. The resulting method is called the *Gaussian elimination method with complete pivoting*. Generally, partial pivoting is preferred because complete pivoting becomes very complicated.

Example 1

Solve the following system of equations using partial pivoting by the Gauss elimination method:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 6 \\ 4x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

In the left column, 4 is the pivot because it is the entry that has the largest absolute value.

$$[A : B] \overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{\left(\frac{1}{4}\right)R_1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & \frac{1}{2} & \frac{1}{4} & -1 \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is 1 in the submatrix formed by deleting the first row and first column.

$$R_3 - \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & -3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 &= 1 \\ x_2 - \frac{1}{2}x_3 &= 4 \\ \frac{1}{2}x_3 &= -3 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} x_3 &= -6 \\ x_2 &= 4 + \frac{1}{2}x_3 = 4 + \frac{1}{2}(-6) = 1 \\ x_1 &= 1 - \frac{1}{2}x_2 - \frac{3}{4}x_3 = 1 - \frac{1}{2}(1) - \frac{3}{4}(-6) = 5 \end{aligned}$$

Hence, the solution is

$$x_1 = 5, \qquad x_2 = 1, \qquad x_3 = -6$$

Example 2

Solve the following system of equations using the Gauss elimination method with partial pivoting.

$$\begin{aligned} x+\ \ y+\ \ z&=7\\ 3x+3y+4z&=24\\ 2x+\ \ y+3z&=16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x}=B$$

$$\begin{bmatrix}1&1&1\\3&3&4\\2&1&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}7\\24\\16\end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc|c}1&1&1&7\\3&3&4&24\\2&1&3&16\end{array}\right]$$

In the left column, 3 is the pivot because it is the entry that has largest absolute value.

$$\overset{R_{12}}{[A:B]\sim\left[\begin{array}{ccc|c}3&3&4&24\\1&1&1&7\\2&1&3&16\end{array}\right]}$$

$$\overset{\left(\frac{1}{3}\right)R_1}{\sim\left[\begin{array}{ccc|c}1&1&\frac{4}{3}&8\\1&1&1&7\\2&1&3&16\end{array}\right]}$$

$$\overset{R_2-R_1,\ R_3-2R_1}{\sim\left[\begin{array}{ccc|c}1&1&\frac{4}{3}&8\\0&0&-\frac{1}{3}&-1\\0&-1&\frac{1}{3}&0\end{array}\right]}$$

This completes the first pass. For the second pass, the pivot is –1 in the submatrix formed by deleting the first row and first column.

$$\overset{R_{23}}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

$$\overset{(-1)R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

The corresponding system of equations is

$$x + y + \frac{4}{3}z = 8$$

$$y - \frac{1}{3}z = 0$$

$$-\frac{1}{3}z = -1$$

Solving these equations by back substitution,

$$z = 3$$

$$y = \frac{1}{3}z = \frac{1}{3}(3) = 1$$

$$x = 8 - y - \frac{4}{3}z = 8 - 1\frac{4}{3}(3) = 3$$

Hence, the solution is

$$x = 3, \qquad y = 1, \qquad z = 3$$

EXERCISE 9.1

Solve the following systems of equations by the Gauss elimination method:

1. $x - y + z = 1$
$-3x + 2y - 3z = -6$
$2x - 5y + 4z = 5$

[**Ans.:** $x = -2, y = 3, z = 6$]

2. $x + 3y - 2z = 5$
$2x + y - 3z = 1$
$3x + 2y - z = 6$

[Ans.: $x = 1, y = 2, z = 1$]

3. $6x + 3y + 6z = 30$
$2x + 3y + 3z = 17$
$x + 2y + 2z = 11$

[Ans.: $x = 1, y = 2, z = 3$]

4. $2x + y + z = 4$
$3y - 3z = 0$
$-y + 2z = 1$

[Ans.: $x = 1, y = 1, z = 1$]

5. $2x + 2y + z = 12$
$3x + 2y + 2z = 8$
$5x + 10y - 8z = 10$

[Ans.: $x = -12.75, y = 14.375, z = 8.75$]

6. $3x + 4y + 5z = 18$
$2x - y + 8z = 13$
$5x - 2y + 7z = 20$

[Ans.: $x = 3, y = 1, z = 1$]

7. $2x + 6y - z = -12$
$5x - y + z = 11$
$4x - y + 3z = 10$

$$\left[\text{Ans.: } x = \frac{113}{69}, y = -\frac{172}{69}, z = \frac{22}{69}\right]$$

9.7 GAUSS–JORDAN METHOD

This method is a modification of the Gauss elimination method. This method solves a given system of equations by transforming the coefficient matrix into a unit matrix.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Applying elementary row transformations to augmented matrix to reduce coefficient matrix to unit matrix,

$$[A : B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$$

The corresponding system of equations is

$$x = d_1$$
$$y = d_2$$
$$z = d_3$$

Hence, the solution is

$$x = d_1,\ y = d_2,\ z = d_3$$

Working Rule

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Example 1

Solve the following system of equations:

$$x + 3y + 2z = 17$$
$$x + 2y + 3z = 16$$
$$2x - y + 4z = 13$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 1 & 2 & 3 & 16 \\ 2 & -1 & 4 & 13 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_2 - R_1, R_3 - 2R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & -1 & 1 & -1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$(-1)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$R_1 - 3R_2, R_3 + 7R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\left(-\frac{1}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - 5R_3, R_2 + R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equations is

$$x = 4$$
$$y = 3$$
$$z = 2$$

Hence, the solution is

$$x = 4, y = 3, z = 2$$

Example 2

Solve the following system of equations:

$$3x - 2y + 5z = 2$$
$$4x + y + 2z = 4$$
$$2x - y + 4z = 7$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$
$$\begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 3 & -2 & 5 & 2 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_1 - R_3$$
$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$
$$R_2 - 4R_1, R_3 - 2R_1$$
$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 0 & 5 & -2 & 24 \\ 0 & 1 & 2 & 17 \end{array}\right]$$

$$\overset{R_{23}}{\sim}\begin{bmatrix} 1 & -1 & 1 & \vdots & -5 \\ 0 & 1 & 2 & \vdots & 17 \\ 0 & 5 & -2 & \vdots & 24 \end{bmatrix}$$

$$\overset{R_1 + R_2, R_3 - 5R_2}{\sim}\begin{bmatrix} 1 & 0 & 3 & \vdots & 12 \\ 0 & 1 & 2 & \vdots & 17 \\ 0 & 0 & -12 & \vdots & -61 \end{bmatrix}$$

$$\overset{\left(-\frac{1}{12}\right)R_3}{\sim}\begin{bmatrix} 1 & 0 & 3 & \vdots & 12 \\ 0 & 1 & 2 & \vdots & 17 \\ 0 & 0 & 1 & \vdots & \frac{61}{12} \end{bmatrix}$$

$$\overset{R_1 - 3R_3, R_2 - 2R_3}{\sim}\begin{bmatrix} 1 & 0 & 0 & \vdots & -\frac{13}{4} \\ 0 & 1 & 0 & \vdots & \frac{41}{6} \\ 0 & 0 & 1 & \vdots & \frac{61}{12} \end{bmatrix}$$

The corresponding system of equations is

$$x = -\frac{13}{4}$$
$$y = \frac{41}{6}$$
$$z = \frac{61}{12}$$

Hence, the solution is

$$x = -\frac{13}{4},\ y = \frac{41}{6},\ z = \frac{61}{12}$$

Example 3

Solve the following system of equations:

$$\begin{aligned} x - 2y &= -4 \\ -5y + z &= -9 \\ 4x - 3z &= -10 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -5 & 1 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ -10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 4 & 0 & -3 & -10 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_3 - 4R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$\left(-\frac{1}{5}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$R_1 + 2R_2, R_3 - 8R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & -\frac{7}{5} & -\frac{42}{5} \end{array}\right]$$

$$\left(-\frac{5}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$R_1 + \left(\frac{2}{5}\right)R_3, R_2 + \left(\frac{1}{5}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

The corresponding system of equations is

$$x = 2$$
$$y = 3$$
$$z = 6$$

Hence, the solution is

$$x = 2, y = 3, z = 6$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 2x - 6y + 8z &= 24 \\ 5x + 4y - 3z &= 2 \\ 3x + y + 2z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 2 & -6 & 8 & 24 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$\left(\frac{1}{2}\right)R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

$$R_2 - 5R_1, R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 19 & -23 & -58 \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$\left(\frac{1}{19}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$R_1 + 3R_2, R_3 - 10R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & \frac{40}{19} & \frac{200}{19} \end{array}\right]$$

$$\left(\frac{19}{40}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$R_1-\left(\frac{7}{19}\right)R_3, R_2+\left(\frac{23}{19}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The corresponding system of equations is

$$x = 1$$
$$y = 3$$
$$z = 5$$

Hence, the solution is

$$x = 1, y = 3, z = 5$$

Example 5

Solve the following system of linear equations:

$$\begin{aligned} 2x + 5y - 3z &= 1 \\ 5x + y + 4z &= 2 \\ 7x + 3y + z &= 4 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 5 & -3 \\ 5 & 1 & 4 \\ 7 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 2 & 5 & -3 & 1 \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$\left(\frac{1}{2}\right)R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

$$R_2 - 5R_1,\ R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{23}{2} & \frac{23}{2} & -\frac{1}{2} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$\left(-\frac{2}{23}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$R_1 - \frac{5}{2}R_2,\ R_3 + \frac{29}{2}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & \frac{9}{23} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & 0 & -3 & \frac{26}{23} \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 1 & \frac{9}{23} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & 0 & 1 & -\frac{26}{69} \end{array}\right]$$

$$R_1 - R_3,\ R_2 + R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{53}{69} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{26}{69} \end{array}\right]$$

The corresponding system of equations is

$$x = \frac{53}{69}$$
$$y = -\frac{1}{3}$$
$$z = -\frac{26}{69}$$

Hence, the solution is

$$x = \frac{53}{69},\ y = -\frac{1}{3},\ z = -\frac{26}{69}$$

EXERCISE 9.2

Solve the following systems of equations by Gauss–Jordan method:

1. $x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2z = 13$

[**Ans.:** $x = 2, y = -1, z = 3$]

2. $2x + 3y - z = 5$
$4x + 4y - 3z = 3$
$2x - 3y + 2z = 2$

[**Ans.:** $x = 1, y = 2, z = 3$]

3. $10x + y + z = 12$
$2x + 10y + z = 13$
$x + y + 5z = 7$

[**Ans.:** $x = 1, y = 1, z = 1$]

4. $2x_1 + x_2 - 3x_3 = 11$
$4x_1 - 2x_2 + 3x_3 = 8$
$-2x_1 + 2x_2 - x_3 = -6$

[**Ans.:** $x_1 = 3, x_2 = -1, x_3 = -2$]

5. $2x_1 + 6x_2 + x_3 = 7$
$x_1 + 2x_2 - x_3 = -1$
$5x_1 + 7x_2 - 4x_3 = 9$

[**Ans.:** $x_1 = 10, x_2 = -3, x_3 = 5$]

6. $2x + y + 4z = 12$
$8x - 3y + 2z = 20$
$4x + 11y - z = 33$

[**Ans.:** $x = 3, y = 2, z = 1$]

7. $x + y + z = 1$
$4x + 3y - z = 6$
$3x + 5y + 3z = 4$

$\left[\text{**Ans.:** } x = 1, y = \frac{1}{2}, z = -\frac{1}{2}\right]$

9.8 GAUSS–JACOBI METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}\ x + a_{12}\ y + a_{13}\ z &= b_1 \\ a_{21}\ x + a_{22}\ y + a_{23}\ z &= b_2 \\ a_{31}\ x + a_{32}\ y + a_{33}\ z &= b_3 \end{aligned}\right\} \quad \ldots (9.1)$$

where $|a_{11}|$, $|a_{22}|$, $|a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \\ y &= \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \\ z &= \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \end{aligned}\right\} \quad \ldots (9.2)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximation and substituting in Eq. (9.2),

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$
$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_0 - a_{23}z_0)$$
$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_0 - a_{32}y_0)$$

Again substituting these values of x, y, z in Eq. (9.2), the next approximation is obtained.

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).
(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 6x + 2y - z &= 4 \\ x + 5y + z &= 3 \\ 2x + y + 4z &= 27 \end{aligned}$$

Solution

Rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{6}(4 - 2y + z) \\ y &= \frac{1}{5}(3 - x - z) \\ z &= \frac{1}{4}(27 - 2x - y) \end{aligned}\right\} \quad \ldots (1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and putting in Eq. (1),

$$x_1 = \frac{2}{3} = 0.67$$

$$y_1 = \frac{3}{5} = 0.6$$

$$z_1 = \frac{27}{4} = 6.75$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{6}\left[4 - 2(0.6) + 6.75\right] = 1.59$$

$$y_2 = \frac{1}{5}\left[3 - 0.67 - 6.75\right] = -0.884$$

$$z_2 = \frac{1}{4}\left[27 - 2(0.67) - 0.6\right] = 6.265$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{6}\left[4 - 2(-0.884) + 6.265\right] = 2.005$$

$$y_3 = \frac{1}{5}\left[3 - 1.59 - 6.265\right] = -0.971$$

$$z_3 = \frac{1}{4}\left[27 - 2(1.59) - (-0.884)\right] = 6.176$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{6}\left[4 - 2(-0.971) + 6.176\right] = 2.01$$

$$y_4 = \frac{1}{5}\left[3 - 2.005 - 6.176\right] = -1.03$$

$$z_4 = \frac{1}{4}\left[27 - 2(2.005) - (-0.971)\right] = 5.99$$

Iteration 5: Putting x_4, y_4, z_4 in Eq. (1),

$$x_5 = \frac{1}{6}\left[4 - 2(-1.03) + 5.99\right] = 2.00$$

$$y_5 = \frac{1}{5}\left[3 - 2.01 - 5.99\right] = -1.00$$

$$z_5 = \frac{1}{4}\left[27 - 2(2.01) - (-1.03)\right] = 6.00$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 2,\ y = -1,\ z = 6$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 8x - y + 2z &= 13 \\ x - 10y + 3z &= 17 \\ 3x + 2y + 12z &= 25 \end{aligned}$$

Solution

Since absolute values of all diagonal elements are large as compared to absolute values of other coefficients, rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{8}(13 + y - 2z) \\ y &= -\frac{1}{10}(17 - x - 3z) \\ z &= \frac{1}{12}(25 - 3x - 2y) \end{aligned}\right\} \quad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as first approximation and putting in Eq. (1),

$$x_1 = \frac{13}{8} = 1.625$$

$$y_1 = -\frac{17}{10} = -1.7$$

$$z_1 = \frac{25}{12} = 2.08$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{8}\left[13 - 1.7 - 2(2.08)\right] = 0.8925$$

$$y_2 = -\frac{1}{10}\left[17 - 1.625 - 3(2.08)\right] = -0.9135$$

$$z_2 = \frac{1}{12}\left[25 - 3(1.625) - 2(-1.7)\right] = 1.9604$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{8}\left[13 - 0.9135 - 2(1.9604)\right] = 1.0207$$

$$y_3 = -\frac{1}{10}\left[17 - 0.8925 - 3(1.9604)\right] = -1.0226$$

$$z_3 = \frac{1}{12}\left[25 - 3(0.8925) - 2(-0.9135)\right] = 2.0124$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{8}\left[13 - 1.0226 - 2(2.0124)\right] = 0.9941$$

$$y_4 = -\frac{1}{10}\left[17 - 1.0207 - 3(2.0124)\right] = -0.9942$$

$$z_4 = \frac{1}{12}\left[25 - 3(1.0207) - 2(-1.0226)\right] = 1.9985$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1,\ y = -1,\ z = 2$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{8}(13 + y - 2z)$	$y = -\frac{1}{10}(17 - x - 3z)$	$z = \frac{1}{12}(25 - 3x - 2y)$
1	$x_0 = 0$ $x_1 = 1.625$	$y_0 = 0$ $y_1 = -1.7$	$z_0 = 0$ $z_1 = 2.08$
2	$x_2 = 0.8925$	$y_2 = -0.9135$	$z_2 = 1.9604$
3	$x_3 = 1.0207$	$y_3 = -1.0226$	$z_3 = 2.0124$
4	$x_4 = 0.9941$	$y_4 = -0.9942$	$z_4 = 1.9985$

EXERCISE 9.3

Solve the following system of equations by using Gauss–Jacobi method:

1. $4x + y + 3z = 17$
$x + 5y + z = 14$
$2x - y + 8z = 12$

[Ans.: $x = 3, y = 2, z = 1$]

2. $10x + y + 2z = 13$
$2x + 10y + 3z = 15$
$x + 3y + 10z = 14$

[Ans.: $x = 1, y = 1, z = 1$]

3. $10x - 2y - 3z = 205$
$2x - 10y + 2z = -154$
$2x + y - 10z = -120$

[Ans.: $x = 32, y = 26, z = 21$]

4. $12x + 2y + z = 27$
$2x + 15y - 3z = 16$
$2x - 3y + 25z = 23$

[Ans.: $x = 2.0148, y = 0.9731, z = 0.8756$]

5. $20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$

[**Ans.:** $x = 1, y = -1, z = 1$]

6. $10x - 5y - 2z = 3$
$4x - 10y + 3z = -3$
$x + 6y + 10z = -3$

[**Ans.:** $x = 0.342, y = 0.285, z = -0.505$]

7. $8x - 3y + 2z = 20$
$4x + 11y - z = 33$
$6x + 3y + 12z = 35$

[**Ans.:** $x = 3.0168, y = 1.9859, z = 0.9118$]

8. $x + y + 54z = 110$
$27x + 6y - z = 85$
$6x + 15y + 2z = 72$

[**Ans.:** $x = 2.425, y = 3.573, z = 1.926$]

9.9 GAUSS—SIEDEL METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \ldots(9.3)$$

where $|a_{11}|, |a_{22}|, |a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right) \\ y &= \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right) \\ z &= \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right) \end{aligned}\right\} \quad \ldots(9.4)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximations and substituting in the equation of x,

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$

Now, substituting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}z_0)$$

Substituting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}y)$$

Iteration 2

Substituting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{a_{11}}(b_1 - a_{12}y_1 - a_{13}z_1)$$

Substituting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_2 - a_{23}z_1)$$

Substituting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{a_{33}}(b_3 - a_{31}x_2 - a_{32}y_2)$$

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).
(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 3x - 0.1y - 0.2z &= 7.85 \\ 0.1x + 7y - 0.3z &= -19.3 \\ 0.3x - 0.2y + 10z &= 71.4 \end{aligned}$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied. Rewriting the equations.

$$\left.\begin{aligned} x &= \frac{1}{3}(7.85 + 0.1y + 0.2z) \\ y &= \frac{1}{7}(-19.3 - 0.1x + 0.3z) \\ z &= \frac{1}{10}(71.4 - 0.3x + 0.2y) \end{aligned}\right\} \quad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{3}(7.85) = 2.6167$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{7}(-19.3 - 0.1x_1 + 0.3z_0) \\ &= \frac{1}{7}\left[-19.3 - 0.1(2.6167) + 0.3(0)\right] \\ &= -2.7945 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{10}(71.4 - 0.3x_1 + 0.2y_1)$$

$$= \frac{1}{10}\left[71.4 - 0.3(2.6167) + 0.2(-2.7945)\right]$$

$$= 7.0056$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{3}(7.85 + 0.1y_1 + 0.2z_1)$$

$$= \frac{1}{3}\left[7.85 + 0.1(-2.7945) + 0.2(7.0056)\right]$$

$$= 2.9906$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{7}(-19.3 - 0.1x_2 + 0.3z_1)$$

$$= \frac{1}{7}\left[-19.3 - 0.1(2.9906) + 0.3(7.0056)\right]$$

$$= -2.4996$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{10}(71.4 - 0.3x_2 + 0.2y_2)$$

$$= \frac{1}{10}\left[71.4 - 0.3(2.9906) + 0.2(-2.4996)\right]$$

$$= 7.0003$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{3}(7.85 + 0.1y_2 + 0.2z_2)$$

$$= \frac{1}{3}\left[7.85 + 0.1(-2.4996) + 0.2(7.0003)\right]$$

$$= 3.000$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{7}(-19.3 - 0.1x_3 + 0.3z_2)$$

$$= \frac{1}{7}\left[-19.3 - 0.1(3) + 0.3(7.0003)\right]$$

$$= -2.4999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{10}(71.4 - 0.3x_3 + 0.2y_3)$$

$$= \frac{1}{10}\left[71.4 - 0.3(3) + 0.2(-2.4999)\right]$$

$$= 7.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 3,\ y = -2.5,\ z = 7$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 5x + \ \ y - \ \ z &= 10 \\ 2x + 4y + \ \ z &= 14 \\ x + \ \ y + 8z &= 20 \end{aligned}$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied. Rewriting the equations,

$$x = \frac{1}{5}(10 - y + z)$$

$$y = \frac{1}{4}(14 - 2x - z)$$

$$z = \frac{1}{8}(20 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{5}(10) = 2$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{4}(14 - 2x_1 - z_0)$$

$$= \frac{1}{4}\left[14 - 2(2) - 0\right]$$

$$= 2.5$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(20 - x_1 - y_1) \\ &= \frac{1}{8}(20 - 2 - 2.5) \\ &= 1.9375 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{5}(10 - y_1 + z_1) \\ &= \frac{1}{5}(10 - 2.5 + 1.9375) \\ &= 1.8875 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{4}(14 - 2x_2 - z_1) \\ &= \frac{1}{4}[14 - 2(1.8875) - 1.9375] \\ &= 2.0719 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(20 - x_2 - y_2) \\ &= \frac{1}{8}(20 - 1.8875 - 2.0719) \\ &= 2.0050 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{5}(10 - y_2 + z_2) \\ &= \frac{1}{5}(10 - 2.0719 + 2.0050) \\ &= 1.9866 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{4}(14 - 2x_3 - z_2) \\ &= \frac{1}{4}[14 - 2(1.9866) - 2.005] \\ &= 2.0055 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{8}(20 - x_3 - y_3)$$

$$= \frac{1}{8}(20 - 1.9866 - 2.0055)$$

$$= 2.0009$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$x_4 = \frac{1}{5}(10 - y_3 + z_3)$$

$$= \frac{1}{5}(10 - 2.0055 + 2.0009)$$

$$= 1.9991$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$y_4 = \frac{1}{4}(14 - 2x_4 - z_3)$$

$$= \frac{1}{4}[14 - 2(1.9991) - 2.0009]$$

$$= 2.0002$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$z_4 = \frac{1}{8}(20 - x_4 - y_4)$$

$$= \frac{1}{8}(20 - 1.9991 - 2.0002)$$

$$= 2.0001$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 2, y = 2, z = 2$$

Example 3

Solve the following system of linear equations:

$$\begin{aligned} 8x + y + z &= 5 \\ x + 8y + z &= 5 \\ x + y + 8z &= 5 \end{aligned}$$

[Summer 2015, Winter 2013]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{8}(5 - y - z)$$

$$y = \frac{1}{8}(5 - x - z)$$

$$z = \frac{1}{8}(5 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x.

$$x_1 = \frac{1}{8}(5) = 0.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(5 - x_1 - z_0) \\ &= \frac{1}{8}(5 - 0.625 - 0) \\ &= 0.5469 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(5 - x_1 - y_1) \\ &= \frac{1}{8}(5 - 0.625 - 0.5469) \\ &= 0.4785 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(5 - y_1 - z_1) \\ &= \frac{1}{8}(5 - 0.5469 - 0.4785) \\ &= 0.4968 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(5 - x_2 - z_1) \\ &= \frac{1}{8}(5 - 0.4968 - 0.4785) \\ &= 0.5031 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(5 - x_2 - y_2) \\ &= \frac{1}{8}(5 - 0.4968 - 0.5031) \\ &= 0.5 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation x,

$$\begin{aligned} x_3 &= \frac{1}{8}(5 - y_2 - z_2) \\ &= \frac{1}{8}(5 - 0.5031 - 0.5) \\ &= 0.4996 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{8}(5 - x_3 - z_2) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5) \\ &= 0.5001 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{8}(5 - x_3 - y_3) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5001) \\ &= 0.5 \end{aligned}$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 0.5,\ y = 0.5,\ z = 0.5$$

Example 4

Use the Gauss–Siedel method to solve

$$\begin{aligned} 6x + y + z &= 105 \\ 4x + 8y + 3z &= 155 \\ 5x + 4y - 10z &= 65 \end{aligned}$$

[Summer 2015]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{6}(105 - y - z)$$

$$y = \frac{1}{8}(155 - 4x - 3z)$$

$$z = -\frac{1}{10}(65 - 5x - 4y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{6}(105) = 17.5$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(155 - 4x_1 - 3z_0) \\ &= \frac{1}{8}\left[155 - 4(17.5) - 3(0)\right] \\ &= 10.625 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= -\frac{1}{10}(65 - 5x_1 - 4y_1) \\ &= -\frac{1}{10}\left[65 - 5(17.5) - 4(10.625)\right] \\ &= 6.5 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{6}(105 - y_1 - z_1) \\ &= \frac{1}{6}(105 - 10.625 - 6.5) \\ &= 14.6458 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(155 - 4x_2 - 3z_1) \\ &= \frac{1}{8}\left[155 - 4(14.6458) - 3(6.5)\right] \\ &= 9.6146 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= -\frac{1}{10}(65 - 5x_2 - 4y_2) \\ &= -\frac{1}{10}[65 - 5(14.6458) - 4(9.6146)] \\ &= 4.6687 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{6}(105 - y_2 - z_2) \\ &= \frac{1}{6}(105 - 9.6146 - 4.6687) \\ &= 15.1195 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{8}(155 - 4x_3 - 3z_2) \\ &= \frac{1}{8}[155 - 4(15.1195) - 3(4.6687)] \\ &= 10.0645 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= -\frac{1}{10}(65 - 5x_3 - 4y_3) \\ &= -\frac{1}{10}[65 - 5(15.1195) - 4(10.0645)] \\ &= 5.0856 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of y,

$$\begin{aligned} x_4 &= \frac{1}{6}(105 - y_3 - z_3) \\ &= \frac{1}{6}(105 - 10.0645 - 5.0856) \\ &= 14.975 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{8}(155 - 4x_4 - 3z_3) \\ &= \frac{1}{8}[155 - 4(14.975) - 3(5.0856)] \\ &= 9.9804 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= -\frac{1}{10}(65 - 5x_4 - 4y_4) \\ &= -\frac{1}{10}[65 - 5(14.975) - 4(9.9804)] \\ &= 4.9797 \end{aligned}$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of x,

$$\begin{aligned} x_5 &= \frac{1}{6}(105 - y_4 - z_4) \\ &= \frac{1}{6}(105 - 9.9804 - 4.9797) \\ &= 15.0067 \end{aligned}$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$\begin{aligned} y_5 &= \frac{1}{8}(155 - 4x_5 - 3z_4) \\ &= \frac{1}{8}[155 - 4(15.0067) - 3(4.9797)] \\ &= 10.0043 \end{aligned}$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= -\frac{1}{10}(65 - 5x_5 - 4y_5) \\ &= -\frac{1}{10}[65 - 5(15.0067) - 4(10.0043)] \\ &= 5.0051 \end{aligned}$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 15,\ y = 10,\ z = 5$$

Example 5

Solve the following system of equations:

$$\begin{aligned} 25x + 2y - 3z &= 48 \\ 3x + 27y - 2z &= 56 \\ x + 2y + 23z &= 52 \end{aligned}$$

starting with (1, 1, 0).

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied. Rewriting the equations,

$$x = \frac{1}{25}(48 - 2y + 3z)$$

$$y = \frac{1}{27}(56 - 3x + 2z)$$

$$z = \frac{1}{23}(52 - x - 2y)$$

Iteration 1: It is given that $x_0 = 1$, $y_0 = 1$, $z_0 = 0$. Putting $y = y_0$, $z = z_0$ in the equation of x,

$$\begin{aligned} x_1 &= \frac{1}{25}(48 - 2y_0 + 3z_0) \\ &= \frac{1}{25}[48 - 2(1) + 3(0)] \\ &= 1.84 \end{aligned}$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{27}(56 - 3x_1 + 2z_0) \\ &= \frac{1}{27}[56 - 3(1.84) + 2(0)] \\ &= 1.8696 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{23}(52 - x_1 - 2y_1) \\ &= \frac{1}{23}[52 - 1.84 - 2(1.8696)] \\ &= 2.0183 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{25}(48 - 2y_1 + 3z_1) \\ &= \frac{1}{25}[48 - 2(1.8696) + 3(2.0183)] \\ &= 2.0126 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{27}(56 - 3x_2 + 2z_1)$$

$$= \frac{1}{27}\left[56 - 3(2.0126) + 2(2.0183)\right]$$
$$= 1.9999$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{23}(52 - x_2 - 2y_2)$$
$$= \frac{1}{23}\left[52 - 2.0126 - 2(1.9999)\right]$$
$$= 1.9994$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{25}(48 - 2y_2 + 3z_2)$$
$$= \frac{1}{25}\left[48 - 2(1.9999) + 3(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{27}(56 - 3x_3 + 2z_2)$$
$$= \frac{1}{27}\left[56 - 3(1.9999) + 2(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{23}(52 - x_3 - 2y_3)$$
$$= \frac{1}{23}\left[52 - 1.9999 - 2(1.9999)\right]$$
$$= 2.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 2, y = 2, z = 2$$

Example 6

Solve the following system of equations, by the Gauss–Seidel method:

$$\begin{aligned} 2x + \ \ y + 6z &= \ 9 \\ 8x + 3y + 2z &= 13 \\ \ x + 5y + \ \ z &= \ 7 \end{aligned}$$

[Summer 2015]

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 8x + 3y + 2z &= 13 \\ x + 5y + z &= 7 \\ 2x + y + 6z &= 9 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{8}(13 - 3y - 2z)$$

$$y = \frac{1}{5}(7 - x - z)$$

$$z = \frac{1}{6}(9 - 2x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{8}(13) = 1.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{5}(7 - x_1 - z_0) \\ &= \frac{1}{5}(7 - 1.625 - 0) \\ &= 1.075 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{6}(9 - 2x_1 - y_1) \\ &= \frac{1}{6}\left[9 - 2(1.625) - 1.075\right] \\ &= 0.7792 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(13 - 3y_1 - 2z_1) \\ &= \frac{1}{8}\left[13 - 3(1.075) - 2(0.7792)\right] \\ &= 1.0271 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{5}(7 - x_2 - z_1) \\ &= \frac{1}{5}(7 - 1.0271 - 0.7792) \\ &= 1.0387 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{6}(9 - 2x_2 - y_2) \\ &= \frac{1}{6}\left[9 - 2(1.0271) - 1.0387\right] \\ &= 0.9845 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{8}(13 - 3y_2 - 2z_2) \\ &= \frac{1}{8}\left[13 - 3(1.0387) - 2(0.9845)\right] \\ &= 0.9894 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{5}(7 - x_3 - z_2) \\ &= \frac{1}{5}(7 - 0.9894 - 0.9845) \\ &= 1.0052 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{6}(9 - 2x_3 - y_3) \\ &= \frac{1}{6}\left[9 - 2(0.9894) - 1.0052\right] \\ &= 1.0027 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{8}(13 - 3y_3 - 2z_3) \\ &= \frac{1}{8}\left[13 - 3(1.0052) - 2(1.0027)\right] \\ &= 0.9974 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{5}(7 - x_4 - z_3) \\ &= \frac{1}{5}(7 - 0.9974 - 1.0027) \\ &= 1 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{6}(9 - 2x_4 - y_4) \\ &= \frac{1}{6}[9 - 2(0.9974) - 1] \\ &= 1.0009 \end{aligned}$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1, y = 1, z = 1$$

Example 7

Solve the following system of equations:

$$\begin{aligned} x + 2y + z &= 0 \\ 3x + y - z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

starting with (1, 1, 1).

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations,

$$\begin{aligned} 3x + y - z &= 0 \\ x + 2y + z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{3}(-y + z)$$

$$y = \frac{1}{2}(-x - z)$$

$$z = \frac{1}{4}(3 - x + y)$$

Iteration 1: Assuming $x_0 = 1$, $y_0 = 1$, $z_0 = 1$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{3}(-y_0 + z_0)$$
$$= \frac{1}{3}(-1+1)$$
$$= 0$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{2}(-x_1 - z_0)$$
$$= \frac{1}{2}(-0-1)$$
$$= -0.5$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{4}(3 - x_1 + y_1)$$
$$= \frac{1}{4}(3-0-0.5)$$
$$= 0.625$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{3}(-y_1 + z_1)$$
$$= \frac{1}{3}\left[-(-0.5)+0.625\right]$$
$$= 0.375$$

Putting $x = x_2$, $z = z_1$, in the equation of y,

$$y_2 = \frac{1}{2}(-x_2 - z_1)$$
$$= \frac{1}{2}(-0.375-0.625)$$
$$= -0.5$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{4}(3 - x_2 + y_2)$$
$$= \frac{1}{4}(3-0.375-0.5)$$
$$= 0.5313$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{3}(-y_2 + z_2)$$

$$= \frac{1}{3}\left[-(-0.5) + 0.5313\right]$$

$$= 0.3438$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{2}(-x_3 - z_2)$$

$$= \frac{1}{2}(-0.3438 - 0.5313)$$

$$= -0.4376$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{4}(3 - x_3 + y_3)$$

$$= \frac{1}{4}(3 - 0.3438 - 0.4376)$$

$$= 0.5547$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$x_4 = \frac{1}{3}(-y_3 + z_3)$$

$$= \frac{1}{3}\left[-(-0.4376) + 0.5547\right]$$

$$= 0.3307$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$y_4 = \frac{1}{2}(-x_4 - z_3)$$

$$= \frac{1}{2}(-0.3307 - 0.5547)$$

$$= -0.4427$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$z_4 = \frac{1}{4}(3 - x_4 + y_4)$$

$$= \frac{1}{4}(3 - 0.3307 - 0.4427)$$

$$= 0.5566$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of z,

$$x_5 = \frac{1}{3}(-y_4 + z_4)$$

$$= \frac{1}{3}\left[-(-0.4427) + 0.5566\right]$$

$$= 0.3331$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$y_5 = \frac{1}{2}(-x_5 - z_4)$$

$$= \frac{1}{2}(-0.3331 - 0.5566)$$

$$= -0.4449$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$z_5 = \frac{1}{4}(3 - x_5 + y_5)$$

$$= \frac{1}{4}(3 - 0.3331 - 0.4449)$$

$$= 0.5555$$

Iteration 6: Putting $y = y_5$, $z = z_5$ in the equation of z,

$$x_6 = \frac{1}{3}(-y_5 + z_5)$$

$$= \frac{1}{3}\left[-(-0.4449) + 0.5555\right]$$

$$= 0.3335$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$y_6 = \frac{1}{2}(-x_6 - z_5)$$

$$= \frac{1}{2}(-0.3335 - 0.5555)$$

$$= -0.4445$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$z_6 = \frac{1}{4}(3 - x_6 + y_6)$$

$$= \frac{1}{4}(3 - 0.3335 - 0.4445)$$

$$= 0.5555$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 0.333,\ y = -0.444,\ z = 0.555$$

Example 8

Solve the following system of equations:

$$\begin{aligned} 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \\ 54x + y + z &= 110 \end{aligned}$$

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 54x + y + z &= 110 \\ 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{54}(110 - y - z)$$

$$y = -\frac{1}{15}(72 - 2x - 6z)$$

$$z = -\frac{1}{27}(85 + x - 6y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{54}(110) = 2.037$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= -\frac{1}{15}(72 - 2x_1 - 6z_0) \\ &= -\frac{1}{15}\left[72 - 2(2.037) - 6(0)\right] \\ &= -4.5284 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = -\frac{1}{27}(85 + x_1 - 6y_1)$$

$$= -\frac{1}{27}[85 + 2.037 - 6(-4.5284)]$$

$$= -4.2299$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$x_2 = \frac{1}{54}(110 - y_1 - z_1)$$

$$= \frac{1}{54}[110 - (-4.5284) - (-4.2299)]$$

$$= 2.1992$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = -\frac{1}{15}(72 - 2x_2 - 6z_1)$$

$$= -\frac{1}{15}[72 - 2(2.1992) - 6(-4.2299)]$$

$$= -6.1987$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = -\frac{1}{27}(85 + x_2 - 6y_2)$$

$$= -\frac{1}{27}[85 + 2.1992 - 6(-6.1987)]$$

$$= -4.6071$$

Iteration 3: Putting $y = y_2$, $z = z_2$, in the equation of x,

$$x_3 = \frac{1}{54}(110 - y_2 - z_2)$$

$$= \frac{1}{54}[110 - (-6.1987) - (-4.6071)]$$

$$= 2.2371$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = -\frac{1}{15}(72 - 2x_3 - 6z_2)$$

$$= -\frac{1}{15}[72 - 2(2.2371) - 6(-4.6071)]$$

$$= -6.3446$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = -\frac{1}{27}(85 + x_3 - 6y_3)$$

$$= -\frac{1}{27}[85 + 2.2371 - 6(-6.3446)]$$

$$= -4.6409$$

Iteration 4: Putting $y = y_3$, $z = z_3$, in the equation of x,

$$x_4 = \frac{1}{54}(110 - y_3 - z_3)$$

$$= \frac{1}{54}[110 - (-6.3446) - (-4.6409)]$$

$$= 2.2405$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$y_4 = -\frac{1}{15}(72 - 2x_4 - 6z_3)$$

$$= -\frac{1}{15}[72 - 2(2.2405) - 6(-4.6409)]$$

$$= -6.3576$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$z_4 = -\frac{1}{27}(85 + x_4 - 6y_4)$$

$$= -\frac{1}{27}[85 + 2.2405 - 6(-6.3576)]$$

$$= -4.6439$$

Iteration 5: Putting $y = y_4$, $z = z_4$, in the equation of x,

$$x_5 = \frac{1}{54}(110 - y_4 - z_4)$$

$$= \frac{1}{54}[110 - (-6.3576) - (-4.6439)]$$

$$= 2.2408$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$y_5 = -\frac{1}{15}(72 - 2x_5 - 6z_4)$$

$$= -\frac{1}{15}[72 - 2(2.2408) - 6(-4.6439)]$$

$$= -6.3588$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$z_5 = -\frac{1}{27}(85 + x_5 - 6y_5)$$
$$= -\frac{1}{27}[85 + 2.2408 - 6(-6.3588)]$$
$$= -4.6442$$

Iteration 6: Putting $y = y_5$, $z = z_5$, in the equation of x,

$$x_6 = \frac{1}{54}(110 - y_5 - z_5)$$
$$= \frac{1}{54}[110 - (-6.3588) - (-4.6442)]$$
$$= 2.2408$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$y_6 = -\frac{1}{15}(72 - 2x_6 - 6z_5)$$
$$= -\frac{1}{15}[72 - 2(2.2408) - 6(-4.6442)]$$
$$= -6.3589$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$z_6 = -\frac{1}{27}(85 + x_6 - 6y_6)$$
$$= -\frac{1}{27}[85 + 2.2408 - 6(-6.3589)]$$
$$= -4.6442$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 2.2408,\ y = -6.3589,\ z = -4.6442$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{54}(110 - y - z)$	$y = -\frac{1}{15}(72 - 2x - 6z)$	$z = -\frac{1}{27}(85 + x - 6y)$
1	$x_0 = 0$ $x_1 = 2.037$	$y_0 = 0$ $y_1 = -4.5284$	$z_0 = 0$ $z_1 = -4.2299$
2	$x_2 = 2.1992$	$y_2 = -6.1987$	$z_2 = -4.6071$
3	$x_3 = 2.2371$	$y_3 = -6.3446$	$z_3 = -4.6409$
4	$x_4 = 2.2405$	$y_4 = -6.3576$	$z_4 = -4.6439$
5	$x_5 = 2.2408$	$y_5 = -6.3588$	$z_5 = -4.6442$
6	$x_6 = 2.2408$	$y_6 = -6.3589$	$z_6 = -4.6442$

EXERCISE 9.4

Solve the following system of equations by using Gauss–Seidel method:

1. $54x + y + z = 110$
$2x + 15y + 6z = 72$
$-x + 6y + 27z = 85$

[**Ans.:** $x = 1.92,\ y = 3.57,\ z = 2.42$]

2. $20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$

[**Ans.:** $x = 1,\ y = -1,\ z = 1$]

3. $10x + y + z = 12$
$2x + 10y + z = 13$
$2x + 2y + 10z = 14$

[**Ans.:** $x = 1,\ y = 1,\ z = 1$]

4. $27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

[**Ans.:** $x = 2.43,\ y = 3.57,\ z = 1.92$]

5. $28x + 4y - z = 32$
$2x + 17y + 4z = 35$
$x + 3y + 10z = 24$

[**Ans.:** $x = 0.99,\ y = 1.51,\ z = 1.85$]

Points to Remember

Gauss Elimination Method

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.
(iv) Write the corresponding linear system of equations from the echelon form.
(v) Solve the corresponding linear system of equations by back substitution.

Gauss–Jordan Method

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Gauss–Jacobi Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).
(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Gauss–Siedel Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).
(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

CHAPTER 10

Roots of Algebraic and Transcendental Equations

Chapter Outline

10.1 Introduction
10.2 Bisection Method
10.3 Regula Falsi Method
10.4 Newton–Raphson Method
10.5 Secant Method

10.1 INTRODUCTION

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, where a_0, a_1, a_2, ..., a_n are constants and n is a positive integer, is called an *algebraic polynomial of degree n* if $a_0 \neq 0$. The equation $f(x) = 0$ is called an algebraic equation if $f(x)$ is an algebraic polynomial, e.g., $x^3 - 4x - 9 = 0$. If $f(x)$ contains functions such as trigonometric, logarithmic, exponential, etc., then $f(x) = 0$ is called a *transcendental equation*, e.g., $2x^3 - \log(x + 3)\tan x + e^x = 0$.

In general, an equation is solved by factorization. But in many cases, the method of factorization fails. In such cases, numerical methods are used. There are some methods to solve the equation $f(x) = 0$ such as

(i) Bisection method
(ii) Regula Falsi method
(iii) Newton–Raphson method
(iv) Secant method

10.2 BISECTION METHOD

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be two real values of x at P and Q respectively such that $f(x_1)$ is positive and $f(x_0)$ is negative or vice versa (Fig. 10.1). Then there is one root of the equation $f(x) = 0$ between x_0 and x_1. Now, this interval $[x_0, x_1]$ is divided into two sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$, where $x_2 = \dfrac{x_0 + x_1}{2}$.

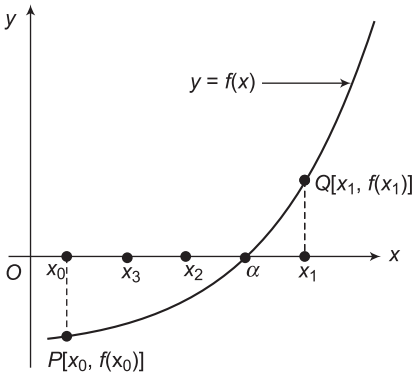


Fig. 10.1

If $f(x_0)$ and $f(x_2)$ are of opposite signs then the interval $[x_0, x_2]$ is divided into $[x_0, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_0 + x_2}{2}$. However, if $f(x_0)$ and $f(x_2)$ are of the same sign then $f(x_1)$ and $f(x_2)$ will be opposite signs and the interval $[x_1, x_2]$ is divided into $[x_1, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_1 + x_2}{2}$. This process is continued till the desired accuracy is obtained.

Example 1

Find the positive root of $x^3 - 2x - 5 = 0$, correct up to two decimal places.

Solution

Let $f(x) = x^3 - 2x - 5$

$f(1) = -6$ and $f(2) = -1, f(3) = 16$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = 5.625$$

Since $f(2.5) > 0$ and $f(2) < 0$, the root lies between 2.5 and 2.

$$x_2 = \frac{2.5+2}{2} = 2.25$$

$$f(x_2) = f(2.25) = 1.8906$$

Since $f(2.25) > 0$ and $f(2) < 0$, the root lies between 2.25 and 2.

$$x_3 = \frac{2.25+2}{2} = 2.125$$
$$f(x_3) = f(2.125) = 0.3457$$

Since $f(2.125) > 0$ and $f(2) < 0$, the root lies between 2.125 and 2.

$$x_4 = \frac{2.125+2}{2} = 2.0625$$
$$f(x_4) = f(2.0625) = -0.3513$$

Since $f(2.0625) < 0$ and $f(2.125) > 0$, the root lies between 2.0625 and 2.125.

$$x_5 = \frac{2.0625+2.125}{2} = 2.09375$$
$$f(x_5) = f(2.09375) = -0.0089$$

Since $f(2.09375) < 0$ and $f(2.125) > 0$, the root lies between 2.09375 and 2.125.

$$x_6 = \frac{2.09375+2.125}{2} = 2.109375$$
$$f(x_6) = f(2.109375) = 0.1668$$

Since $f(2.109375) > 0$ and $f(2.09375) < 0$, the root lies between 2.109375 and 2.09375.

$$x_7 = \frac{2.109375+2.09375}{2} = 2.10156$$

Since x_6 and x_7 are same up to two decimal places, the positive root is 2.10.

Example 2

Find a root of $x^3 - 5x + 3 = 0$ *by the bisection method correct up to four decimal places.* **[Summer 2015]**

Solution

Let $f(x) = x^3 - 5x + 3$

$f(0) = 3$ and $f(1) = -1$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$
$$f(x_1) = f(0.5) = 0.625$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$
$$f(x_2) = f(0.75) = -0.3281$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = 0.1191$$

Since $f(0.625) > 0$ and $f(0.75) < 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.1125$$

Since $f(0.6875) < 0$ and $f(0.625) > 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875 + 0.625}{2} = 0.65625$$
$$f(x_5) = f(0.65625) = 0.00137$$

Since $f(0.65625) > 0$ and $f(0.6875) < 0$, the root lies between 0.65625 and 0.6875.

$$x_6 = \frac{0.65625 + 0.6875}{2} = 0.67188$$
$$f(x_6) = f(0.67188) = -0.0561$$

Since $f(0.67188) < 0$ and $f(0.65625) > 0$, the root lies between 0.67188 and 0.65625.

$$x_7 = \frac{0.67188 + 0.65625}{2} = 0.66407$$
$$f(x_7) = f(0.66407) = -0.02750$$

Since $f(0.66407) < 0$ and $f(0.65625) > 0$, the root lies between 0.66407 and 0.65625.

$$x_8 = \frac{0.66407 + 0.65625}{2} = 0.66016$$
$$f(x_8) = f(0.66016) = -0.01309$$

Since $f(0.66016) < 0$ and $f(0.65625) > 0$, the root lies between 0.66016 and 0.65625.

$$x_9 = \frac{0.66016 + 0.65625}{2} = 0.65821$$
$$f(x_9) = f(0.65821) = -0.00589$$

Since $f(0.65821) < 0$ and $f(0.65625) > 0$, the root lies between 0.65821 and 0.65625.

$$x_{10} = \frac{0.65821 + 0.65625}{2} = 0.65723$$
$$f(x_{10}) = f(0.65723) = -0.0023$$

Since $f(0.65723) < 0$ and $f(0.65625) > 0$, the root lies between 0.65723 and 0.65625.

$$x_{11} = \frac{0.65723 + 0.65625}{2} = 0.65674$$
$$f(x_{11}) = f(0.65674) = -0.00044$$

Since $f(0.65674) < 0$ and $f(0.65625) > 0$, the root lies between 0.65674 and 0.65625.

$$x_{12} = \frac{0.65674 + 0.65625}{2} = 0.6565$$
$$f(x_{12}) = f(0.6565) = 0.00044$$

Since $f(0.6565) > 0$ and $f(0.65674) < 0$, the root lies between 0.6565 and 0.65674.

$$x_{13} = \frac{0.6565 + 0.65674}{2} = 0.6566$$
$$f(x_{13}) = f(0.6566) = 0.00075$$

Since $f(0.6566) > 0$ and $f(0.65674) < 0$, the root lies between 0.6566 and 0.65674.

$$x_{14} = \frac{0.6566 + 0.65674}{2} = 0.65667$$

Since x_{13} and x_{14} are same up to four decimal places, the root is 0.6566.

Example 3

Perform the five iterations of the bisection method to obtain a root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution

Let $f(x) = x^3 - x - 1$

$f(1) = -1$ and $f(2) = 5$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

$$x_1 = \frac{1 + 2}{2} = 1.5$$
$$f(x_1) = f(1.5) = 0.875$$

Since $f(1.5) > 0$ and $f(1) < 0$, the root lies between 1.5 and 1.

$$x_2 = \frac{1.5 + 1}{2} = 1.25$$
$$f(x_2) = f(1.25) = -0.2968$$

Since $f(1.25) < 0$ and $f(1.5) > 0$, the root lies between 1.25 and 1.5.

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$
$$f(x_3) = f(1.375) = 0.2246$$

Since $f(1.375) > 0$ and $f(1.25) < 0$, the root lies between 1.375 and 1.25.

$$x_4 = \frac{1.375 + 1.25}{2} = 1.3125$$
$$f(x_4) = f(1.3125) = -0.0515$$

Since $f(1.3125) < 0$ and $f(1.375) > 0$, the root lies between 1.3125 and 1.375.

$$x_5 = \frac{1.3125 + 1.375}{2} = 1.3438$$

Hence, the root is 1.3438 up to five iterations.

Example 4

Find the approximate solution of $x^3 + x - 1 = 0$ *correct to three decimal places.* **[Winter 2013]**

Solution

Let $f(x) = x^3 + x - 1$

$f(0) = -1$ and $f(1) = 1$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0 + 1}{2} = 0.5$$
$$f(x_1) = f(0.5) = -0.375$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$
$$f(x_2) = f(0.75) = 0.1719$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = -0.1309$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = 0.01245$$

Since $f(0.6875) > 0$ and $f(0.625) < 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875 + 0.625}{2} = 0.6563$$
$$f(x_5) = f(0.6563) = -0.0644$$

Since $f(0.6563) < 0$ and $f(0.6875) > 0$, the root lies between 0.6563 and 0.6875.

$$x_6 = \frac{0.6563 + 0.6875}{2} = 0.6719$$
$$f(x_6) = f(0.6719) = -0.0248$$

Since $f(0.6719) < 0$ and $f(0.6875) > 0$, the root lies between 0.6719 and 0.6875.

$$x_7 = \frac{0.6719 + 0.6875}{2} = 0.6797$$
$$f(x_7) = f(0.6797) = -0.0141$$

Since $f(0.6797) < 0$ and $f(0.6875) > 0$, the root lies between 0.6797 and 0.6875.

$$x_8 = \frac{0.6797 + 0.6875}{2} = 0.6836$$
$$f(x_8) = f(0.6836) = 0.0031$$

Since $f(0.6836) > 0$ and $f(0.6797) < 0$, the root lies between 0.6836 and 0.6797.

$$x_9 = \frac{0.6836 + 0.6797}{2} = 0.6817$$
$$f(x_9) = f(0.6817) = -0.0015$$

Since $f(0.6817) < 0$ and $f(0.6836) > 0$, the root lies between 0.6817 and 0.6836.

$$x_{10} = \frac{0.6817 + 0.6836}{2} = 0.6827$$
$$f(x_{10}) = f(0.6827) = 0.00089$$

Since $f(0.6827) > 0$ and $f(0.6817) < 0$, the root lies between 0.6827 and 0.6817.

$$x_{11} = \frac{0.6827 + 0.6817}{2} = 0.6822$$

Since x_{10} and x_{11} are same up to three decimal points, the root is 0.682.

Example 5

Find a root of the equation $x^3 - 4x - 9 = 0$ using the bisection method in four stages.

Solution

Let $f(x) = x^3 - 4x - 9$

$f(2) = -9$ and $f(3) = 6$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = -3.375$$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

$$x_2 = \frac{2.5+3}{2} = 2.75$$
$$f(x_2) = f(2.75) = 0.7969$$

Since $f(2.75) > 0$ and $f(2.5) < 0$, the root lies between 2.75 and 2.5.

$$x_3 = \frac{2.75+2.5}{2} = 2.625$$
$$f(x_3) = f(2.625) = -1.4121$$

Since $f(2.625) < 0$ and $f(2.75) > 0$, the root lies between 2.625 and 2.75.

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Hence, the root is 2.6875 up to four stages.

Example 6

Find the negative root of $x^3 - 7x + 3$ by the bisection method up to three decimal places.

Solution

Let $f(x) = x^3 - 7x + 3$

$f(-2) = 9$ and $f(-3) = -3$

Since $f(-2) > 0$ and $f(-3) < 0$, the root lies between –2 and –3.

$$x_1 = \frac{-2-3}{2} = -2.5$$
$$f(x_1) = f(-2.5) = 4.875$$

Since $f(-2.5) > 0$ and $f(-3) < 0$, the root lies between –2.5 and –3.

$$x_2 = \frac{-2.5-3}{2} = -2.75$$
$$f(x_2) = f(-2.75) = 1.4531$$

Since $f(-2.75) > 0$ and $f(-3) < 0$, the root lies between –2.75 and –3.

$$x_3 = \frac{-2.75-3}{2} = -2.875$$
$$f(x_3) = f(-2.875) = -0.6387$$

Since $f(-2.875) < 0$ and $f(-2.75) > 0$, the root lies between –2.875 and –2.75.

$$x_4 = \frac{-2.875 - 2.75}{2} = -2.8125$$
$$f(x_4) = f(-2.8125) = 0.4402$$

Since $f(-2.8125) > 0$ and $f(-2.875) < 0$, the root lies between –2.8125 and –2.875.

$$x_5 = \frac{-2.8125 - 2.875}{2} = -2.8438$$
$$f(x_5) = f(-2.8438) = -0.0918$$

Since $f(-2.8438) < 0$ and $f(-2.8125) > 0$, the root lies between –2.8438 and –2.8125.

$$x_6 = \frac{-2.8438 - 2.8125}{2} = -2.8282$$
$$f(x_6) = f(-2.8282) = 0.1754$$

Since $f(-2.8282) > 0$ and $f(-2.8438) < 0$, the root lies between –2.8282 and –2.8438.

$$x_7 = \frac{-2.8282 - 2.8438}{2} = -2.836$$
$$f(x_7) = f(-2.836) = 0.0423$$

Since $f(-2.836) > 0$ and $f(-2.8438) < 0$, the root lies between –2.836 and –2.8438.

$$x_8 = \frac{-2.836 - 2.8438}{2} = -2.8399$$
$$f(x_8) = f(-2.8399) = -0.0246$$

Since $f(-2.8399) < 0$ and $f(-2.836) > 0$, the root lies between –2.8399 and –2.836.

$$x_9 = \frac{-2.8399 - 2.836}{2} = -2.838$$
$$f(x_9) = f(-2.838) = 0.0081$$

Since $f(-2.838) > 0$ and $f(-2.8399) < 0$, the root lies between –2.838 and –2.8399.

$$x_{10} = \frac{-2.838 - 2.8399}{2} = -2.8389$$

Since x_9 and x_{10} are same up to three decimal places, the negative root is –2.838.

Example 7

Perform three iterations of the bisection method to obtain the root of the equation $2 \sin x - x = 0$, *correct up to three decimal places.*

[Summer 2015]

Solution

Let $f(x) = 2 \sin x - x$

$f(1) = 0.6829$ and $f(2) = -0.1814$

Since $f(1) > 0$ and $f(2) < 0$, the root lies between 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$

$$f(x_1) = f(1.5) = 0.4949$$

Since $f(1.5) > 0$ and $f(2) < 0$, the root lies between 1.5 and 2.

$$x_2 = \frac{1.5+2}{2} = 1.75$$

$$f(x_2) = f(1.75) = 0.2179$$

Since $f(1.75) > 0$ and $f(2) < 0$, the root lies between 1.75 and 2.

$$x_3 = \frac{1.75+2}{2} = 1.875$$

Hence, the root is 1.875 up to three iterations.

Example 8

Solve $x = \cos x$ by the bisection method correct to two decimal places.

[Summer 2014]

Solution

Let $f(x) = x - \cos x$

$f(0) = -1$ and $f(1) = 0.4597$

Since $f(0) < 1$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = -0.3776$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = 0.0183$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.186$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.0853$$

Since $f(0.6875) < 0$ and $f(0.75) > 0$, the root lies between 0.6875 and 0.75.

$$x_5 = \frac{0.6875 + 0.75}{2} = 0.71875$$
$$f(x_5) = f(0.71875) = -0.0338$$

Since $f(0.71875) < 0$ and $f(0.75) > 0$, the root lies between 0.71875 and 0.75.

$$x_6 = \frac{0.71875 + 0.75}{2} = 0.7344$$
$$f(x_6) = f(0.7344) = -0.0078$$

Since $f(0.7344) < 0$ and $f(0.75) > 0$, the root lies between 0.7344 and 0.75.

$$x_7 = \frac{0.7344 + 0.75}{2} = 0.7422$$
$$f(x_7) = f(0.7422) = 0.0052$$

Since $f(0.7422) > 0$ and $f(0.7344) < 0$, the root lies between 0.7422 and 0.7344.

$$x_8 = \frac{0.7422 + 0.7344}{2} = 0.7383$$
$$f(x_8) = f(0.7383) = -0.0013$$

Since $f(0.7383) < 0$ and $f(0.7422) > 0$, the root lies between 0.7383 and 0.7422.

$$x_9 = \frac{0.7383 + 0.7422}{2} = 0.74025$$
$$f(x_9) = f(0.74025) = 0.00195$$

Since $f(0.74025) > 0$ and $f(0.7383) < 0$, the root lies between 0.74025 and 0.7383.

$$x_{10} = \frac{0.74025 + 0.7383}{2} = 0.7393$$
$$f(x_{10}) = f(0.7393) = 0.0004$$

Since $f(0.7393) > 0$ and $f(0.7383) < 0$, the root lies between 0.7393 and 0.7383.

$$x_{11} = \frac{0.7393 + 0.7383}{2} = 0.7388$$

Since x_{10} and x_{11} are the same up to two decimal places, the root is 0.73.

Example 9

Find a real root between 0 *and* 1 *of the equation* $e^{-x} - x = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = e^{-x} - x$

$f(0) = 1$ and $f(1) = -0.63$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.1065$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.2776$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.0897$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625+0.5}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = 7.28 \times 10^{-3}$$

Since $f(0.5625) > 0$ and $f(0.625) < 0$, the root lies between 0.5625 and 0.625.

$$x_5 = \frac{0.5625+0.625}{2} = 0.5938$$

$$f(x_5) = f(0.5938) = -0.0416$$

Since $f(0.5938) < 0$ and $f(0.5625) > 0$, the root lies between 0.5938 and 0.5625.

$$x_6 = \frac{0.5938+0.5625}{2} = 0.5782$$

$$f(x_6) = f(0.5782) = -0.0173$$

Since $f(0.5782) < 0$ and $f(0.5625) > 0$, the root lies between 0.5782 and 0.5625.

$$x_7 = \frac{0.5782+0.5625}{2} = 0.5704$$

$$f(x_7) = f(0.5704) = -5.1007 \times 10^{-3}$$

Since $f(0.5704) < 0$ and $f(0.5625) > 0$, the root lies between 0.5704 and 0.5625.

$$x_8 = \frac{0.5704 + 0.5625}{2} = 0.5665$$

$$f(x_8) = f(0.5665) = 1.008 \times 10^{-3}$$

Since $f(0.5665) > 0$ and $f(0.5704) < 0$, the root lies between 0.5665 and 0.5704.

$$x_9 = \frac{0.5665 + 0.5704}{2} = 0.5685$$

$$f(x_9) = f(0.5685) = -2.1256 \times 10^{-3}$$

Since $f(0.5685) < 0$ and $f(0.5665) > 0$, the root lies between 0.5685 and 0.5665.

$$x_{10} = \frac{0.5685 + 0.5665}{2} = 0.5675$$

$$f(x_{10}) = f(0.5675) = -5.5898 \times 10^{-4}$$

Since $f(0.5675) < 0$ and $f(0.5665) > 0$, the root lies between 0.5675 and 0.5665.

$$x_{11} = \frac{0.5675 + 0.5665}{2} = 0.567$$

Since x_{10} and x_{11} are the same up to three decimal places, the root is 0.567.

Example 10

Find the root of $\cos x - xe^x = 0$ *in four steps.*

Solution

Let $f(x) = \cos x - xe^x$

$f(0) = 1$ and $f(1) = -2.18$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0 + 1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.0532$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.8561$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.3567$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625 + 0.5}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = -0.1413$$

Since $f(0.5625) < 0$ and $f(0.5) > 0$, the root lies between 0.5625 and 0.5.

$$x_5 = \frac{0.5625 + 0.5}{2} = 0.53125$$

Hence, the root is 0.53125 in four steps.

EXERCISE 10.1

Find a positive root of the following equations correct to four decimal places using bisection method:

1. $x^3 - 4x - 9 = 0$

[Ans.: 2.7065**]**

2. $x^3 + 3x - 1 = 0$

[Ans.: 0.3222**]**

3. $x^3 + x^2 - 1 = 0$

[Ans.: 0.7549**]**

4. $x^4 - x^3 - x^2 - 6x - 4 = 0$

[Ans.: 2.5528**]**

5. $3x = \sqrt{1 + \sin x}$

[Ans.: 0.3918**]**

6. $3x = \cos x + 1$

[Ans.: 0.6071**]**

7. $x - \cos x = 0$

[Ans.: 0.7391**]**

8. $xe^x = 1$

[Ans.: 0.5671**]**

9. $x \log_{10} x = 1.2$ lying between 2 and 3

[Ans.: 2.7406**]**

10.3 REGULA FALSI METHOD

This method resembles the bisection method. In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs, i.e., the graph of $y = f(x)$ crosses the x-axis between these points. Hence, a root lies between x_0 and x_1 and $f(x_0)\, f(x_1) < 0$ (Fig.10.2).

The equation of the chord joining the points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

In this method, the curve PQ is replaced by the chord PQ and the point of intersection of the chord with the x-axis is taken as an approximation to the root.

If x_2 is the point of intersection of the x-axis and the line joining $P[x_0, f(x_0)$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1.

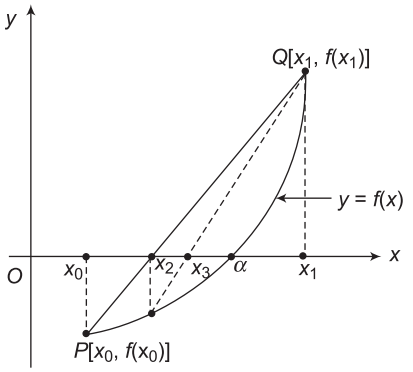


Fig. 10.2

Using slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_0)}{x_2 - x_0} = \frac{0 - f(x_0)}{x_2 - x_0}$$

$$x_2 - x_0 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy. This iteration process is known as the method of false position or *regula falsi method*.

Example 1

Find a positive root of $x^3 - 4x + 1$ *correct up to three decimal places.* **[Summer 2015]**

Solution

Let $f(x) = x^3 - 4x + 1$

$f(0) = 1$ and $f(1) = -2$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $x_1 = 1$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1-0}{-2-1}(1)$$

$$= 0.3333$$

$$f(x_2) = f(0.3333) = -0.2962$$

Since $f(0.3333) < 0$ and $f(0) > 0$, the root lies between 0.3333 and 0, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.3333 - 0}{-0.2962 - 1}(1)$$

$$= 0.2571$$

$$f(x_3) = f(0.2571) = -0.0114$$

Since $f(0.2571) < 0$ and $f(0) > 0$, the root lies between 0.2571 and 0, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.2571 - 0}{-0.0114 - 1}(1)$$

$$= 0.2542$$

$$f(x_4) = f(0.2542) = -0.0004$$

Since $f(0.2542) < 0$ and $f(0) > 0$, the root lies between 0.2542 and 0, i.e., x_4 and x_0.

$$x_5 = x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.2542 - 0}{-0.0004 - 1}(1)$$

$$= 0.2541$$

Since x_4 and x_5 are same up to three decimal places, a positive root is 0.254.

Example 2

Find the root of the equation $2x - \log_{10}x = 7$, *which lies between* 3.5 *and* 4, *correct up to five places of decimal.*

Solution

Let $f(x) = 2x - \log_{10}x - 7$

$f(3.5) = -0.54407$ and $f(4) = 0.39794$

Since $f(3.5) < 0$ and $f(4) > 0$, the root lies between 3.5 and 4.

Let $x_0 = 3.5$, $x_1 = 4$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 3.5 - \frac{4 - 3.5}{0.39794 + 0.54407}(-0.54407)$$

$$= 3.78878$$

$$f(x_2) = f(3.78878) = -0.00094$$

Since $f(3.78878) < 0$ and $f(4) > 0$, the root lies between 3.78878 and 4, i.e., x_2 and x_1.

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 3.78878 - \frac{4 - 3.78878}{0.39794 + 0.00094}(-0.00094)$$

$$= 3.78928$$

$$f(x_3) = f(3.78928) = 0.000003$$

Since $f(3.78928) > 0$ and $f(3.78878) < 0$, the root lies between 3.78928 and 3.78878, i.e., x_3 and x_2.

$$x_4 = x_2 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2)$$

$$= 3.78878 - \frac{3.78928 - 3.78878}{0.000003 + 0.00094}(-0.00094)$$

$$= 3.78928$$

Since x_3 and x_4 are same up to five decimal places, the root is 3.78928.

Example 3

Find a real root of the equation $x \log_{10} x = 1.2$ *by the regula falsi method.* **[Summer 2015]**

Solution

Let $f(x) = x \log_{10} x - 1.2$

$f(2) = -0.5979$ and $f(3) = 0.2314$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 2$, $x_1 = 3$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3-2}{0.2314 + 0.5979}(-0.5979)$$

$$= 2.721$$

$$f(x_2) = f(2.721) = -0.0171$$

Since $f(2.721) < 0$ and $f(3) > 0$, the root lies between 2.721 and 3, i.e., x_2 and x_1.

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 2.721 - \frac{3 - 2.721}{0.2314 + 0.0171}(-0.0171)$$

$$= 2.7402$$

$$f(x_3) = f(2.7402) = -0.0004$$

Since $f(2.7402) < 0$ and $f(3) > 0$, the root lies between 2.7402 and 3, i.e., x_3 and x_1.

$$x_4 = x_1 - \frac{x_3 - x_1}{f(x_3) - f(x_1)} f(x_1)$$

$$= 3 - \frac{2.7042 - 3}{-0.0004 - 0.2314}(0.2314)$$

$$= 2.7406$$

Since x_3 and x_4 are same up to three decimal places, a real root is 2.740.

Example 4

Solve the equation $x \tan x = -1$, *starting with* $x_0 = 2.5$ and $x_1 = 3$, *correct up to three decimal places.*

Solution

Let $f(x) = x \tan x + 1$

$f(2.5) = -0.8676$ and $f(3) = 0.5724$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

Let $x_0 = 2.5$, $x_1 = 3$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{3 - 2.5}{0.5724 + 0.8676}(-0.8676)$$

$$= 2.8013$$

$$f(x_2) = f(2.8013) = 0.0082$$

Since $f(2.8013) > 0$ and $f(2.5) < 0$, the root lies between 2.8013 and 2.5, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{2.8013 - 2.5}{0.0082 + 0.8676}(-0.8676)$$

$$= 2.7985$$

$$f(x_3) = f(2.7985) = 0.0003$$

Since $f(2.7985) > 0$ and $f(2.5) < 0$, the root lies between 2.7985 and 2.5, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{2.7985 - 2.5}{0.0003 + 0.8676}(-0.8676)$$

$$= 2.7984$$

Since x_3 and x_4 are same up to three decimal places, the root is 2.798.

Example 5

Find the real root of the equation $\log_{10} x - \cos x = 0$, *correct to four decimal places.*

Solution

Let $f(x) = \log_{10} x - \cos x$

$f(1) = -0.5403$ and $f(1.5) = 0.10535$

Since $f(1) < 0$ and $f(1.5) > 0$, the root lies between 1 and 1.5.

Let $x_0 = 1$, $x_1 = 1.5$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1 - \frac{1.5 - 1}{0.10535 + 0.5403}(-0.5403)$$

$$= 1.41842$$

$$f(x_2) = f(1.41842) = 0.00002$$

Since $f(1.41842) > 0$ and $f(1) < 0$, the root lies between 1.41842 and 1, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 1 - \frac{1.41842 - 1}{0.00002 + 0.5403}(-0.5403)$$

$$= 1.41840$$

Since x_2 and x_3 are same up to four decimal places, the real root is 1.4184.

Example 6

Find the smallest root of an equation $x - e^{-x} = 0$ correct to three significant digits. **[Summer 2015]**

Solution

Let $f(x) = x - e^{-x}$

$f(0) = -1$ and $f(1) = 0.6321$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $x_1 = 1$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1 - 0}{0.6321 + 1}(-1)$$

$$= 0.6127$$

$$f(x_2) = f(0.6127) = 0.0708$$

Since $f(0.6127) > 0$ and $f(0) < 0$, the root lies between 0.6127 and 0, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.6127 - 0}{0.0708 + 1}(-1)$$

$$= 0.5722$$

$$f(x_3) = f(0.5722) = 0.0079$$

Since $f(0.5722) > 0$ and $f(0) < 0$, the root lies between 0.5722 and 0, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$
$$= 0 - \frac{0.5722 - 0}{0.0079 + 1}(-1)$$
$$= 0.5677$$
$$f(x_4) = f(0.5677) = 0.0009$$

Since $f(0.5677) > 0$ and $f(0) < 0$, the root lies between 0.5677 and 0, i.e., x_4 and x_0.

$$x_5 = x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0)$$
$$= 0 - \frac{0.5677 - 0}{0.0009 + 1}(-1)$$
$$= 0.5672$$

Since x_4 and x_5 are same up to three significant digits, a positive root is 0.567.

Example 7

Find the root of the equation $\cos x - xe^x = 0$ *correct up to three decimal places, lying between* 0.5 *and* 0.7.

Solution

Let $f(x) = \cos x - xe^x$

$f(0.5) = 0.0532$ and $f(0.7) = -0.6448$

Since $f(0.5) > 0$ and $f(0.7) < 0$, the root lies between 0.5 and 0.7.

Let $x_0 = 0.5$, $x_1 = 0.7$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 0.5 - \frac{0.7 - 0.5}{-0.6448 - 0.0532}(0.0532)$$
$$= 0.5152$$
$$f(x_2) = f(0.5152) = 0.0078$$

Since $f(0.5152) > 0$ and $f(0.7) < 0$, the root lies between 0.5152 and 0.7, i.e., x_2 and x_1.

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$
$$= 0.5152 - \frac{0.7 - 0.5152}{-0.6448 - 0.0078}(0.0078)$$
$$= 0.5174$$
$$f(x_3) = f(0.5174) = 0.0011$$

Since $f(0.5174) > 0$ and $f(0.7) < 0$, the root lies between 0.5174 and 0.7, i.e., x_3 and x_1.

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 0.5174 - \frac{0.7 - 0.5174}{-0.6448 - 0.0011}(0.0011)$$

$$= 0.5177$$

Since x_3 and x_4 are same up to three decimal places, the root is 0.517.

EXERCISE 10.2

Find a real root of the following equations correct to three decimal places using regula falsi method:

1. $x^3 + x - 1 = 0$

[Ans.: 0.682**]**

2. $x^3 - 4x - 9 = 0$

[Ans.: 2.707**]**

3. $x^3 - 5x - 7 = 0$

[Ans.: 2.747**]**

4. $xe^3 = 3$

[Ans.: 1.050**]**

5. $e^{-x} - \sin x = 0$

[Ans.: 0.5885**]**

6. $2x = \cos x + 3$

[Ans.: 1.524**]**

7. $x^2 - \log_e x = 12$

[Ans.: 3.646**]**

8. $e^x = 3x$

[Ans.: 1.512**]**

10.4 NEWTON—RAPHSON METHOD

Let $f(x) = 0$ be the given equation and x_0 be an approximate root of the equation. If $x_1 = x_0 + h$ be the exact root then $f(x_1) = 0$.

i.e., $f(x_0 + h) = 0$

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0 \qquad \text{[By Taylor's series]}$$

Since h is small, neglecting h^2 and higher powers of h,

$$f(x_0)+h\,f'(x_0)=0$$

$$h=-\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ $$x_1=x_0+h=x_0-\frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is obtained.

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

In general, $$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

This equation is known as the *Newton–Raphson formula* or *Newton's iteration formula*.

10.4.1 Geometrical Interpretation

Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 10.3). The equation of the tangent at $P_0[x_0, f(x_0)]$ is

$$y-f(x_0)=f'(x_0)(x-x_0)$$

This line cuts the x-axis at x_1.

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

which is a first approximation to the root α.

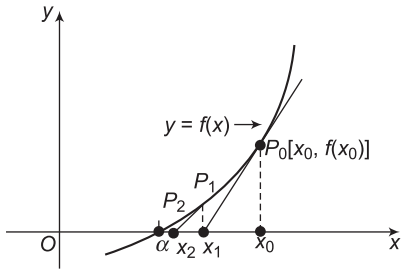


Fig. 10.3

If P_1 is the point corresponding to x_1 on the curve then the tangent at P_1 will cut the x-axis at x_2 which is nearer to α and is the second approximation to the root. Repeating this process, the root α is approached quite rapidly. Thus, this method consists of replacing the part of the curve between the point P_0 and the x-axis by means of the tangent to the curve at P_0.

10.4.2 Convergence of the Newton–Raphson Method

By the Newton–Raphson method,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=\phi(x_n) \qquad \text{...(10.1)}$$

The Newton–Raphson method converges if $|\phi'(x)| < 1$.

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = 1 - \left[\frac{[f'(x)]^2 - f(x)\, f''(x)}{[f'(x)]^2}\right] = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

$$|\phi'(x)| = \left|\frac{f(x)\, f''(x)}{[f'(x)]^2}\right|$$

Hence, the Newton–Raphson method converges if

$$\left|\frac{f(x)\, f''(x)}{[f'(x)]^2}\right| < 1$$

$$|f(x)\, f''(x)| < [f'(x)]^2 \qquad ...(10.2)$$

If α is the actual root of $f(x) = 0$, a small interval should be selected in which $f(x)$, $f'(x)$ and $f''(x)$ are all continuous and the condition given by Eq. (10.2) is satisfied.

Hence, the Newton–Raphson method always converges provided the initial approximation x_0 is taken very close to the actual root α.

10.4.3 Rate of Convergence of the Newton—Raphson Method

Let α be exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root. If $\in_n$ and $\in_{n+1}$ are the corresponding errors then

$$x_n = \alpha + \in_n$$

$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (10.1),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$\in_{n+1} = \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$= \in_n - \frac{f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha) + \cdots} \qquad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha)} \qquad [\because \quad f(\alpha) = 0]$$

Neglecting the derivatives of order higher than two,

$$\begin{aligned}
\epsilon_{n+1} &= \epsilon_n - \frac{\epsilon_n\, f'(\alpha) + \frac{\epsilon_n^2}{2} f''(\alpha)}{f'(\alpha) + \epsilon_n\, f''(\alpha)} \\
&= \frac{1}{2}\left[\frac{\epsilon_n^2\, f''(\alpha)}{f'(\alpha) + \epsilon_n\, f''(\alpha)}\right] \\
&= \frac{\epsilon_n^2}{2}\left[\frac{\frac{f''(\alpha)}{f'(\alpha)}}{1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)}}\right] \\
&\approx \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} \qquad \text{...(10.3)}
\end{aligned}$$

Equation 10.3 shows that the error at each stage is proportional to the square of the error in the previous stage. Hence, the Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Example 1

Find the root of the equation $x^3 + x - 1 = 0$, *correct up to four decimal places.*

Solution

Let $f(x) = x^3 + x - 1$

$f(0) = -1$ and $f(1) = 1$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3x^2 + 1$$

By the Newton–Raphson method,

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
f(x_0) &= f(1) = 1 \\
f'(x_0) &= f'(1) = 4 \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 1 - \frac{1}{4} \\
&= 0.75
\end{aligned}$$

$$f(x_1) = f(0.75) = 0.171875$$
$$f'(x_1) = f'(0.75) = 2.6875$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.75 - \frac{0.171875}{2.6875}$$
$$= 0.68605$$
$$f(x_2) = f(0.68605) = 0.00894$$
$$f'(x_2) = f'(0.68605) = 2.41198$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.68605 - \frac{0.00894}{2.41198}$$
$$= 0.68234$$
$$f(x_3) = f(0.68234) = 0.000028$$
$$f'(x_3) = f'(0.68234) = 2.39676$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.68234 - \frac{0.000028}{2.39676}$$
$$= 0.68233$$

Since x_3 and x_4 are same up to four decimal places, the root is 0.6823.

Example 2

Find a root of $x^4 - x^3 + 10x + 7 = 0$, *correct up to three decimal places between* –2 *and* –1 *by the Newton–Raphson method.*

Solution

Let $f(x) = x^4 - x^3 + 10x + 7$

The root lies between –2 and –1.

Let $x_0 = -2$

$$f'(x) = 4x^3 - 3x^2 + 10$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(-2) = 11$$

$$f'(x_0) = f'(-2) = -34$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -2 - \frac{11}{(-34)}$$

$$= -1.6765$$

$$f(x_1) = f(-1.6765) = 2.8468$$

$$f'(x_1) = f'(-1.6765) = -17.2802$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -1.6765 - \frac{2.8468}{(-17.2802)}$$

$$= -1.5118$$

$$f(x_2) = f(-1.5118) = 0.561$$

$$f'(x_2) = f'(-1.5118) = -10.6777$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -1.5118 - \frac{0.561}{(-10.6777)}$$

$$= -1.4593$$

$$f(x_3) = f(-1.4593) = 0.0497$$

$$f'(x_3) = f(-1.4593) = -8.8193$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= -1.4593 - \frac{0.0497}{(-8.8193)}$$

$$= -1.4537$$

$$f(x_4) = f(-1.4537) = 0.0008$$

$$f'(x_4) = f'(-1.4537) = -8.6278$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= -1.4537 - \frac{0.0008}{(-8.6278)}$$

$$= -1.4536$$

Since x_4 and x_5 are same up to three decimal places, a root is –1.453.

Example 3

Find the root of $x^4 - x - 10 = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = x^4 - x - 10$

$f(1) = -10$, and $f(2) = 4$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$f'(x) = 4x^3 - 1$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(2) = 4$$
$$f'(x_0) = f'(2) = 31$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{4}{31}$$
$$= 1.871$$
$$f(x_1) = f(1.871) = 0.3835$$
$$f'(x_1) = f'(1.871) = 25.1988$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.871 - \frac{0.3835}{25.1988}$$
$$= 1.8558$$
$$f(x_2) = f(1.8558) = 5.2922 \times 10^{-3}$$
$$f'(x_2) = f'(1.8558) = 24.5655$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.8558 - \frac{5.2922 \times 10^{-3}}{24.5655}$$
$$= 1.8556$$

Since x_2 and x_3 are same up to three decimal places, the root is 1.855.

Example 4

Find the real root of $x \log_{10} x - 1.2 = 0$, *correct up to three decimal places.* **[Summer 2015]**

Solution

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2,\ f(2) = -0.5979$ and $f(3) = 0.2314$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 3$

$$f'(x) = \log_{10} x + x\frac{1}{x \log_e 10} = \log_{10} x + \log_{10} e = \log_{10} x + 0.4343$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(3) = 0.2314$$

$$f'(x_0) = f'(3) = 0.9114$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{0.2314}{0.9114}$$

$$= 2.7461$$

$$f(x_1) = f(2.7461) = 4.759 \times 10^{-3}$$

$$f'(x_1) = f'(2.7461) = 0.8730$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7461 - \frac{4.759 \times 10^{-3}}{0.8730}$$

$$= 2.7406$$

$$f(x_2) = f(2.7406) = -4.0202 \times 10^{-5}$$

$$f'(x_2) = f'(2.7406) = 0.8721$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7406 - \frac{(-4.0202 \times 10^{-5})}{0.8721}$$

$$= 2.7406$$

Since x_2 and x_3 are the same up to three decimal places, the real root is 2.7406.

Example 5

Find a root between 0 *and* 1 *of the equation* $e^x \sin x = 1$, *correct up to four decimal places.*

Solution

Let $f(x) = e^x \sin x - 1$

$f(0) = -1$ and $f(1) = 1.28$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$

$$f'(x) = e^x (\cos x + \sin x)$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(0) = -1$$

$$f'(x_0) = f'(0) = 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{(-1)}{1}$$

$$= 1$$

$$f(x_1) = f(1) = 1.2874$$

$$f'(x_1) = f'(1) = 3.7560$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{1.2874}{3.7560}$$

$$= 0.6572$$

$$f(x_2) = f(0.6572) = 0.1787$$

$$f'(x_2) = f'(0.6572) = 2.7062$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6572 - \frac{0.1787}{2.7062}$$

$$= 0.5912$$

$$f(x_3) = f(0.5912) = 6.6742 \times 10^{-3}$$
$$f'(x_3) = f'(0.5912) = 2.5063$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.5912 - \frac{6.6742 \times 10^{-3}}{2.5063}$$
$$= 0.5885$$
$$f(x_4) = f(0.5885) = -8.1802 \times 10^{-5}$$
$$f'(x_4) = f'(0.5885) = 2.4982$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$
$$= 0.5885 - \frac{-8.1802 \times 10^{-5}}{2.4982}$$
$$= 0.5885$$

Since x_4 and x_5 are the same up to four decimal places, the root is 0.5885.

Example 6

Find the real root of the equation $3x = \cos x + 1$, *correct up to four decimal places.*

Solution

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2$ and $f(1) = 1.4597$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3 + \sin x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(1) = 1.4597$$
$$f'(x_0) = f'(1) = 3.8415$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{1.4597}{3.8415}$$
$$= 0.62$$

$$f(x_1) = f(0.62) = 0.0461$$
$$f'(x_1) = f'(0.62) = 3.5810$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.62 - \frac{0.0461}{3.5810}$$
$$= 0.6071$$
$$f(x_2) = f(0.6071) = -5.8845 \times 10^{-6}$$
$$f'(x_2) = f'(0.6071) = 3.5705$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.6071 - \frac{-5.8845 \times 10^{-6}}{3.5705}$$
$$= 0.6071$$

Since x_2 and x_3 are the same up to four decimal places, the real root is 0.6071.

Example 7

Find the real positive root of the equation $x \sin x + \cos x = 0$, *which is near* $x = \pi$ *correct up to four significant digits.* **[Summer 2015]**

Solution

Let $f(x) = x \sin x + \cos x$

Let $x_0 = \pi$

$$f'(x) = x \cos x + \sin x - \sin x = x \cos x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(\pi) = -1$$
$$f'(x_0) = f'(\pi) = -\pi$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= \pi - \frac{(-1)}{(-\pi)}$$
$$= 2.82328$$

$$f(x_1) = f(2.82328) = -0.06618$$
$$f'(x_1) = f'(2.823287) = -2.68145$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.82328 - \frac{(-0.06618)}{(-2.68145)}$$
$$= 2.7986$$
$$f(x_2) = f(2.7986) = -0.00056$$
$$f'(x_2) = f'(2.7986) = -2.63559$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.7986 - \frac{(-0.00056)}{(-2.63559)}$$
$$= 2.79839$$
$$f(x_3) = f(2.79839) = -0.0001$$
$$f'(x_3) = f'(2.79839) = -2.63519$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 2.79839 - \frac{(-0.0001)}{(-2.63519)}$$
$$= 2.79839$$

Since x_3 and x_4 are same up to four decimal point, the root is 2.7983.

Example 8

Find the positive root of $x = \cos x$ using Newton's method correct to three decimal places.

Solution

Let $f(x) = x - \cos x$

$f(0) = -1$ and $f(1) = 0.4597$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$f'(x) = 1 + \sin x$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(1) = 0.4597$$

$$f'(x_0) = f'(1) = 1.8415$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{0.4597}{1.8415}$$

$$= 0.7504$$

$$f(x_1) = f(0.7504) = 0.019$$

$$f'(x_1) = f'(0.7504) = 1.6819$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7504 - \frac{0.019}{1.6819}$$

$$= 0.7391$$

$$f(x_2) = f(0.7391) = 0.00002$$

$$f'(x_2) = f'(0.7391) = 1.6736$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7391 - \frac{0.00002}{1.6736}$$

$$= 0.7391$$

Since x_2 and x_3 are same up to three decimal places, the root is 0.739.

Example 9

Derive the iteration formula for $\sqrt{N}$ *and, hence, find*

(i) $\sqrt{28}$ **[Summer 2015]**

(ii) $\sqrt{65}$ **[Winter 2014]**

(iii) $\sqrt{3}$ **[Winter 2014]**

correct up to three decimal places.

Solution

Let $x = \sqrt{N}$

$x^2 - N = 0$

Let $f(x) = x^2 - N$

$f'(x) = 2x$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

This is the iteration formula for $\sqrt{N}$.

(i) For $N = 28$, $f(x) = x^2 - 28$

$f(5) = -3$ and $f(6) = 8$

Since $f(5) < 0$ and $f(6) > 0$, the root lies between 5 and 6.

Let $x_0 = 5$

$$x_{n+1} = \frac{x_n^2 + 28}{2x_n}$$

$$x_1 = \frac{x_0^2 + 28}{2x_0} = 5.3$$

$$x_2 = \frac{x_1^2 + 28}{2x_1} = 5.2915$$

$$x_3 = \frac{x_2^2 + 28}{2x_2} = 5.2915$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{28} = 5.2915$$

(ii) For $N = 65$, $f(x) = x^2 - 65$

$f(8) = -1$ and $f(9) = 16$

Since $f(8) < 0$ and $f(9) > 0$, the root lies between 8 and 9.

Let $x_0 = 8$

$$x_{n+1} = \frac{x_n^2 + 65}{2x_n}$$

$$x_1 = \frac{x_0^2 + 65}{2x_0} = 8.0625$$

$$x_2 = \frac{x_1^2 + 65}{2x_1} = 8.0623$$

Since x_1 and x_2 are same up to three decimal places,

$$\sqrt{65} = 8.0623$$

(iii) For $N = \sqrt{3}, f(x) = x^2 - 3$

$f(1) = -2$ and $f(2) = 1$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

$$x_1 = \frac{x_0^2 + 3}{2x_0} = 1.75$$

$$x_2 = \frac{x_1^2 + 3}{2x_1} = 1.7321$$

$$x_3 = \frac{x_2^2 + 3}{2x_2} = 1.7321$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{3} = 1.7321$$

Example 10

Find an iterative formula for $\sqrt[k]{N}$, where N is a positive number and hence, evaluate (i) $\sqrt[3]{11}$, and (ii) $\sqrt[3]{58}$ **[Summer 2015]**

Solution

Let $x = \sqrt[k]{N}$

$$x^k - N = 0$$

Let $f(x) = x^k - N$

$f'(x) = kx^{k-1}$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^k - N}{k\,x_n^{k-1}}$$

$$= \frac{(k-1)x_n^k + N}{k\,x_n^{k-1}}$$

This is the iterative formula for $\sqrt[k]{N}$.

(i) When $N = 11$ and $k = 3$,

$f(x) = x^3 - 11$

$f(2) = -3$ and $f(3) = 16$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 3$

$$x_{n+1} = \frac{2x_n^3 + 11}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 11}{3x_0^2} = 2.4074$$

$$x_2 = \frac{2x_1^3 + 11}{3x_1^2} = 2.2376$$

$$x_3 = \frac{2x_2^3 + 11}{3x_2^2} = 2.2240$$

$$x_4 = \frac{2x_3^3 + 11}{3x_3^2} = 2.2240$$

Since x_3 and x_4 are same up to four decimal places,

$$\sqrt[3]{11} = 2.2240$$

(ii) When $N = 58$ and $k = 3$,

$f(x) = x^3 - 58$

$f(3) = -31$ and $f(4) = 6$

Since $f(3) < 0$ and $f(4) > 0$, the root lies between 3 and 4.

Let $x_0 = 4$

$$x_{n+1} = \frac{2x_n^3 + 58}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 58}{3x_0^2} = 3.875$$

$$x_2 = \frac{2x_1^3 + 58}{3x_1^2} = 3.8709$$

$$x_3 = \frac{2x_2^3 + 58}{3x_2^2} = 3.8709$$

Since x_2 and x_3 are same up to four decimal places,

$$\sqrt[3]{58} = 3.8709$$

EXERCISE 10.3

I. Find the roots of the following equations.

1. $x^3 - x - 1 = 0$

[Ans.: 1.3247]

2. $x^3 + 2x^2 + 50x + 7 = 0$

[Ans.: −0.1407]

3. $x^3 - 5x + 3 = 0$

[Ans.: 0.6566]

4. $x^4 - x - 9 = 0$

[Ans.: 1.8134]

5. $\cos x - xe^x = 0$

[Ans.: 0.5177]

6. $x \log_{10} x = 4.772393$

[Ans.: 6.0851]

7. $x - 2\sin x = 0$

[Ans.: 1.8955]

8. $x \tan x = 1.28$

[Ans.: 6.4783]

9. $\cos x = x^2$

[Ans.: 0.8241]

II. Find the values of the following:

1. $\sqrt{35}$

[Ans.: 5.916]

2. $\sqrt[3]{24}$

[Ans.: 2.884]

3. $\dfrac{1}{\sqrt{14}}$

[Ans.: 0.2673]

10.5 SECANT METHOD

The Newton–Raphson method requires the evaluation of two functions (the function and its derivative) per iteration. For complicated expressions, the method takes a large amount of time. Hence, it is desirable to have a method that converges as fast as the Newton–Raphson method but involves only evaluation of the function.

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be the approximate roots of the equation $f(x) = 0$ and $f(x_0)$ and $f(x_1)$ are their function values respectively. If x_2 is the point of intersection of the x-axis and the line joining points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1 (Fig. 10.4).

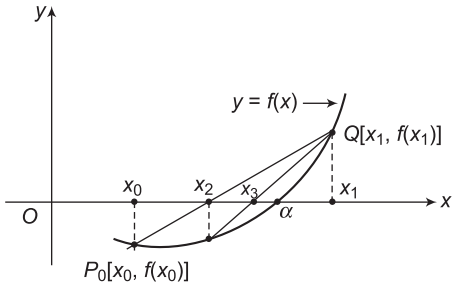


Fig. 10.4

Using the slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$x_2 - x_1 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Using x_1 and x_2, the process is repeated to obtain x_3.

In general,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \qquad n \geq 1$$

This method is similar to the regula falsi method. This method starts with two initial approximations x_0 and x_1 and calculates x_2 by the same formula as in the regula falsi

method but proceeds to the next iteration without considering any root bracketing, i.e., the condition $f(x_0)\, f(x_1) < 0$.

Convergence of the Secant Method

By the Secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \ldots(10.4)$$

Let α be the exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root.

If $\in_n$, $\in_{n-1}$, are the corresponding error then

$$x_n = \alpha + \in_n$$
$$x_{n-1} = \alpha + \in_{n-1}$$
$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (10.4),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$\in_{n+1} = \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$= \in_n - \frac{(\in_n - \in_{n-1}) \left[f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots \right]}{(\in_n - \in_{n-1}) f'(\alpha) + \frac{1}{2} (\in_n^2 - \in_{n-1}^2) f''(\alpha) + \cdots} \quad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right) \left[\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) \right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \frac{1}{2} \left(\in_n^2 - \in_{n-1}^2 \right) f''(\alpha)} \left[\begin{array}{l} \because\ f(\alpha) = 0 \\ \text{and neglecting} \\ \text{higher order} \\ \text{derivatives of } f(\alpha) \end{array} \right]$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right) \in_n f'(\alpha) \left[1 + \frac{\in_n}{2} \frac{f''(\alpha)}{f(\alpha)} \right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \left[1 + \frac{\left(\in_n + \in_{n-1}\right)}{2} \frac{f''(\alpha)}{f(\alpha)} \right]}$$

$$= \in_n - \frac{\in_n \left(1 + c \in_n\right)}{1 + c \left(\in_n + \in_{n-1}\right)}$$

$$\text{where } c = \frac{1}{2} \frac{f''(\alpha)}{f(\alpha)} \qquad \ldots(10.5)$$

$$\in_{n+1} = \frac{\in_n + c \in_n^2 + c \in_n \in_{n-1} - \in_n - c \in_n^2}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$= \frac{c \in_n \in_{n-1}}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$\approx c \in_n \in_{n-1} \qquad \left[\because\ 1 + c\left(\in_n + \in_{n-1}\right) \approx 1\right] \qquad ...(10.6)$$

Equation (10.5) is a nonlinear difference equation which can be solved by letting $\in_{n+1} = A \in_n^p$ or $\in_n = A \in_{n-1}^p$.

$$\therefore \qquad \in_{n-1} = \in_n^{\frac{1}{p}} A^{-\frac{1}{p}}$$

Substituting in Eq. (10.6),

$$A \in_n^p = c \in_n \in_n^{\frac{1}{p}} A^{-\frac{1}{p}}$$

$$\in_n^p = c\, A^{-\left(1+\frac{1}{p}\right)} \in_n^{1+\frac{1}{p}}$$

Equating the power of $\in_n$ on both the sides,

$$p = 1 + \frac{1}{p}$$

$$p^2 - p - 1 = 0$$

$$p = \frac{1}{2}(1 \pm \sqrt{5})$$

Taking the positive sign only,

$$p = 1.618$$

$$\in_{n+1} = A \in_n^{1.618}$$

Hence, the rate of convergence of the secant method is 1.618 which is lesser than the Newton–Raphson method. The secant method evaluates the function only once in each iteration, whereas the Newton–Raphson method evaluates two functions $f(x)$ and $f'(x)$ in each iteration. Hence, the secant method is more efficient than the Newton–Raphson method.

Example 1

Find the approximate root of $x^3 - 2x - 1 = 0$, *starting from* $x_0 = 1.5$, $x_1 = 2$, *correct upto three decimal places.*

Solution

Let $f(x) = x^3 - 2x - 1$

$$x_0 = 1.5,\ x_1 = 2$$

$$f(x_0) = f(1.5) = -0.625 \text{ and } f(x_1) = f(2) = 3$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2 - 1.5}{3 + 0.625}(3)$$

$$= 1.5862$$

$$f(x_2) = f(1.5862) = -0.1815$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.5862 - \frac{1.5862 - 2}{-0.1815 - 3}(-0.1815)$$

$$= 1.6098$$

$$f(x_3) = f(1.6098) = -0.0479$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.6098 - \frac{1.6098 - 1.5862}{-0.0479 + 0.1815}(-0.0479)$$

$$= 1.6183$$

$$f(x_4) = f(1.6183) = 0.0016$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.6183 - \frac{1.6183 - 1.6098}{0.0016 + 0.0479}(0.0016)$$

$$= 1.6181$$

Since x_4 and x_5 are same up to three decimal places, the root is 1.618.

Example 2

Find the approximate root of the equation $x^3 + x^2 - 3x - 3 = 0$, correct up to five decimal places.

Solution

Let $f(x) = x^3 + x^2 - 3x - 3 = 0$

Let $x_0 = 1, x_1 = 2$

$f(x_0) = f(1) = -4$ and $f(x_1) = f(2) = 3$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2-1}{3+4}(3)$$

$$= 1.57143$$

$$f(x_2) = f(1.57143) = -1.36442$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.57143 - \frac{1.57143 - 2}{-1.36442 - 3}(-1.36442)$$

$$= 1.70541$$

$$f(x_3) = f(1.70541) = -0.24775$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.70541 - \frac{1.70541 - 1.57143}{-0.24775 + 1.36442}(-0.24775)$$

$$= 1.73514$$

$$f(x_4) = f(1.73514) = 0.0293$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.73514 - \frac{1.73514 - 1.70541}{0.0293 + 0.24775}(0.0293)$$

$$= 1.732$$

$$f(x_5) = f(1.732) = -0.00048$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 1.732 - \frac{1.732 - 1.73514}{-0.00048 - 0.0293}(-0.00048)$$

$$= 1.73205$$

$$f(x_6) = f(1.73205) = -0.000008$$

$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6)$$
$$= 1.73205 - \frac{1.73205 - 1.732}{-0.000008 + 0.00048}(-0.000008)$$
$$= 1.73205$$

Since x_6 and x_7 are same up to five decimal places, the root is 1.73205.

Example 3

Find the root of $x \log_{10} x - 1.9 = 0$, *correct up to three decimal places with* $x_0 = 3$ *and* $x_1 = 4$.

Solution

Let $f(x) = x \log_{10} x - 1.9$

$x_0 = 3, x_1 = 4$

$f(x_0) = f(3) = -0.4686$ and $f(x_1) = f(4) = 0.5082$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$
$$= 4 - \frac{4 - 3}{0.5082 + 0.4686}(0.5082)$$
$$= 3.4797$$
$$f(x_2) = f(3.4797) = -0.0156$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$
$$= 3.4797 - \frac{3.4797 - 4}{-0.0156 - 0.5082}(-0.0156)$$
$$= 3.4952$$
$$f(x_3) = f(3.4952) = -0.0005$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$
$$= 3.4952 - \frac{3.4952 - 3.4797}{-0.0005 + 0.0156}(-0.0005)$$
$$= 3.4957$$

Since x_3 and x_4 are same up to three decimal places, the root is 3.495.

Example 4

Find the positive solution of $x - 2 \sin x = 0$, *correct up to three decimal places starting from* $x_0 = 2$ *and* $x_1 = 1.9$. **[Summer 2014]**

Solution

Let $f(x) = x - 2 \sin x$

$x_0 = 2, x_1 = 1.9$

$f(x_0) = f(2) = 0.1814$ and $f(x_1) = f(1.9) = 0.0074$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1.9 - \frac{1.9 - 2}{0.0074 - 0.1814}(0.0074)$$

$$= 1.8957$$

$$f(x_2) = f(1.8957) = 0.00034$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.8957 - \frac{1.8957 - 1.9}{0.00034 - 0.0074}(0.00034)$$

$$= 1.8955$$

Since x_2 and x_3 are same up to three decimal places, the positive root is 1.895.

Example 5

Solve $xe^x - 1 = 0$, *correct up to three decimal places between* 0 *and* 1.

Solution

Let $f(x) = xe^x - 1$

Let $x_0 = 0, x_1 = 1$

$f(x_0) = f(0) = -1$ and $f(x_1) = f(1) = 1.7183$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{1.7183+1}(1.7183)$$

$$= 0.3679$$

$$f(x_2) = f(0.3679) = -0.4685$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3679 - \frac{0.3679-1}{-0.4685-1.7183}(-0.4685)$$

$$= 0.5033$$

$$f(x_3) = f(0.5033) = -0.1675$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.5033 - \frac{0.5033-0.3679}{-0.1675+0.4685}(-0.1675)$$

$$= 0.5786$$

$$f(x_4) = f(0.5786) = 0.032$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5786 - \frac{0.5786-0.5033}{0.032+0.1675}(0.032)$$

$$= 0.5665$$

$$f(x_5) = f(0.5665) = -0.0018$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 0.5665 - \frac{0.5665-0.5786}{-0.0018-0.032}(-0.0018)$$

$$= 0.5671$$

$$f(x_6) = f(0.5671) = -0.0001$$

$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6)$$

$$= 0.5671 - \frac{0.5671-0.5665}{-0.0001+0.0018}(-0.0001)$$

$$= 0.5671$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.567.

Example 6

Find the root of $\cos x - xe^x = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = \cos x - xe^x$

Let $x_0 = 0, x_1 = 1$

$f(x_0) = f(0) = 1$ and $f(x_1) = f(1) = -2.178$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{-2.178-1}(-2.178)$$

$$= 0.3147$$

$$f(x_2) = f(0.3147) = 0.5198$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3147 - \frac{0.3147-1}{0.5198+2.178}(0.5198)$$

$$= 0.4467$$

$$f(x_3) = f(0.4467) = 0.2036$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.4467 - \frac{0.4467-0.3147}{0.2036-0.5198}(0.2036)$$

$$= 0.5317$$

$$f(x_4) = f(0.5317) = -0.0429$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5317 - \frac{0.5317-0.4467}{-0.0429-0.2036}(-0.0429)$$

$$= 0.5169$$

$$f(x_5) = f(0.5169) = 0.0026$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$
$$= 0.5169 - \frac{0.5169 - 0.5317}{0.0026 + 0.0429}(0.0026)$$
$$= 0.5177$$
$$f(x_6) = f(0.5177) = 0.0002$$
$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6)$$
$$= 0.5177 - \frac{0.5177 - 0.5169}{0.0002 - 0.0026}(0.0002)$$
$$= 0.5178$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.517.

EXERCISE 10.4

Find a real root of the following equations correct upto three decimal places using secant method:

1. $x^3 - 2x^2 + 3x - 4 = 0$

[Ans.: 1.650**]**

2. $x^3 + 3x^2 - 3 = 0$

[Ans.: 0.879**]**

3. $e^x - 4x = 0$

[Ans.: 0.357**]**

4. $\sin x = e^x - 3x$

[Ans.: 0.360**]**

5. $2x - 7 - \log_{10} x = 0$

[Ans.: 3.789**]**

6. $e^x \tan x = 1$

[Ans.: 3.183**]**

7. $3x - 6 = \log_{10} x$

[Ans.: 2.108**]**

Points to Remember

Bisection Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs. The first approximation to the root is

$$x_2 = \frac{x_0 + x_1}{2}$$

If $f(x_0)$and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2 and the next approximation x_3 is obtained as

$$x_3 = \frac{x_0 + x_2}{2}$$

This process is repeated till the root is obtained to the desired accuracy.

Regula Falsi Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs.

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy.

Newton–Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Secant Method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

The rate of convergence of the secant method is 1.618.

CHAPTER 11
Eigenvalues by Power and Jacobi Methods

Chapter Outline

11.1 Introduction
11.2 Eigenvalues and Eigenvectors
11.3 Nature of Eigenvalues of Special Types of Matrices
11.4 Relations between Eigenvalues and Eigenvectors
11.5 Power Method
11.6 Jacobi Method

11.1 INTRODUCTION

Eigenvalues and eigenvectors are important concepts in linear algebra. They are derived from the German word 'eigen' which means proper or characteristic. Eigenvectors are nonzero vectors that get mapped into scalar multiples of themselves under a linear operator. These are useful in solving systems of differential equations, analyzing population growth models, and are also useful in quantum mechanics and economics.

11.2 EIGENVALUES AND EIGENVECTORS

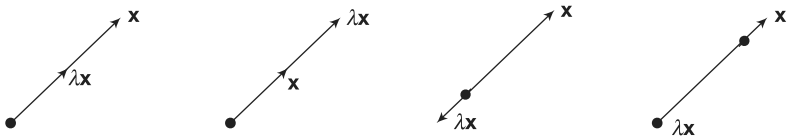
Any nonzero vector $\mathbf{x}$ is said to be a characteristic vector or eigenvector of a square matrix A, if there exists a number λ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $A = [a_{ij}]_{n \times n}$ is an n-rowed square matrix and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

Also, λ is said to be characteristic root or characteristic value of eigenvalue of the matrix A.

Depending on the sign and the magnitude of the eigenvalue λ corresponding to $\mathbf{x}$, the linear operator $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches eigenvector $\mathbf{x}$ by a factor λ. If λ is negative, the direction of the eigenvector reverses.



(i) $0 \le \lambda \le 1$ (ii) $\lambda \ge 1$ (iii) $-1 \le \lambda \le 0$ (iv) $\lambda \le -1$

Fig. 11.1

Now, $$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x}$$

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The matrix $A - \lambda I$ is called the characteristic matrix of A, where I is the unit matrix of order n.

$$\det (A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} - \lambda \end{vmatrix}$$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A.

The equation $\det (A - \lambda I) = 0$ is called the characteristic equation of A and the roots of this equation are called the eigenvalues of the matrix A. The set of all eigenvectors is called the eigenspace of A corresponding to λ. The set of all eigenvalues of A is called the spectrum of A.

Notes

(i) The sum of the eigenvalues of a matrix is equal to the sum of its principal diagonal elements.

(ii) The product of all the eigenvalues of a matrix is equal to the determinant of the matrix.

11.3 NATURE OF EIGENVALUES OF SPECIAL TYPES OF MATRICES

(i) The eigenvalues of a triangular matrix are the diagonal elements of the matrix.

(ii) The eigenvalues of a real symmetric matrix are real.

(iii) The eigenvalues of a skew real symmetric matrix are either purely imaginary or zero.

(iv) The eigenvalues of a Hermitian matrix are real.

(v) The eigenvalues of a skew Hermitian matrix are either purely imaginary or zero.

(vi) The eigenvalues of a unitary matrix are of unit modulus.

(vii) The eigenvalues of an orthogonal matrix are of unit modulus.

11.4 RELATIONS BETWEEN EIGENVALUES AND EIGENVECTORS

(i) If $\mathbf{x}$ is an eigenvector of a matrix A corresponding to the eigenvalue λ then $k\mathbf{x}$ is also an eigenvector of A corresponding to the same eigenvalue λ, where k is any nonzero scalar.

(ii) If $\mathbf{x}$ is an eigenvector of a matrix A then $\mathbf{x}$ cannot correspond to more than one eigenvalue of A.

(iii) The eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.

(iv) If two or more eigenvalues are equal then the corresponding eigenvectors may or may not be linearly independent.

(v) The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

(vi) Any two eigenvectors corresponding to two distinct eigenvalues of a unitary matrix are orthogonal .

(vii) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

(viii) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

(ix) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $\lambda \pm \mathrm{k}$ is an eigenvalue of $A \pm kI$ and $\mathbf{x}$ is a corresponding eigenvector.

(x) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $k\lambda$ is an eigenvalue of the matrix kA and $\mathbf{x}$ is a corresponding eigenvector.

(xi) If λ is an eigenvalue of a matrix A then λ is also an eigenvalue of the matrix A^T. Matrices A and A^T need not have the same eigenvectors.

11.5 POWER METHOD

The eigenvalues of an $n \times n$ matrix are obtained by solving the characteristic equation

$$\lambda^n + c_{n-1}\,\lambda^{n-1} + c_{n-2}\,\lambda^{n-2} + \cdots + c_0 = 0$$

It is difficult and time-consuming to solve such polynomial equations for large values of n.

Moreover, numerical methods for approximating roots of polynomial equations of high degree are sensitive to rounding errors. The power method is an iterative method to find the numerically largest (dominant) eigenvalue and the corresponding eigenvectors of the matrix A.

This method can be applied only if the eigenvalues are real and corresponding eigenvectors are linearly independent.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, where λ_1 is the dominant eigenvalue of A

i.e., $$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ be the linearly independent eigenvectors corresponding to $\lambda_1, \lambda_2, ..., \lambda_n$ then any arbitrary vector $\mathbf{x}_0$ can be expressed as

$$\mathbf{x}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

Premultiplying by A,

$$\begin{aligned} A\mathbf{x}_0 &= c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_nA\,\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \qquad [\because\ A\mathbf{x} = \lambda\mathbf{x}] \end{aligned}$$

Similarly,

$$\begin{aligned} A^2\mathbf{x}_0 &= c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 + \cdots + c_n\lambda_n^2\,\mathbf{x}_n \\ A^k\mathbf{x}_0 &= c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + \cdots + c_n\lambda_n^k\mathbf{x}_n \\ &= \lambda_1^k\left[c_1\mathbf{x}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right] \end{aligned}$$

For the large value of k,

$$\lim_{k\to 0}\left(\frac{\lambda_i}{\lambda_1}\right)^k = 0, \qquad i = 1,2,...,n$$

$$A^k\,\mathbf{x}_0 = \lambda_1^k\,c_1\,\mathbf{x}_1$$

where $c_1\mathbf{x}_1$ is the eigenvector corresponding to λ_1

Notes

(i) The numerically smallest eigenvalue of a matrix A is reciprocal of the numerically largest eigenvalue of A^{-1}.

(ii) The power method converges quickly if $\frac{|\lambda_2|}{|\lambda_1|}$ is small, and slowly if $\frac{|\lambda_2|}{|\lambda_1|}$ is close to 1.

(iii) If $\mathbf{x}$ is an eigenvector of a matrix A then its corresponding eigenvalue is given by

$$\lambda = \frac{\mathbf{x}\cdot A\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} = \frac{\mathbf{x}^TA\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

This quotient is called the Rayleigh quotient.

In cases for which the power method generates a good approximation of a dominant eigenvector, the Rayleigh quotient provides a correspondingly good approximation of the dominant eigenvalue.

Example 1

Find the dominant eigenvalue of $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$.

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 0.56 \\ 1 \end{bmatrix} = 9\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 0.56 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 6.80 \end{bmatrix} = 6.8\begin{bmatrix} 0.61 \\ 1 \end{bmatrix} = 6.8\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 0.61 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.22 \\ 7.05 \end{bmatrix} = 7.05\begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = 7.05\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.2 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = 7\mathbf{x}_4$$

Hence, the dominant eigenvalue is 7 and the corresponding eigenvector is $\begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$.

Example 2

Find the largest eigenvalue of the matrix $A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$.

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -0.67 \end{bmatrix} = 3\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -0.67 \end{bmatrix} = \begin{bmatrix} 6.35 \\ -4.68 \end{bmatrix} = 6.35\begin{bmatrix} 1 \\ -0.74 \end{bmatrix} = 6.35\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -0.74 \end{bmatrix} = \begin{bmatrix} 6.7 \\ -4.96 \end{bmatrix} = 6.7\begin{bmatrix} 1 \\ -0.74 \end{bmatrix} = 6.7\mathbf{x}_3$$

Hence, the largest eigenvalue is 6.7 and the corresponding eigenvector is $\begin{bmatrix} 1 \\ -0.74 \end{bmatrix}$.

Example 3

Find the largest eigenvalue of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 4\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 1.75 \end{bmatrix} = 4.5\begin{bmatrix} 1 \\ 0.39 \end{bmatrix} = 4.5\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.39 \end{bmatrix} = \begin{bmatrix} 4.78 \\ 2.17 \end{bmatrix} = 4.78\begin{bmatrix} 1 \\ 0.45 \end{bmatrix} = 4.78\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 2.35 \end{bmatrix} = 4.9\begin{bmatrix} 1 \\ 0.48 \end{bmatrix} = 4.9\mathbf{x}_4$$

$$A\mathbf{x}_4 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.48 \end{bmatrix} = \begin{bmatrix} 4.96 \\ 2.44 \end{bmatrix} = 4.96\begin{bmatrix} 1 \\ 0.49 \end{bmatrix} = 4.96\mathbf{x}_5$$

$$A\mathbf{x}_5 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 4.98 \\ 2.47 \end{bmatrix} = 4.98\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 4.98\mathbf{x}_6$$

$$A\mathbf{x}_6 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2.5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 5\mathbf{x}_7$$

Hence, the largest eigenvalue is 5 and the corresponding eigenvector is $\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$.

Example 4

Find the largest eigenvalue of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. **[Summer 2015]**

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = 2\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.5\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = 2.8\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.43 \\ 1.86 \end{bmatrix} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = 3.43\mathbf{x}_4$$

$$A\mathbf{x}_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 2.74 \\ -3.41 \\ 2.08 \end{bmatrix} = 3.41\begin{bmatrix} 0.8 \\ -1 \\ 0.61 \end{bmatrix} = 3.41\mathbf{x}_5$$

$$A\mathbf{x}_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.8 \\ -1 \\ 0.61 \end{bmatrix} = \begin{bmatrix} 2.6 \\ -3.41 \\ 2.22 \end{bmatrix} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = 3.41\mathbf{x}_6$$

$$A\mathbf{x}_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 2.52 \\ -3.41 \\ 2.3 \end{bmatrix} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = 3.41\mathbf{x}_7$$

$$A\mathbf{x}_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 2.48 \\ -3.41 \\ 2.34 \end{bmatrix} = 3.41\begin{bmatrix} 0.73 \\ -1 \\ 0.69 \end{bmatrix} = 3.41\mathbf{x}_8$$

$$A\mathbf{x}_8 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.73 \\ -1 \\ 0.69 \end{bmatrix} = \begin{bmatrix} 2.46 \\ -3.42 \\ 2.38 \end{bmatrix} = 3.42\begin{bmatrix} 0.72 \\ -1 \\ 0.7 \end{bmatrix} = 3.42\mathbf{x}_9$$

$$A\mathbf{x}_9 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.72 \\ -1 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 2.44 \\ -3.42 \\ 2.4 \end{bmatrix} = 3.42\begin{bmatrix} 0.71 \\ -1 \\ 0.7 \end{bmatrix} = 3.42\mathbf{x}_{10}$$

$$A\mathbf{x}_{10} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71 \\ -1 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.41 \\ 2.4 \end{bmatrix} = 3.41 \begin{bmatrix} 0.71 \\ -1 \\ 0.7 \end{bmatrix} = 3.41\mathbf{x}_{11}$$

Hence, the largest eigenvalue is 3.41 and the corresponding eigenvector is $\begin{bmatrix} 0.71 \\ -1 \\ 0.7 \end{bmatrix}$.

Example 5

Determine the largest eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$.

[Summer 2014]

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = 6\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 2.36 \\ 8.03 \end{bmatrix} = 8.03 \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = 8.03\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 0.24 \\ 5.99 \end{bmatrix} = 5.99 \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = 5.99\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ -0.08 \\ 6.26 \end{bmatrix} = 6.26 \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = 6.26\mathbf{x}_4$$

$$A\mathbf{x}_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.36 \\ 0.28 \\ 6.95 \end{bmatrix} = 6.95 \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix} = 6.95\mathbf{x}_5$$

$$A\mathbf{x}_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.22 \\ 0.52 \\ 7.16 \end{bmatrix} = 7.16 \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = 7.16\mathbf{x}_6$$

$$A\mathbf{x}_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0.52 \\ 7.07 \end{bmatrix} = 7.07 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = 7.07\mathbf{x}_7$$

$$A\mathbf{x}_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix} = 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = 7.01\mathbf{x}_8$$

Hence, the largest eigenvalue is $7.01 \approx 7$ and the corresponding eigenvector is $\begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$.

Example 6

Find the dominant eigenvalue of $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ *choosing* $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
Compute the Rayleigh quotient and an error bound at this stage.

[Winter 2012]

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.33 \\ 0.67 \\ 1 \end{bmatrix} = 6\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.33 \\ 0.67 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.99 \\ 3.68 \\ 4.67 \end{bmatrix} = 4.67 \begin{bmatrix} 0.21 \\ 0.79 \\ 1 \end{bmatrix} = 4.67\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.21 \\ 0.79 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 4.16 \\ 4.79 \end{bmatrix} = 4.79 \begin{bmatrix} 0.13 \\ 0.87 \\ 1 \end{bmatrix} = 4.79\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.13 \\ 0.87 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.39 \\ 4.48 \\ 4.87 \end{bmatrix} = 4.87 \begin{bmatrix} 0.08 \\ 0.92 \\ 1 \end{bmatrix} = 4.87\mathbf{x}_4$$

$$A\mathbf{x}_4 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.08 \\ 0.92 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.24 \\ 4.68 \\ 4.92 \end{bmatrix} = 4.92 \begin{bmatrix} 0.05 \\ 0.95 \\ 1 \end{bmatrix} = 4.92\mathbf{x}_5$$

$$A\mathbf{x}_5 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.95 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 4.8 \\ 4.95 \end{bmatrix} = 4.95 \begin{bmatrix} 0.03 \\ 0.97 \\ 1 \end{bmatrix} = 4.95\mathbf{x}_6$$

$$A\mathbf{x}_6 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.03 \\ 0.97 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 4.88 \\ 4.97 \end{bmatrix} = 4.97 \begin{bmatrix} 0.02 \\ 0.98 \\ 1 \end{bmatrix} = 4.97\mathbf{x}_7$$

$$A\mathbf{x}_7 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.98 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.06 \\ 4.92 \\ 4.98 \end{bmatrix} = 4.98 \begin{bmatrix} 0.01 \\ 0.99 \\ 1 \end{bmatrix} = 4.98\mathbf{x}_8$$

$$A\mathbf{x}_8 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.99 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.03 \\ 4.96 \\ 4.99 \end{bmatrix} = 4.99 \begin{bmatrix} 0.01 \\ 0.99 \\ 1 \end{bmatrix} = 4.99\mathbf{x}_9$$

Hence, the dominant eigenvalue is $4.99 \approx 5$ and the corresponding eigenvector is $\begin{bmatrix} 0.01 \\ 0.99 \\ 1 \end{bmatrix}$.

$$\text{Rayleigh Quotient } \lambda = \frac{\mathbf{x}_8 \cdot A\mathbf{x}_8}{\mathbf{x}_8 \cdot \mathbf{x}_8} = \frac{\mathbf{x}_8^T A\mathbf{x}_8}{\mathbf{x}_8^T \mathbf{x}_8}$$

$$= \frac{[0.01 \;\; 0.99 \;\; 1] \begin{bmatrix} 0.03 \\ 4.96 \\ 4.99 \end{bmatrix}}{[0.01 \;\; 0.99 \;\; 1] \begin{bmatrix} 0.01 \\ 0.99 \\ 1 \end{bmatrix}}$$

$$= \frac{9.9007}{1.9802}$$
$$= 4.9998$$
$$\text{Error} = 4.9998 - 4.99 = 0.0098$$

Example 7

Determine the largest eigenvalue of $A = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution

Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$A\mathbf{x}_0 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \\ 3 \end{bmatrix} = 10\begin{bmatrix} -0.1 \\ 1 \\ 0.3 \end{bmatrix} = 10\mathbf{x}_1$$

$$A\mathbf{x}_1 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -0.1 \\ 1 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 2.3 \\ 0.3 \\ 1 \end{bmatrix} = 2.3\begin{bmatrix} 1 \\ 0.13 \\ 0.43 \end{bmatrix} = 2.3\mathbf{x}_2$$

$$A\mathbf{x}_2 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.13 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 0.85 \\ 10.56 \\ 3.56 \end{bmatrix} = 10.56\begin{bmatrix} 0.08 \\ 1 \\ 0.34 \end{bmatrix} = 10.56\mathbf{x}_3$$

$$A\mathbf{x}_3 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.08 \\ 1 \\ 0.34 \end{bmatrix} = \begin{bmatrix} 2.28 \\ 2.14 \\ 1.58 \end{bmatrix} = 2.28\begin{bmatrix} 1 \\ 0.94 \\ 0.69 \end{bmatrix} = 2.28\mathbf{x}_4$$

$$A\mathbf{x}_4 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.94 \\ 0.69 \end{bmatrix} = \begin{bmatrix} 2.7 \\ 11.63 \\ 4.63 \end{bmatrix} = 11.63\begin{bmatrix} 0.23 \\ 1 \\ 0.4 \end{bmatrix} = 11.63\mathbf{x}_5$$

$$A\mathbf{x}_5 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.23 \\ 1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 2.37 \\ 3.7 \\ 2.09 \end{bmatrix} = 3.7\begin{bmatrix} 0.64 \\ 1 \\ 0.56 \end{bmatrix} = 3.7\mathbf{x}_6$$

$$A\mathbf{x}_6 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.64 \\ 1 \\ 0.56 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 7.96 \\ 3.48 \end{bmatrix} = 7.96\begin{bmatrix} 0.33 \\ 1 \\ 0.44 \end{bmatrix} = 7.96\mathbf{x}_7$$

$$A\mathbf{x}_7 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.33 \\ 1 \\ 0.44 \end{bmatrix} = \begin{bmatrix} 2.43 \\ 4.74 \\ 2.43 \end{bmatrix} = 4.74\begin{bmatrix} 0.51 \\ 1 \\ 0.51 \end{bmatrix} = 4.74\,\mathbf{x}_8$$

$$A\mathbf{x}_8 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.51 \\ 1 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 2.53 \\ 6.61 \\ 3.04 \end{bmatrix} = 6.61\begin{bmatrix} 0.38 \\ 1 \\ 0.46 \end{bmatrix} = 6.61\,\mathbf{x}_9$$

$$A\mathbf{x}_9 = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.38 \\ 1 \\ 0.46 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.26 \\ 2.6 \end{bmatrix} = 5.26\begin{bmatrix} 0.47 \\ 1 \\ 0.49 \end{bmatrix} = 5.26\,\mathbf{x}_{10}$$

$$A\mathbf{x}_{10} = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.47 \\ 1 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 2.49 \\ 6.19 \\ 2.9 \end{bmatrix} = 6.19\begin{bmatrix} 0.4 \\ 1 \\ 0.47 \end{bmatrix} = 6.19\,\mathbf{x}_{11}$$

$$A\mathbf{x}_{11} = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.4 \\ 1 \\ 0.47 \end{bmatrix} = \begin{bmatrix} 2.48 \\ 5.47 \\ 2.67 \end{bmatrix} = 5.47\begin{bmatrix} 0.45 \\ 1 \\ 0.49 \end{bmatrix} = 5.47\,\mathbf{x}_{12}$$

$$A\mathbf{x}_{12} = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.45 \\ 1 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 2.51 \\ 5.99 \\ 2.84 \end{bmatrix} = 5.99\begin{bmatrix} 0.42 \\ 1 \\ 0.47 \end{bmatrix} = 5.99\,\mathbf{x}_{13}$$

$$A\mathbf{x}_{13} = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.42 \\ 1 \\ 0.47 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.67 \\ 2.73 \end{bmatrix} = 5.67\begin{bmatrix} 0.43 \\ 1 \\ 0.48 \end{bmatrix} = 5.67\,\mathbf{x}_{14}$$

$$A\mathbf{x}_{14} = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.43 \\ 1 \\ 0.48 \end{bmatrix} = \begin{bmatrix} 2.49 \\ 5.78 \\ 2.77 \end{bmatrix} = 5.78\begin{bmatrix} 0.43 \\ 1 \\ 0.48 \end{bmatrix} = 5.78\,\mathbf{x}_{14}$$

Hence, the largest eigenvalue is 5.78 and the corresponding eigenvector is $\begin{bmatrix} 0.43 \\ 1 \\ 0.48 \end{bmatrix}$.

Example 8

Find numerically smallest eigenvalue of the given matrix using power method, correct up to three decimal places. **[Winter 2014]**

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}.$$

Solution

The numerically smallest eigenvalue of a matrix A is the reciprocal of numerically largest eigenvalue of A^{-1}.

$$A^{-1} = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}$$

Let $B = A^{-1}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$B\mathbf{x}_0 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.13 \\ 0.36 \end{bmatrix} = 0.36 \begin{bmatrix} 0.111 \\ 0.361 \\ 1 \end{bmatrix} = 0.36\,\mathbf{x}_1$$

$$B\mathbf{x}_1 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.111 \\ 0.361 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.053 \\ 0.099 \\ 0.169 \end{bmatrix} = 0.169 \begin{bmatrix} 0.314 \\ 0.586 \\ 1 \end{bmatrix} = 0.169\,\mathbf{x}_2$$

$$B\mathbf{x}_2 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.314 \\ 0.586 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.048 \\ 0.099 \\ 0.215 \end{bmatrix} = 0.215 \begin{bmatrix} 0.223 \\ 0.46 \\ 1 \end{bmatrix} = 0.215\,\mathbf{x}_3$$

$$B\mathbf{x}_3 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.223 \\ 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.051 \\ 0.101 \\ 0.194 \end{bmatrix} = 0.194 \begin{bmatrix} 0.263 \\ 0.521 \\ 1 \end{bmatrix} = 0.194\,\mathbf{x}_4$$

$$B\mathbf{x}_4 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.263 \\ 0.521 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.099 \\ 0.203 \end{bmatrix} = 0.203 \begin{bmatrix} 0.246 \\ 0.488 \\ 1 \end{bmatrix} = 0.203\,\mathbf{x}_5$$

$$B\mathbf{x}_5 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.246 \\ 0.488 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.101 \\ 0.199 \end{bmatrix} = 0.199 \begin{bmatrix} 0.251 \\ 0.508 \\ 1 \end{bmatrix} = 0.199\,\mathbf{x}_6$$

$$B\mathbf{x}_6 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.251 \\ 0.508 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.099 \\ 0.2 \end{bmatrix} = 0.2 \begin{bmatrix} 0.25 \\ 0.495 \\ 1 \end{bmatrix} = 0.2\,\mathbf{x}_7$$

$$B\mathbf{x}_7 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.495 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix} = 0.2 \begin{bmatrix} 0.25 \\ 0.5 \\ 1 \end{bmatrix} = 0.2\,\mathbf{x}_8$$

$$B\mathbf{x}_8 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix} = 0.2 \begin{bmatrix} 0.25 \\ 0.5 \\ 1 \end{bmatrix} = 0.2\,\mathbf{x}_9$$

The numerically largest eigenvalue of $B = A^{-1}$ is 0.2. Hence, the numerically smallest eigenvalue of A is $\dfrac{1}{0.2} = 5$.

EXERCISE 11.1

Find the dominant eigenvalue and corresponding eigenvector of the following matrices by the power method:

1. $\begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$

$$\left[\text{Ans.: } -3, \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}\right]$$

2. $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

$$\left[\text{Ans.: } 4.6179, \begin{bmatrix} 1 \\ 0.618 \end{bmatrix}\right]$$

3. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\text{Ans.: } 5.3727, \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}\right]$$

4. $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\text{Ans.: } 4, \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}\right]$$

5. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

$$\left[\text{Ans.: } 11.66, \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}\right]$$

6. $\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$

$$\left[\text{Ans.: } 13, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

7. $\begin{bmatrix} 3 & 2 & 4 \\ -1 & 4 & 10 \\ 1 & 3 & -1 \end{bmatrix}$ $\left[\text{Ans.: } 7.4, \begin{bmatrix} 0.8 \\ 1 \\ 0.4 \end{bmatrix}\right]$

8. $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ $\left[\text{Ans.: } 6, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$

11.6 JACOBI METHOD

The Jacobi method is used to find all the eigenvalues and corresponding eigenvectors of a real symmetric matrix A. We know that the eigenvalues of a real symmetric matrix of order n has n mutually orthogonal real eigenvectors and there exists an orthogonal matrix S such that S^TAS is a diagonal matrix, whose diagonal elements are the eigenvalues of A. Also, the columns of S are the eigenvectors of A, corresponding to respective eigenvalues.

Let a_{ij} be the numerically largest nondiagonal element of a real symmetric matrix A.

$$A = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

Let S be the most general orthogonal rotation matrix.

$$S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let $B = S^TAS$ be the similar transformation which results in a diagonal matrix.

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos^2\theta + a_{ij}\sin 2\theta + a_{jj}\sin^2\theta & \frac{1}{2}(a_{jj} - a_{ii})\sin 2\theta + a_{ij}\cos 2\theta \\ \frac{1}{2}(a_{jj} - a_{ii})\sin 2\theta + a_{ij}\cos 2\theta & a_{ii}\sin^2\theta - a_{ij}\sin 2\theta + a_{jj}\cos^2\theta \end{bmatrix}$$

The matrix B reduces to a diagonal matrix if

$$\frac{1}{2}(a_{jj} - a_{ii}) \sin 2\theta + a_{ij} \cos 2\theta = 0$$

$$\frac{\sin 2\theta}{\cos 2\theta} = -\frac{a_{ij}}{\frac{1}{2}(a_{jj} - a_{ii})}$$

i.e., $$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}}$$

$\therefore$ $$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\,a_{ij}}{a_{ii} - a_{jj}}\right) \qquad \text{if } a_{ii} \neq a_{jj}$$

$$= \frac{\pi}{4} \quad \text{if } a_{ii} = a_{jj} \text{ and } a_{ij} > 0$$

$$= -\frac{\pi}{4} \quad \text{if } a_{ii} = a_{jj} \text{ and } a_{ij} < 0$$

The diagonal elements of B are eigenvalues of A and the columns of S are the corresponding eigenvectors of A.

The Jacobi method can be extended to higher order symmetric matrices by *similarity transformation*. If a_{ij} is the numerically largest term among the nondiagonal elements of a matrix A then the orthogonal rotation matrix S_1 is selected such that all its diagonal elements are 1 and all nondiagonal elements are 0 except $a_{ii} = \cos\,\theta$, $a_{ij} = -\sin\,\theta$, $a_{ji} = \sin\,\theta$, $a_{jj} = \cos\,\theta$

where $$\tan 2\theta = \frac{2\,a_{ij}}{a_{ii} - a_{jj}}$$

$$S_1 = \begin{array}{c} \begin{matrix} & & & i^{\text{th}}\text{ column} & j^{\text{th}}\text{ column} & & \\ & & & \downarrow & \downarrow & & \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cos\theta & -\sin\theta & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \sin\theta & \cos\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \end{array} \begin{matrix} \\ \\ \leftarrow i^{\text{th}}\text{ row} \\ \\ \leftarrow j^{\text{th}}\text{ row} \\ \\ \end{matrix}$$

$$B_1 = S_1^T\, AS_1$$

If B_1 is not a diagonal matrix, the procedure is repeated to find S_2 and B_2.

$$B_2 = S_2^T\ B_1 S_2$$
$$= S_2^T\ S_1^T\ A S_1\ S_2$$

Proceeding similarly,

$$B_n = S_n^T\ S_{n-1}^T \cdots S_1^T\ A\ S_1 \cdots S_{n-1}\ S_n$$
$$= (S_1 S_2 \cdots S_n)^T A (S_1 S_2 \cdots S_n)$$
$$= S^T A S$$

where $S = S_1 S_2 \cdots S_n$

If B_n is a diagonal matrix, its diagonal elements are the eigenvalues of A and the columns of S are the corresponding eigenvectors of A.

Example 1

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Solution

Let $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(3)}{2-2} = \infty$$

$$\theta = \frac{\pi}{4}$$

$$S = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$B = S^T A S$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence, the eigenvalues of A are the diagonal elements of B, i.e., 5 and –1 and the corresponding eigenvectors are columns of S, i.e., $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 2

Find the eigenvalues of $A = \begin{bmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$.

Solution

Let $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(\sqrt{3})}{6-4} = \sqrt{3}$$

$$\theta = \frac{\pi}{6}$$

$$S = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$B = S^T AS$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$$

Hence, the eigenvalues of A are the diagonal elements of B, i.e., 7 and 3 and the corresponding eigenvectors are columns of S, i.e., $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$.

Example 3

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

Solution

The largest nondiagonal element is $|a_{23}| = |a_{32}| = 1$.

Let $S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}} = \frac{2(-1)}{3-3} = -\frac{2}{0} = \infty$$

$$\theta = \frac{\pi}{4}$$

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$B_1 = S_1^T A S_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Hence, the eigenvalues of A are the diagonal elements of B_1, i.e., 1, 2, 4 and the corresponding eigenvectors are columns of S_1, i.e.,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Example 4

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution

The largest nondiagonal element is $a_{13} = a_{31} = 1$.

Let $S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2(1)}{2-2} = \frac{2}{0} = \infty$$

$$\theta = \frac{\pi}{4}$$

$$S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$B_1 = S_1^T A S_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the eigenvalues of A are diagonal elements of B_1, i.e., 3, 2, 1 and the corresponding eigenvectors are columns of S_1, i.e., $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example 5

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$.

Solution

The largest nondiagonal element is $a_{13} = a_{31} = 2$.

Let $S_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2(2)}{1-1} = \frac{4}{0} = \infty$$

$$\theta = \frac{\pi}{4}$$

$$S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$B_1 = S_1^T A S_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

B_1 is not a diagonal matrix. The largest nondiagonal element of B_1 is $a_{12} = a_{21} = 2$.

Let $S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the rotation matrix.

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(2)}{3-3} = \frac{4}{0} = \infty$$

$$\theta = \frac{\pi}{4}$$

$$S_2 = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_2 = S_2^T\ B_1\ S_2$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

After 2 rotations, A is reduced to the diagonal matrix B_2. Hence, the eigenvalues of A are diagonal elements of B_2, i.e., 5, 1, –1.

$$S = S_1\ S_2$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence, the corresponding eigenvectors are columns of S, i.e., $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

EXERCISE 11.2

Find the eigenvalues and corresponding eigenvectors of the following matrices by the Jacobi's method:

1. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 1, 3, \text{and} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$

2. $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 5,\ 3,\ \text{and}\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$

3. $\begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 10,\ -5,\ \text{and}\ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

4. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 2,\ 4,\ \text{and}\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$

5. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 1,\ 2,\ 3,\ \text{and}\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right]$

6. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 8,\ 8,\ 2,\ \text{and}\ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right]$

7. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\left[\textbf{Ans.:}\ 2,\ 3,\ 6,\ \text{and}\ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right]$

Points to Remember

Power Method

The power method is an iterative method to find the numerically largest (dominant) eigenvalue and the corresponding eigenvectors of the matrix A.

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix A then λ_1 is the dominant eigenvalue of A

if $$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

If $\mathbf{x}$ is an eigenvector of a matrix A then its corresponding eigenvalue is given by

$$\lambda = \frac{\mathbf{x} \cdot A\mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

This quotient is called the Rayleigh quotient.

Jacobi Method

The Jacobi method is used to find all the eigenvalues and corresponding eigenvectors of a real symmetric matrix A. If a_{ij} is the numerically largest term among the nondiagonal elements of a matrix A then the orthogonal rotation matrix S_1 is selected such that all its diagonal elements are 1 and all nondiagonal elements are 0 except $a_{ii} = \cos\theta$, $a_{ij} = -\sin\theta$, $a_{ji} = \sin\theta$, $a_{jj} = \cos\theta$

where $$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}}$$

$$S_1 = \begin{array}{c} \begin{matrix} & & & i^{\text{th}}\text{ column} & j^{\text{th}}\text{ column} & & \\ & & & \downarrow & \downarrow & & \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cos\theta & -\sin\theta & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \sin\theta & \cos\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \end{array} \begin{matrix} \\ \\ \leftarrow i^{\text{th}}\text{ row} \\ \\ \leftarrow j^{\text{th}}\text{ row} \\ \\ \end{matrix}$$

$$B_1 = S_1^T AS_1$$

If B_1 is not a diagonal matrix, the procedure is repeated to find S_2 and B_2.

CHAPTER 12

Numerical Solution of Ordinary Differential Equations

Chapter Outline

12.1 Introduction
12.2 Euler's Method
12.3 Modified Euler's Method
12.4 Runge–Kutta Methods

12.1 INTRODUCTION

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain given conditions. The analytical method of solutions of differential equations can be applied to solve only a selected class of differential equations. In many physical and engineering problems, these methods cannot be used and, hence, numerical methods are used to solve such differential equations.

Consider the first-order differential equations

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

A number of numerical methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. Picard's and Taylor's series methods belong to the former class of solutions, whereas those of Euler, Runge–Kutta, Milne, etc., belong to the latter class. In these

later methods, the values of y are calculated in short steps for equal intervals of x and are, therefore, termed step-by-step methods. In the Euler and Runge–Kutta methods, the interval length h should be kept small and, hence, these methods can be applied for tabulating y over a limited range only. If, however, the function values are desired over a wide range, the Milne method may be used. These later methods require starting values which are found by Picard's or Taylor series or Runge–Kutta methods. In this chapter, we will study the Euler and Runge–Kutta methods only.

12.2 EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

The solution of the differential equation is represented by the curve as shown in Fig. 12.1. The point $P_0(x_0, y_0)$ lies on the curve.

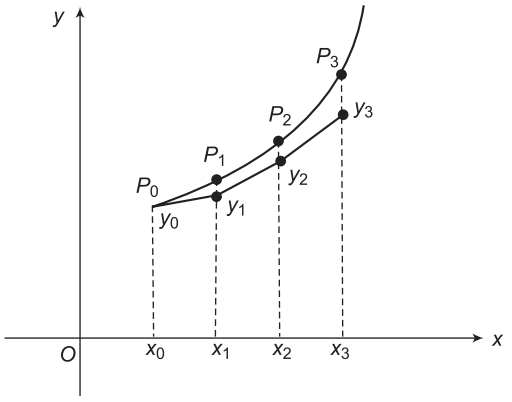


Fig. 12.1

At $x = x_0$, $\left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0)$

The equation of the tangent to the curve at the point (x_0, y_0) is given by

$$y - y_0 = \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)(x - x_0)$$

$$= f(x_0, y_0)\,(x - x_0)$$

$$y = y_0 + f(x_0, y_0)\,(x - x_0)$$

If the point x_1 is very close to x_0, the curve is approximated by the tangent line in the interval (x_0, x_1). Hence, the value of y on the curve is approximately equal to the value of y on the tangent at the point (x_0, y_0) corresponding to $x = x_1$.

$\therefore$ $$y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$$

$$= y_0 + h\,f(x_0, y_0) \qquad \text{where } h = x_1 - x_0$$

At $x = x_1$, $\left.\frac{dy}{dx}\right|_{x=x_1} = f(x_1, y_1)$

Again the curve is approximated by the tangent line through the point (x_1, y_1).

$$y_2 = y_1 + h\,f(x_1, y_1)$$

Hence, $$y_{n+1} = y_n + h\,f(x_n, y_n)$$

This formula is known as *Euler's formula*. In this method, the actual curve is approximated by a sequence of short straight lines. As the step size h increases, the straight line deviates much from the actual curve.

Hence, accuracy cannot be obtained.

Example 1

Using Euler's method, find $y(0.2)$ *given* $\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = y - \frac{2x}{y}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.1} = 2$$

$$x_1 = 0.1$$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1 f(0, 1)$$
$$= 1 + 0.1\left[1 - \frac{2(0)}{1}\right]$$
$$= 1.1$$

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.1 + 0.1 f(0.1, 1.1)$$
$$= 1.1 + 0.1\left[1.1 - \frac{2(0.1)}{1.1}\right]$$
$$= 1.1918$$

Hence, $y_2 = y(0.2) = 1.1918$

Example 2

Find the value of y for $\frac{dy}{dx} = x + y, y(0) = 1$ *when* $x = 0.1, 0.2$ *with step size* $h = 0.05$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.05} = 4$$

$$x_1 = 0.05, \quad x_2 = 0.1, \quad x_3 = 0.15$$

$$\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ &= 1 + 0.05 f(0, 1) \\ &= 1 + 0.05(0 + 1) \\ &= 1.05 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h f(x_1, y_1) \\ &= 1.05 + 0.05 f(0.05, 1.05) \\ &= 1.05 + 0.05(0.05 + 1.05) \\ &= 1.105 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h f(x_2, y_2) \\ &= 1.105 + 0.05 f(0.1, 1.105) \\ &= 1.105 + 0.05(0.1 + 1.105) \\ &= 1.16525 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + h f(x_3, y_3) \\ &= 1.16525 + 0.05 f(0.15, 1.16525) \\ &= 1.16525 + 0.05(0.15 + 1.16525) \\ &= 1.231 \end{aligned}$$

Hence, $y_2 = y(0.1) = 1.105$

$y_4 = y(0.2) = 1.231$

Example 3

Solve the initial-value problem $\frac{dy}{dx} = x\sqrt{y}, y(1) = 1$ *and, hence, find* $y(1.5)$ *by taking* $h = 0.1$ *using Euler's method.* **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x\sqrt{y}$$

Given: $x_0 = 1, \quad y_0 = 1, \quad h = 0.1, \quad x = 1.5$

$$n = \frac{x - x_0}{h} = \frac{1.5 - 1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$
\begin{aligned}
y_1 &= y_0 + h f(x_0, y_0) \\
&= 1 + 0.1 f(1,1) \\
&= 1 + 0.1(1\sqrt{1}) \\
&= 1.1 \\
y_2 &= y_1 + h f(x_1, y_1) \\
&= 1.1 + 0.1 f(1.1, 1.1) \\
&= 1.1 + 0.1(1.1\sqrt{1.1}) \\
&= 1.2154 \\
y_3 &= y_2 + h f(x_2, y_2) \\
&= 1.2154 + 0.1 f(1.2, 1.2154) \\
&= 1.2154 + 0.1(1.2\sqrt{1.2154}) \\
&= 1.3477 \\
y_4 &= y_3 + h f(x_3, y_3) \\
&= 1.3477 + 0.1 f(1.3, 1.3477) \\
&= 1.3477 + 0.1(1.3\sqrt{1.3477}) \\
&= 1.4986 \\
y_5 &= y_4 + h f(x_4, y_4) \\
&= 1.4986 + 0.1 f(1.4, 1.4986) \\
&= 1.4986 + 0.1(1.4\sqrt{1.4986}) \\
&= 1.67
\end{aligned}
$$

Hence, $y_5 = y(1.5) = 1.67$

Example 4

Using Euler's method, find the approximate value of y at $x = 1.5$ *taking* $h = 0.1$. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{\sqrt{xy}}$ *and* $y(1) = 2$.

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y - x}{\sqrt{xy}}$$

Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.1, \quad x = 1.5$

$$n = \frac{x - x_0}{h} = \frac{1.5 - 1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$
\begin{aligned}
y_1 &= y_0 + h f(x_0, y_0) \\
&= 2 + 0.1 f(1, 2) \\
&= 2 + 0.1\left[\frac{2-1}{\sqrt{1(2)}}\right] \\
&= 2.0707 \\
y_2 &= y_1 + h f(x_1, y_1) \\
&= 2.0707 + 0.1(f(1.1, 2.0707) \\
&= 2.0707 + 0.1\left[\frac{2.0707 - 1.1}{\sqrt{1.1(2.0707)}}\right] \\
&= 2.1350 \\
y_3 &= y_2 + h f(x_2, y_2) \\
&= 2.1350 + 0.1 f(1.2, 2.1350) \\
&= 2.1350 + 0.1\left[\frac{2.1350 - 1.2}{\sqrt{1.2(2.1350)}}\right] \\
&= 2.1934 \\
y_4 &= y_3 + h f(x_3, y_3) \\
&= 2.1934 + 0.1 f(1.3, 2.1934) \\
&= 2.1934 + 0.1\left[\frac{2.1934 - 1.3}{\sqrt{1.3(2.1934)}}\right] \\
&= 2.2463 \\
y_5 &= y_4 + h f(x_4, y_4) \\
&= 2.2463 + 0.1 f(1.4, 2.2463) \\
&= 2.2463 + 0.1\left[\frac{2.2463 - 1.4}{\sqrt{1.4(2.2463)}}\right] \\
&= 2.2940
\end{aligned}
$$

Hence, $y_5 = y(1.5) = 2.2940$

Example 5

Using Euler's method, find the approximate value of y *at* $x = 1$ *taking* $h = 0.2$. *Given* $\frac{dy}{dx} = x^2 + y^2$ *and* $y(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y^2$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.2, \quad x = 1$

$$n = \frac{x - x_0}{h} = \frac{1 - 0}{0.2} = 5$$

$$x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8$$

$$\begin{aligned}
y_1 &= y_0 + h f(x_0, y_0) \\
&= 1 + 0.2 f(0, 1) \\
&= 1 + 0.2[(0)^2 + (1)^2] \\
&= 1.2 \\
y_2 &= y_1 + h f(x_1, y_1) \\
&= 1.2 + 0.2 f(0.2, 1.2) \\
&= 1.2 + 0.2\left[(0.2)^2 + (1.2)^2\right] \\
&= 1.496 \\
y_3 &= y_2 + h f(x_2, y_2) \\
&= 1.496 + 0.2 f(0.4, 1.496) \\
&= 1.496 + 0.2\left[(0.4)^2 + (1.496)^2\right] \\
&= 1.9756 \\
y_4 &= y_3 + h f(x_3, y_3) \\
&= 1.9756 + 0.2 f(0.6, 1.9756) \\
&= 1.9756 + 0.2\left[(0.6)^2 + (1.9756)^2\right] \\
&= 2.8282 \\
y_5 &= y_4 + h f(x_4, y_4) \\
&= 2.8282 + 0.2 f(0.8, 2.8282) \\
&= 2.8282 + 0.2\left[(0.8)^2 + (2.8282)^2\right] \\
&= 4.5559
\end{aligned}$$

Hence, $y_5 = y(1) = 4.5559$

Example 6

Given $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with the initial condition* $y = 1$ *at* $x = 0$. *Find* y *at* $x = 0.1$ *in five steps.*

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y - x}{y + x}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad n = 5, \quad x = 0.1$

$$h = \frac{x - x_0}{n} = \frac{0.1 - 0}{5} = 0.02$$

$$x_1 = 0.02, \quad x_2 = 0.04, \quad x_3 = 0.06, \quad x_4 = 0.08$$

$$\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ &= 1 + 0.02 f(0, 1) \\ &= 1 + 0.02\left(\frac{1-0}{1+0}\right) \\ &= 1.02 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h f(x_1, y_1) \\ &= 1.02 + 0.02 f(0.02, 1.02) \\ &= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right) \\ &= 1.0392 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h f(x_2, y_2) \\ &= 1.0392 + 0.02 f(0.04, 1.0392) \\ &= 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right) \\ &= 1.0577 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + h f(x_3, y_3) \\ &= 1.0577 + 0.02 f(0.06, 1.0577) \\ &= 1.0577 + 0.02\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right) \\ &= 1.0756 \end{aligned}$$

$$\begin{aligned} y_5 &= y_4 + h f(x_4, y_4) \\ &= 1.0756 + 0.02 f(0.08, 1.0756) \\ &= 1.0756 + 0.02\left(\frac{1.0756 - 0.08}{1.0756 + 0.08}\right) \\ &= 1.0928 \end{aligned}$$

Hence, $y_5 = y(0.1) = 1.0928$

EXERCISE 12.1

Solve the following differential equations using Euler's method:

1. $\frac{dy}{dx} = xy$ with $y(0) = 2$, $h = 0.2$ at $x = 1$

[Ans.: 2.9186]

2. $\frac{dy}{dx} = \frac{y - x}{x}$ with $y(1) = 2$ at $x = 2$ taking $h = 0.2$

[Ans.: 2.6137]

3. $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ taking $h = 0.1$ at $x = 1.3$ and $x = 1.5$

[Ans.: 1.0268, 1.0889]

4. $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$ taking $h = 0.1$ at $x = 0.2$

[Ans.: 1.231]

5. $\frac{dy}{dx} = 1 - 2xy$ with $y(0) = 0$ taking $h = 0.2$ at $x = 0.6$

[Ans.: 0.5226]

6. $\frac{dy}{dx} = x + \sqrt{y}$ with $y(2) = 4$ taking $h = 0.2$ at $x = 3$

[Ans.: 8.7839]

7. $\frac{dy}{dx} = x + y + xy$ with $y(0) = 1$ taking $h = 0.025$ at $x = 0.1$

[Ans.: 1.1117]

8. $\frac{dy}{dx} = 1 - y^2$ with $y(0) = 0$ taking $h = 0.2$ at $x = 1$

[Ans.: 0.8007]

12.3 MODIFIED EULER'S METHOD

The Euler's method is very easy to implement but it cannot give accurate solutions. A very small step size is required to get any meaningful result. Since the starting point of each sub-interval is used to find the slope of the solution curve, the solution would be correct only if the function is linear. In the modified Euler's method, the arithmetic average of the slopes is used to approximate the solution curve.

In the modified Euler's method, $y_1^{(0)}$ is first calculated from the Euler's method.

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

This value is improved by making use of average slopes at (x_0, y_0) and $(x_1, y_1^{(0)})$. The first approximation to y_1 is written as

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

This value of $y_1^{(1)}$ is further improved by the equation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

which is the second approximation to y_1.

In general,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2,...$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The procedure will be terminated depending on the accuracy required. If two consecutive values of $y_1^{(k)}$ and $y_1^{(k+1)}$ are equal, $y_1 = y_1^{(k)}$.

Now, $y_2^{(0)}$ is calculated from the Euler's method.

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

Better approximation to y_2 is obtained as

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

This procedure is repeated till two approximation to y_2 are equal. Proceeding in the same manner, other values, i.e., y_3, y_4, etc., can be calculated.

Example 1

Determine the value of y when $x = 0.1$ *correct up to four decimal places by taking* $h = 0.05$. *Given that* $y(0) = 1$ *and* $\dfrac{dy}{dx} = x^2 + y$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x_1 = 0.05$

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + 0.05(1) = 1.05$$

First approximation to y_1

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.05)\right]$$

$$= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.05\}\right]$$

$$= 1.0513$$

Second approximation to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.0513)\right]$$
$$= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.0513\}\right]$$
$$= 1.0513$$

Since the values of $y_1^{(1)}$ and $y_1^{(2)}$ are equal,

$$y_1 = y(0.05) = 1.0513$$

(ii) Now, $x_1 = 0.05, \quad y_1 = 1.0513, \quad h = 0.05, \quad x_2 = 0.1$

$$f(x_1, y_1) = (0.05)^2 + 1.0513 = 1.0538$$
$$y_2^{(0)} = y_1 + h f(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.1040$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + f(0.1, 1.1040)\right]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1040\}\right]$$
$$= 1.1055$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + f(0.1, 1.1055)\right]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1055\}\right]$$
$$= 1.1055$$

Since the values of $y_2^{(1)}$ and $y_2^{(2)}$ are equal,

$$y_2 = y(0.1) = 1.1055$$

Example 2

Using the modified Euler's method, solve $\frac{dy}{dx} = 1 - y$ *with the initial condition* $y(0) = 0$ *at* $x = 0.1, 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 1 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad x_1 = 0.1, \quad h = x_1 - x_0 = 0.1$

$$f(x_0, y_0) = 1 - 0 = 1$$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 0 + 0.1(1) = 0.1$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.1)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.1)\right] \\ &= 0.095 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.095)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.095)\right] \\ &= 0.0953 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0953)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0953)\right] \\ &= 0.0952 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned} y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0952)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0952)\right] \\ &= 0.0952 \end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.1) = 0.0952$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.0952, \quad h = 0.1, \quad x_2 = 0.2$

$$f(x_1, \quad y_1) = 1 - 0.0952 = 0.9048$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 0.0952 + 0.1(0.9048) = 0.1857$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1857)\right]$$

$$= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1857)\right]$$

$$= 0.1812$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1812)\right]$$

$$= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1812)\right]$$

$$= 0.1814$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1814)\right]$$

$$= 0.952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1814)\right]$$

$$= 0.1814$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(0.2) = 0.1814$$

Example 3

Apply the modified Euler's method to solve the initial-value problem $y' = x + y$ *with* $y(0) = 0$ *choosing* $h = 0.2$ *and compute* y *for* $x = 0.2$, $x = 0.4$. **[Winter 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.2, \quad x_1 = 0.2$

$$f(x_0, y_0) = 0 + 0 = 0$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = 0 + 0.2(0) = 0$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0) \right] \\ &= 0.02 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.02) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.02) \right] \\ &= 0.022 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.022) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.022) \right] \\ &= 0.0222 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned} y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)}) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.0222) \right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.0222) \right] \\ &= 0.0222 \end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.2) = 0.0222$$

(ii) Now, $x_1 = 0.2, \quad y_1 = 0.0222, \quad h = 0.2, \quad x_2 = 0.4$

$$f(x_1, y_1) = 0.2 + 0.0222 = 0.2222$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 0.0222 + \frac{0.2}{2}(0.2222) = 0.0444$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0444)\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0444)\right]$$

$$= 0.0889$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0889)\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0889)\right]$$

$$= 0.0933$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0933)\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0933)\right]$$

$$= 0.0938$$

Fourth approximation to y_2

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0938)\right]$$

$$= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0938)\right]$$

$$= 0.0938$$

Since the value of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(0.4) = 0.0938$$

Example 4

Use modified Euler's method to find the value of y satisfying the equation $\frac{dy}{dx} = \log(x+y)$ *for* $x = 1.2$ *and* $x = 1.4$*, correct up to four decimal placed by taking* $h = 0.2$*. Given that* $y(1) = 2$*.*

Solution

$$\frac{dy}{dx} = f(x,y) = \log(x+y)$$

(i) Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.2, \quad x_1 = 1.2$

$$f(x_0, y_0) = \log(1+2) = 1.0986$$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 2 + 0.2(1.0986) = 2.2197$$

First approximation to y_1

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2197)\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2197)\right]$$

$$= 2.2328$$

Second approximation to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2328)\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2328)\right]$$

$$= 2.2332$$

Third approximation to y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2332)\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2332)\right]$$

$$= 2.2332$$

Since the values of $y_1^{(2)}$ and $y_1^{(3)}$ are equal,

$$y_1 = y(1.2) = 2.2332$$

(ii) Now, $x_1 = 1.2, \quad y_1 = 2.2332, \quad h = 0.2, \quad x_2 = 1.4$

$$f(x_1, y_1) = \log(1.2 + 2.2332) = 1.2335$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 2.2332 + 0.2(1.2335) = 2.4799$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4799)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4799)\right]$$

$$= 2.4291$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4921)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4921)\right]$$

$$= 2.4924$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4924)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4924)\right]$$

$$= 2.4924$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(1.4) = 2.4924$$

Example 5

Solve $\dfrac{dy}{dx} = 2 + \sqrt{xy}$ *with* $x_0 = 1.2$, $y_0 = 1.6403$ *by Euler's modified method for* $x = 1.6$, *correct up to four decimal places by taking* $h = 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2 + \sqrt{xy}$$

(i) Given: $x_0 = 1.2, \quad y_0 = 1.6403, \quad h = 0.2, \quad x_1 = 1.4$

$$f(x_0, y_0) = 2 + \sqrt{(1.2)\,(1.6403)} = 3.4030$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1.6403 + 0.2\,(3.4030) = 2.3209$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3209)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)\,(2.3209)}\right\}\right] \\ &= 2.3609 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3609)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)\,(2.3609)}\right\}\right] \\ &= 2.3624 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3624)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)\,(2.3624)}\right\}\right] \\ &= 2.3625 \end{aligned}$$

Fourth approximation to y_1

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$
$$= 1.6403 + \frac{0.2}{2}\left[3.4030 + f(1.4, 2.3625)\right]$$
$$= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3625)}\right\}\right]$$
$$= 2.3625$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(1.4) = 2.3625$$

(ii) Now, $x_1 = 1.4, \quad y_1 = 2.3625, \quad h = 0.2, \quad x_2 = 1.6$

$$f(x_1, y_1) = 2 + \sqrt{(1.4)\,(2.3625)} = 3.8187$$
$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 2.3625 + 0.2\,(3.8187) = 3.1262$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1262)\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1262)}\right\}\right]$$
$$= 3.1680$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1680)\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6) + 3.1680)}\right\}\right]$$
$$= 3.1695$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1695)\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1695)}\right\}\right]$$
$$= 3.1696$$

Fourth approximation to y_2

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1696)\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1696)}\right\}\right]$$

$$= 3.1696$$

Since the values of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(1.6) = 3.1696$$

EXERCISE 12.2

Solve the following differential equations by the modified Euler's method:

1. $\frac{dy}{dx} = x + 3y$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$
   [Ans.: 1.3548**]**

2. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$
   [Ans.: 0.9137**]**

3. $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$
   [Ans.: 1.1104**]**

4. $\frac{dy}{dx} = -xy^2$ with $y(0) = 2$ for $x = 0.2$ by taking $h = 0.1$
   [Ans.: 1.9238**]**

5. $\frac{dy}{dx} = 1 + \frac{y}{x}$ with $y(1) = 2$ for $x = 1.2$
   [Ans.: 2.6182**]**

6. $\frac{dy}{dx} = x + \sqrt{y}$ with $y(0) = 1$ for $x = 0.2$
   [Ans.: 1.2309**]**

7. $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ for $x = 1.1$ taking $h = 0.05$
   [Ans.: 1.0073**]**

8. $\frac{dy}{dx} = y - x$ with $y(0) = 2$ for $x = 0.2$
   [Ans.: 2.4222**]**

12.4 RUNGE—KUTTA METHODS

Runge–Kutta methods do not require the determination of higher order derivatives. These methods require only the function values at different points on the sub-interval. The main advantage of Runge–Kutta methods is the self-starting feature and, consequently, the ease of programming. One disadvantage of Runge–Kutta methods is the requirement that the function must be evaluated at different values of x and y in every step of the function. This repeated determination of the function may result in a less efficient method with respect to computing time than other methods of comparable accuracy in which previously determined values of the dependent variable are used in the subsequent steps.

12.4.1 First-Order Runge—Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

By Euler's method,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Expanding LHS by Taylor's series,

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \cdots$$

Euler's method is known as the first-order Runge–Kutta method.

12.4.2 Second-Order Runge—Kutta Method (Heun Method)

The second order Rungta-Kutta method is given by the equations

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\ (x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

12.4.3 Third-Order Runge—Kutta Method

The third-order Runge–Kutta method is given by the equations

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_n + h,\, y_n + 2k_2 - k_1)$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$y_{n+1} = y_n + k$$

12.4.4 Fourth-Order Runge—Kutta Method

This method is mostly used and is often referred to as the Runge–Kutta method only without reference of the order. The fourth-order Runge–Kutta method is given by the equations

$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = h\,f(x_n + h, y_n + k_3)$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_{n+1} = y_n + k$$

Example 1

Given that $y = 1.3$ *when* $x = 1$ *and* $\dfrac{dy}{dx} = 3x + y$. *Use the second-order Runge–Kutta method (i.e., Heun method) to approximate* y *when* $x = 1.2$. *Use a step size of* 0.1. **[Winter 2012]**

Solution

$$\frac{dy}{dx} = f(x, y) = 3x + y$$

(i) Given: $x_0 = 1, \quad y_0 = 1.3, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$
$$= 0.1\,f(1, 1.3)$$
$$= 0.1\left[3(1) + 1.3\right]$$
$$= 0.43$$
$$k_2 = h\,f(x_0 + h, y_0 + k_1)$$
$$= 0.1\,f(1 + 0.1, 1.3 + 0.43)$$

$$= 0.1\, f(1.1, 1.73)$$
$$= 0.1\,[3(1.1) + 1.73]$$
$$= 0.503$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.43 + 0.503)$$
$$= 0.4665$$
$$y_1 = y_0 + k$$
$$= 1.3 + 0.4665$$
$$= 1.7665$$

(ii) Now, $x_1 = 1.1, \quad y_1 = 1.7665, \quad h = 0.1, \quad n = 1$

$$k_1 = h\, f(x_1, y_1)$$
$$= 0.1\, f(1.1, 1.7665)$$
$$= 0.1\,[3(1.1) + 1.7665]$$
$$= 0.5067$$
$$k_2 = h\, f(x_1 + h, y_1 + k_1)$$
$$= 0.1\, f(1.1 + 0.1, 1.7665 + 05067)$$
$$= 0.1\, f(1.2, 2.2732)$$
$$= 0.1[3(1.2) + 2.2732)]$$
$$= 0.5873$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.5067 + 0.5873)$$
$$= 0.5470$$
$$y_2 = y_1 + k$$
$$= 1.7665 + 0.5470$$
$$= 2.3135$$

Hence, $y_2 = y(1.2) = 2.3135$

Example 2

Use the second-order Runge–Kutta method to find an approximate value of y given that $\frac{dy}{dx} = x - y^2$ *and* $y(0) = 1$ *at* $x = 0.2$ *taking* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x - y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h f(x_0, y_0) \\ &= 0.1\, f(0, 1) \\ &= 0.1\left[0 - (1)^2\right] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= h f(x_0 + h, y_0 + k_1) \\ &= 0.1\, f[0 + 0.1, 1 + (-0.1)] \\ &= 0.1\, f(0.1, 0.9) \\ &= 0.1\left[0.1 - (0.9)^2\right] \\ &= -0.071 \end{aligned}$$

$$\begin{aligned} k &= \frac{1}{2}(k_1 + k_2) \\ &= \frac{1}{2}(-0.1 - 0.071) \\ &= -0.0855 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + k \\ &= 1 - 0.0855 \\ &= 0.9145 \end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9145, \quad h = 0.1, \quad n = 1$

$$\begin{aligned} k_1 &= h f(x_1, y_1) \\ &= 0.1\, f(0.1, 0.9145) \\ &= 0.1\left[0.1 - (0.9145)^2\right] \\ &= -0.0736 \end{aligned}$$

$$\begin{aligned} k_2 &= h f(x_1 + h, y_1 + k_1) \\ &= 0.1\, f\left[0.1 + 0.1, 0.9145 - 0.0736\right] \\ &= 0.1\, f\,(0.2, 0.8408) \\ &= 0.1\left[0.2 - (0.8408)^2\right] \\ &= -0.0507 \end{aligned}$$

$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(-0.0736 - 0.0507)$$
$$= -0.0622$$
$$y_2 = y_1 + k$$
$$= 0.9145 - 0.0622$$
$$= 0.8523$$

Hence, $y_2 = y(0.2) = 0.8523$

Example 3

Obtain the values of y at $x = 0.1, 0.2$ *using the Runge–Kutta method of third order for the differential equation* $\frac{dy}{dx} + y = 0,\ y(0) = 1.$

Solution

$$\frac{dy}{dx} = f(x, y) = -y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h\, f(x_0, y_0)$$
$$= 0.1\, f(0, 1)$$
$$= 0.1\,(-1)$$
$$= -0.1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right)$$
$$= 0.1\, f(0.05, 0.95)$$
$$= 0.1(-0.95)$$
$$= -0.095$$
$$k_3 = h\, f(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= 0.1\, f[0 + 0.1, 1 + 2(-0.095) + 0.1]$$
$$= 0.1\, f(0.1, 0.91)$$
$$= 0.1(-0.91)$$
$$= -0.091$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}[-0.1 + 4(-0.095) - 0.091]$$
$$= -0.0952$$
$$y_1 = y_0 + k$$
$$= 1 - 0.0952$$
$$= 0.9048$$

Hence, $y_1 = y(0.1) = 0.9048$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9048, \quad h = 0.1, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= 0.1\, f(0.1, 0.9048)$$
$$= 0.1(-0.9048)$$
$$= -0.0905$$
$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 0.9048 - \frac{0.0905}{2}\right)$$
$$= 0.1\, f(0.15, 0.8596)$$
$$= 0.1(-0.8596)$$
$$= -0.086$$
$$k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1)$$
$$= 0.1\, f[0.1 + 0.1, 0.9048 + 2(-0.086) + 0.0905]$$
$$= 0.1\, f(0.2, 0.8233)$$
$$= 0.1\,(-0.8233)$$
$$= -0.0823$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}[-0.0905 + 4(-0.086) - 0.0823]$$
$$= -0.0861$$
$$y_2 = y_1 + k$$
$$= 0.9048 - 0.0861$$
$$= 0.8187$$

Hence, $y_2 = y(0.2) = 0.8187$

Example 4

Apply the third-order Runge–Kutta method to the initial-value problem $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$ *over the interval* $(0, 0.2)$ *taking* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.1\, f(0, 1)$$
$$= 0.1(0 - 1)$$
$$= -0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right)$$
$$= 0.1\, f(0.05, 0.95)$$
$$= 0.1\left[(0.05)^2 - 0.95\right]$$
$$= -0.0948$$

$$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= 0.1\, f\left[0 + 0.1, 1 + 2(-0.0948) + 0.1\right]$$
$$= 0.1\, f(0.1, 0.9104)$$
$$= 0.1\left[(0.1)^2 - 0.9104\right]$$
$$= -0.09$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}\left[-0.1 + 4(-0.0948) - 0.09\right]$$
$$= -0.0949$$

$$y_1 = y_0 + k$$
$$= 1 - 0.0949$$
$$= 0.9051$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9051, \quad h = 0.1, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= 0.1 f(0.1, 0.9051)$$
$$= 0.1\left[(0.1)^2 - 0.9051\right]$$
$$= -0.0895$$
$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1 f\left(0.1 + \frac{0.1}{2}, 0.9051 - \frac{0.0895}{2}\right)$$
$$= 0.1 f(0.15, 0.8604)$$
$$= 0.1\left[(0.15)^2 - 0.8604\right]$$
$$= -0.0838$$
$$k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1)$$
$$= 0.1 f\left[0.1 + 0.1, 0.9051 + 2(-0.0838) + 0.0895\right]$$
$$= 0.1 f(0.2, 0.827)$$
$$= 0.1\left[(0.2)^2 - 0.827\right]$$
$$= -0.0787$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}\left[-0.0895 + 4(-0.0838) - 0.0787\right]$$
$$= -0.0839$$
$$y_2 = y_1 + k$$
$$= 0.9051 - 0.0839$$
$$= 0.8212$$

Example 5

Solve the differential equation $\frac{dy}{dx} = x + y$, *with the fourth-order Runge–Kutta method, where* $y(0) = 1$, *with* $x = 0$ *to* $x = 0.2$ *with* $h = 0.1$. **[Winter 2012]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.1 f(0, 1)$$
$$= 0.1 (0 + 1)$$
$$= 0.1$$
$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$
$$= 0.1 f(0.05, 1.05)$$
$$= 0.1 (0.05 + 1.05)$$
$$= 0.11$$
$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$$
$$= 0.1 f(0.05, 1.055)$$
$$= 0.1(0.05 + 1.055)$$
$$= 0.1105$$
$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.1 f(0 + 0.1, 1 + 0.1105)$$
$$= 0.1 f(0.1, 1.1105)$$
$$= 0.1(0.1 + 1.1105)$$
$$= 0.1211$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.11) + 2(0.1105) + 0.1211]$$
$$= 0.1103$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1103$$
$$= 1.1103$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1103, \quad h = 0.1, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= 0.1 f(0.1, 1.1103)$$
$$= 0.1(0.1 + 1.1103)$$
$$= 0.1210$$

$$
\begin{aligned}
k_2 &= h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.1103 + \frac{0.1210}{2}\right) \\
&= 0.1\, f(0.15, 1.1708) \\
&= 0.1\,(0.15 + 1.1708) \\
&= 0.1321 \\
k_3 &= h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.1103 + \frac{0.1321}{2}\right) \\
&= 0.1\, f(0.15, 1.1764) \\
&= 0.1\,(0.15 + 1.1764) \\
&= 0.1326 \\
k_4 &= h f(x_1 + h, y_1 + k_3) \\
&= 0.1\, f(0.1 + 0.1, 1.1103 + 0.1326) \\
&= 0.1\, f(0.2, 1.2429) \\
&= 0.1(0.2 + 1.2429) \\
&= 0.1443 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.1210 + 2(0.1321) + 2(0.1326) + 0.1443\right] \\
&= 0.1325 \\
y_2 &= y_1 + k \\
&= 1.1103 + 0.1325 \\
&= 1.2428
\end{aligned}
$$

Example 6

Using the Runge–Kutta method of fourth-order, solve $10\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$, $y(0) = 1$ *at* $x = 0.1$ *and* $x = 0.2$ *taking* $h = 0.1$. **[Summer 2015]**

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = \frac{x^2 + y^2}{10} = 0.1(x^2 + y^2)$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.1\, f(0, 1)$$
$$= 0.1(0.1)(0 + 1)$$
$$= 0.01$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.01}{2}\right)$$
$$= 0.1\, f(0.05, 1.005)$$
$$= 0.1(0.1)\left[(0.05)^2 + (1.005)^2\right]$$
$$= 0.0101$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0101}{2}\right)$$
$$= 0.1\, f(0.05, 1.0051)$$
$$= 0.1(0.1)\left[(0.05)^2 + (1.0051)^2\right]$$
$$= 0.0101$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.1\, f(0 + 0.1, 1 + 0.0101)$$
$$= 0.1\, f(0.1, 1.0101)$$
$$= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right]$$
$$= 0.0103$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.01 + 2(0.0101) + 2(0.0101) + 0.0103\right]$$
$$= 0.0101$$

$$y_1 = y_0 + k$$
$$= 1 + 0.0101$$
$$= 1.0101$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.0101, \quad h = 0.1, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= h f(0.1, 1.0101)$$
$$= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right]$$
$$= 0.0103$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0103}{2}\right)$$
$$= 0.1\, f(0.15, 1.0153)$$
$$= 0.1(0.1)\left[(0.15)^2 + (1.0153)^2\right]$$
$$= 0.0105$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0105}{2}\right)$$
$$= 0.1\, f(0.15, 1.0154)$$
$$= 0.1(0.1)\left[(0.15)^2 + (1.0154)^2\right]$$
$$= 0.0105$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$
$$= 0.1\, f(0.1 + 0.1, 1.0101 + 0.0105)$$
$$= 0.1\, f(0.2, 1.0206)$$
$$= 0.1\,(0.1)\left[(0.2)^2 + (1.0206)^2\right]$$
$$= 0.0108$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.0103 + 2(0.0105) + 2(0.0105) + 0.0108\right]$$
$$= 0.0105$$

$$y_2 = y_1 + k$$
$$= 1.0101 + 0.0105$$
$$= 1.0206$$

Example 7

Use the fourth order Runge–Kutta method to find the value of y at $x = 1$, *given that* $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$ *with* $h = 0.5$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.5, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.5\ f(0, 1)$$
$$= 0.5\left(\frac{1-0}{1+0}\right)$$
$$= 0.5$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$
$$= 0.5\ f(0.25, 1.25)$$
$$= 0.5\left(\frac{1.25-0.25}{1.25+0.25}\right)$$
$$= 0.3333$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right)$$
$$= 0.5\ f(0.25, 1.1667)$$
$$= 0.5\left(\frac{1.1667-0.25}{1.1667+0.25}\right)$$
$$= 0.3235$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.5\ f(0+0.5, 1+0.3235)$$
$$= 0.5(0.5, 1.3235)$$
$$= 0.5\left(\frac{1.3235-0.5}{1.3235+0.5}\right)$$
$$= 0.2258$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.5 + 2(0.3333) + 2(0.3235) + 0.2258]$$
$$= 0.3399$$
$$y_1 = y_0 + k$$
$$= 1 + 0.3399$$
$$= 1.3399$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3399, \quad h = 0.5, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.5\ f(0.5, 1.3399)$$
$$= 0.5\left(\frac{1.3399 - 0.5}{1.3399 + 0.5}\right)$$
$$= 0.2282$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.5 f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.2282}{2}\right)$$
$$= 0.5\ f(0.75, 1.454)$$
$$= 0.5\left(\frac{1.454 - 0.75}{1.454 + 0.75}\right)$$
$$= 0.1597$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.1597}{2}\right)$$
$$= 0.5\ f(0.75, 1.4198)$$
$$= 0.5\left(\frac{1.4198 - 0.75}{1.4198 + 0.75}\right)$$
$$= 0.1543$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3)$$
$$= 0.5\ f(0.5 + 0.5, 1.3399 + 0.1543)$$
$$= 0.5\ f(1, 1.4942)$$
$$= 0.5\left(\frac{1.4942 - 1}{1.4942 + 1}\right)$$
$$= 0.0991$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.2282 + 2(0.1597) + 2(0.1543) + 0.0991\right]$$
$$= 0.1592$$
$$y_2 = y_1 + k$$
$$= 1.3399 + 0.1592$$
$$= 1.4991$$

Hence, $y_2 = y(1) = 1.4991$

Example 8

Using the fourth order Runge–Kutta method, find y at x = 0.1 for differential equation $\frac{dy}{dx} = 3e^x + 2y,\ y(0) = 0$ *by taking* $h = 0.1$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = 3e^x + 2y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$
$$= 0.1\,f(0, 0)$$
$$= 0.1(3e^0 + 0)$$
$$= 0.3$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right)$$
$$= 0.1\,f(0.05, 0.15)$$
$$= 0.1\left[3e^{0.05} + 2(0.15)\right]$$
$$= 0.3454$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3454}{2}\right)$$
$$= 0.1\,f(0.05, 0.1727)$$

$$= 0.1\left[3e^{0.05} + 2(0.1727)\right]$$
$$= 0.3499$$
$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$
$$= 0.1\,f(0 + 0.1, 0 + 0.3499)$$
$$= 0.1\,f(0.1, 0.3499)$$
$$= 0.1\left[3e^{0.1} + 2(0.3499)\right]$$
$$= 0.4015$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.3 + 2(0.3454) + 2(0.3499) + 0.4015\right]$$
$$= 0.3487$$
$$y_1 = y_0 + k$$
$$= 0 + 0.3487$$
$$= 0.3487$$

Hence, $y_2 = y(0.1) = 0.3487$

Example 9

Determine $y(0.1)$ *and* $y(0.2)$ *correct to four decimal places from* $\dfrac{dy}{dx} = 2x + y,\ y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$
$$= 0.1\,f(0, 1)$$
$$= 0.1\left[2(0) + 1\right]$$
$$= 0.1$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$
$$= 0.1\,f(0.05, 1.05)$$

$$= 0.1[2(0.05) + 1.05]$$
$$= 0.115$$
$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.115}{2}\right)$$
$$= 0.1\, f(0.05, 1.0575)$$
$$= 0.1[2(0.05) + 1.0575]$$
$$= 0.11575$$
$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.1\, f(0 + 0.1, 1 + 0.11575)$$
$$= 0.1\, f(0.1, 1.11575)$$
$$= 0.1[2(0.1) + 1.11575]$$
$$= 0.13158$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.115) + 2(0.11575) + 0.13158]$$
$$= 0.1155$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1155$$
$$= 1.1155$$

Hence, $y_1 = y(0.1) = 1.1155$

(ii) Given: $x_1 = 0.1, \quad y_1 = 1.1155, \quad h = 0.1, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= 0.1\, f(0.1, 1.1155)$$
$$= 0.1[2(0.1) + 1.1155]$$
$$= 0.13165$$
$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.1155 + \frac{0.13165}{2}\right)$$
$$= 0.1\, f(0.15, 1.1813)$$
$$= 0.1[2(0.15) + 1.1813]$$
$$= 0.14813$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.1155 + \frac{0.14813}{2}\right)$$
$$= 0.1\, f(0.15, 1.18965)$$
$$= 0.1[2(0.15) + 1.18965]$$
$$= 0.149$$
$$k_4 = h(x_1 + h, y_1 + k_3)$$
$$= 0.1\, f(0.1 + 0.1, 1.1155 + 0.149)$$
$$= 0.1\, f(0.2, 1.2645)$$
$$= 0.1[2(0.2) + 1.2645]$$
$$= 0.16645$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.13165 + 2(0.14813) + 2(0.149) + 0.16645]$$
$$= 0.1487$$
$$y_2 = y_1 + k$$
$$= 1.1155 + 0.1487$$
$$= 1.2642$$

Hence, $y_2 = y(0.2) = 1.2642$

Example 10

Apply the Runge–Kutta method of fourth order to find an approximate value of y at $x = 0.6$ $\dfrac{dy}{dx} = \sqrt{x+y}$, $y(0.4) = 0.41$ *in two steps.*

Solution

$$\frac{dy}{dx} = f(x, y) = \sqrt{x+y}$$

(i) Given: $x_0 = 0.4, \quad y_0 = 0.41, \quad h = 0.1, \quad n = 0$

$$k_1 = hf(x_0, y_0)$$
$$= 0.1\, f(0.4, 0.41)$$
$$= 0.1\sqrt{0.4 + 0.41}$$
$$= 0.09$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 f\left(0.4 + \frac{0.1}{2}, 0.41 + \frac{0.09}{2}\right)$$
$$= 0.1\, f(0.45, 0.455)$$
$$= 0.1\sqrt{0.45 + 0.455}$$
$$= 0.0951$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0.4 + \frac{0.1}{2}, 0.41 + \frac{0.0951}{2}\right)$$
$$= 0.1\, f(0.45, 0.4576)$$
$$= 0.1\sqrt{0.45 + 0.4576}$$
$$= 0.0953$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.1\, f(0.4 + 0.1, 0.41 + 0.0953)$$
$$= 0.1\, f(0.5, 0.5053)$$
$$= 0.1\sqrt{0.5 + 0.5053}$$
$$= 0.1003$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.09 + 2(0.0951) + 2(0.0953) + 0.1003]$$
$$= 0.0952$$
$$y_1 = y_0 + k$$
$$= 0.41 + 0.0952$$
$$= 0.5052$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 0.5052, \quad h = 0.1, \quad n = 1$

$$k_1 = h\, f(x_1, y_1)$$
$$= 0.1\, f(0.5, 0.5052)$$
$$= 0.1\sqrt{0.5 + 0.5052}$$
$$= 0.1003$$
$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0.5 + \frac{0.1}{2}, 0.5052 + \frac{0.1003}{2}\right)$$

$$= 0.1\, f(0.55, 0.5554)$$
$$= 0.1\sqrt{0.55 + 0.5554}$$
$$= 0.1051$$
$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0.5 + \frac{0.1}{2}, 0.5052 + \frac{0.1051}{2}\right)$$
$$= 0.1\, f(0.55, 0.5578)$$
$$= 0.1\sqrt{0.55 + 0.5578}$$
$$= 0.1053$$
$$k_4 = h\, f(x_1 + h, y_1 + k_3)$$
$$= 0.1\, f(0.5 + 0.1, 0.5052 + 0.1053)$$
$$= 0.1\, f(0.6, 0.6105)$$
$$= 0.1\sqrt{0.6 + 0.6105}$$
$$= 0.1100$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.1003 + 2(0.1051) + 2(0.1053) + 0.1100\right]$$
$$= 0.1052$$
$$y_2 = y_1 + k$$
$$= 0.5052 + 0.1052$$
$$= 0.6104$$

Hence, $y_2 = y(0.6) = 0.6104$

Example 11

Solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x + y}$, $x_1 = 0$, $y_1 = 1$ *for the interval* (0, 1) *choosing* $h = 0.5$ *by the Runge–Kutta method of fourth order.*

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = \frac{1}{x + y}$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.5, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.5 f(0, 1)$$
$$= 0.5\left(\frac{1}{0+1}\right)$$
$$= 0.5$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$
$$= 0.5\ f(0.25, 1.25)$$
$$= 0.5\left(\frac{1}{0.25+1.25}\right)$$
$$= 0.3333$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right)$$
$$= 0.5\ f(0.25, 1.1666)$$
$$= 0.5\left(\frac{1}{0.25+1.1666}\right)$$
$$= 0.3529$$

$$k_4 = h f(x_0 + h, y_0 + k_2)$$
$$= 0.5\ f(0 + 0.5, 1 + 0.3529)$$
$$= 0.5\ f(0.5, 1.3529)$$
$$= 0.5\left(\frac{1}{0.5+1.3529}\right)$$
$$= 0.2698$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.5 + 2(0.3333) + 2(0.3529) + 0.2698]$$
$$= 0.3570$$

$$y_1 = y_0 + k$$
$$= 1 + 0.3570$$
$$= 1.3570$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3570, \quad h = 0.5, \quad n = 1$

$$k_1 = h f(x_1, y_1)$$
$$= 0.5\, f(0.5, 1.3570)$$
$$= 0.5\left(\frac{1}{0.5+1.3570}\right)$$
$$= 0.2692$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.5\, f\left(0.5 + \frac{0.5}{2}, 1.3570 + \frac{0.2692}{2}\right)$$
$$= 0.5\, f(0.75, 1.4916)$$
$$= 0.5\left(\frac{1}{0.75+1.4916}\right)$$
$$= 0.2230$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.5\, f\left(0.5 + \frac{0.5}{2}, 1.3570 + \frac{0.2230}{2}\right)$$
$$= 0.5\, f(0.75, 1.4685)$$
$$= 0.5\left(\frac{1}{0.75+1.4685}\right)$$
$$= 0.2253$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$
$$= 0.5\, f(0.5+0.5, 1.3570+0.2253)$$
$$= 0.5\, f(1, 1.5823)$$
$$= 0.5\left(\frac{1}{1+1.5823}\right)$$
$$= 0.1936$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.2692 + 2(0.2230) + 2(0.2253) + 0.1936\right]$$
$$= 0.2265$$

$$y_2 = y_1 + k$$
$$= 1.3570 + 0.2265$$
$$= 1.5835$$

Example 12

Apply the Runge–Kutta method of fourth order to find an approximate value of y at $x = 0.2$ *if* $\frac{dy}{dx} = x + y^2$, *given that* $y = 1$ *when* $x = 0$ *in steps of* $h = 0.1$. **[Summer 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h f(x_0, y_0)$$
$$= 0.1 f(0, 1)$$
$$= 0.1(0 + 1^2)$$
$$= 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$
$$= 0.1 f(0.05, 1.05)$$
$$= 0.1\left[0.05 + (1.05)^2\right]$$
$$= 0.1152$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1152}{2}\right)$$
$$= 0.1 f(0.05, 1.0576)$$
$$= 0.1\left[0.05 + (1.0576)^2\right]$$
$$= 0.1168$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$
$$= 0.1 f(0 + 0.1, 1 + 0.1168)$$
$$= 0.1 f(0.1, 1.1168)$$
$$= 0.1\left[0.1 + (1.1168)^2\right]$$
$$= 0.1347$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.1152) + 2(0.1168) + 0.1347]$$
$$= 0.1164$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1164$$
$$= 1.1164$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1164, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 1.1164)$$
$$= 0.1[0.1 + (1.1164)^2]$$
$$= 0.1346$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1346}{2}\right)$$
$$= 0.1\,f(0.15, 1.1837)$$
$$= 0.1[0.15 + (1.1837)^2]$$
$$= 0.1551$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1551}{2}\right)$$
$$= 0.1\,f(0.15, 1.1939)$$
$$= 0.1[0.15 + (1.1939)^2]$$
$$= 0.1575$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3)$$
$$= 0.1\,f(0.1 + 0.1, 1.1164 + 0.1575)$$
$$= 0.1\,f(0.2, 1.2739)$$
$$= 0.1[0.2 + (1.2739)^2]$$
$$= 0.1822$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1346 + 2(0.1551) + 2(0.1575) + 0.1822]$$
$$= 0.157$$
$$y_2 = y_1 + k$$
$$= 1.1164 + 0.157$$
$$= 1.2734$$

Hence, $y_2 = y(0.2) = 1.2734$

EXERCISE 12.3

Solve the following differential equations by the Runga–Kutta method:

1. $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[Ans.: 1.2424]

2. $\frac{dy}{dx} = xy$ with $y(1) = 2$ at $x = 1.2, x = 1.4$

[Ans.: 2.4921, 3.2311]

3. $\frac{dy}{dx} = x^2 + y^2$ with $x_0 = 1,\ y_0 = 1.5,\ h = 0.1$ at $x = 1.2$

[Ans.: 2.5043]

4. $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, at $x = 0.2$ and $x = 0.4$

[Ans.: 1.8310, 2.0214]

5. $\frac{dy}{dx} = \frac{y - x}{y + x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[Ans.: 1.1678]

6. $\frac{dy}{dx} = 1 + y^2$ with $x_0 = 0, y_0 = 0$ at $x = 0.2, 0.4$ and 0.6

[Ans.: 0.2027, 0.4228, 0.6891]

7. $\frac{dy}{dx} = xy^2$ with $x_0 = 2, y_0 = 1$ for $x = 2.2$ taking $h = 0.2$

[Ans.: -1.7241]

8. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$ taking $h = 0.1$

[Ans.: 0.8512]

9. $\frac{dy}{dx} = \frac{x-y}{xy}$ with $x_0 = 1, y_0 = 1$ at $x = 1.1$

[Ans.: 1.0045**]**

10. $\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.1, 0.2, 0.3,$ and 0.4

[Ans.: 1.0911, 1.1677, 1.2352, 1.2902**]**

Points to Remember

Euler's Method

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Modified Euler's Method

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2, \ldots$$

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

Runge–Kutta Methods

1. First-Order Runge–Kutta Method

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \cdots$$

2. Second-Order Runge–Kutta Method (Heun Method)

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

3. Third-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_n + h, y_n + 2k_2 - k_1)$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$y_{n+1} = y_n + k$$

4. Fourth-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_n + h, y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

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