Engineering Mathematics-I

(M101)

Fourth Edition

WBUT–2015

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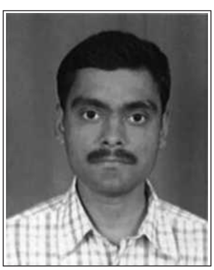

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Dr Kar and Karmakar have jointly published two other books, **Engineering Mathematics II** and **Engineering Mathematics III,** for WBUT with McGraw Hill Education (India).

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Dedicated

To my teacher

Dr Sanjib Kr Datta

and

My beloved family members

Sourav Kar

To the Holy Mother

Maa Sarada

Subrata Karmakar

Contents

Preface

The highest form of pure thought is in Mathematics.

Plato (428 BC–348 BC)

Since mathematics forms the basis of any branch of engineering and technology, the West Bengal University of Technology has made a commendable endeavour by introducing different syllabi for mathematics at different semesters in the B.Tech. level. The current book has been written as per the latest WBUT syllabus for the first-year, first-semester B.Tech. students. Our main objective of writing this book is to help students build upon the fundamental concepts which are also required for subjects studied in the higher semesters. Each and every topic of the book is lucidly explained and illustrated with different kinds of examples. Also, stepwise clarifications of different methods of solving problems are given.

Salient Features

- Full coverage of the WBUT syllabus (2010 Regulation)
- Lucid explanation of topics like Matrix, Infinite Series, Vector Algebra, Vector Calculus, Calculus of Functions of Several Variables
- Stepwise solutions to examples
- Solved WBUT questions from 2001-2009 incorporated within each chapter
- Solutions of WBUT examination papers from 2010-2014 are placed at the end of the book
- Rich pedagogy:
    - 400 Solved Examples
    - 315 Short and Long Answer Type Questions
    - 220 Multiple Choice Questions

Chapter Organisation

The contents of the book are divided into nine chapters.

In **Chapter 1,** we first represent the fundamentals of matrices along with the notations and algebraic operations applicable on them. Here, we also discuss the determinant of a square matrix, singular and non-singular matrices, and the method of computing the inverse of a matrix along with its properties, orthogonal matrix and trace of a matrix. **Chapter 2** deals with the concept of the rank of a matrix, matrix inversion method, Cramer's rule, consistency and inconsistency of a system of homogeneous and nonhomogeneous linear simultaneous equations, Eigen values and Eigen vectors, and the Cayley–Hamilton theorem and its applications.

Chapter 3 discusses successive differentiation and Leibnitz's theorem along with its applications. In **Chapter 4,** we present the very well-known three mean-value theorems, namely, Rolle's, Lagrange's and Cauchy's mean-value theorems along with their wide range of applications in various fields. The series expansion theorems and formulas, namely, Taylor's and Maclaurin's series expansion, are also discussed in this chapter.

Chapter 5 explains the concept of reduction formulas for integration and its applications. In **Chapter 6,** we introduce the concept of functions of several variables. Also, we describe the methods of differentiations and their applications towards optimisations of the functions. **Chapter 7** deals with line integrals, double integrals and triple integrals.

Chapter 8 basically covers preliminary ideas of real sequences and illustrative ideas of infinite series. **Chapter 9** has been divided into three parts. In the first part of this chapter, we discuss vector algebra. The second part of the chapter deals with vector differentiations, gradient, divergence and curl. In the third part of the chapter, we give theorems on vector integrations (Green's theorem, Divergence theorem, Stokes' theorem) and their applications to physical problems.

At the end of the each chapter, various kinds of solved examples covering all the topics, including 2001–2009 solved WBUT questions, are given. Numerous short and long-answer-type question and multiple-choice questions are given in the exercises of every chapter. Solutions of 2010 to 2014 WBUT examination papers are provided at the end of the book.

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Last, but not the least, we are very thankful to the entire team at McGraw Hill Education (India) for publishing this book within a short span of time.

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ROADMAP TO THE SYLLABUS

Engineering Mathematics-I

This text is suitable for the Subject Code M101.

Module I

Matrix

Determinant of a square matrix; Minors and Cofactors; Laplace's method of expansion of a determinant; Product of two determinants; Adjoint of a determinant; Jacobi's theorem on adjoint of a determinant; Singular and nonsingular matrices; Adjoint of a matrix; Inverse of a nonsingular matrix and its properties; Orthogonal matrix and its properties, Trace of a matrix

Rank of a matrix and its determination using elementary row and column operations; Solution of simultaneous linear equations by matrix inversion method; Consistency and inconsistency of a system of homogeneous and inhomogeneous linear simultaneous equations; Eigen values and Eigen vectors of a square matrix (of order 2 or 3); Eigen values of AP^{TP}; kA; AP^{-1P}; Cayley–Hamilton theorem and its applications

GO TO

CHAPTER 1 MATRIX I
CHAPTER 2 MATRIX II

Module II

Successive Differentiation

Higher-order derivatives of a function of single variable; Leibnitz's theorem (statement only and its application; problems of the type of recurrence relations in derivatives of different orders and also to find $((y_n)_0)$

Mean-Value Theorems and Expansion of Functions

Rolle's theorem and its application; Mean-value theorems—Lagrange's and Cauchy's theorems and their application; Taylor's theorem with Lagrange's and Cauchy's form of remainders and its application; Expansions of functions by Taylor's and Maclaurin's theorems; Maclaurin's infinite series expansion of the functions $\sin x$; $\cos x$; e^x; $\log(1+x)$; $(a+x)^n$, n being an integer or a fraction (assuming that the remainder $R_n \to 0$ as $n \to \infty$ in each case)

Reduction Formula

Reduction formulae both for indefinite and definite integrals of types
$\int \sin^n x$; $\int \cos^n x$; $\int \sin^m x \cos^n x$; $\int \cos^m x \sin nx$; $\int dx/(x^2 + a^2)^n$, m, n are positive integers

GO TO
CHAPTER 3 SUCCESSIVE DIFFERENTIATION
CHAPTER 4 MEAN-VALUE THEOREMS AND EXPANSION OF FUNCTIONS
CHAPTER 5 REDUCTION FORMULA

Module III

Calculus of Functions of Several Variables

Introduction to functions of several variables with examples; Knowledge of limit and continuity; Partial derivatives and related problems; Homogeneous functions and Euler's theorem and related problems up to three variables; Chain rules; Differentiation of implicit functions; Total differentials and their related problems; Jacobians up to three variables and related problems; Maxima, minima and saddle points of functions and related problems; Concept of line integrals; Double and triple integrals

GO TO
CHAPTER 6 CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES
CHAPTER 7 LINE INTEGRAL, DOUBLE INTEGRAL AND TRIPLE INTEGRAL

Module IV

Infinite Series

Preliminary ideas of sequence; Infinite series and their convergence/divergence; Infinite series of positive terms; Tests for convergence: Comparison test; Cauchy's Root test; D' Alembert's Ratio test and Raabe's test (statements and related problems on these tests); Alternating series; Leibnitz's Test (statement; definition) illustrated by simple example; Absolute convergence and Conditional convergence

GO TO
CHAPTER 8 INFINITE SERIES

Module V

Vector Algebra and Vector Calculus

Scalar and vector fields—definition and terminologies; Dot and cross products; Scalar and vector triple products and related problems; Equation of straight line; Plane and sphere; Vector function of a scalar variable; Differentiation of a vector function; Scalar and vector point functions; Gradient of a scalar point function; divergence and curl of a vector point function; Directional derivative (related problems on these topics) Green's theorem; Gauss Divergence Theorem and Stokes' theorem (statements and applications)

GO TO **CHAPTER 9** VECTOR ANALYSIS

CHAPTER

1

Matrix I

1.1 INTRODUCTION

Matrix algebra is a very essential part of mathematics. It has a wide range of applications in various branches of science and technology. Besides direct applications, we also borrow the concept of matrix notations for representing various systems in a compact manner.

In this chapter, we first represent the fundamentals of matrices along with the notations and algebraic operations applicable on them. Next we discuss symmetric and skew-symmetric matrices with the help of the transpose property.

Here, we shall also discuss a very important characteristic of matrices, namely, '*determinant*', which is very useful for dealing with physical problems in science and technology. Here, we give different methods for computing determinants along with the various algebraic operations.

Next, we describe the concept of singular and nonsingular matrices and the method of computing the inverse of a matrix along with its properties.

In the last part, orthogonal matrix and trace of a matrix have been illustrated lucidly.

Definition: A rectangular array of mn elements a_{ij} into m rows and n colunms where the elements a_{ij} belong to a field F enclosed by a pair of brackets, is said to be a **matrix** of order $m \times n$ over the field F. The $m \times n$ matrix is of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ or } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

F is said to be the *field of scalars*. In particular, F is the field of real or complex numbers. The matrix is denoted by $\left(a_{ij}\right)_{m \times n}$.

For example,

(i) $\begin{pmatrix} 3 & 1 \\ 5 & 6 \\ 8 & 0 \end{pmatrix}_{3\times 2}$ is a matrix with 3 rows and 2 columns over a real field

(ii) $\begin{pmatrix} 5 & 1 & 9 \\ -4 & 6 & 1 \end{pmatrix}_{2\times 3}$ is a matrix with 2 rows and 3 columns over a real field

(iii) $\begin{pmatrix} 2 & -1 & 9 \\ 4 & 6 & 5 \\ 8 & -7 & 2 \end{pmatrix}_{3\times 3}$ is a matrix with 3 rows and 3 columns over a real field

1.2 DIFFERENT TYPE OF MATRICES

1) **Zero Matrix or Null Matrix:** A matrix is called zero matrix if every element of it is 0. A null matrix of order $m \times n$ is denoted by $O_{m\times n}$.

For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero matrix of order 3×2 and it is denoted by $O_{3\times 2}$.

Also, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a zero matrix of order 3×3 and it is denoted by $O_{3\times 3}$.

2) **Square Matrix:** A matrix with equal number of rows and columns is called a square matrix.

For example, $\begin{pmatrix} 4 & 5 \\ 2 & -9 \end{pmatrix}$ is a square matrix of order 2×2.

3) **Diagonal Matrix:** A square matrix is said to be a diagonal matrix if the elements other than the diagonal elements are all zero.

For example, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a diagonal matrix of order 3×3.

4) **Scalar Matrix:** A diagonal matrix is said to be a scalar matrix if all the diagonal elements are the same scalar.

For example, $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix of order 3×3.

5) **Identity Matrix:** A scalar matrix with diagonal elements equal to 1 is called an identity matrix.

For example, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3×3. It is denoted by I_3.

6) Triangular Matrix:

a) A square matrix (a_{ij}) is said to be an **upper triangular matrix** if all the elements below the diagonal are 0. That is, $a_{ij} = 0, \ i > j$.

For example, $\begin{pmatrix} 4 & 6 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ is an upper triangular matrix of order 3×3.

b) A square matrix (a_{ij}) is said to be a **lower triangular matrix** if all the elements above the diagonal are 0. That is, $a_{ij} = 0, \ i < j$.

For example, $\begin{pmatrix} 4 & 0 & 0 \\ 7 & 1 & 0 \\ -9 & 8 & 1 \end{pmatrix}$ is a lower triangular matrix of order 3×3.

7) Row Matrix and Column Matrix:

Any matrix $A = \left(a_{ij}\right)_{m \times n}$ is called a row matrix if $m = 1$, i.e., the matrix has only one row.

So its form is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \end{pmatrix}_{1 \times n}$.

Any matrix $A = \left(a_{ij}\right)_{m \times n}$ is called a column matrix if $n = 1$, i.e., the matrix has only one column.

So its form is $A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \dots \\ a_{m1} \end{pmatrix}_{m \times 1.}$

For example, $\begin{pmatrix} 2 & -3 & 0 & 5 & 8 \end{pmatrix}$ is a row matrix.

For example, $A = \begin{pmatrix} 3 \\ 0 \\ -6 \\ -3 \\ 1 \end{pmatrix}$ is a column matrix.

1.3 ALGEBRAIC OPERATIONS ON MATRICES

1.3.1. Equality of Two Matrices

Two matrices A and B are said to be equal if A and B have the same order and their corresponding elements are equal.

Thus if $A = \left(a_{ij}\right)_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then $A = B$ if and only if $a_{ij} = b_{ij}$ for $i = 1, 2, \dots m; j = 1, 2, \dots n$.

For example, $A=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ are equal if $a=1, b=2, c=3,$ $d=4, e=5, f=6.$

1.3.2. Multiplication by a Scalar

The product of an $m\times n$ matrix $A=\left(a_{ij}\right)_{m\times n}$ by a scalar c where $c\in F$, the field of scalars is a matrix $B=\left(b_{ij}\right)_{m\times n}$ defined by

$$b_{ij}=ca_{ij},\quad i=1,2,\ldots m;\quad j=1,2,\ldots n \text{ and is written as } B=cA.$$

For example, let $A=\begin{pmatrix} 2 & -1 & 4 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$ then $2A=\begin{pmatrix} 4 & -2 & 8 \\ -6 & 2 & 0 \\ 10 & 0 & 2 \end{pmatrix}$.

1.3.3. Addition of Matrices

If $A=\left(a_{ij}\right)_{m\times n}$ and $B=(b_{ij})_{m\times n}$ then their sum (or difference) $A\pm B$ is the matrix $C=\left(c_{ij}\right)_{m\times n}$ where $c_{ij}=a_{ij}\pm b_{ij}, i=1,2,\ldots m;\ \ j=1,2,\ldots n.$

Two matrices of the same order are said to be **conformable for addition.**

For example,

(i) If $A=\begin{pmatrix} 5 & -1 & 8 \\ 4 & 2 & 9 \\ -7 & 5 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} -7 & 5 & 4 \\ -9 & 2 & 5 \\ 3 & 8 & 5 \end{pmatrix}$

then $A+B=\begin{pmatrix} 5-7 & -1+5 & 8+4 \\ 4-9 & 2+2 & 9+5 \\ -7+3 & 5+8 & 1+5 \end{pmatrix}=\begin{pmatrix} -2 & 4 & 12 \\ -5 & 4 & 14 \\ -4 & 13 & 6 \end{pmatrix}$

(ii) If $A=\begin{pmatrix} 2+i & 4+5i \\ 3-5i & -2+8i \end{pmatrix}$ and $B=\begin{pmatrix} -8+7i & 3i \\ -4+6i & 7+i \end{pmatrix}$

then $A-B=\begin{pmatrix} 10-6i & 4+2i \\ 7-11i & -9+7i \end{pmatrix}$ where $i=\sqrt{-1}$.

Properties:

a) Matrix addition is commutative, i.e., $A+B=B+A$.

b) Matrix addition is associative, i.e., $A+(B+C)=(A+B)+C$.

c) Scalar multiplication is distributive over matrix addition, i.e., $c(A+B)=cA+cB$.

d) For any matrix $A_{m\times n}$, $0\cdot A_{m\times n}=O_{m\times n}$.

e) For any matrix $A_{m\times n}$, $A_{m\times n}+O_{m\times n}=A_{m\times n}$.

1.3.4. Multiplication of matrices

If $A = \left(a_{ij}\right)_{m\times n}$ is of order $m\times n$ and $B = (b_{ij})_{n\times p}$ is a matrix of order $n\times p$ then the product AB is a matrix of order $m\times p$ and $AB = C = \left(c_{ij}\right)_{m\times p}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1,2,\ldots m; \quad j = 1,2,\ldots p.$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$

Here number of columns of $A = 3 =$ number of rows of B. So multiplication is possible and the product AB is a 2×2 matrix, given by

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

Observation:

a) **The *ij*-th element of the product *AB* is obtained by multiplying the corresponding elements of the *i*-th row of *A* and the *j*-th column of *B* and adding such products.**

b) **If the number of columns of *A* is not equal to the number of rows of *B* then *AB* is not defined.**

Example 1 Let us consider two matrices $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \\ 0 & 5 \end{pmatrix}_{3\times 2}$

Since number of columns of $A = 3 =$ number of rows of B, the product AB is defined and is given by a 2×2 matrix

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & -2 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 0\times 3 + (-1)\times 0 & 1\times 0 + 0\times(-2) + (-1)\times 5 \\ 0\times 1 + 2\times 3 + 3\times 0 & 0\times 0 + 2\times(-2) + 3\times 5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 6 & 11 \end{pmatrix}_{2\times 2}$$

Again, since number of columns of $B = 2 =$ number of rows of A, the product BA is defined and is given by a 3×3 matrix

$$BA = \begin{pmatrix} 1 & 0 \\ 3 & -2 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 0\times 0 & 1\times 0 + 0\times 2 & 1\times(-1) + 0\times 3 \\ 3\times 1 + (-2)\times 0 & 3\times 0 + (-2)\times 2 & 3\times(-1) + (-2)\times 3 \\ 0\times 1 + 5\times 0 & 0\times 0 + 5\times 2 & 0\times(-1) + 5\times 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & -4 & -9 \\ 0 & 10 & 15 \end{pmatrix}_{3\times 3}.$$

Properties:

(1) In general, matrix multiplication is not commutative, i.e., $AB \neq BA$.

From the previous example, it is obvious that $AB \neq BA$. Also, in this case the orders of AB and BA are different.

But in the next example we will see the fact that $AB \neq BA$ even when the orders of AB and BA are same.

Example 2 Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}_{2\times 2}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix}_{2\times 2}$.

Then $AB = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} -5 & 12 \\ 18 & -11 \end{pmatrix}_{2\times 2}$

and $BA = \begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ -6 & -23 \end{pmatrix}_{2\times 2}$

Here though the orders of AB and BA are same, $AB \neq BA$.

But in some special cases, AB may be same as BA, i.e., $AB = BA$, which follows from the next example:

Example 3 Let $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}_{2\times 2}$ and $B = \begin{pmatrix} 3 & -2 \\ 0 & 5 \end{pmatrix}_{2\times 2}$.

Then $AB = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -12 \\ 0 & 15 \end{pmatrix}_{2\times 2}$

and $BA = \begin{pmatrix} 3 & -2 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -12 \\ 0 & 15 \end{pmatrix}_{2\times 2}$

Here, $AB = BA$, which proves the above stated fact.

(2) Matrix multiplication is associative, i.e., $A(BC) = (AB)C$.

(3) Matrix multiplication is distributive over matrix addition,

i.e., $A(B+C) = AB + AC$, provided both sides are defined.

(4) Matrix addition is distributive over matrix multiplication,

i.e., $(B+C)A = BA + CA$, provided both sides are defined.

(5) For any square matrix A of order $n\times n$, $A \cdot I_n = I_n \cdot A = A$.

(6) For any square matrix A of order $n\times n$, $A \cdot O_{n,n} = O_{n,n} \cdot A = O_{n,n}$.

(7) The product of two non null matrices may result to a null matrix. This will be evident from the following example.

Example 4 Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, we see that the product of two non null matrices results to a null matrix.

Example 5 **Let A and B are two square matrices of the same order, examine if the following holds**

$$(A + B)^2 = A^2 + 2AB + B^2$$

Sol. Since, A and B are square matrices, A^2 and B^2 are both defined. Since, A and B are square matrices of same order, $A + B$ and AB are both defined. Now,

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B)\\ &= A(A + B) + B(A + B)\\ &= A^2 + AB + BA + B^2\end{aligned}$$

Since, in general matrix multiplication is not commutative, i.e., $AB \neq BA$, we have

$$(A + B)^2 \neq A^2 + 2AB + B^2$$

But, if the product of A and B are commutative i.e., $AB = BA$, then

$$(A + B)^2 = A^2 + 2AB + B^2$$

holds good

1.4 TRANSPOSE OF A MATRIX

The transpose of a matrix $A = \left(a_{ij}\right)_{m\times n}$ is a matrix $A^T = \left(a_{ji}\right)_{n\times m}$ obtained by converting rows into corresponding columns and vice-versa.

For example, let us consider $A = \begin{pmatrix} 1 & 5 & -1 \\ -2 & 0 & 3 \end{pmatrix}_{2\times 3}$

Then $A^T = \begin{pmatrix} 1 & 5 & -1 \\ -2 & 0 & 3 \end{pmatrix}_{2\times 3}^T = \begin{pmatrix} 1 & -2 \\ 5 & 0 \\ -1 & 3 \end{pmatrix}_{3\times 2}$

Observation:
The number of rows of A^T = number of columns of A.
and the number of columns of A^T = number of rows of A.

Properties:

(a) $(A^T)^T = A$
(b) $(A \pm B)^T = A^T \pm B^T$
(c) $(cA)^T = cA^T$, **where c is a scalar**
(d) $(cA + dB)^T = cA^T + dB^T$, **where c and d are scalars**
(e) $(AB)^T = B^T A^T$

Verification of the Above Properties

Let us consider the two matrices, $A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$

(a) Here, $A^T = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$

Now $\left(A^T\right)^T = \begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix} = A.$

So the property (a) is verified.

(b) Here, $A + B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 7 & 8 \\ 2 & 5 & 1 \\ 5 & 4 & 5 \end{pmatrix}.$

So, $(A+B)^T = \begin{pmatrix} -1 & 7 & 8 \\ 2 & 5 & 1 \\ 5 & 4 & 5 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & 5 \\ 7 & 5 & 4 \\ 8 & 1 & 5 \end{pmatrix}.$

Again, $A^T + B^T = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix}^T + \begin{pmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}^T$

$$= \begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 0 \\ 7 & -1 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 5 \\ 7 & 5 & 4 \\ 8 & 1 & 5 \end{pmatrix}.$$

Hence, $(A+B)^T = A^T + B^T$.

Similarly, we can show $(A-B)^T = A^T - B^T$.

Therefore, the property (b) is verified.

(c) Here, $cA = \begin{pmatrix} -c & 2c & c \\ c & 3c & 2c \\ 5c & 4c & c \end{pmatrix}.$

So $(cA)^T = \begin{pmatrix} -c & 2c & c \\ c & 3c & 2c \\ 5c & 4c & c \end{pmatrix}^T = \begin{pmatrix} -c & c & 5c \\ 2c & 3c & 4c \\ c & 2c & c \end{pmatrix}$

$$= c\begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix} = c \cdot A^T.$$

Hence, the property (c) is verified.

(d) Here, $cA + dB = \begin{pmatrix} -c & 2c & c \\ c & 3c & 2c \\ 5c & 4c & c \end{pmatrix} + \begin{pmatrix} 2d & 5d & 7d \\ d & 2d & -d \\ 0 & 0 & 4d \end{pmatrix}$

$$= \begin{pmatrix} -c+2d & 2c+5d & c+7d \\ c+d & 3c+2d & 2c-d \\ 5c & 4c & c+4d \end{pmatrix}.$$

So, $(cA+dB)^T = \begin{pmatrix} -c+2d & 2c+5d & c+7d \\ c+d & 3c+2d & 2c-d \\ 5c & 4c & c+4d \end{pmatrix}^T = \begin{pmatrix} -c+2d & c+d & 5c \\ 2c+5d & 3c+2d & 4c \\ c+7d & 2c-d & c+4d \end{pmatrix}$

$$= \begin{pmatrix} -c & c & 5c \\ 2c & 3c & 4c \\ c & 2c & c \end{pmatrix} + \begin{pmatrix} 2d & d & 0 \\ 5d & 2d & 0 \\ 7d & -d & 4d \end{pmatrix} = \begin{pmatrix} -c & c & 5c \\ 2c & 3c & 4c \\ c & 2c & c \end{pmatrix} + \begin{pmatrix} 2d & d & 0 \\ 5d & 2d & 0 \\ 7d & -d & 4d \end{pmatrix}$$

$$= c\begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix} + d\begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 0 \\ 7 & -1 & 4 \end{pmatrix} = cA^T + dB^T.$$

Hence, the property (d) is verified.

(e) $$(AB)^T = \left[\begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix} . \begin{pmatrix} 2 & 5 & 7 \\ 1 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} 0 & -1 & -5 \\ 5 & 11 & 8 \\ 14 & 33 & 35 \end{pmatrix}^T = \begin{pmatrix} 0 & 5 & 14 \\ -1 & 11 & 33 \\ -5 & 8 & 35 \end{pmatrix}.$$

Again, $$B^T \cdot A^T = \begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 0 \\ 7 & -1 & 4 \end{pmatrix} . \begin{pmatrix} -1 & 1 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 5 & 14 \\ -1 & 11 & 33 \\ -5 & 8 & 35 \end{pmatrix}.$$

So, $(AB)^T = B^T A^T$ and the property (e) is verified.

1.5 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

A square matrix A is said to be symmetric if $A^T = A$, i.e., $A = (a_{ij})_{n\times n}$ is symmetric if $a_{ij} = a_{ji}$.

Examples of symmetric matrices are

$$\begin{pmatrix} 2 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 9 \end{pmatrix}, \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \begin{pmatrix} 1+i & 2-3i & 4+5i & 7i \\ 2-3i & 2 & -6i & 3-5i \\ 4+5i & -6i & 4i & -7+i \\ 7i & 3-5i & -7+i & 2-i \end{pmatrix}$$

A square matrix A is said to be skew-symmetric if $A^T = -A$, i.e., $A = (a_{ij})_{n\times n}$ is skew-symmetric if $a_{ij} = -a_{ji}$.

Examples of skew-symmetric matrices are

$$\begin{pmatrix} 0 & 5 & 0 \\ -5 & 0 & 6 \\ 0 & -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 9 & -3 \\ -9 & 0 & 6 \\ 3 & -6 & 0 \end{pmatrix}.$$

Properties:

(i) If A and B are two symmetric matrices of the same order then their addition $A+B$ is also symmetric.

(ii) Suppose A and B are two symmetric matrices of the same order, then their product AB is also symmetic, provided $AB = BA$.

(iii) Diagonal elements of any skew-symmetric matrix are all zero.

Theorem 1.1: For any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

Proof: Since $\left(A + A^T\right)^T = (A)^T + \left(A^T\right)^T = A^T + A = A + A^T$, we can say that $A + A^T$ is symmetric.

Again $\left(A - A^T\right)^T = (A)^T - \left(A^T\right)^T = A^T - A = -(A - A^T)$, therefore $A - A^T$ is skew-symmetric.

Hence, the theorem is proved.

Theorem 1.2: Any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof: For any square matrix A, we can write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

From the last theorem, we know $A + A^T$ is symmetric, so $\frac{1}{2}(A + A^T)$ is also symmetric.

Also, we have from the last theorem that $A - A^T$ is skew-symmetric and so $\frac{1}{2}(A - A^T)$ is also skew-symmetric.

Therefore, A can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Hence, the theorem is proved.

For example, let us express $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 7 & 8 \end{pmatrix}$ as the sum of symmetric and skew-symmetric matrices.

Here, $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 7 & 8 \end{pmatrix}$ and so $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 7 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 5 & 9 & 8 \end{pmatrix}$

Now $A + A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 5 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 10 \\ 5 & 8 & 16 \\ 10 & 16 & 16 \end{pmatrix}$

and $A - A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 7 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 5 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$.

From the last theorem, we have $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew-symmetric.

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, we can write

$$A = \frac{1}{2}\begin{pmatrix} 2 & 5 & 10 \\ 5 & 8 & 16 \\ 10 & 16 & 16 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\Rightarrow A = \frac{1}{2}\begin{pmatrix} 2 & 5 & 10 \\ 5 & 8 & 16 \\ 10 & 16 & 16 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

Hence, A can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Theorem 1.3: The product of a matrix with its transpose results in a symmetric matrix.

Proof: Beyond the scope of the book.

Example 6 Verify that the product of the matrix $\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$ and its transpose results to a symmetric matrix.

Sol. Here we have

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$$

and so

$$A^T = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$$

Now,

$$A.A^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 13 & 23 & 33 \\ 23 & 41 & 59 \\ 33 & 59 & 85 \end{pmatrix}$$

which is a symmetric matrix.

1.6 SOME SPECIAL TYPES OF MATRICES

(1) Idempotent Matrix: Any square matrix A is called an idempotent matrix if $A^2 = A$.

For example, let $A = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$

Then $A^2 = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}.\begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix} = A.$$

So A is idempotent.

2) Nilpotent Matrix:

Any square matrix A is called a nilpotent matrix of index p if $A^p = O$, where p is the least positive integer.

For example, let $A = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}$

Then $A^2 = \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix} . \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$

So A is a nilpotent matrix of index 2.

3) Involutary Matrix:

A square matrix A is said to be involutary if $A^2 = I$.

For example, let us show that $A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$ is involutary.

Here,

$$A^2 = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the matrix A is involutary.

Example 7 If A and B are two matrices such that $AB = A$ and $BA = B$, then prove that A^T and B^T are idempotent.

Sol. Since $AB = A$, we have

$(AB)^T = A^T \Rightarrow B^T A^T = A^T$ ...(1)

Again $BA = B$, so from (1)

$(BA)^T A^T = A^T \Rightarrow A^T \left(B^T A^T\right) = A^T$

$\Rightarrow A^T A^T = A^T$ (by(1))

$\Rightarrow \left(A^T\right)^2 = A^T$

So, it is proved that A^T is idempotent.

Similarly, since $BA = B$, we have

$$(BA)^T = B^T \Rightarrow A^T B^T = B^T \qquad ...(2)$$

Again, $AB = A$, so from (1)

$$(AB)^T B^T = B^T \Rightarrow B^T \left(A^T B^T\right) = B^T$$

$$\Rightarrow B^T B^T = B^T \text{ (by(1))}$$

$$\Rightarrow \left(B^T\right)^2 = B^T$$

So, it is also proved that B^T is idempotent.

1.7 DETERMINANT OF A SQUARE MATRIX

Definition Let M be the set of all square matrices of order $n \times n$ with real or complex entries then the determinant of any matrix of M is a function from the set M to any scalar field of real or complex numbers, i.e., determinant of a square matrix is a function which assigns to each matrix a scalar value.

Determinant of a square matrix $A = (a_{ij})_{n\times n}$ is denoted by $\det A$ or $|A|$ and we say the order of the determinant is n.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, then $\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

For example, let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, Then $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is a determinant of order 3.

1.8 PROPERTIES OF THE DETERMINANTS

Property 1: When any two rows or columns of a determinant are identical then the value of the determinant is zero.

For example, $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$, since $1st$ and $3rd$ rows are identical.

$\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix} = 0$, since $1st$ and $2nd$ columns are identical.

Property 2: If all the entries in any one of the rows or columns are zero then the value of the determinant is zero.

For example, $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0,$ since all the elements in the $3rd$ row are zero.

$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{vmatrix} = 0,$ since all the elements in the $2nd$ column are zero.

Property 3: The determinant of any square matrix and its transpose have the same value, i.e., there will be no effect in the value of the determinant if we change the rows into columns and columns into rows.

For example, $\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 9 \\ -2 & 5 & 8 \\ 4 & 7 & 5 \end{vmatrix}$ since rows and columns are interchanged.

Property 4: The value of a determinant alters its sign when any two adjacent rows or columns are interchanged.

For example, $\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} = (-1)\begin{vmatrix} 6 & 5 & 7 \\ 1 & -2 & 4 \\ 9 & 8 & 5 \end{vmatrix}$ since $1st$ and $2nd$ rows are interchanged.

$\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 4 & -2 \\ 6 & 7 & 5 \\ 9 & 5 & 8 \end{vmatrix}$ since $2nd$ and $3rd$ columns are interchanged.

Property 5: If any row or column is multiplied by any scalar then the value of the determinant is multiplied by the same scalar.

For example, $\begin{vmatrix} 1\times 2 & -2 & 4 \\ 6\times 2 & 5 & 7 \\ 9\times 2 & 8 & 5 \end{vmatrix} = 2\times\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix}$ since the all the elements of the $1st$ column are multiplied by 2.

$\begin{vmatrix} 1 & -2 & 4 \\ 6\times 3 & 5\times 3 & 7\times 3 \\ 9 & 8 & 5 \end{vmatrix} = 3\times\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix}$ since the all the elements of the $2nd$ row are multiplied by 3.

Property 6: When the elements of any row (or any column) of a determinant are expressed as a sum of two quantities, then the determinant is the sum of two different determinants containing the terms of the sum as a row (or column) respectively.

For example, $\begin{vmatrix} 1+2 & -2 & 4 \\ 6+3 & 5 & 7 \\ 9+4 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} + \begin{vmatrix} 2 & -2 & 4 \\ 3 & 5 & 7 \\ 4 & 8 & 5 \end{vmatrix}$ since the all the elements of the $1st$ column are expressed as a sum of two quantities.

$$\begin{vmatrix} 1 & -2 & 4 \\ 6+2 & 5+4 & 7+5 \\ 9 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} + \begin{vmatrix} 1 & -2 & 4 \\ 2 & 4 & 5 \\ 9 & 8 & 5 \end{vmatrix}$$ since the all the elements of the $2nd$ row are expressed as a sum of two quantities.

Property 7: If any row (or column) in the determinant is replaced by summation of two or more rows (or columns) then it does not effect the value of the determinant.

For example, $\begin{vmatrix} 2 & -2 & 4 \\ 3 & 5 & 7 \\ 4 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 2+3 & -2+5 & 4+7 \\ 3 & 5 & 7 \\ 4 & 8 & 5 \end{vmatrix}$, $1st$ row is replaced by the sum of $1st$ and $2nd$ rows. $[R_1' \to R_1 + R_2]$

$\begin{vmatrix} 1 & -2 & 4 \\ 6 & 5 & 7 \\ 9 & 8 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -2+4 & 4 \\ 6 & 5+7 & 7 \\ 9 & 8+5 & 5 \end{vmatrix}$, $2nd$ column is replaced by the sum of $2nd$ and $3rd$ columns. $[C_2' \to C_2 + C_3]$.

Property 8: If any row (or column) in the determinant is a scalar multiple of any other row (or column) then the value of the determinant is zero.

For example, $\begin{vmatrix} 2 & -2 & 4 \\ 3 & 5 & 6 \\ 4 & 8 & 8 \end{vmatrix} = 0$, since $3rd$ column is two multiples of the $1st$ column.

$\begin{vmatrix} 1 & -2 & 4 \\ 3 & -6 & 12 \\ 9 & 8 & 5 \end{vmatrix} = 0$, $2nd$ row is three multiples of the $1st$ row.

1.9 MINORS AND COFACTORS OF A DETERMINANT

Let $A = (a_{ij})_{n\times n}$ and its determinant is given by $\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$.

1.9.1 Minor

The minor of any entry a_{ij} of $\det A$ is defined to be that determinant obtained by deleting the corresponding row and column intersecting at the entry a_{ij}. The minor of a_{ij} is denoted by M_{ij} and is given by deleting i-th row and j-th column from $\det A$ as follows:

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

Observation If $\det A$ is of order n then minor of any element a_{ij} is of order $(n-1)$

1.9.2 Cofactor

The cofactor of any element $a_{ij} = (-1)^{i+j} \times$ (minor of any element a_{ij}). The cofactor of any element a_{ij} is denoted by A_{ij}.

So, the cofactor corresponding to a_{ij} is given by $A_{ij} = (-1)^{i+j} \times M_{ij}$.

Example 8

Let $\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Minor of 1 (a_{11} term) is obtained by deleting $1st$ row and $1st$ column from $\det A$ as $M_{11} = 4$.

Cofactor of 1 (a_{11} term) is given by $A_{11} = (-1)^{1+1} \times M_{11} = (-1)^{1+1} \times 4 = 4$.

Minor of 2 (a_{12} term) is obtained by deleting $1st$ row and $2nd$ column from $\det A$ as $M_{12} = 3$.

Cofactor of 2 (a_{12} term) is given by $A_{12} = (-1)^{1+2} \times M_{12} = (-1)^{1+2} \times 3 = -3$.

Minor of 3 (a_{21} term) is obtained by deleting $2nd$ row and $1st$ column from $\det A$ as $M_{21} = 2$.

Cofactor of 3 (a_{21} term) is given by $A_{21} = (-1)^{2+1} \times M_{21} = (-1)^{2+1} \times 2 = -2$.

Minor of 4 (a_{22} term) is obtained by deleting $2nd$ row and $2nd$ column from $\det A$ as $M_{22} = 1$.

Cofactor of 4 (a_{22} term) is given by $A_{22} = (-1)^{2+2} \times M_{22} = (-1)^{2+2} \times 1 = 1$.

Example 9

Let $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Minor of 1 (a_{11} term) is obtained by deleting $1st$ row and $1st$ column from $\det A$ as $M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$.

Cofactor of 1 (a_{11} term) is given by $A_{11} = (-1)^{1+1} \times M_{11} = (-1)^{1+1} \times \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$.

Minor of 2 (a_{12} term) is obtained by deleting $1st$ row and $2nd$ column from $\det A$ as $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$.

Cofactor of 2 (a_{12} term) is given by $A_{12} = (-1)^{1+2} \times M_{12} = (-1)^{1+2} \times \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$.

Minor of 4 (a_{21} term) is obtained by deleting $2nd$ row and $1st$ column from the $\det A$ as $M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$.

Cofactor of 4 (a_{21} term) is given by $A_{21} = (-1)^{2+1} \times M_{21} = (-1)^{2+1} \times \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$.

1.10 EXPANSION OF A DETERMINANT

1.10.1 Case 1

Let $A = (a_{ij})_{1\times 1} = (a_{11})$; then $\det A = |a_{11}| = a_{11}$.

For example, let $A = (2)$, then $\det A = |2| = 2$.

1.10.2 Case 2

Let $A = (a_{ij})_{2\times 2}$; then $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

(i) Row-wise Expansion:

Expanding about the $1st$ row, we get the value of the determinant as

$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{12} \times (\text{cofactor of } a_{12}) = a_{11} \times A_{11} + a_{12} \times A_{12}$

Similarly, expanding about the $2nd$ row we get the value of the determinant as

$\det A = a_{21} \times (\text{cofactor of } a_{21}) + a_{22} \times (\text{cofactor of } a_{22}) = a_{21} \times A_{21} + a_{22} \times A_{22}$.

(ii) Column-wise Expansion:

Expanding about the $1st$ column, we get the value of the determinant as

$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{21} \times (\text{cofactor of } a_{21}) = a_{11} \times A_{11} + a_{21} \times A_{21}$.

Similarly, expanding about the $2nd$ column, we get the value of the determinant as

$\det A = a_{12} \times (\text{cofactor of } a_{12}) + a_{22} \times (\text{cofactor of } a_{22}) = a_{12} \times A_{12} + a_{22} \times A_{22}$.

Note:

Any $2nd$ order determinant can be evaluated directly as $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} \times a_{22}) - (a_{12} \times a_{21})$.

Example 10 Let $\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Here, $a_{11} = 1$ and the corresponding cofactor $A_{11} = 4$.

$a_{12} = 2$ and the corresponding cofactor $A_{12} = -3$

$a_{21} = 3$ and the corresponding cofactor $A_{21} = -2$
$a_{22} = 4$ and the corresponding cofactor $A_{22} = 1$

(i) Row-wise expansion:

Expanding about the $1st$ row, we get

$$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{12} \times (\text{cofactor of } a_{12}) = a_{11} \times A_{11} + a_{12} \times A_{12}$$
$$= 1 \times 4 + 2 \times (-3) = -2.$$

Similarly, expanding about the $2nd$ row, we get

$$\det A = a_{21} \times (\text{cofactor of } a_{21}) + a_{22} \times (\text{cofactor of } a_{22}) = a_{21} \times A_{21} + a_{22} \times A_{22}$$
$$= 3 \times (-2) + 4 \times 1 = -2.$$

(ii) Column-wise expansion:

Expanding about the $1st$ column, we get

$$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{21} \times (\text{cofactor of } a_{21}) = a_{11} \times A_{11} + a_{21} \times A_{21}$$
$$= 1 \times 4 + 3 \times (-2) = -2.$$

Similarly, expanding about the $2nd$ column, we get

$$\det A = a_{12} \times (\text{cofactor of } a_{12}) + a_{22} \times (\text{cofactor of } a_{22}) = a_{12} \times A_{12} + a_{22} \times A_{22}$$
$$= 2 \times (-3) + 4 \times 1 = -2.$$

1.10.3 Case 3

Let $A = (a_{ij})_{3\times 3}$; then $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

(i) Row-wise Expansion:

Expanding about the $1st$ row, we get the value of the determinant as

$$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{12} \times (\text{cofactor of } a_{12}) + a_{13} \times (\text{cofactor of } a_{13})$$
$$= a_{11} \times A_{11} + a_{12} \times A_{12} + a_{13} \times A_{13}.$$

Similarly, expansions can be done about the $2nd$ row and $3rd$ row.

(ii) Column-wise Expansion:

Expanding about the $1st$ column, we get the value of the determinant as

$$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{21} \times (\text{cofactor of } a_{21}) + a_{31} \times (\text{cofactor of } a_{31})$$
$$= a_{11} \times A_{11} + a_{21} \times A_{21} + a_{31} \times A_{31}.$$

Similarly, expansion can be done about the $2nd$ column and $3rd$ column.

Example 11 Let us consider $\det A = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{vmatrix}$

(i) Row-wise expansion:

Expanding about the $1st$ row, we get

$$\det A = a_{11} \times (\text{cofactor of } a_{11}) + a_{12} \times (\text{cofactor of } a_{12}) + a_{13} \times (\text{cofactor of } a_{13})$$
$$= a_{11} \times A_{11} + a_{12} \times A_{12} + a_{13} \times A_{13}$$

$$= 1\times(-1)^{1+1}\begin{vmatrix}3 & 0\\4 & 5\end{vmatrix} + 0\times(-1)^{1+2}\begin{vmatrix}2 & 0\\0 & 5\end{vmatrix} + 2\times(-1)^{1+3}\begin{vmatrix}2 & 3\\0 & 4\end{vmatrix}$$

$$= 15 - 0 + 16 = 31.$$

Similarly, expanding about the $2nd$ row and $3rd$ row we can get $\det A = 31$.

(ii) Column-wise expansion:

Expanding about the $1st$ column, we get

$$\det A = a_{11}\times(\text{cofactor of } a_{11}) + a_{21}\times(\text{cofactor of } a_{21}) + a_{31}\times(\text{cofactor of } a_{31})$$

$$= a_{11}\times A_{11} + a_{21}\times A_{21} + a_{31}\times A_{31}$$

$$= 1\times(-1)^{1+1}\begin{vmatrix}3 & 0\\4 & 5\end{vmatrix} + 2\times(-1)^{2+1}\begin{vmatrix}0 & 2\\4 & 5\end{vmatrix} + 0\times(-1)^{3+1}\begin{vmatrix}0 & 2\\3 & 0\end{vmatrix}$$

$$= 15 + 16 + 0 = 31$$

Similarly, expanding about the $2nd$ column and $3rd$ column we can get $\det A = 31$.

1.10.4 Generalisation of the above Cases

$$A = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} \text{ then } \det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$$

can be evaluated by expanding the determinant about any one of the n rows or n columns.

So we can conclude the facts by stating the following theorems.

Theorem 1.4: The determinant of any square matrix can be evaluated by adding the products of the elements of any row or column and their corresponding cofactors.

Theorem 1.5: If we consider the products of the elements of any row (or column) and the cofactors corresponding to the other row (or column), then summation of such products are always zero.

Proof: Beyond of scope of the book.

Example 12 Let us consider $\det A = \begin{vmatrix}1 & 0 & 2\\2 & 3 & 0\\0 & 4 & 5\end{vmatrix}$.

Summation of the products of the elements of the $1st$ row and the cofactors corresponding to the $2nd$ row

$$= a_{11}\times(\text{cofactor of } a_{21}) + a_{12}\times(\text{cofactor of } a_{22}) + a_{13}\times(\text{cofactor of } a_{23})$$

$$= a_{11}\times A_{21} + a_{12}\times A_{22} + a_{13}\times A_{23}$$

$$= 1\times(-1)^{2+1}\begin{vmatrix}0 & 2\\4 & 5\end{vmatrix} + 0\times(-1)^{2+2}\begin{vmatrix}1 & 2\\0 & 5\end{vmatrix} + 2\times(-1)^{2+3}\begin{vmatrix}1 & 0\\0 & 4\end{vmatrix}$$

$$= 8 + 0 - 8 = 0.$$

1.11 LAPLACE METHOD OF EXPANSION OF A DETERMINANT

1.11.1 Complementary Minor

Consider $A = (a_{ij})_{n\times n}$. Then $D = \det A = \left|a_{ij}\right|_{n\times n}$ is a determinant of order n. If we delete r number of rows and r number of columns from D, the remaining determenant is of order $(n-r)$ and is said to be a minor of order $(n-r)$ of D.

When all the rows and columns of a minor M are deleted from the determinant D then the remaining determinant is called a complimentary minor of M.

Consider $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$.

Now if we delete the $2nd$, $3rd$ rows and $3rd, 4th$ column from D, we have the minor as

$$M_{23,\,34} = \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}.$$

Also, the complementary minor of $M_{23,\,34}$ is obtained by deleting all the rows and columns corresponding to $M_{23,\,34}$ in D.

So, complementary minor of $M_{23,\,34} = \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$, (obtained by deleting the $1st$, $4th$ rows and $1st$, $2nd$ columns from D).

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and $\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$ are also the examples of complementary minors.

$\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$ and $\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$ are also the examples of complementary minors.

1.11.2 Algebraic Complement of a Minor

Consider M to be a minor of order r obtained by $i_1, i_2, ..., i_r$ rows and $j_1, j_2, ..., j_r$ columns and $\bar{M}$ be its complementory minor.

Then the algebraic complement of M is $(-1)^{(i_1+i_2+\cdots+i_r)+(j_1+,j_2+\cdots+j_r)} \times \bar{M}$.

Consider $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$. Then

(i) Algebraic complement of $\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}$ is $(-1)^{(1+4)+(1+2)} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$

(ii) Algebraic complement of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is $(-1)^{(1+2)+(1+2)} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$

(iii) Algebraic complement of $\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$ is $(-1)^{(3+4)+(1+2)} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$

1.11.3 Laplace Expansion

The value of a determinant $D = |a_{ij}|_{n\times n}$ can be expressed as the sum of the products of all minors of order r and their respective algebraic complements.

Let us try to expand $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$ by the minors of order 2 selecting from the first two rows as

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \times (-1)^{(1+2)+(1+2)} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \times (-1)^{(1+2)+(1+3)} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \times (-1)^{(1+2)+(2+3)} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \times (-1)^{(1+2)+(2+4)} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \times (-1)^{(1+2)+(3+4)} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}.$$

Note: The above can be expanded considering minors of any order and forming from any set of rows. But it is always easy to expand like above.

Laplace Expansion can be done in terms of any sets of columns also.

Example 13 Let us calculate the determinant

$D = \begin{vmatrix} 2 & -3 & 1 & 5 \\ 6 & 2 & 7 & -2 \\ 1 & 9 & 8 & 3 \\ -5 & 4 & 2 & 7 \end{vmatrix}$ using Laplace expansion.

Expanding $D = \begin{vmatrix} 2 & -3 & 1 & 5 \\ 6 & 2 & 7 & -2 \\ 1 & 9 & 8 & 3 \\ -5 & 4 & 2 & 7 \end{vmatrix}$ as above, we have

$$D = \begin{vmatrix} 2 & -3 \\ 6 & 2 \end{vmatrix} \times (-1)^{(1+2)+(1+2)} \begin{vmatrix} 8 & 3 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix} \times (-1)^{(1+2)+(1+3)} \begin{vmatrix} 9 & 3 \\ 4 & 7 \end{vmatrix}$$
$$+ \begin{vmatrix} 2 & 5 \\ 6 & -2 \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} 9 & 8 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} -3 & 1 \\ 2 & 7 \end{vmatrix} \times (-1)^{(1+2)+(2+3)} \begin{vmatrix} 1 & 3 \\ -5 & 7 \end{vmatrix}$$
$$+ \begin{vmatrix} -3 & 5 \\ 2 & -2 \end{vmatrix} \times (-1)^{(1+2)+(2+4)} \begin{vmatrix} 1 & 8 \\ -5 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 7 & -2 \end{vmatrix} \times (-1)^{(1+2)+(3+4)} \begin{vmatrix} 1 & 9 \\ -5 & 4 \end{vmatrix}$$

$= 22\times 50 - 8\times 51 + 15\times 14 - 23\times 22 + 4\times 42 - 37\times 49$

$= 1249.$

1.12 PRODUCT OF DETERMINANTS

Let $D_1 = |a_{ij}|_{n\times n}$ and $D_2 = |b_{ij}|_{n\times n}$ be two determinants of order n. Then the product $D_1 \times D_2$ is a determinant of order n.

1.12.1 Four Processes of Multiplication

1) Row–Columnwise Multiplication:

$$D_1 \times D_2 = |a_{ij}|_{n\times n} \times |b_{ij}|_{n\times n}.$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} + ..a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + ..a_{1n}b_{n2} & \dots & a_{11}b_{1n} + a_{12}b_{2n} + ..a_{1n}b_{nn} \\ a_{21}b_{11} + a_{22}b_{21} + ..a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + ..a_{2n}b_{n2} & \dots & a_{21}b_{1n} + a_{22}b_{2n} + ..a_{2n}b_{nn} \\ \dots & \dots & \dots & \dots \\ a_{n1}b_{11} + a_{n2}b_{21} + ..a_{nn}b_{n1} & a_{n1}b_{12} + a_{n2}b_{22} + ..a_{nn}b_{n2} & \dots & a_{n1}b_{1n} + a_{n2}b_{2n} + ..a_{nn}b_{nn} \end{vmatrix}$$

2) Row-Row-wise Multiplication:

$$D_1 \times D_2 = |a_{ij}|_{n\times n} \times |b_{ij}|_{n\times n}.$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} + ..a_{1n}b_{1n} & a_{11}b_{21} + a_{12}b_{22} + ..a_{1n}b_{2n} & \dots & a_{11}b_{n1} + a_{12}b_{n2} + ..a_{1n}b_{nn} \\ a_{21}b_{11} + a_{22}b_{12} + ..a_{2n}b_{1n} & a_{21}b_{21} + a_{22}b_{22} + ..a_{2n}b_{2n} & \dots & a_{21}b_{n1} + a_{22}b_{n2} + ..a_{2n}b_{nn} \\ \dots & \dots & \dots & \dots \\ a_{n1}b_{11} + a_{n2}b_{12} + ..a_{nn}b_{1n} & a_{n1}b_{21} + a_{n2}b_{22} + ..a_{nn}b_{2n} & \dots & a_{n1}b_{n1} + a_{n2}b_{n2} + ..a_{nn}b_{nn} \end{vmatrix}.$$

3) Column-Row wise multiplication:

$$D_1 \times D_2 = |a_{ij}|_{n\times n} \times |b_{ij}|_{n\times n}.$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11} + a_{21}b_{12} + ..a_{n1}b_{1n} & a_{12}b_{11} + a_{22}b_{12} + ..a_{n2}b_{1n} & \dots & a_{1n}b_{11} + a_{2n}b_{12} + ..a_{nn}b_{1n} \\ a_{11}b_{21} + a_{21}b_{22} + ..a_{n1}b_{2n} & a_{12}b_{21} + a_{22}b_{22} + ..a_{n2}b_{2n} & \dots & a_{1n}b_{21} + a_{2n}b_{22} + ..a_{nn}b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{11}b_{n1} + a_{22}b_{n2} + ..a_{n1}b_{nn} & a_{12}b_{1n} + a_{22}b_{n} + ..a_{n2}b_{nn} & \dots & a_{1n}b_{n1} + a_{2n}b_{n2} + ..a_{nn}b_{nn} \end{vmatrix}.$$

4) Column-column wise multiplication:

$D_1 \times D_2 = |a_{ij}|_{n\times n} \times |b_{ij}|_{n\times n}$.

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11}+a_{21}b_{21}+..a_{n1}b_{n1} & a_{12}b_{11}+a_{22}b_{21}+..a_{n2}b_{n1} & \dots & a_{1n}b_{11}+a_{21}b_{n2}+..a_{nn}b_{n1} \\ a_{11}b_{12}+a_{21}b_{22}+..a_{n1}b_{n2} & a_{12}b_{12}+a_{22}b_{22}+..a_{n2}b_{n2} & \dots & a_{1n}b_{12}+a_{21}b_{22}+..a_{nn}b_{n2} \\ \dots & \dots & \dots & \dots \\ a_{11}b_{1n}+a_{21}b_{2n}+..a_{n1}b_{nn} & a_{12}b_{1n}+a_{22}b_{2n}+..a_{n2}b_{nn} & \dots & a_{1n}b_{1n}+a_{21}b_{2n}+..a_{nn}b_{nn} \end{vmatrix}$$

Example 14 Let us consider $D_1 = \begin{vmatrix} 1 & 0 & 2 \\ -5 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$ and $D_2 = \begin{vmatrix} 2 & 1 & 0 \\ 4 & -3 & 1 \\ 3 & 0 & 7 \end{vmatrix}$

Then $D_1 \times D_2 = \begin{vmatrix} 1 & 0 & 2 \\ -5 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 0 \\ 4 & -3 & 1 \\ 3 & 0 & 7 \end{vmatrix}$

(i) By row–columnwise multiplication, we have

$$D_1 \times D_2 = \begin{vmatrix} 1\times2+0\times4+2\times3 & 1\times1+0\times(-3)+2\times0 & 1\times0+0\times1+2\times7 \\ (-5)\times2+3\times4+0\times3 & (-5)\times1+3\times(-3)+0\times0 & (-5)\times0+3\times1+0\times7 \\ 0\times2+(-2)\times4+1\times3 & 0\times1+(-2)\times(-3)+1\times0 & 0\times0+(-2)\times1+1\times7 \end{vmatrix}$$

$$= \begin{vmatrix} 10 & 1 & 14 \\ 2 & -14 & 3 \\ -5 & 6 & 5 \end{vmatrix} = -1541.$$

(ii) By row-row-wise multiplication, we have

$$D_1 \times D_2 = \begin{vmatrix} 1\times2+0\times1+2\times0 & 1\times4+0\times(-3)+2\times1 & 1\times3+0\times0+2\times7 \\ (-5)\times2+3\times1+0\times0 & (-5)\times4+3\times(-3)+0\times1 & (-5)\times3+3\times0+0\times7 \\ 0\times2+(-2)\times1+1\times0 & 0\times4+(-2)\times(-3)+1\times1 & 0\times3+(-2)\times0+1\times7 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 6 & 17 \\ -7 & -29 & -15 \\ -2 & 7 & 7 \end{vmatrix} = -1541.$$

So we have the same result in each of the cases.

Similarly, if we apply the other two methods of multiplication, we will get the same result.

Theorem 1.6: Let A and B be two square matrices of order $n\times n$. Then

(i) $\det(cA) = c^n \det(A)$

(ii) $\det(AB) = \det(A)\cdot\det(B)$

1.13 ADJOINT OF A DETERMINANT

Let us consider $D = |a_{ij}|_{n\times n} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$ and A_{ij} be the cofactor of a_{ij} in D.

Then adjoint of D is defined as

$$\bar{D} = |A_{ij}|_{n\times n} = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix}.$$

For example, let $D = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{vmatrix}$. Then the adjoint of D is

$$\bar{D} = \begin{vmatrix} \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 15 & -10 & 8 \\ 8 & 5 & -4 \\ -6 & 4 & 3 \end{vmatrix}.$$

Statement 1.3.1 **Jacobi's Theorem on Adjoint of a Determinant**

Let $D \neq 0$ be a determinant of order n and $\bar{D}$ be its adjoint. Then $\bar{D} = D^{n-1}$.

Proof: Beyond the scope of this book.

Corollary: **For $n = 3$, the Jacobis theorem becomes $\bar{D} = D^2$.**

Proof: Here, $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and adjoint of D is $\bar{D} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$, where A_{ij} is the cofactor of a_{ij} in D.

Now $D \times \bar{D} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \times \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

$$= \begin{vmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}$$

$$= D^3.$$

So, $D \times \bar{D} = D^3$

$\Rightarrow \bar{D} = D^2$ (for $D \neq 0$).

1.14 SINGULAR AND NON-SINGULAR MATRICES

Any square matrix $A = (a_{ij})_{n\times n}$ is said to be nonsingular iff $\det(A) = |a_{ij}|_{n\times n} \neq 0$. Otherwise, it is singular.

For example,

(i) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix}$ is a nonsingular matrix since its determinant is nonzero

(ii) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a singular matrix since its determinant is zero

1.15 ADJOINT OF A MATRIX

Let us consider any square matrix

$$A = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and A_{ij} be the cofactor of a_{ij} in $\det A$. Then adjoint of the matrix A is denoted by $adj(A)$ and is defined as the transpose of the matrix $(A_{ij})_{n\times n}$.

So, $adj(A) = (A_{ij})^T_{n\times n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T$.

For example, let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix}$. Then the adjoint of the matrix A is

$$adj(A) = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 15 & -10 & 8 \\ 8 & 5 & -4 \\ -6 & 4 & 3 \end{pmatrix}^T = \begin{pmatrix} 15 & 8 & -6 \\ -10 & 5 & 4 \\ 8 & -4 & 3 \end{pmatrix}.$$

1.15.1 Properties

(1) For any square matrix A, $adj(A^T) = [adj(A)]^T$.

(2) For any square matrix A of order $n \times n$,
$adj(cA) = c^{n-1}[adj(A)]$, where c is any scalar.

Theorem 1.7: For any square matrix A of order $n \times n$, always $A \cdot [adj(A)] = [adj(A)] \cdot A = \det A \cdot I_n$

Proof: Beyond of scope of the book.

Example 15 From the previous example, we have

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} \text{ and } adj(A) = \begin{pmatrix} 15 & 8 & -6 \\ -10 & 5 & 4 \\ 8 & -4 & 3 \end{pmatrix}.$$

Also, it is easy to check that $\det A = 31$.

Now $A \cdot [adj(A)] = \begin{pmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{pmatrix} = 31 \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det A \cdot I_3$.

Again $[adj(A)] \cdot A = \begin{pmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{pmatrix} = 31 \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det A \cdot I_3$.

Hence, the above theorem is verified.

1.16 INVERSE OF A NONSINGULAR MATRIX

If for a nonsingular square matrix A of order n there exists a non-singular square matrix B of order n such that $AB = BA = I_n$, then B is said to be the inverse of A. If inverse exists for A then we say the matrix A is inverible and inverse of A is denoted by A^{-1}.

Now from the last theorem we have $A.[adj(A)] = [adj(A)].A = \det A.I_n$.

So, $A \cdot \dfrac{1}{\det A}[adj(A)] = \dfrac{1}{\det A}[adj(A)] \cdot A = I_n$

Therefore, from the definition we can say $\dfrac{1}{\det A}[adj(A)]$ is the inverse of the matrix A.

Hence $A^{-1} = \dfrac{1}{\det A}[adj(A)]$ and it satisfies $A \cdot A^{-1} = A^{-1} \cdot A = I_n$.

Note:
So it is obvious from the above that inverse exists for a matrix A iff $\det A \neq 0$, i.e., iff A is non-singular.

Example 16 Let us consider $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix}$.

From the previous example, we have $\det A = 31$ and $adj(A) = \begin{pmatrix} 15 & 8 & -6 \\ -10 & 5 & 4 \\ 8 & -4 & 3 \end{pmatrix}$.

Since $\det A = 31 \neq 0$, the given matrix is nonsingular and so A^{-1} exists.

Now $A^{-1} = \dfrac{1}{\det A}[adj(A)] = \dfrac{1}{31}\begin{pmatrix} 15 & 8 & -6 \\ -10 & 5 & 4 \\ 8 & -4 & 3 \end{pmatrix}$.

Theorem 1.8: Inverse of a nonsingular square matrix is always unique.

Proof: Beyond the scope of the syllabus.

1.16.1 Properties

(1) If A^{-1} exists for A then $\left(A^{-1}\right)^{-1} = A$.

(2) If A and B are two invertible matrices then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(3) If A is invertible then A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Let us verify the above properties:

Example 17 Let us consider two matrices $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Here, $\det A = 2 \neq 0$ and $\det B = 3 \neq 0$.

Also $adj(A) = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$

and $adj(B) = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$.

So $A^{-1} = \dfrac{1}{\det A}[adj(A)] = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$

and $B^{-1} = \dfrac{1}{\det A}[adj(B)] = \dfrac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$.

Verification of Property (1)

We have from above, $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ and $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$.

Now $\det(A^{-1}) = 1$ and $adj(A^{-1}) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$.

So $\left(A^{-1}\right)^{-1} = \dfrac{1}{\det(A^{-1})}[adj(A^{-1})] = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = A.$

Hence, the property is verified.

Verification of Property (2)

We have from above $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Also, $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$ and $B^{-1} = \dfrac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$.

Now $AB = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 12 \end{pmatrix}$

Since $\det(AB) = 6 \neq 0$, AB is invertible.

$adj(AB) = \begin{pmatrix} 12 & -3 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 12 & -2 \\ -3 & 1 \end{pmatrix}$.

So $(AB)^{-1} = \dfrac{1}{\det(AB)}[adj(AB)] = \dfrac{1}{6}\begin{pmatrix} 12 & -2 \\ -3 & 1 \end{pmatrix}$.

Again $B^{-1}A^{-1} = \dfrac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \times \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \dfrac{1}{6}\begin{pmatrix} 12 & -2 \\ -3 & 1 \end{pmatrix}$

Hence $(AB)^{-1} = B^{-1}A^{-1}$ which verifies property (2).

Verification of Property (3)

We have $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$, $adj(A) = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$, $\det A = 2$ and $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$.

So, $\left(A^{-1}\right)^T = \dfrac{1}{2}\begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$.

Now $A^T = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, $\det(A^T) = 2 \neq 0$, so A^T is also invertible

and $adj(A^T) = (adj(A))^T = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$

So $(A^T)^{-1} = \dfrac{1}{\det(A^T)}[adj(A^T)] = \dfrac{1}{2}\begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$

Hence, $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ and so property (3) is verified.

1.17 ORTHOGONAL MATRIX

Any square matrix $A = (a_{ij})_{n\times n}$ is called orthogonal if it satisfies $AA^T = I_n$.

Theorem 1.9: For an orthogonal matrix $A_{n\times n}$, always $A^T A = I_n$.

Proof: Beyond the scope of the book.

From the above, we can say for any orthogonal matrix of order $n\times n$, $AA^T = A^T A = I_n$.

For example, let us consider the matrix $A=\frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}$

So $A^T = \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$.

Now $AA^T = \frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}$

$$=\frac{1}{9}\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=I_{3.}$$

So $AA^T = I_3$. Therefore, A is an orthogonal matrix.

Again, $A^T A = \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}$

$$=\frac{1}{9}\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=I_{3.}$$

Therefore, $AA^T = A^T A = I_3$.

Theorem 1.10: Any orthogonal matrix A is nonsingular and its determinant is given by $\det(A)=\pm 1$. [WBUT 2003]

Proof: For any orthogonal matrix A, **we have** $AA^T = I_n$.

So, $\det(AA^T)=\det(I_n)=1$.

$\Rightarrow \det A\cdot\det A^T = 1$

$\Rightarrow (\det A)^2 = 1$

$\Rightarrow \det A = \pm 1 \neq 0.$

So the orthogonal matrix A is nonsingular and its determinant is given by $\det(A)=\pm 1$. Hence, the theorem is proved.

Example 18 We have from the previous example that the matrix

$A=\frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix}$ is orthogonal.

Here, $\det A = 1$.

1.17.1 Inverse of an Orthogonal Matrix

From the above, it is clear that for any orthogonal matrix A, A^{-1} exists since it is nonsingular.

Also, we have for any orthogonal matrix A of order $n \times n$, $AA^T = A^T A = I_n$.

So, by the definition of inverse, we can say $A^{-1} = A^T$.

Example 19 From the previous two examples, $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is orthogonal and $\det A = 1$.

So, A^{-1} exists and is given by $A^{-1} = A^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$.

Theorem 1.11: Transpose and inverse of an orthogonal matrix is again orthogonal.

Proof: Since for any orthogonal matrix A of order $n \times n$, $AA^T = A^T A = I_n$.

Then $AA^T = A^T A = I_n$

$\Rightarrow A^T A = AA^T = I_n$

$\Rightarrow A^T (A^T)^T = (A^T)^T A^T = I_n$.

Hence by the definition, A^T is orthogonal.

Since for any orthogonal matrix A, $A^{-1} = A^T$ holds, A^{-1} is also orthogonal.

Note: The product of two orthogonal matrices is again orthogonal.

1.18 TRACE OF A MATRIX

Let A be any square matrix of order $n \times n$, i.e., $A = (a_{ij})_{n\times n}$. Then trace of A, denoted by $tr\ A$, is the sum of the principal diagonal elements of A.

Consider $A = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$,

then $tr\ A = a_{11} + a_{22} + \cdots + a_{nn}$.

Properties:

(1) $tr\ A + tr\ B = tr\ (A+B)$.

(2) $tr\ A^T = tr\ A$

(3) $tr\,(AB) = tr\ (BA)$.

Proof: Beyond the scope of this book.

Example 20 Let $A = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$

then $tr\ A = 1 + 3 = 4$ and $tr\ B = 1 + 4 = 5$

So, $tr\,A + tr\,B = 4 + 5 = 9.$

Now, $A + B = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$

So, $tr\,(A + B) = 2 + 7 = 9$

Hence $tr\,A + tr\,B = tr\,(A + B),$ which verifies the **property (1)**.

Again $A^T = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ and $tr\,A^T = 1 + 3 = 4.$

So, $tr\,A^T = tr\,A,$ which verifies the **property (2)**.

Here $AB = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 24 \end{pmatrix}$

So, $tr\,(AB) = 1 + 24 = 25$

Again $BA = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 9 \\ 16 & 12 \end{pmatrix}$

So, $tr\,(BA) = 13 + 12 = 25.$

Hence $tr\,(AB) = tr\,(BA),$ which verifies the **property (3)**.

WORKED-OUT EXAMPLES

Example 1.1 Find if it is possible to form AB and BA, stating with reasons where the operations do not hold when,

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$$ [WBUT-2004]

Sol. The order of matrices A and B are respectively (2×3) and (3×2).

Since the number of columns of the matrix A and the number of rows of the matrix B are same, therefore AB is possible and is given by the following 2×2 matrix:

$$AB = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 8-6+1 & 12-5 \\ 6+21-1 & 9+5 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 26 & 14 \end{pmatrix}.$$

Again, since the number of columns of the matrix B and number of rows of the matrix A are same, therefore BA is also possible and is given by following 3×3 matrix

$$BA = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 8+9 & 4-21 & -2+3 \\ -12 & -6 & -3 \\ -4+15 & -2-35 & 1+5 \end{pmatrix}$$

$$=\begin{pmatrix} 17 & -17 & 1 \\ -12 & -6 & -3 \\ 11 & -37 & 6 \end{pmatrix}$$

Example 1.2 Prove that $P^t AP$ is a symmetric or a skew-symmetric matrix according to whether A is symmetric or skew-symmetric. [WBUT-2009]

Sol. Let A be a symmetric matrix, i.e., $A^t = A$ and $B = P^t AP$.

Now,

$$B^t = (P^t AP)^t = P^t A^t (P^t)^t = P^t AP = B$$

Therefore, $P^t AP$ is a symmetric matrix.

Again, let A be a skew-symmetric matrix, i.e , $A^t = -A$ and $B = P^t AP$.

Now,

$$B^t = (P^t AP)^t = P^t A^t (P^t)^t = P^t (-A)P = -P^t AP = -B$$

Therefore, $P^t AP$ is a skew-symmetric matrix.

Example 1.3 Find the matrices A and B such that

$$A+2B=\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \text{ and } 2A-B=\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

Sol. Here,

$$A+2B=\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \quad ...(1)$$

and

$$2A-B=\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \quad ...(2)$$

Multiplying **(2)** by 2 and adding to **(1)**, we have

$$(A+2B)+2(2A-B)=\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}+2\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$or, 5A=\begin{pmatrix} 3 & 7 \\ 7 & 0 \end{pmatrix}$$

$$or, A=\begin{pmatrix} \frac{3}{5} & \frac{7}{5} \\ \frac{7}{5} & 0 \end{pmatrix}$$

From **(2)** we have,

$$B = 2\begin{pmatrix} \frac{3}{5} & \frac{7}{5} \\ \frac{7}{5} & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{-6}{5} & 1 \end{pmatrix}$$

Example 1.4 Find the matrices A and B such that

$A + 3B = 2I_3$ and $3A - B = 4A^T$

Sol. Here, $A + 3B = 2I_3$ ...(1)

and

$3A - B = 4A^T$ ...(2)

Multiplying **(2)** by 3 and adding to **(1)**, we have

$(A + 3B) + 3(3A - B) = 2I_3 + 12A^T$

or, $10A = 2I_3 + 12A^T$

or, $5A - I_3 = 6A^T$ ...(3)

Therefore, transposing both sides,

$6\left(A^T\right)^T = \left(5A - I_3\right)^T$

or, $6A = 5A^T - I_3$

or, $6A + I_3 = 5A^T$ ...(4)

Multiplying **(3)** by 5 and **(1)** by 6 and subtracting, we have

$5\left(5A - I_3\right) - 6\left(6A + I_3\right) = 30A^T - 30A^T$

or, $-11A - 11I_3 = 0$

So, $A = -I_3$ ...(5)

From (1) and (5), we obtain

$-I_3 + 3B = 2I_3$

or, $B = I_3$.

Hence,

$$A = -I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } B = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 1.5 If $A = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$ then show that A is an idempotent matrix.

[WBUT-2003]

Sol. A matrix A is said to be an idempotent matrix if $A^2 = A$.

Here,

$$A^2 = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-5 & -1-3+5 & 1+3-5 \\ -3-9+15 & 3+9-15 & -3-9+15 \\ 5-15+25 & 5+15-25 & -5-15+25 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix} = A$$

Therefore, A is an idempotent matrix.

Example 1.6 Show that the matrix $\begin{pmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{pmatrix}$ is a nilpotent matrix.

[WBUT-2005]

Sol. Let, $A = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{pmatrix}$

Now,

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-4 & -1-3+4 & 1+3-4 \\ -3-9+12 & 3+9-12 & -3-9+12 \\ -4-12+16 & 4+12-16 & -4-12+16 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Therefore, A is a nilpotent matrix.

Example 1.7 If A is an idempotent matrix then show that $B = I - A$ is also idempotent. Hence, show that $AB = BA = O$.

Sol. Since A is an idempotent matrix, we have $A^2 = A$.

Here, we are to show $B^2 = B$.

Now $B^2 = (I-A)^2 = (I-A)\cdot(I-A)$

$= I - I\cdot A - A\cdot I + A\cdot A = I - A - A + A^2$

$= I - 2A + A \,(\text{since} A^2 = A)$

$= I - A = B$

So, $B = I - A$ is also idempotent.

Also,

$AB = A.(I-A) = A\cdot I - A\cdot A = A - A^2 = A - A = O$

and

$BA = (I-A)\cdot A = I\cdot A - A\cdot A = A - A^2 = A - A = O.$

Example 1.8 Show that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$ [WBUT-2002]

Sol. Here,

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

$$= abcd\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$ [Dividing first, second, third, fourth rows by a, b, c, d respectively]

$$= abcd\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

$[R_1' \to R_1 + R_2 + R_3 + R_4]$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix} \quad [C_2' \to C_2 - C_1,\ C_3' \to C_3 - C_1,\ C_4' \to C_4 - C_1]$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{[Expanding the determinant about its first row]}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Example 1.9 Evaluate $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$ by Laplace expansion method.

[WBUT-2003, 2007]

Sol. Expanding the determinant by Laplace method in terms of minors of second order, we have,

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

$$= (-1)^{1+2+1+2}\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}\begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} + (-1)^{1+2+1+3}\begin{vmatrix} 0 & b \\ -a & d \end{vmatrix}\begin{vmatrix} -d & f \\ -e & 0 \end{vmatrix}$$

$$+(-1)^{1+2+1+4}\begin{vmatrix} 0 & c \\ -a & e \end{vmatrix}\begin{vmatrix} -d & 0 \\ -e & -f \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} a & b \\ 0 & d \end{vmatrix}\begin{vmatrix} -b & f \\ -c & 0 \end{vmatrix}$$

$$+(-1)^{1+2+2+4}\begin{vmatrix} a & c \\ 0 & e \end{vmatrix}\begin{vmatrix} -b & 0 \\ -c & -f \end{vmatrix} + (-1)^{1+2+3+4}\begin{vmatrix} b & c \\ d & e \end{vmatrix}\begin{vmatrix} -b & -d \\ -c & -e \end{vmatrix}$$

$$= a^2f^2 - abef + adcf + dacf - abef + (be-cd)^2$$

$$= a^2f^2 - 2abef + 2adcf + (be-cd)^2$$

$$= (af - be + cd)^2$$

Example 1.10 If A_i, B_i and C_i be the cofactors of a_i, b_i, c_i $(i = 1, 2, 3)$ in

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then show that } \begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} = 2\Delta^2$$

[WBUT-2003]

Sol. By Jacobi's theorem for $3rd$ order determinant, we have

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Now,

$$\begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & C_1+A_1 & A_1+B_1 \\ B_2 & C_2+A_2 & A_2+B_2 \\ B_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} + \begin{vmatrix} C_1 & C_1+A_1 & A_1+B_1 \\ C_2 & C_2+A_2 & A_2+B_2 \\ C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & C_1+A_1 & A_1 \\ B_2 & C_2+A_2 & A_2 \\ B_3 & C_3+A_3 & A_3 \end{vmatrix} [C_3' \to C_3 - C_1] + \begin{vmatrix} C_1 & A_1 & A_1+B_1 \\ C_2 & A_2 & A_2+B_2 \\ C_3 & A_3 & A_3+B_3 \end{vmatrix} [C_2' \to C_2 - C_1]$$

$$= \begin{vmatrix} B_1 & C_1 & A_1 \\ B_2 & C_2 & A_2 \\ B_3 & C_3 & A_3 \end{vmatrix} [C_2' \to C_2 - C_3] + \begin{vmatrix} C_1 & A_1 & B_1 \\ C_2 & A_2 & B_2 \\ C_3 & A_3 & B_3 \end{vmatrix} [C_3' \to C_3 - C_2]$$

$$= (-1)\begin{vmatrix} B_1 & A_1 & C_1 \\ B_2 & A_2 & C_2 \\ B_3 & A_3 & C_3 \end{vmatrix} + (-1)\begin{vmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{vmatrix}$$

[Interchanging $2nd$ and $3rd$ column in the first and interchanging $1st$ and $2nd$ column in the second determinant]

$$= (-1)^2\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} + (-1)^2\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

[Interchanging $1st$ and $2nd$ column in the first and interchanging $2nd$ and $3rd$ column in the second determinant]

$$= 2\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 2\Delta^2.$$

Example 1.11

(i) Define symmetric and skew-symmetric determinants.

(ii) Show that every skew-symmetric determinant of odd order is zero.

[WBUT-2004]

Sol. **(i)** If A be a symmetric matrix then $\det A$ is called a **symmetric determinant**.

If A be a skew-symmetric matrix then $\det A$ is called a **skew-symmetric determinant**.

(ii) Let us consider a skew-symmetric matrix $A = (a_{ij})_{n\times n}$ of odd order n.

Then the skew-symmetric determinant of odd order n is given by

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} -a_{11} & -a_{21} & \dots & -a_{n1} \\ -a_{12} & -a_{22} & \dots & -a_{n2} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & -a_{nn} \end{vmatrix} \text{ since } a_{ij} = -a_{ji}$$

$$= (-1)^n \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix} \text{ [Taking common (–1) from each row]}$$

i.e., $\det A = (-1)^n \cdot \det\left(A^T\right)$

Since n is odd, we have

$\det A = -\det A$

i.e., $2\det A = 0$

i.e., $\det A = 0$

Hence any skew-symmetric determinant of odd order is zero.

Example 1.12 Prove that

$$\begin{vmatrix} a+1 & a & a & a \\ a & a+2 & a & a \\ a & a & a+3 & a \\ a & a & a & a+4 \end{vmatrix} = 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)$$ [WBUT-2004]

Sol. Here, $\begin{vmatrix} a+1 & a & a & a \\ a & a+2 & a & a \\ a & a & a+3 & a \\ a & a & a & a+4 \end{vmatrix}$

$$= (1.2.3.4)\begin{vmatrix} 1+\frac{a}{1} & \frac{a}{1} & \frac{a}{1} & \frac{a}{1} \\ \frac{a}{2} & 1+\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{3} & \frac{a}{3} & 1+\frac{a}{3} & \frac{a}{3} \\ \frac{a}{4} & \frac{a}{4} & \frac{a}{4} & 1+\frac{a}{4} \end{vmatrix}$$ [Dividing *first*, *second*, *third*, *fourth* rows by 1, 2, 3, 4 respectively]

$$= 24\begin{vmatrix} 1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & 1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & 1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & 1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} \\ \frac{a}{2} & 1+\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{3} & \frac{a}{3} & 1+\frac{a}{3} & \frac{a}{3} \\ \frac{a}{4} & \frac{a}{4} & \frac{a}{4} & 1+\frac{a}{4} \end{vmatrix}$$

$[R_1' \to R_1 + R_2 + R_3 + R_4]$

$$= 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{a}{2} & 1+\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{3} & \frac{a}{3} & 1+\frac{a}{3} & \frac{a}{3} \\ \frac{a}{4} & \frac{a}{4} & \frac{a}{4} & 1+\frac{a}{4} \end{vmatrix}$$

$$= 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{2} & 1 & 0 & 0 \\ \frac{a}{3} & 0 & 1 & 0 \\ \frac{a}{4} & 0 & 0 & 1 \end{vmatrix} \quad [C_2' \to C_2 - C_1,\ C_3' \to C_3 - C_1,\ C_4' \to C_4 - C_1]$$

$$= 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{[Expanding the determinant about its first row]}$$

$$= 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right).$$

Example 1.13 Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

[WBUT-2004, 2008, 2009]

Sol. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix} C_2' \to C_2 - C_1, C_3' \to C_3 - C_1$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a+b+c)(a-b-c) \\ b^2 & (c+a+b)(c+a-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} (b+c)^2 & (a-b-c) & (a-b-c) \\ b^2 & (c+a-b) & 0 \\ c^2 & 0 & (a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} 2bc & -2c & -2b \\ b^2 & (c+a-b) & 0 \\ c^2 & 0 & (a+b-c) \end{vmatrix} R_1' \to R_1 - (R_2 + R_3)$$

$$= \frac{(a+b+c)^2}{bc}\begin{vmatrix} 2bc & -2cb & -2bc \\ b^2 & (c+a-b)b & 0 \\ c^2 & 0 & (a+b-c)c \end{vmatrix} C_2' \to bC_2, C_3' \to cC_3$$

$$= \frac{(a+b+c)^2}{bc} 2bc \begin{vmatrix} 1 & -1 & -1 \\ b^2 & (c+a-b)b & 0 \\ c^2 & 0 & (a+b-c)c \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} 1 & 0 & 0 \\ b^2 & (cb+ab) & b^2 \\ c^2 & c^2 & (ac+bc) \end{vmatrix} C_2' \to C_2 + C_1, C_3' \to C_3 + C_1$$

$$= 2(a+b+c)^2 \begin{vmatrix} (cb+ab) & b^2 \\ c^2 & (ac+bc) \end{vmatrix}$$ [Expanding the determinant about its first row].

$$= 2(a+b+c)^2 (abc^2 + b^2c^2 + a^2bc + ab^2c - b^2c^2)$$

$$= 2abc(a+b+c)^2 (a+b+c)$$

$$= 2abc(a+b+c)^3$$

Example 1.14 Prove without expanding $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = 0$ [WBUT-2005].

Sol. Here $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-ac-a^2+bc \\ 0 & c-a & c^2-ab-a^2+bc \end{vmatrix} R_2' \to R_2 - R_1, R_3' \to R_3 - R_1$$

$$= \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & (b-a)(a+b+c) \\ 0 & c-a & (c-a)(a+b+c) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2-bc \\ 0 & 1 & (a+b+c) \\ 0 & 1 & (a+b+c) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & (a+b+c) \\ 1 & (a+b+c) \end{vmatrix} \text{ [Expanding the determinant about first column]}$$

$$= (b-a)(c-a)(a+b+c)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= (b-a)(c-a)(a+b+c)\times 0$$

$$= 0$$

Example 1.15 Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$

[WBUT-2006]

Sol. Let $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Then expanding we have $D = -(a^3+b^3+c^3-3abc)$

Now **adjoint** of D is given by

$$\overline{D} = \begin{vmatrix} \begin{vmatrix} c & a \\ a & b \end{vmatrix} & -\begin{vmatrix} b & a \\ c & b \end{vmatrix} & \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ -\begin{vmatrix} b & c \\ a & b \end{vmatrix} & \begin{vmatrix} a & c \\ c & b \end{vmatrix} & -\begin{vmatrix} a & b \\ c & a \end{vmatrix} \\ \begin{vmatrix} b & c \\ c & a \end{vmatrix} & -\begin{vmatrix} a & c \\ b & a \end{vmatrix} & \begin{vmatrix} a & b \\ b & c \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

Now from **Jacobi's theorem** for $3rd$ order determinant, we have $\overline{D} = D^2$

i.e., $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \left[-(a^3+b^3+c^3-3abc)\right]^2$

$= (a^3+b^3+c^3-3abc)^2.$

Hence the result is proved.

Example 1.16 Prove that $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

[WBUT-2006]

Sol. Here $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

$$= \begin{vmatrix} b^2+c^2 & b^2 & c^2 \\ a^2 & c^2+a^2 & c^2 \\ a^2 & b^2 & a^2+b^2 \end{vmatrix} \text{[Transposing the determinant]}$$

$$= \begin{vmatrix} c^2 & b^2 & c^2 \\ -c^2 & c^2+a^2 & c^2 \\ a^2-b^2 & b^2 & a^2+b^2 \end{vmatrix} C_1' \to C_1 - C_2$$

$$= \begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2+a^2 & c^2 \\ -2b^2 & b^2 & a^2+b^2 \end{vmatrix} C_1' \to C_1 - C_3$$

$$= -2\begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & c^2+a^2 & c^2 \\ b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & a^2 & c^2 \\ b^2 & 0 & a^2+b^2 \end{vmatrix} C_2' = C_2 - C_1$$

$$= -2\left[-b^2\begin{vmatrix} c^2 & c^2 \\ b^2 & a^2+b^2 \end{vmatrix} + c^2\begin{vmatrix} c^2 & a^2 \\ b^2 & 0 \end{vmatrix}\right] \text{[Expanding the determinant about its first row.]}$$

$$= -2\left[-b^2\{c^2(a^2+b^2) - b^2c^2\} - c^2a^2b^2\right]$$

$$= -2\{-c^2a^2b^2 - c^2a^2b^2\}$$

$$= 4a^2b^2c^2$$

Example 1.17 Prove that

$$\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a).$$

[WBUT-2008]

Sol.

$$\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (x-a)^2-(z-a)^2 & (x-b)^2-(z-b)^2 & (x-c)^2-(z-c)^2 \\ (y-a)^2-(z-a)^2 & (y-b)^2-(z-b)^2 & (y-c)^2-(z-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} \quad \begin{matrix} R_1' \to R_1 - R_3, \\ R_2' \to R_2 - R_3 \end{matrix}$$

$$= \begin{vmatrix} (x+z-2a)(x-z) & (x+z-2b)(x-z) & (x+z-2c)(x-z) \\ (y+z-2a)(y-z) & (y+z-2b)(y-z) & (y+z-2c)(y-z) \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

$$= (x-z)(y-z)\begin{vmatrix} (x+z-2a) & (x+z-2b) & (x+z-2c) \\ (y+z-2a) & (y+z-2b) & (y+z-2c) \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

$$= (x-z)(y-z)\begin{vmatrix} (x-y) & (x-y) & (x-y) \\ (y+z-2a) & (y+z-2b) & (y+z-2c) \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} \quad R_1' \to R_1 - R_2$$

$$= (x-z)(y-z)(x-y)\begin{vmatrix} 1 & 1 & 1 \\ (y+z-2a) & (y+z-2b) & (y+z-2c) \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$$

$$= (x-z)(y-z)(x-y)\begin{vmatrix} 1 & 0 & 0 \\ (y+z-2a) & 2(a-b) & 2(a-c) \\ (z-a)^2 & (z-b)^2-(z-a)^2 & (z-c)^2-(z-a)^2 \end{vmatrix}$$

$C_2' \to C_2 - C_1,\ C_3' \to C_3 - C_1$

$$= (x-z)(y-z)(x-y)\begin{vmatrix} 2(a-b) & 2(a-c) \\ (z-b)^2-(z-a)^2 & (z-c)^2-(z-a)^2 \end{vmatrix}$$

[Expanding the determinant about its first row.]

$$= (x-z)(y-z)(x-y)\begin{vmatrix} 2(a-b) & 2(a-c) \\ (a-b)(2z-a-b) & (a-c)(2z-a-c) \end{vmatrix}$$

$$= 2(x-z)(y-z)(x-y)(a-b)(a-c)\begin{vmatrix} 1 & 1 \\ (2z-a-b) & (2z-a-c) \end{vmatrix}$$

$$= 2(x-z)(y-z)(x-y)(a-b)(a-c)[(2z-a-c)-(2z-a-b)]$$

$$= 2(x-z)(y-z)(x-y)(a-b)(a-c)(b-c)$$

$$= 2(z-x)(y-z)(x-y)(a-b)(b-c)(c-a)$$

Example 1.18 Solve the equation $\begin{vmatrix} x+p & q & r \\ q & x+r & p \\ r & p & x+q \end{vmatrix} = 0$

Sol. $$\begin{vmatrix} x+p & q & r \\ q & x+r & p \\ r & p & x+q \end{vmatrix} = 0$$

$$or, \begin{vmatrix} x+p+q+r & x+p+q+r & x+p+q+r \\ q & x+r & p \\ r & p & x+q \end{vmatrix} = 0\,[R_1' \to R_1 + R_2 + R_3]$$

$$or, (x+p+q+r)\begin{vmatrix} 1 & 1 & 1 \\ q & x+r & p \\ r & p & x+q \end{vmatrix} = 0$$

$$or, (x+p+q+r)\begin{vmatrix} 1 & 0 & 0 \\ q & x+r-q & p-q \\ r & p-r & x+q-r \end{vmatrix} = 0[C_2' \to C_2 - C_{1,}\ C_3' \to C_3 - C_1]$$

$$or, (x+p+q+r)\begin{vmatrix} x+r-q & p-q \\ p-r & x+q-r \end{vmatrix} = 0$$

$$or, (x+p+q+r)(x^2 - p^2 - q^2 - r^2 + pq + qr + rp) = 0$$

Therefore,

$$x = -(p+q+r) \text{ or } x = \pm\sqrt{p^2 + q^2 + r^2 - pq - qr - rp}$$

Example 1.19 Prove that

$$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} = \left[\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}\right]^2$$

Sol. Let us consider $D = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$

Then $D^2 = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$

$$or,\ D^2 = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} \text{[Multiplying row–wise]} \qquad ...(1)$$

Again adjoint of D is given by

$$\overline{D} = \begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix}$$

Since by **Jacobi's theorem** for $3rd$ order determinant,

$\overline{D} = D^2$

we obtain

$$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \left[\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}\right]^2 \qquad ...(2)$$

Combining (1) and (2), we have

$$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} = \left[\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}\right]^2$$

Example 1.20 Prove that $\begin{vmatrix} 1 & bcd & b+c+d & a^2 \\ 1 & cda & c+d+a & b^2 \\ 1 & dab & d+a+b & c^2 \\ 1 & abc & a+b+c & d^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

Sol. Here, $\begin{vmatrix} 1 & bcd & b+c+d & a^2 \\ 1 & cda & c+d+a & b^2 \\ 1 & dab & d+a+b & c^2 \\ 1 & abc & a+b+c & d^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & bcd & a+b+c+d-a & a^2 \\ 1 & cda & b+c+d+a-b & b^2 \\ 1 & dab & c+d+a+b-c & c^2 \\ 1 & abc & d+a+b+c-d & d^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bcd & a+b+c+d & a^2 \\ 1 & cda & b+c+d+a & b^2 \\ 1 & dab & c+d+a+b & c^2 \\ 1 & abc & d+a+b+c & d^2 \end{vmatrix} - \begin{vmatrix} 1 & bcd & a & a^2 \\ 1 & cda & b & b^2 \\ 1 & dab & c & c^2 \\ 1 & abc & d & d^2 \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 1 & bcd & 1 & a^2 \\ 1 & cda & 1 & b^2 \\ 1 & dab & 1 & c^2 \\ 1 & abc & 1 & d^2 \end{vmatrix} - \frac{1}{abcd}\begin{vmatrix} a & abcd & a^2 & a^3 \\ b & bcda & b^2 & b^3 \\ c & cdab & c^2 & c^3 \\ d & dabc & d^2 & d^3 \end{vmatrix}$$

[Multiplying first, second, third, fourth rows by a, b, c, d respectively]

$$= 0 - \frac{abcd}{abcd}\begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \text{[Interchanging 1}st\text{ and 2}nd\text{ columns]}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

Example 1.21 Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Sol.

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} [C_1' \to C_1 - bC_3, C_2' \to C_2 + aC_3]$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix} [R_3' \to R_3 - bR_1]$$

$$= (1+a^2+b^2)^2[1-a^2+b^2+2a^2]$$

$$= (1+a^2+b^2)^3$$

Example 1.22 Show that $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}^2 = \begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$

Sol. $$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}^2 = \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} \text{[By row–wise multiplication]}$$

$$= \begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$$

Example 1.23 Show that $\begin{vmatrix} 1 & n & n^2 & n^3 \\ 1 & (n+1) & (n+1)^2 & (n+1)^3 \\ 1 & (n+2) & (n+2)^2 & (n+2)^3 \\ 1 & (n+3) & (n+2)^2 & (n+3)^3 \end{vmatrix} = 12$

Sol. $$\begin{vmatrix} 1 & n & n^2 & n^3 \\ 1 & (n+1) & (n+1)^2 & (n+1)^3 \\ 1 & (n+2) & (n+2)^2 & (n+2)^3 \\ 1 & (n+3) & (n+2)^2 & (n+3)^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & (n+1)-n & (n+1)^2-n(n+1) & (n+1)^3-n(n+1)^2 \\ 1 & (n+2)-n & (n+2)^2-n(n+2) & (n+2)^3-n(n+2)^2 \\ 1 & (n+3)-n & (n+2)^2-n(n+3) & (n+3)^3-n(n+3)^2 \end{vmatrix}$$

$[C_2' \to C_2 - nC_1,\ C_3' \to C_3 - nC_2,\ C_4' \to C_4 - nC_3]$

$$= \begin{vmatrix} 1 & n+1 & (n+1)^2 \\ 2 & 2(n+2) & 2(n+2)^2 \\ 3 & 3(n+3) & 3(n+3)^2 \end{vmatrix}$$

$$= 2\cdot 3\cdot \begin{vmatrix} 1 & (n+1) & (n+1)^2 \\ 1 & (n+2) & (n+2)^2 \\ 1 & (n+3) & (n+3)^2 \end{vmatrix} \quad [R_2' \to R_2 - R_1;\ R_3' \to R_3 - R_1]$$

$$= 2\cdot 3 \begin{vmatrix} 1 & (n+1) & (n+1)^2 \\ 0 & 1 & 2n+3 \\ 0 & 2 & 4n+8 \end{vmatrix}$$

$$= 2\cdot 3\cdot[4n+8-2(2n+3)] = 12$$

Example 1.24 Without expanding, prove that $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$ [WBUT-2007]

Sol. Here, $D = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

$$= -\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} \text{[Transposing]}$$

$= -D$

$or,\ 2D = 0$

$or,\ D = 0.$

Example 1.25 If the matrix $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ \lambda & -3 & 0 \end{pmatrix}$ is singular then find the value of λ.

[WBUT-2004]

Sol. Since the matrix is singular, we have, $\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ \lambda & -3 & 0 \end{vmatrix} = 0$

$or,\ -1(-3\lambda) - 2(6) = 0$

$or,\ 3\lambda = 12$

$or,\ \lambda = 4.$

Example 1.26 Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$ if it exists.

Sol. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$.

Then $\det A = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = 20 \neq 0.$

Since $\det A \neq 0$, A^{-1} exists.

Now the adjoint of the matrix A is given by

$$adj(A) = \begin{pmatrix} \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{pmatrix}.$$

Hence $A^{-1} = \dfrac{1}{\det A} adj(A)$

$$A^{-1} = \frac{1}{20} \cdot \begin{pmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & \frac{-7}{20} & \frac{-2}{5} \\ \frac{-9}{10} & \frac{3}{10} & \frac{1}{5} \end{pmatrix}$$

Example 1.27 For the matrix $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$, prove that

$A^3 - 6A^2 + 12A - 10I = O.$

Hence, find A^{-1}.

Sol. Here $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$.

Therefore,

$$A^2 = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -10 & 21 \\ 2 & -4 & 8 \\ -1 & -4 & 15 \end{pmatrix}$$

and

$$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & -10 & 21 \\ 2 & -4 & 8 \\ -1 & -4 & 15 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} -8 & -24 & 78 \\ 0 & -14 & 36 \\ -6 & -12 & 52 \end{pmatrix}$$

Now

$$A^3 - 6A^2 + 12A - 10I = \begin{pmatrix} -8 & -24 & 78 \\ 0 & -14 & 36 \\ -6 & -12 & 52 \end{pmatrix} - 6\begin{pmatrix} 1 & -10 & 21 \\ 2 & -4 & 8 \\ -1 & -4 & 15 \end{pmatrix}$$

$$+12\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Hence, we have

$$A^3 - 6A^2 + 12A - 10I = O.$$

Now we are to find A^{-1}.

$$A^3 - 6A^2 + 12A - 10I = O$$

$$or,\ A(A^2 - 6A + 12I) = 10I$$

$$or,\ A.\left[\frac{1}{10}(A^2 - 6A + 12I)\right] = I$$

Hence by the definition of inverse, we have

$$A^{-1} = \left[\frac{1}{10}(A^2 - 6A + 12I)\right]$$

$$A^{-1} = \frac{1}{10}\left[\begin{pmatrix} 1 & -10 & 21 \\ 2 & -4 & 8 \\ -1 & -4 & 15 \end{pmatrix} - 6\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix} + 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\text{i.e., } A^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

Example 1.28 Determine the values of a, b, c for which the matrix $\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ is orthogonal.

Sol. Let

$$A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$$

If A is an orthogonal matrix then $AA^T = I$
which implies

$$\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}\begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$or, \begin{pmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equating the corresponding entries of $1st$ row, we have

$$4b^2+c^2 = 1 \qquad ...(1)$$

$$2b^2-c^2 = 0 \qquad ...(2)$$

$$a^2+b^2+c^2 = 1 \qquad ...(3)$$

Adding (1) and (2), we get

$$6b^2 = 1 \Rightarrow b^2 = \frac{1}{6} \Rightarrow b = \pm\frac{1}{\sqrt{6}}$$

Putting $b = \pm\frac{1}{\sqrt{6}}$ in (2), we get

$$c^2 = 2b^2 = \frac{1}{3} \Rightarrow c = \pm\frac{1}{\sqrt{3}}$$

Putting the value of b and c in (3), we obtain

$$a^2 = 1-b^2-c^2 = 1-\frac{1}{6}-\frac{1}{3} = \frac{1}{2} \Rightarrow a = \pm\frac{1}{\sqrt{2}}$$

Therefore,

$$a = \pm\frac{1}{\sqrt{2}}; b = \pm\frac{1}{\sqrt{6}}; c = \pm\frac{1}{\sqrt{3}}$$

Example 1.29 Show that the matrix $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ is orthogonal and hence find its inverse.

Sol. Let $A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Now,

$$AA^T = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^T$$

$$= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since $AA^T = I$, A is an orthogonal matrix.
Again for any othogonal matrix A, we know $A^{-1} = A^T$. Hence,

$$A^{-1} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Example 1.30 If A be a skew-symmetric and $(I+A)$ be a nonsingular matrix then show that $B = (I-A)(I+A)^{-1}$ is orthogonal.

[WBUT-2009]

Sol. A square matrix is orthogonal if

$$A \cdot A^T = I_n$$

Now,

$$B \cdot B^T = \{(I-A)(I+A)^{-1}\}\{(I-A)(I+A)^{-1}\}^T$$

$$= (I-A)(I+A)^{-1}\{(I+A)^{-1}\}^T(I-A)^T, \text{ since} (AB)^T = B^T A^T$$

$$= (I-A)(I+A)^{-1}\{(I+A)^T\}^{-1}(I^T - A^t), \text{ since } (A^T)^{-1} = (A^{-1})^T$$

$$= (I-A)(I+A)^{-1}(I^T + A^T)^{-1}(I^T - A^T)$$

$$= (I-A)(I+A)^{-1}(I + A^T)^{-1}(I - A^T), \text{ since } I^T = I$$

$$= (I - A)(I + A)^{-1}(I - A)^{-1}(I + A), \text{ since } A^T = -A$$

$$= (I - A)\{(I - A)(I + A)\}^{-1}(I + A), \text{ since } (AB)^{-1} = B^{-1}\cdot A^{-1}$$

Again,

$$(I - A)(I + A) = I + A - A - A^2$$
$$= (I + A)I - (I + A)A = (I + A)(I - A)$$

Hence, from above

$$B.B^T = (I - A)\{(I + A)(I - A)\}^{-1}(I + A)$$
$$= (I - A)(I - A)^{-1}(I + A)^{-1}(I + A)$$
$$= I \cdot I = I$$

Therefore, $B = (I - A)(I + A)^{-1}$ is orthogonal.

EXERCISES

Short and Long Answer Type Questions

1. Find the matrices A and B if

$$A + 2B = \begin{pmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{pmatrix} \text{ and } 2A - B = \begin{pmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\left[\textbf{Ans : } A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{pmatrix}\right]$$

2. For the matrices $A = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, verify the following:

   a) $(A + B)^T = A^T + B^T$

   b) $(AB)^T = B^T \cdot A^T$

3. Express the following matrices as the sum of symmetric and skew-symmetric matrices.

   a) $\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

$$\left[\textbf{Ans : }\text{Symmertric matrix: } \begin{pmatrix} 2 & \frac{7}{2} \\ \frac{7}{2} & 2 \end{pmatrix}, \text{ Skew-symmertric matrix: } \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{-1}{2} & 0 \end{pmatrix}\right]$$

b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$\left[\textbf{Ans:} \text{ Symmertric matrix: } \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}, \text{ Skew-symmertric matrix: } \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\right]$

c) $\begin{pmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{pmatrix}$

$\left[\textbf{Ans:} \text{ Symmertric matrix: } \begin{pmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{pmatrix}, \text{ Skew-symmertric matrix: } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$

4. If A is a skew-symmetric matrix then show that A^2 is symmetric. Also, verify this with the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$$

5. If $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, prove that

$A(\theta)A(\varphi) = A(\varphi)A(\theta) = A(\theta+\varphi)$ [WBUT-2006]

6. If $A = \begin{pmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{pmatrix}$ then show that $A^2 = I$, i.e., A is involutory.

7. If $A = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & 5 & 5 \end{pmatrix}$ then show that $A^2 = A$, i.e., A is idempotent.

8. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$ then prove that $A^2 - 23A - 40I = O.$

9. Show that

$$\begin{pmatrix} u & v & w \end{pmatrix} \times \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = au^2 + bv^2 + cw^2 + 2huv + 2fvw + 2gwu$$

10. Prove without expanding the following determinants:

a) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$

b) $\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}.$

c) $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix} = 0.$

d) $\begin{vmatrix} 1+p & 1 & 1 \\ 1 & 1+q & 1 \\ 1 & 1 & 1+r \end{vmatrix} = pqr\left(1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right).$

e) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (b-c)(c-a)(a-b)(bc+ca+ab).$

11. Prove that $(\lambda + p + q + r)$ is a factor of $\begin{vmatrix} \lambda + p & q & r \\ q & \lambda + r & p \\ r & p & \lambda + q \end{vmatrix}.$

12. Solve the following for x:

(i) $\begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+1 & 3 \\ 3 & -6 & x+1 \end{vmatrix} = 0$ $\left[\textbf{Ans:}\ x = -1, -1 \pm i\sqrt{7}\right]$

(ii) $\begin{vmatrix} x^3 - a^3 & x^2 & x \\ b^3 - a^3 & b^2 & b \\ c^3 - a^3 & c^2 & c \end{vmatrix} = 0.$ $\left[\textbf{Ans:}\ x = b, c, \frac{a^3}{bc}\right]$

13. Prove that $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ if $(a+b+c) = 0.$

14. If $A + B + C = \pi$, then prove that $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0$

15. Show that $(x+3)$ is a factor of the determinant $\begin{vmatrix} x+3 & 4 & 5 \\ 5 & x+3 & 5 \\ 5 & -4 & x+3 \end{vmatrix}$

16. Show that the matrix $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is an orthogonal matrix.

17. For $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$, show that $(I - A)(I + A)^{-1}$ is an orthogonal matrix.

18. Verify (i) $(AB)^{-1} = B^{-1} \cdot A^{-1}$ (ii) $(A^T)^{-1} = \left(A^{-1}\right)^T$ for the matrices

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 0 & 4 \\ -2 & 0 & 0 \end{pmatrix} \text{and } B = \begin{pmatrix} 4 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & 4 & 0 \end{pmatrix}$$

19. If $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 4 \\ -2 & 0 & 8 \end{pmatrix}$, find the matrix $B_{3\times3}$ for which $AB = I_3$.

20. Examine whether the following matrices A and B are conformable for addition and multiplication. If so, find $A + B$, AB, BA.

(i) $A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 6 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 2 & 5 \end{pmatrix}$

$$\left[\textbf{Ans : } AB = \begin{pmatrix} 7 & 11 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 3 & 12 & 6 \\ 0 & 0 & 0 \\ 2 & 34 & 4 \end{pmatrix}\right]$$

(ii) $A = \begin{pmatrix} a & 0 & b \\ b & c & 0 \\ 0 & c & a \end{pmatrix}$, $B = \begin{pmatrix} -a & b \\ 0 & 0 \\ -c & a \end{pmatrix}$

$$\left[\textbf{Ans : } AB = \begin{pmatrix} -a^2 - bc & 2ab \\ -ab & b^2 \\ -ac & a^2 \end{pmatrix}\right]$$

21. If $A + I = \begin{pmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{pmatrix}$ then evaluate $(A + I)(A - I)$ where I is the identity matrix of order 3.

$$\left[\textbf{Ans : } \begin{pmatrix} -12 & -12 & 9 \\ -6 & -13 & -4 \\ 3 & -6 & -18 \end{pmatrix}\right]$$

22. Find the matrices A and B if

$$2A+3B=\begin{pmatrix}8 & 3\\ 7 & 6\end{pmatrix},\ A+B^T=\begin{pmatrix}3 & 1\\ 3 & 3\end{pmatrix}$$

$$\left[\textbf{Ans : } A=\begin{pmatrix}1 & 0\\ 2 & 3\end{pmatrix} \text{ and } B=\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}\right]$$

23. If $A+B=2B^T$ and $3A+2B=I_3$, find the matrices A and B.

$$\left[\textbf{Ans : } A=\begin{pmatrix}\frac{1}{5} & 0 & 0\\ 0 & \frac{1}{5} & 0\\ 0 & 0 & \frac{1}{5}\end{pmatrix}, B=\begin{pmatrix}\frac{1}{5} & 0 & 0\\ 0 & \frac{1}{5} & 0\\ 0 & 0 & \frac{1}{5}\end{pmatrix}\right]$$

24. Find the matrices A and B such that

$$3A-B^T=2I_3 \text{ and } 2A+B=\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

$$\left[\textbf{Ans : } A=\begin{pmatrix}\frac{3}{5} & \frac{-2}{5} & \frac{1}{5}\\ \frac{3}{5} & \frac{3}{5} & 0\\ \frac{1}{5} & 0 & \frac{3}{5}\end{pmatrix} \text{ and } B=\begin{pmatrix}\frac{-1}{5} & \frac{9}{5} & \frac{3}{5}\\ \frac{-6}{5} & \frac{-1}{5} & 0\\ \frac{3}{5} & 0 & \frac{-1}{5}\end{pmatrix}\right]$$

25. If $A=\begin{pmatrix}0 & 1 & 2\\ 2 & 0 & 1\\ 1 & 2 & 0\end{pmatrix}$ show that $A^3-6A-9I_3=O$. Hence obtain a matrix B such that $BA=I_3$

26. Find all real matrices $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ such that $A^2=O$

$$\left[\textbf{Ans : }\begin{pmatrix}a & b\\ c & -a\end{pmatrix}, \text{ where } a^2+bc=0.\right]$$

27. Express the following matrices as the sum of symmetric and skew-symmetric matrices.

i) $\begin{pmatrix}1 & a & 1\\ b & 1 & 1\\ 1 & 1 & c\end{pmatrix}$ ii) $\begin{pmatrix}\alpha & \alpha & \beta\\ \beta & \beta & \gamma\\ \gamma & \gamma & \alpha\end{pmatrix}$

28. Show that $\begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$ is idempotent.

29. Show that $\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ is nilpotent of index 3.

30. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, prove by the method of induction that

$$A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$$

where n is any positive integer.

31. Prove without expanding the following:

a) $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$

b) $\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$

c) $\begin{vmatrix} p^3 & p^2 & p & 1 \\ (p+1)^3 & (p+1)^3 & (p+1)^3 & 1 \\ (p+2)^3 & (p+1)^3 & (p+1)^3 & 1 \\ (p+3)^3 & (p+1)^3 & (p+1)^3 & 1 \end{vmatrix} = 12$

d) $\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & a^2+c^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix} = (ab+bc+ca)^2$

e) $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$

f) $\begin{vmatrix} y+z & x-z & x-y \\ y-z & z+x & y-x \\ z-y & z-x & x+y \end{vmatrix} = 8xyz$

g) $\begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix} = (y-z)(z-x)(x-y)$

h) $\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = (a-b)(b-c)(c-a)(abc+1)$

i) $\begin{vmatrix} a+b & 0 & b \\ c & b+c & 0 \\ 0 & a & c+a \end{vmatrix} = (a+b+c)(ab+bc+ca)$

j) $\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$

k) $\begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

l) $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$

32. Using product of determinants, show that

a) $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$

b) $\begin{vmatrix} a^2+2bc & c^2+2ab & b^2+2ca \\ b^2+2ca & a^2+2bc & c^2+2ab \\ c^2+2ab & b^2+2ca & a^2+2bc \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$

c) $\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix} = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

[WBUT-2008]

33. Using Laplace method of expansion, prove the following:

a) $\begin{vmatrix} x & y & -u & -v \\ y & x & v & u \\ u & v & x & y \\ -v & -u & y & x \end{vmatrix} = (x^2+v^2-y^2-u^2)^2$

b) $\begin{vmatrix} a & b & c & d \\ -a & b & c & d \\ -a & -b & c & d \\ -a & -b & -c & d \end{vmatrix} = 8abcd$

c) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix} = 0$

d) $\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2)$

e) $\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix} = 1 - ax - by - cz$

34. Prove that the determinant $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$ is a perfect square.

35. Verify

a) $adj(A^T) = (adjA)^T$ and

b) $A \cdot adj(A) = adj(A) \cdot A = \det A \cdot I$

for the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 4 & 6 \\ 1 & 2 & 1 \end{pmatrix}$.

36. Find the adjoint of the following matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}$ $\left[\textbf{Ans:} \begin{pmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{pmatrix} \right]$

b) $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ $\left[\textbf{Ans:} \begin{pmatrix} 2 & 4 & -6 \\ 0 & -4 & 0 \\ -4 & -4 & 4 \end{pmatrix} \right]$

37. Find the matrix A such that $\det A = 2$ and $adj\,A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

$\left[\textbf{Ans:}\ A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \right]$

38. Find the inverse of the following matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ $\left[\textbf{Ans:} \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix}\right]$

b) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$ $\left[\textbf{Ans:} \frac{1}{11}\begin{pmatrix} 7 & 2 & -3 \\ -2 & 1 & 4 \\ 1 & 5 & -2 \end{pmatrix}\right]$

39. A and B are real orthogonal matrices of the same order and $\det A + \det B = 0$. Show that $A+B$ is a singular matrix.

40. Determine the value of a, b, c so that $\begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ is orthogonal.

$\left[\textbf{Ans:}\ a = \pm\frac{1}{\sqrt{2}},\quad b = \pm\frac{1}{\sqrt{6}},\quad c = \pm\frac{1}{\sqrt{3}}\right]$

Multiple-Choice Questions

1. The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{pmatrix}$ is a

a) symmetric matrix b) skew-symmetric matrix
c) diagonal matrix d) none of these

2. If A is a non-null square matrix then $A + A^T$ is a
a) symmetric matrix b) skew-symmetric matrix
c) null matrix d) none of these

3. If A is a non-null square matrix then $A - A^T$ is a
a) symmetric matrix b) skew-symmetric matrix
c) null matrix d) none of these

4. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $(A^2)^T =$

a) $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ b) I_2

c) $2A$ d) none of these

5. $(2A+3B)^T$ is equal to

a) $2A + 3B^T$ b) $2A^T + 3B^T$

c) $4A^T + 9B^T$ d) none of these

6. $(AB)^T$ is equal to

a) $A^T + B^T$ b) $A^T B^T$

c) $B^T A^T$ d) none of these

7. If $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $A \cdot A^t =$

a) I_2 b) A

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ d) none of these

8. If $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ then $A^2 + 7I =$

a) O b) $2A$ c) $3A$ d) $5A$

9. If $\begin{pmatrix} 2 & k \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$ then the value of k is

a) -5 b) 0 c) 5 d) -1

10. If A is a symmetric as well as skew-symmetric then A is a/an

a) diagonal matrix b) null matrix

c) Identity matrix d) none of these

11. If A is an idempotent matrix then $I - A$ is a/an

a) nilpotent matrix b) idempotent matrix

c) involutory matrix d) none of these

12. If A and B are two square matrices of same order such that $(A+B)^2 = A^2 + B^2 + 2AB$ then

a) $A = B^T$ b) $A^2 = B$

c) $AB = BA$ d) none of these

13. The cofactor of x in the determinant $\begin{vmatrix} x & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 2 \end{vmatrix}$ is

a) -2 b) 4 c) 2 d) 0

14. The adjoint of the determinant $\begin{vmatrix} 2 & 1 \\ 3 & 6 \end{vmatrix}$ is

a) $\begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$ b) $\begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix}$ c) $\begin{vmatrix} -6 & 3 \\ 1 & -2 \end{vmatrix}$ d) $\begin{vmatrix} 6 & -3 \\ -1 & 2 \end{vmatrix}$

15. The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$ is

a) 1 b) -1 c) 2 d) 0

16. If ω is the cube root of unity then the value of the determinant $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$ is

a) ω^2 b) ω c) $1+\omega$ d) 0

17. The value of a skew-symmetric determinant of odd order is always

a) 0 b) 1 c) -1 d) none of these

18. The roots of the equation $\begin{vmatrix} x+1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & x-3 \end{vmatrix} = 0$ are

a) 1, 2, 3 b) $-1, 2, 3$ c) $1, -2, 3$ d) $-1, -2, 3$

19. $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix} = 0$ if

a) a, b, c are in AP b) a, b, c are in GP

c) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP d) none of these

20. If α, β are the roots of the equation $x^2 - 3x + 2 = 0$ then $\begin{vmatrix} 0 & \alpha & \beta \\ \beta & 0 & \alpha \\ 1 & -\alpha & \alpha \end{vmatrix} =$

a) 6 b) $\frac{3}{2}$ c) -6 d) 3 [WBUT-2007]

21. If $\begin{vmatrix} 1 & 2 & 3 \\ 4 & a & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$ then the value of a is

a) 5 b) either -2 or 1 c) 1 d) not -2

22. If $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = kabc$, then $k =$

a) 3 b) 1 c) 4 d) 2

23. The value of $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix}$ is

a) 2 b) 0 c) 405 d) -1 [WBUT-2005]

24. If $\det(A_{3\times 3}) = 4$ then $\det(2A_{3\times 3})$ is equal to

a) 32 b) 16 c) 8 d) 4

25. $adj(2A_{3\times3})$ is equal to

a) $32.adj(A)$ b) $8.adj(A)$ c) $4.adj(A)$ d) $2.adj(A)$

26. $adj(A^T)$ is equal to

a) $3.adj(A)$ b) $adj\left(A^{-1}\right)$ c) $adj(A)$ d) $[adj(A)]^T$

27. For a $3rd$ order determinant D, its adjoint determinant is equal to

a) D^2 b) D c) D^3 d) D^4

28. The trace of $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

a) 7 b) 5 c) 6 d) 4.

29. The trace of A^T is same as

a) trace of A b) trace of A^{-1}

c) [trace of $A]^T$ d) none of these

30. For an orthogonal matrix A, A^{-1} is same as

a) A b) A^T c) $adj\,A$ d) none of these

31. For any nonsingular matrix A, $\left(A^T\right)^{-1}$ is same as

a) $\left(A^{-1}\right)^T$ b) A^T c) A d) none of these

32. For any orthogonal matrix A, $\det A$ is equal to

a) 0 b) 1 c) ± 1 d) -1

Answers:

1. (a)	2. (a)	3. (b)	4. (c)	5. (b)	6. (c)	7. (a)	8. (d)	9. (d)
10. (b)	11. (b)	12. (c)	13. (c)	14. (d)	15. (d)	16. (d)	17. (a)	18. (b)
19. (b)	20. (c)	21. (a)	22. (c)	23. (b)	24. (a)	25. (c)	26. (d)	27. (a)
28. (c)	29. (a)	30. (b)	31. (a)	32. (c)				

CHAPTER

2

Matrix II

2.1 INTRODUCTION

In this chapter, first we deal with the concept of the rank of a matrix and also the process of determination of rank. Next, we discuss the matrix inversion method, Cramer's rule and also the consistency and inconsistency of a system of homogeneous and nonhomogeneous linear simultaneous equations.

Then we represent the methods of determination for Eigen values and Eigen vectors of a square matrix and also the Cayley–Hamilton theorem and its applications.

In the last part, we discuss the diagonalisation of a square matrix which is included as further-reading, material for interested students.

2.2 RANK OF A MATRIX

Let A be a nonzero matrix of order $m\times n$. The rank of A is defined to be r if r is the greatest positive integer such that A has at least one nonzero minor of order r.

Important Observations

(i) The rank of a null matrix is zero.

(ii) Rank of n-th order identity matrix is n.

(iii) If the rank of A be r, every minor of order greater than r is zero.

(iv) For a nonzero $m\times n$ matrix A, $0 <$ rank $A < \min\{m, n\}$.

(v) For an n-th order square matrix A, the rank of A is n if $\det(A) \neq 0$ and rank of A is less than n if $\det(A) = 0$.

(vi) Rank of A = Rank of A^T.

Example 1

Let $A=\begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}$. Here, $\det(A)\neq 0$. Since highest-order nonzero minor is of order 2, rank of $A=2$.

Example 2

Let $A=\begin{pmatrix}0 & 2 & 3 & 4\\ 0 & 5 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$. Here, the highest-order minors are $3rd$ order minors, and they are

$$\begin{pmatrix}0 & 2 & 3\\ 0 & 5 & 1\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 2 & 4\\ 0 & 5 & 1\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 4\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}2 & 3 & 4\\ 5 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}.$$

They are all singular matrices.

So, rank of $A<3$. So we have to search for at least one $2nd$ order nonzero minor if it exists.

Now, we have $\begin{pmatrix}2 & 3\\ 5 & 1\end{pmatrix}$ as a 2nd order nonzero minor since $\begin{vmatrix}2 & 3\\ 5 & 1\end{vmatrix}\neq 0.$

Hence, rank of $A=2$

2.3 ELEMENTARY ROW AND COLUMN OPERATIONS

Let A be a nonzero matrix of order $m\times n$. Elementary row (or column) operations on A are of the following three kinds:

(i) Interchanging of any two rows (or columns) of A.

[**Notation:** R_{ij} (or C_{ij}) stands for interchanging of the ith row and jth row (or of the i th column and jth column).]

(ii) Multiplication of a row (or column) by a nonzero quantity.

[**Notation:** $d\cdot R_i$ (or $d\cdot C_i$) stands for multiplication of the ith row by *d* (or of the i th column by *d*).]

(iii) Addition of scalar multiple of a row (or column) to another row (or column).

[**Notation:** $R_i+d\cdot R_j$ (or $C_i+d\cdot C_j$) stands for addition of the *d* multiple of the jth row to the ith row (or d multiple of the jth column to the ith column).]

Example 3

$$\begin{pmatrix}2 & -2 & 4 & 3\\ 3 & 2 & 5 & 0\\ 1 & 1 & 3 & -4\end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix}3 & 2 & 5 & 0\\ 2 & -2 & 4 & 3\\ 1 & 1 & 3 & -4\end{pmatrix},$$

$$\begin{pmatrix} 2 & -2 & 4 & 3 \\ 3 & 2 & 5 & 0 \\ 1 & 1 & 3 & -4 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 2 & 4 & -2 & 3 \\ 3 & 5 & 2 & 0 \\ 1 & 3 & 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 3 & 2 & 5 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{2R_2} \begin{pmatrix} 2 & -2 & 4 \\ 6 & 4 & 10 \\ 1 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 3 & 2 & 5 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{3C_1} \begin{pmatrix} 6 & -2 & 4 \\ 9 & 2 & 5 \\ 3 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 & 3 \\ 2 & 5 & 0 \\ 1 & 3 & -4 \end{pmatrix} \xrightarrow{R_2+2R_3} \begin{pmatrix} -2 & 4 & 3 \\ 4 & 11 & -8 \\ 1 & 3 & -4 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 4 & 3 \\ 2 & 5 & 0 \\ 1 & 3 & -4 \end{pmatrix} \xrightarrow{C_1+2C_3} \begin{pmatrix} 4 & 4 & 3 \\ 2 & 5 & 0 \\ -7 & 3 & -4 \end{pmatrix}$$

2.4 ROW EQUIVALENT AND COLUMN EQUIVALENT MATRICES

Suppose a matrix $B_{m\times n}$ is obtained by performing a finite number of elementary row (or column) operations on another matrix $A_{m\times n}$. Then $A_{m\times n}$ and $B_{m\times n}$ are said to be row equivalent (or column equivalent).

Example 4

$$A = \begin{pmatrix} 2 & -2 & 4 & 3 \\ 3 & 2 & 5 & 0 \\ 1 & 1 & 3 & -4 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 3 & 2 & 5 & 0 \\ 2 & -2 & 4 & 3 \\ 1 & 1 & 3 & -4 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_3} \begin{pmatrix} 5 & 4 & 11 & -8 \\ 2 & -2 & 4 & 3 \\ 1 & 1 & 3 & -4 \end{pmatrix} \xrightarrow{R_2+3R_3} \begin{pmatrix} 3 & 2 & 5 & 0 \\ 5 & 1 & 13 & 9 \\ 1 & 1 & 3 & -4 \end{pmatrix} = B.$$

A and B are row equivalent.

Example 5

$$A = \begin{pmatrix} 2 & -2 & 4 & 3 \\ 3 & 2 & 5 & 0 \\ 1 & 1 & 3 & -4 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 2 & 4 & -2 & 3 \\ 3 & 5 & 2 & 0 \\ 1 & 3 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{C_2+2C_3} \begin{pmatrix} 2 & 0 & -2 & 3 \\ 3 & 9 & 2 & 0 \\ 1 & 5 & 1 & -4 \end{pmatrix} \xrightarrow{C_1+(-1)C_2} \begin{pmatrix} -2 & 4 & -2 & 3 \\ -2 & 5 & 2 & 0 \\ -2 & 3 & 1 & -4 \end{pmatrix} = B.$$

A and B are column equivalent.

2.5 ROW-REDUCED ECHELON MATRIX

A matix A is called a row-reduced echelon matrix (or row-echelon matrix) if

(i) all nonzero rows precede all zero rows of A

(ii) in a nonzero row, the first nonzero element is 1 (called the leading 1)

(iii) all the columns which contain the leading 1 of some row have all other elements zero

(iv) for each nonzero row, if the leading element of row *i* occurs in column p_i then $p_1 < p_2 < p_3 < \ldots$

Example 6

The following matrices are row-reduced echelon matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.1: A matrix can be made row equivalent to a row-reduced echelon matrix by elementary row operations.

Proof: Beyond the scope of this book.

Theorem 2.2: Two row equivalent matrices have the same rank.

Proof: Beyond the scope of this book.

Theorem 2.3: If a row-reduced echelon matrix *A* has *r* nonzero rows then rank $A = r$.

Proof: Beyond the scope of this book.

2.6 DETERMINATION OF RANK OF MATRIX BY ELIMENTARY OPERATIONS

Steps:

Step 1 Apply elementary row operations on the matrix.

Step 2 Convert the matrix to a row-reduced echelon matrix.

Step 3 Count the number of nonzero rows.

Step 4 The value obtained in Step 3 is the rank.

Example 7 Let us find the rank of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$. [WBUT 2006]

We apply elementary row operations on A to reduce it to a row-echelon matrix.

$$A=\begin{pmatrix}1&1&1\\1&-1&-1\\3&1&1\end{pmatrix}\xrightarrow{R_2-R_1,\ R_3-3R_1}\begin{pmatrix}1&1&1\\0&-2&-2\\0&-2&-2\end{pmatrix}$$

$$\xrightarrow{R_3-R_2,\ R_1+\frac{1}{2}R_2}\begin{pmatrix}1&0&0\\0&-2&-2\\0&0&0\end{pmatrix}\xrightarrow{\left(-\frac{1}{2}\right)R_2}\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}=B$$

The row-reduced echelon matrix B has the two nonzero rows.
So, Rank $B=2$ and hence Rank $A=2$.

Example 8 Find the rank of $A=\begin{pmatrix}1&2&1&0\\3&6&12&9\\0&0&5&8\\1&2&2&1\end{pmatrix}$.

We apply elementary row operations on A to reduce it to a row-echelon matrix

$$A=\begin{pmatrix}1&2&1&0\\3&6&12&9\\0&0&5&8\\1&2&2&1\end{pmatrix}\xrightarrow{R_2-3R_1,\ R_4-R_1}\begin{pmatrix}1&2&1&0\\0&0&9&9\\0&0&5&8\\0&0&1&1\end{pmatrix}$$

$$\xrightarrow{\frac{1}{9}R_2}\begin{pmatrix}1&2&1&0\\0&0&1&1\\0&0&5&8\\0&0&1&1\end{pmatrix}\xrightarrow{R_4-R_2,\ R_3-5R_1,\ R_1-R_2}\begin{pmatrix}1&2&0&-1\\0&0&1&1\\0&0&0&3\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2}\begin{pmatrix}1&2&0&-1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}\xrightarrow{R_2-R_3,\ R_1-R_3}\begin{pmatrix}1&2&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}=B.$$

The row-reduced echelon matrix B has 3 non-zero rows.
So, Rank $B=3$ and hence Rank $A=3$.

2.7 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY MATRIX INVERSION METHOD

Let us consider the system of n linear equations involving n unknowns:

$$a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2$$

....

$$a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, called the coefficient matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$.

Then the above system of equations can be written in the form $AX = B$.
Now if $\det(A) \neq 0$, then the system $AX = B$ has the unique solution $X = A^{-1}B$.

Example 9 Let us solve the following system of equations:

$$\begin{aligned} 2x - 3y + 4z &= -4 \\ x + z &= 0 \\ -y + 4z &= 2 \end{aligned}$$

[WBUT 2005]

Here, the coefficient matrix $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$

Then the given system of equations can be written as $AX = B$.

Now $\det(A) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{vmatrix} = 10 \neq 0.$

So the system has a unique solution $X = A^{-1}B$.

Here, $\operatorname{adj}(A) = \begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$.

Now $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$, or, $A^{-1} = \dfrac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$.

Hence $X = A^{-1}B$

or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$

$$= \frac{1}{10}\begin{pmatrix} -10 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

So the solutions are $x = -1,\ y = 2,\ z = 1$.

2.8 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY CRAMER'S RULE

Let us consider the system of n linear equations involving n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

....

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, called the coefficient matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$.

Then the above system of equations can be written in the form $AX = B$.

Now if $\det(A) \neq 0$, then there exists a unique solution of the system $AX = B$ and is given by

$$x_1 = \frac{\det A_1}{\det A} = \frac{1}{\det A}\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{1}{\det A}\begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix}$$

....

$$x_n = \frac{\det A_n}{\det A} = \frac{1}{\det A}\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix}$$

Observations:

(1) When $\det(A) \neq 0$ then the system $AX = B$ has a unique solution.

(2) When $\det(A) = 0$ and at least one of $\det A_1, \det A_2, \ldots, \det A_n$ is nonzero, then the above system has no solution.

(3) When $\det(A) = 0$ and $\det A_1 = \det A_2 = \cdots = \det A_n = 0$ then the above system has an infinite number of solutions.

Example 10 Let us solve the following system by Cramer's rule:

$$x+2y+3z=6$$
$$2x+4y+z=7$$
$$3x+2y+9z=14.$$

If we write the above equations in the form $AX=B$ then

the coefficient matrix $A=\begin{pmatrix}1&2&3\\2&4&1\\3&2&9\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $B=\begin{pmatrix}6\\7\\14\end{pmatrix}$

Here, $\det A=\begin{vmatrix}1&2&3\\2&4&1\\3&2&9\end{vmatrix}=-20\neq 0.$

So the system has a unique solution.

Now $\det A_1=\begin{vmatrix}6&2&3\\7&4&1\\14&2&9\end{vmatrix}=-20.$

$$\det A_2=\begin{vmatrix}1&6&3\\2&7&1\\3&14&9\end{vmatrix}=-20$$

$$\det A_3=\begin{vmatrix}1&2&6\\2&4&7\\3&2&14\end{vmatrix}=-20$$

So, $x=\dfrac{\det A_1}{\det A}=\dfrac{-20}{-20}=1.$

$$y=\frac{\det A_2}{\det A}=\frac{-20}{-20}=1.$$

$$z=\frac{\det A_2}{\det A}=\frac{-20}{-20}=1.$$

Hence the solution is $(1,1,1)$, which is unique.

2.9 SYSTEM OF HOMOGENEOUS AND NONHOMOGENEOUS LINEAR EQUATIONS

Let us consider the following system of m linear equations with n unknowns

$$a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2$$
....
$$a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m$$

The above system is called a homogeneous system of linear equations if all the $b_i^{'s}$ are zero and the system is called a nonhomogeneous system of linear equations if at least one b_i is nonzero.

2.9.1. Matrix Representation of a System of Homogeneous and Nonhomogeneous Linear Equations

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, called the co-efficient matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$.

Then the above system of equations can be written in the form $AX = B$.

Here, $\overline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$ is called the augmented matrix.

The system $AX = B$ is homogeneous for $B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$, otherwise the system is nonhomogeneous.

So, the homogeneous system can be written in the form $AX = O$, where O is the null matrix or zero matrix.

Example 11 The following is a homogeneous system of 3 linear equations with 4 unknowns:

$$\begin{aligned} 2x - 3y + 4z &= 0 \\ x + z + w &= 0 \\ -y + 4z - 2w &= 0 \end{aligned}$$

Here, the coefficient matrix $A = \begin{pmatrix} 2 & -3 & 4 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 4 & -2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

and augmented matrix $\overline{A} = \begin{pmatrix} 2 & -3 & 4 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 \end{pmatrix}$.

Example 12 The following is a nonhomogeneous system of 3 linear equations with 3 unknowns:

$$5x-4y+4z=2$$
$$x+z=0$$
$$3x-2y+4z=0$$

Here the coefficient matrix $A=\begin{pmatrix}5 & -4 & 4\\ 1 & 0 & 1\\ 3 & -2 & 4\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, $B=\begin{pmatrix}2\\ 0\\ 0\end{pmatrix}$

and augmented matrix $\bar{A}=\begin{pmatrix}2 & -3 & 4 & 2\\ 1 & 0 & 1 & 0\\ 0 & -1 & 4 & 0\end{pmatrix}$.

2.10 CONSISTENCY AND INCONSISTECY OF THE SYSTEM OF LINEAR EQUATIONS

We say the system of linear equations is consistent if it has a solution. On the other hand, the system is called inconsistent if it has no solution.

For example,

i) The following system

$$5x-4y=2$$
$$x-y=0$$

has the solution (2, 2). So the system is consistent.

ii) The following system

$$8x-4y=2$$
$$2x-y=1$$

has no solution. So the system is inconsistent.

2.11 EXISTENCE OF THE SOLUTION OF HOMOGENEOUS SYSTEM

Let us consider the following homogeneous system of m linear equations with n unknowns:

$$a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0$$
$$a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0$$

It is very important to keep in mind that the homogeneous system is always **consistent,** since it has always a solution of the form $(x_1,\ x_2\ \ldots,x_n)=(0,0,\ldots,0)$.

This solution is called a **trivial solution**. The solutions other than the trivial are known as **nontrivial solutions.**

Example 13 The following homogeneous system of 2 linear equations with 3 unknowns

$$3x-2y+z=0$$
$$x+y-3z=0$$

has a trivial solution (0, 0, 0). Also it is very interesting to see that (1, 2, 1), (2, 4, 2), (3, 6, 3), etc., are also the solutions of the system. In fact the system has a solution of the form of $k(1,2,1)$. Actually, these are nontrivial solutions.

Theorem 2.4: In a homogeneous system with *m* equations and *n* unknowns, if the number of equations are less than the number of unknowns (i.e., *m* < *n*) then the system has a nontrivial (non-zero) solution. In fact, there exists infinitely many solutions.

Proof: Beyond the scope of the book.

Example 14 Let us solve the following homogeneous system:

$$x+2y-z=0$$
$$2x+y-2z=0$$

First of all, it is clear that it has a trivial (or, zero) solution (0, 0, 0). So the system is consistent.

Here number of equations $(m)=2$ and number of unknowns $(n)=3$. So, $m<n$. Therefore, nontrivial solution also exists.

Now if we write the system as $AX=O$, then $A=\begin{pmatrix}1 & 2 & -1\\ 2 & 1 & -2\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ and $O=\begin{pmatrix}0\\ 0\end{pmatrix}$, null matrix.

Here, we apply elementary row operations to convert A to a row-echelon matrix.

$$A=\begin{pmatrix}1 & 2 & -1\\ 2 & 1 & -2\end{pmatrix}\xrightarrow{R_2-2R_1}\begin{pmatrix}1 & 2 & -1\\ 0 & -3 & 0\end{pmatrix}$$

$$\xrightarrow{\left(\frac{1}{3}\right)R_2}\begin{pmatrix}1 & 2 & -1\\ 0 & 1 & 0\end{pmatrix}\xrightarrow{R_1-2R_2}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.$$

So the given system is equivalent to $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

i.e.,

$$x-z=0$$
$$y=0$$

Let we choose $z = k$, then $x = k$, where k is any arbitrary constant.

Therefore, the solution is $(x, y, z) = (k, 0, k) = k(1, 0, 1)$, which is nontrivial.

Hence, the system has infinitely many solutions.

Theorem 2.5: In a homogeneous system with *n* equations and *n* unknowns, if the rank of the coefficient matrix is less than *n* then the system has a non-trivial (nonzero) solution. In fact, there exists infinitely many solutions.

Proof: Beyond the scope of the book.

Example 15 Solve the following homogeneous system:

$$x + y + 3z = 0$$
$$2x + y + z = 0$$
$$3x + 2y + 4z = 0$$

First of all, it is clear that it has a trivial (or, zero) solution (0, 0, 0). So the system is consistent.

Here number of equations $(m) = 3$ and number of unknowns $(n) = 3$.

Now if we write the system as $AX = O$ then $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{pmatrix}$,

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, null matrix.

Here, we apply elementary row operations to convert A to a row-echelon matrix.

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{pmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 - 3R_1} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1)R_2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, the rank of $A = 2$ $(< 3,$ number of unknowns$)$.

Therefore, nontrivial solution exists.

So the given system is equivalent to $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e.,

$$x - 2z = 0$$
$$y + 5z = 0$$

Let us choose $z = k$, then $x = 2k$ and $y = -5k$ where k is any arbitrary constant.

Therefore, the solution is $(x, y, z) = (2k, -5k, 2k) = k(2, -5, 2)$, which is nontrivial.

Hence, the system has infinitely many solutions.

2.12 EXISTENCE OF THE SOLUTION OF A NON-HOMOGENEOUS SYSTEM

Let us consider the following non-homogeneous system of m linear equations with n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

If we write the above system of equations in the form of $AX = B$

then coefficient matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$ and

augmented matrix $\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$.

Theorem 2.6: A nonhomogeneous system $AX = B$ is consistent iff rank A = rank $\bar{A}$. In other words, solution exists for a nonhomogeneous system $AX = B$ iff rank A = rank $\bar{A}$, otherwise the system has no solution.

Proof: Beyond the scope of the book.

Example 16 Let us consider the system

$$x + 2y = 5$$
$$2x + 5y = 11$$
$$3x + 7y = 17$$

Here, the coefficient matrix is $A=\begin{pmatrix}1&2\\2&5\\3&7\end{pmatrix}$ and the augmented matrix is $\bar{A}=\begin{pmatrix}1&2&5\\2&5&11\\3&7&17\end{pmatrix}$.

Now we apply elementary row operations to the augmented matrix $\bar{A}$

$$\bar{A}=\begin{pmatrix}1&2&5\\2&5&11\\3&7&17\end{pmatrix}\xrightarrow{R_2-2R_1,\ R_3-3R_1}\begin{pmatrix}1&2&5\\0&1&1\\0&1&2\end{pmatrix}$$

$$\xrightarrow{R_1-2R_2,\ R_3-R_2}\begin{pmatrix}1&0&3\\0&1&1\\0&0&1\end{pmatrix}\xrightarrow{R_1-3R_3, R_2-R_3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

So from the above, A is row equivalent to $\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$ and $\bar{A}$ is row equivalent to $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

and therefore rank of $A=2$ and rank of $\bar{A}=3$.

Hence rank A $\neq$ rank $\bar{A}$ and correspondingly, the system is not consistent, i.e., the system has no solution.

Theorem 2.7: Consider a nonhomogeneous system $AX = B$ with m linear equations and n unknowns which is consistent (i.e., rank A = rank $\bar{A}$). Then the following cases hold.

i) The sytem has a unique solution (i.e., only one solution) if rank A = rank $\bar{A}=n$ when $m=n$ or $m>n$.

ii) The sytem has infinitely many solutions if a) rank A = rank $\bar{A}<n$ when $m=n$ or $m>n$ and b) rank A = rank $\bar{A}\le m$ when $m<n$.

Proof: Beyond the scope of the book.

Example 17 Let us consider the system

$$x+y+z=1$$

$$2x+y+2z=1$$

$$x+2y+3z=0$$

Here, the coefficient matrix is $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and

the augmented matrix is $\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$.

Now we apply elementary row operations to the augmented matrix $\bar{A}$.

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix} \xrightarrow{R_2-2R_1,\, R_3-3R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow{(-1)R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{R_3-R_2,\, R_1-3R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

$$\xrightarrow{\left(\frac{1}{2}\right)R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1-R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

So from the above, A is row equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and $\bar{A}$ is row equivalent to $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

and therefore, rank of $A = 3$ and rank of $\bar{A} = 3$.

Hence, rank A = rank $\bar{A}$ $= 3$ (i.e., rank A = rank $\bar{A}$ = number of unknowns) and according to **case (i)** of the above theorem, the system is consistent and the system has a unique solution.

Now the above system is equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

i.e.,

$x = 1$

$y = 1$

$z = -1$

Hence we have the unique solution (1, 1, –1) for the given system.

Example 18 Let us consider the system

$$x+y+z=6$$

$$x+2y+3z=14$$

$$x+4y+7z=30$$

Here, the coefficient matrix is $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 7\end{pmatrix}$ and

the augmented matrix is $\bar{A}=\begin{pmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 1 & 4 & 7 & 30\end{pmatrix}$.

Now we apply elementary row operations to the augmented matrix $\bar{A}$

$$\bar{A}=\begin{pmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 1 & 4 & 7 & 30\end{pmatrix}\xrightarrow{R_2-R_1,\ R_3-R_1}\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 8\\ 0 & 3 & 6 & 24\end{pmatrix}$$

$$\xrightarrow{R_3-3R_2,\ R_1-R_2}\begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 8\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

So from the above, A is row equivalent to $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$

and $\bar{A}$ is row equivalent to $\begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 8\\ 0 & 0 & 0 & 0\end{pmatrix}$.

and therefore, rank of $A=2$ and rank of $\bar{A}=2$.

Hence, rank A = rank $\bar{A}$ $=2<3$ (i.e., rank A = rank $\bar{A}$ < number of unknowns) and according to **case (ii) (a)** of the above theorem, the system is consistent and the system has infinitely many solutions.

Now the above system is equivalent to $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-2\\ 8\\ 0\end{pmatrix}$

i.e.,

$$x-z=-2$$

$$y+z=8$$

Let $z = k$, then $x = k - 2$ and $y = -k + 8$, where k is any arbitary constant.

So $(x, y, z) = (k - 2, -k + 8, k) = (-2, 8, 0) + k(1, -1, 1)$.

Hence we have infinitely many solutions for the given system.

Example 19 Let us consider the system

$$x + 2y + z = 2$$

$$2x + 5y + 3z = 5$$

Here, the coefficient matrix is $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \end{pmatrix}$ and

the augmented matrix is $\overline{A} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 3 & 5 \end{pmatrix}$.

Now we apply elementary row operations to the augmented matrix $\overline{A}$

$$\overline{A} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

So from the above, A is row equivalent to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

and $\overline{A}$ is row equivalent to $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

and therefore, rank of $A = 2$ and rank of $\overline{A} = 2$.

Hence, rank A = rank $\overline{A}$ $= 2$ (i.e., rank A = rank $\overline{A}$ $\leq$ number of equations) and according to **case (ii) (b)** of the above theorem, the system is consistent and the system has infinitely many solutions.

Now the above system is equivalent to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

i.e.,

$$x - z = 0$$

$$y + z = 1$$

Let $z = k$, then $x = k$ and $y = -k + 1$, where k is any arbitrary constant.

So $(x, y, z) = (k, -k + 1, k) = (0, 1, 0) + k(1, -1, 1)$.

Hence, we have infinitely many solutions for the given system.

2.13 EIGEN VALUES AND EIGEN VECTORS

2.13.1 Characteristic Polynomial and Characteristic Equation

Let us consider an $n \times n$ matrix A. Then the **characteristic polynomial** $f_A(\lambda)$ is defined as

$$\det(A-\lambda I_n) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

where $A = (a_{ij})_{n \times n}$.

It is obvious from the definition that the characteristic polynomial $f_A(\lambda)$ is of nth degree and is of the form

$$f_A(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \dots + b_n,$$

where $b_0 (\neq 0), b_1, b_2, ..., b_n$ are constants.

Note: It can be easily shown that the constant term appears in the polynomial (i.e., the term b_n) is equal to $\det(A)$.

The **characteristic equation** is defined as

$$\det(A-\lambda I_n) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0.$$

i.e., $f_A(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \dots + b_n = 0.$

So the degree of the equation is n, which is same as the order of the matrix A.

Example 20 Let us find the characteristic polynomial and characteristic equation of the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

Here, characteristic polynomial is

$$f_A(\lambda) = \det(A-\lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 1.$$

It is very important to note that in the polynomial (-1) is the constant term which is equal to $\det A = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1$.

The characteristic equation is $f_A(\lambda) = 0$. i.e., $\lambda^2 - 6\lambda - 1 = 0$.

The degree of the equation is 2, which is same as order of the matrix A.

2.13.2 Cayley–Hamilton Theorem

If A be a square matrix of order $n \times n$ then A satisfies its own characteristic equation.

From the above theorem, we can say that if the characteristic equation of a square matrix A of order $n \times n$ is

$$b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n = 0,$$

then we have

$$b_0A^n + b_1A^{n-1} + b_2A^{n-2} + \cdots + b_nI_n = O_n.$$

where O_n is the null matrix of order n.

For example,

In the last example, the chracteristic equation is $\lambda^2 - 6\lambda - 1 = 0$.

So by Cayley–Hamilton theorem we have $A^2 - 6A - I_2 = O_2$.

2.13.3 Determination of Inverse of a Matrix using Cayley–Hamilton Theorem

Let us consider a nonsingular square matrix A of order $n \times n$ and its chracteristic equation is $b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n = 0$.

Then by Cayley–Hamilton theorem, we have

$$b_0A^n + b_1A^{n-1} + b_2A^{n-2} + \cdots b_{n-1}A + b_nI_n = O_n \qquad ...(1)$$

Now since A is nonsingular, $b_n = \det(A) \neq 0$ and so b_n^{-1} exists.

So from the above equation (1),

$$b_0A^n + b_1A^{n-1} + b_2A^{n-2} + \cdots b_{n-1}A = -b_nI_n$$

$$A\left(b_0A^{n-1} + b_1A^{n-2} + b_2A^{n-3} + \cdots b_{n-1}I_n\right) = -b_nI_n$$

$$A\left[-b_n^{-1}\left(b_0A^{n-1} + b_1A^{n-2} + b_2A^{n-3} + \cdots b_{n-1}I_n\right)\right] = I_n$$

So,

$$A^{-1} = -b_n^{-1}\left(b_0A^{n-1} + b_1A^{n-2} + b_2A^{n-3} + \cdots b_{n-1}I_n\right)$$

Example 21 Let us find the characteristic equation of $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{pmatrix}$ and using Cayley–Hamilton theorem, find A^{-1}.

The characteristic equation of A is $\det(A - \lambda I_n) = 0$

i.e., $\begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 3-\lambda & -2 \\ 1 & 0 & -2-\lambda \end{vmatrix} = 0$

i.e., $\lambda^3 - 3\lambda^2 - 4\lambda + 10 = 0.$

By Cayley–Hamilton theorem,

$$A^3 - 3A^2 - 4A + 10I_3 = O_3$$

i.e., $A^3 - 3A^2 - 4A = -10I_3$

i.e., $A\left(A^2 - 3A - 4I_3\right) = -10I_3$

i.e., $A\left[-\frac{1}{10}\left(A^2 - 3A - 4I_3\right)\right] = I_3$

i.e., $A^{-1} = \left[-\frac{1}{10}\left(A^2 - 3A - 4I_3\right)\right]$

$$A^2 = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ -2 & 9 & -2 \\ 0 & -1 & 4 \end{pmatrix}$$

So, $(A^2 - 3A - 4I_3) = \begin{pmatrix} 4 & -5 & 2 \\ -2 & 9 & -2 \\ 0 & -1 & 4 \end{pmatrix} - \begin{pmatrix} 6 & -3 & 0 \\ 0 & 9 & -6 \\ 3 & 0 & -6 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$$= \begin{pmatrix} -6 & -2 & 2 \\ -2 & -4 & 4 \\ -3 & -1 & 6 \end{pmatrix}.$$

Hence, $A^{-1} = \left[-\frac{1}{10}\left(A^2 - 3A - 4I_3\right)\right] = \frac{1}{15}\begin{pmatrix} 6 & 2 & -2 \\ 2 & 4 & -4 \\ 3 & 1 & -6 \end{pmatrix}.$

2.13.4 Eigen Values of a Matrix

Roots of the characteristic equation of a square matrix A are called the eigen values of A.

Eigen values are also known as *characteristic roots*.

For example,

let $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$

Then the characteristic equation is $\det(A - \lambda I_2) = 0.$

i.e., $\begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = 0$

i.e., $(1-\lambda)(2-\lambda) = 0$

i.e., $\lambda = 1, 2.$

So, 1 and 2 are the eigen values.

Theorem 2.8: 0 is always an eigen value for a singular matrix.

Proof: Let A be any singular matrix of order $n \times n$ and its characteristic equation is $b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n = 0.$

Now since A is singular, $b_n = \det A = 0.$

So, $b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda = 0$

i.e., $\lambda(b_0\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-1}) = 0$

Therefore, $\lambda = 0$ is an eigen value.

For example,

let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$

Clearly, $\det A = 0$. So, A is singular.

Then the characteristic equation is $\det(A - \lambda I_3) = 0.$

i.e., $\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & 2-\lambda & 3 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

i.e., $\lambda(1-\lambda)(2-\lambda) = 0$

i.e., $\lambda = 0, 1, 2.$

So, 0 is an eigen value.

Theorem 2.9: The diagonal elements are the eigen values for any diagonal matrix.

Proof: Let us consider an $n \times n$ diagonal matrix, $A = \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n \end{pmatrix}.$

Then its characteristic equation is $\begin{vmatrix} d_1-\lambda & 0 & \ldots & 0 \\ 0 & d_2-\lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n-\lambda \end{vmatrix} = 0$

i.e., $(d_1 - \lambda)(d_2 - \lambda)\ldots(d_n - \lambda) = 0$

i.e., $\lambda = d_1, d_2, \ldots, d_n.$

Hence, the theorem is proved.

For example,

let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$

The diagonal elements are 1, 2, and 3.

Then the characteristic equation is $\det(A-\lambda I_3)=0$.

i.e., $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}=0$

i.e., $(1-\lambda)(2-\lambda)(3-\lambda)=0$

i.e., $\lambda=1, 2, 3.$

So, the eigen values are 1, 2, and 3.

2.13.5 Eigen Vectors of a Matrix

Let us consider a square matrix A of order $n\times n.$ Now if an n-dimensional non-null vector $\hat{x}=\left(x_1, x_2, \ldots, x_n\right)$ satisfies the equation

$A\hat{x}=\lambda\hat{x}$, where λ is any scalar then $\hat{x}$ is called an eigen vector of the matrix A.

Now if $A\hat{x}=\lambda\hat{x}$ holds, then $(A-\lambda I_n)\,\hat{x}=O.$

This is nothing but a homogeneous system of n equations with n unknowns. Since the system has a non-null solution,

we have $\det(A-\lambda I_n)=0.$

This leads to the conclusion that the scalar λ is the eigen value corresponding to the eigen vector $\hat{x}$.

Theorem 2.10: There exists a unique eigen value corresponding to a eigen vector.

Proof: Beyond the scope of the book.

Theorem 2.11: If $\hat{x}_1$ and $\hat{x}_2$ are two eigen vectors corresponding to two distinct eigen values then $\hat{x}_1$ and $\hat{x}_2$ are independent.

Proof: Beyond the scope of the book.

Example 22 Let us find the eigen values and eigen vectors of the matrix $A=\begin{pmatrix} 4 & 6 \\ 2 & 8 \end{pmatrix}$.

The characteristic equation is $\begin{vmatrix} 4-\lambda & 6 \\ 2 & 8-\lambda \end{vmatrix}=0$

i.e., $\lambda^2-12\lambda+20=0$

i.e., $(\lambda-10)(\lambda-2)=0$

So, the eigen values are $\lambda=10$ and 2, say $\lambda_1=10$ and $\lambda_2=2.$

Let $\hat{X}_1=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to $\lambda_1=10.$

Then $A\hat{X}_1 = \lambda_1 \hat{X}_1$

i.e., $\begin{pmatrix} 4 & 6 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 10\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

i.e.,

$$4x_1 + 6x_2 = 10x_1$$
$$2x_1 + 8x_2 = 10x_2$$

i.e.,

$$-6x_1 + 6x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

The above system is equivalent to $x_1 - x_2 = 0$.

Now if $x_2 = c$, then $x_1 = c$, where c is an arbitrary real number.

So the eigen vectors are $\hat{X}_1 = \begin{pmatrix} c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to the eigen value $\lambda_1 = 10$.

Let $\hat{X}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to $\lambda_2 = 2$.

Then $A\hat{X}_2 = \lambda_2 \hat{X}_2$

i.e., $\begin{pmatrix} 4 & 6 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

i.e.,

$$4x_1 + 6x_2 = 2x_1$$
$$2x_1 + 8x_2 = 2x_2$$

i.e.,

$$2x_1 + 6x_2 = 0$$
$$2x_1 + 6x_2 = 0$$

The above system is equivalent to $x_1 + 3x_2 = 0$.

Now if $x_2 = c$, then $x_1 = -3c$, where c is an arbitrary real number.

So the eigen vectors are $\hat{X}_2 = \begin{pmatrix} -3c \\ c \end{pmatrix} = c\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ corresponding to the eigen value $\lambda_2 = 2$.

Theorem 2.12: Suppose λ be an eigen value of an $n \times n$ square matrix A. Then the following hold:

(1) λ is also an eigen value of A^T.

(2) $c\lambda$ is also an eigen value of cA for any scalar c.

(3) λ^n is an eigen value of A^n.

(4) λ^{-1} is an eigen value of A^{-1}.

Proof:

(1) Since λ is an eigen value of an $n\times n$ square matrix A,

we have $(A-\lambda I_n)^T = A^T - \lambda I_n^T = A^T - \lambda I_n$ {since $I_n^T = I_n$ }

So, $\det[(A-\lambda I_n)^T] = \det(A^T - \lambda I_n)$

i.e., $\det(A-\lambda I_n) = \det(A^T - \lambda I_n)$.

Since λ is an eigen value of an $n\times n$ square matrix A, $\det(A-\lambda I_n) = 0$.

So, $\det(A^T - \lambda I_n) = 0$.

This proves the fact that λ is also an eigen value of A^T.

(2) Since λ is an eigen value of an $n\times n$ square matrix A, $\det(A-\lambda I_n) = 0$.

Now $\det(cA - c\lambda I_n) = \det\left[c(A-\lambda I_n)\right] = c^n \det(A-\lambda I_n) = 0$.

This proves the fact that $c\lambda$ is also an eigen value of cA for any scalar c.

(3) Let us consider X be the eigen vector corresponding to the eigen value λ.

Then $AX = \lambda X$.

So, $A^2 X = A(\lambda X) = \lambda(AX) = \lambda(\lambda X)$

i.e., $A^2 X = \lambda^2 X$.

Therefore λ^2 is an eigen value of A^2.

Again $A^3 X = A(A^2 X) = A(\lambda^2 X) = \lambda^2 (AX) = \lambda^2 (\lambda X)$

i.e., $A^3 X = \lambda^3 X$.

So, λ^3 is an eigen value of A^3.

Proceeding in the similar manner we have $A^n X = \lambda^n X$.

This proves the fact that λ^n is an eigen value of A^n.

(4) Let us consider X be the eigen vector corresponding to the eigen value λ.

Then $AX = \lambda X$.

So, $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$

$\Rightarrow I_n X = \lambda(A^{-1}X)$

$\Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow A^{-1}X = \lambda^{-1}X$.

This proves the fact that λ^{-1} is an eigen value of A^{-1}.

Theorem 2.13: For an idempotent matrix, the eigen values are either 0 or 1.

Proof: Let λ be an idempotent matrix A and X be an eigen vector corresponding to λ.

Then we have $A^2 = A$ and $AX = \lambda X$.

So, $A(AX) = A(\lambda X)$

$\Rightarrow A^2 X = \lambda(AX)$

$\Rightarrow AX = \lambda(\lambda X)$

$\Rightarrow AX = \lambda^2 X$.

So, $\lambda X = \lambda^2 X$

$\Rightarrow (\lambda^2 - \lambda)X = O$

$\Rightarrow (\lambda^2 - \lambda) = 0$

$\Rightarrow \lambda(\lambda - 1) = 0$

$\Rightarrow \lambda = 0, 1$.

This proves the theorem.

The following topic is included as advanced reading for interested students.

2.14 DIAGONALISATION OF A SQUARE MATRIX

2.14.1 Similar Matrices

Any matrix A of order n is said to be similar to another matrix B of the same order if there exists a nonsingular $n \times n$ matrix P such that $B = P^{-1}AP$.

It is easy to prove that if A is similar to B then B also is similar to A and vice-versa. In this case, we say two matrices A and B of the same order are similar.

Theorem 2.14: Two similar matrices have the same eigen values.

Proof: Let A and B are two similar matrices of order n. Then for a $n \times n$ nonsingular matrix P, we have $B = P^{-1}AP$.

Now the characteristic polynomial of B is

$$\det\left(B - \lambda I_n\right) = \det\left(P^{-1}AP - \lambda I_n\right) \qquad ...(1)$$

Again

$$P^{-1}\left(\lambda I_n\right)P = P^{-1}\lambda\left(I_n P\right) = \lambda\left(P^{-1}P\right) = \lambda I_n.$$

So using the above result, from (1)

$$\det(B - \lambda I_n) = \det\left[P^{-1}AP - P^{-1}\left(\lambda I_n\right)P\right]$$

$$= \det\left[P^{-1}(A-\lambda I_n)P\right]$$

$$= \det\left(P^{-1}\right)\cdot\det\left(A-\lambda I_n\right)\cdot\det P$$

Therefore,

$$\det(B-\lambda I_n) = \det\left(P^{-1}P\right)\cdot\det\left(A-\lambda I_n\right)$$

$$= \det\left(I_n\right)\cdot\det\left(A-\lambda I_n\right) = \det\left(A-\lambda I_n\right).$$

Since A and B have the same characteristic polynomial, they have the same characteristic equations and correspondingly they have the same eigen values.

Note: The converse of the above theorem is not always true, i.e., the matrices having the same eigen values need not be always similar.

Definition: A matrix A of order $n\times n$ is said to be diagonalisable if and only if A is similar to an $n\times n$ diagonal matrix.

i.e., if $D = P^{-1}AP,$ where P is an $n\times n$ nonsingular matrix and $n\times n$ diagonal matrix D is given by

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Since A and $P^{-1}AP$ have the same eigen values and also the eigen values of any diagonal matrix are its diagonal elements, we can say that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the distinct eigen values of the matrix A.

Theorem 2.15: A matrix A of order $n\times n$ is said to be diagonalisable if and only if there exist n eigen vectors of A which are linearly independent.

Proof: Beyond the scope of the book.

Theorem 2.16: If the eigen values of a matrix A of order $n\times n$ are all distinct and real, then A is diagonalisable.

Proof: Beyond the scope of the book.

2.14.2 Steps for Diagonalisation of any Square Matrix

Here we consider a square matrix A of order 3×3 and which has distinct eigen values.

Step (1) Find all three distinct eigen values of A. Suppose they are $\lambda_1, \lambda_2, \lambda_3$.

Step (2) Find all three eigen vectors of A corresponding to $\lambda_1, \lambda_2, \lambda_3$. Suppose

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

are the eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively.

Step (3) Form the nonsingular matrix $P = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$.

Step (4) The matrix $P^{-1}AP$ is the diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Example 23 Let us show that $A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ is diagonalisable, and find P such that $P^{-1}AP$ is a diagonal matrix.

The characteristic equation is $\det(A - \lambda I_3) = 0$.

i.e., $\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda-2)(\lambda+1) = 0$

$\Rightarrow \lambda = 1, 2, -1.$

So, the eigen values are 1, 2, and -1.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to $\lambda = 1$.

Then $AX = \lambda X$.

$$\Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

i.e.,

$$\begin{aligned} x_1 + x_2 - 2x_3 &= x_1 \\ -x_1 + 2x_2 + x_3 &= x_2 \\ x_2 - x_3 &= x_3 \end{aligned}$$

i.e.,

$$\begin{aligned} x_2 - 2x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

So the system is equivalent to

$$x_2 - 2x_3 = 0$$
$$-x_1 + x_2 + x_3 = 0$$

Now if we set $x_3 = k_1$, then $x_2 = 2k_1$ and $x_1 = 3k_1$, where k_1 is any arbitrary constant.

So the eigen vector corresponding to the eigen value $\lambda = 1$ is $\begin{pmatrix} 3k_1 \\ 2k_1 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Similarly, the eigen vectors corresponding to the eigen value $\lambda = 2$ and $\lambda = -1$ are $k_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $k_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ respectively, where k_2, k_3 are arbitrary constants.

Since all the three eigen values of A are distinct, the eigen vectors are linearly independent and correspondingly A is diagonalisable.

So, we choose $P = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

Also $\det P = 6 \neq 0$, so P is nonsingular.

Here $adj(P) = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 2 & -2 \\ -3 & 2 & 7 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & -3 \\ -2 & 2 & 2 \\ -1 & -2 & 7 \end{pmatrix}$

So, $P^{-1} = \dfrac{1}{6} \begin{pmatrix} 3 & 0 & -3 \\ -2 & 2 & 2 \\ -1 & -2 & 7 \end{pmatrix}$.

Now

$$P^{-1}AP = \frac{1}{6} \begin{pmatrix} 3 & 0 & -3 \\ -2 & 2 & 2 \\ -1 & -2 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D$$

where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, a diagonal matrix with the eigen values as its diagonal.

WORKED-OUT EXAMPLES

Example 2.1 Find the rank of the matrix $\begin{pmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{pmatrix}$

[WBUT-2002, 2008].

Sol. Let $A = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{pmatrix}$

Applying elementary row operations on the matrix A, we have,

$$A = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{pmatrix} \xrightarrow{R_2+2R_1,\ R_4+R_1}$$

$$\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{pmatrix} \xrightarrow{R_4+2R_2}$$

$$\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1)R_1 \frac{1}{2}R_2}$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2-R_3,\ R_1-R_3}$$

$$\begin{pmatrix} 1 & -2 & 0 & -5 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{4}R_2}$$

$$\begin{pmatrix} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+2R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

The row-reduced echelon matrix B has the 3 nonzero rows.

So, rank of $B = 3$ and hence rank of $A = 3$.

Example 2.2 Find the rank of the matrix $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix}$ [WBUT-2005]

Sol. Let, $A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

Applying elementary row operations on the matrix A, we get,

$$A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix} \xrightarrow{R_2 - 3R_1,\, R_3 - R_1}$$

$$\begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{pmatrix} \xrightarrow{\frac{-1}{6}R_2\ \frac{-1}{2}R_3}$$

$$\begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2,\, R_1 - 3R_2}$$

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

The row-reduced echelon matrix B has the 2 nonzero rows.

So, rank of $B = 2$ and hence rank of $A = 2$.

Example 2.3 Find the rank of the rectangular matrix $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$

[WBUT-2006]

Sol. Let $A = \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$

Applying elementary row operations on the matrix A, we get,

$$A=\begin{pmatrix}1&3&2&4&1\\0&0&2&2&0\\2&6&2&6&2\\3&9&1&10&6\end{pmatrix}\xrightarrow{R_3-2R_1,\ R_4-3R_1}$$

$$\begin{pmatrix}1&3&2&4&1\\0&0&2&2&0\\0&0&-2&-2&0\\0&0&-5&-2&3\end{pmatrix}\xrightarrow{\frac{1}{2}R_2,\ \left(\frac{-1}{2}\right)R_3}$$

$$\begin{pmatrix}1&3&2&4&1\\0&0&1&1&0\\0&0&1&1&0\\0&0&-5&-2&3\end{pmatrix}\xrightarrow{R_3-R_2},$$

$$\begin{pmatrix}1&3&2&4&1\\0&0&1&1&0\\0&0&0&0&0\\0&0&-5&-2&3\end{pmatrix}\xrightarrow{R_{34}}$$

$$\begin{pmatrix}1&3&2&4&1\\0&0&1&1&0\\0&0&-5&-2&3\\0&0&0&-0&0\end{pmatrix}\xrightarrow{R_3-5R_2,\ R_1-2R_2}.$$

$$\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&3&3\\0&0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{3}R_3}$$

$$\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}\xrightarrow{R_1-2R_3,\ R_2-R_3},$$

$$\begin{pmatrix}1&3&0&0&-1\\0&0&1&0&-1\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}=B$$

The row-reduced echelon matrix B has the 3 nonzero rows.

So, rank of $B=3$ and hence rank of $A=3$.

Example 2.4 Using elementary row operations find the inverse of the matrix

$$A=\begin{pmatrix}1 & 1 & 2\\ 2 & 4 & 4\\ 3 & 3 & 7\end{pmatrix}.$$

Sol. Let us apply elementary row operations on the matrix $(A \mid I_3)$

$$(A \mid I_3)=\begin{pmatrix}1 & 1 & 2 & 1 & 0 & 0\\ 2 & 4 & 4 & 0 & 1 & 0\\ 3 & 3 & 7 & 0 & 0 & 1\end{pmatrix}\xrightarrow{R_2-2R_1,\ R_3-3R_1}$$

$$\begin{pmatrix}1 & 1 & 2 & 1 & 0 & 0\\ 0 & 2 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & -3 & 0 & 1\end{pmatrix}\xrightarrow{\frac{1}{2}R_2}$$

$$\begin{pmatrix}1 & 1 & 2 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0\\ 0 & 0 & 1 & -3 & 0 & 1\end{pmatrix}\xrightarrow{R_1-R_2}$$

$$\begin{pmatrix}1 & 0 & 2 & 2 & \frac{-1}{2} & 0\\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0\\ 0 & 0 & 1 & -3 & 0 & 1\end{pmatrix}\xrightarrow{R_1-2R_3}$$

$$\begin{pmatrix}1 & 0 & 0 & 8 & \frac{-1}{2} & -2\\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0\\ 0 & 0 & 1 & -3 & 0 & 1\end{pmatrix}=(I_3 \mid A^{-1})$$

Therefore,

$$A^{-1}=\begin{pmatrix}8 & \frac{-1}{2} & -2\\ -1 & \frac{1}{2} & 0\\ -3 & 0 & 1\end{pmatrix}$$

Example 2.5 Solve the system of equations by matrix inversion method:

$$x + y - z = 6$$
$$2x - 3y + z = 1$$
$$2x - 4y + 2z = 1$$

Sol. The above system of equations can be written as $AX = B$

$$\text{i.e., } \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$$

Therefore,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & -4 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} 3$$

So,

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & -4 & 2 \end{vmatrix} = -2 \neq 0$$

Since $\det A \neq 0$, A^{-1} exists, and the system has a unique solution $X = A^{-1}B$.

Now,

$$\text{adj } A = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ -4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -2 & -2 & -2 \\ 2 & 4 & 6 \\ -2 & -3 & -5 \end{pmatrix}^T$$

$$= \begin{pmatrix} -2 & 2 & -2 \\ -2 & 4 & -3 \\ -2 & 6 & -5 \end{pmatrix}$$

Now,

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{-1}{2}\begin{pmatrix} -2 & 2 & -2 \\ -2 & 4 & -3 \\ -2 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & \frac{3}{2} \\ 1 & -3 & \frac{5}{2} \end{pmatrix}$$

Therefore,

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & \frac{3}{2} \\ 1 & -3 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ \frac{11}{2} \\ \frac{11}{2} \end{pmatrix}$$

Therefore, the solution of the system of equations is

$$x = 6;\ y = \frac{11}{2};\ z = \frac{11}{2}.$$

Example 2.6 If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$, show that $AB = 6I_3$.

Utilize this result to solve the following system of equations

$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

[WBUT-2009].

Sol. Here,

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+2 & 1-2+1 & 1-1 \\ 2-4+2 & 1+4+1 & 1-1 \\ 6-6 & 3-3 & 3+3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$$

So, the first part is proved.

Now, to solve the system

$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

first we write the system in the following form

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

i.e., $BX = C$ ...(1)

where,

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } C = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

Now from the above relation $AB = 6I_3$, we have

$$\left(\frac{1}{6}A\right) \cdot B = I_3$$

So, from the definition of inverse, we conclude that B^{-1} exists and it is given by

$$B^{-1} = \frac{1}{6}A$$

$$B^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}.$$

Since B^{-1} exists, the solution of the system (1) is given by

$$X = B^{-1}C$$

$$\text{i.e., } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Hence, the solution of the given system is

$x = 1, y = 1, z = 2.$

Example 2.7 Solve by Cramer's rule

$$2x - y = 3$$
$$3y - 2z = 5$$
$$-2z + x = 4$$

[WBUT-2008]

Sol. If we write the above equations in the form $AX = B$ then

the coefficient matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$.

The determinant of the coefficient matrix is

$$\det A = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{vmatrix} = -10 \neq 0$$

Therefore, the system of equations is consistent and the system has a unique solution.

Now

$$\det A_1 = \begin{vmatrix} 3 & -1 & 0 \\ 5 & 3 & -2 \\ 4 & 0 & -2 \end{vmatrix} = -20$$

$$\det A_2 = \begin{vmatrix} 2 & 3 & 0 \\ 0 & 5 & -2 \\ 1 & 4 & -2 \end{vmatrix} = -10$$

$$\det A_3 = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & 5 \\ 1 & 0 & 4 \end{vmatrix} = 10$$

So, the solution of the system of equations by **Cramer's rule** is given by

$$x = \frac{\det A_1}{\det A} = \frac{-20}{-10} = -1$$

$$y = \frac{\det A_2}{\det A} = \frac{-10}{-10} = 1$$

$$z = \frac{\det A_3}{\det A} = \frac{10}{-10} = -1.$$

Example 2.8 Solve by Cramer's rule

$$3x + y + z = 4$$

$$x - y + 2z = 6$$

$$x + 2y - z = -3$$

[WBUT-2009]

Sol. If we write the above equations in the form $AX = B$ then

the coefficient matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$.

The determinant of the coefficient matrix is

$$\det A = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = -3 \neq 0$$

Therefore, the system of equations is consistent and the system has a unique solution.

Now,

$$\det A_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = -3$$

$$\det A_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3$$

$$\det A_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = -6$$

So, by **Cramer's rule**, the solution of the system is given by

$$x = \frac{\det A_1}{\det A} = \frac{-3}{-3} = 1$$

$$y = \frac{\det A_2}{\det A} = \frac{3}{-3} = -1$$

$$z = \frac{\det A_3}{\det A} = \frac{-6}{-3} = 2.$$

Example 2.9 Investigate for what value of λ and μ the following equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

have i) no solution, ii) a unique solution, and iii) an infinite number of solutions.

[WBUT-2004]

Sol. If we write the above equations in the form $AX = B$ then the coefficient matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 6 \\ 10 \\ \mu \end{pmatrix}$.

The determinant of the coefficient matrix is

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}$$

$$= 1(2\lambda - 6) - 1(\lambda - 3) + 1(2 - 2)$$

$$= \lambda - 3$$

Now,

$$\det A_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix}$$

$$= 6(2\lambda - 6) - 1(10\lambda - 3\mu) + 1(20 - 2\mu)$$

$$= 2\lambda + \mu - 16$$

$$\det A_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix}$$

$$= 1(10\lambda - 3\mu) - 6(\lambda - 3) + 1(\mu - 10)$$

$$= 4\lambda - 2\mu + 8$$

$$\det A_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix}$$

$$= 1(2\mu - 20) - 1(\mu - 10)$$

$$= \mu - 10$$

[**Note:** The following cases will be discussed according to the **observations of Section 3.8**]

Case (i): The system of equations have **no solution** when $\det A = 0 \Rightarrow \lambda = 3$ and at least one of $\det A_1$, $\det A_2$, $\det A_3$ is nonzero,

i.e., when $\lambda = 3$ and at least one of

$$2\lambda + \mu - 16 \neq 0,\ 4\lambda - 2\mu + 8 \neq 0,\ \mu - 10 \neq 0$$

i.e., when $\lambda = 3$ and $\mu \neq 10$.

Case (ii): The system of equations have a **unique solution** when $\det A \neq 0$, i.e., when $\lambda \neq 3$.

Case (iii): The system of equations have **an infinite number of solutions** when

$$\det A = 0 \Rightarrow \lambda = 3 \text{ and } \det A_1 = \det A_2 = \det A_3 = 0$$

i.e., when $\lambda = 3$ and $2\lambda + \mu - 16 = 0,\ 4\lambda - 2\mu + 8 = 0,\ \mu - 10 = 0$

i.e., when $\lambda = 3$ and $\mu = 10$.

Example 2.10 Determine the nature of the solution without solving the homogeneous system of equations:

$$x + y + 3z = 0$$
$$2x + y + z = 0$$
$$3x + 2y + 4z = 0$$

Sol. The determinant of the coefficient matrix A is $\det A = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

$$= 1(4-2) - 1(8-3) + 3(4-3) = 2 - 5 + 3 = 0.$$

Since the determinant of the coefficient matrix is zero for the given homogeneous system of equations, the system has infinitely many nontrivial solutions.

Example 2.11 Solve by the consistency of the following system of equations and solve if possible

$$x + y + z = 1$$
$$2x + y + 2z = 2$$
$$3x + 2y + 3z = 5$$

[WBUT-2006, 2008]

Sol. The system of linear equations can be written in matrix form as $AX = B$

i.e., $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

The coefficient matrix of the system of equations is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}$$

and the augmented matrix is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix}.$$

Applying elementary row operations on the augmented matrix $\overline{A}$, we have,

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 - 3R_1}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 - R_2,\ R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{(-1)R_2,\ \frac{1}{2}R_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here, rank of $\overline{A}$ is 3 and rank of A is 2.

Since rank of $\overline{A} \neq$ rank of A, the given system of equations is inconsistent.

In other words, the system does not have any solution.

Example 2.12 For what value of k do the following equations

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$ have solutions? Solve them completely in each case.

[WBUT-2003]

Sol. The system of linear equations can be written in matrix form as $AX = B$

$$\text{i.e., } \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ k \\ k^2 \end{pmatrix}$$

The coefficient matrix is $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{pmatrix}$ and the augmented matrix is

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right).$$

Applying elementary row operations on the augmented matrix $\overline{A}$, we have

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right) \xrightarrow{R_2-2R_1,\, R_3-4R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{array}\right) \xrightarrow{R_3-3R_2,\, R_1+R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & k-1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array}\right)$$

The coefficient matrix is equivalent to the matrix $\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Therefore, the rank of A is 2.

The system of equations have solutions if rank $A =$ rank $\overline{A}$.

The matrix $\overline{A}$ has rank 2 if and only if $k^2-3k+2=0 \Rightarrow k=1, 2.$

For, $k=1$, the system of equations becomes

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

The above system is equivalent to

$$x+3z=0$$

$$-y+2z=-1$$

Putting $z=k_1$, we get $x=-3k_1$ and $y=2k_1+1$, where k_1 is an arbitrary constant.

So the solution is $(x, y, z)=(-3k_1, 2k_1+1, k_1)$. In this case, the number of solutions is infinite.

For $k = 2$, the system of equations becomes

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The above system is equivalent to

$$x + 3z = 1$$

$$-y + 2z = 0$$

Putting $z = k_2$, we get $x = -3k_2$ and $y = 2k_2$, where k_2 is an arbitrary constant.

So the solution is $(x, y, z) = (-3k_2, 2k_2, k_2)$. In this case also, the number of solutions is infinite.

Example 2.13 Determine the values of a and b so that the system of equations

$$2x + 3y + 4z = 9$$

$$x - 2y + az = 5$$

$$3x + 4y + 7z = b$$

have i) a unique solution, ii) many solutions, and iii) no solution.

Sol. If we write the system of linear equations in the matrix form as $AX = B$ then the coefficient matrix of the system of linear equations is

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -2 & a \\ 3 & 4 & 7 \end{pmatrix}$$

and the augmented matrix is

$$\overline{A} = \begin{pmatrix} 2 & 3 & 4 & 9 \\ 1 & -2 & a & 5 \\ 3 & 4 & 7 & b \end{pmatrix}$$

The system of equations have unique solution when the determinant of the coefficient matrix is not equal to zero.

$$\det A = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & a \\ 3 & 4 & 7 \end{vmatrix}$$

$$= 2(-14 - 4a) - 3(7 - 3a) + 4(4 + 6) = a - 9$$

Therefore, for $\det A \neq 0 \Rightarrow a \neq 9$ the system of equations have a **unique solution.**

When $a = 9,$ the augmented matrix becomes

$$\overline{A} = \begin{pmatrix} 2 & 3 & 4 & 9 \\ 1 & -2 & 9 & 5 \\ 3 & 4 & 7 & b \end{pmatrix}$$

Applying elementary row operations on the matrix $\overline{A},$ we have

$$\overline{A} = \begin{pmatrix} 2 & 3 & 4 & 9 \\ 1 & -2 & 9 & 5 \\ 3 & 4 & 7 & b \end{pmatrix} \xrightarrow{R_{12}}$$

$$\begin{pmatrix} 1 & -2 & 9 & 5 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 7 & b \end{pmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 + 3R_1}$$

$$\begin{pmatrix} 1 & -2 & 9 & 5 \\ 0 & 7 & 14 & -1 \\ 0 & 10 & 20 & b-15 \end{pmatrix} \xrightarrow{\left(\frac{1}{7}\right)R_2}$$

$$\begin{pmatrix} 1 & -2 & 9 & 5 \\ 0 & 1 & 2 & \frac{-1}{7} \\ 0 & 10 & 20 & b-15 \end{pmatrix} \xrightarrow{R_1 + 2R_2,\ R_3 - 10R_2}$$

$$\begin{pmatrix} 1 & 0 & 13 & \frac{33}{7} \\ 0 & 1 & 2 & \frac{-1}{7} \\ 0 & 0 & 0 & b - 15 + \frac{10}{7} \end{pmatrix}$$

The system of equations is consistent when rank $A =$ rank $\overline{A}$ and this is possible for

$$b - 15 + \frac{10}{7} = 0$$

i.e., $b = \dfrac{95}{7}.$

In this case, rank $A =$ rank $\overline{A} = 2,$ which is less then the number of unknowns $(= 3)$ and the system has **infinitely many solutions.**

Again, if

$$b - 15 + \frac{10}{7} \neq 0 \Rightarrow b \neq \frac{95}{7}.$$

then rank $A = 2$ and rank $\overline{A} = 3$, i.e., rank $A \neq$ rank $\overline{A}$

and so the system of equations is inconsistent and correspondingly, the system has **no solution.**

Summarizing the above, we have

i) **the system of equations has a unique solutions when $a \neq 9$**

ii) **the system of equations has infinitely many solutions when $a = 9$ and $b = \frac{95}{7}$**

iii) **the system of equations has no solution when $a = 9$ and $b \neq \frac{95}{7}$**

Example 2.14 Solve the system of equations

$$x + 2y + z - 3u = 1$$

$$2x + 4y + 3z + u = 3$$

$$3x + 6y + 4z - 2u = 4 \text{ if possible .}$$

Sol. The coefficient matrix of the system of equations is

$$A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{pmatrix}$$

and the augmented matrix is

$$\overline{A} = \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{pmatrix}$$

Applying elemetary row operation on the augmented matrix $\overline{A}$, we have

$$\overline{A} = \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{pmatrix} \xrightarrow{R_2 - 2R_1,\ R_3 - 3R_1}$$

$$\begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2,\ R_1 - R_2}$$

$$\begin{pmatrix} 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here, the equivalent matrix has two nonzero rows and rank $A =$ rank $\overline{A} = 2$. So the system of equations is consistent.

Since, rank $A =$ rank $\overline{A} = 2 <$ Number of unknowns ($= 4$),

the system of equations has infinitely many solutions.

The equivalent system of equations becomes

$$\begin{pmatrix} 1 & 2 & 0 & -10 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

or, $x + 2y - 10u = 0$

$$z + 7u = 1$$

Taking, $u = k_1$, $y = k_2$, we have $x = 10k_1 - 2k_2$, $z = 1 - 7k_1$, where k_1 and k_2 are arbitrary constants.

Hence, the solution is given by

$(x, y, z) = (10k_1 - 2k_2, k_2, 1 - 7k_1, k_1)$

where k_1 and k_2 are arbitrary constants.

Example 2.15 If $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ find all eigen values of A and obtain all the eigen vectors corresponding to its eigen values. [WBUT-2004]

Sol. The characteristic equation of the matrix A is

$\det(A - \lambda I_3) = 0$

or, $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)\{(3-\lambda)(2-\lambda) - 2\} - 2\{(2-\lambda) - 1\} + \{2 - (3-\lambda)\} = 0$

or, $(2-\lambda)\{6 - 5\lambda + \lambda^2 - 2\} - 2(1-\lambda) + (-1+\lambda) = 0$

or, $(2-\lambda)(\lambda^2 - 5\lambda + 4) - 3 + 3\lambda = 0$

or, $2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 3 + 3\lambda = 0$

or, $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

or, $(\lambda-1)(\lambda^2-6\lambda+5)=0$

or, $(\lambda-1)(\lambda-5)(\lambda-1)=0$

Therefore, the eigen values are $\lambda=1,1,5$.

Let $X_1=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda=1$.

Therefore, we have,

$$AX_1=1\cdot X_1$$

or, $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

or, $2x+2y+z=x$

$x+3y+z=y$

$x+2y+2z=z$

or, $x+2y+z=0$

$x+2y+z=0$

$x+2y+z=0$

So the above system is equivalent to

$$x+2y+z=0.$$

Let $y=k_1$ and $z=k_2$ then $x=-2k_1-k_2$ where k_1 and k_2 are arbitrary constants.

Therefore, the eigen vector corresponding to the eigen value $\lambda=1$ is given by

$$X_1=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -2k_1-k_2 \\ k_1 \\ k_2 \end{pmatrix}$$

$$=k_1\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}+k_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Let $X_2=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda=5$.

Therefore, we have,

$$AX_2 = 5X_2$$

$$\text{or, } \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{or, } \quad 2x+2y+z = 5x$$

$$x+3y+z = 5y$$

$$x+2y+2z = 5z$$

$$\text{or, } \quad -3x+2y+z = 0$$

$$x-2y+z = 0$$

$$x+2y-3z = 0$$

Here, the determinant of the coeficient matrix is

$$\Delta = \begin{vmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{vmatrix} = 0$$

Therefore, the system of homogeneous equations have nontrivial solutions. The solutions are

$$\frac{x}{4} = \frac{y}{4} = \frac{z}{4} = k$$

Therefore, the eigen vector corresponding to $\lambda = 5$ is

$$X_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4k \\ 4k \\ 4k \end{pmatrix} = 4k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 2.16 Verify that the matrix $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$ satisfies its own characteristic equation. If possible, find A^{-1}. [WBUT-2002]

Sol. The characteristic equation of the matrix A is

$$\det(A-\lambda I) = 0$$

$$\text{or, } \begin{vmatrix} -\lambda & -1 & 2 \\ 1 & -\lambda & 3 \\ 2 & 3 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 9) + 1(-\lambda - 6) + 2(3 + 2\lambda) = 0$$

$$\Rightarrow -\lambda^3 + 9\lambda - \lambda - 6 + 6 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda = 0$$

$$\Rightarrow \lambda^3 - 12\lambda = 01$$

Now,

$$A^2 = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -3 \\ 6 & 8 & 2 \\ 3 & -2 & 13 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 3 & 6 & -3 \\ 6 & 8 & 2 \\ 3 & -2 & 13 \end{pmatrix}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -12 & 24 \\ 12 & 0 & 36 \\ 24 & 36 & 0 \end{pmatrix}$$

$$A^3 - 12A = \begin{pmatrix} 0 & -12 & 24 \\ 12 & 0 & 36 \\ 24 & 36 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Since we have $A^3 - 12A = O$, the matrix A satifies its characteristic equation (1).

Again from (1), we have $\lambda(\lambda^2 - 12) = 0$ which implies $\lambda = 0$ is an eigen-value, i.e., the matrix A is singular.

Therefore, A^{-1} does not exist.

Example 2.17 If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then verify that A satisfies its own characteristic equation. Hence, find A^{-1} and A^9. [WBUT-2007, 2008]

Sol. The characteristic equation of the matrix A is

$$\det(A - \lambda I) = 0$$

$$\text{or, } \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

or, $\lambda^3 - 2\lambda + 1 = 0.$

Now

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

Therefore,

$$A^3 - 2A + I$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Since

$$A^3 - 2A + I = O \qquad \text{...(1)}$$

the matrix A satisfies its own characteristic equation.

Now, from (1), we have

$$A^3 - 2A + I = O$$

or, $A(A^2 - 2I) = -I$

i.e., $A \cdot \left[-(A^2 - 2I)\right] = I$

So, from the definition of inverse, we have

$$A^{-1} = -(A^2 - 2I) = 2I - A^2$$

$$\text{i.e., } A^{-1} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Again, from (1), we obtain

$$A^3 - 2A + I = O$$

or, $A^3 = 2A - I$

or, $A^9 = (A^3)^3 = (2A-I)^3 = 8A^3 - 12A^2 + 6A - I$

$$= 8(2A-I) - 12A^2 + 6A - I = 22A - 12A^2 - 9I$$

or, $A^9 = 22\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} - 12\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -24 & 20 \\ 0 & -55 & 34 \\ 0 & 34 & -21 \end{pmatrix}$$

Example 2.18 Show that the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$ satisfies the Cayley–Hamilton theorem. [WBUT-2007]

Sol. The characteristic equation of the matrix is

$$\det(A - \lambda I) = 0$$

or, $\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$

or, $(-\lambda)\{(1-\lambda)(4-\lambda)\} + \{3 + 2(1-\lambda)\} = 0$

or, $(-\lambda)(\lambda^2 - 5\lambda + 4) + (5 - 2\lambda) = 0$

or, $-\lambda^3 + 5\lambda^2 - 4\lambda - 2\lambda + 5 = 0$

or, $\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0$

Now,

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ 5 & 5 & 14 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ 5 & 5 & 14 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 5 & 14 \\ -3 & 4 & 12 \\ -13 & 19 & 61 \end{pmatrix}$$

So,

$$A^3 - 5A^2 + 6A - 5I$$

$$= \begin{pmatrix} -5 & 5 & 14 \\ -3 & 4 & 12 \\ -13 & 19 & 61 \end{pmatrix} - 5\begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ 5 & 5 & 14 \end{pmatrix} + 6\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Therefore, the matrix A satisfies its own characteristic equation.

Example 2.19 Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$ [WBUT-2008].

Sol. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$

The characteristic equation of A is

$$\det(A - \lambda I) = 0$$

$$\text{or} \begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

$$\text{or, } (1-\lambda)\{(-2-\lambda)(6-\lambda)+12\} + 1\{2(6-\lambda)-12\} + 2\{-6+3(2+\lambda)\} = 0$$

$$\text{or, } (1-\lambda)\{-12-4\lambda+\lambda^2+12\} + \{12-2\lambda-12\} + 2\{-6+6+3\lambda\} = 0$$

$$\text{or, } (1-\lambda)(\lambda^2-4\lambda) + 4\lambda = 0$$

$$\text{or, } \lambda\{(1-\lambda)(\lambda-4)+4\} = 0$$

$$\text{or, } \lambda\{\lambda-\lambda^2+4\lambda-4+4\} = 0$$

$$\text{or, } \lambda^2(\lambda-5) = 0$$

Therefore, the eigen values of the matrix A are $\lambda = 0, 0, 5$.

Let $X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 0$.

Therefore, we have

$$AX = 0 \cdot X$$

$$\text{or, } \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{or, } x - y + 2z = 0$$

$$2x - 2y + 4z = 0$$

$$3x - 3y + 6z = 0$$

The above system is equivalent to

$$x - y + 2z = 0$$

Let $y = k_1$ and $z = k_2$, then $x = k_1 - 2k_2$ where k_1 and k_2 are arbitrary constants.

Therefore, the eigen vector corresponding to the eigen value $\lambda = 0$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 - 2k_2 \\ k_1 \\ k_2 \end{pmatrix}$$

$$= k_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Let $X_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 5$.

Therefore, we have

$$AX_2 = 5X_2$$

$$\text{or, } \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{or, } \quad x - y + 2z = 5x$$

$$2x - 2y + 4z = 5y$$

$$3x - 3y + 6z = 5z$$

$$\text{or, } -4x - y + 2z = 0$$

$$2x - 7y + 4z = 0$$

$$3x - 3y + z = 0$$

This is a homogeneous system and the determinant of the coeficient matrix is

$$\begin{vmatrix} -4 & -1 & 2 \\ 2 & -7 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 0.$$

Therefore, the system of homogeneous equations has nontrivial solutions and the solutions are

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{30} = k,$$

where k is any arbitrary constant.

Therefore the eigen vector corresponding to the eigen value $\lambda = 5$

$$X_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10k \\ 20k \\ 30k \end{pmatrix} = 10k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Example 2.20 Determine the eigen vectors of $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and then diagonalise A with the help of the basis of eigen vectors. [WBUT-2003]

Sol. The characteristic equation of the matrix A is

$$\det(A - \lambda I) = 0$$

or, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)(2-\lambda) - 4 = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $(\lambda - 1)(\lambda - 6) = 0$

So, the eigen values of the matrix A are $\lambda = 1, 6$.

Now let $X_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ be an eigen vector corresponding to the eigen value $\lambda = 1$, then

$$AX_1 = 1 \cdot X_1$$

or, $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

or, $4x + 4y = 0$

$x + y = 0$

The above system is equivalent to

$x+y=0$

Taking $x=k_1$, we have $y=-k_1$, where k_1 is any arbitrary constant.

Therefore, the eigen vector corresponding to the eigen value $\lambda=1$

$$X_1=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} k_1 \\ -k_1 \end{pmatrix}=k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Again let $X_2=\begin{pmatrix} x \\ y \end{pmatrix}$ be an eigen vector corresponding to the eigen value $\lambda=6$, then

$$AX_2=6X_2$$

$$\text{or, } \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 6x \\ 6y \end{pmatrix}$$

$$\text{or, } -x+4y=0$$

$$x-4y=0$$

The above is equivalent to

$x-4y=0$

Taking $y=k_2$, we have $x=4k_2$, where k_2 is any arbitrary constant.
Therefore, the eigen vector corresponding to the eigen value $\lambda=6$

$$X_2=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 4k_2 \\ k_2 \end{pmatrix}=k_2\begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Since both the two eigen values of A are distinct, the eigen vectors are linearly independent and correspondingly A is diagonalisable.

So, we choose $P=\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$,

Also, $\det P=5\neq 0$, so P is nonsingular.

Here, $\operatorname{adj}(P)=\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}^T=\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$

So, $P^{-1}=\dfrac{1}{5}\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$.

Now

$$P^{-1}AP=\frac{1}{5}\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}=D$$

where $D=\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, a diagonal matrix with the eigen values as its diagonal.

EXERCISES

Short and Long Answer Type Questions

1. Find the rank of the following matrices:

(a) $\begin{pmatrix} 5 & 4 & 5 \\ 4 & 5 & 7 \\ 5 & 7 & 10 \end{pmatrix}$

[**Ans :** Rank is 2]

(b) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

[**Ans :** Rank is 3]

(c) $\begin{pmatrix} 2 & 1 & 0 & 4 \\ 1 & 3 & 2 & 1 \\ -1 & 2 & 4 & 3 \end{pmatrix}$

[**Ans :** Rank is 3]

(d) $\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$

[**Ans :** Rank is 3]

(e) $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix}$ [WBUT-2005]

[**Ans :** Rank is 2]

(f) $\begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$

[**Ans :** Rank is 2]

(g) $\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$

[**Ans :** Rank is 2]

(h) $\begin{pmatrix} 1 & -1 & 2 & 0 & 4 \\ 2 & 3 & 1 & 5 & 2 \\ 1 & 3 & -1 & 0 & 3 \\ 1 & 7 & -4 & 1 & 1 \end{pmatrix}$

[**Ans :** Rank is 4]

(i) $\begin{pmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -2 \end{pmatrix}$

[**Ans :** Rank is 1]

2. Find all values of μ for which the rank of the following matrix is 2.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \mu \\ 1 & 1 & 6 & \mu+1 \end{pmatrix}$$

[**Ans :** $\mu = 1$].

3. Using elementary row operations, find the inverse of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 6 & 4 & 1 \end{pmatrix}$$

$$\left[\textbf{Ans :} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ \frac{-1}{3} & \frac{-4}{3} & 1 \end{pmatrix} \right]$$

4. Using elementary row operations, find the matrix A if

$$A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -3 & 4 \end{pmatrix}.$$

$$\left[\textbf{Ans :} \begin{pmatrix} 1 & 1 & -1 \\ 5 & 9 & -8 \\ 3 & 6 & -5 \end{pmatrix} \right]$$

5. Find the eigen values of the following matrices:

(a) $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

[**Ans :** 6, -1]

(b) $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

[**Ans :** 6, 1].

6. Find the eigen values of the matrix

$$\begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[**Ans :** $-6, -2, 7, 1$]

7. Prove that the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Mention the corresponding eigen value.

[**Ans :** Eigen value is 5]

8. Prove that the vector $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ corresponding to the eigen value 3.

9. Verify the Cayley–Hamilton theorem for the following matrices.

a) $\begin{pmatrix} 2 & 1 & 3 \\ -1 & 3 & -7 \\ 1 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 & 1 \\ 2 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 4 & -8 \\ 5 & 1 & 8 \end{pmatrix}$

10. If a matrix A is invertible and its eigen values are $\lambda_1, \lambda_2, \ldots \lambda_n$ and $B = A^{-1}$, show that the eigen values of B are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$. [WBUT-2006]

11. Find the rank of the matrix $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$ [WBUT-2006]

[**Ans :** Rank is 3]

12. Find all values of λ for which the rank of the matrix $\begin{pmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ 1 & 1 & 1 & \lambda \end{pmatrix}$ is less than 4.

$\left[\textbf{Ans :} \lambda = 1, \frac{-1}{3}\right]$

13. Find the rank of the matrix $\begin{pmatrix} a & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{pmatrix}$ where $aa' + bb' + cc' = 0$.

14. Solve the following system of equations by matrix inversion method:

a) $2x - y + 3z = 4$
$x + 2y + 2z = 5$
$3x - y + 4z = 6$

[**Ans :** $x = 1,\ y = 1,\ z = 1$]

b) $x + y + z = 1$
$x + 2y + 3z = 16$
$x + 3y + 4z = 22$

[**Ans :** $x = 1,\ y = 3,\ z = 3$]

c) $x + 2y + 3z = 6$
$2x + 4y + z = 7$
$3x + 2y + 9z = 14$

[**Ans :** $x = 1,\ y = 1,\ z = 1$]

15. Solve the following system of equations by Cramer's rule:

a) $x + y - z = 6$
$2x - 3y + z = -1$
$3x - 4y + 2z = -1$

[**Ans :** $x = 3,\ y = 2,\ z = -1$]

b) $-x + y + z = 2$
$2x - y + 3z = 4$
$3x + 2y - 6z = 1$

$\left[\textbf{Ans :}\ x = \frac{9}{7},\ y = \frac{59}{28},\ z = \frac{33}{28}\right]$

16. Examine the consistency of the following system of equations and if possible, solve:

a) $2x - y + z = 1$
$x + y + 2z = -1$
$3x + 2y - z = 4$

[**Ans :** Consistent and unique solution $x = 1,\ y = 0,\ z = -1$]

b) $4x - 2y + 6z = 8$
$x + y - 3z = -1$
$15x - 3y + 9z = 21$

[**Ans :** Consistent and infinitely many solutions $x = 1,\ y = 3k - 2,\ z = k$]

c) $$\begin{aligned} x-y+2z&=4\\ 3x+y+4z&=6\\ x+y+z&=2 \end{aligned}$$

[**Ans :** Inconsistent]

d) $$\begin{aligned} x-4y+7z&=8\\ 3x+8y-2z&=6\\ 7x-8y+26z&=31 \end{aligned}$$

[**Ans :** Inconsistent]

e) $$\begin{aligned} x-4y-z&=3\\ 3x+y-2z&=7\\ 2x-3y+z&=10 \end{aligned}$$

$\left[\textbf{Ans: } \text{Consistent and unique solution } x=\frac{62}{17}, y=\frac{-5}{17}, z=\frac{31}{17}\right]$

f) $$\begin{aligned} 2x+y+4z&=4\\ x-3y-z&=5\\ 3x-2y+2z&=-1\\ -8x+3y-8z&=-2 \end{aligned}$$

[**Ans :** Consistent and unique solution $x=1,\ y=2,\ z=0$]

17. Examine whether the following homogeneous system of equations have nontrivial solutions and find them if they exist.

a) $$\begin{aligned} x+2y+3z&=0\\ 2x+3y+z&=0\\ x+y+2z&=0 \end{aligned}$$

[**Ans :** Only trivial solution]

b) $$\begin{aligned} x+2y+3z&=0\\ 3x+4y+5z&=0\\ 2x+3y+4z&=0 \end{aligned}$$

[**Ans :** Nontrivial solution $x=k,\ y=-2k,\ z=k$]

c) $$\begin{aligned} x-2y+z-w&=0\\ x+y-2z+3w&=0\\ 4x+y-5z+8w&=0 \end{aligned}$$

$\left[\textbf{Ans: } \text{Nontrivial solution } x=k_1-\frac{5}{3}k_2,\ y=k_1-\frac{4}{3}k_2,\ z=k_1,\ w=k_2\right]$

18. Find the values of a and b so that the following system of equations have (i) a unique solution, ii) no solution, and iii) infinitely many solutions.

a) $$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+az=b$$

[**Ans:** i) Unique solution for $a \neq 5, b=k$ (any constant) ii) No solution for $a=5, b \neq 9$ iii) Infinitely many solutions for $a=5, b=9$]

b) $$x+y+z=b$$
$$2x+y+3z=b+1$$
$$5x+2y+az=b^2$$

[**Ans :** i) unique solution for $a \neq 8, b=k$ (any constant) ii) no solution for $a=8, b \neq 3, -1$ iii) infinitely many solutions for $a=8, b=3, -1$]

c) $$3x-2y+z=b$$
$$5x-8y+9z=3$$
$$2x+y+az=-1$$

$$\left[\begin{array}{l}\textbf{Ans: } \text{i) Unique solution for } a \neq -3,\, b=k\,(\text{any constant}) \\ \quad \text{ii) No solution for } a=-3,\, b \neq \dfrac{1}{3} \\ \quad \text{iii) Infinitely many solutions for } a=-3,\, b=\dfrac{1}{3}\end{array}\right]$$

19. Find the eigen values and corresponding eigen vectors of the following matrices:

a) $$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\left[\begin{array}{l}\textbf{Ans: } \text{Eigen values}: \lambda = 1, 1, 4., \\ \text{Eigen vectors}: \text{for } \lambda = 1,\, k_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{and for } \lambda = 1,\ k_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\end{array}\right]$$

b) $$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

$$\left[\begin{array}{l}\textbf{Ans: } \text{Eigen values}: \lambda = 1, -1, 3, \\ \text{Eigen vectors}: \text{for } \lambda = 1,\, k_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{for } \lambda = -1,\, k_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{for } \lambda = 3,\, k_3\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}\end{array}\right]$$

20. Verify Cayley–Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and find A^{-1} and A^4.

21. If $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, by Cayley–Hamilton theorem show that $A^{-1} = A^3$.

22. Find the inverse of the following matrices by finding the characteristic equation (using Cayley–Hamilton theorem):

(a) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$\left[\textbf{Ans: } A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right]$$

(b) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$

$$\left[\textbf{Ans: } A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{7}{18} & \frac{-1}{18} & \frac{2}{9} \\ \frac{1}{9} & \frac{5}{18} & \frac{1}{9} \end{pmatrix}\right]$$

(c) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$ [WBUT-2005]

$$\left[\textbf{Ans: } A^{-1} = \begin{pmatrix} \frac{-2}{9} & \frac{5}{9} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} & 0 \\ \frac{5}{9} & \frac{1}{9} & \frac{-1}{3} \end{pmatrix}\right]$$

(d) $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

$$\left[\textbf{Ans: } A^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\right]$$

23. Determine eigen vectors of A and then diagonalise A with the help of the basis of eigen vectors:

(a) $A = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

24. Find the matrices P so that $P^{-1}AP$ is a diagonal matrix (i.e., find P which diagonalises the following matrices):

(i) $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$

$$\left[\textbf{Ans : } P = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

(ii) $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

$$\left[\textbf{Ans : } P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \right]$$

(iii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$

$$\left[\textbf{Ans : } P = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & -6 \\ 1 & 2 & 4 \end{pmatrix} \right]$$

Multiple-Choice Questions

1. The rank of the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ is

a) 2 b) 3 c) 4 d) none of these

2. For what value of λ does the system of equations $x + y + z = 1$; $x + 2y - z = 2$; $5x + 7y + \lambda z = 4$ have a unique solution?

a) $\lambda \neq 2$ b) $\lambda \neq 1$ c) $\lambda \neq 3$ d) $\lambda \neq 4$

3. The value of a for which rank of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 5 & a & 3 \\ 0 & 3 & 1 \end{pmatrix}$ is less than 3?

a) $\frac{3}{4}$ b) $\frac{3}{5}$ c) $\frac{3}{2}$ d) 1.

4. The value of k for which the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & k & 1 \\ 10 & 1 & 0 \end{pmatrix}$ is 2 is

a) 1 b) 0 c) -1 d) 2

5. The system of equations $x+2y-z=2;\ 4x+8y-4z=8$ has

a) infinite many solutions b) no solution
c) a unique solution d) none of these

6. The rank of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 6 & 6 & 3 \end{pmatrix}$ is

a) 2 b) 3 c) 1 d) none of these

7. The equation $x-y=0$ has

a) no solution b) exactly one solution
c) exactly two solutions d) infinite number of solutions

8. All the eigen values of any nilpotent matrix are

a) 0 b) 1 c) 2 d) none of these

9. The sum of the eigen values of $A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ is

a) 6 b) 5 c) 4 d) 7

[**Hint :** The trace of any square matrix is equal to the sum of the eigen values.]

10. The system of equations $x+y-3z=0;\ 3x-y-z=0;\ 2x+y-4z=0$ has

a) a nontrivial solution b) a trivial solution
c) no solution d) none of these

11. The eigen values of $A=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ are

a) 3, 2, 4 b) 5, 4, 6 c) 4, 3, 5 d) 7, 2, 9

12. The eigen values of $A=\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ are

a) 2, 4 b) 0, 4 c) 0, 2 d) 0, 0.

13. One of the eigen values of $A=\begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 2 \end{pmatrix}$ is

a) 16 b) 15 c) 0 d) 14.

14. λ be an eigen value of an $n\times n$ square matrix A then λ^{-1} is an eigen value of

a) A^2 b) $2A$ c) A^T d) A^{-1}

15. λ be an eigen value of an $n\times n$ square matrix A then 2λ is an eigen value of

a) A^2 b) $2A$ c) A^T d) A^{-1}

16. λ be an eigen value of an $n\times n$ square matrix A, then λ is also an eigen value of

a) A^2 b) $2A$ c) A^T d) A^{-1}

17. λ be an eigen value of an $n\times n$ square matrix A then λ^2 is an eigen value of

a) A^2 b) $2A$ c) A^T d) A^{-1}

18. If the sum of the eigen values of the matrix $A=\begin{pmatrix} 4 & 0 & 9 \\ 0 & a & 0 \\ 7 & 0 & 2 \end{pmatrix}$ is 7, then a is

a) 7 b) 1 c) 8 d) 2

19. If trace of a 3×3 matrix is 12, and two of its eigen values are 4, 6 then the third eigen value is

a) 2 b) 1 c) 8 d) 3

20. If $A^2 = A$ then its eigen values are either

a) 0 or 2 b) 0 or 1 c) 2 or 1 d) none of these.

Answers:

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (d)	8. (a)
9. (a)	10. (a)	11. (a)	12. (b)	13. (c)	14. (d)	15. (b)	16. (c)
17. (a)	18. (b)	19. (a)	20. (b)				

CHAPTER

3

Successive Differentiation

3.1 INTRODUCTION

Suppose we have a differentiable function $y = f(x)$ defined over an interval I. Then its first-order derivative is denoted by

$$\frac{dy}{dx}, f'(x), \frac{d}{dx}\big(f(x)\big), y', y_1.$$

Now suppose the first-order derivative is again differentiable on a certain interval. Then the second-order derivative is denoted by

$$\frac{d^2y}{dx^2}, f''(x), \frac{d^2}{dx^2}\big(f(x)\big), y'', y_2.$$

In this way we can find the higher-order derivatives, differentiating the functions again and again, if they exist.

Basically, this leads to the formation of the present chapter named as successive differentiation.

3.2 SUCCESSIVE DIFFERENTIATION

Successive differentiation of a function means differentiation of a function successively or repeatedly.

Suppose any function is given and you are to find its 100-th derivative, if it exists. Now the question is whether you can find it without having prior knowledge of the 1st, 2nd, ..., 99-th derivatives of the function. This means when you are finding derivatives of higher orders, you should know all of its previous-order derivatives. This is a very laborious and time-consuming job.

So, to meet the above difficulty, if we can find a general formula for the n-th order derivative of a particular function, if it exists, then by putting simply the value of n, we can get the derivative of any order as we require.

Keeping similarity with the notations as before, we denote the n-th order derivative of a function $y = f(x)$ by

$$\frac{d^n y}{dx^n},\ f^{(n)}(x),\ \frac{d^n}{dx^n}\big(f(x)\big),\ y^{(n)},\ y_n.$$

3.3 *n*-TH DERIVATIVE OF SOME IMPORTANT FUNCTIONS

(a) Let us consider $y = (ax+b)^m$, m is any number.

Then

$$y_1 = m\cdot(ax+b)^{m-1}\cdot a$$
$$y_2 = m(m-1)\cdot(ax+b)^{m-2}\cdot a^2$$
$$= m(m-\overline{2-1})\cdot(ax+b)^{m-2}\cdot a^2$$
$$y_3 = m(m-1)(m-2)\cdot(ax+b)^{m-3}\cdot a^3$$
$$= m(m-1)(m-\overline{3-1})\cdot(ax+b)^{m-3}\cdot a^3$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = m(m-1)(m-2)\dots(m-\overline{n-1})\cdot(ax+b)^{m-n}\cdot a^n,$$

if $n < m, m$ is any number.

Especially when m is any $+ve$ integer and $n < m$,

$$y_n = m(m-1)(m-2)\dots(m-\overline{n-1})\cdot(ax+b)^{m-n}\cdot a^n$$
$$= \frac{m!}{(m-n)!}\cdot a^n\cdot(ax+b)^{m-n}.$$

When $n = m$,

$$y_n = m(m-1)(m-2)\dots1\cdot(ax+b)^{m-m}\cdot a^m$$
$$= m!\cdot a^m$$
$$= n!\cdot a^n.$$

When $n > m$ and m is any $+ve$ integer,

$$y_n = 0.$$

Example 1 If $y = (2x+3)^6$, let us find y_4, y_6 and y_7.

Here, $m = 6$, a $+ve$ integer.

In the first case, $n = 4$. So $n < m$. Then

$$y_n = \frac{m!}{(m-n)!}\cdot a^n\cdot(ax+b)^{m-n}.$$

$$y_4 = \frac{6!}{(6-4)!} \cdot 2^4 \cdot (2x+3)^{6-4}$$
$$= \frac{6!}{2!} \cdot 2^4 \cdot (2x+3)^2$$

In the second case, $n = 6$. So $n = m$. Then

$$y_n = n! \cdot a^n.$$
$$y_6 = 6! \cdot 2^6.$$

In the third case, $n = 7$. So $n > m$. Then

$$y_7 = 0.$$

Example 2 If $y = x^{-2}$, let us find y_4.

Here, $m = -2$, a $-ve$ integer and $n = 4$.

$$y_n = m(m-1)(m-2)\ldots(m-\overline{n-1}) \cdot x^{m-n}$$
$$y_4 = (-2)(-2-1)(-2-2)(-2-3)x^{-2-4}$$
$$= 120 \cdot x^{-6}$$

Example 3 If $y = x^{\frac{3}{4}}$, let us find y_3.

Here, $m = \frac{3}{4}$, a fraction and $n = 3$.

$$y_n = m(m-1)(m-2)\ldots(m-\overline{n-1}) \cdot x^{m-n}$$
$$y_4 = \left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)\left(\frac{3}{4}-2\right)x^{\frac{3}{4}-3}$$

(b) Let us consider $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

Then

$$y_1 = (-1) \cdot (ax+b)^{-2} \cdot a$$
$$= (-1) \cdot (ax+b)^{-1-1} \cdot a$$
$$y_2 = (-1)(-2) \cdot (ax+b)^{-3} \cdot a^2$$
$$= (-1)^2 \cdot 2! \cdot (ax+b)^{-2-1} \cdot a^2$$
$$y_3 = (-1)(-2)(-3) \cdot (ax+b)^{-4} \cdot a^3$$
$$= (-1)^3 \cdot 3! \cdot (ax+b)^{-3-1} \cdot a^3$$

....

$$y_n = (-1)^n \cdot n! \cdot (ax+b)^{-n-1} \cdot a^n$$
$$= \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}.$$

Example 4 If $y = \frac{1}{3x+5}$

then

$$y_n = \frac{(-1)^n \cdot n! \cdot 3^n}{(3x+5)^{n+1}}$$

$$y_4 = \frac{(-1)^4 \cdot 4! \cdot 3^4}{(3x+5)^5}.$$

Example 5 If $y = \frac{1}{2x-3}$

then

$$y_n = \frac{(-1)^n \cdot n! \cdot 2^n}{(2x-3)^{n+1}}$$

$$y_3 = \frac{(-1)^3 \cdot 3! \cdot 2^3}{(2x-3)^4}.$$

(c) Let us consider $y = \log(ax+b)$

Then

$$y_1 = \frac{1}{ax+b} \cdot a = (ax+b)^{-1} \cdot a$$

$$\begin{aligned}
y_2 &= (-1) \cdot (ax+b)^{-2} \cdot a^2 \\
&= (-1) \cdot 1! \cdot (ax+b)^{-2} \cdot a^2 \\
&= (-1)^{2-1} \cdot (2-1)! \cdot (ax+b)^{-2} \cdot a^2 \\
y_3 &= (-1)(-2) \cdot (ax+b)^{-3} \cdot a^3 \\
&= (-1)^2 \cdot 2! \cdot (ax+b)^{-3} \cdot a^3 \\
&= (-1)^{3-1} \cdot (3-1)! \cdot (ax+b)^{-3} \cdot a^3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
y_n &= (-1)^{n-1} \cdot (n-1)! \cdot (ax+b)^{-n} \cdot a^n \\
&= \frac{(-1)^{n-1} \cdot (n-1)! \cdot a^n}{(ax+b)^n}.
\end{aligned}$$

Alternative Method

$$y = \log(ax+b)$$

so, $y_1 = \frac{1}{ax+b} \cdot a$

$$y_n = (n-1)\text{th derivative of } y_1$$
$$= (n-1)\text{th derivative of} \left(\frac{1}{ax+b} \cdot a \right)$$
$$= a \cdot \left\{ (n-1)\text{th derivative of} \left(\frac{1}{ax+b} \right) \right\}$$

i.e., $y_n = a \cdot \dfrac{(-1)^{n-1} \cdot (n-1)! \cdot a^{n-1}}{(ax+b)^n}$

$$= \frac{(-1)^{n-1} \cdot (n-1)! \cdot a^n}{(ax+b)^n}$$

Example 6 If $y = \log \dfrac{x^3}{2x+1}$

then

$$y = 3\log x - \log(2x+1)$$
$$y_n = 3 \cdot \frac{(-1)^{n-1} \cdot (n-1)!}{x^n} - \frac{(-1)^{n-1} \cdot (n-1)! \cdot 2^n}{(2x+1)^n}$$
$$y_6 = 3 \cdot \frac{(-1)^{6-1} \cdot (6-1)!}{x^6} - \frac{(-1)^{6-1} \cdot (6-1)! \cdot 2^6}{(2x+1)^6}$$
$$= 5! \left(\frac{2^6}{(2x+1)^6} - \frac{3}{x^6} \right).$$

(d) Let us consider $y = e^{(ax+b)}$

Then

$$y_1 = e^{(ax+b)} \cdot a$$
$$y_2 = e^{(ax+b)} \cdot a^2$$
$$y_3 = e^{(ax+b)} \cdot a^3$$
$$\cdots\cdots\cdots\cdots$$
$$y_n = e^{(ax+b)} \cdot a^n$$

Example 7 If $y = e^{(2x+3)}$ then

$$y_n = e^{(2x+3)} \cdot 2^n$$
$$y_5 = e^{(2x+3)} \cdot 2^5$$
$$= 2^5 \cdot e^{(2x+3)}.$$

(e) Let us consider $y = \cos(ax+b)$

Then

$$y_1 = -\sin(ax+b)\cdot a$$

$$= a\cdot\cos\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = -a^2\cdot\sin\left(\frac{\pi}{2}+ax+b\right)$$

$$= a^2\cdot\cos\left(2\cdot\frac{\pi}{2}+ax+b\right)$$

$$y_3 = -a^3\cdot\sin\left(2\cdot\frac{\pi}{2}+ax+b\right)$$

$$= a^3\cdot\cos\left(3\cdot\frac{\pi}{2}+ax+b\right)$$

....

$$y_n = a^n\cdot\cos\left(n\cdot\frac{\pi}{2}+ax+b\right).$$

Example 8 If $y=\sin^2(3x+4)$, find y_n and y_4.

Here,

$$y=\sin^2(3x+4)$$

$$=\frac{1}{2}[1-\cos 2(3x+4)]$$

$$=\frac{1}{2}[1-\cos(6x+8)]$$

$$y_n = \frac{1}{2}[1-\cos(6x+8)]_n$$

$$=-\frac{6^n}{2}\cdot\cos\left(n\cdot\frac{\pi}{2}+6x+8\right).$$

$$y_4 = -\frac{6^4}{2}\cdot\cos\left(4\cdot\frac{\pi}{2}+6x+8\right)$$

$$=-\frac{6^4}{2}\cos(6x+8)$$

(f) Let us consider $y=\sin(ax+b)$ [WBUT 2003]

Then

$$y_1 = \cos(ax+b)\cdot a$$

$$= a\cdot\sin\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = a^2 \cdot \cos\left(\frac{\pi}{2} + ax + b\right)$$

$$= a^2 \cdot \sin\left(2 \cdot \frac{\pi}{2} + ax + b\right)$$

$$y_3 = a^3 \cdot \cos\left(2 \cdot \frac{\pi}{2} + ax + b\right)$$

$$= a^3 \cdot \sin\left(3 \cdot \frac{\pi}{2} + ax + b\right)$$

....

$$y_n = a^n \cdot \sin\left(n \cdot \frac{\pi}{2} + ax + b\right).$$

Example 9 If $y = \sin(2x+3)\cos(2x+3)$, find y_n and y_8.

Here,

$$y = \sin(2x+3)\cos(2x+3)$$

$$= \frac{1}{2}\sin 2(2x+3)$$

$$= \frac{1}{2}\sin(4x+6)$$

$$y_n = \frac{1}{2}\left[\sin(4x+6)\right]_n$$

$$= \frac{4^n}{2} \cdot \sin\left(n \cdot \frac{\pi}{2} + 4x + 6\right).$$

$$y_8 = \frac{4^8}{2} \cdot \sin\left(8 \cdot \frac{\pi}{2} + 4x + 6\right)$$

$$= \frac{4^8}{2} \cdot \sin(4x+6)$$

3.4 n-TH ORDER DERIVATIVE OF PRODUCT OF TWO FUNCTIONS OF SAME VARIABLE

First, we recall the multiplication rule of the differentiation for finding $1st$ order derivative of the product of two functions.

Suppose u and v are two functions of x and $1st$ order derivative exists for them then $1st$ order derivative of $y = u \cdot v$ is given by

$$y_1 = u_1 \cdot v + u \cdot v_1.$$

Also, we can find the higher order derivatives of the product $u \cdot v$, differentiating repeatedly if they exist.

Now the question is whether we can have any general formula for finding directly the n-th order derivative of the product of two functions of the same variable. The answer is yes and it is given by the following theorem:

Leibnitz's Theorem

Suppose u and v are two functions of x and the n-th order derivative exists for both of them. Then the n-th order derivative of the product $y = u.v$ is given by

$$\boldsymbol{y_n = (u \cdot v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + u v_n.}$$

Proof: Beyond the scope of the book.

Note:

If we put $n = 1$ in the above formula, we obtain $y_1 = (u \cdot v)_1 = u_1 \cdot v + u \cdot v_1$, which is the multiplication rule of differentiation for finding $1st$ order derivative of the product of two functions. So it is clear that **Leibnitz's Theorem** is nothing but the generalised multiplication rule for finding higher-order derivatives.

Selection of *u* and *v*

In general, we can choose any function of the product as u or v, but if we carefully see formula, the order of the derivative of v increases term by term. So, among the two functions, which ever has more priority of vanishing at the higher order derivatives should be taken as v. Basically, the reason behind it is to make the calculation easier and, of course, to save time too.

Example 10 If $y = x^2 \log x$ then let us find y_n.

Here, we set $u = \log x$ and $v = x^2$.

Then

$$u_n = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$$

and

$$v_1 = 2x,\ v_2 = 2,\ v_3 = v_4 = \cdots = 0$$

i.e., $v_n = 0$, for $n > 2$

Here, it is obvious from the above that x^2 has the more priority over $\log x$ of vanishing at the higher order derivatives and due to that factor we have choosen x^2 as v.

Now using Leibnitz's theorem, we find the n-th derivative of $y = u \cdot v = \log x \cdot \left(x^2\right)$ as

$$y_n = (u \cdot v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + u v_n$$

i.e., $y_n = \left\{\log x \cdot \left(x^2\right)\right\}_n$

$$= \frac{(-1)^{n-1} \cdot (n-1)!}{x^n} \cdot (x^2) + {}^nC_1 \frac{(-1)^{n-2} \cdot (n-2)!}{x^{n-1}} \cdot (2x) + {}^nC_2 \frac{(-1)^{n-3} \cdot (n-3)!}{x^{n-2}} \cdot (2).$$

Example 11 If $y = e^x \sin x$ then let us find y_n.

Here, we set $u = e^x$ and $v = \sin x$.

Then

$$u_n = e^x \text{ and } v_n = \sin\left(\frac{n\pi}{2} + x\right)$$

Here, it is obvious from the above that no function has the priority over another of vanishing at the higher order derivatives and due to that we can choose u and v arbitrarily.

Now using Leibnitz's theorem, we find the *n*-th derivative of $y = u \cdot v = e^x \sin x$ as

$$y_n = (u \cdot v)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + u v_n$$

i.e., $y_n = \left\{e^x \cdot \sin x\right\}_n$

$$= e^x \cdot \sin x + {}^nC_1 \cdot e^x \cdot \sin\left(\frac{\pi}{2} + x\right) + {}^nC_2 \cdot e^x \cdot \sin\left(\frac{2\pi}{2} + x\right) + \cdots + e^x \cdot \sin\left(\frac{n\pi}{2} + x\right)$$

WORKED-OUT EXAMPLES

Example 3.1 If $y = \dfrac{x^2}{(x-1)(x-2)(x-3)}$, find y_n. [WBUT 2001]

Sol. Let us consider

$$y = \frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Substituting $x = 1,\ 2,\ 3$, we have respectively

$$1^2 = A(1-2)(1-3)$$

i.e., $A = \dfrac{1}{2}$

$$2^2 = B(2-1)(2-3)$$

i.e., $B = -4$.

and

$$3^2 = C(3-1)(3-2)$$

i.e., $C = \dfrac{9}{2}$

Therefore,

$$y=\frac{x^2}{(x-1)(x-2)(x-3)}=\frac{\frac{1}{2}}{(x-1)}-\frac{4}{(x-2)}+\frac{\frac{9}{2}}{(x-3)}$$

since $y_n=\dfrac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}$ when $y=\dfrac{1}{ax+b}$, we get from the above,

$$y_n=\frac{1}{2}\frac{(-1)^n\cdot n!}{(x-1)^{n+1}}-4\cdot\frac{(-1)^n\cdot n!}{(x-2)^{n+1}}+\frac{9}{2}\frac{(-1)^n\cdot n!}{(x-3)^{n+1}}.$$

Example 3.2 If $y=e^{ax}\cos bx$, prove that

$$y_n=\left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

Sol. Here, we have

$$y=e^{ax}\cos bx$$

Now differentiating w.r.t x,

$$y_1=a\cdot e^{ax}\cos bx-e^{ax}\cdot b\sin bx$$
$$=e^{ax}(a\cos bx-b\sin bx)$$

Consider $a=r\cos\theta$ and $b=r\sin\theta$ then

$r^2=a^2+b^2$ and $\theta=\tan^{-1}\dfrac{b}{a}$

So,

$$y_1=e^{ax}(r\cos\theta\cos bx-r\sin\theta\sin bx)$$
$$=r\cdot e^{ax}\cdot\cos(bx+\theta)$$

Again differentiating w.r.t x, we have

$$y_2=r\cdot e^{ax}\cdot a\cdot\cos(bx+\theta)-r\cdot e^{ax}\cdot\sin(bx+\theta)\cdot b$$
$$=r\cdot e^{ax}\cdot r\cos\theta\cdot\cos(bx+\theta)-r\cdot e^{ax}\cdot\sin(bx+\theta)\cdot r\sin\theta$$
$$=r^2\cdot e^{ax}\cdot\{\cos(bx+\theta)\cdot\cos\theta-\sin(bx+\theta)\cdot\sin\theta\}$$
$$=r^2\cdot e^{ax}\cdot\cos(bx+2\theta).$$

Proceeding similarly as above

$$y_n=r^n\cdot e^{ax}\cdot\cos(bx+n\theta)$$
$$y_n=\left(a^2+b^2\right)^{\frac{n}{2}}\cdot e^{ax}\cdot\cos\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

Example 3.3 If $y = e^{ax}\sin bx$, prove that

$$y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

Sol. Follow Example 3.2.

Example 3.4 If $y = x^{n-1}\log x$, prove that

$$y_n = \frac{(n-1)!}{x}.$$

Sol. Here, we have

$$y = x^{n-1}\log x$$

Now differentiating w.r.t. x,

$$y_1 = (n-1)\cdot x^{n-2}\cdot\log x + x^{n-1}\cdot\frac{1}{x}$$

i.e., $xy_1 = (n-1)\cdot x^{n-1}\cdot\log x + x^{n-1}$

i.e., $xy_1 = (n-1)\cdot y + x^{n-1}$

Now applying Leibnitz's rule, we differentiate $(n-1)$ times,

$$[y_1\cdot x]_{n-1} = [(n-1)\cdot y]_{n-1} + [x^{n-1}]_{n-1}$$

i.e., $\{y_1\}_{n-1}\cdot x + {}^{n-1}C_1\{y_1\}_{n-2}\cdot 1 = (n-1)y_{n-1} + (n-1)!$

i.e., $y_n\cdot x + (n-1)\cdot y_{n-1} = (n-1)y_{n-1} + (n-1)!$

i.e., $y_n\cdot x = (n-1)!$

Hence

$$y_n = \frac{(n-1)!}{x}.$$

Example 3.5 If $y = 2\cos x(\sin x - \cos x)$ then show that $(y_{10})_0 = 2^{10}$.

Sol. It is given that

$$y = 2\cos x(\sin x - \cos x)$$

$$= 2\cos x\sin x - 2\cos^2 x$$

$$= \sin 2x - \cos 2x - 1$$

Differentiating n times, we have

$$y_n = \{\sin 2x\}_n - \{\cos 2x\}_n$$

$$= 2^n \sin\left(n\frac{\pi}{2} + 2x\right) - 2^n \cos\left(n\frac{\pi}{2} + 2x\right)$$

$$= 2^n \left\{\sin\left(n\frac{\pi}{2} + 2x\right) - \cos\left(n\frac{\pi}{2} + 2x\right)\right\}$$

Now putting $x = 0$, we have

$$(y_n)_0 = 2^n \left\{\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right\}$$

So

$$(y_{10})_0 = 2^{10} \left\{\sin\left(\frac{10\pi}{2}\right) - \cos\left(\frac{10\pi}{2}\right)\right\}$$

$$= 2^{10} \{\sin 5\pi - \cos 5\pi\}$$

$$= 2^{10} \{0 - (-1)^5\}$$

Hence

$$(y_{10})_0 = 2^{10}.$$

Example 3.6 If $y = \dfrac{1}{x^2 - a^2}$, show that $y_n = \dfrac{(-1)^n \cdot n!}{2a}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}}\right]$

Sol. Here,

$$y = \frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)}$$

$$= \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right]$$

So differentiating n times, we have

$$y_n = \frac{1}{2a}\left[\left\{\frac{1}{(x-a)}\right\}_n - \left\{\frac{1}{(x+a)}\right\}_n\right]$$

$$= \frac{1}{2a}\left[\frac{(-1)^n \cdot n!}{(x-a)^{n+1}} - \frac{(-1)^n \cdot n!}{(x+a)^{n+1}}\right]$$

Hence,

$$y_n = \frac{(-1)^n \cdot n!}{2a}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right].$$

Example 3.7 If $y=\dfrac{1}{x^2+a^2}$, show that $y_n=\dfrac{(-1)^n\cdot n!}{2ia}\left[\dfrac{1}{(x-ia)^{n+1}}-\dfrac{1}{(x+ia)^{n+1}}\right]$

Sol. Here,

$$y=\frac{1}{x^2+a^2}=\frac{1}{(x+ia)(x-ia)}$$

$$=\frac{1}{2ia}\left[\frac{1}{(x-ia)}-\frac{1}{(x+ia)}\right]$$

So differentiating n times, we have

$$y_n=\frac{1}{2ia}\left[\left\{\frac{1}{(x-ia)}\right\}_n-\left\{\frac{1}{(x+ia)}\right\}_n\right]$$

$$=\frac{1}{2ia}\left[\frac{(-1)^n\cdot n!}{(x-ia)^{n+1}}-\frac{(-1)^n\cdot n!}{(x+ia)^{n+1}}\right]$$

Hence,

$$y_n=\frac{(-1)^n\cdot n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right].$$

Example 3.8 If $f(x)=\tan x$ and n is a $+ve$ integer, prove with the help of Leibnitz's theorem that $f^n(0)-{}^nC_2 f^{(n-2)}(0)+{}^nC_4 f^{(n-4)}(0)-\cdots=\sin\left(\dfrac{n\pi}{2}\right)$

[WBUT 2001]

Sol. We have

$$f(x)=\tan x=\frac{\sin x}{\cos x}$$

or, $f(x)\cdot\cos x=\sin x$.

Applying Leibnitz's theorem, we differentiate n times w.r.t x,

$$\{f(x)\cdot\cos x\}_n=\{\sin x\}_n$$

$$f^n(x)\cos x-{}^nC_1 f^{(n-1)}(x)(-\sin x)+{}^nC_2 f^{(n-2)}(x)(-\cos x)$$

$$+{}^nC_3 f^{(n-3)}(x)\sin x+{}^nC_4 f^{(n-4)}(x)\cos x+\cdots=\sin\left(\frac{n\pi}{2}+x\right)$$

Now, putting $x=0$ in the above, we get

$$f^n(0)-{}^nC_2 f^{(n-2)}(0)+{}^nC_4 f^{(n-4)}(0)-\cdots=\sin\left(\frac{n\pi}{2}\right).$$

Example 3.9 If $x+y=1$, prove that the n-th derivative of $x^n y^n$ is

$$n!\left\{y^n-\left({}^nC_1\right)^2 y^{n-1}x+\left({}^nC_2\right)^2 y^{n-2}x^2-\left({}^nC_3\right)^2 y^{n-3}x^3+\cdots(-1)^n x^n\right\}.$$

[WBUT 2002]

Sol. Let $u=x^n$ and $v=y^n=(1-x)^n$

Then $u_n=n!,\ \ v_n=(-1)^n\cdot n!$.

Also $u_r=\dfrac{n!}{(n-r)!}x^{n-r},\ \ r<n$

So,

$$(u\cdot v)_n=u_n v+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+\cdots+uv_n$$

$$\left(x^n y^n\right)_n=\left\{x^n(1-x)^n\right\}_n$$

$$=n!(1-x)^n+{}^nC_1\frac{n!}{1!}\cdot x\cdot n(1-x)^{n-1}(-1)$$

$$+{}^nC_2\frac{n!}{2!}\cdot x^2\cdot n(n-1)(1-x)^{n-2}(-1)^2+\cdots+x^n(-1)^n\cdot n!$$

$$=n!\left\{(1-x)^n-\left({}^nC_1\right)^2(1-x)^{n-1}x+\left({}^nC_2\right)^2(1-x)^{n-2}x^2\right.$$

$$\left.-\left({}^nC_3\right)^2(1-x)^{n-3}x^3+\cdots(-1)^n x^n\right\}$$

So,

$$\left(x^n y^n\right)_n=n!\left\{y^n-\left({}^nC_1\right)^2 y^{n-1}x+\left({}^nC_2\right)^2 y^{n-2}x^2\right.$$

$$\left.-\left({}^nC_3\right)^2 y^{n-3}x^3+\cdots(-1)^n x^n\right\}.$$

Example 3.10 If $y=\tan^{-1}x$ then show that

(i) $\left(1+x^2\right)y_1=1$

(ii) $\left(1+x^2\right)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0$

Find also the value of y_n at $x=0$. [WBUT 2003, 2005]

Sol. If we differentiate $y=\tan^{-1}x$ w.r.t x, then

$$y_1=\frac{1}{1+x^2}$$

i.e., $y_1\left(1+x^2\right)=1$ ...(1)

Applying Leibnitz's theorem, we differentiate n times w.r.t x,

$$\left[y_1\left(1+x^2\right)\right]_n=[1]_n$$

$$\Rightarrow \{y_1\}_n\cdot\left(1+x^2\right)+{}^nC_1\{y_1\}_{n-1}\cdot(2x)+{}^nC_2\{y_1\}_{n-2}\cdot(2)=0$$

$$\Rightarrow \left(1+x^2\right)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0 \quad ...(2)$$

Now we will find $(y_n)_0$.

From (1),

$$y_1=\frac{1}{1+x^2}$$

i.e., $(y_1)_0=1$

Also, from (1),

$$y_2=-\frac{2x}{\left(1+x^2\right)^2}$$

i.e., $(y_2)_0=0$.

Now putting $x=0$ in (2),

$$(y_{n+1})_0+n(n-1)(y_{n-1})_0=0$$

i.e., $(y_{n+1})_0=-n(n-1)(y_{n-1})_0$ ...(3)

If we put $n=3, 5, 7, \ldots$ in the above then

$$(y_4)_0=-3\cdot 2\cdot(y_2)_0=0, \text{ since } (y_2)_0=0$$

$$(y_6)_0=-5\cdot 4\cdot(y_4)_0=0, \text{ since } (y_4)_0=0$$

$$(y_8)_0=-7\cdot 6\cdot(y_6)_0=0, \text{ since} (y_6)_0=0$$

and so on.

Therefore, $(y_n)_0=0$, when n is even.

Again putting $n=2, 4, 6, \ldots$ in the relation (3), we have

$$(y_3)_0=-2\cdot 1\cdot(y_1)_0=(-1)\cdot 2!, \text{ since } (y_1)_0=1$$

$$(y_5)_0=-4\cdot 3\cdot(y_3)_0=(-1)^2\cdot 4!, \text{ since } (y_3)_0=(-1)\cdot 2!$$

$$(y_7)_0=-6\cdot 5\cdot(y_5)_0=(-1)^3\cdot 6!, \text{ since} (y_5)_0=(-1)^2\cdot 4!$$

Therefore, $(y_n)_0=(-1)^{\frac{n-1}{2}}\cdot(n-1)!$, when n is odd.

Hence, we can say

$$(y_n)_0 = (-1)^{\frac{n-1}{2}} \cdot (n-1)!, \text{ when } n \text{ is odd}$$

$$= 0, \text{ when } n \text{ is even.}$$

Example 3.11 Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right).$$ [WBUT 2003, 2008].

Sol. Here we are to find

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{d^n}{dx^n}\left(\frac{1}{x}\cdot \log x\right)$$

$$= \left(\frac{1}{x}\cdot \log x\right)_n$$

We set $u = \frac{1}{x}$, then $u_n = \frac{(-1)^n \cdot n!}{x^{n+1}}$ (using **(b)** of **Art. 3.3**) and

if $v = \log x$, then $v_n = \frac{(-1)^{n-1}\cdot (n-1)!}{x^n}$ (using **(c)** of **Art. 3.3**)

Now using Leibnitz's theorem, we have

$$(u\cdot v)_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \cdots + uv_n$$

i.e., $$\left\{\frac{1}{x}\cdot \log x\right\}_n = \frac{(-1)^n \cdot n!}{x^{n+1}}\cdot \log x + {}^nC_1 \frac{(-1)^{n-1}\cdot (n-1)!}{x^n}\cdot \frac{1}{x}$$

$$+ {}^nC_2 \frac{(-1)^{n-2}\cdot (n-2)!}{x^{n-1}}\cdot\left(-\frac{1}{x^2}\right) + \cdots + \frac{1}{x}\cdot \frac{(-1)^{n-1}\cdot (n-1)!}{x^n}$$

i.e., $$\left\{\frac{1}{x}\cdot \log x\right\}_n = \frac{(-1)^n \cdot n!}{x^{n+1}}\log x - \frac{(-1)^n\cdot n!}{x^{n+1}} - \frac{(-1)^n \cdot n!}{x^{n+1}}\cdot\frac{1}{2}$$

$$- \cdots - \frac{(-1)^n\cdot n!}{x^{n+1}}\cdot \frac{1}{n}.$$

i.e., $$\left\{\frac{1}{x}\cdot \log x\right\}_n = (-1)^n \frac{n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right).$$

i.e., $$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right).$$

Example 3.12 Show that if $u = \sin ax + \cos ax$ then

$$D^n u = a^n \left\{1 + (-1)^n \sin 2ax\right\}^{\frac{1}{2}}, \text{ where } D \equiv \frac{d}{dx}.$$ [WBUT 2003].

Sol. It is given that

$$u = \sin ax + \cos ax$$

Now differentiating n times, we have

$$u_n = \{\sin ax\}_n + \{\cos ax\}_n$$

Using **(e)** and **(f)** of **Art. 3.3**,

$$u_n = a^n \sin\left(\frac{n\pi}{2} + ax\right) + a^n \cos\left(\frac{n\pi}{2} + ax\right)$$

$$= a^n \left\{\sin\left(\frac{n\pi}{2} + ax\right) + \cos\left(\frac{n\pi}{2} + ax\right)\right\}$$

Squaring both sides, we have

$$(u_n)^2 = a^{2n} \left\{\sin\left(\frac{n\pi}{2} + ax\right) + \cos\left(\frac{n\pi}{2} + ax\right)\right\}^2$$

$$= a^{2n} \left\{1 + 2\sin\left(\frac{n\pi}{2} + ax\right) \cdot \cos\left(\frac{n\pi}{2} + ax\right)\right\}$$

$$= a^{2n} \left\{1 + \sin 2\left(\frac{n\pi}{2} + ax\right)\right\}$$

$$= a^{2n} \{1 + \sin(n\pi + 2ax)\}$$

$$= a^{2n} \left\{1 + (-1)^n \sin 2ax\right\}$$

Hence,

$$u_n = a^n \left\{1 + (-1)^n \sin 2ax\right\}^{\frac{1}{2}}.$$

Example 3.13 If $y = \cos\left(m \sin^{-1} x\right)$ then prove that

$$\left(1 - x^2\right) y_{n+2} - (2n+1) x y_{n+1} + \left(m^2 - n^2\right) y_n = 0.$$

Also, find $(y_n)_0.$ [WBUT 2004]

Sol. We have

$$y = \cos\left(m \sin^{-1} x\right)$$

i.e., $\cos^{-1} y = m \sin^{-1} x$

Now differentiating w.r.t x,

$$-\frac{1}{\sqrt{1-y^2}} \cdot y_1 = m \cdot \frac{1}{\sqrt{1-x^2}}$$

Squaring both sides, we have

$$(y_1)^2 \cdot \left(1-x^2\right) = m^2\left(1-y^2\right) \qquad \text{...(1)}$$

Differentiating (1) w.r.t x,

$$2 \cdot y_1 \cdot y_2 \cdot \left(1-x^2\right) + (y_1)^2 \cdot (-2x) = m^2(-2yy_1)$$

Cancelling $-2y_1$ from both sides,

$$y_2.(1-x^2) - xy_1 + m^2 y = 0. \qquad \text{...(2)}$$

Now Applying Leibnitz's rule, we differentiate (2), n times

$$\left[y_2 \cdot \left(1-x^2\right)\right]_n - [y_1 \cdot x]_n + \left[m^2 y\right]_n = 0.$$

i.e., $\left[\{y_2\}_n \cdot \left(1-x^2\right) + {}^nC_1\{y_2\}_{n-1} \cdot (-2x) + {}^nC_2\{y_2\}_{n-2} \cdot (-2)\right]$

$$-\left[\{y_1\}_n \cdot x + {}^nC_1\{y_1\}_{n-1} \cdot 1\right] + m^2 y_n = 0$$

i.e., $\left[y_{n+2} \cdot \left(1-x^2\right) - 2nx \cdot y_{n+1} - n(n-1)y_n\right] - [x \cdot y_{n+1} + n \cdot y_n] + m^2 y_n = 0$

i.e., $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + \left(m^2 - n^2\right)y_n = 0$ ...(3)

Now we will find $(y_n)_0$.

Putting $x = 0$ in $y = \cos\left(m \sin^{-1} x\right)$, we get $(y)_0 = 1$.

We have from (1), by putting $x = 0$

i.e., $(y_1)_0 = 0$

Again from (2), by putting $x = 0$

i.e., $(y_2)_0 = -m^2$.

Now putting $x = 0$ in (3),

$$(y_{n+2})_0 + \left(m^2 - n^2\right)(y_n)_0 = 0$$

i.e., $(y_{n+2})_0 = \left(n^2 - m^2\right)(y_n)_0$ ...(4)

If we put $n = 1, 3, 5, \dots$ in the above then

$$(y_3)_0 = \left(1^2 - m^2\right)\cdot(y_1)_0 = 0, \text{ since } (y_1)_0 = 0$$

$$(y_5)_0 = \left(3^2 - m^2\right)\cdot(y_3)_0 = 0, \text{ since } (y_3)_0 = 0$$

$$(y_7)_0 = \left(5^2 - m^2\right)\cdot(y_5)_0 = 0, \text{ since } (y_5)_0 = 0$$

and so on.

Therefore, $(y_n)_0 = 0$, when n is odd.

Again putting $n = 2, 4, 6, \dots$ in the relation (4), we have

$$(y_4)_0 = \left(2^2 - m^2\right)\cdot(y_2)_0 = -m^2\left(2^2 - m^2\right), \text{ since } (y_2)_0 = -m^2$$

$$(y_6)_0 = \left(4^2 - m^2\right)\cdot(y_4)_0 = -m^2\left(2^2 - m^2\right)\left(4^2 - m^2\right),$$

since $(y_4)_0 = -m^2\left(2^2 - m^2\right)$

Similarly, $(y_8)_0 = \left(6^2 - m^2\right)\cdot(y_6)_0 = -m^2\left(2^2 - m^2\right)\left(4^2 - m^2\right)\left(6^2 - m^2\right)$

and so on.

Therefore, $(y_n)_0 = -m^2\left(2^2 - m^2\right)\left(4^2 - m^2\right)\left(6^2 - m^2\right)\dots\left[(n-2)^2 - m^2\right]$.

when n is even

Hence we can say

$(y_n)_0 = 0$, when n is odd

$= -m^2\left(2^2 - m^2\right)\left(4^2 - m^2\right)\left(6^2 - m^2\right)\dots\left[(n-2)^2 - m^2\right]$,

when n is even.

Example 3.14 If $y = \left(x^2 - 1\right)^n$ then prove that

$\left(x^2 - 1\right)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$ [WBUT 2006]

Sol. We have

$$y = \left(x^2 - 1\right)^n$$

Now differentiating w.r.t x,

$$y_1 = n\cdot\left(x^2 - 1\right)^{n-1}\cdot 2x$$

i.e., $\left(x^2-1\right)\cdot y_1 = 2nx\cdot\left(x^2-1\right)^n$

i.e., $\left(x^2-1\right)\cdot y_1 = 2nxy$

Again differentiating w.r.t x,

$\left(x^2-1\right)\cdot y_2 + 2xy_1 = 2n(y+xy_1)$

i.e., $\left(x^2-1\right)\cdot y_2 + 2x(1-n)y_1 - 2ny = 0$

Now Applying Leibnitz's rule, we differentiate n times

$$\left[y_2\cdot\left(x^2-1\right)\right]_n + 2(1-n)[y_1\cdot x]_n - 2n[y]_n = 0.$$

i.e., $\left[\{y_2\}_n\cdot\left(x^2-1\right) + {}^nC_1\{y_2\}_{n-1}\cdot(2x) + {}^nC_2\{y_2\}_{n-2}\cdot(2)\right]$

$\qquad +2(1-n)\left[\{y_1\}_n\cdot x + {}^nC_1\{y_1\}_{n-1}\cdot 1\right] - 2ny_n = 0$

i.e., $\left\{y_{n+2}\cdot\left(x^2-1\right) + 2nx\cdot y_{n+1} + n(n-1)y_n\right\}$

$\qquad +2(1-n)\{x\cdot y_{n+1} + n\cdot y_n\} - 2ny_n = 0$

i.e., $\left(x^2-1\right)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

Example 3.15 If $y_n = \dfrac{d^n}{dx^n}\left(x^n \log x\right)$ then prove that

$y_n = ny_{n-1} + (n-1)!$ [WBUT 2007]

Sol. Here,

$$y_n = \frac{d^n}{dx^n}\left(x^n \log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\cdot\left\{\frac{d}{dx}\left(x^n \log x\right)\right\}$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(nx^{n-1}\log x + x^n\cdot\frac{1}{x}\right)$$

So,

$$y_n = \frac{d^{n-1}}{dx^{n-1}}\left(nx^{n-1}\log x\right) + \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right)$$

$$= n\cdot\frac{d^{n-1}}{dx^{n-1}}\left(x^{n\;1}\log x\right) + (n-1)!$$

i.e., $y_n = ny_{n-1} + (n-1)!$.

Example 3.16 If $y = a\cos(\log x) + b\sin(\log x)$, prove that

$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left(n^2+1\right)y_n = 0$ [WBUT 2007]

Sol. Here

$$y = a\cos(\log x) + b\sin(\log x)$$

Now differentiating w.r.t x,

$$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

i.e., $xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again differentiating w.r.t x,

$$y_1 + xy_2 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

i.e., $y_1 + xy_2 = \left(-\frac{1}{x}\right)\{a\cos(\log x) + b\sin(\log x)\}$

So,

$$xy_1 + x^2 y_2 = -y$$

i.e., $x^2 y_2 + xy_1 + y = 0$

Now Applying Leibnitz's rule, we differentiate n times.

$$\left[y_2\cdot x^2\right]_n + [y_1\cdot x]_n + [y]_n = 0.$$

i.e., $\left[\{y_2\}_n\cdot x^2 + {}^nC_1\cdot\{y_2\}_{n-1}(2x) + {}^nC_2\{y_2\}_{n-2}\cdot(2)\right]$

$$+\left[\{y_1\}_n\cdot x + {}^nC_1\{y_1\}_{n-1}\cdot 1\right] + y_n = 0$$

i.e., $\left\{y_{n+2}\cdot x^2 + 2nx\cdot y_{n+1} + n(n-1)y_n\right\} + \{x\cdot y_{n+1} + n\cdot y_n\} + y_n = 0$

Hence

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left(n^2+1\right)y_n = 0$$

Example 3.17 With the help of result obtained by differentiating n times x^{2n} in two different ways, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2\cdot(n-1)^2}{1^2\cdot 2^2} + \frac{n^2\cdot(n-1)^2\cdot(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \cdots = \frac{(2n)!}{(n!)^2}.$$

Sol. Let $y = x^{2n}$ then

$$y_n = \frac{(2n)!}{(2n-n)!} x^{2n-n} = \frac{(2n)!}{n!} x^n. \qquad ...(1)$$

Now again we can write

$$y = x^{2n} = x^n \cdot x^n$$

Now applying Leibnitz's rule, we differentiate n times.

$$y_n = \left[x^n \cdot x^n\right]_n$$

$$= \left\{x^n\right\}_n \cdot x^n + {}^nC_1 \left\{x^n\right\}_{n-1} \cdot n \cdot x^{n-1} + {}^nC_2 \left\{x^n\right\}_{n-2} \cdot n(n-1) \cdot x^{n-2}$$

$$+ {}^nC_3 \left\{x^n\right\}_{n-3} \cdot n(n-1)(n-2) \cdot x^{n-3} + \cdots$$

i.e. $y_n = n! \cdot x^n + n \cdot \frac{n!}{1!} \cdot x \cdot n \cdot x^{n-1} + \frac{n(n-1)}{2} \cdot \frac{n!}{2!} \cdot x^2 \cdot n(n-1) \cdot x^{n-2}$

$$+ \frac{n(n-1)(n-2)}{3!} \cdot \frac{n!}{3!} \cdot x^3 \cdot n(n-1)(n-2) \cdot x^{n-3} + \cdots$$

i.e. $y_n = n! \cdot x^n \left\{1 + \frac{n^2}{1^2} + \frac{n^2 \cdot (n-1)^2}{1^2 \cdot 2^2} + \frac{n^2 \cdot (n-1)^2 \cdot (n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \cdots\right\}$ ...(2)

From (1) and (2), we have

$$n! \cdot x^n \left\{1 + \frac{n^2}{1^2} + \frac{n^2 \cdot (n-1)^2}{1^2 \cdot 2^2} + \frac{n^2 \cdot (n-1)^2 \cdot (n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \cdots\right\} = \frac{(2n)!}{n!} x^n$$

Hence,

$$1 + \frac{n^2}{1^2} + \frac{n^2 \cdot (n-1)^2}{1^2 \cdot 2^2} + \frac{n^2 \cdot (n-1)^2 \cdot (n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \cdots = \frac{(2n)!}{(n!)^2}.$$

EXERCISES

Short and Long Answer Type Questions

1. Find the n-th derivative y_n of the following functions:

(i) $y = (a-bx)^{\frac{1}{2}}$

$$\left[\textbf{Ans:} (-1)^n b^n \left\{\frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2^n}\right\} \cdot (a-bx)^{-n+\frac{1}{2}}\right]$$

(ii) $y = x^{\frac{-1}{2}}$

$$\left[\textbf{Ans:} (-1)^n \left\{\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n}\right\} \cdot x^{-n-\frac{1}{2}}\right]$$

(iii) $y = x^{\frac{3}{4}}$

$$\left[\textbf{Ans}: (-1)^{n-1}\frac{3\cdot1\cdot5\cdot9\ldots(4n-7)}{4^n}\cdot x^{\frac{3-4n}{4}}\right]$$

(iv) $y = \dfrac{1}{(5x-7)}$

$$\left[\textbf{Ans}: \frac{(-1)^n\cdot n!\cdot 5^n}{(5x-7)^{n+1}}\right]$$

(v) $y = a^{3-4x}$

$$\left[\textbf{Ans}: a^3\cdot(-1)^n\cdot 4^n(\log a)^n\cdot a^{-4x}\right]$$

(vi) $y = \log(2x+9)$

$$\left[\textbf{Ans}: \frac{(-1)^{n-1}\cdot(n-1)!\cdot 2^n}{(2x+9)^n}\right]$$

(vii) $y = \log\dfrac{a+x}{a-x}$

$$\left[\textbf{Ans}: (n-1)!\left\{\frac{1}{(a-x)^n}-\frac{(-1)^n}{(a+x)^n}\right\}\right]$$

(viii) $y = \sin^3 x.$

$$\left[\textbf{Ans}: \frac{1}{4}\left\{3\sin\left(\frac{n\pi}{2}+x\right)-3^n\sin\left(\frac{n\pi}{2}+3x\right)\right\}\right]$$

(ix) $y = e^{2x}\sin x\sin 2x$

$$\left[\textbf{Ans}: \frac{1}{2}e^x\left\{2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}+x\right)+10^{\frac{n}{2}}\cos\left(3x+n\tan^{-1}3\right)\right\}\right]$$

(x) $y = \dfrac{1}{(x-1)^3(x-2)}$

$$\left[\textbf{Ans}: (-1)^{n-1}n!\left\{\frac{(n+2)(n+1)}{2(x-1)^{n+3}}+\frac{(n+1)}{(x-1)^{n+2}}+\frac{1}{2(x-1)^{n+1}}+\frac{1}{(x-2)^{n+1}}\right\}\right]$$

(xi) $y = \dfrac{x^2}{(x-1)(x-2)(x-3)}$ [WBUT 2001]

$$\left[\textbf{Ans}: \frac{1}{2}\frac{(-1)^n\cdot n!}{(x-1)^{n+1}}-4\cdot\frac{(-1)^n\cdot n!}{(x-2)^{n+1}}+\frac{9}{2}\frac{(-1)^n\cdot n!}{(x-3)^{n+1}}\right]$$

(xii) $y=\dfrac{1}{x^2+16}$ [WBUT 2001]

$$\left[\textbf{Ans: } \frac{(-1)^n\cdot n!}{8i}\left[\frac{1}{(x-4i)^{n+1}}-\frac{1}{(x+4i)^{n+1}}\right]\right]$$

2. Find the n-th derivative y_n of the following functions using Leibnitz's theorem:

(i) $y=e^x\log x$

$$\left[\textbf{Ans: } e^x\left\{\log x+{}^nC_1\cdot\frac{1}{x}-{}^nC_2\cdot\frac{1}{x^2}+{}^nC_3\cdot\frac{2!}{x^3}+\cdots+\frac{(-1)^{n-1}(n-1)!}{x^n}\right\}\right]$$

(ii) $y=x^3\cos x$

$$\left[\begin{aligned}\textbf{Ans: } & x^3\left(\cos\frac{n\pi}{2}+x\right)+{}^nC_1\cdot\cos\left\{\frac{(n-1)\pi}{2}+x\right\}\cdot 3x^2\\ & +{}^nC_2\cdot\cos\left\{\frac{(n-2)\pi}{2}+x\right\}\cdot 6x+{}^nC_3\cdot\cos\left\{\frac{(n-3)\pi}{2}+x\right\}\cdot 6\end{aligned}\right]$$

(iii) $y=x^ne^x$

$$\left[\textbf{Ans: } e^x\left[x^n+{}^nC_1\cdot nx^{n-1}+{}^nC_2\cdot n(n-1)x^{n-2}+\cdots+n!\right]\right]$$

(iv) $y=x^n(1-x)^n$

$$\left[\textbf{Ans: } n!\left[(1-x)^n+\left({}^nC_1\right)^2\cdot x(1-x)^{n-1}+\left({}^nC_2\right)^2\cdot x^2(1-x)^{n-2}+\cdots+x^n\right]\right]$$

(v) $y=e^x\cos x$

$$\left[\textbf{Ans: } e^x\left[\cos x+{}^nC_1.\cos\left\{\frac{\pi}{2}+x\right\}+{}^nC_2.\cos\left\{2.\frac{\pi}{2}+x\right\}+\ldots+\cos\left\{\frac{n\pi}{2}+x\right\}\right]\right]$$

(vi) $y=x^2\tan^{-1}x$

$$\left[\begin{aligned}\textbf{Ans: } y_n &=(-1)^{n-1}(n-3)!\sin^{n-2}\theta\\ &\{(n-1)(n-2)\sin n\theta\cos^2\theta\\ &-2n(n-2)\sin(n-1)\theta\cos\theta\\ &+n(n-1)\sin(n-2)\theta\},\text{ where }\cot\theta=x\end{aligned}\right]$$

(vii) $y=x\log\dfrac{x-1}{x+1}$.

$$\left[\textbf{Ans: } y_n=(-1)^n(n-2)!\left\{\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right\}\right]$$

3. If $y=2\cos x(\sin x-\cos x)$, then show that $(y_{10})_0=2^{10}$. [WBUT 2001]

4. If $y = e^{\tan^{-1} x}$, prove that

(i) $\left(1+x^2\right)y_2 + (2x-1)y_1 = 0$

(ii) $\left(1+x^2\right)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$

5. If $f(x) = \tan x$ and n is a $+ve$ integer, prove with the help of Leibnitz's theorem that $f^n(0) - {}^nC_2 f^{(n-2)}(0) + {}^nC_4 f^{(n-4)}(0) - \cdots = \sin\left(\frac{n\pi}{2}\right)$ [WBUT 2001]

6. If $y = \sin\left(m\sin^{-1} x\right)$, prove that

(i) $\left(1-x^2\right)y_2 - xy_1 + m^2 y = 0$

(ii) $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 - m^2\right)y_n = 0$

7. If $y = \left(\sin^{-1} x\right)^2$, prove that

(i) $\left(1-x^2\right)y_2 - xy_1 - 2 = 0$

(ii) $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

8. If $y = \tan^{-1}x$ then show that

(i) $\left(1+x^2\right)y_1 = 1$

(ii) $\left(1+x^2\right)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

Find also the value of y_n at $x = 0$. [WBUT 2003, 2005]

9. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that

(i) $\left(x^2-1\right)y_2 + xy_1 - m^2 y = 0$

(ii) $\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2 - m^2\right)y_n = 0$

10. If $x + y = 1$, prove that the n-th derivative of $x^n y^n$ is

$$n!\left\{y^n - \left({}^nC_1\right)^2 y^{n-1}x + \left({}^nC_2\right)^2 y^{n-2}x^2 - \left({}^nC_3\right)^2 y^{n-3}x^3 + \cdots (-1)^n x^n\right\}.$$

[WBUT 2002]

11. If $y = \sqrt{\frac{1+x}{1-x}}$, prove that $\left(1-x^2\right)y_n - \{2(n-1)x+1\}y_{n-1} - (n-1)(n-2)y_{n-2} = 0$

12. If $y=\log\left(x+\sqrt{1+x^2}\right)$, prove that $\left(x^2-1\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

13. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right)=(-1)^n\dfrac{n!}{x^{n+1}}\left(\log x-1-\dfrac{1}{2}-\dfrac{1}{3}-\cdots-\dfrac{1}{n}\right)$.

[WBUT 2003, 2008]

14. If $y=\dfrac{x^3}{x^2-1}$, prove that $(y_n)_0=\begin{Bmatrix}0, & \text{if } n \text{ is even}\\ -n!, & \text{if } n \text{ is odd}\end{Bmatrix}$ for $n>1$.

15. If $y=\left(x^2-1\right)^n$ then prove that $\left(x^2-1\right)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

[WBUT 2006, 2009]

16. If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $\left(1-x^2\right)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2y_n=0$

17. If $y=e^{m\cos^{-1}x}$, prove that $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+m^2\right)y_n=0$

18. If $y_n=\dfrac{d^n}{dx^n}\left(x^n\log x\right)$ then prove that $y_n=ny_{n-1}+(n-1)!$ [WBUT 2007]

19. If $\cos^{-1}\left(\dfrac{y}{b}\right)=\log\left(\dfrac{x}{n}\right)^n$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$

20. If $y=a\cos(\log x)+b\sin(\log x)$, prove that

$x^2y_{n+2}+(2n+1)xy_{n+1}+\left(n^2+1\right)y_n=0$ [WBUT 2007]

Multiple-Choice Questions

1. If $y=e^{-3x}$ then y_n is given by

a) e^{-3x} b) $(-3)^ne^{-3x}$ c) $(-3)^n$ d) none.

2. If $y=3^{-5x}$ then y_n is given by

a) $(-1)^n5^n(\log 3)^n3^{-5x}$ b) $(-1)^n(\log 3)^n3^{-5x}$
c) $5^n(\log 3)^n3^{-5x}$ d) none

3. The n-th derivative of $(ax+b)^{10}$ when $n>10$ is

a) a^{10} b) $10!a^{10}$ c) 0 d) 10! [WBUT 2007]

4. If $y = x\cos x$ then y_n is given by

a) $x\cos\left(n\frac{\pi}{2}+x\right)$ b) $x\cos\left(n\frac{\pi}{2}+x\right)+n\cos\left(\overline{n-1}\frac{\pi}{2}+x\right)$

c) $\cos\left(n\frac{\pi}{2}+x\right)+n\cos\left(\overline{n-1}\frac{\pi}{2}+x\right)$ d) none

5. If $y = \sin 4x + \cos 4x$ then $(y_n)_0$ is given by

a) $(-1)^n 4^n$ b) 4^n c) 0 d) none

6. If $y = \cos^2 4x$ then $(y_4)_0$ is given by

a) 1 b) 4 c) 0 d) none

7. If $y = e^{ax+b}$ then $(y_6)_0$ is given by

a) ae^b b) a^6e^b c) a^6e^{a+b} d) none

8. If $y = \log\sqrt{\frac{1+x}{1-x}}$ then $(y_n)_0$ is given by

a) 0 b) 1, when n is even and -1, when n is odd

c) -1, when n is even and 0, when n is odd d) none

9. If $y = \sqrt{a^2 - x^2}$ then $\left(yy_2 + y_1^2\right)$ is

a) 2 b) 1 c) 0 d) none

10. If $y = \frac{x^n}{x-1}$ then $(y_n)_0$ is

a) $-(n!)$ b) $(-1)^n$ c) $(n!)$ d) none

11. If $y = e^{3x}\sin 4x$ then y_n is given by

a) $5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$ b) $5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)e^{3x}$

c) $\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)e^{3x}$ d) none

12. If $y = \cos^2 4x + x^4$ then $(y_n)_0$ is given by

a) 120 b) $4^n \sin\left(4 + n\tan^{-1}\frac{4}{3}\right)$

c) 0 d) none

13. If $y = \cos x\cos 3x$ then $(y_5)_0$ is given by

a) 3 b) 15 c) 0 d) none

14. If $y = \log(2+3x)$ then $(y_6)_0$ is given by

a) $\dfrac{-(5!)\cdot 2^6}{3^6}$ b) $\dfrac{-(5!)\cdot 3^6}{2^6}$ c) 1 d) none

15. If $y = \dfrac{1}{(3x+5)}$ then $(y_4)_0$ is given by

a) $\dfrac{3^4}{5^5}$ b) $\dfrac{-(4!)\cdot 3^4}{5^5}$ c) $\dfrac{4!\cdot 3^4}{5^5}$ d) none

Answers:

1. b	2. a	3. c	4. b	5. b	6. d	7. b	8. c	9. d
10. a	11. b	12. d	13. c	14. b	15. c			

CHAPTER

4

Mean Value Theorems and Expansion of Functions

4.1 INTRODUCTION

There are some real-valued functions being continuous and derivable on a certain interval, which possess some special properties at any point lying in between boundary points of that interval. Mean-value theorems are such theorems which involve some particular results as stated above.

Basically, in this chapter we discuss the very well-known three mean-value theorems, namely, **Rolle's, Lagrange's and Cauchy's mean-value theorems** along with their wide range of applications in various fields.

In this chapter, we also deal with some series expansion theorems and formulas, namely, **Taylor's and Maclaurin's series expansion** and their application towards some standard functions like e^x, $\sin x, \log(1+x)$, etc.

4.2 ROLLE'S THEOREM

4.2.1 Statement

Let $f: I \to R$ be a real-valued function where $I = [a, b]$ and *f* satisfies the following conditions:

i) *f* is continious in the closed interval $[a, b]$

ii) f is derivable in the open interval (a, b), i.e., $f'(x)$ exists for $x \in (a, b)$ and

iii) $f(a) = f(b)$

Then there exists at least one value of x (say c), $c \in (a, b)$, i.e., $a < c < b$ such that, $f'(c) = 0$. [WBUT-2003]

Proof: Since the function f is continuous in the closed interval $[a, b]$, it is also bounded there. Let us consider that m and M are the greatest lower bound $(g.l.b)$ and least upper bound $(l.u.b)$ respectively for the function f.

Then there exists two points c and d in $[a, b]$ such that $f(c) = m$ and $f(d) = M$.

Now two cases may arise:

Case i) $m = M$

In this case, the function $f(x) = m$ is constant for all $[x \in a, b]$ and correspondingly, the derivative $f'(x) = 0$ for all $[x \in a, b]$.

Hence, the result is proved.

Case ii) $m \neq M$

Since in this case $f(a) = f(b)$ and $m \neq M$, at least one of m and M is different from $f(a)$ or $f(b)$.

Suppose $m \neq f(a)$; then $f(c) \neq f(a) \Rightarrow c \neq a$

And $m \neq f(b)$; then $f(c) \neq f(b) \Rightarrow c \neq b$

So c is neither a nor b. Therefore, $a < c < b$

By hypothesis, f is derivable in the open interval (a, b), i.e., $f'(c)$ exists for $a < c < b$.

Now it remains to prove $f'(c) = 0$.

For this purpose, first consider $f'(c) < 0$. Then there exists an interval $(c, c + h_1)$, $h_1 > 0$, for every point $x \in (c, c + h_1)$, $f(x) < f(c) = m$, which contradicts the fact that m is the greatest lower bound $(g \cdot l \cdot b)$

Next we consider $f'(c) > 0$. Then there exists an interval $(c - h_2, c)$, $h_2 > 0$, for every point $x \in (c - h_2, c)$, $f(x) < f(c) = m$, which is again a contradiction as before.

So the only possiblity is that $f'(c) = 0$ for $a < c < b$.

Hence, the theorem is proved.

Note: The theorem asserts the existence of at least one value c, where $f'(c) = 0$.

So, there may be more than one value of c for which the derivative vanishes.

4.2.2 Geomertical Interpretation

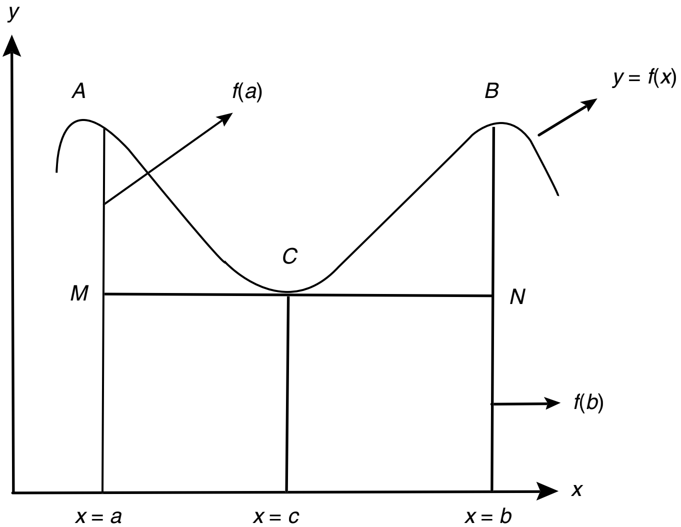

Figure 4.1 Rolle's Theorem

From **Fig. 4.1,** it is clear that at the points $\boldsymbol{A}$ $(a, f(a))$ and $\boldsymbol{B}$ $(b, f(b))$, the ordinates are same for the continuous graph $y = f(x)$, i.e., the values at the points $x=a$ and $x = b$ (which are $f(a)$ and $f(b)$ respectively) are equal.

Since the function f is derivable in the open interval (a, b), a tangent exists at each point of the graph except the extreme points $\boldsymbol{A}$ $(a, f(a))$ and $\boldsymbol{B}$ $(b, f(b))$. Now, we can see in the graph that there exists a point $\boldsymbol{C}$ $(c, f(c))$ in between two extreme points $\boldsymbol{A}$ and $\boldsymbol{B}$, at which the tangent MN is parallel to the x-axis, i.e., gradient of the tangent at $\boldsymbol{C}$ $(c, f(c))$ is zero. It implies that $f'(c) = 0$.

Correspondingly, we have a point $x = c$ in between the points $x = a$ and $x = b$, such that $f'(c) = 0$.

4.2.3 Important Observation

The conditions of Rolle's theorem are only sufficient, they are not neccessary.

This will be followed by three important examples:

Example 1 Verify Rolle's theorem for the function $f(x) = 1 - x^2$ for $-1 \le x \le 1$.

Sol. Here, we are to examine three conditions.

i) Since, $f(x)$ is a polynomial in x and all polynomials in x are continious functions for all values of $x \in R$,

$f(x) = 1 - x^2$ is continuous for all x, where $-1 \le x \le 1$.

ii) Due to the same reason as above, $f(x)$ is also derivable for all x, where $-1 \le x \le 1$.

Moreover, $f'(x) = -2x$, which exists for all values of x in $-1 < x < 1$.

iii) $f(-1) = f(1) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied by $f(x) = 1 - x^2$.

Now

$f'(c) = 0 \Rightarrow -2c = 0 \Rightarrow c = 0.$

Definitely, $c = 0$ lies between -1 and 1, i.e., $-1 < c < 1$.

Hence, we can conclude that $f(x)$ satisfies all the conditions and as such there exists $c \in (-1, 1)$ such that $f'(c) = 0$.

Therefore, Rolle's theorem holds good.

Example 2 Verify Rolle's theorem for the function $f(x) = |x|,\ -1 \le x \le 1$.

[WBUT-2003, 2009]

Sol. Here, $f(x)$ is defined as

$$f(x) = -x, \text{ for } -1 \le x < 0,$$

$$= x \text{ for } 0 \le x < 1.$$

i) $f(x)$ is continious for all x in $-1 \le x \le 1$ except at $x = 0$. Now,

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x = 0$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (-x) = 0$$

and $f(0) = 0$

so, $f(x)$ is also continious at $x = 0$.

Therefore, $f(x)$ is continuous for all x in $-1 \le x \le 1$.

ii) Here,

$$f'(x) = -1, \text{ for } -1 < x < 0,$$

$$= 1 \text{ for } 0 < x < 1.$$

So, $f(x)$ is derivable for all x in $-1 < x < 1$ except at $x = 0$.

Now we check the derivability of $f(x)$ at $x = 0$.

Since,

$$f'(0+) = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} = 1$$

and $f'(0-) = \lim_{h \to 0-} \dfrac{f(h) - f(0)}{h} = -1$

i.e., $f'(0+) \neq f'(0-)$

so, $f(x)$ is not derivable at $x = 0$.

Therefore, $f(x)$ is not derivable for all x in $-1 < x < 1$.

iii) $f(-1) = f(1) = 1$.

Since all the conditions of Rolle's theorem is not satisfied by $f(x)$, Rolle's theorem is not applicable here.

Here, we can observe that there exists no such c, $-1 < c < 1$ for which $f'(c) = 0$, i.e., Rolle's theorem does not hold since the condition (ii) is violated.

Example 3 Verify Rolle's theorem for the function $f(x) = \dfrac{1}{x} + \dfrac{1}{2-x}$, in $0 \le x \le 2$.

Sol.

(i) $f(x)$ is not continious for all x in $0 \le x \le 2$ (since it is continuous for $0 < x < 2$).

(ii) $f(x)$ is derivable in $0 < x < 2$, since

$f'(x) = \dfrac{1}{(2-x)^2} - \dfrac{1}{x^2}$ exists for $0 < x < 2$.

(iii) $f(0)$ and $f(2)$ are not defined. So, $f(0) \neq f(2)$.

All the conditions are not satisfied by the function $f(x)$, so Rolle's theorem is not applicable here.

But it is interesting to see that

$$f'(c) = \frac{1}{(2-c)^2} - \frac{1}{c^2} = 0$$

$\Rightarrow c = 1$, which lies between 0 and 2

i.e., there exists c where $0 < c < 2$ such that $f'(c) = 0$.

So, the result of Rolle's theorem is still true, though all the conditions are not satisfied.

Conclusion from Examples 1 to 3:

From Example 1 We have observed that all the conditions are satisfied, so Rolle's theorem holds good.

From Example 2 We have observed that some of the conditions are violated, so Rolle's theorem does not hold.

From Example 3 We have observed that some of the conditions are violated but Rolle's theorem is still true.

Summing up the above, we conclude that if all the conditions are satisfied by $f(x)$ in $[a, b]$ then the result $f'(c) = 0$, where $a < c < b$ surely occurs. But if any of the conditions are violated by $f(x)$, the result $f'(c) = 0$, where $a < c < b$ may still be true but not in at all times. In the latter case, we can say that Rolle's theorem is not applicable.

So it is clear that the conditions of Rolle's theorem are only sufficient, by no way they are neccessary.

4.3 LAGRANGE'S MEAN VALUE THEOREM (LAGRANGE'S MVT)

4.3.1 Statement

Let $f : I \to R$ be a real-valued function where $I = [a, b]$ and f satisfies the following conditions:

i) f is continuous in the closed interval $[a, b]$

ii) f is derivable in the open interval (a, b), i.e., $f'(x)$ exists for $x \in (a, b)$.

Then there exists at least one value of x (say c), $c \in (a, b)$, i.e., $a < c < b$ such that

$$\boxed{\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for } a < c < b.}$$ [WBUT 2002, 2004]

Proof: Let us construct a function

$$\phi(x) \text{ as } \quad \phi(x) = f(x) + k \cdot x \text{ for } x \in a, b] \tag{1}$$

where the constant k is to be determined such that $\phi(a) = \phi(b)$.

So,

$$\phi(a) = \phi(b) \Rightarrow f(a) + k \cdot a = f(b) + k \cdot b$$

$$\Rightarrow k = -\frac{f(b) - f(a)}{b - a}. \tag{2}$$

Now since $f(x)$ is continuous in $[a, b]$, $\phi(x)$ is continuous there and also since $f(x)$ is derivable in (a, b), $\phi(x)$ is derivable there.

Also $\phi(a) = \phi(b)$.

Therefore, $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$. So, there exists a value $x = c$, $a < c < b$ such that $\phi'(c) = 0$.

Therefore, by (1),

$$\phi'(c) = f'(c) + k = 0 \Rightarrow f'(c) = -k$$

Using (2) in the above, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ for } a < c < b.$$

Hence, the theorem is proved.

Note: If we consider $f(a) = f(b)$ then from the above $f'(c) = 0$ for $a < c < b$. So, Lagrange's MVT becomes Rolle's theorem.

4.3.2 Geomertical Interpretation

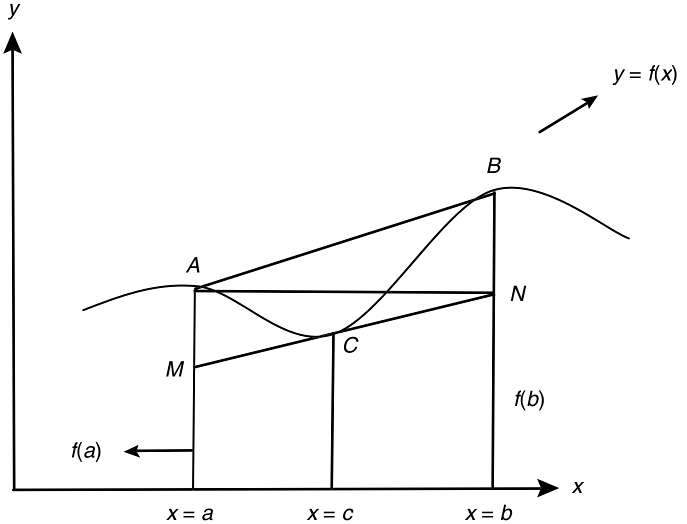

Figure 4.2 Lagrange's MVT

Figure 4.2 represents a curve $y = f(x)$ which is continuous in $[a, b]$ and derivable in (a, b). Now we consider the chord ***AB*** joining the two points ***A*** $(a, f(a))$ and ***B*** $(b, f(b))$ of the curve.

So, gradient of the chord ***AB*** is

$$\frac{BN}{AN} = \frac{f(b) - f(a)}{b - a}.$$

Again since the function is derivable everywhere in (a, b), a tangent exists at every point between the extreme points ***A*** and ***B***.

Now we draw a tangent ***MN*** which is parallel to the chord ***AB*** and touches the curve at the point ***C*** $(c, f(c))$. Here the point ***C*** lies between the points ***A*** and ***B*** and correspondingly $x = c$ lies between $x = a$ and $x = b$.

Then gradient of the tangent ***MN*** is $f'(c)$ (Since here at ***C***, $x = c$).

Since the chord ***AB*** and the tangent ***MN*** are parallel, their gradients are same. So, we have

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ where } a < c < b.$$

4.3.3 Other Forms of Lagrange's Mean-Value Theorem

(1) If in the statement of Section 4.3.1 we consider $b = a + h,\ h > 0$, then the point c where $a < c < a + h$ can be repesented as

$c = a + \theta h$, where $0 < \theta < 1$ and correspondingly Lagrange's MVT in the interval $[a, a + h]$ can be written as

$$\boxed{f(a+h) = f(a) + hf'(a+\theta h), \text{where } 0 < \theta < 1.}$$

(2) Now if we put $a = 0$ and $h = x$ in the above form then Lagrange's MVT in the interval $[0, x]$ can be written as

$$\boxed{f(x) = f(0) + x f'(\theta x), \text{ where } 0 < \theta < 1.}$$

Example 4 Verify Lagrange's MVT for the function $f(x) = x^2 + 3x + 2$ for $1 \le x \le 2$.

Sol. Here, we are to examine two conditions.

i) Since $f(x)$ is a polynomial in x and all polynomials in x are continuous functions for all values of $x \in R$,

$f(x) = x^2 + 3x + 2$ is continuous for all x, where $1 \le x \le 2$.

ii) Due to the same reason as above, $f(x)$ is also derivable for all x, where $1 \le x \le 2$.

Moreover, $f'(x) = 2x + 3$, which exists for all values of x in $1 \le x \le 2$.

Since all the conditions of Lagrange's MVT are satisfied by $f(x) = x^2 + 3x + 2$ in $1 \le x \le 2$,

there should exist $c \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{2-1} = f'(c).$$

Now the above implies

$$\frac{12-6}{1} = 2c + 3.$$

i.e., $c = \dfrac{3}{2}$.

Here, $c = \frac{3}{2}$ lies between 1 and 2.

Hence Lagrange's MVT is verified for the given function.

Example 5 Verify Lagrange's mean-value theorem for $f(x) = \cos x$, where $0 \le x \le \frac{\pi}{2}$

Sol. Here we are to examine two conditions.

i) $f(x) = \cos x$ is continuous for all values of x, $0 \le x \le \frac{\pi}{2}$.

ii) $f'(x) = -\sin x$ exists for all values of x, $0 < x < \frac{\pi}{2}$.

Since all the conditions of Lagrange's MVT are satisfied by $f(x) = \cos x$ in $0 \le x \le \frac{\pi}{2}$,

there should exist $c \in \left(0, \frac{\pi}{2}\right)$ such that

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = f'(c).$$

Now the above implies

$$\frac{\cos\frac{\pi}{2} - \cos 0}{\frac{\pi}{2} - 0} = -\sin c$$

$$\Rightarrow \sin c = \frac{2}{\pi}$$

$$\Rightarrow c = \sin^{-1}\left(\frac{2}{\pi}\right)$$

Here, $c = \sin^{-1}\left(\frac{2}{\pi}\right)$ lies between 0 and $\frac{\pi}{2}$.

Hence Lagrange's MVT is verified for the given function.

4.3.4 Applications of Lagrange's Form of Mean-Value Theorem

Example 6 (Estimation of numerical value)

(i) Estimate the numerical value of $\sqrt[4]{17}$ using Lagrange's MVT.

Sol. Let us consider the function $f(x) = x^{\frac{1}{4}}$ in $[16, 17]$.

i) $f(x) = x^{\frac{1}{4}}$ is a continuous function for all values of $[x \in 16, 17]$

ii) $f'(x) = \frac{1}{4} x^{\frac{-3}{4}}$ exists for all values of $x \in (16, 17)$.

Since all the conditions of Lagrange's MVT are satisfied by $f(x)$, there should exist such a $c \in (16, 17)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow f(17) - f(16) = (17 - 16) f'(c)$$

Taking $h = b - a = 17 - 16 = 1, \quad c = a \quad \theta h = 16 \quad \theta, \quad 0 < \theta < 1$

$$\Rightarrow f(17) - f(16) = (17 - 16) f'(16 \quad \theta), 0 < \theta < 1$$

$$\Rightarrow \sqrt[4]{17} - \sqrt[4]{16} = \frac{1}{4} \cdot (16 \quad \theta)^{\frac{-3}{4}}, 0 < \theta < 1$$

Since

$$(16 \quad \theta)^{\frac{-3}{4}} < (16)^{\frac{-3}{4}} = \frac{1}{8}$$

We have from above

$$\sqrt[4]{17} - 2 < \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

i.e., $\sqrt[4]{17} < 2 + \frac{1}{32} = 2\frac{1}{32}$

Hence, the estimate is

$$2 < \sqrt[4]{17} < 2\frac{1}{32}.$$

(ii) Estimate the value of $\log \frac{4}{3}$ using Lagrange's MVT.

Sol. Let us consider the function $f(x) = \log x$ in $[3, 4]$.

i) $f(x)$ is continuous for all values of $[x \in 3, 4]$.

ii) $f'(x) = \dfrac{1}{x}$ which exists for all values of $x \in (3,4)$.

Therefore, all the conditions of Lagrange's MVT is satisfied by $f(x)$, and there should exist $c \in (3, 4)$ such that

$$\frac{\log 4 - \log 3}{(4-3)} = f'(c) \text{ for } 3 < c < 4$$

$$\log\left(\frac{4}{3}\right) = \frac{1}{c} \text{ for } 3 < c < 4$$

Since

$$3 < c < 4 \Rightarrow \frac{1}{4} < \frac{1}{c} < \frac{1}{3}$$

The estimate is given by

$$\frac{1}{4} < \log\left(\frac{4}{3}\right) < \frac{1}{3}.$$

Example 7 (Proof of some standard inequalities using Lagrange's MVT)

(i) Using Lagrange's MVT, prove

$$\frac{x}{1+x} < \log(1+x) < x \text{ if } x > 0.$$

Sol. Let $f(x) = \log(1+x)$ for $x > 0$.

It is obvious that $f(x)$ satisfies all the conditions of Lagrange's MVT in $[0, x]$.

From **Section 4.3.3**, we have **Lagrange's MVT** in the interval $[0, x]$ as

$$f(x) = f(0) + x f'(\theta x), \text{ where } 0 < \theta < 1.$$

Here $f(0) = \log 1 = 0$ and $f'(x) = \dfrac{1}{1+x}$.

So from above

$$\log(1+x) = 0 + x\frac{1}{1+\theta x}, \text{ where } 0 < \theta < 1.$$

$$\text{or, } \log(1+x) = \frac{x}{1+\theta x}$$

Now we have

$$0 < \theta < 1$$

$$\text{or, } 0 < \theta x < x, \text{ since } x > 0$$

$$\text{or, } 1 < 1 + \theta x < 1 + x$$

$$\text{or, } \frac{1}{1+x} < \frac{1}{1+\theta x} < 1$$

Since $x > 0$ we have,

$$\frac{1}{1+x} \cdot x < \frac{1}{1+\theta x} \cdot x < 1 \cdot x$$

$$\text{or, } \frac{x}{1+x} < \frac{x}{1+\theta x} < x$$

Again $\log(1+x) = \dfrac{x}{1+\theta x}$, therefore

$$\text{or, } \frac{x}{1+x} < \log(1+x) < x.$$

(ii) Using Lagrange's MVT, prove $0 < \dfrac{1}{x}\log\dfrac{e^x - 1}{x} < 1$ [WBUT 2002]

Sol. Let $f(x) = e^x$.

It is obvious that $f(x)$ satisfies all the conditions of Lagrange's MVT in $[0, x]$.

From **Section 4.3.3**, we have **Lagrange's MVT** in the interval $[0, x]$ as

$f(x) = f(0) + x f'(\theta x)$, where $0 < \theta < 1$.

Here $f(0) = e^0 = 1$ and $f'(x) = e^x$.

So from above

$e^x = 1 + xe^{\theta x}$, where $0 < \theta < 1$.

$$\text{or, } e^{\theta x} = \frac{e^x - 1}{x}$$

$$\text{or, } \log(e^{\theta x}) = \log\frac{e^x - 1}{x}$$

$$\text{or, } \theta x = \log\frac{e^x - 1}{x}$$

$$\text{or, } \theta = \frac{1}{x}\log\frac{e^x - 1}{x}$$

Since $0 < \theta < 1$, we have

$$0 < \frac{1}{x}\log\frac{e^x - 1}{x} < 1$$

(iii) Using Lagrange's MVT, prove $1+\dfrac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1+\dfrac{x}{2}, -1 < x < 0$

[WBUT 2004]

Sol. Let $f(x) = \sqrt{1+x}$.

It is obvious that $f(x)$ satisfies all the conditions of Lagrange's MVT

From **Section 4.3.3**, we have **Lagrange's MVT** in the interval $[0, x]$ as

$f(x) = f(0) + x f'(\theta x)$, where $0 < \theta < 1$.

Here $f(0) = \sqrt{1+0} = 1$ and $f'(x) = \dfrac{1}{2\sqrt{1+x}}$.

So from above

$\sqrt{1+x} = 1 + x\dfrac{1}{2\sqrt{1+\theta x}}$, where $0 < \theta < 1$.

Now we have

$0 < \theta < 1$

or, $0 > \theta x > x$, since $x < 0$

or, $1 > 1+\theta x > 1+x$

or, $1 > \sqrt{1+\theta x} > \sqrt{1+x}$

or, $1 < \dfrac{1}{\sqrt{1+\theta x}} < \dfrac{1}{\sqrt{1+x}}$

Since $x < 0$,

$1 \cdot x \cdot \dfrac{1}{2} > \dfrac{1}{\sqrt{1+\theta x}} \cdot x \cdot \dfrac{1}{2} > \dfrac{1}{\sqrt{1+x}} \cdot x \cdot \dfrac{1}{2}$

or, $\dfrac{x}{2\sqrt{1+x}} < \dfrac{x}{2\sqrt{1+\theta x}} < \dfrac{x}{2}$

or, $1+\dfrac{x}{2\sqrt{1+x}} < 1+\dfrac{x}{2\sqrt{1+\theta x}} < 1+\dfrac{x}{2}$

Again

$\sqrt{1+x} = 1 + x\dfrac{1}{2\sqrt{1+\theta x}}$

Hence

$1+\dfrac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1+\dfrac{x}{2}, -1 < x < 0$

Example 8 **(Proof of some well-known properties of function using Lagrange's MVT)**

(i) Suppose $f'(x)=0$ in $[a,b]$. Then using Lagrange's MVT, prove that $f(x)$ is constant in $[a,b]$.

Sol. Let us consider two arbitrary points x_1 and x_2 so that $a < x_1 < x_2 < b$.

Now since $f'(x)=0$ in $[a,b]$, the function $f(x)$ is derivable and so continuous in $[a,b]$.

Then the function $f(x)$ is also continuous and derivable in $[x_1, x_2]$.

Therefore, we can apply Lagrange's MVT on $f(x)$ in $[x_1, x_2]$, and applying we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \text{ for } x_1 < c < x_2.$$

Now since $f'(x)=0$ in $[a,b]$, also $f'(x)=0$ in $[x_1, x_2]$

and so, $f'(c)=0$ for $x_1 < c < x_2$.

Therefore

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \Rightarrow f(x_2)=f(x_1)$$

Since x_1 and x_2 are two arbitrary points and we have the same functional value for them, i.e., $f(x_1)=f(x_2)$.

Hence we can conclude that $f(x)$ is constant in $[a,b]$.

(ii) Suppose $f'(x)=g'(x)$ in $[a,b]$ then using Lagrange's MVT prove that $f(x) = g(x)+$ constant in $[a,b]$.

Sol. Let $\phi(x)=f(x)-g(x)$.

Then $\phi^{'}(x)=f^{'}(x)-g^{'}(x)=0.$

Now in the last example using Lagrange's MVT we have proved that if $f'(x)=0$ in $[a,b]$, then $f(x)=$ **constant** in $[a,b]$.

Therefore,

$\phi(x) =$ **constant in** $[a,b]$

i.e., $f(x)-g(x) =$ **constant in** $[a,b]$

$f(x)=g(x)+$ **constant in** $[a,b]$.

(iii) Suppose $f(x)$ is continuous in $[a,b]$ then using Lagrange's MVT, prove that $f(x)$ is increasing if $f'(x)>0$ and $f(x)$ is decreasing if $f'(x)<0$

Sol. Let us consider two arbitrary points x_1 and x_2 so that $a<x_1<x_2<b$.

Now since the function $f(x)$ is derivable, it is continuous in $[a,b]$.

Then the function $f(x)$ is also continuous and derivable in $[x_1, x_2]$.

Therefore, we can apply Lagrange's MVT on $f(x)$ in $[x_1, x_2]$, and applying we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c) \text{ for } x_1<c<x_2.$$

In the first case, since $f'(x)>0$ in $[a,b]$, also $f'(x)>0$ in $[x_1, x_2]$

and so, $f'(c)>0$ for $x_1<c<x_2$.

Therefore,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}>0 \Rightarrow f(x_2)>f(x_1)$$

So, $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$ and hence the function is increasing.

In the 2nd case, since $f'(x)<0$ in $[a,b]$, also $f'(x)<0$ in $[x_1, x_2]$

and so, $f'(c)<0$ for $x_1<c<x_2$.

Therefore,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}<0 \Rightarrow f(x_2)<f(x_1)$$

So, $x_1<x_2 \Rightarrow f(x_2)<f(x_1)$ and hence the function is decreasing.

4.4 CAUCHY'S MEAN-VALUE THEOREM (CAUCHY'S MVT)

4.4.1 Statement

Let $f: I \to R$ and $g: I \to R$ be two real-valued functions where $I=[a,b]$ and *f* and *g* satisfy the following conditions,

i) *f* and *g* are both continuous in the closed interval $[a,b]$;

ii) *f* and *g* are both derivable in the open interval (a,b);

iii) $g'(x) \neq 0$ for all values of *x* in $a<x<b$;

Then there exists at least one value of x (say c) $c \in (a, b)$, i.e., $a < c < b$ such that,

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \text{ for } a < c < b.}$$ [WBUT 2001, 2003]

Proof: Let us construct a function $\phi(x)$ as

$$\phi(x) = f(x) + k \cdot g(x) \text{ for } x \in [a, b] \tag{1}$$

where the constant k is to be determined such that $\phi(a) = \phi(b)$.

So,

$$\phi(a) = \phi(b) \Rightarrow f(a) + k \cdot g(a) = f(b) + k \cdot g(b)$$

$$\Rightarrow k = -\frac{f(b)-f(a)}{g(b)-g(a)}. \tag{2}$$

Here, $g(b) \neq g(a)$, otherwise it satisfies the conditions of Rolle's theorem which results $g'(x) = 0, a < x < b,$ which contradicts the condition (iii) of the theorem. So k is finite.

Now since $f(x)$ and $g(x)$ both are continuous in $[a, b]$, $\phi(x)$ is continuous there and also since $f(x)$ and $g(x)$ are derivable in (a, b), $\phi(x)$ is derivable there.

Also, $\phi(a) = \phi(b)$.

Therefore, $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$. So, there exists a value $x = c$, $a < c < b$ such that $\phi'(c) = 0$.

Therefore by (1),

$$\phi'(c) = f'(c) + k \cdot g'(c) = 0 \Rightarrow \frac{f'(c)}{g'(c)} = -k$$

Using (2) in the above, we have

$$f'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} \text{ for } a < c < b.$$

Hence the theorem is proved.

Note: When, $g(x) = x$, Cauchy's mean-value theorem takes the form of Lagrange's mean-value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ for } a < c < b$$

4.4.2 Other Forms of Cauchy's Mean-Value Theorem

1) **Let $b = a + h$; then the point c where $a < c < a + h$ is repesented as $c = a + \theta h$, where $0 < \theta < 1$**

Cauchy's mean-value theorem in the interval $[a, a+h]$ takes the form

$$\boxed{\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \text{ where } 0<\theta<1}$$

2) **If we set $a=0$ and $h=x$ in the above form then in the interval $[0, x]$, Cauchy's MVT takes the form**

$$\boxed{\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\theta x)}{g'(\theta x)}, \text{ where } 0<\theta<1}$$

Example 9 Verify Cauchy's mean-value theorem for

$f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ in $[1, 2]$

Sol.

i) The functions $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ are both continious in $[1, 2]$;

ii) $f'(x)=\dfrac{1}{2\sqrt{x}}$ and $g'(x)=\dfrac{-1}{2}x^{\frac{-3}{2}}$ which exists for all values of $x \in (1, 2)$;

iii) $g'(x) \neq 0$ for all values of x in $1<x<2$;

Therefore, all the conditions of Cauchy's MVT are satisfied by the given functions and so there should exist such a $c \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

which implies

$$\frac{\sqrt{2}-1}{\frac{1}{\sqrt{2}}-1} = \frac{\frac{1}{2}c^{\frac{-1}{2}}}{\frac{-1}{2}c^{\frac{-3}{2}}}$$

or, $c=\sqrt{2}$

Here, $c=\sqrt{2}$ lies between 1 and 2.

Hence Cauchy's MVT is verified.

Example 10 If $f(x)=e^{x}$ and $g(x)=e^{-x}$, using Cauchy's mean-value theorem, show that θ is independent of both x and h and is equal to $\dfrac{1}{2}$. [WBUT 2003]

Sol.

Since $f(x)=e^{x}$ and $g(x)=e^{-x}$ are continuous and differentiable for all real values of x and $g'(x)=-e^{-x}\neq 0$,

Applying Cauchy's MVT on the functions in $[x, x+h]$ **(see Section 4.4.2)** we have

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)}=\frac{f'(x+\theta h)}{g'(x+\theta h)}, 0<\theta<1$$

$$\text{or, } \frac{e^{x+h}-e^{x}}{e^{-(x+h)}-e^{-x}}=\frac{e^{x+\theta h}}{-e^{-(x+\theta h)}}$$

$$\text{or, } \frac{e^{h}(e^{h}-1)}{1-e^{h}}=-\frac{e^{\theta h}}{e^{-\theta h}}$$

$$\text{or, } e^{h}=e^{2\theta h}$$

$$\text{or, } \theta=\frac{1}{2}.$$

Therefore, θ is independent of x and h and is equal to $\frac{1}{2}$.

4.5 TAYLOR'S THEOREM (GENERALISED MEAN-VALUE THEOREM)

4.5.1 Taylor's Theorem with Lagrange's Form of Remainder

Statement: Let $f : I \to R$ be a real-valued function where $I=[a,b]$ and f satisfies the following conditions:

i) the $(n-1)$-th derivatives of $f(x)$, i.e., $f^{(n-1)}(x)$ is continuous in $[a,b]$;

ii) the n-th derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists in (a,b)

Then there exists at least one value of x (say c) $c\in(a,b)$, i.e., $a<c<b$ such that

$$\boxed{f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^{2}}{2!}f''(a)+\cdots+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_{n}}$$

where the $(n+1)$-th term $R_{n}=\frac{(b-a)^{n}}{n!}f^{(n)}(c)$ is called the Lagrange's form of reminder after n terms.

4.5.2 Other Forms of Taylor's Theorem with Lagrange's Form of Remainder

(1) Let $f: I \to R$ be a real-valued function where $I = [a, a+h]$, $h > 0$ and f satisfies the following conditions:

i) the $(n-1)$-th derivatives of $f(x)$, i.e., $f^{(n-1)}(x)$ is continuous in $[a, a+h]$;

ii) the $f'(x) = \frac{1}{x}$, -th derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists in $(a, a+h)$

Then there exists at least one value of θ, $0 < \theta < 1$ such that,

$$\boxed{\boldsymbol{f(a \quad h) = f(a) \quad hf'(a) \quad \frac{h^2}{2!} f''(a) \quad \cdots \quad \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) \quad R_n}}$$

where $R_n = \frac{h^n}{n!} f^{(n)}(a \quad \theta h), 0 < \theta < 1$

(2) Putting $b = x$, (from **Section 4.5.1**), we have

$$\boxed{\boldsymbol{f(x) = f(a) \quad (x-a) f'(a) \quad \frac{(x-a)^2}{2!} f''(a) \quad \cdots \quad \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \quad R_n}}$$

where $R_n = \frac{(x-a)^n}{n!} f^{(n)}\{a + \theta(x-a)\}, 0 < \theta < 1$

Note: The form (2) is known as Taylor's expansion of $f(x)$ about $x = a$ with the Lagrange's form of reminder.

Basically, this is a finite-series expansion of a function about any point. Sometimes we also call this as the power-series expansion of x about a in the finite form.

Example 11 Using Taylor's theorem, expand $f(x) = \log x$, $1 < x < \infty$ about the point $x = 2$ with the Lagrange's form of remainder after 3 terms.

Sol.

Here, $f(x) = \log x$ for $1 < x < \infty$.

Now, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3} \cdots$ etc. all exist and continuous in $1 < x < \infty$.

From **(2)** of **Section 4.5.2** we have Taylor's expansion of $f(x)$ about the point $x = a$ with the Lagrange's form of reminder after 3 terms as

$$f(x) = f(a) \quad (x-a) f'(a) \quad \frac{(x-a)^2}{2!} f''(a) \quad R_3$$

where $R_3 = \frac{(x-a)^3}{3!} f'''\{a + \theta(x-a)\}, 0 < \theta < 1$

Here $a = 2$, so the expansion becomes

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + R_3$$

where $R_3 = \dfrac{(x-2)^3}{3!} f'''\{2+\theta(x-2)\}, 0 < \theta < 1$

Now

$$f(2) = \log 2, \; f'(2) = \frac{1}{2}, \; f''(2) = \frac{-1}{4}$$

and $f'''\{2+\theta(x-2)\} = \dfrac{2}{\{2+\theta(x-2)\}^3}$

Putting the values, we have

$$f(x) = \log 2 + \frac{1}{2}(x-2) - \frac{1}{4}\frac{(x-2)^2}{2!} + R_3$$

where $R_3 = \dfrac{(x-2)^3}{3!} \dfrac{2}{\{2+\theta(x-2)\}^3}, 0 < \theta < 1$

So,

$$\log x = \log 2 + \frac{1}{2}(x-2) - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{3} \cdot \frac{1}{\{2+\theta(x-2)\}^3} \text{ where } 0 < \theta < 1.$$

4.5.3 Taylor's Theorem with Cauchy's Form of Remainder

Statement: Let $f : I \to R$ be a real-valued function where $I = [a, b]$ and f satisfies the following conditions:

i) the $(n-1)$-th derivatives of $f(x)$, i.e., $f^{(n-1)}(x)$ is continuous in $[a, b]$

ii) the n-th derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists in (a, b)

Then there exists at least one value of x (say c) $c \in (a, b)$, i.e., $a < c < b$ such that,

$$\boxed{f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n}$$

where the $(n+1)$-th term $R_n = \dfrac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c)$ is called the Cauchy's form of remainder after n terms.

4.5.4 Other Forms of Taylor's Theorem with Cauchy's Form of Remainder

Let $f : I \to R$ be a real-valued function where $I = [a, a+h]$, $h > 0$ and f satisfies the following conditions:

i) **the $(n-1)$-th derivatives of $f(x)$, i.e., $f^{(n-1)}(x)$ is continuous in $[a, a+h]$**

ii) **the n-th derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists in $(a, a+h)$**

Then there exists at least one value of θ, $0 < \theta < 1$ such that,

$$\boxed{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n}$$

where $R_n = \dfrac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h), 0 < \theta < 1$

4.6 MACLAURIN'S THEOREM

Statement: Let $f : I \to R$ be a real-valued function where $I = [0, x]$ and f satisfies the following conditions:

i) **the $(n-1)$-th derivatives of $f(x)$ i.e $f^{(n-1)}(x)$ is continuous in $[0, x]$**

ii) **the n-th derivative of $f(x)$ i.e $f^{(n)}(x)$ exists in $(0, x)$**

Then there exists at least one value of θ, $0 < \theta < 1$ such that

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n}$$

where $R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x), 0 < \theta < 1$ **(Lagrange′s form of remainder)**

and $R_n = \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x), 0 < \theta < 1$ **(Cauchy′s form of remainder)**

Observation: Generally, we choose Lagrange's form of remainder if anything is not mentioned.

Note: If we put $a = 0$ in the form (2) of Section 4.5.2, then too we can get the same expression of Maclaurin's theorem as above.

This is also called Maclaurin's finite-series expansion of a function about $x = 0$. Sometimes we call this as the power-series expansion of x in finite form.

Example 12 Expand $f(x) = \sin x$ in a finite series with the Lagrange's form of remainder.

Sol. Here, $f(x) = \sin x$ and so its n-th order derivative is given by

$$f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right).$$

Thus, $f^{(n)}(x)$ exists for every order n and also for every value of x.

Here $f^{(n)}(0) = \sin\frac{n\pi}{2}$.

Now the Maclaurin's finite-series expansion of $f(x)$ with the Lagrange's form of remainder is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where $R_n = \frac{x^n}{n!}f^{(n)}(\theta x), 0 < \theta < 1$

Here,

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(iv)}(0) = 0 \ldots \text{etc.}$$

and $f^{(n)}(\theta x) = \sin\left(\frac{n\pi}{2} + \theta x\right)$.

Putting the values, we have

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-1) + \cdots + \frac{x^{n-1}}{(n-1)!}\sin\frac{(n-1)\pi}{2} + R_n$$

where $R_n = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right), 0 < \theta < 1$

So,

$$\sin x = x - \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}\sin\frac{(n-1)\pi}{2} + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right) \text{where } 0 < \theta < 1$$

4.7 INFINITE SERIES EXPANSION OF FUNCTIONS

In this section, we check whether we can express any function $f(x)$ as an infinite series about the point $x = a$ in the form of

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \quad (1)$$

Now the first question that arises is that for any function can we always get a series of the above form (1). The answer is that we can construct the series iff $f^{(n)}(a)$ exists for each n.

Next we have to find out whether the infinite series (1) will be convergent or not. To answer this question consider n-th partial sum

$$S_n = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

Now if $\lim\limits_{n\to\infty} S_n$ exists and finite $(=S)$ then the series converges to S.

The last and most vital question is that if f satisfies all the conditions of Taylor's theorem with any form of remainder in the interval $[a-h, a+h]$,

i.e.,

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n,$$

then under what conditions will the infinite series (1) be convergent to $f(x)$.

Now, we have from above

$$f(x) = S_n + R_n$$

i.e., $S_n = f(x) - R_n$

Now $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} f(x) - \lim\limits_{n\to\infty} R_n$

$$= f(x) - \lim_{n\to\infty} R_n$$

Again the infinite series (1) will be convergent and converges to $f(x)$ iff

$$\lim_{n\to\infty} S_n = f(x),$$

which is possible from above iff $\lim\limits_{n\to\infty} R_n = 0.$

Hence the infinite series (1) will be convergent and converges to $f(x)$ iff $\lim\limits_{n\to\infty} R_n = 0.$

In this case, we can write

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\cdots$$

4.7.1 Taylor's Infinite-Series Expansion

Statement: Let $f: I \to R$ be a real valued function where $I = [a-h, a+h]$ and f satisfies the following conditions:

i) the n^{th} derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists for all n in $[a-h, a+h]$

ii) $\lim\limits_{n\to\infty} R_n = 0$ where R_n is any form of remainder after n terms in the Taylor's finite expansion of $f(x)$ about $x = a$.

Then we have Taylor's infinite-series expansion of $f(x)$ about $x = a$ as

$$\boxed{f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\cdots}$$

4.7.2 Maclaurin's Infinite-Series Expansion

Statement: Let $f: I \to R$ be a real-valued function where $I = [-h, h]$, $h > 0$ and *f* satisfies the following conditions:

i) the n^{th} derivative of $f(x)$, i.e., $f^{(n)}(x)$ exists for all *n* in $[-h, h]$

ii) $\lim\limits_{n\to\infty} R_n = 0$ where R_n is the any form of reminder after *n* terms in the Maclaurin's finite expansion of $f(x)$.

Then we have Maclaurin's infinite-series expansion of $f(x)$ as

$$\boxed{f(x) = f(0) \quad xf'(0) \quad \frac{x^2}{2!} f''(0) \quad \cdots \quad \frac{x^n}{n!} f^{(n)}(0) \quad \cdots}$$

Note:

If we consider $a = 0$ in the Taylor's infinite series expansion of $f(x)$ then we have the Maclaurin's infinite series expansion of $f(x)$.

4.7.3. Maclaurin's Infinite-Series Expansion of sin *x*

Here

$$f(x) = \sin x$$

Now

$$f^{(n)}(x) = \sin\left(\frac{n\pi}{2} \quad x\right), \text{ for every } n$$

Thus, $f(x) = \sin x$ possesses derivatives of every order for every value of x in any interval $[-h \cdot h]$.

Here

$$f^{(n)}(0) = \sin\frac{n\pi}{2}, \text{ for all } n$$

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(iv)}(0) = 0 \ldots \text{and so on.}$$

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Lagrange's form is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} \quad \theta x\right), 0 < \theta < 1$$

Here

$$0 \le |R_n| \le \left|\frac{x^n}{n!}\sin\left(\frac{n\pi}{2} \quad \theta x\right)\right|$$

$$= \left|\frac{x^n}{n!}\right|\left|\sin\left(\frac{n\pi}{2} \quad \theta x\right)\right|$$

Again

$|\sin x| \le 1$ for all x.

So, we have

$$0 \le |R_n| \le \left|\frac{x^n}{n!}\right| \cdot 1 = \left|\frac{x^n}{n!}\right|$$

$$\text{i.e., } -\left|\frac{x^n}{n!}\right| \le R_n \le \left|\frac{x^n}{n!}\right|$$

Again

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

which implies

$$\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0.$$

Hence from above, we obtain

$$\lim_{n\to\infty} R_n = 0.$$

Therefore, we have Maclaurin's infinite-series expansion of $f(x) = \sin x$ and is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

$$\text{or, } \sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}\right) + \cdots$$

$$\text{or, } \sin x = x - \frac{x^3}{3!} \quad \frac{x^5}{5!} - \cdots$$

4.7.4. Maclaurin's Infinite-Series Expansion of cos *x*

Here

$$f(x) = \cos x$$

Now

$$f^{(n)}(x) = \cos\left(\frac{n\pi}{2} \quad x\right), \text{ for every } n$$

Thus, $f(x) = \cos x$ possesses derivatives of every order for every value of x in any interval $[-h, h]$.

Here

$$f^{(n)}(0) = \cos\frac{n\pi}{2}, \text{ for all } n$$

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0,\ f^{(iv)}(0) = 1\ldots \text{ and so on.}$$

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Lagrange's form is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right), 0 < \theta < 1$$

Here

$$0 \le |R_n| \le \left|\frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$= \left|\frac{x^n}{n!}\right|\left|\cos\left(\frac{n\pi}{2} + \theta x\right)\right|$$

Again

$$|\cos x| \le 1 \text{ for all } x.$$

So, we have

$$0 \le |R_n| \le \left|\frac{x^n}{n!}\right| \cdot 1 = \left|\frac{x^n}{n!}\right|$$

$$\text{i.e., } -\left|\frac{x^n}{n!}\right| \le R_n \le \left|\frac{x^n}{n!}\right|$$

Again

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

which implies

$$\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0.$$

Hence from above, we obtain

$$\lim_{n\to\infty} R_n = 0.$$

Therefore, we have Maclaurin's infinite-series expansion of $f(x)=\cos x$ and is given by

$$f(x) = f(0) \quad \frac{x}{1!}f'(0) \quad \frac{x^2}{2!}f''(0) \quad \frac{x^3}{3!}f'''(0) \quad \frac{x^4}{4!}f^{(iv)}(0)\cdots \quad \frac{x^n}{n!}f^{(n)}(0) \quad \cdots$$

$$\text{or, } \cos x = 1 \quad \frac{x}{1!}(0) \quad \frac{x^2}{2!}(-1) \quad \frac{x^3}{3!}(0) \quad \frac{x^4}{4!}(1) \quad \cdots \quad \frac{x^n}{n!}\cos\left(\frac{n\pi}{2}\right) \quad \cdots$$

$$\text{or, } \cos x = 1 - \frac{x^2}{2!} \quad \frac{x^4}{4!} - \frac{x^6}{6!} \quad \cdots$$

4.7.5 Maclaurin's Infinite-Series Expansion of e^x

[WBUT 2004]

Here

$$f(x) = e^x$$

Now

$$f^{(n)}(x) = e^x, \text{ for every } n$$

Thus, $f(x)=e^x$ possesses derivatives of every order and for every value of x in any interval $[-h, h]$.

Here

$$f^{(n)}(0) = 1, \text{ for all } n$$

$$f(0)=1,\ f'(0)=1,\ f''(0)=1,\ f'''(0)=1,\ f^{(iv)}(0)=1\ldots \text{ and so on.}$$

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Lagrange's form is

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!}e^{(\theta x)},\ 0<\theta<1$$

Here

$$0 \le |R_n| = \left|\frac{x^n}{n!} e^{(\theta x)}\right| = \left|\frac{x^n}{n!}\right| \left|e^{(\theta x)}\right| \le \left|\frac{x^n}{n!}\right| e^{|\theta x|}$$

Since

$0 < \theta < 1 \Rightarrow 0 < \theta x < x$

i.e., $0 < |\theta x| < |x|$

we have

$$e^{|\theta x|} < e^{|x|}.$$

Again

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

which implies

$$\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0.$$

Hence, from above, we obtain

$$\lim_{n\to\infty} R_n = 0.$$

Therefore, we have Maclaurin's infinite-series expansion of $f(x) = e^x$ and it is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

i.e., $e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \cdots + \frac{x^n}{n!}(1) + \cdots$

i.e., $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$

4.7.6 Maclaurin's Infinite-Series Expansion of $\log(1+x)$, $-1 < x \le 1$ [WBUT 2006]

Here

$$f(x) = \log(1+x)$$

So,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Thus, $f(x)=\log(1+x)$ possesses derivatives of every order for every value of x

$f^{(n)}(0)=(-1)^{n-1}(n-1)!.$

Case I: Let $0 \le x \le 1$

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Lagrange's form is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}, 0<\theta<1$$

$$= \frac{(-1)^{n-1}}{n} \cdot \frac{x^n}{(1+\theta x)^n}$$

Now

$$0 \le |R_n| = \left| \frac{(-1)^{n-1}}{n} \cdot \frac{x^n}{(1+\theta x)^n} \right|$$

$$= \left| \frac{(-1)^{n-1}}{n} \right| \left| \frac{x^n}{(1+\theta x)^n} \right|$$

$$= \frac{1}{n} \cdot |x^n| \cdot \frac{1}{(1+\theta x)^n}$$

Since $0 \le x \le 1$, we have $|x^n| \le 1$ and since $0<\theta<1$ and $x \ge 0$, we have

$0<\theta$ and $x \ge 0 \Rightarrow 0 < \theta x$

i.e., $1 < 1+\theta x \Rightarrow \dfrac{1}{1+\theta x} < 1$

So

$$0 \le |R_n| \le \frac{1}{n} \cdot 1 \cdot 1$$

$\to 0$ as $n \to \infty$

Hence $\lim_{n\to\infty} R_n = 0$, when $0 \le x \le 1$.

Case II: Let $-1 < x < 0$

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Cauchy's form is

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x), 0<\theta<1$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{(1+\theta x)}$$

Since

$0 < \theta < 1$ and $x < 0 \Rightarrow 0 > \theta x > 1x$

i.e., $1 > 1 + \theta \cdot x > 1 + x$

we have

$$1 < \frac{1}{1+\theta \cdot x} < \frac{1}{1+x}$$

Again

$-1 < x < 0$ and $0 < \theta < 1 \Rightarrow -\theta < x\theta < 0$

i.e., $1 - \theta < 1 + x\theta$

i.e., $\left(\frac{1-\theta}{1+\theta x}\right) < 1$

Now, $\lim_{n\to\infty} R_n = 0$, $-1 < x < 0$ since

$$0 \le |R_n| = |x|^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{(1+\theta x)} \le |x|^n \cdot 1 \cdot \frac{1}{1+x} [-1 < x < 0]$$

$\to 0$ as $n \to \infty$ since $|x| < 1$.

So in the both cases $\lim_{n\to\infty} R_n = 0$.

Therefore, we have Maclaurin's infinite series expansion of $f(x) = \log(1+x)$, $-1 < \mathrm{x} \le 1$ and is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

$$\text{i.e., } \log(1+x) = \log 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2!) + \frac{x^4}{4!}(-3!) + \cdots$$
$$+ \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \cdots$$

$$\text{i.e., } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1} x^n}{n} + \cdots$$

4.7.7 Maclaurin's Infinite-Series Expansion of $(a+x)^n$ where n is a Positive Integer

Here

$$f(x) = (a+x)^n$$

So,

$$f^{(k)}(x) = n(n-1)(n-2)\ldots(n-k+1)(a+x)^{n-k}$$

$$= \frac{n!}{(n-k)!}(a+x)^{n-k}$$

Then, $f^{(k)}(0) = n(n-1)(n-2)\ldots(n-k+1)\cdot a^{n-k}$

$$= \frac{n!}{(n-k)!}\cdot a^{n-k}$$

So, $f^{(k)}(x)$ exists for all x and

when $k > n$, $f^{(k)}(x) = 0$.

Remainder R_n in Lagranges form is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x).$$

Now $R_k = \dfrac{x^k}{k!} f^{(k)}(\theta x) = 0$ for all $k > n$, since $f^{(k)}(\theta x) = 0$ when $k > n$.

Hence

$$\lim_{n\to\infty} R_n = 0.$$

Therefore, we have Maclaurin's infinite-series expansion of $f(x) = (a+x)^n$, for positive integer n and is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0),$$

since all other terms vanish.

i.e., $$(a+x)^n = a^n + \frac{x}{1!}\left(\frac{n!}{(n-1)!}\cdot a^{n-1}\right) + \frac{x^2}{2!}\left(\frac{n!}{(n-2)!}\cdot a^{n-2}\right)$$
$$+ \frac{x^3}{3!}\cdot\left(\frac{n!}{(n-3)!}.a^{n-3}\right) + \cdots + \frac{x^n}{n!}\cdot\frac{n!}{(n-n)!}\cdot a^{n-n}$$

i.e., $$(a+x)^n = a^n + n\cdot a^{n-1}\cdot x + \frac{n(n-1)}{2}\cdot a^{n-2}\cdot x^2$$
$$+ \frac{n(n-1)(n-2)}{3!}\cdot a^{n-3}\cdot x^3 + \cdots + x^n.$$

$$\text{i.e., } (a+x)^n = a^n + {}^nC_1 \cdot a^{n-1} \cdot x + {}^nC_2 \cdot a^{n-2} \cdot x^2 + {}^nC_3 \cdot a^{n-3} \cdot x^3 + \cdots + {}^nC_r \cdot a^{n-r} \cdot x^r + \cdots + x^n.$$

This is also known as binomial expansion of $(a+x)^n$, where n is any positive integer.

4.7.8 Maclaurin's Infinite-Series Expansion of $(a+x)^n$ where n is a Negative Integer or a Fraction

$$(a+x)^n = a^n + n \cdot a^{n-1} \cdot x + \frac{n(n-1)}{2} \cdot a^{n-2} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot a^{n-3} \cdot x^3 + \cdots + \frac{n(n-1)\ldots(n-\overline{k-1})}{k!} \cdot a^{n-k} \cdot x^k + \cdots [\text{when ever} -a < x < a]$$

This is also known as binomial expansion of $(a+x)^n$ where n is a negative integer or a fraction and $-a < x < a$.

WORKED-OUT EXAMPLES

Example 4.1 Verify Rolle's theorem for

$f(x) = x^2 - 5x + 6$ in $2 \le x \le 3$ [WBUT-2001]

Sol. Here we are to examine three conditions.

i) Since $f(x)$ is a polynomial in x and all polynomials in x are continuous functions for all values of $x \in R$,

$f(x) = x^2 - 5x + 6$ is continious for all x, where $2 \le x \le 3$.

ii) Due to the same reason as above, $f(x)$ is also derivable for all x, where $2 \le x \le 3$.

Moreover, $f'(x) = 2x - 5$, which exists for all values of x in $2 \le x \le 3$.

iii) $f(2) = f(3) = 0$.

Thus all the conditions of Rolle's theorem are satisfied by $f(x) = x^2 - 5x + 6$ in $2 \le x \le 3$ and so there should exist at least a point c, $2 < c < 3$ such that $f'(c) = 0$.

Now

$$f'(c) = 0 \Rightarrow 2c - 5 = 0 \Rightarrow c = \frac{5}{2}$$

Definitely $c = \frac{5}{2}$ lies between 2 and 3, i.e., $2 < c < 3$.

Therefore, Rolle's theorem is verified.

Example 4.2 State Rolle's theorem. Examine whether the theorem is applicable on

$f(x) = x^3 - 6x^2 + 11x - 6$ in $1 \le x \le 3$ [WBUT-2002]

Sol. Here,

$$f(x) = x^3 - 6x^2 + 11x - 6 \text{ in } 1 \le x \le 3$$

i) Since $f(x)$ is a polynomial in x and all polynomials in x are continuous functions for all values of $x \in R$,

$f(x) = x^3 - 6x^2 + 11x - 6$ is continuous for all x, where $1 \le x \le 3$.

ii) Due to the same reason as above, $f(x)$ is also derivable for all x, where $1 \le x \le 3$.

Moreover, $f'(x) = 3x^2 - 12x + 11$, which exists for all values of x in $1 \le x \le 3$.

iii) $f(1) = f(3) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied by $f(x) = x^3 - 6x^2 + 11x - 6$ in $1 \le x \le 3$

and so there should exist at least a point c, $1 < c < 3$ such that $f'(c) = 0$.

Now

$$f'(c) = 0 \Rightarrow 3c^2 - 12c + 11 = 0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm \sqrt{12}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

Definitely $c = 2 + \dfrac{1}{\sqrt{3}}$ lies between 1 and 3, i.e., $1 < c < 3$.

Therefore, Rolle's theorem is applicable.

Example 4.3 Show that Rolle's theorem is not applicable to

$f(x) = \tan x$ in $[0 \cdot \pi]$

although $f(0) = f(\pi)$. [WBUT-2004, 2006]

Sol. Here

$f(x) = \tan x$ in $[0 \cdot \pi]$.

It is obvious that $f(x) = \tan x$ is continuous everywhere in $[0, \pi]$ except at $x = \dfrac{\pi}{2}$ and consequently is not derivable there.

Also, $f(0) = f(\pi)$.

Since all the conditions of Rolle's theorem are not satisfied by $f(x) = \tan x$ in $[0 \cdot \pi]$, Rolle's theorem is not applicable to $f(x) = \tan x$ in $[0 \cdot \pi]$ although $f(0) = f(\pi)$.

Example 4.4 Show that Lagrange's mean-value theorem is not applicable to the function

$$f(x) = \begin{bmatrix} x \sin \frac{1}{x} \text{ when } x \neq 0 \\ 0 \text{ when } x = 0 \end{bmatrix} \text{in } [-1, 1]$$ [WBUT-2005]

Sol. Here we are to check two conditions.

i) Since x is continuous everywhere and $\sin \frac{1}{x}$ is continuous everywhere except at $x = 0$, $f(x) = x \sin \frac{1}{x}$ is continuous everywhere except at $x = 0$.

Now it is easy to show that $\lim\limits_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = f(0)$.

i.e., $f(x)$ is continuous at $x = 0$.

Combining the above two cases, we say that $f(x)$ is continuous everywhere and so $f(x)$ is continuous in $[-1, 1]$.

ii) When $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \left(\frac{1}{x} \right)$$

i.e., $f'(x)$ exists for $x \neq 0$.

Now

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h} \text{ which does not exist.}$$

So, $f(x)$ is not derivable at $x = 0$.

Therefore, $f(x)$ is not derivable in $(-1, 1)$.

Since all the conditions of Lagrange's mean-value theorem are not satisfied, the theorem is not applicable to $f(x)$ in $[-1, 1]$.

Example 4.5 If $f(x)=\sin^{-1}x,\ 0<a<b<1,$ use mean-value theorem to prove that

$$\frac{(b-a)}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{(b-a)}{\sqrt{1-b^2}}$$ [WBUT-2007, 2008]

Sol. Let $f(x)=\sin^{-1}x$ in $[a,b],\ 0<a<b<1.$

Here, $f(x)$ is continious in $[a,b]$ and $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ which exists for all values of x in (a,b).

Therefore, all the conditions of Lagrange's mean-value theorem are satisfied and there exists at least one value of x, say c, $a<c<b$, such that

$$f(b)-f(a)=(b-a)f'(c),\ a<c<b$$

$$\text{or, } \sin^{-1}b-\sin^{-1}a=\frac{(b-a)}{\sqrt{1-c^2}},\ a<c<b \qquad \text{...(1)}$$

Now,

$$a<c<b \Rightarrow a^2<c^2<b^2$$

$$\Rightarrow -a^2>-c^2>-b^2$$

$$\text{or, } 1-a^2>1-c^2>1-b^2$$

$$\text{or, } \sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$$

$$\text{or, } \frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$$

$$\text{or, } \frac{(b-a)}{\sqrt{1-a^2}}<\frac{(b-a)}{\sqrt{1-c^2}}<\frac{(b-a)}{\sqrt{1-b^2}} \qquad \text{...(2)}$$

Therefore, from (1) and (2), we get,

$$\frac{(b-a)}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{(b-a)}{\sqrt{1-b^2}}$$

Hence, the required result is proved.

Example 4.6 Using mean-value theorem, prove that

$$\frac{\pi}{6}+\frac{\sqrt{3}}{15}<\sin^{-1}\left(\frac{3}{5}\right)<\frac{\pi}{6}+\frac{1}{8}$$ [WBUT-2005]

Sol. Let us consider $f(x)=\sin^{-1}x$ in $\left[\frac{1}{2},\frac{3}{5}\right]$

Since $f(x)$ is continuous in $\left[\frac{1}{2},\frac{3}{5}\right]$ and $f'(x)=\frac{1}{\sqrt{1-x^2}}$ exists in $\left(\frac{1}{2},\frac{3}{5}\right)$, all the conditions of Lagrange's mean-value theorem are satisfied.

So, there exists at least one value of x, say c, $\frac{1}{2}<c<\frac{3}{5}$ such that

$$f\left(\frac{3}{5}\right)-f\left(\frac{1}{2}\right)=\left(\frac{3}{5}-\frac{1}{2}\right)f'(c)$$

$$\text{or, } \sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{1}{2}\right)=\frac{1}{10}\cdot\frac{1}{\sqrt{1-c^2}} \qquad \text{...(1)}$$

Now,

$$\frac{1}{2}<c<\frac{3}{5}\Rightarrow -\left(\frac{1}{2}\right)^2>-c^2>-\left(\frac{3}{5}\right)^2$$

$$\Rightarrow 1-\left(\frac{1}{2}\right)^2>1-c^2>1-\left(\frac{3}{5}\right)^2$$

$$\Rightarrow \frac{4}{3}<\frac{1}{1-c^2}<\frac{25}{16}$$

$$\Rightarrow \frac{1}{10}\frac{2}{\sqrt{3}}<\frac{1}{10}\frac{1}{\sqrt{1-c^2}}<\frac{1}{10}\frac{5}{4}$$

$$\Rightarrow \frac{1}{5\sqrt{3}}<\frac{1}{10}\frac{1}{\sqrt{1-c^2}}<\frac{1}{8} \qquad \text{...(2)}$$

There from (1) and (2), we have

$$\frac{1}{5\sqrt{3}}<\sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{1}{2}\right)<\frac{1}{8}$$

$$\text{or, } \frac{1}{5\sqrt{3}}<\sin^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{6}<\frac{1}{8}$$

$$\text{or, } \frac{\pi}{6}+\frac{\sqrt{3}}{15}<\sin^{-1}\left(\frac{3}{5}\right)<\frac{\pi}{6}+\frac{1}{8}$$

Thus, the required result is proved.

Alternative Method of Solution

In the last example **(4.5),** using Lagrange's mean-value theorem, we have proved that

$$\frac{(b-a)}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{(b-a)}{\sqrt{1-b^2}}$$

where $f(x) = \sin^{-1}x,\ 0 < a < b < 1.$

Putting $a = \dfrac{1}{2}$ and $b = \dfrac{3}{5}$ in the above, we have

$$\frac{\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^2}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{1}{2}\right) < \frac{\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\left(\frac{3}{5}\right)^2}}$$

$$\Rightarrow \frac{\left(\frac{1}{10}\right)}{\frac{\sqrt{3}}{2}} < \sin^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{6} < \frac{\left(\frac{1}{10}\right)}{\frac{4}{5}}$$

$$\Rightarrow \frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$$

$$\Rightarrow \frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}.$$

Example 4.7 Using Lagrange's mean-value theorem, prove that

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \quad \text{where } 0 < a < b < 2$$

Sol. Let $f(x) = \tan^{-1}x$ in $[a, b],\ 0 < a < b < 2.$

Here, $f(x)$ is continuous in $[a, b]$ and $f'(x) = \dfrac{1}{1+x^2}$ which exists for all values of x in (a, b).

Therefore, all the conditions of Lagrange's mean-value theorem are satisfied and there exists at least one value of x, say c, $a < c < b$, such that

$$f(b) - f(a) = (b-a)f'(c),\ a < c < b$$

$$\text{or, } \tan^{-1}b - \tan^{-1}a = \frac{(b-a)}{(1+c^2)},\ a < c < b \qquad ...(1)$$

Now,

$$a < c < b$$

$$\text{or, } 1+a^2 < 1+c^2 < 1+b^2$$

$$\text{or, } \frac{1}{1+a^2} > \frac{1}{(1+c^2)} > \frac{1}{1+b^2}$$

$$\text{or, } \frac{(b-a)}{1+a^2} > \frac{(b-a)}{(1+c^2)} > \frac{(b-a)}{1+b^2} \quad \text{...(2)}$$

Therefore, from (1) and (2), we get,

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

Hence the required result is proved.

Example 4.8 Using mean-value theorem, prove that

$$\frac{x}{1+x^2} < \tan^{-1}x < x,\ 0 < x < \frac{\pi}{2}$$

Sol. Let $f(x) = \tan^{-1} x$ for $0 < x < \frac{\pi}{2}$.

It is obvious that $f(x)$ satisfies all the conditions of Lagrange's MVT in $[0, x]$.

From Section 4.3.3, we have Lagrange's MVT in the interval $[0, x]$ as

$f(x) = f(0) + x f'(\theta x)$, where $0 < \theta < 1$.

Here, $f(0) = \tan^{-1}0 = 0$ and $f'(x) = \dfrac{1}{1+x^2}$.

So from above

$$\tan^{-1}x = 0 + x\frac{1}{1+(\theta x)^2}, \text{ where } 0 < \theta < 1.$$

$$\text{or, } \tan^{-1}x = \frac{x}{1+(\theta x)^2}$$

Now we have

$0 < \theta < 1 \Rightarrow 0 < \theta x < x$, since $x > 0$

$$\text{or, } 0 < (\theta x)^2 < x^2 \Rightarrow 1 < 1 + (\theta x)^2 < 1 + x^2$$

$$\text{or, } \frac{1}{1+x^2} < \frac{1}{1+(\theta x)^2} < 1$$

$$\text{or, } \frac{x}{1+x^2} < \frac{x}{1+(\theta x)^2} < x, \quad \text{since } x > 0$$

Again $\tan^{-1}x = \dfrac{x}{1+(\theta x)^2}$, therefore

$$\frac{x}{1+x^2} < \tan^{-1}x < x.$$

Example 4.9 Use MVT to prove that

$$\sin 46^\circ \sim \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right) \qquad \text{[WBUT-2003]}$$

Sol. Let us consider $f(x) = \sin x$ in $[45^\circ, 46^\circ]$.

It is obvious that all the conditions of Lagrange's MVT are satisfied by $f(x) = \sin x$ in $[45^\circ, 46^\circ]$.

Also $f'(x) = \cos x$.

Now Lagrange's mean-value theorem in $[a, a+h]$ is **[see alternative form (1) of Section 4.3.3]**

$$f(a+h) = f(a) + hf'(a+\theta h), \quad 0<\theta<1$$

Putting $a = 45^\circ$ and $h = 1^\circ$, we have

$$f(46^\circ) = f(45^\circ) + 1^\circ \cdot f'(45^\circ + \theta \cdot 1^\circ), 0<\theta<1$$

$$\text{or, } \sin 46^\circ = \sin 45^\circ + 1^\circ \cdot \cos(45^\circ + \theta \cdot 1^\circ), 0<\theta<1$$

$$= \sin 45^\circ + \frac{\pi}{180}\cos(45^\circ + \theta^\circ), 0<\theta<1$$

Since θ is very small,

$$\sin 46^\circ \sim \sin 45^\circ + \frac{\pi}{180}\cos 45^\circ$$

$$\sim \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cdot\frac{1}{\sqrt{2}}$$

$$\sim \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right)$$

Example 4.10 Estimate $\sqrt[3]{28}$ using Lagrange's mean-value theorem.

Sol. Let us consider the function $f(x) = x^{\frac{1}{3}}$ in $[27, 28]$.

Here, $f(x)$ is continuous in $[27, 28]$ and $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$ exists for all values of x in $(27, 28)$.

So all the conditions of Lagrange's MVT are satisfied.

Therefore, by Lagrange's mean-value theorem in the interval $[a, a+h]$, we have

$$f(a+h) = f(a) + hf'(a+\theta h), 0<\theta<1$$

Putting $a = 27,\ \ h = 28 - 27 = 1$

$$f(28) = f(27) + 1 \cdot f'(27 + \theta \cdot 1),\ 0 < \theta < 1$$

$$\text{or, } \sqrt[3]{28} = \sqrt[3]{27} + \frac{1}{3}(27 + \theta)^{\frac{-2}{3}}$$

$$\Rightarrow \sqrt[3]{28} = 3 + \frac{1}{3}\frac{1}{(27+\theta)^{\frac{2}{3}}} < 3 + \frac{1}{3}\frac{1}{(27)^{\frac{2}{3}}}$$

$$\Rightarrow \sqrt[3]{28} < \left(3 + \frac{1}{3 \cdot 9}\right) = 3\frac{1}{27}$$

Again since $27 < 28 \Rightarrow \sqrt[3]{27} < \sqrt[3]{28} \Rightarrow 3 < \sqrt[3]{28}$, we have

$$3 < \sqrt[3]{28} < 3\frac{1}{27}.$$

Example 4.11 If $f'(x) = \dfrac{1}{4 - x^2}$ and $f(0) = 1$, using Lagrange's mean-value theorem, estimate $f(1)$.

Sol. Since, $f'(x) = \dfrac{1}{4 - x^2}$ exists for all x in $(0, 1)$, therefore $f(x)$ is continuous in $[0, 1]$.

Therefore, applying Lagrange's mean-value theorem to $f(x)$ in $[0, 1]$, there exists at least one value of x, say c, $0 < c < 1$ such that

$$f(1) - f(0) = (1 - 0)f'(c),\ 0 < c < 1$$

$$\Rightarrow f(1) - 1 = \frac{1}{4 - c^2},\ 0 < c < 1$$

Since $0 < c < 1$, we have

$$\left[\frac{1}{4 - c^2}\right]_{c=0} < f(1) - 1 < \left[\frac{1}{4 - c^2}\right]_{c=1}$$

$$\text{or, } \frac{1}{4} < f(1) - 1 < \frac{1}{3}$$

$$\text{or, } 1.25 < f(1) < 1.33.$$

The above gives an estimate for $f(1)$.

Example 4.12 Find the values of a, b, c for which the function

$$f(x) = 3; x \le 0$$
$$= -x^2 + 3x + a; 0 < x < 1$$
$$= bx + c; 1 \le x \le 2$$

satisfies the conditions of Lagrange's mean-value theorem.

Sol. The function $f(x)$ will satisfy the conditions of Lagrange's mean-value theorem if

(i) $f(x)$ is continuous in $[0, 2]$

(ii) $f(x)$ is derivable in $(0, 2)$

The function

$$f(x) = 3; x \le 0$$
$$= -x^2 + 3x + a; 0 < x < 1$$
$$= bx + c; 1 \le x \le 2$$

is continuous and derivable for all values of x in $[0, 2]$ except at $x = 0$ and $x = 1$.

Now,

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (-x^2 + 3x + a) = a$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (3) = 3 \text{ and } f(0) = 3$$

Therefore, $f(x)$ is continuous at $x = 0$ if

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0)$$

$$\Rightarrow a = 3 \quad ...(1)$$

Again

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (bx + c) = b + c$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (-x^2 + 3x + a) = a + 2 = 5, \text{ since } a = 3$$

and $f(1) = b + c$

Therefore, $f(x)$ is continuous at $x = 1$ if

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1)$$

$$\Rightarrow b + c = 5 \quad ...(2)$$

Similarly, $f(x)$ is derivable at $x=1$ if

$$\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0-}\frac{f(1+h)-f(1)}{h}$$

$$\text{or, }\lim_{h\to 0+}\frac{b(1+h)+c-(b+c)}{h}=\lim_{h\to 0-}\frac{-(1+h)^2+3(1+h)+a-(b+c)}{h}$$

Putting $a=3$ and $b+c=5$,

$$\text{or, }\lim_{h\to 0+}\frac{bh}{h}=\lim_{h\to 0-}\frac{-(1+h)^2+3(1+h)+3-5}{h}$$

$$\text{or, }\lim_{h\to 0+}\frac{bh}{h}=\lim_{h\to 0-}\frac{-h^2+h}{h}$$

or, $b=1$ ...(3)

Therefore, from (1), (2) and (3), we have $a=3$, $b=1$ and $c=4$.

Example 4.13 Apply Maclaurin's theorem to the function $f(x)=(1+x)^4$ to deduce that

$(1+x)^4=1+4x+6x^2+4x^3+x^4$ [WBUT-2001]

Sol. Maclaurin's theorem with Lagrange's form of remainder is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(iv)}(0)$$

$$+\cdots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\theta x),\quad 0<\theta<1 \qquad ...(1)$$

Here,

$f(x)=(1+x)^4 \quad \Rightarrow f(0)=1$

$f'(x)=4(1+x)^3 \quad \Rightarrow f'(0)=4$

$f''(x)=12(1+x)^2 \quad \Rightarrow f''(0)=12$

$f'''(x)=24(1+x) \quad \Rightarrow f'''(0)=24$

$f^{(iv)}(x)=24 \quad \Rightarrow f^{(iv)}(0)=24$

$f^{(n)}(x)=0$ for all x, when $n>4$

Therefore, putting the above values in (1), we have

$$(1+x)^4 = 1+4x+12\cdot\frac{x^2}{2!}+24\cdot\frac{x^3}{3!}+24\cdot\frac{x^4}{4!}$$

$$= 1+4x+6x^2+4x^3+x^4$$

Example 4.14 If

$$f(h) = f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h),\ 0<\theta<1,\ f(x)=\frac{1}{1+x}$$

and $h=7$, find θ.

Sol. Here

$$f(x)=\frac{1}{1+x} \Rightarrow f(0)=1$$

$$f'(x)=\frac{-1}{(1+x)^2} \Rightarrow f'(0)=-1$$

$$f''(x)=\frac{2}{(1+x)^3} \Rightarrow f''(\theta h)=\frac{2}{(1+\theta h)^3}$$

So, $f(x)$ satisfies the conditions of Maclaurin's theorem for $x\neq -1$ in $[0,h]$.

The given expression

$$f(h) = f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h) \qquad ...(1)$$

is Maclaurin's theorem with Lagrange's form of remainder after 2 terms.

Therefore, putting the values in (1),

$$\frac{1}{1+h} = 1-h+\frac{h^2}{2}\frac{2}{(1+\theta h)^3}$$

For $h=7$, we have

$$\frac{1}{8} = 1-7+\frac{49}{(1+7\theta)^3}$$

$$\text{or, } (1+7\theta)^3 = 8 \quad \Rightarrow 1+7\theta = 2$$

$$\Rightarrow \theta = \frac{1}{7}\text{(Ans.)}$$

Example 4.15 Expand a^x in a finite series with Lagrange's form of remainder.

[WBUT-2002]

Sol. Here,

$$f(x) = a^x = e^{(\log_e a)x} \quad \Rightarrow f(0) = 1$$

$$f'(x) = (\log_e a)e^{(\log_e a)x} \quad \Rightarrow f'(0) = (\log_e a)$$

$$f''(x) = (\log_e a)^2 e^{(\log_e a)x} \quad \Rightarrow f''(0) = (\log_e a)^2$$

....

$$f^{(n)}(x) = (\log_e a)^n e^{(\log_e a)x} \quad \Rightarrow f^{(n)}(\theta x) = (\log_e a)^n e^{(\log_e a)\theta x}$$

So, it is clear from the above that the given function satisfies all the conditions of Maclaurin's theorem.

Now the Maclaurin's series in finite form with Lagrange's form of remainder is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)$$

$$+ \frac{x^n}{n!} f^{(n)}(\theta x), 0 < \theta < 1 \qquad ...(1)$$

Putting the values from above in (1), we have the expansion of $f(x) = a^x$ in a finite series with Lagrange's form of remainder as

$$a^x = 1 + x \cdot (\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \cdots$$

$$+ \frac{x^{n-1}}{(n-1)!}(\log_e a)^{n-1} + \frac{x^n}{n!}(\log_e a)^n e^{(\log_e a)\theta x}$$

Example 4.16 Expand the function $e^x \sin x$ in powers of x in infinite series:

Sol. Here,

$$f(x) = e^x \sin x \Rightarrow f(0) = 0$$

We know from successive differentiation that if $y = e^{ax} \sin bx$ then

$$y_n = (a^2 + b^2)^{\frac{n}{2}} \cdot e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

Therefore, for $f(x) = e^x \sin x,$ we have

$$f^{(n)}(x) = (1^2 + 1^2)^{\frac{n}{2}} \cdot e^x \cdot \sin(x + n \cdot \tan^{-1} 1)$$

$$= (2)^{\frac{n}{2}} \cdot e^x \cdot \sin\left(x + \frac{n\pi}{4}\right)$$

$$= (\sqrt{2})^n \cdot e^x \cdot \sin\left(x + \frac{n\pi}{4}\right), \text{ for each } n.$$

Thus, $f(x) = e^x \sin x$ possesses derivatives of every order and for every value of x in any interval $[-h, h]$.

Here

$$f^{(n)}(0) = (\sqrt{2})^n \cdot \sin\left(\frac{n\pi}{4}\right), \text{ for each } n$$

So,

$$f'(0) = (\sqrt{2}) \cdot \sin\left(\frac{\pi}{4}\right)$$

$$f''(0) = (\sqrt{2})^2 \cdot \sin\left(\frac{2\pi}{4}\right)$$

… and so on.

Now in the Maclaurin's finite expansion, the remainder R_n after n terms in Lagrange's form is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!} \cdot (\sqrt{2})^n \cdot e^{\theta x} \cdot \sin\left(\frac{n\pi}{4} + \theta x\right), 0 < \theta < 1$$

Now,

$$0 \le |R_n| = (\sqrt{2})^n \left|\frac{x^n}{n!}\right| \left|e^{\theta x}\right| \left|\sin\left(\frac{n\pi}{4} + \theta x\right)\right| \qquad ...(1)$$

Since

$$0 < \theta < 1 \Rightarrow 0 < \theta x < x \Rightarrow 0 < |\theta x| < |x|$$

we have

$$\left|e^{\theta x}\right| \le e^{|\theta x|} < e^{|x|}, \text{which is a finite quantity.}$$

and

$$\left|\sin\left(\frac{n\pi}{4} + \theta x\right)\right| \le 1.$$

Also

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x.$$

So from (1), we get

$$\lim_{n\to\infty} R_n = 0.$$

Therefore, we have Maclaurin's infinite-series expansion of $f(x) = e^x \sin x$ and is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

$$\text{i.e., } e^x \sin x = 0 + \frac{x}{1!}\left(\sqrt{2}\sin\frac{\pi}{4}\right) + \frac{x^2}{2!}(\sqrt{2})^2 \sin\left(\frac{2\pi}{4}\right) + \frac{x^3}{3!}(\sqrt{2})^3 \sin\left(\frac{3\pi}{4}\right)$$
$$+ \frac{x^4}{4!}(\sqrt{2})^4 \sin\left(\frac{4\pi}{4}\right) + \cdots$$

$$\text{i.e., } e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

Example 4.17 Apply Maclaurin's theorem to prove that

$$\sin x > x - \frac{1}{6}x^3, \text{ if } 0 < x < \frac{\pi}{2}.$$

Sol. Maclaurin's theorem with Lagrange's form of remainder after 3 terms is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x), \quad 0 < \theta < 1 \qquad \text{...(1)}$$

Let

$$f(x) = \sin x \Rightarrow f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \Rightarrow f'''(\theta x) = -\cos(\theta x)$$

Therefore, putting the above values in (1), we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x), \quad 0 < \theta < 1$$

$$\text{or, } \sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}\{-\cos(\theta x)\}, \quad 0 < \theta < 1$$

$$\text{or, } \sin x = x - \frac{x^3}{6}\cos(\theta x), 0 < \theta < 1 \qquad \text{...(2)}$$

Since

$$0 < \theta < 1 \text{ and } 0 < x < \frac{\pi}{2},$$

we have

$$0 < \theta x < \frac{\pi}{2} \Rightarrow 0 < \cos(\theta x) < 1.$$

Therefore,

$$\frac{x^3}{6}\cos(\theta x) < \frac{x^3}{6} \text{ since } x > 0$$

$$\Rightarrow -\frac{x^3}{6}\cos(\theta x) > -\frac{x^3}{6} \Rightarrow x - \frac{x^3}{6}\cos(\theta x) > x - \frac{x^3}{6}$$

Hence from (2),

$$\sin x > x - \frac{1}{6}x^3$$

EXERCISES

Short and Long Answer Type Questions

1. Show that Rolle's theorem holds for the following functions:

   a) $f(x) = 4 - x^2$ in $[-2, 2]$

   b) $f(x) = (x+2)^3(x-4)$ in $[-2, 4]$

   c) $f(x) = x(x+2)e^{\frac{-x}{2}}$ in $[-2, 0]$

   d) $f(x) = \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   e) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

2. Show that Rolle's theorem is not applicable to the following functions:

   (i) $f(x) = |x-3|$ in $[0, 3]$.

(ii) $f(x)=\sin\left(\frac{1}{x}\right)$ in $[-1,1]$

(iii) $f(x)=3+(x-1)^{\frac{1}{3}}$ in $[-1,1]$

3. If the function $f(x)$ is defined on $[0,1]$ by

$$f(x)=2, \text{ if } 0\le x\le \frac{1}{3}$$
$$=3, \text{ if } \frac{1}{3}<x\le 1.$$

then prove that $f(x)$ satisfies none of the conditions of Rolle's theorem but $f'(x)$ vanishes for each $x\in(0,1)$.

4. If

$$f(x)=\begin{vmatrix}\sin x & \sin\theta & \sin\phi\\ \cos x & \cos\theta & \cos\phi\\ \tan x & \tan\theta & \tan\phi\end{vmatrix},\quad 0<\theta<\phi<\frac{\pi}{2},$$

using Rolle's theorem, show that $f'(\psi)=0$, where $\theta<\psi<\phi$.

5. Verify Lagrange's mean-value theorem for the following functions:

(i) $f(x)=x(x-1)(x-2)$ in $\left[0,\frac{1}{2}\right]$

(ii) $f(x)=\sqrt{x}$ in $[9,16]$

(iii) $f(x)=\begin{bmatrix} x\cos\frac{1}{x} \text{ when } x\ne 0\\ 0 \text{ when } x=0\end{bmatrix}$ in $[-1,1]$

(iv) $f(x)=\log\sin x$ in $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

(v) $f(x)=3-\sqrt[3]{(x-2)^2}$ in $[4,10]$

[**Ans :** (i) yes (ii) yes (iii) no (iv) yes (v) no]

6. Verify Cauchy's MVT for the functions:

a) $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{x}$ in $[1,3]$

b) $f(x)=\log x$ and $g(x)=x$ in $[1,e]$

c) $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ in $[3,6]$

d) $f(x)=x(x-2)$ and $g(x)=x^2$ in $[-1,1]$

[**Ans :** (i) yes (ii) yes (iii) yes (iv) no]

7. If $f(x)=\cos x$, $g(x)=\sin x$ in $[a,b]$, then show that c of Cauchy's mean-value theorem for the two function is $A.M.$ of a and b.

8. Using mean-value theorem, prove the following inequalities:

a) $\dfrac{x}{\sqrt{1-x^2}} \ge \sin^{-1}x \ge x$, if $0 \le x < 1$

b) $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1$, if $x > 0$

c) $x < \log\dfrac{1}{1-x} < \dfrac{x}{1-x}$, if $0 < x < 1$

d) $\dfrac{2x}{1-x^2} > \log\dfrac{1+x}{1-x} > x$, if $0 < x < 1$

e) $\dfrac{2x}{\pi} \le \sin x \le x$, if $0 \le x \le \dfrac{\pi}{2}$

9. Estimate $\cos 61^\circ$ using Lagrange's form of MVT. [**Ans :** 0.4849]

10. Find the approximate value of $\sqrt{9.12}$ using Taylor's theorem. [**Ans :** 3.0199]

11. In the mean-value theorem

$$f(a+h)=f(a)+hf'(a+\theta h),\ 0<\theta<1$$

if

$$f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x$$

and $a=0, h=3$, show that θ has two values.

12. Using mean-value theorem prove the following:

(i) $\sqrt{101}$ lies between 10 and 10.05

(ii) $\dfrac{1}{7} < \log 1.4 < \dfrac{1}{5}$

(iii) $\dfrac{\pi}{4}+\dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4}+\dfrac{1}{6}$

13. Use Taylor's theorem to expand $x^4-3x^3+2x^2-x+1$ in powers of $(x-3)$.

[**Ans :** $16+38(x-3)+29(x-3)^2+9(x-3)^3+(x-3)^4$]

14. Use Maclaurin's theorem to prove the following:

i) $\log(1+x) > x-\dfrac{x^2}{2}$, if $x>0$

ii) $\cos x > 1-\dfrac{x^2}{2}$, if $0<x<\dfrac{\pi}{2}$

iii) $e^x > 1+x+\dfrac{x^2}{2}$, if $x>0$

15. Prove the following by infinite series expansion:

(i) $e^x = e^2\left[1+(x-2)+\dfrac{(x-2)^2}{2!}+\dfrac{(x-2)^3}{3!}+\cdots\right]$ for all $x\in[2-h,2+h]$

(ii) $\log x = (x-1)-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}-\cdots$ for $0<x\le 2$.

(iii) $\sin x = 1-\dfrac{1}{2!}\left(x-\dfrac{\pi}{2}\right)^2+\dfrac{1}{4!}\left(x-\dfrac{\pi}{2}\right)^4-\cdots$

16. Find the binomial expansion of $(1+x)^n$, when n is a positive integer.

[**Ans :** $1+{}^nC_1\cdot x+{}^nC_2\cdot x^2+{}^nC_3\cdot x^3+\cdots+{}^nC_r\cdot x^r+\cdots+x^n$]

Multiple-Choice Questions

1. Maclaurin's expansion for the function $f(x)=\sqrt[4]{x}$ in $[-1,1]$ is

a) applicable
b) not applicable
c) partially suitable
d) none of these

2. Lagrange's MVT is obtained from Cauchy's MVT for the function $f(x)$ and $g(x)$ by putting $g(x)=$

a) x
b) 0
c) 1
d) none of these

3. Which of the following functions does not satisfy the conditions of Rolle's theorem in $[-1,1]$?

a) x^2
b) $\dfrac{1}{x^4+2}$
c) $\dfrac{1}{x}$
d) $\sqrt{x^2+3}$

4. Which of the following pair of functions do not satisfy the conditions of Cauchy's MVT in $[-2, 2]$?

a) x^2, $\log x$ b) $\sin x^2$, x

c) $|x-4|$, x^2+4 d) $1+x^2$, $\dfrac{x}{x^2+4}$

5. $\log(1+x)$ can be expanded in an infinite series on the interval

a) $(-1, 1]$ b) $[-1, 1]$ c) $[-1, 1)$ d) $(-1, 1)$

6. The region of validity of the expansion $\log(1+5x)$ is

a) $-5 < x < 5$ b) $\dfrac{-1}{5} \le x \le \dfrac{1}{5}$

c) $\dfrac{-1}{5} < x < \dfrac{1}{5}$ d) $\dfrac{-1}{5} < x \le \dfrac{1}{5}$

7. If a function $f(x)$ satisfies all the conditions of Rolle's theorem on $[a, b]$ then $f'(x)$ vanishes

a) every where on (a, b) b) at exactly one point of (a, b)

c) at least one point of (a, b) d) none of these

8. Let $f(x)$ be a differentiable function on $(7, 9)$. Then $f(x)$ satisfies the conditions of Lagrange's mean-value theorem on $[7, 9]$ if

a) $f(x)$ is continious on $(7, 9)$ b) $f(x)$ is continious at $[7, 9]$

c) $f(x)$ is continious at $x = 7$ d) none of these

9. If $f(x)$ is continious in $[a, a+h]$ and derivable in $(a, a+h)$ then $f(a+h) - f(a) = hf'(a+\theta h)$, where

a) θ is any real number b) $0 < \theta < 1$

c) $\theta > 1$ d) θ is an integer

10. The region of validity of the expansion $\log_e(1+2x)$ in Maclaurin's infinite series is

a) $-1 < x \le 1$ b) $\dfrac{-1}{2} < x \le \dfrac{1}{2}$

c) $0 \le x \le 2$ d) none of these

11. If the Maclaurin's expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{k} - \cdots \infty$$

then the value of k is

a) $4!$ b) $5!$ c) $-5!$ d) 6

12. For a function $f(x)$ the expression

$$\frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

is known as

a) Lagrange's remainder b) Cauchy's remainder
c) Maclaurin's remainder d) Taylor's remainder

13. $f(x)$ has derivative of every order in a neighbourhood of zero. Then $f(x)$ can be expanded in an infinite series if

a) $f^{(n)}(x) = 0$ for some n and x
b) remainder R_n exists for all n
c) remainder $R_n \to 0$ as $n \to \infty$
d) none of these

14. Which of the following statements is true?

a) Two conditions are necessary for Rolle's theorem.
b) If $f'(c) = 0$, $a < c < b$ then $f(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$.
c) If $f'(c) = 0$, $a < c < b$ then $f(x)$ must be continious in $[a, b]$.
d) Two conditions are necessary for Lagrange's mean-value theorem.

15. Cauchy's mean-value theorem can not be applied on the two functions $f(x) = x^3$ and $g(x) = x^4$ on the interval $[-2, 2]$ because

a) x^4 is not derivable at apoint in the interval $(-2, 2)$
b) $f(x)$ is an odd function
c) $4x^3 = 0$ at $x = 0$
d) $g(x)$ is an even function

Answers:

1. (b) 2. (a) 3. (c) 4. (a) 5. (a) 6. (d) 7. (c) 8. (c) 9. (b)
10. (b) 11. (b) 12. (b) 13. (c) 14. (d) 15. (c)

CHAPTER

5

Reduction Formula

5.1 INTRODUCTION

In the present chapter, we deal with the concept of reduction formula for integration. Basically, reduction formula allow us to express an integration involving higher powers of a function by another integration which involves comparatively lower powers of the same function. Also, by means of reduction formula we are able to compute indefinite as well as definite integrals.

Here in the chapter first we develop reduction formulas for some standard integrations and then we apply those formulas to evaluate the integrations. Each of the items are illustrated with suitable examples.

5.2 REDUCTION FORMULA FOR

(a) $\int \sin^n x \, dx$, where n (>1) is a positive integer

(b) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$, where $n(>1)$ is a positive integer [WBUT 2006]

(a) Let us consider

$$\begin{aligned} I_n &= \int \sin^n x \, dx \\ &= \int \sin^{n-1} x \cdot \sin x \, dx \end{aligned}$$

Integrating by parts taking $\sin^{n-1} x$ as the first function and $\sin x$ as the second function, we have

$$I_n = \sin^{n-1} x \int \sin x\, dx - \int \left\{ \frac{d(\sin^{n-1} x)}{dx} \int \sin x dx \right\} dx$$

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$\Rightarrow I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

i.e., $$\boxed{I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}} \quad ...(1)$$

Therefore, (1) represents the reduction formula for $I_n = \int \sin^n x dx$ where n is any positive integer

(b) Let us consider

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = [I_n]_0^{\pi/2}.$$

Now taking limits on both sides of (1), we have

$$[I_n]_0^{\frac{\pi}{2}} = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n}[I_{n-2}]_0^{\frac{\pi}{2}}$$

i.e., $$\boxed{J_n = \frac{(n-1)}{n} \cdot J_{n-2}} \quad ...(2)$$

Therefore (2) represents the reduction formula for $J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$.

Calculation of the Value of the Definite Integral

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x dx, \text{ where } n(>1) \text{ is a positive integer.}$$

From (2), we have the reduction formula for above as

$$J_n = \frac{(n-1)}{n} \cdot J_{n-2} \quad ...(3)$$

Now replacing n by $n-2$, we get from (3)

$$J_{n-2} = \frac{(n-3)}{(n-2)} \cdot J_{n-4} \quad ...(4)$$

From (3) and (4), we obtain

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot J_{n-4} \qquad \text{...(5)}$$

Again replacing n by $n-2$, we get from (4)

$$J_{n-4} = \frac{(n-5)}{(n-4)} \cdot J_{n-6} \qquad \text{...(6)}$$

From (5) and (6), we have

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdot J_{n-6}$$

Similarly proceeding as above, we have the following cases:

Case (i) n *is even.*

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot J_0$$

Again

$$J_0 = \int_0^{\pi/2} \sin^0 x dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Hence

$$\boxed{\boldsymbol{J_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}, \text{ when } n \text{ is even.}}}$$

Case (ii) n *is odd.*

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot J_1$$

Again

$$J_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1.$$

Hence

$$\boxed{\boldsymbol{J_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \frac{2}{3}, \text{ when } n \text{ is odd.}}}$$

Alternative method of finding reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$, where $n(>1)$ is any positive integer

Let

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= [\sin^{n-1}x\int \sin x\, dx]_0^{\pi/2} - \int_0^{\frac{\pi}{2}}\left\{\frac{d(\sin^{n-1}x)}{dx}\int \sin x\, dx\right\}dx$$

$$= [-\sin^{n-1}x\cos x]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}x\cos^2 x dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x(1-\sin^2 x)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}xdx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n xdx$$

i.e., $J_n = (n-1)J_{n-2} - (n-1)J_n$

$$\Rightarrow J_n = \frac{(n-1)}{n}J_{n-2}$$

Therefore, the reduction formula of $J_n = \int_0^{\frac{\pi}{2}} \sin^n x \quad dx$, where n is any positive integer is

$$J_n = \frac{(n-1)}{n}J_{n-2}.$$

Example 1 Using reduction formula, find $\int \sin^4 x\, dx$.

Sol. If we consider

$$I_n = \int \sin^n x\, dx$$

Then the reduction formula is

$$I_n = \frac{-\sin^{n-1} x\cos x}{n} + \frac{(n-1)}{n}I_{n-2}$$

We are to calculate

$$I_4 = \int \sin^4 x\, dx.$$

Here

$$I_4 = \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}I_2.$$

Again

$$I_2 = \int \sin^2 x\, dx = \frac{-\sin x\cos x}{2} + \frac{1}{2}I_0$$

and

$$I_0 = \int \sin^0 x\, dx = \int dx = x$$

Therefore, we have

$$I_4 = \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}I_2$$

$$= \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}\left\{\frac{-\sin x\cos x}{2} + \frac{1}{2}I_0\right\}$$

$$= \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}\left\{\frac{-\sin x\cos x}{2} + \frac{1}{2}x\right\}$$

$$= \frac{-\sin^3 x\cos x}{4} - \frac{3}{4}\frac{\sin x\cos x}{2} + \frac{3}{8}x$$

Example 2 Using reduction formula, find $\int_0^{\frac{\pi}{2}} \sin^5 x dx$. [WBUT 2006]

Sol. If we consider

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

Then the reduction formula is

$$J_n = \frac{(n-1)}{n}\cdot J_{n-2}$$

We are to calculate

$$J_5 = \int_0^{\frac{\pi}{2}} \sin^5 x\, dx$$

Here,

$$J_5 = \frac{4}{5}\cdot J_3$$

Again

$$J_3 = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}\cdot J_1$$

and

$$J_1 = \int_0^{\frac{\pi}{2}} \sin x dx = 1$$

Therefore,

$$J_5 = \int_0^{\frac{\pi}{2}} \sin^5 x\; dx = \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{15}$$

Example 3 Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$.

Sol. Let us consider

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

So, we are to calculate

$$J_6 = \int_0^{\frac{\pi}{2}} \sin^6 x\, dx.$$

Here $n = 6$, an even integer.

Now we have the value of J_n, when n is even as

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Hence,

$$J_6 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{32}.$$

5.3 REDUCTION FORMULA FOR

(a) $\int \cos^n x\, dx$, where $n (> 1)$ is a positive integer

(b) $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$, where $n (> 1)$ is a positive integer

[WBUT 2008]

(a) Let us consider

$$I_n = \int \cos^n x\, dx$$

$$= \int \cos^{n-1} x \cdot \cos x\, dx$$

Integrating by parts, we have

$$I_n = \cos^{n-1} x \int \cos x\, dx - \int \left\{ \frac{d(\cos^{n-1} x)}{dx} \int \cos x dx \right\} dx$$

$$= \cos^{n-1} x \sin x - \int -(n-1) \cos^{n-2} x \cdot \sin x \sin x dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x - (n-1) \int \cos^n x dx$$

i.e., $I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

i.e., $n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$$\boxed{\Rightarrow I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}} \qquad \text{...(1)}$$

Therefore, (1) represents the reduction formula of $I_n = \int \cos^n x\, dx,$ where n is any positive integer.

(b) Let us consider

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx = [I_n]_0^{\frac{\pi}{2}}$$

Now taking limits on both sides of (1), we have

$$[I_n]_0^{\frac{\pi}{2}} = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n}[I_{n-2}]_0^{\frac{\pi}{2}}$$

$$\text{i.e., } J_n = \frac{(n-1)}{n} \cdot J_{n-2} \qquad \text{...(2)}$$

Therefore, (2) represents the reduction formula for $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx.$

Calculation of the value of the Definite Integral

$J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, where $n(> 1)$ is a positive integer.

It is very interesting to see that

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n \left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

So, the values of two definite integrations $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$ and $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$ are same.

Hence from the last section 5.2, we have the value of $J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ as the following

Case (i) When n *is even.*

$$\boxed{J_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}.}$$

Case (ii) When n *is odd.*

$$\boxed{J_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \frac{2}{3}.}$$

Alternative method of finding Reduction Formula for $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$, where n (>1) is any positive integer.

Let

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x\, dx$$

Using integration by parts,

$$J_n = \left[\cos^{n-1} x \int \cos x dx \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left\{ \frac{d(\cos^{n-1} x)}{dx} \int \cos x dx \right\} dx$$

$$= 0 + \int_0^{\frac{\pi}{2}} (n-1) \cos^{n-2} x \cdot \sin x \cdot \sin x \cdot dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \cdot \sin^2 x dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x (1 - \cos^2 x) dx$$

i.e., $J_n = (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x dx$

i.e., $J_n = (n-1) J_{n-2} - (n-1) J_n$

i.e., $J_n = \dfrac{(n-1)}{n} J_{n-2}$

Therefore the reduction formula of $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\ dx$, where $n(>1)$ is any positive integer is given by

$$J_n = \frac{(n-1)}{n} J_{n-2}.$$

Example 4 Using reduction formula, find $\int \cos^4 x\, dx$.

Sol. If we consider

$$I_n = \int \cos^n x dx$$

then the reduction formula is

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$$

We are to calculate

$$I_4 = \int \cos^4 x\, dx.$$

Here

$$I_4 = \int \cos^4 xdx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2$$

$$I_2 = \int \cos^2 xdx = \frac{\cos x \sin x}{2} + \frac{1}{2} I_0$$

$$I_0 = \int dx = x$$

Therefore,

$$I_4 = \int \cos^4 xdx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left\{\frac{\cos x \sin x}{2} + \frac{1}{2} I_0\right\}$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left\{\frac{\cos x \sin x}{2} + \frac{1}{2} x\right\}$$

Example 5 Using reduction formula, find $\int_0^{\frac{\pi}{2}} \cos^6 xdx$.

Sol. If we consider

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n xdx$$

Then the reduction formula is

$$J_n = \frac{(n-1)}{n} \cdot J_{n-2}$$

We are to calculate

$$J_6 = \int_0^{\frac{\pi}{2}} \cos^6 x\, dx$$

Here,

$$J_6 = \frac{5}{6} \cdot J_4$$

Again

$$J_4 = \frac{3}{4} \cdot J_2$$

and

$$J_2 = \frac{1}{2} \cdot J_0$$

But

$$J_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Therefore,

$$J_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

5.4 REDUCTION FORMULA FOR

(a) $\int \sin^m x \cos^n x dx$, where $m(>1)$ and $n(>1)$ are positive integers

(b) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$, where $m(>1)$ and $n(>1)$ are positive integers [WBUT 2008]

(a) Let

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \cos^{n-1} x(\sin^m x \cos x) \, dx$$

Integrating by parts, we have

$$I_{m,n} = \cos^{n-1} x \int \sin^m x \cos x dx - \int \left\{ \frac{d(\cos^{n-1} x)}{dx} \int \sin^m x \cos x dx \right\} dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int -(n-1)\cos^{n-2} x \sin x \frac{\sin^{m+1} x}{m+1} dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \cos^{n-2} x \sin^2 xdx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \cos^{n-2} x(1-\cos^2 x)dx$$

so, $$I_{m,n} = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} \int \sin^m x \cos^{n-2} xdx - \frac{(n-1)}{(m+1)} \int \sin^m x \cos^n xdx$$

$$\Rightarrow \frac{(m+n)}{(m+1)} I_{m,n} = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{(m+1)} I_{m,n-2}$$

$$\boxed{\Rightarrow I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{(m+n)} + \frac{(n-1)}{(m+n)} I_{m,n-2}} \qquad ...(1)$$

If we write

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^{m-1} x\,(\cos^n x \sin x)\,dx$$

Then we can get

$$\boxed{I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{(m+n)} + \frac{(m-1)}{(m+n)} I_{m-2,n}} \qquad \text{...(2)}$$

Therefore, the reduction formula of $I_{m,n} = \int \sin^m x \cos^n x\,dx$ is given by both the formulas (1) and (2).

(b) Let us consider

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \left[I_{m,n}\right]_0^{\frac{\pi}{2}}.$$

Now taking limits on both sides of (1), we have

$$\left[I_{m,n}\right]_0^{\frac{\pi}{2}} = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{(m+n)}\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{(m+n)}\left[I_{m,n-2}\right]_0^{\frac{\pi}{2}}$$

i.e., $$J_{m,n} = \frac{(n-1)}{(m+n)} J_{m,n-2} \qquad \text{...(3)}$$

Again taking limits on both sides of (2), we have

$$\left[I_{m,n}\right]_0^{\frac{\pi}{2}} = \left[-\frac{\sin^{m-1} x \cos^{n+1} x}{(m+n)}\right]_0^{\frac{\pi}{2}} + \frac{(m-1)}{(m+n)}\left[I_{m-2,n}\right]_0^{\frac{\pi}{2}}$$

i.e., $$J_{m,n} = \frac{(m-1)}{(m+n)} J_{m-2,n} \qquad \text{...(4)}$$

Therefore, both the formulas (3) and (4) represent the reduction formula of $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx.$

Observation:

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^m\left(\frac{\pi}{2} - x\right) \cdot \cos^n\left(\frac{\pi}{2} - x\right) dx$$

$$= J_{n,m}$$

Calculation of the Value of the Definite Integral

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx.$$

Let us consider the reduction formula (4), i.e.,

$$J_{m,n} = \frac{(m-1)}{(m+n)} J_{m-2,n} = \frac{(m-1)}{(n+m)} J_{m-2,n} \qquad ...(5)$$

Now replacing m by $m-2$, we get from (5)

$$J_{m-2,n} = \frac{(m-3)}{(n+m-2)} J_{m-4,n} \qquad ...(6)$$

From (5) and (6), we obtain

$$J_{m,n} = \frac{(m-1)}{(n+m)} \cdot \frac{(m-3)}{(n+m-2)} \cdot J_{m-4,n} \qquad ...(7)$$

Again replacing m by $m-2$, we get from (6)

$$J_{m-4,n} = \frac{(m-5)}{(n+m-4)} J_{m-6,n} \qquad ...(8)$$

From (7) and (8), we have

$$J_{m,n} = \frac{(m-1)}{(n+m)} \cdot \frac{(m-3)}{(n+m-2)} \cdot \frac{(m-5)}{(n+m-4)} J_{m-6,n} \qquad ...(9)$$

Similarly proceeding, we have the following cases:

Case (i) m *is odd and* n *is any (odd or even) integer*

$$J_{m,n} = \frac{(m-1)}{(n+m)} \cdot \frac{(m-3)}{(n+m-2)} \cdot \frac{(m-5)}{(n+m-4)} \cdots \frac{4}{(n+5)} \cdot \frac{2}{(n+3)} J_{1,n}$$

Again

$$J_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx$$

$$= -\left[\frac{\cos^{n+1} x}{(n+1)}\right]_0^{\frac{\pi}{2}} = \frac{1}{(n+1)}.$$

Hence

$$\boxed{\begin{aligned} \mathbf{J_{m,n}} &= \frac{\mathbf{(m-1)}}{\mathbf{(n+m)}} \frac{\mathbf{(m-3)}}{\mathbf{(n+m-2)}} \frac{\mathbf{(m-5)}}{\mathbf{(n+m-4)}} \cdots \frac{\mathbf{4}}{\mathbf{(n+5)}} \frac{\mathbf{2}}{\mathbf{(n+3)}} \frac{\mathbf{1}}{\mathbf{(n+1)}} \\ &= \frac{\mathbf{2 \cdot 4 \cdot 6 \ldots (m-3) \cdot (m-1)}}{\mathbf{(n+1) \cdot (n+3) \ldots (n+m)}}. \end{aligned}}$$

Case (ii) m *and* n *both are even integers*

$$J_{m,n} = \frac{(m-1)}{(n+m)} \cdot \frac{(m-3)}{(n+m-2)} \cdot \frac{(m-5)}{(n+m-4)} \cdots \frac{3}{(n+4)} \cdot \frac{1}{(n+2)} J_{0,n}$$

Again

$$J_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

Since n is even we have from **section 5.3,**

$$J_{0,n} = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.$$

Hence

$$\boxed{\begin{aligned} \boldsymbol{J_{m,n}} &= \left[\frac{(m-1)}{(n+m)} \frac{(m-3)}{(n+m-2)} \cdots \frac{3}{(n+4)} \frac{1}{(n+2)}\right]\left[\frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}\right]. \\ &= \frac{[1\cdot 3\cdot 5\ldots(m-3)\cdot(m-1)][1\cdot 3\cdot 5\ldots(n-3)\cdot(n-1)]}{2\cdot 4\cdot 6\ldots(n+m)} \cdot \frac{\pi}{2} \end{aligned}}$$

Example 6 Using reduction formula, find $\int \sin^3 x \cdot \cos^2 x \, dx$

Sol. The reduction formula of $I_{m,n} = \int \sin^m x \cos^n x \, dx$ is given by

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{(m+n)} + \frac{(n-1)}{(m+n)} I_{m,n-2}$$

Here, $m = 3, n = 2$

Now

$$I_{3,2} = \frac{\cos x \cdot \sin^4 x}{5} + \frac{1}{5} I_{3,0}$$

and $I_{3,0} = \int \sin^3 x \, dx = \int (1-\cos^2 x) \sin x \, dx$

$$= \int \sin x \, dx - \int \cos^2 x \cdot \sin x \, dx$$

$$= -\cos x + \frac{\cos^3 x}{3}$$

Therefore,

$$I_{3,2} = \frac{\cos x \sin^4 x}{5} + \frac{1}{5} I_{3,0}$$

$$= \frac{\cos x \sin^4 x}{5} + \frac{1}{5}\left\{-\cos x + \frac{\cos^3 x}{3}\right\}$$

Example 7 Using reduction formula, find $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$

Sol. Since

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{2\cdot 4\cdot 6\ldots(m-1)}{(n+1)(n+3)(n+5)\ldots(n+m)}$$

when m is odd and n may be odd or even integers.

Here, $m = 3$ (odd), $n=2$ (even)

Therefore,

$$J_{3,2} = \int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$$

$$= \frac{2}{3.5} = \frac{2}{15}$$

Example 8 Using reduction formula, find $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$

Sol. Since

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{1\cdot3\cdot5...(m-1)\cdot1\cdot3\cdot5...(n-1)}{2\cdot4\cdot6...(m+n)}\frac{\pi}{2}$$

when both m, n are even integers.

Here, $m = 2$ (even), $n=2$ (even).

Therefore,

$$J_{2,2} = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^2 x \, dx$$

$$= \frac{1\cdot1}{2\cdot4}\frac{\pi}{2} = \frac{\pi}{16}$$

5.5 REDUCTION FORMULA FOR

(a) $\int \cos^m x \sin nx \, dx$, where m and n are positive integers.

(b) $\int_0^{\frac{\pi}{2}} \cos^m x \sin nx \, dx$, where m and n are positive integers.

(a) Let

$$I_{m,n} = \int \cos^m x \sin nx \, dx$$

Integrating by parts, we have

$$I_{m,n} = \cos^m x \int \sin nx \, dx - \int \left\{ \frac{d\left(\cos^m x\right)}{dx} \int \sin nx \, dx \right\} dx$$

$$= \cos^m x \left(\frac{-\cos nx}{n} \right) - \frac{m}{n} \int \cos^{m-1} x \cdot \sin x \cdot \cos nx \, dx$$

Since

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x \Rightarrow \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

$$I_{m,n} = \cos^m x\left(\frac{-\cos nx}{n}\right) - \frac{m}{n}\int \cos^{m-1} x\{\sin nx \cos x - \sin(n-1)x\}\, dx$$

$$= \frac{-\cos^m x \cos nx}{n} - \frac{m}{n}\left[\int \cos^m x \sin nx dx - \int \cos^{m-1} x \sin(n-1)x dx\right]$$

$$= \frac{-\cos^{mx} x \cos nx}{n} - \frac{m}{n}[I_{m,n} - I_{m-1,n-1}]$$

$$\Rightarrow \frac{(m+n)}{n} I_{m,n} = \frac{-\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = \frac{-\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Therefore, the reduction formula of $I_{m,n} = \int \cos^m x \sin nx\, dx$ where m, n are positive integers is given by

$$\boxed{\boldsymbol{I_{m,n} = \frac{-\cos^m x \cdot \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}}} \qquad ...(1)$$

(b) Let us consider

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx = \left[I_{m,n}\right]_0^{\frac{\pi}{2}}.$$

Now taking limits on both sides of (1), we have

$$\left[I_{m,n}\right]_0^{\frac{\pi}{2}} = \left[\frac{-\cos^m x \cdot \cos nx}{m+n}\right]_0^{\frac{\pi}{2}} + \frac{m}{m+n}\left[I_{m-1,n-1}\right]_0^{\frac{\pi}{2}}$$

i.e., $$\boxed{\boldsymbol{J_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} \cdot J_{m-1,n-1}}} \qquad ...(2)$$

Therefore, (2) represents the reduction formula for $\int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$.

Alternative Method of Finding Reduction Formula for $\int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$

Let

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$$

$$= \left[\cos^m x \int \sin nx dx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left\{\frac{d\left(\cos^m\right)x}{dx} \int \sin nx dx\right\} dx$$

$$= \left[\frac{-\cos^m x \cos nx}{n}\right]_0^{\frac{\pi}{2}} - \frac{m}{n}\left[\int_0^{\frac{\pi}{2}} \cos^m x \sin nx dx - \int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin(n-1)x\, dx\right]$$

$$\Rightarrow \frac{(m+n)}{n} J_{m,\,n} = \left[\frac{-\cos^m x \cos nx}{n}\right]_0^{\frac{\pi}{2}} + \frac{m}{n} J_{m-1,\,n-1}$$

$$\Rightarrow J_{m,\,n} = \left[\frac{-\cos^m x \cos nx}{m+n}\right]_0^{\frac{\pi}{2}} + \frac{m}{m+n} J_{m-1,\,n-1}$$

$$\Rightarrow J_{m,\,n} = \frac{1}{m+n} + \frac{m}{m+n} J_{m-1,\,n-1}$$

Therefore, the reduction formula of $J_{m,\,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$ where m, n are positive integers is given by

$$J_{m,\,n} = \frac{1}{m+n} + \frac{m}{m+n} J_{m-1,\,n-1}$$

Example 9 Using reduction formula, find $\int_0^{\frac{\pi}{2}} \cos^3 x \cdot \sin 2x\, dx$

Sol. The reduction formula of $J_{m,\,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$ where m, n are positive integers is

$$J_{m,\,n} = \frac{1}{m+n} + \frac{m}{m+n} J_{m-1,\,n-1}$$

Here, $m = 3$ and $n = 2$.

Now,

$$J_{3,\,2} = \frac{1}{3+2} + \frac{3}{3+2} J_{2,\,1} = \frac{1}{5} + \frac{3}{5} J_{2,\,1},$$

$$J_{2,\,1} = \frac{1}{2+1} + \frac{2}{2+1} J_{1,\,0} = \frac{1}{3} + \frac{2}{3} J_{1,\,0}$$

and $J_{1,\,0} = \int_0^{\frac{\pi}{2}} 0\, dx = 0$

Therefore,

$$J_{3,\,2} = \frac{1}{5} + \frac{3}{5} J_{2,\,1}$$

$$= \frac{1}{5} + \frac{3}{5} \cdot \frac{1}{3} = \frac{2}{5}$$

5.6 REDUCTION FORMULA FOR

(a) $\int \frac{dx}{(x^2+a^2)^n}$**, where** $n(>1)$ **is a positive integer**

(b) $\int_0^\infty \frac{dx}{(x^2+a^2)^n}$**, where** $n(>1)$ **is a positive integer**

(a) Let us consider

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \int \frac{1}{(x^2+a^2)^n} \cdot 1\, dx$$

Now integrating by parts, we have

$$I_n = \frac{1}{(x^2+a^2)^n} \cdot \int 1\, dx - \int \left[\frac{d}{dx}\left\{\frac{1}{(x^2+a^2)^n}\right\} \cdot \int 1\, dx \right] dx$$

$$= \frac{1}{(x^2+a^2)^n} \cdot (x) + \int \frac{n \cdot 2x}{(x^2+a^2)^{n+1}} \cdot (x)\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \cdot \int \frac{x^2}{(x^2+a^2)^{n+1}}\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \cdot \int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \cdot \left[\int \frac{1}{(x^2+a^2)^n}\, dx - a^2 \cdot \int \frac{1}{(x^2+a^2)^{n+1}}\, dx \right]$$

$$\Rightarrow I_n = \frac{x}{(x^2+a^2)^n} + 2n \cdot \left[I_n - a^2 \cdot I_{n+1} \right]$$

$$\Rightarrow 2n \cdot a^2 \cdot I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1) \cdot I_n$$

Now replacing n by $n-1$, we have from above

$$\boxed{\mathbf{2(n-1) \cdot a^2 \cdot I_n = \frac{x}{(x^2+a^2)^{n-1}} + (2n-3) \cdot I_{n-1}}} \qquad \text{...(1)}$$

So (1) represents the reduction formula for

$I_n = \frac{dx}{(x^2+a^2)^n}$, where $n(>1)$ is a positive integer.

(b) Let us consider

$$J_n = \int_0^\infty \frac{dx}{(x^2+a^2)^n}$$

so, $J_n = [I_n]_0^\infty$.

Now taking limits on both sides of (1), we have

$$2(n-1)\cdot a^2 \cdot [I_n]_0^\infty = \left[\frac{x}{(x^2+a^2)^{n-1}}\right]_0^\infty + (2n-3)\cdot[I_{n-1}]_0^\infty$$

$$2(n-1)\cdot a^2 \cdot J_n = \left[\lim_{x\to\infty}\frac{x}{(x^2+a^2)^{n-1}} - 0\right] + (2n-3)\cdot J_{n-1} \qquad \text{...(2)}$$

Again

$\lim_{x\to\infty}\frac{x}{(x^2+a^2)^{n-1}}$ is an indeterminate form of $\frac{\infty}{\infty}$.

So using L'Hospital's rule, we have

$$\lim_{x\to\infty}\frac{x}{(x^2+a^2)^{n-1}} = \lim_{x\to\infty}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}\left[(x^2+a^2)^{n-1}\right]}$$

$$= \lim_{x\to\infty}\frac{1}{(n-1)\cdot(x^2+a^2)^{n-2}\cdot 2x}$$

i.e., $$\lim_{x\to\infty}\frac{x}{(x^2+a^2)^{n-1}} = 0. \qquad \text{...(3)}$$

Using the result (3) in (2), we have

$$2(n-1)\cdot a^2 \cdot J_n = (2n-3)\cdot J_{n-1}$$

i.e., $$\boxed{\boldsymbol{J_n = \frac{1}{a^2}\cdot\frac{(2n-3)}{(2n-2)}\cdot J_{n-1}}} \qquad \text{...(4)}$$

Hence (4) is the reduction formula for

$J_n = \int_0^\infty \frac{dx}{(x^2+a^2)^n}$, where $n(>1)$ is a positive integer.

Calculation of the Value of the Definite Integral

$J_n = \int_0^\infty \frac{dx}{(x^2+a^2)^n}$, where $n(>1)$ is a positive integer.

From (4), we have the reduction formula for above as

$$J_n = \frac{1}{a^2}\cdot\frac{(2n-3)}{(2n-2)}\cdot J_{n-1} \qquad \text{...(5)}$$

Now replacing n by $n-1$, we get from (5)

$$J_{n-1} = \frac{1}{a^2} \cdot \frac{(2n-5)}{(2n-4)} \cdot J_{n-2} \qquad ...(6)$$

From (5) and (6), we obtain

$$J_n = \frac{1}{a^2} \cdot \frac{(2n-3)}{(2n-2)} \cdot \frac{1}{a^2} \cdot \frac{(2n-5)}{(2n-4)} \cdot J_{n-2}$$

$$\text{i.e., } J_n = \left(\frac{1}{a^2} \cdot \frac{1}{a^2}\right) \cdot \frac{(2n-3)}{(2n-2)} \cdot \frac{(2n-5)}{(2n-4)} \cdot J_{n-2} \qquad ...(7)$$

Again replacing n by $n-1$, we get from (6)

$$J_{n-2} = \frac{1}{a^2} \cdot \frac{(2n-7)}{(2n-6)} \cdot J_{n-3} \qquad ...(8)$$

From (7) and (8), we have

$$J_n = \left(\frac{1}{a^2} \cdot \frac{1}{a^2} \cdot \frac{1}{a^2}\right) \cdot \frac{(2n-3)}{(2n-2)} \cdot \frac{(2n-5)}{(2n-4)} \cdot \frac{(2n-7)}{(2n-6)} \cdot J_{n-3} \qquad ...(9)$$

Similarly proceeding as above

$$J_n = \left(\underbrace{\frac{1}{a^2} \cdot \frac{1}{a^2} \cdots \frac{1}{a^2}}_{(n-1)\text{ terms}}\right) \cdot \frac{(2n-3)}{(2n-2)} \cdot \frac{(2n-5)}{(2n-4)} \cdot \frac{(2n-7)}{(2n-6)} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot J_1$$

$$= \frac{1}{a^{2(n-1)}} \cdot \frac{(2n-3)}{(2n-2)} \cdot \frac{(2n-5)}{(2n-4)} \cdot \frac{(2n-7)}{(2n-6)} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot J_1 \qquad ...(10)$$

Again

$$J_1 = \int_0^\infty \frac{dx}{(x^2 + a^2)}$$

$$= \left[\frac{1}{a} \cdot \tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$= \frac{1}{a} \cdot \frac{\pi}{2}.$$

So putting the value of J_1 in (10), we have

$$\boxed{\begin{aligned} \boldsymbol{J_n} &= \frac{\mathbf{1}}{\boldsymbol{a^{2(n-1)}}} \; \frac{\mathbf{(2n-3)}}{\mathbf{(2n-2)}} \; \frac{\mathbf{(2n-5)}}{\mathbf{(2n-4)}} \; \frac{\mathbf{(2n-7)}}{\mathbf{(2n-6)}} \cdots \frac{\mathbf{5}}{\mathbf{6}} \; \frac{\mathbf{3}}{\mathbf{4}} \; \frac{\mathbf{1}}{\mathbf{2}} \left(\frac{\mathbf{1}}{\boldsymbol{a}} \; \frac{\boldsymbol{\pi}}{\mathbf{2}}\right) \\ &= \frac{\mathbf{1}}{\boldsymbol{a^{2n-1}}} \; \frac{\mathbf{(2n-3)}}{\mathbf{(2n-2)}} \; \frac{\mathbf{(2n-5)}}{\mathbf{(2n-4)}} \; \frac{\mathbf{(2n-7)}}{\mathbf{(2n-6)}} \cdots \frac{\mathbf{5}}{\mathbf{6}} \; \frac{\mathbf{3}}{\mathbf{4}} \; \frac{\mathbf{1}}{\mathbf{2}} \; \frac{\boldsymbol{\pi}}{\mathbf{2}} \end{aligned}}$$

Alternative Method for Computation

$$I_n = \int_0^\infty \frac{dx}{(x^2 + a^2)^n}, \text{ where } n (> 1) \text{ is a positive integer.}$$

Let

$x = a \tan\theta$ then $dx = a\sec^2\theta \cdot d\theta$

Putting the above values in integration, we have

$$I_n = \int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta \cdot d\theta}{(a^2\tan^2\theta + a^2)^n}$$

$$= \frac{1}{a^{2n-1}} \int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta \, d\theta$$

We have from **Section 5.3** if $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx,$ then

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}., \text{ when } n \text{ is even}$$

Since $2n-2$ is always even, using the result of **Section 5.3**, we have

$$I_n = \frac{1}{a^{2n-1}} \int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta \, d\theta$$

$$= \frac{1}{a^{2n-1}} \left[\frac{(2n-3)}{(2n-2)} \cdot \frac{(2n-5)}{(2n-4)} \cdot \frac{(2n-7)}{(2n-6)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

Example 10 Find $\int \frac{dx}{(x^2 + a^2)^2}$.

Sol. We have the reduction formula for

$$I_n = \frac{dx}{(x^2 + a^2)^n}, \text{ where } n(> 1) \text{ is a positive integer.}$$

as

$$2\,(n-1) \cdot a^2 \cdot I_n = \frac{x}{(x^2 + a^2)^{n-1}} + (2n-3) \cdot I_{n-1}.$$

Here we are to find I_2.

By putting $n = 2$ in the above formula we get

$$2(2-1) \cdot a^2 \cdot I_2 = \frac{x}{(x^2 + a^2)^{2-1}} + (2 \cdot 2 - 3) \cdot I_{2-1}.$$

i.e., $$2a^2 \cdot I_2 = \frac{x}{(x^2 + a^2)} + I_1.$$

Again

$$I_1 = \frac{dx}{(x^2 + a^2)} = \frac{1}{a} \tan^{-1} a.$$

Hence

$$I_2 = \frac{1}{2a^2} \cdot \frac{x}{(x^2 + a^2)} + \frac{1}{2a^3} \cdot \tan^{-1} a.$$

WORKED-OUT EXAMPLES

Example 5.1 If $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} \theta \, d\theta$, where n is a positive integer, show that $I_n = \frac{2n}{2n+1} I_{n-1}$.

Use this to evaluate $\int_0^{\frac{\pi}{2}} \sin^7 \theta \, d\theta$. [WBUT 2001]

Sol. Here

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} \theta \, d\,\theta = \int_0^{\frac{\pi}{2}} \sin^{2n}\theta \sin\theta d\theta$$

Integrating by parts, taking $\sin^{2n}\theta$ as the first function and $\sin\theta$ as the second function, we have

$$= \sin^{2n}\theta \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta - \int_0^{\frac{\pi}{2}} \left\{ \frac{d(\sin^{2n}\theta)}{d\theta} \int \sin\theta \, d\theta \right\} d\theta$$

$$= [-sin^n\theta \cos\theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2n \sin^{2n-1}\theta \cos^2\theta d\theta$$

$$= 0 + 2n \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta (1 - \sin^2\theta) \, d\theta$$

$$= 2n \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \, d\theta - 2n \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta d\theta$$

$$= 2nI_{n-1} - 2nI_n$$

$$\Rightarrow (2n+1)\, I_n = 2nI_{n-1}$$

$$\Rightarrow I_n = \frac{2n}{(2n+1)} I_{n-1}.$$

To evaluate $\int_0^{\frac{\pi}{2}} \sin^7 \theta d\theta$ by the above reduction formula, we have

$$I_3 = \int_0^{\frac{\pi}{2}} \sin^7 \theta \, d\theta = \frac{2\cdot 3}{(2\cdot 3+1)} I_2 = \frac{6}{7} I_2$$

Similarly,

$$I_2 = \int_0^{\frac{\pi}{2}} \sin^5 \theta d\theta = \frac{2\cdot 2}{(2\cdot 2+1)} I_1 = \frac{4}{5} I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta = \frac{2\cdot 1}{(2\cdot 1+1)} I_0 = \frac{2}{3} I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}} \sin \theta d\theta = [-\cos\theta]_0^{\pi/2} = 1$$

Therefore, from the above relations, we have

$$I_3 = \int_0^{\frac{\pi}{2}} \sin^7 \theta d\theta = \frac{6}{7} I_2 = \frac{6}{7}\frac{4}{5} I_1 = \frac{6}{7}\frac{4}{5}\frac{2}{3} I_0 = \frac{16}{35}$$

Example 5.2 Show that

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx = \frac{5}{32}\pi$$ [WBUT 2003]

Sol. Let

$x = \sin \theta$

then $dx = \cos\theta \, d\theta$

Then

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^6\theta \cdot \cos\theta}{\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta$$

Now from **Section 5.2**, we have if $J_n = \int_0^{\frac{\pi}{2}} \sin^n x \cdot dx$, then

$$J_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \; n \text{ is even.}$$

Since here $n = 6$, using the result of **Section 5.2,**

$$\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}.$$

Hence

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}} \, dx = \frac{5}{32} \pi.$$

Example 5.3 Prove that if $u_n = \int_0^1 x^n \tan^{-1} x \, dx$ then

$$(n+1) u_n + (n-1) u_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$ [WBUT 2002]

Sol. Here

$$u_n = \int_0^1 x^n \tan^{-1} x dx = \int_0^1 \tan^{-1} x \cdot x^n dx$$

Performing integration by parts, we have

$$u_n = \left[\tan^{-1} x \frac{x^{n+1}}{n+1} \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \frac{x^{n+1}}{n+1} dx$$

$$= \frac{\pi}{4} \cdot \frac{1}{n+1} - \frac{1}{n+1} \int_0^1 \frac{x^2}{1+x^2} x^{n-1} \, dx$$

$$= \frac{\pi}{4} \cdot \frac{1}{n+1} - \frac{1}{n+1} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) x^{n-1} \, dx$$

So,

$$u_n = \frac{\pi}{4} \cdot \frac{1}{n+1} - \frac{1}{n+1} \int_0^1 x^{n-1} \, dx + \frac{1}{n+1} \int_0^1 \frac{1}{1+x^2} x^{n-1} \, dx$$

$$u_n = \frac{\pi}{4}\cdot\frac{1}{n+1} - \frac{1}{n+1}\left[\frac{x^n}{n}\right]_0^1 + \frac{1}{n+1}\left\{\left[x^{n-1}\tan^{-1}x\right]_0^1 - \int_0^1 (n-1)x^{n-2}\tan^{-1} xdx\right\}$$

Hence from above

$$u_n = \frac{\pi}{4}\cdot\frac{1}{n+1} - \frac{1}{n+1}\frac{1}{n} + \frac{1}{n+1}\frac{\pi}{4} - \frac{n-1}{n+1}\int_0^1 x^{n-2}\tan^{-1} xdx$$

i.e., $$u_n = \frac{1}{n+1}\left(\frac{\pi}{2} - \frac{1}{n}\right) - \frac{n-1}{n+1}\int_0^1 x^{n-2}\tan^{-1} x\,dx$$

i.e., $$u_n = \frac{1}{n+1}\left(\frac{\pi}{2} - \frac{1}{n}\right) - \frac{n-1}{n+1}u_{n-2}$$

Therefore,

$$(n+1)\,u_n + (n-1)\,u_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

Example 5.4 If $u_n = \int_0^{\frac{\pi}{4}} \tan^n\theta\,d\theta,$ prove that

$n(u_{n+1} + u_{n-1}) = 1.$ [WBUT 2003]

Sol. Here

$$u_n = \int_0^{\frac{\pi}{4}} \tan^n\theta\,d\theta = \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\cdot\tan^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta\cdot(\sec^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta d\,(\tan\theta) - \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\,d\theta$$

From above, we can write

$$u_n = \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\frac{\pi}{4}} - u_{n-2}$$

i.e., $$u_n = \frac{1}{n-1} - u_{n-2}$$

Replacing n by $n+1$ in the above expression, we have

$$u_{n+1} = \frac{1}{n} - u_{n-1}$$

i.e., $n(u_{n+1} + u_{n-1}) = 1.$

Hence the result is proved.

Example 5.5 If $I_n = \dfrac{\sin n\theta}{\sin \theta}\, d\theta,$ show that

$(n-1)(I_n - I_{n-2}) = 2\sin(n-1)\,\theta.$ [WBUT 2004]

Sol. We have

$$I_n = \frac{\sin n\theta}{\sin\theta}\, d\theta$$

So,

$$I_{n-2} = \int \frac{\sin(n-2)\,\theta}{\sin\theta}\, d\theta$$

Now

$$I_n - I_{n-2} = \int \frac{\sin n\theta}{\sin\theta}\, d\theta - \int \frac{\sin(n-2)\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{\sin n\theta - \sin(n-2)\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{2\cos(n-1)\theta \cdot \sin\theta}{\sin\theta}\, d\theta$$

$$= 2\int \cos(n-1)\,\theta d\theta$$

Therefore

$$I_n - I_{n-2} = 2\frac{\sin(n-1)\,\theta}{n-1}$$

i.e., $(n-1)(I_n - I_{n-2}) = 2\sin(n-1)\,\theta.$

Hence the result is proved.

Example 5.6 If $I_n = \displaystyle\int \frac{\cos n\theta}{\cos\theta}\, d\theta,$ show that

$(n-1)\,(I_n + I_{n-2}) = 2\sin(n-1)\,\theta$

Hence evaluate

$\int (4\cos^2\theta - 3)\, d\theta$ [WBUT 2005]

Sol. We have

$$I_n = \int \frac{\cos n\theta}{\cos\theta}\, d\theta \qquad ...(1)$$

So,

$$I_{n-2} = \int \frac{\cos(n-2)\,\theta}{\cos\theta}\, d\theta$$

Now

$$I_n + I_{n-2} = \int \frac{\cos n\theta}{\cos\theta}\, d\theta + \int \frac{\cos(n-2)\,\theta}{\cos\theta} d\theta$$

$$= \int \frac{\cos n\theta + \cos(n-2)\,\theta}{\cos\theta}\, d\theta$$

$$= \int \frac{2\cos(n-1)\theta \cdot \cos\theta}{\cos\theta}\, d\theta$$

$$= 2\int \cos(n-1)\,\theta d\theta$$

Therefore

$$I_n + I_{n-2} = 2\frac{\sin(n-1)\,\theta}{n-1}$$

i.e., $(n-1)(I_n + I_{n-2}) = 2\sin(n-1)\,\theta.$ ...(2)

Hence, the result is proved.

Now we are to evaluate

$$\int (4\cos^2\theta - 3)\, d\theta$$

using the above result.

For this purpose, if we put $n = 3$ in (1), then

$$I_3 = \int \frac{\cos 3\theta}{\cos\theta}\, d\theta = \int \frac{(4\cos^3\theta - 3\cos\theta)}{\cos\theta}\, d\theta$$

$$= \int (4\cos^2\theta - 3)\, d\theta$$

Basically, we are to find I_3.

Putting $n = 3$ in (2), we have

$$(3-1)(I_3 + I_{3-2}) = 2\sin(3-1)\theta$$

i.e., $(I_3 + I_1) = \sin 2\theta$

Again $I_1 = \int \frac{\cos\theta}{\cos\theta} d\theta = \int d\theta = \theta.$

So,

$$I_3 = \sin 2\theta - \theta$$

i.e., $\int (4\cos^2\theta - 3)\, d\theta = \sin 2\theta - \theta.$

Example 5.7 If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\, dx,\ (\text{n} > 1)$ then prove that

$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$ [WBUT 2008]

Sol. Here we have

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\, dx$$

Integrating by parts, we obtain

$$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx$$

$$= 0 - n\left[\left[-x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}(-\sin x)\, dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \sin x\, dx$$

Therefore,

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

i.e., $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Hence the result is proved.

Example 5.8 If $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx,$ then prove that

$$I_{m,n} = \frac{m}{m+n} I_{m-1,\,n-1}.$$

Hence show that

$$\int_0^{\frac{\pi}{2}} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}.$$

Sol. Here

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \cos nx \, dx \qquad ...(1)$$

$$= \left[\frac{\cos^m x \sin nx}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} m \cos^{m-1} x \cdot (-\sin x) \frac{\sin nx}{n} dx$$

i.e., $$I_{m,n} = \frac{m}{n} \int_0^{\frac{\pi}{2}} \cos^{m-1} x \cdot (\sin nx \sin x) dx \qquad ...(2)$$

Again

$\cos(n-1)x = \cos(nx - x) = \cos nx \cos x + \sin nx \sin x$

Then from (2)

$$I_{m,n} = \frac{m}{n} \int_0^{\frac{\pi}{2}} \cos^{m-1} x \cdot \left[\cos(n-1)x - \cos nx \cos x\right] dx$$

$$= \frac{m}{n} \int_0^{\frac{\pi}{2}} \cos^{m-1} x \cos(n-1)x \, dx - \frac{m}{n} \int_0^{\frac{\pi}{2}} \cos^m x \cos nx \, dx$$

i.e., $I_{m,n} = \frac{m}{n} I_{m-1,\,n-1} - \frac{m}{n} I_{m,n}$

i.e., $\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{m}{n} I_{m-1,\,n-1}$

Hence

$$I_{m,n} = \frac{m}{m+n} I_{m-1,\,n-1}. \qquad ...(3)$$

So the first part is proved.

To prove the second part, let

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \, dx.$$

So it is clear from (1) that $J_n = I_{n,n}$.

Therefore, from (3), we have

$$J_n = I_{n,n} = \frac{n}{n+n} I_{n-1,n-1} = \frac{1}{2} J_{n-1} \qquad \text{...(4)}$$

Now from (4)

$$J_n = \frac{1}{2} J_{n-1} = \frac{1}{2}\cdot\frac{1}{2} J_{n-2}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} J_{n-3}$$

$$\cdots\cdots\cdots\cdots$$

$$J_n = \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}}_{n \text{ terms}} J_{n-n}$$

i.e., $J_n = \dfrac{1}{2^n} J_0$

Again

$$J_0 = \int_0^{\frac{\pi}{2}} \cos^0 x \cos 0 \, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

Hence

$$J_n = \frac{1}{2^n} J_0 = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$$

Example 5.9 If $I_n = \int_0^1 (\sin^{-1} x)^n \, dx$ then prove that

$$I_n + n(n-1) I_{n-2} = \left(\frac{\pi}{2}\right)^n.$$

Sol. Let

$$J_n = \int \left(\sin^{-1} x\right)^n dx$$

Putting $x = \sin v$, i.e., $dx = \cos v dv$, we have from the above,

$$J_n = \int v^n \cos v \, dv$$

$$= v^n \sin v - \int n v^{n-1} \sin v dv$$

$$= v^n \sin v - n\left[v^{n-1}(-\cos v) - (n-1)\int v^{n-2}(-\cos v)\, dv\right]$$

$$J_n = v^n \sin v + n v^{n-1} \cos v - n(n-1)\int v^{n-2} \cos v dv$$

i.e., $J_n = v^n \sin v + nv^{n-1}\cos v - n(n-1)J_{n-2}$

$$= \left(\sin^{-1}x\right)^n \cdot x + n\left(\sin^{-1}x\right)^{n-1}\cdot\sqrt{1-x^2} - n(n-1)J_{n-2}$$

Now

$$I_n = \int_0^1 \left(\sin^{-1}x\right)^n dx = [J_n]_0^1$$

$$= \left[\left(\sin^{-1}x\right)^n \cdot x\right]_0^1 + \left[n\left(\sin^{-1}x\right)^{n-1}\cdot\sqrt{1-x^2}\right]_0^1 - n(n-1)[J_{n-2}]_0^1$$

$$= \left(\frac{\pi}{2}\right)^n + 0 - n(n-1)I_{n-2}$$

Hence

$$I_n + n(n-1)I_{n-2} = \left(\frac{\pi}{2}\right)^n.$$

Example 5.10 If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\cos nx\,dx$ and $J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\sin nx\,dx$ then prove that

$$(m+n)I_{m,n} + mJ_{m-1,n-1} = \sin\frac{n\pi}{2}.$$

Sol. Here we have

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\cos nx\,dx$$

$$= \left[\frac{\sin^m x\sin nx}{n}\right]_0^{\frac{\pi}{2}} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \sin^{m-1}x(\cos x)\sin nx\,dx$$

$$= \frac{1}{n}\left(\sin\frac{n\pi}{2}\right) - \frac{m}{n}\int_0^{\frac{\pi}{2}} \sin^{m-1}x(\sin nx\cos x)dx$$

Again

$$\sin(n-1)x = \sin(nx-x) = \sin nx\cos x - \cos nx\sin x$$

So,

$$I_{m,n} = \frac{1}{n}\left(\sin\frac{n\pi}{2}\right) - \frac{m}{n}\int_0^{\frac{\pi}{2}} \sin^{m-1}x\left(\sin(n-1)x + \cos nx\sin x\right)dx$$

$$= \frac{1}{n}\left(\sin\frac{n\pi}{2}\right) - \frac{m}{n}\int_0^{\frac{\pi}{2}} \sin^{m-1}x \sin(n-1)x\,dx - \frac{m}{n}\int_0^{\frac{\pi}{2}} \sin^m x \cos nx\,dx$$

Therefore,

$$I_{m,\,n} = \frac{1}{n}\left(\sin\frac{n\pi}{2}\right) - \frac{m}{n}J_{m-1,\,n-1} - \frac{m}{n}I_{m,\,n}$$

i.e., $nI_{m,\,n} = \left(\sin\frac{n\pi}{2}\right) - mJ_{m-1,\,n-1} - mI_{m,\,n}$

Hence we have

$$(m+n)I_{m,\,n} + mJ_{m-1,\,n-1} = \sin\frac{n\pi}{2}.$$

Example 5.11 If $I_n = \int_0^1 \left(1-x^2\right)^n dx$ then prove that $(2n+1)I_n = 2nI_{n-1}$.

Sol. Here

$$I_n = \int_0^1 \left(1-x^2\right)^n \cdot 1\,dx$$

$$= \left[\left(1-x^2\right)^n \cdot x\right]_0^1 - n\int_0^1 \left(1-x^2\right)^{n-1} \cdot (-2x)\cdot x\,dx$$

$$= 0 - 2n\int_0^1 \left(1-x^2\right)^{n-1} \cdot \left(-x^2\right) dx$$

$$= -2n\int_0^1 \left(1-x^2\right)^{n-1} \cdot \left[\left(1-x^2\right) - 1\right] dx$$

i.e., $I_n = -2n\int_0^1 \left(1-x^2\right)^n dx + 2n\int_0^1 \left(1-x^2\right)^{n-1} dx$

i.e., $I_n = -2nI_n + 2nI_{n-1}$

i.e., $(2n+1)I_n = 2nI_{n-1}$.

Hence the result is proved.

Example 5.12 If $I_{m,n} = \int_0^1 x^m (1-x)^n\,dx$ then prove that $I_{m,\,n-1} = \frac{n-1}{m+n} I_{m,\,n-2}$.

Sol. Here we have

$$I_{m,\,n-1} = \int_0^1 x^m (1-x)^{n-1}\,dx$$

$$= \int_0^1 \left[(1-x)^{n-1}\right]\cdot\left[x^m\right]dx$$

Integrating by parts,

$$I_{m,\,n-1} = \left[(1-x)^{n-1}\cdot\frac{x^{m+1}}{m+1}\right]_0^1 - (n-1)\int_0^1 (1-x)^{n-2}(-1)\cdot\frac{x^{m+1}}{m+1}\,dx$$

$$= 0 + \frac{n-1}{m+1}\int_0^1 (1-x)^{n-2}\cdot x^{m+1}\,dx$$

$$= -\frac{n-1}{m+1}\int_0^1 (1-x)^{n-2}\cdot x^m(-x)\,dx$$

$$= -\frac{n-1}{m+1}\int_0^1 (1-x)^{n-2}\cdot x^m\cdot\left[(1-x)-1\right]dx$$

$$= -\frac{n-1}{m+1}\int_0^1 (1-x)^{n-1}\cdot x^m + \frac{n-1}{m+1}\int_0^1 (1-x)^{n-2}\cdot x^m$$

Therefore,

$$I_{m,\,n-1} = -\frac{n-1}{m+1}I_{m,\,n-1} + \frac{n-1}{m+1}I_{m,\,n-2}$$

i.e., $$\left(1+\frac{n-1}{m+1}\right)I_{m,\,n-1} = \frac{n-1}{m+1}I_{m,\,n-2}$$

i.e., $$\left(\frac{m+n}{m+1}\right)I_{m,\,n-1} = \frac{n-1}{m+1}I_{m,\,n-2}$$

i.e., $$I_{m,\,n-1} = \frac{n-1}{m+n}I_{m,\,n-2}.$$

Hence the result is proved.

Example 5.13 If $I_n = \int_0^\pi \frac{1-\cos n\theta}{1-\cos\theta}\,d\theta$, where n is a positive integer or zero then prove that

$I_{n+2} + I_n = 2I_{n+1}$.

Sol. Here, we have

$$I_n = \int_0^\pi \frac{1-\cos n\theta}{1-\cos\theta}\,d\theta \text{ and so } I_{n+2} = \int_0^\pi \frac{1-\cos(n+2)\theta}{1-\cos\theta}\,d\theta.$$

Therefore,

$$I_{n+2}+I_n=\int_0^{\pi}\frac{1-\cos(n+2)\theta+1-\cos n\theta}{1-\cos\theta}d\theta$$

$$=\int_0^{\pi}\frac{2-[\cos(n+2)\theta+\cos n\theta]}{1-\cos\theta}d\theta.$$

Since

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right),$$

we obtain from above

$$I_{n+2}+I_n=\int_0^{\pi}\frac{2-2\cos(n+1)\theta\cdot\cos\theta}{1-\cos\theta}d\theta$$

$$=2\int_0^{\pi}\frac{1+[(1-\cos\theta)-1]\cdot\cos(n+1)\theta}{1-\cos\theta}d\theta$$

$$=2\int_0^{\pi}\frac{(1-\cos\theta)\cdot\cos(n+1)\theta}{1-\cos\theta}d\theta+2\int_0^{\pi}\frac{1-\cos(n+1)\theta}{1-\cos\theta}d\theta$$

i.e., $$I_{n+2}+I_n=2\int_0^{\pi}\cos(n+1)\theta d\theta+2I_{n+1}$$

$$=2\left[\frac{\sin(n+1)\theta}{n+1}\right]_0^{\pi}+2I_{n+1}$$

$$=0+2I_{n+1}.$$

Hence

$$I_{n+2}+I_n=2I_{n+1}.$$

Example 5.14 If $I_n=\int_0^1 x^n\sqrt{1-x^2}\,dx$ then prove that

$$I_n=\frac{n-1}{n+2}I_{n-2}$$

Sol. Here

$$I_n=\int_0^1 x^n\sqrt{1-x^2}\,dx$$

$$=\int_0^1 x^{n-1}\left(\sqrt{1-x^2}\cdot x\right)dx$$

$$= \left[x^{n-1} \frac{\left(1-x^2\right)^{\frac{3}{2}}}{-3} \right]_0^1 - (n-1)\int_0^1 x^{n-2} \frac{\left(1-x^2\right)^{\frac{3}{2}}}{-3} dx$$

$$I_n = 0 + \frac{(n-1)}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\sqrt{1-x^2}\, dx$$

$$= \frac{(n-1)}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}\, dx - \frac{(n-1)}{3}\int_0^1 x^n \sqrt{1-x^2}\, dx$$

i.e., $I_n = \dfrac{(n-1)}{3} I_{n-2} - \dfrac{(n-1)}{3} I_n$

i.e., $\left(1 + \dfrac{n-1}{3}\right) I_n = \dfrac{(n-1)}{3} I_{n-2}$

i.e., $\dfrac{(n+2)}{3} I_n = \dfrac{(n-1)}{3} I_{n-2}$

i.e., $I_n = \dfrac{n-1}{n+2} I_{n-2}.$

Hence, the result is proved.

Example 5.15 Show that

$$\int_0^{\frac{\pi}{2}} \cos^n x \sin nx\, dx = \frac{1}{2^{n+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^{n-1}}{(n-1)} + \frac{2^n}{n}\right].$$

where m is any positive integer.

Sol. From **Article 5.5**, we have the reduction formula for

$$J_{m,\,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx\, dx$$

as below

$$J_{m,\,n} = \frac{1}{m+n} + \frac{m}{m+n} \cdot J_{m-1,\,n-1} \qquad ...(1)$$

Now let

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin nx\, dx$$

So it is obvious that

$I_n = J_{n,n}$.

and from (1), we obtain

$$I_n = J_{n,n} = \frac{1}{n+n} + \frac{n}{n+n} \cdot J_{n-1,\,n-1}$$

i.e., $I_n = \dfrac{1}{2n} + \dfrac{1}{2} \cdot I_{n-1}$

Now

$$I_n = \frac{1}{2n} + \frac{1}{2} \cdot \left(\frac{1}{2(n-1)} + \frac{1}{2} \cdot I_{n-2} \right)$$

$$= \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^2} \cdot I_{n-2}$$

$$I_n = \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^2} \cdot \left(\frac{1}{2(n-2)} + \frac{1}{2} \cdot I_{n-3} \right)$$

$$= \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^3(n-2)} + \frac{1}{2^3} \cdot I_{n-3}$$

Proceeding in a similar manner, we obtain

$$I_n = \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^3(n-2)} + \cdots + \frac{1}{2^{n-1} \cdot 2} + \frac{1}{2^{n-1}} \cdot I_1 \qquad \text{...(2)}$$

Again

$$I_1 = \int_0^{\frac{\pi}{2}} \cos x \sin x dx = \frac{1}{2}.$$

Hence from (2), we have

$$I_n = \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^3(n-2)} + \cdots + \frac{1}{2^{n-2} \cdot 3} + \frac{1}{2^{n-1} \cdot 2} + \frac{1}{2^{n-1}} \cdot \frac{1}{2}$$

$$= \frac{1}{2n} + \frac{1}{2^2(n-1)} + \frac{1}{2^3(n-2)} + \cdots + \frac{1}{2^{n-2} \cdot 3} + \frac{1}{2^{n-1} \cdot 2} + \frac{1}{2^n}$$

$$= \frac{1}{2^{n+1}} \left[\frac{2^n}{n} + \frac{2^{n-1}}{(n-1)} + \frac{2^{n-2}}{(n-2)} + \cdots + \frac{2^3}{3} + \frac{2^2}{2} + 2 \right]$$

Hence

$$\int_0^{\frac{\pi}{2}} \cos^n x \sin nx dx = \frac{1}{2^{n+1}} \left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^{n-1}}{(n-1)} + \frac{2^n}{n} \right].$$

EXERCISES

Short and Long Answer Type Questions

1. Obtain the reduction formula for

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx, \text{ where } n\,(>1) \text{ is a positive integer}$$

and evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$. [WBUT 2007, 2008]

$\left[\textbf{Ans :} \text{ 2nd Part. } \frac{8}{15}.\right]$

2. Evaluate:

(i) $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$

$\left[\textbf{Ans :} \text{ (i) } \frac{5\pi}{32}, \text{ (ii) } \frac{16}{35}\right]$

3. Obtain a reduction formula for

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx, \text{ where } n\,(>1) \text{ is a positive integer}$$

and evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$. [WBUT 2006, 2009]

$\left[\textbf{Ans :} \text{ 2nd Part. } \frac{8}{15}.\right]$

4. Evaluate:

(i) $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^9 x\,dx$

$\left[\textbf{Ans :} \text{ (i) } \frac{35\pi}{256}, \text{ (ii) } \frac{428}{315}\right]$

5. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\,dx$, show that

$$I_{n+1} - I_n = \frac{1}{n}\sin 2nx.$$

6. If $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx$, show that

$$I_n = \frac{2n-1}{2n} I_{n-1}.$$

7. If $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta d\theta$, where n is a positive integer, show that

$$I_n = \frac{2n}{2n+1} I_{n-1}.$$

Use this to evaluate $\int_0^{\frac{\pi}{2}} \sin^7\theta d\theta$. [WBUT 2001]

$\left[\textbf{Ans :} \text{ 2nd Part. } \frac{16}{35}.\right]$

8. If $I_n = \int_0^{\frac{\pi}{2}} x\cos^n x dx$, where n is a positive integer, show that

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}.$$

9. Show that

(a) $\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx = \frac{5}{32}\pi$ [WBUT 2003]

[**Hint :** put $x = \sin\theta$]

(b) $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx = \frac{8}{15}$ [**Hint :** put $x = \sin\theta$]

(c) $\int_0^\infty \frac{x^4}{\left(1+x^2\right)^4} dx = \frac{\pi}{32}$ [**Hint :** put $x = \tan\theta$]

10. If $I_n = \int_0^{2\pi} \frac{\cos nx}{1-\cos x} dx$, then prove that

$$I_{n-1} - I_n = 2\pi.$$

11. Obtain the reduction formula for

$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$, where $m\,(>1)$ and $n\,(>1)$ are positive integers.

Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x dx$ [WBUT 2008]

$\left[\textbf{Ans :} \text{ 2nd Part. } \dfrac{35\pi}{1280}.\right]$

12. Evaluate:

(i) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^8 x dx$

$\left[\textbf{Ans :} \text{ (i) } \dfrac{8}{693}, \text{ (ii) } \dfrac{5\pi}{4096}\right]$

13. If $I_n = \int_0^1 \left(\cos^{-1} x\right)^n dx$ then prove that

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$$

14. If $u_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta$, prove that

$n(u_{n+1} + u_{n-1}) = 1.$ [WBUT 2003]

15. If $I_n = \int \dfrac{\sin(2n-1)x}{\sin x} dx$ and $J_n = \dfrac{\sin^2 nx}{\sin^2 x} dx$, where n is any integer then prove that

$J_{n+1} - J_n = I_{n+1}.$

16. If $I_n = \int \dfrac{\sin n\theta}{\sin \theta} d\theta$, show that

$(n-1)(I_n - I_{n-2}) = 2\sin(n-1)\theta.$ [WBUT 2004]

17. If $I_n = \int \dfrac{\cos n\theta}{\cos \theta} d\theta$, show that

$(n-1)(I_n + I_{n-2}) = 2\sin(n-1)\theta$

Hence evaluate $\int\left(4\cos^2\theta-3\right)d\theta$ [WBUT 2005]

18. If $I_n=\int_0^{\frac{\pi}{2}}x^n\sin x\,dx,\ (n>1)$ then prove that

$$I_n+n(n-1)I_{n-2}=n\left(\frac{\pi}{2}\right)^{n-1}.$$ [WBUT 2008]

19. Obtain the reduction formula for $I_n=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$ and hence show that

$$\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx=\frac{(2n)!}{\left(2^n n!\right)^2}\cdot\frac{\pi}{2}.$$

$\left[\textbf{Ans :} \text{ First part } I_n=\frac{2n-1}{2n}I_{n-1}.\right]$

20. Prove that if $u_n=\int_0^1 x^n\tan^{-1}x\,dx$ then

$$(n+1)u_n+(n-1)u_{n-2}=\frac{\pi}{2}-\frac{1}{n}.$$ [WBUT 2002]

21. If $I_n=\int_0^{\frac{\pi}{2}}x^n(\sin x+\cos x)dx,$ show that

$$I_n+n(n-1)I_{n-2}=\left(n+\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^{n-1}.$$

22. If $I_n=\int_0^1 x^n\cot^{-1}x\,dx$ then prove that

$$(n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}+\frac{1}{n}.$$

23. Obtain the reduction formula for

$I_n=\int_0^{\infty}\frac{dx}{(x^2+a^2)^n}$, where $n\ (>1)$ is a positive integer and hence show that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^2}.$$

$$\left[\textbf{Ans:} \text{ First part } I_n = \frac{1}{a^2}\cdot\frac{(2n-3)}{(2n-2)}\cdot I_{n-1}.\right]$$

Multiple-Choice Questions

1. The value of $\int_0^{\frac{\pi}{2}} \sin^6 x dx$ is

a) $\frac{7\pi}{32}$ b) $\frac{7\pi}{16}$ c) $\frac{5\pi}{32}$ d) $\frac{5\pi}{16}$

2. The value of $\int_0^{\frac{\pi}{2}} \cos^7 x dx$ is

a) $\frac{8}{35}$ b) $\frac{16}{35}$ c) $\frac{16\pi}{35}$ d) $\frac{8\pi}{35}$

3. The reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ is

a) $I_n = \frac{n-1}{n} I_{n-2}$ b) $I_n = \frac{n-1}{n} I_{n-1}$

c) $I_n = \frac{n}{n-1} I_{n-2}$ d) $I_n = \frac{n}{n-1} I_{n-1}$ [WBUT 2007]

4. The value of $\int_0^{\frac{\pi}{2}} \sin^n x dx$ is same as

a) $\int_0^{\pi} \cos^n 2x dx$ b) $\int_0^{\pi} \cos^n x dx$

c) $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x dx$ d) $\int_0^{\frac{\pi}{2}} \cos^n x dx$

5. The reduction formula for $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ is

a) $I_{m,n} = \frac{n-1}{m+n} I_{m-2,n}$ b) $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

c) $I_{m,n} = \frac{m-1}{m+n} I_{m,n-2}$ d) $I_{m,n} = \frac{n-1}{m+n} I_{m-2,n-2}$

6. The value of $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ is same as

a) $\int_0^{\frac{\pi}{2}} \cos^{m+1} x \sin^{n-1} x dx$ b) $\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin^{n+1} x dx$

c) $\int_0^{\frac{\pi}{2}} \cos^m x \sin^n x dx$ d) none of these.

7. The reduction formula for $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx dx$ is

a) $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$ b) $I_{m,n} = \frac{1}{m+n} I_{m-1,n-1}$

c) $I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$ d) none of these.

8. The value of $\int_0^{\frac{\pi}{2}} \cos^4 x \sin 3x dx$ is

a) $\frac{13\pi}{35}$ b) $\frac{7}{16}$ c) $\frac{13}{35}$ d) $\frac{7\pi}{16}$

9. The value of $\int_0^{\infty} \frac{dx}{\left(x^2+a^2\right)^2}$ is

a) $\frac{\pi}{a^2}$ b) $\frac{\pi}{4a^2}$ c) $\frac{\pi}{2a^2}$ d) $\frac{\pi}{3a^2}$

10. The reduction formula for $u_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ is

a) $u_n = \frac{1}{n-1} - u_{n-2}$

b) $u_n = \frac{1}{n-1} - u_{n-1}$

c) $u_n = \frac{1}{n-2} - u_{n-2}$

d) $u_n = \frac{1}{n-2} - u_{n-1}$

11. The value of $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x dx$ is

a) $\frac{2}{693}$ b) $\frac{8}{693}$ c) $\frac{4}{693}$ (d) $\frac{8\pi}{693}$

12. The value of $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx$ is

a) $\frac{\pi}{16}$ b) $\frac{1}{16}$ c) $\frac{1}{32}$ d) $\frac{\pi}{32}$

ANSWERS:

1. c	2. b	3. a	4. d	5. b	6. c
7. a	8. c	9. b	10. a	11. b	12. d

CHAPTER

6

Calculus of Functions of Several Variables

6.1 INTRODUCTION

In the earlier two chapters (chapters 3 and 4), we have dealt with the functions of single varable only. But we also require the fuctions of two or more variables to solve various problems in different branches of science and technology. Also, the derivatives and integrations of functions of two or more variables have a wide range of applications.

Basically, in this chapter, we first discuss briefly the limit and continuity of the functions of two or more variables. Then we describe the methods of differentiations and their applications towards the optimisation of the functions.

6.2 FUNCTIONS OF SEVERAL VARIABLES

A real function of a single variable $f: R \to R$ is defined by $y = f(x)$ where $x \in R$ and $y \in R$

Example 1

$y = f(x) = x^2$. Here, y is a function of a single variable x.

A real function of two variables $f: R^2 \to R$ is defined as $z = f(x, y)$ where $(x, y) \in R^2$ and $z \in R$.

Example 2

$z = f(x, y) = x^2 + y^2 + xy$. Here, z is a function of two variables, x and y.

The domain D of a function f of two variables is any closed curve on the two-dimensional plane, namely, rectangular, square, circular, etc.

A real function of three variables $f: R^3 \to R$ is defined as $z = f(x_1, x_2, x_3)$ where $(x_1, x_2, x_3) \in R^3$ and $z \in R$.

Example 3

$z = f(x_1, x_2, x_3) = 2x_1 + 3x_2^2 + 4x_1x_3$. Here, z is a function of three variables, x_1, x_2 and x_3.

A real function of n variables $f : R^n \to R$ is defined as $z = f(x_1, x_2, x_3, \dots, x_n)$ where $(x_1, x_2, x_3, \dots, x_n) \in R^n$ and $z \in R$.

Example 4

$z = f(x_1, x_2, x_3, \dots, x_n) = x_1 + x_2 + x_3 + \cdots + x_n$. Here, z is a function of n variables, $x_1, x_2, x_3, \dots, x_n$.

6.3 LIMIT AND CONTINUITY

To describe the analytical concept of limit of a function, first we define two types of δ-neighbourhood (or δ-nbd) of a point (a, b) in the two-dimensional plane.

(i) Square δ-neighbourhood

Any square region consisting of the points (x, y) and satisfying

$$|x-a| < \delta, |y-b| < \delta, \text{ for } \delta > 0$$

is called the square δ-nbd of the point (a, b).

(ii) Circular δ-neighbourhood

Any circular region consisting of the points (x, y) and satisfying

$$0 < (x-a)^2 + (y-b)^2 < \delta^2 \text{ for } \delta > 0$$

is called the circular δ-nbd of the point (a, b).

Suppose (x, y) be any variable point lying in any neighbourhood of a fixed point (a, b) in a two-dimensional plane. Also, let $f(x, y)$ be a function defined on a certain neighbourhood of the point (a, b). Now we will check whether the function $f(x, y)$ tends to a real value l as (x, y) tends to (a, b).

6.3.1 Limit of a Function of Two Variables

General Definition

Let $f(x, y)$ be a function of two independent variables, x and y. If the function f tends to a real value l as $(x, y) \to (a, b)$ then we write

$$\lim_{(x, y)\to(a, b)} f(x, y) = l$$

or, $f(x, y) \to l$ as $(x, y) \to (a, b)$

or, $\lim\limits_{\substack{x\to a \\ y\to b}} f(x, y) = l$.

where l is called the limit.

Analytical Definition:

Let $f(x, y)$ be a function of two independent variables x and y. The function f is said to tend to limit l as $(x, y) \to (a, b)$ if for an arbitrary small positive number ε, no matter how small, there exists a positive number δ, such that

$|f(x, y) - l| < \varepsilon$

for every point (x, y) which lies in any δ-nbd N of the point (a, b).

N may be a square δ-nbd of the point (a, b), i.e.,

$|x - a| < \delta, |y - b| < \delta$, for $\delta > 0$

or, N may be a circular δ-nbd of the point (a, b), i.e.,

$0 < (x - a)^2 + (y - b)^2 < \delta^2$ for $\delta > 0$

or N may be any other δ-nbd.

Symbolically,

$$\lim_{(x, y) \to (a, b)} f(x, y) = l.$$

Example 5

Find $\displaystyle\lim_{(x, y) \to (2, 3)} (x^2 + y^2 + xy)$.

Sol.

$$\lim_{(x, y) \to (2, 3)} (x^2 + y^2 + xy)$$

$$= \lim_{(x, y) \to (2, 3)} (x^2) + \lim_{(x, y) \to (2, 3)} (y^2) + \lim_{(x, y) \to (2, 3)} (xy)$$

$$= 4 + 9 + 6 = 19.$$

Example 6

Justify the following by analytical definition

$$\lim_{(x, y) \to (0, 0)} \frac{x^2 y^2}{x^2 + y^2} = 0$$

Sol. To prove the existance of the limit, for a given $\varepsilon > 0$ we are to find $\delta > 0$ such that in any δ-nbd N of $(0, 0)$,

$$|f(x, y) - l| < \varepsilon$$

$$\text{or, } \left| \frac{x^2 y^2}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\text{or, } \frac{x^2 y^2}{x^2 + y^2} < \varepsilon. \qquad \text{...(1)}$$

Now

$$x^2 < x^2 + y^2 \text{ and } y^2 < x^2 + y^2$$

So

$$\frac{x^2y^2}{x^2+y^2} < \frac{\left(x^2+y^2\right)\left(x^2+y^2\right)}{x^2+y^2}$$

$$= x^2 + y^2 < \varepsilon \text{ (consider)}$$

if

$$0 < (x-0)^2 + (y-0)^2 < \delta^2, \text{ where } \delta = \sqrt{\varepsilon}.$$

Here we are getting a circular δ-nbd of $(0, 0)$.

So, the condition (1) is satisfied and hence the result is proved.

6.3.2 Observations

1) l is called limit or double limit or simultaneous limit.

2) The definition

$$\lim_{(x,\,y)\to(a,\,b)} f(x, y) = l$$

is equvalent to

$$\lim_{x\to a} f(x, b) = l \text{ or, } \lim_{y\to b} f(a, y) = l$$

3) Uniqueness of the Limit:

Now the variable point (x, y) may approach the fixed point (a, b) by any path (e.g., straight line, parabolic, etc.), but the simultaneous limit should be unique in all the cases.

4) Non-existence of Limit:

If we get different values of the limit choosing different paths, i.e., if the limit is not unique then the simultaneous limit l does not exist.

Example 7 Show that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exist.}$$

Sol. Let us consider the parabolic path $x = my^2$; then as $x \to 0$, $y \to 0$ and we obtain from above

$$\lim_{(x,\,y)\to(0,\,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{2my^4}{(1+m^2)y^4}$$

$$= \lim_{y\to 0} \frac{2m}{1+m^2}$$

$$= \frac{2m}{1+m^2}.$$

which is different for different values of m, i.e., choosing different parabolic paths for different values of m, we get different limits.

So the limit is not unique. Hence, the limit does not exist.

5) Repeated Limits:

A repeated limit of $f(x, y)$ as $y \to b$ and then $x \to a$ is defined as

$$\lim_{x\to a}\lim_{y\to b} f(x, y) = \lim_{x\to a} \varphi(x) = l_1 (say)$$

A repeated limit of $f(x, y)$ as $x \to a$ and then $y \to b$ is defined as

$$\lim_{y\to b}\lim_{x\to a} f(x, y) = \lim_{y\to b} \psi(y) = l_2 (say)$$

These two repeated limits may not be equal.

6) In case the simultaneous limit exists, the repeated limits if they exist are necessarily equal, but the converse is not always true,

i.e., even when both the repeated limits exist and are equal, the simultaneous limit may not exist.

Example 8 Show that for the function

$$f(x, y) = \frac{xy}{x^2 + y^2}, (x, y) \neq (0, 0)$$
$$= 0, (x, y) = (0, 0)$$

the repeated limits are equal but simultaneous limit does not exist.

Sol. The repeated limits are,

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}(0) = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{x\to 0}(0) = 0$$

but, along the path (straight line) $y = mx$

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{xy}{x^2 + y^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2 + m^2x^2}$$

$$= \frac{m}{1+m^2}$$

which is different for different values of m.

Therefore, the repeated limits are equal but the simultaneous limit does not exist.

7) If the repeated limits are not equal, the simultaneous limit cannot exist.

Example 9 Show that for the function

$$f(x, y) = \frac{y-x}{y+x}\frac{1+x}{1+y}, (x, y) \neq (0, 0)$$
$$= 0, (x, y) = (0, 0)$$

the repeated limits are not equal and limit does not exist.

Sol. The repeated limits are,

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\left(\frac{1}{1+y}\right) = 1$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{x\to 0}\left(-\frac{1+x}{1}\right) = -1$$

Here, the repated limits are not equal. So simultaneous limit cannot exist. Also, this is obvious from the following:

Along the path $y = mx$

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{y-x}{y+x}\frac{1+x}{1+y}$$

$$= \lim_{x\to 0} \frac{m-1}{m+1}\frac{1+x}{1+mx}$$

$$= \frac{m-1}{m+1}.$$

which is different for different values of m.

Therefore,the repeated limits are not equal and so simultaneous limit does not exist.

6.3.3 Continuity of a Function of Two Variables

Definition: Let $z = f(x, y)$ be a function of two independent variables x and y. The function f is said to be continuous at a point (a, b) of its domain of definition if the double limit or simultaneous limit $\lim\limits_{(x, y)\to(a, b)} f(x, y)$ exists and is equal to the functional value of $f(x, y)$ at (a, b). i.e.,

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

Example 10 Show that the function

$$f(x, y) = x^2 + y^2 + xy, (x, y) \neq (2, 3)$$

$$= 10, (x, y) = (2, 3)$$

is continuous at $(0, 0)$ but discontinuous at $(2, 3)$.

Sol. First, we find the limit

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} (x^2 + y^2 + xy)$$

$$= 0 + 0 + 0 = 0.$$

Again

$$f(0, 0) = 0 + 0 + 0 = 0$$

So,

$$\lim_{(x, y)\to(0, 0)} f(x, y) = 0 = f(0, 0)$$

Then the function is continuous at $(0, 0)$.

Again

$$\lim_{(x, y)\to(2, 3)} f(x, y) = \lim_{(x, y)\to(2, 3)} (x^2 + y^2 + xy)$$

$$= 4 + 9 + 6 = 19.$$

But

$$f(x, y) = 10, \text{ when } (x, y) = (2, 3)$$

i.e., $f(2, 3) = 10$

So,

$$\lim_{(x, y)\to(2, 3)} f(x, y) \neq f(2, 3).$$

Hence the function is not continuous at $(2, 3)$.

6.3.4 Observations

(1) If at a point, limit does not exist then the function cannnot be continuous there.

(2) A function which is not continuous at a point is called discontinious there.

(3) A function is said to be continuous in a region if it is continuous at every point in the region.

6.4 PARTIAL DERIVATIVES

6.4.1 First-Order Partial Derivatives

(i) Consider $f(x, y)$ to be a function of two independent variables, x and y.

The first-order partial derivative of $f(x, y)$ with respect to x, (treating y as constant) is denoted by

$$\frac{\partial f(x, y)}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y) \text{ or } f_x$$

and is defined as

$$\frac{\partial f(x, y)}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}, \text{provided the limit exists.}$$

Similarly, the first-order partial derivative of $f(x, y)$ with respect to y (treating x as constant) denoted by

$$\frac{\partial f(x, y)}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y) \text{ or } f_y$$

and is defined as

$$\frac{\partial f(x, y)}{\partial y} = f_y = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}, \text{ provided the limit exists.}$$

(ii) Consider $f(x, y, z)$ be a function of three independent variables $x,\ y$ and z.

The first-order partial derivative of $f(x, y, z)$ with respect to x, (treating y and z as constant) is denoted by

$$\frac{\partial f(x, y, z)}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y, z) \text{ or } f_x$$

and is defined as

$$\frac{\partial f(x, y, z)}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}, \text{ provided the limit exists.}$$

The first-order partial derivative of $f(x, y, z)$ with respect to y, (treating x and z as constant) denoted by

$$\frac{\partial f(x, y, z)}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y, z) \text{ or } f_y$$

and is defined as

$$\frac{\partial f(x, y, z)}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}, \text{ provided the limit exists.}$$

Similarly, the first-order partial derivative of $f(x, y, z)$ with respect to z, (treating x and y as constant) denoted by

$$\frac{\partial f(x, y, z)}{\partial z}, \frac{\partial f}{\partial z}, f_z(x, y, z) \text{ or } f_z$$

may be defined.

Note:

(1) Always keep in mind that determination of partial derivative of a function w.r.t. any of its independent variables is equivalent to ordinary derivative of the function w.r.t. the same variable, keeping all other variables as constant.

(2) Partial derivatives may exist at a point where the function may not be even continuous.

Example 11

Let $f(x, y) = x^2 + y^2 + xy + x + y$. Then

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + xy + x + y) \\ &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y) \\ &= 2x + 0 + 1 \cdot y + 1 + 0 \text{ (keeping } y \text{ as constant)} \\ &= 2x + y + 1 \end{aligned}$$

and

$$\begin{aligned} f_y &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + xy + x + y) \\ &= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y) \end{aligned}$$

$= 0 + 2y + x \cdot 1 + 0 + 1$ (keeping x as constant)

$= 2y + x + 1.$

Example 12

From definition, find $f_x(0, 0)$ and $f_y(0, 0)$ for the function

$$f(x, y) = \frac{x^2 + y^2}{x + y}, \text{ if } (x, y) \neq (0, 0)$$

$$= 0, \text{ if } (x, y) = (0, 0)$$

Sol. From definition,

$$\frac{\partial f(x, y)}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \text{ provided the limit exists.}$$

So,

$$\left[\frac{\partial f(x, y)}{\partial x}\right]_{(0,0)} = f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^2}{h} - 0}{h} = \lim_{h \to 0} 1 = 1$$

Again

$$\frac{\partial f(x, y)}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}, \text{ provided the limit exists.}$$

Then,

$$\left[\frac{\partial f(x, y)}{\partial y}\right]_{(0,0)} = f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{k^2}{k} - 0}{k} = 1$$

Example 13

If $z(x + y) = x^2 + y^2$ then show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Sol. Here, it is given that

$$z(x + y) = x^2 + y^2. \qquad \text{...(1)}$$

Differentiating partially w.r.t x, we obtain

$$\frac{\partial}{\partial x}[z(x + y)] = \frac{\partial}{\partial x}\left[x^2 + y^2\right]$$

or, $\frac{\partial z}{\partial x}\cdot(x+y)+z\cdot\frac{\partial}{\partial x}(x+y)=\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(y^2)$

or, $\frac{\partial z}{\partial x}(x+y)+z\cdot(1+0)=2x+0$, keeping y as constant.

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{(2x-z)}{(x+y)}$$

Similarly, differentiating partially w.r.t y, we have

$$\frac{\partial z}{\partial y}=\frac{(2y-z)}{(x+y)}$$

Now

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left(\frac{(2x-z)}{(x+y)}-\frac{(2y-z)}{(x+y)}\right)^2$$

$$=4\left(\frac{x-y}{x+y}\right)^2, \text{ putting the value of } z \text{ from (1)}$$

and

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

$$=4\left(1-\frac{(2x-z)}{(x+y)}-\frac{(2y-z)}{(x+y)}\right), \text{ putting the value of } z \text{ from (1)}$$

$$=4\left(\frac{x-y}{x+y}\right)^2$$

Hence the result is proved.

6.4.2 Second-Order Partial Derivatives

The second-order partial derivative of $f_x(x, y)$ with respect to x is denoted by

$$\frac{\partial}{\partial x}\left(f_x(x,y)\right), \frac{\partial}{\partial x}(f_x), \frac{\partial^2 f(x, y)}{\partial x^2} \text{ or } f_{xx}$$

and is defined at a point (a, b) as

$$\left[\frac{\partial^2 f(x, y)}{\partial x^2}\right]_{(a,b)}=f_{xx}(a, b)=\lim_{h\to 0}\frac{f_x(a+h, b)-f_x(a, b)}{h}, \text{ provided the limit exists.}$$

Similarly, the second-order partial derivative of $f_y(x, y)$ with respect to y is denoted by

$$\frac{\partial}{\partial y}\left(f_y(x, y)\right), \frac{\partial}{\partial y}(f_y), \frac{\partial^2 f(x, y)}{\partial y^2} \text{ or } f_{yy}$$

and is defined at a point (a, b) as

$$\left[\frac{\partial^2 f(x, y)}{\partial y^2}\right]_{(a,b)} = f_{yy}(a, b) = \lim_{k\to 0}\frac{f_y(a, b+k) - f_y(a, b)}{k}, \text{ provided the limit exists.}$$

The second-order partial derivative of $f_x(x, y)$ with respect to y is denoted by

$$\frac{\partial}{\partial y}\left(f_x(x, y)\right), \frac{\partial}{\partial y}(f_x), \frac{\partial^2 f(x, y)}{\partial y \partial x} \text{ or } f_{yx}$$

and is defined at a point (a, b) as

$$\left[\frac{\partial^2 f(x, y)}{\partial y \partial x}\right]_{(a,b)} = f_{yx}(a, b) = \lim_{k\to 0}\frac{f_x(a, b+k) - f_x(a, b)}{k}, \text{ provided the limit exists.}$$

The second-order partial derivative of $f_y(x, y)$ with respect to x is denoted by

$$\frac{\partial}{\partial x}\left(f_y(x, y)\right), \frac{\partial}{\partial x}(f_y), \frac{\partial^2 f(x, y)}{\partial x \partial y} \text{ or } f_{xy}$$

and is defined at a point (a, b) as

$$\left[\frac{\partial^2 f(x, y)}{\partial x \partial y}\right]_{(a,b)} = f_{xy}(a, b) = \lim_{h\to 0}\frac{f_y(a+h, b) - f_y(a, b)}{h}, \text{ provided the limit exists.}$$

Note: f_{xy} and f_{yx} are known as mixed partial derivative.

Example 14

Let us consider the function

$$z = x^2 y + xy^2 + x^2 y^2.$$

Then

$$\frac{\partial z}{\partial x} = 2xy + y^2 + 2xy^2 \text{ and } \frac{\partial z}{\partial y} = x^2 + 2xy + 2x^2 y$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 2y + 2y^2 \text{ and } z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2x + 2x^2$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x + 2y + 4xy$$

$$\text{and } z_{yx} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x + 2y + 4xy$$

Example 15

Let us consider the function is

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, (x, y) \neq (0, 0)$$

$$= 0, (x, y) = (0, 0).$$

From the definition, find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Also examine their equality.

[WBUT 2003]

Sol. By the definition of Section 6.4.2, we have

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \qquad ...(1)$$

Now by the definition of Section 6.4.1,

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k}$$

$$= \lim_{k\to 0} \frac{\dfrac{hk(h^2 - k^2)}{h^2 + k^2} - \dfrac{h\cdot 0(h^2 - 0^2)}{h^2 + 0^2}}{k}$$

$$= \lim_{k\to 0} \frac{h(h^2 - k^2)}{(h^2 + k^2)} = h$$

Also by the definition of Section 6.4.1

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k\to 0} \frac{0 - 0}{k} = 0$$

Using the above two results in (1), we obtain

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$= \lim_{h\to 0} \frac{h - 0}{h} = 1 \qquad ...(2)$$

Again by the definition of Section 6.4.2, we have

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \qquad ...(3)$$

Now by the definition of Section 6.4.1,

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{hk(h^2 - k^2)}{h^2 + k^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{k(h^2 - k^2)}{(h^2 + k^2)} = -k$$

Also, by the definition of Section 6.4.1

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

Using the above two results in (3), we obtain

$$f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{-k-0}{k} = -1 \qquad \text{...(4)}$$

From (2) and (4), we have

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

Hence we have the result.

Note: It is clear from the last example that the mixed partial derivatives may not be same always.

6.4.3 Results on the Equality of Mixed Partial Derivatives

Here we represent the two famous theorems on the equality of mixed partial derivatives without proof.

Theorem 6.1: (Schwarz Theorem)

If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function $f(x, y)$ and f_{yx} is continuous at (a, b) then $f_{xy}(a, b)$ exists and equal to $f_{yx}(a, b)$, i.e., $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof: Beyond the scope of the book.

Theorem 6.2: (Young's Theorem)

If f_x and f_y both exist in a certain neighbourhood of a point (a, b) and if both f_x and f_y are differentiable at the point (a, b) of the domain of definition of a function $f(x, y)$ then $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof: Beyond the scope of the book.

6.5 COMPOSITE FUNCTIONS

Let us consider the function $z = f(x, y)$ where x, y are not independent variables but functions of an independent variable t,

i.e., $x = \varphi(t)$ and $y = \psi(t)$.

Then, the composite function $z = f(x, y)$ is written as

$$z = f(x, y) = f\big(\varphi(t), \psi(t)\big)$$

Let us consider the function $z = f(x, y)$ where x, y are not independent variables but functions of the independent variable u and v.

i.e., $x = \varphi(u, v)$ and $y = \psi(u, v)$.

Then, the composite function $z = f(x, y)$ is written as

$$z = f(x, y) = f\left(\varphi(u, v), \psi(u, v)\right)$$

6.5.1 Partial Derivatives of Composite functions (Chain Rules)

First, we recall the chain rule for an ordinary derivative.

Theorem 6.3:

Let us consider

i) the function $z = f(x)$ to be a differentiable function of x, and

ii) x is not an independent variable but a differentiable function of the independent variable t, i.e., $x = \varphi(t)$.

Then,

$$\boxed{\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}.}$$

Theorem 6.4:

Let us consider

i) the function $z = f(x, y)$ to be a differentiable function of $x, y,$ and

ii) x, y are not independent variables but differentiable functions of the independent variable t, i.e., $x = \varphi(t)$ and $y = \psi(t)$.

Then,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Corollary of Theorem 6.4:

In particular, suppose $x = a + ht$ and $y = b + kt$; where a, b, h, k are constants.

Then from above

$$\frac{dz}{dt} = h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)z$$

and,

$$\boxed{\frac{d^n z}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n z}$$

Theorem 6.5:

Let us consider

i) the function $z = f(u)$ to be a differentiable function of u, and

ii) u is not an independent variable but differentiable functions of independent variables x and y.

Then,

$$\boxed{\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}}$$

Theorem 6.6:

Let us consider

i) the function $z = f(x, y)$ to be a differentiable function of x, y; and

ii) x, y are not independent variables but differentiable functions of independent variables u and v, i.e., $x = \varphi(u, v)$ and $y = \psi(u, v)$.

Then,

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

Theorem 6.7: Let us consider

i) the function $r = f(x, y, z)$ to be a differentiable function of x, y, z, and

ii) x, y, z are not independent variables but differentiable functions of independent variables u, v, w.

Then,

$$\boxed{\begin{aligned} \frac{\partial r}{\partial u} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial r}{\partial v} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial v} \\ \frac{\partial r}{\partial w} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial w} \end{aligned}}$$

Example 16

If $z = \sin(uv)$ where

$$u = 3x^2 \quad \text{and} \quad v = \log x$$

find $\dfrac{dz}{dx}$. [WBUT 2004]

Sol. By chain rule

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}$$

$$\text{or,} \quad \frac{dz}{dx} = \frac{\partial(\sin uv)}{\partial u}\frac{d\left(3x^2\right)}{dx} + \frac{\partial(\sin uv)}{\partial v}\frac{d(\log x)}{dx}$$

$$= v \cdot \cos(uv) \cdot 6x + u \cdot \cos(uv) \cdot \frac{1}{x}$$

$$= \cos(uv)\left(6vx + \frac{u}{x}\right)$$

Example 17

If $f(v^2-x^2, v^2-y^2, v^2-z^2)=0$, where v is a function of x, y, z then show that

$$\frac{1}{x}\frac{\partial v}{\partial x}+\frac{1}{y}\frac{\partial v}{\partial y}+\frac{1}{z}\frac{\partial v}{\partial z}=\frac{1}{v}$$ [WBUT 2005]

Sol. Let $\alpha=v^2-x^2$, $\beta=v^2-y^2$ and $\gamma=v^2-z^2$ then,

$$f(\alpha, \beta, \gamma)=0 \text{ and also } \frac{\partial f}{\partial x}=0 \quad ...(1)$$

Using chain rules, we have,

$$\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial x}+\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial x}=0$$

$$\text{or, } \frac{\partial f}{\partial \alpha}\left(2v\frac{\partial v}{\partial x}-2x\right)+\frac{\partial f}{\partial \beta}\left(2v\frac{\partial v}{\partial x}-0\right)+\frac{\partial f}{\partial \gamma}\left(2v\frac{\partial v}{\partial x}-0\right)=0$$

$$\text{or, } \frac{v}{x}\frac{\partial v}{\partial x}=\frac{\dfrac{\partial f}{\partial \alpha}}{\dfrac{\partial f}{\partial \alpha}+\dfrac{\partial f}{\partial \beta}+\dfrac{\partial f}{\partial \gamma}} \quad ...(2)$$

Similarly, from (1) $\dfrac{\partial f}{\partial y}=0$,

Using chain rules we have,

$$\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial y}+\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial y}=0$$

$$\Rightarrow \frac{v}{y}\frac{\partial v}{\partial y}=\frac{\dfrac{\partial f}{\partial \beta}}{\dfrac{\partial f}{\partial \alpha}+\dfrac{\partial f}{\partial \beta}+\dfrac{\partial f}{\partial \gamma}} \quad ...(3)$$

Similarly, from (1) $\dfrac{\partial f}{\partial z}=0$,

Using chain rules, we have,

$$\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial z}+\frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial z}+\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial z}=0$$

$$\Rightarrow \frac{v}{z}\frac{\partial v}{\partial z}=\frac{\dfrac{\partial f}{\partial \gamma}}{\dfrac{\partial f}{\partial \alpha}+\dfrac{\partial f}{\partial \beta}+\dfrac{\partial f}{\partial \gamma}} \quad ...(4)$$

Adding (2), (3) and (4), we obtain

$$\frac{v}{x}\frac{\partial v}{\partial x}+\frac{v}{y}\frac{\partial v}{\partial y}+\frac{v}{z}\frac{\partial v}{\partial z}=\frac{\dfrac{\partial f}{\partial \alpha}+\dfrac{\partial f}{\partial \beta}+\dfrac{\partial f}{\partial \gamma}}{\dfrac{\partial f}{\partial \alpha}+\dfrac{\partial f}{\partial \beta}+\dfrac{\partial f}{\partial \gamma}}=1$$

or, $\frac{1}{x}\frac{\partial v}{\partial x}+\frac{1}{y}\frac{\partial v}{\partial y}+\frac{1}{z}\frac{\partial v}{\partial z}=v.$

Hence the result is proved.

6.6. HOMOGENEOUS FUNCTION AND EULER'S THEOREM

6.6.1 Homogeneous Functions

Definition: A fuction $f(x, y)$ is said to be a homogeneous function of degree n if

$$f(tx, ty) = t^n f(x, y)$$

Alternatively, A function $f(x, y)$ is said to be a homogeneous function of degree n if

$$f(x, y) = x^n \varphi\left(\frac{y}{x}\right)$$

Definition: A function $f(x, y, z)$ is said to be a homogeneous function of degree n if

$$f(tx, ty, tz) = t^n f(x, y, z)$$

Alternatively, a function $f(x, y, z)$ is said to be a homogeneous function of degree n if

$$f(x, y, z) = x^n \varphi\left(\frac{y}{x}, \frac{z}{x}\right)$$

Generalised Definition: A function $f(x, y, z, \ldots)$ is said to be a homogeneous function of degree n if

$$f(tx, ty, tz, \ldots) = t^n f(x, y, z, \ldots)$$

Alternatively, a function $f(x, y, z)$ is said to be a homogeneous function of degree n if

$$f(x, y, z, \ldots) = x^n \varphi\left(\frac{y}{x}, \frac{z}{x}, \ldots\right)$$

Example 18

Let $f(x, y) = x^2 + y^2$; then

$$\begin{aligned} f(tx, ty) &= t^2 x^2 + t^2 y^2 \\ &= t^2 \cdot f(x, y). \end{aligned}$$

So, this is a homogeneous function of degree 2.

Example 19

Let $f(x, y) = x^2 + y^3 + xy^2$, then

$$f(tx, ty) = t^2x^2 + t^3y^3 + t^3xy^2$$
$$= t^2 \cdot (x^2 + ty^3 + txy^2)$$

So this is not a homogeneous function.

6.6.2 Euler's Theorem

Theorem 6.8: (First Order)

Let $f(x, y)$ be a homogeneous function of degree n. Then

$$\boxed{x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y} = n \cdot f(x, y)}$$

Theorem 6.9: (First Order)

Let $f(x, y, z)$ be a homogeneous function of degree n. Then

$$\boxed{x\frac{\partial f(x, y, z)}{\partial x} + y\frac{\partial f(x, y, z)}{\partial y} + z\frac{\partial f(x, y, z)}{\partial z} = n \cdot f(x, y, z)}$$

Theorem 6.10: (Second Order)

Let $f(x, y)$ be a homogeneous function of degree n. Then

$$\boxed{\left(x\frac{\partial f(x, y)}{\partial x} + y\frac{\partial f(x, y)}{\partial y}\right)^2 = n(n-1) \cdot f(x, y)}$$

Theorem 6.11: (Second Order)

Let $f(x, y, z)$ be a homogeneous function of degree n. Then

$$\boxed{\left(x\frac{\partial f(x, y, z)}{\partial x} + y\frac{\partial f(x, y, z)}{\partial y} + z\frac{\partial f(x, y, z)}{\partial z}\right)^2 = n(n-1) \cdot f(x, y, z)}$$

Example 20 If

$$u = \tan^{-1}\frac{x^3 + y^3}{x - y}$$

Using Euler's theorem prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$ [WBUT 2006]

Sol. Let

$$\tan u = \left(\frac{x^3 + y^3}{x - y}\right) = v(x, y)$$

Now,

$$v(tx, ty) = \frac{t^3(x^3+y^3)}{t(x-y)} = t^2\,\frac{(x^3+y^3)}{(x-y)} = t^2 v(x, y).$$

Therefore, $v(x, y)$ is a homogeneous function of degree 2.

By Euler's theorem

$$x\frac{\partial v(x, y)}{\partial x} + y\frac{\partial v(x, y)}{\partial y} = 2\cdot v(x, y)$$

$$\text{or, } x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2(\tan u)$$

$$\text{or, } \sec^2 u\cdot\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} = 2(\tan u)$$

$$\text{or, } \left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} = 2(\tan u)\cdot(\cos^2 u)$$

$$= 2\sin u\cos u = \sin 2u$$

Hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Example 21 If

$$u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$

then show that,

i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = xf\left(\dfrac{y}{x}\right)$

ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$ [WBUT 2004, 2007]

Sol. **i)** Let, $u = v + w$ where

$$v(x, y) = xf\left(\frac{y}{x}\right) \text{and } w(x, y) = g\left(\frac{y}{x}\right).$$

Now

$$v(tx, ty) = txf\left(\frac{ty}{tx}\right) = t^1\cdot v(x, y)$$

and

$$w(tx, ty) = g\left(\frac{ty}{tx}\right) = t^0\cdot w(x, y)$$

Therefore, v and w are homogeneous functions of degree 1 and 0 respectively.

Therefore, by Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1\cdot v$$

and $$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=0\cdot w$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial(v+w)}{\partial x}+y\frac{\partial(v+w)}{\partial y}$$

$$=\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)+\left(x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}\right)$$

$$=v+0=v=xf\left(\frac{y}{x}\right)$$

Hence

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=xf\left(\frac{y}{x}\right) \qquad ...(1)$$

ii) Differentiating (1) partially w.r.t x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=f\left(\frac{y}{x}\right)+xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or, $$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right) \qquad ...(2)$$

Differentiating (1) partially w.r.t y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

or, $$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=f'\left(\frac{y}{x}\right) \qquad ...(3)$$

Multiplying (2) by x and (3) by y and then adding, we get

$$\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]+\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=xf\left(\frac{y}{x}\right)-yf'\left(\frac{y}{x}\right)+yf'\left(\frac{y}{x}\right)$$

or, $$\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]+xf\left(\frac{y}{x}\right)=xf\left(\frac{y}{x}\right)$$

or, $$\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]=0$$

Hence, the result is proved.

6.7 DIFFERENTIATION OF IMPLICIT FUNCTIONS

6.7.1 Implicit Functions

Definition: Let us consider the equation $F(x, y, z) = 0$ where z is a function of two independent variables x and y. In this case, z is called an implicit function of x and y.

For example, $x^2 z + xyz^2 + xy^2 z = 0$ is an implicit function.

6.7.2 Derivative of Implicit Functions

Theorem 6.12: (Two Variables)

If $F(x, y) = 0$ be an equation of two variables x and y where y is an implicit function of x; then

$$\boxed{\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{F_x}{F_y}, \text{ provided } F_y = \frac{\partial F}{\partial y} \neq 0}$$

Theorem 6.13: (Three Variables)

If $F(x, y, z) = 0$ be an equation of three variables x, y and z where z is an implicit function of x and y then

i) $$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z}, \text{ provided } F_z = \frac{\partial F}{\partial z} \neq 0$$

ii) $$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_y}{F_z}, \text{ provided } F_z = \frac{\partial F}{\partial z} \neq 0$$

Example 22

Find $\frac{dy}{dx}$, if $x^3 + y^3 - 3xy - y^2 = 0$.

Sol. Here $F(x, y) = x^3 + y^3 - 3xy - y^2$, so y is an implicit function of x.

$$\frac{\partial F}{\partial x} = \frac{\partial(x^3 + y^3 - 3xy - y^2)}{\partial x} = 3x^2 - 3y,$$

$$\frac{\partial F}{\partial y} = \frac{\partial(x^3 + y^3 - 3xy - y^2)}{\partial y} = 3y^2 - 3x - 2y$$

Therefore,

$$\frac{dy}{dx} = -\frac{\left(\dfrac{\partial F}{\partial x}\right)}{\left(\dfrac{\partial F}{\partial y}\right)} = -\frac{F_x}{F_y} = \frac{(3y-3x^2)}{(3y^2-3x-2y)}.$$

6.8 TOTAL DIFFERENTIALS

6.8.1 Condition of Differentiability of a Function

Let us consider $z = f(x, y)$ be a function of two independent variables x and y. $f(x, y)$ is said to be differentiable if the increment Δz is expressed as

$$\Delta z = \Delta f = \{f(x+\Delta x, y+\Delta y) - f(x, y)\}$$

$$= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \eta\,\Delta x + \kappa\,\Delta y$$

where $\eta \to 0$ and $\kappa \to 0$ as Δx and $\Delta y \to 0$

6.8.1 First-order Total Differential

The expression

$$\boxed{dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

is called the total differential of z or $f(x, y)$.

Example 23 Show that $z = f(x, y) = xy - 2y$ is differentiable.

Sol. Here,

$$\frac{\partial f}{\partial x} = y \text{ and } \frac{\partial f}{\partial y} = (x-2)$$

Therefore,

$$\Delta z = \Delta f = \{f(x+\Delta x, y+\Delta y) - f(x, y)\}$$

$$= (x+\Delta x)(y+\Delta y) - 2(y+\Delta y) - xy + 2y$$

$$= \Delta xy + x\Delta y + \Delta x\Delta y - 2\Delta y$$

$$= y\Delta x + (x-2)\Delta y + \Delta x\Delta y$$

$$= y\Delta x + (x-2)\Delta y + \left(\frac{1}{2}\Delta y\right)\Delta x + \left(\frac{1}{2}\Delta x\right)\Delta y$$

$$= y\Delta x + (x-2)\Delta y + \eta\Delta x + \kappa\Delta y \text{ where } \eta = \left(\frac{1}{2}\Delta y\right) \text{and } \kappa = \left(\frac{1}{2}\Delta x\right)$$

$$= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \eta\Delta x + \kappa\Delta y \text{ where } \eta = \left(\frac{1}{2}\Delta y\right) \text{and } \kappa = \left(\frac{1}{2}\Delta x\right)$$

Now, $\eta = \left(\frac{1}{2}\Delta y\right) \to 0$ and $\kappa = \left(\frac{1}{2}\Delta x\right) \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Therefore, the function $z = f(x, y) = xy - 2y$ is differentiable.

The total differential is,

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = ydx + (x-2)\,dy$$

Theorem 6.14: Let $z = f(x_1, x_2, x_3)$ then the total differential of z is

$$\boxed{dz = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \frac{\partial z}{\partial x_3}dx_3}$$

Theorem 6.15: Let $f(x_1, x_2, \ldots, x_n) = c$ (constant) then $df = 0$.

6.8.2 Second-order Total Differentials

Theorem 6.16: Let $z = f(x, y)$ have continuous second-order derivatives. Then the second-order differential is

$$\boxed{d^2 z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z}$$

Theorem 6.17: Let $z = f(x_1, x_2, x_3)$ have continuous second-order derivatives. Then the second-order differential is

$$\boxed{d^2 z = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \frac{\partial}{\partial x_3}dx_3\right)^2 z}$$

6.9 JACOBIANS AND THEIR PROPERTIES

6.9.1 Definitions

1) Let $u_1(x, y)$ and $u_2(x, y)$ be two functions of independent variables x and y, having first-order partial derivatives. Then the determinant

$$\boxed{\frac{\partial(u_1, u_2)}{\partial(x, y)} = J\left(\frac{u_1, u_2}{x, y}\right) = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{vmatrix}}$$

is called the Jacobian of u_1, u_2 with respect to x, y.

2) Let $u_1(x, y, z)$, $u_2(x, y, z)$ and $u_3(x, y, z)$ be three functions of independent variables x, y and z, having first-order partial derivatives. Then the determinant

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x, y, z)} = J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{vmatrix}$$

is called the Jacobian of u_1, u_2, u_3 with respect to x, y and z.

3) Let $u_1(x_1, x_2, \dots, x_n)$, $u_2(x_1, x_2, \dots, x_n), \dots u_n(x_1, x_2, \dots, x_n)$ be n functions of independent variables $x_1, x_2, \dots, x_n$ having first-order partial derivatives. Then the determinant

$$J\left(\frac{u_1, u_2, \dots u_n}{x_1, x_2, \dots, x_n}\right) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \dots u_n$ with respect to $x_1, x_2, \dots, x_n$.

6.9.2 Properties of Jacobians

Property 1 **Let $u_1(x, y)$ and $u_2(x, y)$ be two functions of independent variables x and y, having first-order partial derivatives; then**

$$J\left(\frac{u_1, u_2}{x, y}\right) \cdot J\left(\frac{x, y}{u_1, u_2}\right) = 1$$

Property 2 (Chain Rule for Jacobians)

Let $u_1(x, y)$ and $u_2(x, y)$ be two functions of variables x and y having first-order partial derivatives, while x and y are functions of r and s; then

$$J\left(\frac{u_1, u_2}{r, s}\right) = J\left(\frac{u_1, u_2}{x, y}\right) \cdot J\left(\frac{x, y}{r, s}\right)$$

Property 3 (Chain Rule for Jacobians)

Let $u_1(x, y, z)$, $u_2(x, y, z)$ and $u_3(x, y, z)$ be three functions of x, y, z having first-order partial derivatives, while x, y and z are functions of r, s and t; then

$$J\left(\frac{u_1, u_2, u_3}{r, s, t}\right) = J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) \cdot J\left(\frac{x, y, z}{r, s, t}\right)$$

Property 4 (Chain Rule for Jacobians)

Let u_1 and u_2 are functions of α, β, γ and α, β, γ are functions of x and y; then

$$\boxed{J\left(\frac{u_1, u_2}{x, y}\right) = J\left(\frac{u_1, u_2}{\alpha, \beta}\right) J\left(\frac{\alpha, \beta}{x, y}\right) + J\left(\frac{u_1, u_2}{\beta, \gamma}\right) J\left(\frac{\beta, \gamma}{x, y}\right) + J\left(\frac{u_1, u_2}{\gamma, \alpha}\right) J\left(\frac{\gamma, \alpha}{x, y}\right)}$$

Example 24

If $f(x, y) = \dfrac{x+y}{1-xy}$ and $g(x, y) = \tan^{-1}x + \tan^{-1}y$. find $\dfrac{\partial(f, g)}{\partial(x, y)}$. [WBUT 2006].

Sol. Here

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix} \qquad ...(1)$$

Now,

$$\frac{\partial f}{\partial x} = \frac{1+y^2}{(1-xy)^2} \text{ and } \frac{\partial f}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

Also

$$\frac{\partial g}{\partial x} = \frac{1}{1+x^2} \text{ and } \frac{\partial g}{\partial y} = \frac{1}{1+y^2}$$

So from (1), we have

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence the result is proved.

6.10 MAXIMA AND MINIMA

6.10.1 Maxima and Minima of Explicit Functions

Stationary Point or Critical Point

All the points satisfying $f_x(x, y) = 0$ and $f_y(x, y) = 0$ are called stationary or critical points.

Necessary Condition for Maxima and Minima

The necessary condition for $f(x, y)$ to have maxima and minima at (a, b) is $f_x(a, b) = 0$ and $f_y(a, b) = 0$, provided they exist.

Note: The above condition reflects the fact that both the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ being zero at a point does not gurantee that at that point the function will always, have maxima or minima

i.e., the function may not have the extremum at all the stationary points.

Saddle Point

A point (a, b) is called a saddle point of the function $z = f(x, y)$ if it is a point of neither maximum nor minimum though $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Sufficient Condition for Maxima and Minima

Let $f(x, y)$ be a continuous function having second-order partial derivatives. The sufficient condition for $f(x, y)$ to have extremum at (a, b) is

$$\boxed{f_x(a, b) = 0 \text{ and } f_y(a, b) = 0, \text{ provided they exist}}$$

and

$$\boxed{f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2 > 0}$$

and this extreme value is

i) a maxima according as

$$f_{xx}(a, b) < 0 \text{ and } f_{yy}(a, b) < 0$$

ii) a minima according as

$$f_{xx}(a, b) > 0 \text{ and } f_{yy}(a, b) > 0$$

Note:

Let $f(x, y)$ be a continuous function having second-order partial derivatives such that

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0, \text{ provided they exist}$$

Now if (i)

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2 < 0,$$

Then $f(x, y)$ has no extreme value at (a, b), i.e., (a, b) is a saddle point.

and if (ii)

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2 = 0$$

Then $f(x, y)$ may or may not have extreme value at (a, b), i.e., the case is undecided and further investigation is required.

Alternate Conditions of Maxima and Minima

Let $z = f(x, y)$ be a continuous function having second-order partial derivatives.

If $df = 0$ at (a, b) then

i) (a, b) is a point of maximum if $d^2 f < 0$, and

ii) (a, b) is a point of minimum if $d^2 f > 0$.

Example 25

Find the maxima and minima of the function $x^3+y^3-3x-12y+20$. Find also the saddle points. [WBUT 2001, 2005]

Sol. Here, $f(x, y) = x^3+y^3-3x-12y+20$. Then

$$f_x(x, y) = 3x^2-3,\ f_y(x, y) = 3y^2-12,$$

$$f_{xx}(x, y) = 6x,\ \ f_{yy}(x, y) = 6y \text{ and } f_{xy}(x, y) = 0$$

Solving,

$$f_x(x, y) = 3x^2-3 = 0$$

and $f_y(x, y) = 3y^2-12 = 0$

we obtain,

$x = \pm 1$ and $y = \pm 2$

Therefore, the (stationary points) critical points are $(1, 2), (1, -2), (-1, 2)$ and $(-1, -2)$.

Now,

$$f_{xx}(x, y) \cdot f_{yy}(x, y) - \left[f_{xy}(x, y)\right]^2 = 36xy$$

At the point $(1, 2)$

$$f_{xx}(1, 2) \cdot f_{yy}(1, 2) - \left[f_{xy}(1, 2)\right]^2 = 72 > 0,$$

$$f_{xx}(1, 2) = 6 > 0 \text{ and } f_{yy}(1, 2) = 12 > 0$$

Therefore, $f(x, y)$ has minimum at $(1, 2)$ and the minimum value is $f(1, 2) = 2$.

At the point $(-1, 2)$

$$f_{xx}(-1, 2) \cdot f_{yy}(-1, 2) - \left[f_{xy}(-1, 2)\right]^2 = -72 < 0$$

Therefore, $f(x, y)$ has neither maximum nor minimum at $(-1, 2)$.

At the point $(1, -2)$

$$f_{xx}(1, -2) f_{yy}(1, -2) - \left[f_{xy}(1, -2)\right]^2 = -72 < 0$$

Therefore, $f(x, y)$ has neither maximum nor minimum at $(1, -2)$.

At the point $(-1, -2)$

$$f_{xx}(-1, -2) f_{yy}(-1, -2) - \left[f_{xy}(-1, -2)\right]^2 = 72 > 0$$

$$f_{xx}(-1, -2) = -6 < 0 \text{ and } f_{yy}(-1, -2) = -12 < 0$$

Therefore, $f(x, y)$ has maximum at $(-1, -2)$ and the minimum value is $f(-1, -2) = 38$.

We have, from the above, that at the stationary points $(-1, 2)$ and $(1, -2)$, the function does not have any extreme values. So the saddle points are $(-1, 2)$ and $(1, -2)$.

The following topic is included for further reading by interested students.

6.10.2 Maxima and Minima of Implicit Functions: (Lagrange's Multiplier Method)

For Functions of Two Variables:

Let $f(x, y)$ be a function of two variables x and y, subject to the constraint conditions $\phi(x, y) = 0$.

Let $L(x, y) = f(x, y) + \lambda\phi(x, y)$, where λ is called the Lagrangian multiplier.

The critical points can be found by solving

$$\phi(x, y) = 0, \frac{\partial L}{\partial x} = 0 \text{ and } \frac{\partial L}{\partial y} = 0$$

i) the critical point is a point of maxima according as $d^2 f < 0$ where $d^2 f$ is determined considering y is dependent on x

or, the critical point is a point of maxima according as $d^2 L < 0$

ii) the critical point is a point of minima according as $d^2 f > 0$ where $d^2 f$ is determined considering y is dependent on x

or, the critical point is a point of minima according as $d^2 L > 0$

For Functions of Three Variables:

Let $f(x, y, z)$ be a function of two variables x, y and z, subject to the constraint conditions $\phi(x, y, z) = 0$.

Let $L(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$, where λ is called the Lagrangian multiplier.

The critical points can be found by solving

$$\phi(x, y, z) = 0, \frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0 \text{ and } \frac{\partial L}{\partial z} = 0.$$

i) the critical point is a point of maxima according as $d^2 f < 0$ where $d^2 f$ is determined considering z is dependent on x and y

or, the critical point is a point of maxima according as $d^2 L < 0$

ii) the critical point is a point of minima according as $d^2 f > 0$ where $d^2 f$ is determined considering z is dependent on x and y

or, the critical point is a point of minima according as $d^2 L > 0$.

Example 26

Find the optimum value of $f(x, y) = x^2 y^2$, subject to the condition $x + y = 1$ using Lagrangian multiplier method.

Sol. Let

$$\phi(x, y) = x + y - 1 = 0. \qquad ...(1)$$

Now $L(x, y) = f(x, y) + \lambda.\phi(x, y)$, where λ is the Lagrangian multiplier.

So,

$$L(x, y) = f(x, y) + \lambda.\phi(x, y)$$
$$= x^2y^2 + \lambda \cdot (x + y - 1)$$

Then

$$\frac{\partial L}{\partial x} = 2xy^2 + \lambda$$

and

$$\frac{\partial L}{\partial y} = 2x^2y + \lambda$$

Now solving $\frac{\partial L}{\partial x} = 0,\ \frac{\partial L}{\partial y} = 0$ and (1) we have

$$x = \frac{1}{2} \text{ and } y = \frac{1}{2}.$$

So the critical point is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Now

$$d^2 f = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 f$$
$$= \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial y^2}(dy)^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx \cdot dy \qquad ...(2)$$

Again

$$\frac{\partial^2 f}{\partial x^2} = 2y^2,\ \frac{\partial^2 f}{\partial y^2} = 2x^2,\ \frac{\partial^2 f}{\partial x \partial y} = 4xy$$

and $x + y = 1 \Rightarrow dx = -dy$

Putting the values in (2), we have

$$d^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial y^2}(dy)^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx \cdot dy$$
$$= 2y^2(dx)^2 + 2x^2(-dx)^2 + 2 \cdot 4xy \cdot dx \cdot (-dx)$$
$$= (2y^2 + 2x^2 - 8xy)(dx)^2$$

Since

$$\left(d^2 f\right)_{\left(\frac{1}{2}, \frac{1}{2}\right)} = \left(2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} - 8 \cdot \frac{1}{2} \cdot \frac{1}{2}\right)(dx)^2$$
$$= -(dx)^2 < 0$$

the function $f(x, y) = x^2y^2$ attains maximum value at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The maximum value of $f(x, y)$ is $\frac{1}{16}$.

WORKED-OUT EXAMPLES

Example 6.1 Show that the function

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}, (x, y) \neq (0, 0)$$
$$= 0, \qquad (x, y) = (0, 0)$$

is continuous at $(0, 0)$.

Sol. To prove the continuity at $(0, 0)$, we are to show

$$\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0)$$

i.e., $$\lim_{(x, y)\to(0, 0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

Now to prove the existence of the above limit, for a given $\varepsilon > 0$ we are to find $\delta > 0$ such that in any δ-nbd N of $(0, 0)$,

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon$$

or, $$|x||y|\frac{|x^2 - y^2|}{x^2 + y^2} < \varepsilon \qquad ...(1)$$

We know

$|x| < \sqrt{x^2 + y^2}, |y| < \sqrt{x^2 + y^2}$ and $|x^2 - y^2| < x^2 + y^2$

So

$$|x||y|\frac{|x^2 - y^2|}{x^2 + y^2} < \frac{\sqrt{x^2 + y^2} \cdot \sqrt{x^2 + y^2} \cdot (x^2 + y^2)}{x^2 + y^2}$$
$$= x^2 + y^2 < \varepsilon \text{ (consider)}$$

if $0 < (x - 0)^2 + (y - 0)^2 < \delta^2$, where $\delta = \sqrt{\varepsilon}$.

Here we are getting a circular δ-nbd of $(0,0)$.

So, the condition for existence of the limit is satisfied and correspondingly, we have

$$\lim_{(x, y)\to(0, 0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

i.e., $$\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0)$$

Hence the given function is continuous at $(0, 0)$.

Example 6.2 If

$$f(x, y) = \frac{xy}{xy + x - y}, \text{ when } (x, y) \neq (0, 0)$$

$$= 0, \qquad \text{when } (x, y) = (0, 0)$$

Show that both the repeated limits exist and are equal but the double limit does not exist.

Sol. The repeated limits are

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \lim_{y \to 0} \frac{xy}{xy + x - y} = \lim_{x \to 0} \frac{0}{0 + x} = 0$$

and

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} \frac{xy}{xy + x - y} = \lim_{y \to 0} \frac{0}{0 - y} = 0.$$

Therefore we have

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y) = 0$$

Hence the repeated limits exist and are equal.

Along the x-axis (i.e., $y = 0$)

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \frac{0}{x} = 0$$

Along the path $y = x$

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{xy}{xy + x - y} = \lim_{x \to 0} \frac{x \cdot x}{x \cdot x + x - x} = \lim_{x \to 0} \frac{x^2}{x^2} = 1.$$

We see that along two different paths, the limits are different. So, the double limit does not exist.

Example 6.3 If

$$f(x, y) = \frac{x^3 + y^3}{x - y}, \text{ when } x \neq y$$

$$= 0, \text{ when } (x, y) = (0, 0)$$

Examine whether the repeated limits and double limit exist and are equal. Is the function continuous at $(0, 0)$.

Sol. The repeated limits are

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \lim_{y \to 0} \frac{x^3 + y^3}{x - y} = \lim_{x \to 0} \frac{x^3}{x} = 0$$

and

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\lim_{x\to 0} \frac{x^3 + y^3}{x - y} = \lim_{y\to 0} \frac{y^3}{-y} = 0$$

Therefore,

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{y\to 0}\lim_{x\to 0} f(x, y) = 0$$

Hence the repeated limits exist and are equal.

Along the curve $y = x - mx^3$, we have

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{x^3 + y^3}{x - y} = \lim_{x\to 0} \frac{x^3 + \left(x - mx^3\right)^3}{mx^3}$$

$$= \lim_{x\to 0} \frac{x^3 + x^3\left(1 - mx^2\right)^3}{mx^3} = \lim_{x\to 0} \frac{1 + \left(1 - mx^2\right)^3}{m} = \frac{2}{m}$$

which is different for different values of m. So, the limit is not unique. Hence the double limit does not exist at $(0, 0)$ and correspondingly, $f(x, y)$ is not continious at $(0, 0)$.

Example 6.4 Show that for the function

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}, (x, y) \neq (0, 0)$$
$$= 0, \qquad (x, y) = (0, 0)$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$ [WBUT-2008]

Sol. By the definition in Section 6.4.2, we have

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \quad \text{...(1)}$$

Now by the definition of Section 6.4.1,

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k\to 0} \frac{\frac{h^2 k^2}{h^2 + k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{h^2 k}{h^2 + k^2} = 0$$

Also, by the definition of Section 6.4.1

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0 - 0}{k} = 0$$

Using the above two results in (1), we obtain

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0 \qquad ...(2)$$

Again by the definition of Section 6.4.2, we have

$$f_{yx}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} \qquad ...(3)$$

Now by the definition of Section 6.4.1,

$$f_x(0,k)=\lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h}=\lim_{h\to 0}\frac{\frac{h^2k^2}{h^2+k^2}-0}{h}$$

$$=\lim_{h\to 0}\frac{hk^2}{h^2+k^2}=0$$

Also, by the definition of Section 6.4.1

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

Using the above two results in (3), we obtain

$$f_{yx}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0 \qquad ...(4)$$

From (2) and (4), we have

$$f_{xy}(0,0)=f_{yx}(0,0)$$

Hence we have the result.

Example 6.5 If $u=\sqrt{xy}$, find the value of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$ [WBUT-2001]

Sol. Here,

$$\frac{\partial u}{\partial x}=\frac{1}{2\sqrt{x}}\sqrt{y} \text{ and } \frac{\partial u}{\partial y}=\frac{1}{2\sqrt{y}}\sqrt{x}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2}=\frac{-1}{4}\frac{\sqrt{y}}{x^{\frac{3}{2}}} \text{ and } \frac{\partial^2 u}{\partial y^2}=\frac{-1}{4}\frac{\sqrt{x}}{y^{\frac{3}{2}}}$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\frac{-1}{4}\frac{\sqrt{y}}{x^{\frac{3}{2}}}\right)+\left(\frac{-1}{4}\frac{\sqrt{x}}{y^{\frac{3}{2}}}\right)=\frac{-1}{4}\left(\frac{x^2+y^2}{x^{\frac{3}{2}}y^{\frac{3}{2}}}\right)$$

Example 6.6 If

$$f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

verify that $f_{xy} = f_{yx}$.

Sol. Differentiating $f(x, y)$ partially with respect to x, we have

$$f_x = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-y}{x^2}\right) - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{(x^2+y^2)y}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Again differentiating f_x partially with respect to y, we have

$$f_{yx} = 2x\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \qquad ...(1)$$

Now differentiating $f(x, y)$ partially with respect to y, we have

$$f_y = x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) - 2y\tan^{-1}\left(\frac{x}{y}\right) - y^2\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating f_y partially with respect to x, we have

$$f_{xy} = 1 - 2y\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad ...(2)$$

From (1) and (2), it is verified that $f_{xy} = f_{yx}$.

Example 6.7 If $u = f\left(x^2 + 2yz,\ y^2 + 2zx\right)$, show that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0$$ [WBUT-2001]

Sol. Let

$X = x^2 + 2yz$ and $Y = y^2 + 2zx$

Therefore

$$u = f\left(x^2 + 2yz,\ y^2 + 2zx\right) = f(X, Y)$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} = 2x\frac{\partial u}{\partial X} + 2z\frac{\partial u}{\partial Y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} = 2z\frac{\partial u}{\partial X} + 2y\frac{\partial u}{\partial Y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} = 2y\frac{\partial u}{\partial X} + 2x\frac{\partial u}{\partial Y}$$

Therefore,

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z}$$

$$= \left(y^2 - zx\right)\left(2x\frac{\partial u}{\partial X} + 2z\frac{\partial u}{\partial Y}\right) + \left(x^2 - yz\right)\left(2z\frac{\partial u}{\partial X} + 2y\frac{\partial u}{\partial Y}\right)$$

$$+ \left(z^2 - xy\right)\left(2y\frac{\partial u}{\partial X} + 2x\frac{\partial u}{\partial Y}\right)$$

$$= \frac{\partial u}{\partial X}\left(2xy^2 - 2zx^2 + 2x^2z - 2yz^2 + 2yz^2 - 2xy^2\right)$$

$$+ \frac{\partial u}{\partial Y}\left(2y^2z - 2xz^2 + 2x^2y - 2y^2z + 2xz^2 - 2x^2y\right)$$

$$= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}\cdot 0 = 0$$

Example 6.8 If

$$u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$$

then show that

i) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x + y + z}$$

ii) $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u=-\frac{3}{(x+y+z)^2}$$

iii) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$$ [WBUT-2003].

Sol. We know,

$$\left(x^3+y^3+z^3-3xyz\right)=(x+y+z)\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right),$$

where ω is the cube root of unity.

Therefore,

$$u=\log\left(x^3+y^3+z^3-3xyz\right)$$

i.e., $u=\log(x+y+z)+\log\left(x+y\omega+z\omega^2\right)+\log\left(x+y\omega^2+z\omega\right)$

Now,

$$\frac{\partial u}{\partial x}=\frac{1}{(x+y+z)}+\frac{1}{\left(x+y\omega+z\omega^2\right)}+\frac{1}{\left(x+y\omega^2+z\omega\right)} \quad ...(1)$$

$$\frac{\partial u}{\partial y}=\frac{1}{(x+y+z)}+\frac{\omega}{\left(x+y\omega+z\omega^2\right)}+\frac{\omega^2}{\left(x+y\omega^2+z\omega\right)} \quad ...(2)$$

$$\frac{\partial u}{\partial z}=\frac{1}{(x+y+z)}+\frac{\omega^2}{\left(x+y\omega+z\omega^2\right)}+\frac{\omega}{\left(x+y\omega^2+z\omega\right)} \quad ...(3)$$

i) Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{(x+y+z)}+\frac{1+\omega+\omega^2}{\left(x+y\omega+z\omega^2\right)}+\frac{1+\omega+\omega^2}{\left(x+y\omega^2+z\omega\right)}$$

Since $1+\omega+\omega^2=0$, we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\frac{3}{x+y+z}.$$

ii) Now, from (1), we have

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1}{(x+y+z)}\right)+\frac{\partial}{\partial x}\left(\frac{1}{(x+y\omega+z\omega^2)}\right)+\frac{\partial}{\partial x}\left(\frac{1}{(x+y\omega^2+z\omega)}\right)$$

$$=-\frac{1}{(x+y+z)^2}-\frac{1}{\left(x+y\omega+z\omega^2\right)^2}-\frac{1}{\left(x+y\omega^2+z\omega\right)^2} \quad ...(4)$$

Since $\omega^4 = \omega$, from (2), we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{1}{(x+y+z)}\right) + \frac{\partial}{\partial y}\left(\frac{\omega}{\left(x+y\omega+z\omega^2\right)}\right) + \frac{\partial}{\partial y}\left(\frac{\omega^2}{\left(x+y\omega^2+z\omega\right)}\right)$$

$$= -\frac{1}{(x+y+z)^2} - \frac{\omega^2}{\left(x+y\omega+z\omega^2\right)^2} - \frac{\omega}{\left(x+y\omega^2+z\omega\right)^2} \qquad ...(5)$$

Since $\omega^4 = \omega$, from (3), we obtain

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{1}{(x+y+z)}\right) + \frac{\partial}{\partial z}\left(\frac{\omega}{\left(x+y\omega+z\omega^2\right)}\right) + \frac{\partial}{\partial z}\left(\frac{\omega^2}{\left(x+y\omega^2+z\omega\right)}\right)$$

$$= -\frac{1}{(x+y+z)^2} - \frac{\omega}{\left(x+y\omega+z\omega^2\right)^2} - \frac{\omega^2}{\left(x+y\omega^2+z\omega\right)^2} \qquad ...(6)$$

Adding (4), (5) and (6), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2} - \frac{\left(1+\omega+\omega^2\right)}{\left(x+y\omega+z\omega^2\right)^2} - \frac{\left(1+\omega+\omega^2\right)}{\left(x+y\omega^2+z\omega\right)^2}$$

$$\text{or, } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = -\frac{3}{(x+y+z)^2}, \text{ since } \left(1+\omega+\omega^2\right) = 0$$

iii)

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ by (1)}$$

$$= 3\left\{\frac{\partial}{\partial x}\left(\frac{1}{(x+y+z)}\right) + \frac{\partial}{\partial y}\left(\frac{1}{(x+y+z)}\right) + \frac{\partial}{\partial z}\left(\frac{1}{(x+y+z)}\right)\right\}$$

$$= 3\left\{-\frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right\}$$

$$= -\frac{9}{(x+y+z)^2}$$

Example 6.9 If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$ show that

$$y\cdot\frac{\partial z}{\partial u} + x\cdot\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$$

[WBUT-2006, 2009]

Sol. Here $z = f(x, y)$ where $x = e^u \cos v,\ y = e^u \sin v$.

By chain rules

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^u \cos v \cdot \frac{\partial z}{\partial x} + e^u \sin v \cdot \frac{\partial z}{\partial y}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}$$

Therefore,

$$y \cdot \frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v}$$

$$= e^u \sin v \left(e^u \cos v \cdot \frac{\partial z}{\partial x} + e^u \sin v \cdot \frac{\partial z}{\partial y} \right) + e^u \cos v \left(-e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y} \right)$$

$$= \left(e^{2u} \sin v \cos v - e^{2u} \sin v \cos v \right) \frac{\partial z}{\partial x} + \left(e^{2u} \sin v \sin v + e^{2u} \cos v \cos v \right) \frac{\partial z}{\partial y}$$

$$= e^{2u} \left(\sin^2 v + \cos^2 v \right) \frac{\partial z}{\partial y} = e^{2u} \frac{\partial z}{\partial y}$$

Example 6.10 Show that the transformation $u = x - ct,\quad v = x + ct$ reduces the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \quad \text{to the equation} \quad \frac{\partial^2 z}{\partial u \partial v} = 0$$

Sol. Here,

$u = x - ct, v = x + ct$ and $z = z(u, v)$

Now,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial z}{\partial u}(-c) + \frac{\partial z}{\partial v}(c)$$

or, $$\frac{\partial z}{\partial t} = -c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) z$$

or, $$\frac{\partial}{\partial t} = -c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right)$$

Therefore,

$$\frac{\partial^2}{\partial t^2} = c^2 \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right)^2 = c^2 \left[\frac{\partial^2}{\partial u^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2} \right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] \qquad \text{...(1)}$$

Again,

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

or, $\frac{\partial z}{\partial x}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z$

or, $\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)$

Therefore,

$$\frac{\partial^2}{\partial x^2}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)^2=\left[\frac{\partial^2}{\partial u^2}+2\frac{\partial^2}{\partial u\partial v}+\frac{\partial^2}{\partial v^2}\right]$$

$$\Rightarrow\frac{\partial^2 z}{\partial x^2}=\left[\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right] \qquad ...(2)$$

Using (1) and (2) in the given equation

$$\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$$

we obtain

$$c^2\left[\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right]=c^2\left[\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right]$$

Simplifying, we get

$$4\cdot\frac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\frac{\partial^2 z}{\partial u\partial v}=0.$$

Hence, the required result is proved.

Example 6.11 If u be a function of x and y, prove that

a)

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

b)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

where $x=rcos\theta$ and $y=rsin\theta$. [WBUT-2002, 2008]

Sol. Here,

$x=rcos\theta$ and $y=rsin\theta$

a) By chain rules, we have

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta \qquad ...(1)$$

and

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)$$

or, $$\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta \qquad ...(2)$$

Therefore, from (1) and (2), we have

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)^2+\left(-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta\right)^2$$

$$=\cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2+2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2$$

$$+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2-2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

b) From (1), we have the operator $\frac{\partial}{\partial r}$ as

$$\frac{\partial}{\partial r}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y} \qquad ...(A)$$

and from (2), we have the operator $\frac{\partial}{\partial \theta}$ as

$$\frac{\partial}{\partial \theta}=-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y} \qquad ...(B)$$

Now,

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$=\cos\theta\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)$$

$$+\sin\theta\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\text{, using operator (A)}$$

Since $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$,

$$\frac{\partial^2 u}{\partial r^2}=\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2} \qquad ...(3)$$

Similarly,

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)=\frac{\partial}{\partial \theta}\left(-\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta\right)$$

$$=-r\cos\theta\left(\frac{\partial u}{\partial x}\right)-r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)-r\sin\theta\left(\frac{\partial u}{\partial y}\right)+r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$=-r\cos\theta\left(\frac{\partial u}{\partial x}\right)-r\sin\theta\left(\frac{\partial u}{\partial y}\right)-r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)+r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$=-r\left\{\cos\theta\left(\frac{\partial u}{\partial x}\right)+\sin\theta\left(\frac{\partial u}{\partial y}\right)\right\}-r\sin\theta\left(-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)$$

$$+r\cos\theta\left(-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right), \text{ using operator (B)}$$

$$=-r\frac{\partial u}{\partial r}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}-r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x}$$

$$-r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

Since $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$, we get from above

$$\frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial u}{\partial r}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$\text{i.e., } \frac{\partial^2 u}{\partial \theta^2}+r\frac{\partial u}{\partial r}=r^2\left[\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right]$$

$$\text{i.e., } \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2} \qquad ...(4)$$

Adding (3) and (4), we have

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$=\left(\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}\right)$$

$$+\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Hence both the results are proved.

Example 6.12 If $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$$ [WBUT-2009]

Sol. Let

$$\cos u=\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=v(x,y)$$

Now,

$$v(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=t^{\frac{1}{2}}v(x,y).$$

Therefore, $v(x,y)$ is a homogeneous function of degree $\dfrac{1}{2}$.
By Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}\cdot v$$

$$\text{or, } x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=\frac{1}{2}\cdot(\cos u)$$

$$\text{or, } -\sin u\cdot\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}=\frac{1}{2}\cdot(\cos u)$$

$$\text{or, } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cdot\left(\frac{\cos u}{\sin u}\right)=-\frac{1}{2}\cot u$$

Hence

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$$

Example 6.13 If $U=\sin^{-1}\left[\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ show that

$$x^2U_{xx}+2xyU_{xy}+y^2U_{yy}=\frac{\tan U}{144}\left(13+\tan^2 U\right).$$ [WBUT-2001, 2008]

Sol. Here,

$$U = \sin^{-1}\left[\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}\right]^{\frac{1}{2}} \Rightarrow \sin^2 U = \frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}$$

Here, $\sin^2 U$ is a homogeneous function of degree $\left(\frac{-1}{6}\right)$.
Therefore, by Euler's theorem

$$x\frac{\partial\left(\sin^2 U\right)}{\partial x} + y\frac{\partial\left(\sin^2 U\right)}{\partial y} = \frac{-1}{6}\left(\sin^2 U\right)$$

$$\text{or, } x\cdot\left(2\sin U\cos U\cdot\frac{\partial U}{\partial x}\right) + y\cdot\left(2\sin U\cos U\cdot\frac{\partial U}{\partial y}\right) = \frac{-1}{6}\sin^2 U$$

$$\text{or, } x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \left(\frac{1}{2\sin U\cos U}\right)\left(\frac{-1}{6}\sin^2 U\right) = \frac{-1}{12}\tan U \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we have respectively

$$\frac{\partial U}{\partial x} + x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = \frac{-1}{12}\sec^2 U\frac{\partial U}{\partial x} \qquad ...(2)$$

$$x\frac{\partial^2 U}{\partial y\partial x} + \frac{\partial U}{\partial y} + y\frac{\partial^2 U}{\partial y^2} = \frac{-1}{12}\sec^2 U\frac{\partial U}{\partial y} \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, we have

$$\left(x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right) + x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x\partial y} + y^2\frac{\partial^2 U}{\partial y^2}$$

$$= \frac{-1}{12}\sec^2 U\left(x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right)\left(\text{since } \frac{\partial^2 U}{\partial x\partial y} = \frac{\partial^2 U}{\partial y\partial x}\right)$$

$$\text{or, } \frac{-1}{12}\tan U + \left(x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x\partial y} + y^2\frac{\partial^2 U}{\partial y^2}\right) = \frac{-1}{12}\sec^2 U\left(\frac{-1}{12}\tan U\right)$$

$$\text{or, } x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x\partial y} + y^2\frac{\partial^2 U}{\partial y^2} = \frac{1}{144}\tan U\sec^2 U + \frac{1}{12}\tan U$$

$$= \frac{\tan U}{144}\left(\sec^2 U + 12\right) = \frac{\tan U}{144}\left(\tan^2 U + 13\right)$$

Hence

$$x^2 U_{xx} + 2xyU_{xy} + y^2 U_{yy} = \frac{\tan U}{144}(13 + \tan^2 U).$$

Example 6.14 If $f(x, y) = 0$, show that

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{(f_y)^3}$$

Sol. Here $f(x, y) = 0$ defines y as an implicit function of x.

Therefore,

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-f_x}{f_y}\right) = -\frac{f_y\frac{d}{dx}(f_x) - f_x\frac{d}{dx}(f_y)}{(f_y)^2} \quad \text{...(1)}$$

Now,

$$\frac{d}{dx}(f_x) = \frac{\partial}{\partial x}(f_x)\frac{dx}{dx} + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx} = f_{xx} + f_{yx}\left(\frac{-f_x}{f_y}\right) \quad \text{...(2)}$$

and

$$\frac{d}{dx}(f_y) = \frac{\partial}{\partial x}(f_y)\frac{dx}{dx} + \frac{\partial}{\partial y}(f_y)\frac{dy}{dx} = f_{xy} + f_{yy}\left(\frac{-f_x}{f_y}\right) \quad \text{...(3)}$$

Using (2) and (3) in (1) and assuming $f_{xy} = f_{yx}$, we have

$$\frac{d^2y}{dx^2} = -\frac{f_y\left(f_{xx} + f_{yx}\left(\frac{-f_x}{f_y}\right)\right) - f_x\left(f_{xy} + f_{yy}\left(\frac{-f_x}{f_y}\right)\right)}{(f_y)^2}$$

$$= -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{(f_y)^3}$$

Example 6.15 If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Sol. Let

$$f(x, y) = x^3 + y^3 + 3xy - 1 = 0$$

This is an implicit function of x and y. Then,

$$f_x = 3x^2 + 3y,\ f_y = 3y^2 + 3x$$

Therefore,

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(3x^2 + 3y)}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x}$$

Here

$u = x\log(xy)$

Now using chain rule,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \left[1\cdot\log(xy) + x\cdot\frac{1}{xy}\cdot y\right]\cdot 1 + \left(x\cdot\frac{1}{xy}\cdot x\right)\left[-\frac{x^2+y}{y^2+x}\right]$$

$$= 1 + \log(xy) - \frac{x\left(x^2+y\right)}{y\left(y^2+x\right)}$$

Example 6.16 If $f(x, y) = 0$ and $\varphi(x, z) = 0$, show that

$$\frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial z}\cdot\frac{dz}{dy} = \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x}$$

Sol. We have,

$f(x, y) = 0$

Therefore,

$$df = 0$$

or, $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = 0$

$$\text{or, } \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x}\frac{dx}{dy} \qquad \text{...(1)}$$

Again,

$\varphi(x, y) = 0$

Therefore,

$$d\phi = 0$$

or, $\dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial z}dz = 0$

$$\Rightarrow \frac{\partial \phi}{\partial x} = -\frac{\partial \phi}{\partial z}\frac{dz}{dy} \qquad \text{...(2)}$$

Multiplying (1) and (2), we have

$$\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x} = \frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial z}\cdot\frac{dz}{dy}$$

Example 6.17 If $f(p, v, t) = 0$ prove that

$$\left(\frac{dp}{dt}\right)_v \times \left(\frac{dt}{dv}\right)_p \times \left(\frac{dv}{dp}\right)_t = -1 \qquad \text{[WBUT-2003]}$$

Sol. Here,

$$f(p, v, t) = 0$$

Therefore,

$$df = 0 \Rightarrow \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial t}dt = 0 \qquad ...(1)$$

Let p be constant. Then $dp = 0$ and from (1), we get

$$\left(\frac{dt}{dv}\right)_p = -\frac{\frac{\partial f}{\partial v}}{\frac{\partial f}{\partial t}} \qquad ...(2)$$

Let v be constant; then $dv = 0$ and from (1) we get

$$\left(\frac{dp}{dt}\right)_v = -\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial p}} \qquad ...(3)$$

Let t be constant; then $dt = 0$ and from (1) we get

$$\left(\frac{dv}{dp}\right)_t = -\frac{\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial v}} \qquad ...(4)$$

Multiplying (2), (3) and (4), we get

$$\left(\frac{dp}{dt}\right)_v \times \left(\frac{dt}{dv}\right)_p \times \left(\frac{dv}{dp}\right)_t = -1$$

Example 6.18 If $f(x, y, z, w) = 0$ prove that

$$\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x} = 1$$

[WBUT-2005]

Sol. Here,

$$f(x, y, z, w) = 0 \qquad ...(1)$$

represents an implicit function involving 4 variables, x, y, z and w. Using the property of differentiation of implicit functions, we have

$$\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}, \text{ considering } z \text{ and } w \text{ as constants.} \qquad ...(2)$$

Similarly, we obtain

$$\frac{\partial y}{\partial z} = -\frac{f_z}{f_y}, \text{ considering } x \text{ and } w \text{ as constants.} \quad ...(3)$$

$$\frac{\partial z}{\partial w} = -\frac{f_w}{f_z}, \text{ considering } x \text{ and } y \text{ as constants.} \quad ...(4)$$

$$\frac{\partial w}{\partial x} = -\frac{f_x}{f_w}, \text{ considering } y \text{ and } z \text{ as constants.} \quad ...(5)$$

Multiplying (2), (3), (4) and (5), we get

$$\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x} = \left(-\frac{f_y}{f_x}\right) \times \left(-\frac{f_z}{f_y}\right) \times \left(-\frac{f_w}{f_z}\right) \times \left(-\frac{f_x}{f_w}\right) = 1$$

Example 6.19 If z is a function of x and y defined by

$$x^2 + y^2 + z^2 + x + y + z + 1 = 0,$$

find $d^2 z$ at $(1, 0, 1)$.

Sol. Here

$$x^2 + y^2 + z^2 + x + y + z + 1 = 0 \quad ...(1)$$

We have from the first- and second-order total differentials,

$$dz = \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right) z$$

and

$$d^2 z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 z$$

$$= \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2}(dy)^2 \quad ...(2)$$

Here

$$x^2 + y^2 + z^2 + x + y + z + 1 = 0$$

Differentiating (1) partially with respect to x, we have

$$2x + 2z\frac{\partial z}{\partial x} + 1 + \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x+1}{2z+1}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{(2z+1)\cdot 2 - (2x+1)\cdot 2\frac{\partial z}{\partial x}}{(2z+1)^2} = -\frac{(2z+1)^2 \cdot 2 + (2x+1)^2 \cdot 2}{(2z+1)^3}$$

$$\text{So, } \left[\frac{\partial^2 z}{\partial x^2}\right]_{(1,0,1)} = \frac{-4}{3} \quad ...(3)$$

Differentiating (1) partially with respect to y, we have

$$2y+2z\frac{\partial z}{\partial y}+1+\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial z}{\partial y}=-\frac{2y+1}{2z+1}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{(2z+1)\cdot 2-(2y+1)\cdot 2\frac{\partial z}{\partial y}}{(2z+1)^2}=-\frac{(2z+1)^2\cdot 2+(2y+1)^2\cdot 2}{(2z+1)^3}$$

$$\text{So,} \left[\frac{\partial^2 z}{\partial y^2}\right]_{(1,0,1)}=\frac{-20}{27} \qquad \text{...(4)}$$

Also

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{(2y+1)}{(2z+1)^2}\cdot 2\frac{\partial z}{\partial x}=-2\cdot\frac{(2y+1)}{(2z+1)^2}\cdot\frac{2x+1}{2z+1}$$

$$\text{So,} \left[\frac{\partial^2 z}{\partial x\partial y}\right]_{(1,0,1)}=\frac{-2}{9} \qquad \text{...(5)}$$

Using (3), (4) and (5) in (2), we have

$$\left[d^2 z\right]_{(1,0,1)}=\left[\frac{\partial^2 z}{\partial x^2}\right]_{(1,0,1)}(dx)^2+2\left[\frac{\partial^2 z}{\partial x\partial y}\right]_{(1,0,1)}dxdy+\left[\frac{\partial^2 z}{\partial y^2}\right]_{(1,0,1)}(dy)^2$$

$$=\frac{-4}{3}(dx)^2-\frac{4}{9}dxdy-\frac{20}{27}(dy)^2$$

Example 6.20 If $f(u,v)=3uv^2$, $g(u,v)=u^2-v^2$ find the Jacobian $\dfrac{\partial(f,g)}{\partial(u,v)}$.

[WBUT-2004]

Sol. Here

$$f(u,v)=3uv^2,\ g(u,v)=u^2-v^2$$

Therefore

$$\frac{\partial f}{\partial u}=3v^2,\ \frac{\partial f}{\partial v}=6uv$$

and $\dfrac{\partial g}{\partial u}=2u,\ \dfrac{\partial g}{\partial v}=-2v$

Hence

$$\frac{\partial(f,g)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\end{vmatrix}=\begin{vmatrix}3v^2 & 6uv\\ 2u & -2v\end{vmatrix}=-6v^3-12u^2v$$

Example 6.21 Show that the functions

$$u = x + y - z,\ v = x - y + z \text{ and } w = x^2 + y^2 + z^2 - 2yz$$

are dependent. Find the relation between them.

Sol. The given functions u, v and w of independent variables x, y, z will be functionally dependent if $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$, otherwise independent.

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix} \quad [C_3' \to C_3 + C_2]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix} = 0$$

Therefore, the functions u, v, w are functionally dependent.

The relation between them is

$$(u+v)^2 + (u-v)^2 = 4\left(x^2 + y^2 + z^2 - 2yz\right) = 4w$$

$$\text{or, } u^2 + v^2 - 2w = 0$$

Example 6.22 Find the extrema of the following function:

$$f(x, y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$$ [WBUT-2004, 2007, 2009]

Sol. Here,

$$f(x, y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$$

Then

$$f_x = 3x^2 + 3y^2 - 6x,\ f_y = 6xy - 6y,$$

$$f_{xx} = 6x - 6,\ f_{yy} = 6x - 6,\ f_{xy} = 6y$$

Now,

$$f_x = 0 \ \Rightarrow 3x^2 + 3y^2 - 6x = 0 \ \Rightarrow x^2 + y^2 - 2x = 0 \quad \text{...(1)}$$

and $f_y = 0 \quad \Rightarrow 6xy - 6y = 0 \quad \Rightarrow xy - y = 0$ ...(2)

Solving equations (1) and (2) the critical points are $(2, 0), (1, 1), (0, 0)$.

Now,

$$f_{xx}f_{yy} - (f_{xy})^2 = (6x-6)^2 - (6y)^2 = 36\left\{(x-1)^2 - y^2\right\}$$

At the point $(2, 0)$

$$f_{xx}(2, 0)f_{yy}(2, 0) - \left\{f_{xy}(2, 0)\right\}^2 = 36 > 0$$

and $f_{xx}(2, 0) = 6 > 0$, $f_{yy}(2, 0) = 6 > 0$

Therefore, $f(x, y)$ is minimum at $(2, 0)$.

At the point $(1, 1)$

$$f_{xx}(1, 1)f_{yy}(1, 1) - \left\{f_{xy}(1, 1)\right\}^2 = -36 < 0$$

Therefore, $f(x, y)$ has no extreme value at $(1, 1)$

At the point $(0, 0)$

$$f_{xx}(0, 0)f_{yy}(0, 0) - \left\{f_{xy}(0, 0)\right\}^2 = 36 > 0$$

and $f_{xx}(0, 0) = -6 < 0$, $f_{yy}(0, 0) = -6 < 0$

Therefore, $f(x, y)$ is maximum at $(0, 0)$.

Example 6.23 Find the maximum and minimum of the function

$f(x, y) = x^3 + y^3 - 3axy$ [WBUT-2002, 2008]

Sol. Here,

$$f(x, y) = x^3 + y^3 - 3axy$$

Then

$$f_x = 3x^2 - 3ay, \; f_y = 3y^2 - 3ax$$

$$f_{xx} = 6x, \; f_{yy} = 6y, \; f_{xy} = -3a$$

Now,

$f_x = 0 \quad \Rightarrow 3x^2 - 3ay = 0 \quad \Rightarrow x^2 - ay = 0$ ...(1)

and $f_y = 0 \quad \Rightarrow 3y^2 - 3ax = 0 \quad \Rightarrow y^2 - ax = 0$ ...(2)

Solving equations (1) and (2), the critical points are $(0, 0)$ and (a, a)

Now,

$$f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9a^2$$

At the point (a, a)

$$f_{xx}(a, a) f_{yy}(a, a) - \{f_{xy}(a, a)\}^2 = 27a^2 > 0$$

and

$$\begin{aligned} f_{xx}(a, a) &= 6a > 0 \quad \text{if } a > 0 \\ &= 6a < 0 \quad \text{if } a < 0 \\ f_{yy}(a, a) &= 6a > 0 \quad \text{if } a > 0 \\ &= 6a < 0 \quad \text{if } a < 0 \end{aligned}$$

Hence $f(x, y)$ is maximum at (a, a) if $a < 0$ and minimum at (a, a) if $a > 0$.

At the point $(0, 0)$

$$f_{xx}(0, 0) f_{yy}(0, 0) - \{f_{xy}(0, 0)\}^2 = -9a^2 < 0$$

Hence $f(x, y)$ is neither maximum nor minimum at $(0, 0)$.

Example 6.24 Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$

Find also the saddle points.

Sol. Here,

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$

Then

$$f_x(x, y) = 3x^2 - 63 + 12y, \quad f_y(x, y) = 3y^2 - 63 + 12x,$$

$$f_{xx}(x, y) = 6x, \ f_{yy}(x, y) = 6y \text{ and } f_{xy}(x, y) = 12$$

Now to find the critical points, we solve

$$f_x(x, y) = 3x^2 - 63 + 12y = 0 \quad \Rightarrow x^2 - 21 + 4y = 0 \qquad ...(1)$$

and $f_y(x, y) = 3y^2 - 63 + 12x = 0 \quad \Rightarrow y^2 - 21 + 4x = 0$ ...(2)

Subtracting (1) from (2) we obtain,

$$(x - y)(x + y - 4) = 0$$

this implies $(x - y) = 0$ or, $(x + y - 4) = 0$

So from above we have two pairs of equations

$$\left\{ \begin{matrix} (x - y) = 0 \\ x^2 - 21 + 4y = 0 \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} (x + y - 4) = 0 \\ x^2 - 21 + 4y = 0 \end{matrix} \right\}$$

Solving the above, we obtain the (stationary points) critical points as $(-7, -7), (3, 3), (5, -1)$ and $(-1, 5)$.

Now,

$$f_{xx}(x, y) \cdot f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 36xy - 144$$

At the point $(-7, -7)$

$$f_{xx}(-7,-7)\cdot f_{yy}(-7,-7)-\left[f_{xy}(-7,-7)\right]^2 = 1620 > 0,$$

$$f_{xx}(-7,-7) = -42 < 0 \text{ and } f_{yy}(-7,-7) = -42 < 0$$

Therefore, $f(x, y)$ has maximum at $(-7,-7)$ and the maximum value is $f(1, 2) = 2.$

At the point $(3, 3)$

$$f_{xx}(3, 3)\cdot f_{yy}(3, 3)-\left[f_{xy}(3, 3)\right]^2 = 180 > 0$$

$$f_{xx}(3, 3) = 18 > 0 \text{ and } f_{yy}(3, 3) = 18 > 0$$

Therefore, $f(x, y)$ has minimum at $(3, 3)$ and the minimum value is $f(3, 3) = -216.$

At both the the points $(5,-1)$ and $(-1, 5)$

$$f_{xx}(5,-1)f_{yy}(5,-1)-\left[f_{xy}(5,-1)\right]^2 = -324 < 0$$

$$\text{and } f_{xx}(-1, 5)f_{yy}(-1, 5)-\left[f_{xy}(-1, 5)\right]^2 = -324 < 0$$

Therefore, $f(x, y)$ has neither maximum nor minimum at both the the points $(5,-1)$ and $(-1, 5)$. So, these are the saddle points.

Example 6.25 Find the point in the plane $x+2y+3z=13$ nearest to the point $(1, 1, 1)$ using Lagrange's multiplier method. [WBUT-2001,2002]

Sol. Let $P(x, y, z)$ be any point on the plane $x+2y+3z=13$.

The distance between the point $P(x, y, z)$ and $A(1, 1, 1)$ is

$$D = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$

Let us consider

$$f(x, y, z) = D^2(x, y, z) = (x-1)^2+(y-1)^2+(z-1)^2$$

Here, we have to find the point $P(x, y, z)$ such that $f(x, y, z)$ or $D^2(x, y, z)$ is minimum subject to

$$\phi(x, y, z) = x+2y+3z-13$$

Let

$$\begin{aligned} L(x, y, z) &= f(x, y, z)+\lambda\phi(x, y, z) \\ &= (x-1)^2+(y-1)^2+(z-1)^2+\lambda(x+2y+3z-13) \end{aligned}$$

Now,

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x} = 0$$

or, $2(x-1)+\lambda=0$ ...(1)

$$\frac{\partial L}{\partial y}=\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

or, $2(y-1)+2\lambda=0$ ...(2)

$$\frac{\partial L}{\partial z}=\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

or, $2(z-1)+3\lambda=0$ ...(3)

The critical points are found by solving,

$$\frac{\partial L}{\partial x}=0,\ \frac{\partial L}{\partial y}=0,\ \frac{\partial L}{\partial z}=0 \text{ and } \phi(x, y, z)=0$$

Putting the values of x, y, z from (1), (2) and (3) in $\phi(x, y, z)=0,$ we get

$$\left(\frac{-\lambda}{2}+1\right)+2(-\lambda+1)+3\left(\frac{-3\lambda}{2}+1\right)-13=0$$

or, $\lambda=-1$

Putting the value of λ in (1), (2) and (3) we have

$$x=\frac{3}{2},\ y=2,\ z=\frac{5}{2}$$

Therefore the required point on the plane is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$.

Now,

$$d^2 f=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)^2 f$$

$$=\frac{\partial^2 f}{\partial x^2}(dx)^2+\frac{\partial^2 f}{\partial y^2}(dy)^2+\frac{\partial^2 f}{\partial z^2}(dz)^2$$

$$+2\frac{\partial^2 f}{\partial x\partial y}dxdy+2\frac{\partial^2 f}{\partial y\partial z}dydz+2\frac{\partial^2 f}{\partial z\partial x}dzdx$$

Again,

$$\frac{\partial f}{\partial x}=2(x-1),\ \frac{\partial f}{\partial y}=2(y-1),\ \frac{\partial f}{\partial z}=2(z-1)$$

$$\frac{\partial^2 f}{\partial x^2}=2,\ \frac{\partial^2 f}{\partial y^2}=2,\ \frac{\partial^2 f}{\partial z^2}=2$$

and

$$\frac{\partial^2 f}{\partial x\partial y}=0,\ \frac{\partial^2 f}{\partial y\partial z}=0,\ \frac{\partial^2 f}{\partial z\partial x}=0$$

Therefore,

$$d^2 f=2\left\{(dx)^2+(dy)^2+(dz)^2\right\}>0$$

So,

$$\left(d^2 f\right)_{\left(\frac{3}{2}, 2, \frac{5}{2}\right)} = 2\left\{(dx)^2 + (dy)^2 + (dz)^2\right\} > 0$$

Therefore, $f(x, y, z)$ or $D^2(x, y, z)$ is minimum at $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$

Hence $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is the point in the plane $x+2y+3z=13$ nearest to the point $(1,1,1)$.

Example 6.26 If $xyz = a^3$, find the critical points of $xy + yz + zx$ using Lagrange's multiplier method.

Sol. Here

$f(x, y, z) = xy + yz + zx$

subject to

$\phi(x, y, z) = xyz - a^3$

Let,

$L(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = xy + yz + zx + \lambda\left(xyz - a^3\right)$

Now,

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

or, $y + z + \lambda(yz) = 0$ ...(1)

$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

or, $x + z + \lambda(xz) = 0$ ...(2)

$$\frac{\partial L}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

or, $y + x + \lambda(xy) = 0$ ...(3)

The critical points are found by solving,

$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial z} = 0$ and $\phi(x, y, z) = 0$

From (1), (2) and (3), we have

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y} = -\lambda \qquad \text{...(4)}$$

$$\text{or, } 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = -3\lambda \qquad \text{...(5)}$$

From (4) and (5), we get

$$x = \frac{-2}{\lambda}, \; y = \frac{-2}{\lambda}, \; z = \frac{-2}{\lambda}$$

Putting the values of x, y, z in $\phi(x, y, z) = 0$ we get

$x = y = z = a$

Therefore, the critical point is (a, a, a).

Example 6.27 Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\dfrac{8abc}{3\sqrt{3}}$. [WBUT 2002]

Sol. The volume of the rectangular parallelepiped is

$$V = 2x \cdot 2y \cdot 2z = 8xyz$$

The problem is to find the maximum volume subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here,

$$f(xy, z) = V = 8xyz \text{ and } \phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Let,

$$L(x, y, z) = f(xy, z) + \lambda\phi(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Now,

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\text{or, } 8yz + \frac{2\lambda x}{a^2} = 0 \qquad ...(1)$$

$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\text{or, } 8zx + \frac{2\lambda y}{b^2} = 0 \qquad ...(2)$$

$$\frac{\partial L}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

$$\text{or, } 8xy + \frac{2\lambda z}{c^2} = 0 \qquad ...(3)$$

Multiplying (1), (2), (3) by x, y, z respectively and adding we have

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

or, $24xyz + 2\lambda = 0$

or, $\lambda = -12xyz$ ...(4)

From (1), we have

$$8yz + \frac{2\lambda x}{a^2} = 0$$

or, $8yz + \dfrac{2(-12xyz)x}{a^2} = 0$

or, $x^2 = \dfrac{a^2}{3}$

or, $x = \pm\dfrac{a}{\sqrt{3}}$

Since $x > 0$, we consider

$$x = \frac{a}{\sqrt{3}}$$

Similarly from (2) and (3),

$$y = \frac{b}{\sqrt{3}},\ z = \frac{c}{\sqrt{3}} \text{ and } \lambda = \frac{-4abc}{\sqrt{3}}$$

Therefore, the critical point is $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$

Again,

$$d^2L = 2\lambda\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right] + 16(ydzdx + zdxdy + xdydz) \quad ...(5)$$

Also from,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

we get

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = 0$$

or, $\dfrac{1}{\sqrt{3}}\left[\dfrac{dx}{a} + \dfrac{dy}{b} + \dfrac{dz}{c}\right] = 0$

or, $\dfrac{dx}{a} + \dfrac{dy}{b} + \dfrac{dz}{c} = 0$ ...(6)

From (6)

$$\left(\frac{dx}{a} + \frac{dy}{b}\right)^2 = \left(-\frac{dz}{c}\right)^2$$

or, $2\dfrac{dxdy}{ab} = \left(\dfrac{dz}{c}\right)^2 - \left(\dfrac{dx}{a}\right)^2 - \left(\dfrac{dy}{b}\right)^2$

Similarly from (6), we obtain

$$2\frac{dydz}{bc} = \left(\frac{dx}{a}\right)^2 - \left(\frac{dy}{b}\right)^2 - \left(\frac{dz}{c}\right)^2$$

and $2\dfrac{dxdz}{ac} = \left(\dfrac{dy}{b}\right)^2 - \left(\dfrac{dx}{a}\right)^2 - \left(\dfrac{dz}{c}\right)^2$

Therefore using the above results in (5)

$$d^2L = -\frac{16abc}{\sqrt{3}}\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right] < 0$$

Hence at $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$, $f(x, y, z)$ is maximum and correspondingly volume V is maximum.

The maximum volume is given by

$$V = 8xyz = \frac{8abc}{3\sqrt{3}}.$$

EXERCISES

Short and Long Answer Type Questions

1. Show that the function

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}, (x, y) \neq (0, 0)$$
$$= 0, (x, y) = (0, 0)$$

is continuous at $(0, 0)$.

2. Show that the function

$$f(x, y) = \frac{x^3 + y^3}{x - y}, (x, y) \neq (0, 0)$$
$$= 0, (x, y) = (0, 0)$$

is not continuous at $(0, 0)$.

3. Show that the function

$$f(x, y) = xy\frac{x^2 y}{x^4 - y^2}, (x, y) \neq (0, 0)$$
$$= 0, (x, y) = (0, 0)$$

is not continuous at $(0, 0)$.

4. Show that the function

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}, (x, y) \neq (0, 0)$$

$$= 0, (x, y) = (0, 0)$$

is differentiable at $(0, 0)$.

5. a) If $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

b) If $u = f(ax + by)$, prove that

$$b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 0$$

c) If $u = \cot^{-1}\left(\dfrac{y}{x}\right)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

d) If $u = \sqrt{xy}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-1}{4}\left(\frac{y^{\frac{1}{2}}}{x^{\frac{3}{2}}} + \frac{x^{\frac{1}{2}}}{y^{\frac{3}{2}}}\right)$$

e) If $u = \tan(ax + y) - (y - ax)^{\frac{3}{2}}$, prove that

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$$

6. a) If $u = \tan^{-1}\left\{\dfrac{x^{\frac{5}{2}} + y^{\frac{5}{2}}}{\sqrt{x} - \sqrt{y}}\right\}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

b) If $u = \cot^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{4}\sin 2u.$$

c) If $u = \sin^{-1}\left\{\dfrac{x^2 y^2}{x+y}\right\}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$$

d) If $u = \log\left\{\dfrac{x^3+y^3}{x^2 y}\right\}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

e) If $u = x^4 y^4 \sin^{-1}\left(\dfrac{x}{y}\right) + \log x - \log y$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 8x^4 y^4 \sin^{-1}\left(\frac{x}{y}\right)$$

7. a) If $u = \tan^{-1}\dfrac{x^3+y^3}{x+y}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(1-4\sin^2 u\right)\sin 2u$$

b) If $u = \sqrt{x^2+y^2}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

c) If $u = \phi\left(\dfrac{y}{x}\right) + \sqrt{x^2+y^2}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

d) If $u = \sin^{-1}\sqrt{x^2+y^2}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$$

e) If $u = \log\left(\dfrac{x^3+y^3}{x^2+y^2}\right)$, show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-1$$

8. If $f\left(lx+my+nz,\, x^2+y^2+z^2\right)=0,$ prove that

$$(lz-nx)+(mz-ny)\frac{\partial y}{\partial x}+(ly-mx)\frac{\partial y}{\partial z}=0$$

9. If $f(x+y+z,\, xyz)=0,$ prove that

$$x(y-z)\frac{\partial z}{\partial x}+y(z-x)\frac{\partial z}{\partial y}=z(x-y).$$

10. If $u=f\left(x^2-y^2,\, y^2-z^2,\, z^2-x^2\right),$ prove that

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$$

11. a) If $u=x^2+y^2$ and $v=xy,$ prove that $\dfrac{\partial(u,\, v)}{\partial(x,\, y)}=2\left(x^2-y^2\right).$

b) If $u=a\cosh x\cos y,\, v=a\sinh x\sin y,$ prove that

$$\frac{\partial(u,\, v)}{\partial(x,\, y)}=\frac{1}{2}a^2(\cosh 2x-\cos 2y)$$

c) If $u=\dfrac{yz}{x},\, v=\dfrac{xz}{y},\, w=\dfrac{xy}{z},$ show that $\dfrac{\partial(u,\, v,\, w)}{\partial(x,\, y,\, z)}=4.$

d) If $u=x+y-z,\, v=x-y+z,\, w=x^2+y^2+z^2-2yz,$ show that

$$\frac{\partial(u,\, v,\, w)}{\partial(x,\, y,\, z)}=0.$$

e) If $u=\dfrac{x^2+y^2+z^2}{x},\, v=\dfrac{x^2+y^2+z^2}{y},\, w=\dfrac{x^2+y^2+z^2}{z},$ show that

$$\frac{\partial(u,\, v,\, w)}{\partial(x,\, y,\, z)}=\frac{x^2y^2z^2}{\left(x^2+y^2+z^2\right)^2}.$$

12. Find the maxima and minima of the following functions:

a) $x^2+y^4-2x^2+4xy-2y^2$ [**Ans:** Minimun at $\left(\sqrt{2},-\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right)$]

b) $4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$

$$\left[\textbf{Ans : } \text{Minimum at } (0, 0) \text{ and maximun at } \left(\frac{3}{2}, \frac{-3}{2}\right), \left(-\frac{3}{2}, \frac{3}{2}\right)\right]$$

c) $y^3 + y^2 + 3x^2 + 4xy$ $\left[\textbf{Ans : } \text{Minimum at } \left(\frac{2}{3}, \frac{-4}{3}\right)\right]$

d) $xy + \dfrac{8}{x} + \dfrac{8}{y}$ [**Ans :** Minimum at $(2, 2)$]

e) $xy + a^3\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$ [**Ans :** Minimum at (a, a)]

13. Show that the function $f(x, y) = x^2 + 2xy + y^2 + x^3 + y^3 + x^7$ has neither a maximum or minimum at the origin.

14. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$.

[**Ans :** Minimum value is $3a^2$ at (a, a, a)]

15. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = 8$.

[**Ans :** Minimum value is 12 at $(2, 2, 2), (-2, -2, -2), (-2, 2, -2), (2, -2, -2)$]

16. Find the extreme value of $4x + 9y$ subject to the condition $xy = 4$.

[**Ans :** Maximum value is -24 and minimum value is 24]

17. Find the extreme value of $7x^2 + 8xy + y^2$ subject to the condition $x^2 + y^2 = 1$.

[**Ans :** Maximum value is 9 and minimum value is -1]

18. Find the extreme value of $x^2 + y^2$ subject to the condition $3x^2 + 4xy + 6y^2 = 140$.

[**Ans :** Maximum value is 70 and minimum value is 20]

19. Find the minimum distance of the point $(1, 2, 3)$ from the plane $x + y - 4z = 9$.

[**Ans :** Minimum distance is 9]

20. Find the extreme value of $u = a^3x^2 + b^3y^2 + c^3z^2$ where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

[**Ans :** Extreme value is $(a + b + c)^3$]

Multiple-Choice Questions

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} =$

a) 0 b) 1 c) $\frac{1}{2}$ d) none of these

2. $f(x, y) = \dfrac{x^3+x^2y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree

a) $\frac{3}{2}$ b) $\frac{5}{2}$ c) $\frac{1}{2}$ d) 2

3. $f(x, y) = \dfrac{x^3y+x^2y^2}{\sqrt{x^2+x+y}}$ is a homogeneous function of degree

a) 0 b) 2 c) $\frac{1}{2}$ d) none of these

4. If $u(x, y) = \dfrac{x^3+y^3}{\sqrt{x+y}}$ then $xu_x + yu_y =$

a) $\frac{1}{2}$ b) $\frac{5}{2}$ c) $\frac{5}{2}u$ d) $\frac{1}{2}u$

5. If $u(x, y) = x^2 + y^2 + \dfrac{x^{\frac{5}{2}}+y^{\frac{5}{2}}}{\sqrt{x+y}}$ then $xu_x + yu_y =$

a) 2 b) $\frac{3}{2}$ c) $2u$ d) $\frac{3}{2}u$

6. If $u(x, y) = \log\left(x^2+y^2\right)$ then the value u_x at $(1, 1)$ is

a) $\frac{1}{2}$ b) 1 c) 0 d) none of these

7. If $u(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$, then $xu_x + yu_y =$

a) 0 b) -1 c) 2 d) u

8. If $u = \log\left(x^2+y^2\right)$ then $u_{xx} + u_{yy} =$

a) 0 b) $\frac{x}{y}$ c) $\frac{y}{x}$ d) 1

9. $\dfrac{\partial(u, v)}{\partial(x, y)}\dfrac{\partial(x, y)}{\partial(u, v)} =$

a) 1 b) 0 c) -1 d) none of these

10. If $x = r\cos\theta$ and $y = r\sin\theta$ then $\dfrac{\partial(r, \theta)}{\partial(x, y)} =$

a) r b) 1 c) $\dfrac{1}{r}$ d) 0

11. $\sin^{-1}\dfrac{x^3 + y^3}{x + y}$ is a homogeneous function of degree

a) 2 b) 1 c) $\dfrac{1}{2}$ d) none of these

12. If $f(x, y)$ is a homogeneous function of degree 3 then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = kf(x, y)$ where $k =$

a) 3 b) 2 c) 0 d) none of these

13. If $f(x, y)$ is a homogeneous function of degree $\dfrac{1}{2}$ then

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} = kf(x, y)$$

where $k =$

a) $\dfrac{1}{2}$ b) $\dfrac{1}{3}$ c) $\dfrac{-1}{4}$ d) 4

14. If $\phi(x, y) = 0$ then $\dfrac{dy}{dx} =$

a) $\dfrac{\phi_x}{\phi_y}$ b) $\dfrac{\phi_y}{\phi_x}$ c) ϕ_y d) none of these

15. If $f(x, y) = x^2 + y^2$ then $f_{xy}(x, y) =$

a) 1 b) 0 c) 2 d) $x + y$

16. If $f(x, y) = x^2 y$ then $df =$

a) $2x^2dx + dy$ b) $x - 2dy$ c) $x + dy$ d) $2xydx + x^2dy$

17. If a function $f(x, y)$ has maximum or minimum value at the point $(3, 4)$ then $f_x(3, 4)$

a) ≥ 0 b) < 0 c) $= 0$ d) none of these

18. If $f_x(a, b) = f_y(a, b) = 0$ then (a, b) is

a) saddle point
b) point of extreme
c) critical point
d) isolated point

19. The critical point of the function $f(x, y) = xy$ is

a) $(1, 1)$
b) $(1, -1)$
c) $(-1, 1)$
d) $(0, 0)$

20. $f(x, y)$ is such that $f_x(a, b) = f_y(a, b) = 0$. Then (a, b) is a saddle point if

a) $f_{xx}(a, b) f_{yy}(a, b) - \{f_{xy}(a, b)\}^2 = 0$

b) $f_{xx}(a, b) f_{yy}(a, b) - \{f_{xy}(a, b)\}^2 < 0$

c) $f_{xx}(a, b) < 0$

d) none of these

Answers:

1. (a)	2. (b)	3. (d)	4. (c)	5. (c)	6. (b)	7. (a)	8. (a)	9. (a)
10. (c)	11. (d)	12. (a)	13. (a)	14. (d)	15. (b)	16. (d)	17. (b)	18. (c)
19. (d)	20. (b)							

CHAPTER

7

Line Integral, Double Integral and Triple Integral

7.1 DEFINITION OF LINE INTEGRALS

Let $f(x, y)$ be defined over the region R, which contains the curve C. The line integral of $f(x, y)$ over C is defined by

$\int_C f(x, y)\, dx = \lim_{n\to\infty} \sum_{m=1}^{n} f(x_m, y_m)\, \Delta x_m$ where (x_m, y_m) are coordinates of arbitrary n points in the curve C and $\Delta x_m = x_m - x_{m-1}$.

The line integral exists if the limit exists and is finite.

7.1.1 Properties of Line Integrals

1. **Let $z = F(x, y)$ be a continuous function at every point on a plane curve in the xy plane whose parametric equation is $x = \phi(t)$, $y = \psi(t)$ for some real value of t. Then**

   a) $$\boxed{\int_C F(x, y)\, dx = \int_{t_0}^{t_1} F\{\phi(t), \psi(t)\}\, d\phi(t)}$$

   b) $$\boxed{\int_C F(x, y)\, dy = \int_{t_0}^{t_1} F\{\phi(t), \psi(t)\}\, d\psi(t)}$$

2. **Let the equation of the curve be $y = f(x)$, $x_0 \le x \le x_n$; then**

$$\boxed{\int_C F(x, y)\, dx = \int_{x_0}^{x_n} F\{x, f(x)\}\, dx}$$

3. Let the equation of the curve be $x = \phi(y),\ y_0 \le y \le y_n$; then

$$\boxed{\int_C F(x, y)\, dy = \int_{y_0}^{y_n} F\{\phi(y), y\}\, dy}$$

4. If $z = f(x, y)$ and $z = \phi(x, y)$ are integrable along C then $f(x, y) \pm \phi(x, y)$ are also integrable and

$$\boxed{\int_C \{f(x, y) \pm \phi(x, y)\}\, dx = \int_C \{f(x, y)\, dx \pm \int_C \phi(x, y) dx}$$

5. If $arc\,AB = arc\,AC + arc\,CB$ then

$$\boxed{\int_{AB} F(x, y)\, dx = \int_{AC} F(x, y)\, dx + \int_{CB} F(x, y)\, dx}$$

and

$$\boxed{\int_{AB} F(x, y)\, dy = \int_{AC} F(x, y)\, dy + \int_{CB} F(x, y)\, dy}$$

7.1.2 Evaluation of the Line Integral $\int_C \{f(x, y)\, dx + g(x, y)\, dy\}$

1. Let the plane curve C be given by $y = \phi(x),\ a \le x \le b$

Then, $$\boxed{\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_a^b \{f(x, \phi(x))\, dx + g(x, \phi(x))\phi'(x)\, dx\}}$$

2. Let the plane curve C be given by $x = \psi(y),\quad c \le y \le d$

Then, $$\boxed{\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_c^d \{f(\psi(y), y)\psi'(y)\, dy + g(\psi(y), y)\, dy\}}$$

3. Let the parametric equation of the plane curve C be given by $x = x(t),\ y = y(t),\ t_0 \le t \le t_1$

Then,

$$\boxed{\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_{t_0}^{t_1} [\{f(x(t), y(t))\}x'(t)\, dt + \{g(x(t), y(t))\}y'(t)\, dt]}$$

4. If $\{f(x, y)\, dx + g(x, y)\, dy\}$ can be expressed as $\{f(x, y)\, dx + g(x, y)\, dy\} = dU(x, y)$ then

$$\boxed{\int_C \{f(x, y)\,dx + g(x, y)\,dy\} = 0}$$

where C is a closed curve.

Example 1 Evaluate $\int_C \{2xydx + (x^2 - y^2)\,dy\}$ where C is the line segment AB from $A(0, 0)$ to $B(2, 1)$.

Sol. The equation of the line segment AB from $A(0, 0)$ to $B(2, 1)$ is

$$\frac{x-0}{2} = \frac{y-0}{1} \Rightarrow y = \frac{1}{2}x$$

Therefore,

$$\int_C \{2xydx + (x^2 - y^2)\,dy\}$$

$$= \int_0^2 \left\{2x\left(\frac{1}{2}x\right)dx + \left(x^2 - \left(\frac{1}{2}x\right)^2\right)\frac{1}{2}\,dx\right\}$$

$$= \int_0^2 \left(x^2\,dx + \frac{3x^2}{8}\,dx\right)$$

$$= \frac{11}{8}\int_0^2 x^2\,dx = \frac{11}{8}\left[\frac{x^3}{3}\right]_0^2 = \frac{11}{3}$$

Example 2 Evaluate $\int_C xy^2\,dx$ where C is the circle $x^2 + y^2 = 1$.

Sol. The parametric equation of the circle $x^2 + y^2 = 1$ is

$x = \cos\theta,\ y = \sin\theta$ where $0 \le \theta \le 2\pi$

Therefore,

$$\int_C xy^2\,dx = \int_0^{2\pi} \cos\theta\,(\sin^2\theta)\,d(\cos\theta)$$

$$= -\int_0^{2\pi} \sin^3\theta\cos\theta\,d\theta$$

$$= -\int_0^{2\pi} \sin^3\theta\,d(\sin\theta)$$

$$= -\left[\frac{\sin^4\theta}{4}\right]_0^{2\pi}$$

$$= 0$$

7.2 DOUBLE INTEGRALS

The double integral may be considered as the definite integrals of functions of two variables.

Let $f(x, y)$ be defined over the region R.

Then the double integral of $f(x, y)$ over R is defined by

$$\iint_R f(x, y)\, dxdy = \lim_{n\to\infty} \sum_{m=1}^{n} f(x_m, y_m)\Delta x_m \Delta y_m$$

where (x_m, y_m) are coordinates of arbitrary n points in the region R.

The double integral exists if the limit exists and is finite.

Observations

1. **A double integral is improper if either the domain of integration is an infinite region or the integrand has an infinite discontinuity at a point of the region.**
2. **The continuity of $f(x, y)$ over R ensures the existence of the double integral but, the existence of the integral does not always follow the continuity of $f(x, y)$.**

7.2.1 Evaluation of Double Integrals

The most convinient method of evaluation of double integrals is the method of evaluation by iterated integrals. In the first stage, integration is done with respect to exactly one variable, keeping the other variable fixed. And in the second stage, the resulting function is integrated with respect to the remaining variable.

The selection of the proper order of integration is based on the configuration on the domain of integration R.

Case 1: Evaluation of Double Integrals when *R* is Rectangular

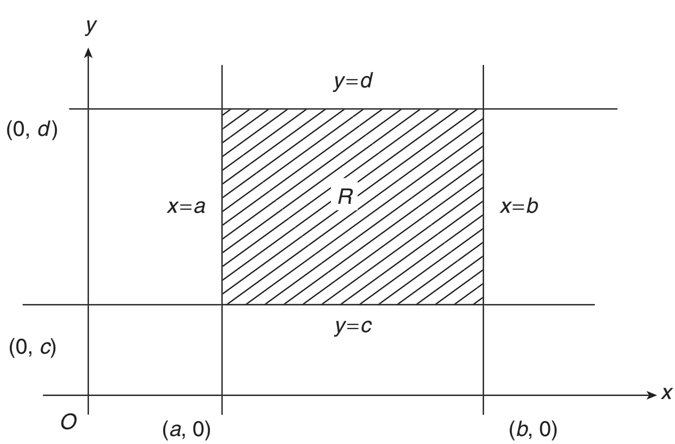

Figure 7.1

Let $R = \{(x, y) : a \le x \le b, c \le y \le d\}$

Then

$$\iint_R f(x, y)\, dxdy = \int_{y=c}^{y=d} \left\{ \int_{x=a}^{x=b} f(x, y)\, dx \right\} dy$$

or

$$\iint_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \left\{ \int_{y=c}^{y=d} f(x, y)\, dy \right\} dx$$

In particular, if $f(x, y) = \phi(x) \times \psi(y)$ then

$$\iint_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \phi(x)\, dx \times \int_{y=c}^{y=d} \psi(y)\, dy$$

Example 3 Evaluate $\displaystyle\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol. Here, $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

$$\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$$

$$= \int_0^1 dy \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x\right]_0^1 dy$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-y^2}}\, dy$$

$$= \frac{\pi}{2} \left[\sin^{-1} y\right]_0^1$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

Case 2: Evaluation of Double Integrals when *R* has Two Linear Boundaries Parallel to the *y*-axis

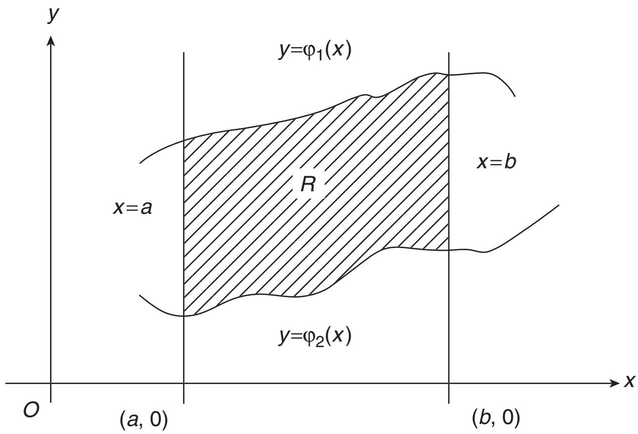

Figure 7.2

Let $R = \{(x, y) : a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$ where $\phi_1(x)$ and $\phi_2(x)$ are continuous functions over $[a, b]$

Then $\iint_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \left\{ \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\, dy \right\} dx$

Example 4 Evaluate $\iint \sqrt{4x^2 - y^2}\, dxdy$ over the triangle formed by the straight lines $y = 0,\ x = 1$ and $y = x.$ [WBUT-2002, 2005]

Sol.

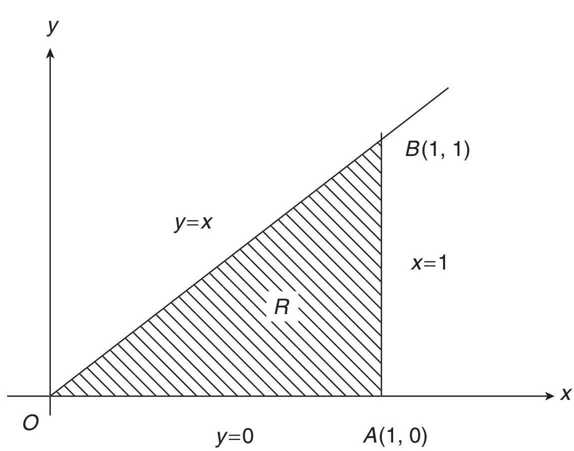

Figure 7.3

The region R is the triangle OAB formed by the lines $y=0,\ x=1$ and $y=x$ and R has two linear boundaries parallel to y-axis.

Here, $R=\{(x, y): 0 \le x \le 1, 0 \le y \le x\}$

Therefore,

$$\iint_R \sqrt{4x^2-y^2}\,dxdy$$

$$=\int_0^1 \left\{\int_0^x \sqrt{4x^2-y^2}\,dy\right\} dx$$

$$=\int_0^1 \left[\frac{y}{2}\sqrt{4x^2-y^2}+\frac{4x^2}{2}\sin^{-1}\frac{y}{2x}\right]_0^x dx$$

$$=\int_0^1 \left[\left(\frac{x}{2}\sqrt{3x^2}+2x^2\sin^{-1}\frac{1}{2}\right)-(2x^2\sin^{-1}0)\right] dx$$

$$=\int_0^1 \left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)x^2\,dx$$

$$=\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\left[\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{18}\left(3\sqrt{3}+2\pi\right)$$

Case 3: Evaluation of Double Integrals when R has Two Linear Boundaries Parallel to x-axis:

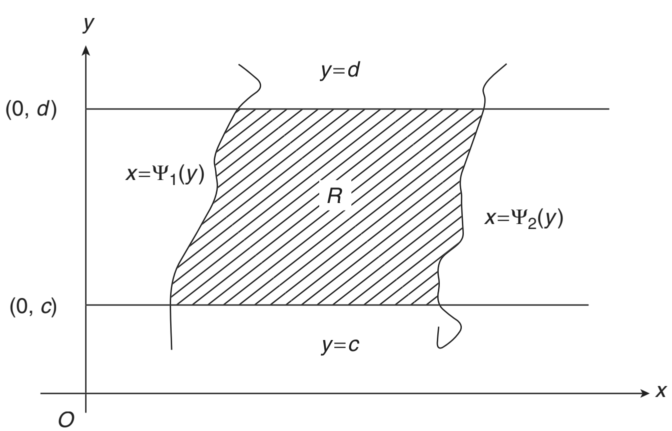

Figure 7.4

Let $R = \{(x, y) : \psi_1(y) \le x \le \psi_2(y), c \le y \le d\}$ where $\psi_1(y)$ and $\psi_2(y)$ are continuous functions over $[c, d]$

Then

$$\iint_R f(x, y)\,dxdy = \int_{y=c}^{y=d} \left\{ \int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x, y)\,dx \right\} dy$$

Example 5 Evaluate $\int_0^1 \int_y^{y^2+1} x^2\,ydxdy$

Sol. Here, $R = \{(x, y) : y \le x \le y^2 + 1, 0 \le y \le 1\}$

Therefore,

$$\int_0^1 \int_y^{y^2+1} x^2\,ydxdy$$

$$= \int_0^1 \left\{ \int_y^{y^2+1} x^2\,ydx \right\} dy$$

$$= \int_0^1 \left[\frac{yx^3}{3} \right]_y^{y^2+1} dy$$

$$= \frac{1}{3}\int_0^1 [y(y^2+1)^3 - y^4]\,dy$$

$$= \frac{1}{3}\int_0^1 (y^7 + 3y^5 + 3y^3 + y - y^4)\,dy$$

$$= \frac{1}{3}\left[\frac{y^8}{8} + \frac{y^6}{2} + \frac{3y^4}{4} + \frac{y^2}{2} - \frac{y^5}{5} \right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{8} + \frac{1}{2} + \frac{3}{4} + \frac{1}{2} - \frac{1}{5} \right] = \frac{67}{120}$$

Case 4: Evaluation of Double Integrals when *R* is Enclosed by a Curve

Let R be a region enclosed by a closed curve $\phi(x, y) = 0$.

Let

$$\phi(x, y) = 0$$

be reduced to either $y = f_1(x)$ or $x = f_2(y)$.

Suppose y is determined in terms of x and limits of y so obtained are $y = y_1, y = y_2.$

Putting $y = 0$ in the equation of the curve $\phi(x, y) = 0$ gives the limits of x, say $x = x_1, x = x_2$

Then, evaluate the integral according to the order of integration by the way the limits are determined

$$\iint_R f(x, y)\, dxdy = \int_{x=x_1}^{x=x_2} \left\{ \int_{y=y_1}^{y=y_2} f(x, y)\, dy \right\} dx$$

Example 6 Evaluate $\iint_R xydxdy$, where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$.

Sol.

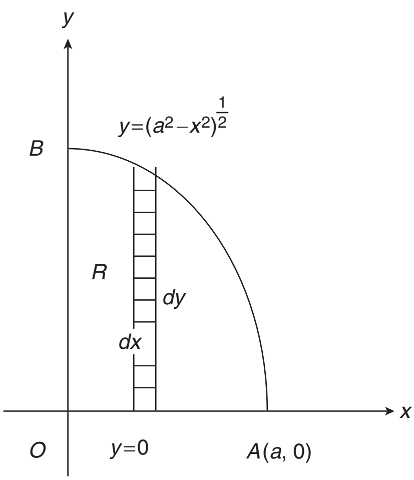

Figure 7.5

Here, the region R is enclosed by the first quadrant of the circle $x^2 + y^2 = a^2$.

Now,

$$x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$$

Therefore,

$$0 \leq y \leq \sqrt{a^2 - x^2}$$

Now putting $y = 0$ in $x^2 + y^2 = a^2$, we have

$0 \le x \le a$

Therefore,

$$\iint_R xy\,dx\,dy = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} xy\,dy$$

$$= \int_0^a x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

7.2.2 Transformation of Double Integrals

Let us consider the integral

$$\iint_R f(x, y)\,dx\,dy$$

where $f(x, y)$ is defined over the region R.

Let us take the transformation

$x = \phi(u, v)$ and $y = \psi(u, v)$

Then the Jacobian of the transformation is defined by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The transformation is invertible if $J \neq 0$.

If $x = \phi(u, v)$ and $y = \psi(u, v)$

is an invertible transformation then

$$\boxed{\iint_R f(x, y)\,dx\,dy = \iint_{R_1} f(\phi(u, v), \psi(u, v))\, J \cdot du\,dv}$$

where R_1 is the region in the new coordinate system.

Example 7 Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$

by transforming to polar coordinates.

Sol.

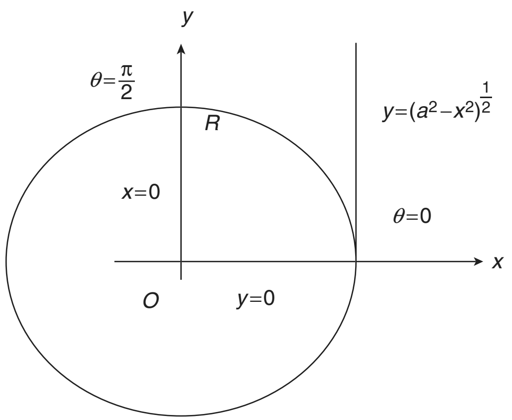

Figure 7.6

Let $x = r\cos\theta$, $y = r\sin\theta$ be the transform from Cartesian to polar coordinates.

The Jacobian of the transformation is

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0$$

The domain of integration $R = \left\{(x, y) : 0 \le x \le a\,; 0 \le y \le \sqrt{a^2 - x^2}\right\}$, i.e., the first quadrant of the circle $x^2 + y^2 = a^2$.

Under the transformation the domain of integration is $R_1 = \left\{(r, \theta) : 0 \le r \le a; 0 \le \theta \le \frac{\pi}{2}\right\}$

Therefore,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin^2\theta \cdot r dr d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \times \int_0^a r^4 dr$$

$$= \frac{\pi}{4} \times \frac{a^5}{5} = \frac{\pi a^5}{20}$$

7.3 TRIPLE INTEGRALS

The concept of triple integrals may be considered as the definite integral of functions of three variables.

Let $f(x, y, z)$ be defined over the closed three dimensional region R of volume V. Then the triple integral of $f(x, y, z)$ over R is defined by

$$\iiint_R f(x, y, z)\, dxdydz = \lim_{n\to\infty} \sum_{m=1}^{n} f(x_m, y_m, z_m)\Delta x_m \Delta y_m \Delta z_m$$

where (x_m, y_m, z_m) are coordinates of arbitrary n points in the volume V.

The triple integral exists if the limit exists and is finite.

Observations

The triple integral is said to be improper if either *R* is of infinite volume or *f*(*x,y,z*) has a singularity over *R*.

7.3.1 Evaluation of Triple Integrals

The most convenient method of evaluation of triple integrals is the method of evaluation by iterated integrals in three stages. In the first stage, the integral turns to be double integral with respect to exactly one variable, and in the second stage, the resulting integral is integrated with respect to the remaining variables as a double integral.

The selection of the proper order of integration is based on the configuration on the domain of integration R.

Case 1: Evaluation of Triple Integral when *R* is a Rectangular Parallelpiped

Let $R = \{(x, y, z) : x_1 \le x \le x_2\ ; y_1 \le y \le y_2 ; z_1 \le z \le z_2\}$

Then $$\iiint_R f(x, y, z)\, dxdydz = \int_{x=x_1}^{x=x_2} \left[\int_{y=y_1}^{y=y_2} \left\{ \int_{z=z_1}^{z=z_2} f(x, y, z) dz \right\} dy \right] dx$$

In particular, if $f(x, y, z) = f_1(x) \times f_2(y) \times f_3(z)$ then

$$\iiint_R f(x, y, z)\, dxdydz = \int_{x=x_1}^{x=x_2} f_1(x)\, dx \times \int_{y=y_1}^{y=y_2} f_2(y)\, dy \times \int_{z=z_1}^{z=z_2} f_3(z)\, dz$$

Example 8 Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dxdydz$

Sol.

Here,

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z} dxdydz = \int_0^1 e^x dx \times \int_0^1 e^y dy \times \int_0^1 e^z dz$$

$$= [e^x]_0^1 \times \left[e^y\right]_0^1 \times \left[e^z\right]_0^1$$

$$= (e-1)\times(e-1)\times(e-1)$$

$$= (e-1)^3$$

Case 2: Evaluation of Triple Integral when Exactly Two Variables have Constant Limits

Let $R = \{(x, y, z) : x_1 \le x \le x_2 ; y_1 \le y \le y_2 \; ; z_1 \le z \le z_2\}$ and let x_1, x_2, y_1 and y_2 are constants. Then

$$\iiint_R f(x, y, z)\; dxdydz = \int_{x=x_1}^{x=x_2}\left[\int_{y=y_1}^{y=y_2}\left\{\int_{z=z_1}^{z=z_2} f(x, y, z)\; dz\right\}dy\right]dx$$

or

$$\iiint_R f(x, y, z)\; dxdydz = \int_{y=y_1}^{y=y_2}\left[\int_{x=x_1}^{x=x_2}\left\{\int_{z=z_1}^{z=z_2} f(x, y, z)dz\right\}dx\right]dy$$

Example 9 Evaluate $\int_0^a\int_0^x\int_0^y x^3y^2\; zdzdydx$

Sol.

$$\int_0^a\int_0^x\int_0^y x^3y^2 zdzdydx = \int_0^a\left[\int_0^x\left\{\int_0^y x^3y^2 zdz\right\}dy\right]dx$$

$$= \int_0^a\left[\int_0^x x^3y^2\left[\frac{z^2}{2}\right]_0^y dy\right]dx$$

$$= \frac{1}{2}\int_0^a\left[\int_0^x x^3y^4 dy\right]dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{x^3y^5}{5}\right]_0^x dx = \frac{1}{10}\int_0^a x^8\; dx = \frac{a^9}{90}$$

Case 3: Evaluation of Triple Integral when Exactly One Variable has a Constant Limit

Let $R = \{(x, y, z) : a \le x \le b; \phi_1(x) \le y \le \phi_2(x); \psi_1(x, y) \le z \le \psi_2(x, y)\}$

Then

$$\iiint_R f(x, y, z) dxdydz = \int_{x=a}^{b} \left[\int_{y=\phi_1(x)}^{\phi_2(x)} \left\{ \int_{z=\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz \right\} dy \right] dx$$

Case 4: Evaluation of Triple Integrals when *R* is Enclosed by a Curve

Let the domain of integration R is enclosed by the surface $\phi(x, y, z) = 0$.

Let

$\phi(x, y, z) = 0$

is expressed as

$z = \Psi(x, y)$

Thus the limits of z are determined as $z = z_1$ and $z = z_2$

Putting $z = 0$ in the equation of the curve $\phi(x, y, z) = 0$, express

$y = \rho(x)$

Thus the limits of y are determined as $y = y_1$ and $y = y_2$

Putting $z = y = 0$ in the equation of the curve $\phi(x, y, z) = 0$, the limits of x are determined as $x = x_1$ and $x = x_2$

Therefore, we evaluate the triple integral following the order of integration as

$$\iiint_R f(x, y, z)\, dxdydz = \int_{x=x_1}^{x=x_2} \left[\int_{y=y_1}^{y=y_2} \left\{ \int_{z=z_1}^{z=z_2} f(x, y, z)\, dz \right\} dy \right] dx$$

Example 10 Evaluate $\iiint (x^2 + y^2 + z^2)\, dxdydz$ where R is the region bounded by $x = 0; y = 0; z = 0$ and $x + y + z = a (a > 0)$

Sol.

Here

$x + y + z = a (a > 0)$

or, $z = a - x - y$

Therefore, the lower and upper limits of z are $z = 0$ and $z = a - x - y$

Putting $z = 0$ in $x + y + z = a (a > 0)$, we have

$x + y = a$

or, $y = a - x$

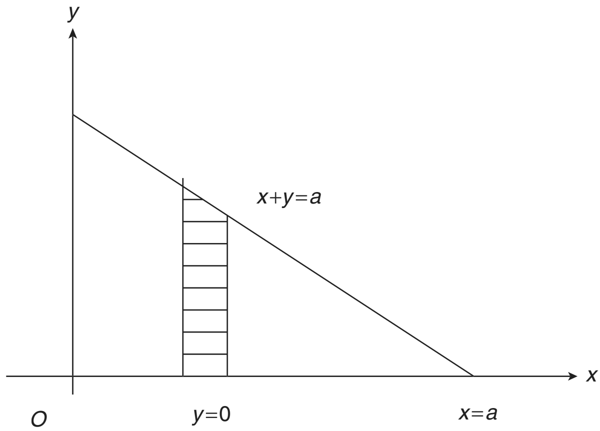

Figure 7.7

Therefore, the lower and upper limits of y are $y = 0$ and $y = a - x$

Putting $z = 0;\ y = 0$ in $x + y + z = a(a > 0)$, we have

$x = a$

Therefore, the lower and upper limits of x are $x = 0$ and $x = a$.

Therefore,

$$\iiint (x^2 + y^2 + z^2)\, dxdydz = \int_{x=0}^{a} dx \int_{y=0}^{a-x} dy \int_{z=0}^{a-x-y} (x^2 + y^2 + z^2)\, dz$$

$$= \int_{x=0}^{a} dx \int_{y=0}^{a-x} dy \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^{a-x-y}$$

$$= \int_{x=0}^{a} dx \int_{y=0}^{a-x} \left[x^2(a-x) - x^2 y + (a-x)y^2 - y^3 + \frac{(a-x-y)^3}{3} \right] dy$$

$$= \int_{x=0}^{a} \left[x^2(a-x)y - \frac{x^2 y^2}{2} + \frac{(a-x)y^3}{3} - \frac{y^4}{4} - \frac{(a-x-y)^4}{12} \right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} \left[\frac{x^2(a-x)^2}{2} + \frac{(a-x)^4}{6} \right] dx$$

$$= \left[\frac{1}{2} a^2 \frac{x^3}{3} - \frac{ax^4}{4} + \frac{x^6}{10} - \frac{(a-x)^6}{30} \right]_0^a$$

$$= \frac{a^5}{20}$$

7.3.2 Transformation of Triple Integrals

Let us consider the integral

$$\iiint_R f(x, y, z)\, dxdydz$$

where $f(x, y, z)$ is defined over the region R.

Let us take the transformation

$$x = \phi(u, v, w),\ y = \psi(u, v, w) \text{ and } z = \rho(u, v, w)$$

Then the Jacobian of the transformation is defined by

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The transformation is invertible if $J \neq 0$.

If $x = \phi(u; v; w),\ y = \psi(u; v; w)\ and\ z = \rho(u; v; w)$

is an invertible transformation then

$$\iiint_R f(x, y, z)\, dxdydz = \iiint_{R_1} f(\phi(u,v,w), \Psi(u,v,w), \rho(u,v,w))\, Jdudvdw$$

where R_1 is the region in new coordinate system

Example 11 Evaluate $\iiint (x^2 + y^2 + z^2)\, dxdydz$ taken over the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$.

Sol. Let us transform the given integral into spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi;\ y = r\sin\theta\sin\phi;\ z = r\cos\theta$$

Then

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = r^2 \sin\theta \neq 0$$

Under this transformation, the domain of integration is $R_1 = \{(r, \theta, \phi) : 0 \le r \le 1; 0 \le \theta \le \pi; 0 \le \phi \le 2\pi\}$

Therefore,

$$\iiint (x^2 + y^2 + z^2)\, dxdydz$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2 (r^2 \sin\theta)\, d\theta d\phi dr$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{1} r^4 dr$$

$$= [\phi]_0^{2\pi} \times [-\cos\theta]_0^{\pi} \times \left[\frac{r^5}{5}\right]_0^1$$

$$= \frac{2\pi}{5}$$

WORKED-OUT EXAMPLES

Example 7.1 Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(x+y)\, dxdy$ [WBUT-2001, 2009]

Sol.

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(x+y)\, dxdy$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{\pi} \sin(x+y)\, dx\right] dy$$

$$= \int_0^{\frac{\pi}{2}} -[\cos(x+y)]_0^{\pi}\, dy$$

$$= -\int_0^{\frac{\pi}{2}} [\cos(\pi+y) - \cos y]\, dy$$

$$= 2\int_0^{\frac{\pi}{2}} \cos y dy$$

$$= 2[\sin y]_0^{\frac{\pi}{2}} = 2$$

Example 7.2 Evaluate $\iint xy(x+y)\, dxdy$ over the area bounded by $y = x^2$ and $y = x$ [WBUT-2001]

Sol.

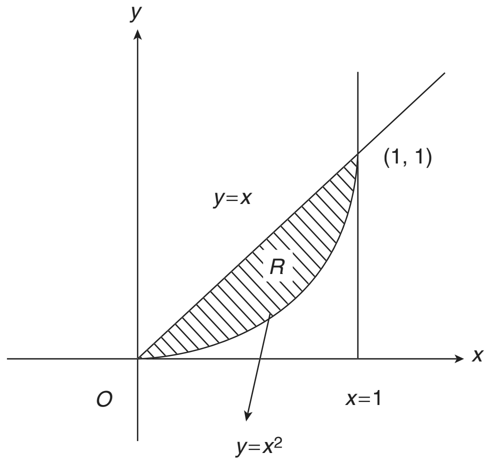

Figure 7.8

The region R is shown in Fig. 7.8 by the shaded portion.

Therefore,

$$\iint_R xy\,(x+y)\,dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} xy\,(x+y)\,dxdy - \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,(x+y)\,dxdy$$

$$= \int_{x=0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_0^x dx - \int_{x=0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_0^{x^2} dx$$

$$= \frac{5}{6}\int_{x=0}^{1} x^4\,dx - \int_{x=0}^{1}\left[\frac{x^6}{2}+\frac{x^7}{3}\right]dx$$

$$= \frac{1}{6}-\left(\frac{1}{14}+\frac{1}{24}\right)$$

$$= \frac{3}{56}$$

Example 7.3 Evaluate $\displaystyle\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dydx$ by changing to polar coordinates.

[WBUT-2002]

Sol. Let $x = r\cos\theta,\ y = r\sin\theta$ be the transform from cartesian to polar coordinates.

The Jacobian of the transformation is

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0$$

The domain of integration $R = \{(x, y) : 0 \le x \le a; 0 \le y \le \sqrt{a^2 - x^2}\}$ i.e., the first quardant of the circle $x^2 + y^2 = a^2$.

Under the transformation the domain of integration is $R_1 = \left\{(r, \theta) : 0 \le r \le a; 0 \le \theta \le \frac{\pi}{2}\right\}$

Therefore,

$$\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2)\, dydx = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^2 \cdot rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\frac{\pi}{2}} d\theta$$

$$= \frac{a^4}{4} \times \frac{\pi}{2} = \frac{\pi a^4}{8}$$

Example 7.4 Evaluate $\int_0^a \int_0^x \int_0^y x^3 y^2\, zdzdydx$ [WBUT-2003]

Sol.

$$\int_0^a \int_0^x \int_0^y x^3 y^2 zdzdydx$$

$$= \int_0^a \left[\int_0^x \left\{\int_0^y x^3 y^2 zdz\right\} dy\right] dx$$

$$= \int_0^a \left[\int_0^x \left[\frac{x^3 y^2 z^2}{2}\right]_0^y dy\right] dx$$

$$= \int_0^a \left[\int_0^x \frac{x^3 y^4}{2}\, dy\right] dx$$

$$= \int_0^a \left[\frac{x^3 y^5}{10} \right]_0^x dx$$

$$= \frac{1}{10} \int_0^a x^8 \, dx = \frac{a^9}{90}$$

Example 7.5 Evaluate $\iiint (x+y+z+1)^4 \, dxdydz$ over the region defined by $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$ [WBUT-2003]

Sol.

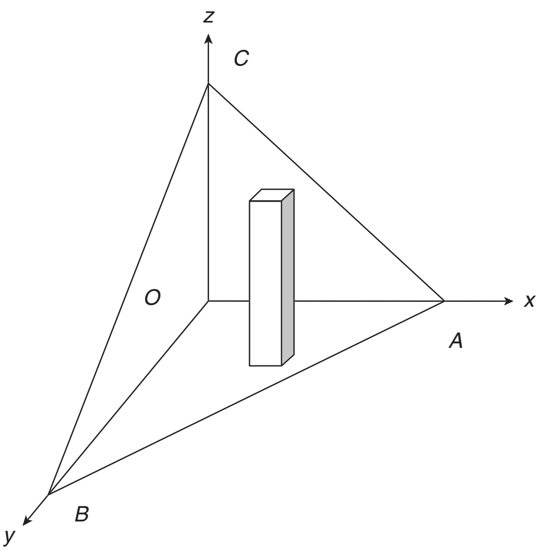

Figure 7.9

The plane $x+y+z=1$ cuts X, Y and Z axes at $A(1, 0, 0), B(0, 1, 0)$ and $C(0, 0, 1)$.

Therefore, the upper limit and lower limit of z are 0 and $1-x-y$.

Putting $z=0$ in $x+y+z=1$ we have the upper limit and lower limit of y are 0 and $1-x$.

Putting $z=0$ and $y=0$, we have the upper limit and lower limit of x are 0 and 1.

Therefore,

$$\iiint (x+y+z+1)^4 \, dxdydz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^4 \, dxdydz$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^5}{5} \right]_0^{1-x-y} dxdy$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{5}\left[32-(x+y+1)^5 \right] dxdy$$

$$= \frac{1}{5}\int_0^1 \left[32y - \frac{(x+y+1)^6}{6} \right]_0^{1-x} dx$$

$$= \frac{1}{5}\int_0^1 \left[32(1-x) - \frac{32}{3} + \frac{(x+1)^6}{6} \right] dx$$

$$= \frac{1}{5}\left[32x - 32\frac{x^2}{2} - \frac{32}{3}x + \frac{(x+1)^7}{42} \right]_0^1$$

$$= \frac{1}{5}\left[32 - 16 - \frac{32}{3} + \frac{64}{21} \right] = \frac{117}{70}$$

Example 7.6 Evaluate $\iint_R \frac{1}{\sqrt{x^2+y^2}} dxdy$ where $R = \{|x| \le 1; |y| \le 1\}$

[WBUT-2004]

Sol. Here $R = \{|x| \le 1; |y| \le 1\} = \{(x, y) : -1 \le x \le 1; -1 \le y \le 1\}$

Therefore,

$$\iint_R \frac{1}{\sqrt{x^2+y^2}} dxdy = \int_{x=-1}^{1} \int_{y=-1}^{1} \frac{1}{\sqrt{x^2+y^2}} dxdy$$

Since $\frac{1}{\sqrt{x^2+y^2}}$ is an even function

$$= 4\int_0^1 \int_0^1 \frac{1}{\sqrt{x^2+y^2}} dxdy$$

$$= 4\int_0^1 \left[\log\,(y+\sqrt{x^2+y^2} \right]_0^1 dx$$

$$= 4\int_0^1 \left[\log\,(1+\sqrt{1+x^2}) - \log x \right] dx$$

$$= 4\left\{ \left[x\log\,(1+\sqrt{1+x^2}) \right]_0^1 - \int_0^1 \frac{x^2}{(1+\sqrt{1+x^2})\sqrt{1+x^2}} dx - \left[x\log x - x \right]_0^1 \right.$$

$$= 4\left\{\log(1+\sqrt{2}) - \int_0^1 \left[\frac{\sqrt{1+x^2}}{x^2} - \frac{x^2}{\sqrt{1+x^2}}\right] dx + 1\right\}$$

$$= 4\left\{\log(1+\sqrt{2}) + 1 - \int_0^1 \left(1 - \frac{1}{\sqrt{1+x^2}}\right) dx\right\}$$

$$= 4\{\log(1+\sqrt{2}) + 1 - \left[x - \log(x+\sqrt{1+x^2})\right]_0^1$$

$$= 4\,\{\log(1+\sqrt{2}) + 1 - (1 - \log(1+\sqrt{2})\}$$

$$= 8\log(1+\sqrt{2})$$

Example 7.7 Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = 1$ [WBUT-2006]

Sol. Let $x = r\cos\theta,\ y = r\sin\theta$ be the transform from Cartesian to polar coordinates.

The Jacobian of the transformation is

$$j = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0$$

The domain of integration $R = \{(x, y) : 0 \le x \le a; 0 \le y \le \sqrt{a^2 - x^2}\}$, i.e., the first quardant of the circle $x^2 + y^2 = a^2$.

Under the transformation, the domain of integration is

$$R_1 = \left\{(r, \theta) : 0 \le r \le a; 0 \le \theta \le \frac{\pi}{2}\right\}$$

Therefore,

$$\iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dxdy = \iint_{R_1} \sqrt{\frac{1-r^2}{1+r^2}}\, rdrd\theta$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\frac{\pi}{2}} \sqrt{\frac{1-r^2}{1+r^2}}\, rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} d\theta \int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, rdr$$

$$= \frac{\pi}{2}\int_{r=0}^{1}\frac{(1-r^2)}{\sqrt{1-r^4}}\,rdr \;= \frac{\pi}{2}\int_{r=0}^{1}\frac{rdr}{\sqrt{1-r^4}} - \frac{\pi}{2}\int_{r=0}^{1}\frac{r^3dr}{\sqrt{1-r^4}}$$

$$= \frac{\pi}{2}\left[\sin^{-1}r^2\right]_0^1 \frac{-\pi}{3} = \frac{\pi^2}{4} - \frac{\pi}{3}$$

Example 7.8 Evaluate $\int_C (3xydx - y^2dy)$ where C is the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$ [WBUT-2007]

Sol. Here, $y = 2x^2 \Rightarrow dy = 4xdx$

Therefore,

$$\int_C (3xydx - y^2\,dy)$$

$$= \int_{x=0}^{1}\{3x\cdot 2x^2dx - (2x^2)^2\cdot 4xdx\}$$

$$= \int_{x=0}^{1}(6x^3 - 16x^5)\,dx$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{-7}{6}$$

EXERCISES

Short and Long Answer Type Questions

1 Evaluate $\int_C [(x^2 + xy)\,dx + (x^2 + y^2)\,dy]$ where C is the square formed by the lines $x = \pm 1,\ y = \pm 1$. [**Ans :** 0]

2. Evaluate $\int_C [(\cos x \sin y - xy)\,dx + \sin x \cos y\,dy]$ where C is the circle $x^2 + y^2 = 1$ $\left[\textbf{Ans :} \frac{-14}{3}\right]$

3. Evaluate $\int_C \{(x^2 + y^2)\,dx - 2\,xydy\}$, where C is the rectangle in the xy plane bounded by $x = 0, x = a; y = 0, y = a.$

[**Ans :** $-2\,ab^2$]

4. Evaluate the line integral $\int_C \{(2xy - x^2)\, dx + (x + y^2)\, dy\}$, where C is the closed curve of the region bounded by $y = x^2,\ y^2 = x$.

$\left[\textbf{Ans :} \frac{1}{30}\right]$

5. Evaluate the following double integrals:

a) $\int_0^4 \int_0^1 xy\,(x - y)\, dydx$ **[Ans : 8]**

b) $\int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2}\; dxdy$ $\left[\textbf{Ans :} \frac{a^3\pi}{6}\right]$

c) $\int_0^\pi \int_0^{a\cos\theta} r\sin\theta\; drd\theta$ $\left[\textbf{Ans :} \frac{a^2}{6}\right]$

d) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\; dydx$ $\left[\textbf{Ans :} \frac{3}{35}\right]$

e) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1 + x^2 + y^2}$ $\left[\textbf{Ans :} \frac{\pi}{4}\log\left(\sqrt{2} + 1\right)\right]$

6. Evaluate $\iint \sqrt{a^2 - x^2 - y^2}\, dxdy$ over the semicircle $x^2 + y^2 = a^2$ in the positive quardant. $\left[\textbf{Ans :} \frac{a^5}{5}\right]$

7. Transform the integral to Cartesian form and hence evaluate $\int_0^\pi \int_0^a r^3 \sin\theta \cos\theta\, drd\theta$

[Ans : 0]

8. Evaluate the following integrals:

a) $\int_0^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)\; dxdydz$ **[Ans : 6]**

b) $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^2 xyzdzdydx$ $\left[\textbf{Ans :} \frac{3}{8}\right]$

c) $\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} xyzdzdydx$ $\left[\textbf{Ans :} \frac{13}{9} - \frac{1}{6}\log 3\right]$

9. Evaluate $\iiint_R (x+y+z)\, dxdydz$ where $R = \{(x, y, z) : 0 \le x \le 1; 1 \le y \le 2; 2 \le z \le 3\}$

$\left[\textbf{Ans :} \frac{9}{2}\right]$

10. Evaluate $\iint_R (x+y+z)\, dxdydz$ over the tetrahedron bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $x+y+z=1$. $\left[\textbf{Ans :} \frac{1}{8}\right]$

11. Evaluate $\iint_R ydxdy$ where R is the region of the first quardant bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ [WBUT-2008]

12. Evaluate $\iiint z^2 dxdydz$ over the region defined by $z \ge 0,\ x^2+y^2+z^2 \le a^2$

[WBUT-2004]

Multiple-Choice Questions

1. The value of the integral $\int_{(0,0)}^{(1,1)} [(x+y)\, dx+(y-x)\, dy]$ is independent of the path.

a) True b) False

2. The value of the integral $\iint_R dxdy$ where $R = \{(x, y) : |x|+|y| \le 1\}$ is

a) 2 b) $\sqrt{2}$ c) 4 d) 3

3. $\int_0^1\int_0^1\int_0^1 xyzdxdydz$ is equal to

a) 1 b) 0 c) $\frac{1}{8}$ d) 4

4. The value of the line integral $\int_C \frac{1}{2}(xdx+ydy)$ along any closed curve C is

a) $\frac{1}{2}$ b) 1 c) 0 d) none of these

5. The value of $\int_C (xdx-dy)$, where C is the line joining (0, 1) to (1, 0) is

a) $\frac{3}{2}$ b) $\frac{1}{2}$ c) 0 d) $\frac{2}{3}$

6. $\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} r^3 dr d\theta =$

a) $\frac{\pi}{64}$ b) $\frac{\pi}{32}$ c) $\frac{3\pi}{64}$ d) $\frac{5\pi}{64}$

7. $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy =$

a) $\frac{1}{3}$ b) 0 c) $\frac{2}{3}$ d) none of these

8. The value of the integral $\int_C xdy$ where C is the arc cut off from the parabola $y^2 = x$ from the point (0, 0) to (1, −1) is

a) $\frac{-1}{3}$ b) $\frac{1}{3}$ c) 0 d) none of these

Answers:

1. (b) 2. (a) 3. (c) 4. (c) 5. (a) 6. (c) 7. (c) 8. (a)

CHAPTER

8

Infinite Series

8.1 INTRODUCTION

This chapter basically deals with preliminary ideas of real sequences and illustrative ideas of infinite series.

The first few sections elaborate the ideas of a sequence; different types of sequences, including bounded and monotone sequences, and their convergence and divergence. Each of the items are illustrated with various kinds of examples.

In the later sections, we discuss the different kinds of infinite series including alternating series and also the tests of convergence of these series. Here too, useful examples are cited to illustrate the facts.

Further, solutions of some important problems given in university examinations are provided in the last section.

8.2 PRELIMINARY IDEAS OF SEQUENCES

A sequence in R or a real sequence is a mapping $f : N \to R$ where N is the set of natural numbers and R is the set of real numbers. So for each $n \in N$, there exists $f(n)$ and the sequence is denoted by $\{f(n)\}$.

We often denote a sequence by $\{a_n\}$ or $\{x_n\}$, etc. A sequence is also denoted by $\{a_1, a_2, a_3, \ldots\}$.

Example 1

Let $f : N \to R$ is defined by $f(n) = n^3, n \in N;$ then $f(1) = 1^3, f(2) = 2^3, f(3) = 3^3, \ldots$

The sequence is denoted by $\{n^3\}$ or $\{1^3, 2^3, 3^3, \ldots\}$.

Note: For our convenience, we replace $\{f(n)\}$ by $\{a_n\}$ where a_n is the n-th term of the sequence.

8.3 DIFFERENT TYPES OF SEQUENCES:

1) **Finite Sequence:** A sequence $\{a_n\}$ having a finite number of terms is called a finite sequence, for example,
   $\{2, 5, 6, 9\}$ is a finite sequence of four terms.
2) **Infinite Sequence:** A sequence $\{a_n\}$ having infinite number of terms is called an infinite sequence, for example,
   $\{n\} = \{1, 2, 3, \ldots\}$ is an infinite sequence.
3) **Harmonic Sequence:** A sequence $\{a_n\}$ where $a_n = \frac{1}{n}$, $n \in N$ is a well-known harmonic sequence.
4) **Constant Sequence:** A sequence $\{a_n\}$ where $a_n = k$, $n \in N$ for any real number k is called a constant sequence.

8.4 BOUNDED SEQUENCE

The real sequence $\{a_n\}$ is said to be a bounded sequence if there exits real numbers m and M such that $m \le a_n \le M$, for example,

$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is a bounded sequence since $0 \le \frac{1}{n} \le 1.$

(a) A sequence $\{a_n\}$ is bounded above if there exists a real number M such that $a_n \le M$ for all $n \in N$. M is called the *upper bound* of the sequence.

Example 2

The sequence $\{a_n\} = \left\{\frac{n-1}{2n}\right\}$ is bounded above since $\frac{n-1}{2n} < \frac{1}{2}$ for all $n \in N$.

Therefore $M = \frac{1}{2}$ is the upper bound.

(b) A sequence $\{a_n\}$ is bounded below if there exists a real number m such that $a_n \ge m$ for all $n \in N$, and m is called the *lower bound* of the sequence.

Example 3

The sequence $\{a_n\} = \{n^2\}$ is bounded below since $n^2 \ge 1$ for all $n \in N$. Therefore, $m = 1$ is the lower bound.

8.5 MONOTONE SEQUENCE

(a) A sequence $\{a_n\}$ is said to be monotonic increasing if and only if $a_{n+1} \geq a_n$ for all $n \in N$.

Example 4

The sequence $\{a_n\} = \left\{\frac{n}{n+1}\right\}$ is a monotonic increasing sequence since

$$\frac{(n+1)}{(n+2)} - \frac{n}{(n+1)} = \frac{1}{(n+1)(n+2)} > 0, \text{ for all } n \in N.$$

(b) A sequence $\{a_n\}$ is said to be monotonic decreasing if and only if $a_{n+1} \leq a_n$ for all $n \in N$.

Example 5

The sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$ is a monotonic decreasing sequence since

$$\frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0, \text{ for all } n \in N.$$

Observations

1) **A sequence $\{a_n\}$ is said to be strictly monotonic increasing if and only if $a_{n+1} > a_n$ for all $n \in N$.**

Example 6

The sequence $\{a_n\} = \left\{\frac{n}{n+1}\right\}$ is a strictly monotonic increasing sequence.

2) **A sequence $\{a_n\}$ is said to be strictly monotonic decreasing if and only if $a_{n+1} < a_n$ for all $n \in N$.**

Example 7

The sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$ is a strictly monotonic decreasing sequence.

3) **A sequence $\{a_n\}$ is said to be monotone if $\{a_n\}$ is either monotonic increasing or monotonic decreasing.**
4) **If a sequence $\{a_n\}$ is monotonic increasing then $\{-a_n\}$ is monotonic decreasing.**
5) **A sequence $\{a_n\}$ need not always be monotone.**

Example 8

$\{2, 0, 2, 0, 2, 0 \ldots\}$ is neither monotonic increasing nor monotonic decreasing.

8.6 LIMIT OF A SEQUENCE

A real number l is said to be a limit of a sequence $\{a_n\}$ if for any pre-assigned positive ε, however small, there exists a natural number n_0 depending on ε such that

$|a_n - l| < \varepsilon$ for all $n \geq n_0$.

We write $\lim\limits_{n \to \infty} a_n = l$.

Observation

To establish the limit l of a sequence $\{a_n\}$, we take ε as an arbitrary positive number and then find some positive integer n_0 such that the numerical magnitude of the difference $(a_n - l)$ is less than ε for every a_n where $n > n_0$.

Example 9

To establish $\lim\limits_{n \to \infty} \dfrac{2n+1}{n+1} = 2$, let us consider a pre-assigned positive number ε, however small, such that

$$\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon$$

$$\Rightarrow \frac{1}{n+1} < \varepsilon$$

$$\Rightarrow n+1 > \frac{1}{\varepsilon}$$

$$\Rightarrow n > \frac{1}{\varepsilon} - 1.$$

Taking $n_0 = \left[\dfrac{1}{\varepsilon} - 1 \right]$, i.e., the integral part of $\left(\dfrac{1}{\varepsilon} - 1 \right)$ we have

$$\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon \text{ for all } n \geq n_0.$$

8.7 CONVERGENT SEQUENCE

A sequence $\{a_n\}$ is called a convergent sequence if it has a finite real number l as its limit. We say the sequence $\{a_n\}$ converges to l.

Example 10

The sequence $\left\{\frac{n^2+1}{n^2}\right\}$ is a convergent sequence and converges to the limit 1.

Observations

1) **A convergent sequence has at most one limit.**
2) **A convergent sequence is bounded.**
3) **Every bounded sequence is not convergent.**

8.8 DIVERGENT SEQUENCE

If for any pre-asssigned positive number K, however large, there exists a natural number n_0 such that $a_n > K$ for all $n \geq n_0$, then $\{a_n\}$ is said to diverge to ∞.

We write

$$\lim_{n\to\infty} a_n = \infty.$$

If for any pre-asssigned positive number K, however large, there exists a natural number n_0 such that $a_n < -K$ for all $n \geq n_0$ then $\{a_n\}$ is said to diverge to $-\infty$.

We write

$$\lim_{n\to\infty} a_n = -\infty.$$

A sequence $\{a_n\}$ is said to be a divergent sequence if $\{a_n\}$ either diverges to ∞ or diverges to $-\infty$.

Example 11 The sequence $\{n^2\}$ is a divergent sequence and diverges to the limit ∞, since $\lim_{n\to\infty} n^2 = \infty$.

Note: There are also some sequences which are neither convergent nor divergent, known as **oscillatory sequences.**

Example 12 $\{(-2)^n\} = \{-2, 2, -2, 2, \ldots\}$ is a common example of an oscillatory sequence.

8.9 INFINITE SERIES

Consider $\{a_n\}$ be a sequence of real numbers. Then $a_1 + a_2 + a_3 + \cdots + a_n + \cdots \infty$ is said to be the infinite series generated by the sequence $\{a_n\}$. The infinite series is denoted by $\sum_{n=1}^{\infty} a_n$.

Example 13 $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is a series generated by the sequence $\{a_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

8.10 CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. The sequence $\{S_n\}$ is called the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. The infinite series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent according to whether $\{S_n\}$ is convergent or divergent.

If $\lim_{n\to\infty} S_n = S$ then S is the sum of the series $\sum_{n=1}^{\infty} a_n$ and if $\lim_{n\to\infty} S_n = \infty$ (or $-\infty$) then the infinite series is said to diverge to ∞ (or, $-\infty$).

Example 14

Let $\{a_n\} = \left\{\frac{1}{n(n+1)}\right\}$ be a sequence of real numbers, $n \in N$.

So,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

The sequence of partial sums of the series is $\{S_n\}$ where

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n-1}\right)$$

$$= 1 - \frac{1}{n-1}$$

From the above, we have

$$\lim_{n\to\infty} S_n = 1$$

Therefore, the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and converges to 1.

Example 15

Let $\{a_n\} = \{n^3\}$ be a sequence of real numbers, $n \in N$.

So,

$$\sum_{n=1}^{\infty} a_n = 1^3 + 2^3 + 3^3 + \cdots$$

The sequence of partial sums of the series is $\{S_n\}$ where

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

From the above, we have

$$\lim_{n\to\infty} S_n = \infty$$

Therefore, the infinite series $\sum_{n=1}^{\infty} a_n$ is divergent and diverges to ∞.

8.10.1 The *p*-Series

The infinite series $\sum_{n=1}^{\infty} a_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^P} + \cdots \infty = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is convergent, since it is a p-series where $p = 2$.

(2) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent, since it is a p-series where $p = 1$.

8.10.2 Geometric Series

A series of the form $1 + r + r^2 + r^3 + \cdots + r^n + \cdots$ is called a geometric series with common ratio r.

The above series is

(i) convergent for $|r| < 1$, i.e., $-1 < r < 1$

(ii) divergent for $r \ge 1$

(iii) oscillatory for $r \le -1$

8.11 PROPERTIES OF CONVERGENCE OF INFINITE SERIES

1) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent infinite series and they converge to p and q respectively then the series $\sum_{n=1}^{\infty}(c \cdot a_n + d \cdot b_n)$ is also convergent and converges to $c \cdot p + d \cdot q$.

2) If an infinite series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

Example 16

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so we have $\lim_{n\to\infty} \frac{1}{n^2} = 0.$

The converse is not true.

Example 17

$\lim_{n\to\infty} \frac{1}{n} = 0,$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

3) If $\lim_{n\to\infty} a_n \neq 0$ for an infinite series $\sum_{n=1}^{\infty} a_n$ then the series cannot be convergent.

4) If the sequence of partial sum $\{S_n\}$ is not bounded then $\{S_n\}$ being a monotone increasing sequence, diverges to ∞.

In this case, the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

5) Addition or removal of a finite number of terms does not effect the convergence of an infinite series.

8.11.1 Series of Positive Terms

An infinite series $\sum_{n=1}^{\infty} a_n$ is called a series of positive terms if $a_n > 0$ for all $n \in N$.

Theorem 8.1: The necessary and sufficient condition for an infinite series of positive terms $\sum_{n=1}^{\infty} a_n$ to be convergent is that the sequence of partial sums $\{S_n\}$ is bounded.

Proof: Beyond the scope of the book.

Theorem 8.2: An infinite series of positive terms either converges or diverges to ∞. This kind of series cannot diverge to $-\infty$ and cannot be oscillatory.

Proof: Beyond the scope of the book.

8.12 DIFFERENT TESTS OF CONVERGENCE OF INFINITE SERIES

8.12.1 Comparison Test

Consider $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series of positive terms.

If for all $n \geq m$, $\frac{a_n}{b_n} \leq k$, k being a fixed positive number then

i) $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} b_n$ is convergent

ii) $\sum_{n=1}^{\infty} b_n$ is divergent if $\sum_{n=1}^{\infty} a_n$ is divergent

8.12.2 Limit Form of Comparison Test

Consider $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series of positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a nonzero finite number.

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and diverge together. [WBUT-2008]

Example 18

Consider the following two series:

$$\sum_{n=1}^{\infty} a_n = \frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots\infty$$

and

$$\sum_{n=1}^{\infty} b_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\infty$$

Here

$$a_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{(n+2)}{2(n+1)^2} \quad \text{and} \quad b_n = \frac{1}{n}$$

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n(n+2)}{2(n+1)^2}=\frac{1}{2}\text{, a nonzero finite value.}$$

Since $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, by comparison test the series $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example 19

Test the convergence of the infinite series $\sum_{n=1}^{\infty} a_n$ where $a_n=(n^3+1)^{\frac{1}{3}}-n.$

[WBUT 2003, 2007]

Sol. Here, we have

$$a_n=(n^3+1)^{\frac{1}{3}}-n$$

$$=\left\{n^3\left(1+\frac{1}{n^3}\right)\right\}^{\frac{1}{3}}-n$$

$$=n\cdot\left\{\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}\right\}-n$$

$$\text{i.e., } a_n=n\left\{1+\frac{1}{3}\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots\infty\right\}-n$$

$$=\frac{1}{n^2}\left\{\frac{1}{3}-\frac{1}{9}\frac{1}{n^3}+\cdots\infty\right\}$$

Let us consider the series $\sum_{n=1}^{\infty} b_n$ where $b_n=\frac{1}{n^2}$

Now

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left\{\frac{1}{3}-\frac{1}{9}\frac{1}{n^3}+\cdots\infty\right\}=\frac{1}{3}\text{, a nonzero finite value.}$$

Since, $\sum_{n=1}^{\infty} b_n$ is a convergent series, by comparison test, $\sum_{n=1}^{\infty} a_n$ is also convergent.

Example 20

Test the convergence of the infinite series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots\infty$$ [WBUT 2005].

Sol. We know that addition and removal of finite number of terms does not affect the convergence of an infinite series.

So, removing the first term from the given series, the resulting series is

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$$

where $a_n = \dfrac{n^n}{(n+1)^{n+1}}$.

Let us consider the series $\sum_{n=1}^{\infty} b_n$, where $b_n = \dfrac{1}{n}$. Now

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{n+1}}{(n+1)^{n+1}} = 1, \text{ a nonzero finite value.}$$

Since, $\sum_{n=1}^{\infty} b_n$ is a divergent series, by comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent.

Therefore, correspondingly the given series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$$

is also divergent.

8.12.3 D'Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l, \text{ any real value.}$$

Then, the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l<1$

ii) diverges if $l>1$

iii) the test fails if l = 1

Example 21

Examine the convergence of the infinite series

$$\frac{1^2\cdot 2^2}{1!}+\frac{2^2 3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\cdots\infty$$

Sol. If we write the given series in the form of $\sum_{n=1}^{\infty} a_n$ then

$$a_n=\frac{n^2(n+1)^2}{n!} \text{ and } a_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}.$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+2)^2(n!)}{n^2(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)n^2}$$

$$=0<1.$$

Therefore, by D' Alembert's ratio test, the series is convergent.

Example 22

Examine the convergence of the infinite series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\cdots$$ [WBUT 2002, 2007]

Sol. If we write the given series in the form of $\sum_{n=1}^{\infty} a_n$ then

$$a_n=\left(\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right)^2$$

$$\text{and so } a_{n+1}=\left(\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)\cdot(2n+3)}\right)^2$$

$$\text{Now, }\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)\cdot(2n+3)}\right)^2}{\left(\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right)^2}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{2n+3}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1.$$

Therefore, by D' Alembert's ratio test, the given series is convergent.

8.12.4 Cauchy's Root Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = l.$

Then the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l < 1$
ii) diverges if $l > 1$
iii) the test fails if $l = 1$ [WBUT 2004]

Example 23

Examine the convergence of the infinite series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$ [WBUT 2001]

Sol. Let us consider

$$\sum_{n=1}^{\infty} a_n = \left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$

Then

$$a_n = \left\{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right\}^{-n}$$

Now,

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left\{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right\}^{-n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-1}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right) \left\{ \left(1+\frac{1}{n}\right)^{n} - 1 \right\} \right]^{-1}$$

$$= (e-1)^{-1} < 1.$$

Therefore, by Cauchy's root test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 24

Examine the convergence of the infinite series $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$ [WBUT 2004]

Sol. Let

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

then

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}.$$

Now,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}} \right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} \right]$$

$$= \frac{1}{e} < 1.$$

Therefore, by Cauchy's root test, $\sum_{n=1}^{\infty} a_n$ is convergent.

8.12.5 Raabe's Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=l.$

Then the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l>1$

ii) diverges if $l<1$

iii) the test fails if $l=1$

Example 25

Examine the convergence of the infinite series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\cdots$$

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infinite series, by removing the first term of the given series, let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7}+\cdots$$

Then

$$a_n = \frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}\cdot\frac{1}{(2n-1)}$$

and so $a_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-3)(2n-1)}{2\cdot 4\cdot 6\ldots(2n-2)2n}\cdot\frac{1}{(2n+1)}$

Now

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n-1)^2}{2n(2n+1)}=1$$

Therefore, D' Alembert's ratio test fails.

But

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{2n(2n+1)}{(2n-1)^2}-1\right]$$

$$= \lim_{n\to\infty}\frac{6n^2-n}{(2n-1)^2} = \frac{3}{2}>1$$

Therefore, by Raabe's test, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

8.13 ALTERNATING SERIES

The infinite series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n > 0$ for all $n \in N$ is called an alternating series.

Example 26

Let us consider the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Here $a_n > 0$ for all $n \in N$. So this is an alternating series.

8.13.1 Test of Convergence of Alternating Series (Leibnitz's Test)

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alterenating series with $a_n > 0$ for all $n \in N$. Then the series converges if

i) $a_{n+1} < a_n$, i.e., $\{a_n\}$ is a monotonic decreasing sequence

ii) $\lim_{n\to\infty} a_n = 0$

[WBUT-2009]

Example 27

Examine the convergence of the alternating series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$$

Sol. If we write the series in the form of $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ then

$$a_n = \frac{(n+1)}{n} \text{ and } a_{n+1} = \frac{n+2}{n+1}.$$

Now

$$a_n - a_{n+1} = \frac{(n+1)}{n} - \frac{n+2}{n+1}$$

$$= \frac{1}{n(n+1)} > 0 \text{ for all } n$$

So $a_n > a_{n+1}$, i.e., $\{a_n\}$ is a monotonic decreasing sequence.

But

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)}{n} = 1$$

i.e., $\lim\limits_{n\to\infty} a_n \neq 0.$

Therefore, by Leibnitz's test, the alternating series is not convergent.

Example 28

Examine the convergence of the alternating series $\sum\limits_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$. [WBUT 2002]

Sol. Since $\cos n\pi = (-1)^n$, the alternating series can be written as

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}.$$

Then,

$$a_n = \frac{1}{n^2+1} \text{ and so } a_{n+1} = \frac{1}{(n+1)^2+1}$$

Now

$$a_n - a_{n+1} = \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \frac{(2n+1)}{(n^2+1)((n+1)^2+1)} > 0$$

The above implies that $\{a_n\}$ is a monotonic decreasing sequence.

$$\text{Also } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

Therefore, $\sum\limits_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ is convergent by Leibnitz's test.

8.14 ABSOLUTE CONVERGENCE

Let $\sum\limits_{n=1}^{\infty} a_n$ be an infinite series. The series is said to be absolutely convergent if $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series. The alternating series is said to be absolutely convergent if

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}a_n\right| = \sum_{n=1}^{\infty}|a_n|$$

is convergent. [WBUT-2009]

Note: An absolutely convergent series is convergent, but the converse is not always true.

Example 29

Examine the absolute convergence of the alternating series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$.

Sol. Since $\cos n\pi = (-1)^n$, the alternating series can be written as

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}.$$

Here, $a_n = \frac{1}{n^2}$. Then

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

But the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2\ (>1)$. So the series is convergent.

Correspondingly, $\sum_{n=1}^{\infty}|a_n|$ is also convergent.

Hence the given series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$ is absolutely convergent.

8.15 CONDITIONAL CONVERGENCE

Let $\sum_{n=1}^{\infty}a_n$ be an infinite series. The series is said to be conditionally convergent if $\sum_{n=1}^{\infty}a_n$ is convergent but not absolutely convergent, i.e., $\sum_{n=1}^{\infty}|a_n|$ is not convergent.

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent but $\sum_{n=1}^{\infty}|a_n|$ is not convergent. [WBUT-2009]

Example 30

Examine the conditional convergence of the alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\infty$$

Sol. This is an alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

Here

$$a_n=\frac{1}{n} \text{ and so } a_{n+1}=\frac{1}{n+1}$$

Let us apply Leibnitz's test for checking convergence.

Now

$$a_n-a_{n+1}=\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}>0$$

This proves that $\{a_n\}$ is a monotonic decreasing sequence.

Again, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0.$

Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is a convergent series.

Now we consider $\sum_{n=1}^{\infty}|a_n|$ which is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}.$$

This is a p-series with $p=1$, and so the series is divergent.

Since, $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent and $\sum_{n=1}^{\infty}|a_n|$ is divergent,

$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent.

WORKED-OUT EXAMPLES

Example 8.1 Examine the convergence of the infinite series

$$\sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$ [WBUT 2008]

Sol. Let

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

This is an infinite series of positive terms. Here

$$a_n = \left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

$$= \frac{\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}$$

$$= \frac{2}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}$$

Let us consider the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is convergent since it is a p-series for $p = 2$.

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^2}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}.$$

$= 1$, a nonzero finite value.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent series, by comparison test, we can conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

is also convergent.

Example 8.2 Examine the convergence of the infinite series

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

Sol. Let us consider

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

where

$$a_n = \frac{x^n}{n} \text{ and } a_{n+1} = \frac{x^{n+1}}{n+1}.$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}\cdot n}{x^n(n+1)} = x.$$

Then by D'Alembert's ratio test, we have

i) If $x<1$, the infinite series is convergent

ii) If $x>1$, the infinite series is divergent

iii) If $x=1$, the test fails

For $x=1$, the series becomes

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, the infinite series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ diverges for $x \geq 1$ and converges for $x<1$.

Example 8.3 Examine the convergence of the infinite series

$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$ [WBUT-2009]

Sol. We know that addition or removal of a finite number of terms does not alter the convergence of an infinite series.

So, removing the first term, we have the series of the form

$$\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$

Suppose we write the series in the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$$

Then

$$a_n = \frac{x^n}{n^2+1} \text{ and so } a_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}.$$

Here,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \Big/ \frac{x^n}{n^2+1}$$

$$= x \cdot \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = x$$

So by D'Alembert's ratio test, we can conclude that

i) If $x < 1$, the infinite series is convergent

ii) If $x > 1$, the infinite series is divergent

iii) If $x = 1$, the test fails

For $x = 1$, the series becomes

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \sum_{n=1}^{\infty} b_n$$

Here, $b_n = \dfrac{1}{n^2+1}$.

Consider the series $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent. Now

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1, \text{ a nonzero finite number.}$$

Therefore, by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is a convergent series.

Hence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$ is convergent for $x \le 1$ and divergent for $x > 1$.

Example 8.4 Examine the convergence of the infinite series $\sum_{n=1}^{\infty}\frac{n!\cdot 2^n}{n^n}$. [WBUT 2003]

Sol. Let us consider

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n!\cdot 2^n}{n^n}$$

Then

$$a_n=\frac{n!2^n}{n^n}\text{ and so }a_{n+1}=\frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$$

Here

$$\lim\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}\Big/\frac{n!2^n}{n^n}$$

$$=\lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^n}=\frac{2}{e}<1$$

Therefore by D'Alembert's ratio test, the series $\sum_{n=1}^{\infty}\frac{n!\cdot 2^n}{n^n}$ is convergent.

Example 8.5 Test the convergence of the series

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$ [WBUT-2001]

Sol. Let us consider the given infinite series

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$$

then

$$a_n=\frac{1}{\sqrt{n}}\sin\frac{1}{n}$$

Let us consider the series

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$$

Then

$$b_n=\frac{1}{n^{\frac{3}{2}}}$$

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}\sin\frac{1}{n}}{\frac{1}{n^{\frac{3}{2}}}}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1$$

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

is a convergent series, by comparison test

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{1}{n}$$

is a convergent series.

Example 8.6 Test the convergence of the series

$$1 + \frac{\sqrt{2}-1}{1!} + \frac{\left(\sqrt{2}-1\right)^2}{2!} + \frac{\left(\sqrt{2}-1\right)^3}{3!} + \cdots \infty$$ [WBUT-2001]

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infinite series, by removing the first term of the given series, let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\left(\sqrt{2}-1\right)^n}{n!}$$

Then

$$a_n = \frac{\left(\sqrt{2}-1\right)^n}{n!} \text{ and } a_{n+1} = \frac{\left(\sqrt{2}-1\right)^{n+1}}{(n+1)!}$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{\left(\sqrt{2}-1\right)^{n+1}}{(n+1)!}}{\dfrac{\left(\sqrt{2}-1\right)^n}{n!}} = \lim_{n\to\infty} \frac{\left(\sqrt{2}-1\right)}{(n+1)} = 0 < 1$$

Therefore, by D' Alembert's ratio test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\left(\sqrt{2}-1\right)^n}{n!}$$

is convergent.

CHAPTER

8

Infinite Series

8.1 INTRODUCTION

This chapter basically deals with preliminary ideas of real sequences and illustrative ideas of infinite series.

The first few sections elaborate the ideas of a sequence; different types of sequences, including bounded and monotone sequences, and their convergence and divergence. Each of the items are illustrated with various kinds of examples.

In the later sections, we discuss the different kinds of infinite series including alternating series and also the tests of convergence of these series. Here too, useful examples are cited to illustrate the facts.

Further, solutions of some important problems given in university examinations are provided in the last section.

8.2 PRELIMINARY IDEAS OF SEQUENCES

A sequence in R or a real sequence is a mapping $f : N \to R$ where N is the set of natural numbers and R is the set of real numbers. So for each $n \in N$, there exists $f(n)$ and the sequence is denoted by $\{f(n)\}$.

We often denote a sequence by $\{a_n\}$ or $\{x_n\}$, etc. A sequence is also denoted by $\{a_1, a_2, a_3, \ldots\}$.

Example 1

Let $f : N \to R$ is defined by $f(n) = n^3, n \in N;$ then $f(1) = 1^3$, $f(2) = 2^3$, $f(3) = 3^3, \ldots$

The sequence is denoted by $\{n^3\}$ or $\{1^3, 2^3, 3^3, \ldots\}$.

Note: For our convenience, we replace $\{f(n)\}$ by $\{a_n\}$ where a_n is the n-th term of the sequence.

8.3 DIFFERENT TYPES OF SEQUENCES:

1) **Finite Sequence:** A sequence $\{a_n\}$ having a finite number of terms is called a finite sequence, for example,
   $\{2, 5, 6, 9\}$ is a finite sequence of four terms.

2) **Infinite Sequence:** A sequence $\{a_n\}$ having infinite number of terms is called an infinite sequence, for example,
   $\{n\}=\{1, 2, 3, \ldots\}$ is an infinite sequence.

3) **Harmonic Sequence:** A sequence $\{a_n\}$ where $a_n = \frac{1}{n}$, $n \in N$ is a well-known harmonic sequence.

4) **Constant Sequence:** A sequence $\{a_n\}$ where $a_n = k$, $n \in N$ for any real number k is called a constant sequence.

8.4 BOUNDED SEQUENCE

The real sequence $\{a_n\}$ is said to be a bounded sequence if there exits real numbers m and M such that $m \le a_n \le M$, for example,

$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is a bounded sequence since $0 \le \frac{1}{n} \le 1$.

(a) A sequence $\{a_n\}$ is bounded above if there exists a real number M such that $a_n \le M$ for all $n \in N$. M is called the *upper bound* of the sequence.

Example 2

The sequence $\{a_n\}=\left\{\frac{n-1}{2n}\right\}$ is bounded above since $\frac{n-1}{2n} < \frac{1}{2}$ for all $n \in N$.

Therefore $M = \frac{1}{2}$ is the upper bound.

(b) A sequence $\{a_n\}$ is bounded below if there exists a real number m such that $a_n \ge m$ for all $n \in N$, and m is called the *lower bound* of the sequence.

Example 3

The sequence $\{a_n\}=\{n^2\}$ is bounded below since $n^2 \ge 1$ for all $n \in N$. Therefore, $m = 1$ is the lower bound.

8.5 MONOTONE SEQUENCE

(a) A sequence $\{a_n\}$ is said to be monotonic increasing if and only if $a_{n+1} \geq a_n$ for all $n \in N$.

Example 4

The sequence $\{a_n\} = \left\{\dfrac{n}{n+1}\right\}$ is a monotonic increasing sequence since

$$\frac{(n+1)}{(n+2)} - \frac{n}{(n+1)} = \frac{1}{(n+1)(n+2)} > 0, \text{ for all } n \in N.$$

(b) A sequence $\{a_n\}$ is said to be monotonic decreasing if and only if $a_{n+1} \leq a_n$ for all $n \in N$.

Example 5

The sequence $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ is a monotonic decreasing sequence since

$$\frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0, \text{ for all } n \in N.$$

Observations

1) **A sequence $\{a_n\}$ is said to be strictly monotonic increasing if and only if $a_{n+1} > a_n$ for all $n \in N$.**

Example 6

The sequence $\{a_n\} = \left\{\dfrac{n}{n+1}\right\}$ is a strictly monotonic increasing sequence.

2) **A sequence $\{a_n\}$ is said to be strictly monotonic decreasing if and only if $a_{n+1} < a_n$ for all $n \in N$.**

Example 7

The sequence $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ is a strictly monotonic decreasing sequence.

3) **A sequence $\{a_n\}$ is said to be monotone if $\{a_n\}$ is either monotonic increasing or monotonic decreasing.**
4) **If a sequence $\{a_n\}$ is monotonic increasing then $\{-a_n\}$ is monotonic decreasing.**
5) **A sequence $\{a_n\}$ need not always be monotone.**

Example 8

$\{2, 0, 2, 0, 2, 0\ldots\}$ is neither monotonic increasing nor monotonic decreasing.

8.6 LIMIT OF A SEQUENCE

A real number l is said to be a limit of a sequence $\{a_n\}$ if for any pre-assigned positive ε, however small, there exists a natural number n_0 depending on ε such that

$|a_n - l| < \varepsilon$ for all $n \geq n_0$.

We write $\lim\limits_{n\to\infty} a_n = l$.

Observation

To establish the limit l of a sequence $\{a_n\}$, we take ε as an arbitrary positive number and then find some positive integer n_0 such that the numerical magnitude of the difference $(a_n - l)$ is less than ε for every a_n where $n > n_0$.

Example 9

To establish $\lim\limits_{n\to\infty} \dfrac{2n+1}{n+1} = 2$, let us consider a pre-assigned positive number ε, however small, such that

$$\left|\frac{2n+1}{n+1} - 2\right| < \varepsilon$$

$$\Rightarrow \frac{1}{n+1} < \varepsilon$$

$$\Rightarrow n+1 > \frac{1}{\varepsilon}$$

$$\Rightarrow n > \frac{1}{\varepsilon} - 1.$$

Taking $n_0 = \left[\dfrac{1}{\varepsilon} - 1\right]$, i.e., the integral part of $\left(\dfrac{1}{\varepsilon} - 1\right)$ we have

$$\left|\frac{2n+1}{n+1} - 2\right| < \varepsilon \text{ for all } n \geq n_0.$$

8.7 CONVERGENT SEQUENCE

A sequence $\{a_n\}$ is called a convergent sequence if it has a finite real number l as its limit. We say the sequence $\{a_n\}$ converges to l.

Example 10

The sequence $\left\{\frac{n^2+1}{n^2}\right\}$ is a convergent sequence and converges to the limit 1.

Observations

1) **A convergent sequence has at most one limit.**
2) **A convergent sequence is bounded.**
3) **Every bounded sequence is not convergent.**

8.8 DIVERGENT SEQUENCE

If for any pre-asssigned positive number K, however large, there exists a natural number n_0 such that $a_n > K$ for all $n \geq n_0$, then $\{a_n\}$ is said to diverge to ∞.

We write

$$\lim_{n\to\infty} a_n = \infty.$$

If for any pre-asssigned positive number K, however large, there exists a natural number n_0 such that $a_n < -K$ for all $n \geq n_0$ then $\{a_n\}$ is said to diverge to $-\infty$.

We write

$$\lim_{n\to\infty} a_n = -\infty.$$

A sequence $\{a_n\}$ is said to be a divergent sequence if $\{a_n\}$ either diverges to ∞ or diverges to $-\infty$.

Example 11 The sequence $\{n^2\}$ is a divergent sequence and diverges to the limit ∞, since $\lim_{n\to\infty} n^2 = \infty$.

Note: There are also some sequences which are neither convergent nor divergent, known as **oscillatory sequences.**

Example 12 $\{(-2)^n\} = \{-2, 2, -2, 2, \ldots\}$ is a common example of an oscillatory sequence.

8.9 INFINITE SERIES

Consider $\{a_n\}$ be a sequence of real numbers. Then $a_1 + a_2 + a_3 + \cdots + a_n + \cdots \infty$ is said to be the infinite series generated by the sequence $\{a_n\}$. The infinite series is denoted by $\sum_{n=1}^{\infty} a_n$.

Example 13 $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is a series generated by the sequence $\{a_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

8.10 CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. The sequence $\{S_n\}$ is called the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. The infinite series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent according to whether $\{S_n\}$ is convergent or divergent.

If $\lim_{n \to \infty} S_n = S$ then S is the sum of the series $\sum_{n=1}^{\infty} a_n$ and if $\lim_{n \to \infty} S_n = \infty$ (or $-\infty$) then the infinite series is said to diverge to ∞ (or, $-\infty$).

Example 14

Let $\{a_n\} = \left\{\frac{1}{n(n+1)}\right\}$ be a sequence of real numbers, $n \in N$.

So,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

The sequence of partial sums of the series is $\{S_n\}$ where

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n-1}\right)$$

$$= 1 - \frac{1}{n-1}$$

From the above, we have

$$\lim_{n \to \infty} S_n = 1$$

Therefore, the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and converges to 1.

Example 15

Let $\{a_n\} = \{n^3\}$ be a sequence of real numbers, $n \in N$.

So,

$$\sum_{n=1}^{\infty} a_n = 1^3 + 2^3 + 3^3 + \cdots$$

The sequence of partial sums of the series is $\{S_n\}$ where

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

From the above, we have

$$\lim_{n\to\infty} S_n = \infty$$

Therefore, the infinite series $\sum_{n=1}^{\infty} a_n$ is divergent and diverges to ∞.

8.10.1 The *p*-Series

The infinite series $\sum_{n=1}^{\infty} a_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^P} + \cdots \infty = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is convergent, since it is a p-series where $p = 2$.

(2) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent, since it is a p-series where $p = 1$.

8.10.2 Geometric Series

A series of the form $1 + r + r^2 + r^3 + \cdots + r^n + \cdots$ is called a geometric series with common ratio r.

The above series is

(i) convergent for $|r| < 1$, i.e., $-1 < r < 1$

(ii) divergent for $r \ge 1$

(iii) oscillatory for $r \le -1$

8.11 PROPERTIES OF CONVERGENCE OF INFINITE SERIES

1) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent infinite series and they converge to p and q respectively then the series $\sum_{n=1}^{\infty}(c \cdot a_n + d \cdot b_n)$ is also convergent and converges to $c \cdot p + d \cdot q$.

2) If an infinite series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

Example 16

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so we have $\lim_{n\to\infty} \frac{1}{n^2} = 0.$

The converse is not true.

Example 17

$\lim_{n\to\infty} \frac{1}{n} = 0,$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

3) If $\lim_{n\to\infty} a_n \neq 0$ for an infinite series $\sum_{n=1}^{\infty} a_n$ then the series cannot be convergent.

4) If the sequence of partial sum $\{S_n\}$ is not bounded then $\{S_n\}$ being a monotone increasing sequence, diverges to ∞.

In this case, the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

5) Addition or removal of a finite number of terms does not effect the convergence of an infinite series.

8.11.1 Series of Positive Terms

An infinite series $\sum_{n=1}^{\infty} a_n$ is called a series of positive terms if $a_n > 0$ for all $n \in N$.

Theorem 8.1: The necessary and sufficient condition for an infinite series of positive terms $\sum_{n=1}^{\infty} a_n$ to be convergent is that the sequence of partial sums $\{S_n\}$ is bounded.

Proof: Beyond the scope of the book.

Theorem 8.2: An infinite series of positive terms either converges or diverges to ∞. This kind of series cannot diverge to $-\infty$ and cannot be oscillatory.

Proof: Beyond the scope of the book.

8.12 DIFFERENT TESTS OF CONVERGENCE OF INFINITE SERIES

8.12.1 Comparison Test

Consider $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series of positive terms.

If for all $n \geq m$, $\frac{a_n}{b_n} \leq k$, k being a fixed positive number then

i) $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} b_n$ is convergent

ii) $\sum_{n=1}^{\infty} b_n$ is divergent if $\sum_{n=1}^{\infty} a_n$ is divergent

8.12.2 Limit Form of Comparison Test

Consider $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series of positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a nonzero finite number.

Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and diverge together. [WBUT-2008]

Example 18

Consider the following two series:

$$\sum_{n=1}^{\infty} a_n = \frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots\infty$$

and

$$\sum_{n=1}^{\infty} b_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\infty$$

Here

$$a_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{(n+2)}{2(n+1)^2} \quad \text{and} \quad b_n = \frac{1}{n}$$

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n(n+2)}{2(n+1)^2}=\frac{1}{2}, \text{ a nonzero finite value.}$$

Since $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, by comparison test the series $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example 19

Test the convergence of the infinite series $\sum_{n=1}^{\infty} a_n$ where $a_n=(n^3+1)^{\frac{1}{3}}-n$.

[WBUT 2003, 2007]

Sol. Here, we have

$$a_n=(n^3+1)^{\frac{1}{3}}-n$$

$$=\left\{n^3\left(1+\frac{1}{n^3}\right)\right\}^{\frac{1}{3}}-n$$

$$=n\cdot\left\{\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}\right\}-n$$

$$\text{i.e., } a_n=n\left\{1+\frac{1}{3}\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots\infty\right\}-n$$

$$=\frac{1}{n^2}\left\{\frac{1}{3}-\frac{1}{9}\frac{1}{n^3}+\cdots\infty\right\}$$

Let us consider the series $\sum_{n=1}^{\infty} b_n$ where $b_n=\frac{1}{n^2}$

Now

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left\{\frac{1}{3}-\frac{1}{9}\frac{1}{n^3}+\cdots\infty\right\}=\frac{1}{3}, \text{ a nonzero finite value.}$$

Since, $\sum_{n=1}^{\infty} b_n$ is a convergent series, by comparison test, $\sum_{n=1}^{\infty} a_n$ is also convergent.

Example 20

Test the convergence of the infinite series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots\infty$$ [WBUT 2005].

Sol. We know that addition and removal of finite number of terms does not affect the convergence of an infinite series.

So, removing the first term from the given series, the resulting series is

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$$

where $a_n = \dfrac{n^n}{(n+1)^{n+1}}$.

Let us consider the series $\sum_{n=1}^{\infty} b_n$, where $b_n = \dfrac{1}{n}$. Now

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{n+1}}{(n+1)^{n+1}} = 1, \text{ a nonzero finite value.}$$

Since, $\sum_{n=1}^{\infty} b_n$ is a divergent series, by comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent.

Therefore, correspondingly the given series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$$

is also divergent.

8.12.3 D'Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l, \text{ any real value.}$$

Then, the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l < 1$

ii) diverges if $l > 1$

iii) the test fails if $l = 1$

Example 21

Examine the convergence of the infinite series

$$\frac{1^2\cdot 2^2}{1!}+\frac{2^2 3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\cdots\infty$$

Sol. If we write the given series in the form of $\sum_{n=1}^{\infty} a_n$ then

$$a_n=\frac{n^2(n+1)^2}{n!} \text{ and } a_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}.$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+2)^2(n!)}{n^2(n+1)!}$$

$$=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)n^2}$$

$$=0<1.$$

Therefore, by D' Alembert's ratio test, the series is convergent.

Example 22

Examine the convergence of the infinite series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\cdots$$ [WBUT 2002, 2007]

Sol. If we write the given series in the form of $\sum_{n=1}^{\infty} a_n$ then

$$a_n=\left(\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right)^2$$

and so $$a_{n+1}=\left(\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)\cdot(2n+3)}\right)^2$$

Now, $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)\cdot(2n+3)}\right)^2}{\left(\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right)^2}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{2n+3}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1.$$

Therefore, by D' Alembert's ratio test, the given series is convergent.

8.12.4 Cauchy's Root Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = l.$

Then the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l < 1$
ii) diverges if $l > 1$
iii) the test fails if $l = 1$ [WBUT 2004]

Example 23

Examine the convergence of the infinite series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$ [WBUT 2001]

Sol. Let us consider

$$\sum_{n=1}^{\infty} a_n = \left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Then

$$a_n = \left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}$$

Now,

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right\}^{-1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^{n} - 1\right\}\right]^{-1}$$

$$= (e-1)^{-1} < 1.$$

Therefore, by Cauchy's root test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 24

Examine the convergence of the infinite series $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$ [WBUT 2004]

Sol. Let

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

then

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}.$$

Now,

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}\right]$$

$$= \frac{1}{e} < 1.$$

Therefore, by Cauchy's root test, $\sum_{n=1}^{\infty} a_n$ is convergent.

8.12.5 Raabe's Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms and $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=l.$

Then the series $\sum_{n=1}^{\infty} a_n$

i) converges if $l>1$

ii) diverges if $l<1$

iii) the test fails if $l=1$

Example 25

Examine the convergence of the infinite series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\cdots$$

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infinite series, by removing the first term of the given series, let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7}+\cdots$$

Then

$$a_n = \frac{1\cdot 3\cdot 5\ldots(2n-3)}{2\cdot 4\cdot 6\ldots(2n-2)}\cdot\frac{1}{(2n-1)}$$

and so $a_{n+1} = \dfrac{1\cdot 3\cdot 5\ldots(2n-3)(2n-1)}{2\cdot 4\cdot 6\ldots(2n-2)2n}\cdot\dfrac{1}{(2n+1)}$

Now

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n-1)^2}{2n(2n+1)} = 1$$

Therefore, D' Alembert's ratio test fails.

But

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{2n(2n+1)}{(2n-1)^2}-1\right]$$

$$= \lim_{n\to\infty}\frac{6n^2-n}{(2n-1)^2} = \frac{3}{2} > 1$$

Therefore, by Raabe's test, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

8.13 ALTERNATING SERIES

The infinite series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n > 0$ for all $n \in N$ is called an alternating series.

Example 26

Let us consider the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Here $a_n > 0$ for all $n \in N$. So this is an alternating series.

8.13.1 Test of Convergence of Alternating Series (Leibnitz's Test)

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alterenating series with $a_n > 0$ for all $n \in N$. Then the series converges if

i) $a_{n+1} < a_n$, i.e., $\{a_n\}$ is a monotonic decreasing sequence

ii) $\lim_{n\to\infty} a_n = 0$

[WBUT-2009]

Example 27

Examine the convergence of the alternating series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$$

Sol. If we write the series in the form of $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ then

$$a_n = \frac{(n+1)}{n} \text{ and } a_{n+1} = \frac{n+2}{n+1}.$$

Now

$$a_n - a_{n+1} = \frac{(n+1)}{n} - \frac{n+2}{n+1}$$

$$= \frac{1}{n(n+1)} > 0 \text{ for all } n$$

So $a_n > a_{n+1}$, i.e., $\{a_n\}$ is a monotonic decreasing sequence.

But

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)}{n} = 1$$

i.e., $\lim\limits_{n\to\infty} a_n \neq 0.$

Therefore, by Leibnitz's test, the alternating series is not convergent.

Example 28

Examine the convergence of the alternating series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$. [WBUT 2002]

Sol. Since $\cos n\pi = (-1)^n$, the alternating series can be written as

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}.$$

Then,

$$a_n = \frac{1}{n^2+1} \text{ and so } a_{n+1} = \frac{1}{(n+1)^2+1}$$

Now

$$a_n - a_{n+1} = \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \frac{(2n+1)}{(n^2+1)((n+1)^2+1)} > 0$$

The above implies that $\{a_n\}$ is a monotonic decreasing sequence.

$$\text{Also } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ is convergent by Leibnitz's test.

8.14 ABSOLUTE CONVERGENCE

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. The series is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series. The alternating series is said to be absolutely convergent if

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}a_n\right| = \sum_{n=1}^{\infty}|a_n|$$

is convergent. [WBUT-2009]

Note: An absolutely convergent series is convergent, but the converse is not always true.

Example 29

Examine the absolute convergence of the alternating series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$.

Sol. Since $\cos n\pi = (-1)^n$, the alternating series can be written as

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}.$$

Here, $a_n = \frac{1}{n^2}$. Then

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

But the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2\ (>1)$. So the series is convergent.

Correspondingly, $\sum_{n=1}^{\infty}|a_n|$ is also convergent.

Hence the given series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$ is absolutely convergent.

8.15 CONDITIONAL CONVERGENCE

Let $\sum_{n=1}^{\infty}a_n$ be an infinite series. The series is said to be conditionally convergent if $\sum_{n=1}^{\infty}a_n$ is convergent but not absolutely convergent, i.e., $\sum_{n=1}^{\infty}|a_n|$ is not convergent.

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent but $\sum_{n=1}^{\infty}|a_n|$ is not convergent. [WBUT-2009]

Example 30

Examine the conditional convergence of the alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\infty$$

Sol. This is an alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

Here

$$a_n=\frac{1}{n} \text{ and so } a_{n+1}=\frac{1}{n+1}$$

Let us apply Leibnitz's test for checking convergence.

Now

$$a_n-a_{n+1}=\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}>0$$

This proves that $\{a_n\}$ is a monotonic decreasing sequence.

Again, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0.$

Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is a convergent series.

Now we consider $\sum_{n=1}^{\infty}|a_n|$ which is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}.$$

This is a p-series with $p=1$, and so the series is divergent.

Since, $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent and $\sum_{n=1}^{\infty}|a_n|$ is divergent,

$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent.

WORKED-OUT EXAMPLES

Example 8.1 Examine the convergence of the infinite series

$$\sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$ [WBUT 2008]

Sol. Let

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

This is an infinite series of positive terms. Here

$$a_n = \left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

$$= \frac{\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}$$

$$= \frac{2}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}$$

Let us consider the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is convergent since it is a p-series for $p = 2$.

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^2}{\left\{\sqrt{n^4+1}+\sqrt{n^4-1}\right\}}.$$

$= 1$, a nonzero finite value.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent series, by comparison test, we can conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$$

is also convergent.

Example 8.2 Examine the convergence of the infinite series

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

Sol. Let us consider

$$\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

where

$$a_n=\frac{x^n}{n} \text{ and } a_{n+1}=\frac{x^{n+1}}{n+1}.$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{x^{n+1}\cdot n}{x^n(n+1)}=x.$$

Then by D'Alembert's ratio test, we have

i) If $x<1,$ the infinite series is convergent

ii) If $x>1,$ the infinite series is divergent

iii) If $x=1,$ the test fails

For $x=1,$ the series becomes

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, the infinite series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ diverges for $x\geq 1$ and converges for $x<1.$

Example 8.3 Examine the convergence of the infinite series

$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$ [WBUT-2009]

Sol. We know that addition or removal of a finite number of terms does not alter the convergence of an infinite series.
So, removing the first term, we have the series of the form

$$\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$

Suppose we write the series in the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$$

Then

$$a_n = \frac{x^n}{n^2+1} \text{ and so } a_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}.$$

Here,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \Big/ \frac{x^n}{n^2+1}$$

$$= x \cdot \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = x$$

So by D'Alembert's ratio test, we can conclude that

i) If $x<1$, the infinite series is convergent

ii) If $x>1$, the infinite series is divergent

iii) If $x=1$, the test fails

For $x=1$, the series becomes

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\cdots = \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \sum_{n=1}^{\infty} b_n$$

Here, $b_n = \dfrac{1}{n^2+1}$.

Consider the series $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent. Now

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1, \text{ a nonzero finite number.}$$

Therefore, by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is a convergent series.

Hence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$ is convergent for $x \le 1$ and divergent for $x>1$.

Example 8.4 Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{n^n}$. [WBUT 2003]

Sol. Let us consider

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{n^n}$$

Then

$$a_n = \frac{n!2^n}{n^n} \text{ and so } a_{n+1} = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$$

Here

$$\lim \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \bigg/ \frac{n!2^n}{n^n}$$

$$= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

Therefore by D'Alembert's ratio test, the series $\sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{n^n}$ is convergent.

Example 8.5 Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{1}{n}$$ [WBUT-2001]

Sol. Let us consider the given infinite series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{1}{n}$$

then

$$a_n = \frac{1}{\sqrt{n}} \sin\frac{1}{n}$$

Let us consider the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

Then

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \sin\frac{1}{n}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

is a convergent series, by comparison test

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{1}{n}$$

is a convergent series.

Example 8.6 Test the convergence of the series

$$1+\frac{\sqrt{2}-1}{1!}+\frac{\left(\sqrt{2}-1\right)^2}{2!}+\frac{\left(\sqrt{2}-1\right)^3}{3!}+\cdots\infty$$ [WBUT-2001]

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infinite series, by removing the first term of the given series, let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\left(\sqrt{2}-1\right)^n}{n!}$$

Then

$$a_n = \frac{\left(\sqrt{2}-1\right)^n}{n!} \text{ and } a_{n+1} = \frac{\left(\sqrt{2}-1\right)^{n+1}}{(n+1)!}$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\dfrac{\left(\sqrt{2}-1\right)^{n+1}}{(n+1)!}}{\dfrac{\left(\sqrt{2}-1\right)^n}{n!}} = \lim_{n\to\infty}\frac{\left(\sqrt{2}-1\right)}{(n+1)} = 0 < 1$$

Therefore, by D' Alembert's ratio test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\left(\sqrt{2}-1\right)^n}{n!}$$

is convergent.

Example 8.7 Test the convergence of the series

$$\frac{1^2+2}{1^4}x+\frac{2^2+2}{2^4}x^2+\frac{3^2+2}{3^4}x^3+\cdots\infty$$ [WBUT-2002]

Sol. Let the given infinite series be

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{(n^2+2)}{n^4}x^n$$

Then

$$a_n=\frac{(n^2+2)}{n^4}x^n \text{ and } a_{n+1}=\frac{\{(n+1)^2+2\}}{(n+1)^4}x^{n+1}$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{\{(n+1)^2+2\}}{(n+1)^4}x^{n+1}}{\frac{(n^2+2)}{n^4}x^n}=\lim_{n\to\infty}\frac{(n+1)^2+2}{n^2+2}\frac{n^4}{(n+1)^4}x$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2+\frac{2}{n^2}}{1+\frac{2}{n^2}}\frac{1}{\left(1+\frac{1}{n}\right)^4}x=x$$

By D'Alembert's ratio test,

(i) If $x<1$, the infinite series is convergent

(ii) If $x>1$, the infinite series is divergent

(iii) If $x=1$, the test fails

For $x=1$, the infinite series becomes

$$\frac{1^2+2}{1^4}+\frac{2^2+2}{2^4}+\frac{3^2+2}{3^4}+\cdots\infty$$

Here,

$$a_n=\frac{(n^2+2)}{n^4}$$

Consider the series

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1\text{(a nonzero finite number).}$$

Since

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a convergent series, by comparison test

$$\sum_{n=1}^{\infty}\frac{(n^2+2)}{n^4}$$

is also a convergent series.

Therefore,

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{(n^2+2)}{n^4}x^n$$

is convergent for $x\le 1$ and divergent for $x>1$.

Example 8.8 Examine the convergence of the series

$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots\infty$$ [WBUT-2004]

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infnite series, by removing the first term of the given series, Let the resulting infinite series be

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{x^n}{n^2+1}$$

Then

$$a_n=\frac{x^n}{n^2+1}\text{ and so }a_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{x^{n+1}}{(n+1)^2+1}}{\dfrac{x^n}{n^2+1}}=\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}x$$

$$=\lim_{n\to\infty}\frac{n^2+1}{n^2+2n+2}x=\lim_{n\to\infty}\frac{1+\dfrac{1}{n^2}}{1+\dfrac{2}{n}+\dfrac{2}{n^2}}x=x$$

Therefore, by D'Alembert's ratio test,

i) If $x<1$, the infinite series is convergent

ii) If $x>1$, the infinite series is divergent

iii) If $x=1$, the test fails

For $x=1$, the series becomes

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Consider the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \text{ (a nonzero finite number).}$$

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent series, by comparison test

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

is also a convergent series.

Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$$

is convergent for $x \le 1$ and divergent for $x>1$.

Example 8.9 Test the convergence of the series

$$\frac{\sqrt{1}}{a\cdot 1^{\frac{3}{2}}+b} + \frac{\sqrt{2}}{a\cdot 2^{\frac{3}{2}}+b} + \frac{\sqrt{3}}{a\cdot 3^{\frac{3}{2}}+b} + \cdots\infty \quad (a>0)$$

[WBUT-2004]

Sol. Let the infinite series be

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{a\cdot n^{\frac{3}{2}}+b}$$

where

$$a_n = \frac{\sqrt{n}}{a\cdot n^{\frac{3}{2}}+b}$$

Consider the divergent series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{\sqrt{n}}{a\cdot n^{\frac{3}{2}}+b}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{a\cdot n^{\frac{3}{2}}+b}$$

$$= \lim_{n\to\infty} \frac{1}{a+\dfrac{b}{n^{\frac{3}{2}}}} = \frac{1}{a} \quad \text{(a nonzero finite number)}$$

Therefore, by comparison test

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{a\cdot n^{\frac{3}{2}}+b}$$

i.e, $\dfrac{\sqrt{1}}{a\cdot 1^{\frac{3}{2}}+b} + \dfrac{\sqrt{2}}{a\cdot 2^{\frac{3}{2}}+b} + \dfrac{\sqrt{3}}{a\cdot 3^{\frac{3}{2}}+b} + \cdots\infty$

is a divergent series.

Example 8.10 Test the convergence of the series

$$1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\cdots\infty,\ \ x\neq 1$$ [WBUT-2004, 2009]

Sol. Since addition or removal of finite number of terms does not effect the convergence of an infinite series, by removing the first term of the given series,

Let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2}x^n.$$

Then

$$a_n = \frac{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2}x^n$$

and so $a_{n+1} = \dfrac{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2(2n+2)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2(2n+3)^3}x^{n+1}$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{2^2\cdot4^2\cdot6^2\ldots(2n)^2(2n+2)^2}{3^2\cdot5^2\cdot7^2\ldots(2n+1)^2(2n+3)^3}x^{n+1}}{\dfrac{2^2\cdot4^2\cdot6^2\ldots(2n)^2}{3^2\cdot5^2\cdot7^2\ldots(2n+1)^2}x^n}$$

$$=\lim_{n\to\infty}\frac{(2n+2)^2}{(2n+3)^2}x=x$$

Therefore, by D'Alembert's ratio test,

i) If $x<1$, the infinite series is convergent

ii) If $x>1$, the infinite series is divergent

Example 8.11 Test the convergence of the series

$$\sin\left(\frac{1}{1^{\frac{3}{2}}}\right)+\sin\left(\frac{1}{2^{\frac{3}{2}}}\right)+\sin\left(\frac{1}{3^{\frac{3}{2}}}\right)+\sin\left(\frac{1}{4^{\frac{3}{2}}}\right)+\cdots\infty$$ [WBUT-2005]

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\sin\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Then

$$a_n=\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

Consider the p-series for $p=\dfrac{3}{2}$ as

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$$

where,

$$b_n=\frac{1}{n^{\frac{3}{2}}}$$

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)}{\left(\frac{1}{n^{\frac{3}{2}}}\right)}=1\text{, a nonzero finite value.}$$

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

is a convergent series, by comparison test,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

is also convergent.

Example 8.12 Test the convergence of the series $\sum_{n=1}^{\infty} n^4 e^{-n^2}$ [WBUT-2005]

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^4 e^{-n^2}.$$

Then

$$a_n = n^4 e^{-n^2} \text{ and so } a_{n+1} = (n+1)^4 e^{-(n+1)^2}$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^4 e^{-(n+1)^2}}{n^4 e^{-n^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \frac{1}{e^{(n+1)^2 - n^2}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 \frac{1}{e^{2n+1}} = 0 < 1$$

Therefore, by D'Alembert's ratio test, the series is convergent.

Example 8.13 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} x^n;\ x > 0$

[WBUT-2006]

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} x^n.$$

Then

$$a_n = \frac{n^2-1}{n^2+1} x^n \text{ and } a_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)^2-1}{(n+1)^2+1}x^{n+1}}{\frac{n^2-1}{n^2+1}x^n}=\lim_{n\to\infty}\frac{(n^2+2n)(n^2+1)}{(n^2+2n+2)(n^2-1)}x$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n^2}\right)}{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)\left(1-\frac{1}{n^2}\right)}x=x$$

Therefore, by D'Alembert's ratio test, the series is

(i) convergent if $x<1$

(ii) divergent if $x>1$ and

(iii) the test fails for $x=1$

For $x=1$, the series becomes

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$$

Now,

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2-1}{n^2+1}=1\neq 0$$

So the series is divergent for $x=1$.

Hence, the series is convergent for $x<1$ and divergent for $x\geq 1$.

Example 8.14 For what values of x is the following series convergent?

$$\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\cdots\infty$$ [WBUT-2006]

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{x^n}{(2n-1)(2n+1)}.$$

Then

$$a_n=\frac{x^n}{(2n-1)(2n+1)}\text{ and so }a_{n+1}=\frac{x^{n+1}}{(2n+1)(2n+3)}$$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{x^{n+1}}{(2n+1)(2n+3)}}{\dfrac{x^n}{(2n-1)(2n+1)}}$$

$$=\lim_{n\to\infty}\frac{(2n-1)}{(2n+3)}x=x$$

Therefore, by D'Alembert's ratio test, the series is

(i) convergent if $x<1$

(ii) divergent if $x>1$ and

(iii) the test fails for $x=1$

For $x=1$, the series becomes

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

Consider another series

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

where,

$$b_n=\frac{1}{n^2}$$

Now,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\dfrac{1}{(2n-1)(2n+1)}}{\dfrac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{(2n-1)(2n+1)}$$

$$=\lim_{n\to\infty}\frac{1}{\left(2-\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)}=\frac{1}{4}\quad\text{(a nonzero finite value)}$$

Since

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

being a p-series for $p=2$, is a convergent series, By comparison test

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

is also convergent.

Therefore, the given series is convergent for $x \le 1$.

Example 8.15 Examine the convergence of the series $\sum_{n=0}^{\infty} \left(\frac{nx}{n+1} \right)^n$

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infnite series, by removing the first term of the given series,

Let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{nx}{n+1} \right)^n$$

Then

$$a_n = \left(\frac{nx}{n+1} \right)^n$$

Now,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{ \left(\frac{nx}{n+1} \right)^n \right\}^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{nx}{n+1} \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right) x = x$$

Therefore, by Cauchy's root test, the series is

(i) convergent for $x < 1$

(ii) divergent for $x > 1$

(iii) test fails for $x = 1$

For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^n$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0$$

Therefore, the series is divergent in this case.

Hence the given series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 8.16 Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{(1+nx)^n}{n^n}$

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(1+nx)^n}{n^n}.$$

Then $a_n = \dfrac{(1+nx)^n}{n^n}$

Now,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{\frac{(1+nx)^n}{n^n}\right\}^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1+nx}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} + x\right) = x$$

Therefore, by Cauchy's root test, the series is

(i) convergent for $x < 1$

(ii) divergent for $x > 1$

(iii) test fails for $x = 1$

For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1+n}{n}\right)^n$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1+n}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

Therefore, the series is divergent in this case.

Hence, the series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 8.17 Examine the convergence of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty$$

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infnite series, by removing the first term of the given series, Let the resulting series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^n x^n.$$

Then

$$a_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

Now,

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)x\right] = x$$

Therefore, by Cauchy's root test, the series is

(i) convergent for $x < 1$

(ii) divergent for $x > 1$

(iii) test fails for $x = 1$

For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^n$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^n}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{1}{e} \neq 0$$

Therefore, the series is divergent in this case.

Hence, the series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 8.18 Examine the convergence of the series $\sum_{n=2}^{\infty}\frac{1+n\log n}{n^2+5}$

Sol. Since the addition or removal of finite number of terms does not effect the convergence of an infnite series, by adding $\frac{1+1\cdot\log 1}{1^2+5}$ as the first term to the given series, Let the resulting series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1+n\log n}{n^2+5}.$$

Then

$$a_n = \frac{1+n\log n}{n^2+5}$$

Consider the well-known divergent series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

where

$$b_n = \frac{1}{n}$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1+n\log n}{n^2+5}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n+n^2\log n}{n^2+5}$$

$$= \lim_{n\to\infty} \frac{\dfrac{1}{n}+\log n}{1+\dfrac{5}{n^2}} = \infty$$

Therefore, by comparison test, the series

$$\sum_{n=1}^{\infty} \frac{1+n\log n}{n^2+5}$$

is divergent.

Example 8.19 Examine the convergence of the series

$$\frac{1}{2}\frac{3^2}{4^2} + \frac{1\cdot 3\cdot 5^2}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7^2}{2\cdot 4\cdot 6\cdot 8^2} + \cdots \infty$$

Sol. Since addition or removal of a finite number of terms does not affect the convergence of an infinite series, by removing the first term of the given series, let the resulting infinite series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)^2}{2\cdot 4\cdot 6\ldots 2n(2n+2)^2}$$

Then, $$a_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)^2}{2\cdot 4\cdot 6\ldots 2n(2n+2)^2}$$

and so $a_{n+1} = \dfrac{1\cdot 3\cdot 5\ldots(2n+1)(2n+3)^2}{2\cdot 4\cdot 6\ldots(2n+2)(2n+4)^2}$

Now,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+3)^2(2n+2)}{(2n+4)^2(2n+1)}$$

$$= \lim_{n\to\infty}\frac{\left(2+\dfrac{3}{n}\right)^2\left(2+\dfrac{2}{n}\right)}{\left(2+\dfrac{4}{n}\right)^2\left(2+\dfrac{1}{n}\right)} = 1$$

So, it is obvious from the above that D'Alembert's ratio test fails.

Therefore, we apply Raabe's test.

Now,

$$\lim_{n\to\infty} n\left\{\frac{a_n}{a_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{(2n+4)^2(2n+1)}{(2n+3)^2(2n+2)}-1\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{4n^3+6n^2-2n}{(2n+3)^2(2n+2)}\right\}$$

$$= \lim_{n\to\infty}\frac{4+\dfrac{6}{n}-\dfrac{2}{n^2}}{\left(2+\dfrac{2}{n}\right)\left(2+\dfrac{3}{n}\right)^2} = \frac{4}{8} = \frac{1}{2} < 1$$

Therefore, by Raabe's test, the given series is divergent.

Example 8.20 Test the convergence of the following series:

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \cdots\infty,\ x > 0$$

Sol. Let the infinite series be of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{2^2\cdot 4^2\ldots(2n)^2}{3\cdot 4\cdot 5\cdot 6\ldots(2n+2)}x^{2n+2}$$

where

$$a_n = \frac{2^2\cdot 4^2\ldots(2n)^2}{3\cdot 4\cdot 5\cdot 6\ldots(2n+2)}x^{2n+2}$$

and $a_{n+1} = \dfrac{2^2 \cdot 4^2 \ldots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \ldots (2n+2)(2n+3)(2n+4)} x^{2n+4}$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+3)(2n+4)} x^2 = x^2$$

By D'Alembert's ratio test, the series is

(i) convergent if $x^2 < 1$, i.e., $0 < x < 1$

(ii) divergent if $x^2 > 1$ i.e., $x > 1$

(iii) The test fails for $x = 1$.

For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^2 \cdot 4^2 \ldots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \ldots (2n+2)}$$

Then

$$a_n = \frac{2^2 \cdot 4^2 \ldots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \ldots (2n+2)}$$

and $a_{n+1} = \dfrac{2^2 \cdot 4^2 \ldots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \ldots (2n+2)(2n+3)(2n+4)}$

Now,

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right)$$

$$= \lim_{n\to\infty} \frac{n(6n+8)}{(2n+2)^2} = \frac{6}{4} = \frac{3}{2} > 1$$

Therefore, by Raabe's test, the series is convergent for $x = 1$.

Example 8.21 Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi}{2}}{(n+1)}$

Sol. Since

$$\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$$

the given series can be represented as

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n+1)}$$

So, this is an alternating series and we apply Leibnitz's test for testing its convergence.

Here

$$a_n = \frac{1}{(n+1)} \text{ and so } a_{n+1} = \frac{1}{(n+2)}$$

Now,

$$a_n - a_{n+1} = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)} > 0 \text{ for all } n \in N$$

Since, $a_{n+1} < a_n$, so $\{a_n\}$ is monotonically decreasing.
Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Therefore, by Leibnitz's test the alternating series is convergent.

Example 8.22 Test the convergence of the series

$$\frac{1}{1\cdot 2^3} - \frac{1^2+2^2}{2\cdot 3^3} + \frac{1^2+2^2+3^2}{3\cdot 4^3} - \frac{1^2+2^2+3^2+4^2}{4\cdot 5^3} + \cdots\infty$$

Sol. The given series can be represented as

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1^2+2^2+3^2+4^2+\cdots n^2)}{n(n+1)^3}$$

Then

$$a_n = \frac{n(n+1)(2n+1)}{6n(n+1)^3} = \frac{(2n+1)}{6(n+1)^2}$$

So, this is an alternating series and we apply Leibnitz's test for testing its convergence.

Now,

$$\frac{a_{n+1}}{a_n} = \frac{(2n+3)(n+1)^2}{(2n+1)(n+2)^2}$$

$$= \frac{2n^3+7n^2+8n+3}{2n^3+9n^2+12n+4}$$

$$= 1 - \frac{2n^2+4n+1}{2n^3+9n^2+12n+4} < 1$$

Since $a_{n+1} < a_n$, so $\{a_n\}$ is monotonically decreasing.
Also

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n+1)}{6(n+1)^2} = 0$$

Hence by Leibnitz's test, the alternating series is convergent.

Example 8.23 Test the convergence of the following two series:

a) $\sum_{n=1}^{\infty}\frac{6}{e^n}$ and b) $\sum_{n=1}^{\infty}\frac{1}{(\ln 3)^n}$

Sol.

a) The series can be written as

$$\sum_{n=1}^{\infty}\frac{6}{e^n}=\frac{6}{e}+\frac{6}{e^2}+\frac{6}{e^3}+\frac{6}{e^4}+\cdots\infty$$

$$=\frac{6}{e}\left(1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\cdots\infty\right).$$

Now

$$1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\cdots\infty \qquad \ldots(1)$$

is a geometric series whose common ratio is $r=\frac{1}{e}<1$.

Hence the series (1) is convergent and converges to $\frac{1}{1-\frac{1}{e}}=\frac{e}{e-1}$.

Consequently, the series $\sum_{n=1}^{\infty}\frac{6}{e^n}$ is also convergent and converges to $\frac{6}{e}\left(\frac{e}{e-1}\right)=\frac{6}{e-1}$.

b) The series can be written as

$$\sum_{n=1}^{\infty}\frac{1}{(\ln 3)^n}=\frac{1}{\ln 3}+\frac{1}{(\ln 3)^2}+\frac{1}{(\ln 3)^3}+\cdots\infty$$

$$=\frac{1}{\ln 3}\left(1+\frac{1}{\ln 3}+\frac{1}{(\ln 3)^2}+\cdots\infty\right)$$

Now

$$1+\frac{1}{\ln 3}+\frac{1}{(\ln 3)^2}+\cdots\infty \qquad \ldots(2)$$

is a geometric series whose common ratio $r=\frac{1}{\ln 3}>1$.

Therefore, the series (2) is divergent and consequently, the series $\sum_{n=1}^{\infty}\frac{1}{(\ln 3)^n}$ is also divergent.

Example 8.24 Show that the following series is divergent:

$1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots\infty$

Sol. The given series is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n(n+1)$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n(n+1) = \infty$$

Therefore, the given series is divergent.

Example 8.25 Test the convergence of the following series:

$\dfrac{6}{1\cdot 3\cdot 5}+\dfrac{8}{3\cdot 5\cdot 7}+\dfrac{10}{5\cdot 7\cdot 9}+\cdots\infty$ [WBUT-2008]

Sol. Let the given series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+4}{(2n-1)(2n+1)(2n+3)}.$$

Then

$$a_n = \frac{2n+4}{(2n-1)(2n+1)(2n+3)}$$

Consider the convergent series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

where

$$b_n = \frac{1}{n^2}$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{2n+4}{(2n-1)(2n+1)(2n+3)}}{\dfrac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{(2n+4)n^2}{(2n-1)(2n+1)(2n+3)}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{4}{n}\right)}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} = \frac{1}{4} \text{(a nonzero finite value)}$$

Therefore, by comparison test

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n+4}{(2n-1)(2n+1)(2n+3)}$$

is convergent.

Example 8.26 Test the convergence of the following series:

$$1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots\infty$$ [WBUT-2008]

Sol. Let the given series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^p}{n!}.$$

Then

$$a_n = \frac{n^p}{n!} \text{ and so } a_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^p}{(n+1)!}}{\frac{n^p}{n!}} = \lim_{n\to\infty} \frac{(n+1)^p\, n!}{n^p\,(n+1)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \frac{1}{n+1} = 0 < 1$$

Therefore, by D' Alembert's ratio test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^p}{n!}$ is convergent.

Example 8.27 Test the convergence of the alternating series:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\infty$$ [WBUT-2009]

Sol. The alternating series can be written as

$$\sum_{n=}^{\infty}(-1)^{n-1}a_n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}.$$

Here, $a_n = \dfrac{1}{n^2}$ and so $a_{n+1} = \dfrac{1}{(n+1)^2}$.

Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} < 1 \text{ for all } n \in N.$$

Since, $a_{n+1} < a_n$, so $\{a_n\}$ is monotonically decreasing.

Also

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$$

Hence, by Leibnitz's test, the alternating series is convergent.

Example 8.28 Show that the series $\sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$ is absolutely convergent.

[WBUT-2004, 2009]

Sol. Let the given series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.$$

Then

$$a_n = \frac{\cos nx}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{\cos nx}{n^2}\right|.$$

Since $|\cos nx| < 1$ for all n, x we have

$$|a_n| = \frac{1}{n^2}|\cos nx| < \frac{1}{n^2}.$$

Now we consider the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is a p-series with $p=2$ (>1) and so convergent.

Here

$$b_n = \frac{1}{n^2}$$

Therefore, for all $n \in N$,

$$|a_n| < b_n \Rightarrow \frac{|a_n|}{b_n} < 1.$$

So, by comparison test, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Hence, the given series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is absolutely convergent.

Example 8.29 Prove that the infinite series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n+1} \frac{x^n}{n} + \cdots$$

is absolutely convergent when $|x| < 1$ and conditionally convergent when $x = 1$.

[WBUT 2001, 2007]

Sol. Let the given series be

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \ldots(1)$$

where

$$a_n = (-1)^{n+1} \frac{x^n}{n}.$$

Now we consider the series

$$\sum_{n=1}^{\infty} |a_n| \qquad \ldots(2)$$

where

$$|a_n| = \left| (-1)^{n+1} \frac{x^n}{n} \right| = \frac{|x|^n}{n}.$$

So,

$$|a_{n+1}| = \left| (-1)^{n+2} \frac{x^{n+1}}{n+1} \right| = \frac{|x|^{n+1}}{n+1}.$$

Therefore

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{n}{(n+1)}|x|$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}|x|=|x|.$$

Then by D'Alembert's ratio test, we have

i) If $|x|<1,$ the infinite series (2) is convergent

ii) If $|x|>1,$ the infinite series (2) is divergent

Hence, the given series (1) is absolutely convergent for $|x|<1.$

For $x=1,$ the series becomes

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

which is conditionally convergent. (For proof see the example 30 of Article 8.15).

Example 8.30 Test the convergence of the series:

$$1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots\infty$$ [WBUT-2003]

Sol. The given series can be written as

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}.$$

So this is an alternating series and

$$a_n=\frac{1}{n^p}\text{ and }a_{n+1}=\frac{1}{(n+1)^p}.$$

Then

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}=\frac{n^p}{(n+1)^p}<1\text{ for }p>0\text{ and }n\in N.$$

Since, $a_{n+1}<a_n$ for $p>0,$ so $\{a_n\}$ is monotonically decreasing for $p>0.$

Also, for $p > 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$$

Hence by Leibnitz's test, the alternating series is convergent for $p > 0$.

But for $p < 0$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \infty.$$

Therefore, the series can't be convergent for $p < 0$.

EXERCISES

Short and Long Answer Type Questions

(A) Test the convergence of the following series:

1) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\infty$

[**Ans :** Convergent]

2) $\frac{1^2\cdot 2^2}{1!} + \frac{2^2\cdot 3^2}{2!} + \frac{3^2\cdot 4^2}{3!} + \cdots\infty$

[**Ans :** Convergent]

3) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

[**Ans :** Convergent]

4) $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \frac{1}{1+2^{-4}} + \cdots\infty$

[**Ans :** Divergent]

5) $\frac{1}{2} + \frac{3}{2^2} + \frac{7}{2^3} + \frac{15}{2^4} + \cdots\infty$

[**Ans :** Divergent]

6) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

[**Ans :** Convergent]

7) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$ [WBUT-2004]

[**Ans :** Convergent]

8) $\sum_{n=1}^{\infty} \frac{n!2^n}{n^n}$

[**Ans :** Convergent]

9) $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots \infty$ [WBUT-2002, 2007]

[**Ans :** Convergent]

10) $\sum_{n=1}^{\infty} \frac{n^3 - n + 1}{n!}$

[**Ans :** Convergent]

11) $\sum_{n=1}^{\infty} \frac{n+1}{n^6}$

[**Ans :** Convergent]

12) $\sum_{n=1}^{\infty} \frac{n^4}{e^{n^2}}$

[**Ans :** Convergent]

(B) Examine the convergence of the following series for different values of x:

13) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

[**Ans :** Convergent if $-1 \le x < 1$, divergent if $x \ge 1$ or $x < -1$]

14) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}, x > 0$

[**Ans :** Convergent if $x > 1$ or $0 < x < 1$]

15) $\sum_{n=1}^{\infty} \left(\frac{nx}{n+1}\right)^n$

[**Ans :** Convergent if $x < 1$, divergent if $x \ge 1$]

16) $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}} x^n$

[**Ans :** Convergent if $x < 1$, divergent if $x \ge 1$]

17) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos^2 nx}{n^{\frac{3}{2}}}$$

is absolutely convergent.

18) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(\log n)^2}$$

is absolutely convergent.

(C) Test the convergence of the following series:

1) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

[**Ans :** convergent]

2) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

[**Ans :** convergent]

3) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

[**Ans :** divergent]

4) $\displaystyle\frac{1^2+2}{1^4}x+\frac{2^2+2}{2^4}x^2+\frac{3^2+2}{3^4}x^3+\cdots\infty$ [WBUT-2002]

[**Ans :** Convergent if $x \le 1$, divergent if $x > 1$]

5) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

[**Ans :** divergent]

6) $\displaystyle 1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots\infty$ [WBUT-2004]

[**Ans :** Convergent if $x \le 1$, divergent if $x > 1$]

7) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$

$\left[\textbf{Ans} : \text{Convergent if } p > \frac{1}{2}, \text{ divergent if } p \le \frac{1}{2}\right]$

8) $\displaystyle 1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\cdots\infty,\ x \ne 1$ [WBUT-2004, 2009]

[**Ans :** Convergent if $x < 1$, divergent if $x > 1$]

9) $\displaystyle\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+\frac{5^p}{4^q}+\cdots\infty\ (p, q > 0)$

[**Ans :** Convergent if $q > p+1$, divergent if $q \le p+1$]

10) $\dfrac{x}{1\cdot 3}+\dfrac{x^2}{3\cdot 5}+\dfrac{x^3}{5\cdot 7}+\cdots\infty$ [WBUT-2006]

[**Ans :** Convergent if $x \le 1$, divergent if $x > 1$]

11) $\dfrac{5}{2}-\dfrac{7}{4}+\dfrac{9}{6}-\dfrac{11}{8}+\cdots\infty$

[**Ans :** Not convergent]

12) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}}{(n+1)}$

[**Ans :** Convergent]

13) $1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots\infty$

[**Ans :** Divergent]

14) $x^2+\dfrac{2^2}{3\cdot 4}x^4+\dfrac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6+\dfrac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8+\cdots\infty,\ x>0$

[**Ans :** Convergent if $0 < x \le 1$, divergent if $x > 1$]

15) $\displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right\}^{-n}$

[**Ans :** Convergent]

16) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$ [WBUT-2001]

[**Ans :** convergent]

17) $\dfrac{1}{2}+\dfrac{2}{3}x+\left(\dfrac{3}{4}\right)^2x^2+\left(\dfrac{4}{5}\right)^3x^3+\cdots\infty$

[**Ans :** Convergent if $x < 1$, divergent if $x \ge 1$]

18) $\dfrac{a+x}{1!}+\dfrac{(a+x)^2}{2!}+\dfrac{(a+3x)^3}{3!}+\cdots\infty,\ x>0$

$\left[\textbf{Ans}:\ \text{Convergent if } 0<x<\dfrac{1}{e}, \text{ divergent if } x\ge\dfrac{1}{e}\right]$

19) $1+\dfrac{1}{2\cdot 1^2+1}+\dfrac{1}{2\cdot 2^2+1}+\dfrac{1}{2\cdot 3^2+1}+\cdots\infty$

[**Ans :** Convergent]

20) $\dfrac{2^3}{1^p+3^p}+\dfrac{3^3}{2^p+5^p}+\dfrac{4^3}{3^p+7^p}+\cdots\infty$

[**Ans :** Convergent if $p > 4$, divergent if $p \le 4$]

21) $1+\frac{1}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\cdots\infty,\ x>0$

[**Ans :** Convergent if $0<x<e$, divergent if $x\geq e$]

(D) Examine the convergence of the following alternating series:

22) $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\cdots\infty$

[**Ans :** Convergent]

23) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\infty$ [WBUT-2009]

[**Ans :** Convergent]

24) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^n}{n^2}$

[**Ans :** Divergent]

25) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n^2-n},\ 0<x<1$

[**Ans :** Convergent]

26) $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{5}+1}+\cdots\infty$

[**Ans :** Convergent]

27) $\frac{1}{6}-\frac{2}{11}+\frac{3}{16}-\frac{4}{21}+\cdots\infty$

[**Ans :** Not convergent]

28) $\frac{u}{u+1}-\frac{u^2}{u^2+1}+\frac{u^3}{u^3+1}-\frac{u^4}{u^4+1}+\cdots\infty,\ 0<u<1$

[**Ans :** Convergent]

(E) Prove that the following series are absolutely convergent:

29) $1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\cdots\infty$

30) $x-\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}-\frac{4^4x^4}{4!}+\cdots\infty,\ x\neq\pm\frac{1}{e}$

(F) Prove that the following series are conditionally convergent:

31) $1-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}-\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{5}}-\cdots\infty$

32) $\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\frac{4}{17}+\frac{5}{26}-\cdots\infty$

Multiple Choice Questions

1. The series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if

   a) $p<1$ b) $p>0$ c) $p>1$ d) $p<0$

2. The series $\frac{1}{\sqrt[5]{1}}+\frac{1}{\sqrt[5]{2}}+\frac{1}{\sqrt[5]{3}}+\frac{1}{\sqrt[5]{4}}+\cdots\infty$ is

   a) divergent b) convergent c) oscillatory d) none of these

3. The series $\frac{1}{\sqrt[3]{1^5}}+\frac{1}{\sqrt[3]{2^5}}+\frac{1}{\sqrt[3]{3^5}}+\frac{1}{\sqrt[3]{4^5}}+\cdots\infty$ is

   a) convergent b) oscillatory c) divergent d) none of these

4. The sequence $\{(-1)^n\cdot 2^n\}$ is

   a) monotone b) bounded c) convergent d) oscillatory infinitely

5. The sequence $\left\{\frac{1}{n}\sin n\right\}$ is

   a) oscillatory
   b) divergent to ∞
   c) convergent with limit 1
   d) convergent with limit 0

6. The sequence $\left\{\frac{1}{4+3n}\right\}$ is

   a) decreasing and unbounded
   b) increasing and bounded
   c) decreasing and bounded
   d) none of these

7. The sequence $\left\{3+(-1)^n\frac{1}{n}\right\}$ is

   a) oscillatory
   b) monotone
   c) convergent
   d) bounded but not convergent

8. The sequence $\left\{\frac{n^3+1}{n^2}\right\}$ is

   a) bounded
   b) divergent to ∞
   c) convergent
   d) none of these

9. Which of the following sequence is convergent?

a) $\left\{\frac{1}{4^n}+2n\right\}$ b) $\left\{\frac{n^4+1}{n^3}\right\}$

c) $\left\{\frac{1}{2^n}+\frac{1}{4^n}\right\}$ d) $\{1+(-1)^n\}$

10. The series $\sum_{n=1}^{\infty}\frac{n+1}{n}$ is

a) convergent b) divergent to ∞
c) oscillatory d) none of these

11. If $\sum_{n=1}^{\infty} a_n$ is convergent then

a) $\{a_n\}$ is monotone b) $\{a_n\}$ is convergent with limit 0

c) $\lim_{n\to\infty} a_n = 1$ d) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$.

12. The series $\sum_{n=1}^{\infty}\frac{1}{(n+1)^n}$ is

a) divergent to ∞ b) convergent
c) oscillatory d) none of these

13. The series $\sum_{n=1}^{\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$ is

a) convergent b) divergent
c) oscillatory d) none of these.

14. The series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n^p}\right)$ is convergent if

a) $p>2$ b) $p>1$ c) $p\le 2$ d) $p>0$

15. Which of the following infinite series is convergent?

a) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+4\right)$ b) $\sum_{n=1}^{\infty}\left(\frac{8}{5}\right)^n$

c) $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\infty$ d) $\sum_{n=1}^{\infty}\frac{3n^2+1}{4n^3+1}$

16. The infinite series

$$\frac{1}{1+x^2}-\frac{1}{2+x^2}+\frac{1}{3+x^2}-\frac{1}{4+x^2}+\cdots\infty$$

is convergent

a) only for $-1 \le x \le 1$
b) only for $x = 0$
c) for no real values of x
d) for all real values of x

17. The infinite series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\infty$$

is

a) absolutely convergent
b) oscillatory
c) conditionally convergent
d) none of these

18. The series $\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)$ is

a) convergent
b) divergent
c) neither convergent nor divergent
d) none of these

19. Let $\sum a_n$ be an infinite series of positive terms. If $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \frac{4}{3}$ then $\sum a_n$ is

a) convergent
b) divergent
c) oscillate infinitely
d) none of these

20. If $a_n > 0$ for all n and $a_1 \ge a_2 \ge a_3 \ge \cdots$ and $\lim_{n\to\infty}(a_n)a_n = 0$ then the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$

a) oscillates infinitely
b) is divergent to $-\infty$
c) is convergent
d) none of these

Answers:

1. (c)	2. (a)	3. (a)	4. (d)	5. (d)	6. (c)	7. (c)	8. (b)	9. (c)
10. (b)	11. (b)	12. (b)	13. (b)	14. (a)	15. (c)	16. (d)	17. (c)	18. (a)
19. (b)	20. (c)							

CHAPTER

9

Vector Analysis

9.1 INTRODUCTION

Vectors are very important part of any branch of science and technology. Vectors and their differentiation and integrations have wide range of applications for solving problems in many practical situations. Basically, we shall divide the chapter into three parts.

In the **first part of this chapter**, we discuss **Vector Algebra** which includes **various kinds of vectors**, different terminologies, different kinds of **products of vectors**, equations of straight line, plane and sphere in vector form and of course their applications too.

In the **second part of the chapter**, we deal with vector differentiations, **gradient, divergence, curl,** directional derivative along with their applications.

In the **third part of the chapter**, we give theorems on vector integrations (**Green's theorem, Divergence theorem, Stoke's theorem**) and their applications to physical problems.

PART-I (VECTOR ALGEBRA)

9.2 SCALARS AND VECTORS

Any physical quantity which has magnitude only is known as a **scalar**. The examples of scalars are area, volume, mass, speed, etc.

Any physical quantity which has magnitude as well as direction is known as a **vector.** The examples of vectors are displacement, velocity, force, etc.

Generally, a vector is represented by a directed line segment. Any vector from point A to point B is denoted by $\overrightarrow{AB}$. Let us consider any vector $\overrightarrow{AB} = \vec{a}$, where the length of the line segment AB is a (which is always positive). Then the magnitude or absolute value of the vector is denoted by $\left|\overrightarrow{AB}\right|$, and is given by $\left|\overrightarrow{AB}\right| = |\vec{a}| = a$.

9.2.1 Different Kinds of Vectors

Here we give a few definitions.

(1) Like Vectors: Vectors having the same direction are known as like vectors.

(2) Null Vector or Zero Vector: Any vector having the magnitude zero is known as a null vector or zero vector. It is generally denoted by $\vec{0}$ or θ.

(3) Unit Vector: Any non-null vector of unit length (or, having the unit magnitude) is known as a unit vector.

Let $\vec{a}$ be any non-null vector. Then the unit vector in the direction of $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ or $\frac{\vec{a}}{a}$.

In the three-dimentional Cartesian coordinate system, the unit vectors along the x-axis, y-axis and z-axis are $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively. These are called *fundamental unit vectors*.

(4) Equal Vectors: Two vectors $\vec{a}$ and $\vec{b}$ are called equal if they have the same magnitudes as well as the same direction. Then we write $\vec{a} = \vec{b}$. Two parallel vectors having the same magnitude are equal.

(5) Negative of a Vector: Let $\overrightarrow{AB} = \vec{a}$ be a vector. Then a vector having the same magnitude but opposite direction is known as the negative of $\overrightarrow{AB}$ and is given by $\overrightarrow{BA} = -\overrightarrow{AB} = -\vec{a}$.

(6) Position Vector: The position of a point P with respect to any arbitrary point O is represented by a vector $\overrightarrow{OP}$. Here, O is called the initial point or the vector origin and the vector $\overrightarrow{OP}$ is called the position vector w.r.t O.

9.2.2 Addition and Subtraction of Vectors

Additon of Vectors using Triangle Law

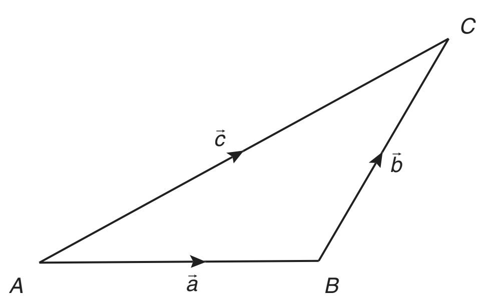

Figure 9.1 Triangle Law

Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$. Then by Triangle Law, **(see Fig. 9.1)** we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\text{or, } \vec{a} + \vec{b} = \vec{c}.$$

Additon of Vectors using Parallelogram Law

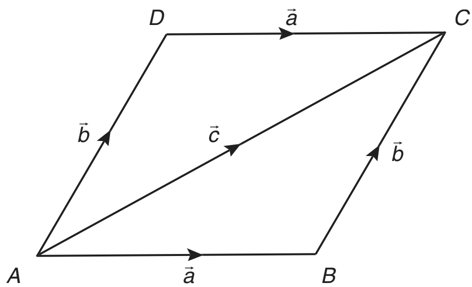

Figure 9.2 Parallelogram Law

Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$. Then by Parallelogram Law **(see Fig. 9.2),** we have

$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

$$\text{or, } \vec{a} + \vec{b} = \vec{c}$$

Properties of Vector Addition

(i) Addition of vectors is commutative, i.e., $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

(ii) Addition of vectors is associative, i.e., $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Subtraction of Vectors

Let $\overrightarrow{AB} = \vec{a}$; $\overrightarrow{BC} = \vec{b}$; then their subtraction is given by

$$\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AB} + (-\overrightarrow{BC})$$

$$\text{i.e., } \vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

9.2.3 Scalar Multiplication of a Vector

Multiplication of a vector $\vec{a}$ by any scalar m, $+ve$ or $-ve$, is denoted by the vector $m\vec{a}$.

Now $|m\vec{a}| = |m||\vec{a}|$, i.e., magnitude of the vector $m\vec{a}$ is $|m|$ multiple of magnitude of the vector $\vec{a}$.

Direction of the vector $m\vec{a}$ is same or opposite of the direction of the vector $\vec{a}$, accordingly as m is positive or negative.

In particular, if $m = 0$ then $m\vec{a} = 0 \cdot \vec{a} = \vec{0}$.

Properties

(i) $(m+n)\,\vec{a} = m\vec{a} + n\vec{a}$

(ii) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

(iii) $m(n\vec{a}) = mn\vec{a} = m(n\vec{a})$

9.2.4 Collinear Vectors

Any set of vectors having the same or different magnitudes is said to be collinear if all of them have the same directions.

In particular, when two vectors $\vec{a}$ and $\vec{b}$ are collinear then we can write $\vec{a} = \lambda \cdot \vec{b}$ for any scalar λ.

Here, we state a theorem on the collinearity of three points.

Theorem 9.1: Any set of three distinct points A, B and C will be collinear (i.e., lie on the same line) iff there exists three scalars α, β, γ (not all zero) such that

$$\alpha\ \vec{a} + \beta\ \vec{b} + \gamma\ \vec{c} = \vec{0} \quad \text{and} \quad \alpha + \beta + \gamma = 0$$

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of A, B and C respectively w.r.t a vector origin.

Proof: Beyond the scope of the book.

9.2.5 Coplanar Vectors

Any set of vectors are called coplanar if all of them are parallel to the same plane.

Here, we state a theorem on the coplanarity of four points.

Theorem 9.2: Any set of four distinct points A, B, C and D (no three of them are collinear) will be coplanar (i.e., lie on the same plane) iff there exists four scalars α, β, γ, δ (not all zero) such that

$$\alpha\ \vec{a} + \beta\ \vec{b} + \gamma\ \vec{c} + \delta\ \vec{d} = \vec{0} \quad \text{and} \quad \alpha + \beta + \gamma + \delta = 0$$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are the position vectors of A, B, C and D respectively w.r.t a vector origin.

Proof: Beyond the scope of the book.

9.2.6 Resolution of Vectors in Rectangular Cartesian coordinate System

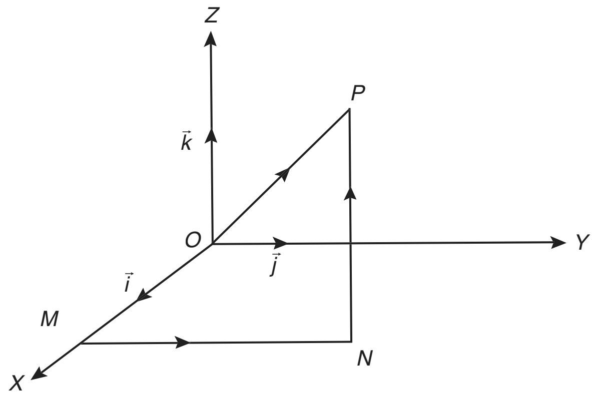

Figure 9.3

Let the unit vectors along the three axes OX, OY and OZ be $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively and $P(x, y, z)$ be any point.

From **Fig. 9.3**, it clear that PN is perpendicular to the XY plane and MN is parallel to Y-axis.

So, $OM = x$, $MN = y$, $NP = z$ and correspondingly, $\overrightarrow{OM} = x\hat{i}$, $\overrightarrow{MN} = y\hat{j}$, $\overrightarrow{NP} = z\hat{k}$.

Now from ΔOMN, $\overrightarrow{ON} = x\hat{i} + y\hat{j}$.

So from ΔONP, we have $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

Hence we can say for any point $P(x, y, z)$, the position vector $\overrightarrow{OP}$ w.r.t some vector origin O is given by $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, where the vector components of $\overrightarrow{OP}$ along the directions of X-axis, Y-axis, and Z-axis are respectively $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$.

Now $\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2 + z^2}$.

So, the unit vector along the direction of $\overrightarrow{OP}$ is $\dfrac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$.

Observation:

Let us consider the two points A and B whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2). So the position vectors of A and B are $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ respectively.

So the vector $\overrightarrow{AB}$, joining two points A and B is given by

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

9.3 SCALAR PRODUCT OR DOT PRODUCT OF VECTORS

9.3.1 Definition

The scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$, where θ is the smallest angle between their directions, is defined by

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta$$

where $|\vec{a}|$ and $\left|\vec{b}\right|$ are magnitudes of $\vec{a}$ and $\vec{b}$ respectively **(see Fig. 9.4).**

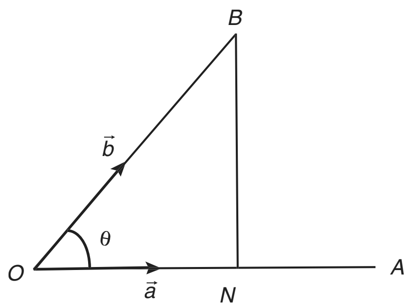

Figure 9.4

9.3.2 Geometrical Interpretations

The scalar product of two vectors is nothing but the product of the length of one vector and the projection of the other to the former one.

9.3.3 Properties of Scalar Product

1) **The scalar product of two vectors always yields a scalar quantity.**
2) **The scalar product of two vectors is commutative, i.e.,** $\vec{a}\ \ \vec{b} = \vec{b}\ \ \vec{a}$.
3) **Two non-null vectors** $\vec{a}$ **and** $\vec{b}$ **are perpendicular if and only if** $\vec{a}\ \ \vec{b} = 0$.
4) $\vec{a}^2 = \vec{a}\ \ \vec{a} = |\vec{a}|\ \ |\vec{a}|\ \ \cos 0 = |\vec{a}|^2$
5) **Here,** $\hat{i} \cdot \hat{i} = \hat{i}^2 = \left|\hat{i}\right|^2 = 1$. **Similarly,** $\hat{j}^2 = \hat{k}^2 = 1$. **Again,** $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$, **where** $\hat{i}$, $\hat{j}$, $\hat{k}$ **respectively are the unit vectors along the three coordinate axes.**
6) **Let** $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, **and** $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

   Then, $\vec{a}\ \ \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\ \ (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = a_1b_1 + a_2b_2 + a_3b_3$

7) The angle between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a}\ \vec{b}}{|\vec{a}||\vec{b}|}\right)$$

$$= \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

8) Distributive property:

$$\vec{a}\ (\vec{b} + \vec{c}) = \vec{a}\ \vec{b} + \vec{a}\ \vec{c}$$

9) Component of $\vec{a}$ along $\vec{b}$ is given by

$$|\vec{a}|\ \cos\theta = |\vec{a}|\ \frac{\vec{a}\ \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\vec{a}\ \vec{b}}{b}$$

10) If $\vec{a}\ \vec{b} = \vec{a}\ \vec{c}$ then $\vec{a}\ (\vec{b} - \vec{c}) = 0$ implies the following facts.
$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.
i.e., $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.

Example 1

If $|\vec{\alpha}| = 3$ and $|\vec{\beta}| = 4$, then find the values of the scalar μ for which the vectors $\vec{\alpha} + \mu\vec{\beta}$ and $\vec{\alpha} - \mu\vec{\beta}$ will be perpendicular to each other. [WBUT 2005]

Sol. The vectors $\vec{\alpha} + \mu\vec{\beta}$ and $\vec{\alpha} - \mu\vec{\beta}$ will be perpendicular to each other if

$$(\vec{\alpha} + \mu\vec{\beta}) \cdot (\vec{\alpha} - \mu\vec{\beta}) = 0$$

$$\text{i.e., } |\vec{\alpha}|^2 - \mu^2 |\vec{\beta}|^2 = 0$$

$$\text{or, } \mu^2 = \frac{|\vec{\alpha}|^2}{|\vec{\beta}|^2} = \frac{9}{16}$$

$$\text{or, } \mu = \pm\frac{3}{4}$$

9.4 VECTOR OR CROSS PRODUCTS OF VECTORS

9.4.1 Definition

The cross product, or vector product, of two vectors $\vec{a}$ and $\vec{b}$, where θ is the smallest angle between their directions, is defined by

$$\bar{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$$

where $|\vec{a}|$ and $|\vec{b}|$ are magnitudes of $\vec{a}$ and $\vec{b}$ respectively and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and the direction of $\hat{n}$ is same as the direction of the motion of a right-handed screw rotating from $\vec{a}$ to $\vec{b}$ **(see Fig. 9.5).**

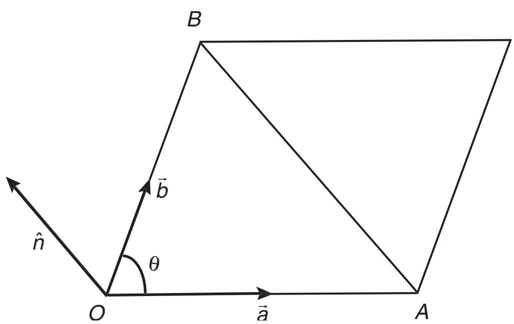

Figure 9.5

9.4.2 Geometrical Interpretations

We have $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$.

So, $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = OA\cdot OB\cdot\sin\theta = 2\times$ area of ΔOAB = area of the parallelogram with the adjacent sides OA and OB **(see Fig. 9.5).**

Hence, $\vec{a}\times\vec{b}$ represents the vector area of the parallelogram whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$.

9.4.3 Properties of Cross Products of Vectors

1) **The cross product of two vectors always yields a vector quantity.**
2) **Let $\vec{a}$ and $\vec{b}$ be any two vectors. Then, $(\vec{a}\times\vec{b}) = -(\vec{b}\times\vec{a})$, i.e., cross product is non-commutative.**
3) **Two non-null vectors $\vec{a}$ and $\vec{b}$ are parallel or collinear if and only if $\vec{a}\times\vec{b} = 0$.**
4) **For any vector $\vec{a}$, we have $\vec{a}\times\vec{a} = \vec{0}$.**
5) **Here, $\hat{i}\in\hat{i} = \hat{j}\in\hat{j} = \hat{k}\in\hat{k} = \vec{0}$. Again $\hat{i}\in\hat{j} = \hat{k}$, $\hat{j}\in\hat{k} = \hat{i}$, $\hat{k}\in\hat{i} = \hat{j}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively are the unit vectors along the three coordinate axes.**

6) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
Then,

$$\vec{a} \in \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

7) The unit vector $\hat{n}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$, is given by

$$\hat{n} = \frac{\vec{a} \in \vec{b}}{\left|\vec{a} \in \vec{b}\right|}$$

8) The angle between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \sin^{-1}\left(\frac{\left|\vec{a} \times \vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|}\right).$$

9) Distributive property:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

10) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then $\vec{a} \times (\vec{b} - \vec{c}) = 0$ implies the following facts.

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a}$ is parallel to $\vec{b} - \vec{c}$

i.e., $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}$ is parallel to $\vec{b} - \vec{c}$.

Example 2

Find a unit vector perpendicular to each of the vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ and obtain the angle between them.

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - \hat{k}$.

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(1-2) - \hat{j}(-2-6) + \hat{k}(2+3) = -\hat{i} + 8\hat{j} + 5\hat{k}$$

So,

$$\left|\vec{a} \times \vec{b}\right| = \left|-\hat{i} + 8\hat{j} + 5\hat{k}\right| = \sqrt{1^2 + 8^2 + 5^2} = \sqrt{90}$$

Therefore, the unit vector is given by

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} = \frac{-\hat{i}+8\hat{j}+5\hat{k}}{\sqrt{90}}$$

Now, the angle is given by

$$\theta = \sin^{-1}\frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|} = \sin^{-1}\frac{\sqrt{90}}{\sqrt{2^2+1^2+2^2}\sqrt{3^2+1^2+1^2}}$$

$$= \sin^{-1}\frac{3\sqrt{10}}{3\sqrt{11}} = \sin^{-1}\left(\frac{\sqrt{10}}{\sqrt{11}}\right).$$

Example 3

Show that $\left(\vec{a}\times\vec{b}\right)^2 = \vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2$.

Sol. For any vector $\vec{a}$, we have $\vec{a}^2 = \left|\vec{a}\right|^2$. So,

$$\left(\vec{a}\times\vec{b}\right)^2 = \left|\vec{a}\times\vec{b}\right|^2$$

$$= \left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\sin^2\theta$$

$$= \left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot(1-\cos^2\theta)$$

$$= \left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2-\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cdot\cos^2\theta$$

$$= \vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2.$$

9.5 SCALAR TRIPLE PRODUCTS

9.5.1 Definition

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Then the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as $\vec{a}\cdot(\vec{b}\times\vec{c})$.

It is always a scalar quantity and is denoted by $[\vec{a}\vec{b}\vec{c}]$.

Geometrically, it represents the volume of a parallelepiped whose coterminus edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$.

9.5.2 Properties of Scalar Triple Product

1) **Let** $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ **and** $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ **then**

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2) $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\vec{c}\vec{a}] = [\vec{c}\vec{a}\vec{b}] = -[\vec{a}\vec{c}\vec{b}] = -[\vec{b}\vec{a}\vec{c}] = -[\vec{c}\vec{b}\vec{a}]$.

3) $[\hat{i}, \hat{j}, \hat{k}] = [\hat{j}, \hat{k}, \hat{i}] = [\hat{k}, \hat{i}, \hat{j}] = 1$, **where** $\hat{i}$, $\hat{j}$, $\hat{k}$ **respectively are the unit vectors along the three coordinate axes.**

4) **The three nonzero vectors** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **are coplanar if and only if**

$$\left[\vec{a}\vec{b}\vec{c}\right] = 0 \text{ i.e., } \vec{a}\ \ (\vec{b}\times\vec{c}) = 0$$

9.6 VECTOR TRIPLE PRODUCTS

9.6.1 Definition

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors in three dimensions. Then the vector triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\ \ \vec{c})\ \ \vec{b} - (\vec{a}\ \ \vec{b})\ \ \vec{c}$$

9.6.2 Properties of Vector Triple Products

1) **Let** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **be three nonzero vectors. Then,** $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

2) **If any two of the nonzero vectors** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **are parallel or equal then,**

$$\vec{a}\times(\vec{b}\times\vec{c}) = 0$$

3) $(\vec{b}\times\vec{c})\times\vec{a} = -\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\ \ \vec{b})\ \ \vec{c} - (\vec{a}\ \ \vec{c})\ \ \vec{b}$.

9.7 STRAIGHT LINE

9.7.1 Equation of a Line Passing Through a Given Point and Parallel to a Given Vector

Let A be the given point whose position vector is $\vec{a}$ w.r.t the origin O, and also suppose the line is parallel to the given vector $\vec{b}$. Let P be any point on the line and its position vector is given by $\vec{r}$.

Then the equation of the required line in vector form is given by $\vec{r} = \vec{a} + t\vec{b}$, where t is any scalar.

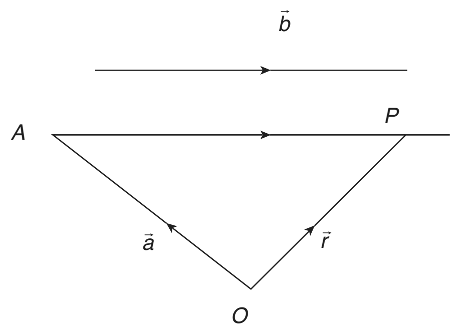

Figure 9.6

Note: The equation of the line passing through the origin is $\vec{r} = t\vec{b}$, since $\vec{a} = \vec{0}$.

Example 4 Find the equation of the line through the point $(-2, 1, 0)$ and parallel to the vector $5\hat{i} - 3\hat{j} + 4\hat{k}$.

Sol. Here, the given point is $(-2, 1, 0)$, whose position vector is $\vec{a}$ and the vector $\vec{b} = 5\hat{i} - 3\hat{j} + 4\hat{k}$ represents the point $(5, -3, 4)$.

Also, $\vec{r}$ is the position vector of any arbitrary point (x, y, z).

We have from the above section, the equation of the line as $\vec{r} = \vec{a} + t\vec{b}$ for any scalar t.

Therefore,

$$(x, y, z) = (-2, 1, 0) + t(5, -3, 4)$$

which implies

$$\frac{x+2}{5} = \frac{y-1}{-3} = \frac{z-0}{4} = t$$

Hence the required equation of the line is

$$\frac{x+2}{5} = \frac{y-1}{-3} = \frac{z}{4}.$$

9.7.2 Equation of a Line Passing Through Two Points

Let A and B be the given points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively w.r.t the origin O. Let P be any point on the line and its position vector is given by $\vec{r}$.

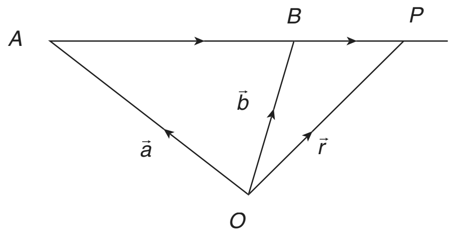

Figure 9.7

Then the equation of the required line in vector form is given by

$$\vec{r} = t\vec{a} + (1-t)\vec{b} \qquad \ldots(1)$$

where t is any scalar.

Note: Let the coordinate of the points A and B are (x_2, y_2, z_2) and (x_1, y_1, z_1) w.r.t the rectangular Cartesian coordinate system. Also, $P(x, y, z)$ be any point on the line. Then, from (1) we have

$$(x, y, z) = t(x_2, y_2, z_2) + (1-t)(x_1, y_1, z_1)$$

which gives

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t.$$

This is the equation of a line through two given points in three-dimensional Cartesian coordinate system.

Example 5 Find the equation of the line through the points $(2, 3, 4)$ and $(3, 4, 5)$.

Sol. Here, $(x_1, y_1, z_1) \equiv (2, 3, 4)$ and $(x_2, y_2, z_2) \equiv (3, 4, 5)$.

Hence the required equation of the line is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

or, $\dfrac{x-2}{3-2} = \dfrac{y-3}{4-3} = \dfrac{z-4}{5-4}$

or, $x-2 = y-3 = z-4$.

9.8 PLANE

9.8.1 Equation of a Plane Perpendicular to the Unit Vector $\hat{n}$ and Passing Through a Point whose Position Vector is $\vec{a}$

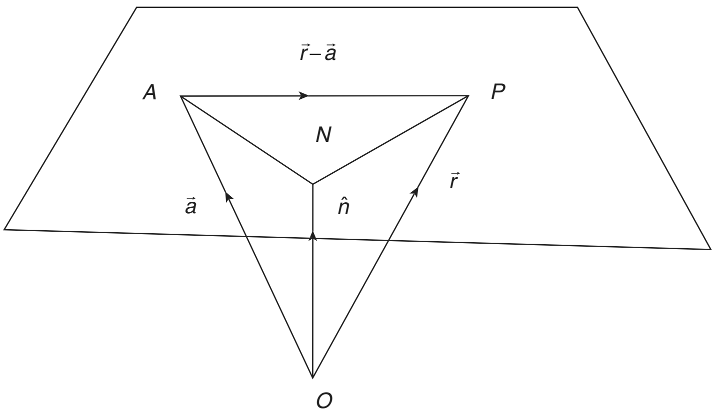

Figure 9.8

Let A be the given point whose position vector is $\vec{a}$. Suppose $\overrightarrow{ON}$ is perpendicular to the plane and $\left|\overrightarrow{ON}\right| = p$, length of the perpendicular from the origin. Also, consider P to be any point on the plane whose position vector is $\vec{r}$. Here $\overrightarrow{ON} = p \cdot \hat{n}$ and $\overrightarrow{OP} = \vec{r}$.

Then the required equation of the plane is given by $\boldsymbol{\vec{r} \cdot \vec{n} = p}$. This is known as *normal form* of the equation of the plane.

Note: If the plane passes through the origin then the equation becomes $\vec{r} \cdot \vec{n} = 0$.

9.8.2 Equation of a Plane Passing Through a Point whose Position Vector is $\vec{a}$ and Parallel to Two Vectors $\vec{b}$ and $\vec{c}$

Let P be any point on the plane whose position vector is $\vec{r}$.

Then the required equation of the plane is $\boldsymbol{[\vec{r}\vec{b}\vec{c}] = [\vec{a}\vec{b}\vec{c}]}$.

9.8.3 Equation of a Plane Passing Through Three Given Points

Let the position vectors of three points be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then $\vec{b}-\vec{a}$ and $\vec{c}-\vec{a}$ lie in the same plane. So, $\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)$ is perpendicular to the plane.

Let P be any point on the plane whose position vector is $\vec{r}$. Then also $\vec{r}-\vec{a}$ is perpendicular to $(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$.

So,

$$(\vec{r}-\vec{a})\cdot\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0$$

which implies

$$\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{u}}=[\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}\vec{\boldsymbol{c}}]$$

where

$$\vec{\boldsymbol{u}}=\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}+\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}+\vec{\boldsymbol{c}}\times\vec{\boldsymbol{a}}.$$

is the required equation of the plane.

Example 6 Find the equation of the plane through the points $A(2,-1,4)$, $B(3,4,7)$ and $C(-2,3,-1)$.

Sol. Here $\overrightarrow{AB}=(1,5,3)$ and $\overrightarrow{AC}=(-4,4,-5)$.

Let $P(x,y,z)$ be any point on the plane.

Here $\overrightarrow{AB}\times\overrightarrow{AC}=(-37,-7,24)$ is perpendicular to the plane and $\overrightarrow{AP}=(x-2,\ y+1,\ z-4)$ lies on the plane.

So, $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{AB}\times\overrightarrow{AC}$. Therefore

$$\overrightarrow{AP}\cdot\{\overrightarrow{AB}\times\overrightarrow{AC}\}=0$$

or, $(-37,-7,24)\cdot(x-2,\ y+1,\ z-4)=0$

or, $-37(x-2)-7(y+1)+24(z-4)=0.$

This is the required equation of the plane.

9.8.4 Distance of a Point from a Plane

Let the position vector of the given point be $\vec{a}$ and the equation of the plane be $\vec{r}\cdot\vec{n}=p$, where $\vec{n}$ is normal to the plane.

Then the required distance is

$$\frac{|\boldsymbol{p}-\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{n}}|}{|\vec{\boldsymbol{n}}|}.$$

9.9 SPHERE

9.9.1 General Equation of a Sphere

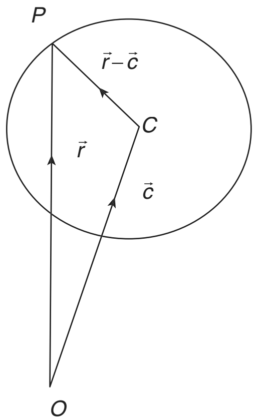

Figure 9.9

Let the radious of the sphere be a, and $\vec{c}$ be the position vector of its centre C. Also, let P be any point on the sphere whose position vector is $\vec{r}$.

It is clear from the figure that $\overrightarrow{CP} = \vec{r} - \vec{c}$.

Again $\left|\overrightarrow{CP}\right|^2 = a^2$. Therefore

$(\vec{r} - \vec{c})^2 = a^2$

i.e., $\boldsymbol{(\vec{r})^2 - 2 \cdot \vec{r} \cdot \vec{c} + (\vec{c})^2 - a^2 = 0}$

This is the required equation of the sphere.

Note:

(1) If the centre of the sphere is the origin, i.e., $\vec{c} = \vec{0}$ then the equation of the sphere becomes

$(\vec{r})^2 = a^2$.

(2) If the origin lies on the sphere then $(\vec{c})^2 = a^2$, and correspondingly the equation becomes

$(\vec{r})^2 - 2 \cdot \vec{r} \cdot \vec{c} = 0$.

9.9.2 Equation of the Sphere with Given Diameter Ends

Let $\vec{a}$ and $\vec{b}$ be the position vectors of the ends of the diameter of the sphere.

Then the equation of the sphere is given by

$$(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b})=0$$

where $\vec{r}$ is the position vector of any arbitrary point on the sphere.

WORKED-OUT EXAMPLES

Example 9.1 Given two vectors $\vec{\alpha}=3\hat{i}-\hat{j}$ and $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, express $\vec{\beta}$ in the form of $\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

[WBUT-2005]

Sol. Since $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$ then

$\vec{\beta}_1=k\vec{\alpha}$ and $\vec{\beta}_2\cdot\vec{\alpha}=0$ where k is any scalar

Now,

$\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$ then $\vec{\beta}=k\vec{\alpha}+\vec{\beta}_2$

Therefore,

$$\vec{\beta}\cdot\vec{\alpha}=k\vec{\alpha}\cdot\vec{\alpha}+\vec{\beta}_2\cdot\vec{\alpha}$$

$$\text{or, } \vec{\beta}\cdot\vec{\alpha}=k|\vec{\alpha}|^2$$

$$\text{or, } k=\frac{\vec{\beta}.\vec{\alpha}}{|\vec{\alpha}|^2}$$

$$\text{or, } k=\frac{(2\hat{i}+\hat{j}-3\hat{k})(3\hat{i}-\hat{j})}{(3)^2+1^2}=\frac{5}{10}=\frac{1}{2}$$

Therefore,

$$\vec{\beta}_1=k\vec{\alpha}=\frac{1}{2}(3\hat{i}-\hat{j})$$

and

$$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1=(2\hat{i}+\hat{j}-3\hat{k})-\frac{1}{2}(3\hat{i}-\hat{j})=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

Example 9.2 Let $\vec{\alpha},\vec{\beta},\vec{\gamma}$ be unit vectors satisfying $\vec{\alpha}\cdot\vec{\beta}=0$ and $\vec{\alpha}\cdot\vec{\gamma}=0$. If the angle between $\vec{\beta}$ and $\vec{\gamma}$ is $\frac{\pi}{6}$ then show that $\vec{\alpha}=\pm 2\,(\vec{\beta}\times\vec{\gamma})$.

Sol. Here $|\vec{\alpha}|=|\vec{\beta}|=|\vec{\gamma}|=1$

Since,

$\vec{\alpha}\cdot\vec{\beta}=0$ and $\vec{\alpha}\cdot\vec{\gamma}=0$.

Therefore $\vec{\alpha}$ is perpendicular to $\vec{\beta}$ and $\vec{\gamma}$

Therefore,

$\vec{\alpha} = t(\vec{\beta} \times \vec{\gamma})$, t being a scalar

Therefore,

$$|\vec{\alpha}|^2 = |t(\vec{\beta} \times \vec{\gamma})|^2$$

$$\text{or, } 1 = t^2 \left\{ |\vec{\beta}||\vec{\gamma}| \sin \frac{\pi}{6} \right\}^2$$

$$\text{or, } 1 = t^2 \left\{ 1 \cdot 1 \cdot \frac{1}{2} \right\}^2$$

$$\text{or, } t^2 = 2^2 \Rightarrow t = \pm 2$$

Hence

$$\vec{\alpha} = \pm 2(\vec{\beta} \times \vec{\gamma})$$

Example 9.3 If $\hat{a}, \hat{b}, \hat{c}$ are unit vectors such that $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angles which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$.

Sol.

Here,

$$\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$$

Therefore,

$$(\hat{a} \cdot \hat{c}) \cdot \hat{b} - (\hat{a} \cdot \hat{b}) \cdot \hat{c} = \frac{1}{2}\hat{b}$$

Equating coefficients of $\hat{b}$ and $\hat{c}$, we get

$$(\hat{a} \cdot \hat{c}) = \frac{1}{2} \qquad \ldots(1)$$

and

$$(\hat{a} \cdot \hat{b}) = 0 \qquad \ldots(2)$$

From (1),

$|\hat{a}||\hat{c}| \cos\theta = \frac{1}{2}$ where θ is the angle between $\hat{a}$ and $\hat{c}$

$$\text{or, } \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \text{ since } |\hat{a}| = |\hat{c}| = 1$$

From (2),

$|\hat{a}||\hat{b}|\cos\phi = 0$ where ϕ is the angle between $\hat{a}$ and $\hat{b}$

$$\text{or, } \phi = \cos^{-1} 0 = \frac{\pi}{2}$$

Therefore, the angle between $\hat{a}$ and $\hat{b}$ is $\frac{\pi}{2}$ and the angle between $\hat{a}$ and $\hat{c}$ is $\frac{\pi}{3}$.

Example 9.4 Given three vectors a, b, c, prove that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$. [WBUT-2005]

Sol. Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Now,

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - c_1b_3) + \hat{k}(b_1c_2 - c_1b_2)$$

Therefore,

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & -(b_1c_3 - c_1b_3) & (b_1c_2 - c_1b_2) \end{vmatrix}$$

$$= \hat{i}\,[a_2(b_1c_2 - c_1b_2) + a_3(b_1c_3 - c_1b_3)]$$

$$-\hat{j}\,[a_1(b_1c_2 - c_1b_2) - a_3(b_2c_3 - b_3c_2)]$$

$$+\hat{k}[-a_1(b_1c_3 - c_1b_3) - a_2(b_2c_3 - b_3c_2)]$$

$$= \hat{i}[(a_2c_2 + a_3c_3 + a_1c_1)b_1] + \hat{j}[(a_2c_2 + a_3c_3 + a_1c_1)b_2]$$

$$+\hat{k}[(a_2c_2 + a_3c_3 + a_1c_1)b_3] - \hat{i}[(a_2b_2 + a_3b_3 + a_1b_1)c_1]$$

$$-\hat{j}[(a_2b_2 + a_3b_3 + a_1b_1)c_2] - \hat{k}[(a_2b_2 + a_3b_3 + a_1b_1)c_3]$$

$$= (a_2c_2 + a_3c_3 + a_1c_1)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$-(a_2b_2 + a_3b_3 + a_1b_1)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$-\{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}).$$

$$= (\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$$

Example 9.5 Find the constant m such that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{c} = 3\hat{i} + m\hat{j} + 5\hat{k}$ are coplanar. [WBUT-2004].

Sol.

The three nonzero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if $[\vec{a}\vec{b}\vec{c}] = 0$, which implies

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$$

or, $2(10 + 3m) + 1(5 + 9) + 1(m - 6) = 0$

or, $7m + 28 = 0$

or, $m = -4$

Example 9.6 If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, show that

$[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = [\vec{a}, \vec{b}, \vec{c}]^2$. [WBUT 2006, 2009]

Sol.

$$[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]$$

$$= \{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\}(\vec{c} \times \vec{a})$$

$$= \{(\overrightarrow{p} \cdot \overrightarrow{c}) \cdot \overrightarrow{b} - (\overrightarrow{p} \cdot \overrightarrow{b}) \cdot \overrightarrow{c}\}(\overrightarrow{c} \times \overrightarrow{a}), \text{ where } \overrightarrow{p} = (\overrightarrow{a} \times \overrightarrow{b})$$

$$= (\vec{p} \cdot \vec{c}) \cdot \vec{b} \cdot (\vec{c} \times \vec{a}) - (\vec{p} \cdot \vec{b}) \cdot \vec{c} \cdot (\vec{c} \times \vec{a})$$

$$= (\vec{p} \cdot \vec{c}) \cdot [\vec{b}, \vec{c}, \vec{a}] - (\vec{p} \cdot \vec{b})[\vec{c}, \vec{c}, \vec{a}]$$

$$= -\{(\vec{a} \times \vec{b}) \cdot \vec{c}\}[\vec{b}, \vec{a}, \vec{c}] - 0, \text{ since } [\vec{c}, \vec{c}, \vec{a}] = 0$$

$$= \{\vec{c} \cdot (\vec{a} \times \vec{b})\}[\vec{a}, \vec{b}, \vec{c}]$$

$$= [\vec{c}, \vec{a}, \vec{b}][\vec{a}, \vec{b}, \vec{c}]$$

$$= [\vec{a}, \vec{b}, \vec{c}][\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, \vec{b}, \vec{c}]^2$$

PART-II (VECTOR DIFFERENTIATION AND GRADIENT, DIVERGENCE, CURL)

9.10 VECTOR FUNCTION OF A SCALAR VARIABLE

9.10.1 Definition

If for each value of a scalar variable t, there corresponds a unique vector $\vec{\mathbf{f}}$ then $\vec{\mathbf{f}}$ is called a *vector function* of the scalar variable t and is denoted by $\vec{\mathbf{f}}(t)$.

If the components of a vector function $\vec{\mathbf{f}}(t)$ along the coordinate axes be $\vec{\mathbf{f}}_1(t)$, $\vec{\mathbf{f}}_2(t)$, $\vec{\mathbf{f}}_3(t)$ then the vector function $\vec{\mathbf{f}}(t)$ is written as

$$\vec{\mathbf{f}}(t) = \vec{\mathbf{f}}_1(t)\hat{i} + \vec{\mathbf{f}}_2(t)\hat{j} + \vec{\mathbf{f}}_3(t)\hat{k}.$$

The position vector of a point P in three-dimensional space with respect to a vector origin is a function of a scalar variable t and is denoted by,

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

9.10.2 Limit and Continuity of a Vector Function

Limit of a Vector Function

A Vector function $\vec{\mathbf{f}}(t)$ is said to tend to a limit $\vec{a}$ as t tends to s, if for any pre-assigned positive number ε, there exists a small positive number δ, such that

$$\left|\vec{\mathbf{f}}(t) - \vec{a}\right| < \varepsilon, \text{ when } |t - s| < \delta$$

Limit of a vector function is denoted by

$$\lim_{t \to s} \vec{\mathbf{f}}(t) = \vec{a}$$

Continuity of a Vector Function

A vector function $\vec{\mathbf{f}}(t)$ is said to be continuous at a point s if

$$\lim_{t \to s} \vec{\mathbf{f}}(t) = \vec{\mathbf{f}}(s)$$

9.11 DIFFERENTIATION OF VECTOR FUNCTIONS

The ordinary derivative of a single-valued function $\vec{\mathbf{f}}(t)$ with respect to t is defined as

$$\frac{d\vec{\mathbf{f}}(t)}{dt} = \lim_{\delta t \to 0} \frac{\vec{\mathbf{f}}(t + \delta t) - \vec{\mathbf{f}}(t)}{\delta t}$$

provided the limit exists.

9.11.1 General Rules for Differentiation

1) $\dfrac{d(\vec{a}\pm\vec{b})}{dt}=\dfrac{d\vec{a}}{dt}\pm\dfrac{d\vec{b}}{dt}$

2) $\dfrac{d(\vec{a}\cdot\vec{b})}{dt}=\vec{a}\cdot\dfrac{d\vec{b}}{dt}+\vec{b}\cdot\dfrac{d\vec{a}}{dt}$

3) $\dfrac{d(\vec{a}\times\vec{b})}{dt}=\vec{a}\times\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}\times\vec{b}$

4) $\dfrac{d}{dt}[\vec{a},\vec{b},\vec{c}]=\left[\dfrac{d\vec{a}}{dt},\vec{b},\vec{c}\right]+\left[\vec{a},\dfrac{d\vec{b}}{dt},\vec{c}\right]+\left[\vec{a},\vec{b},\dfrac{d\vec{c}}{dt}\right]$

5) $\dfrac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\}=\dfrac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c})+\vec{a}\times\left(\dfrac{d\vec{b}}{dt}\times\vec{c}\right)+\vec{a}\times\left(\vec{b}\times\dfrac{d\vec{c}}{dt}\right)$

6) $\dfrac{d(\phi\vec{a})}{dt}=\phi\dfrac{d\vec{a}}{dt}$ where ϕ is any scalar

9.12 SCALAR AND VECTOR POINT FUNCTION

If for every position of a point in space, a physical quantity has one or more definite values assigned to it then it is said to be a *point function*. If the point function has only one value at each point then the function is called a *single-valued function.*

9.12.1 Scalar Point Function

f is said to be a **scalar point function** of $\vec{r}$ if for every value of $\vec{r}$, there corresponds a definite scalar quantity f. The scalar point function is denoted by

$f(\vec{r})$ or $f(x, y, z)$

where $\vec{r}$ is the position vector corresponding to any point $P(x, y, z)$ in the space. The scalar point function will constitute a scalar field, for example,

$f(x, y, z)=x^2+y+xyz.$

9.12.2 Vector Point Function

$\vec{\mathbf{F}}$ is said to be a **vector point function** of $\vec{r}$ if for every value of $\vec{r}$, there corresponds a definite vector quantity $\vec{\mathbf{F}}$. The vector point function is denoted by

$\vec{\mathbf{F}}(\vec{r})$ or $\vec{\mathbf{F}}(x, y, z)$

where $\vec{r}$ is the position vector corresponding to any point $P(x, y, z)$ in the space. The vector point function will constitute a vector field, for example,

$\vec{\mathbf{F}}(x, y, z)=x^2\hat{i}+yz\hat{j}+z^2\hat{k}.$

9.13 GRADIENT OF A SCALAR POINT FUNCTION

9.13.1 Definition

Let $\phi(x, y, z)$ be a scalar point function differentiable at each point in a certain region R of space. Then the gradient of ϕ, denoted by grad ϕ is defined by,

$$\textbf{grad}\ \phi = \vec{\nabla}\phi(x, y, z)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi(x, y, z)$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

9.13.2 Properties of Gradient of a Scalar Point Function

1) The necessary and sufficient condition for a scalar point function $\phi(x, y, z)$ to be a constant is

$$\vec{\nabla}\phi(x, y, z) = 0$$

2) If $\phi(x, y, z)$ and $\psi(x, y, z)$ are two scalar point functions then,

$$\vec{\nabla}\{\phi(x, y, z) \pm \psi(x, y, z)\} = \vec{\nabla}\phi(x, y, z) \pm \vec{\nabla}\psi(x, y, z)$$

3) If $\phi(x, y, z)$ and $\psi(x, y, z)$ are two scalar point functions then,

$$\vec{\nabla}\{\phi(x, y, z)\cdot\psi(x, y, z)\} = \phi(x, y, z)\vec{\nabla}\psi(x, y, z) + \psi(x, y, z)\vec{\nabla}\phi(x, y, z)$$

4) If $\phi(x, y, z)$ and $\psi(x, y, z)$ are two scalar point functions then,

$$\vec{\nabla}\left(\frac{\phi(x, y, z)}{\psi(x, y, z)}\right) = \frac{\psi(x, y, z)\vec{\nabla}\phi(x, y, z) - \phi(x, y, z)\vec{\nabla}\psi(x, y, z)}{\{\psi(x, y, z)\}^2}$$

5) If c is a constant and $\phi(x, y, z)$ is a scalar point function, then

$$\vec{\nabla}\left(c\phi(x, y, z)\right) = c\left(\vec{\nabla}\phi(x, y, z)\right)$$

Example 7 If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

i) $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

ii) $\vec{\nabla}(r^n) = nr^{n-2}\vec{r}$ [WBUT 2004]

Sol. Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, so $r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$.

i) $$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\hat{i}\frac{1}{r^2}\frac{x}{r} - \hat{j}\frac{1}{r^2}\frac{y}{r} - \hat{k}\frac{1}{r^2}\frac{z}{r}$$

$$= \frac{-1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}$$

ii) Since $r^2 = x^2 + y^2 + z^2$, we have

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\vec{\nabla}(r^n) = \hat{i}\frac{\partial}{\partial x}(r^n) + \hat{j}\frac{\partial}{\partial y}(r^n) + \hat{k}\frac{\partial}{\partial z}(r^n)$$

$$= \hat{i}\cdot nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\cdot nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\cdot nr^{n-1}\frac{\partial r}{\partial z}$$

$$= \hat{i}\cdot nr^{n-1}\frac{x}{r} + \hat{j}\cdot nr^{n-1}\frac{y}{r} + \hat{k}\cdot nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\vec{r}$$

9.14 LEVEL SURFACE

Let $f(\vec{r})$ or $f(x, y, z)$ be a scalar point function over a region in space. Then the points (x, y, z) in the region satisfying the equation $f(x, y, z) = c$ constitute a family of surfaces.This family of surfaces is called a level surface determined by f, for example, $x^3 + y^2 - 2z = 5$ is a level surface, determined by the function $f(x, y, z) = x^3 + y^2 - 2z$ for $c = 5$.

9.15 DIRECTIONAL DERIVATIVE OF A SCALAR POINT FUNCTION

9.15.1 Definition

Let $\phi(x, y, z)$ be a scalar point function possessing first-order derivatives. Then the directional derivative of $\phi(x, y, z)$ at $P(x, y, z)$ along the unit vector $\hat{a}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is given by

$$\nabla\phi \quad \vec{a} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\ (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).$$

$$= a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

9.15.2 Observations

1) **Directional derivative is the rate of change of ϕ at (x, y, z) in the direction of $\hat{a}$.**

2) **The directional derivative along along any straight line can be expressed in terms of those along the coordinate axes.**

3) **The directional derivative at $P(x, y, z)$ along the reverse direction will be**

$$\frac{d\phi}{ds} = -a_1\frac{\partial\phi}{\partial x} - a_2\frac{\partial\phi}{\partial y} - a_3\frac{\partial\phi}{\partial z}$$

Example 8 Find the directional derivative of $\phi(x, y, z) = xy^2z + x^2z$ at $(1,1,2)$ in the direction $(2\hat{i} + \hat{j} - 2\hat{k})$.

Sol. Here,

$$\vec{\nabla}\phi(x, y, z) = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= (y^2z + 2xz)\hat{i} + 2xyz\hat{j} + (xy^2 + x^2)\hat{k}$$

Now, $\vec{\nabla}\phi(1, 1, 2) = 6\hat{i} + 4\hat{j} + 2\hat{k}$.

The unit vector in the direction $(2\hat{i} + \hat{j} - 2\hat{k})$ is

$$\hat{a} = \frac{(2\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

The directional derivative is

$$\vec{\nabla}\phi(1, 1, 2)\cdot\hat{a} = (6\hat{i} + 4\hat{j} + 2\hat{k})\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = 4$$

9.15.3 Properties of Directional Derivative of a Scalar Point Function

Theorem 9.3: The directional derivative of a scalar field ϕ at a point $P(x, y, z)$ in the direction of the unit vector $\hat{a}$ is given by

$$\frac{d\phi}{ds} = \vec{\nabla}\phi\cdot\hat{a}$$

where s is the distance of the point $P(x, y, z)$ from some fixed point in the direction of $\vec{a}$.

Theorem 9.4: The directional derivatives of a scalar point function $\phi(x, y, z)$ in the directions of X, Y and Z-axes are

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \text{ and } \frac{\partial \phi}{\partial z} \text{ respectively.}$$

Theorem 9.5: Let $\hat{n}$ be a unit normal vector to the level surface $\phi(x, y, z) = c$ at a point $P(x, y, z)$, n being the distance of $P(x, y, z)$ measured from a fixed point in the direction of $\hat{n}$. Then,

$$\vec{\nabla}\phi = \left(\frac{d\phi}{dn}\right)\hat{n}$$

i.e., the direction of $\vec{\nabla}\phi$ is normal to the level surface.

Theorem 9.6: The directional derivative of a scalar field function $\phi(x, y, z)$ is maximum along the normal to the level surface $\phi(x, y, z) = c$, and the maximum value is

$$\left|\vec{\nabla}\phi\right| = \left|\left(\frac{d\phi}{dn}\right)\hat{n}\right| = \left|\frac{d\phi}{dn}\right|$$

Example 9 Find the maximum value of the directional derivative of $\phi(x, y, z) = x^2 + z^2 - y^2$ at the point $(1, 3, 2)$.

Sol. Here,

$$\vec{\nabla}\phi(x, y, z) = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= 2x\hat{i} - 2y\hat{j} + 2z\hat{k}.$$

At $(1, 3, 2)$, $\vec{\nabla}\phi(1, 3, 2) = 2\hat{i} - 6\hat{j} + 4\hat{k}$ is the direction in which the directional derivative is maximum.

The maximum value of the directional derivative is

$$\left|\vec{\nabla}\phi(1, 3, 2)\right| = \left|2\hat{i} - 6\hat{j} + 4\hat{k}\right| = \sqrt{2^2 + 6^2 + 4^2} = 2\sqrt{14}$$

9.16 TANGENT PLANE AND NORMAL TO A LEVEL SURFACE

Let $\phi(x, y, z) = c$ be the equation of a level surface. Then,

i) the equation of tangent plane at $P(x, y, z)$ is

$$(X - x)\frac{\partial \phi}{\partial x} + (Y - y)\frac{\partial \phi}{\partial y} + (Z - z)\frac{\partial \phi}{\partial z} = 0$$

ii) the equation of the normal at $P(x, y, z)$ is

$$\frac{(X-x)}{\frac{\partial \phi}{\partial x}} = \frac{(Y-y)}{\frac{\partial \phi}{\partial y}} = \frac{(Z-z)}{\frac{\partial \phi}{\partial z}}.$$

Example 10 Find the equation of the tangent plane to the surface $xyz = 4$ at the point $(1, 2, 2)$. Find also the equation of the normal line at that point.

Sol. The equation of the level surface is $\phi(x, y, z) \equiv xyz - 4 = 0$.

Now,

$$\frac{\partial \phi}{\partial x} = yz, \frac{\partial \phi}{\partial y} = zx, \frac{\partial \phi}{\partial z} = xy$$

and $\frac{\partial \phi}{\partial x}(1, 2, 2) = 4, \frac{\partial \phi}{\partial y}(1, 2, 2) = 2$ and $\frac{\partial \phi}{\partial z}(1, 2, 2) = 2.$

The equation of the tangent plane at $(1, 2, 2)$ is

$$(X-x)\frac{\partial \phi}{\partial x} + (Y-y)\frac{\partial \phi}{\partial y} + (Z-z)\frac{\partial \phi}{\partial z} = 0$$

or, $(x-1)4 + (y-2)2 + (z-2)2 = 0$

or, $4x + 2y + 2z - 12 = 0$

The equation of the normal at $(1, 2, 2)$ is

$$\frac{(X-x)}{\frac{\partial \phi}{\partial x}} = \frac{(Y-y)}{\frac{\partial \phi}{\partial y}} = \frac{(Z-z)}{\frac{\partial \phi}{\partial z}}$$

or, $\frac{(x-1)}{4} = \frac{(y-2)}{2} = \frac{(z-2)}{2}$

Example 11 Find a unit normal to the surface $x^2 - y^2 + z = 2$ at $(1, -1, 2)$

Sol. The surface is given by $\phi(x, y, z) = x^2 - y^2 + z - 2$.

Therefore,

$$\vec{\nabla}\phi(x, y, z) = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) = 2x\hat{i} - 2y\hat{j} + \hat{k}$$

and $\vec{\nabla}\phi(1, -1, 2) = 2\hat{i} + 2\hat{j} + \hat{k}$.

The unit normal to the surface is

$$\frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}.$$

Another unit normal is $-\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$ in the other side of the surface.

9.17 DIVERGENCE AND CURL OF A VECTOR POINT FUNCTION

9.17.1 Definition

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any continuously differentiable vector point function.

The **divergence of the vector point function** is defined by

$$\textbf{div}\ \vec{F} = \vec{\nabla}\ \ \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\ (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

div $\vec{F}$ can also be written in the summation form as

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \vec{F} = \sum \hat{i} \cdot \frac{\partial \vec{F}}{\partial x},$$

the summation being taken over all $\hat{i}$, $\hat{j}$ and $\hat{k}$.

The **curl of the vector point function** is defined by

$$\textbf{curl}\ \vec{F} = \vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

curl $\vec{F}$ can also be written in the summation form as

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \vec{F} = \sum \hat{i} \cdot \times \frac{\partial \vec{F}}{\partial x},$$

the summation being taken over all $\hat{i}$, $\hat{j}$ and $\hat{k}$.

9.17.2 Properties of Divergence and Curl

1) If $\vec{F}$ and $\vec{G}$ be two differentiable vector point functions then,

i) $\text{div}(\vec{F} \pm \vec{G}) = \text{div}\ \vec{F} \pm \text{div}\ \vec{G}$

ii) $\text{curl}\ (\vec{F} \pm \vec{G}) = \text{curl}\ \vec{F} \pm \text{curl}\ \vec{G}$

Proof:

(i) $\text{div}\ (\vec{F} \pm \vec{G}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(\vec{F} \pm \vec{G})$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) = \sum \hat{i} \cdot \left(\frac{\partial \vec{F}}{\partial x} \pm \frac{\partial \vec{G}}{\partial x}\right)$$

$$= \sum \hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \pm \sum \hat{i} \cdot \frac{\partial \vec{G}}{\partial x}$$

$$= \vec{\nabla} \cdot \vec{F} \pm \vec{\nabla} \cdot \vec{G} = \text{div } \vec{F} \pm \text{div } \vec{G}.$$

(ii) curl $(\vec{F} \pm \vec{G}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times (\vec{F} \pm \vec{G})$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) = \sum \hat{i} \times \left(\frac{\partial \vec{F}}{\partial x} \pm \frac{\partial \vec{G}}{\partial x} \right)$$

$$= \sum \hat{i} \times \frac{\partial \vec{F}}{\partial x} \pm \sum \hat{i} \times \frac{\partial \vec{G}}{\partial x}$$

$$= \text{curl } \vec{F} \pm \text{curl } \vec{G}$$

2) If $\vec{F}$ be vector point function and ϕ be a scalar point function. Then

i) div $(\phi\vec{F}) = \vec{\nabla}\phi \quad \vec{F} + \phi$div $\vec{F}$

ii) curl$(\phi\vec{F}) = \vec{\nabla}\phi \times \vec{F} + \phi \quad$ curl$\vec{F}$ [WBUT-2004]

Proof

(i) div $(\phi\vec{F}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)(\phi\vec{F})$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}(\phi\vec{F})$$

$$= \sum \left\{ \hat{i}\left(\frac{\partial \phi}{\partial x}\vec{F} \right) \right\} + \sum \left\{ \hat{i}\left(\phi\frac{\partial \vec{F}}{\partial x} \right) \right\}$$

$$= \left(\sum \hat{i}\frac{\partial \phi}{\partial x} \right)\vec{F} + \sum \left(\hat{i}\frac{\partial \vec{F}}{\partial x} \right)$$

$$= \vec{\nabla}\phi \cdot \vec{F} + \phi \text{ div } \vec{F}$$

(ii) curl $(\phi\vec{F}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times (\phi\vec{F})$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\phi\vec{F})$$

$$= \sum \left\{ \hat{i} \times \left(\frac{\partial \phi}{\partial x}\vec{F} \right) \right\} + \sum \left\{ \hat{i} \times \left(\phi\frac{\partial \vec{F}}{\partial x} \right) \right\}$$

$$= \sum \left\{ \left(\hat{i}\frac{\partial \phi}{\partial x} \right) \times \vec{F} \right\} + \sum \left\{ \phi\left(\hat{i} \times \frac{\partial \vec{F}}{\partial x} \right) \right\}$$

$$= \sum \left(\hat{i}\frac{\partial \phi}{\partial x} \right) \times \vec{F} + \phi \sum \left(\hat{i} \times \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \vec{\nabla}\phi \times \vec{F} + \phi \cdot \text{curl } \vec{F}$$

3) If $\vec{F}$ and $\vec{G}$ be two differentiable vector point functions, then

i) $\textbf{grad}(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{G}\times \textbf{curl } \vec{F} + \vec{F}\times \textbf{curl } \vec{G}$

ii) $\textbf{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot \textbf{curl}\vec{F} - \vec{F}\cdot \textbf{curl}\vec{G}$

iii) $\textbf{curl } (\vec{F}\times\vec{G}) = \vec{F}\cdot \textbf{div } \vec{G} - \vec{G}\cdot \textbf{div}\vec{F} + (\vec{G}\cdot\vec{\nabla})\cdot\vec{F} - (\vec{F}\cdot\vec{\nabla})\cdot\vec{G}$ [WBUT-2003].

Proof:

(i) grad $(\vec{F}\cdot\vec{G})$

$$= \vec{\nabla}(\vec{F}\cdot\vec{G}) = \sum \hat{i}\frac{\partial}{\partial x}(\vec{F}\cdot\vec{G})$$

$$= \sum \hat{i}\left(\vec{F}\frac{\partial \vec{G}}{\partial x} + \vec{G}\cdot\frac{\partial \vec{F}}{\partial x} \right) = \sum \left(\vec{F}\frac{\partial \vec{G}}{\partial x} \right)\hat{i} + \sum \left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x} \right)\hat{i}$$

$$= \left(\vec{F}\cdot\sum \hat{i}\frac{\partial}{\partial x} \right)\vec{G} + \vec{F}\times\sum\left(\hat{i}\times\frac{\partial \vec{G}}{\partial x} \right) + \left(\vec{G}\cdot\sum \hat{i}\frac{\partial}{\partial x} \right)\vec{F} + \vec{G}\times\sum\left(\hat{i}\times\frac{\partial \vec{F}}{\partial x} \right)$$

$$= (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{G}\times \text{curl } \vec{F} + \vec{F}\times \text{curl } \vec{G}$$

(ii) div $(\vec{F}\times\vec{G}) = \sum \hat{i}\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$

$$= \sum \hat{i}\cdot\left\{ \frac{\partial \vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial \vec{G}}{\partial x} \right\}$$

$$= \sum\left\{ \hat{i}\left(\frac{\partial \vec{F}}{\partial x}\times\vec{G} \right) \right\} + \sum\left\{ \hat{i}\left(\vec{F}\times\frac{\partial \vec{G}}{\partial x} \right) \right\}$$

$$= \sum\left\{ \left(\hat{i}\times\frac{\partial \vec{F}}{\partial x} \right)\cdot\vec{G} \right\} - \sum\left\{ \hat{i}\cdot\left(\frac{\partial \vec{G}}{\partial x}\times\vec{F} \right) \right\}$$

$$= \left\{ \sum\left(\hat{i}\times\frac{\partial \vec{F}}{\partial x} \right)\cdot\vec{G} \right\} - \sum\left\{ \left(\hat{i}\times\frac{\partial \vec{\mathbf{G}}}{\partial x} \right)\cdot\vec{F} \right\}$$

$$= \vec{G} \text{ curl } \vec{F} - \vec{F} \text{ curl } \vec{G}$$

(iii) curl $(\vec{F}\times\vec{G}) = \nabla\times(\vec{F}\times\vec{G})$

$$= \sum\left\{ \hat{i}\times\frac{\partial(\vec{F}\times\vec{G})}{\partial x} \right\} = \sum\left\{ \hat{i}\times\left(\frac{\partial \vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial \vec{G}}{\partial x} \right) \right\}$$

$$= \sum\left\{\hat{i}\times\left(\frac{\partial \vec{F}}{\partial x}\times\vec{G}\right)\right\}+\sum\{\hat{i}\times(\vec{F}\times\vec{F})\}$$

$$= \sum\left\{(\hat{i}\cdot\vec{G})\frac{\partial \vec{F}}{\partial x}-\left(\hat{i}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{G}\right\}+\sum\left\{\left(\hat{i}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{F}-(\hat{i}\cdot\vec{F})\frac{\partial \vec{G}}{\partial x}\right\}$$

$$= \sum\left\{(\hat{i}\cdot\vec{G})\frac{\partial \vec{F}}{\partial x}\right\}-\sum\left\{\left(\hat{i}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{G}\right\}+\sum\left\{\left(\hat{i}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{F}\right\}-\sum\left\{(\hat{i}\cdot\vec{F})\frac{\partial \vec{G}}{\partial x}\right\}$$

$$= \left\{\vec{G}\sum\hat{i}\frac{\partial}{\partial x}\right\}\vec{F}-\left\{\sum\hat{i}\frac{\partial \vec{F}}{\partial x}\right\}\vec{G}+\left\{\sum\left(\hat{i}\cdot\frac{\partial \vec{G}}{\partial x}\right)\right\}\vec{F}-\left\{\vec{F}\sum\hat{i}\frac{\partial}{\partial x}\right\}\vec{G}$$

$$= \vec{F}\cdot\text{div }\vec{G}-\vec{G}\cdot\text{div }\vec{F}+(\vec{G}\cdot\vec{\nabla})\cdot\vec{F}-(\vec{F}\cdot\vec{\nabla})\cdot\vec{G}$$

9.17.3 Irrotational and Solenoidal Vectors

A vector point function $\vec{F}$ is said to be **irrotational** if curl $\vec{F}=0$ and **solenoidal** if div $\vec{F}=0$.

Example 12 If the vectors $\vec{A}$ and $\vec{B}$ are irrotational then show that the vectors $\vec{A}\times\vec{B}$ is solenoidal. [WBUT 2004, 2006]

Sol. If the vector functions $\vec{A}$ and $\vec{B}$ are irrotational then

curl $\vec{A}=0$ and curl $\vec{B}=0$

Now,

div $(\vec{A}\times\vec{B})=\vec{\nabla}\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot\text{curl }\vec{A}-\vec{A}\cdot\text{curl }\vec{B}=\vec{B}\cdot 0-\vec{A}\cdot 0=0$

Since, div $(\vec{A}\times\vec{B})=0$ therefore $(\vec{A}\times\vec{B})$ is solenoidal.

9.17.4 Some Results on Second-Order Differential Operators

Laplacian Operator

The Laplacian operator is defined as

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

If $\varphi(x, y, z)$ be a scalar point function then

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$ is a scalar quantity.

If $\vec{v}(x, y, z)$ be a vector point function then

$$\nabla^2\vec{v}=\frac{\partial^2\vec{v}}{\partial x^2}+\frac{\partial^2\vec{v}}{\partial y^2}+\frac{\partial^2\vec{v}}{\partial z^2}$$ is a vector quantity.

The equation $\nabla^2\varphi = 0$ is known as the **Laplace Equation**.

i) Let $\varphi(x, y, z)$ be a scalar point function then, div $(\text{grad } \phi) = \vec{\nabla} \ \vec{\nabla}\phi = \nabla^2\phi$

Proof:

$$\text{div}(\text{grad}\phi) = \text{div}\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \right)$$

$$= \sum \hat{i}\frac{\partial}{\partial x}\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} \right)\phi = \nabla^2\phi$$

ii) Let $\varphi(x, y, z)$ be a scalar point function then,
curl$(\text{grad } \phi) = \vec{\nabla} \times \vec{\nabla}\phi = \vec{0}$, [WBUT-2005]

Proof:

$$\text{curl}(\text{grad } \phi)$$

$$= \text{curl}\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \right)$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \right)$$

$$= \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y} \right)\hat{i} + \left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z} \right)\hat{j} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x} \right)\hat{k}$$

$$= \vec{0}$$

iii) Let $\vec{F}(x, y, z) = \vec{F}_1\hat{i} + \vec{F}_2\hat{j} + \vec{F}_3\hat{k}$ be a vector point function then,
div$(\text{curl } \vec{F}) = \vec{\nabla}(\vec{\nabla} \times \vec{F}) = 0$

Proof:

$$\text{div}(\text{curl } \vec{F})$$

$$= \text{div curl } \{\vec{F}_1\hat{i} + \vec{F}_2\hat{j} + \vec{F}_3\hat{k}\}$$

$$= \text{div}\left\{ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right)\hat{k} \right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right)$$

$$= 0$$

iv) Let $\vec{F}(x, y, z) = \vec{F}_1\hat{i} + \vec{F}_2\hat{j} + \vec{F}_3\hat{k}$ be a vector point function then,
$\text{curl}\left(\text{curl}\ \vec{F}\right) = \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\vec{F}) - \nabla^2\vec{F}$

Proof:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$$

$$= \sum \hat{i}\times\frac{\partial}{\partial x}\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x} + \hat{j}\times\frac{\partial\vec{F}}{\partial y} + \hat{k}\times\frac{\partial\vec{F}}{\partial z}\right)$$

$$= \sum \hat{i}\times\left(\hat{i}\times\frac{\partial^2\vec{F}}{\partial x^2} + \hat{j}\times\frac{\partial^2\vec{F}}{\partial x\partial y} + \hat{k}\times\frac{\partial^2\vec{F}}{\partial x\partial z}\right)$$

$$= \sum\left[\left(\hat{i}\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right)\cdot\hat{i} - \hat{i}\cdot\hat{i}\left(\frac{\partial^2\vec{F}}{\partial x^2}\right) + \left(\left(\hat{i}\ \frac{\partial^2\vec{F}}{\partial x\partial y}\right)\hat{j} - (\hat{i}\cdot\hat{j})\frac{\partial^2\vec{F}}{\partial x\partial y}\right)\right.$$

$$\left. + \left(\left(\hat{i}\ \frac{\partial^2\vec{F}}{\partial x\partial z}\right)\hat{k} - (\hat{i}\cdot\hat{k})\frac{\partial^2\vec{F}}{\partial x\partial z}\right)\right]$$

Using the formula $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$ and since $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$, we have from above

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$$

$$= \sum\left(\hat{i}\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right)\cdot\hat{i} + \left(\hat{i}\ \frac{\partial^2\vec{F}}{\partial x\partial y}\right)\hat{j} + \left(\hat{i}\ \frac{\partial^2\vec{F}}{\partial x\partial z}\right)\hat{k} - \sum\frac{\partial^2\vec{F}}{\partial x^2}$$

$$= \sum\hat{i}\ \frac{\partial}{\partial x}\cdot\left(\hat{i}\cdot\frac{\partial\vec{F}}{\partial x} + \hat{j}\ \frac{\partial\vec{F}}{\partial y} + \hat{k}\ \frac{\partial\vec{F}}{\partial z}\right) - \nabla^2\vec{F}$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}$$

WORKED-OUT EXAMPLES

Example 9.7 If $\phi = xy + yz + zx$ and $\vec{A} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$ then find a) $\vec{A}\cdot\vec{\nabla}\phi$ b) $\phi\cdot(\vec{\nabla}\cdot\vec{A})$ c) $\vec{\nabla}\phi\ \times\vec{A}$ at the point $(3, -1, 2)$

Sol.

Here,

$\phi = xy + yz + zx$

Now,

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy + yz + zx)$$

$$= (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 y\hat{i} + y^2 z\hat{j} + z^2 x\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y^2 z) + \frac{\partial}{\partial z}(z^2 x)$$

$$= 2(xy + yz + zx)$$

At the point $(3, -1, 2)$

$\vec{\nabla}\phi\ (3, -1, 2) = \hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{A}(3, -1, 2) = -9\hat{i} + 2\hat{j} + 12\hat{k}$

Therefore,

a) $\vec{A}\cdot\vec{\nabla}\phi = (-9\hat{i} + 2\hat{j} + 12\hat{k})\cdot(\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$$= -9 + 10 + 24 = 25$$

b) $\phi\cdot(\vec{\nabla}\cdot\vec{A}) = 2(xy + yz + zx)\cdot(xy + yz + zx) = 2(xy + yz + zx)^2$

Therefore,

$$\phi\cdot(\vec{\nabla}\cdot\vec{A})(3, -1, 2) = 2(-3 - 2 + 6)^2 = 2$$

c) $\vec{\nabla}\phi \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 5 & 2 \\ -9 & 2 & 12 \end{vmatrix}$

$$= \hat{i}(60 - 4) - \hat{j}(12 + 18) + \hat{k}(2 + 45)$$

$$= 56\hat{i} - 30\hat{j} + 47\hat{k}$$

Example 9.8 If $\vec{A} = 2x^2\hat{i} - 2yz\hat{j} + xz^2\hat{k}$ and $f = 2z - x^3 y$ find a) $\vec{A}\cdot\text{grad } f$ b) $\vec{A}\times\text{grad } f$ at $(1, -1, 1)$

Sol.

Here,

$$\text{grad } f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2z - x^3 y)$$

$$= -3x^2 y\hat{i} - x^3\hat{j} + 2\hat{k}$$

Therefore,

a) $\vec{A}\cdot \text{grad}\, f = (2x^2\hat{i} - 2yz\hat{j} + xz^2\hat{k})(-3x^2y\hat{i} - x^3\hat{j} + 2\hat{k})$

$$= -6x^4z + 2x^3yz + 2xz^2$$

At $(1,-1,1)$

$\vec{A}\cdot \text{grad}\ f = \mathbf{6}$

b) $\vec{A}\cdot\times\text{grad}\, f = \begin{vmatrix} i & j & k \\ 2x^2 & -2yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix}$

$$= \hat{i}(-4yz + x^4z^2) - \hat{j}(4x^2 + 3x^3yz^2) + \hat{k}(-2x^5 - 6x^2y^2z)$$

At $(1,-1,1)$

$\vec{A}\cdot\times\text{grad}\ f = 5\hat{i} - \hat{j} - 8\hat{k}$

Example 9.9 Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$

[WBUT-2001, 2009]

Sol.

Here,

$$\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)$$

$$+\hat{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

Therefore,

$$\text{div}\ \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\{(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\}$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

and

$$\text{curl}\ \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\{(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$= (-3x + 3x)\hat{i} + (-3y + 3y)\hat{j} + (-3z + 3z)\hat{k} = 0$$

Example 9.10 Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $(2, 1, 3)$ in the direction of the vector $\hat{i} - 2\hat{k}$. [WBUT-2001]

Sol. Here,

$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$

Therefore,

$$\vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)(2x^2 + 3y^2 + z^2) = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$$

At the point $(2, 1, 3)$

$$\vec{\nabla} f(2, 1, 3) = 8i + 6j + 6k$$

If $\hat{a}$ is the unit vector in the direction of $\hat{i} - 2\hat{k}$, then

$$\hat{a} = \frac{\hat{i} - 2\hat{k}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}$$

Therefore, the required directional derivative is

$$\vec{\nabla} f(2, 1, 3) \cdot \hat{a} = (8i + 6j + 6k)\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k} \right) = \frac{-4}{\sqrt{5}}$$

Example 9.11 Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the scalar function ϕ such that $\vec{A} = \vec{\nabla}\phi$. [WBUT-2002, 2004]

Sol.

Here,

$$\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z) \right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3) \right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3) \right]$$

$$= \hat{i}(-1 + 1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x) = 0$$

Therefore, $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational.

Since

$$\vec{A} = \vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

therefore,

$$\frac{\partial\phi}{\partial x} = (6xy + z^3),\ \frac{\partial\phi}{\partial y} = (3x^2 - z),\ \frac{\partial\phi}{\partial z} = (3xz^2 - y)$$

Now,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

or, $d\phi = (6xy + z^3)\,dx + (3x^2 - z)\,dy + (3xz^2 - y)\,dz$

or, $d\phi = (6xydx + 3x^2dy) + (z^3dx + 3xz^2dz) - (xdy + ydz)$

or, $d\phi = 3[yd(x^2) + x^2dy] + [z^3dx + xd(z^3)] - (xdy + ydz)$

or, $d\phi = 3d(x^2y) + d(xz^3) - d(yz)$

Integrating, we get

$\phi(x, y, z) = 3x^2y + xz^3 - yz + c$ where c is arbitrary constant.

Example 9.12 Show that curl grad $f = 0$ where $f(x, y, z) = x^2y + 2xy + z^2$.

[WBUT-2003]

Sol. Here,

$f(x, y, z) = x^2y + 2xy + z^2$

Now,

$$\text{grad } f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y + 2xy + z^2)$$

$$= (2xy + 2)\hat{i} + (x^2 + 2x)\hat{j} + 2z\hat{k}$$

Therefore,

$$\text{curl grad } f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy+2) & (x^2+2x) & 2z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2x + 2 - 2x - 2) = 0$$

Example 9.13 In what direction from the point $(1, 1, -1)$ is the directional derivative of $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$ a maximum? Obtain the magnitude of the directional derivative. [WBUT-2003, 2006, 2007]

Sol. Directional derivative of ϕ is maximum in the direction of $\vec{\nabla}\phi$.

Here,

$$\phi(x,y,z) = x^2 - 2y^2 + 4z^2$$

Therefore,

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 - 2y^2 + 4z^2)$$

$$= 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

At $(1, 1, -1)$

$$\vec{\nabla}\phi \cdot (1, 1, -1) = 2\hat{i} - 4\hat{j} - 8\hat{k}$$

Therefore, the directional derivative is maximum along $2\hat{i} - 4\hat{j} - 8\hat{k}$.

The magnitude of the directional derivative is

$$\left|\vec{\nabla}\phi\right| = \sqrt{2^2 + (-4)^2 + (-8)^2} = 2\sqrt{21}$$

Example 9.14 In what direction from the point (1, 2, 3) is the directional derivative of $f = x^2 - y^2 + 2z^2$ a maximum? Also find the value of this maximum directional derivative. [WBUT-2004]

Sol. Directional derivative of f is maximum in the direction of $\vec{\nabla}f$.

Here,

$$f(x, y, z) = x^2 - y^2 + 2z^2$$

Therefore,

$$\vec{\nabla}f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 - y^2 + 2z^2)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

At $(1, 2, 3)$

$$\vec{\nabla}\phi \cdot (1, 2, 3) = 2i - 4j + 12k$$

Therefore, the directional derivative is maximum along $2\hat{i} - 4\hat{j} + 12\hat{k}$.

The magnitude of the directional derivative is

$$\left|\vec{\nabla}f\right| = \sqrt{2^2 + (-4)^2 + (12)^2} = \sqrt{162}$$

Example 9.15 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that grad $f(r) \times \vec{r} = \theta$ where θ is the null vector. [WBUT-2005]

Sol. Here, $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Now,

$$\text{grad } f(r) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Since f is a function of r

Therefore,

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}$$

$$= f'(r)\frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}} \cdot 2x$$

$$= \frac{xf'(r)}{r}$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y} = \frac{yf'(r)}{r}$$

and

$$\frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z} = \frac{zf'(r)}{r}$$

Therefore,

$$\text{grad } f(r) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{xf'(r)}{r}\hat{i} + \frac{yf'(r)}{r}\hat{j} + \frac{zf'(r)}{r}\hat{k}$$

$$= \frac{f'(r)}{r}(xi + yj + zk)$$

$$= \frac{f'(r)}{r}\vec{r}$$

Now,

$$\text{grad } f(r) \times \vec{r} = \left(\frac{f'(r)}{r}\vec{r}\right) \times \vec{r}$$

$$= \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \theta$$

Example 9.16 If $\vec{v} = \vec{w} \times \vec{r}$, prove that $\vec{w} = \frac{1}{2}\text{curl } \vec{v}$ where $\vec{w}$ is a constant vector.

Sol. Let,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$

Therefore,

$\text{curl } \vec{v} = \text{curl } \vec{w} \times \vec{r}$

$$= \vec{\nabla} \times \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times (w_2 z - w_3 y)\hat{i} + (w_3 x - w_1 z)\hat{j} + (w_1 y - w_2 x)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (w_2 z - w_3 y) & (w_3 x - w_1 z) & (w_1 y - w_2 x) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(w_3 x - w_1 z) - \frac{\partial}{\partial z}(w_3 x - w_1 z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_2 z - w_3 y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y)\right]$$

$$= \hat{i}(w_1 + w_1) + \hat{j}(w_2 + w_2) + \hat{k}(w_3 + w_3)$$

$$= 2[w_1\hat{i} + w_2\hat{j} + w_3\hat{k}] = 2\vec{w}$$

Therefore,

$$\vec{w} = \frac{1}{2}\text{curl } \vec{v}$$

Example 9.17 Find a unit normal to the surface $x^2 y + 2xz = 4$ at the point $(2, -2, 3)$.

Sol. Let

$$\phi(x, y, z) = x^2 y + 2xz - 4$$

Now,

$$\vec{\nabla}\phi(x, y, z) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 y + 2xz - 4)$$

$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

At the point $(2, -2, 3)$, the normal to the given surface is $-2\hat{i} + 4\hat{j} + 4\hat{k}$

The unit normal to the surface is

$$\frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{(-2)^2 + 4^2 + 4^2}} = \pm\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

Example 9.18 Determine the constant a so that the vector $\vec{v} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

Sol. Since the vector $\vec{v}$ is solenoidal $\text{div } \vec{v} = 0$

$$\text{or,}\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\{(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}\}=0$$

$$\text{or,}\ \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$$

$$\text{or,}\ 1+1+a=0$$

$$\text{or,}\ a=-2$$

Example 9.19 A particle moves on a curve $x=2t^2$, $y=t^2-4t$ and $z=3t-5$ where t denotes time. Find the components of velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$. [WBUT-2002, 2009]

Sol. Let $\vec{r}$ be the position vector of any point on the given curve. Then

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

Therefore, velocity of the particle is

$$\vec{v}=\frac{d\vec{r}}{dt}=4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$$

At $t=1$, the velocity is

$$[\vec{v}]_{t=1}=4\hat{i}-2\hat{j}+3\hat{k}$$

The acceleration of the particle is

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}=4\hat{i}+2\hat{j}$$

At $t=1$, the acceleration is

$$\vec{a}]_{t=1}=4\hat{i}+2\hat{j}$$

Let

$$\vec{\alpha}=\hat{i}-3\hat{j}+2\hat{k}$$

Therefore, the component of velocity along $\vec{\alpha}$ is

$$\frac{\vec{v}\cdot\vec{\alpha}}{|\vec{\alpha}|}=\frac{(4\hat{i}-2\hat{j}+3\hat{k})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{14}}=\frac{8\sqrt{14}}{7}$$

Therefore, the component of acceleration along $\vec{\alpha}$ is

$$\frac{\vec{a}\cdot\vec{\alpha}}{|\vec{\alpha}|}=\frac{(4\hat{i}+2\hat{j})(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{14}}=\frac{-\sqrt{14}}{7}$$

Example 9.20 Evaluate $[\vec{r},\vec{r}',\vec{r}'']$ where $\vec{r}=a\cos u\hat{i}+a\sin u\hat{j}+bu\hat{k}$.

[WBUT-2008]

Sol.

Here

$$\vec{r}=a\cos u\hat{i}+a\sin u\hat{j}+bu\hat{k}.$$

$$\vec{r}' = \frac{d\vec{r}}{du} = -a\sin u\hat{i} + a\cos u\hat{j} + b\hat{k}$$

$$\vec{r}'' = \frac{d^2\vec{r}}{du^2} = -a\cos u\hat{i} - a\sin u\hat{j}$$

Therefore,

$$[\vec{r}, \vec{r}', \vec{r}''] = \begin{vmatrix} a\cos u & a\sin u & bu \\ -a\sin u & a\cos u & b \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

$$= -a\cos u\{ba\sin u - bua\cos u\} + a\sin u\{ba\cos u + bua\sin u\}$$

$$= -ba^2\cos u\sin u + ba^2u\cos^2 u + ba^2\cos u\sin u + ba^2u\sin^2 u$$

$$= ba^2u(\cos^2 u + \sin^2 u) = ba^2u$$

Example 9.21 Find the angle between the surfaces $x^3 + y^3 + z^3 - 3xyz = 5$ and $x^2y + y^2z + z^2x - 5xyz = 8$ at the point (1,0,1). [WBUT-2008]

Sol.

Let

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz - 5$$

$$\phi(x, y, z) = x^2y + y^2z + z^2x - 5xyz - 8$$

Now,

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= 3(x^2 - yz)\hat{i} + 3(y^2 - xz)\hat{j} + 3(z^2 - xy)\hat{k}$$

$$\vec{\nabla}f(1, 0, 1) = 3\hat{i} - 3\hat{j} + 3\hat{k}$$

$$\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= (2xy + z^2 - 5yz)\hat{i} + (x^2 + 2yz - 5xz)\hat{j} + (y^2 + 2xz - 5xy)\hat{k}$$

$$\vec{\nabla}\phi(1, 0, 1) = \hat{i} - 4\hat{j} + 2\hat{k}$$

Let θ be the angle between the surfaces. Then

$$\theta = \cos^{-1}\frac{\vec{\nabla}f \cdot \vec{\nabla}\phi}{|\vec{\nabla}f||\vec{\nabla}\phi|}$$

$$= \cos^{-1}\frac{(3\hat{i} - 3\hat{j} + 3\hat{k})(\hat{i} - 4\hat{j} + 2\hat{k})}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 4^2 + 2^2}}$$

$$= \cos^{-1}\frac{21}{\sqrt{27}\sqrt{21}} = \cos^{-1}\frac{\sqrt{7}}{3}$$

Example 9.22 Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point $(1, 2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$ [WBUT-2008]

Sol.

Here,

$$f(x, y, z) = x^2yz + 4xz^2$$

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= (2xyz + 4z^2)\,\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

Therefore,

$$\vec{\nabla}f(1.2, -1) = -\hat{j} - 6\hat{k}$$

Therefore, the directional derivative in the direction $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$\vec{\nabla}f(1.2, -1)\frac{2i - j - 2k}{|2i - j - 2k|} = (-j - 6k)\frac{1}{3}(2i - j - 2k) = \frac{13}{3}$$

Example 9.23 Find the equation of the tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.

Sol. Here,

$$\phi(x, y, z) = 2xz^2 - 3xy - 4x - 7$$

Therefore,

$$\frac{\partial \phi}{\partial x} = 2z^2 - 3y - 4,\ \frac{\partial \phi}{\partial y} = -3x,\ \frac{\partial \phi}{\partial z} = 4xz$$

At the point $(1, -1, 2)$.

$$\frac{\partial \phi}{\partial x}(1, -1, 2) = 7,\ \frac{\partial \phi}{\partial y}(1, -1, 2) = -3,\ \frac{\partial \phi}{\partial z}(1, -1, 2) = 8$$

The equation of the tangent plane at the point $(1, -1, 2)$ is given by

$$\frac{\partial \phi}{\partial x}(x - 1) + \frac{\partial \phi}{\partial y}(y + 1) + \frac{\partial \phi}{\partial z}(z - 2) = 0$$

or, $7(x - 1) - 3(y + 1) + 8(z - 2) = 0$

or, $7x - 3y + 8z = 26$

The equation of the normal to the surface at $(1, -1, 2)$ is given by

$$\frac{(x - 1)}{\frac{\partial \phi}{\partial x}} = \frac{(y + 1)}{\frac{\partial \phi}{\partial y}} = \frac{(z - 2)}{\frac{\partial \phi}{\partial z}}$$

or, $\dfrac{(x - 1)}{7} = \dfrac{(y + 1)}{-3} = \dfrac{(z - 2)}{8}$

Example 9.24 Prove $\frac{d}{dt}\left[\vec{r}, \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}\right] = \left[\vec{r}, \frac{d\vec{r}}{dt}, \frac{d^3\vec{r}}{dt^3}\right]$

Sol. Now,

$$\frac{d}{dt}\left[\vec{r}, \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}\right]$$

$$= \left[\frac{d\vec{r}}{dt}, \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}, \frac{d^2\vec{r}}{dt^2}, \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}, \frac{d\vec{r}}{dt}, \frac{d^3\vec{r}}{dt^3}\right]$$

$$= \left[\vec{r}, \frac{d\vec{r}}{dt}, \frac{d^3\vec{r}}{dt^3}\right] \text{since} \left[\frac{d\vec{r}}{dt}, \frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}\right] = 0, \left[\vec{r}, \frac{d^2\vec{r}}{dt^2}, \frac{d^2\vec{r}}{dt^2}\right] = 0$$

Example 9.25 Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

Sol. Since, $(1, -1, 2)$ lies on the surface $ax^2 - byz = (a+2)x$.

Therefore,

$a + 2b = a + 2 \Rightarrow b = 1$

Let the given surfaces are,

$f(x, y, z) = ax^2 - byz - (a+2)x$

and

$\phi(x, y, z) = 4x^2y + z^3 - 4$

Therefore,

$$\vec{\nabla} f(x, y, z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\{ax^2 - byz - (a+2)x\}$$

$$= \{2ax - (a+2)\}\hat{i} - z\hat{j} - y\hat{k}$$

$$\vec{\nabla}\phi(x, y, z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4)$$

$$= 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

Now,

$\vec{\nabla} f(1, -1, 2) = (a-2)\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{\nabla}\phi(1, -1, 2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$

Since the surfaces are orthogonal

$\vec{\nabla} f(1, -1, 2)\vec{\nabla}\phi(1, -1, 2) = 0$

or, $\{(a-2)\hat{i} - 2\hat{j} + \hat{k}\}\{-8\hat{i} + 4\hat{j} + 12\hat{k}\} = 0$

or, $-8(a-2) - 8 + 12 = 0$

or, $a = \frac{5}{2}$

Therefore, $a = \frac{5}{2}$ and $b = 1$.

Example 9.26 Show that $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ is irrotational. Find a scalar function ϕ such that $\vec{F} = \vec{\nabla}\phi$.

Sol.

Here,

$\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

Now,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y\sin z - \sin x) & (x\sin z + 2yz) & (xy\cos z + y^2) \end{vmatrix}$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - (y\cos z - y\cos z)\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Therefore, $\vec{F}$ is irrotational.

Since,

$\vec{F} = \vec{\nabla}\phi.$

or, $(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

Therefore,

$\frac{\partial\phi}{\partial x} = (y\sin z - \sin x), \frac{\partial\phi}{\partial y} = (x\sin z + 2yz), \frac{\partial\phi}{\partial z} = (xy\cos z + y^2)$

Now,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$$= (6xydx + 3x^2dy) + (z^3dx + 3xz^2dz) - (zdy + ydz)$$

$$= 3d(x^2y) + d(xz^3) - d(yz)$$

Integrating, we get,

$\phi(x, y, z) = 3x^2y + xz^3 - yz + c$

where c is an arbitrary constant.

PART-III (VECTOR INTEGRATION)

9.18 GREEN'S THEOREM IN A PLANE

9.18.1 Cartesian Form

Let us consider two continuous functions $M(x, y)$ and $N(x, y)$ of x and y possessing continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R on the two-dimensional xy plane bounded by a closed curve C.

Then, Green's theorem states that

$$\oint_C \{M(x, y)\, dx + N(x, y)\, dy\} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where the line integral along the curve C is taken in the anticlockwise direction.

9.18.2 Vector Form

Let $\vec{F} = M(x, y)\,\hat{i} + N(x, y)\,\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$ where M and N have continuous partial derivatives in a region R on the xy plane bounded by a closed curve C.

Then,Green's theorem states that

$$\oint_C \vec{F} d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dxdy$$

Note: Using Green's theorem, we are able to transform a double integral over a closed region into a line integral along the boundary of the region and vice-versa.

Example 13 Verify Green's theorem in the plane for $\oint_C \{(xy + y^2)\, dx + x^2 dy\}$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

[WBUT 2001, 2003]

Sol. We have Green's theorem as

$$\oint_C \{M(x, y)\, dx + N(x, y)\, dy\} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

So, for the given problem

$M(x, y) = (xy + y^2)$ and $N(x, y) = x^2$.

Then

$\frac{\partial M}{\partial y} = x + 2y$ and $\frac{\partial N}{\partial x} = 2x$.

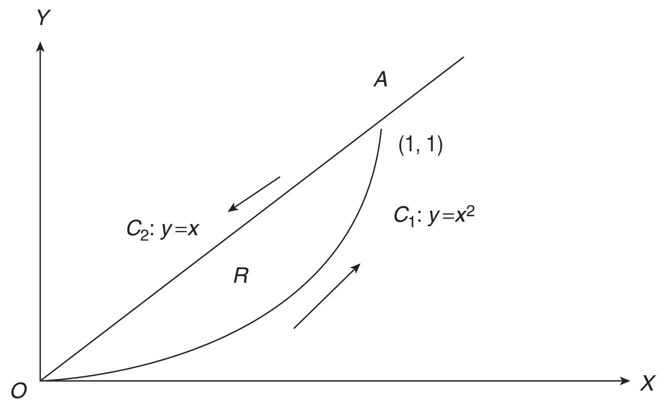

Figure 9.10

From **Fig. 9.10**, it is clear that

$$\oint_C (Mdx + Ndy) = \oint_{C_1} (Mdx + Ndy) + \oint_{C_2} (Mdx + Ndy)$$

where $C_1 : y = x^2$ and $C_2 : y = x$. So,

$$\oint_C (Mdx + Ndy) = \oint_{C_1} \{(xy + y^2)dx + x^2 dy\} + \oint_{C_2} \{(xy + y^2)dx + x^2 dy\}$$

Now, on the curve $C_1 : y = x^2$, $dy = 2x \cdot dx$ and on the curve $C_2 : y = x$, $dy = dx$.

Then from above

$$\oint_C (Mdx + Ndy)$$

$$= \int_0^1 \{(x \cdot x^2 + x^4)\, dx + x^2 \cdot 2xdx\} + \int_1^0 \{(x \cdot x + x^2)\, dx + x^2 dx\}$$

$$= \int_0^1 (3x^3 + x^4)\, dx + \int_1^0 3x^2 dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 + [x^3]_1^0 = -\frac{1}{20}$$

Now, we find the double integral

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{y=x^2}^{y=x} \{2x - (x + 2y)\}\, dxdy$$

$$= \int_0^1 \int_{y=x^2}^{y=x} (x-2y)\,dydx = \int_0^1 [xy - y^2]_{y=x^2}^{y=x}\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx = \frac{-1}{20}$$

In both the cases, we have the same value for the integrals. Hence the Green's theorem is verified.

Example 14 Evaluate using Green's theorem

$$\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y dy\}$$

where C is the circle $x^2 + y^2 = 1$. [WBUT 2004]

Sol. Let $M = (\cos x \sin y - xy)$ and $N = \sin x \cos y$.

Now by Green's theorem, we have

$$\oint_C \{M(x, y)\, dx + N(x, y)\, dy\} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Now,

$$\frac{\partial M}{\partial y} = \cos x \cos y - x \text{ and } \frac{\partial N}{\partial x} = \cos x \cos y$$

So,

$$\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y dy\}$$

$$= \iint_R (\cos x \cos y - \cos x \cos y + x)\, dxdy$$

$$= \iint_R x dx dy \qquad \ldots(1)$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

Then (1) becomes

$$\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y dy\}$$

$$= \iint_R x dx dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\cos\theta \cdot r d\theta dr$$

$$= \left(\int_{\theta=0}^{2\pi} \cos\theta d\theta\right)\left(\int_{r=0}^{1} r^2\right)$$

$$= 0 \times \frac{1}{3} = 0$$

9.19 GAUSS' DIVERGENCE THEOREM

9.19.1 Cartesian Form

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$. Then Gauss' divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_C (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

where V is the volume enclosed by the closed surface S.

9.19.2 Vector Form

Let $\vec{F}$ be a vector point function possessing continuous first-order partial derivatives in the volume V bounded by a closed surface S. Then Gauss' divergence theorem states that

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} \cdot dS$$

where $\hat{n}$ is the outward drawn unit normal vector to the surface S.

Example 15 Evaluate the volume integral $\iiint_V \vec{\nabla} \cdot \vec{F} dV$

where $\vec{F} = (x^2 - z^2)\hat{i} + 2xy\hat{j} + (y^2 + z)\hat{k}$ bounded by the planes $x = y = z = 0$ and $x = y = z = 1$.

Sol. Here,

$$\vec{F} = (x^2 - z^2)\hat{i} + 2xy\hat{j} + (y^2 + z)\hat{k}$$

and

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - z^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(y^2 + z)$$

$$= 2x + 2x + 1 = 4x + 1$$

Therefore,

$$\iiint_V \nabla \cdot F dV = \int_0^1 \int_0^1 \int_0^1 (4x + 1)\, dxdydz$$

$$= \int_0^1 \int_0^1 [2x^2 + x]_0^1 dydz$$

$$= \int_0^1 \int_0^1 3dydz = 3$$

Example 16 Verify the divergence theorem for the vector function $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over a cube bounded by $x = 0, x = 1;$ $y = 0, y = 1;$ $z = 0, z = 1.$ [WBUT2002]

Sol.

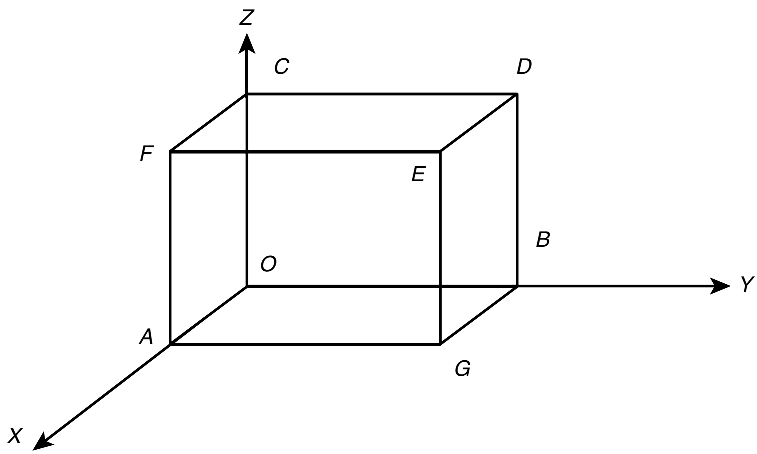

Figure 9.13

Here in **Fig. 9.13,**

$x = 0$ and $x = 1$ are the equations of the planes $OBDC$ and $AGEF$.

$y = 0$ and $y = 1$ are the equations of the planes $OAFC$ and $BGED$.

$z = 0$ and $z = 1$ are the equations of the planes $OAGB$ and $CFED$.

From the divergence theorem,

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} \cdot dS$$

Here**,**

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

and

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

Then

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \int_0^1\int_0^1\int_0^1 (4z - y)\, dxdydz$$

$$= \int_0^1\int_0^1 [2z^2 - yz]_0^1 \, dxdy = \int_0^1\int_0^1 (2-y)\, dxdy$$

$$= \int_0^1\int_0^1 dx \cdot \int_0^1 (2-y)\, dy = \frac{3}{2}.$$

Again, let S_1 be side $FEGA$, S_2 be the side $BDCO$, S_3 be side $BDEG$, S_4 be the side $OAFC$, S_5 be side $DCFE$, S_6 be the side $BGAO$. Then

$$\iint_S \vec{F}\cdot\hat{n}dS = \iint_{S_1} \vec{F}\cdot\hat{n}dS + \iint_{S_2} \vec{F}\cdot\hat{n}dS + \iint_{S_3} \vec{F}\cdot\hat{n}dS + \iint_{S_4} \vec{F}\cdot\hat{n}dS + \iint_{S_5} \vec{F}\cdot\hat{n}dS + \iint_{S_6} \vec{F}\cdot\hat{n}dS$$

On the surface S_1, $x=1$ and the normal $\hat{n}=\hat{i}$, so

$$\iint_{S_1} \vec{F}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{y=0}^{1} (4z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}dydz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} 4zdydz = \int_{y=0}^{1} [2z^2]_0^1 \, dy = \int_{y=0}^{1} 2dy = 2.$$

On the surface S_2, $x=0$ and the normal $\hat{n}=-\hat{i}$, so

$$\iint_{S_2} \vec{F}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{y=0}^{1} (0\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\, dydz = 0$$

On the surface S_3, $y=1$ and the normal $\hat{n}=\hat{j}$, so

$$\iint_{S_3} \vec{F}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{x=0}^{1} (4xz\hat{i} - \hat{j} + z\hat{k})\cdot(\hat{j})\, dxdz$$

$$= \int_{z=0}^{1}\int_{x=0}^{1} (-1)\, dxdz = -1$$

On the surface S_4, $y=0$ and the normal $\hat{n}=-\hat{j}$, so

$$\iint_{S_4} \vec{F}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{x=0}^{1} (4xz\hat{i} - 0\cdot\hat{j} + z\hat{k})\cdot(-\hat{j})\, dxdz = 0$$

On the surface S_5, $z=1$ and the normal $\hat{n}=\hat{k}$, so

$$\iint_{S_5} \vec{F}\cdot\hat{n}dS = \int_{y=0}^{1}\int_{x=0}^{1} (4x\hat{i} - y^2\cdot\hat{j} + y\hat{k})\cdot(\hat{k})\, dxdy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} ydxdy = \frac{1}{2}$$

On the surface S_6, $z=0$ and the normal $\hat{n}=-\hat{k}$, so

$$\iint_{S_6} \vec{F}\cdot\hat{n}dS = \int_{y=0}^{1}\int_{x=0}^{1} (4x\hat{i} - y^2\cdot\hat{j} + 0\hat{k})\cdot(-\hat{k})\,dxdy = 0$$

Therefore,

$$\iint_S \vec{F}\cdot\hat{n}dS = 2-1+\frac{1}{2}=\frac{3}{2}$$

So, we have

$$\iiint_V \vec{\nabla}\cdot\vec{F}dV = \iint_S \vec{F}\cdot\hat{n}dS$$

Hence the theorem is verified.

9.20 STOKE'S THEOREM

Let $\vec{F}$ be a continuously differentiable vector point function and S be the surface bounded by a closed curve C. Then Stoke's theorem states that

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}ds$$

where the curve C is described in the anticlockwise sense and $\hat{n}$ being the unit normal at any point of S is drawn with a similar sense, in which a right-handed screw would move when rotated in the sense of description of C.

Example 17 Verify Stoke's theorem for

$$\vec{F}=(x^2+y^2)\hat{i}-2xy\hat{j}$$

taken around the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$.

[WBUT 2003]

Sol.

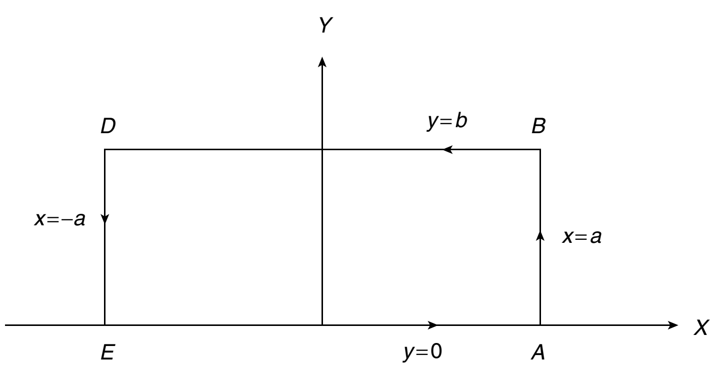

Figure 9.16

Stoke's theorem states that

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\cdot ds$$

Here,

$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$$

then

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & 2xy & 0 \end{vmatrix} = -4y\hat{k}$$

Now for the surface S, $\hat{n} = \hat{k}$

$$\text{curl } \vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$$

Therefore,

$$\iint_S \text{curl } \vec{F}\cdot\hat{n}\cdot ds = \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy = -4ab^2.$$

Again

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \{(x^2+y^2)\hat{i} - 2xy\hat{j}\}\{dx\cdot\hat{i} + dy\cdot\hat{j}\}$$

$$= \oint_C \{(x^2+y^2)dx - 2xydy\}$$

Now from **Fig. 9.16,** it is obvious that

$$\oint_C \{(x^2+y^2)dx - 2xydy\} = \oint_{EA} \{(x^2+y^2)dx - 2xydy\} + \oint_{AB} \{(x^2+y^2)dx - 2xydy\}$$

$$+ \oint_{BD} \{(x^2+y^2)dx - 2xydy\} + \oint_{DE} \{(x^2+y^2)dx - 2xydy\} \qquad ...(1)$$

$$\oint_{EA} \{(x^2+y^2)dx - 2xydy\} = \int_{-a}^{a} x^2dx (\text{since } y=0,\ dy=0)$$

$$= \frac{2a^3}{3}$$

$$\oint_{AB} \{(x^2+y^2)dx - 2xydy\} = \int_0^b (-2ay)dy \ (\text{since } x=a,\ dx=0)$$

$$= -ab^2$$

$$\oint_{BD} \{(x^2+y^2)dx - 2xydy\} = \int_a^{-a} (x^2+b^2)dx \text{ (since } y=b,\, dy=0)$$

$$= -\frac{2a^3}{3} - 2ab^2$$

$$\oint_{DE} \{(x^2+y^2)dx - 2xydy\} = \int_b^0 2ay \cdot dy \text{ (since} x=-a,\, dx=0)$$

$$= -ab^2$$

So from (1), we have

$$\oint_C \{(x^2+y^2)dx - 2xydy\} = -4ab^2$$

Therefore,

$$\oint_C \vec{F}\cdot d\vec{r} = -4ab^2 = \oint_S \text{curl } \vec{F}\cdot\hat{n}\cdot ds$$

Hence Stoke's theorem is verified.

Example 18 Apply Stoke's theorem to evaluate $\oint_C (ydx + zdy + xdz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$. [WBUT-2001]

Sol.

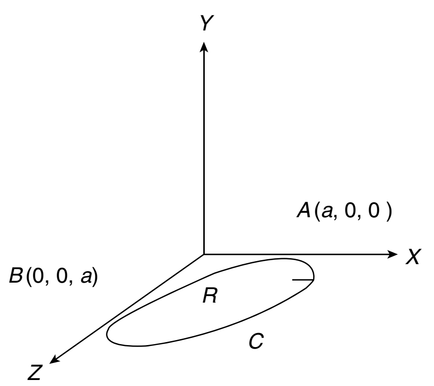

Figure 9.17

Since the intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$ is a circle, here the curve C is a circle with diameter AB where A and B have coordinates $(a, 0, 0)$ and $(0, 0, a)$ respectively.

Therefore, the radius of the circle is $\dfrac{a}{\sqrt{2}}$.

We can write

$$\oint_C (ydx + zdy + xdz) = \oint_C (y\hat{i} + z\hat{j} + x\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \oint_C \vec{F}\cdot d\vec{r} \qquad ...(1)$$

where $\vec{F} = (y\hat{i} + z\hat{j} + x\hat{k})$ and $d\vec{r} = (dx\hat{i} + dy\hat{j} + dz\hat{k})$

By Stoke's theorem, we have

$$\oint_C \vec{F}d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}ds \qquad ...(2)$$

The unit normal to the surface is

$$\hat{n} = \frac{1\cdot\hat{i} + 1\cdot\hat{k}}{\sqrt{2}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$$

By (1) and (2), we can write

$$\oint_C (ydx + zdy + xdz) = \iint_S \text{curl}(y\hat{i} + z\hat{j} + x\hat{k})\cdot\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right)ds \qquad ...(3)$$

Now,

$$\text{curl } \vec{F} = \text{curl}(y\hat{i} + z\hat{j} + x\hat{k}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\hat{i} + \hat{j} + \hat{k}) \qquad ...(4)$$

Therefore, from (3) and (4) we have

$$\oint_C (ydx + zdy + xdz)$$

$$= -\iint_S (\hat{i} + \hat{j} + \hat{k})\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right)ds$$

$$= \frac{-2}{\sqrt{2}}\iint_S ds,$$

since $\iint_S ds =$ area of circle bounded by $C = \pi\left(\frac{a}{\sqrt{2}}\right)^2$

$$= \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2$$

$$= \frac{-\pi a^2}{\sqrt{2}}$$

WORKED-OUT EXAMPLES

Example 9.27 Verify Green's theorem for $\oint_C [(3x-8y^2)\,dx+(4y-6xy)\,dy]$

where C is the boundary of the region bounded by $x=0$, $y=0$, and $x+y=1$.

[WBUT-2002]

Sol.

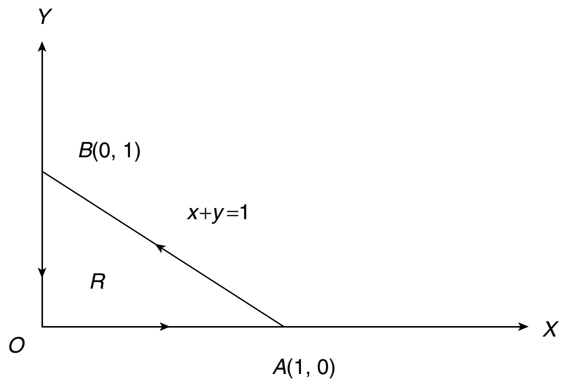

Figure 9.11

Green's theorem states that

$$\oint_C (Mdx+Ndy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

where R is the region bounded by the closed curve C constisting of the lines OA, AB and BO, where A and B are the points $(1,0)$ and $(0,1)$ respectively.

Here,

$$M(x,y)=(3x-8y^2),\ N(x,y)=(4y-6xy)\ \text{and}\ \frac{\partial N}{\partial x}=-6y,\frac{\partial M}{\partial y}=-16y$$

Now,

$$\oint_C (Mdx+Ndy)=\oint_C [(3x-8y^2)\,dx+(4y-6xy)\,dy]$$

$$=\int_{OA}[(3x-8y^2)\,dx+(4y-6xy)\,dy]+\int_{AB}[(3x-8y^2)\,dx+(4y-6xy)\,dy]$$

$$+\int_{BO}[(3x-8y^2)\,dx+(4y-6xy)\,dy]$$

On OA, $y=0$, so $dy=0$. On AB, $x+y=1$, so $dy=-dx$. On BO, $x=0$, so $dx=0$.

Therefore,

$$\oint_C (Mdx+Ndy)$$

$$=\int_{x=0}^{1} 3xdx+\int_{x=1}^{0}[\{3x-8(1-x)^2\}\,dx+\{4(1-x)-6x(1-x)\}(-dx)]+\int_{y=1}^{0} 4\cdot ydy$$

$$=3\left(\frac{1}{2}\right)+\int_{x=1}^{0}(-14x^2+29x-12)\,dx+4\left(\frac{-1}{2}\right)=\frac{5}{3}$$

Again,

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\iint_R(-6y+16y)\,dxdy$$

$$=\int_{x=0}^{1}\int_{y=0}^{1-x}10ydxdy=\int_{x=0}^{1}\frac{1}{2}(1-x)^2\,dx=\frac{5}{3}$$

Hence Green's theorem is verified.

Example 9.28 Verify Green's theorem in the plane for $\oint_C (x^2dx+xydy)$ where C is the square in the xy plane given by $x=0, y=0, x=a, y=a(a>0)$.

[WBUT-2005]

Sol.

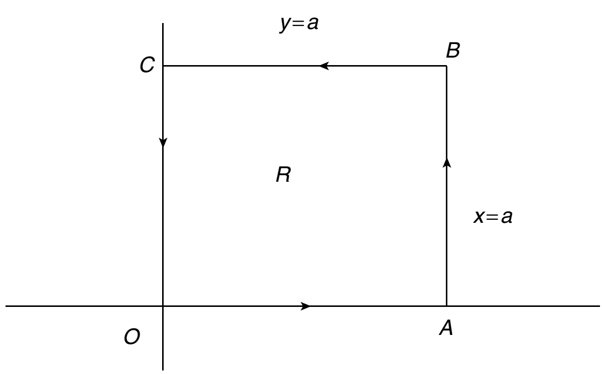

Figure 9.12

Green's theorem states that

$$\oint_C (Mdx+Ndy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

where R is the region bounded by the closed curve C.

Here,

$$M(x, y) = x^2, N(x, y) = xy \text{ and } \frac{\partial N}{\partial x} = y, \frac{\partial M}{\partial y} = 0$$

Now,

$$\oint_C (Mdx + Ndy) = \oint_C (x^2dx + xydy)$$

$$= \int_{OA} (x^2dx + xydy) + \int_{AB} (x^2dx + xydy) + \int_{BC} (x^2dx + xydy) + \int_{CO} (x^2dx + xydy)$$

On OA, $y = 0$, so $dy = 0$. On AB, $x = a$, so $dx = 0$. On BC, $y = a$, so $dy = 0$. On CO, $x = 0$, so $dx = 0$

Therefore,

$$\oint_C (Mdx + Ndy) = \int_{x=0}^{a} x^2dx + \int_{y=0}^{a} aydy + \int_{x=a}^{0} x^2dx + \int_{y=a}^{0} 0 \cdot ydy$$

$$= \left[\frac{x^3}{3}\right]_0^a + a\left[\frac{y^2}{2}\right]_0^a + \left[\frac{x^3}{3}\right]_a^0$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{2}$$

Again,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (y - 0)\, dxdy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a} ydxdy = \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_0^a dx$$

$$= \int_{x=0}^{a} \frac{a^2}{2} dx = \frac{a^3}{2}$$

Hence, Green's theorem is verified.

Example 9.29 Verify Gauss's divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9, z = 0, z = 2.$ [WBUT-2003, 2007]

Sol.

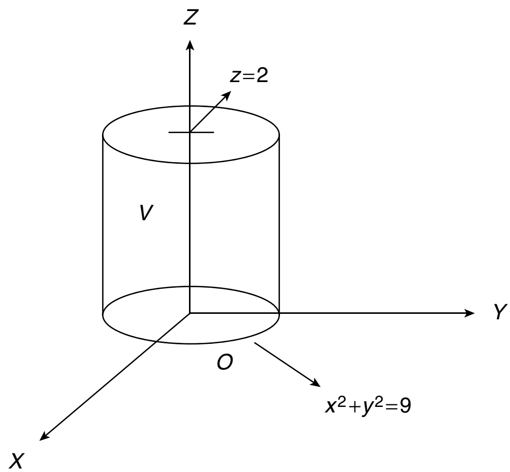

Figure 9.14

Gauss's divergence theorem states that

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

where the volume V bounded by a closed surface S and $\hat{n}$ is the outward drawn unit normal vector to the surface S.

Here, V is the volume bounded by surface $S: x^2+y^2=9, z=0, z=2.$

Now

$$\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$$

So

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(y\hat{i} + x\hat{j} + z^2\hat{k}) = 2z$$

For a particular $z,\ x^2+y^2=9$ is a circle. Therefore, $-3 \le x \le 3.$

and for a particular value of $x,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

Therefore,

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dv = \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=0}^{2} 2z\,dx\,dy\,dz = \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left[\frac{z^2}{2}\right]_0^2 dx\,dy$$

$$= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\,dx\,dy = \int_{x=-3}^{3} 4[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx$$

$$= 4\int_{x=-3}^{3} 2\sqrt{9-x^2}\,dx = 16\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 16\left[\frac{9}{2}\sin^{-1}1\right] = 36\pi \qquad \ldots(1)$$

Now

$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_{S_1}\vec{F}\cdot\hat{n}ds + \iint_{S_2}\hat{F}\cdot\hat{n}ds + \iint_{S_3}\vec{F}\cdot\hat{n}ds \qquad \ldots(2)$$

where S_1 is the circular base in the plane $z=0$, S_2 is the circular top in the plane $z=2$ and S_3 is the curved surface of the cylinder, given by $x^2+y^2=9$.

In the integral $\iint_{S_1}\vec{F}\cdot\hat{n}ds$, $\hat{n}$ is normal to S_1, so $\hat{n}=\hat{k}$ and $z=0$. Therefore

$$\iint_{S_1}\vec{F}\cdot\hat{n}ds = \iint_{S_1}\left(y\hat{i}+x\hat{j}+0\right)\cdot\hat{k}\cdot\hat{k}dxdy = 0. \qquad \ldots(3)$$

In the integral $\iint_{S_2}\vec{F}\cdot\hat{n}ds$, $\hat{n}$ is normal to S_2, so $\hat{n}=\hat{k}$ and $z=2$. Therefore

$$\iint_{S_2}\vec{F}\cdot\hat{n}ds = \iint_{S_2}\left(y\hat{i}+x\hat{j}+2^2\cdot\hat{k}\right)\cdot\hat{k}dxdy$$

$$= 4\iint_{S_2}dxdy = 4\cdot(\text{Area of } S_2)$$

$$= 4\cdot\pi(3)^2 = 36\pi \qquad \ldots(4)$$

Again S_3 is represented by $x^2+y^2-9=0$. So, $\vec{\nabla}(x^2+y^2-9)$ is normal vector on S_3. So the unit normal vector,

$$\hat{n} = \frac{\vec{\nabla}(x^2+y^2-9)}{\left|\vec{\nabla}(x^2+y^2-9)\right|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{36}}[\text{since } x^2+y^2=9]$$

$$= \frac{x\hat{i}+y\hat{j}}{3}$$

Therefore,

$$\iint_{S_3}\vec{F}\cdot\hat{n}ds = \iint_{S_3}\left(y\hat{i}+x\hat{j}+z^2\cdot\hat{k}\right)\frac{x\hat{i}+y\hat{j}}{3}ds$$

$$= \frac{2}{3}\iint_{S_3}xyds$$

On S_3, $x^2+y^2=9$, let $x=3\cos\theta$, $y=3\sin\theta$ and $ds=3dzd\theta$. So, for the entire surface, θ varies from 0 to 2π and z varies from 0 to 2

$$\iint_{S_3}\vec{F}\cdot\hat{n}ds=\frac{2}{3}\iint_{S_3}xyds$$

$$=\frac{2}{3}\int_{\theta=0}^{2\pi}\int_{z=0}^{2}27\sin\theta\cos\theta d\theta dz$$

$$=18\int_{\theta=0}^{2\pi}\sin2\theta d\theta z]_0^2$$

$$=36\left[-\frac{\cos2\theta}{2}\right]_0^{2\pi}=0 \qquad \ldots(5)$$

Using (3), (4) and (5) in (1),

$$\iint_S\vec{F}\cdot\hat{n}ds=0+36\pi+0=36\pi. \qquad \ldots(6)$$

From (1) and (6), it is clear that Gauss's divergence theorem is verified.

Example 9.30 Evaluate by divergence theorem

$$\iint_S\{x^2dydz+y^2dzdx+2z(xy-x-y)dxdy\}$$

where S is the surface of the cube $0\le x\le1, 0\le y\le1, 0\le z\le1$. [WBUT-2005]

Sol.

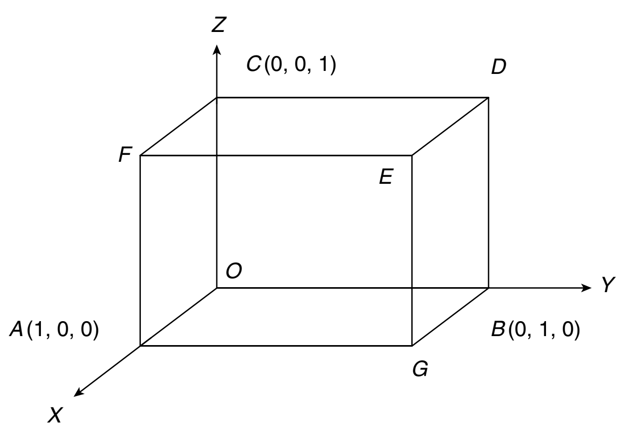

Figure 9.15

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then the divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

where V is the volume enclosed by the closed surface S.

Here,

$$F_1 = x^2,\ F_2 = y^2,\ F_3 = 2z(xy - x - y)$$

and $\dfrac{\partial F_1}{\partial x} = 2x,\ \dfrac{\partial F_2}{\partial y} = 2y,\ \dfrac{\partial F_3}{\partial z} = 2(xy - x - y)$

Therefore,

$$\iint_S \{x^2 dydz + y^2 dzdx + 2z(xy - x - y)dxdy\}$$

$$\iiint_V = \{2x + 2y + 2(xy - x - y)\}\ dxdydz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} \{2x + 2y + 2(xy - x - y)\}\ dxdydz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 2xydxdydz = \int_{x=0}^{1} \int_{y=0}^{1} 2xy[z]_0^1\, dxdy$$

$$= \int_{x=0}^{1} 2x \left[\frac{y^2}{2} \right]_0^1 dx = \int_{x=0}^{1} xdx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}$$

Example 9.31 Using divergence theorem, evaluate $\iint_S \vec{u} \cdot \vec{n} ds$ where $\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the sphere $x^2 + y^2 + z^2 = 9$ and $\vec{n}$ is outward normal to S. [WBUT-2006]

Sol. Let V be the volume of the sphere $x^2 + y^2 + z^2 = 9$ with a radius of 3.

By the divergence theorem,

$$\iint_S \vec{u} \cdot \vec{n} ds = \iiint_V \vec{\nabla} \cdot \vec{u} dv$$

Now,

$$\vec{\nabla} \cdot \vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

Therefore,

$$\iint_S \vec{u}\cdot\vec{n}ds = \iiint_V \vec{\nabla}\cdot\vec{u}dv = \iiint_V 3dv = 3\iiint_V dxdydz$$

$$= 3\times(\text{volume of the sphere of radius} = 3 \text{ units})$$

$$= 3\times\left(\frac{4}{3}\pi 3^3\right) = 108\pi$$

Example 9.32 Use Stoke's theorem to prove $\text{div curl } \vec{F} = 0.$ [WBUT-2002]

Sol.

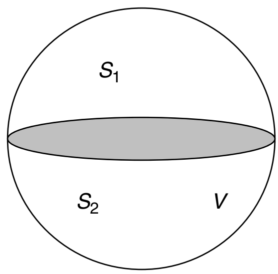

Figure 9.18

Let V be any volume enclosed by a closed surface $S.$

Let us divide V by a plane into two surfaces S_1 and $S_2,$ and let C denote the common closed curve bounding both the portions.

Therefore, by Stoke's theorem

$$\iiint_V \text{div curl } \vec{F}dv = \iint_S \text{curl } \vec{F}\cdot\hat{n}ds$$

$$= \iint_{S_1} \text{curl } \vec{F}\cdot\hat{n}ds + \iint_{S_2} \text{curl } \vec{F}\cdot\hat{n}ds$$

$$= \int_C \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r} = 0$$

where negative sign is taken in the second integral as it is traversed in the direction opposite to that of the first.

Since the above result is true for every volume element $V,$ we have

$$\text{div curl } \vec{F} = 0$$

Example 9.33 Verify Stoke's theorem for $\vec{A} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

[WBUT-2004]

Sol.

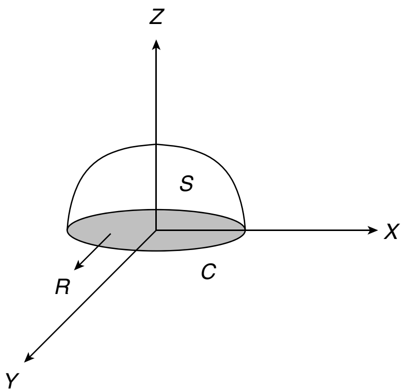

Figure 9.19

Stoke's theorem states that for any vector function $\vec{A}$

$$\oint_C \vec{A}d\vec{r} = \iint_S \text{curl}\ \vec{A}\cdot\hat{n}ds,$$

where $\hat{n}$ being the outward drawn unit normal at any point of S.

The boundary of C of S is a circle in xy plane whose equation is $x^2+y^2=9, z=0$.

Let the parametric equation of C be $x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$.

Therefore,

$$\oint_C \vec{A}d\vec{r} = \oint_C (2y\hat{i} + 3x\hat{j} - z^2\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (2ydx + 3xdy - z^2dz)$$

$$= 9\int_{t=0}^{2\pi} (-2\sin^2 t + 3\cos^2 t)dt = 9\pi \qquad \ldots(1)$$

Now,

$$\text{curl}\ \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(3-2) = \hat{k}$$

Therefore,

$$\iint_S \text{curl}\ \vec{A}\cdot\hat{n}ds = \iint_R \hat{k}\cdot\hat{n}\frac{dxdy}{\hat{k}\cdot\hat{n}} = \iint_R dxdy$$

where R is the region enclosed by the circle $x^2 + y^2 = 9$. So,

$$\iint_R dxdy = (\text{Area of the circle } x^2 + y^2 = 9)$$

$$= \pi\cdot(3)^2 = 9\pi.$$

Therefore,

$$\iint_S \text{curl}\,\vec{A}\cdot\hat{n}ds = 9\pi.$$

From (1) and (2), it is clear that Stoke's theorem is verified.

Example 9.34 Verify Stokes theorem for $\vec{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

[WBUT-2006]

Sol.

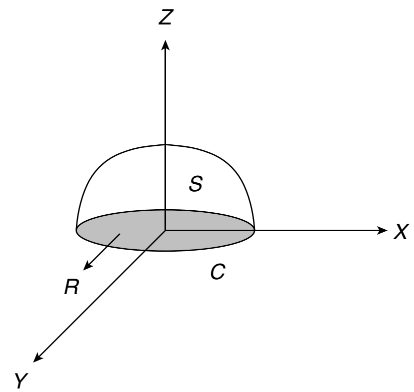

Figure 9.20

Stoke's theorem states that for any vector function $\vec{F}$

$$\oint_C \vec{F}d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}ds,$$

where $\hat{n}$ being the outward drawn unit normal at any point of S.

Here, the boundary C of S is a circle in the xy plane whose equation is $x^2+y^2=1, z=0$.

Let the parametric equation of C is

$x=\cos t,\ y=\sin t,\ z=0,\ 0\le t\le 2\pi$

Now

$$\oint_C \vec{F}d\vec{r} = \oint_C \{(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}\}(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C \{(2x-y)dx-yz^2dy-y^2zdz\}$$

$$= \oint_C (2x-y)dx \text{ [since, on } C, z=0, dz=0]$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt = \int_0^{2\pi} \left(-\sin 2t + \frac{1-\cos 2t}{2}\right)dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi \qquad \ldots(1)$$

Again,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$$

Therefore,

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} ds$$

$= \iint_R \text{curl } \vec{F} \cdot \hat{n} \dfrac{dxdy}{\hat{n} \cdot \hat{k}}$ where R is the region bounded by the circle C

$$= \iint_R (\hat{k} \cdot \hat{n}) \frac{dxdy}{\hat{n} \cdot \hat{k}} = \iint_R dxdy$$

$$= \pi(1)^2 (\text{area of the circle } x^2 + y^2 = 1) = \pi \qquad \ldots(2)$$

Therefore by (1) and (2), Stoke's theorem is verified.

Example 9.35 Verify Stoke's theorem for $\vec{A} = (y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$ over the surface of the cube $x = y = z = 0$ and $x = y = z = 2$ above xy plane.

[WBUT-2007]

Sol.

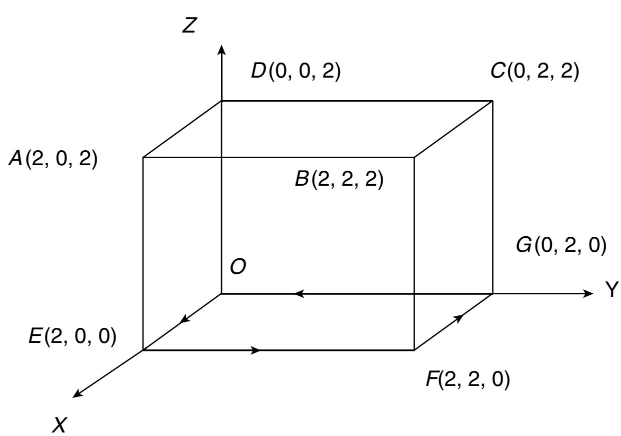

Figure 9.21

Let us denote S_1, S_2, S_3, S_4 and S_5 as the five faces $ABFE$, $DCGO$, $ABCD$, $BCGF$ and $ADOE$ respectively above XOY plane.

Therefore, the boundary of the surface is the square $EFGO$.

Stoke's theorem states that

$$\iint_{S_1+S_2+S_3+S_4+S_5} \text{curl}\,\vec{A}\cdot\hat{n}ds = \int_{EFGO} \vec{A}\cdot d\vec{r} \qquad \ldots(1)$$

Now,

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y-z+2) & (yz+4) & -xz \end{vmatrix}$$

$$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$$

Therefore,

$$\iint_{S_1} \text{curl}\,\vec{A}\cdot\hat{n}ds = \iint_{S_1} \{-y\hat{i}+(z-1)\hat{j}-\hat{k}\}\cdot\hat{i}\cdot ds, \text{ since on } S_1, \hat{n}=\hat{i}$$

$$= \int_0^2\int_0^2 -ydydz = \int_0^2 -y[z]_0^2 dy = -2\left[\frac{y^2}{2}\right]_0^2 = -4$$

Similarly,

$$\iint_{S_2} \text{curl}\,\vec{A}\cdot\hat{n}ds = \iint_{S_2} \{-y\hat{i}+(z-1)\hat{j}-\hat{k}\}\cdot(-\hat{i})\cdot ds, \text{ since on } S_2, \hat{n}=-\hat{i}$$

$$= \int_0^2\int_0^2 ydydz = 4$$

$$\iint_{S_3} \text{curl}\,\vec{A}\cdot\hat{n}ds = \iint_{S_3} \{-y\hat{i}+(z-1)\hat{j}-\hat{k}\}\cdot(\hat{k})\cdot ds, \text{ since on } S_3, \hat{n}=\hat{k}$$

$$= \int_0^2\int_0^2 -dxdy = -4$$

$$\iint_{S_4} \text{curl}\,\vec{A}\cdot\hat{n}ds = \iint_{S_4} \{-y\hat{i}+(z-1)\hat{j}-\hat{k}\}\cdot(\hat{j})\cdot ds, \text{ since on } S_4, \hat{n}=\hat{j}$$

$$= \int_0^2\int_0^2 (z-1)dxdz = 0$$

$$\iint_{S_5} \text{curl } \vec{A} \cdot \hat{n} ds = \iint_{S_5} \{-y\hat{i} + (z-1)\hat{j} - \hat{k}\} \cdot (-\hat{j}) \cdot ds \text{ since on } S_5, \hat{n} = -\hat{j}$$

$$= \int_0^2 \int_0^2 -(z-1) dx dz = 0$$

Therefore,

$$\iint_{S_1+S_2+S_3+S_4+S_5} \text{curl } \vec{A} \cdot \hat{n} ds = \iint_{S_1} \text{curl } \vec{A} \cdot \hat{n} ds + \iint_{S_2} \text{curl } \vec{A} \cdot \hat{n} ds$$

$$+ \iint_{S_3} \text{curl } \vec{A} \cdot \hat{n} ds + \iint_{S_4} \text{curl } \vec{A} \cdot \hat{n} ds + \iint_{S_5} \text{curl } \vec{A} \cdot \hat{n} ds$$

$$= -4 + 4 - 4 + 0 + 0 = -4 \qquad \ldots(2)$$

Now,

$$\int_{EFGO} \vec{A} \cdot d\vec{r} = \int_{EFGO} \{(y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}\} \{dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{EFGO} \{(y - z + 2)dx + (yz + 4)dy - xzdz\}$$

$$= \int_{OE} \{(y - z + 2)dx + (yz + 4)dy - xzdz\}$$

$$+ \int_{EF} \{(y - z + 2)dx + (yz + 4)dy - xzdz\}$$

$$+ \int_{FG} \{(y - z + 2)dx + (yz + 4)dy - xzdz\}$$

$$+ \int_{GO} \{(y - z + 2)dx + (yz + 4)dy - xzdz\}$$

$$= \int_{x=0}^{2} 2dx + \int_{y=0}^{2} 4dy + \int_{x=2}^{0} 4dx + \int_{y=2}^{0} 4dy = -4 \qquad \ldots(3)$$

Therefore, From (1) and (2) we conclude that Stoke's theorem (1) is satisfied.

Example 9.36 If $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path C given by $x = t$, $y = t^2$ and $z = t^3$. [WBUT-2002].

Sol. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

Therefore,

$$\int_C \vec{A}\cdot d\vec{r} = \int_C [(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}](dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C [(3x^2+6y)dx - 14yzdy + 20xz^2dz]$$

$$= \int_0^1 [(3t^2+6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2dt)]$$

$$= \int_0^1 [9t^2 - 28t^6 + 60t^9]dt$$

$$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$$

$$= 3 - 4 + 6 = 5.$$

EXERCISES

Short and Long Answer Type Questions

1. Show that the vector $9\hat{i} + \hat{j} - 6\hat{k}$ is perpendicular to the vector $4\hat{i} - 6\hat{j} + 5\hat{k}$.

2. Determine λ so that $\lambda\hat{i} - 4\hat{j} + 3\hat{k}$ and $3\hat{i} + \lambda\hat{j} - 2\hat{k}$ are perpendicular.

   [**Ans :** $\lambda = -6$]

3. Find the angle between the vectors $\hat{i} - 2\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} - 2\hat{k}$.

   $\left[\textbf{Ans :} \cos^{-1}\left(\frac{4}{3}\right)\right]$

4. Find a vector of magnitude 5, perpendicular to both the vectors $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$.

   $\left[\textbf{Ans :} \frac{-5}{\sqrt{3}}(i + j + k)\right]$

5. If $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, show that

   a) $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = \frac{-3}{2}$

   b) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

6. If $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$, show that $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -25$.

7. If $|\vec{a}|=3, |\vec{b}|=4,$ find the values of μ for which the vectors $\vec{a}+\mu\vec{b}$ and $\vec{a}-\mu\vec{b}$ will be perpendicular to each other. $\left[\textbf{Ans}: \frac{3}{4}\right]$

8. Prove that the following vectors are coplanar:

 a) $\hat{i}-2\hat{j}+\hat{k}, 2\hat{i}+\hat{j}-3\hat{k}, 3\hat{i}+\hat{j}+\hat{k}$

 b) $\hat{i}-2\hat{j}+3\hat{k}, -2\hat{i}+3\hat{j}-4\hat{k}, -\hat{j}+2\hat{k}$

9. Determine the value of the constant λ so that the vectors $2i-j+k, i+2j+\lambda k$ and $3i-4j+k$ are coplanar.

10. Prove that

 a) $[\vec{a}\times\vec{b}, \vec{c}\times\vec{d}, \vec{e}\times\vec{f}]=[\vec{a},\vec{b},\vec{e}][\vec{c},\vec{d},\vec{f}]-[\vec{a},\vec{b},\vec{f}][\vec{c},\vec{d},\vec{e}]$

 b) $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]=2[\vec{a},\vec{b},\vec{c}]$

 c) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

11. If $\vec{a}$ and $\vec{c}$ are perpendicular to each other then show that $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are also perpendicular to each other.

12. Find the volume of the parallelepiped whose edges are along $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-3\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$.

13. If $\vec{\alpha}\times\vec{\beta}=\vec{\gamma}$, $\vec{\beta}\times\vec{\gamma}=\vec{\alpha}$ and $\vec{\gamma}\times\vec{\alpha}=\vec{\beta}$ then show that $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are mutually perpendicular.

14. If $\vec{\alpha}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$ and $\vec{\beta}=(2t-3)\hat{i}+\hat{j}-t\hat{k}$, find $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$ at $t=2$.

15. If $\vec{r}=\vec{a}\cos nt+\vec{b}\sin nt$ then prove that $\vec{r}\times\frac{d\vec{r}}{dt}=n(\vec{a}\times\vec{b})$ and $\frac{d^2\vec{r}}{dt^2}+n^2\vec{r}=0$.

16. If $\vec{a},\vec{b},\vec{c},\vec{d}$ be vectors such that $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ then show that $(\vec{a}-\vec{d})$ and $(\vec{b}-\vec{c})$ are collinear.

17. Show that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.

18. If $\vec{r}=(2x^2y-x^4)\hat{i}+(e^{xy}-y\sin x)\hat{j}+(x^2\cos y)\hat{k}$ then show that the value of $\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2}$ at $(1,0)$ is $-(\hat{i}+12\hat{j}+12\hat{k})$.

19. If $\vec{\alpha}=x^2yz\hat{i}-2xz^3\hat{j}+xz^2\hat{k}$ and $\vec{\beta}=2z\hat{i}+y\hat{j}-x^2\hat{k}$ then show that the value of $\frac{\partial^2}{\partial x\partial y}(\vec{\alpha}\times\vec{\beta})$ at $(1,0,-2)$ is $-4(\hat{i}+2\hat{j})$

20. Find the directional derivative of $\phi(x,y,z)=x^2yz+4xz^2$ at the point $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$.

21. Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at $(4,-3,2)$.

22. Find the equations of the tangent line and normal plane to the curve of intersection of the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.
23. Show that the vector $y\hat{i}+x\hat{j}$ is both solenoidal and irrotational.
24. If $f(x,y,z)$ be a scalar point function such that $\vec{\nabla}^2 f(x,y,z)=0$, then show that $\vec{\nabla}^2 f$ is irrotational as well as solenoidal.
25. If $\vec{r}=xi+yj+zk$ and $r=|\vec{r}|$, prove that

    a) $\vec{\nabla}(\vec{r}\cdot r^{-2})=\dfrac{1}{r^2}$

    b) $\text{div}(\vec{a}\times\vec{r})=0$ where $\vec{a}$ is a constant vector.

    c) $\text{div}(\text{grad } r^n)=n(n+1)r^{n-2}$

    d) $\text{div}\left(\dfrac{\vec{a}\times\vec{r}}{\vec{r}^{\,n}}\right)=0$

    e) $\text{curl}(r^{-1}\vec{r})=0$

    f) $\text{curl}(r^3\vec{r})=0$
26. Verify Green's theorem in the plane for $\int_C [(y-\sin x)dx+\cos x dy]$ where C is the triangle whose vertices are $(0,0)$, $\left(\dfrac{\pi}{2},0\right)$ and $\left(\dfrac{\pi}{2},1\right)$.
27. Evaluate by Green's theorem in the plane for $\int_C (e^{-x}\sin y dx+e^{-x}\cos y dy)$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $\left(\pi,\dfrac{\pi}{2}\right)$ and $\left(0,\dfrac{\pi}{2}\right)$

    [**Ans :** $2(e^{-\pi}-1)$]
28. Verify Stoke's theorem for $\vec{F}=(x^2+y^2)\hat{i}-2xy\hat{j}$ taken round the rectangle bounded $x=\pm a$, $y=0$, $y=b$.
29. Evaluate by Stoke's theorem $\int_C (yzdx+zxdy+xydz)$, where C is the curve $x^2+y^2=1$, $z=y^2$.

    [**Ans :** 0]
30. Verify Gauss's divergence theorem for $\vec{F}=y\hat{i}+x\hat{j}+z^2\hat{k}$ over the cylindrical region bounded by $x^2+y^2=a^2$, $z=0$ and $z=h$.
31. Evaluate $\iint_S \vec{F}\cdot\hat{n}ds$ where $\vec{F}=3xz\hat{i}+y^2\hat{j}-3yz\hat{k}$ and S is the surface of the cube bounded by $x=0$, $x=2$; $y=0$, $y=2$; $z=0$, $z=2$ and $\hat{n}$ is the outward drawn unit normal to the surface S.
32. Evaluate $\iiint_V \vec{F}dv$ where $\vec{F}=2z\hat{i}-x\hat{j}+y\hat{k}$ and V is the region bounded by the surfaces $x=0$, $y=0$, $x=2$, $y=4$, $z=x^2$ and $z=2$.

    $\left[\textbf{Ans :} \dfrac{32}{15}(3\hat{i}+5\hat{j})\right]$

33. Evaluate $\iiint_V \vec{F}dv$ where $\vec{F} = (2x^2 - 3z)i - 2xyj - 4xk$ and V is the region bounded by the surfaces $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$.

$\left[\textbf{Ans :} \frac{8}{3}\right]$

34. Evaluate $\int_C (\sin z dx - \cos x dy + \sin y dz)$ by Stoke's theorem where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

[**Ans :** 2]

35. Evaluate $\iint_S \{(z^2 - x)dydz - xydxdz + 3zdxdz\}$ where S is the surface of the closed region bounded by $y^2 = 4 - z$ and the planes $x = 0, x = 3, z = 0$.

[**Ans :** 16]

Multiple-Choice Questions

1. The value of λ for which the vectors $\vec{a} = \lambda\hat{i} - 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + \lambda\hat{j} - 2\hat{k}$ are perpendicular to each other is

   a) –6 b) 6 c) 3 d) 2

2. A unit vector perpendicular to each of the vectors $i + j$ and $j + k$ is

   a) $\frac{1}{\sqrt{3}}(i + j + k)$ b) $\frac{1}{\sqrt{3}}(i - j + k)$ c) $\frac{1}{\sqrt{3}}(i + j - k)$ d) $\frac{1}{\sqrt{3}}(-i + j + k)$

3. If for the vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ then $\vec{a}$ and $\vec{b}$ are

   a) parallel b) collinear c) perpendicular d) none of these

4. If $\vec{a}$ and $\vec{b}$ are two mutually perpendicular vectors then $|\vec{a} \times \vec{b}| =$

   a) $\vec{a} \cdot \vec{b}$ b) $|\vec{a} \cdot \vec{b}|$ c) $|\vec{a}||\vec{b}|$ d) 0

5. $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) =$

   a) $2(\vec{a} \cdot \vec{b})$ b) $\vec{a} \times \vec{b}$ c) $2(\vec{a} \times \vec{b})$ d) 0

6. The value of λ for which the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 2\hat{i} - 4\hat{j}$ and $\vec{c} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are coplanar is

   a) $\frac{3}{5}$ b) $\frac{1}{5}$ c) $\frac{2}{5}$ d) $\frac{5}{3}$

7. If $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = 2\hat{i} - \hat{k}$, then $(\vec{a} \times \vec{b}) \cdot \vec{a} =$

   a) $\hat{i} + \hat{j} + \hat{k}$ b) $\hat{i} + \hat{k}$ c) 0 d) 2

8. The vector $x\hat{i} + y\hat{j} + z\hat{k}$ is perpendicular to the vector $2\hat{i} + 5\hat{j} + 11\hat{k}$ when

   a) $x = 2, y = 3, z = -11$ b) $x = 2, y = -3, z = 11$
   c) $x = -2, y = 3, z = 11$ d) $x = 2, y = -3, z = 1$

9. The angle between the vectors $2\hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}+4\hat{k}$ is

a) $\cos^{-1}\left(\frac{1}{15}\right)$ b) $\cos^{-1}\left(\frac{2}{5}\right)$ c) $\cos^{-1}\left(\frac{2}{15}\right)$ d) none of these

10. If $|\vec{a}|^2=|\vec{b}|^2$, then the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are

a) parallel b) perpendicular

c) inclined at an angle 30° with each other d) none of these

11. If $\vec{a}=3t^2\hat{i}+t\hat{j}-t^3\hat{k}$ and $\vec{b}=\sin t\hat{i}-2\cos t\hat{j}$, then $\frac{d}{dt}(\vec{a}\times\vec{b})$ at $t=\frac{\pi}{2}$ is

a) $\frac{\pi^3}{4}\hat{i}-\frac{3}{4}\pi^2\hat{j}-\hat{k}$ b) $\frac{\pi^3}{4}\hat{i}+\frac{3}{4}\pi^2\hat{j}+\hat{k}$

c) $\frac{\pi^3}{4}\hat{i}-\frac{3}{4}\pi^2\hat{j}+\hat{k}$ d) none of these

12. If $\vec{r}=2t\hat{i}-t^2\hat{j}+\frac{1}{3}t^3\hat{k}$ then the value of $\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}$ at $t=1$ is

a) $2\hat{i}-4\hat{j}+4\hat{k}$ b) $2\hat{i}+4\hat{j}+4\hat{k}$ c) $-2\hat{i}+4\hat{j}+4\hat{k}$ d) $2\hat{i}+4\hat{j}-4\hat{k}$

13. If $f(x,y,z)=x^3+3yz^2$ then $\text{grad} f$ at $(1,1,1)$ is

a) $3\hat{i}+3\hat{j}+3\hat{k}$ b) $\hat{i}+\hat{j}+2\hat{k}$ c) $3\hat{i}+3\hat{j}+6\hat{k}$ d) none of these

14. A normal vector to the plane $x+2y+3z-1=0$ is

a) $2\hat{i}+\hat{j}+3\hat{k}$ b) $\hat{i}+2\hat{j}+3\hat{k}$ c) $\hat{i}-2\hat{j}-3\hat{k}$ d) $\hat{i}+2\hat{j}-3\hat{k}$

15. If $\phi=x^3+3yz^2$ then $\vec{\nabla}^2\varphi$ is

a) $x+6y$ b) $6x+y$ c) $6x+6y$ d) $x+y$

16. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then $\text{curl}\,\vec{r}=$

a) $3\hat{i}$ b) 0 c) $\hat{i}+\hat{j}+\hat{k}$ d) none of these

17. The magnitude of the vector drawn in the direction perpendicular to the surface $x^2+2y^2+z^2=7$ at (1, –1, 2) is

a) $\frac{2}{3}$ b) $\frac{3}{2}$ c) 3 d) 6

18. If $r^2=x^2+y^2+z^2$ then $\vec{\nabla}^2(\ln r)=$

a) $\frac{1}{r^2}$ b) $\frac{1}{r}$ c) r^2 d) r

19. If $\vec{F}=xy\hat{i}+yz\hat{j}+zx\hat{k}$ then $\text{curl}\,\vec{F}=$

a) $x\hat{i}+y\hat{j}+z\hat{k}$ b) $y\hat{i}+z\hat{j}+\hat{k}$ c) $-y\hat{i}-z\hat{j}-x\hat{k}$ d) $y\hat{i}+z\hat{j}+x\hat{k}$

20. The value of t for which $\vec{F}=(txy-z^3)\hat{i}+(t-2)x^2\hat{j}+(1-t)xz^2\hat{k}$ is irrotational is

a) 0 b) 4 c) –4 d) 2

21. The value of a for which $\vec{F}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal is

a) 2 b) –2 c) 0 d) none of these

22. The directional derivative of $\phi=xyz$ at $(1,1,1)$ in the direction $\hat{j}$ is

a) $\sqrt{3}$ b) 1 c) 2 d) 4

23. If $\vec{v}=\vec{w}\times\vec{r}$ where $\vec{w}$ is a constant vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then div $\vec{v}=$

a) 1 b) 0 c) 2 d) none of these

24. The value of $\text{grad}(x+y-z)d\vec{r}$ from $(0,1,-1)$ to $(1,2,0)$ is

a) 3 b) –1 c) 1 d) 2

25. If $\phi(x, y, z)=c$ represent the equation of a surface then normal to this surface is

a) grad ϕ b) div(grad ϕ) c) curl(grad ϕ) d) none of these

Answers:

1. (a)	2. (b)	3. (c)	4. (c)	5. (c)	6. (d)	7. (c)	8. (d)
9. (c)	10. (b)	11. (a)	12. (a)	13. (c)	14. (b)	15. (c)	16. (b)
17. (d)	18. (a)	19. (c)	20. (b)	21. (b)	22. (b)	23. (b)	24. (a)
25. (a)							

SOLUTION OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-1 (NEW) 2010

MATHEMATICS-I (M 101)

Time Alloted: 3 Hours *Full Marks: 70*

GROUP-A (Multiple-Choice Type)

1. Choose the correct alternative for any ten of the following: **($10 \times 1 = 10$)**

*(i) If α, β are the roots of the equation $x^2 - 3x + 2 = 0$ then

$$\begin{vmatrix} 0 & \alpha & \beta \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{vmatrix} \text{ is}$$

(a) 6 (b) $\frac{3}{2}$ (c) -6 (d) 3

Solution: Since α, β are the roots of the equation, we have

$$\alpha + \beta = 3 \quad \text{and} \quad \alpha\beta = 2.$$

Now

$$\begin{vmatrix} 0 & \alpha & \beta \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{vmatrix} = -\alpha\beta(\alpha + \beta) = -6$$

Hence, the correct alternative is $\boxed{\text{(c) } -6}$.

*(ii) If $y = e^{ax+b}$ then $(y_5)_0 =$

(a) ae^b (b) a^5e^b (c) a^be^{ax} (d) none of these

Solution: Here, $y = e^{ax+b} \Rightarrow y_5 = a^5e^{ax+b}$; therefore $(y_5)0 = a^5e^b$.

Hence, the correct alternative is $\boxed{\text{(b) } \alpha^5e^b}$.

*(iii) If Rolle's theorem is applied to $f(x) = x(x^2 - 1)$ in [0, 1] then $c =$

(a) 1 (b) 0 (c) $-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

Solution: If Rolle's theorem is applied to $f(x) = x(x^2 - 1)$, we have

$$f'(c) = 0 \Rightarrow 3c^2 - 1 = 0 \Rightarrow c = \pm\frac{1}{\sqrt{3}}.$$

Level of difficulty:- *Low, **Medium, ***High.

Since by the conditions of Rolle's theorem $0 < c < 1$, we take $c = \frac{1}{\sqrt{3}}$.

Hence, the alternative option is $\boxed{\text{(d) } \frac{1}{\sqrt{3}}}$.

**(iv) If $u + v = x$, $uv = y$ then $\frac{\partial(u,v)}{\partial(x,y)} =$

(a) $\frac{1}{u-v}$ (b) uv (c) $u + v$ (d) $\frac{u}{v}$.

Solution: We know

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}$$

Now

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ v & u \end{vmatrix} = u - v.$$

So, $\frac{\partial(u,v)}{\partial(x,y)}$ is $\boxed{\text{(a) } \frac{1}{u-v}}$.

*(v) The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \, d\theta$ is

(a) $\frac{6.4.2}{7.5.3.1}$ (b) $\frac{6!}{7!}$ (c) 0 (d) $\frac{2.(6.4.2)}{7.5.3.1}$

Solution: The correct alternative is $\boxed{\text{(c) } 0}$, since $\sin^7\theta$ is an odd function of θ.

*(vi) The sequence $\left\{(-1)^n \frac{1}{n}\right\}$ is

(a) convergent (b) oscillatory (c) divergent (d) none of these

Solution: The correct alternative is $\boxed{\text{(b) oscillatory}}$.

**(vii) If $\vec{\alpha} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{\beta} = 2\hat{i} - \hat{k}$ then $(\vec{\alpha} \times \vec{\beta}).\,\vec{\alpha}$ is equal to

(a) $\hat{i} + \hat{j} + \hat{k}$ (b) $\hat{i} + \hat{k}$

(c) $\hat{i} - \hat{k}$ (d) 0

Solution: Here,

$$\vec{\alpha} \times \vec{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 2\hat{i} + 5\hat{j} + 4\hat{k}.$$

Therefore,

$$(\vec{\alpha} \times \vec{\beta}) \cdot \vec{\alpha} = 2.3 + 5.(-2) + 4.1 = 0.$$

Hence the correct alternative is $\boxed{\text{(d) } 0}$.

***(viii) The matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is

(a) symmetric (b) skew-symmetric

(c) singular (d) orthogonal

Solution: Here,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the correct alternative is $\boxed{\text{(d) orthogonal}}$.

**(ix) The value of t for which

$$\vec{f} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + tz)\,\hat{k}$$

is solenoidal is

(a) 2 (b) -2 (c) 0 (d) 1

Solution: The correct alternative is $\boxed{\text{(b) } -2}$. See Example 9.18.

***(x) The distance between two parallel planes $x + 2y - z = 4$ and $2x + 4y - 2z = 3$ is

(a) $\frac{5}{\sqrt{24}}$ (b) $\frac{5}{24}$ (c) $\frac{11}{\sqrt{24}}$ (d) none of these

Solution: The correct alternative is $\boxed{\text{(a) } \frac{5}{\sqrt{24}}}$.

**(xi) In the MV Theorem,

$$f(h) = f(0) + hf'(\theta h); \quad 0 < \theta < 1.$$

If $f(x) = \dfrac{1}{1+x}$ and $h = 3$ then value of θ is

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{2}}$ (d) none of these

Solution: Here, $f(x) = \dfrac{1}{1+x}$ and $h = 3$, so by given relation

$$f(h) = f(0) + hf'(\theta h); \quad 0 < \theta < 1$$

or $$\frac{1}{1+h} = 1 + h\left[-\frac{1}{(1+\theta h)^2}\right]$$

Putting $h = 3$, we have

$$\frac{1}{4} = 1 - \frac{3}{(1+3\theta)^2} \Rightarrow (1+3\theta)^2 = 4$$

$$\Rightarrow \quad 1 + 3\theta = \pm 2 \Rightarrow \theta = \frac{1}{3}, -1$$

Since $0 < \theta < 1$, we take $\theta = \frac{1}{3}$.

Hence the correct alternative is $\boxed{\text{(b) } \frac{1}{3}}$.

*(xii) The series $\sum \frac{1}{n^p}$ is convergent if

(a) $p \geq 1$ (b) $p > 1$ (c) $p < 1$ (d) $p \leq 1$

Solution: The correct alternative is $\boxed{\text{(b) } p > 1}$.

GROUP-B (Short-Answer type Questions)

Answer any three of the following: **$(3 \times 5 = 15)$**

**2. If $y = (x^2 - 1)^n$ then show that

$$(x^2 - 1)\, y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

Hence, find $y_n(0)$.

Solution: See Example 3.14.

**3. Using MVT, prove that

$$x > \tan^{-1} x > \frac{x}{1+x^2},\ 0 < x < \frac{\pi}{2}.$$

Solution: See Example 4.8.

***4. Show that

$$\begin{bmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{bmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

Solution: See Example 1.8.

***5. Test the nature of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots$$

Solution: See Example 22 of Chapter 8.

**6. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then show that

$$\left[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}\right] = \left[\vec{a} \;\; \vec{b} \;\; \vec{c}\right]^2$$

Solution: See Example 9.6.

*7. If $u = \tan^{-1} \dfrac{x^2 - y^2}{x - y}$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u.$$

Solution: See Example 20 of Chapter 6.

GROUP-C (Long-Answer type Questions)

Answer any three of the following: **(3 × 15 = 45)**

***8. (a) Determine the conditions under which the system of equations

$$x + y + z = 1, x + 2y - z = b, 5x + 7y + az = b^2$$

admits

(i) only one solution

(ii) no solution

(iii) many solutions

Solution: If we write the system of linear equations in the matrix form as $AX = B$ then the coefficient matrix of the system of linear equations is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{pmatrix}$$

And the augmented matrix is

$$\bar{A} = (A|B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & a & b^2 \end{pmatrix}$$

The system of equations has only one solution when the determinant of the coefficient matrix is not equal to zero.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{vmatrix} = a - 1$$

Therefore, for det $A \neq 0 \Rightarrow a \neq 1$ the system of equations has only one solution.

For $a = 1$, the system has either no solution or many solutions. When $a = 1$, the augmented matrix becomes

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & 1 & b^2 \end{pmatrix}$$

Appling elementary row operations on the matrix $\bar{A}$, we have

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & 1 & b^2 \end{pmatrix} \xrightarrow{R_2 - R_1, R_3 - 5R_1}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & -4 & b^2 - 5 \end{pmatrix} \xrightarrow{R_1 - R_2, R_3 - 2R_1}$$

$$\begin{pmatrix} 1 & 0 & 3 & -b+2 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & 0 & b^2 - 2b - 3 \end{pmatrix}$$

The system of equations is consistent when Rank A = Rank $\bar{A}$ and this is possible for

$$b^2 - 2b - 3 = 0$$

i.e., $$b = -1, 3.$$

In this case, Rank A = Rank $\bar{A}$ = 2, which is less then number of unknowns (= 3) and the system has **infinitely many solutions.**

Again, if

$$b^2 - 2b - 3 \neq 0 \Rightarrow b \neq -1, 3.$$

then Rank $A = 2$ and Rank $\bar{A} = 3$, i.e., Rank $A \neq$ Rank $\bar{A}$ and so the system of equations is inconsistent and corestpondingly, the system has **no solution.**

Summarising the above,

(i) the system of equations has **only one solution** when a $\neq$ 1

(ii) the system of equations has **infinitely many solutions** when $a = 1$, $b = -1$ or $a = 1$, $b = 3$

(iii) the system of equations has **no solution** when $a = 1$ and $b \neq -1, 3$

***(b) Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: The characteristic equation of A is

$$\det(A - \lambda I) = 0$$

or, $$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or, $$(1-\lambda)\{(2-\lambda)^2 - 1\} = 0$$

or, $$(1-\lambda)^2 (3-\lambda) = 0$$

or, $$\lambda = 1, 1, 3.$$

Therefore, the eigen values of the matrix A are $\lambda = 1, 1, 3$.

Let $X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 1$.

Then, by the difinition of eigen vector, we have

$$AX = 1.X$$

or, $$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or, $$\begin{aligned} 2x + y + z &= x \\ x + 2y + z &= y \\ z &= z \end{aligned}$$

The above system is equivalent to

$$x + y + z = 0$$

Let $y = k_1$ and $z = k_2$, then $x = -k_1 - k_2$ where k_1 and k_2 are arbitray constants.

Therefore, the eigen vector corresponding to the eigen value $\lambda = 1$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{pmatrix}$$

$$= k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Again, let $X_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 3$.

Therefor, we have

$$AX_2 = 3X_2$$

or,
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or,
$$\begin{aligned} 2x + y + z &= 3x \\ x + 2y + z &= 3y \\ z &= 3z \end{aligned}$$

or,
$$\begin{aligned} -x + y + z &= 0 \\ x - y + z &= 0 \\ 2z &= 0 \end{aligned}$$

The above system is equivalent to

$$x - y + z = 0, z = 0$$

which imples the system

$$x - y = 0, z = 0$$

Let $y = k_1$, then $x = k_1$ where k_1 is any arbitrary constant.

Therefore, the eigen vector corresponding to the eigen value $\lambda = 3$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_1 \\ 0 \end{pmatrix}$$

$$= k_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

**(c) Find whether the following series is convergent:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

Solution: See Example 23 of Chapter 8.

*9. (a) If $f(x) = x^2$, $g(x) = x^3$ on [1, 2], is Cauchy's mean value theorem applicable? It so find ξ.

Solution: (i) The functions $f(x) = x^2$ and $g(x) = x^3$ are both being polynomials, continuous in [1, 2];

(ii) $f'(x) = 2x$ and $g'(x) = 3x^2$ which exists for all values of $x \in (1, 2)$; and

(iii) $g'(x) \neq 0$ for all values of x in $1 < x < 2$.

Since all the conditions of Cauchy's MVT are satisfied by the given functions, Cauchy's MVT is applicable and so there should exist such a $\xi \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(\xi)}{g'(\xi)}$$

which implies

$$\frac{4-1}{8-1} = \frac{2\xi}{3\xi^2} \Rightarrow \xi = \frac{14}{9}.$$

So, $\xi = \dfrac{14}{9}$, which lies between 1 and 2.

**(b) If $I_n = \displaystyle\int \frac{\cos n\theta}{\cos\theta}\, d\theta$, show that

$$(n-1)(I_n + I_{n-2}) = 2\sin(n-1)\theta.$$

Hence, evaluate $\displaystyle\int (4\cos^2\theta - 3)\, d\theta$.

Solution: See Example 5.6.

**(c) If $r = |\vec{r}\,|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$\vec{\nabla}(r^n) = n \cdot r^{n-2} \cdot \vec{r}.$$

Solution: See Example 7 of Chapter 9.

*10. (a) Find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ where $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = \sin\theta$.

Solution: By chain rule for Jacobians, we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}. \tag{1}$$

Now by the definition of a Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} = 12xy.$$

Also, we have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 0 & \cos\theta \end{vmatrix} = \cos^2\theta.$$

Putting the above results in (1), we obtain

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 12xy \cdot \cos^2\theta$$

$$= 12 \cdot r\cos\theta \cdot \sin\theta \cdot \cos^2\theta = 12r\sin\theta\cos^3\theta.$$

**(b) Verify Green's theorem for

$$\vec{F} = (xy + y^2)\hat{i} + x^2\hat{j}$$

where the curve C is bounded by $y = x$ and $y = x^2$.

Solution: See Example 13 of Chapter 9.

***(c) Evaluate:

$$\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx.$$

Solution:

$$\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx = \int_0^a \left[\int_0^x \left\{ \int_0^y x^3 y^2 z dz \right\} dy \right] dx$$

$$= \int_0^a \left[\int_0^x x^3 y^2 \left[\frac{z^2}{2} \right]_0^y dy \right] dx$$

$$= \frac{1}{2} \int_0^a \left[\int_0^x x^3 y^4 dy \right] dx$$

$$= \frac{1}{2} \int_0^a \left[\frac{x^3 y^5}{5} \right]_0^x dx = \frac{1}{10} \int_0^a x^8 dx = \frac{a^9}{90}.$$

***11. (a) Find the maxima and minima of the function

$$x^3 + y^3 - 3x + 12y + 20.$$

Also, find the saddle point.

Solution: See Example 25 of Chapter 6.

**(b) State Cayley–Hamilton theorem and verify the same for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Find A^{-1} and A^8.

Solution: The characteristic equation of A is det $(A - \lambda I_n) = 0$

i.e., $$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

i.e., $$\lambda^2 - 5 = 0.$$

By Cayley–Hamilton theorem, we know that every square matrix satisfies its own characteristic eqation. Therefore,

$$A^2 - 5I_2 = O \tag{1}$$

i.e., $$A^2 = 5I_2 \Rightarrow A\left(\frac{1}{5}A\right) = I_2$$

i.e., $$A^{-1} = \frac{1}{5}A$$

Therefore,

$$A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Again, by (1), we have

$$A^2 = 5I_2$$

$$\Rightarrow \quad (A^2)^4 = (5I_2)^4$$

$$\Rightarrow \quad A^8 = 5^4(I_2)^4 = 625I_2$$

$$\Rightarrow \quad A^8 = 625\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}.$$

(c) Show that Curl $\vec{\nabla} f = 0$, where $f(x, y, z) = x^2y + 2xy + z^2$.

Solution: See Example 9.12.

***12. (a) Given the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0 \quad , & (x, y) = (0,0) \end{cases}$$

Find from the definition $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Is $f_{xy} = f_{yx}$?

Solution: See Example 15 of Chapter 6.

***(b) Integrate by changing the order of integration:

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$$

Solution: By the given form of integration, it is clear that we have to integrate first w.r.t y which varies from $y = \dfrac{x^2}{a}$ to $y = 2a - x$ and then we are to integrate w.r.t x which varies from $x = 0$ to $x = a$.

Here, $y = \dfrac{x^2}{a} \Rightarrow x^2 = ay$ (representing a parabola) and $y = 2a - x$ (representing a strainght line) itersects at the point (a, a).

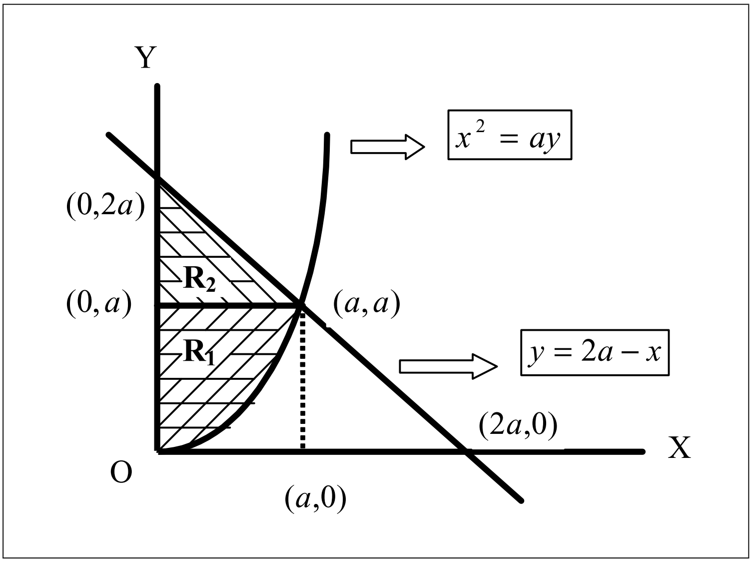

Fig. Q. 1

The region of integration is the area bounded by $x^2 = ay$ (parabola), $y = 2a - x$ (straight line) and y-axis as shown in the figure by the shaded portion.

Now when we change the order of integration, i.e., first we integrate w.r.t x and then w.r.t y, for taking limits we subdivide the region of integration ito two parts as R_1 and R_2. So we take the integration separately into the two subregions R_1 and R_2.

Here, in the region R_1,

- x varies from y-axis to the parabola $x^2 = ay (\Rightarrow x = \sqrt{ay})$, i.e., x varies from 0 to $\sqrt{ay}$

- y varies from x-axis to the line $y = a$, i.e., y varies from 0 to a

  Now, in the region R_2,
- x varies from y-axis to the straight line $y = 2a - x$ ($\Rightarrow x = 2a - y$), i.e., x varies from 0 to $2a - y$
- y varies from the line $y = a$ to the line $y = 2a$, i.e., y varies from a to $2a$

Changing the order of integration, we have

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx = \underbrace{\int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy}_{\text{(Region } R_1\text{)}} + \underbrace{\int_a^{2a} \int_0^{2a-y} xy\,dx\,dy}_{\text{(Region } R_2\text{)}}$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{a}{2}\int_0^a y^2 dy + \frac{1}{2}\int_a^{2a} y(2a-y)^2\,dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\int_a^{2a} [4a^2 y - 4ay^2 + y^3]\,dy$$

$$= \frac{a^4}{6} + \frac{1}{2}\left\{4a^2\left[\frac{y^2}{2}\right]_a^{2a} - 4a\left[\frac{y^3}{3}\right]_a^{2a} + \left[\frac{y^4}{4}\right]_a^{2a}\right\}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left\{6a^4 - \frac{28}{3}a^4 + \frac{15}{4}a^4\right\} = \frac{3}{8}a^4.$$

**(c) If $F(p, v, t) = 0$, show that

$$\left(\frac{dp}{dt}\right)_{v\text{ constant}} \times \left(\frac{dv}{dp}\right)_{t\text{ constant}} \times \left(\frac{dt}{dv}\right)_{p\text{ constant}} = -1.$$

Soltion: See Example 6.17.

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-1 (NEW) 2011

MATHEMATICS-I (M 101)

Time Alloted: 3 Hours *Full Marks: 70*

GROUP-A (Multiple Choice Type Questions)

1. Choose the correct alternatives for any ten of the following: (10 × 1 = 10)

(i) The least upper bound of the sequence $\left\{\frac{n}{n+1}\right\}$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

Solution: The correct alternative is $\boxed{\text{(c) } 1}$

(ii) The value of $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix}$ is

(a) 2000 (b) 0
(c) 45 (d) none of these

Solution: The correct alternative is $\boxed{\text{(b) } 0}$

(iii) If $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ is the characteristic equation of a square matrix A then A^{-1} is equal to

(a) $A^2 - 6A + 9I$ (b) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I$

(c) $A^2 - 6A + 9$ (d) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}$

Solution: The correct alternative is $\boxed{\text{(b) } \frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I}$

(iv) If $x = r\cos\theta$, $y = r\sin\theta$, then $\frac{\partial(r, \theta)}{\partial(x, y)}$ is

(a) r (b) 1

(c) $\frac{1}{r}$ (d) none of these

Solution: The correct alternative is $\boxed{\text{(c) } \frac{1}{r}}$

(v) $f(x, y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x}$ is a homogeneous function of degree

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) 2

Solution: The correct alternative is $\boxed{\text{(b) } -\frac{1}{2}}$

(vi) If $\vec{\alpha} \cdot (\vec{\beta} \times \vec{\gamma}) = 0$, then $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are

(a) coplanar (b) independent
(c) collinear (d) none of these

Solution: The correct alternative is $\boxed{\text{(a) coplanar}}$

(vii) The nth derivative of $(ax + b)^{10}$ is (where $n > 10$)

(a) a^{10} (b) $10!a^{10}$ (c) 0 (d) 10!

Solution: The correct alternative is $\boxed{\text{(c) } 0}$

(viii) If for any two vectors $\vec{a}$ and $\vec{b}$,

$$\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$$

then $\vec{a}$ and $\vec{b}$ are

(a) parallel (b) collinear
(c) perpendicular (d) none of these

Solution: The correct alternative is $\boxed{\text{(d) orthogonal}}$

(ix) If $A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then $A =$

(a) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ (c) $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Solution: The correct alternative is $\boxed{\text{(b) } \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}$

(x) The reduction formula of $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$ is

(a) $I_n = \left(\frac{n-1}{n}\right) I_{n-1}$ (b) $I_n = \left(\frac{n}{n-1}\right) I_n$

(c) $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ (d) none of these

Solution: The correct alternative is $\boxed{\text{(c) } I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$

(xi) The series $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ is

(a) convergent (b) divergent

(c) oscillatory (d) none of these

Solution: The correct alternative is $\boxed{\text{(b) divergent}}$

(xii) Lagrange's Mean Value Theorem is obtained from Cauchy's Theorem for two functions $f(x)$ and $g(x)$ by putting $g(x) =$

(a) 1 (b) x^2 (c) x (d) $\frac{1}{x}$

Solution: The correct alternative is $\boxed{\text{(c) } x}$

GROUP-B (Short-Answer Type Questions)

Answer any three of the following: **($3 \times 5 = 15$)**

2. Prove that every square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution: See Theorem 1.2 of Page 1.10.

3. By Laplace's method, prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$$

(consider minors of order 2).

Solution: Here, we expand the given determinant by Laplace's method of expansion in terms of a minor of order 2 considering the first two rows as follows:

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \times (-1)^{(1+2)+(1+2)} \begin{vmatrix} a & b \\ -b & a \end{vmatrix} + \begin{vmatrix} a & c \\ -b & d \end{vmatrix} \times (-1)^{(1+2)+(1+3)} \begin{vmatrix} -d & b \\ c & a \end{vmatrix}$$

$$+ \begin{vmatrix} a & d \\ -b & -c \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} -d & a \\ c & -b \end{vmatrix} + \begin{vmatrix} b & c \\ a & d \end{vmatrix} \times (-1)^{(1+2)+(2+3)} \begin{vmatrix} -c & b \\ -d & a \end{vmatrix}$$

$$+ \begin{vmatrix} b & d \\ a & -c \end{vmatrix} \times (-1)^{(1+2)+(2+4)} \begin{vmatrix} -c & a \\ -d & -b \end{vmatrix} + \begin{vmatrix} c & d \\ d & -c \end{vmatrix} \times (-1)^{(1+2)+(3+4)} \begin{vmatrix} -c & -d \\ -d & c \end{vmatrix}$$

$$\begin{aligned}
&= (a^2 + b^2)(a^2 + b^2) + (ad + bc)(ad + bc) \\
&\quad + (-ac + bd)(-ac + bd) + (bd - ac)(bd - ac) \\
&\quad + (bc + ad)(bc + ad) + (c^2 + d^2)(c^2 + d^2) \\
&= (a^2 + b^2)^2 + 2(ad + bc)^2 + 2(-ac + bd)^2 \\
&\qquad + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2[(ad + bc)^2 + (-ac + bd)^2] \\
&\qquad + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2[a^2 d^2 + b^2 c^2 - 2adbc + a^2 c^2 \\
&\qquad + b^2 d^2 - 2acbd] + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2(a^2 + b^2)(c^2 + d^2) + (c^2 + d^2)^2 \\
&= (a^2 + b^2 + c^2 + d^2)^2
\end{aligned}$$

4. If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$ then prove that

$$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

Solution: Here, we have

$$2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$$

or, $$y^{\frac{1}{m}} - 2x + \frac{1}{y^{\frac{1}{m}}} = 0$$

or, $$\left(y^{\frac{1}{m}}\right)^2 - 2x\left(y^{\frac{1}{m}}\right) + 1 = 0$$

Applying the rule for finding solution of the above quadratic equation, we get

$$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{(2x)^2 - 4.1.1}}{2} = \left(x \pm \sqrt{x^2 - 1}\right)$$

$$\Rightarrow \quad y = \left(x \pm \sqrt{x^2 - 1}\right)^m \tag{i}$$

Differentiating (i) w.r.t. x, we have

$$y_1 = m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1} \cdot \left(1 \pm \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 - 1}} \cdot 2x\right)$$

$$= m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1} \cdot \left(\frac{\sqrt{x^2 - 1} \pm x}{\sqrt{x^2 - 1}}\right)$$

$$= \pm m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1} \cdot \frac{\left(x \pm \sqrt{x^2 - 1}\right)}{\sqrt{x^2 - 1}}$$

i.e., $$y_1 = \pm m\frac{\left(x \pm \sqrt{x^2 - 1}\right)^m}{\sqrt{x^2 - 1}} = \pm\frac{my}{\sqrt{x^2 - 1}} \tag{ii}$$

Squaring (ii) and simplifying, we get

$$(y_1)^2(x^2 - 1) = m^2y^2 \tag{iii}$$

Again differentiating (iii) w.r.t. x, we have

$$2y_1y_2(x^2 - 1) + (y_1)^2\,2x = m^2\,2y \cdot y_1$$

$$\Rightarrow \quad y_2(x^2 - 1) + y_1x - m^2y = 0 \tag{iv}$$

Now applying Leibnitz's theorem, we differentiate (iv) n times w.r.t. x,

$$\{y_2(x^2-1)\}_n + \{y_1 x\}_n - \{m^2 y\}_n = 0$$

$$\Rightarrow \quad [\{y_2\}_n \cdot (x^2-1) + {}^nC_1\{y_2\}_{n-1}\cdot(2x) + {}^nC_2\{y_2\}_{n-2}\cdot(2)] + [\{y_1\}_n \cdot x + {}^nC_1\{y_1\}_{n-1}\cdot 1] - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

5. If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution: See Example 21 of Page 6.19.

6. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (xdy - ydx)$.

Solution: We know that Green's theorem states the following:

$$\oint_C \{M(x,y)dx + N(x,y)dy\} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy \tag{i}$$

where the region R on the two-dimensional xy plane is bounded by a simple closed curve C and the line integral along the curve C is taken in the anti-clockwise direction.

Here, comparing LHS of (i) with $\oint_C (xdy - ydx)$, we have

$$M = -y,\ N = x \Rightarrow \frac{\partial M}{\partial y} = -1,\ \frac{\partial N}{\partial x} = 1 \tag{ii}$$

Therefore using (ii) in (i), we get

$$\begin{aligned}\oint_C (xdy - ydx) &= \iint_R [1-(-1)]\,dx\,dy \\ &= 2\iint_R dx\,dy \\ &= 2 \times [\text{Area bounded by } C]\end{aligned}$$

$$\Rightarrow \quad \text{Area bounded by } C = \frac{1}{2}\oint_C (xdy - ydx)$$

Hence, the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (xdy - ydx)$.

GROUP-C (Long-Answer Type Questions)

Answer any three of the following: **($3 \times 15 = 45$)**

7. (i) If

$$f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right),$$

verify $f_{xy} = f_{yx}$.

Solution: See Example 6.6 of Page 6.34.

(ii) State Rolle's theorem and examine if you can apply the same for $f(x) = \tan x$ in $[0, \pi]$.

Solution: See Example 4.3 of Page 4.33.

(iii) Find the value of λ and μ for which

$$\begin{aligned} x + y + z &= 3 \\ 2x - y + 3z &= 4 \\ 5x - y + \lambda z &= \mu \end{aligned}$$

has (a) a unique solution, (b) many solutions (c) no solution.

Solution: If we write the system of linear equations in the matrix form as $AX = B$ then the coefficient matrix of the system of linear equations is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & \lambda \end{pmatrix}$$

and the augmented matrix is

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & \lambda & \mu \end{pmatrix}$$

The system of equations has a unique solution when the determinant of the coefficient matrix is not equal to zero.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & \lambda \end{vmatrix}$$

$$= 1(-\lambda + 3) - 1(2\lambda - 15) + 1(-2 + 5) = -3\lambda + 21$$

Therefore, for $\det A \neq 0 \Rightarrow -3\lambda + 21 \neq 0 \Rightarrow \lambda \neq 7$, the system of equations have **unique solution**.

When $\lambda = 7$, the augmented matrix becomes

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & \mu \end{pmatrix}$$

Applying elementary row operations on the matrix $\bar{A}$, we have

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & \mu \end{pmatrix} \xrightarrow{R_2 - 2R_1,\, R_3 - 5R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & \mu - 15 \end{pmatrix} \xrightarrow{R_3 - 2R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix} \xrightarrow{\left(-\frac{1}{3}\right)R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{4}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix}$$

The system of equations is consistent when Rank A = Rank $\bar{A}$ and this is possible for

$$\mu - 11 = 0 \Rightarrow \mu = 11.$$

In this case, Rank A = Rank $\bar{A}$ = 2, which is less then number of unknowns $(=3)$ and the system has **infinitely many solutions**.

Again, if

$$\mu - 11 \neq 0 \Rightarrow \mu \neq 11.$$

then Rank A = 2 and Rank $\bar{A}$ = 3, i.e., Rank $A \neq$ Rank $\bar{A}$, and so the system of equations is inconsistent and correspondingly the system has **no solution**.

Summarizing the above, the system of equations has

(a) a **unique solution** when $\lambda \neq 7$

(b) **infinitely many solutions** when $\lambda = 7$ and $\mu = 11$

(c) **no solution** when $\lambda = 7$ and $\mu \neq 11$.

8. (i) Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$$

Find also the saddle points.

Solution: See Example 6.24 of Page 6.51.

(ii) State Leibnitz's test for alternating series and apply it to examine the convergence of

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty$$

Solution: See Example 8.27 of Page 8.42.

(iii) Applying Lagrange's Mean Value Theorem, prove that

$$\frac{x}{1+x} \le \log(1 + x) \le x, \text{ for all } x > 0.$$

Solution: See Example 7 of Page 4.11.

9. (i) If $y = e^{m\sin^{-1}x}$, show that

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0.$$

Hence, find y_n when $x = 0$.

Solution: Here, we are given that

$$y = e^{m\sin^{-1}x} \tag{i}$$

Differentiating (i) w.r.t. x, we have

$$y_1 = e^{m\sin^{-1}x} \cdot m\left(\frac{1}{\sqrt{1-x^2}}\right)$$

i.e.,

$$y_1 = \frac{my}{\sqrt{1-x^2}} \tag{ii}$$

Squaring (ii) and simplifying, we get

$$(y_1)^2(1 - x^2) = m^2y^2 \tag{iii}$$

Again differentiating (iii) w.r.t. x, we have

$$2y_1y_2(1 - x^2) + (y_1)^2(-2x) = m^2 2y \cdot y_1$$

$$\Rightarrow \quad y_2(1 - x^2) - y_1x - m^2y = 0 \tag{iv}$$

Now applying Leibnitz's theorem, we differentiate (iv) n times w.r.t. x,

$$\{y_2(1 - x^2)\}_n - \{y_1x\}_n - \{m^2y\}_n = 0.$$

$$\Rightarrow \quad [\{y_2\}_n \cdot (1 - x^2) + {}^nC_1\{y_2\}_{n-1} \cdot (-2x) + {}^nC_2\{y_2\}_{n-2} \cdot (-2)]$$
$$-[\{y_1\}_n \cdot x + {}^nC_1\{y_1\}_{n-1} \cdot 1] - m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad \text{(v)}$$

Calculation of y_n when $x = 0$, i.e., $(y_n)_0$:

Putting $x = 0$ in (v), we have

$$(y_{n+2})_0 = (n^2 + m^2)\,(y_n)_0$$

Replacing n by $n - 2$, we get

$$(y_n)_0 = [(n-2)^2 + m^2]\,(y_{n-2})_0 \quad \text{(vi)}$$

Replacing n by $n - 2$ in (vi), we get

$$(y_{n-2})_0 = [(n-4)^2 + m^2]\,(y_{n-4})_0 \quad \text{(vii)}$$

Using (vii) in (vi),

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2]\,(y_{n-4})_0$$

Similarly, we have

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2]\,[(n-6)^2 + m^2]\,(y_{n-6})_0 \quad \text{(viii)}$$

Proceeding in a similar manner we have from (viii), when n is odd as the following:

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2] \ldots [3^2 + m^2]\,[1^2 + m^2]\,(y_1)_0 \quad \text{(ix)}$$

From (ii), we have $(y_1)_0 = m$. Using this in (ix), we get

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2] \ldots [3^2 + m^2]\,[1^2 + m^2]\,m, \text{ when } n \text{ is odd.}$$

Also proceeding in a similar manner we have from (viii), when n is even as the following:

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2] \ldots [4^2 + m^2]\,[2^2 + m^2]\,(y_2)_0 \quad \text{(x)}$$

From (iv), we have $(y_2)_0 = m^2$. Using this in (x), we get

$$(y_n)_0 = [(n-2)^2 + m^2]\,[(n-4)^2 + m^2] \ldots [4^2 + m^2]\,[2^2 + m^2]m^2, \text{ when } n \text{ is even.}$$

(ii) Prove that $\left[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\; \vec{c}+\vec{a}\right] = 2\left[\vec{a}\;\vec{b}\;\vec{c}\right]$, where $\vec{a}\;\; \vec{b}\;\; \vec{c}$ are three vectors.

Solution: Using the definition of scalar triple product, we write

$$\left[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\; \vec{c}+\vec{a}\right] = \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]$$

$$= \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}+\vec{c}\right)\times\vec{c}+\left(\vec{b}+\vec{c}\right)\times\vec{a}\right]$$

$$= \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}\times\vec{c}\right)+\left(\vec{c}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right)\right]$$

$$= (\vec{a}+\vec{b})\cdot\left[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})\right],$$

since $\quad \vec{c}\times\vec{c}=\vec{0}$

$$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})$$
$$+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}\;\vec{b}\;\vec{c}\right]+\left[\vec{a}\;\vec{b}\;\vec{a}\right]+\left[\vec{a}\;\vec{c}\;\vec{a}\right]+\left[\vec{b}\;\vec{b}\;\vec{c}\right]$$
$$+\left[\vec{b}\;\vec{b}\;\vec{a}\right]+\left[\vec{b}\;\vec{c}\;\vec{a}\right]$$

By the property of scalar triple product of vectors, we have $\left[\vec{a}\;\vec{b}\;\vec{a}\right] = 0$, $\left[\vec{a}\;\vec{c}\;\vec{a}\right]=0$, $\left[\vec{b}\;\vec{b}\;\vec{c}\right]=0$, $\left[\vec{b}\;\vec{b}\;\vec{a}\right]=0$ (since two vectors in the product are same) and $\left[\vec{b}\;\vec{c}\;\vec{a}\right] = \left[\vec{a}\;\vec{b}\;\vec{c}\right]$.

Using this in the above, we get

$$\left[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}\right] = \left[\vec{a}\;\vec{b}\;\vec{c}\right]+0+0+0+0+\left[\vec{a}\;\vec{b}\;\vec{c}\right] = 2\left[\vec{a}\;\vec{b}\;\vec{c}\right]$$

(iii) Find the directional derivative of $f = xyz$ at $(1,1,1)$ in the direction $2\hat{i}-\hat{j}-2\hat{k}$.

Solution: Here, it is given that $f = xyz$. Then

$$\vec{\nabla}f = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)f$$
$$= \left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)$$
$$= yz\hat{i}+xz\hat{j}+xy\hat{k}$$

So, $$\left[\vec{\nabla}f\right]_{(1,1,1)} = \hat{i}+\hat{j}+\hat{k}$$

Here we are to find the directional derivative in the direction $2\hat{i}-\hat{j}-2\hat{k}$. The unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$ is given by

$$\hat{a} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$$

Then the required directional derivative of $f = xyz$ at (1, 1, 1) in the direction of $2\hat{i}-\hat{j}-2\hat{k}$ is given by

$$\left[\vec{\nabla}f\right]_{(1,1,1)}\cdot\hat{a} = (\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}\right) = -\frac{1}{3}$$

10. (i) Prove that

$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: See Example 1.16 of Page 1.43.

(ii) State the Divergence Theorem of Gauss. Verify divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$, $z = 2$.

Solution: See Example 9.29 of Page 9.58.

(iii) Test the series for convergence:

$$\frac{1^p}{2^q} + \frac{2^p}{3^q} + \frac{3^p}{4^q} + \ldots$$

Solution: Let us consider the given series as

$$\sum_{n=1}^{\infty} a_n = \frac{1^p}{2^q} + \frac{2^p}{3^q} + \frac{3^p}{4^q} + \ldots$$

Then

$$a_n = \frac{n^p}{(n+1)^q} = \frac{n^p}{n^q\left(1+\frac{1}{n}\right)^q} = \frac{1}{n^{q-p}\left(1+\frac{1}{n}\right)^q}$$

Let us consider another series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{q-p}}$$

which is convergent for $q - p > 1$ and divergent for $q - p \le 1$.

Now we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^q} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is convergent for $q - p > 1$ and divergent for $q - p \le 1$, by comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent for $q - p > 1$ and divergent for $q - p \le 1$.

11. (i) Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Hence obtain $\int_0^{\frac{\pi}{2}} \sin^9 x\,dx$.

Solution: See Section 5.2 of Page 5.1.

(ii) Given two vectors $\vec{\alpha} = 3\vec{i} - \vec{j}$, $\vec{\beta} = 2\vec{i} + \vec{j} - 3\vec{k}$. Express $\vec{\beta}$ in the form $\vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution: See Example 9.1 of Page 9.17.

(iii) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the scalar function ϕ, such that $\vec{A} = \vec{\nabla}\phi$.

Solution: See Example 9.11 of Page 9.36.

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-1 (NEW) 2012

MATHEMATICS-II (M 101)

Time Alloted: 3 Hours *Full Marks: 70*

GROUP-A (Multiple Choice Type Questions)

1. Choose the correct alternatives for any ten of the following: (10 × 1 = 10)

*(i) The sequence $\left\{(-1)^n \frac{1}{n}\right\}$ is

(a) convergent (b) oscillatory

(c) divergent (d) none of these

*(ii) The matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is

(a) symmetric (b) skew-symmetric

(c) singular (d) orthogonal

**(iii) The value of t for which

$$\vec{f} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + tz)\,\hat{k}$$

is solenoidal is

(a) 2 (b) −2 (c) 0 (d) 1

*(iv) The series $\sum \frac{1}{n^p}$ is convergent if

(a) $p \geq 1$ (b) $p \leq 1$ (c) $p > 1$ (d) $p < 1$

**(v) The two eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

Level of difficulty:- *Low, **Medium, ***High.

are 2 and −2. The third eigenvalue is

(a) 1 (b) 0 (c) 3 (d) 2

*(vi) If Rolle's theorem is applied to $f(x) = x(x^2 - 1)$ in [0, 1] then $c =$

(a) 1 (b) 0 (c) $-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

*(vii) If $u = \frac{x^3 + y^3}{\sqrt{x^2 + y^2}}$, find the value of n so that $xu_x + yu_y = nu$.

(a) 0 (b) 2

(c) $\frac{1}{2}$ (d) none of these

**(viii) The n^{th} derivative of $\sin(5x + 3)$ is

(a) $5^n \cos(5x + 3)$ (b) $5^n \sin(\frac{n\pi}{2} + 5x + 3)$

(c) $5^n \cos(\frac{n\pi}{2} + 5x + 3)$ (d) none of these

**(ix) The value of $\int_C (xdx - dy)$ where C is a line joining (0, 1) to (1, 0) is

(a) 0 (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

*(x) The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 \theta d\theta$ is

(a) 0 (b) $\frac{6.4.2}{7.5.3.1}$

(c) $\frac{6!}{7!}$ (d) none of these

*(xi) If the characteristic equation of a matrix A is $X^3 + 3X^2 + 5X + 9 = 0$ then determinant of the matix is

(a) 7 (b) 5 (c) 6 (d) 9

*(xii) Let A and B be two square matrices and A^{-1}, B^{-1} exist. Then $(AB)^{-1}$ is

a) $A^{-1}B^{-1}$ b) $B^{-1}A^{-1}$

c) AB d) none of these

Answers

(i) (b)	(ii) (d)	(iii) (b)	(iv) (c)
(v) (d)	(vi) (d)	(vii) (b)	(viii) (b)
(ix) (b)	(x) (a)	(xi) (d)	(xii) (b)

GROUP B (Short Answer Type Questions)

Answer any three of the following: **($3 \times 15 = 45$)**

**2. Verify Rolle's theorem for the function

$$f(x) = |x|, -1 \le x \le 1$$

Solution: See Example 2 of Page 4.4.

***3. A and B are orthogonal matrices and $|A| + |B| = 0$. Prove that $A + B$ is singular.

Solution: Since A and B are orthogonal, $|A| = \pm 1 \neq 0$, $|B| = \pm 1 \neq 0$. Also, $A^TA = AA^T = I$ and $B^TB = BB^T = I$.

Let us consider $C = A + B$. Then

$$C^T = (A + B)^T = A^T + B^T$$

$$\Rightarrow \quad C^TA = A^TA + B^TA$$

$$\Rightarrow \quad C^TA = I + B^TA$$

$$\Rightarrow \quad BC^TA = BI + BB^T A$$

$$\Rightarrow \quad BC^TA = B + IA$$

$$\Rightarrow \quad BC^TA = B + A = A + B$$

Now,

$$|A + B| = |BC^TA| = |B||C^T||A|$$
$$= -|A|\,|C|\,|A|\text{, since } |A| + |B| = 0 \text{ and } |C^T| = |C|$$
$$= -\{|A|\}^2\,|C| = -|C|\text{, since } |A| = \pm 1$$
$$= -|A + B|$$

$$\therefore \quad 2\,|A + B| = 0$$

$$\therefore \quad |A + B| = 0.$$

Hence, $A + B$ is singular.

**4. Find the n^{th} derivative of $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$

Solution: Let us consider

$$y = \frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow \quad x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Substituting $x = 1, 2, 3$ we have respectively

$$2 = A(1-2)(1-3)$$

i.e., $A = 1$

$$5 = B(2-1)(2-3)$$

i.e., $B = -5.$

and

$$10 = C(3-1)(3-2)$$

i.e., $C = 5$

Therefore,

$$y = \frac{x^2}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{5}{(x-3)}$$

since $y_n = \dfrac{(-1)^n.n!.a^n}{(ax+b)^{n+1}}$ when $y = \dfrac{1}{ax+b}$, we get from the above,

$$y_n = \frac{(-1)^n.n!}{(x-2)^{n+1}} - 5\frac{(-1)^n.n!}{(x-2)^{n+1}} + 5\frac{(-1)^n.n!}{(x-3)^{n+1}}$$

**5. Let

$$f(x, y) = \begin{cases} \dfrac{xy}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Solution: We have

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \tag{1}$$

Now,

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{\dfrac{hk}{h+k^2} - \dfrac{h.0}{h+0^2}}{k}$$

$$= \lim_{k\to 0} \frac{h}{h+k^2} = 1$$

and

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Using the above two results in (1), we obtain

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{1-0}{h} = \lim_{h\to 0} \frac{1}{h}$$

which does not exist. Again, we have

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \tag{2}$$

Now,

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{\frac{hk}{h+k^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{k}{h+k^2} = \frac{1}{k}$$

and

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Using the above two results in (2), we obtain

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f(0,k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{1}{k} - 0}{k} = \lim_{k\to 0} \frac{1}{k^2}$$

which also does not exist.

**6. Find div $\vec{F}$, and curl $\vec{F}$ where

$$\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$$

Solution: See Example 9.9 of Page 9.35.

GROUP C (Long Answer Type Questions)

Answer any three of the following: **(3 × 15 = 45)**

***7. (a) If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: Here,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 10x + 4$$

**(b) Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2-\beta\gamma \\ 1 & \beta & \beta^2-\gamma\alpha \\ 1 & \gamma & \gamma^2-\alpha\beta \end{vmatrix} = 0$

Solution: See Example 1.14 of Page 1.41.

**(c) If $v = f(x^2+2yz, y^2+2zx)$, prove that

$$(y^2-zx)\frac{\partial v}{\partial x} + (x^2-yz)\frac{\partial v}{\partial y} + (z^2-xy)\frac{\partial v}{\partial z} = 0$$

Solution: See Example 6.7 of Page 6.35.

***8. (a) If $\theta = t^n e^{\frac{-r^2}{4t}}$, find what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

Solution: Since $\theta = t^n e^{\frac{-r^2}{4t}}$, we have

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}} \cdot \left(\frac{-2r}{4t}\right) = -\frac{\theta r}{2t}.$$

Now,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\cdot\left(-\frac{\theta r}{2t}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{\theta r^3}{2t}\right)$$

$$= -\frac{1}{r^2}\frac{r^3}{2t}\frac{\partial \theta}{\partial r} - \frac{1}{r^2}\frac{3r^2\theta}{2t} = \frac{1}{r^2}\frac{r^3}{2t}\left(-\frac{\theta r}{2t}\right) - \frac{1}{r^2}\frac{3r^2\theta}{2t}$$

$$= \frac{\theta r^2}{4t^2} - \frac{3\theta}{2t}$$

Again,

$$\frac{\partial \theta}{\partial t} = nt^{t-1}e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}} \cdot \left(\frac{r^2}{4t^2}\right) = \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}$$

By the given condition, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

$$\frac{\theta r^2}{4t^2} - \frac{3\theta}{2t} = \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}$$

which implies $n = -\frac{3}{2}$

**(b) Using the mean-value theorem, prove that

$$0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$$

Solution: See Example 7 (ii) of Page 4.12.

**(c) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$ $(n > 1)$ then show that

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Solution: See Example 5.7 of Page 5.27.

**9. (a) State D'Alembert's ratio test for convergence of an infinite series. Examine the convergence or divergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 \cdots$$

Solution: See Example 22 of Page 8.12.

**(b) If $y = e^{\tan^{-1} x}$ then show that

$$(1 + x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$

Solution:

We have

$$y = e^{\tan^{-1} x}$$

Now, differentiating w.r.t. x,

$$y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} = \frac{y}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2)y_1 = y \tag{1}$$

Again, differentiating (1) w.r.t. x,

$$(1 + x^2)y_2 + (2x)y_1 = y_1$$

$$\Rightarrow \quad (1 + x^2)y_1 + (2x - 1)y_1 = 0 \tag{2}$$

Now, differentiating (2) n times by Leibnitz's rule, we have

$$[y_2.(1 + x^2)]_n + [y_1.(2x - 1)]_n = 0.$$

i.e.,
$$[\{y_2\}_n.(1 + x^2) + {}^nC_1\{y_2\}_{n-1}.(2x) + {}^nC_2\{y_2\}_{n-2}.(2)]$$
$$+[\{y_1\}_n.(2x - 1) + {}^nC_1\{y_1\}_{n-1}.2] = 0$$

i.e., $[y_{n+2}.(1 + x^2) + 2nx.y_{n+1}. + n(n-1)y_n] + [(2x - 1).y_{n+1} + 2n.y_n] = 0$

i.e., $(1 + x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n + 1)y_n = 0$

**(c) Find the extreme value of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$

Solution: See Example 25 of Page 6.27.

***10. (a) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then verify that A satisfies its own characteristic equation.

Hence, find A^{-1} and A^9.

Solution: See Example 2.17 of Page 2.48.

**(b) If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$ then show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - \sin^2 u) \sin 2u$$

Solution: We have

$$u = \tan^{-1} \frac{x^3 + y^3}{x - y} \Rightarrow \tan u = \left(\frac{x^3 + y^3}{x - y}\right) = v(x, y), \text{ (say)}$$

Here,

$$v(tx, ty) = \frac{t^3(x^3 + y^3)}{t(x - y)} = t^2 \frac{x^3 + y^3}{(x - y)} = t^2 v(x, y).$$

Therefore, $v(x, y)$ is a homogeneous function of degree 2. Now, by Euler's theorem,

$$x \frac{\partial v(x, y)}{\partial x} + y \frac{\partial v(x, y)}{\partial y} = 2.v(x, y)$$

or, $$\frac{\partial(\tan u)}{\partial x} + y \frac{\partial(\tan u)}{\partial y} = 2 (\tan u)$$

or, $$\sec^2 u. \left\{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right\} = 2 (\tan u)$$

or, $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \tag{1}$$

Now, differentiating (1) partially w.r.t. x we get,

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \frac{\partial u}{\partial x}$$

or, $$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = 2(\cos 2u - 1)\frac{\partial u}{\partial x} \quad (2)$$

Again, differentiating (1) partially w.r.t. y we get,

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial y}$$

or, $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y} \quad (3)$$

Multiplying (2) by x and (3) by y and then adding we get,

$$\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right] = (2\cos 2u - 1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

or, $$\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right] = (2\cos 2u - 1)\sin 2u = (1 - 4\sin^2 u)\sin 2u$$

Hence, the result is proved.

***(c) Given the system of equations:

$x_1 + 4x_2 + 2x_3 = 1,\ 2x_1 + 7x_2 + 5x_3 = k,\ 4x_1 + mx_2 + 10x_3 = 2k + 1.$

Find for what values of k and m, the system has (i) an unique solution, (ii) no solution, and (iii) many solutions.

Solution: If we write the system of linear equations in the matrix form as $AX = B$ then the coefficient matrix of the system of linear equations is

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & m & 10 \end{pmatrix}$$

and the augmented matrix is

$$\bar{A} = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & k \\ 4 & m & 10 & 2k+1 \end{pmatrix}$$

The system of equations has a unique solution when the determinant of the coefficient matrix is not equal to zero.

$$\det A = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & m & 10 \end{vmatrix} = -m + 14$$

Therefore, for det $A \neq 0 \Rightarrow m \neq 14$, the system of equations has a **unique solution**.

When $m = 14$, the augmented matrix becomes

$$\bar{A} = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & k \\ 4 & 14 & 10 & 2k+1 \end{pmatrix}$$

Applying elementary row operations on the matrix $\bar{A}$, we have

$$\bar{A} = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & k \\ 4 & 14 & 10 & 2k+1 \end{pmatrix} \xrightarrow{R_2 - 2R_1,\, R_3 - 4R_1} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & k-2 \\ 0 & -2 & 2 & 2k-3 \end{pmatrix} \xrightarrow{R_3 - 2R_2}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & k-2 \\ 0 & 0 & 0 & (2k-3)-2(k-2) \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & k-2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here, Rank $A = 2$ and Rank $\bar{A} = 3$, i.e., Rank $A \neq$ Rank $\bar{A}$. So the system of equations is inconsistent and correspondingly, the system has **no solution**.

Summarizing the above, we have

i) the system of equations has a unique solution when $m \neq 14$

ii) the system of equations has no solution when $m = 14$

iii) it is not possible that the system of equations has many solutions

**11. (a) Show that $\vec{\nabla} r^n = nr^{n-2}\vec{r} = nr^{n-2}\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: See Example 7 of Page 9.23.

**(b) Evaluate $\iint \sqrt{4x^2 - y^2}\, dxdy$ over the triangle formed by the straight lines $y = 0$, $x = 1$, and $y = x$.

Solution: See Example 4 of Page 7.6.

***(c) Verify Stokes' theorem for

$$\vec{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k},$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: See Example 9.34 of Page 9.65.

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-I (NEW) 2013

MATHEMATICS-I (M 101)

Time Alloted: 3 Hours *Full Marks: 70*

GROUP-A (Multiple Choice Type Questions)

1. Choose the correct alternatives for any ten of the following: (10 × 1 = 10)

*(i) The value of the determinant $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix}$ is

(a) 0 (b) 10 (c) 100 (d) 1000

*(ii) The equation $x + y + z = 0$ has

(a) infinite solutions (b) no solution

(c) unique solution (d) two solutions

**(iii) The value of $\int_1^0 \int_0^1 (x+y)\,dxdy =$

(a) 2 (b) 3 (c) 1 (d) 0

*(iv) $f(x, y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x}$ is a homogeneous function of degree

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) 2

**(v) In the MVT

$$f(h) = f(0) + hf'(\theta h),\ 0 < \theta < 1$$

if $f(x) = \dfrac{1}{1+x}$ and $h = 3$ if then the value of θ is

(a) 1 (b) $\frac{1}{3}$

(c) $\frac{1}{\sqrt{2}}$ (d) none of these

Level of difficulty:- *Low, **Medium, ***High.

**(vi) If $y = e^{ax+b}$, then $(y_5)_0 =$

(a) ae^b (b) a^5e^b

(c) $a^b e^{ax}$ (d) none of these

*(vii) The series $\sum \frac{1}{(2n+1)^n}$ is

(a) convergent (b) divergent

(c) oscillatory (d) none of these

*(viii) $\int_0^{\frac{\pi}{2}} \cos^6 x dx$ is equal to

(a) $\frac{7\pi}{12}$ (b) $\frac{5\pi}{32}$ (c) $\frac{\pi}{32}$ (d) $\frac{3\pi}{16}$

**(ix) If $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$ then the vectors $\vec{a}, \vec{b}, \vec{c}$, are

(a) colinear (b) coplanar

(c) orthogonal (d) none of these

*(x) If $u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$, then the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(a) 0 (b) $2u(x, y)$

(c) $u(x, y)$ (d) none of these.

(xi) The centre of the sphere given by the equation

$$a(x^2 + y^2 + z^2) + 2bx + 2cy + 2dz + w = 0$$

is

(a) $\left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a}\right)$ (b) $(-b, -c, -d)$

(c) $\left(-\frac{b}{2a}, -\frac{c}{2a}, -\frac{d}{2a}\right)$ (d) $\left(\frac{b}{2a}, \frac{c}{2a}, \frac{d}{2a}\right)$

Answers

(i) (a)	(ii) (a)	(iii) (c)	(iv) (b)
(v) (b)	(vi) (b)	(vii) (a)	(viii) (b)
(ix) (b)	(x) (a)	(xi) (a)	

GROUP B (Short Answer Type Questions)

Answer any three of the following: **(3 × 5 = 15)**

*2. Prove that every square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution: See Theorem 1.2 of Page 1.10.

**3. Show that

$$\vec{f} = (6xy + z^2)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

is irrotational. Hence, find a scalar function ϕ such that $\vec{f} = \vec{\nabla}\phi$.

Solution: See Example 9.11 of Page 9.36.

**4. Using mean-value theorem, prove that

$$x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}\ ,\ 0 < x < 1$$

Solution: Let $f(x) = \sin^{-1} x$ in $[0, x]$ where $0 < x < 1$. Then, $f(x)$ is continuous in $[0, x]$ and $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ exists in (0, 1).

Hence, by Lagrange's MVT, we have,

$$f(x) = f(0) + xf'(\theta x),\ 0 < \theta < 1$$

or, $$\sin^{-1} x = \frac{x}{\sqrt{1-\theta^2 x^2}} \tag{i}$$

Now,

$$0 < \theta < 1$$

or, $$0 < \theta x < x \quad [x > 0]$$

or, $$1 > 1 - \theta^2 x^2 > 1 - x^2$$

or, $$1 > \sqrt{1-\theta^2 x^2} > \sqrt{1-x^2}$$

or, $$1 < \frac{1}{\sqrt{1-\theta^2 x^2}} < \frac{1}{\sqrt{1-x^2}}$$

or, $$x < \frac{x}{\sqrt{1-\theta^2 x^2}} < \frac{x}{\sqrt{1-x^2}} \tag{ii}$$

Therefore, from (i) and (ii), we have,

$$x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, 0 < x < 1$$

**5. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint (xdy - ydx)$

Solution: See Problem 6 of Page SQP 2.6.

**6. Prove that the function

$$f(x, y) = x^2 - 2xy + y^2 - x^3 - y^3 + x^5$$

has neither maxima nor minima at the origin.

Solution: Here,

$$f(x, y) = x^2 - 2xy + y^2 - x^3 - y^3 + x^5$$

and

$$f_x = 5x^4 - 3x^2 + 2x - 2y \text{ and } f_y = -3y^2 + 2y - 2x$$
$$f_{xx} = 20x^3 - 6x + 2, f_{xy} = -2, f_{yy} = -6y + 2$$

Since

$$f_x(0, 0) = 0 = f_y(0, 0)$$

therefore, (0, 0) is a stationary point. Also,

$$f_{xx}(0, 0) = 2, f_{xy}(0, 0) = -2, f_{yy}(0, 0) = 2$$

We have,

$$f_{xx}(0, 0) f_{yy}(0, 0) - (f_{xy})^2 (0, 0) = 4 - 4 = 0$$

Hence, $f(x, y)$ has neither maxima nor minima at the origin.

GROUP C (Long Answer Type Questions)

Answer any three of the following: **(3 × 15 = 45)**

*7. (a) If $\vec{f} = |\vec{r}|$ where, $\vec{r} = \widehat{xi} + \widehat{yj} + \widehat{zk}$, prove that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

Solution: See Example 7 of Page 9.23.

**(b) Prove that

$$\begin{vmatrix} b^{2+}c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: See Example 1.16 of Pages 1.43.

*(c) If $y = \cos^{-1}(m\sin^{-1} x)$ then prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$$

Solution: See Example 3.13 of Page 3.17.

***8.** (a) If the vector functions $\vec{F}$ and $\vec{G}$ are irrotational, prove that $\vec{F} \times \vec{G}$ is solenoidal.

Solution: See Example 12 of Page 9.31.

**(b) If $f(x,y)=x^2\tan^{-1}\left(\dfrac{y}{x}\right)-y^2\tan^{-1}\left(\dfrac{x}{y}\right)$, verify that $f_{xy}=f_{yx}$.

Solution: See Example 6.6 of Page 6.34.

**(c) Find the maxima and minima of the function

$$x^3+y^3-3x+12y+20$$

Also, find the saddle point.

Solution: See Example 25 of Page 6.27

**9. (a) Evaluate $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$ by the Laplace expansion method.

Solution: See Example 1.9 of Page 1.36.

**(b) Verify Green's theorem for

$$\oint_C [(3x-8y^2)dx+(4y-6x)dy]$$

where C is the region bounded by $x=0$, $y=0$, and $x+y=1$.

Solution: See Example 9.27 of Page 9.56.

***(c) For what values of λ and μ does the system of equations

$$\begin{aligned} x+y+z&=6 \\ x+2y+3z&=10 \\ x+2y+\lambda z&=\mu \end{aligned}$$

has (i) a unique solution, (ii) no solution, and (iii) infinite solutions.

Solution: See Example 2.9 of Page 2.37.

**10. (a) If $u_n=\int_0^{\frac{\pi}{2}}\tan^n\theta\,d\theta$ then prove that

$$n(u_{n+1}+u_{n-1})=1$$

Solution: See Example 5.4 of Page 5.24.

***(b) Prove that if $0 < a < b$,

$$\frac{(b-a)}{(1+b^2)} < \tan^{-1} b - \tan^{-1} a < \frac{(b-a)}{(1+a^2)}$$

Hence, show that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

Solution: For first part, see Example 4.7 of Page 4.37.

Let $a = 1,\ b = \frac{4}{3}$ so that $0 < a < b < 1$ is satisfied. Substituting these values in the above result, we have,

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1} 1 < \frac{\frac{4}{3}-1}{1+(1)^2}$$

or,
$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{4}{3}-1}{1+(1)^2}$$

or,
$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

**(c) Test the convergence of the series

$$\frac{6}{1.3.5} + \frac{8}{3.5.7} + \frac{10}{5.7.9} + \cdots$$

Solution: See Example 8.25 of Page 8.41.

**11. (a) State Leibnitz's theorem for convergence of an alternating series. Hence, test the convergence of the following series:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

Solution: For the statement of Leibnitz's theorem, see Article 8.13.1 of Page 8.16. For the solution of the problem, see Example 8.27 of Page 8.42.

**(b) If $z = f(x, y)$, where, $x = e^u \cos v$ and $y = e^v \sin u$, show that

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$$

Solution: See Example 6.9 of Page 6.37.

(c) Evaluate $\int_a^0 \int_0^x \int_0^{x+y} e^{x+y+z}\, dxdydz$

Solution: Here,

$$I = \int_a^0 \int_0^x \int_0^{x+y} e^{x+y+z}\, dxdydz = \int_a^0 \left\{ \int_0^x e^{x+y} \left[\int_0^{x+y} e^z\, dz \right] dy \right\} dx$$

$$= \int_a^0 \left\{ \int_0^x e^{x+y} \left[e^z \right]_0^{x+y} dy \right\} dx = \int_a^0 \left\{ \int_0^x e^{x+y} [e^{x+y} - 1] dy \right\} dx$$

$$= \int_a^0 \left\{ \int_0^x [e^{2(x+y)} - e^{x+y}] dy \right\} dx = \int_a^0 \left\{ e^{2x} \left[\frac{e^{2y}}{2} \right]_0^x - e^x [e^y]_0^x \right\} dx$$

$$= \int_a^0 \left\{ e^{2x} \left[\frac{e^{2x}}{2} - \frac{1}{2} \right] - e^x [e^x - 1] \right\} dx$$

$$= \int_a^0 \left\{ \frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x \right\} dx$$

$$= \left[\frac{e^{4x}}{8} \right]_a^0 - \left[\frac{3e^{2x}}{4} \right]_a^0 + [e^x]_a^0$$

$$= \frac{3}{8} - \frac{e^{4a}}{8} + \frac{3e^{2a}}{4} - e^a$$

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-I (NEW) 2014

MATHEMATICS-I (M 101)

Time Alloted: 3 Hours *Full Marks: 70*

GROUP-A (Multiple Choice Type Questions)

***1. Answer any ten questions:** **$(10 \times 1 = 10)$**

(i) $\int_0^{\frac{\pi}{2}} \sin^5\theta d\theta =$

(a) $\frac{8}{15}$ (b) $\frac{8\pi}{15}$ (c) $\frac{8}{15}$ (d) $\frac{4}{15}$

*(ii) If $u(x, y) = yf\left(\frac{x^2}{y^2}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

(a) 0 (b) $2u(x, y)$ (c) $u(x, y)$ (d) 2

**(iii) The value of $\int_C (xdx - dy)$, where C is the line joining (0, 1), to (1, 0) is

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{2}{3}$

**(iv) The component of the vector $2\hat{i} + 5\hat{j} + 7\hat{k}$ on $\hat{i} - 2\hat{j} + 2\hat{k}$ is

(a) $\sqrt{78}$ (b) 3 (c) 6 (d) 2

*(v) The value of t for which $(x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + tz)\hat{k}$ is solenoidal is

(a) 2 (b) -2 c) 0 d) 1

*(vi) If $x = r\cos\theta$ and $y = r\sin\theta$ then, $\frac{\partial(r, \theta)}{\partial(x, y)} =$

(a) r (b) 1 (c) $\frac{1}{r}$ (d) 0

*(vii) $f(x, y) = \frac{x^3 + y^3}{\sqrt{x^2 + y^2}}$ is a homogeneous function of degree

(a) 0 (b) 2 (c) 1 (d) $\frac{1}{2}$

Level of difficulty:- *Low, **Medium, ***High.

**(viii) If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$ then A is

(a) idempotent (b) nilpotent
(c) involutary (d) none of these

*(ix) If $y = \tan^{-1} x$ then

(a) $(1 + x^2)y_1 = 1$ (b) $(1 + x^2)y_2 = 1$
(c) $(1 + x^2)y_1 = 0$ (d) $(1 + x^2)y_1 = 2$

*(x) If A is a real skew-symmetric matrix such that $A^2 + I = 0$ then A is

(a) singular (b) a unit matrix
(c) orthogonal (d) none of these

*(xi) The sequence $\left\{1, \frac{1}{3}, \frac{1}{3^2}, \cdots, \frac{1}{3^n} \cdots \infty\right\}$ is

(a) divergent (b) oscillatory
(c) convergent (d) none of these

*(xii) For a function $f(x)$, the expression $\frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ is known as

(a) Lagrange's remainder (b) Cauchy's remainder
(c) Maclaurin's remainder (d) Taylor's remainder

Answers

(i) (a), (c)	(ii) (c)	(iii) (a)	(iv) (d)
(v) (b)	(vi) (c)	(vii) (b)	(viii) (a)
(ix) (a)	(x) (d)	(xi) (c)	(xii) (b)

GROUP B (Short Answer Type Questions)

Answer any three Questions: **(3 × 5 = 15)**

**2. Using the Laplace method of expansion, prove that

$$\begin{vmatrix} x & y & -u & -v \\ y & x & v & u \\ u & v & x & y \\ -v & -u & y & x \end{vmatrix} = (x^2 + v^2 - y^2 - u^2)^2$$

Solution: Using the Laplace method of expansion, we have

$$\begin{vmatrix} x & y & -u & -v \\ y & x & v & u \\ u & v & x & y \\ -v & -u & y & x \end{vmatrix} = \begin{vmatrix} x & y \\ y & x \end{vmatrix} \times (-1)^{(1+2)+(1+2)} \begin{vmatrix} x & y \\ y & x \end{vmatrix}$$

$$+ \begin{vmatrix} x & -u \\ y & v \end{vmatrix} \times (-1)^{(1+2)+(1+3)} \begin{vmatrix} v & y \\ -u & x \end{vmatrix}$$

$$+ \begin{vmatrix} x & -v \\ y & u \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} v & x \\ -u & y \end{vmatrix}$$

$$+ \begin{vmatrix} -u & -v \\ v & u \end{vmatrix} \times (-1)^{(1+2)+(3+4)} \begin{vmatrix} u & v \\ -v & -u \end{vmatrix}$$

$$+ \begin{vmatrix} x & -v \\ y & u \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} v & x \\ -u & y \end{vmatrix}$$

$$+ \begin{vmatrix} y & -u \\ x & v \end{vmatrix} \times (-1)^{(1+2)+(2+3)} \begin{vmatrix} u & y \\ -v & x \end{vmatrix}$$

$$= (x^2 - y^2)^2 - (yu + xv)^2 + (xu + vy)^2 + (v^2 - u^2)^2$$

$$= (x^2 + v^2 - y^2 - u^2)^2$$

**3. For what values of x is the following infinite seriens convergent?

$$\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}, x > 0$$

Solution: Let

$$a_n = \frac{(n+1)^n x^n}{n^{n+1}}, x > 0$$

Applying the Cauchy root test,

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{(n+1)^n x^n}{n^{n+1}} \right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)x}{n.n^{\frac{1}{n}}} \right) = \lim_{n\to\infty} \left(\frac{\left(1+\frac{1}{n}\right)x}{n^{\frac{1}{n}}} \right) = x$$

Therefore, by Cauchy root test, the series is convergent when $x < 1$ and divergent when $x > 1$. The test fails when $x = 1$.

When $x = 1$,

$$a_n = \frac{(n+1)^n}{n^{n+1}} = \left(1+\frac{1}{n}\right)^n \frac{1}{n}$$

Consider a series $\sum_{n=1}^{\infty} b_n$, where $b_n = \dfrac{1}{n}$.

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

Since $\sum_{n=1}^{\infty} b_n$ is a divergent series. So by comparison test, the series is divergent.

Therefore, the infinite series is convergent for $x < 1$.

*4. If α, β, γ are the angles which a vector makes with the coordinate axes, prove that

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$$

Solution: Let

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

then,

$$\cos \alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}\,;\ \cos \beta = \frac{y}{\sqrt{x^2+y^2+z^2}}\,;\ \cos \gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Now,

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = (1-\cos^2\alpha) + (1-\cos^2\beta) + (1-\cos^2\gamma)$$

$$= 3 - \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 3 - 1 = 2$$

**5. If $y = x^{n-1} \log x$, using Leibnitz's theorem, show that $y_n = \dfrac{(n-1)!}{x}$.

Solution: See Example 3.4 of Page 3.11.

**6. Using Green's theorem, evaluate $\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y dy\}$ where C is the circle $x^2 + y^2 = 1$.

Solution: See Example 14 of Page 9.48

GROUP C (Long Answer Type Questions)

Answer any three of the following: **(3 × 15 = 45)**

**7. (a) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then show that $A^2 - 4A - 5I_3 = 0$. Hence, find A^{-1}.

Solution: Here,

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Therefore,

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Since

$$A^2 - 4A - 5I_3 = 0$$

or, $$A^{-1}(A^2 - 4A - 5I_3) = 0$$

or, $$A - 4I_3 - 5A^{-1} = 0$$

or, $$A^{-1} = \frac{1}{5}[A - 4I_3]$$

or, $$A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

**(b) If $y = e^{m \sin^{-1} x}$ then show that

(i) $(1 - x^2)y_2 - xy_1 - m^2y = 0$

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also, find $(y_n)_0$.

Solution: See Example 9(i) of Pages SQP 2.9.

**(c) Is Rolle's theorem applicable to the function $f(x) = (x - p)^m(x - q)^n$, $x \in [p, q]$ where m, n are positive integers? If so, find the constant c of Rolle's theorem, where c has its usual meaning.

Solution: Here,

$$f(x) = (x - p)^m(x - q)^n, x \in [p, q]$$

where m, n are positive integers. Now,

(i)

$$f(p) = 0 = f(q)$$

(ii) Since $f(x)$ is a polynomial, $f(x)$ is continuous in $[p, q]$

(iii)
$$f'(x) = m(x-p)^{m-1}(x-q)^n + n(x-p)^m(x-q)^{n-1}$$

exists in (p, q).

Therefore, Rolle's theorem is applicable to the function $f(x)$ and there exists $c \in (p, q)$ such that

$$f'(c) = 0$$

or, $$m(c-p)^{m-1}(c-q)^n + n(c-p)^m(c-q)^{n-1} = 0$$

or, $$(c-p)^{m-1}(c-q)^{n-1}\{m(c-q) + n(c-p)\} = 0$$

or, $$\{m(c-q) + n(c-p)\} = 0$$

or, $$c = \frac{mq+np}{m+n}$$

**8. (a) State D' Alembert's ratio test. Applying this test, examine the convergence of the following series:

$$1 + \frac{2^\alpha}{2!} + \frac{3^\alpha}{3!} + \frac{4^\alpha}{4!} + \cdots\infty, \alpha > 0$$

Solution: See Example 8.26 of Page 8.42.

**(b) Show that

$$\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$$

Solution: See Problem 9(ii) of Page SQP 2.10.

**(c) If $f(v^2 - x^2, v^2 - y^2, v^2 - z^2) = 0$, where v is a function of x, y, z then show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$$

Solution: See Example 17 of Page 6.16.

***9. (a) Determine the conditions under which the system of equations

$$\begin{aligned} x + y + z &= 1 \\ x + 2y - z &= k \\ 5x + 7y + az &= k^2 \end{aligned}$$

admits (i) only one solution, (ii) no solution, and (iii) many solutions.

Solution: See Problem 8(a) of Page SQP1.5.

***(b) Verify the divergence theorem for the vector function $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over a cube bounded by $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$.

Solution: See Example 16 of Page 9.50.

**(c) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx (n > 1)$ then prove that

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Solution: See Example 5.7 of Page 5.27.

**10. (a) Verify Lagrange's mean-value theorem at $[-1, 1]$ for

$$f(x) = x \sin\frac{1}{x}, x \neq 0$$
$$= 0, x = 0$$

Solution: See Example 4.4 of Page 4.34.

***(b) If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right) \text{ and } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Solution: See Example 21 of Page 6.19.

**(c) Find the rank of the following matrix $\begin{bmatrix} 2 & 3 & 16 & 5 \\ 4 & 5 & 6 & 7 \\ 2 & 0 & 1 & 3 \\ 8 & 8 & 23 & 15 \end{bmatrix}$.

Solution: Let us apply elementary row operations on the matrix; then

$$\begin{bmatrix} 2 & 3 & 16 & 5 \\ 4 & 5 & 6 & 7 \\ 2 & 0 & 1 & 3 \\ 8 & 8 & 23 & 15 \end{bmatrix} \xrightarrow{R_1 + R_2 + R_3} \begin{bmatrix} 2 & 3 & 16 & 5 \\ 4 & 5 & 6 & 7 \\ 8 & 8 & 23 & 15 \\ 8 & 8 & 23 & 15 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{bmatrix} 2 & 3 & 16 & 5 \\ 4 & 5 & 6 & 7 \\ 8 & 8 & 23 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & 3 & 16 & 5 \\ 0 & -1 & -26 & -3 \\ 0 & -4 & -41 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[(-1)R_3]{(-1)R_2} \begin{bmatrix} 2 & 3 & 16 & 5 \\ 0 & 1 & 26 & 3 \\ 0 & 4 & 41 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - 4R_2]{R_1 - 3R_2} \begin{bmatrix} 2 & 0 & -62 & -4 \\ 0 & 1 & 26 & 3 \\ 0 & 0 & -63 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\left(\frac{-1}{7}\right)R_3]{\left(\frac{1}{2}\right)R_1}\begin{bmatrix}1 & 0 & -31 & -2\\ 0 & 1 & 26 & 3\\ 0 & 0 & 9 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow[R_2-3R_3]{R_1+2R_3}\begin{bmatrix}1 & 0 & -13 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 9 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since the number of nonzero rows of the reduced matrix is 3, therefore, the rank of the matrix is 3.

** 11. (a) Find the extremum of the following function:

$$x^3 + y^3 - 3axy$$

Solution: See Example 6.23 of Page 6.50.

**(b) Show that $\vec{\nabla}\,\phi$ is irrotational, where $\phi = x^2y + 2xy + z^2$.

Solution: Now,

$$\begin{aligned}\vec{\nabla}\,\phi &= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\\ &= (2xy + 2y)\,\hat{i} + (x^2 + 2x)\,\hat{j} + 2z\hat{k}\end{aligned}$$

and

$$\begin{aligned}\text{curl}\left(\vec{\nabla}\phi\right) &= \begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (2xy+2y) & (x^2+2x) & 2z\end{vmatrix}\\ &= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2x+2-2x-2)\\ &= 0\end{aligned}$$

Therefore, $\vec{\nabla}\,\phi$ is irrotational.

**(c) Evaluate $\int_0^a\int_0^x\int_0^y x^3y^2z\,dxdydz$

Solution: See Example 7.4 of Page 7.19.

SOLUTIONS OF UNIVERSITY QUESTIONS (W.B.U.T.)

B.TECH SEM-1 (NEW) 2011

MATHEMATICS-I (M 101)

Time Alloted: 3 Hours | *Full Marks: 70*

GROUP-A (Multiple Choice Type Questions)

1. Choose the correct alternatives for any ten of the following: **($10 \times 1 = 10$)**

*(i) The least upper bound of the sequence $\left\{\dfrac{n}{n+1}\right\}$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) 2

Solution: The correct alternative is $\boxed{\text{(c) } 1}$

*(ii) The value of $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix}$ is

(a) 2000 (b) 0

(c) 45 (d) none of these

Solution: The correct alternative is $\boxed{\text{(b) } 0}$

**(iii) If $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ is the characteristic equation of a square matrix A then A^{-1} is equal to

(a) $A^2 - 6A + 9I$ (b) $\dfrac{1}{4}A^2 - \dfrac{3}{2}A + \dfrac{9}{4}I$

(c) $A^2 - 6A + 9$ (d) $\dfrac{1}{4}A^2 - \dfrac{3}{2}A + \dfrac{9}{4}$

Solution: The correct alternative is $\boxed{\text{(b) } \dfrac{1}{4}A^2 - \dfrac{3}{2}A + \dfrac{9}{4}I}$

*(iv) If $x = r\cos\theta$, $y = r\sin\theta$, then $\dfrac{\partial(r, \theta)}{\partial(x, y)}$ is

(a) r (b) 1

(c) $\dfrac{1}{r}$ (d) none of these

Level of difficulty:- *Low, **Medium, ***High.

Solution: The correct alternative is $\boxed{\text{(c) } \frac{1}{r}}$

*(v) $f(x, y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x}$ is a homogeneous function of degree

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) 2

Solution: The correct alternative is $\boxed{\text{(b) } -\frac{1}{2}}$

**(vi) If $\vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma}) = 0$, then $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are

(a) coplanar (b) independent

(c) collinear (d) none of these

Solution: The correct alternative is $\boxed{\text{(a) coplanar}}$

**(vii) The nth derivative of $(ax + b)^{10}$ is (where $n > 10$)

(a) a^{10} (b) $10!a^{10}$ (c) 0 (d) 10!

Solution: The correct alternative is $\boxed{\text{(c) } 0}$

***(viii) If for any two vectors $\vec{a}$ and $\vec{b}$,

$$\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$$

then $\vec{a}$ and $\vec{b}$ are

(a) parallel (b) collinear

(c) perpendicular (d) none of these

Solution: The correct alternative is $\boxed{\text{(d) orthogonal}}$

**(ix) If $A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then $A =$

(a) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ (c) $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Solution: The correct alternative is $\boxed{\text{(b) } \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}$

*(x) The reduction formula of $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ is

(a) $I_n = \left(\frac{n-1}{n}\right) I_{n-1}$ (b) $I_n = \left(\frac{n}{n-1}\right) I_n$

(c) $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ (d) none of these

Solution: The correct alternative is $\boxed{\text{(c) } I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$

**(xi) The series $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ is

(a) convergent (b) divergent

(c) oscillatory (d) none of these

Solution: The correct alternative is $\boxed{\text{(b) divergent}}$

*(xii) Lagrange's Mean Value Theorem is obtained from Cauchy's Theorem for two functions $f(x)$ and $g(x)$ by putting $g(x) =$

(a) 1 (b) x^2 (c) x (d) $\frac{1}{x}$

Solution: The correct alternative is $\boxed{\text{(c) } x}$

GROUP-B (Short-Answer Type Questions)

Answer any three of the following: **(3 × 5 = 15)**

*2. Prove that every square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution: See Theorem 1.2 of Page 1.10.

***3. By Laplace's method, prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$$

(consider minors of order 2).

Solution: Here, we expand the given determinant by Laplace's method of expansion in terms of a minor of order 2 considering the first two rows as follows:

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \times (-1)^{(1+2)+(1+2)} \begin{vmatrix} a & b \\ -b & a \end{vmatrix} + \begin{vmatrix} a & c \\ -b & d \end{vmatrix} \times (-1)^{(1+2)+(1+3)} \begin{vmatrix} -d & b \\ c & a \end{vmatrix}$$

$$+ \begin{vmatrix} a & d \\ -b & -c \end{vmatrix} \times (-1)^{(1+2)+(1+4)} \begin{vmatrix} -d & a \\ c & -b \end{vmatrix} + \begin{vmatrix} b & c \\ a & d \end{vmatrix} \times (-1)^{(1+2)+(2+3)} \begin{vmatrix} -c & b \\ -d & a \end{vmatrix}$$

$$+ \begin{vmatrix} b & d \\ a & -c \end{vmatrix} \times (-1)^{(1+2)+(2+4)} \begin{vmatrix} -c & a \\ -d & -b \end{vmatrix} + \begin{vmatrix} c & d \\ d & -c \end{vmatrix} \times (-1)^{(1+2)+(3+4)} \begin{vmatrix} -c & -d \\ -d & c \end{vmatrix}$$

$$\begin{aligned}
&= (a^2 + b^2)(a^2 + b^2) + (ad + bc)(ad + bc) \\
&\quad + (-ac + bd)(-ac + bd) + (bd - ac)(bd - ac) \\
&\quad + (bc + ad)(bc + ad) + (c^2 + d^2)(c^2 + d^2) \\
&= (a^2 + b^2)^2 + 2(ad + bc)^2 + 2(-ac + bd)^2 \\
&\qquad + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2[(ad + bc)^2 + (-ac + bd)^2] \\
&\qquad + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2[a^2d^2 + b^2c^2 - 2adbc + a^2c^2 \\
&\qquad + b^2d^2 - 2acbd] + (c^2 + d^2)^2 \\
&= (a^2 + b^2)^2 + 2(a^2 + b^2)(c^2 + d^2) + (c^2 + d^2)^2 \\
&= (a^2 + b^2 + c^2 + d^2)^2
\end{aligned}$$

***4. If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$ then prove that

$$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

Solution: Here, we have

$$2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$$

or, $$y^{\frac{1}{m}} - 2x + \frac{1}{y^{\frac{1}{m}}} = 0$$

or, $$\left(y^{\frac{1}{m}}\right)^2 - 2x\left(y^{\frac{1}{m}}\right) + 1 = 0$$

Applying the rule for finding solution of the above quadratic equation, we get

$$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{(2x)^2 - 4.1.1}}{2} = \left(x \pm \sqrt{x^2 - 1}\right)$$

$\Rightarrow$ $$y = \left(x \pm \sqrt{x^2 - 1}\right)^m \qquad \text{(i)}$$

Differentiating (i) w.r.t. x, we have

$$y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1} \cdot \left(1 \pm \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-1}} \cdot 2x\right)$$

$$= m\left(x \pm \sqrt{x^2-1}\right)^{m-1} \cdot \left(\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right)$$

$$= \pm m\left(x \pm \sqrt{x^2-1}\right)^{m-1} \cdot \frac{\left(x \pm \sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

i.e., $$y_1 = \pm m \frac{\left(x \pm \sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \pm \frac{my}{\sqrt{x^2-1}} \qquad \text{(ii)}$$

Squaring (ii) and simplifying, we get

$$(y_1)^2 (x^2 - 1) = m^2 y^2 \qquad \text{(iii)}$$

Again differentiating (iii) w.r.t. x, we have

$$2y_1 y_2 (x^2 - 1) + (y_1)^2 2x = m^2 2y \cdot y_1$$

$\Rightarrow$ $$y_2(x^2 - 1) + y_1 x - m^2 y = 0 \qquad \text{(iv)}$$

Now applying Leibnitz's theorem, we differentiate (iv) n times w.r.t. x,

$$\{y_2(x^2-1)\}_n + \{y_1 x\}_n - \{m^2 y\}_n = 0$$

$$\Rightarrow \quad [\{y_2\}_n \cdot (x^2-1) + {}^nC_1\{y_2\}_{n-1}\cdot(2x) + {}^nC_2\{y_2\}_{n-2}\cdot(2)] + [\{y_1\}_n \cdot x + {}^nC_1\{y_1\}_{n-1}\cdot 1] - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

**5. If $u = xf\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right)$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution: See Example 21 of Page 6.19.

**6. Show that the area bounded by a simple closed curve C is given by $\dfrac{1}{2}\displaystyle\oint_C (xdy - ydx)$.

Solution: We know that Green's theorem states the following:

$$\oint_C \{M(x, y)dx + N(x, y)dy\} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy \qquad \text{(i)}$$

where the region R on the two-dimensional xy plane is bounded by a simple closed curve C and the line integral along the curve C is taken in the anti-clockwise direction.

Here, comparing LHS of (i) with $\displaystyle\oint_C (xdy - ydx)$, we have

$$M = -y, N = x \Rightarrow \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1 \qquad \text{(ii)}$$

Therefore using (ii) in (i), we get

$$\begin{aligned}\oint_C (xdy - ydx) &= \iint_R [1-(-1)]\,dx\,dy \\ &= 2\iint_R dx\,dy \\ &= 2 \times [\text{Area bounded by } C]\end{aligned}$$

$$\Rightarrow \quad \text{Area bounded by } C = \frac{1}{2}\oint_C (xdy - ydx)$$

Hence, the area bounded by a simple closed curve C is given by $\dfrac{1}{2}\displaystyle\oint_C (xdy - ydx)$.

GROUP-C (Long-Answer Type Questions)

Answer any three of the following: **($3 \times 15 = 45$)**

***7. (i) If

$$f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right),$$

verify $f_{xy} = f_{yx}$.

Solution: See Example 6.6 of Page 6.34.

*(ii) State Rolle's theorem and examine if you can apply the same for $f(x) = \tan x$ in $[0, \pi]$.

Solution: See Example 4.3 of Page 4.33.

***(iii) Find the value of λ and μ for which

$$\begin{aligned} x + y + z &= 3 \\ 2x - y + 3z &= 4 \\ 5x - y + \lambda z &= \mu \end{aligned}$$

has (a) a unique solution, (b) many solutions (c) no solution.

Solution: If we write the system of linear equations in the matrix form as $AX = B$ then the coefficient matrix of the system of linear equations is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & \lambda \end{pmatrix}$$

and the augmented matrix is

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & \lambda & \mu \end{pmatrix}$$

The system of equations has a unique solution when the determinant of the coefficient matrix is not equal to zero.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & \lambda \end{vmatrix}$$

$$= 1(-\lambda + 3) - 1(2\lambda - 15) + 1(-2 + 5) = -3\lambda + 21$$

Therefore, for $\det A \neq 0 \Rightarrow -3\lambda + 21 \neq 0 \Rightarrow \lambda \neq 7$, the system of equations have **unique solution**.

When $\lambda = 7$, the augmented matrix becomes

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & \mu \end{pmatrix}$$

Applying elementary row operations on the matrix $\bar{A}$, we have

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & \mu \end{pmatrix} \xrightarrow{R_2 - 2R_1,\, R_3 - 5R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & \mu - 15 \end{pmatrix} \xrightarrow{R_3 - 2R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix} \xrightarrow{\left(-\frac{1}{3}\right)R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{4}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \mu - 11 \end{pmatrix}$$

The system of equations is consistent when Rank A = Rank $\bar{A}$ and this is possible for

$$\mu - 11 = 0 \Rightarrow \mu = 11.$$

In this case, Rank A = Rank $\bar{A}$ = 2, which is less then number of unknowns $(= 3)$ and the system has **infinitely many solutions**.

Again, if

$$\mu - 11 \neq 0 \Rightarrow \mu \neq 11.$$

then Rank A = 2 and Rank $\bar{A}$ = 3, i.e., Rank $A \neq$ Rank $\bar{A}$, and so the system of equations is inconsistent and correspondingly the system has **no solution**.

Summarizing the above, the system of equations has

(a) a **unique solution** when $\lambda \neq 7$

(b) **infinitely many solutions** when $\lambda = 7$ and $\mu = 11$

(c) **no solution** when $\lambda = 7$ and $\mu \neq 11$.

**8. (i) Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$$

Find also the saddle points.

Solution: See Example 6.24 of Page 6.51.

**(ii) State Leibnitz's test for alternating series and apply it to examine the convergence of

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$$

Solution: See Example 8.27 of Page 8.42.

*(iii) Applying Lagrange's Mean Value Theorem, prove that

$$\frac{x}{1+x} \le \log(1 + x) \le x, \text{ for all } x > 0.$$

Solution: See Example 7 of Page 4.11.

*9. (i) If $y = e^{m\sin^{-1}x}$, show that

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0.$$

Hence, find y_n when $x = 0$.

Solution: Here, we are given that

$$y = e^{m\sin^{-1}x} \tag{i}$$

Differentiating (i) w.r.t. x, we have

$$y_1 = e^{m\sin^{-1}x} \cdot m\left(\frac{1}{\sqrt{1-x^2}}\right)$$

i.e.,
$$y_1 = \frac{my}{\sqrt{1-x^2}} \tag{ii}$$

Squaring (ii) and simplifying, we get

$$(y_1)^2(1 - x^2) = m^2y^2 \tag{iii}$$

Again differentiating (iii) w.r.t. x, we have

$$2y_1y_2(1 - x^2) + (y_1)^2(-2x) = m^2 2y \cdot y_1$$

$$\Rightarrow \quad y_2(1 - x^2) - y_1x - m^2y = 0 \tag{iv}$$

Now applying Leibnitz's theorem, we differentiate (iv) n times w.r.t. x,

$$\{y_2(1 - x^2)\}_n - \{y_1x\}_n - \{m^2y\}_n = 0.$$

$$\Rightarrow \quad [\{y_2\}_n \cdot (1 - x^2) + {}^nC_1\{y_2\}_{n-1} \cdot (-2x) + {}^nC_2\{y_2\}_{n-2} \cdot (-2)]$$
$$-[\{y_1\}_n \cdot x + {}^nC_1\{y_1\}_{n-1} \cdot 1] - m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad \text{(v)}$$

Calculation of y_n when $x = 0$, i.e., $(y_n)_0$:

Putting $x = 0$ in (v), we have

$$(y_{n+2})_0 = (n^2+m^2)(y_n)_0$$

Replacing n by $n-2$, we get

$$(y_n)_0 = [(n-2)^2 + m^2](y_{n-2})_0 \quad \text{(vi)}$$

Replacing n by $n-2$ in (vi), we get

$$(y_{n-2})_0 = [(n-4)^2 + m^2](y_{n-4})_0 \quad \text{(vii)}$$

Using (vii) in (vi),

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2](y_{n-4})_0$$

Similarly, we have

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2][(n-6)^2 + m^2](y_{n-6})_0 \quad \text{(viii)}$$

Proceeding in a similar manner we have from (viii), when n is odd as the following:

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2] \ldots [3^2 + m^2][1^2 + m^2](y_1)_0 \quad \text{(ix)}$$

From (ii), we have $(y_1)_0 = m$. Using this in (ix), we get

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2] \ldots [3^2 + m^2]$$
$$[1^2 + m^2]m, \text{ when } n \text{ is odd.}$$

Also proceeding in a similar manner we have from (viii), when n is even as the following:

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2] \ldots [4^2 + m^2][2^2 + m^2](y_2)_0 \quad \text{(x)}$$

From (iv), we have $(y_2)_0 = m^2$. Using this in (x), we get

$$(y_n)_0 = [(n-2)^2 + m^2][(n-4)^2 + m^2] \ldots [4^2 + m^2]$$
$$[2^2 + m^2]m^2, \text{ when } n \text{ is even.}$$

*(ii) Prove that $\left[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}\right] = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$, where $\vec{a}$ $\vec{b}$ $\vec{c}$ are three vectors.

Solution: Using the definition of scalar triple product, we write

$$\left[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}\right] = \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]$$

$$= \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}+\vec{c}\right)\times\vec{c} + \left(\vec{b}+\vec{c}\right)\times\vec{a}\right]$$

$$= \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}\times\vec{c}\right) + \left(\vec{c}\times\vec{c}\right) + \left(\vec{b}\times\vec{a}\right) + \left(\vec{c}\times\vec{a}\right)\right]$$

$$= \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right)\right],$$

since $\vec{c}\times\vec{c}=\vec{0}$

$$= \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)+\vec{a}\cdot\left(\vec{b}\times\vec{a}\right)+\vec{a}\cdot\left(\vec{c}\times\vec{a}\right)+\vec{b}\cdot\left(\vec{b}\times\vec{c}\right) + \vec{b}\cdot\left(\vec{b}\times\vec{a}\right)+\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$= \left[\vec{a}\,\vec{b}\,\vec{c}\right]+\left[\vec{a}\,\vec{b}\,\vec{a}\right]+\left[\vec{a}\,\vec{c}\,\vec{a}\right]+\left[\vec{b}\,\vec{b}\,\vec{c}\right] +\left[\vec{b}\,\vec{b}\,\vec{a}\right]+\left[\vec{b}\,\vec{c}\,\vec{a}\right]$$

By the property of scalar triple product of vectors, we have $\left[\vec{a}\,\vec{b}\,\vec{a}\right]=0$, $\left[\vec{a}\,\vec{c}\,\vec{a}\right]=0$, $\left[\vec{b}\,\vec{b}\,\vec{c}\right]=0$, $\left[\vec{b}\,\vec{b}\,\vec{a}\right]=0$ (since two vectors in the product are same) and $\left[\vec{b}\,\vec{c}\,\vec{a}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

Using this in the above, we get

$$\left[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\; \vec{c}+\vec{a}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]+0+0+0+0+\left[\vec{a}\,\vec{b}\,\vec{c}\right]=2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$$

*(iii) Find the directional derivative of $f = xyz$ at $(1, 1, 1)$ in the direction $2\hat{i}-\hat{j}-2\hat{k}$.

Solution: Here, it is given that $f = xyz$. Then

$$\vec{\nabla}f = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)f$$

$$= \left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)$$

$$= yz\hat{i}+xz\hat{j}+xy\hat{k}$$

So, $\left[\vec{\nabla}f\right]_{(1,1,1)} = \hat{i}+\hat{j}+\hat{k}$

Here we are to find the directional derivative in the direction $2\hat{i}-\hat{j}-2\hat{k}$. The unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$ is given by

$$\hat{a} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$$

Then the required directional derivative of $f = xyz$ at (1, 1, 1) in the direction of $2\hat{i}-\hat{j}-2\hat{k}$ is given by

$$\left[\vec{\nabla}f\right]_{(1,1,1)}\cdot\hat{a} = \left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}\right) = -\frac{1}{3}$$

**10. (i) Prove that

$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: See Example 1.16 of Page 1.43.

***(ii) State the Divergence Theorem of Gauss. Verify divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$, $z = 2$.

Solution: See Example 9.29 of Page 9.58.

**(iii) Test the series for convergence:

$$\frac{1^p}{2^q} + \frac{2^p}{3^q} + \frac{3^p}{4^q} + \ldots$$

Solution: Let us consider the given series as

$$\sum_{n=1}^{\infty} a_n = \frac{1^p}{2^q} + \frac{2^p}{3^q} + \frac{3^p}{4^q} + \ldots$$

Then

$$a_n = \frac{n^p}{(n+1)^q} = \frac{n^p}{n^q\left(1+\frac{1}{n}\right)^q} = \frac{1}{n^{q-p}\left(1+\frac{1}{n}\right)^q}$$

Let us consider another series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{q-p}}$$

which is convergent for $q - p > 1$ and divergent for $q - p \leq 1$.

Now we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^q} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is convergent for $q - p > 1$ and divergent for $q - p \leq 1$, by comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent for $q - p > 1$ and divergent for $q - p \leq 1$.

*11. (i) Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Hence obtain $\int_0^{\frac{\pi}{2}} \sin^9 x\,dx$.

Solution: See Section 5.2 of Page 5.1.

**(ii) Given two vectors $\vec{\alpha} = 3\vec{i} - \vec{j}$, $\vec{\beta} = 2\vec{i} + \vec{j} - 3\vec{k}$. Express $\vec{\beta}$ in the form $\vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution: See Example 9.1 of Page 9.17.

**(iii) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the scalar function ϕ, such that $\vec{A} = \vec{\nabla}\phi$.

Solution: See Example 9.11 of Page 9.36.