Linear Algebra and Vector Calculus (2110015)

GTU-December 2014

Second Edition

About the Authors

Ravish R Singh is presently Vice Principal at Shree L R Tiwari College of Engineering, Thane. He obtained a BE degree from University of Mumbai in 1991, an MTech degree from IIT Bombay in 2001 and a PhD degree from Faculty of Technology, University of Mumbai, in 2013. He has published several books with McGraw Hill Education (India) on varied subjects like *Engineering Mathematics (I and II), Applied Mathematics, Electrical Engineering, Electrical and Electronics Engineering*, etc., for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Uttar Pradesh Technical University. Dr Singh is a member of IEEE, ISTE, and IETE, and has published research papers in national and international journals. His fields of interest include Circuits, Signals and Systems, and Engineering Mathematics

Mukul Bhatt is presently Assistant Professor, Department of Humanities and Sciences, at Thakur College of Engineering and Technology, Mumbai. She obtained her MSc (Mathematics) degree from HNB Garhwal University in 1992. She has published several books with McGraw Hill Education (India) such as *Engineering Mathematics (I and II)* and *Applied Mathematics* for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Uttar Pradesh Technical University. She has seventeen years of teaching experience at various levels in engineering colleges in Mumbai and her fields of interest include Integral Calculus, Complex Analysis, and Operations Research. She is a member of ISTE.

Linear Algebra and Vector Calculus (2110015)

GTU-December 2014

Second Edition

Ravish R Singh
Shree L R Tiwari College of Engineering
Thane

Mukul Bhatt
Thakur College of Engineering and Technology
Mumbai

McGraw Hill Education (India) Private Limited
NEW DELHI

McGraw Hill Education Offices
New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

McGraw Hill Education (India) Private Limited

Published by McGraw Hill Education (India) Private Limited
P-24, Green Park Extension, New Delhi 110 016

Linear Algebra and Vector Calculus (For GTU)

ISBN-13: 978-93-392-1914-7
ISBN-10: 93-392-1914-7

Managing Director: *Kaushik Bellani*

Head—Higher Education (Publishing and Marketing): *Vibha Mahajan*
Senior Publishing Manager (SEM & Tech. Ed.): *Shalini Jha*
Assistant Sponsoring Editor: *Koyel Ghosh*
Editorial Executive: *Piyali Chatterjee*
Manager—Production Systems: *Satinder S Baveja*
Assistant Manager—Editorial Services: *Sohini Mukherjee*
Senior Production Executive: *Anuj Kr Shriwastava*

Assistant General Manager (Marketing)—Higher Education: *Vijay Sarathi*
Assistant Product Manager (SEM & Tech. Ed.): *Tina Jajoriya*
Senior Graphic Designer—Cover: *Meenu Raghav*

General Manager—Production: *Rajender P Ghansela*
Manager—Production: *Reji Kumar*

Typeset at The Composers, 260, C.A. Apt., Paschim Vihar, New Delhi 110 063 and printed at
Pashupati Printers Pvt. Ltd., 1/429/16, Gali No. 1, Friends Colony, Industrial Area, G.T. Road,
Shahdara, Delhi 110 095

Cover Printer: SDR Printers

R**

To

Our parents in law

Shri Uday Pratap Singh

Shrimati Shaila Singh

Ravish R Singh

Late Shri Dataram Bhatt

Shrimati Sateshwaridevi Bhatt

Mukul Bhatt

Contents

Preface

Linear Algebra and Vector Calculus is a key area in the study of an engineering course. It is the study of numbers, structures, and associated relationships using rigorously defined literal, numerical and operational symbols. A sound knowledge of the subject develops analytical skills, thus enabling engineering graduates to solve numerical problems encountered in daily life, as well as apply vector principles to physical problems, particularly in the area of engineering.

Rationale

We have observed that many students who opt for engineering find it difficult to conceptualise the subject since very few available texts have syllabus compatibility and the right pedagogy. Feedback received from students and teachers have highlighted the need for a comprehensive textbook on linear algebra and vector calculus that covers all topics of first-year engineering along with suitable solved problems. This book—an outcome of our vast experience of teaching undergraduate students of engineering—provides a solid foundation in vector principles, enabling students to solve mathematical, scientific, and associated engineering principles.

Users

This book is meant for the first-year engineering students of Gujarat Technological University (GTU) studying the subject Linear Algebra and Vector Calculus (2110015). The structuring of the book takes into account all the topics in the GTU syllabi in a student-friendly manner.

Intent

An easy-to-understand and student-friendly text, it presents concepts in adequate depth using a step-by-step problem-solving approach. The text is well supported with a plethora of solved examples at varied difficulty levels, practice problems and engineering applications. It is intended that students will gain logical understanding from solved problems and then by solving similar problems themselves.

Features

Each topic has been thoroughly covered from the examination point of view. The theory part of the text is explained in a lucid manner. For each topic, problems of all possible combinations have been worked out. This is followed by an exercise with answers. Objective type questions provided in each chapter help students in mastering concepts. Salient features of the book are summarised below:

- Exactly in-sync with the latest GTU syllabus of Linear Algebra and Vector Calculus (2110015)

- Lucid writing style and tutorial approach throughout the book, i.e., teach-by-examples
- Offers extensive opportunities to students for practice and self-evaluation through numerous step-by-step solved examples and exercises
- Application-based problems for better comprehension of concepts
- Solved GTU 2014, 2013 and 2012 examination papers
- Exam-oriented rich pedagogy includes
  - **80 Illustrations**
  - **400 Solved Examples**
  - **300 Exercise Problems**

Organization

The content of the book is spread over seven chapters.

Chapter 1 gives an in-depth account of matrices and systems of linear equations.
Chapter 2 discusses vector spaces.
Chapter 3 presents linear transformations.
Chapter 4 gives an overview of inner product spaces.
Chapter 5 deals with eigenvalues and eigenvectors.
Chapter 6 covers vector functions.
Chapter 7 explains vector calculus.

Apart from these, solved GTU question papers of 2012, 2013 and 2014 have been provided at the end of the book.

Acknowledgements

We are grateful to the following reviewers who reviewed various chapters of the script and generously shared their valuable comments:

Adalja Divya Ghanshambhai	*Marwadi Education Foundation Group of Institutions, Rajkot*
Urvi Trivedi	*SAL Engineering College, Ahmedabad*
Mahendra A Patel	*Government Engineering College, Gandhinagar*
C M Barasara	*Atmiya Institute of Technology and Science, Rajkot*

We would also like to thank all the staff at McGraw Hill Education (India), especially Vibha Mahajan, Shalini Jha, Koyel Ghosh, Piyali Chatterjee, Sohini Mukherjee, Satinder Singh Baveja and Anuj Kr. Shriwastava for coordinating with us during the editorial, copyediting and production stages of this book.

Our acknowledgements would be incomplete without a mention of the contribution of all our family members. We extend a heartfelt thanks to them for always motivating and supporting us throughout the project.

Constructive suggestions for the improvement of the book will always be welcome.

Ravish R Singh
Mukul Bhatt

Publisher's Note:
Remember to write to us. We look forward to receiving your feedback, comments and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Please mention the title and authors' name as the subject. In case you spot piracy of this book, please do let us know.

Roadmap to the Syllabus
Linear Algebra and Vector Calculus (2110015)

Unit 1

Systems of Linear Equations and Matrices

[illegible]

GO TO

Chapter 1 – **Matrices and Systems of Linear Equations**

Unit 2

Linear Combinations and Linear Independence

[illegible] R^n [illegible]

Vector Spaces

[illegible]

GO TO

Chapter 2 – **Vector Spaces**

Unit 3

Linear Transformations

[illegible]

Eigenvalues and Eigenvectors

[illegible]

GO TO

Chapter 3 – **Linear Transformation**
Chapter 5 – **Eigenvalues and Eigenvector**

Unit 4

Inner Product Spaces

[illegible] R^n [illegible]

GO TO

Chapter 4 – **Inner Product Spaces**

Chapter 5 – **Eigenvalues and Eigenvectors**

Unit 5

Vector Functions

[illegible]

GO TO

Chapter 6 – **Vector Functions**

Unit 6

Vector Calculus

[illegible]

GO TO

Chapter 7 – **Vector Calculus**

Matrices and Systems of Linear Equations

Chapter 1

1.1 INTRODUCTION

A matrix is a rectangular table of elements which may be numbers or any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformation and record data that depend on multiple parameters. There are many applications of matrices in mathematics, viz. graph theory, probability theory, statistics, computer graphics, geometrical optics, etc.

1.2 MATRIX

A set of *mn* elements (real or complex) arranged in a rectangular array of *m* rows and *n* columns is called a matrix of order *m* by *n*, written as $m \times n$.

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The matrix can also be expressed in the form $A = [a_{ij}]_{m\times n}$, where a_{ij} is the element in the i^{th} row and j^{th} column, written as $(i, j)^{\text{th}}$ element of the matrix.

1.3 SOME DEFINITIONS ASSOCIATED WITH MATRICES

(1) *Row Matrix*

A matrix having only one row and any number of columns, is called a row matrix or row vector, e.g.

$$[2 \quad 5 \quad -3 \quad 4]$$

(2) *Column Matrix*

A matrix, having only one column and any number of rows, is called a column matrix or column vector, e.g.

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

(3) *Zero or Null Matrix*

A matrix, whose all the elements are zero, is called a zero or null matrix and is denoted by **0**, e.g.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) *Square Matrix*

A matrix, in which the number of rows is equal to the number of columns, is called a square matrix, e.g.

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & -5 \\ 2 & 6 & 8 \end{bmatrix}$$

(5) *Diagonal Matrix*

A square matrix, all of whose non-diagonal elements are zero and at least one diagonal element is non-zero, is called a diagonal matrix. e.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(6) *Unit or Identity Matrix*

A diagonal matrix, all of whose diagonal elements are unity, is called a unit or identity matrix and is denoted by I, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) *Scalar Matrix*

A diagonal matrix, all of whose diagonal elements are equal, is called a scalar matrix, e.g.

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(8) *Upper Triangular Matrix*

A square matrix, in which all the elements below the diagonal are zero, is called an upper triangular matrix, e.g.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 8 \end{bmatrix}$$

(9) *Lower Triangular Matrix*

A square matrix, in which all the elements above the diagonal are zero, is called a lower triangular matrix, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 6 & 8 \end{bmatrix}$$

(10) *Trace of a Matrix*

The sum of all the diagonal elements of a square matrix is called the trace of a matrix,

e.g.
$$A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\text{Trace of } A = 2 + 6 + 3 = 11$$

(11) *Transpose of a Matrix*

A matrix obtained by interchanging rows and columns of a matrix is called transpose of a matrix and is denoted by A' or A^T, e.g.

If
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -4 & 1 \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \end{bmatrix}$$

i.e. if $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$

(12) *Determinant of a Matrix*

If A is a square matrix, then determinant of A is represented as $|A|$ or $\det(A)$.

If
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{then } \det(A) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

(13) *Singular and Non-Singular Matrices*

A square matrix A is called singular if det $(A) = 0$ and non-singular if det $(A) \neq 0$.

1.4 SOME SPECIAL MATRICES

(1) *Symmetric Matrix*

A square matrix $A = [a_{ij}]_{m\times m}$ is called symmetric if $a_{ij} = a_{ji}$ for all i and j, i.e. $A = A^T$, e.g.

$$\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 1 & i & -3i \\ i & -2 & 4 \\ -3i & 4 & 3 \end{bmatrix}$$

(2) *Skew Symmetric Matrix*

A square matrix $A = [a_{ij}]_{m\times m}$ is called skew symmetric if $a_{ij} = -a_{ji}$ for all i and j, i.e. $A = -A^T$.
Thus, the diagonal elements of a skew symmetric matrix are all zero, e.g.

$$\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

Example 1: Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Solution: Let A be a square matrix.

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) = P + Q$$

where, $$P = \frac{1}{2}\left(A + A^T\right)$$

and $$Q = \frac{1}{2}\left(A - A^T\right)$$

Now, $$P^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left[A^T + \left(A^T\right)^T\right]$$

$$= \frac{1}{2}\left(A^T + A\right) = P$$

Hence, P is a symmetric matrix.

Also, $$Q^T = \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left[A^T - \left(A^T\right)^T\right]$$

$$= \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right) = -Q$$

Hence, Q is a skew symmetric matrix.
Thus, every matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Uniqueness Let $A = R + S$, where R is a symmetric and S is a skew symmetric matrix.

$$A^T = (R+S)^T = R^T + S^T = R - S$$

Now, $$\frac{1}{2}\left(A + A^T\right) = \frac{1}{2}[(R+S)+(R-S)] = R = P$$

and $$\frac{1}{2}\left(A - A^T\right) = \frac{1}{2}[(R+S)-(R-S)] = S = Q$$

Hence, representation $A = P + Q$ is unique.

Example 2: Express the matrix $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution: $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}$

Let $$P = \frac{1}{2}\left(A + A^T\right)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 4 & 15 \\ 4 & -4 & -2 \\ 15 & -2 & 26 \end{bmatrix}$$

$$Q = \frac{1}{2}\left(A - A^T\right)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 6 & -1 \\ -6 & 0 & -6 \\ 1 & 6 & 0 \end{bmatrix}$$

We know that P is a symmetric and Q is a skew symmetric matrix.

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 2 & 4 & 15 \\ 4 & -4 & -2 \\ 15 & -2 & 26 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 6 & -1 \\ -6 & 0 & -6 \\ 1 & 6 & 0 \end{bmatrix}$$

(3) *Conjugate of a Matrix*

A matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$, e.g.

$$A = \begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$$

(4) *Transposed Conjugate of a Matrix*

The conjugate of the transpose of a matrix A is called the transposed conjugate or conjugate transpose of A and is denoted by A^θ, e.g.

$$A^\theta = (\bar{A})^T = (\overline{A^T})$$

e.g., If $A = \begin{bmatrix} 1-2i & 2+3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1-2i & 4-5i & 8 \\ 2+3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$

Then, $A^\theta = \begin{bmatrix} 1+2i & 4+5i & 8 \\ 2-3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

(5) *Hermitian Matrix*

A square matrix $A = [a_{ij}]$ is called Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j, i.e. $A = A^\theta$, e.g.,

$$\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$$

(6) *Skew Hermitian Matrix*

A square matrix $A = [a_{ij}]$ is called skew Hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j, i.e. $A = -A^\theta$. Hence, diagonal elements of a skew Hermitian matrix must be either purely imaginary or zero, e.g.

$$\begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$$

Example 1: Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Solution: Let A be a square matrix.

$$A = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right) = P + Q$$

where, $$P = \frac{1}{2}\left(A + A^{\theta}\right)$$

and $$Q = \frac{1}{2}\left(A - A^{\theta}\right)$$

Now, $$P^{\theta} = \frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left[A^{\theta} + \left(A^{\theta}\right)^{\theta}\right]$$

$$= \frac{1}{2}\left(A^{\theta} + A\right) = P$$

Hence, P is a Hermitian matrix.

Also, $$Q^{\theta} = \frac{1}{2}\left(A - A^{\theta}\right)^{\theta} = \frac{1}{2}\left[A^{\theta} - \left(A^{\theta}\right)^{\theta}\right]$$

$$= \frac{1}{2}\left(A^{\theta} - A\right) = -Q$$

Hence, Q is a skew Hermitian matrix

Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Uniqueness Let $A = R + S$ where R is a Hermitian and S is skew Hermitian matrix.

$$A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S$$

Now, $$\frac{1}{2}\left(A + A^{\theta}\right) = \frac{1}{2}[(R+S) + (R-S)] = R = P$$

and $$\frac{1}{2}\left(A - A^{\theta}\right) = \frac{1}{2}[(R+S) - (R-S)] = S = Q$$

Hence, representation $A = P + Q$ is unique.

Example 2: Express the matrix $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrix.

Solution:

$$A=\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$$

$$A^{\theta}=(\bar{A})^{T}=\begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

Let
$$P=\frac{1}{2}\left(A+A^{\theta}\right)=\frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}+\begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}$$

$$Q=\frac{1}{2}\left(A-A^{\theta}\right)=\frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}-\begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

We know that P is a Hermitian and Q is a skew Hermitian matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$$

Example 3: Show that every square matrix can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices.

Solution: Let A be a square matrix.

$$A=\frac{1}{2}\left(A+A^{\theta}\right)+i\frac{1}{2i}\left(A-A^{\theta}\right)=P+iQ$$

where,
$$P=\frac{1}{2}\left(A+A^{\theta}\right) \quad \text{and} \quad Q=\frac{1}{2i}\left(A-A^{\theta}\right)$$

Now,
$$P^{\theta}=\frac{1}{2}\left(A+A^{\theta}\right)^{\theta}=\frac{1}{2}\left[A^{\theta}+\left(A^{\theta}\right)^{\theta}\right]$$

$$=\frac{1}{2}\left(A^{\theta}+A\right)=P$$

Hence, P is a Hermitian matrix.

Also,
$$Q^\theta = \left[\frac{1}{2i}\left(A - A^\theta\right)\right]^\theta = -\frac{1}{2i}\left[A^\theta - \left(A^\theta\right)^\theta\right]$$
$$= -\frac{1}{2i}\left(A^\theta - A\right) = \frac{1}{2i}\left(A - A^\theta\right) = Q$$

Hence, Q is a Hermitian matrix.

Thus, every square matrix can be expressed as $P + iQ$ where P and Q are Hermitian matrices.

Uniqueness Let $A = R + iS$ where R and S are Hermitian matrices.

$$A^\theta = \left(R + iS\right)^\theta = R^\theta + (iS)^\theta = R - iS$$

Now,
$$\frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left[(R+iS) + (R-iS)\right] = R = P$$

and
$$\frac{1}{2}\left(A - A^\theta\right) = \frac{1}{2}\left[(R+iS) - (R-iS)\right] = iS = iQ$$

Hence, representation $A = P + iQ$ is unique.

Example 4: Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $P + iQ$ where P and Q are both Hermitian.

Solution:
$$A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}$$

Let
$$P = \frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix} + \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$$
$$= \frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}$$

$$Q=\frac{1}{2i}\left(A-A^{\theta}\right)=\frac{1}{2i}\left\{\begin{bmatrix}2i & -3 & 1-i\\ 0 & 2+3i & 1+i\\ -3i & 3+2i & 2-5i\end{bmatrix}-\begin{bmatrix}-2i & 0 & 3i\\ -3 & 2-3i & 3-2i\\ 1+i & 1-i & 2+5i\end{bmatrix}\right\}$$

$$=\frac{1}{2i}\begin{bmatrix}4i & -3 & 1-4i\\ 3 & 6i & -2+3i\\ -1-4i & 2+3i & -10i\end{bmatrix}$$

We know that P and Q are Hermitian matrices.

$$A=P+iQ=\frac{1}{2}\begin{bmatrix}0 & -3 & 1+2i\\ -3 & 4 & 4-i\\ 1-2i & 4+i & 4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}4i & -3 & 1-4i\\ 3 & 6i & -2+3i\\ -1-4i & 2+3i & -10i\end{bmatrix}$$

Example 5: Prove that every Hermitian matrix can be written as $P + iQ$ where P is a real symmetric and Q is a real skew symmetric matrix.

Solution: Let A be a Hermitian matrix.

$$A^{\theta} = A$$

$$A=\frac{1}{2}\left(A+\bar{A}\right)+i\frac{1}{2i}\left(A-\bar{A}\right)=P+iQ$$

where, $P=\frac{1}{2}(A+\bar{A})$ and $Q=\frac{1}{2i}\left(A-\bar{A}\right)$ are real matrices.

Now,
$$P^{T}=\left[\frac{1}{2}\left(A+\bar{A}\right)\right]^{T}=\frac{1}{2}\left[A^{\theta}+\bar{A}\right]^{T}$$

$$=\frac{1}{2}\left[\left(\bar{A}\right)^{T}+\bar{A}\right]^{T}=\frac{1}{2}\left[\left\{\left(\bar{A}\right)^{T}\right\}^{T}+\left(\bar{A}\right)^{T}\right]$$

$$=\frac{1}{2}\left(\bar{A}+A^{\theta}\right)=\frac{1}{2}\left(\bar{A}+A\right)=P$$

Hence, P is a real symmetric matrix.

Also,
$$Q^{T}=\left[\frac{1}{2i}\left(A-\bar{A}\right)\right]^{T}=\frac{1}{2i}\left[A^{\theta}-\bar{A}\right]^{T}$$

$$=\frac{1}{2i}\left[\left(\bar{A}\right)^{T}-\bar{A}\right]^{T}=\frac{1}{2i}\left[\left\{\left(\bar{A}\right)^{T}\right\}^{T}-\left(\bar{A}\right)^{T}\right]=\frac{1}{2i}\left(\bar{A}-A^{\theta}\right)$$

$$=\frac{1}{2i}\left(\bar{A}-A\right)=-\frac{1}{2i}\left(A-\bar{A}\right)=-Q$$

Hence, Q is a real skew symmetric matrix.

Thus, every Hermitian matrix can be written as $P + iQ$, where P is a real symmetric matrix and Q is a real skew symmetric matrix.

Example 6: Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix.

Solution: $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

Let
$$P = \frac{1}{2}\left(A + \bar{A}\right) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$Q = \frac{1}{2i}\left(A - \bar{A}\right) = \frac{1}{2i}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 0 & -2i & 2i \\ 2i & 0 & -6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

We know that P is a real symmetric matrix and Q is a real skew symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -i & i \\ i & 0 & -3i \\ -i & 3i & 0 \end{bmatrix}$$

Example 7: Prove that every skew Hermitian matrix can be written as $P + iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Solution: Let A be a skew Hermitian matrix.

$$A^{\theta} = -A$$

$$A = \frac{1}{2}\left(A+\bar{A}\right) + i\frac{1}{2i}\left(A-\bar{A}\right) = P + iQ$$

where, $P = \frac{1}{2}\left(A+\bar{A}\right)$ and $Q = \frac{1}{2i}\left(A-\bar{A}\right)$ are real matrices.

Now,
$$P^T = \left[\frac{1}{2}\left(A+\bar{A}\right)\right]^T = \frac{1}{2}\left[-A^{\theta}+\bar{A}\right]^T$$

$$= \frac{1}{2}\left[-\left(\bar{A}\right)^T+\bar{A}\right]^T = \frac{1}{2}\left[-\left\{\left(\bar{A}\right)^T\right\}^T+\left(\bar{A}\right)^T\right]$$

$$= \frac{1}{2}\left(-\bar{A}+A^{\theta}\right) = \frac{1}{2}\left(-\bar{A}-A\right)$$

$$= -\frac{1}{2}\left(A+\bar{A}\right) = -P$$

Hence, P is a real skew symmetric matrix.

$$Q^T = \left[\frac{1}{2i}\left(A-\bar{A}\right)\right]^T = \frac{1}{2i}\left[-A^{\theta}-\bar{A}\right]^T$$

$$= \frac{1}{2i}\left[-\left(\bar{A}\right)^T-\bar{A}\right]^T = \frac{1}{2i}\left[-\left\{\left(\bar{A}\right)^T\right\}^T-\left(\bar{A}\right)^T\right]$$

$$= \frac{1}{2i}\left(-\bar{A}-A^{\theta}\right) = \frac{1}{2i}\left(-\bar{A}+A\right)$$

$$= \frac{1}{2i}\left(A-\bar{A}\right) = Q$$

Hence, Q is a real symmetric matrix.
Thus, every skew Hermitian matrix can be written as $P + iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Example 8: Express the skew Hermitian matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$, where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Solution:
$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$$

$$\overline{A}=\begin{bmatrix}-2i & 2-i & 1+i\\ -2-i & i & -3i\\ -1+i & -3i & 0\end{bmatrix}$$

Let $$P=\frac{1}{2}\left(A+\overline{A}\right)=\frac{1}{2}\left\{\begin{bmatrix}2i & 2+i & 1-i\\ -2+i & -i & 3i\\ -1-i & 3i & 0\end{bmatrix}+\begin{bmatrix}-2i & 2-i & 1+i\\ -2-i & i & -3i\\ -1+i & -3i & 0\end{bmatrix}\right\}$$

$$=\frac{1}{2}\begin{bmatrix}0 & 4 & 2\\ -4 & 0 & 0\\ -2 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 2 & 1\\ -2 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$Q=\frac{1}{2i}\left(A-\overline{A}\right)=\frac{1}{2i}\left\{\begin{bmatrix}2i & 2+i & 1-i\\ -2+i & -i & 3i\\ -1-i & 3i & 0\end{bmatrix}-\begin{bmatrix}-2i & 2-i & 1+i\\ -2-i & i & -3i\\ -1+i & -3i & 0\end{bmatrix}\right\}$$

$$=\frac{1}{2i}\begin{bmatrix}4i & 2i & -2i\\ 2i & -2i & 6i\\ -2i & 6i & 0\end{bmatrix}=\begin{bmatrix}2 & 1 & -1\\ 1 & -1 & 3\\ -1 & 3 & 0\end{bmatrix}$$

We know that P is a real skew symmetric matrix and Q is a real symmetric matrix.

$$A=P+iQ=\begin{bmatrix}0 & 2 & 1\\ -2 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}+\begin{bmatrix}2i & i & -i\\ i & -i & 3i\\ -i & 3i & 0\end{bmatrix}$$

(7) *Unitary Matrix*

A square matrix A is called unitary if $AA^\theta = A^\theta A = I$.

Example 1: Prove that matrix A is unitary and hence find A^{-1}.

(i) $A=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1+i}{2} & \frac{1-i}{2}\end{bmatrix}$ (ii) $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$.

Solution: (i) $$A=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1+i}{2} & \frac{1-i}{2}\end{bmatrix}$$

$$A^T=\begin{bmatrix}\frac{1+i}{2} & \frac{1+i}{2}\\ \frac{-1+i}{2} & \frac{1-i}{2}\end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$AA^{\theta} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1-i^2-i^2+1 & 1-i^2+i^2-1 \\ 1-i^2-1+i^2 & 1-i^2+1-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

$$A^{-1} = A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \qquad [\because AA^{\theta} = I]$$

(ii) $$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{T} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ -i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

For unitary matrix,

$$A^{-1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(8) *Orthogonal Matrix*

A square matrix A is called orthogonal if $AA^T = A^TA = I$.

Example 1: Verify if the following matrices are orthogonal and hence find their inverse:

(i) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: (i) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is an orthogonal matrix.

$$A^{-1} = A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \qquad [\because AA^T = I]$$

(ii) $A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is an orthogonal matrix.
For an orthogonal matrix,

$$A^{-1} = A^T = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: Find l, m, n and A^{-1} if $A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ is orthogonal.

Solution: Since the matrix A is orthogonal,

$$AA^T = I$$

$$\begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}\begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4m^2+n^2 & 2m^2-n^2 & -2m^2+n^2 \\ 2m^2-n^2 & l^2+m^2+n^2 & l^2-m^2-n^2 \\ -2m^2+n^2 & l^2-m^2-n^2 & l^2+m^2+n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding components,

$$4m^2 + n^2 = 1 \qquad \dots (1)$$

$$2m^2 - n^2 = 0 \qquad \dots (2)$$

$$l^2 + m^2 + n^2 = 1 \qquad \dots (3)$$

Solving Eqs. (1), (2) and (3),

$$l^2 = \frac{1}{2}, \quad l = \pm\frac{1}{\sqrt{2}}$$

$$m^2 = \frac{1}{6}, \quad m = \pm\frac{1}{\sqrt{6}}$$

$$n^2 = \frac{1}{3}, \quad n = \pm\frac{1}{\sqrt{3}}$$

$$A^{-1} = A^T = \begin{bmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \mp\frac{1}{\sqrt{6}} \\ \pm\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} \end{bmatrix}$$

Example 3: If $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal, find the relationship among $l_1, m_1, n_1, \ldots n_3$.

Solution: Since the matrix A is orthogonal,

$$AA^T = I$$

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 & l_2 l_3 + m_2 m_3 + n_2 n_3 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding components,

$$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$$

and $$l_1 l_2 + m_1 m_2 + n_1 n_2 = l_1 l_3 + m_1 m_3 + n_1 n_3 = l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$$

Exercise 1.1

1. Express the following matrices as the sum of a symmetric matrix and a skew symmetric matrix:

(i) $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix}$

2. Express the following matrices as the sum of a Hermitian matrix and a skew Hermitian matrix.

(i) $\begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4 \\ -i & 3-i & 1-i \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4+i & 6i \\ 6 & 5-i & 4 \\ 0 & 1-i & 8i \end{bmatrix}$

3. Express the following matrices as $P + iQ$, where P and Q are both Hermitian.

(i) $\begin{bmatrix} 2 & 3-i & 1+2i \\ i & 0 & 1 \\ 1+2i & 1 & 3i \end{bmatrix}$

(ii) $\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

4. Express the following Hermitian matrices as $P + iQ$, where P is a real symmetric matrix and Q is a real skew symmetric matrix.

(i) $\begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$

5. Express the following skew Hermitian matrices as $P + iQ$, where P is a real and skew symmetric matrix and Q is a real and symmetric matrix.

(i) $\begin{bmatrix} 0 & 2-3i & 1+i \\ -2-3i & 2i & 2-i \\ -1+i & -2-i & -i \end{bmatrix}$

(ii) $\begin{bmatrix} i & 2i & -1+3i \\ 2i & 2i & 2-i \\ 1+3i & -2-i & 3i \end{bmatrix}$

6. Show that the following matrices are unitary.

(i) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$

(iii) $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

7. Show that following matrices are orthogonal and hence find their inverses.

(i) $\frac{1}{9}\begin{bmatrix} 8 & -4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

8. Find l, m, n and A^{-1} if

$$A = \begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix} \text{ is}$$

orthogonal.

9. Find a, b, c if $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.

[**Ans.:** $a = 1, b = -8, c = 4$]

10. If (a_r, b_r, c_r) where $r = 1, 2, 3$ be the direction cosines of the three mutually perpendicular lines referred to an orthogonal coordinate system, then prove that $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is orthogonal.

1.5 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any non-zero number
(iii) The addition or subtraction of k items the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij} : Interchange of i^{th} and j^{th} row
(ii) kR_i : Multiplication of i^{th} row by a non zero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i and $C_i + kC_j$ respectively.

1.5.1 *Elementary Matrices*

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

1.5.2 *Equivalence of Matrices*

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A, then A is called the equivalent to B. Symbolically, we can write $A \sim B$.

1.5.3 *Echelon Form of a Matrix*

A matrix A is said to be in row echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a non-zero row.
(ii) The first non-zero number from the left of a non-zero row is a 1. This is called a leading 1.
(iii) For each non-zero row, the leading 1 appears to the right and below any leading 1 in the preceding rows.

The following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A matrix A is said to be in reduced row echelon form if each column that contains a leading 1 in row echelon form of the matrix A has zeros everywhere else in that column.

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1: In each part determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(i) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -6 & 4 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

(i) The given matrix is in reduced row echelon form and row echelon form since it satisfies properties (i), (ii), (iii) and columns containing leading 1 have zero everywhere else.

(ii) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (iii).

(iii) The given matrix is in row echelon form since it satisfies properties (i), (ii) and (iii).

(iv) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (i).

Example 2: Find a row echelon form of the following matrices:

(i) $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

Solution: (i)

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

R_{13}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$R_2 - 2R_1,\ R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 6 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

R_{23}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

$(-1)R_2$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

$R_3 + 3R_2,\ R_4 + 7R_2$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -7 & -26 \end{bmatrix}$$

R_{34}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -26 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\left(-\frac{1}{7}\right)R_3$$

$$\sim\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\left(-\frac{1}{8}\right)R_4$$

$$\sim\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii)
$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$$

$$R_2 + R_1,\ R_4 - 2R_1$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 6 & -5 \end{bmatrix}$$

$$R_{23}$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 0 & 5 \\ 0 & -1 & 6 & -5 \end{bmatrix}$$

$$R_3 - 2R_2,\ R_4 + R_2$$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -4 & 7 \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

$$\left(-\frac{1}{4}\right)R_3$$

$$\sim\begin{bmatrix}1 & 2 & -3 & 1\\0 & 1 & 2 & -1\\0 & 0 & 1 & -\frac{7}{4}\\0 & 0 & 8 & -6\end{bmatrix}$$

$$R_4-8R_3$$

$$\sim\begin{bmatrix}1 & 2 & -3 & 1\\0 & 1 & 2 & -1\\0 & 0 & 1 & -\frac{7}{4}\\0 & 0 & 0 & 8\end{bmatrix}$$

$$\left(\frac{1}{8}\right)R_4$$

$$\sim\begin{bmatrix}1 & 2 & -3 & 1\\0 & 1 & 2 & -1\\0 & 0 & 1 & -\frac{7}{4}\\0 & 0 & 0 & 1\end{bmatrix}$$

Example 3: Find the reduced row echelon form of the matrices of Example 2.

(i) $\begin{bmatrix}0 & -1 & 2 & 3\\2 & 3 & 4 & 5\\1 & 3 & -1 & 2\\3 & 2 & 4 & 1\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2 & -3 & 1\\-1 & 0 & 3 & 4\\0 & 1 & 2 & -1\\2 & 3 & 0 & -3\end{bmatrix}$

Solution: (i) The row echelon form of the matrix is

$$\begin{bmatrix}1 & 3 & -1 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & \frac{26}{7}\\0 & 0 & 0 & 1\end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 - \frac{26}{7}R_4,\ R_2 + 3R_4,\ R_1 - 2R_4$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 + 2R_3,\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 + \frac{7}{4}R_4,\ R_2 + R_4,\ R_1 - R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_3,\ R_1 + 3R_3$$
$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$R_1 - 2R_2$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.6 SYSTEM OF NON-HOMOGENEOUS LINEAR EQUATIONS

A system of m non-homogeneous linear equations in n variables $x_1, x_2, \dots x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is called coefficient matrix of order $m \times n$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector of order $n \times 1$.

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $m \times 1$.

1.6.1 *Solutions of System of Linear Equations: Gaussian Elimination and Gauss–Jordan Elimination Method*

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix
$$[A:B]=\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the augmented matrix of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to either row echelon form or reduced row echelon form.

Reducing the augmented matrix to row echelon form is called Gaussian elimination method. Reducing the augmented matrix to reduced row echelon form is called Gauss–Jordan elimination method.

The Gaussian elimination method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the row echelon form of the augmented matrix by using elementary row operations.
Step 3: Write the corresponding linear system of equations from row echelon form.
Step 4: Solve the corresponding linear system of equations by back substitution.

The Gauss–Jordan elimination method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the reduced row echelon form of the augmented matrix by using elementary row operations.
Step 3: For each non-zero row of the matrix, solve the corresponding system of equations for the variables associated with the leading one in that row.

Note: The linear system has a unique solution if $\det(A) \neq 0$

Example 1: Solve each of the following systems by Gaussian elimination method.

(i)
$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} 4x - 2y + 6z &= 8 \\ x + y - 3z &= -1 \\ 15x - 3y + 9z &= 21 \end{aligned}$$

(iii)
$$\begin{aligned} 3x + y - 3z &= 13 \\ 2x - 3y + 7z &= 5 \\ 2x + 19y - 47z &= 32 \end{aligned}$$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$(-2)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

Solving these equations,

$$x = 1, y = 2$$

Hence, $x = 1$, $y = 2$, $z = 3$ is the solution of the system.

(ii) The matrix form of the system is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_2 - 4R_1, \; R_3 - 15R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2, \; \left(-\frac{1}{18}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x + y - 3z = -1$$

$$y - 3z = -2$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variable z is called a free variable. Assigning the free variable z an arbitrary value t,

$$y = 3t - 2$$
$$x = -1 - 3t + 2 + 3t = 1$$

Hence, $x = 1$, $y = 3t - 2$, $z = t$ is the solution of the system where t is a parameter.

(iii) The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & -3 \\ 2 & -3 & 7 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 32 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{3}\right)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & -\frac{11}{3} & 9 & -\frac{11}{3} \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$\left(-\frac{3}{11}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & 1 \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$R_3 - \frac{55}{3}R_2$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & -1 & \big| & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & \big| & 1 \\ 0 & 0 & 0 & \big| & 5 \end{bmatrix}$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = 5$$

Hence, the system is inconsistent and has no solution.

Example 2: Solve the following system for x, y and z.

$$-\frac{1}{x} + \frac{3}{y} + \frac{4}{z} = 30$$
$$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9$$
$$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10$$

Solution: The matrix form of the system is

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\begin{bmatrix} -1 & 3 & 4 & \big| & 30 \\ 3 & 2 & -1 & \big| & 9 \\ 2 & -1 & 2 & \big| & 10 \end{bmatrix}$$

Reducing the augmented matrix to row echelon form,

$$(-1)R_1$$

$$\sim \begin{bmatrix} 1 & -3 & -4 & \big| & -30 \\ 3 & 2 & -1 & \big| & 9 \\ 2 & -1 & 2 & \big| & 10 \end{bmatrix}$$

$$R_2 - 3R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right]$$

$$\left(\frac{1}{11}\right)R_2,\ \left(\frac{1}{5}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The corresponding system of equations is

$$\frac{1}{x} - \frac{3}{y} - \frac{4}{z} = -30$$

$$\frac{1}{y} + \frac{1}{z} = 9$$

$$\frac{1}{z} = 5$$

Solving these equations,

$$x = \frac{1}{2},\ y = \frac{1}{4},\ z = \frac{1}{5}$$

Hence, $x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{5}$ is the solution of the system.

Example 3: Solve the following system of non-linear equations for the unknown angles α, β and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$ and $0 \le \gamma < \pi$.

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$

$$4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$$

$$6\sin\alpha - 3\cos\beta + \tan\gamma = 9$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

$$R_2 - 4R_1,\ \ R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right)R_2,\ \left(-\frac{1}{8}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} \sin\alpha - \frac{1}{2}\cos\beta + \frac{3}{2}\tan\gamma &= \frac{3}{2} \\ \cos\beta - 2\tan\gamma &= -1 \\ \tan\gamma &= 0 \end{aligned}$$

Solving these equations,

$$\begin{aligned} \gamma &= 0 \\ \cos\beta &= -1 \quad \Rightarrow \quad \beta = \pi \end{aligned}$$

$$\begin{aligned}\sin\alpha &= \frac{1}{2}\cos\beta - \frac{3}{2}\tan\gamma + \frac{3}{2}\\ &= \frac{1}{2}(-1) - \frac{3}{2}(0) + \frac{3}{2} = 1\\ \alpha &= \frac{\pi}{2}\end{aligned}$$

Hence, $\alpha = \frac{\pi}{2}, \beta = \pi, \gamma = 0$ is the solution of the system.

Example 4: Investigate for what values of λ and μ the equations

$$\begin{aligned}x + 2y + z &= 8\\ 2x + 2y + 2z &= 13\\ 3x + 4y + \lambda z &= \mu\end{aligned}$$

have (i) no solution, (ii) a unique solution, and (iii) many solutions.

Solution: The matrix form of the system is

$$\begin{bmatrix}1 & 2 & 1\\ 2 & 2 & 2\\ 3 & 4 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}8\\ 13\\ \mu\end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c}1 & 2 & 1 & 8\\ 2 & 2 & 2 & 13\\ 3 & 4 & \lambda & \mu\end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 8\\ 0 & -2 & 0 & -3\\ 0 & -2 & \lambda-3 & \mu-24\end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 8\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & -2 & \lambda-3 & \mu-24\end{array}\right]$$

$$R_3 + 2R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & \dfrac{3}{2} \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right]$$

(i) If $\lambda = 3$ and $\mu \neq 21$, the system is inconsistent and has no solution.
(ii) If $\lambda \neq 3$ and μ has any value, the system is consistent and has a unique solution.
(iii) If $\lambda = 3$ and $\mu = 21$, the system is consistent and has infinite (many) solutions.

Example 5: Determine the values of λ for which the following equations are consistent. Also, solve the system for these values of λ.

$$\begin{aligned} x + 2y + z &= 3 \\ x + y + z &= \lambda \\ 3x + y + 3z &= \lambda^2 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1,\ R_3 - 3R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda-3 \\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$(-1)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$R_3 + 5R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array}\right]$$

The equations will be consistent if $\lambda^2 - 5\lambda + 6 = 0$, i.e. $\lambda = 3$ or $\lambda = 2$.

Case I: When $\lambda = 3$,

$$\begin{aligned} x + 2y + z &= 3 \\ y \quad &= 0 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x = 3 - 2(0) - t = 3 - t$$

Hence, $x = 3 - t$, $y = 0$, $z = t$ is the solution of the system where t is a parameter.

Case II: When $\lambda = 2$,

$$\begin{aligned} x + 2y + z &= 3 \\ y \quad &= 1 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x = 3 - 2(1) - t = 1 - t$$

Hence, $x = 1 - t$, $y = 1$, $z = t$ is the solution of the system where t is a parameter.

Example 6: Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= \alpha \\ 4x + 5y + 6z &= \beta \\ 5x + 6y + 7z &= \gamma \end{aligned}$$

is consistent only if α, β and λ are in arithmetic progression (A.P.)

Solution: The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1, \; R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta - \alpha \\ 2 & 2 & 2 & \gamma - \alpha \end{array}\right]$$

$$\underset{\sim}{R_{12}}\left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta-\alpha \\ 3 & 4 & 5 & \alpha \\ 2 & 2 & 2 & \gamma-\alpha \end{array}\right]$$

$$\underset{\sim}{R_2-3R_1,\ R_3-2R_1}\left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \alpha-2\beta+\gamma \end{array}\right]$$

The system of equations is consistent if,

$$\alpha-2\beta+\gamma=0$$

$$\beta=\frac{\alpha+\gamma}{2}$$

i.e. α, β and γ are in arithmetic progression (A.P.)

Example 7: Show that if $\lambda \neq 0$, the system of equations

$$\begin{aligned} 2x+\ \ y\ \ \ \ \ \ &= a \\ x+\lambda y-\ z &= b \\ y+2z &= c \end{aligned}$$

has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda = 0$, $a = 1$, $b = 1$, $c = -1$.

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The system has a unique solution if $\det(A) \neq 0$

$$\begin{aligned} \det(A) = 2(2\lambda+1)-1\ (2+0) &\neq 0 \\ 4\lambda &\neq 0 \\ \lambda &\neq 0 \end{aligned}$$

Hence, the system of equations has a unique solution if $\lambda \neq 0$ for any value of a, b, c.

If $\lambda = 0$, the system is either inconsistent or has an infinite number of solutions.

For $\lambda = 0$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 1 & 0 & -1 & b \\ 0 & 1 & 2 & c \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_{12}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\underset{\sim}{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\underset{\sim}{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

The system is consistent if $c - a + 2b = 0$
The corresponding system of equations is

$$x - z = b$$
$$y + 2z = a - 2b$$

Assigning the free variable z any arbitrary value t,

$$y = a - 2b - 2t$$
$$x = b + t$$

Hence, $x = b + t$, $y = a - 2b - 2t$, $z = t$ is the solution of the system where t is a parameter.

When $a = 1$, $b = 1$, $c = -1$

$$x = 1 + t$$
$$y = -1 - 2t$$
$$z = t$$

Example 8: Solve each of the following systems by Gauss–Jordan elimination method:

(i) $x_1 + x_2 + 2x_3 = 8$
$-x_1 - 2x_2 + 3x_3 = 1$
$3x_1 - 7x_2 + 4x_3 = 10$

(ii) $2x_1 + 2x_2 + 2x_3 = 0$
$-2x_1 + 5x_2 + 2x_3 = 1$
$8x_1 + x_2 + 4x_3 = -1$

(iii) $x - y + 2z - w = -1$
$2x + y - 2z - 2w = -2$
$-x + 2y - 4z + w = 1$
$3x - 3w = -3$

(iv) $-2y + 3z = 1$
$3x + 6y - 3z = -2$
$6x + 6y + 3z = 5$

(v) $x_1 - 2x_2 - x_3 + 3x_4 = 1$
$2x_1 - 4x_2 + x_3 = 5$
$x_1 - 2x_2 + 2x_3 - 3x_4 = 4$

(vi) $2x - y + z = 9$
$3x - y + z = 6$
$4x - y + 2z = 7$
$-x + y - z = 4$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 + R_1,\ R_3 - 3R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$(-1)R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_3 + 10R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$\left(-\frac{1}{52}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 + 5R_3,\ R_1 - 2R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equations is

$$x_1 = 3$$
$$x_2 = 1$$
$$x_3 = 2$$

Hence, $x_1 = 3, x_2 = 1, x_3 = 2$ is the solution of the system.

(ii) The matrix form of the system is

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$$R_2 + 2R_1,\ R_3 - 8R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right]$$

$$R_3 + R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{7}\right)R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{3}{7}x_3 = -\frac{1}{7}$$
$$x_2 + \frac{4}{7}x_3 = \frac{1}{7}$$

Since leading ones are in columns 1 and 2, x_1 and x_2 are called leading variables whereas x_3 is a free variable. Assigning the free variable x_3 any arbitrary value t,

$$x_1 = -\frac{1}{7} - \frac{3}{7}t$$
$$x_2 = \frac{1}{7} - \frac{4}{7}t$$

Hence, $x_1 = -\frac{1}{7} - \frac{3}{7}t,\ x_2 = \frac{1}{7} - \frac{4}{7}t,\ x_3 = t$ is the solution of the system where t is a parameter.

(iii) The matrix form of the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1,\ R_4 - 3R_1$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - 3R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x - \quad\quad\quad w &= -1 \\ y - 2z \quad\quad &= 0 \end{aligned}$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variables z and w are called free variables. Assigning the free variables z and w any arbitrary values t_1 and t_2 respectively,

$$x = -1 + t_2$$

and

$$y = 2t_1$$

Hence, $x = -1 + t_2,\ y = 2t_1,\ z = t_1,\ w = t_2$ is the solution of the system where t_1 and t_2 are parameters.

(iv) The matrix form of the system is

$$\begin{bmatrix} 0 & -2 & 3 \\ 3 & 6 & -3 \\ 6 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\overset{\left(\frac{1}{3}\right)R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_3 - 6R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$\left(-\frac{1}{2}\right)R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$R_3 + 6R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = 6$$

Hence, the system is inconsistent and has no solution.

(v) The matrix form of the system is

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$R_2 - 2R_1,\ R_3 - R_1$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$R_3 - R_2$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\left(\frac{1}{3}\right)R_2$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + R_2$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 - 2x_2 + \qquad x_4 &= 2 \\ x_3 - 2x_4 &= 1 \end{aligned}$$

The leading ones are in columns 1 and 3. Hence, the variables x_1 and x_3 are called leading variables whereas the variables x_2 and x_4 are called free variables. Assigning the free variables x_2 and x_4 any arbitrary values t_1 and t_2 respectively,

$$\begin{aligned} x_1 &= 2 + 2t_1 - t_2 \\ x_3 &= 1 + 2t_2 \end{aligned}$$

Hence, $x_1 = 2 + 2t_1 - t_2,\ x_2 = t_1,\ x_3 = 1 + 2t_2,\ x_4 = t_2$ is the solution of the system where t_1 and t_2 are the parameters.

(vi) The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\underset{\sim}{R_{14}} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$\underset{\sim}{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$\underset{\sim}{R_2 - 3R_1,\ R_3 - 4R_1,\ R_4 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 2 & -2 & 18 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$

$$\underset{\sim}{R_{24}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 3 & -2 & 23 \\ 0 & 2 & -2 & 18 \end{array}\right]$$

$$\underset{\sim}{R_3 - 3R_2,\ R_4 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = -16$$

Hence, the system is inconsistent and has no solution.

Exercise 1.2

1. Solve the following systems of equations by Gaussian elimination method:

(i)
$$\begin{aligned} 2x-3y-z&=3\\ x+2y-z&=4\\ 5x-4y-3z&=-2 \end{aligned}$$

(ii)
$$\begin{aligned} x+2y-z&=1\\ x+y+2z&=9\\ 2x+y-z&=2 \end{aligned}$$

(iii)
$$\begin{aligned} 6x+y+z&=-4\\ 2x-3y-z&=0\\ -x-7y-2z&=7 \end{aligned}$$

(iv)
$$\begin{aligned} 2x-y-z&=2\\ x+2y+z&=2\\ 4x-7y-5z&=2 \end{aligned}$$

(v)
$$\begin{aligned} 2x_1+x_2+2x_3+x_4&=6\\ 6x_1-6x_2+6x_3+12x_4&=36\\ 4x_1+3x_2+3x_3-3x_4&=1\\ 2x_1+2x_2-x_3+x_4&=10 \end{aligned}$$

Ans.:
(i) inconistent
(ii) consistent
$x=2,\ y=1, z=3$
(iii) consistent
$x=-1,\ y=-2,\ z=-4$
(iv) consistent
$x=\dfrac{6+t}{5},\ y=\dfrac{2-3t}{5},\ z=t$
(v) consistent
$x_1=2,\ x_2=1,\ x_3=-1,\ x_4=3$

2. Solve the following system of equations by Gauss–Jordan elimination method:

(i)
$$\begin{aligned} x+2y+z&=-1\\ 6x+y+z&=-4\\ 2x-3y-z&=0\\ -x-7y-2z&=7\\ x-y&=1 \end{aligned}$$

(ii)
$$\begin{aligned} x+y+z&=6\\ x-2y+2z&=5\\ 3x+y+z&=8\\ 2x-2y+3z&=7 \end{aligned}$$

(iii)
$$\begin{aligned} 2x_1+x_2+5x_4&=4\\ 3x_1-2x_2+2x_3&=2\\ 5x_1-8x_2-4x_3&=1 \end{aligned}$$

Ans.:
(i) consistent
$x=-1,\ y=-2,\ z=4$
(ii) consistent
$x=-1,\ y=-2,\ z=3$
(iii) inconsistent

3. Investigate for what values of λ and μ, the system of simultaneous equations

$$\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu \end{aligned}$$

have (i) no solution, (ii) a unique solution, and (iii) infinite number of solutions.

Ans.:
(i) $\lambda=3, \mu\neq 10$
(ii) $\lambda\neq 3$, any value of μ
(iii) $\lambda=3, \mu=10$

4. Investigate for what values of k the equations

$$\begin{aligned} x+y+z &= 1 \\ 2x+y+4z &= k \\ 4x+y+10z &= k^2 \end{aligned}$$

have infinite number of solutions.

[**Ans.:** $k = 1, 2$]

5. Determine the values of λ for which the following system of equations.

$$\begin{aligned} 3x-y+\lambda z &= 0 \\ 2x+y+z &= 2 \\ x-2y-\lambda z &= -1 \end{aligned}$$

will fail to have a unique solution. For this value of λ, are the equations consistent?

$$\left[\textbf{Ans.: } \lambda = -\frac{7}{2}, \text{no solution}\right]$$

6. Find for what values λ, the set of equations

$$\begin{aligned} 2x-3y+6z-5t &= 3 \\ y-4z+t &= 1 \\ 4x-5y+8z-9t &= \lambda \end{aligned}$$

has (i) no solution, and (ii) infinite number of solutions and find the solutions of the equations when they are consistent.

$$\left[\begin{aligned} &\textbf{Ans.: } \text{(i) } \lambda \neq 7, \\ &\text{(ii) } \lambda = 7, x = 3k_1 + k_2 + 3, \\ &y = 4k_1 - k_2 + 1, z = k_1, \\ &t = k_2 \end{aligned}\right]$$

1.7 SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

A system of m homogeneous linear equations in n variables $x_1, x_2, \ldots x_n$ or simply a linear system, is a set of m linear equations each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = \mathbf{0}$$

where A is any matrix of order $m \times n$, $\mathbf{x}$ is a vector of order $n \times 1$ and $\mathbf{0}$ is a null vector of order $m \times 1$. The matrix A is called coefficient matrix of the system of equations.

1.7.1 *Solutions of a System of Linear Equations*

For a system of m linear equations in n variables, there are two possibilities of the solutions to the system.

(i) The system has exactly one solution, i.e. $x_1 = 0, x_2 = 0 \ldots, x_n = 0$. This solution is called the trivial solution.

(ii) The system has infinite solutions.

Note: The system of equations has a non-trivial solution if $\det(A) = 0$.

Example 1: Solve the following systems of equations by the Gauss–Jordan elimination method.

(i) $$\begin{aligned} 3x - y - z &= 0 \\ x + y + 2z &= 0 \\ 5x + y + 3z &= 0 \end{aligned}$$

(ii) $$\begin{aligned} x + y - z + w &= 0 \\ x - y + 2z - w &= 0 \\ 3x + y \quad + w &= 0 \end{aligned}$$

(iii) $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 \quad &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\begin{aligned}
&R_{12} \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & -1 & -1 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right] \\
&R_2 - 3R_1,\ R_3 - 5R_1 \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -4 & -7 & 0 \\ 0 & -4 & -7 & 0 \end{array}\right] \\
&\left(-\frac{1}{4}\right)R_2,\ \left(-\frac{1}{4}\right)R_3 \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \end{array}\right] \\
&R_3 - R_2 \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x + \frac{1}{4}z = 0$$

$$y + \frac{7}{4}z = 0$$

Solving for the leading variables,

$$x = -\frac{1}{4}z$$

$$y = -\frac{7}{4}z$$

Assigning the free variable z an arbitrary value t,

$$x = -\frac{1}{4}t$$

$$y = -\frac{7}{4}t$$

Hence, $x = -\frac{1}{4}t,\ y = -\frac{7}{4}t$ is the non-trivial solution of the system where t is a parameter.

(ii) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to the reduced row echelon form,

$$\begin{matrix} R_2 - R_1, R_3 - 3R_1 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & -2 & 3 & -2 & 0 \end{array}\right] \end{matrix}$$

$$\begin{matrix} \left(-\frac{1}{2}\right)R_2, \left(-\frac{1}{2}\right)R_3 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \end{array}\right] \end{matrix}$$

$$\begin{matrix} R_3 - R_2 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{matrix}$$

$$\begin{matrix} R_1 - R_2 \\ \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{matrix}$$

The corresponding system of equations is

$$\begin{aligned} x + \frac{1}{2}z \qquad &= 0 \\ y - \frac{3}{2}z + w &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x &= -\frac{1}{2}z \\ y &= \frac{3}{2}z - w \end{aligned}$$

Assigning the free variables z and w arbitrary values t_1 and t_2 respectively,

$$x = -\frac{1}{2}t_1$$

$$y = \frac{3}{2}t_1 - t_2$$

Hence, $x = -\frac{1}{2}t_1,\ y = \frac{3}{2}t_1 - t_2,\ z = t_1,\ w = t_2$ is the non-trivial solution of the system where t_1 and t_2 are parameters.

(iii) The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} \left(\frac{1}{2}\right)R_3 \\ \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_2 + R_3 \\ \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_1 - 2R_2 \\ \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The corresponding system of equations is

$$\begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$$

Hence, the system has a trivial solution, i.e. $x = 0$, $y = 0$, $z = 0$.

Example 2: Show that the following non-linear system has 18 solutions if $0 \le \alpha \le 2\pi, 0 \le \beta \le 2\pi$ and $0 \le \gamma < 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -1 & -5 & 5 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{array}\right]$$

$$R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$(-1)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 + 3R_3,\ R_1 - 3R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\sin\alpha = 0$$
$$\cos\beta = 0$$
$$\tan\gamma = 0$$

From these equations,

$$\alpha = 0, \pi, 2\pi$$
$$\beta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad [\because \alpha, \beta \text{ and } \gamma \text{ lie between 0 and } 2\pi]$$
$$\gamma = 0, \pi, 2\pi$$

Hence, there are $3 \cdot 2 \cdot 3 = 18$ possible solutions which satisfy the system of equations.

Example 3: For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

$$\begin{aligned} x+2y+3z &= \lambda x \\ 3x+\ y+2z &= \lambda y \\ 2x+3y+\ z &= \lambda z \end{aligned}$$

Solution: The system of equations is

$$\begin{aligned} (1-\lambda)x+2y+3z &= 0 \\ 3x+(1-\lambda)y+2z &= 0 \\ 2x+3y+(1-\lambda)z &= 0 \end{aligned}$$

The matrix form of the system is

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system will possess a non-trivial solution if $\det(A) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)[(1-\lambda)^2-6]-2(3-3\lambda-4)+3(9-2+2\lambda) &= 0 \\ (1-\lambda)(\lambda^2-2\lambda-5)+2+6\lambda+21+6\lambda &= 0 \\ \lambda^2-2\lambda-5-\lambda^3+2\lambda^2+5\lambda+12\lambda+23 &= 0 \\ -\lambda^3+3\lambda^2+15\lambda+18 &= 0 \end{aligned}$$

$$\lambda = 6, \quad \lambda = -1.5 \pm 0.866\,i$$

For real value of λ, i.e. $\lambda = 6$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 3 & -5 & 2 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 1 & -8 & 7 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$\underset{R_{12}}{\sim}\left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ -5 & 2 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$\underset{R_2+5R_1,\, R_3-2R_1}{\sim}\left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & -38 & 38 & 0 \\ 0 & 19 & -19 & 0 \end{array}\right]$$

$$\underset{\left(-\frac{1}{38}\right)R_2,\, \left(\frac{1}{19}\right)R_3}{\sim}\left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\underset{R_3-R_2}{\sim}\left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_1+8R_2}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x-z=0$$
$$y-z=0$$

Solving for the leading variables,

$$x=z$$
$$y=z$$

Assigning the free variable z an arbitrary value t,

$$x=t$$
$$y=t$$

Hence, $x=t, y=t, z=t$ is the non-trivial solution of the system where t is a parameter.

Example 4: If the following system has a non-trivial solution, then prove that $a+b+c=0$ or $a=b=c$ and hence find the solution in each case.

$$ax+by+cz=0$$
$$bx+cy+az=0$$
$$cx+ay+bz=0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a non-trivial solution if det $(A)=0$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

$$a(bc-a^2)-b(b^2-ac)+c(ab-c^2)=0$$
$$-a^3+b^3+c^3-3abc=0$$
$$-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$$
$$a+b+c=0$$

or

$$a^2+b^2+c^2-ab-bc-ca=0$$
$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$$
$$a-b=0, b-c=0, c-a=0$$
$$a=b, b=c, c=a$$
$$a=b=c$$

Hence, the system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.
The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} a & b & c & 0 \\ b & c & a & 0 \\ c & a & b & 0 \end{array}\right]$$

$R_3+R_1+R_2$

$$\sim\left[\begin{array}{ccc:c} a & b & c & 0 \\ b & c & a & 0 \\ a+b+c & a+b+c & a+b+c & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} ax+ \quad by \quad + \quad cz &= 0 \\ bx+ \quad cy \quad + \quad az &= 0 \\ (a+b+c)x+(a+b+c)y + (a+b+c)z &= 0 \end{aligned}$$

(i) When $a+b+c=0,$ we have only two equations.

$$ax+by+cz=0$$
$$bx+cy+az=0$$

$$\frac{x}{\begin{vmatrix} b & c \\ c & a \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ b & a \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = t$$

$$\frac{x}{ab-c^2} = -\frac{y}{a^2-bc} = \frac{z}{ac-b^2} = t$$

Hence, $x=(ab-c^2)t, y=(bc-a^2)t, z=(ac-b^2)t$ is the solution of the system where t is a parameter.

(ii) When $a = b = c$, we have only one equation.

$$x+y+z=0$$

Let $$y=t_1$$ $$z=t_2$$

Then $$x=-t_1-t_2$$

Hence, $x=-t_1-t_2, y=t_1, z=t_2$ is the solution of the system where t_1 and t_2 are parameters.

Example 5: Discuss for all values of k, the system of equations

$$\begin{aligned} 2x+ \quad 3ky+(3k+4)z &= 0 \\ x+ (k+4)y+(4k+2)z &= 0 \\ x+2(k+1)y+(3k+4)z &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} \\ &= (k-8)(-k+2) + 5k(k-2) \\ &= (k-2)(-k+8+5k) \\ &= 4(k-2)(k+2) \end{aligned}$$

(i) When $k \neq \pm 2$, $\det(A) \neq 0$, the system has a trivial solution, i.e. $x = 0, y = 0, z = 0$.

(ii) When $k = \pm 2$, $\det(A) = 0$, the system has non-trivial solutions.

Case I: When $k = 2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & -6 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 6R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x = 0$$

$$y + \frac{10}{6}z = 0$$

Solving for the leading variables,

$$x = 0$$
$$y = -\frac{10}{6}z$$

Assigning the free variable z any arbitrary value t,

$$y = -\frac{10}{6}t = -\frac{5}{3}t$$

Hence, $x = 0,\ y = -\frac{5}{3}t,\ z = t$ is the solution of the system where t is a parameter.

Case II: When $k = -2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(-\frac{1}{10}\right)R_2, \left(-\frac{1}{4}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x - 4z = 0$$
$$y - \ \ z = 0$$

Solving for the leading variables,

$$x = 4z$$
$$y = z$$

Assigning the free variable z any arbitrary value t,

$$x = 4t$$
$$y = t$$

Hence, $x = 4t,\ y = t,\ z = t$ is the solution of the system where t is a parameter.

Exercise 1.3

1. Solve the following equations:

(i) $$\begin{aligned} x - y + z &= 0 \\ x + 2y + z &= 0 \\ 2x + y + 3z &= 0 \end{aligned}$$

(ii) $$\begin{aligned} x - 2y + 3z &= 0 \\ 2x + 5y + 6z &= 0 \end{aligned}$$

(iii) $$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

(iv) $$\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x - 4y + 5z &= 0 \end{aligned}$$

(v) $$\begin{aligned} 7x + y - 2z &= 0 \\ x + 5y - 4z &= 0 \\ 3x - 2y + z &= 0 \\ 2x - 7y + 5z &= 0 \end{aligned}$$

(vi) $$\begin{aligned} 3x + 4y - z - 9w &= 0 \\ 2x + 3y + 2z - 3w &= 0 \\ 2x + y - 14z - 12w &= 0 \\ x + 3y + 13z + 3w &= 0 \end{aligned}$$

(vii) $$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

(viii) $$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 4x_2 + 5x_3 &= 0 \end{aligned}$$

[**Ans.:** (*i*) $x = 0,\ y = 0,\ z = 0$

(*ii*) $x = -3t,\ y = 0,\ z = t$

(*iii*) $x = \frac{211}{9}t,\ y = 4t,\ z = \frac{7}{9}t,$ $w = t$

(*iv*) $x = -t,\ y = t,\ z = t$

(*v*) $x = \frac{3}{17}t,\ y = \frac{13}{17}t,\ z = t$

(*vi*) $x = 11t,\ y = -8t,\ z = t,$ $w = 0$

(*vii*) $x_1 = -\frac{1}{3}t,\ x_2 = \frac{2}{3}t,$ $x_3 = -\frac{2}{3}t,\ x_4 = t$

(*viii*) $x_1 = -x_2 = -x_3 = t$]

2. For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

(i) $$\begin{aligned} 3x + y - \lambda z &= 0 \\ 4x - 2y - 3z &= 0 \\ 2\lambda x + 4y - \lambda z &= 0 \end{aligned}$$

(ii) $$\begin{aligned} (1-\lambda)x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_1 + (1-\lambda)x_2 + 2x_3 &= 0 \\ 2x_1 + 3x_2 + (1-\lambda)x_3 &= 0 \end{aligned}$$

[**Ans.:**

(i) Non-trivial solution $\lambda = 1, -9$

For $\lambda = 1,\ x = -t,\ y = -t,\ z = -2t$

For $\lambda = -9,\ x = -3t,\ y = -9t,\ z = 2t$

(ii) $\lambda = 6,\ x = y = z = t$]

3. Show that the system of equations $2x - 2y + z = \lambda x$, $2x - 3y + 2z = \lambda y$, $-x + 2y = \lambda z$ can posses a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

$$\left[\begin{array}{l} \textbf{Ans.:} \text{ For } \lambda = 1,\ x = 2t_2 - t_1 \\ \qquad y = t_2,\ z = t_1 \\ \text{For } \lambda = -3,\ x = -t, \\ \qquad y = -2t,\ z = t \end{array}\right]$$

1.8 INVERSE OF A MATRIX

If A be any n-rowed square matrix, then a matrix B, if it exists such that

$$AB = BA = \mathrm{I}_n$$

is called the inverse of A,

i.e., $$B = A^{-1}$$

We will explain a few terms associated with matrices before finding the inverse of a matrix.

(1) *Minor of an Element of a Determinant*

If $$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

minor of a determinant is a determinant obtained by removing the row and columns of $\det(A)$ passing through the element, e.g.

$$\text{Minor of the element } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Minor of the element } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\text{Minor of the element } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(2) *Cofactor of an Element of a Determinant*

If $$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

cofactor of an element a_{ij} of a determinant is the minor multiplied by $(-1)^{i+j}$, e.g.

$$\text{Cofactor of element } a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of the element } a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of the element } a_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(3) *Adjoint of a Square Matrix*

The transpose of the matrix of the cofactors is called the adjoint of the matrix.

Let A be a non-singular n-rowed square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The matrix formed by the cofactors of the elements of A is

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

The transpose of this matrix of cofactors is called the adjoint of A and is denoted by adj A.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Theorem 1.1: If A is a non-singular square matrix of order n, then

(i) $A(\text{adj } A) = (\text{adj } A)A = |A| \, I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$

1.8.1 *Inverse of a Matrix by Determinant Method*

If A is an $n \times n$ singular square matrix, then inverse of A is

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A$$

Theorem 1.2: Every invertible matrix possesses a unique inverse.

Theorem 1.3: The necessary and sufficient condition for a square matrix A to possess an inverse is that $\det(A) \neq 0$, i.e. A is non-singular.

Theorem 1.4: The inverse of a product is the product of the inverses taken in the reverse order.

$$(AB)^{-1} = B^{-1} A^{-1}$$

Theorem 1.5: If A is an $n \times n$ non-singular matrix, then

$$(A^{-1})^T = (A^T)^{-1}$$

Example 1: Find the adjoint of the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: The cofactors of elements of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \quad ; \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 7$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \quad ; \quad A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 \quad ; \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 7$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad ; \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{bmatrix}$

$$\text{adj } A = \text{transpose of the matrix of cofactors}$$

$$= \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Example 2: Find adj (adj A), where $A = \frac{1}{9}\begin{bmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{bmatrix}$

Solution:

$$A = \frac{1}{9}\begin{bmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{bmatrix}$$

We know that

$$\text{adj (adj } A) = |A|^{n-2} A$$

and

$$|kA| = k^n |A|$$

Here, $n = 3$

$$|A| = \left(\frac{1}{9}\right)^3 \begin{vmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{vmatrix}$$

$$= \frac{1}{729}[(-1)(-16+7)-(-8)(16+56)+4(4+32)]$$

$$= \frac{1}{729} \times 729$$

$$= 1$$

$$\text{adj (adj } A) = |A|^{3-2} A$$

$$= A$$

Example 3: If $A = \begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and adj (adj A) = A, find a.

Solution:
$$A = \begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

We know that

$$\text{adj (adj } A) = |A|^{n-2} A$$

Here, $n = 3$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 1(0-4) - 2(a-4) + 1(a-0) = -a+4$$

$$\text{adj (adj } A) = (-a+4)^1 A$$

$$= (-a+4)A$$

But
$$\text{adj (adj } A) = A$$

$$(-a+4)A = A$$

$$-a+4 = 1$$

$$a = 3$$

Example 4: Find the inverses of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: (i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} 0 & 4 & -4 \\ -1 & -1 & 3 \\ 2 & -2 & 2 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1(2-2) - 2(0-4) + 1(0-4) = 4$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A = \frac{1}{4} \begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1(3+6) + 1(-1-4) = -11$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A = -\frac{1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

(iii) $$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of cofactors of elements of $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 5: Find the matrix A, if adj $A = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$.

Solution: adj $A = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$

We know that

$$|\text{adj}(A)| = |A|^{n-1}$$

Here, $n = 3$

$$|\text{adj}(A)| = |A|^2$$

Now, $|\text{adj}(A)| = -2(15-11) - 1(10-22) + 3(-2+6) = 16$

Thus, $|A|^2 = 16$

$$|A| = 4$$

The matrix of cofactors of elements of adj $A = \begin{bmatrix} 4 & 12 & 4 \\ 8 & 4 & 4 \\ 20 & 16 & 8 \end{bmatrix}$

$$\text{adj (adj } A) = \begin{bmatrix} 4 & 8 & 20 \\ 12 & 4 & 16 \\ 4 & 4 & 8 \end{bmatrix}$$

$$(\text{adj } A)^{-1} = \frac{1}{|\text{adj } (A)|}\text{adj}(\text{adj } A) = \frac{1}{16}\begin{bmatrix} 4 & 8 & 20 \\ 12 & 4 & 16 \\ 4 & 4 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Since $A(\text{adj } A) = |A| I_3$

$$A = |A|(\text{adj} A)^{-1} = 4 \cdot \frac{1}{4}\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Example 6: Find the matrix A if

$$\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} A \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$$

Solution: Let

$$B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$$

Then

$$BAC = D$$

$$AC = B^{-1}D$$

$$A = B^{-1}DC^{-1}$$

$$B^{-1} = \frac{1}{|B|}\text{adj} B = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

$$C^{-1} = \frac{1}{|C|}\text{adj } C = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Hence,

$$A = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & -16 \\ 88 & 56 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ 11 & 7 \end{bmatrix}$$

Example 7: If $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} \frac{q+r}{2} & \frac{r-p}{2} & \frac{q-p}{2} \\ \frac{r-q}{2} & \frac{r+p}{2} & \frac{p-q}{2} \\ \frac{q-r}{2} & \frac{p-r}{2} & \frac{p+q}{2} \end{bmatrix}$, prove that ABA^{-1} is a diagonal matrix.

Solution:

$$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\det(A)=0(0-1)-1(0-1)+1(1-0)=2\neq 0$$

Hence, A^{-1} exists.

The matrix of cofactors of elements of $A=\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{adj } A=\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1}=\frac{1}{\det(A)}\text{adj } A=\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now,

$$AB=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{q+r}{2} & \frac{r-p}{2} & \frac{q-p}{2} \\ \frac{r-q}{2} & \frac{r+p}{2} & \frac{p-q}{2} \\ \frac{q-r}{2} & \frac{p-r}{2} & \frac{p+q}{2} \end{bmatrix}=\begin{bmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{bmatrix}$$

$$ABA^{-1}=\begin{bmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 2p & 0 & 0 \\ 0 & 2q & 0 \\ 0 & 0 & 2r \end{bmatrix}=\begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$$

Hence, ABA^{-1} is a diagonal matrix.

Example 8: Show that $[\text{diag}\,(\alpha, \beta, \gamma)]^{-1} = \text{diag}\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$ if $\alpha\beta\gamma \neq 0$.

Solution: Let $A = \text{diag}(\alpha, \beta, \gamma) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$

$$\text{adj}\, A = \begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{vmatrix} = \alpha\beta\gamma$$

If $\alpha\beta\gamma \neq 0$, then
$$A^{-1} = \frac{1}{\det(A)} \text{adj}\, A$$

$$= \frac{1}{\alpha\beta\gamma} \begin{bmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

$$= \text{diag}\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$$

Example 9: Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

Solution: Let $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$

The matrix of cofactors of elements of $A = \begin{bmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{vmatrix} = 1 + \tan^2\frac{\theta}{2} = \sec^2\frac{\theta}{2}$$

$$A^{-1} = \frac{1}{\det(A)}\text{adj } A = \frac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\dfrac{\theta}{2} \\ -\tan\dfrac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}\dfrac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1 & -\tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} & 1 \end{bmatrix}$

$$= \frac{1}{\sec^2\dfrac{\theta}{2}}\begin{bmatrix} 1 - \tan^2\dfrac{\theta}{2} & -\tan\dfrac{\theta}{2} - \tan\dfrac{\theta}{2} \\ \tan\dfrac{\theta}{2} + \tan\dfrac{\theta}{2} & -\tan^2\dfrac{\theta}{2} + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} & -2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} \\ 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} & \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 10: Find the inverses of $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix}$ and hence, find inverse of $C = \begin{bmatrix} 1+ab & a & 0 \\ b & 1+ab & a \\ 0 & b & 1 \end{bmatrix}$

Solution: $A=\begin{bmatrix}1 & a & 0\\ 0 & 1 & a\\ 0 & 0 & 1\end{bmatrix}$

The matrix of cofactors of elements of $A=\begin{bmatrix}1 & 0 & 0\\ -a & 1 & 0\\ a^2 & -a & 1\end{bmatrix}$

$$\text{adj } A=\begin{bmatrix}1 & -a & a^2\\ 0 & 1 & -a\\ 0 & 0 & 1\end{bmatrix}$$

$$\det(A)=\begin{vmatrix}1 & a & 0\\ 0 & 1 & a\\ 0 & 0 & 1\end{vmatrix}=1$$

$$A^{-1}=\frac{1}{\det(A)}\text{adj } A$$

$$=\begin{bmatrix}1 & -a & a^2\\ 0 & 1 & -a\\ 0 & 0 & 1\end{bmatrix}$$

Replacing a by b, A^T becomes $\begin{bmatrix}1 & 0 & 0\\ b & 1 & 0\\ 0 & b & 1\end{bmatrix}$ which is equal to the matrix B.

Hence, replacing a by b in the transpose of A^{-1}, we get

$$B^{-1}=\begin{bmatrix}1 & 0 & 0\\ -b & 1 & 0\\ b^2 & -b & 1\end{bmatrix}$$

Now,

$$C=\begin{bmatrix}1+ab & a & 0\\ b & 1+ab & a\\ 0 & b & 1\end{bmatrix}=AB$$

$$C^{-1}=(AB)^{-1}=B^{-1}A^{-1}$$

$$=\begin{bmatrix}1 & 0 & 0\\ -b & 1 & 0\\ b^2 & -b & 1\end{bmatrix}\begin{bmatrix}1 & -a & a^2\\ 0 & 1 & -a\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a & a^2 \\ -b & 1+ab & -a^2b-a \\ b^2 & -ab^2-b & a^2b^2+ab+1 \end{bmatrix}$$

Example 11: Find the inverse of the matrix $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and if $A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix}$, show that SAS^{-1} is diag. (2, 3, 1).

Solution: $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The matrix of cofactors of elements of $S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{adj } S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\det(S) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(-1) + 1(1) = 2$$

$$S^{-1} = \frac{1}{\det(S)} \text{adj } S = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$SA = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 4 & 4 \\ 6 & 0 & 6 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$SAS^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \text{diag.}\,(2, 3, 1)$$

1.8.2 *Inverse of a Matrix by Elementary Transformation (Gauss–Jordan Elimination method)*

Let A be any non-singular matrix. Then $A = IA$. Applying suitable elementary row transformation to A on the L.H.S and to I on the R.H.S, so that A reduces to I and I reduces to any matrix B.

Hence, $I = BA$

$$B = A^{-1}$$

Example 1: Find the inverses of the following matrices by elementary transformation (Gauss–Jordan elimination method):

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$

Solution: (i) Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

$$A = I_3 A$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to reduced row echelon form,

R_{13}

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 - 4R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$\left(-\frac{1}{5}\right)R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix} A$$

$(-1)R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_2 - 3R_3,\ R_1 - 4R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -\frac{4}{5} & -\frac{19}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$$I_3 = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

(ii) Let
$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$

$$A = I_4 A$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to reduced row echelon form,

$$R_3 - 2R_1,\ R_4 - 3R_1$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & -3 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 - 3R_2,\ R_4 - R_2$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$$(-1)R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 + R_4,\ R_1 - 2R_4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -3 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$$I_4 = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

Exercise 1.4

1. Find the inverses of the following matrices by the determinant method:

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

Ans.: (i) $\frac{1}{6}\begin{bmatrix} -4 & 3 & 1 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iii) $\frac{1}{4}\begin{bmatrix} 0 & -1 & 2 \\ 4 & -1 & -2 \\ -4 & 3 & 2 \end{bmatrix}$

2. Find the inverse of the matrix

$$S=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix} \text{ and}$$

$$A=\frac{1}{2}\begin{bmatrix}3 & -2 & -1\\ -1 & 4 & 1\\ 1 & 2 & 5\end{bmatrix} \text{ and show that}$$

SAS^{-1} is the diagonal matrix of diag. (3, 2, 1).

3. If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, show that $A^3=A^{-1}$.

4. If $A=\begin{bmatrix}8 & 4 & 3\\ 2 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}$, prove that

$$A^{-2}=\frac{1}{9}\begin{bmatrix}2 & -4 & -5\\ -10 & 47 & -11\\ -9 & -54 & 27\end{bmatrix}$$

5. If $A=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & -1\\ 2 & 2 & 3\end{bmatrix}$, find A^{-1} if it exists. Hence, find the inverse of

$$B=\begin{bmatrix}3 & 6 & -3\\ 0 & 3 & -3\\ 6 & 6 & 9\end{bmatrix}$$

$$\left[\textbf{Ans.: } A^{-1}=\frac{1}{3}\begin{bmatrix}5 & -8 & -1\\ -2 & 5 & 1\\ -2 & 2 & 1\end{bmatrix},\ B^{-1}=\frac{1}{9}\begin{bmatrix}5 & -8 & -1\\ -2 & 5 & 1\\ -2 & 2 & 1\end{bmatrix}\right]$$

6. If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}$ and

$$B=\begin{bmatrix}2 & 5 & 3\\ 3 & 1 & 2\\ 1 & 2 & 1\end{bmatrix},$$

show that $(AB)^{-1}=B^{-1}A^{-1}$.

7. Find the matrix A if

$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}A\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}-2 & 4\\ 3 & -1\end{bmatrix}$$

$$\left[\textbf{Ans.:}\begin{bmatrix}24 & 13\\ -34 & -18\end{bmatrix}\right]$$

8. Find the inverse of A if

$$\begin{bmatrix}1 & 0 & 0\\ 2 & -1 & 0\\ -2 & 1 & 1\end{bmatrix}A\begin{bmatrix}1 & -2 & 9\\ 0 & 1 & -6\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\left[\textbf{Ans.:}\begin{bmatrix}-21 & 11 & 9\\ 14 & -7 & -6\\ -2 & 1 & 1\end{bmatrix}\right]$$

9. Using elementary row transformations, find the inverses of the following matrices:

(i) $\begin{bmatrix}8 & 4 & 3\\ 2 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}$ (ii) $\begin{bmatrix}5 & -1 & 5\\ 0 & 2 & 0\\ -5 & 3 & -15\end{bmatrix}$

(iii) $\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$

(iv) $\begin{bmatrix}1 & -1 & 1\\ 2 & 2 & -3\\ 1 & -4 & 9\end{bmatrix}$ (v) $\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 0\\ 0 & 1 & 2\end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

(viii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$

Ans.:

(i) $\frac{1}{3}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$ (ii) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & 3 \end{bmatrix}$ (iv) $\frac{1}{17}\begin{bmatrix} 6 & 5 & 1 \\ -21 & 8 & 5 \\ -10 & 3 & 4 \end{bmatrix}$

(v) $\frac{1}{4}\begin{bmatrix} 6 & -1 & -9 \\ -4 & 2 & 6 \\ 2 & -1 & -1 \end{bmatrix}$

(vi) $\begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

(vii) $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

(viii) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

(ix) $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$

(x) $\begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

10. Find the matrix A if

$$A^{-1} = \begin{bmatrix} -1 & -3 & -3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Ans.: $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

1.9 RANK OF A MATRIX

The positive integer r is said to be the rank of a matrix A if it possesses the following properties:

(i) There is at least one minor of order r which is non-zero.

(ii) Every minor of order greater than r is zero.
Rank of matrix A is denoted by $\rho(A)$.

Theorem 1.6: The rank of a matrix remains unchanged by elementary transformations.

Theorem 1.7: The rank of the transpose of a matrix is same as that of the original matrix.

Theorem 1.8: The rank of the product of two matrices cannot exceed the rank of either matrix.

$$\rho(AB) \leq \rho(A) \quad \text{or} \quad \rho(AB) \leq \rho(B)$$

1.9.1 *Rank of a Matrix by Determinant Method*

(1) The rank of a matrix is less than or equal to r, if all $(r+1)$ rowed minors of the matrix are zero.

(2) The rank of a matrix is greater than or equal to r, if at least one minor of order r is not equal to zero.

(3) The rank of a null matrix is zero.

(4) The rank of a non-singular square matrix is always equal to its order.

e.g. consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

Therefore, the rank of A is less than 3. There is at least one minor of A of order 2, i.e. $\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} \neq 0$, Hence, the rank of A, i.e., $\rho(A) = 2$

Example 1: Find the ranks of the following matrices by determinant method:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Solution: (i) Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\ &= 2(12-2) - 3(16-1) + 4(8-3) \\ &= -5 \\ &\neq 0 \end{aligned}$$

A is a non-singular matrix of order 3.

Hence, $\rho(A) = 3$

(ii) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} \\ &= 1(21-20) - 2(14-12) + 3(10-9) \\ &= 0 \end{aligned}$$

Therefore, the rank of A is less than 3. The minor of order 2 is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$.

Hence, $\rho(A) = 2$

(iii) Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{vmatrix} \\ &= 4(-6+6) - 2(-12+12) + 3(-8+8) \\ &= 0 \end{aligned}$$

Therefore, the rank of A is less than 3.
Consider all the minors of order 2, i.e.,

$$\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 3 \\ -2 & -\frac{3}{2} \end{vmatrix} = 0$$

All the minors of order 2 are zero. Therefore, the rank of A is less than 2.

Hence, $\rho(A) = 1$

(iv) Let $$A = \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Consider all the minors of order 3, i.e.

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & -2 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & -4 \\ 4 & 3 & 5 \\ -2 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & 5 \\ -1 & -2 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & -4 \\ 2 & 3 & 5 \\ -1 & 6 & -7 \end{vmatrix} = -120$$

One minor of rank 3 is not equal to zero.
Hence, $\rho(A) = 3$

Example 2: For what value of x, will the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{bmatrix}$ be of rank

(i) equal to 3 (ii) less than 3

Solution: $$\det(A) = \begin{vmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{vmatrix}$$

$$= (3-x)[(4-x)(1-x) - 0] - 2(1-x) + 2(-4+8-2x)$$
$$= (3-x)(4-x)(1-x) + 2(3-x)$$
$$= (3-x)(x^2 - 5x + 6)$$
$$= (3-x)(x-3)(x-2)$$
$$= -(x-3)^2(x-2)$$

(i)
$$\rho(A) = 3 \quad \text{if} \quad \det(A) \neq 0$$
$$(x-3)^2(x-2) \neq 0$$
$$x \neq 2, 3$$

(ii)
$$\rho(A) < 3 \quad \text{if} \det(A) = 0$$
$$(x-3)^2(x-2) = 0$$
$$x = 2, 3$$

Example 3: Find the value of p for which the following matrix A will be of

(i) rank one (ii) rank two (iii) rank three

$$A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$

Solution:
$$\det(A) = \begin{vmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{vmatrix}$$
$$= 3(9-p^2) - p(3p-p^2) + p(p^2-3p)$$
$$= (3-p)(9+3p-p^2-p^2)$$
$$= (p-3)(2p^2-3p-9)$$
$$= (p-3)(p-3)(2p+3)$$
$$= (p-3)^2(2p+3)$$

(i) If $p = 3$

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$\det(A) = 0$ and all the minors of order 2 are zero.

Hence, $\rho(A) = 1$

Rank of A will be 1 if $p = 3$

(ii) Rank of A will be 2 if $\det(A) = 0$ but $p \neq 3$

$$(p-3)^2(2p+3) = 0 \quad \text{but} \quad p \neq 3$$
$$2p+3 = 0$$
$$p = -\frac{3}{2}$$

(iii) Rank of A will be 3 if $\det(A) \neq 0$

$$(p-3)^2(2p+3) \neq 0$$

$$p \neq 3 \quad p \neq -\frac{3}{2}$$

Example 4: Determine the value of b such that the rank of A is 3 where

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

Rank of A will be 3 if $\det(A) = 0$ and at least one minor of A of order 3 must be non-zero.
By elementary transformation, $C_2 - C_1$ and $C_3 + C_1$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ b & 2-b & b+2 & 2 \\ 9 & 0 & b+9 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ b & 2-b & b+2 & 2 \\ 9 & 0 & b+9 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 2-b & b+2 & 2 \\ 0 & b+9 & 3 \end{vmatrix}$$

$$= 0 - 3(2-b) + (2-b)(b+9)$$

$$= (2-b)(b+6)$$

Now $\rho(A) = 3 < 4$ when $\det(A) = 0$

$$(2-b)(b+6) = 0$$

$$b = 2, -6$$

For $b = 2$, one of the minor of order 3,

$$\begin{vmatrix} 4 & -3 & 1 \\ 2 & 2 & 2 \\ 9 & 2 & 3 \end{vmatrix} = -42 \neq 0$$

Hence, $\rho(A) = 3$.

1.9.2 *Rank of a Matrix by Row Echelon Form*

The rank of a matrix in row echelon form is equal to the number of non-zero rows of the matrix, e.g.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A is in row echelon form and the number of non-zero rows is two. Hence, the rank of the matrix is two.

i.e. $$\rho(A) = 2$$

Example 1: Find the ranks of the following matrices by reducing to row echelon form:

(i) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Solution: (i) Let $$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

$$R_{13}$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$$R_3 - 5R_1$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$$

$$R_3 - 8R_2$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

$$\left(-\frac{1}{12}\right)R_3$$

$$\sim\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

The equivalent matrix is in row echelon form.
Number of non-zero rows = 3

$$\rho(A) = 3$$

(ii) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_2 + 2R_1,\ R_3 - R_1$$

$$\sim\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_{24}$$

$$\sim\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$$

$$R_3 + 2R_2,\ \ R_4 - 3R_2$$

$$\sim\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equivalent matrix is in row echelon form.
Number of non-zero rows = 2

$$\rho(A) = 2$$

(iii) Let

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\underset{R_{13}}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\underset{R_3-3R_1,\, R_{24}}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{R_3-4R_2,\, R_4-2R_2}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

$$\underset{R_4+2R_3}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

$$\underset{\left(-\frac{1}{23}\right)R_4}{\sim}\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The equivalent matrix is in row echelon form.
Number of non-zero rows = 4

$$\rho(A) = 4$$

1.9.3 *Rank of Matrix by Reduction to Normal Form*

Theorem 1.9: Any matrix of order $m \times n$ can be reduced to the form $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ by elementary transformation where r is the rank of the matrix. This form is known as normal form or first canonical form of a matrix.

Corollary:

(1) The rank of a matrix A of order $m \times n$ is r if and only if it can be reduced to the normal form $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ by elementary transformations.

(2) If A be an $m \times n$ matrix of rank r, then there exists non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Note: P and Q are not unique.

Example 1: Find the ranks of the following matrices by reducing to normal form:

(i) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$.

Solution:

(i) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

$$R_2 - 3R_1,\ R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$$R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 - 2C_1,\ C_3 + C_1,\ C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(-\frac{1}{2}\right)C_2, \left(\frac{1}{3}\right)C_3, \left(-\frac{1}{10}\right)C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 - C_2, C_4 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\rho(A) = 2$$

(ii) Let

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$C_2 + C_1, C_3 - 2C_1, C_4 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$R_{24}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

$$C_4 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix}$$

$$R_3 - 3R_2,\ R_4 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

$$C_{34}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

$$\left(-\frac{1}{2}\right)C_3, \left(-\frac{1}{8}\right)C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$R_4 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim [I_4]$$

$$\rho(A) = 4$$

(iii) Let
$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_2 + R_1,\ R_3 - R_1}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{C_2 - 2C_1,\ C_3 - 3C_1,\ C_4 + C_1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_3 + 2R_2,\ R_4 - R_2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{C_4 + 2C_2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{\left(-\frac{1}{2}\right)R_3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{R_4 - R_3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 - C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim [I_3 \quad \mathbf{0}]$$

$$\rho(A) = 3$$

Example 2: Find non-singular matrices P and Q such that PAQ is in the normal form and hence, find $\rho(A)$ for the following matrices:

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$

Solution: (i) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - R_1,\ R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(-\frac{1}{2}\right)R_2,\ \left(-\frac{1}{2}\right)R_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 - C_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - R_2, R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = PAQ$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 2$$

(ii) Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\left(-\frac{1}{2}\right) C_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$\left(\frac{1}{3}\right)R_2, \left(\frac{1}{3}\right)R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = PAQ$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\rho(A) = 2$$

(iii) Let $$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 + C_2, C_4 + C_2$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - C_1, C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = PAQ$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & -3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 2$$

Exercise 1.5

1. Find the ranks of A, B, AB and verify that rank of the product of two matrices cannot exceed the rank of either matrix.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$$

2. Find the possible values of p, for which the following matrix A will have (i) rank 1 (ii) rank 2 (iii) rank 3

$$A = \begin{bmatrix} p & p & 2 \\ 2 & p & p \\ p & 2 & p \end{bmatrix}$$

[**Ans.:** (i) $p = 2$ (ii) $p = -2$ (iii) $p \neq -1, p \neq 2$]

3. Find the rank of

$$A = \begin{bmatrix} x-1 & x+1 & x \\ -1 & x & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ where } x \text{ is real.}$$

[**Ans.:** 3]

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & x & 1 \\ -1 & -1 & x \end{bmatrix}$, prove that rank of A is 3, where x is a real number.

5. Find the value of λ for which rank of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 5 \\ 7 & 2 & \lambda \end{bmatrix}$

(i) is less than 3 (ii) equal to 3

6. Find the ranks of the following matrices by reducing to row echelon form:

(i) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$

$\begin{bmatrix} \textbf{Ans.:} \text{(i) 2 (ii) 1 (iii) 4} \\ \text{(iv) 2 (v) 4 (vi) 2} \end{bmatrix}$

7. Find the ranks of the following matrices by reducing to normal form:

(i) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -2 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 3 & 0 \\ 9 & 8 & 0 & 8 \end{bmatrix}$

$\begin{bmatrix} \textbf{Ans.:} \text{(i) 2 (ii) 4 (iii) 3} \\ \text{(iv) 3 (v) 4 (vi) 3} \end{bmatrix}$

8. Find non-singular matrices P and Q such that PAQ is in normal form. Also find their ranks.

(i) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Ans.:

(i) $P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank $= 3$

$$\left[\begin{array}{l}
\text{(ii) } P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \\
Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{rank} = 2 \\
\text{(iii) } P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}, \\
Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \text{rank} = 2 \\
\text{(iv) } P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ 1 & 1 & -1 \end{bmatrix}, \\
Q = \begin{bmatrix} 1 & -2 & -\frac{3}{5} \\ 0 & 1 & -\frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}, \text{rank} = 3
\end{array}\right]
\left[\begin{array}{l}
\text{(v) } P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}, \\
\begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{rank} = 3 \\
\text{(vi) } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix}, \\
Q = \begin{bmatrix} 1 & -1 & -\frac{4}{18} & \frac{1}{45} \\ 0 & 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 1 & \frac{2}{18} & \frac{4}{45} \\ 0 & 0 & 0 & \frac{1}{40} \end{bmatrix}, \text{rank} = 3
\end{array}\right]$$

1.10 APPLICATIONS OF SYSTEMS OF LINEAR EQUATIONS

Linear systems are used to model a wide variety of problems. Constructing models for mechanical systems, electrical networks, Indian economy, chemical equations, etc., are some of the applications of linear systems.

Example 1: A 500 N ball is supported by three cables as shown in Fig 1.1. Find the tension in each cable.

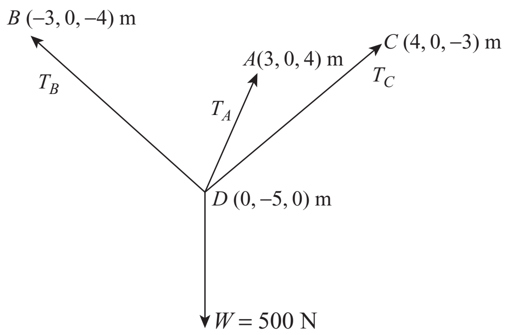


Fig. 1.1

Solution: Writing forces in standard vector form,

$$\mathbf{T_A} = T_A\left(\frac{3\hat{i}+5\hat{j}+4\hat{k}}{5\sqrt{2}}\right) = \frac{3}{5\sqrt{2}}\hat{i} + \frac{5}{5\sqrt{2}}\hat{j} + \frac{4}{5\sqrt{2}}\hat{k}$$

$$\mathbf{T_B} = T_B\left(\frac{-3\hat{i}+5\hat{j}-4\hat{k}}{5\sqrt{2}}\right) = \frac{-3}{5\sqrt{2}}\hat{i} + \frac{5}{5\sqrt{2}}\hat{j} - \frac{4}{5\sqrt{2}}\hat{k}$$

$$\mathbf{T_C} = T_C\left(\frac{4\hat{i}+5\hat{j}-3\hat{k}}{5\sqrt{2}}\right) = \frac{4}{5\sqrt{2}}\hat{i} + \frac{5}{5\sqrt{2}}\hat{j} - \frac{3}{5\sqrt{2}}\hat{k}$$

Since $\Sigma F_x = 0,$ $$\frac{3}{5\sqrt{2}}T_A - \frac{3}{5\sqrt{2}}T_B + \frac{4}{5\sqrt{2}}T_C = 0$$

$$3T_A - 3T_B + 4T_C = 0 \qquad \cdots(1)$$

Since $\Sigma F_y = 0,$ $$\frac{5}{5\sqrt{2}}T_A + \frac{5}{5\sqrt{2}}T_B + \frac{5}{5\sqrt{2}}T_C = 500$$

$$T_A + T_B + T_C = 500\sqrt{2} \qquad \cdots(2)$$

Since $\Sigma F_z = 0,$ $$\frac{4}{5\sqrt{2}}T_A - \frac{4}{5\sqrt{2}}T_B - \frac{3}{5\sqrt{2}}T_C = 0$$

$$4T_A - 4T_B - 3T_C = 0 \qquad \cdots(3)$$

The matrix form of the system of linear equations is

$$\begin{bmatrix} 3 & -3 & 4 \\ 1 & 1 & 1 \\ 4 & -4 & -3 \end{bmatrix}\begin{bmatrix} T_A \\ T_B \\ T_C \end{bmatrix} = \begin{bmatrix} 0 \\ 500\sqrt{2} \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 3 & -3 & 4 & 0 \\ 1 & 1 & 1 & 500\sqrt{2} \\ 4 & -4 & -3 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{R_{12}}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 500\sqrt{2} \\ 3 & -3 & 4 & 0 \\ 4 & -4 & -3 & 0 \end{array}\right]$$

$$\underset{R_2-3R_1,\, R_3-4R_1}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 500\sqrt{2} \\ 0 & -6 & 1 & -1500\sqrt{2} \\ 0 & -8 & -7 & -2000\sqrt{2} \end{array}\right]$$

$$\underset{-\frac{1}{6}R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 500\sqrt{2} \\ 0 & 1 & -\frac{1}{6} & 250\sqrt{2} \\ 0 & -8 & -7 & -2000\sqrt{2} \end{array}\right]$$

$$\underset{R_3+8R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 500\sqrt{2} \\ 0 & 1 & -\frac{1}{6} & 250\sqrt{2} \\ 0 & 0 & -\frac{25}{3} & 0 \end{array}\right]$$

$$\underset{-\frac{3}{25}R_3}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 500\sqrt{2} \\ 0 & 1 & -\frac{1}{6} & 250\sqrt{2} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$T_A + T_B + T_C = 500\sqrt{2}$$

$$T_B - \frac{1}{6}T_C = 250\sqrt{2}$$

$$T_C = 0$$

Solving these equations,

$$T_A = 353.55\,N$$

$$T_B = 353.55\,N$$

$$T_C = 0$$

Example 2: Find the currents I_1, I_2 and I_3 in the circuit shown in Fig 1.2.

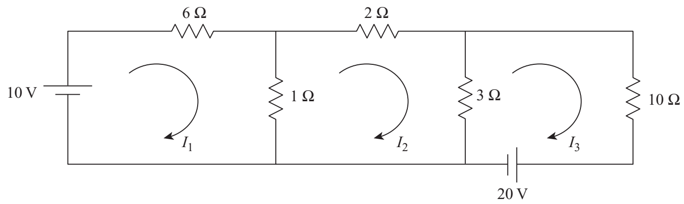


Fig. 1.2

Solution: Applying Kirchhoff's voltage law to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$

$$7I_1 - I_2 = 10 \qquad ...(1)$$

Applying Kirchoff's voltage law to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$

$$I_1 - 6I_2 + 3I_3 = 0 \qquad ...(2)$$

Applying Kirchoff's voltage law to Mesh 3,

$$-3(I_3 - I_2) - 10I_3 - 20 = 0$$

$$3I_2 - 13I_3 = 20 \qquad ...(3)$$

The matrix form of the system of linear equations is

$$\begin{bmatrix} 7 & -1 & 0 \\ 1 & -6 & 3 \\ 0 & 3 & -13 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 20 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 7 & -1 & 0 & 10 \\ 1 & -6 & 3 & 0 \\ 0 & 3 & -13 & 20 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -6 & 3 & 0 \\ 7 & -1 & 0 & 10 \\ 0 & 3 & -13 & 20 \end{array}\right]$$

$$R_2 - 7R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -6 & 3 & 0 \\ 0 & 41 & -21 & 10 \\ 0 & 3 & -13 & 20 \end{array}\right]$$

$$\frac{1}{41}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -6 & 3 & 0 \\ 0 & 1 & -\frac{21}{41} & \frac{10}{41} \\ 0 & 3 & -13 & 20 \end{array}\right]$$

$$R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & 3 & 0 \\ 0 & 1 & -\dfrac{21}{41} & \dfrac{10}{41} \\ 0 & 0 & -\dfrac{470}{41} & \dfrac{790}{41} \end{array}\right]$$

$$-\frac{41}{470}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & 3 & 0 \\ 0 & 1 & -\dfrac{21}{41} & \dfrac{10}{41} \\ 0 & 0 & 1 & -\dfrac{790}{470} \end{array}\right]$$

The corresponding system of equations is

$$I_1 - 6I_2 + 3I_3 = 0$$

$$I_2 - \frac{21}{41}I_3 = \frac{10}{41}$$

$$I_3 = -\frac{790}{470}$$

Solving these equations,

$$I_1 = 1.34 \text{ A}$$
$$I_2 = -0.62 \text{ A}$$
$$I_3 = -1.68 \text{ A}$$

Exercise 1.6

1. Figure 1.3 shows a tripod carrying a load of 500 kN. Supports *A*, *B* and *C* are co-planar in the *x*-*z* plane. Find force in each member in the tripod. The joints are ball-and-socket type.

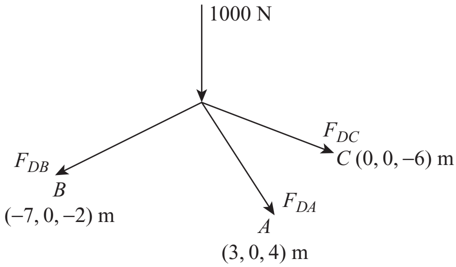


Fig. 1.3

$$\left[\begin{array}{l} \textbf{Ans.:} F_{DA} = 585.9N \\ F_{DC} = 322.4N \\ F_{DB} = 282.3N \end{array}\right]$$

2. Find currents I_1, I_2 and I_3 in the circuit shown in Fig 1.4.

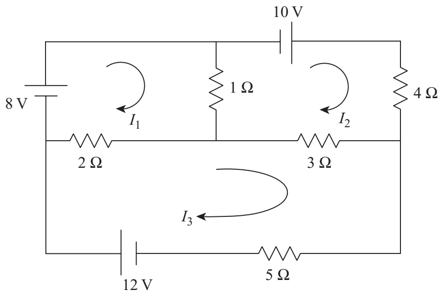


Fig. 1.4

$$\left[\begin{array}{r}\textbf{Ans.:} I_1 = 6.01 A \\ I_2 = 3.27 A \\ I_3 = 3.38 A\end{array}\right]$$

Vector Spaces

Chapter 2

2.1 INTRODUCTION

Vector space is a system consisting of a set of generalized vectors and a field of scalars, having the same rules for vector addition and scalar multiplication as physical vectors and scalars. The operations of vector addition and scalar multiplication have to satisfy certain axioms. An example of a vector space is the Euclidean vector space where every element is represented by a list of n real numbers, scalars are real numbers, addition is component wise and scalar multiplication is multiplication on each term separately. Vector spaces are characterized by their dimension which gives the number of independent directions in the space. They are useful in mathematics, science and engineering.

2.2 EUCLIDEAN VECTOR SPACE

Euclidean vector space or simply n-space is the space of all n-tuples of real numbers, $(u_1, u_2, \dots, u_n)$. It is commonly denoted as R^n.

2.2.1 *Vectors in R^n*

An ordered set of n real numbers $(u_1, u_2, u_3, \dots, u_n)$ represent a vector $\mathbf{u}$ in the vector space R^n. The real number u_k is called the kth component or coordinate of $\mathbf{u}$. This vector $\mathbf{u}$ represents a point in n-dimensional space R^n.

When $n = 2$ or 3, the vector $\mathbf{u}$ represents a point in two-dimensional or three-dimensional space respectively.

2.2.2 *Vector Addition and Multiplication by Scalars*

If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are two vectors in R^n then vector addition of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

and if k is any scalar, the scalar multiple is defined by,

$$k\mathbf{u} = (ku_1, ku_2, \dots, ku_n)$$

Properties of Vectors in R^n

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k_1, k_2 are scalars then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $k_1(k_2\mathbf{u}) = (k_1k_2)\mathbf{u}$
6. $k_1(\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$
7. $(k_1 + k_2)\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

2.2.3 Inner (dot) Product in R^n

Inner (dot) product of two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + +u_2v_2 + \cdots + u_nv_n$$

Properties of Inner Product in R^n

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k is any scalar then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(k\,\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} \geq 0$. Also $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$

2.2.4 Norm or Length in R^n

The norm or length of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n is denoted by $\|\mathbf{u}\|$ and is defined by

$$\begin{aligned}\|\mathbf{u}\| &= (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \\ &= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\end{aligned}$$

Properties of Length in R^n

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n and k is any scalar then

1. $\|\mathbf{u}\| \geq 0$
2. $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
3. $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$
4. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$, Triangle inequality.

2.2.5 *Distance in R^n*

The distance between the points $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \end{aligned}$$

Properties of Distance in R^n

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n then

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$, Triangle inequality.

2.2.6 *Angle between Vectors in R^n*

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors in R^n and if θ is the angle between them then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Theorem 2.1: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$$

Proof:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \qquad \ldots(2.1) \end{aligned}$$

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \qquad \ldots(2.2) \end{aligned}$$

Subtracting Eq. (2.2) from (2.1),

$$\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}$$

Hence,

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$$

2.2.7 *Orthogonality in R^n*

Two vectors **u** and **v** in R^n are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$

2.2.8 *Pythagorean Theorem in R^n*

If **u** and **v** are orthogonal (perpendicular) vectors in R^n then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof: Since **u** and **v** are orthogonal,

$$\mathbf{u}\cdot\mathbf{v} = 0 \qquad \text{...(2.3)}$$

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2 \quad \text{[using Eq. (2.3)]}\end{aligned}$$

2.2.9 *Cauchy–Schwarz Inequality in R^n*

If **u** and **v** are vectors in R^n then

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

Example 1: Normalize $\mathbf{v} = \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{4}\right)$ and generate a unit vector.

Solution:

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{1}{4}\right)^2}\\ &= \sqrt{\frac{1}{4}+\frac{4}{9}+\frac{1}{16}}\\ &= \frac{\sqrt{109}}{12}\end{aligned}$$

Dividing each component of **v** by $\|\mathbf{v}\|$ to obtain normalized vector of **v**,

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{6}{\sqrt{109}}, \frac{8}{\sqrt{109}}, -\frac{3}{\sqrt{109}}\right)$$

The normalized vector $\hat{\mathbf{v}}$ is a unit vector since $\|\hat{\mathbf{v}}\| = 1$

Example 2: Find the vector **x** that satisfies $2\mathbf{x} - 6\mathbf{v} = \mathbf{w} + \mathbf{x}$ where $\mathbf{u} = (-3, -1, 1, 0)$, $\mathbf{v} = (2, 0, 5, 3)$, $\mathbf{w} = (-2, 4, 1, 7)$

Solution:

$$\begin{aligned}2\mathbf{x} - 6(2,0,5,3) &= (-2,4,1,7)+\mathbf{x}\\ \mathbf{x} &= (-2,4,1,7)+(12,0,30,18)\\ &= (10,4,31,25)\end{aligned}$$

Example 3: Let **u** = (4, 1, 2, 3), **v** = (0, 3, 8, –2) and **w** = (3, 1, 2, 2). Evaluate

(i) $\|\mathbf{u}\|+\|\mathbf{v}\|$ (ii) $\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\|$ (iii) $\left\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\right\|$ (iv) $\|\mathbf{u}+\mathbf{v}\|$

Solution: (i)

$$\begin{aligned}\|\mathbf{u}\|+\|\mathbf{v}\| &= \sqrt{16+1+4+9}+\sqrt{0+9+64+4}\\ &= \sqrt{30}+\sqrt{77}\end{aligned}$$

(ii)

$$\begin{aligned}3\mathbf{u}-5\mathbf{v}+\mathbf{w} &= 3(4,1,2,3)-5(0,3,8,-2)+(3,1,2,2)\\ &= (12-0+3,\ 3-15+1,\ 6-40+2,\ 9+10+2)\\ &= (15,-11,-32,21)\end{aligned}$$

$$\begin{aligned}\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\| &= \sqrt{(15)^2+(-11)^2+(-32)^2+(21)^2}\\ &= \sqrt{1811}\end{aligned}$$

(iii)

$$\|\mathbf{w}\| = \sqrt{9+1+4+4} = \sqrt{18}$$

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{18}}(3,1,2,2)$$

$$\left\|\frac{\mathbf{1}}{\|\mathbf{w}\|}\mathbf{w}\right\| = \sqrt{\frac{1}{18}(9+1+4+4)} = \sqrt{1} = 1$$

This concludes that if **w** is a non-zero vector, then

$$\left\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\right\| = \frac{1}{\|\mathbf{w}\|}\cdot\|\mathbf{w}\| = 1$$

i.e. $\frac{1}{\|\mathbf{w}\|}\mathbf{w}$ has Euclidean norm 1.

Example 4: Find the Euclidean inner product of **u** = (3, 1, 4, –5) and **v** = (1, 0, –2, –3).

Solution:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (3,1,4,-5)\cdot(1,0,-2,-3)\\ &= (3)(1)+(1)(0)+(4)(-2)+(-5)(-3)\\ &= 3-8+15\\ &= 10\end{aligned}$$

Example 5: Find two vectors in R^2 with Euclidean norm 1 whose Euclidean inner product with (3, –1) is zero.

Solution: Let $\mathbf{u} = (u_1, u_2)$ be a vector in R^2 such that

$$(3, -1) \cdot (u_1, u_2) = 0$$

$$3u_1 - u_2 = 0 \qquad ...(1)$$

and

$$\|\mathbf{u}\| = 1$$

$$\sqrt{u_1^2 + u_2^2} = 1$$

$$u_1^2 + u_2^2 = 1$$

$$u_1^2 + 9u_1^2 = 1 \qquad \text{[using equation (1)]}$$

$$u_1 = \pm\frac{1}{\sqrt{10}}$$

$$u_2 = \pm\frac{3}{\sqrt{10}}$$

$$\mathbf{u} = \left(\pm\frac{1}{\sqrt{10}}, \pm\frac{3}{\sqrt{10}}\right)$$

Example 6: Let R^3 have the Euclidean inner product. For which values of k are **u** and **v** orthogonal?

(i) $\mathbf{u} = (k, k, 1)$, $\mathbf{v} = (k, 5, 6)$
(ii) $\mathbf{u} = (2, 1, 3)$, $\mathbf{v} = (1, 7, k)$
(iii) $\mathbf{u} = (1, k, -3)$, $\mathbf{v} = (2, -5, 4)$

Solution: If **u** and **v** are orthogonal then $\mathbf{u} \cdot \mathbf{v} = 0$

(i)

$$\mathbf{u} \cdot \mathbf{v} = 0$$

$$(k, k, 1) \cdot (k, 5, 6) = 0$$

$$k^2 + 5k + 6 = 0$$

$$(k+2)(k+3) = 0$$

$$k = -2, -3$$

(ii)

$$\mathbf{u} \cdot \mathbf{v} = 0$$

$$(2, 1, 3) \cdot (1, 7, k) = 0$$

$$2 + 7 + 3k = 0$$

$$3k = -9$$

$$k = -3$$

(iii)

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= 0 \\ (1, k, -3)\cdot(2, -5, 4) &= 0 \\ 2 - 5k - 12 &= 0 \\ 5k &= -10 \\ k &= -2 \end{aligned}$$

Example 7: Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to the vectors $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 1, 0)$

Solution: Let $\mathbf{w} = (w_1, w_2, w_3)$ be a vector in R^3 such that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{w} &= 0, \\ (1, 1, 1)\cdot(w_1, w_2, w_3) &= 0, \\ w_1 + w_2 + w_3 &= 0 \qquad \text{...(1)}, \end{aligned}$$

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} &= 0 \\ (1, 1, 0)\cdot(w_1, w_2, w_3) &= 0 \\ w_1 + w_2 &= 0 \qquad \text{...(2)}, \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{w}\| &= 1 \\ \sqrt{w_1^2 + w_2^2 + w_3^2} &= 1 \\ w_1^2 + w_2^2 + w_3^2 &= 1 \qquad \text{...(3)} \end{aligned}$$

Solving equations (1) and (2),

$$w_3 = 0$$

Substituting $w_2 = -w_1$ and $w_3 = 0$ in equation (3),

$$\begin{aligned} w_1^2 + w_1^2 &= 1 \\ w_1 &= \pm\frac{1}{\sqrt{2}} \\ w_1 &= \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \\ w_2 &= -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \end{aligned}$$

$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Example 8: Show that the zero vector is the only vector orthogonal to every vector in R^n.

Solution: Let $\mathbf{u}$ is orthogonal to every vector in R^n. Then $\mathbf{u}$ is orthogonal to itself.

$$\mathbf{u}\cdot\mathbf{u} = 0$$

$$u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2 = 0 \quad \text{where} \quad \mathbf{u} = (u_1, u_2, \ldots, u_n)$$
$$u_i^2 = 0, \quad i = 1, 2, \ldots, n$$
$$u_i = 0, \quad i = 1, 2, \ldots, n$$
$$\mathbf{u} = (0, 0, \ldots, 0)$$

Hence, zero vector is the only vector orthogonal to every vector in R^n.

Example 9: Determine k such that $\|\mathbf{u}\| = \sqrt{39}$ where $u = (1, k, -2, 5)$

Solution:
$$\|\mathbf{u}\| = \sqrt{39}$$
$$\|\mathbf{u}\|^2 = 39$$

$$1^2 + k^2 + (-2)^2 + (5)^2 = 39$$
$$k^2 + 30 = 39$$
$$k^2 = 9$$
$$k = \pm 3$$

Example 10: Find Euclidean distance between $\mathbf{u} = (3, -5, 4)$ and $\mathbf{v} = (6, 2, -1)$.

Solution:
$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$
$$= \sqrt{(3-6)^2 + (-5-2)^2 + (4+1)^2}$$
$$= \sqrt{9 + 49 + 25}$$
$$= \sqrt{83}.$$

Example 11: If **u** and **v** are orthogonal unit vectors, what is the distance between **u** and **v**?

Solution: Since **u** and **v** are orthogonal unit vectors, $\mathbf{u} \cdot \mathbf{v} = 0$, $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 1$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$
$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$
$$= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$$
$$= \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$$
$$= 1 - 0 + 1$$
$$= 2$$

Hence, $d(\mathbf{u}, \mathbf{v}) = \sqrt{2}$

Example 12: Let R^2 have the Euclidean inner product. Find the cosine of the angle θ between the vectors **u** = (4, 3, 1, –2) and **v** = (–2, 1, 2, 3).

Solution:

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

$$= \frac{(4,3,1,-2)\cdot(-2,1,2,3)}{\sqrt{16+9+1+4}\,\sqrt{4+1+4+9}}$$

$$= \frac{-8+3+2-6}{\sqrt{30}\sqrt{18}}$$

$$= -\frac{9}{3\sqrt{60}}$$

$$= -\frac{3}{2\sqrt{15}}$$

Example 13: Verify Cauchy–Schwarz inequality for the vectors **u** = (–3, 1, 0), **v** = (2, –1, 3).

Solution: Cauchy–Schwarz inequality states that

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

$$\mathbf{u}\cdot\mathbf{v} = (-3,1,0)\cdot(2,-1,3)$$

$$= -6-1 = -7$$

$$|\mathbf{u}\cdot\mathbf{v}| = |-7| = 7$$

$$\|\mathbf{u}\| = \sqrt{9+1} = \sqrt{10}$$

$$\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{10}\sqrt{14} = \sqrt{140}$$

$$7 < \sqrt{140}$$

$$|\mathbf{u}\cdot\mathbf{v}| < \|\mathbf{u}\|\|\mathbf{v}\|$$

Hence, the inequality is verified.

Exercise 2.1

1. Find **u** + **v**, 3**u** – 2**v**, **u** – 2**v** + 3**w** if

(i) **u** = (–1, 2, 1), **v** = (2, 1, 3), **w** = (0, 3, –1)

(ii) **u** = (–3, 2, 1, 0), **v** = (4, 7, –3, 2), **w** = (5, –2, 8, 1)

Ans.:
(i) (1, 3, 4), (–7, 4, –3), (–5, 9, –8)
(ii) (1, 9, –2, 2), (–17, –8, 9, –4), (4, –18, 31, –1)

2. Find a, b, c for the following if
$\mathbf{u} = (1, -2, 3)$, $\mathbf{v} = (-3, -1, 3)$,
$\mathbf{w} = (a, -1, b)$, $\mathbf{x} = (3, c, 2)$

(i) $\mathbf{w} = \frac{1}{2}\mathbf{u}$ (ii) $\mathbf{w} + \mathbf{v} = \mathbf{u}$

(iii) $\mathbf{w} + \mathbf{x} = \mathbf{v}$

[**Ans.:**
(i) $a = \frac{1}{2}, b = \frac{3}{2}$
(ii) $a = 4, b = 0$
(iii) $a = -6, b = 1, c = 0.$]

3. Find a, b, c, d such that
$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + d\mathbf{x} = (0, 5, 6, -3)$
where,
$\mathbf{u} = (-1, 3, 2, 0)$, $\mathbf{v} = (2, 0, 4, -1)$,
$\mathbf{w} = (7, 1, 1, 4)$ and $\mathbf{x} = (6, 3, 1, 2)$

[**Ans.:** $a = 1, b = 1, c = -1, d = 1$]

4. Find the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ for the following:

(i) $\mathbf{u} = (4, 8, 2)$, $\mathbf{v} = (0, 1, 3)$

(ii) $\mathbf{u} = (3, 1, 4, -5)$, $\mathbf{v} = (2, 2, -4, -3)$

(iii) $\mathbf{u} = (-1, 1, 0, 4, -3)$,
$\mathbf{v} = (-2, 2, 0, 2, -1)$

[**Ans.:** (i) 14 (ii) 7 (iii) 15]

5. Find a such that $\|(1, a, -3, 2)\| = 5$

[**Ans.:** $a = \pm\sqrt{11}$]

6. Evaluate the following if $\mathbf{u} = (0, 2, 3, 1)$, $\mathbf{v} = (2, 0, -1, -1)$, $\mathbf{w} = (-3, -1, -2, 0)$

(i) $\|\mathbf{u} + \mathbf{v}\|$ (ii) $\|2\mathbf{u} + 3\mathbf{v} + 4\mathbf{w}\|$

(iii) $\frac{1}{\|\mathbf{w}\|}\mathbf{w}$ (iv) $\left\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\right\|$

[**Ans.:**
(i) $\sqrt{12}$ (ii) $\sqrt{27}$
(iii) $\frac{1}{\sqrt{14}}\mathbf{w}$ (iv) 1]

7. If $\mathbf{u}$ is a non-zero vector in R^n, show that $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector in the direction of $\mathbf{u}$.

8. Find the cosine of the angle between

(i) $\mathbf{u} = (2, 3, 1)$, $\mathbf{v} = (3, -2, 0)$

(ii) $\mathbf{u} = (1, 2, -1, 3)$, $\mathbf{v} = (0, 0, -1, -2)$

[**Ans.:** (i) 0 (ii) $-\frac{1}{\sqrt{3}}$]

9. Find the distance between the following:

(i) (0, 2, 3), (1, 2, –4)

(ii) (3, 4, 0, 1), (2, 2, 1, –1)

[**Ans.:** (i) $\sqrt{50}$ (ii) $\sqrt{10}$]

10. Find the constant a such that
$\mathbf{u} \cdot \mathbf{v} = 0$ where $\mathbf{u} = (a, 2, 1, a)$ and
$\mathbf{v} = (a, -1, -2, -3)$.

[**Ans.:** $a = -1, 4$]

11. Determine whether the given vectors are orthogonal.

(i) $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$

(ii) $\mathbf{u} = (4, 2, 6, -8)$, $\mathbf{v} = (-2, 3, -1, -1)$

(iii) $\mathbf{u} = (1, 2, 3, -4)$, $\mathbf{v} = (0, -3, 1, 0)$

[**Ans.:** (i) yes (ii) yes (iii) no]

12. For which value of k are $\mathbf{u} = (2, k, 3)$ and $\mathbf{v} = (1, -2, 1)$ orthogonal?

[**Ans.:** $k = \frac{5}{2}$]

13. Find a, b, c not all zero so that
$\mathbf{u} = (a, b, c)$ is orthogonal to both
$\mathbf{v} = (1, 2, 1)$ and $\mathbf{w} = (1, -1, 1)$

[**Ans.:** $a = 1, b = 0, c = -1$]

14. Verify Cauchy–Schwarz inequality:

(i) $\mathbf{u} = (-4, 2, 1)$,
$\mathbf{v} = (8, -4, -2)$

(ii) $\mathbf{u} = (0, -2, 2, 1)$,
$\mathbf{v} = (-1, -1, 1, 1)$

[**Ans.:** (i) yes (ii) yes]

2.3 VECTOR SPACES

Let V be a non-empty set of objects on which the operations of addition and multiplication by scalars are defined. Here addition means a rule for assigning to each pair of objects $\mathbf{u}$, $\mathbf{v}$ in V a unique object $\mathbf{u} + \mathbf{v}$ and scalar multiplication means a rule for assigning to each scalar k and each object $\mathbf{u}$ in V a unique object $k\mathbf{u}$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k_1, k_2 then V is called a vector space and the objects in V are called vectors.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. There is an object $\mathbf{0}$ in V, called zero vector, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each object $\mathbf{u}$ in V, there exists an object $-\mathbf{u}$ in V called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k_1 is any scalar and $\mathbf{u}$ is an object in V, then $k_1\mathbf{u}$ is in V.
7. $k_1 (\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$
8. If k_1, k_2 are scalars and $\mathbf{u}$ is an object in V, then $(k_1 + k_2)\,\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$.
9. $k_1(k_2\mathbf{u}) = (k_1 k_2)\mathbf{u}$
10. $1\,\mathbf{u} = \mathbf{u}$

The operations of addition and scalar multiplication in these axioms are not always defined as standard vector operations (addition and scalar multiplication) on Euclidean space R^n.

The scalars may be real numbers or complex numbers. When the scalars are real numbers, the vector space is called *real vector space*, and when the scalars are complex numbers, the vector space is called *complex vector space*.

Some standard vector spaces are as follows:

(i) The set R^n under standard vector addition and scalar multiplication.

(ii) The set P_n of all polynomials of degree $\le n$ together with the zero polynomial under addition and scalar multiplication of polynomials.

(iii) The set M_{mn} of all $m \times n$ matrices of real numbers under matrix addition and scalar multiplication.

(iv) The set $F[a, b]$ of all real-valued functions defined on the interval $[a, b]$ under addition and scalar multiplication of functions.

(v) The set $F[-\infty, \infty]$ of all real-valued functions defined for all real numbers under addition and scalar multiplication of functions.

Example 1: Determine whether the given set V is closed under the given operations:

(i) V is the set of all ordered triples of real numbers of the form $(0, y, z)$;

$$(0, y, z) + (0, y', z') = (0, y + y', z + z')$$
$$k(0, y, z) = (0, 0, kz)$$

(ii) V is the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a = d$ under matrix addition and scalar multiplication.

Solution:

(i) (a) $(0, y, z) + (0, y', z') = (0, y + y', z + z')$

Since y, y', z, z' are real numbers, $y + y', z + z'$ are also real numbers. Therefore, $(0, y + y', z + z')$ is in V.

Hence, V is closed under the addition operation.

(b) $k(0, y, z) = (0, 0, kz)$

If z is a real number then kz is also a real number. Therefore, $(0, 0, kz)$ is in V.

Hence, V is closed under multiplication operation.

(ii) (a) Let $\mathbf{u} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ where $a_1 = d_1$ and $a_2 = d_2$ be two objects in V.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

If $a_1 = d_1$, $a_2 = d_2$, then $a_1 + a_2 = d_1 + d_2$.

Therefore, $\mathbf{u} + \mathbf{v}$ is also an object in V.

Hence, V is closed under matrix addition.

(b) Let k be some scalar.

$$k\mathbf{u} = k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$$

If $a_1 = d_1$, then $ka_1 = kd_1$. Therefore, $k\mathbf{u}$ is also an object in V.

Hence, V is closed under scalar multiplication.

Example 2: Determine whether the set V of all pairs of real numbers (x, y) with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $k(x, y) = (kx, ky)$ is a vector space.

Solution: Let $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ and $\mathbf{w} = (x_3, y_3)$ are objects in V and k_1, k_2 are some scalars.

1. $\mathbf{u} + \mathbf{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$

   Since x_1, x_2, y_1, y_2 are real numbers $x_1 + x_2 + 1$ and $y_1 + y_2 + 1$ are also real numbers. Therefore, $\mathbf{u} + \mathbf{v}$ is also an object in V.

2. $\mathbf{u}+\mathbf{v}=(x_1+x_2+1,\ y_1+y_2+1)$
$$=(x_2+x_1+1,\ y_2+y_1+1)$$
$$=\mathbf{v}+\mathbf{u}$$

Hence, vector addition is commutative.

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(x_1,\ y_1)+[(x_2,\ y_2)+(x_3,\ y_3)]$
$$\begin{aligned}&=(x_1,\ y_1)+(x_2+x_3+1,\ y_2+y_3+1)\\&=[x_1+(x_2+x_3+1)+1,\ y_1+(y_2+y_3+1)+1]\\&=[(x_1+x_2+1)+x_3+1,\ (y_1+y_2+1)+y_3+1]\\&=(x_1+x_2+1,\ y_1+y_2+1)+(x_3,\ y_3)\\&=(\mathbf{u}+\mathbf{v})+\mathbf{w}\end{aligned}$$

Hence, vector addition is associative.

4. Let $(a,\ b)$ be an object in V such that
$$(a,b)+\mathbf{u}=\mathbf{u}$$
$$(a,b)+(x_1,\ y_1)=(x_1,\ y_1)$$
$$(a+x_1+1,\ b+y_1+1)=(x_1,\ y_1)$$
$$a+x_1+1=x_1 \quad , \quad b+y_1+1=y_1$$
$$a=-1 \quad , \quad b=-1$$

Also, $\mathbf{u}+(a,b)=\mathbf{u}$
Hence, $(-1,-1)$ is the zero vector in V.

5. Let $(a,\ b)$ be an object in V such that
$$\mathbf{u}+(a,b)=(-1,-1)$$
$$(x_1,\ y_1)+(a,b)=(-1,-1)$$
$$(x_1+a+1,\ y_1+b+1)=(-1,-1)$$
$$x_1+a+1=-1 \quad , \quad y_1+b+1=-1$$
$$a=-x_1-2 \quad , \quad b=-y_1-2$$

Also, $(a,\ b)+\mathbf{u}=(-1,-1)$
Hence, $(-x_1-2,\ -y_1-2)$ is the negative of $\mathbf{u}$ in V.

6. $k_1\mathbf{u}=k_1(x_1,\ y_1)$
$$=(k_1x_1,\ k_1y_1)$$

Since k_1x_1 and k_1y_1 are real numbers, $k_1\mathbf{u}$ is an object in V
Hence, V is closed under scalar multiplication.

7. $k_1(\mathbf{u}+\mathbf{v})=k_1(x_1+x_2+1,\ y_1+y_2+1)$
$$=(k_1x_1+k_1x_2+k_1,\ k_1y_1+k_1y_2+k_1)$$
$$\neq k_1\mathbf{u}+k_1\mathbf{v}$$

V is not distributive under scalar multiplication.
Hence, V is not a vector space.

Example 3: Determine whether the set R^+ of all positive real numbers with operations

$$x + y = xy \text{ and } kx = x^k \text{ is a vector space.}$$

Solution: Let x, y and z be positive real numbers in R^+ and k_1, k_2 are some scalars

1. $x + y = xy$, is also a positive real number
   R^+ is closed under vector addition.
2. $x + y = xy = yx = y + x$
   Vector addition is commutative.
3. $x+(y+z) = x(y+z) = x(yz) = (xy)z = (x+y)z = (x+y)+z$
   Vector addition is associative.
4. Let a be an object in R^+ such that
   $$a + x = x$$
   $$ax = x$$
   $$a = 1$$
   Also $x + a = x$
   Hence, $\mathbf{0} = 1$ is the zero vector in V.
5. Let a be an object in R^+ such that
   $$x + a = 1$$
   $$xa = 1$$
   $$a = \frac{1}{x}$$
   Also, $a + x = 1$
   Hence, $\frac{1}{x}$ is the negative of x in R^+.
6. If k_1 is real then $k_1 x = x^{k_1}$ is a positive real number for all x in R^+.
   R^+ is closed under scalar multiplication.
7. $k_1(x + y) = k_1(xy) = (xy)^{k_1}$
   $$= x^{k_1} y^{k_1} = (k_1 x)(k_1 y) = k_1 x + k_1 y$$
   Scalar multiplication is distributive with respect to vector addition in R^+.
8. $(k_1 + k_2)x = x^{k_1+k_2} = x^{k_1} x^{k_2}$
   $$= (k_1 x)(k_2 x) = k_1 x + k_2 x$$
   Scalar multiplication is distributive with respect to scalar addition in R^+.
9. $k_1(k_2 x) = k_1 x^{k_2} = (x^{k_2})^{k_1}$
   $$= x^{k_2 k_1} = x^{k_1 k_2} = (k_1 k_2)x$$
   Scalar and vector multiplications are compatible with each other.
10. $1x = x^1 = x$

All axioms are satisfied by R^+ under given operations. Hence, R^+ is a vector space under given operations.

Example 4: Why are the following sets not vector spaces under the given operations? Justify your answer.

(i) The set of all pairs of real numbers (x, y) with the operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$.

(ii) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$

Solution: (i) 1 is a scalar.

$$1(x, y) = (2x, 2y) \neq (x, y)$$

Axiom 10 fails. Hence, given set is not a vector space.

(ii) $(x_1, y_1, z_1)+\{(x_2, y_2, z_2)+(x_3, y_3, z_3)\}$

$$\begin{aligned}
&= (x_1, y_1, z_1)+(z_2+z_3, y_2+y_3, x_2+x_3)\\
&= \{z_1+(x_2+x_3), y_1+(y_2+y_3), x_1+(z_2+z_3)\}\\
&= \{(z_1+x_2)+x_3, (y_1+y_2)+y_3, (x_1+z_2)+z_3\}\\
&= (x_1+z_2, y_1+y_2, z_1+x_2)+(z_3, y_3, x_3)\\
&= \{(z_1, y_1, x_1)+(x_2, y_2, z_2)\}+(z_3, y_3, x_3)
\end{aligned}$$

Given set is not associative under vector addition. Axiom 3 fails. Hence, the given set is not a vector space.

Example 5: Check whether $V = R^2$ is a vector space with respect to the operations $(x_1, y_1)+(x_2, y_2) = (x_1+x_2-2, y_1+y_2-3)$ and $k(x, y) = (kx+2k-2, ky -3k+3)$, k is a real number.

Solution: Let $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ and $\mathbf{w} = (x_3, y_3)$ are objects in R^2 and k_1, k_2 are some real scalars.

1. $\mathbf{u}+\mathbf{v} = (x_1, y_1)+(x_2, y_2)$

$$= (x_1+x_2-2, y_1+y_2-3) \text{ which is also in } R^2.$$

R^2 is closed under vector addition.

2. $\mathbf{u}+\mathbf{v} = (x_1+x_2-2, y_1+y_2-3)$

$$\begin{aligned}
&= (x_2+x_1-2, y_2+y_1-3)\\
&= (x_2, y_2)+(x_1, y_1)\\
&= \mathbf{v}+\mathbf{u}
\end{aligned}$$

Vector addition is commutative.

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (x_1, y_1)+\{(x_2, y_2)+(x_3, y_3)\}$

$$\begin{aligned}
&= (x_1, y_1)+(x_2+x_3-2, y_2+y_3-3)\\
&= (x_1+(x_2+x_3-2)-2, y_1+(y_2+y_3-3)-3)\\
&= ((x_1+x_2-2)+x_3-2, (y_1+y_2-3)+y_3-3)\\
&= (x_1+x_2-2, y_1+y_2-3)+(x_3, y_3)\\
&= \{(x_1, y_1)+(x_2, y_2)\}+(x_3, y_3)\\
&= (\mathbf{u}+\mathbf{v})+\mathbf{w}
\end{aligned}$$

Vector addition is commutative.

4. Let (a, b) be an object in R^2 such that

$$(a, b) + \mathbf{u} = \mathbf{u}$$
$$(a, b) + (x_1, y_1) = (x_1, y_1)$$
$$(a + x_1 - 2, b + y_1 - 3) = (x_1, y_1)$$
$$a + x_1 - 2 = x_1 \quad , \quad b + y_1 - 3 = y_1$$
$$a = 2 \quad , \quad b = 3$$

Also, $\mathbf{u} + (a, b) = \mathbf{u}$
Hence, (2, 3) is the zero vector in V.

5. Let (a, b) be an object in R^2 such that

$$\mathbf{u} + (a, b) = (2, 3)$$
$$(x_1, y_1) + (a, b) = (2, 3)$$
$$(x_1 + a - 2, y_1 + b - 3) = (2, 3)$$
$$x_1 + a - 2 = 2 \quad , \quad y_1 + b - 3 = 3$$
$$a = -x_1 + 4 \quad , \quad b = -y_1 + 6$$

Also, $(a, b) + \mathbf{u} = (2, 3)$
Hence, $(-x_1 + 4, -y_1 + 6)$ is the negative of $\mathbf{u}$ in V.

6. If k_1 is a real number then $k_1(x_1, y_1) = (k_1x_1 + 2k_1 - 2, k_1y - 3k_1 + 3)$ is also in R^2. R^2 is closed under scalar multiplication.

7. $k_1(\mathbf{u} + \mathbf{v}) = k_1\left\{(x_1, y_1) + (x_2, y_2)\right\}$

$$= k_1(x_1 + x_2 - 2, y_1 + y_2 - 3)$$
$$= \left(k_1(x_1 + x_2 - 2) + 2k_1 - 2, k_1(y_1 + y_2 - 3) - 3k_1 + 3\right)$$
$$= (k_1x_1 + 2k_1 - 2 + k_1x_2 - 2k_1, k_1y_1 - 3k_1 + 3 + k_1y_2 - 3k_1)$$
$$\neq k_1\mathbf{u} + k_1\mathbf{v}$$

Scalar multiplication is not distributive with respect to vector addition in R^2.
Hence, R^2 is not a vector space.

Exercise 2.2

1. Determine whether the given set V is closed under the given operations.

(i) The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

(ii) The set of all polynomials of the form $a_0 + a_1x + a_2x^2$ where a_0, a_1, a_2 are real numbers and $a_2 = a_3 + 1$ with operations defined as

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$
$$k(a_1 + a_2x + a_3x^2)$$
$$= (ka_0) + (ka_1)x + (ka_2)x^2$$

(iii) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with the standard matrix addition and scalar multiplication.

[**Ans. :** (i) yes (ii) no (iii) no]

2. Determine which sets are vector spaces under the given operations:

(i) The set of all ordered triples of real numbers (x, y, z) with the operations

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_2, y_1 + y_2, z_2)$$
$$k(x, y, z) = (kx, ky, kz)$$

(ii) The set of all ordered triples of real numbers of the form $(0, 0, z)$ with the operations

$$(0, 0, z_1) + (0, 0, z_2) = (0, 0, z_1 + z_2)$$
$$k(0, 0, z) = (0, 0, kz)$$

(iii) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with the operations defined as

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$$
$$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$$

(iv) The set of all ordered pairs of real numbers (x, y), where $x \le 0$, with the usual operations in R^2

[**Ans. :** (i) no (ii) yes (iii) yes (iv) no]

3. Show that the set V of all pairs of real numbers of the form $(1, x)$ with the operations defined as

$$(1, x_1) + (1, x_2) = (1, x_1 + x_2)$$
$$k(1, x) = (1, kx)$$

is a vector space.

4. Show that the set M_{nn} of all $n \times n$ matrices with real entries is a vector space under the matrix addition and scalar multiplication.

2.4 SUBSPACES

A non-empty subset W of a vector space V is called a subspace of V if W is itself a vector space under the operations defined on V.

Note: Every vector space has at least two subspaces, itself and the subspace $\{\mathbf{0}\}$. The subspace $\{\mathbf{0}\}$ is called the zero subspace consisting only of the zero vector.

Since W is the part of a vector space V, most of the axioms are true for W as they are true for V. The following theorem shows that to prove W a subspace of a vector space V, we need to verify only the closure property with respect to the operations defined on V.

Theorem 2.2: If W is a non-empty subset of vector space V, then W is a subspace of V if and only if the following axioms hold:

Axiom 1: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W then $\mathbf{u} + \mathbf{v}$ is in W.

Axiom 2: If k is any scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Example 1: Show that $W = \{(x, y) \mid x = 3y\}$ is a subspace of R^2. State all possible subspaces of R^2.

Solution: Let $\mathbf{u} = \{(x_1, y_1) \mid x_1 = 3y_1\}$ and $\mathbf{v} = \{(x_2, y_2) \mid x_2 = 3y_2\}$ are in W and k is any scalar.

Axiom 1:
$$\mathbf{u} + \mathbf{v} = (x_1, y_1) + (x_2, y_2)$$
$$= (x_1 + x_2, y_1 + y_2)$$

But
$$x_1 = 3y_1 \text{ and } x_2 = 3y_2$$
$$\therefore x_1 + x_2 = 3(y_1 + y_2)$$
$$\mathbf{u} + \mathbf{v} = \{(x_1 + x_2, y_1 + y_2) \mid x_1 + x_2 = 3(y_1 + y_2)\}$$

Thus, $\mathbf{u} + \mathbf{v}$ is in W.

Axiom 2:
$$k\mathbf{u} = k(x_1, y_1)$$
$$= (kx_1, ky_1)$$

But
$$x_1 = 3y_1$$
$$\therefore kx_1 = 3(ky_1)$$
$$k\mathbf{u} = \{(kx_1, ky_1) \mid kx_1 = 3\,(ky_1)\}$$

Thus, $k\mathbf{u}$ is in W.
Hence, W is a subspace of R^2.
All possible subspaces of R^2 are

(i) $\{\mathbf{0}\}$ (ii) R^2 (iii) Lines passing through the origin.

Example 2: Check whether the following are subspaces of R^3. Justify your answer. State all possible subspaces of R^3.

(i) $W = \{(x, 0, 0) \mid x \in R\}$

(ii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$

(iii) $W = \{(x, y, z) \mid y = x + z + 1\}$

Solution: (i) Let $\mathbf{u} = \{(x_1, 0, 0) \mid x_1 \in R\}$ and $\mathbf{v} = (x_2, 0, 0) \mid x_2 \in R\}$ be in W, and k be any scalar.

Axiom 1:
$$\mathbf{u} + \mathbf{v} = (x_1, 0, 0) + (x_2, 0, 0)$$
$$= (x_1 + x_2, 0, 0)$$

Since R is closed under addition, $x_1 + x_2$ is in R.
Thus, $\mathbf{u} + \mathbf{v}$ is in W.

Axiom 2:
$$k\mathbf{u} = k(x_1, 0, 0)$$
$$= (kx_1, 0, 0)$$

Since R is closed under scalar multiplication, kx_1 is in R.
Thus, $k\mathbf{u}$ is in W.
Hence, W is a subspace of R^3.

(ii) Let $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 0, 1)$ be two vectors of the set W satisfying the condition $x^2 + y^2 + z^2 \le 1$.

Axiom 1:
$$\mathbf{u}+\mathbf{v}=(1,0,0)+(0,0,1)$$
$$=(1,0,1)$$

Here $x^2+y^2+z^2=2>1$. Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^3.

(iii) Let $\mathbf{u}=\{(x_1,y_1,z_1)\mid y_1=x_1+z_1+1\}$ and $\mathbf{v}=\{(x_2,y_2,z_2)\mid y_2=x_2+z_2+1\}$ be in W.

Axiom 1:
$$\mathbf{u}+\mathbf{v}=(x_1,y_1,z_1)+(x_2,y_2,z_2)$$
$$=(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$$

But
$$y_1=x_1+z_1+1,\ y_2=x_2+z_2+1$$
$$\therefore\ y_1+y_2=(x_1+z_1+1)+(x_2+z_2+1)$$
$$=(x_1+x_2)+(z_1+z_2)+2$$

Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^3.
All possible subspaces of R^3 are (i) $\{\mathbf{0}\}$ (ii) Lines passing through the origin. (iii) Planes through the origin (iv) R^3.

Example 3: Show that the set of solution vectors of a homogenous linear system $A\mathbf{x}=\mathbf{0}$ of m equations in n unknowns, is a subspace of R^n.

Solution: Let W be the set of solution vectors of $A\mathbf{x}=\mathbf{0}$.
Case I: If system has only a trivial solution ($\mathbf{x}=\mathbf{0}$) then W has at least one vector $\mathbf{0}$ and hence is a subspace of R^3.
Case II: In case of non-trivial solution, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solution vectors in W and k is any scalar.

Axiom 1:
$$A(\mathbf{x}_1+\mathbf{x}_2)=A\mathbf{x}_1+A\mathbf{x}_2$$
$$=\mathbf{0}+\mathbf{0}\qquad [\because A\mathbf{x}_1=\mathbf{0},\ A\mathbf{x}_2=\mathbf{0}]$$
$$=\mathbf{0}$$

Thus, $\mathbf{x}_1+\mathbf{x}_2$ is also a solution vector in W.

Axiom 2:
$$A(k\mathbf{x}_1)=k(A\mathbf{x}_1)\qquad [\because k \text{ is a scalar}]$$
$$=\mathbf{0}$$

Thus, $k\mathbf{x}_1$ is also a solution vector in W.
Hence, W is a subspace of R^n.

Example 4: Show that the following sets are the subspaces of the respective real vector space V under the standard operations:

(i) $W=\{a_0+a_1x+a_2x^2+a_3x^3\mid a_0=0\}$, $\qquad V=P_3$

(ii) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a+b+c+d=0 \right\}$, $V = M_{22}$

(iii) $W = \{A_{nn} \mid AB = BA \quad \text{for a fixed } B_{nn}\}$, $V = M_{nn}$

(iv) $W = \{f \mid f(x) = a_1 + a_2 \sin x, \quad \text{where } a_1 \text{ and } a_2 \text{ are real numbers}\}$,
$V = F(-\infty, \infty)$

Solution: (i) Let $\mathbf{p}_1 = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2 + b_3x^3$ be in W such that $a_0 = 0$, $b_0 = 0$ and k is any scalar.

Axiom 1: $\mathbf{p}_1 + \mathbf{p}_2 = (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3)$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$

But $$a_0 = 0, b_0 = 0$$

$$\therefore a_0 + b_0 = 0$$

Thus, $\mathbf{p}_1 + \mathbf{p}_2$ is in W.

Axiom 2: $k\,\mathbf{p}_1 = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$

But $$a_0 = 0$$

$$\therefore ka_0 = 0$$

Thus, $k\mathbf{p}_1$ is in W.
Hence, W is a subspace of P_3.

(ii) Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be in M_{22} such that $a_1 + b_1 + c_1 + d_1 = 0$, $a_2 + b_2 + c_2 + d_2 = 0$ and k is any scalar.

Axiom 1: $A_1 + A_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

But $a_1 + b_1 + c_1 + d_1 = 0$, $a_2 + b_2 + c_2 + d_2 = 0$

$$\therefore (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) + (d_1 + d_2)$$
$$= (a_1 + b_1 + c_1 + d_1) + (a_2 + b_2 + c_2 + d_2) = 0$$

Thus, $A_1 + A_2$ is in W.

Axiom 2: $k\,A_1 = \begin{bmatrix} k\,a_1 & k\,b_1 \\ k\,c_1 & k\,d_1 \end{bmatrix}$

But $a_1 + b_1 + c_1 + d_1 = 0$

$$\therefore ka_1 + kb_1 + kc_1 + kd_1 = k(a_1 + b_1 + c_1 + d_1) = 0$$

Thus, kA_1 is in W.
Hence, W is a subspace of M_{22}.

(iii) Let A_1 and A_2 be in W such that $A_1B = BA_1$, $A_2B = BA_2$ and k be any real scalar.

Axiom 1: $$(A_1 + A_2)B = A_1B + A_2B$$
$$= BA_1 + BA_2$$
$$= B(A_1 + A_2)$$

Thus, $A_1 + A_2$ is in W.

Axiom 2: $$(kA_1)B = k(A_1B)$$
$$= k(BA_1)$$
$$= B(kA_1) \qquad [\because k \text{ is a scalar}]$$

Thus, kA_1 is in W.
Hence, W is a subspace of M_{nn}.

(iv) Let $f_1(x) = a_1 + a_2 \sin x$ and $f_2(x) = b_1 + b_2 \sin x$ be in W where a_1, a_2, b_1, b_2 are real numbers and k be any scalar.

Axiom 1: $$f_1(x) + f_2(x) = (a_1 + a_2 \sin x) + (b_1 + b_2 \sin x)$$
$$= (a_1 + b_1) + (a_2 + b_2) \sin x$$

Since a_1, b_1, a_2, b_2 are real numbers, $(a_1 + b_1)$ and $(a_2 + b_2)$ are also real numbers.
Thus, $f_1(x) + f_2(x)$ is in W.

Axiom 2: $$k f_1(x) = k(a_1 + a_2 \sin x)$$
$$= ka_1 + ka_2 \sin x$$

Since k is a real scalar, ka_1 and ka_2 are real numbers.
Thus, $k f_1(x)$ is in W.
Hence, W is a subspace of $F(-\infty, \infty)$.

Example 5: State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector space V under the standard operations:

(i) $W = \{(x, y) \mid x^2 = y^2\}$, $\quad V = R^2$

(ii) $W = \{(x, y) \mid xy \geq 0\}$, $\quad V = R^2$

(iii) $W = \{A_{n\times n} \mid A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}\}$, $\quad V = M_{nn}$

(iv) $W = \{f \mid f(x) \leq 0, \forall x\}$, $\quad V = F(-\infty, \infty)$

(v) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3, \forall x$ where a_0, a_1, a_2 and a_3 are integers$\}$, $V = P_3$

Solution: (i) Let $\mathbf{u} = (-1, 1)$ and $\mathbf{v} = (2, 2)$ be two vectors of the set W such that $x^2 = y^2$.

Axiom 1: $$\mathbf{u} + \mathbf{v} = (-1, 1) + (2, 2)$$
$$= (1, 3)$$

Here $1^2 \neq 3^2$. Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^2.

(ii) Let $\mathbf{u}=(-2,-3)$ and $\mathbf{v}=(3,1)$ be two vectors of the set W such that $xy \geq 0$.

Axiom 1: $\mathbf{u}+\mathbf{v}=(-2,-3)+(3,1)$

$$=(1,-2)$$

Here $1(-2)=-2<0$. Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^2.

(iii) From the definition of W, it is clear that W is the set of all non-singular matrices of order n so that $A\mathbf{x}=\mathbf{0}$ has only trivial solution ($\mathbf{x}=\mathbf{0}$)

Let $A_1=\begin{bmatrix} 3 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{n\times n}$ and $A_2=\begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{n\times n}$

are two matrices in W such that $|A_1| \neq 0$ and $|A_2| \neq 0$.

Axiom 1:
$$A_1+A_2=\begin{bmatrix} 5 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}$$

$|A_1+A_2|=0$. Thus, A_1+A_2 is not in W.

W is not closed under addition and hence is not a subspace of M_{nn}.

(iv) W is the set of all negative functions of x. Let $f(x)$ is in W such that $f(x) \leq 0$

Axiom 2: If $k=-2$, then

$$k\,f(x)=-2f(x)>0 \quad [\because f(x) \leq 0]$$

Thus, $kf(x)$ is not in W.
W is not closed under scalar multiplication and hence is not a subspace of $F(-\infty, \infty)$.

(v) Let $\mathbf{u}=a_0+a_1x+a_2x^2+a_3x^3$ be in W, where a_0, a_1, a_2, a_3, are integers.

Axiom 2: If $k=\frac{1}{2}$, then

$$\frac{1}{2}\mathbf{u}=\frac{a_0}{2}+\frac{a_1}{2}x+\frac{a_2}{2}x^2+\frac{a_3}{2}x^3$$

Since $\frac{a_0}{2}, \frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}$ are not necessarily to be integers, $\frac{1}{2}\mathbf{u}$ is not in W.

W is not closed under scalar multiplication and hence is not a subspace of P_3.

2.5 LINEAR COMBINATION

A vector $\mathbf{v}$ is called a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Note: If $r = 1$, then $\mathbf{v} = k_1\mathbf{v}_1$. This shows that a vector $\mathbf{v}$ is a linear combination of a single vector $\mathbf{v}_1$ if it is a scalar multiple of $\mathbf{v}_1$.

Vector Expressed as a Linear Combination of Given Vectors

The method to check if a vector $\mathbf{v}$ is a linear combination of the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is as follows:

1. Express $\mathbf{v}$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2.4}$$

2. If the system of equations in (1) is consistent then $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. If it is inconsistent, then v is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$.

Note: To express $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, solve the system of equations in (2.4) directly to determine scalars $k_1, k_2, \ldots, k_r$.

Example 1: Which of the following are linear combinations of $\mathbf{v}_1 = (0, -2, 2)$ and $\mathbf{v}_2 = (1, 3, -1)$?

(i) (3, 1, 5) (ii) (0, 4, 5)

Solution: Let $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$

(i) $(3, 1, 5) = k_1(0, -2, 2) + k_2(1, 3, -1)$

$$= (k_2, -2k_1 + 3k_2, 2k_1 - k_2)$$

Equating corresponding components,

$$k_2 = 3$$
$$-2k_1 + 3k_2 = 1$$
$$2k_1 - k_2 = 5$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\overset{R_{12}}{\sim}\left[\begin{array}{rr|r} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{array}\right]$$

$$\overset{\left(-\frac{1}{2}\right)R_1}{\sim}\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{array}\right]$$

$$\overset{R_3-2R_1}{\sim}\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right]$$

$$\overset{R_3-2R_2}{\sim}\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

The system of equations is consistent.
Hence, $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
The corresponding system of equations is

$$k_1 - \frac{3}{2}k_2 = -\frac{1}{2}$$
$$k_2 = 3$$

Solving these equations,

$$k_1 = 4, k_2 = 3$$

Hence, $$\mathbf{v} = 4\mathbf{v}_1 + 3\mathbf{v}_2$$

(ii) $(0, 4, 5) = k_1(0, -2, 2) + k_2(1, 3, -1)$
$= (k_2, -2k_1 + 3k_2, 2k_1 - k_2)$

Equating corresponding components,

$$k_2 = 0$$
$$-2k_1 + 3k_2 = 4$$
$$2k_1 - k_2 = 5$$

The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_{12}} \left[\begin{array}{rr|r} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{array}\right]$$

$$\underset{\sim}{\left(-\frac{1}{2}\right)R_1} \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -2 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{array}\right]$$

$$\underset{\sim}{R_3 - 2R_1} \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{array}\right]$$

$$\underset{\sim}{R_3 - 2R_2} \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array}\right]$$

From the last row of the matrix

$$0k_1 + 0k_2 = 9$$

The system of equations is inconsistent.
Hence, $\mathbf{v}$ is not a linear combination of $\mathbf{v}_1$, and $\mathbf{v}_2$.

Example 2: Which of the following are linear combinations of

$$A_1 = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(i) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Solution: Let $A = k_1A_1 + k_2\, A_2 + k_3\, A_3$

(i)
$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = k_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 4k_1 + k_2 & -k_2 + 2k_3 \\ -2k_1 + 2k_2 + k_3 & -2k_1 + 3k_2 + 4k_3 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} 4k_1 + \ \ k_2 \qquad &= 6 \\ -\ k_2 + 2k_3 &= -8 \\ -2k_1 + 2k_2 + \ \ k_3 &= -1 \\ -2k_1 + 3k_2 + 4k_3 &= -8 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{4}\right)R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right]$$

$$R_3 + 2R_1,\ R_4 + 2R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & -1 & 2 & -8 \\ 0 & \frac{5}{2} & 1 & 2 \\ 0 & \frac{7}{2} & 4 & -5 \end{array}\right]$$

$$R_3 + \frac{5}{2}R_2,\ R_4 + \frac{7}{2}R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & 11 & -33 \end{array}\right]$$

$$\left(\frac{1}{6}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 11 & -33 \end{array}\right]$$

$$(-1)R_2,\ R_4 - 11R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent.

Hence, A is a linear combination of A_1, A_2, A_3. The corresponding system of equations is

$$\begin{aligned} k_1 + \frac{1}{4}k_2 \qquad &= \frac{3}{2} \\ k_2 - 2k_3 &= 8 \\ k_3 &= -3 \end{aligned}$$

Solving these equations,

$$k_1 = 1,\ k_2 = 2,\ k_3 = -3$$

Hence,

$$A = A_1 + 2A_2 - 3A_3$$

(ii)

$$\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix} = k_1\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4k_1 + k_2 & -k_2 + 2k_3 \\ -2k_1 + 2k_2 + k_3 & -2k_1 + 3k_2 + 4k_3 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} 4k_1 + \ k_2 \qquad &= -1 \\ -\ k_2 + 2k_3 &= -5 \\ -2k_1 + 2k_2 + \ k_3 &= 7 \\ -2k_1 + 3k_2 + 4k_3 &= 1 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & -1 \\ 0 & -1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ -2 & 3 & 4 & 1 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{4}\right)R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ -2 & 3 & 4 & 1 \end{array}\right]$$

$$R_3+2R_1,\ R_4+2R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -1 & 2 & 5 \\ 0 & \frac{5}{2} & 1 & \frac{13}{2} \\ 0 & \frac{7}{2} & 4 & \frac{1}{2} \end{array}\right]$$

$$R_3+\frac{5}{2}R_2,\ R_4+\frac{7}{2}R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -1 & 2 & 5 \\ 0 & 0 & 6 & 19 \\ 0 & 0 & 11 & 18 \end{array}\right]$$

$$\left(\frac{1}{6}\right)R_3,\ (-1)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & \frac{19}{6} \\ 0 & 0 & 11 & 18 \end{array}\right]$$

$$R_4-11R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{101}{6} \end{array}\right]$$

From the last row of the matrix

$$0k_1 + 0k_2 + 0k_3 = -\frac{101}{6}$$

The system of equations is inconsistent.
Hence, A is not a linear combination of A_1, A_2, and A_3.

Example 3: Express the vector $\mathbf{v} = (6, 11, 6)$ as a linear combination of $\mathbf{v}_1 = (2, 1, 4)$, $\mathbf{v}_2 = (1, -1, 3)$ and $\mathbf{v}_3 = (3, 2, 5)$.

Solution: Let $\mathbf{v} = k_1\,\mathbf{v}_1 + k_2\,\mathbf{v}_2 + k_3\,\mathbf{v}_3$

$$\begin{aligned}(6, 11, 6) &= k_1(2, 1, 4) + k_2(1, -1, 3) + k_3(3, 2, 5)\\ &= (2k_1 + k_2 + 3k_3,\ k_1 - k_2 + 2k_3,\ 4k_1 + 3k_2 + 5k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}2k_1 + k_2 + 3k_3 &= 6\\ k_1 - k_2 + 2k_3 &= 11\\ 4k_1 + 3k_2 + 5k_3 &= 6\end{aligned}$$

Solving these equations,

$$k_1 = 4,\ k_2 = -5,\ k_3 = 1$$

Hence, $\mathbf{v} = 4\mathbf{v}_1 - 5\mathbf{v}_2 + \mathbf{v}_3$

Example 4: Express the polynomial $\mathbf{p} = -9 - 7x - 15x^2$ as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, $\mathbf{p}_3 = 3 + 2x + 5x^2$.

Solution: Let $\mathbf{p} = k_1\,\mathbf{p}_1 + k_2\,\mathbf{p}_2 + k_3\,\mathbf{p}_3$

$$\begin{aligned}-9 - 7x - 15x^2 &= k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2)\\ &= (2k_1 + k_2 + 3k_3) + (k_1 - k_2 + 2k_3)x + (4k_1 + 3k_2 + 5k_3)x^2\end{aligned}$$

Equating corresponding coefficients,

$$\begin{aligned}2k_1 + k_2 + 3k_3 &= -9\\ k_1 - k_2 + 2k_3 &= -7\\ 4k_1 + 3k_2 + 5k_3 &= -15\end{aligned}$$

Solving these equations,

$$k_1 = -2,\ k_2 = 1,\ k_3 = -2$$

Hence, $\mathbf{p} = -2\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3$

Example 5: Express the matrix $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ as a linear combination of

$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution: Let
$$A = k_1 A_1 + k_2 A_2 + k_3 A_3$$
$$\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} = k_1 \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} k_1 + k_2 + 2k_3 & -k_1 + k_2 + 2k_3 \\ -k_3 & 3k_1 + 2k_2 + k_3 \end{bmatrix}$$

Equating corresponding components,
$$\begin{aligned} k_1 + \ k_2 + 2k_3 &= \ 5 \\ -k_1 + \ k_2 + 2k_3 &= \ 1 \\ -k_3 &= -1 \\ 3k_1 + 2k_2 + \ k_3 &= \ 9 \end{aligned}$$

Solving these equations,
$$k_1 = 2, k_2 = 1, k_3 = 1$$

Hence, $A = 2A_1 + A_2 + A_3$

Example 6: For which value of λ will the vector $\mathbf{v} = (1, \lambda, 5)$ be the linear combination of vectors $\mathbf{v}_1 = (1, -3, 2)$ and $\mathbf{v}_2 = (2, -1, 1)$?

Solution: Let
$$\mathbf{v} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2$$
$$(1, \lambda, 5) = k_1(1, -3, 2) + k_2(2, -1, 1)$$
$$= (k_1 + 2k_2, -3k_1 - k_2, 2k_1 + k_2)$$

Equating corresponding components,

$$k_1 + 2k_2 = 1 \quad \text{...(1)}$$
$$-3k_1 - k_2 = \lambda \quad \text{...(2)}$$
$$2k_1 + k_2 = 5 \quad \text{...(3)}$$

$\mathbf{v}$ will be the linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if the above system of equations is consistent.
Solving equations (1) and (3),
$$k_1 = 2, \quad k_2 = -1$$

Substituting k_1, k_2 in equation (2),
$$\lambda = -5$$

2.6 SPAN

The set of all the vectors that are the linear combination of the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is called span of S and is denoted by
$$\text{span } S \quad \text{or} \quad \text{span } \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

Theorem 2.3: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in a vector space V then

(i) The span S is a subspace of V.
(ii) The span S is the smallest subspace of V that contains the set S. Any other subspace W of V that contains the set S must contain span S, i.e. span $S \subseteq W$.

Theorem 2.4: If S_1 and S_2 are two sets of vectors in a vector space V then

$$\text{span } S_1 = \text{span } S_2$$

if and only if each vector in S_1 is a linear combination of those in S_2 and vice versa. i.e., $S_1 \subset \text{span } S_2$ and $S_2 \subset \text{span } S_1$.

Vectors spanning the vector space: The method to check if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ span the vector space V is as follows:

1. Choose an arbitrary vector $\mathbf{b}$ in V.
2. Express $\mathbf{b}$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r. \qquad \text{...(2.5)}$$

3. If the system of the equations in (2.5) is consistent for all choices of $\mathbf{b}$ then vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ span V. If it is inconsistent for some choices of $\mathbf{b}$, vectors do not span V.

Note: (i) If coefficient matrix A of (2.5) is a non-singular matrix, i.e. det $(A) \neq 0$, then the system of equations in (2.5) is consistent for all choices of $\mathbf{b}$ and hence the given vectors span V.

(ii) If det $(A) = 0$ then the system of equations in (2.5) is inconsistent for some choices of $\mathbf{b}$ and hence given vectors do not span V.

Example 1: Let V be a vector space. For a non-empty set A, prove that $A \subset \text{span } A$.

Solution: Let $A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$

Each vector $\mathbf{v}_i$ of A can be expressed as

$$\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_r$$

This shows that each vector of A can be written as a linear combination of the vectors of A.

Hence, $A \subset \text{span } A$.

Example 2: Find a condition on a, b, c so that the vector $\mathbf{v} = (a, b, c)$ is in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (2, 1, 0)$, $\mathbf{v}_2 = (1, -1, 2)$ and $\mathbf{v}_3 = (0, 3, -4)$

Solution: The vector $\mathbf{v}$ will be in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$

$$\begin{aligned}(a, b, c) &= k_1(2, 1, 0) + k_2(1, -1, 2) + k_3(0, 3, -4)\\ &= (2k_1 + k_2, k_1 - k_2 + 3k_3, 2k_2 - 4k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned} 2k_1 + k_2 \quad\quad &= a \\ k_1 - k_2 + 3k_3 &= b \\ 2k_2 - 4k_3 &= c \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 2 & 1 & 0 & a \\ 1 & -1 & 3 & b \\ 0 & 2 & -4 & c \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 3 & b \\ 2 & 1 & 0 & a \\ 0 & 2 & -4 & c \end{array}\right]$$

$$R_2 - 2R_1, \left(\frac{1}{2}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 3 & b \\ 0 & 3 & -6 & a-2b \\ 0 & 1 & -2 & \frac{c}{2} \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 3 & b \\ 0 & 1 & -2 & \frac{a-2b}{3} \\ 0 & 1 & -2 & \frac{c}{2} \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 3 & b \\ 0 & 1 & -2 & \frac{a-2b}{3} \\ 0 & 0 & 0 & \frac{3c-2a+4b}{3} \end{array}\right]$$

The system will be consistent if $\frac{3c-2a+4b}{3} = 0$ i.e., $3c - 2a + 4b = 0$

Hence, $\mathbf{v}$ will be in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if $3c - 2a + 4b = 0$.

Example 3: Determine whether the following vectors span the vector space R^3.

(i) $\mathbf{v}_1 = (2, 2, 2), \mathbf{v}_2 = (0, 0, 3), \mathbf{v}_3 = (0, 1, 1)$

(ii) $\mathbf{v}_1 = (3, 1, 4), \mathbf{v}_2 = (2, -3, 5), \mathbf{v}_3 = (5, -2, 9), \mathbf{v}_4 = (1, 4, -1)$

Solution: Let $\mathbf{b} = (b_1, b_2, b_3)$ be an arbitrary vector in R^3 and can be expressed as a linear combination of the given vectors.

(i)
$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$\begin{aligned}(b_1, b_2, b_3) &= k_1(2, 2, 2) + k_2(0, 0, 3) + k_3(0, 1, 1)\\ &= (2k_1, 2k_1 + k_3, 2k_1 + 3k_2 + k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}2k_1 \qquad\qquad &= b_1\\ 2k_1 + \qquad k_3 &= b_2\\ 2k_1 + 3k_2 + k_3 &= b_3\end{aligned}$$

Coefficient matrix,
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

The coefficient matrix is a square matrix.

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix}\\ &= 2(-3) = -6 \neq 0\end{aligned}$$

The system of equations is consistent for all choices of vector **b**.
Hence, the given vectors span R^3.

(ii)
$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$\begin{aligned}(b_1, b_2, b_3) &= k_1(3, 1, 4) + k_2(2, -3, 5) + k_3(5, -2, 9) + k_4(1, 4, -1)\\ &= (3k_1 + 2k_2 + 5k_3 + k_4, k_1 - 3k_2 - 2k_3 + 4k_4, 4k_1 + 5k_2 + 9k_3 - k_4)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}3k_1 + 2k_2 + 5k_3 + k_4 &= b_1\\ k_1 - 3k_2 - 2k_3 + 4k_4 &= b_2\\ 4k_1 + 5k_2 + 9k_3 - k_4 &= b_3\end{aligned}$$

The coefficient matrix is not a square matrix.
The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\overset{R_{12}}{\sim}\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 3 & 2 & 5 & 1 & b_1 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right]$$

$$\overset{R_2-3R_1,\quad R_3-4R_1}{\sim}\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 11 & 11 & -11 & b_1-3b_2 \\ 0 & 17 & 17 & -17 & b_3-4b_2 \end{array}\right]$$

$$\overset{\left(\frac{1}{11}\right)R_2,\left(\frac{1}{17}\right)R_3}{\sim}\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \dfrac{b_1-3b_2}{11} \\ 0 & 1 & 1 & -1 & \dfrac{b_3-4b_2}{17} \end{array}\right]$$

$$\overset{R_3-R_2}{\sim}\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \dfrac{b_1-3b_2}{11} \\ 0 & 0 & 0 & 0 & \dfrac{-17b_1+7b_2+11b_3}{187} \end{array}\right]$$

If $-17b_1+7b_2+11b_3 \neq 0$, the system of the equations is inconsistent. Thus, the system of the equations does not have a solution for all choices of the vector **b**. Hence, the given vectors do not span R^3.

Example 4: Determine whether the following polynomials span P_2:

(i) $\mathbf{p}_1 = 1-x+2x^2, \quad \mathbf{p}_2 = 5-x+4x^2, \quad \mathbf{p}_3 = -2-2x+2x^2.$

(ii) $\mathbf{p}_1 = 2+x^2, \mathbf{p}_2 = 1-x+2x^2, \mathbf{p}_3 = 2+x, \mathbf{p}_4 = 4+x+x^2.$

Solution: Let $b = b_1 + b_2 x + b_3 x^2$ be an arbitrary polynomial in P_2 and can be expressed as a linear combination of given polynomials.

(i) $$\mathbf{b} = k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3$$

$$\begin{aligned} b_1 + b_2x + b_3x^2 &= k_1(1-x+2x^2) + k_2(5-x+4x^2) + k_3(-2-2x+2x^2) \\ &= (k_1+5k_2-2k_3) + (-k_1-k_2-2k_3)x + (2k_1+4k_2+2k_3)x^2 \end{aligned}$$

Equating corresponding coeffiencts,

$$\begin{aligned} k_1 + 5k_2 - 2k_3 &= b_1 \\ -k_1 - k_2 - 2k_3 &= b_2 \\ 2k_1 + 4k_2 + 2k_3 &= b_3 \end{aligned}$$

Coefficient matrix, $A = \begin{bmatrix} 1 & 5 & -2 \\ -1 & -1 & -2 \\ 2 & 4 & 2 \end{bmatrix}$

The coefficient matrix is a square matrix.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 5 & -2 \\ -1 & -1 & -2 \\ 2 & 4 & 2 \end{vmatrix} \\ &= 1(-2+8) - 5(-2+4) - 2(-4+2) \\ &= 0 \end{aligned}$$

The system of equations is inconsistent for some choices of **b**.
Hence, the given polynomials do not span P_2.

(ii) $$\mathbf{b} = k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 + k_4\mathbf{p}_4$$

$$\begin{aligned} b_1 + b_2x + b_3x^2 &= k_1(2+x^2) + k_2(1-x+2x^2) + k_3(2+x) + k_4(4+x+x^2) \\ &= (2k_1+k_2+2k_3+4k_4) + (-k_2+k_3+k_4)x + (k_1+2k_2+k_4)x^2 \end{aligned}$$

Equating corresponding coefficients,

$$\begin{aligned} 2k_1 + k_2 + 2k_3 + 4k_4 &= b_1 \\ -k_2 + k_3 + k_4 &= b_2 \\ k_1 + 2k_2 + k_4 &= b_3 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 2 & 1 & 2 & 4 & b_1 \\ 0 & -1 & 1 & 1 & b_2 \\ 1 & 2 & 0 & 1 & b_3 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{13}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_3 \\ 0 & -1 & 1 & 1 & b_2 \\ 2 & 1 & 2 & 4 & b_1 \end{array}\right]$$

$$R_3 - 2R_1$$

$$\sim\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_3 \\ 0 & -1 & 1 & 1 & b_2 \\ 0 & -3 & 2 & 2 & b_1 - 2b_3 \end{array}\right]$$

$$R_3 - 3R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_3 \\ 0 & -1 & 1 & 1 & b_2 \\ 0 & 0 & -1 & -1 & b_1 - 2b_3 - 3b_2 \end{array}\right]$$

The system of equations is consistent for all choices of **b**.
Hence, the given polynomials span P_2.

Example 5: Determine whether the following matrices span M_{22}.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution: Let $\mathbf{b} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ be an arbitrary matrix in M_{22} and can be expressed as the linear combination of given matrices.

$$\mathbf{b} = k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & k_2 + k_3 + k_4 \\ k_3 + k_4 & k_4 \end{bmatrix}$$

Equating corresponding coefficients,

$$k_1 + k_2 + k_3 + k_4 = b_1$$
$$k_2 + k_3 + k_4 = b_2$$
$$k_3 + k_4 = b_3$$
$$k_4 = b_4$$

The system of equations is consistent for all choices of b_1, b_2, b_3, b_4 i.e., **b**.
Hence, the given matrices span M_{22}.

Example 6: Let $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(i) (2, 3, –7, 3) (ii) (1, 1, 1, 1) (iii) (0, 0, 0, 0)

Solution: The vector **v** will be in span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let $$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

(i) $$\begin{aligned}(2, 3, -7, 3) &= k_1(2, 1, 0, 3) + k_2(3, -1, 5, 2) + k_3(-1, 0, 2, 1)\\ &= (2k_1 + 3k_2 - k_3, k_1 - k_2, 5k_2 + 2k_3, 3k_1 + 2k_2 + k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}2k_1 + 3k_2 - k_3 &= 2\\ k_1 - k_2 &= 3\\ 5k_2 + 2k_3 &= -7\\ 3k_1 + 2k_2 + k_3 &= 3\end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2\\ 1 & -1 & 0 & 3\\ 0 & 5 & 2 & -7\\ 3 & 2 & 1 & 3\end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_{12}}\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3\\ 2 & 3 & -1 & 2\\ 0 & 5 & 2 & -7\\ 3 & 2 & 1 & 3\end{array}\right]$$

$$R_2 - 2R_1, R_4 - 3R_1$$
$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3\\ 0 & 5 & -1 & -4\\ 0 & 5 & 2 & -7\\ 0 & 5 & 1 & -6\end{array}\right]$$

$$R_3 - R_2, R_4 - R_2$$
$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3\\ 0 & 5 & -1 & -4\\ 0 & 0 & 3 & -3\\ 0 & 0 & 2 & -2\end{array}\right]$$

$$\left(\frac{1}{5}\right)R_2, \left(\frac{1}{3}\right)R_3, \left(\frac{1}{2}\right)R_4$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & -\frac{1}{5} & \vdots & -\frac{4}{5} \\ 0 & 0 & 1 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$$

$$R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & -\frac{1}{5} & \vdots & -\frac{4}{5} \\ 0 & 0 & 1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

The system of equations is consistent. Thus, **v** can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Hence, **v** is in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(ii)
$$(1, 1, 1, 1) = k_1(2, 1, 0, 3) + k_2(3, -1, 5, 2) + k_3(-1, 0, 2, 1)$$
$$= (2k_1 + 3k_2 - k_3, k_1 - k_2, 5k_2 + 2k_3, 3k_1 + 2k_2 + k_3)$$

Equating corresponding components,

$$\begin{aligned} 2k_1 + 3k_2 - k_3 &= 1 \\ k_1 - k_2 &= 1 \\ 5k_2 + 2k_3 &= 1 \\ 3k_1 + 2k_2 + k_3 &= 1 \end{aligned}$$

The augmented matrix of the system is

$$\begin{bmatrix} 2 & 3 & -1 & \vdots & 1 \\ 1 & -1 & 0 & \vdots & 1 \\ 0 & 5 & 2 & \vdots & 1 \\ 3 & 2 & 1 & \vdots & 1 \end{bmatrix}$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 \\ 2 & 3 & -1 & \vdots & 1 \\ 0 & 5 & 2 & \vdots & 1 \\ 3 & 2 & 1 & \vdots & 1 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 5 & -1 & -1 \\ 0 & 5 & 2 & 1 \\ 0 & 5 & 1 & -2 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 5 & -1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & -1 \end{array}\right]$$

$$\left(\frac{1}{5}\right)R_2,\ \left(\frac{1}{3}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 2 & -1 \end{array}\right]$$

$$R_4 - 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{7}{3} \end{array}\right]$$

From the last row of the matrix

$$0k_1 + 0k_2 + 0k_3 = -\frac{7}{3}$$

The system of equations is inconsistent. Thus, $\mathbf{v}$ cannot be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Hence, $\mathbf{v}$ is not in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(iii) $(0, 0, 0, 0) = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$

Thus, $\mathbf{v}$ can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.
Hence, $\mathbf{v}$ is in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Example 7: Let $\mathbf{f} = \cos^2 x$ and $\mathbf{g} = \sin^2 x$. Which of the following lie in the space spanned by **f** and **g**?

(i) $\cos 2x$ (ii) $\sin x$

Solution: (i)
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1\mathbf{f} + (-1)\mathbf{g}$$

$\cos 2x$ can be expressed as a linear combination of **f** and **g**.
Hence, $\cos 2x$ lies in the space spanned by **f** and **g**.

(ii) $\sin x$ cannot be expressed as a linear combination of $\cos^2 x$ and $\sin^2 x$.
Hence, $\sin x$ does not lie in the space spanned by **f** and **g**.

Example 8: Find an equation for the plane spanned by the vectors $\mathbf{v}_1 = (-1, 1, 1)$ and $\mathbf{v}_2 = (3, 4, 4)$.

Solution: Let $\mathbf{v} = (x, y, z)$ be an arbitrary vector on the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$
$$(x, y, z) = k_1(-1, 1, 1) + k_2(3, 4, 4)$$
$$= (-k_1 + 3k_2, k_1 + 4k_2, k_1 + 4k_2)$$

Equating corresponding components,

$$-k_1 + 3k_2 = x$$
$$k_1 + 4k_2 = y$$
$$k_1 + 4k_2 = z$$

Eliminating k_1, k_2 from the above equations,
$y = z$, which is the required plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Example 9: Find the parametric equations of the line spanned by the vector $\mathbf{v}_1 = (3, -2, 5)$.

Solution: Let $\mathbf{v} = (x, y, z)$ be an arbitrary point on the line spanned by the vector $\mathbf{v}_1$.

$$\mathbf{v} = k\, v_1$$
$$(x, y, z) = k(3, -2, 5)$$
$$= (3k, -2k, 5k)$$

Equating corresponding components,

$$x = 3k, \quad y = -2k, \quad z = 5k$$

which is the parametric equations of the line spanned by the vector $\mathbf{v}_1$, where k is a parameter.

Example 10: Show that $\mathbf{v}_1 = (1, 6, 4), \mathbf{v}_2 = (2, 4, -1), \mathbf{v}_3 = (-1, 2, 5)$ and $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$ span the same subspace of R^3.

Solution: Here we need to prove that span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = span $\{\mathbf{w}_1, \mathbf{w}_2\}$, i.e., $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in span $\{\mathbf{w}_1, \mathbf{w}_2\}$ and $\mathbf{w}_1, \mathbf{w}_2$ are in span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(i) Let

$$\begin{aligned}\mathbf{v}_1 &= k_1\mathbf{w}_1 + k_2\mathbf{w}_2\\ (1, 6, 4) &= k_1(1, -2, -5) + k_2(0, 8, 9)\\ &= (k_1, -2k_1 + 8k_2, -5k_1 + 9k_2)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}k_1 \qquad &= 1\\ -2k_1 + 8k_2 &= 6\\ -5k_1 + 9k_2 &= 4\end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 1 & 0 & 1\\ -2 & 8 & 6\\ -5 & 9 & 4\end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 2R_1, R_3 + 5R_1$$

$$\sim\left[\begin{array}{rr|r} 1 & 0 & 1\\ 0 & 8 & 8\\ 0 & 9 & 9\end{array}\right]$$

$$\left(\frac{1}{8}\right)R_2, \left(\frac{1}{9}\right)R_3$$

$$\sim\left[\begin{array}{rr|r} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 1 & 1\end{array}\right]$$

$$R_3 - R_2$$

$$\sim\left[\begin{array}{rr|r} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{array}\right]$$

The system of equations is consistent.

Hence, $\mathbf{v}_1$ is in the span $\{\mathbf{w}_1, \mathbf{w}_2\}$.

Let $\mathbf{v}_2 = b_1\,\mathbf{w}_1 + b_2\,\mathbf{w}_2$

$$\begin{aligned}(2, 4, -1) &= b_1(1, -2, -5) + b_2(0, 8, 9)\\ &= (b_1, -2b_1 + 8b_2, -5b_1 + 9b_2)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned} b_1 &= 2\\ -2b_1 + 8b_2 &= 4\\ -5b_1 + 9b_2 &= -1\end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 1 & 0 & 2\\ -2 & 8 & 4\\ -5 & 9 & -1\end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\begin{array}{c} R_2 + 2R_1,\ R_3 + 5R_1\\ \sim\left[\begin{array}{rr|r} 1 & 0 & 2\\ 0 & 8 & 8\\ 0 & 9 & 9\end{array}\right]\end{array}$$

Proceeding as in the previous part, this matrix reduces to the form

$$\sim\left[\begin{array}{rr|r} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{array}\right]$$

The system of equations is consistent.

Hence, $\mathbf{v}_2$ is in the span$\{\mathbf{w}_1, \mathbf{w}_2\}$.

Let $\mathbf{v}_3 = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$

$$(-1, 2, 5) = c_1(1, -2, -5) + c_2(0, 8, 9) = (c_1, -2c_1 + 8c_2, -5c_1 + 9c_2)$$

Equating corresponding components,

$$\begin{aligned} c_1 \qquad &= -1\\ -2c_1 + 8c_2 &= 2\\ -5c_1 + 9c_2 &= 5\end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 1 & 0 & -1 \\ -2 & 8 & 2 \\ -5 & 9 & 5 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 2R_1,\ R_3 + 5R_1$$

$$\sim \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 8 & 0 \\ 0 & 9 & 0 \end{array}\right]$$

Proceeding as in the previous part, this matrix reduces to the form,

$$\sim \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The system of equations is consistent.
Hence, $\mathbf{v}_3$ is in the span $\{\mathbf{w}_1, \mathbf{w}_2\}$

(ii) Let $$\mathbf{w}_1 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$\begin{aligned}(1, -2, -5) &= k_1(1, 6, 4) + k_2(2, 4, -1) + k_3(-1, 2, 5) \\ &= (k_1 + 2k_2 - k_3,\ 6k_1 + 4k_2 + 2k_3,\ 4k_1 - k_2 + 5k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 + 2k_2 - k_3 &= 1 \\ 6k_1 + 4k_2 + 2k_3 &= -2 \\ 4k_1 - k_2 + 5k_3 &= -5 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 6 & 4 & 2 & -2 \\ 4 & -1 & 5 & -5 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 6R_1,\ R_3 - 4R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -8 & 8 & -8 \\ 0 & -9 & 9 & -9 \end{array}\right]$$

$$\left(-\frac{1}{8}\right)R_2, \left(\frac{1}{9}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right]$$

$$R_3 + R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system of equations is consistent.
Hence, $\mathbf{w}_1$ is in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Let
$$\mathbf{w}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$
$$(0, 8, 9) = c_1(1, 6, 4) + c_2(2, 4, -1) + c_3(-1, 2, 5)$$
$$= (c_1 + 2c_2 - c_3,\ 6c_1 + 4c_2 + 2c_3,\ 4c_1 - c_2 + 5c_3)$$

Equating corresponding components,

$$\begin{aligned} c_1 + 2c_2 - \ \ c_3 &= 0 \\ 6c_1 + 4c_2 + 2c_3 &= 8 \\ 4c_1 - \ \ c_2 + 5c_3 &= 9 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 6 & 4 & 2 & 8 \\ 4 & -1 & 5 & 9 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 6R_1,\ R_3 - 4R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -8 & 8 & 8 \\ 0 & -9 & 9 & 9 \end{array}\right]$$

$$\left(-\frac{1}{8}\right)R_2, \left(\frac{1}{9}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$R_3 + R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system of equations is consistent.
Hence, $\mathbf{w}_2$ is in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
From part (i) and (ii), we conclude that

$$\text{span } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span } \{\mathbf{w}_1, \mathbf{w}_2\}$$

Hence, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ span the same subspace of R^3.

Exercise 2.3

1. Which of the following are subspaces of R^3 and R^4 under the standard operations?

(i) $W = \{(x, y, z) \mid x = z = 0\}$

(ii) $W = \{(x, y, z) \mid z > 0\}$

(iii) $W = \{(x, y, z) \mid x = -z\}$

(iv) $W = \{(x, y, z) \mid y = z = 1\}$

(v) $W = \{(x, y, z) \mid y = 2x + 1\}$

(vi) $W = \{(x_1, x_2, x_3, x_4) \mid x_3 = x_1 + 2x_2 \text{ and } x_4 = x_1 - 3x_2\}$

[Ans. : (i), (iii), (vi)]

2. Which of the following are subspaces of P_2 and P_3 under the standard operations?

(i) $W = \{a_0 + a_1 x + a_2 x^2 \mid a_0 = 2\}$

(ii) $W = \{a_0 + a_1 x + a_2 x^2 \mid a_0 = a_1 + a_2\}$

(iii) $W = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 + a_2 + a_3 = 0\}$

[Ans. : (ii), (iii)]

3. Which of the following are subspaces of M_{22} and M_{23} under the standard operations?

(i) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, c = 0 \right\}$

(ii) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, b = a, c = d = -a \right\}$

(iii) $W = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \middle| \, a = 2c + 1 \right\}$

(iv) $W = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \middle| \, a + c = 0 \text{ and } b + d + f = 0 \right\}$

[Ans. : (i), (ii), (iv)]

4. Which of the following are subspaces of M_{nn} under standard operations?

(i) $W = \{A_{nn} \mid A_{nn} \text{ is upper triangular}\}$

(ii) $W = \{A_{nn} \mid \det(A) = 1\}$

(iii) $W = \{A_{nn} \mid A^T = -A\}$

(iv) $W = \{A_{nn} \mid \det(A) = 0\}$

[Ans. : (i), (iii)]

5. Which of the following are subspaces of $F(-\infty, \infty)$ under standard operations?

(i) $W = \{\mathbf{f} \mid f(x) \text{ is constant}\}$

(ii) $W = \{\mathbf{f} \mid f(0) = 2\}$

(iii) $W = \{\mathbf{f} \mid f(0) = 0\}$

(iv) $W = \{\mathbf{f} \mid f(x)$ is integrable on the interval $[a, b]\}$

(v) $W = \{\mathbf{f} \mid f(x^2) = [f(x)]^2\}$

[Ans. : (i), (iii), (iv)]

6. Which of the following are the linear combinations of $\mathbf{v}_1 = (1, -3, 2)$ and $\mathbf{v}_2 = (2, -1, 1)$?

(i) $\mathbf{v} = (1, 7, -4)$ (ii) $\mathbf{v} = (2, -5, 4)$
(iii) $\mathbf{v} = (0, 1, 4)$

[Ans. : (i), (ii)]

7. Which of the following are the linear combinations of $\mathbf{p}_1 = 5 - 2x + x^2$, $\mathbf{p}_2 = -3x + 2x^2$ and $\mathbf{p}_3 = 3 + x$?

(i) $\mathbf{v} = 0$ (ii) $\mathbf{v} = -3 + 4x + x^2$
(iii) $2 - 5x + 3x^2$

[Ans. : (i), (ii)]

8. Which of the following are the linear combinations of

$$A_1 = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

and $A_3 = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$?

(i) $A = \begin{bmatrix} 3 & 9 \\ -4 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

[Ans. : (i)]

9. Express the vector $\mathbf{v} = (1, -2, 5)$ as a linear combination of the vectors $\mathbf{v}_1 = (1, 1, 1,)$, $\mathbf{v}_2 = (1, 2, 3)$, and $\mathbf{v}_3 = (2, -1, 1)$.

[Ans. : $\mathbf{v} = -6\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3$]

10. Express the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$

as a linear combination of

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

[Ans. : $A = 3A_1 - 2A_2 - A_3$]

11. For which value of λ will the vector $\mathbf{v} = (1, -2, \lambda)$ be a linear combination of the vectors $\mathbf{v}_1 = (3, 0, -2)$ and $\mathbf{v}_2 = (2, -1, -5)$?

[Ans. : $\lambda = -8$]

12. Find a condition for which the vector $\mathbf{v} = (a, b, c)$ is a linear combination of the vectors $\mathbf{v}_1 = (1, -3, 2)$ and $\mathbf{v}_2 = (2, -1, 1)$.

[Ans. : $a - 3b - 5c = 0$]

13. Determine whether the following vectors span the vector space R^3:

(i) $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (0, 0, 1)$

(ii) $\mathbf{v}_1 = (1, 2, 5)$, $\mathbf{v}_2 = (1, 3, 7)$, $\mathbf{v}_3 = (1, -1, -1)$

(iii) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

(iv) $\mathbf{v}_1 = (1, 2, 6)$, $\mathbf{v}_2 = (3, 4, 1)$, $\mathbf{v}_3 = (4, 3, 1)$, $\mathbf{v}_4 = (3, 3, 1)$

[Ans. : (i) yes (ii) no (iii) no (iv) yes]

14. Which of the following vectors span the vector space R^4?

(i) $\mathbf{v}_1 = (1, 2, 1, 0)$, $\mathbf{v}_2 = (1, 1, -1, 0)$, $\mathbf{v}_3 = (0, 0, 0, 1)$

(ii) $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (1, 2, -1, 1)$, $\mathbf{v}_3 = (0, 0, 1, 1)$ $\mathbf{v}_4 = (2, 1, 2, 1)$

[Ans. : (ii)]

15. Determine whether the polynomials $\mathbf{p}_1 = 1 + 2x + x^3$, $\mathbf{p}_2 = 2 - x + x^2$, $\mathbf{p}_3 = 2 + x^3$, $\mathbf{p}_4 = 2 - 5x + x^2 - x^3$ span $\mathbf{p}_3$?

[Ans. : no]

16. Let $\mathbf{v}_1 = (1, 0, 0, 1)$, $\mathbf{v}_2 = (1, -1, 0, 0)$, $\mathbf{v}_3 = (0, 1, 2, 1)$. Which of the following vectors are in the span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(i) $(0, 1, 1, 0)$ (ii) $(-1, 4, 2, 2)$
(iii) $(2, -1, 3, 1)$

[Ans. : (iii)]

2.7 LINEAR DEPENDENCE AND INDEPENDENCE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0} \tag{2.6}$$

If the homogeneous system obtained from (2.6) has only a trivial solution (i.e., $k_1 = 0$, $k_2 = 0, \ldots, k_r = 0$) then S is called a linearly independent set. If the system has a non-trivial solution (i.e., at least one k is non-zero) then S is called a linearly dependent set.

Note: If the determinant of the coefficient matrix of (2.6) is zero then vectors are linearly dependent, otherwise they are linearly independent.

Theorem 2.5: A set S of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is

(i) Linearly dependent if and only if at least one vector of S can be expressed as a linear combination of the remaining vectors in S.
(ii) Linearly independent if and only if no vector of S can be expressed as a linear combination of the remaining vectors in S.
(iii) Linearly dependent if S contains zero vector as $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r$.

Theorem 2.6: A set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ with exactly two vectors is linearly dependent if one vector is a scalar multiple of the other vector,

$$\text{i.e., } \mathbf{v}_1 = k_1\mathbf{v}_2 \text{ or } \mathbf{v}_2 = k_2\mathbf{v}_1$$

Theorem 2.7: If S_1 and S_2 are two finite set of vectors such that S_1 is a subset of S_2 $(S_1 \subset S_2)$ then if

(i) S_1 is linearly dependent then S_2 is also linearly dependent
(ii) S_2 is linearly independent then S_1 is also linearly independent, i.e. every subset of a linearly independent set is linearly independent

Theorem 2.8: The set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in R^n is linearly dependent if $r > n$ i.e., number of unknowns is more than the number of equations in the homogeneous system obtained from Eq. (2.6).

Example 1: Which of the following sets of vectors in R^3 and R^4 are linearly dependent?

(i) (4, –1, 2), (– 4, 10, 2), (4, 0, 1)
(ii) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)
(iii) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)
(iv) (1, 1, 2, 1), (1, 0, 0, 2), (4, 6, 8, 6), (0, 3, 2, 1)

Solution: Let $\mathbf{v}_1 = (4, -1, 2)$, $\mathbf{v}_2 = (-4, 10, 2)$, $\mathbf{v}_3 = (4, 0, 1)$
Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$

$$k_1(4, -1, 2) + k_2(-4, 10, 2) + k_3(4, 0, 1) = (0, 0, 0)$$
$$(4k_1 - 4k_2 + 4k_3, -k_1 + 10k_2, 2k_1 + 2k_2 + k_3) = (0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} 4k_1 - 4k_2 + 4k_3 &= 0 \\ -k_1 + 10k_2 \qquad &= 0 \\ 2k_1 + 2k_2 + k_3 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 4 & -4 & 4 & 0 \\ -1 & 10 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(\frac{1}{4}\right)R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 10 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$$R_2 + R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right]$$

$$\left(\frac{1}{9}\right)R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{1}{9} & 0 \\ 0 & 4 & -1 & 0 \end{array}\right]$$

$$R_3 - 4R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{9} & 0 \end{array}\right]$$

The system has a trivial solution.
Hence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

(ii) Let $\mathbf{v}_1 = (-2, 0, 1)$, $\mathbf{v}_2 = (3, 2, 5)$, $\mathbf{v}_3 = (6, -1, 1)$, $\mathbf{v}_4 = (7, 0, -2)$
Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$.

$$k_1(-2, 0, 1) + k_2(3, 2, 5) + k_3(6, -1, 1) + k_4(7, 0, -2) = (0, 0, 0)$$

$$(-2k_1 + 3k_2 + 6k_3 + 7k_4,\ 2k_2 - k_3,\ k_1 + 5k_2 + k_3 - 2k_4) = (0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} -2k_1 + 3k_2 + 6k_3 + 7k_4 &= 0 \\ 2k_2 - k_3 &= 0 \\ k_1 + 5k_2 + k_3 - 2k_4 &= 0 \end{aligned}$$

The number of unknowns, $r = 4$
The number of equations, $n = 3$

$$r > n$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

(iii) Let $\mathbf{v}_1 = (0, 0, 2, 2)$, $\mathbf{v}_2 = (3, 3, 0, 0)$, $\mathbf{v}_3 = (1, 1, 0, -1)$

Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$

$$k_1(0, 0, 2, 2) + k_2(3, 3, 0, 0) + k_3(1, 1, 0, -1) = (0, 0, 0, 0)$$
$$(3k_2 + k_3, 3k_2 + k_3, 2k_1, 2k_1 - k_3) = (0, 0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} 3k_2 + k_3 &= 0 \\ 3k_2 + k_3 &= 0 \\ 2k_1 &= 0 \\ 2k_1 - k_3 &= 0 \end{aligned}$$

Solving these equations,

$$k_1 = 0, k_2 = 0, k_3 = 0$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(iv) Let $\mathbf{v}_1 = (1, 1, 2, 1)$, $\mathbf{v}_2 = (1, 0, 0, 2)$, $\mathbf{v}_3 = (4, 6, 8, 6)$, $\mathbf{v}_4 = (0, 3, 2, 1)$

Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$

$$k_1(1, 1, 2, 1) + k_2(1, 0, 0, 2) + k_3(4, 6, 8, 6) + k_4(0, 3, 2, 1) = (0, 0, 0, 0)$$
$$(k_1 + k_2 + 4k_3, k_1 + 6k_3 + 3k_4, 2k_1 + 8k_3 + 2k_4, k_1 + 2k_2 + 6k_3 + k_4) = (0, 0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_2 + 4k_3 &= 0 \\ k_1 + 6k_3 + 3k_4 &= 0 \\ 2k_1 + 8k_3 + 2k_4 &= 0 \\ k_1 + 2k_2 + 6k_3 + k_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 1 & 0 & 6 & 3 & 0 \\ 2 & 0 & 8 & 2 & 0 \\ 1 & 2 & 6 & 1 & 0 \end{array}\right]$$

Reducing augmented matrix to row echelon form,

$$\xrightarrow{R_2 - R_1,\ R_3 - 2R_1,\ R_4 - R_1} \sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_2,\ \left(\frac{1}{2}\right)R_3} \sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2,\ R_4 - R_2} \sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 4 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{\left(-\frac{1}{2}\right)R_3,\ \left(\frac{1}{4}\right)R_4} \sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - R_3} \sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a non-trivial solution.
Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Example 2: Which of the following sets of polynomials in P_2 are linearly dependent?

(i) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$ (ii) $2 + x + x^2, x + 2x^2, 2 + 2x + 3x^2$

Solution: Let $\mathbf{p}_1 = 2 - x + 4x^2$, $\mathbf{p}_2 = 3 + 6x + 2x^2$, $\mathbf{p}_3 = 2 + 10x - 4x^2$

Consider,
$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$$
$$k_1(2 - x + 4x^2) + k_2(3 + 6x + 2x^2) + k_3(2 + 10x - 4x^2) = 0$$
$$(2k_1 + 3k_2 + 2k_3) + (-k_1 + 6k_2 + 10k_3)x + (4k_1 + 2k_2 - 4k_3)x^2 = 0$$

Equating corresponding components,
$$2k_1 + 3k_2 + 2k_3 = 0$$
$$-k_1 + 6k_2 + 10k_3 = 0$$
$$4k_1 + 2k_2 - 4k_3 = 0$$

The augmented matrix of the system is
$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

R_{12}
$$\sim \left[\begin{array}{ccc|c} -1 & 6 & 10 & 0 \\ 2 & 3 & 2 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right]$$

$(-1)R_1, \left(\frac{1}{2}\right)R_3$
$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 2 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$R_2 - 2R_1, R_3 - 2R_1$
$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 15 & 22 & 0 \\ 0 & 13 & 18 & 0 \end{array}\right]$$

$\left(\frac{1}{15}\right)R_2$
$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 1 & \frac{22}{15} & 0 \\ 0 & 13 & 18 & 0 \end{array}\right]$$

$$R_3 - 13R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 1 & \frac{22}{15} & 0 \\ 0 & 0 & -\frac{16}{15} & 0 \end{array}\right]$$

The system has a trivial solution.
Hence, the given polynomials are linearly independent.

(ii) Let $\mathbf{p}_1 = 2 + x + x^2$, $\mathbf{p}_2 = x + 2x^2$, $\mathbf{p}_3 = 2 + 2x + 3x^2$

Consider, $$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$$

$$k_1(2+x+x^2) + k_2(x+2x^2) + k_3(2+2x+3x^2) = 0$$

$$(2k_1 + 2k_3) + (k_1 + k_2 + 2k_3)x + (k_1 + 2k_2 + 3k_3)x^2 = 0$$

Equating corresponding components,

$$\begin{aligned} 2k_1 \qquad + 2k_3 &= 0 \\ k_1 + \; k_2 + 2k_3 &= 0 \\ k_1 + 2k_2 + 3k_3 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$$

Reducing augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$$

$$R_2 - R_1,\; R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a non-trivial solution.
Hence, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are linearly dependent.

Example 3: Which of the following sets of matrices in M_{22} are linearly dependent?

(i) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

Solution: Let $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

Consider,
$$k_1A_1 + k_2A_2 + k_3A_3 = \mathbf{0}$$
$$k_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + k_2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} k_1 + k_2 & k_1 + k_3 \\ k_1 & k_1 + 2k_2 + 2k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding components,
$$\begin{aligned} k_1 + k_2 \quad\quad &= 0 \\ k_1 + \quad\quad k_3 &= 0 \\ k_1 \quad\quad\quad\quad &= 0 \\ k_1 + 2k_2 + 2k_3 &= 0 \end{aligned}$$

The augmented matrix of the system is
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,
$$R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$
$$R_3 - R_2,\ R_4 + R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$(-1)R_2, R_4 + 3R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a trivial solution.
Hence, A_1, A_2, A_3 are linearly independent.

(ii) Let $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

Consider, $$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + k_2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + k_3\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} + k_4\begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 + 2k_4 & k_1 + 3k_3 + 6k_4 \\ k_1 + k_3 + 4k_4 & 2k_1 + 2k_2 + 2k_3 + 6k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_2 \qquad + 2k_4 &= 0 \\ k_1 \qquad + 3k_3 + 6k_4 &= 0 \\ k_1 \qquad + k_3 + 4k_4 &= 0 \\ 2k_1 + 2k_2 + 2k_3 + 6k_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 3 & 6 & 0 \\ 1 & 0 & 1 & 4 & 0 \\ 2 & 2 & 2 & 6 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1, R_3 - R_1, R_4 - 2R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right]$$

$$R_4 + R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a non-trivial solution.
Hence, A_1, A_2, A_3, A_4 are linearly depdendent.

Example 4: Which of the following sets of vectors are linearly dependent?

(i) $\mathbf{v}_1 = (-1, 2, 4)$ and $\mathbf{v}_2 = (5, -10, -20)$ in R^3

(ii) $\mathbf{p}_1 = 1 - 2x + x^2$ and $\mathbf{p}_2 = 4 - x + 3x^2$ in P_2.

(iii) $A_1 = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}.

Solution: (i) $(5, -10, -20) = -5\,(-1, 2, 4)$

$$\mathbf{v}_2 = -5\mathbf{v}_1$$

$\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$.
Hence, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

(ii) $\mathbf{p}_1 = 1 - 2x + x^2$, $\mathbf{p}_2 = 4 - x + 3x^2$

Neither $\mathbf{p}_1$ is a scalar multiple of $\mathbf{p}_2$ nor $\mathbf{p}_2$ is a scalar multiple of $\mathbf{p}_1$.
Hence, $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent.

(iii) $\begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} = -1\begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$

$$A_1 = -1A_2$$

A_1 is a scalar multiple of A_2.
Hence, A_1 and A_2 are linearly dependent.

Example 5: Show that the vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$ and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4. Express each vector as a linear combination of the other two.

Solution: Consider,

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \qquad \text{...(1)}$$

$$k_1(0, 3, 1, -1) + k_2(6, 0, 5, 1) + k_3(4, -7, 1, 3) = (0, 0, 0, 0)$$

$$(6k_2 + 4k_3, 3k_1 - 7k_3, k_1 + 5k_2 + k_3, -k_1 + k_2 + 3k_3) = (0, 0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} 6k_2 + 4k_3 &= 0 \\ 3k_1 \qquad - 7k_3 &= 0 \\ k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + k_2 + 3k_3 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_{13}} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

$$\underset{\sim}{R_2 - 3R_1,\ R_4 + R_1} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -15 & -10 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 6 & 4 & 0 \end{array}\right]$$

$$\underset{\sim}{\left(-\frac{1}{15}\right)R_2} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 6 & 4 & 0 \end{array}\right]$$

$$\underset{\sim}{R_3 - 6R_2,\ R_4 - 6R_2} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has a non-trivial solution.
Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.
The corresponding system of equations is

$$k_1 + 5k_2 + k_3 = 0$$

$$k_2 + \frac{2}{3}k_3 = 0$$

Solving these equations,

$$k_2 = -\frac{2}{3}k_3$$
$$k_1 = \frac{7}{3}k_3$$

Substituting in equation (1),

$$\frac{7}{3}k_3\mathbf{v}_1 - \frac{2}{3}k_3\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$
$$\frac{7}{3}\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

$$\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3$$
$$\mathbf{v}_2 = \frac{7}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_3$$
$$\mathbf{v}_3 = -\frac{7}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2$$

Example 6: For what real values of λ are the vectors $\mathbf{v}_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \mathbf{v}_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$ in R^3 linearly dependent?

Solution: Consider,
$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

$$k_1\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + k_2\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + k_3\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0,0,0)$$
$$\left(\lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3, -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3, -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3\right) = (0,0,0)$$

Equating corresponding components,

$$\begin{aligned}\lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 &= 0\\ -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3 &= 0\\ -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3 &= 0\end{aligned} \qquad \text{...(1)}$$

The vectors are linearly dependent (i.e. non-trivial solution) if determinant of the coefficient matrix of (1) is zero.

$$\text{Coefficient matrix,}\quad A=\begin{bmatrix}\lambda & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \lambda\end{bmatrix}$$

$$\det(A)=\begin{vmatrix}\lambda & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \lambda\end{vmatrix}$$

$$=\lambda\left(\lambda^2-\frac{1}{4}\right)+\frac{1}{2}\left(-\frac{\lambda}{2}-\frac{1}{4}\right)-\frac{1}{2}\left(\frac{1}{4}+\frac{\lambda}{2}\right)$$

$$=\lambda^3-\frac{3\lambda}{4}-\frac{1}{4}$$

If $\det(A)=0$, then

$$\lambda^3-\frac{3\lambda}{4}-\frac{1}{4}=0$$

$$\lambda=1,-\frac{1}{2}$$

Hence, for $\lambda=1$ and $\lambda=-\frac{1}{2}$, the set of vectors is linearly dependent.

2.7.1 *Geometrical Interpretation of Linear Dependence and Independence*

(i) Two vectors are linearly dependent if $\mathbf{v}_1=k_1\mathbf{v}_2$ or $\mathbf{v}_2=k_2\mathbf{v}_1$ otherwise, they are linearly independent. Geometrically, it states that the two vectors in R^2 or R^3 are linearly dependent if they lie on the same line with their starting points at the origin.

(ii) Three vectors are linearly dependent if at least one vector is the linear combination of the remaining two, i.e. $\mathbf{v}_1=k_1\mathbf{v}_2+k_2\mathbf{v}_3$ or any two vectors are scalar multiple of the third vector i.e. $\mathbf{v}_2=k_1\mathbf{v}_1$, $\mathbf{v}_3=k_2\mathbf{v}_1$. This shows that $\mathbf{v}_1$ lies in the plane spanned by

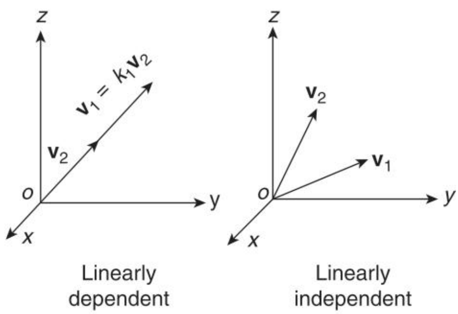


Fig. 2.1

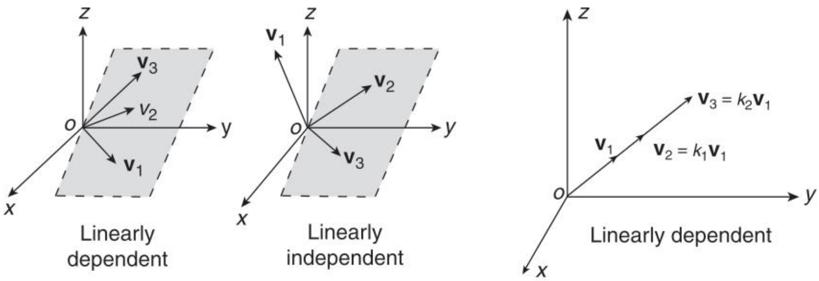


Fig. 2.2

$\mathbf{v}_2$ and $\mathbf{v}_3$ or all three vectors lie on the same line. Geometrically, it states that the three vectors in R^3 are linearly dependent if either they lie on the same plane or they lie on the same line with their starting points at the origin.

Example 1: If $\mathbf{v}_1 = (4, 6, 8)$, $\mathbf{v}_2 = (2, 3, 4)$, $\mathbf{v}_3 = (-2, -3, -4)$ are three vectors in R^3 that have initial points at the origin. Do they lie on the same line?

Solution:

$$\begin{aligned}\mathbf{v}_1 &= (4, 6, 8)\\ &= 2(2, 3, 4)\\ &= 2\mathbf{v}_2\\ \mathbf{v}_3 &= (-2, -3, -4)\\ &= -1(2, 3, 4)\\ &= (-1)\mathbf{v}_2\end{aligned}$$

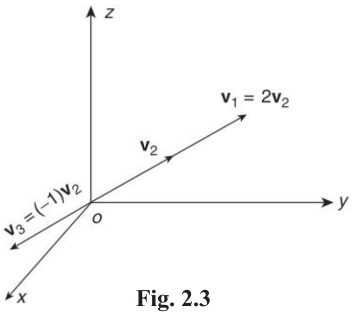


Fig. 2.3

Since $\mathbf{v}_1$ and $\mathbf{v}_3$ are scalar multiples of $\mathbf{v}_2$, they lie on the same line.

Example 2: Show that there is no line containing the points (1, 1), (3, 5), (–1, 6) and (7, 2)

Solution: Since none of the points is a scalar multiple of the other, they do not lie on the same line.

2.7.2 *Linear Dependence and Independence of Functions*

If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are $(n-1)$ times differentiable functions on the interval $(-\infty, \infty)$ then the Wronskian of these functions is

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem 2.9: If the Wronskian of $(n - 1)$ times differentiable functions on the interval $(-\infty, \infty)$ is not identically zero on this interval then these functions are linearly independent.

Note: If the Wronskian of the functions is identically zero on the interval $(-\infty, \infty)$, then no conclusion can be made about the linear dependence or independence of the functions.

Example 1: Which of the following sets of functions $F(-\infty, \infty)$ are linearly independent?

(i) $x, \sin x$ (ii) $1, e^x, e^{2x}$ (iii) $e^x, xe^x, x^2 e^x$ (iv) $6, 3\sin^2 x, 2\cos^2 x$
(v) $\sin(x + 1), \sin x, \cos x$ (vi) $(3 - x)^2, x^2 - 6x, 5$

Solution: The Wronskian of the functions is

$$W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$
$$= x\cos x - \sin x$$

Since, this function is not zero for all values of x in the interval $(-\infty, \infty)$, the given functions are linearly independent.

(ii) The Wronskian of the functions is

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$
$$= 1(4e^{3x} - 2e^{3x}) - e^x(0) + e^{2x}(0)$$
$$= 2e^{3x}$$

Since this function is not zero for all values of x in the interval $(-\infty, \infty)$, the given functions are linearly independent.

(iii) The Wronskian of the functions is

$$W = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & (x+2)e^x & (x^2+4x+2)e^x \end{vmatrix}$$
$$= e^x\left[(x+1)e^x \cdot (x^2+4x+2)e^x - (x^2+2x)e^x \cdot (x+2)e^x\right]$$
$$- xe^x\left[(x^2+4x+2)e^{2x} - (x^2+2x)e^{2x}\right] + x^2e^x\left[(x+2)e^{2x} - (x+1)e^{2x}\right]$$
$$= e^{3x}(2 - x^2)$$

Since this function is not zero for all values of x in the interval $(-\infty, \infty)$, the given functions are linearly independent.

(iv) The Wronskian of the function is

$$W = \begin{vmatrix} 6 & 3\sin^2 x & 2\cos^2 x \\ 0 & 6\sin x\cos x & -4\cos x\sin x \\ 0 & 6\cos 2x & -4\cos 2x \end{vmatrix}$$
$$= 6(-24\sin x\cos x\cos 2x + 24\cos x\sin x\cos 2x)$$
$$= 0$$

No conclusion can be made about the linear independence of these functions. From trigonometric identity,

$$6 = 6\sin^2 x + 6\cos^2 x$$
$$= 2(3\sin^2 x) + 3(2\cos^2 x)$$

This shows that 6 can be expressed as a linear combination of 3 $\sin^2 x$ and 2 $\cos^2 x$. Hence, the given functions are linearly dependent.

Note: Appropriate identities can be used directly to show linear dependence without using Wronskian.

(v) $\sin(x+1) = \sin x\cos 1 + \cos x\sin 1$

$$= k_1 \sin x + k_2 \cos x, \text{ where } k_1 = \cos 1,\ k_2 = \sin 1$$

This shows that the function sin $(x + 1)$ can be expressed as a linear combination of $\sin x$ and $\cos x$.

Hence, the given functions are linearly dependent.

(vi) Let $\mathbf{f}_1 = (3-x)^2 = 9 - 6x + x^2,\ \mathbf{f}_2 = x^2 - 6x,\ \mathbf{f}_3 = 5$
Here,

$$\mathbf{f}_1 = 9 - 6x + x^2$$
$$= \frac{9}{5}\cdot 5 + x^2 - 6x$$
$$= \frac{9}{5}\mathbf{f}_3 + \mathbf{f}_2$$

This shows that $\mathbf{f}_1$ can be expressed as a linear combination of $\mathbf{f}_2$ and $\mathbf{f}_3$. Hence, the given functions are linearly dependent.

Exercise 2.4

1. Which of the following sets of vectors in R^3 are linearly dependent?

(i) (1, 2, –1), (3, 2, 5)

(ii) (4, –6, 2), (2, –3, 1)

(iii) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

(iv) (1, –1, 1), (2, 1, 1), (3, 0, 2)

(v) (–2, 0, 1), (8, –1, 3), (6, –1, 1), (3, 2, 5)

[**Ans. :** (ii), (iv), (v)]

2. Which of the following sets of vectors in R^4 are linearly dependent?

   (i) (1, 2, –1, 0), (1, 3, 1, 2), (4, 2, 1, 0), (6, 1, 0, 1)

   (ii) (2, –1, 3, 2), (1, 3, 4, 2), (3, –5, 2, 2)

   (iii) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)

   (iv) (1, 0, 2, 1), (3, 1, 2, 1), (4, 6, 2, 4), (–6, 0, –3, 0)

[**Ans. :** (ii), (iv)]

3. Which of the following sets of vectors in P_2 are linearly dependent?

   (i) $1+2x+4x^2, 3+7x+10x^2$

   (ii) $3+x+x^2, 2-x+5x^2, 4-3x^2$

   (iii) $1+2x+3x^2, 3-2x+x^2, 1-6x-5x^2$

   (iv) $1+x+4x^2, x+4x^2, 1-2x-3x^2, 5-x+6x^2$

[**Ans. :** (iii), (iv)]

4. Show that $S = \{1-x-x^3, -2+3x+x^2+2x^3, 1+x^2+5x^3\}$ is linearly independent in P_3.

5. Which of the following sets of vectors in M_{22} are linearly dependent?

   (i) $\begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}$

   (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

   (iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

   (iv) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -6 & 0 \\ -3 & 0 \end{bmatrix}$

[**Ans. :** (ii), (iv)]

6. Show that the following vectors form a linearly dependent set in respective vector spaces. Express each vector as a linear combination of the other two.

   (i) $\mathbf{v}_1 = (3, 1, -4)$, $\mathbf{v}_2 = (2, 2, -3)$, $\mathbf{v}_3 = (0, -4, 1)$ in R^3

   (ii) $\mathbf{v}_1 = (1, 0, 2, 1)$, $\mathbf{v}_2 = (3, 1, 2, 1)$, $\mathbf{v}_3 = (4, 6, 2, -4)$, $\mathbf{v}_4 = (-6, 0, -3, -4)$ in R^4

   (iii) $\mathbf{p}_1 = 2 + x + x^2$, $\mathbf{p}_2 = x + 2x^2$, $\mathbf{p}_3 = 2 + 2x + 3x^2$

   (iv) $A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 4 & 6 \\ 8 & 6 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$

$$\left[\begin{array}{l} \textbf{Ans.:} \text{ (i) } \mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 \\ \text{(ii) } \mathbf{v}_3 = -2\mathbf{v}_1 + 6\mathbf{v}_2 + 2\mathbf{v}_4 \\ \text{(iii) } \mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2 \\ \text{(iv) } A_3 = 3A_1 + A_2 + A_4 \end{array}\right]$$

7. For what values of λ are the vectors (–1, 0, –1), (2, 1, 2) and (1, 1, λ) in R^3 linearly dependent?

[**Ans. :** $\lambda = 1$]

8. If the following vectors in R^3 have their initial points at the origin then check if they lie on the same plane.

   (i) $\mathbf{v}_1 = (2, -2, 0)$, $\mathbf{v}_2 = (6, 1, 4)$, $\mathbf{v}_3 = (2, 0, -4)$

   (ii) $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (3, -2, 1)$, $\mathbf{v}_3 = (1, -6, -5)$

[**Ans. :** (i) no (ii) yes]

9. If the following vectors in R^3 have their initial points at the origin then check if they lie on the same line.

   (i) (–1, –2, –3), (3, 9, 0), (6, 0, –1)

   (ii) (–2, –1, 1), (6, 3, –3), (–4, –2, 2)

[**Ans. :** (i) no (ii) yes]

10. Which of the following sets of functions in $F(-\infty, \infty)$ are linearly independent?

   (i) 1, $\sin x$, $\sin 2x$ (ii) 1, x, e^x

   (iii) $\cos 2x$, $\sin^2 x$, $\cos^2 x$ (iv) 1, x, x^2

[**Ans. :** (i), (ii), (iv)]

2.8 BASIS

The set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a basis for V if

(i) S is linearly independent
(ii) S spans V

Note: Basis for a vector space is not unique.

Theorem 2.10: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V then every vector in V can be expressed as a linear combination of the vectors in S in exactly one way.

Examples on Standard or Natural Basis

Example 1: Show that the vectors $\mathbf{e}_1 = \mathbf{i} = (1, 0, 0)$, $\mathbf{e}_2 = \mathbf{j} = (0, 1, 0)$ and $\mathbf{e}_3 = \mathbf{k} = (0, 0, 1)$ form a basis for R^3.

Solution: Let $\mathbf{b} = (b_1, b_2, b_3)$ be an arbitrary vector in R^3 and can be expressed as a linear combination the of given vectors.

$$\mathbf{b} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3$$

$$(b_1, b_2, b_3) = k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1)$$

$$= (k_1, k_2, k_3)$$

Equating corresponding components,

$$k_1 = b_1,\ k_2 = b_2,\ k_3 = b_3$$

Since for each choice of b_1, b_2, b_3 some scalars k_1, k_2, k_3 exist, the given vectors span R^3.

Now consider,

$$k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{e}_3 = \mathbf{0}$$

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$$

$$(k_1, k_2, k_3) = (0, 0, 0)$$

Equating corresponding components,

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0$$

Thus, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent.

Hence, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ form a basis for R^3 and is known as standard or natural basis for R^3.

Note: In general, the set $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0), \ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)$ form a basis for R^n and is known as standard or natural basis for R^n.

Example 2: Show that the set $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n.

Solution: Each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

which is a linear combination of the vectors $1, x, x^2, \ldots, x^n$. Thus, the set S spans P_n.

Now consider,

$$k_1 + k_2 x + k_3 x^2 + \cdots + k_{n+1} x^n = 0$$

Equating corresponding components,

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0,\ \ldots,\ k_{n+1} = 0$$

Thus, the set S is linearly independent.

Hence, the set S is a basis for P_n and is known as a standard or natural basis for P_n.

Example 3: Show that the set $S = \{v_1, v_2, v_3, v_4\}$ is a basis for the vector space M_{22} where, $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: Let $\mathbf{b} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ be an arbitrary vector in M_{22} and can be expressed as the linear combination of the given vectors.

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

Equating corresponding components,

$$k_1 = b_1,\ k_2 = b_2,\ k_3 = b_3,\ k_4 = b_4$$

Since for each choice of b_1, b_2, b_3, b_4 some scalars k_1, k_2, k_3, k_4 exist, the set S spans M_{22}.

Now consider,

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$$

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding components,

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0,\ k_4 = 0$$

Thus, the set S is linearly independent.

Hence, the set S is a basis for M_{22} and is known as standard or natural basis for M_{22}.

Note: In general, the standard basis for M_{mn} consists of mn different matrices with single entry as 1 and remaining entries as 0.

Examples on Basis

Example 4: Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3, where $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$ and $\mathbf{v}_3 = (3, 3, 3)$

Solution: Let $\mathbf{b} = (b_1, b_2, b_3)$ be an arbitrary vector in R^3 and can be expressed as a linear combination of given vectors.

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \qquad ...(1)$$

$$\begin{aligned}(b_1, b_2, b_3) &= k_1(1, 0, 0) + k_2(2, 2, 0) + k_3(3, 3, 3)\\ &= (k_1 + 2k_2 + 3k_3, 2k_2 + 3k_3, 3k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}k_1 + 2k_2 + 3k_3 &= b_1\\ 2k_2 + 3k_3 &= b_2\\ 3k_3 &= b_3\end{aligned} \qquad ...(2)$$

Coefficient matrix, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 6 \neq 0$$

Since the determinant of the coefficient matrix obtained from equation (2) is non-zero, the set S spans V.

To prove that S is linearly independent, we need to show that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \qquad ...(3)$$

has only a trivial solution, i.e. $k_1 = k_2 = k_3 = 0$. Comparing equation (3) with equation (1), we observe that the coefficient matrix of the equations (2) and (3) is same.

Since determinant of the coefficient matrix of equation (3) is non-zero, the system has only trivial solution.

Hence, S is linearly independent and spans R^3 and so is a basis for R^3.

Note: 1. To show the set of vectors S to be a basis of a vector space V, it is sufficient to the prove that the determinant of the coefficient matrix obtained from equation (2) is non-zero.

2. If the determinant of the coefficient matrix is zero, S does not span V and hence S is not a basis of V.

Example 5: Determine whether the following set of vectors form a basis for R^3.
(i) (1, 1, 1), (1, 2, 3), (2, –1, 1)
(ii) (1, 1, 2), (1, 2, 5), (5, 3, 4)

Solution: Let $\mathbf{b} = \{b_1, b_2, b_3\}$ be an arbitrary vector in R^3 and can be expressed as a linear combination of given vectors.

(i)
$$\{b_1, b_2, b_3\} = k_1(1, 1, 1) + k_2(1, 2, 3) + k_3(2, -1, 1)$$
$$= (k_1 + k_2 + 2k_3,\ k_1 + 2k_2 - k_3,\ k_1 + 3k_2 + k_3)$$

$$\begin{aligned} k_1 + \ \ k_2 + 2k_3 &= b_1 \\ k_1 + 2k_2 - \ \ k_3 &= b_2 \\ k_1 + 3k_2 + \ \ k_3 &= b_3 \end{aligned}$$

Coefficient matrix,
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix}$$
$$= 1(2+3) - 1(1+1) + 2(3-2)$$
$$= 5 \neq 0$$

Hence, the given set of vectors forms a basis for R^3.

(ii)
$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 2, 5) + k_3(5, 3, 4)$$
$$= (k_1 + k_2 + 5k_3,\ k_1 + 2k_2 + 3k_3,\ 2k_1 + 5k_2 + 4k_3)$$

$$\begin{aligned} k_1 + \ \ k_2 + 5k_3 &= b_1 \\ k_1 + 2k_2 + 3k_3 &= b_2 \\ 2k_1 + 5k_2 + 4k_3 &= b_3 \end{aligned}$$

Coefficient matrix,
$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix}$$
$$= 1(8-15) - 1(4-6) + 5(5-4)$$
$$= 0$$

Hence, the given set of vectors does not form a basis for R^3.

Example 6: Determine whether the following set of vectors forms a basis for P_2.

(i) $-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2$

(ii) $1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x$

Solution: Let $\mathbf{b} = b_1 + b_2x + b_3x^2$ be an arbitrary polynomial in P_2 and can be written as a linear combination of the given vectors.

(i) $$b_1 + b_2 x + b_3 x^2 = k_1(-4 + x + 3x^2) + k_2(6 + 5x + 2x^2) + k_3(8 + 4x + x^2)$$
$$= (-4k_1 + 6k_2 + 8k_3) + (k_1 + 5k_2 + 4k_3)x + (3k_1 + 2k_2 + k_3)x^2$$

Equating corresponding coefficients,

$$-4k_1 + 6k_2 + 8k_3 = b_1$$
$$k_1 + 5k_2 + 4k_3 = b_2$$
$$3k_1 + 2k_2 + k_3 = b_3$$

Coefficient matrix, $$A = \begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$$
$$= -4(5-8) - 6(1-12) + 8(2-15)$$
$$= -26 \neq 0$$

Hence, the given set of vectors forms a basis for P_2.

(ii) $$b_1 + b_2x + b_3x^2 = k_1(1 - 3x + 2x^2) + k_2(1 + x + 4x^2) + k_3(1 - 7x)$$
$$= (k_1 + k_2 + k_3) + (-3k_1 + k_2 - 7k_3)x + (2k_1 + 4k_2)x^2$$

Equating corresponding components,

$$k_1 + k_2 + k_3 = b_1$$
$$-3k_1 + k_2 - 7k_3 = b_2$$
$$2k_1 + 4k_2 = b_3$$

Coefficient matrix, $$A = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix}$$
$$= 1(28) - 1(14) + 1(-12-2)$$
$$= 0$$

Hence, the given set of vectors does not form a basis for P_2.

Example 7: Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for M_{22} where $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$

Solution: Let $\mathbf{b} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ be an arbitrary vector in M_{22} and can be expressed as the linear combination of the given vectors.

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = k_1\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + k_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & 2k_1 - k_2 + 2k_3 \\ k_1 - k_2 + 3k_3 - k_4 & -2k_1 + k_3 + 2k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 \qquad\qquad\qquad &= b_1 \\ 2k_1 - k_2 + 2k_3 \qquad\quad &= b_2 \\ k_1 - k_2 + 3k_3 - \ k_4 &= b_3 \\ -2k_1 \qquad + \ k_3 + 2k_4 &= b_4 \end{aligned}$$

Coefficient matrix, $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= -1(6+1) - 2(-2) = -3 \neq 0$$

Hence, S is a basis for M_{22}.

Example 8: Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$. Show that (i) $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V. (ii) Find a basis for V.

Solution: (i) From trigonometry, we have

$$\cos^2 x - \sin^2 x = \cos 2x$$

i.e., $$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3 \qquad \text{...(1)}$$

This shows that $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Therefore, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

Hence, S is not a basis for V.

(ii) Since from equation (1), any one vector can be expressed as the linear combination of the remaining two, any two of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ will form a basis for V.

Example 9: Let V be the space spanned by $\mathbf{v}_1 = \sin x$, $\mathbf{v}_2 = \cos x$, $\mathbf{v}_3 = x$. Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for V.

Solution: It is given that S spans V. To prove S linearly independent, we need to show that the Wronskian, W of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is non-zero.

$$W = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1' & v_2' & v_3' \\ v_1'' & v_2'' & v_3'' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x & x \\ \cos x & -\sin x & 1 \\ -\sin x & -\cos x & 0 \end{vmatrix}$$

$$= \sin x(\cos x) - \cos x(\sin x) + x(-\cos^2 x - \sin^2 x)$$

$$= -x$$

This function is not zero for all values of x. This shows that S is linearly independent. Hence, S forms a basis for V.

Basis for the Subspace Span (S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set in a vector space V then S is a basis for the subspace span (S).

2.9 FINITE DIMENSIONAL VECTOR SPACE

A vector space V is called finite dimensional if the number of vectors in its basis are finite. Otherwise, V is called infinite dimensional.

Theorem 2.11: If basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a finite dimensional vector space V has n vectors then

(i) Every set in V having more than n vectors is linearly dependent
(ii) Every set in V having less than n vectors does not span V

Theorem 2.12: From the above theorem, we conclude that all the bases for a finite-dimensional vector space have the same number of vectors.

2.9.1 *Dimension*

The number of vectors in a basis of a non-zero finite dimensional vector space V is known as the dimension of V and is denoted by dim (V).

Note: Dimensions of some standard vector spaces can be found directly from their standard basis.

(i) $\dim(R^n) = n$
(ii) $\dim(P_n) = n + 1$
(iii) $\dim(M_{mn}) = mn$
(iv) $\dim\{\mathbf{0}\} = 0$ [$\because$ **0** is linearly dependent, vector space $\{\mathbf{0}\}$ has no basis.]

Theorem 2.13: If $\dim(V) = n$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set in V with exactly n vectors then S is a basis for V if either S is linearly independent or S spans V.

Theorem 2.14: Let S be a non-empty set of vectors in a vector space V.

(i) If S is a linearly independent set then $S \cup \{\mathbf{v}\}$ is also linearly independent if the vector **v** does not belong to the span (S).
(ii) If **v** is a vector in S that can be expressed as a linear combination of other vectors in S then

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

Theorem 2.15: If W is a subspace of a finite dimensional vector space V then

(i) W is finite dimensional and $\dim(W) \le \dim(V)$; if $\dim(W) = \dim(V)$ then $W = V$.
(ii) Every basis for W is part of a basis for V.

2.10 BASIS AND DIMENSION FOR SOLUTION SPACE OF THE HOMOGENEOUS SYSTEMS

Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of m equations in n unknowns. The basis and dimension for the solution space of this system can be found as follows:

1. Solve the homogeneous system using Gaussian elimination method. If the system has only a trivial solution then the solution space is $\{\mathbf{0}\}$, which has no basis and hence the dimension of the solution space is zero.
2. If the solution vector **x** contains arbitrary constants (parameters) $t_1, t_2, \ldots, t_p$, express **x** as a linear combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ with $t_1, t_2, \ldots, t_p$ as coefficients.

$$\text{i.e. } \mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_p\mathbf{x}_p$$

3. The set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ form a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ and hence the dimension of the solution space is p.

Note: If the row echelon form has r non-zero rows then dimension of the solution space is $p = n - r$ where n represents the number of unknowns.

Example 1: Determine the dimension and a basis for the solution space of the system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ -2x_1 - 2x_2 + 4x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr:r} 1 & 1 & -2 & 0 \\ -2 & -2 & 4 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 2R_1,\ R_3 + R_1$$

$$\sim \left[\begin{array}{rrr:r} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + x_2 - 2x_3 = 0$$

Solving for the leading variables,

$$x_1 = -x_2 + 2x_3$$

Assigning the free variables x_2 and x_3 arbitrary values t_1 and t_2 respectively, $x_1 = -t_1 + 2t_2, x_2 = t_1, x_3 = t_2$ is the solution of the system.

The solution vector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 + 2t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

$$= t_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

Hence, $$\text{Basis} = \{\mathbf{x}_1, \mathbf{x}_2\} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\text{Dimension} = 2$$

Example 2: Find the dimension and a basis for the solution space of the system

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{3}\right)R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 - 5R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$\left(-\frac{3}{8}\right)R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 = 0$$

$$x_2 + \frac{1}{4}x_3 + x_4 = 0$$

Solving for the leading variables,

$$x_1 = -\frac{1}{3}x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4$$

$$x_2 = -\frac{1}{4}x_3 - x_4$$

Assigning the free variables x_3 and x_4 arbitrary values t_1 and t_2 respectively.

$$\begin{aligned} x_2 &= -\frac{1}{4}t_1 - t_2 \\ x_1 &= -\frac{1}{3}\left(-\frac{1}{4}t_1 - t_2\right) - \frac{1}{3}t_1 - \frac{1}{3}t_2 \\ &= -\frac{1}{4}t_1 \end{aligned}$$

Hence, $x_1 = -\frac{1}{4}t_1,\ x_2 = -\frac{1}{4}t_1 - t_2,\ x_3 = t_1,\ x_4 = t_2$ is the solution of the system.

The solution vector is

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{4}t_1 \\ -\frac{1}{4}t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} \\ &= t_1\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \\ &= t_1\mathbf{x}_1 + t_2\mathbf{x}_2 \end{aligned}$$

Hence,

$$\begin{aligned} \text{Basis} &= \{\mathbf{x}_1, \mathbf{x}_2\} \\ &= \left\{\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\} \end{aligned}$$

$$\text{Dimension} = 2$$

Example 3: Find the dimension and a basis for the solution space of the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 2 & 5 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right]$$

$$R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 26 & 0 \end{array}\right]$$

$$\left(\frac{1}{26}\right) R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_2 + 7x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

Hence, $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ is the solution of the system.
The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \{\mathbf{0}\}$$

Hence, the solution space has no basis and dimension = 0.

Example 4: Determine the dimension and basis for the following subspaces of R^3 and R^4.

(i) the plane $3x - 2y + 5z = 0$
(ii) the line $x = 2t, y = -t, z = 4t$
(iii) all vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$

Solution: (i) $3x - 2y + 5z = 0$
Solving for x,

$$x = \frac{2}{3}y - \frac{5}{3}z$$

Assigning y and z arbitrary values t_1 and t_2 respectively,

$$x = \frac{2}{3}t_1 - \frac{5}{3}t_2$$

Any vector **x** lying on the plane is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t_1 - \frac{5}{3}t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

Thus, $\mathbf{x}_1$ and $\mathbf{x}_2$ span the given plane. Also, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent as they are not scalar multiples of each other.

Hence, $$\text{Basis} = \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Dimension} = 2$$

(ii) Any vector **x** lying on the line $x = 2t, y = -t, z = 4t$ is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = t\mathbf{x}_1$$

Thus, $\mathbf{x}_1$ spans the given line and is also linearly independent as it is a non-zero vector.

$$\text{Hence, Basis} = \{\mathbf{x}_1\} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Dimension = 1

(iii) Let

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ a-b \\ a+b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= a\mathbf{x}_1 + b\mathbf{x}_2$$

Thus, $\mathbf{x}_1$ and $\mathbf{x}_2$ span the given set of vectors. Also, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent as one is not the scalar multiple of another.

Hence,

$$\text{Basis} = \{\mathbf{x}_1, \mathbf{x}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Dimension = 2

Example 5: Find a basis and dimension of

$$W = \{(a_1, a_2, a_3, a_4) \in R^4 \mid a_1 + a_2 = 0, a_2 + a_3 = 0, a_3 + a_4 = 0\}$$

Solution:

$$a_1 + a_2 = 0 \;\Rightarrow\; a_2 = -a_1$$

$$a_2 + a_3 = 0 \;\Rightarrow\; a_3 = -a_2 = a_1$$

$$a_3 + a_4 = 0 \;\Rightarrow\; a_4 = -a_3 = -a_1$$

Any vector $\mathbf{x}$ in W is

$$\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_1 \\ a_1 \\ -a_1 \end{bmatrix}$$

$$= a_1\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= a_1\mathbf{x}_1$$

Thus, $\mathbf{x}_1$ spans W and is also linearly independent as it is a non-zero vector.

$$\text{Hence, Basis} = \{\mathbf{x}_1\} = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$\text{Dimension} = 1$$

Example 6: Find the dimension and a basis for the following subspaces of P_2 and P_3.

(i) all polynomials of the form $a_0 + a_1 x + a_2x^2 + a_3 x^3$, where $a_0 = 0$
(ii) all polynomials of the form $ax^3 + bx^2 + cx + d$, where $b = 3a - 5d$ and $c = d + 4a$

Solution: (i) Let $\mathbf{p}$ be any polynomial in the given subspace of P_2.

$$\begin{aligned}\mathbf{p} &= a_0 + a_1x + a_2x^2 + a_3x^3 \\ &= a_1x + a_2x^2 + a_3x^3 \quad [\because a_0 = 0]\end{aligned}$$

Thus, the vectors x, x^2 and x^3 span the given subspace of P_2. Also, x, x^2 and x^3 are linearly independent which can be verified as follows:

Let $k_1x + k_2x^2 + k_3x^3 = 0$

Equating corresponding coefficients,

$$k_1 = k_2 = k_3 = 0.$$

Thus, x, x^2 and x^3 are linearly independent.

Hence,

$$\text{Basis} = \{x, x^2, x^3\}$$
$$\text{Dimension} = 3$$

(ii) Let $\mathbf{p}$ be any polynomial in the given subspace of P_3.

$$\begin{aligned}\mathbf{p} &= ax^3 + bx^2 + cx + d \\ &= ax^3 + (3a - 5d)x^2 + (d + 4a)x + d \\ &= a(x^3 + 3x^2 + 4x) + d(-5x^2 + x + 1) \\ &= a\mathbf{p}_1 + d\mathbf{p}_2\end{aligned}$$

Thus, $\mathbf{p}_1$ and $\mathbf{p}_2$ span the given subspace of P_3. Also, $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent as one is not the scalar multiple of another.

Hence,

$$\begin{aligned}\text{Basis} &= \{\mathbf{p}_1, \mathbf{p}_2\} \\ &= \left\{(x^3 + 3x^2 + 4x), (-5x^2 + x + 1)\right\}\end{aligned}$$

$$\text{Dimension} = 2$$

2.11 REDUCTION AND EXTENSION TO BASIS

Theorem 2.16: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of non-zero vectors in a vector space V.

(i) If S spans V then S can be reduced to a basis for V by removing some vectors from S and dim $(V) < n$.
(ii) If S is linearly independent then S can be extended to a basis for V by adding some vectors into S and dim $(V) > n$.

2.11.1 *Reduction to Basis*

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of non-zero vectors in a real vector space V.
If V = span S and dim $(V) < n$ then S can be reduced to a basis for V as follows:

1. Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$ ...(2.7)
2. Construct the augmented matrix of the homogeneous system obtained from Eq. (2.7). Reduce the homogeneous system to row echelon form.
3. The vectors corresponding to the columns containing the leading 1's form a basis for V.

Note: By changing the order of vectors in S, other possible bases can be found.

2.11.2 *Extension to Basis*

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a linearly independent set of vectors in a real vector space V. If dim $(V) = n > m$ then S can be extended to a basis for V as follows:

1. Form the set $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n.
2. Follow all the steps (1 to 3) of 2.11.1.

Note: By changing the order of standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in S', other possible bases can be found.

Example 1: Reduce $S = \{(1, 0, 0), (0, 1, -1), (0, 4, -3), (0, 2, 0)\}$ to obtain a basis for W = span S

Solution: Consider,

$$k_1(1, 0, 0) + k_2(0, 1, -1) + k_3(0, 4, -3) + k_4(0, 2, 0) = (0, 0, 0)$$
$$(k_1, k_2 + 4k_3 + 2k_4, -k_2 - 3k_3) = (0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} k_1 \qquad\qquad &= 0 \\ k_2 + 4k_3 + 2k_4 &= 0 \\ -k_2 - 3k_3 \qquad &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -1 & -3 & 0 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_3 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

The leading 1's appear in columns 1, 2 and 3.
Hence, Basis = {(1, 0, 0), (0, 1, –1), (0, 4, –3)}

Example 2: Reduce $S = \{1 - 2x + x^2 + x^3,\ 1 + x^2,\ -2x + x^3,\ 3 - 4x + 3x^2 + 2x^3\}$ to obtain a basis for the subspace of P_3, W = span S. What is the dimension of W?

Solution: Consider,

$$k_1(1-2x+x^2+x^3)+k_2(1+x^2)+k_3(-2x+x^3)+k_4(3-4x+3x^2+2x^3)=0$$

$$(k_1+k_2+3k_4)+(-2k_1-2k_3-4k_4)x+(k_1+k_2+3k_4)x^2+(k_1+k_3+2k_4)=0$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 + k_2 + \quad\quad 3k_4 &= 0 \\ -2k_1 - \quad 2k_3 - 4k_4 &= 0 \\ k_1 + k_2 + \quad\quad 3k_4 &= 0 \\ k_1 + \quad k_3 + 2k_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 0 \\ -2 & 0 & -2 & -4 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 2R_1,\ R_3 - R_1,\ R_4 - R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$R_4 + R_3$$

$$\sim\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading 1's appear in columns 1 and 2.

Hence, Basis = $\{1 - 2x + x^2 + x^3,\ 1 + x^2\}$

Dimension = 2

Example 3: Find a basis for the subspace of P_2 spanned by the vectors $1 + x$, x^2, $-2 + 2x^2$, $-3x$.

Solution: Consider,

$$k_1(1+x) + k_2x^2 + k_3(-2+2x^2) + k_4(-3x) = 0$$

$$(k_1 - 2k_3) + (k_1 - 3k_4)x + (k_2 + 2k_3)x^2 = 0$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 \quad - 2k_3 \quad\quad &= 0 \\ k_1 \quad\quad\quad - 3k_4 &= 0 \\ k_2 + 2k_3 \quad\quad &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$$

$$R_{23}$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_3$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

The leading 1's appear in columns 1, 2 and 3.
Hence, Basis = $\{1 + x, x^2, -2 + 2x^2\}$

Example 4: Reduce $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right\}$ to obtain a basis for the subspace of M_{22}, $W = \text{span } S$. What is the dimension of W.

Solution: Consider,

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + k_4\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + k_3 - k_4 & k_2 + k_3 + k_4 \\ k_2 + k_3 + k_4 & k_1 + k_3 - k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 \quad + k_3 - k_4 &= 0 \\ k_2 + k_3 + k_4 &= 0 \\ k_2 + k_3 + k_4 &= 0 \\ k_1 \quad + k_3 - k_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_4 - R_1$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 - R_2}{\sim}\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading 1's appear in columns 1 and 2.

Hence, $\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

Example 5: Find standard basis vector/vectors that can be added to the following set of vectors to produce a basis for R^3 and R^4.

(i) $\mathbf{v_1} = (-1, 2, 3), \mathbf{v_2} = (1, -2, -2)$

(ii) $\mathbf{v_1} = (1, -4, 2, -3), \mathbf{v_2} = (-3, 8, -4, 6)$

Solution: (i) Form a set $S = \{ \mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ are the standard basis vectors of R^3.

Since the set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ spans R^3, the set S also spans R^3.

Consider,
$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$$

$$k_1(-1, 2, 3) + k_2(1, -2, -2) + k_3(1, 0, 0) + k_4(0, 1, 0) + k_5(0, 0, 1) = (0, 0, 0)$$
$$(-k_1 + k_2 + k_3,\ 2k_1 - 2k_2 + k_4,\ 3k_1 - 2k_2 + k_5) = (0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} -k_1 + k_2 + k_3 \qquad &= 0 \\ 2k_1 - 2k_2 + \quad k_4 \quad &= 0 \\ 3k_1 - 2k_2 + \qquad k_5 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{(-1)R_1}{\sim}\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \end{array}\right]$$

$$R_{23}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

The leading 1's appear in columns 1, 2 and 3.
Hence, Basis = $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$

Note: By changing the order of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in S, other possible bases can be found.

(ii) Form a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ where $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$ and $\mathbf{e}_4 = (0, 0, 0, 1)$ are the standard basis vectors of R^4.

Since the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ span R^4, the set S also spans R^4.

Consider,

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 + k_6\mathbf{e}_4 = \mathbf{0}$$

$$k_1(1, -4, 2, -3) + k_2(-3, 8, -4, 6) + k_3(1, 0, 0, 0) + k_4(0, 1, 0, 0) + k_5(0, 0, 1, 0) + k_6(0, 0, 0, 1) = (0, 0, 0, 0)$$

$$(k_1 - 3k_2 + k_3, -4k_1 + 8k_2 + k_4, 2k_1 - 4k_2 + k_5, -3k_1 + 6k_2 + k_6) = (0, 0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} k_1 - 3k_2 + k_3 \qquad\qquad &= 0 \\ -4k_1 + 8k_2 + \quad k_4 \qquad &= 0 \\ 2k_1 - 4k_2 + \qquad k_5 \quad &= 0 \\ -3k_1 + 6k_2 + \qquad\quad k_6 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrrrrr|r} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 4R_1,\ R_3 - 2R_1,\ R_4 + 3R_1$$

$$\sim \left[\begin{array}{rrrrrr|r} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left(-\frac{1}{4}\right)R_2,\ \left(\frac{1}{2}\right)R_3,\ \left(-\frac{1}{3}\right)R_4$$

$$\sim \left[\begin{array}{rrrrrr|r} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - R_2$$

$$\sim \left[\begin{array}{rrrrrr|r} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{3} & 0 \end{array}\right]$$

$$R_4 - R_3$$

$$\sim \left[\begin{array}{rrrrrr|r} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{3} & 0 \end{array}\right]$$

$(4)R_3, (-2)R_4$

$$\sim \left[\begin{array}{cccccc|c} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right]$$

The leading 1's appear in columns 1, 2, 4 and 5.
Hence, Basis = $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2, \mathbf{e}_3\}$

Note: By changing the order of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ in S, other possible bases can be found.

Exercise 2.5

1. Determine whether the following set of vectors form a basis for the indicated vector spaces:

(i) (1, 3), (1, –1) for R^2

(ii) (1, –1), (2, 3), (–1, 5) for R^2

(iii) (1, 2, 1), (2, 1, 0), (1, –1, 2) for R^3

(iv) (2, –3, 1), (4, 1, 1), (0, –7, 1) for R^3

(v) (0, 0, 1, 1), (–1, 1, 1, 2), (1, 1, 0, 0), (2, 1, 2, 1) for R^4

(vi) (1, –1, 0, 2), (1, –1, 2, 0), (–3, 1, –1, 2) for R^4

[**Ans.:** (i) yes (ii) no (iii) yes (iv) no (v) yes (vi) no]

2. Determine whether the following set of vectors form a basis for P_2 and P_3:

(i) $1 - x^2$, $1+2x + x^2$, $-3x + 2x^2$

(ii) $1 + x + 2x^2$, $2 + 2x + 4x^2$, $-3 + 2x - x^2$

(iii) $1 + x + x^2 + x^3$, $3 + x + 2x^2 + x^3$, $2 + 3x + x^2 + 2x^3$, $2 + 2x + x^2 + x^3$

(iv) $3x + 2x^2 + x^3$, $1 + 2x^3$, $4 + 6x + 8x^2 + 6x^3$, $1 + x + 2x^2 + x^3$

[**Ans.:** (i) yes (ii) no (iii) yes (iv) no]

3. Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for M_{22} where

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$$

4. Determine the dimension and a basis for the solution space of the systems:

(i)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + 2x_2 + x_3 - 3x_4 &= 0 \\ 2x_1 + 4x_2 + 4x_3 - x_4 &= 0 \\ 3x_1 + 6x_2 + 7x_3 + x_4 &= 0 \end{aligned}$$

(iii)
$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 &= 0 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 &= 0 \\ x_3 - x_4 - x_5 &= 0 \end{aligned}$$

$$\left[\textbf{Ans.: } \text{(i) Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{Dim} = 2 \right]$$

[(ii) Basis $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{11}{2} \\ 0 \\ -\frac{5}{2} \\ 1 \end{bmatrix} \right\}$, Dim = 2

(iii) Basis $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, Dim = 2]

5. Determine the dimension and basis for the following subspaces of R^3 and R^4:

(i) the plane $2x - 3y + 4z = 0$

(ii) the line $x = -t$, $y = 2t$, $z = -3t$

(iii) all vectors of the form (a, b, c, d), where $d = a + b$

(iv) all vectors of the form $a + bx + cx^2$, where $a = 2c - 3b$

[**Ans.:**

(i) Basis $= \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$, Dim = 2

(ii) Basis $= \left\{ \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \right\}$, Dim = 1

(iii) Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, Dim = 3

(iv) Basis $= \left\{ x^2 + 2, x - 3 \right\}$, Dim = 2]

6. Reduce the following sets to obtain a basis for the subspace of the indicated vector space:

(i) $S = \{(1, -3, 2), (2, 4, 1), (3, 1, 3), (1, 1, 1)\}$ for R^3

(ii) $S = \{1 + x + x^2, 1 + 2x + 3x^2, 2 - x + x^2, 4 + 3x - 2x^2\}$

(iii) $S = \left\{ \begin{bmatrix} 1 & -2 \\ 5 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}, \begin{bmatrix} 3 & 8 \\ -3 & -5 \end{bmatrix} \right\}$

[**Ans.:** (i) $\{(1, -3, 2), (2, 4, 1), (1, 1, 1)\}$

(ii) $\{1 + x + x^2, 1 + 2x + 3x^2, 2 - x + x^2, 4 + 3x - 2x^2\}$

(iii) $\left\{ \begin{bmatrix} 1 & -2 \\ 5 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \right\}$]

7. Find a basis for the subspace of R^4 spanned by the following vectors:

$\mathbf{v}_1 = (1, 1, 0, -1)$

$\mathbf{v}_2 = (0, 1, 2, 1)$

$\mathbf{v}_3 = (1, 0, 1, -1)$

$\mathbf{v}_4 = (1, 1, -6, -3)$

$\mathbf{v}_5 = (-1, -5, 1, 0)$

[**Ans.:** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$]

8. Find a basis and dimension for the subspace of P_3 spanned by the following polynomials:

$\mathbf{p}_1 = 1 - 2x + x^2 + x^3$

$\mathbf{p}_2 = 1 + x^2$

$\mathbf{p}_3 = -2x + x^3$

$\mathbf{p}_4 = 3 - 4x + 3x^2 + 2x^3$

[**Ans.:** Basis = $\{\mathbf{p}_1, \mathbf{p}_2\}$, Dim = 2]

9. Find standard basis vector/vectors that can be added to the following set of vectors to produce a basis for R^3 and R^4.

(i) $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

(ii) $\mathbf{v}_1 = (1, -2, 5, -3)$, $\mathbf{v}_2 = (2, 3, 1, -4)$

[**Ans.:** one possible basis (i) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4\}$]

2.12 COORDINATE VECTOR RELATIVE TO A BASIS

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V then any vector $\mathbf{v}$ in V can be expressed as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

The scalars $k_1, k_2, \dots, k_n$ are called the coordinates of $\mathbf{v}$ relative to the basis S and the vector $(k_1, k_2, \dots, k_n)$ in R^n is called the coordinate vector of $\mathbf{v}$ relative to S. This vector is denoted by

$$(\mathbf{v})_S = (k_1, k_2, \dots, k_n)$$

Note: The coordinate vectors depend on the order in which the basis vectors are written. If the order of the basis vectors is changed, a corresponding change of order occurs in the coordinate vectors.

Example 1: Find the coordinate vector of v relative to the basis S.

(i) $\mathbf{v} = (1, 1)$; $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (2, -4)$, $\mathbf{v}_2 = (3, 8)$

(ii) $\mathbf{v} = (5, -12, 3)$; $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

Solution: (i) Let

$$\begin{aligned}(\mathbf{v})_S &= (k_1, k_2)\\ \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2\\ (1, 1) &= k_1(2, -4) + k_2(3, 8)\\ &= (2k_1 + 3k_2, -4k_1 + 8k_2)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}2k_1 + 3k_2 &= 1\\ -4k_1 + 8k_2 &= 1\end{aligned}$$

Solving these equations,

$$k_1 = \frac{5}{28}, k_2 = \frac{3}{14}$$

Hence, $(\mathbf{v})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$

(ii) Let

$$\begin{aligned}(\mathbf{v})_S &= (k_1, k_2, k_3)\\ \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3\\ (5, -12, 3) &= k_1(1, 2, 3) + k_2(-4, 5, 6) + k_3(7, -8, 9)\\ &= (k_1 - 4k_2 + 7k_3, 2k_1 + 5k_2 - 8k_3, 3k_1 + 6k_2 + 9k_3)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 - 4k_2 + 7k_3 &= 5 \\ 2k_1 + 5k_2 - 8k_3 &= -12 \\ 3k_1 + 6k_2 + 9k_3 &= 3 \end{aligned}$$

Solving these equations,

$$k_1 = -2,\ k_2 = 0,\ k_3 = 1$$

Hence, $(\mathbf{v})_S = (-2, 0, 1)$

Example 2: Find the coordinate vector of $\mathbf{p} = 2 - x + x^2$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$.

Solution: Let

$$\begin{aligned} (\mathbf{p})_S &= (k_1, k_2, k_3) \\ \mathbf{p} &= k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 \\ 2 - x + x^2 &= k_1(1+x) + k_2(1+x^2) + k_3(x+x^2) \\ &= (k_1 + k_2) + (k_1 + k_3)x + (k_2 + k_3)x^2 \end{aligned}$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 + k_2 \qquad &= 2 \\ k_1 + \qquad k_3 &= -1 \\ k_2 + k_3 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = 0,\ k_2 = 2,\ k_3 = -1$$

Hence, $(\mathbf{p})_S = (0, 2, -1)$

Example 3: Find the coordinate vector of A relative to the basis $S = \{A_1, A_2, A_3, A_4\}$, where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution: Let $(A)_S = (k_1, k_2, k_3, k_4)$

$$A = k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = k_1\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 + k_2 & k_1 + k_2 \\ k_3 & k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} -k_1 + k_2 \qquad &= 2 \\ k_1 + k_2 \qquad &= 0 \\ k_3 \quad &= -1 \\ k_4 &= 3 \end{aligned}$$

Solving these equations,

$$k_1 = -1, k_2 = 1, k_3 = -1, k_4 = 3$$

Hence, $(A)_S = (-1, 1, -1, 3)$

Example 4: The vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (3, 0, -1)$ form a basis of V. Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $S_2 = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1\}$ are different orderings of these vectors. Determine the vector $\mathbf{v}$ in V having following coordinate vectors.

(i) $(\mathbf{v})_{S_1} = (3, -1, 8)$

(ii) $(\mathbf{v})_{S_2} = (3, -1, 8)$

Solution: (i) $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $k_1 = 3$, $k_2 = -1$, $k_3 = 8$

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \\ &= 3(1, -1, 1) - 1(0, 1, 2) + 8(3, 0, -1) \\ &= (3 + 24, -3 - 1, 3 - 2 - 8) \\ \mathbf{v} &= (27, -4, -7) \end{aligned}$$

(ii) $S_2 = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1\}$ and $k_1 = 3$, $k_2 = -1$, $k_3 = 8$

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_3 + k_2\mathbf{v}_2 + k_3\mathbf{v}_1 \\ &= 3(3, 0, -1) - 1(0, 1, 2) + 8(1, -1, 1) \\ &= (9 + 8, -1 - 8, -3 - 2 + 8) \\ \mathbf{v} &= (17, -9, 3) \end{aligned}$$

In this example we observe that on changing the order of vectors in the basis, we get two different vectors in V corresponding to same coordinate vectors.

Coordinate Matrices

If $(\mathbf{v})_S = (k_1, k_2, \ldots, k_n)$ is the coordinate vector of $\mathbf{v}$ relative to the basis S then the coordinate matrix of $\mathbf{v}$ relative to the basis S is defined as

$$[\mathbf{v}]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

2.13 CHANGE OF BASIS

The basis for a vector space V is not unique. Sometimes it is required to change the basis for a vector space. To change the basis for V, it is necessary to know the relationship between the coordinates (coordinate matrices) of a vector $\mathbf{v}$ in V relative to both the bases.

Relationship between the Coordinate Matrices Relative to Different Bases

Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be bases of vector space V. If $\mathbf{v}$ is any vector in V then

$$\mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_n\mathbf{w}_n$$

$$[\mathbf{v}]_{S_2} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

Now,
$$\begin{aligned}[\mathbf{v}]_{S_1} &= \left[k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_n\mathbf{w}_n\right]_{S_1} \\ &= [k_1\mathbf{w}_1]_{S_1} + [k_2\mathbf{w}_2]_{S_1} + \cdots + [k_n\mathbf{w}_n]_{S_1} \\ &= k_1[\mathbf{w}_1]_{S_1} + k_2[\mathbf{w}_2]_{S_1} + \cdots + k_n[\mathbf{w}_n]_{S_1}\end{aligned}$$

Let the coordinate vector of $\mathbf{w}_i$ relative to S_1 be

$$[\mathbf{w}_i]_{S_1} = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$$

$$[\mathbf{v}]_{S_1} = k_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + k_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + k_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} k_1a_{11} + k_2a_{12} + \cdots + k_na_{1n} \\ k_1a_{21} + k_2a_{22} + \cdots + k_na_{2n} \\ \vdots \qquad\qquad \vdots \\ k_1a_{n1} + k_2a_{n2} + \cdots + k_na_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$[\mathbf{v}]_{S_1} = P[\mathbf{v}]_{S_2}$$

The matrix P is called the transition matrix from S_2 to S_1. The columns of P are the coordinate matrices of the new basis vectors relative to the old basis i.e.

$$P = \left[[\mathbf{w}_1]_{S_1} [\mathbf{w}_2]_{S_1} \cdots [\mathbf{w}_n]_{S_1}\right]$$

Theorem 2.17: If P is the transition matrix from a basis S_2 to a basis S_1 for a finite dimensional vector space V then

(i) P is invertible
(ii) P^{-1} is the transition matrix from S_1 to S_2
(iii) For every vector $\mathbf{v}$ in the vector space V, we have

$$[\mathbf{v}]_{S_1} = P[\mathbf{v}]_{S_2}$$

and $$[\mathbf{v}]_{S_2} = P^{-1}[\mathbf{v}]_{S_1}$$

Example 1: Find $\mathbf{v}$ if the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (1, 2, 3)$, $\mathbf{v}_3 = (1, 1, 0)$ and $[\mathbf{v}]_S = \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}$

Solution: $(\mathbf{v})_S = (6, -1, 4)$

$$\begin{aligned} \mathbf{v} &= 6\mathbf{v}_1 - \mathbf{v}_2 + 4\mathbf{v}_3 \\ &= 6(2, -1, 3) - (1, 2, 3) + 4(1, 1, 0) \\ &= (12 - 1 + 4, -6 - 2 + 4, 18 - 3) \\ \mathbf{v} &= (15, -4, 15) \end{aligned}$$

Example 2: Find $\mathbf{p}$ if the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$ and $[\mathbf{p}]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

Solution: $(\mathbf{p})_S = (3, 0, 4)$

$$\begin{aligned} \mathbf{p} &= 3\mathbf{p}_1 + 0\mathbf{p}_2 + 4\mathbf{p}_3 \\ &= 3 + 4x^2 \end{aligned}$$

Example 3: Find A if the basis $S = \{A_1, A_2, A_3, A_4\}$, where

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [A]_S = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$$

Solution: $(A)_S = (-8, 7, 6, 3)$

$$\begin{aligned} A &= -8A_1 + 7A_2 + 6A_3 + 3A_4 \\ &= -8\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8+7 & -8+7 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 15 & -1 \\ 6 & 3 \end{bmatrix} \end{aligned}$$

Example 4: Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (0, 6)$, $\mathbf{v}_1 = (2, 1)$, $\mathbf{v}_2 = (-1, 4)$

(i) Find the transition matrix from S_2 to S_1.
(ii) Find the transition matrix from S_1 to S_2.

Solution: (i) The transition matrix P from S_2 to S_1 is

$$P = \left[[\mathbf{v}_1]_{S_1} [\mathbf{v}_2]_{S_1} \right]$$

Let

$$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_1 &= k_1 \boldsymbol{u}_1 + k_2 \boldsymbol{u}_2 \\ (2, 1) &= k_1(1, -1) + k_2(0, 6) \\ &= (k_1, -k_1 + 6k_2) \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 &= 2 \\ -k_1 + 6k_2 &= 1 \\ k_2 &= \frac{1}{2} \end{aligned}$$

$$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

Let

$$[\mathbf{v}_2]_{S_1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{aligned}\mathbf{v}_2 &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2\\(-1, 4) &= c_1(1, -1) + c_2(0, 6)\\&= (c_1, -c_1 + 6c_2)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}c_1 &= -1\\-c_1 + 6c_2 &= 4\\c_2 &= \frac{1}{2}\end{aligned}$$

$$[\mathbf{v}_2]_{S_1} = \begin{bmatrix}-1\\ \frac{1}{2}\end{bmatrix}$$

Hence, $$P = \begin{bmatrix}2 & -1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

(ii) The transition matrix from S_1 to S_2 is P^{-1}.

$$P^{-1} = \frac{1}{\det(P)}\text{adj } P$$

$$\text{adj } P = \begin{bmatrix}\frac{1}{2} & 1\\ -\frac{1}{2} & 2\end{bmatrix}$$

$$\det(P) = \frac{3}{2}$$

$$P^{-1} = \frac{2}{3}\begin{bmatrix}\frac{1}{2} & 1\\ -\frac{1}{2} & 2\end{bmatrix}$$

Example 5: Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{u}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-3\\4\end{bmatrix}$

(i) Find the transition matrix from S_2 to S_1.
(ii) Find the transition matrix from S_1 to S_2.
(iii) Find $[\mathbf{w}]_{S_1}$ where $\mathbf{w} = \begin{bmatrix}3\\-5\end{bmatrix}$.
(iv) Find $[\mathbf{w}]_{S_2}$ using (iii).

Solution: (i) The transition matrix P from S_2 to S_1 is

$$P = \left[[\mathbf{v}_1]_{S_1} \; [\mathbf{v}_2]_{S_1} \right]$$

Let $$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\mathbf{v}_1 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Equating corresponding components,

$$k_1 = 2, k_2 = 1$$

$$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $$[\mathbf{v}_2]_{S_1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbf{v}_2 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Equating corresponding components,

$$c_1 = -3, c_2 = 4$$

$$[\mathbf{v}_2]_{S_1} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Hence, $$P = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

(ii) The transition matrix from S_1 to S_2 is P^{-1}.

$$P^{-1} = \frac{1}{\det(P)} \operatorname{adj} P$$

$$\operatorname{adj} P = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\det(P) = 11$$

$$P^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

(iii) Let
$$[\mathbf{w}]_{S_1} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\mathbf{w} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Equating corresponding components,

$$k_1 = 3, k_2 = -5$$

$$[\mathbf{w}]_{S_1} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

In a vector space V if P is the transition matrix from S_2 to S_1 then for any vector $\mathbf{w}$ in V

$$[\mathbf{w}]_{S_1} = P[\mathbf{w}]_{S_2}$$

or
$$[\mathbf{w}]_{S_2} = P^{-1}[\mathbf{w}]_{S_1}$$

$$= \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix}$$

Hence,
$$[\mathbf{w}]_{S_2} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$$

Example 6: Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for P_2, where $\mathbf{u}_1 = 1 + x^2$, $\mathbf{u}_2 = -2 + x$, $\mathbf{u}_3 = 3 + x$, $\mathbf{v}_1 = x + 2x^2$, $\mathbf{v}_2 = 3 + x^2$, $\mathbf{v}_3 = x$.

(i) Find the transition matrix from S_1 to S_2

(ii) Find $[\mathbf{w}]_{S_2}$ using transition matrix, where

$$\mathbf{w} = 5 + 4x - x^2$$

Solution: The transition matrix P from S_1 to S_2 is

$$P = \left[[\mathbf{u}_1]_{S_2} \; [\mathbf{u}_2]_{S_2} \; [\mathbf{u}_3]_{S_2} \right]$$

Let
$$[\mathbf{u}_1]_{S_1} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_1 &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \\ 1 + x^2 &= k_1(x + 2x^2) + k_2(3 + x^2) + k_3(x) \\ &= 3k_2 + (k_1 + k_3)x + (2k_1 + k_2)x^2 \end{aligned}$$

Equating corresponding coefficients,

$$\begin{aligned} 3k_2 \qquad &= 1 \\ k_1 + \qquad k_3 &= 0 \\ 2k_1 + k_2 \qquad &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = \frac{1}{3}, k_2 = \frac{1}{3}, k_3 = -\frac{1}{3}$$

$$[\mathbf{u}_1]_{S_2} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Let
$$[\mathbf{u}_2]_{S_2} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_2 &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 \\ -2 + x &= 3b_2 + (b_1 + b_3)x + (2b_1 + b_2)x^2 \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} 3b_2 \qquad &= -2 \\ b_1 + \qquad b_3 &= 1 \\ 2b_1 + b_2 \qquad &= 0 \end{aligned}$$

Solving these equations,

$$b_1 = \frac{1}{3}, b_2 = -\frac{2}{3}, b_3 = \frac{2}{3}$$

$$[\mathbf{u}_2]_{S_2} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

Let $$[\mathbf{u}_3]_{S_2} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{u}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$3 + x = 3c_2 + (c_1 + c_3)x + (2c_1 + c_2)x^2$$

Equating corresponding coefficients,

$$\begin{aligned} 3c_2 \quad &= 3 \\ c_1 + \quad c_3 &= 1 \\ 2c_1 + c_2 \quad &= 0 \end{aligned}$$

Solving these equations,

$$c_1 = -\frac{1}{2}, c_2 = 1, c_3 = \frac{3}{2}$$

$$[\mathbf{u}_3]_{S_2} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \end{bmatrix}$$

(ii) **w** is a vector in P_2 and P is the transition matrix from S_1 to S_2.

$$[\mathbf{w}]_{S_2} = P[\mathbf{w}]_{S_1}$$

Let $$[\mathbf{w}]_{S_1} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\mathbf{w} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3$$

$$5 + 4x - x^2 = k_1(1 + x^2) + k_2(-2 + x) + k_3(3 + x)$$

$$= (k_1 - 2k_2 + 3k_3) + (k_2 + k_3)x + k_1x^2$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 - 2k_2 + 3k_3 &= 5 \\ k_2 + k_3 &= 4 \\ k_1 &= -1 \end{aligned}$$

Solving these equations,

$$k_1 = -1, k_2 = \frac{6}{5}, k_3 = \frac{14}{5}$$

$$[\mathbf{w}]_{S_1} = \begin{bmatrix} -1 \\ \frac{6}{5} \\ \frac{14}{5} \end{bmatrix}$$

$$[\mathbf{w}]_{S_2} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{2}{3} & 1 \\ -\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 \\ \frac{6}{5} \\ \frac{14}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} + \frac{2}{5} - \frac{7}{5} \\ -\frac{1}{3} - \frac{4}{5} + \frac{14}{5} \\ \frac{1}{3} + \frac{4}{5} + \frac{21}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ \frac{16}{3} \end{bmatrix}$$

Example 7: Consider the basis $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$. If the transition matrix P from S_2 to S_1 is

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

determine S_2.

Solution: The transition matrix P from S_2 to S_1 is

$$P = \left[[\mathbf{v}_1]_{S_1} \, [\mathbf{v}_2]_{S_1} \, [\mathbf{v}_3]_{S_1} \right]$$

Given
$$P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Comparing both the matrices,

$$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 \\ &= (1, 0, 1) + 2(1, 1, 0) - (0, 0, 1) \\ &= (1 + 2, 2, 1 - 1) \\ &= (3, 2, 0) \end{aligned}$$

$$[\mathbf{v}_2]_{S_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3 \\ &= (1, 0, 1) + (1, 1, 0) - (0, 0, 1) \\ &= (2, 1, 0) \end{aligned}$$

$$[\mathbf{v}_3]_{S_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_3 &= 2\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \\ &= 2(1, 0, 1) + (1, 1, 0) + (0, 0, 1) \\ &= (3, 1, 3) \end{aligned}$$

Hence, $S_2 = \{(3, 2, 0), (2, 1, 0), (3, 1, 3)\}$

Example 8: Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for P_1, where $\mathbf{v}_1 = x$, $\mathbf{v}_2 = -1 + x$.

If the transition matrix from S_1 to S_2 is $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, determine S_1.

Solution: The transition matrix from S_1 to S_2 is

$$P = \left[[\mathbf{u}_1]_{S_2} \, [\mathbf{u}_2]_{S_2} \right]$$

Given $$P = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Comparing both the matrices,

$$[\mathbf{u}_1]_{S_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_1 &= 2\mathbf{v}_1 - \mathbf{v}_2 \\ &= 2x - (-1 + x) \\ &= 1 + x \end{aligned}$$

$$[\mathbf{u}_2]_{S_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_2 &= 3\mathbf{v}_1 + 2\mathbf{v}_2 \\ &= 3x + 2(-1 + x) \\ &= -2 + 5x \end{aligned}$$

Hence, $S_1 = \{(1 + x), (-2 + 5x)\}$

Example 9: Consider bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (0, 1)$

If the transition matrix from S_1 to S_2 is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, determine S_2.

Solution: To determine S_2, we need the transition matrix from S_2 to S_1. Since the transition matrix from S_1 to S_2 is

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

The transition matrix from S_2 to S_1 will be P^{-1}.

$$P^{-1} = \frac{1}{\det(P)} \text{adj } P$$

$$\text{adj } P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det(P) = 1$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Also,

$$P^{-1} = \left[[\mathbf{v}_1]_{S_1} \, [\mathbf{v}_2]_{S_1} \right]$$

Comparing both the matrices,

$$[\mathbf{v}_1]_{S_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_1 &= 1\mathbf{u}_1 - 1\mathbf{u}_2 \\ &= (1, 2) - (0, 1) \\ &= (1, 1) \end{aligned}$$

$$[\mathbf{v}_2]_{S_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_2 &= -\mathbf{u}_1 + 2\mathbf{u}_2 \\ &= -(1, 2) + 2(0, 1) \\ &= (-1, 0) \end{aligned}$$

Hence, $S_2 = \{(1, 1), (-1, 0)\}$

Exercise 2.6

1. Find the coordinate vector of **v** relative to the basis S for R^3.

(i) $\mathbf{v} = (3, 1, -4)$ and $S = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$

(ii) $\mathbf{v} = (2, -1, 3)$ and $S = \{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$

[**Ans.:** (i) $(\mathbf{v})_S = (3, -2, -5)$
(ii) $(\mathbf{v})_S = (3, -2, 1)$]

2. Find the coordinate vector of $\mathbf{p} = 4 - 3x + 2x^2$ relative to the basis $S = \{1 + x + x^2, 1 + x, 1\}$ for P_3.

[**Ans. :** $(\mathbf{p})_S = (2, -5, 7)$]

3. Find the coordinate vector of $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$ relative to the basis

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

in M_{22}.

[**Ans. :** $(\mathbf{A})_S = (-7, 11, -21, 30)$]

4. Find the vector **v** if the coordinate matrix $[\mathbf{v}]_S$ is given with respect to the basis S for vector space V.

(i) $S = \{(0, 1, -1), (1, 0, 0), (1, 1, 1)\}$

for R^2 and $[\mathbf{v}]_S = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

(ii) $S = \{1 + x^2, 1 + x, x + x^2\}$ for P_2

and $[\mathbf{v}]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

(iii)

$$S = \left\{ \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \right\}$$

for M_{22} and $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

$$\left[\textbf{Ans.: } \text{(i)}\,(3,1,3)\ \ \text{(ii)}\,2-3x+x^2\ \ \text{(iii)}\begin{bmatrix}-1 & 0\\ 9 & 7\end{bmatrix}\right]$$

5. Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{u}_1 = (1, -2)$, $\mathbf{u}_2 = (3, -4)$, $\mathbf{v}_1 = (1, 3)$, $\mathbf{v}_2 = (3, 8)$.

(i) Find the transition matrix from S_2 to S_1.

(ii) Find the transition matrix from S_1 to S_2.

$$\left[\textbf{Ans.: } \text{(i)}\begin{bmatrix}-\frac{13}{2} & -18\\ \frac{5}{2} & 7\end{bmatrix}\ \ \text{(ii)}\begin{bmatrix}-14 & -36\\ 5 & 13\end{bmatrix}\right]$$

6. Consider the bases $S_1 = \{(1, 2), (0, 1)\}$ and $S_2 = \{(1, 1), (2, 3)\}$.

(i) Find the transition matrix from S_1 to S_2.

(ii) Find $[\mathbf{w}]_{S_2}$ using transition matrix, where $\mathbf{w} = (1, 5)$

$$\left[\textbf{Ans .: } \text{(i)}\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\ \ \text{(ii)}\,[\mathbf{w}]_{S_2} = \begin{bmatrix}-7\\ 4\end{bmatrix}\right]$$

7. Consider the bases $S_1 = \{1 + x^2, -1, x + 2x^2\}$ and $S_2 = \{-1+x, 1+2x-x^2, x\}$ for P_2.

(i) Find the transition matrix from S_2 to S_1.

(ii) Find $[\mathbf{w}]_{S_1}$ using transition matrix, where $\mathbf{w} = 1 + 3x + 8x^2$

$$\left[\textbf{Ans.: } \text{(i)}\begin{bmatrix}-2 & -5 & -2\\ -1 & -6 & -2\\ 1 & 2 & 1\end{bmatrix}\ \ \text{(ii)}\,[w]_{S_1} = \begin{bmatrix}2\\ 1\\ 3\end{bmatrix}\right]$$

8. Consider the bases

$$S_1 = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\}$$

and

$$S_2 = \left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}$$

for M_{22}.

(i) Find the transition matrix from S_1 to S_2.

(ii) Find $[\mathbf{w}]_{S_2}$ using transition matrix, where $\mathbf{w} = \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$

$$\left[\textbf{Ans.: } \text{(i)}\begin{bmatrix}0 & 1 & 2 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & -2 & 0\end{bmatrix}\ \ \text{(ii)}\,[\mathbf{w}]_{S_2} = \begin{bmatrix}2\\ -2\\ 1\\ -1\end{bmatrix}\right]$$

9. Consider the bases $S_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S_2 = \{-1 + x, 1 + x\}$. If the transition matrix from S_2 to S_1 is $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$, determine S_1.

[**Ans.:** $S_1 = \{-5 + x, 3 - x\}$]

2.14 ROW SPACE, COLUMN SPACE AND NULL SPACE

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
be an $m \times n$ matrix. The vectors
$$\mathbf{r}_1 = [a_{11} \;\; a_{12} \ldots a_{1n}]$$
$$\mathbf{r}_2 = [a_{21} \;\; a_{22} \ldots a_{2n}]$$
$$\vdots$$
$$\mathbf{r}_m = [a_{m1} \; a_{m2} \ldots a_{mn}]$$
in R^n are called row vectors of A, and the vectors
$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$
in R^m are called column vectors of A.

(1) *Row Space*

The subspace of R^n spanned by the row vectors of A is called the row space of A.

(2) *Column Space*

The subspace of R^m spanned by the column vectors of A is called the column space of A.

(3) *Null Space*

The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ is called the null space of A.

2.14.1 *Basis for Row Space*

Theorem 2.18: Elementary row transformations do not change the row space and null space of a matrix.

Note: If a matrix A is reduced to row echelon form B then the row spaces of A and B are same.

Theorem 2.19: If B is the row echelon form of A then the row vectors of B with leading 1's (i.e. non-zero row vectors) form a basis for the row space of B, and hence form a basis for the row space of A.

Note:

(i) A basis for the row space of a matrix A may not consist entirely of row vectors.

(ii) A basis for the row space of A consisting entirely of row vectors of A can be obtained by finding the basis for column space of A^T.

2.14.2 *Basis for Column Space*

Theorem 2.20: If A and B are row equivalent matrices then

(i) A set of column vectors of matrix A is linearly independent if and only if the corresponding column vectors of B are linearly independent
(ii) A set of column vectors of matrix A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B

Theorem 2.21: If B is the row echelon form of a matrix A then

(i) The column vectors containing the leading 1's of row vectors form a basis for the column space of B
(ii) The column vectors of A corresponding to the column vectors of B containing the leading 1's form a basis for the column space of A

2.14.3 *Basis for Null Space*

The basis for the null space of A is the basis for the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$. This method has been discussed in 2.10.

Example 1: Find a basis for the row and column spaces of A.

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2 - 2R_1, \quad R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix}$$

$$R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(-\frac{1}{7}\right)R_2$$

$$\sim\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Basis for the row space of A = Non-zero rows of B $= \left\{(1, 4, 5, 2), \left(0, 1, 1, \frac{4}{7}\right)\right\}$

The leading 1's appear in columns 1 and 2.

Hence, basis for the column space of $A = \left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}\right\}$

Example 2: Find a basis for the row and column spaces of

$$A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2 - 2R_1,\ R_3 - 2R_1,\ R_4 + R_1$$

$$\sim\begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 - R_2$$

$$\sim\begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Basis for the row space of A = non-zero rows of B = {(1, 4, 5, 4), (0, 1, –2, –6), (0, 0, 1, 5)}

The leading 1's appear in columns 1, 2 and 3.

Hence, basis for the column space of $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$

Example 3: Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

consisting entirely the row vectors of A.

Solution: We know that

Row space of A = Column space of A^T

∴ Basis for the row space of A = Transpose of the basis for the column space of A^T.

$$A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$$

Reducing the matrix A^T to row echelon form,

$$R_2 - 4R_1,\ R_3 - 5R_1,\ R_4 - 6R_1,\ R_5 - 9R_1$$

$$A \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -14 & 4 & -5 \\ 0 & -14 & 4 & -5 \\ 0 & -14 & 4 & -5 \\ 0 & -28 & 8 & -10 \end{bmatrix}$$

$$R_3 - R_2,\ R_4 - R_2,\ R_5 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -14 & 4 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(-\frac{1}{14}\right)R_2$$

$$\sim\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{4}{14} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The leading 1's appear in columns 1 and 2.

$$\text{Basis for the column space of } A^T = \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$

Hence, basis for the row space of $A = \{(1, 4, 5, 6, 9), (3, -2, 1, 4, -1)\}$

Example 4: Find a basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

consisting of vectors that are not entirely the column vectors of A.

Solution: We know that

Column space of A = Row space of A^T

$\therefore$ Basis for the column space of A = Transpose of the basis for the row space of A^T

$$A^T = \begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & -1 & 2 & 1 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Reducing the matrix A^T to row echelon form,

$$R_2 + 2R_1,\ R_3 - 7R_1$$

$$\sim\begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & -3 & -24 & -15 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

$$\begin{aligned} &R_3 + 3R_2 \\ &\sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\left(\frac{1}{5}\right)R_4 \\ &\sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{bmatrix} = B \end{aligned}$$

Basis for the row space of A^T = Non-zero rows of B

$$= \left\{(1, 1, 3, 2), (0, 1, 8, 5), \left(0, 0, 1, \frac{3}{5}\right)\right\}$$

Hence, basis for the column space of $A = \left\{\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{5} \end{bmatrix}\right\}$

Example 5: Find a basis for the space spanned by the vectors $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (0, 0, 1, 1)$, $\mathbf{v}_3 = (-2, 0, 2, 2)$, $\mathbf{v}_4 = (0, -3, 0, 3)$.

Solution: The space spanned by these vectors is the row space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$\begin{aligned} &R_3 + 2R_1 \\ &\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix} \end{aligned}$$

$$\left(\frac{1}{2}\right)R_3, \left(\frac{1}{3}\right)R_4$$

$$\sim\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$R_2 + R_3, R_4 + R_3$$

$$\sim\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_3 - R_2$$

$$\sim\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$(-1)R_3$$

$$\sim\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_4 - R_3$$

$$\sim\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

Basis for the given space = Basis for the row space of A = Non-zero rows of B = {(1, 1, 0, 0), (0, 1, 2, 2), (0, 0, 1, 1), (0, 0, 0, 1)}

Example 6: Find a basis for the space spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

Solution: The space spanned by these vectors is the column space of the matrix

$$A=\begin{bmatrix}1&2&3&3&5\\2&1&2&3&3\\1&2&3&3&5\\2&1&2&3&3\end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2-2R_1,\ R_3-R_1,\ R_4-2R_1$$

$$\sim\begin{bmatrix}1&2&3&3&5\\0&-3&-4&-3&-7\\0&0&0&0&0\\0&-3&-4&-3&-7\end{bmatrix}$$

$$R_4-R_2$$

$$\sim\begin{bmatrix}1&2&3&3&5\\0&-3&-4&-3&-7\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$\left(-\frac{1}{3}\right)R_2$$

$$\sim\begin{bmatrix}1&2&3&3&5\\0&1&\frac{4}{3}&1&\frac{7}{3}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}=B$$

The leading 1's appear in columns 1 and 2.

Basis for the column space $=\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}1\\2\\1\\2\end{bmatrix},\begin{bmatrix}2\\1\\2\\1\end{bmatrix}\right\}$ which is also the basis for the space spanned by the given vectors.

Theorem 2.22: A system of linear equations $A\mathbf{x}=\mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Example 1: Determine whether **b** is in the column space of A, and if so, express **b** as a linear combination of the column vectors of A if

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Solution: The system of equations formed by A and **b** is

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1,\ R_3 + R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & -2 & 2 & 2 \end{array}\right]$$

$$R_3 + R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right) R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x_1 &= 2 + x_2 - x_3 \\ x_2 &= -1 \qquad + x_3 \end{aligned}$$

Assigning the free variable x_3 arbitrary value t,

$$x_2 = -1+t, \quad x_1 = 2+(-1+t)-t = 1$$

Thus, $x_1 = 1, x_2 = t - 1, x_3 = t$ is the solution of the system.
Since the system is consistent, **b** is in the column space of A.

Now,
$$\mathbf{b} = A\mathbf{x}$$
$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t-1 \\ t \end{bmatrix}$$
$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (t-1)\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Example 2: Determine whether **b** is in the column space of A, and if so, express **b** as a linear combination of the column vectors of A if

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

Solution: The system of equations formed by A and **b** is

$$A\mathbf{x} = \mathbf{b}$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 9R_1, R_3 - R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array}\right]$$

$$\left(\frac{1}{12}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{8}{12} & -\frac{44}{12} \\ 0 & 2 & 0 & -6 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{8}{12} & -\frac{44}{12} \\ 0 & 0 & \frac{8}{6} & \frac{8}{6} \end{array}\right]$$

$$\left(\frac{6}{8}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{8}{12} & -\frac{44}{12} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 - x_2 + \quad x_3 &= \quad 5 \\ x_2 - \frac{8}{12}x_3 &= -\frac{44}{12} \\ x_3 &= \quad 1 \end{aligned}$$

Solving these equations,

$$x_1 = 1, x_2 = -3, x_3 = 1$$

Since the system is consistent, **b** is in the column space of A.

Now,

$$\begin{aligned} \mathbf{b} &= A\mathbf{x} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 3: Determine whether **b** is in the column space of A, and if so, express **b** as a linear combination of the column vectors of A if

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Solution: The system of equations formed by A and **b** is

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x_1 + 0x_2 + 0x_3 = 3$$

This shows that the system is inconsistent and hence **b** is not in the column space of A.

Example 4: Find a basis for the null space of

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

Solution: The null space of A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 5R_1,\ R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 - x_2 + 3x_3 = 0$$
$$x_2 - 19x_3 = 0$$

Solving for the leading variables,

$$x_1 = x_2 - 3x_3$$
$$x_2 = 19x_3$$

Assigning the free variable x_3 arbitrary value t

$$x_1 = 19t - 3t = 16t,\ x_2 = 19t,\ x_3 = t$$

Null space consists vectors of the type

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 19t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} = t\mathbf{v}_1$$

Hence, $$\text{basis} = \{\mathbf{v}_1\} = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

2.15 RANK AND NULLITY

In the previous section we observed that in the row echelon form of a matrix, the number of non-zero rows (i.e. rows containing the leading 1's) form a basis for the row space of A and vectors corresponding to the columns containing the leading 1's form a basis for the column space of A.

Thus, dimension of row space = number of rows containing the leading 1's

and dimension of column space = number of columns containing the leading 1's.

This concludes that for any matrix A

$$\text{Dimension of row space} = \text{Dimension of column space}$$

2.15.1 *Rank*

The dimension of row/column space of a matrix A (or the number of non-zero rows in the row echelon form of A) is called the rank of A and is denoted by $\rho(A)$.

Note: If A is an $m \times n$ matrix then

$$\text{rank}(A) \leq \min(m, n)$$

Thus, the largest possible value of rank $(A) = \min(m, n)$ where $\min(m, n)$ means the smaller of the m and n.

e.g. if A is of order 5×3 then

$$\begin{aligned}\text{The largest possible value of rank}(A) &= \min(5, 3)\\ &= 3\end{aligned}$$

2.15.2 *Nullity*

The dimension of the null space of a matrix A is called the nullity of A and is denoted by nullity (A).

2.15.3 *Dimension Theorem*

Theorem 2.23: If A is an $m \times n$ matrix then

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (number of columns)}$$

Theorem 2.24: If A is an $m \times n$ matrix then nullity (A) represents the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$

Example 1: Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.

Solution:

$$\begin{aligned}\text{The number of parameters} &= \text{nullity}(A)\\ &= n - \text{rank}(A)\\ &= 7 - 3\\ &= 4\end{aligned}$$

Example 2: Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Reducing matrix A to row echelon form,

$$\overset{\left(\frac{1}{2}\right)R_1}{\sim}\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2-4R_1}{\sim}\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{Rank }(A) &= \text{Number of non-zero rows} = 1 \\ \text{nullity }(A) &= n - \text{rank }(A) \\ &= 3 - 1 = 2 \end{aligned}$$

Exercise 2.7

1. Find a basis for the null space of

(i) $A = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & 4 \\ -8 & 4 & 8 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 2 & 2 & -2 & -1 \\ 2 & 6 & 2 & -4 & 1 \\ 1 & 4 & 0 & -3 & 0 \end{bmatrix}$

$$\left[\begin{array}{l} \textbf{Ans.:} \text{(i)} \left\{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \\ \text{(ii)} \left\{\begin{bmatrix} -2 \\ \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\} \end{array}\right]$$

2. Find a basis for the row space of

$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix}$ consisting of vectors that are

(i) row vectors of A

(ii) not entirely row vectors of A

$$\left[\begin{array}{l} \textbf{Ans.:} \text{(i)} \{(1, 2, -1), (1, 9, -1)\} \\ \text{(ii)} \{(1, 0, -1), (0, 1, 0)\} \end{array}\right]$$

3. Find a basis for the column space of

$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix}$ consisting of vectors that are

(i) column vectors of A

(ii) not entirely column vectors of A

$$\left[\textbf{Ans.:} \text{(i)} \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -7 \end{bmatrix}\right\} \text{(ii)} \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}\right]$$

4. Find a basis for the space spanned by the vectors

(i) $\mathbf{v}_1 = (-1, 1, -2, 0)$, $\mathbf{v}_2 = (3, 3, 6, 0)$, $\mathbf{v}_3 = (2, -1, 3, 2)$

(ii) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix},$

$\mathbf{v}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{l} \textbf{Ans. :} \text{(i)} \left\{ \begin{array}{l} (1, -1, 2, 0), (0, 1, 0, 0), \\ \left(0, 0, 1, -\frac{1}{6}\right) \end{array} \right\} \\ \quad \text{(ii)} \left\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \right\} \end{array}\right]$$

5. Determine whether **b** is in the column space of A, and if so, express **b** as a linear combination of the column vectors of A if

(i) $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

$$\left[\begin{array}{l} \textbf{Ans. :} \text{(i)} \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \\ \text{(ii)} \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = -26\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \\ \qquad -7\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \end{array}\right]$$

6. Find the largest possible value of rank (A) and the smallest possible value of nullity (A) in each of the following:

(i) A is 3×3

(ii) A is 4×5

(iii) A is 5×4

[**Ans. :** (i) 3, 0 (ii) 4, 1 (iii) 4, 0]

7. Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×9 matrix of rank 3.

[**Ans. :** 6]

8. Find the rank and nullity of the matrix:

(i) $A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

[**Ans. :** (i) 2, 1 (ii) 2, 2]

Linear Transformations

Chapter 3

3.1 INTRODUCTION

Often it is necessary to transform data from one measurement scale to another e.g., the conversion of temperature from degree centigrade to degree Farenheit is given by $°F = 1.8°C + 32$. This is a linear transformation. Hence, linear transformation is a function that converts one type of data into another type of data. Linear transformation from R^n to R^m is referred to as Euclidean linear transformation whereas, linear transformation from vector space V to vector space W is referred to as general linear transformation. This is useful in many applications in physics, engineering and various branches of mathematics.

3.2 EUCLIDEAN LINEAR TRANSFORMATION

It is a function that associates each element of R^n with exactly one element of R^m. It is represented by $T: R^n \to R^m$ and we say that T maps R^n into R^m. Here, the domain of transformation T is R^n and the codomain of transformation T is R^m.

Consider a linear transformation $T: R^n \to R^m$ defined by

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \qquad \ldots(3.1)$$

In matrix form,

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \ldots(3.2)$$

$$\mathbf{w} = A\mathbf{x} \qquad \ldots(3.3)$$

The matrix A is called the standard matrix of the linear transformation. Eq (3.1) can also be represented as,

$$T(x_1, x_2, \dots, x_n) = (w_1, w_2, \dots, w_n)$$

If $T: R^n \to R^m$ is multiplication by A, then standard matrix of T is also denoted by T_A.

3.3 LINEAR TRANSFORMATIONS

Let V and W be two vector spaces. A linear transformation $(T: V \to W)$ is a function T from V to W such that

(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(b) $T(k\,\mathbf{u}) = k\,T(\mathbf{u})$

for all vectors **u** and **v** in V and all scalars k.
If $V = W$, the linear transformation $T: V \to V$ is called a linear operator.

3.3.1 *Some Standard Linear Transformations*

(1) *Zero Transformation*

Let V and W be two vector spaces. The function T from V to W defined by

$$T(\mathbf{v}) = \mathbf{0}$$

for every vector **v** in V is a linear transformation from V to W.
Let **u** and **v** are in V.

$$T(\mathbf{u}) = \mathbf{0}$$
$$T(\mathbf{v}) = \mathbf{0}$$
$$T(\mathbf{u} + \mathbf{v}) = \mathbf{0} \quad [\because \mathbf{u} + \mathbf{v} \text{ is in } V]$$
$$T(k\mathbf{u}) = \mathbf{0} \quad [\because k\mathbf{u} \text{ is in } V]$$

Thus, $\quad T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
and $\quad T(k\mathbf{u}) = k\,T(\mathbf{u})$

Hence, T is a linear transformation and is called zero transformation.

(2) *Identity Operator*

Let V be a vector space. The function I from V to V defined by

$$I(\mathbf{v}) = \mathbf{v}$$

is a linear transformation from V to V.
Let **u** and **v** are in V.

$$I(\mathbf{v}) = \mathbf{v}$$
$$I(\mathbf{u}) = \mathbf{u}$$

$$I(\mathbf{u}+\mathbf{v}) = \mathbf{u}+\mathbf{v} = I(\mathbf{u})+I(\mathbf{v})$$
$$I(k\mathbf{u}) = k\mathbf{u} = kI(\mathbf{u})$$

Hence, I is a linear transformation and is called identity operator on V.

3.3.2 *Properties of Linear Transformations*

Theorem 3.1: If $T: V \to W$ is a linear transformation then

(a) $T(\mathbf{0}) = \mathbf{0}$
(b) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V
(c) $T(\mathbf{v}-\mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ for all $\mathbf{v}$ and $\mathbf{w}$ in V
(d) $T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n)$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in V and $k_1, k_2, \ldots, k_n$ are all scalars.

3.4 LINEAR OPERATORS (TYPES OF LINEAR TRANSFORMATIONS)

3.4.1 *Reflection Operators*

An operator on R^2 or R^3 that maps each vector into its symmetric image about some line or plane is called a reflection operator. Let $T: R^2 \to R^2$ be a reflection operator defined by

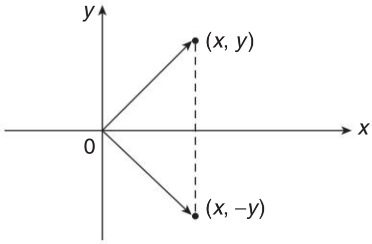
$$T(x, y) = (x, -y)$$

that maps each vector into its symmetric image about the x-axis.
In matrix form,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Some of the basic reflection operators are given in Table 3.1

Fig. 3.1

Table 3.1

Operator	Equation	Standard Matrix
Reflection about the x-axis on R^2	$T(x, y) = (x, -y)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis on R^2	$T(x, y) = (-x, y)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(*contd.*)

Table 3.1 (contd.)

Operator	Equation	Standard Matrix
Reflection about the line $y = x$ on R^2	$T(x, y) = (y, x)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection about the xy-plane on R^3	$T(x, y, z) = (x, y, -z)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane on R^3	$T(x, y, z) = (x, -y, z)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane on R^3	$T(x, y, z) = (-x, y, z)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.4.2 *Projection Operators*

An operator on R^2 or R^3 that maps each vector into its orthogonal projection on a line or plane through the origin is called a projection operator. Let $T : R^2 \to R^2$ be a projection operator defined by

$$T(x, y) = (x, 0)$$

that maps each vector into its orthogonal projection on the x-axis.
In matrix form,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Some of the basic projection operators on R^2 and R^3 are given in Table 3.2.

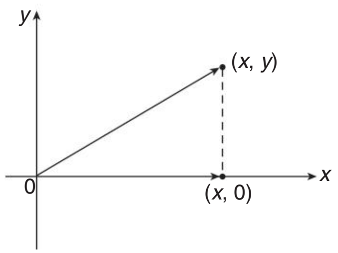


Fig. 3.2

Table 3.2

Operator	Equations	Standard Matrix
Orthogonal projection on the x-axis on R^2	$T(x, y) = (x, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis on R^2	$T(x, y) = (0, y)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Orthogonal projection on the xy-plane on R^3	$T(x, y, z) = (x, y, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane on R^3	$T(x, y, z) = (x, 0, z)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane on R^3	$T(x, y, z) = (0, y, z)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.4.3 *Rotation Operators*

An operator on R^2 that rotates each vector counterclockwise through a fixed angle θ is called a rotation operator. Let $T : R^2 \rightarrow R^2$ be a rotation operator defined by

$$T(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

that rotates each vector counterclockwise through a fixed angle θ.

In matrix form,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Similarly, a rotation operator on R^3 rotates each vector about some rotation axis through a fixed angle θ.

Some of the rotation operators on R^2 and R^3 are given in Table 3.3.

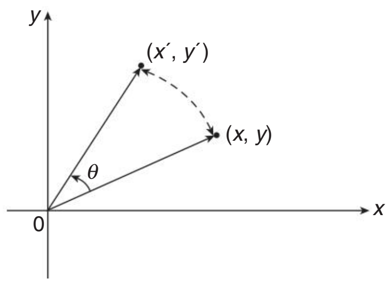


Fig. 3.3

Table 3.3

Operator	**Equations**	**Standard Matrix**
Rotation through an angle θ on R^2	$T(x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive x-axis through an angle θ on R^3	$T(x, y, z) = (x, y\cos\theta - z\sin\theta,\ y\sin\theta + z\cos\theta)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ on R^3	$T(x, y, z) = (x\cos\theta + z\sin\theta,\ y, -x\sin\theta + z\cos\theta)$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ on R^3	$T(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta, z)$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.4.4 *Dilation Operators*

An operator on R^2 or R^3 that stretches each vector uniformly away from the origin in all directions is called a dilation operator. Let $T : R^2 \to R^2$ be a dilation operator defined by

$$T(x, y) = (kx, ky),\ k \geq 1$$

that stretches each vector by a factor k.

In matrix form,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Some of the dilation operators on R^2 and R^3 are given in Table 3.4.

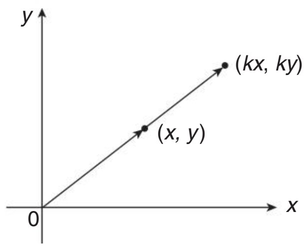


Fig. 3.4

Table 3.4

Operator	Equations	Standard Matrix
dilation with factor k on R^2 $(k \geq 1)$	$T(x, y) = (kx, ky)$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
dilation with factor k on R^3 $(k \geq 1)$	$T(x, y, z) = (kx, ky, kz)$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

3.4.5 *Contraction Operators*

An operator on R^2 or R^3 that compresses each vector uniformly toward the origin from all directions is called a contraction operator. Let $T : R^2 \rightarrow R^2$ be a contraction operator defined by

$$T(x, y) = (kx, ky),\ 0 \leq k \leq 1$$

that compresses each vector by a factor k.

In matrix form,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

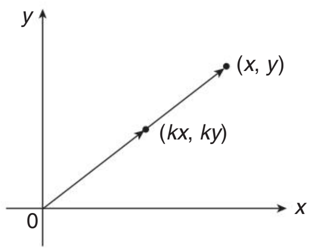


Fig. 3.5

Some of the contraction operators on R^2 and R^3 are given in Table 3.5.

Table 3.5

Operator	Equations	Standard Matrix
Contraction with factor k on R^2 $(0 \leq k \leq 1)$	$T(x, y) = (kx, ky)$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Contraction with factor k on R^3 $(0 \leq k \leq 1)$	$T(x, y, z) = (kx, ky, kz)$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

3.4.6 *Shear Operators*

An operator on R^2 or R^3 that moves each point parallel to the x-axis by the amount ky is called a shear in the x-direction. Similarly, an operator on R^2 or R^3 that moves each vector parallel to y-axis by the amount kx is called a shear in the y-direction. Let $T : R^2 \to R^2$ be a shear operator in the x-direction defined by

$$T(x, y) = (x + ky, y)$$

that moves each point parallel to the x-axis by the amount ky.

In matrix form,

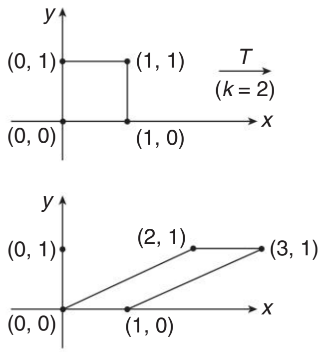
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

When the shear in the x-direction with $k = 2$ is applied to a square with vertices (0, 0), (0, 1), (1, 1) and (1, 0), it is transformed to a parallelogram with vertices (0, 0), (2, 1), (3, 1) and (1, 0).

Some of the shear operators on R^2 are given in Table 3.6.



Shear in x-direction

Fig. 3.6

Table 3.6

Operator	**Equations**	**Standard Matrix**
Shear in the x-direction on R^2	$T(x, y) = (x + ky, y)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction on R^2	$T(x, y) = (x , y + kx)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Example 1: Show that the following functions are linear transformations.

(i) $T : R^2 \to R^2$, where $T(x, y) = (x + 2y, 3x - y)$

(ii) $T : R^3 \to R^2$, where $T(x, y, z) = (2x - y + z, y - 4z)$

Solution: (i)Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be the vectors in R^2 and k be any scalar.

$$T(\mathbf{u}) = (x_1 + 2y_1, 3x_1 - y_1)$$
$$T(\mathbf{v}) = (x_2 + 2y_2, 3x_2 - y_2)$$

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (x_1, y_1)+(x_2, y_2)\\ &= (x_1+x_2, y_1+y_2)\end{aligned}$$

$$\begin{aligned}T(\mathbf{u}+\mathbf{v}) &= (x_1+x_2+2y_1+2y_2, 3x_1+3x_2-y_1-y_2)\\ &= (x_1+2y_1+x_2+2y_2, 3x_1-y_1+3x_2-y_2)\\ &= (x_1+2y_1, 3x_1-y_1)+(x_2+2y_2, 3x_2-y_2)\\ &= T(\mathbf{u})+T(\mathbf{v})\end{aligned}$$

$$\begin{aligned}k\mathbf{u} &= k(x_1, y_1) = (kx_1, ky_1)\\ T(k\mathbf{u}) &= (kx_1+2ky_1, 3kx_1-ky_1)\\ &= k(x_1+2y_1, 3x_1-y_1)\\ &= kT(\mathbf{u})\end{aligned}$$

Hence, T is a linear transformation.

(ii) Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ be the vectors in R^3 and k be any scalar.

$$\begin{aligned}T(\mathbf{u}) &= (2x_1-y_1+z_1, y_1-4z_1)\\ T(\mathbf{v}) &= (2x_2-y_2+z_2, y_2-4z_2)\end{aligned}$$

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (x_1, y_1, z_1)+(x_2, y_2, z_2)\\ &= (x_1+x_2, y_1+y_2, z_1+z_2)\end{aligned}$$

$$\begin{aligned}T(\mathbf{u}+\mathbf{v}) &= (2x_1+2x_2-y_1-y_2+z_1+z_2, y_1+y_2-4z_1-4z_2)\\ &= (2x_1-y_1+z_1+2x_2-y_2+z_2, y_1-4z_1+y_2-4z_2)\\ &= (2x_1-y_1+z_1, y_1-4z_1)+(2x_2-y_2+z_2, y_2-4z_2)\\ &= T(\mathbf{u})+T(\mathbf{v})\end{aligned}$$

$$\begin{aligned}k\mathbf{u} &= k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)\\ T(k\mathbf{u}) &= (2kx_1-ky_1+kz_1, ky_1-4kz_1)\\ &= k(2x_1-y_1+z_1, y_1-4z_1)\\ &= kT(\mathbf{u})\end{aligned}$$

Hence, T is a linear transformation.

Example 2: Determine whether the following functions are linear transformations.

(i) $T: P_2 \to P_3$, where $T(p(x)) = x\,p(x)$

(ii) $T: P_2 \to P_2$, where $T(a_0+a_1x+a_2x^2) = (a_0+1)+(a_1+1)x+(a_2+1)x^2$

Solution: (i) Let $\mathbf{p}_1 = a_0+a_1x+a_2x^2$ and $\mathbf{p}_2 = b_0+b_1x+b_2x^2$ be the two polynomials in P_2 and k be any scalar.

$$\begin{aligned}T(\mathbf{p}_1) &= a_0x+a_1x^2+a_2x^3\\ T(\mathbf{p}_2) &= b_0x+b_1x^2+b_2x^3\end{aligned}$$

$$\begin{aligned}\mathbf{p}_1+\mathbf{p}_2 &= (a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\\ &= (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2\end{aligned}$$

$$\begin{aligned}T(\mathbf{p}_1+\mathbf{p}_2) &= (a_0+b_0)x+(a_1+b_1)x^2+(a_2+b_2)x^3\\ &= a_0x+a_1x^2+a_2x^3+b_0x+b_1x^2+b_2x^3\\ &= T(\mathbf{p}_1)+T(\mathbf{p}_2)\end{aligned}$$

$$\begin{aligned}k\,\mathbf{p}_1 &= k(a_0+a_1x+a_2x^2)\\ &= a_0k+a_1kx+a_2kx^2\end{aligned}$$

$$\begin{aligned}T(k\mathbf{p}_1) &= x(a_0k+a_1kx+a_2kx^2)\\ &= a_0kx+a_1kx^2+a_2kx^3\\ &= k(a_0x+a_1x^2+a_2x^3)\\ &= kT(\mathbf{p}_1)\end{aligned}$$

Hence, T is a linear transformation.

(ii) Let $\mathbf{p}_1 = a_0 + a_1x + a_2x^2$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$ be the two polynomials in P_2 and k be any scalar.

$$\begin{aligned}T(\mathbf{p}_1) &= (a_0+1)+(a_1+1)x+(a_2+1)x^2\\ T(\mathbf{p}_2) &= (b_0+1)+(b_1+1)x+(b_2+1)x^2\end{aligned}$$

$$\begin{aligned}\mathbf{p}_1+\mathbf{p}_2 &= (a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\\ &= (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2\end{aligned}$$

$$\begin{aligned}T(\mathbf{p}_1+\mathbf{p}_2) &= (a_0+b_0+1)+(a_1+b_1+1)x+(a_2+b_2+1)x^2\\ &= (a_0+1)+(a_1+1)x+(a_2+1)x^2+b_0+b_1x+b_2x^2\\ &= T(\mathbf{p}_1)+\mathbf{p}_2\\ &\neq T(\mathbf{p}_1)+T(\mathbf{p}_2)\end{aligned}$$

Hence, T is not a linear transformation.

Example 3: Determine whether the following functions are linear transformations:

(i) $T: M_{mn} \to M_{nm}$, where $T(A) = A^T$

(ii) $T: M_{22} \to R$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$

(iii) $T: M_{nn} \to R$, where $T(A) = \det(A)$

(iv) $T: M_{nn} \to R$, where $T(A) = \text{tr}\,(A)$

Solution: (i) Let A_1 and A_2 be two matrices in M_{mn} and k be any scalar.

$$T(A_1) = A_1^T$$
$$T(A_2) = A_2^T$$
$$T(A_1 + A_2) = (A_1 + A_2)^T = A_1^T + A_2^T = T(A_1) + T(A_2)$$
$$T(kA_1) = (kA_1)^T = kA_1^T = kT(A_1)$$

Hence, T is a linear transformation.

(ii) Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be two matrices in M_{22} and k be any scalar.

$$T(A_1) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) = a_1^2 + b_1^2$$
$$T(A_2) = T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = a_2^2 + b_2^2$$
$$A_1 + A_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$
$$\begin{aligned} T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right) &= (a_1 + a_2)^2 + (b_1 + b_2)^2 \\ &= a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2 \\ &= (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(a_1a_2 + b_1b_2) \\ &\neq T(A_1) + T(A_2) \end{aligned}$$

Hence, T is not a linear transformation.

(iii) Let A_1 and A_2 be two matrices in M_{nn} and k be any scalar.

$$T(A_1) = \det(A_1)$$
$$T(A_2) = \det(A_2)$$
$$\begin{aligned} T(A_1 + A_2) &= \det(A_1 + A_2) \\ &\neq \det(A_1) + \det(A_2) \\ &\neq T(A_1) + T(A_2) \end{aligned}$$

Hence, T is not a linear transformation.

(iv) Let A_1 and A_2 be two matrices in M_{nn} and k be any scalar.

$$T(A_1) = \text{tr}(A_1) = \sum_{i=1}^{n} a_{ii}$$
$$T(A_2) = \text{tr}(A_2) = \sum_{i=1}^{n} b_{ii}$$

$$
\begin{aligned}
T(A_1 + A_2) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\
&= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\
&= T(A_1) + T(A_2) \\
T(kA_1) &= \sum_{i=1}^{n} k a_{ii} \\
&= k \sum_{i=1}^{n} a_{ii} \\
&= kT(A_1)
\end{aligned}
$$

Hence, T is a linear transformation.

3.5 LINEAR TRANSFORMATIONS FROM IMAGES OF BASIS VECTORS

A linear transformation is completely determined by the images of any set of basis vectors. If $T : V \to W$ is a linear transformation and if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is any basis for V then any vector $\mathbf{v}$ in V is expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

The linear transformation $T(\mathbf{v})$ is given by,

$$
\begin{aligned}
T(\mathbf{v}) &= T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) \\
&= k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n)
\end{aligned}
$$

Example 1: Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 0)$ and let $T : R^2 \to R^2$ be the linear transformation such that $T(\mathbf{v}_1) = (1, -2)$ and $T(\mathbf{v}_2) = (-4, 1)$.
Find a formula for $T(x_1, x_2)$ and use the formula to find $T(5, -3)$.

Solution: Let $\mathbf{v} = (x_1, x_2)$ be an arbitrary vector in R^2 and can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$
\begin{aligned}
\mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \\
(x_1, x_2) &= k_1(1, 1) + k_2(1, 0) \\
&= (k_1 + k_2, k_1)
\end{aligned}
$$

Equating corresponding components,

$$
\begin{aligned}
k_1 + k_2 &= x_1 \\
k_1 \quad\;\; &= x_2
\end{aligned}
$$

Hence, $k_1 = x_2$, $k_2 = x_1 - x_2$

$$\therefore \mathbf{v} = x_2\mathbf{v}_1 + (x_1 - x_2)\mathbf{v}_2$$

$$T(\mathbf{v}) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

$$\begin{aligned} T(x_1, x_2) &= x_2T(\mathbf{v}_1) + (x_1 - x_2)T(\mathbf{v}_2) \\ &= x_2(1, -2) + (x_1 - x_2)(-4, 1) \\ &= (x_2, -2x_2) + (-4x_1 + 4x_2, x_1 - x_2) \\ &= (-4x_1 + 5x_2, x_1 - 3x_2) \end{aligned}$$

$$\begin{aligned} T(5, -3) &= (-4(5) + 5(-3), 5 - 3(-3)) \\ &= (-35, 14) \end{aligned}$$

Example 2: Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$ and let $T : R^2 \to R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1, 2, 0)$ and $T(\mathbf{v}_2) = (0, -3, 5)$.
Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2, -3)$.

Solution: Let $\mathbf{v} = (x_1, x_2)$ be an arbitrary vector in R^2 and can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \\ (x_1, x_2) &= k_1(-2, 1) + k_2(1, 3) \\ &= (-2k_1 + k_2, k_1 + 3k_2) \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} -2k_1 + k_2 &= x_1 \\ k_1 + 3k_2 &= x_2 \end{aligned}$$

Solving these equations,

$$k_1 = -\frac{3}{7}x_1 + \frac{1}{7}x_2$$

$$k_2 = \frac{1}{7}x_1 + \frac{2}{7}x_2$$

$$\therefore \mathbf{v} = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)\mathbf{v}_1 + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)\mathbf{v}_2$$

$$T(\mathbf{v}) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

$$\begin{aligned} T(x_1, x_2) &= \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)T(\mathbf{v}_1) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)T(\mathbf{v}_2) \\ &= \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1, 2, 0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0, -3, 5) \end{aligned}$$

$$= \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{6}{7}x_1 + \frac{2}{7}x_2, 0\right) + \left(0, -\frac{3}{7}x_1 - \frac{6}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$$

$$= \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$$

$$= \frac{1}{7}(3x_1 - x_2, -9x_1 - 4x_2, 5x_1 + 10x_2)$$

$$T(2, -3) = \frac{1}{7}(9, -6, -20) = \left(\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7}\right)$$

Example 3: Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3 where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$ and $\mathbf{v}_3 = (1, 0, 0)$ and let $T : R^3 \rightarrow R^3$ be the linear operator such that $T(\mathbf{v}_1) = (2, -1, 4)$, $T(\mathbf{v}_2) = (3, 0, 1)$, $T(\mathbf{v}_3) = (-1, 5, 1)$. Find a formula for $T(x_1, x_2, x_3)$ and use that formula to find $T(2, 4, -1)$.

Solution: Let $\mathbf{v} = (x_1, x_2, x_3)$ be an arbitrary vector in R^3 and can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$(x_1, x_2, x_3) = k_1(1, 1, 1) + k_2(1, 1, 0) + k_3(1, 0, 0)$$

$$= (k_1 + k_2 + k_3, k_1 + k_2, k_1)$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_2 + k_3 &= x_1 \\ k_1 + k_2 \quad\quad &= x_2 \\ k_1 \quad\quad\quad\quad &= x_3 \end{aligned}$$

Hence,

$$\begin{aligned} k_1 &= x_3 \\ k_2 &= x_2 - x_3 \\ k_3 &= x_1 - x_2 \end{aligned}$$

$$\therefore \mathbf{v} = x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3$$

$$T(\mathbf{v}) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + k_3T(\mathbf{v}_3)$$

$$T(x_1, x_2, x_3) = x_3T(\mathbf{v}_1) + (x_2 - x_3)\,T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3)$$

$$= x_3(2, -1, 4) + (x_2 - x_3)(3, 0, 1) + (x_1 - x_2)(-1, 5, 1)$$

$$= (2x_3, -x_3, 4x_3) + (3x_2 - 3x_3, 0, x_2 - x_3) + (-x_1 + x_2, 5x_1 - 5x_2, x_1 - x_2)$$

$$= (-x_1 + 4x_2 - x_3, 5x_1 - 5x_2 - x_3, x_1 + 3x_3)$$

$$T(2, 4, -1) = (-2 + 4(4) - (-1), 5(2) - 5(4) - (-1), 2 + 3(-1))$$

$$= (15, -9, -1)$$

Example 4: Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and let $T : R^2 \to P_2$ be the linear transformation such that $T(\mathbf{v}_1) = 2 - 3x + x^2$ and $T(\mathbf{v}_2) = 1 - x^2$. Find $T\begin{bmatrix} a \\ b \end{bmatrix}$ and then find $T\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Solution: Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be an arbitrary vector in R^2 and can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} k_1 + 2k_2 \\ k_1 + 3k_2 \end{bmatrix}$$

Equating corresponding components,

$$k_1 + 2k_2 = a$$
$$k_1 + 3k_2 = b$$

Solving these equations,

$$k_1 = 3a - 2b$$
$$k_2 = b - a$$

$$\therefore \mathbf{v} = (3a - 2b)\mathbf{v}_1 + (b - a)\mathbf{v}_2$$

$$T(\mathbf{v}) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = (3a - 2b)(2 - 3x + x^2) + (b - a)(1 - x^2)$$
$$= (6a - 4b + b - a) + (-9a + 6b)x + (3a - 2b - b + a)x^2$$
$$= (5a - 3b) + (-9a + 6b)x + (4a - 3b)x^2$$

$$T\begin{bmatrix} -1 \\ 2 \end{bmatrix} = [5(-1) - 3(2)] + [(-9)(-1) + 6(2)]x + [4(-1) - 3(2)]x^2$$
$$= -11 + 21x - 10x^2$$

Example 5: Let $T: M_{22} \to R$ be a linear transformation for which $T(\mathbf{v}_1) = 1$, $T(\mathbf{v}_2) = 2$, $T(\mathbf{v}_3) = 3$, $T(\mathbf{v}_4) = 4$ where $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Find $T\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $T\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

Solution: Let $\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary vector in M_{22} and can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$.

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & k_2 + k_3 + k_4 \\ k_3 + k_4 & k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 &= a \\ k_2 + k_3 + k_4 &= b \\ k_3 + k_4 &= c \\ k_4 &= d \end{aligned}$$

Solving these equations,

$$\begin{aligned} k_1 &= a - b \\ k_2 &= b - c \\ k_3 &= c - d \\ k_4 &= d \end{aligned}$$

$$\therefore \mathbf{v} = (a-b)\mathbf{v}_1 + (b-c)\mathbf{v}_2 + (c-d)\mathbf{v}_3 + d\mathbf{v}_4$$

$$T(\mathbf{v}) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + k_3T(\mathbf{v}_3) + k_4T(\mathbf{v}_4)$$

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b)(1) + (b-c)(2) + (c-d)(3) + d(4)$$

$$= a + b + c + d$$

$$T\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = 1 + 3 + 4 + 2 = 10$$

3.6 COMPOSITION OF LINEAR TRANSFORMATION

Let T_1: $U \rightarrow V$ and T_2: $V \rightarrow W$ be linear transformations. The application of T_1 followed by T_2 produces a transformation from U to W. This transformation is called the composition of T_2 with T_1 and is denoted by $T_2 \circ T_1$

$$(T_2 \circ T_1)\ (\mathbf{u})\ = T_2(T_1(\mathbf{u}))$$

where **u** is a vector in U.

Note 1: The domain of T_2 (which is V) consists of range of T_1.

Note 2: If $[T_1] = A$ and $[T_2] = B$, then $T_1(\mathbf{u}) = A(\mathbf{u})$

$$\therefore (T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u})) = T_2(A\mathbf{u})$$
$$= B(A\mathbf{u}) = (BA)\mathbf{u}$$

This shows that $$[T_2 \circ T_1] = BA = [T_2][T_1]$$

Theorem 3.2: If T_1: $U \to V$ and T_2: $V \to W$ are linear transformations then $(T_2 \circ T_1)$: $U \to W$ is also a linear transformation.

Compositions can be defined for more than two linear transformations. If T_1: $U \to V$, T_2: $V \to W$ and T_3: $W \to U$ are three linear transformations then the composition $T_3 \circ T_2 \circ T_1$ is given by

$$(T_3 \circ T_2 \circ T_1)\ (\mathbf{u}) = T_3(T_2(T_1(\mathbf{u})))$$

Example 1: Find domain and codomain of $T_2 \circ T_1$ and find $(T_2 \circ T_1)(x, y)$.

(i) $T_1\,(x, y) = (2x, 3y)$, $T_2\,(x, y) = (x - y, x + y)$
(ii) $T_1\,(x, y) = (x - y, y + z, x - z)$, $T_2\,(x, y, z) = (0, x + y + z)$

Solution: (i) $T_1\,(x, y) = (2x, 3y)$.
$T_1 : R^2 \to R^2$ is a linear transformation from R^2 to R^2.

$$T_2\,(x, y) = (x - y, x + y)$$

$T_2 : R^2 \to R^2$ is a linear transformation from R^2 to R^2.

Hence $T_2 \circ T_1$ is a linear transformation from R^2 to R^2.

Domain of $T_2 \circ T_1 = R^2$

Codomain of $T_2 \circ T_1 = R^2$

$$[T_2 \circ T_1] = [T_2][T_1]$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix}$$
$$(T_2 \circ T_1)\ (x, y) = (2x - 3y,\ 2x + 3y)$$

(ii) $T_1\,(x, y) = (x - y, y + z, x - z)$
T_1: $R^2 \to R^3$ is a linear transformation from R^2 to R^3.

$$T_2\,(x, y, z) = (0, x + y + z)$$

T_2: $R^3 \to R^2$ is a linear transformation from R^3 to R^2.

Hence, $T_2 \circ T_1$ is a linear transformation from R^2 to R^2.

Domain of $T_2 \circ T_1 = R^2$

Codomain of $T_2 \circ T_1 = R^2$

$$[T_2 \circ T_1] = [T_2][T_1]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$(T_2 \circ T_1)(x, y) = (0, 2x)$$

Example 2: Find the domain and codomain of $T_3 \circ T_2 \circ T_1$ and find $(T_3 \circ T_2 \circ T_1)(x, y)$ where $T_1(x, y) = (x+y, y, -x)$, $T_2(x, y, z) = (0, x+y+z, 3y)$, $T_3(x, y, z) = (3x+2y, 4z - x - 3y)$.

Solution: $T_1(x, y) = (x + y, y, -x)$

$T_1: R^2 \to R^3$ is a linear transformation from R^2 to R^3.

$$T_2(x, y, z) = (0, x + y + z, 3y)$$

$T_2: R^3 \to R^3$ is a linear transformation from R^3 to R^3.

$$T_3(x, y, z) = (3x + 2y, 4z - x - 3y)$$

$T_3: R^3 \to R^2$ is a linear transformation from R^3 to R^2.

Hence, $T_3 \circ T_2 \circ T_1$ is a linear transformation from R^2 to R^2.

Domain of $T_3 \circ T_2 \circ T_1 = R^2$

Codomain of $T_3 \circ T_2 \circ T_1 = R^2$

$$[T_3 \circ T_2 \circ T_1] = \begin{bmatrix} 3 & 2 & 0 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 6 \end{bmatrix}$$

$$(T_3 \circ T_2 \circ T_1)(x, y) = (4y, 6y)$$

Example 3: Let $T_1: P_2 \to P_2$ and $T_2: P_2 \to P_2$ be the linear transformation given by, $T_1(p(x)) = p(x + 1)$ and $T_2(p(x)) = x\,p(x)$.

Find $(T_2 \circ T_1)\ (a_0 + a_1x + a_2x^2)$.

Solution:

$$T_1(p(x)) = p(x + 1)$$

$$T_2(p(x)) = x\,p(x)$$

$$\begin{aligned}(T_2 \circ T_1)(a_0 + a_1x + a_2x^2) &= T_2\left(T_1(a_0 + a_1x + a_2x^2)\right) \\ &= T_2(a_0 + a_1(x+1) + a_2(x+1)^2) \\ &= x(a_0 + a_1(x+1) + a_2(x+1)^2) \\ &= a_0x + a_1x(x+1) + a_2x(x+1)^2 \\ &= a_0x + a_1x^2 + a_1x + a_2x(x^2 + 2x + 1) \\ &= a_0x + a_1x^2 + a_1x + a_2x^3 + 2a_2x^2 + a_2x \\ &= (a_0 + a_1 + a_2)x + (a_1 + 2a_2)x^2 + a_2x^3\end{aligned}$$

Example 4: Let $T_1: M_{22} \to R$ and $T_2: M_{22} \to M_{22}$ be the linear transformations given by $T_1(A) = \text{tr}(A)$ and $T_2(A) = A^{\text{T}}$. Find $(T_1 \circ T_2)(A)$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution:

$$T_1(A) = \text{tr}(A) = a + d$$

$$T_2(A) = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$(T_1 \circ T_2)(A) = T_1(T_2(A)) = T_1(A^T) = \text{tr}(A^T)$$
$$= a + d$$

Example 5: Let $T: R^3 \to R^3$ be the orthogonal projection of R^3 on to the xy-plane. Show that $T \circ T = T$.

Solution: $T: R^3 \to R^3$ be the orthogonal projection of R^3 onto the xy-plane defined by

$$T(x, y, z) = (x, y, 0)$$

The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[T \circ T] = [T][T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $T \circ T = T$

Example 6: Find the standard matrix of the stated composition of linear operators on R^3.

(i) A rotation of 45° about the y-axis, followed by a dilation with the factor $k = \sqrt{2}$.

(ii) A rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction with the factor $k = \frac{1}{4}$.

Solution: (i) Let T_1 be a rotation about the y-axis on R^3.

$$T_1(x, y, z) = (x\cos\theta + z\sin\theta, y, -x\sin\theta + z\cos\theta)$$

$$[T_1] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

For $\theta = 45°$,

$$[T_1] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let T_2 be a dilation with the factor k on R^3.

$$T_2(x, y, z) = (kx, ky\ kz)$$

$$[T_2] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

For $k = \sqrt{2}$,

$$[T_2] = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

The linear transformation of the stated composition of these linear operators on R^3 is given by

$$T = T_2 \circ T_1$$

The standard matrix of T is

$$\begin{aligned} [T] &= [T_2][T_1] \\ &= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

(ii) Let T_1 be a rotation about the x-axis on R^3.

$$T_1 = (x, y, z) = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$$

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

For $\theta = 30°$,

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Let T_2 be a rotation about the z-axis on R^3.

$$T_2(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$$

$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For $\theta = 30°$,

$$[T_2] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let T_3 be a contraction with factor k on R^3.

$$T_3(x, y, z) = (kx, ky, kz)$$

$$[T_3] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

For $k = \frac{1}{4}$,

$$[T_3] = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

The linear transformation of the stated composition of these linear operators on R^3 is given by,

$$T = T_3 \circ T_2 \circ T_1$$

The standard matrix of T is

$$[T] = [T_3][T_2][T_1]$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & -\frac{\sqrt{3}}{16} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$$

Exercise 3.1

1. Which of the following are linear transformations? Justify.

(i) $T: R^2 \to R^2$, where $T(x, y) = (x + y, x)$

(ii) $T: R^2 \to R$, where $T(x, y) = xy$

(iii) $T: R^2 \to R^3$, where $T(x, y) = (x + 1, 2y, x + y)$

(iv) $T: R^3 \to R^2$, where $T(x, y, z) = (|x|, 0)$

(v) $T: R^2 \to R^2$, where $T(x, y) = (x^2, y^2)$

(vi) $T: R^3 \to R^2$, where $T(x, y, z) = (x - y + 2z, 2x + y - z, -x - 2y)$

Ans.:
(i) Linear
(ii) Non-linear
(iii) Non-linear
(iv) Non-linear
(v) Non-linear
(vi) Linear

2. Determine whether the function is a linear transformation. Justify your answer.

(i) $T: P_2 \to P_2$, where $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x + 1) + a_2 (x + 1)^2$

(ii) $T: P_1 \to P_2$, where $T(p(x)) = x\,p(x) + x^2 + 1$

(iii) $T: P_1 \to P_2$, where $T(p(x)) = x\,p(x) + p(0)$

(iv) $T: P_1 \to P_2$, where $T(ax + b) = ax^2 + (a - b)x$

Ans.:
(i) Linear
(ii) Non-linear
(iii) Linear
(iv) Linear

3. Determine whether the function is a linear transformation. Justify your answer.

(i) $T: M_{22} \to M_{23}$, where B is a fixed 2×3 matrix and $T(A) = AB$

(ii) $T : M_{22} \to M_{22}$, where

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & c-d \\ c+d & 2a \end{bmatrix}$$

(iii) $T : V \to R$, where V is an inner product space and $T(u) = \|u\|$

$$\left[\textbf{Ans.: } \text{(i) Linear (ii) Linear (iii) Non-linear}\right]$$

4. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$ and $\mathbf{v}_3 = (3, 3, 4)$ and let $T : R^3 \to R^2$ be the linear transformation such that

$$T(\mathbf{v}_1) = (1,0), T(\mathbf{v}_2) = (-1,1),$$
$$T(\mathbf{v}_3) = (0,1)$$

Find a formula for $T(x_1, x_2, x_3)$ and use that formula to find $T(7, 13, 7)$.

$$\left[\textbf{Ans.: } (-41x_1 + 9x_2 + 24x_3, 14x_1 - 3x_2 - 8x_3), (-2, 3)\right]$$

5. Let $T : R^2 \to P_2$ be a linear transformation for which $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 - 2x$ and $T\begin{bmatrix} 3 \\ -1 \end{bmatrix} = x + 2x^2$. Find $T\begin{bmatrix} a \\ b \end{bmatrix}$ and $T\begin{bmatrix} -7 \\ 9 \end{bmatrix}$.

$$\left[\textbf{Ans.: } \left(\frac{a+3b}{4}\right) - \left(\frac{a+7b}{4}\right)x + \left(\frac{a-b}{2}\right)x^2, \quad 5 - 14x - 8x^2\right]$$

6. Let $T : P_2 \to P_2$ be a linear transformation for which

$$T(1+x) = 1 + x^2,$$
$$T(x+x^2) = x - x^2, T(1+x^2) = 1 + x + x^2$$

Find $T(a + bx + cx^2)$ and $T(4 - x + 3x^2)$.

$$\left[\textbf{Ans.: } a + cx + \left(\frac{3a-b-c}{2}\right)x^2, \quad 4 + 3x + 5x^2\right]$$

7. Let $T : M_{22} \to R$ be a linear transformation. Show that there are scalars a, b, c and d such that

$$T\begin{bmatrix} w & x \\ y & z \end{bmatrix} = aw + bx + cy + dz$$

for all $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ in M_{22}.

8. Let $T_1 : R^2 \to M_{22}$ and $T_2 : R^2 \to R^2$ be the linear transformations given by $T_1\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b & b \\ 0 & a-b \end{bmatrix}$ and

$$T_2 = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c+d \\ -d \end{bmatrix}.$$

Find $(T_1 \circ T_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(T_1 \circ T_2)\begin{bmatrix} x \\ y \end{bmatrix}$.

$$\left[\textbf{Ans.: } \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 2x & -y \\ 0 & 2x+2y \end{bmatrix}\right]$$

9. Find the domain and codomain of $T_2 \circ T_1$, and find $(T_2 \circ T_1)\ (x, y)$.

(a) $T_1\ (x, y) = (x - 3y, 0)$, $T_2\ (x, y) = (4x - 5y, 3x - 6y)$

(b) $T_1\ (x, y) = (2x, -3y, x + y)$, $T_2\ (x, y, z) = (x - y, y + z)$

$$\left[\begin{array}{l}\textbf{Ans.: } \text{domain: } R^2, \text{codomain: } R^2, (2x-3y, 2x+3y) \\ \text{domain: } R^2, \text{codomain: } R^2, (4x-12y, 3x-9y)\end{array}\right]$$

10. Let $T_1 : P_1 \to P_2$ and $T_2 : P_2 \to P_2$ be the linear transformations given by
$T_1(p(x)) = x\,p(x)$ and
$T_2(p(x)) = p(2x+4)$.

Find $(T_2 \circ T_1)(a_0 + a_1 x)$.

[**Ans.:** $a_0(2x + 4) + a_1(2x + 4)^2$]

11. Find the standard matrix of the linear operator T: $R^2 \to R^2$ that first dilates a vector with factor $k = 2$, then rotates the resulting vector by an angle of 45°, and then reflects that vector about the y-axis.

$$\left[\textbf{Ans.: } \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\right]$$

12. Find the standard matrix of the linear operator T: $R^3 \to R^3$ that first rotates a vector about the x-axis by 270°, then rotates the resulting vector about the y-axis by 90°, and then rotates that vector about the z-axis by 180°.

$$\left[\textbf{Ans.: } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}\right]$$

3.7 KERNEL (NULL SPACE) AND RANGE OF A LINEAR TRANSFORMATION

Let V and W be two vector spaces and let T: $V \to W$ be a linear transformation. The kernel or null space of T, denoted by $\ker(T)$ or $N(T)$, is the set of all vectors in V that T maps into the zero vector, **0**. The range of T, denoted by $R(T)$, is the set of all vectors in W that are images of at least one vector in V under T.

Theorem 3.3: If T: $V \to W$ is a linear transformation then

(i) The kernel of T is a subspace of V
(ii) The range of T is a subspace of W

3.7.1 *Rank and Nullity of a Linear Transformation*

If T: $V \to W$ is a linear transformation then the rank of T is the dimension of the range of T and is denoted by rank (T). The nullity of T is the dimension of the kernel of T and is denoted by nullity (T).

Theorem 3.4: If A is $m \times n$ matrix and $T_A : R^n \to R^m$ is multiplication by A then the kernel of T_A is the null space of A and the range of T_A is the column space of A.

Hence, $$\text{nullity } (T_A) = \text{nullity } (A) \text{ and rank } (T_A) = \text{rank } (A)$$

From Theorem 3.4, we can conclude that

$$\text{Basis for } \ker(T) = \text{Basis for the Null space of } A, \text{ i.e. } [T]$$

and $$\text{Basis for } R(T) = \text{Basis for the column space of } A, \text{ i.e. } [T]$$

3.7.2 *Dimension Theorem for Linear Transformation*

Theorem 3.5: If T: $V \to W$ is a linear transformation from a finite dimensional vector space V to a vector space W then

$$\text{rank }(T) + \text{nullity }(T) = \dim V$$

Example 1: Let $T: R^2 \to R^2$ be the linear operator defined by

$$T(x, y) = (2x - y, -8x + 4y)$$

(i) Find a basis for ker (T).
(ii) Find a basis for $R(T)$.

Solution: (i) The basis for ker (T) is the basis for the solution space of the homogeneous system

$$\begin{aligned} 2x - \quad y &= 0 \\ -8x + 4y &= 0 \quad \Rightarrow \quad 2x - y = 0 \end{aligned}$$

Let
$$y = t$$
$$x = \frac{1}{2}t$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Hence,
$$\text{basis for ker }(T) = \{\mathbf{v}_1\} = \left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\}$$

(ii) The basis for the range of T is the basis for the column space of $[T]$.

$$[T] = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$$

Reducing the matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$
$$\sim \begin{bmatrix} 1 & -\frac{1}{2} \\ -8 & 4 \end{bmatrix}$$
$$R_2 + 8R_1$$
$$\sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

The leading 1 appears in column 1.

Hence, basis for $R(T)$ = basis for column space of $[T]$

$$= \left\{ \begin{bmatrix} 2 \\ -8 \end{bmatrix} \right\}$$

Example 2: Let $T : R^4 \rightarrow R^3$ be the linear transformation given by the formula $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4)$

(i) Find a basis for ker (T).
(ii) Find a basis for $R(T)$.
(iii) Verify the dimension theorem.

Solution: (i) The basis for ker (T) is the basis for the solution space of the homogeneous system

$$\begin{aligned} 4x_1 + x_2 - 2x_3 - 3x_4 &= 0 \\ 2x_1 + x_2 + x_3 - 4x_4 &= 0 \\ 6x_1 - 9x_3 + 9x_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right]$$

Reducing the augmented matrix to row-echelon form,

$$\left(\frac{1}{4}\right)R_1, \left(\frac{1}{3}\right)R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 2 & 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - 2R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 2 & -\frac{5}{2} & 0 \\ 0 & -\frac{1}{2} & -2 & \frac{9}{2} & 0 \end{array}\right]$$

$$2R_2, 2R_3$$

$$\sim\left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & -1 & -4 & 9 & 0 \end{array}\right]$$

$$R_3 + R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right)R_4$$

$$\sim\left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{3}{4}x_4 &= 0 \\ x_2 + 4x_3 - 5x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x_1 &= -\frac{1}{4}x_2 + \frac{1}{2}x_3 + \frac{3}{4}x_4 \\ x_2 &= -4x_3 + 5x_4 \end{aligned}$$

Assigning the free variable x_3 arbitrary value t,

$$\begin{aligned} x_2 &= -4t \\ x_1 &= -\frac{1}{4}(-4t) + \frac{1}{2}t = \frac{3}{2}t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}t \\ -4t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix} = t\mathbf{v}_1$$

Hence, basis for $\ker(T) = \{\mathbf{v}_1\} = \left\{\begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix}\right\}$

dimension for ker (T) = dim (ker (T)) = 1

(ii) The basis for the range of T is the basis for the column space of $[T]$.

$$[T] = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}$$

Reducing [T] to row echelon form,

$$\sim \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The leading 1's appear in columns 1, 2 and 4.

Hence, basis for $R(T)$ = basis for column space of $[T] = \left\{\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix}\right\}$

$$\dim\left(R(T)\right) = 3$$

(iii)

$$\text{rank } (T) = \dim\left(R(T)\right) = 3$$
$$\text{nullity } (T) = \dim\left(\ker (T)\right) = 1$$

$$\text{rank } (T) + \text{nullity } (T) = 3 + 1 = 4 = \dim R_4$$

Hence, the dimension theorem is verified.

Example 3: Let T be a multiplication by the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}$$

(i) Find a basis for the range of T.
(ii) Find a basis for the kernel of T.
(iii) Find the rank and nullity of A.
(iv) Find the rank and nullity of T.
(v) Verify the dimension theorem.

Solution: (i) The basis for the range of T is the basis for the column space of A.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2 - 3R_1, \quad R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$(-1)R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The leading 1's appear in columns 1 and 2.

Hence, basis for $R(T)$ = basis for column space of A

$$= \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right\}$$

$$\dim\left(R(T)\right) = 2$$

(ii) The basis for the kernel of T is the basis for the solution space of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 0 \\ 3x_1 + 5x_2 + 13x_3 &= 0 \\ -2x_1 - x_2 - 4x_3 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 3 & 5 & 13 & 0 \\ -2 & -1 & -4 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x_1 &= -2x_2 - 5x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Assigning the free variable x_3 arbitrary value t,

$$\begin{aligned} x_2 &= -2t \\ x_1 &= -2(-2t) - 5t = -t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{v}_1$$

Hence, basis for kernel $(T) = \{\mathbf{v}_1\} = \left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$

$$\dim(\ker(T)) = 1$$

(iii) rank $(A) = \dim(R(T)) = 2$
nullity $(A) = \dim(\ker(T)) = 1$

(iv) rank $(T) =$ rank $(A) = 2$
nullity $(T) =$ rank $(A) = 2$

(v) rank $(T) +$ nullity $(T) = 2 + 1 = 3$
For standard matrix A, number of columns $= n = 3$

Hence, dimension theorem i.e., rank(T) + nullity $(T) = n$, is verified.

Example 4: Let P: $P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = x\,p(x)$.

(i) Find a basis for the kernel of T.
(ii) Find a basis for the range of T.
(iii) Verify the dimension theorem.

Solution: Let $p(x) = a_0 + a_1 x + a_2 x^2$

$$T(p(x)) = a_0 x + a_1 x^2 + a_2 x^3$$

(i) The basis for kernel (T) is the basis for the solution space of $T(p(x)) = 0$

$$a_0x + a_1x^2 + a_2x^3 = 0$$

Comparing the coefficients of powers of x,

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, there is no basis for the kernel of T.

$$\dim(\ker(T)) = 0$$

(ii) Every vector in range T has the form

$$a_0x + a_1x^2 + a_2x^3$$

Hence, the vectors x, x^2 and x^3 span the range of T. Since these vectors are linearly independent, they form a basis for the range of T.

$$\text{Basis for } R(T) = \{x, x^2, x^3\}$$

$$\dim(R(T)) = 3$$

(iii)

$$\text{rank }(T) = \dim(R(T)) = 3$$

$$\text{nullity }(T) = \dim(\ker(T)) = 0$$

$$\text{rank }(T) + \text{nullity }(T) = 3 + 0 = 3 = \dim P_2$$

Hence, the dimension theorem is verified.

Example 5: Let $T : P_2 \to R^2$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2) = (a_0 - a_1, a_1 + a_2)$$

(i) Find a basis for ker (T).
(ii) Find a basis for $R(T)$.
(iii) Verify the dimension theorem.

Solution: (i) The basis for ker (T) is the basis for the solution space of the homogeneous system

$$a_0 - a_1 = 0$$
$$a_1 + a_2 = 0$$

Let

$$a_2 = t$$
$$a_1 = -t$$
$$a_0 = -t$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = t\,\mathbf{v}_1$$

Hence, basis for $\ker(T) = \{\mathbf{v}_1\} = \{-1 - x + x^2\}$

(ii) The basis for the range of T is the basis for the column space of $[T]$.

$$[T] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The leading 1's appear in columns 1 and 2.

Hence, basis for $R(T)$ = basis for column space of $[T]$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

(iii)

$$\text{rank}(T) = \dim\left(R(T)\right) = 2$$

$$\text{nullity}(T) = \dim\left(\ker(T)\right) = 1$$

$$\text{rank}(T) + \text{nullity}(T) = 2 + 1 = 3 = \dim P_2$$

Hence, dimension theorem is verified.

Example 6: Let $T: M_{22} \to M_{22}$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$$

(i) Find a basis for $\ker(T)$.

(ii) Find a basis for $R(T)$.

Solution: (i) The basis for ker (T) is the basis for the solution space of the homogeneous system

$$\begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} a + b \qquad\quad &= 0 \\ b + c \qquad &= 0 \\ a + \qquad\quad d &= 0 \\ b + \quad d &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{R_3 - R_1}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_3 + R_2,\, R_4 - R_2}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_4 + R_3}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{\left(\frac{1}{2}\right)R_4}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} a+b \qquad\quad &= 0 \\ b+c \qquad &= 0 \\ c+d &= 0 \\ d &= 0 \end{aligned}$$

Solving these equations, $a = 0$, $b = 0$, $c = 0$, $d = 0$

$$\ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

Hence, the kernel of T has no basis.

(ii)
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

This shows that each vector in $R(T)$ is the linear combination of four independent matrices.

$$\text{Hence, basis} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

Example 7: Let W be the vector space of all symmetric 2×2 matrices and Let $T : W \to P_2$ be the linear transformation defined by

$$T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2$$

Find the rank and nullity of T.

Solution: The nullity of T is easier to find directly than the rank. To find ker (T),

$$T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a-b) - (b-c)x + (c-a)x^2 = 0$$

Equating corresponding coefficients,

$$a - b = 0$$
$$b - c = 0$$
$$c - a = 0$$

Hence, $\quad a = b = c$

Let $\quad a = b = c = t$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} t & t \\ t & t \end{bmatrix} = t\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = t\mathbf{v}_1$$

$$\ker(T) = \{\mathbf{v}_1\} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Hence, basis for $\quad \ker(T) = \{\mathbf{v}_1\} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

$$\dim(\ker(T)) = 1$$
$$\text{nullity}(T) = 1$$

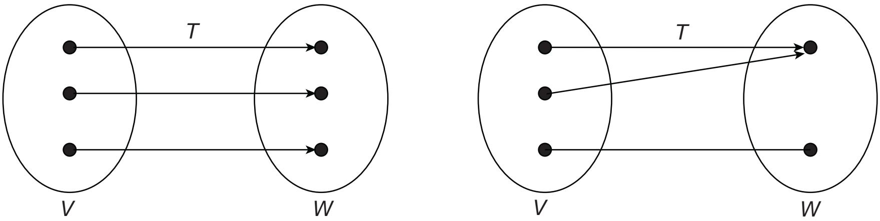
We know that

$$\text{rank}(T) + \text{nullity}(T) = \dim W = 3$$
$$\therefore \text{rank}(T) = 3 - 1 = 2$$

3.7.3 *One-to-one Transformation*

Let V and W be two vector spaces. A linear transformation $T : V \to W$ is one-to-one if T maps distinct vectors in V to distinct vectors in W.

A one-to-one transformation is also called injective transformation.



(i) T is one-to-one (ii) T is not one-to-one

Fig. 3.7

Theorem 3.6: A linear transformation $T : V \to W$ is one-to-one if and only if ker $(T) = \{\mathbf{0}\}$.

Theorem 3.7: A linear transformation $T : V \to W$ is one-to-one if and only if dim (ker (T)) = 0, i.e., nullity $(T) = 0$.

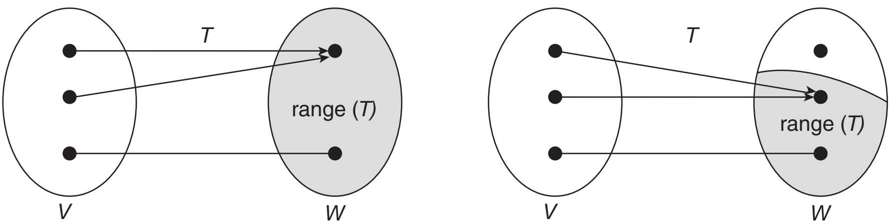
Theorem 3.8: A linear transformation $T : V \to W$ is one-to-one if and only if rank $(T) = \dim V$.

Theorem 3.9: If A is an $m \times n$ matrix and T_A: $R^n \to R^m$ is multiplication by A then T_A is one-to-one if and only if rank $(A) = n$.

Theorem 3.10: If A is an $n \times n$ matrix and T_A: $R^n \to R^n$ is multiplication by A then T_A is one-to-one if and only if A is an invertible matrix.

3.7.4 *Onto Transformation*

Let V and W be two vector spaces. A linear transformation $T : V \to W$ is onto if the range of T is W, i.e., T is onto if and only if for every $\mathbf{w}$ in W, there is a $\mathbf{v}$ in V such that $T(\mathbf{v}) = W$. An onto transformation is also called surjective transformation.



(i) T is onto (ii) T is not onto

Fig. 3.8

Theorem 3.11: A linear transformation $T : V \to W$ is onto if and only if rank $(T) = \dim W$.

Theorem 3.12: If A is an $m \times n$ matrix and T_A: $R^n \to R^m$ is multiplication by A then T_A is onto if and only if rank $(A) = m$.

Theorem 3.13: Let $T : V \rightarrow W$ be a linear transformation and let dim V = dim W

(i) If T is one-to-one, then it is onto.
(ii) If T is onto, then it is one-to-one.

3.7.5 *Bijective Transformation*

If a transformation $T : V \rightarrow W$ is both one-to-one and onto then it is called bijective transformation.

3.7.6 *Isomorphism*

A bijective transformation from V to W is known as an isomorphism between V and W.

Theorem 3.14: Let V be a finite dimensional real vector space. If dim $(V) = n$, then there is an isomorphism from V to R^n.

Theorem 3.15: Let V and W be a finite dimensional vector spaces. If dim (V) = dim (W) then V and W are isomorphic.

Example 1: In each case, determine whether the linear transformation is one-to-one, onto, or both or neither.

(i)	$T : R^2 \rightarrow R^2$,	where	$T(x, y) = (x + y, x - y)$
(ii)	$T : R^2 \rightarrow R^3$,	where	$T(x, y) = (x - y, y - x, 2x - 2y)$
(iii)	$T : R^3 \rightarrow R^2$,	where	$T(x, y, z) = (x + y + z, x - y - z)$
(iv)	$T : R^3 \rightarrow R^3$,	where	$T(x, y, z) = (x + 3y, y, z + 2x)$

Solution: (i) (a) A linear transformation is one-to-one if and only if ker $(T) = \{\mathbf{0}\}$

Let
$$T(x, y) = \mathbf{0}$$
$$(x + y, x - y) = (0, 0)$$
$$x + y = 0$$
$$x - y = 0$$

Solving these equations,
$$x = 0$$
$$y = 0$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) A linear transformation is onto if $R(T) = W$

Let $\mathbf{v} = (x, y)$ and $\mathbf{w} = (a, b)$ be in R^2, where a and b are real numbers such that $T(\mathbf{v}) = \mathbf{w}$.
$$T(x, y) = (a, b)$$
$$(x + y, x - y) = (a, b)$$
$$x + y = a$$
$$x - y = b$$

Solving these equations,

$$x = \frac{a+b}{2}$$

$$y = \frac{a-b}{2}$$

Thus, for every $\mathbf{w} = (a, b)$ in R^2, there exists a $\mathbf{v} = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$ in R^2.
Hence, T is onto.

(ii) (a) Let $$T(x, y) = \mathbf{0}$$

$$(x - y, y - x, 2x - 2y) = (0, 0, 0)$$

$$x - y = 0$$

$$y - x = 0$$

$$2x - 2y = 0$$

$$\therefore x = y$$

Let $$y = t$$

$$x = t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) Let $\mathbf{v} = (x, y)$ be in R^2 and $\mathbf{w} = (a, b, c)$ be in R^3, where a, b, c are real numbers such that $T(\mathbf{v}) = \mathbf{w}$.

$$T(x, y) = (a, b, c)$$

$$(x - y, y - x, 2x - 2y) = (a, b, c)$$

$$x - y = a$$

$$y - x = b \quad \Rightarrow \quad x - y = -b$$

$$2x - 2y = c \quad \Rightarrow \quad x - y = \frac{c}{2}$$

$$\therefore a = -b = \frac{c}{2}$$

Thus, $T(\mathbf{v}) = \mathbf{w}$ only when $a = -b = \frac{c}{2}$, not for all values of a, b and c.
Hence, T is not onto.

(iii) (a) Let $$T(x, y, z) = \mathbf{0}$$

$$(x + y + z, x - y - z) = (0, 0)$$

$$x + y + z = 0$$

$$x - y - z = 0$$

Solving these equations,

$$x = 0$$

Let
$$z = t$$
$$y = -t$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$
$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) Let $\mathbf{v} = (x, y, z)$ be in R^3 and $\mathbf{w} = (a, b)$ be in R^2, where a, b are real numbers such that $T(\mathbf{v}) = \mathbf{w}$.
$$T(x, y, z) = (a, b)$$
$$(x + y + z, x - y - z) = (a, b)$$
$$x + y + z = a$$
$$x - y - z = b$$

Solving these equations,
$$x = \frac{a+b}{2}$$
Let
$$z = t$$
$$y = \frac{a-b-2t}{2}$$

Thus, for every $\mathbf{w} = (a, b)$ in R^2, there exists a $\mathbf{v} = \left(\frac{a+b}{2}, \frac{a-b-2t}{2}, t\right)$ in R^2. Hence, T is onto.

(iv) (a) Let
$$T(x, y, z) = \mathbf{0}$$
$$(x + 3y, y, z + 2x) = (0, 0, 0)$$
$$x + 3y = 0$$
$$y = 0$$
$$z + 2x = 0$$

Solving these equations,
$$x = 0$$
$$y = 0$$
$$z = 0$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) Let $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (a, b, c)$ be in R^3, where a, b, c are real numbers such that $T(\mathbf{v}) = \mathbf{w}$.
$$T(x, y, z) = (a, b, c)$$
$$(x + 3y, y, z + 2x) = (a, b, c)$$

$$\begin{aligned} x+3y&=a \\ y&=b \\ z+2x&=c \end{aligned}$$

Solving these equations,

$$\begin{aligned} x&=a-3b \\ y&=b \\ z&=c-2a+6b \end{aligned}$$

Thus, for every $\mathbf{w}=(a, b, c)$ in R^2, there exists a $\mathbf{v}=(a-3b, b, c-2a+6b)$ in R^2. Hence, T is onto.

Example 2: In each case, determine whether multiplication by A is one-to-one, onto, both or neither.

(i) $A=\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 1 & 5 \\ 4 & -2 \\ 5 & 3 \end{bmatrix}$ (iii) $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution: (i)

$$A=\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2-2R_1, \quad R_3+3R_1$$

$$\sim\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

rank (A) = number of non-zero rows = 1
$\neq 2$, (number of columns)

Hence, A is not one-to-one.

Also, rank $(A) \neq 3$, (number of rows)

Hence, A is not onto.

(ii)

$$A=\begin{bmatrix} 1 & 5 \\ 4 & -2 \\ 5 & 3 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_2-4R_1, R_3-5R_1$$

$$\sim\begin{bmatrix} 1 & 5 \\ 0 & -22 \\ 0 & -22 \end{bmatrix}$$

$$R_3 - R_2$$
$$\sim \begin{bmatrix} 1 & 5 \\ 0 & -22 \\ 0 & 0 \end{bmatrix}$$

$$\left(-\frac{1}{22}\right)R_2$$
$$\sim \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{rank}\ (A) &= \text{number of non-zero rows} \\ &= 2\ (\text{number of columns}) \end{aligned}$$

Hence, A is one-to-one.

Also, $\quad \text{rank}\ (A) \neq 3\ (\text{number of rows})$

Hence, A is not onto.

(iii) $$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Reducing the matrix A to row echelon form,

$$R_3 - R_1$$
$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$R_3 - R_2$$
$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\left(-\frac{1}{2}\right)R_3$$
$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{rank}\ (A) &= \text{number of non-zero rows} \\ &= 3\ (\text{number of columns}) \end{aligned}$$

Hence, A is one-to-one.

Also, $\quad \text{rank}\,(A) = 3$ (number of rows)

Hence, A is onto.

Example 3: In each case, determine whether the linear transformation is one-to-one, onto, both or neither.

(i) $T : P_2 \to P_2$, where $T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1) + (a_2 + 2a_1)x$

(ii) $T : P_2 \to P_2$, where $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 (x + 1) + a_2 (x + 1)^2$

(iii) $T : R^2 \to P_1$, where $T(a, b) = a + (a + b)x$

(iv) $T : P_2 \to R^3$, where $T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ a + b - 3c \\ c - a \end{bmatrix}$

Solution: (i) (a) Let $T(a_0 + a_1 x + a_2 x^2) = \mathbf{0}$

$$(a_0 + a_1) + (a_2 + 2a_1)\, x = 0$$

$$a_0 + a_1 = 0$$

$$a_2 + 2a_1 = 0$$

Let

$$a_2 = t$$

$$a_1 = -\frac{1}{2}t$$

$$a_0 = \frac{1}{2}t$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one to one.

(b) $\dim\left(\ker(T)\right) = 1 = \text{nullity}\,(T)$

From the dimension theorem,

$$\begin{aligned} \text{rank}\,(T) &= \dim P_2 - \text{nullity}\,(T) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Dimension of $W\,(P_2) = 3$.

$$\text{rank}\,(T) \neq \dim W$$

Hence, T is not onto.

(ii) (a) Let $$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2 = \mathbf{0}$$

i.e., $$a_0 + a_1x + a_1 + a_2x^2 + 2a_2x + a_2 = 0$$

$$(a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2x^2 = 0$$

$$a_0 + a_1 + a_2 = 0$$

$$a_1 + 2a_2 = 0$$

$$a_2 = 0$$

Solving these equations,

$$a_0 = 0$$

$$a_1 = 0$$

$$a_2 = 0$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) $\dim(\ker(T)) = 0 = \text{nullity}(T)$
From the dimension theorem,

$$\text{rank}(T) = \dim P_2 - \text{nullity}(T)$$

$$= 3 - 0$$

$$= 3$$

Dimension of $W(P_2) = 3$.

$$\therefore \text{rank}(T) = \dim W$$

Hence, T is onto.

(iii) (a) Let $$T(a, b) = a + (a + b)x = \mathbf{0}$$

$$a = 0$$

$$a + b = 0$$

Solving these equations,

$$a = 0$$

$$b = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) dim (ker (T)) = 0 = nullity (T)
From the dimension theorem,

$$\begin{aligned} \text{rank } (T) &= \dim R^2 - \text{nullity } (T) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

Dimension of $W(P_1) = 2$.

$$\text{rank } (T) = \dim W$$

Hence, T is onto.

(iv) (a) Let $T(a + bx + cx^2) = \mathbf{0}$

$$\begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2a-b &= 0 \\ a+b-3c &= 0 \\ c-a &= 0 \end{aligned}$$

$$\therefore a = \frac{b}{2} = c$$

Let

$$\begin{aligned} c &= t \\ b &= 2t \\ a &= t \end{aligned}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) dim (ker (T)) = 1 = nullity (T)
From the dimension theorem,

$$\begin{aligned} \text{rank } (T) &= \dim P_2 - \text{nullity } (T) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Dimension of $W(R^3) = 3$.

$$\text{rank } (T) \neq \dim \text{W}$$

Hence, T is not onto.

Example 4: In each case, determine whether linear transformation is one-to-one, onto, both or neither.

(i) $T: M_{22} \to M_{22}$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(ii) $T: M_{22} \to M_{22}$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2d & 0 \\ 0 & 0 \end{bmatrix}$

(iii) $T: M_{22} \to R^3$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix}$

Solution: (i) (a) Let $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$d = 0$$
$$-b = 0 \Rightarrow b = 0$$
$$-c = 0 \Rightarrow c = 0$$
$$a = 0$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) $\dim(\ker(T)) = 0 = \text{nullity}(T)$
From the dimension theorem,

$$\begin{aligned} \text{rank}(T) &= \dim M_{22} - \text{nullity}(T) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

Dimension of $W(M_{22}) = 4$

$$\therefore \text{rank}(T) = \dim W$$

Hence T, is onto.

(ii) (a) Let $$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$$

$$\begin{bmatrix} 2d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2d = 0$$

$$\therefore d = 0$$

Let $$a = t_1$$

$$b = t_2$$

$$c = t_3$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 0 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) $\dim(\ker(T)) = 3 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned} \text{rank}(T) &= \dim M_{22} - \text{nullity}(T) \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

Dimension of $W(M_{22}) = 4$.

$$\therefore \text{rank}(T) \neq \dim W$$

Hence, T is not onto.

(iii) (a) Let $$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$$

$$\begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a+b = 0$$

$$b+c = 0$$

$$c+d = 0$$

Let $$d = t$$

$$c = -t$$

$$b = t$$

$$a = -t$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one to one.

(b) $\dim(\ker(T)) = 1 = \text{nullity}(T)$
From the dimension theorem,

$$\begin{aligned} \text{rank}(T) &= \dim M_{22} - \text{nullity}(T) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Dimension of $W(R^3) = 3$.

$$\therefore \text{rank}(T) = \dim W$$

Hence, T is onto.

3.8 INVERSE LINEAR TRANSFORMATIONS

If $T : V \rightarrow W$ is a linear transformation then the range of T is the subspace of W consisting of all images of vectors in V under T. If T is one-to-one then each vector **w** in $R(T)$ is the image of a unique vector **u** in V. Hence, inverse linear transformation $T^{-1}: W \rightarrow V$ maps **w** back into **v**.

Theorem 3.16: If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are one-to-one transformations, then

(i) $T_2 \circ T_1$ is one-to-one.

(ii) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

The standard matrix of the inverse of a composition is the product of the inverses of the standard matrices of the individual operators in the reverse order.

Example 1: Let $T : R^3 \rightarrow R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ 2x_2 - x_3,\ 2x_1 + 3x_2)$$

Determine whether T is one-to-one. If so, find $T^{-1}(x_1, x_2, x_3)$.

Solution: The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

$$\det[T] = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{vmatrix}$$
$$= 1(0+3)+1(0+2)+1(-4)$$
$$= 3+2-4$$
$$= 1 \neq 0$$

Hence, the matrix is invertible and T is one-to-one. The standard matrix of T^{-1} is found by elementary row transformation.

Consider, $[T] = I\,[T]$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}[T]$$

$R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}[T]$$

$\left(\frac{1}{2}\right)R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 0 & 1 \end{bmatrix}[T]$$

$R_1 + R_2,\ R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & -\frac{5}{2} & 1 \end{bmatrix}[T]$$

$2R_3$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ -4 & -5 & 2 \end{bmatrix}[T]$$

$$R_2 + \frac{1}{2}R_3,\ R_1 - \frac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} [T]$$

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [T^{-1}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 3x_2 - x_3 \\ -2x_1 - 2x_2 + x_3 \\ -4x_1 - 5x_2 + 2x_3 \end{bmatrix}$$

Expressing in horizontal notation,

$$T^{-1}(x_1, x_2, x_3) = (3x_1 + 3x_2 - x_3, -2x_1 - 2x_2 + x_3, -4x_1 - 5x_2 + 2x_3)$$

Example 2: Let $T: R^3 \to R^3$ be a multiplication by A. Determine whether T has an inverse. If so, find $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ where $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Solution:

$$\det(A) = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= 1(0-1) - 4(0+1) - 1(1+2)$$

$$= -1 - 4 - 3$$

$$= -8 \neq 0$$

The matrix A is invertible. Hence, T has an inverse.
The inverse can be found by elementary row transformation.
Consider, $A = I\,A$

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 - R_1,\ R_3 + R_1$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -2 & 2 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

$$R_3 - 5R_2,\ R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & 1 \end{bmatrix} A$$

$$\left(\frac{1}{4}\right)R_3$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \end{bmatrix} A$$

$$R_2 + R_3,\ R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8}x_1 + \frac{1}{8}x_2 - \frac{3}{4}x_3 \\ \frac{1}{8}x_1 + \frac{1}{8}x_2 + \frac{1}{4}x_3 \\ -\frac{3}{8}x_1 + \frac{5}{8}x_2 + \frac{1}{4}x_3 \end{bmatrix}$$

Example 3: Let $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ be the linear operators given by the formula

$$T_1(x, y) = (x + y, x - y) \text{ and } T_2(x, y) = (2x + y, x - 2y)$$

(i) Show that T_1 and T_2 are one-to-one.

(ii) Find formulas for $T_1^{-1}(x, y)$ and $T_2^{-1}(x, y)$ and $(T_2 \circ T_1)^{-1}(x, y)$.

(iii) Verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

Solution: (i) T_1 and T_2 are one-to-one if $\ker(T_1) = \{\mathbf{0}\}$ and $\ker(T_2) = \{\mathbf{0}\}$

$$T_1\{x, y\} = (x + y, x - y) = (0, 0)$$

$$x + y = 0$$
$$x - y = 0$$

Solving these equations,

$$x = 0$$
$$y = 0.$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \ker(T_1) = \{\mathbf{0}\}$$

$$T_2(x, y) = (2x + y, x - 2y) = (0, 0)$$

$$2x + y = 0$$
$$x - 2y = 0$$

Solving these equations,

$$x = 0$$
$$y = 0$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\therefore \ker(T_2) = \{\mathbf{0}\}$$

Hence, T_1 and T_2 are one-to-one.

(ii) The standard matrix of T_1 is

$$[T_1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The standard matrix of T_1^{-1} is

$$\left[T_1^{-1}\right] = [T_1]^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$T_1^{-1}(x, y) = \left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right)$$

The standard matrix of T_2 is

$$[T_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

The standard matrix of T_2^{-1} is

$$\left[T_2^{-1}\right] = \left[T_2\right]^{-1} = -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$T_2^{-1}(x, y) = \left(\frac{2}{5}x + \frac{1}{5}y, \frac{1}{5}x - \frac{2}{5}y\right)$$

(iii) $$(T_2 \circ T_1) = [T_2][T_1] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

$$[T_2 \circ T_1]^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

$$(T_2 \circ T_1)^{-1}(x, y) = \left(\frac{3}{10}x - \frac{1}{10}y, \frac{1}{10}x + \frac{3}{10}y\right)$$

$$T_1^{-1} \circ T_2^{-1} = [T_1]^{-1}[T_2]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

$$\therefore [T_2 \circ T_1]^{-1} = T_1^{-1} \circ T_2^{-1}$$

Exercise 3.2

1. Let $T : R^2 \to R^2$ be the linear transformation defined by

$T(x, y) = (x, 0)$

(i) Which of the following vectors are in ker (T)?

(a) $(0, 2)$ (b) $(2, 2)$

(ii) Which of the following vectors are in $R(T)$?

(a) $(3, 0)$ (b) $(3, 2)$

(iii) Find ker (T).

(iv) Find $R(T)$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) (a)} \\ \text{(ii) (a)} \\ \text{(iii) } \{(0, x)\} \\ \text{(iv) } \{(x, 0)\}\end{array}\right]$$

2. Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(x, y, z) = (x + 2y - z, \; y + z, x + y - 2z)$$

(i) Find a basis and the dimension for the range of T.

(ii) Find a basis and the dimension for the kernel of T.

(iii) Verify the dimension theorem.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \{(1,0,1), (0,1,-1)\}, 2 \\ \text{(ii) } \{(3,-1,1)\}, 1\end{array}\right]$$

3. Let $T : R^4 \to R^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, \; x_1 + x_2 + 3x_3 - 3x_4)$$

(i) Find a basis for $R(T)$.

(ii) Find a basis for ker (T).

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \{(1,1,1), (0,1,2)\} \\ \text{(ii) } \{(2,1,-1,0), (1,2,0,1)\}\end{array}\right]$$

4. Let $T : R^3 \to R^4$ be the linear transformation defined by

$$T(x, y, z) = (x + y + z, x + 2y - 3z, \; 2x + 3y - 2z, 3x + 4y - z).$$

(i) Find a basis and the dimension for $R(T)$.

(ii) Find a basis and the dimension for ker (T).

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \{(1,1,2,3), (0,1,1,1)\}, 2 \\ \text{(ii) } \{(-5,4,1)\}, 1\end{array}\right]$$

5. Let T be a multiplication by the matrix A where

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

(i) Find a basis for the range of T.

(ii) Find a basis for the kernel of T.

(iii) Find the rank and nullity of T.

(iv) Find the rank and nullity of A.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ \text{(ii) } \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \text{(iii) Rank } (T) = 1, \text{ nullity } (T) = 2 \\ \text{(iv) Rank } (A) = 1, \text{ nullity } (A) = 2\end{array}\right]$$

6. Let $T: P_2 \to P_2$ be the linear transformation defined by

$T(ax^2 + bx + c) = (a + c)x^2 + (b + c)\,x$

(i) Find a basis for ker (T).

(ii) Find a basis for the range of T.

$$\left[\begin{array}{l}\textbf{Ans.:}\ \text{(i)}\ \{-x^2 - x + 1\} \\ \qquad\quad \text{(ii)}\ \{x^2, x\}\end{array}\right]$$

7. Determine whether any of the following vectors are in the range of

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

(i) $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

[**Ans. :** (ii) and (iv)]

8. Let $P: M_{22} \to M_{22}$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & 0 \\ 0 & c-d \end{bmatrix}$$

(i) Determine whether any of the following matrices are in ker (T).

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

(ii) Determine whether any of the following matrices are in $R(T)$.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$

[**Ans.:** (i) (b) and (d) (ii) (b) and (d)]

9. Find the rank and nullity of the given linear transformations and determine whether T is one-to-one or onto.

(i) $T: R^2 \to R^2$, where $T(x, y) = (x, x + y)$

(ii) $T: R^3 \to R^2$, where $T(x, y, z) = (x - z, z - y)$

(iii) $T: R^2 \to R^3$, where $T(x, y) = (x + y, 2x + y, x)$

(iv) $T: R^3 \to R^1$, where $T(x, y, z) = 0$

(v) $T: P_2 \to P_2$, where $T(a_0 + a_1x + a_2x^2) = a_0 x$

(vi) $T: P_2 \to P_2$, where $T(a_0 + a_1x + a_2x^2) = 0$

(vii) $T: P_2 \to P_2$, where $T(a_0 + a_1x + a_2x^2) = (a_2 - a_1)\,x^2 + (a_1 - a_0)\,x$

(viii) $T: M_{22} \to M_{22}$, where

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & 0 \\ 0 & c-d \end{bmatrix}$$

(ix) $T: M_{22} \to R^1$, where

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b + 2c - 3d$$

(x) $T: P_2 \to M_{22}$, where

$$T(ax^2 + bx + c) = \begin{bmatrix} a & 2b \\ 0 & a \end{bmatrix}$$

(xi) $T: R^3 \to M_{22}$, where

$$T(a, b, c) = \begin{bmatrix} a-b & b-c \\ a+b & b+c \end{bmatrix}$$

$$\left[\begin{array}{ll}\textbf{Ans.:} & \text{(i) nullity } (T) = 0, \\ & \quad \text{rank}(T) = 2, \\ & \quad \text{one-to-one and onto} \\ & \text{(ii) nullity } (T) = 1, \\ & \quad \text{rank}(T) = 2, \\ & \quad \text{not one-to-one but onto} \\ & \text{(iii) nullity } (T) = 0, \\ & \quad \text{rank}(T) = 2, \\ & \quad \text{one-to-one but not onto}\end{array}\right]$$

[(iv) nullity $(T) = 3$,
rank$(T) = 0$,
neither one-to-one nor onto
(v) nullity $(T) = 2$,
rank$(T) = 1$,
neither one-to-one nor onto
(vi) nullity $(T) = 3$,
rank$(T) = 0$,
neither one-to-one nor onto
(vii) nullity $(T) = 1$,
rank$(T) = 2$,
neither one-to-one nor onto
(viii) nullity $(T) = 2$,
rank$(T) = 2$,
neither one-to-one nor onto
(ix) nullity $(T) = 3$,
rank$(T) = 1$,
not one-to-one but onto
(x) nullity $(T) = 1$,
rank$(T) = 2$,
neither one-to-one nor onto
(xi) nullity $(T) = 0$,
rank$(T) = 3$,
one-to-one but not onto.]

10. In each case, let $R^3 \to R^3$ be a multiplication of A, Find $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$.

(i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

[**Ans.:**
(i) $\begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \end{bmatrix}$
(ii) $\begin{bmatrix} 3x_1 + 3x_2 - x_3 \\ -2x_1 - 2x_2 + x_3 \\ -4x_1 - 5x_2 + 2x_3 \end{bmatrix}$]

11. Let $T_1 : P_2 \to P_3$ and $T_2 : P_3 \to P_3$ be the linear transformations defined by

$T_1(p(x)) = x\ p(x)$ and $T_2(p(x)) = p(x+1)$

(i) Find formulas for $T_1^{-1}(p(x)), T_2^{-1}(p(x))$ and $(T_2 \circ T_1)^{-1}(p(x))$.

(ii) Verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

$\left[\textbf{Ans.:} \dfrac{p(x)}{x};\ p(x-1);\ \dfrac{p(x-1)}{x}\right]$

12. Let $T : P_1 \to R^2$ be the function defined by

$T(p(x)) = (p(0), p(1))$

(i) Find $T(1 - 2x)$.

(ii) Show that T is one-to-one.

(iii) Find $T^{-1}(2, 3)$.

[**Ans.:** (i) $(1, -1)$ (iii) $(2 + x)$

3.9 THE MATRIX OF A LINEAR TRANSFORMATION

Let $T : V \to W$ be a linear transformation of an n-dimensional vector space V to an m-dimensional vector space W ($n \neq 0$ and $m \neq 0$) and let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be bases for V and W respectively.
If A be the standard matrix of this transformation then

$$A = \left[[T(\mathbf{v}_1)]_{S_2} \middle| [T(\mathbf{v}_2)]_{S_2} \middle| \dots \middle| [T(\mathbf{v}_n)]_{S_2} \right]$$

satisfies
$$A[\mathbf{v}]_{S_1} = \left[T(\mathbf{v})\right]_{S_2} \qquad \dots(3.4)$$

for every vector $\mathbf{v}$ in V.

where $[\mathbf{v}]_{S_1}$ and $[T(\mathbf{v})]_{S_2}$ are the coordinate vectors of $\mathbf{v}$ and $T(\mathbf{v})$ w.r.t. the respective bases S_1 and S_2. The matrix A in Eq. (3.4) is called the matrix of T w.r.t. the bases S_1 and S_2. Figure 3.9 gives the graphical interpretation of Eq. (3.4).

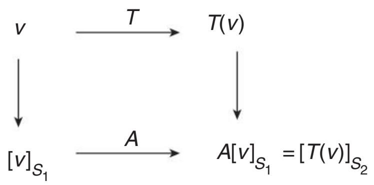


Fig. 3.9

The matrix of a linear transformation $T : V \to W$ w.r.t. the bases $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ for V and W, respectively is calculated as follows:

Step 1: Calculate $T(\mathbf{v}_j)$ for $j = 1, 2, \dots, n$.
Step 2: Find the coordinate vector $[T(\mathbf{v}_j)]_{S_2}$ w.r.t. the basis S_2 by expressing $T(\mathbf{v}_j)$ as a linear combination of the vectors in S_2.
Step 3: The matrix A of T w.r.t. the bases S_1 and S_2 is formed by choosing $\left[T(\mathbf{v}_j)\right]_{S_2}$ as the j^{th} column of A.

$$A = \left[[T(\mathbf{v}_1)]_{S_2} \middle| [T(\mathbf{v}_2)]_{S_2} \middle| \dots \middle| [T(\mathbf{v}_n)]_{S_2} \right]$$

The matrix A is denoted by the symbol $[T]_{S_2, S_1}$.

$$[T]_{S_2, S_1} = \left[[T(\mathbf{v}_1)]_{S_2} \middle| [T(\mathbf{v}_2)]_{S_2} \middle| \dots \middle| [T(\mathbf{v}_n)]_{S_2} \right]$$

and $[T]_{S_2, S_1} [\mathbf{v}]_{S_1} = \left[T(\mathbf{v})\right]_{S_2}$

3.9.1 *Matrices of Linear Operators*

If $T : V \to V$ is a linear operator and $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is the basis for V then the matrix of the linear operator is

$$[T]_{S_1} = \left[[T(\mathbf{v}_1)]_{S_1} \middle| [T(\mathbf{v}_2)]_{S_1} \middle| \dots \middle| [T(\mathbf{v}_n)]_{S_1} \right]$$

and $[T]_{S_1} [\mathbf{v}]_{S_1} = [T(\mathbf{v})]_{S_1}$

3.9.2 *Matrices of Identity Operators*

For identity operator $I : V \to V$, the matrix w.r.t. basis $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the $n \times n$ identity matrix,

i.e.
$$[I]_{S_1} = I$$

and
$$[I]_{S_1}[\mathbf{v}]_{S_1} = [I(\mathbf{v})]_{S_1} = [\mathbf{v}]_{S_1}$$

Theorem 3.17: If $T: R^n \to R^m$ is a linear transformation and if S_1 and S_2 are the standard bases for R^n and R^m respectively then

$$[T]_{S_2, S_1} = [T]$$

i.e., the matrix of T w.r.t. the standard bases is the standard matrix of T.

Example 1: Let $T: R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Find the matrix of the transformation T w.r.t. the bases $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2 and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for R^3, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution:

$$T(\mathbf{v}_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$$T(\mathbf{v}_2) = T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -5(5) + 13(2) \\ -7(5) + 16(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Expressing these vectors as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$T(\mathbf{v}_1) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$

$$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + k_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} k_1 - k_2 \\ 2k_2 + k_3 \\ -k_1 + 2k_2 + 2k_3 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 - k_2 \quad &= 1 \\ 2k_2 + k_3 &= -2 \\ -k_1 + 2k_2 + 2k_3 &= -5 \end{aligned}$$

Solving these equations,

$$k_1 = 1,\ k_2 = 0,\ k_3 = -2$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3$$

$$[T(\mathbf{v}_1)]_{S_2} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$T(\mathbf{v}_2) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$

$$\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + k_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} k_1 - k_2 \\ 2k_2 + k_3 \\ -k_1 + 2k_2 + 2k_3 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 - k_2 \quad &= 2 \\ 2k_2 + k_3 &= 1 \\ -k_1 + 2k_2 + 2k_3 &= -3 \end{aligned}$$

Solving these equations,

$$k_1 = 3,\ k_2 = 1,\ k_3 = -1$$

$$T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

$$[T(\mathbf{v}_2)]_{S_2} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

The matrix of the transformation w.r.t. the bases S_1 and S_2 is

$$[T]_{S_2,S_1} = \left[[T(\mathbf{v}_1)]_{S_2} \middle| [T(\mathbf{v}_2)]_{S_2} \right] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Example 2: Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(x, y, z) = (x + 2y + z,\ 2x - y,\ 2y + z)$$

Find the matrix of transformation T w.r.t.

(i) S_1 (ii) S_1 and S_2 (iii) S_2 and S_1 (iv) S_2

where $S_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

and $S_2 = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

Solution: (i) $T(x, y, z) = (x + 2y + z, 2x - y, 2y + z)$

Since S_1 is the standard bases for R^3, the matrix of T w.r.t. S_1 is the standard matrix of T.

$$[T]_{S_1} = [T] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(ii)

$$T(\mathbf{v}_1) = T(1, 0, 0) = (1, 2, 0)$$
$$T(\mathbf{v}_2) = T(0, 1, 0) = (2, -1, 2)$$
$$T(\mathbf{v}_3) = T(0, 0, 1) = (1, 0, 1)$$

Expressing $T(\mathbf{v}_1)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$T(\mathbf{v}_1) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$
$$(1, 2, 0) = k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1)$$
$$= (k_1, k_2, k_1 + k_2 + k_3)$$

Equating corresponding components,

$$k_1 = 1$$
$$k_2 = 2$$
$$k_1 + k_2 + k_3 = 0$$

Solving these equations

$$k_1 = 1, k_2 = 2, k_3 = -3$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3$$

$$[T(\mathbf{v}_1)]_{S_2} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Expressing $T(\mathbf{v}_2)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$T(\mathbf{v}_2) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$
$$(2, -1, 2) = k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1)$$
$$= (k_1, k_2, k_1 + k_2 + k_3)$$

Equation corresponding components,

$$\begin{aligned} k_1 \qquad\qquad &= 2 \\ k_2 \qquad &= -1 \\ k_1 + k_2 + k_3 &= 2 \end{aligned}$$

Solving these equations,

$$k_1 = 2,\ k_2 = -1,\ k_3 = 1$$

$$T(\mathbf{v}_2) = 2\mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$$

$$[T(\mathbf{v}_2)]_{S_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Expressing $T(\mathbf{v}_3)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$\begin{aligned} T(\mathbf{v}_3) &= k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 \\ (1, 0, 1) &= k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1) \\ &= (k_1, k_2, k_1 + k_2 + k_3) \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 \qquad\qquad &= 1 \\ k_2 \qquad &= 0 \\ k_1 + k_2 + k_3 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = 1,\ k_2 = 0,\ k_3 = 0$$

$$T(\mathbf{v}_3) = \mathbf{w}_1$$

$$[T(\mathbf{v}_3)]_{S_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{aligned} [T]_{S_2, S_1} &= \Big[[T(\mathbf{v}_1)]_{S_2} \,\Big|\, [T(\mathbf{v}_2)]_{S_2} \,\Big|\, [T(\mathbf{v}_3)]_{S_2} \Big] \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \end{aligned}$$

(iii) Since S_1 is the standard bases for R^3,

$$\begin{aligned} [T(\mathbf{w}_1)]_{S_1} &= [T(\mathbf{w}_1)] \\ [T(\mathbf{w}_2)]_{S_1} &= [T(\mathbf{w}_2)] \\ [T(\mathbf{w}_3)]_{S_1} &= [T(\mathbf{w}_3)] \end{aligned}$$

$$T(\mathbf{w}_1) = T(1, 0, 1) = (2, 2, 1)$$
$$T(\mathbf{w}_2) = T(0, 1, 1) = (3, -1, 3)$$
$$T(\mathbf{w}_3) = T(0, 0, 1) = (1, 0, 1)$$

Thus $[T]_{S_1, S_2} = \left[[T(\mathbf{w}_1)]_{S_1} \,\middle|\, [T(\mathbf{w}_2)]_{S_1} \,\middle|\, [T(\mathbf{w}_3)]_{S_1}\right]$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

(iv) Expressing $T(\mathbf{w}_1)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$

$$T(\mathbf{w}_1) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$
$$(2, 2, 1) = k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1)$$
$$= (k_1, k_2, k_1 + k_2 + k_3)$$

Equating corresponding components,

$$k_1 = 2$$
$$k_2 = 2$$
$$k_1 + k_2 + k_3 = 1$$

Solving these equations,

$$k_1 = 2,\ k_2 = 2,\ k_3 = -3$$
$$T(\mathbf{w}_1) = 2\mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3$$
$$[T(\mathbf{w}_1)]_{S_2} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

Expressing $T(\mathbf{w}_2)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$T(\mathbf{w}_2) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$
$$(3, -1, 3) = k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1)$$
$$= (k_1, k_2, k_1 + k_2 + k_3)$$

Equating corresponding components,

$$k_1 = 3$$
$$k_2 = -1$$
$$k_1 + k_2 + k_3 = 3$$

Solving these equations,

$$k_1 = 3,\ k_2 = -1,\ k_3 = 1$$

$$T(\mathbf{w}_2) = 3\mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$$

$$[T(\mathbf{w}_2)]_{S_2} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Expressing $T(\mathbf{w}_3)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$\begin{aligned} T(\mathbf{w}_3) &= k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 \\ (1, 0, 1) &= k_1(1, 0, 1) + k_2(0, 1, 1) + k_3(0, 0, 1) \\ &= (k_1, k_2, k_1 + k_2 + k_3) \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 \quad\quad\quad &= 1 \\ k_2 \quad\, &= 0 \\ k_1 + k_2 + k_3 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = 1,\ k_2 = 0,\ k_3 = 0$$

$$T(\mathbf{w}_3) = \mathbf{w}_1$$

$$[T(\mathbf{w}_3)]_{S_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$[T]_{S_2} = \left[[T(\mathbf{w}_1)]_{S_2} \,\middle|\, [T(\mathbf{w}_2)]_{S_2} \,\middle|\, [T(\mathbf{w}_3)]_{S_2} \right]$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

Example 3: Let $T : P_2 \to P_2$ be the linear operator defined by

$$T(p(x)) = p(2x + 1)$$

i.e., $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(2x + 1) + a_2(2x + 1)^2$

(i) Find $[T]_S$ w.r.t. the basis $S = \{1, x, x^2\}$.
(ii) Compute $T(2 - 3x + 4x^2)$.

Solution: (i) $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(2x + 1) + a_2(2x + 1)^2$

$$\begin{aligned} T(1) &= 1 \\ T(x) &= 2x + 1 = 1 + 2x \\ T(x^2) &= (2x + 1)^2 = 4x^2 + 4x + 1 = 1 + 4x + 4x^2 \end{aligned}$$

Since S is the standard basis,

$$[T(1)]_S = [T(1)] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_S = [T(x)] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [T(x^2)]_S = [T(x^2)] = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

Thus,

$$[T]_S = \left[[T(1)]_S \mid [T(x)]_S \mid [T(x^2)]_S \right]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii) The coordinate vector relative to S for the vector $\mathbf{p} = 2 - 3x + 4x^2$ is

$$[\mathbf{p}]_S = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$$\left[T(2 - 3x + 4x^2)\right]_S = [T(\mathbf{p})]_S = [T]_S [\mathbf{p}]_S$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix}$$

$$T(2 - 3x + 4x^2) = 3 + 10x + 16x^2.$$

Example 4: Let $T: P_1 \to P_2$ be the linear transformation defined by

$$T(p(x)) = x\,p(x)$$

(i) Find the matrix of T w.r.t. the bases

$$S_1 = \{\mathbf{v}_1, \mathbf{v}_2\} \text{ and } S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

where $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{w}_1 = x + 1$, $\mathbf{w}_2 = x - 1$, $\mathbf{w}_3 = x^2$.

(ii) If $p(x) = 3x - 2$, compute $T(p(x))$ directly and using matrix obtained in (i).

Solution: (i) $\qquad T(p(x)) = x\,p(x)$

$$T(\mathbf{v}_1) = T(1) = x \cdot 1 = x$$
$$T(\mathbf{v}_2) = T(x) = x \cdot x = x^2$$

Expressing $T(\mathbf{v}_1)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$T(\mathbf{v}_1) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3$$
$$x = k_1(x+1) + k_2(x-1) + k_3(x^2)$$
$$= (k_1 - k_2) + (k_1 + k_2)x + k_3x^2$$

Equating corresponding components,

$$\begin{aligned} k_1 - k_2 \quad &= 0 \\ k_1 + k_2 \quad &= 1 \\ k_3 &= 0 \end{aligned}$$

Solving these equations,

$$k_1 = \frac{1}{2}, k_2 = \frac{1}{2}, k_3 = 0$$

$$T(\mathbf{v}_1) = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2$$

$$\left[T(\mathbf{v}_1)\right]_{S_2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Expressing $T(\mathbf{v}_2)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$,

$$\begin{aligned} T(\mathbf{v}_2) &= k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 \\ x^2 &= k_1(x+1) + k_2(x-1) + k_3(x^2) \\ &= (k_1 - k_2) + (k_1 + k_2)x + k_3x^2 \end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_1 - k_2 \quad &= 0 \\ k_1 + k_2 \quad &= 0 \\ k_3 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = 0, k_2 = 0, k_3 = 1$$

$$T(\mathbf{v}_2) = \mathbf{w}_3$$

$$\left[T(\mathbf{v}_2)\right]_{S_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $[T]_{S_2, S_1} = \left[[T(\mathbf{v}_1)]_{S_2} \; \middle| [T(\mathbf{v}_2)]_{S_2} \; \right]$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) Direct computation

$T(p(x)) = x\ p(x) = x(3x-2) = 3x^2 - 2x$

Computation using matrix obtained in part (i):

The coordinate vector relative to S_2 for the vector $p(x) = 3x - 2 = -2 + 3x$ is

$$[p(x)]_{S_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$[T(3x-2)]_{S_2} = [T(p(x))]_{S_2} = [T]_{S_2, S_1} [p(x)]_{S_2}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} T(3x-2) &= (-1)(x+1) + (-1)(x-1) + 3x^2 \\ &= -x - 1 - x + 1 + 3x^2 \\ &= 3x^2 - 2x \end{aligned}$$

Example 5: Let $T: M_{22} \to M_{22}$ be defined by $T(A) = A^T$. Let

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $S_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

be bases for M_{22}. Find the matrix of T w.r.t S_1 and S_2.

Solution: Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ be the bases for M_{22}.

$$T(\mathbf{v}_1) = (\mathbf{v}_1)^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T(\mathbf{v}_2) = (\mathbf{v}_2)^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T(\mathbf{v}_3) = (\mathbf{v}_3)^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$T(\mathbf{v}_4) = (\mathbf{v}_4)^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Expressing $T(\mathbf{v}_1)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$,

$$T(\mathbf{v}_1) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 + k_4\mathbf{w}_4$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = k_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + k_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & k_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_3 & k_3 \end{bmatrix} + \begin{bmatrix} k_4 & 0 \\ 0 & k_4 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_4 & k_1 + k_2 \\ k_3 & k_3 + k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 \quad\quad + k_4 &= 1 \\ k_1 + k_2 \quad\quad &= 0 \\ k_3 \quad &= 0 \\ k_3 + k_4 &= 0 \end{aligned}$$

Solving these equations,

$$k_1 = 1, k_2 = -1, k_3 = 0, k_4 = 0$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - \mathbf{w}_2$$

$$[T(\mathbf{v}_1)]_{S_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Expressing $T(\mathbf{v}_2)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$,

$$T(\mathbf{v}_2) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 + k_4\mathbf{w}_4$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} k_1 + k_4 & k_1 + k_2 \\ k_3 & k_3 + k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 + \quad\quad k_4 &= 0 \\ k_1 + k_2 \quad\quad &= 0 \\ k_3 \quad &= 1 \\ k_3 + k_4 &= 0 \end{aligned}$$

Solving these equations,

$$k_1 = 1, k_2 = -1, k_3 = 1, k_4 = -1$$

$$T(\mathbf{v}_2) = \mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3 - \mathbf{w}_4$$

$$[T(\mathbf{v}_2)]_{S_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Expressing $T(\mathbf{v}_3)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$,

$$T(\mathbf{v}_3) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 + k_4\mathbf{w}_4$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k_1 + k_4 & k_1 + k_2 \\ k_3 & k_3 + k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_4 &= 0 \\ k_1 + k_2 &= 1 \\ k_3 &= 0 \\ k_3 + k_4 &= 0 \end{aligned}$$

Solving these equations,

$$k_1 = 0, k_2 = 1, k_3 = 0, k_4 = 0$$

$$T(\mathbf{v}_3) = \mathbf{w}_2$$

$$[T(\mathbf{v}_3)]_{S_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Expressing $T(\mathbf{v}_4)$ as linear combinations of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$,

$$T(\mathbf{v}_4) = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 + k_4\mathbf{w}_4$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_1 + k_4 & k_1 + k_2 \\ k_3 & k_3 + k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} k_1 + k_4 &= 0 \\ k_1 + k_2 &= 0 \\ k_3 &= 0 \\ k_3 + k_4 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = -1, k_2 = 1, k_3 = 0, k_4 = 1$$

$$T(\mathbf{v}_4) = -\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_4$$

$$[T(\mathbf{v}_4)]_{S_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$[T]_{S_2, S_1} = \left[[T(\mathbf{v}_1)]_{S_2} \,\middle|\, [T(\mathbf{v}_2)]_{S_2} \,\middle|\, [T(\mathbf{v}_3)]_{S_2} \,\middle|\, [T(\mathbf{v}_4)]_{S_2}\right]$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Example 6: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix of $T: R^2 \to R^2$ w.r.t. the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$

(i) Find $[T(\mathbf{v}_1)]_S$ and $[T(\mathbf{v}_2)]_S$.

(ii) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

(iii) Find $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

(iv) Calculate $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Solution: (i) $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ is the matrix of T w.r.t. the basis S.

$$A = [T]_S = \left[[T(\mathbf{v}_1)]_S \,\middle|\, [T(\mathbf{v}_2)]_S\right] = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

Hence,

$$[T(\mathbf{v}_1)]_S = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[T(\mathbf{v}_2)]_S = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(ii) From part (i),

$$T(\mathbf{v}_1) = \mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$T(\mathbf{v}_2) = 3\mathbf{v}_1 + 5\mathbf{v}_2 = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 29 \end{bmatrix}$$

(iii) Let $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be any vector in R^2.

Expressing $\mathbf{v}$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + k_2\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} k_1 - k_2 \\ 3k_1 + 4k_2 \end{bmatrix}$$

Equating corresponding components,

$$k_1 - k_2 = x_1$$

$$3k_1 + 4k_2 = x_2$$

Solving these equations,

$$k_1 = \frac{4x_1 + x_2}{7}$$

$$k_2 = \frac{-3x_1 + x_2}{7}$$

$$\mathbf{v} = \left(\frac{4x_1 + x_2}{7}\right)\mathbf{v}_1 + \left(\frac{-3x_1 + x_2}{7}\right)\mathbf{v}_2$$

$$T(\mathbf{v}) = \left(\frac{4x_1 + x_2}{7}\right)T(\mathbf{v}_1) + \left(\frac{-3x_1 + x_2}{7}\right)T(\mathbf{v}_2)$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \left(\frac{4x_1 + x_2}{7}\right)\begin{bmatrix} 3 \\ -5 \end{bmatrix} + \left(\frac{-3x_1 + x_2}{7}\right)\begin{bmatrix} -2 \\ 29 \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{4x_1 + x_2}{7}\right) - 2\left(\frac{-3x_1 + x_2}{7}\right) \\ -5\left(\frac{4x_1 + x_2}{7}\right) + 29\left(\frac{-3x_1 + x_2}{7}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{7}x_1 + \frac{1}{7}x_2 \\ -\frac{107}{7}x_1 + \frac{24}{7}x_2 \end{bmatrix}$$

(iv) $$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}\frac{18}{7}(1)+\frac{1}{7}(1)\\-\frac{107}{7}(1)+\frac{24}{7}(1)\end{bmatrix}=\begin{bmatrix}\frac{19}{7}\\-\frac{83}{7}\end{bmatrix}$$

Example 7: Let $T: P_1 \to P_2$ be a linear transformation. The matrix of T w.r.t. the basis $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is

$$A=\begin{bmatrix}1 & 0\\2 & 1\\-1 & -2\end{bmatrix}$$

where $\mathbf{v}_1 = x + 1$, $\mathbf{v}_2 = x - 1$, $\mathbf{w}_1 = x^2 + 1$, $\mathbf{w}_2 = x$, $\mathbf{w}_3 = x - 1$

(i) Find $[T(\mathbf{v}_1)]_{S_2}$ and $[T(\mathbf{v}_2)]_{S_2}$.
(ii) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.
(iii) Find $T(a_0 + a_1x)$.
(iv) Calculate $T(2x + 1)$.

Solution: (i) A is the matrix of T w.r.t. the basis S_1 and S_2.

$$A=[T]_{S_2,S_1}=\left[[T(\mathbf{v}_1)]_{S_2}\,\middle|\,[T(\mathbf{v}_2)]_{S_2}\right]=\begin{bmatrix}1 & 0\\2 & 1\\-1 & -2\end{bmatrix}$$

Hence, $$[T(\mathbf{v}_1)]_{S_2}=\begin{bmatrix}1\\2\\-1\end{bmatrix},\quad [T(\mathbf{v}_2)]_{S_2}=\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

(ii) From part (i),

$T(\mathbf{v}_1) = \mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3 = (x^2 + 1) + 2x - (x - 1) = x^2 + x + 2$
$T(\mathbf{v}_2) = \mathbf{w}_2 - 2\mathbf{w}_3 = x - 2(x - 1) = -x + 2$

(iii) Let $\mathbf{v} = p(x) = a_0 + a_1x$ be any vector in P_1. Expressing $\mathbf{v}$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\begin{aligned}\mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2\\ a_0 + a_1x &= k_1(x+1) + k_2(x-1)\\ &= (k_1 - k_2) + (k_1 + k_2)x\end{aligned}$$

Equating corresponding components,

$$\begin{aligned}k_1 - k_2 &= a_0\\ k_1 + k_2 &= a_1\end{aligned}$$

Solving these equations,

$$k_1 = \frac{a_1 + a_0}{2}$$

$$k_2 = \frac{a_1 - a_0}{2}$$

$$\therefore \mathbf{v} = \left(\frac{a_1 + a_0}{2}\right)\mathbf{v}_1 + \left(\frac{a_1 - a_0}{2}\right)\mathbf{v}_2$$

$$T(\mathbf{v}) = \left(\frac{a_1 + a_0}{2}\right)T(\mathbf{v}_1) + \left(\frac{a_1 - a_0}{2}\right)T(\mathbf{v}_2)$$

$$T(a_0 + a_1x) = \left(\frac{a_1 + a_0}{2}\right)(x^2 + x + 2) + \left(\frac{a_1 - a_0}{2}\right)(-x + 2)$$

$$= \left(\frac{a_1 + a_0}{2}\right)x^2 + \left(\frac{a_1 + a_0}{2}\right)x + (a_1 + a_0) - \left(\frac{a_1 - a_0}{2}\right)x + (a_1 - a_0)$$

$$= \left(\frac{a_1 + a_0}{2}\right)x^2 + a_0x + 2a_1$$

(iv) $T(2x+1) = \left(\frac{2+1}{2}\right)x^2 + (1)x + 2(2) = \frac{3}{2}x^2 + x + 4$

Example 8: Let the matrix of $T : R^3 \rightarrow R^2$ w.r.t. the bases $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ be

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

where $$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(i) Find $[T(\mathbf{v}_1)]_{S_2}, [T(\mathbf{v}_2)]_{S_2}, [T(\mathbf{v}_3)]_{S_2}$.

(ii) Find $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)$

(iii) Find $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$

(iv) Find $T\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$

Solution: (i) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ is the matrix of T w.r.t. the bases S_1 and S_2.

$$A = [T]_{S_2, S_1} = \Big[[T(\mathbf{v}_1)]_{S_2} \,\Big|\, [T(\mathbf{v}_2)]_{S_2} \,\Big|\, [T(\mathbf{v}_3)]_{S_2} \Big]$$

Hence, $[T(\mathbf{v}_1)]_{S_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, [T(\mathbf{v}_2)]_{S_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, [T(\mathbf{v}_3)]_{S_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(ii) From part (i),

$$T(\mathbf{v}_1) = \mathbf{w}_1 - \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$T(\mathbf{v}_2) = 2\mathbf{w}_1 + \mathbf{w}_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$T(\mathbf{v}_3) = \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(iii) Let $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in R^3.

Expressing $\mathbf{v}$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$,

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -k_1 + k_3 \\ k_1 + k_2 \\ k_2 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} -k_1 \quad\quad + k_3 &= x_1 \\ k_1 + k_2 \quad\quad &= x_2 \\ k_2 \quad\quad &= x_3 \end{aligned}$$

Solving these equations,

$$\begin{aligned} k_1 &= x_2 - x_3 \\ k_2 &= x_3 \\ k_3 &= x_1 + x_2 - x_3 \\ \therefore \mathbf{v} &= (x_2 - x_3)\mathbf{v}_1 + x_3\mathbf{v}_2 + (x_1 + x_2 - x_3)\mathbf{v}_3 \\ T(\mathbf{v}) &= (x_2 - x_3)T(\mathbf{v}_1) + x_3T(\mathbf{v}_2) + (x_1 + x_2 - x_3)T(\mathbf{v}_3) \end{aligned}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = (x_2 - x_3)\begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 3 \end{bmatrix} + (x_1 + x_2 - x_3)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_3 + x_1 + x_2 - x_3 \\ 3x_2 - 3x_3 + 3x_3 + 2x_1 + 2x_2 - 2x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 - 2x_3 \end{bmatrix}$$

(iv) $T\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2+1+2(-1) \\ 2(2)+5(1)-2(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

3.9.3 *Matrices of Compositions and Inverse Transformations*

Theorem 3.18: If T_1: $U \rightarrow V$ and T_2: $V \rightarrow W$ are linear transformations and if S_1, S_3, and S_2 are bases for U, V and W respectively then

$$[T_2 \circ T_1]_{S_2, S_1} = [T_2]_{S_2, S_3}[T_1]_{S_3, S_1}$$

Theorem 3.19: If $T : V \rightarrow V$ is a linear operator and if S is a basis for V then the following are equivalent:

(a) T is one-to-one.
(b) $[T]_S$ is invertible.

When (a) and (b) hold,

$$\left[T^{-1}\right]_S = [T]_s^{-1}$$

Theorem 3.20: If $T : V \rightarrow W$ is a linear transformation and if S_1 and S_2 are bases for V and W respectively then T is invertible if and only if $[T]_{S_2,S_1}$ is invertible. In this case,

$$\left([T]_{S_2, S_1}\right)^{-1} = [T^{-1}]_{S_1, S_2}$$

Example 1: Let T_1: $P_1 \rightarrow P_2$ be the linear transformation defined by

$$T_1(a_0 + a_1x) = 2a_0 - 3a_1x^2$$

and let $T_2 : P_2 \rightarrow P_3$ be the linear transformation defined by

$$T_2(a_0 + a_1x + a_2x^2) = 3a_0x + 3a_1x^2 + 3a_2x^3$$

Let $S_1 = \{1, x\}$, $S_3 = \{1, x, x^2\}$, and $S_2 = \{1, x, x^2, x^3\}$

Find $[T_2 \circ T_1]_{S_2, S_1}$, $[T_2]_{S_2, S_3}$ and $[T_1]_{S_3, S_1}$

Solution: $T_1(1) = 2$

$$T_1(x) = -3x^2$$
$$T_2(1) = 3x$$
$$T_2(x) = 3x^2$$
$$T_2(x^2) = 3x^3$$

Since S_3 is the standard basis for P_2,

$$[T_1(1)]_{S_3} = [T_1(1)]$$
$$[T_1(x)]_{S_3} = [T_1(x)]$$

Thus,
$$[T_1]_{S_3,\,S_1} = \Big[[T_1(1)]_{S_3} \,\Big|\, [T_1(x)]_{S_3}\Big]$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -3 \end{bmatrix}$$

Since S_2 is the standard basis for P_3,

$$[T_2(1)]_{S_2} = [T_2(1)]$$
$$[T_2(x)]_{S_2} = [T_2(x)]$$
$$[T_2(x^2)]_{S_2} = [T_2(x^2)]$$
$$[T_2]_{S_2,S_3} = \Big[[T_2(1)]_{S_2} \,\Big|\, [T_2(x)]_{S_2} \,\Big|\, [T_2(x^2)]_{S_2}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[T_2 \circ T_1]_{S_2,\,S_1} = [T_2]_{S_2,\,S_3}[T_1]_{S_3,\,S_1}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & 0 \\ 0 & -9 \end{bmatrix}$$

Example 2: $T_1 : R^2 \to P_1$ and $T_2 : P_1 \to P_2$ be the linear transformations defined by $T_1\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ and $T_2(p(x)) = x\,p(x)$

Let $S_1 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $S_3 = \{1, x\}$ and $S_2 = \{1, x, x^2\}$

(i) Find $[T_2 \circ T_1]_{S_2, S_1}$ (ii) Find $[T_1^{-1}]_{S_1, S_3}$

Solution:

$$T_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + x$$

$$T_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

$$T_2(1) = x$$

$$T_2(x) = x^2$$

Since S_3 is the standard basis for P_1,

$$\left[T_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{S_3} = \left[T_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

$$\left[T_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_{S_3} = \left[T_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$[T_1]_{S_3, S_1} = \left[\left[T_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{S_3} \middle| \left[T_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_{S_3}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since S_2 is the standard basis for P_2,

$$[T_2(1)]_{S_2} = [T_2(1)]$$

$$[T_2(x)]_{S_2} = [T_2(x)]$$

Thus,

$$[T_2]_{S_2, S_3} = \left[[T_2(1)]_{S_2} \middle| [T_2(x)]_{S_2}\right]$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T_2 \circ T_1]_{S_2, S_1} = [T_2]_{S_2, S_3} [T_1]_{S_3, S_1}$$
$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(ii)
$$\left[T_1^{-1}\right]_{S_1, S_3} = \left([T_1]_{S_3, S_1}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Exercise 3.3

1. Let $T : R^2 \to R^2$ be the linear transformation defined by

$$T(x, y) = (x - 2y, x + 2y)$$

Let $S_1 = \{(1, -1), (0, 1)\}$ be a basis for R^2 and let S_2 be the standard basis for R^2. Find the matrix of T w.r.t.

(i) S_1 (ii) S_1 and S_2
(iii) S_2 and S_1 (iv) S_2

$$\left[\textbf{Ans.:} \quad \text{(i)} \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}\right]$$

2. Let $T : R^2 \to R^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 2x + y \\ x + y \end{bmatrix}$$

Let S_1 and S_2 be the standard bases for R^2 and R^3 respectively. Also, let

$$S_1' = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \text{ and } S_2' = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

be bases for R^2 and R^3, respectively. Find the matrix of T w.r.t.

(i) S_1 and S_2 (ii) S_1' and S_2'

$$\left[\textbf{Ans.:} \quad \text{(i)} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} \frac{7}{3} & -\frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}\right]$$

3. Let $T : R^2 \to R^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$$

Find the matrix $[T]_{S_2, S_1}$ w.r.t. the bases $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\textbf{Ans.:} \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}\right]$$

4. Let $T : R^2 \to R^2$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

and let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find $[T]_s$.

$$\left[\textbf{Ans.:} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right]$$

5. Let $T : P_2 \to P_2$ be the linear operator defined by

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x-1) + a_2(x-1)^2$$

Find the matrix of T w.r.t. the standard basis $S = \{1, x. x^2\}$ for P_2.

$$\left[\textbf{Ans.:} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

6. Let $T : P_1 \to P_3$ be defined by $T(p(x)) = x^2 p(x)$.

Let $S_1 = \{x, 1\}$ and $S_2 = \{x, x + 1\}$ be bases for P_1.

Let $S_1' = \{x^3, x^2, x, 1\}$ and $S_2' = \{x^3, x^2 - 1, x, x+1\}$ be bases for P_3. Find the matrix of T w.r.t.

(i) S_1 and S_1' (ii) S_2 and S_2'

$$\left[\textbf{Ans.:} \quad \text{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}\right]$$

7. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be the matrix of $T : P_2 \to P_2$ w.r.t. the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = 3x + 3x^2$, $\mathbf{v}_2 = -1 + 3x + 2x^2$, $\mathbf{v}_3 = 3 + 7x + 2x^2$

(i) Find $[T(\mathbf{v}_1)]_s$, $[T(\mathbf{v}_2)]_s$ and $[T(\mathbf{v}_3)]_s$.

(ii) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$.

$$\left[\begin{array}{l}\textbf{Ans.:} \\ \text{(i)} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} \\ \text{(ii)}\ 16 + 51x + 19x^2, -6 - 5x + 5x^2, \\ \quad 7 + 40x + 15x^2 \end{array}\right]$$

8. Let $A=\begin{bmatrix}1 & 3 & -1\\ 2 & 0 & 5\\ 6 & -2 & 4\end{bmatrix}$ be the matrix of $T: P_2 \to P_2$ w.r.t. the bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = 3x + 3x^2$, $\mathbf{v}_2 = -1 + 3x + 2x^2$, $\mathbf{v}_3 = 3 + 7x + 2x^2$

(i) Find $[T(\mathbf{v}_1)]_s$, $[T(\mathbf{v}_2)]_s$ and $[T(\mathbf{v}_3)]_s$.

(ii) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$ and $T(\mathbf{v}_3)$.

$$\left[\textbf{Ans.:}\ \begin{bmatrix}1\\2\\6\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}, \begin{bmatrix}-1\\5\\4\end{bmatrix} \quad \begin{matrix}16+51x+19x^2,\\ -6-5x+5x^2,\\ 7+40x+15x^2\end{matrix}\right]$$

9. Let $T_1 : P_1 \to R^2$ and $T_2 : R^2 \to R^2$ be the linear transformation defined by

$$T_1(p(x)) = \begin{bmatrix}p(0)\\ p(1)\end{bmatrix} \text{ and } T_2\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}a-2b\\ 2a-b\end{bmatrix}$$

Let $S_1 = \{1, x\}$, $S_2 = S_3 = \{\mathbf{e}_1, \mathbf{e}_2\}$

Find $[T_2 \circ T_1]_{S_2, S_1}$.

$$\left[\textbf{Ans.:}\ \begin{bmatrix}-1 & -2\\ 1 & -1\end{bmatrix}\right]$$

10. Let $T: P_2 \to P_2$ be a linear transformation defined by

$$T(p(x)) = p(x+2)$$

Let $S_1 = \{1, x, x^2\}$,
$S_2 = \{1, x+2, (x+2)^2\}$

Find $T^{-1}(p(x))$

$$\left[\textbf{Ans.:}\ p(x-2)\right]$$

3.10 EFFECT OF CHANGE OF BASES ON LINEAR OPERATORS

The matrix of a linear operator $T: V \to V$ depends on the basis for V. A basis for V is chosen such that it produces the simplest possible matrix for T such as a diagonal or a triangular matrix.

If $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ are the bases for vector space V, then the transition matrix from S_2 to S_1 is given by,

$$P = \left[[\mathbf{w}_1]_{S_1} \middle| [\mathbf{w}_2]_{S_1} \middle| \ldots \middle| [\mathbf{w}_n]_{S_1}\right]$$

Theorem 3.21: Let $T: V \to V$ be a linear operator of vector space V, and let S_1 and S_2 be bases for vector space V. Then

$$[T]_{S_2} = P^{-1}[T]_{S_1} P$$

where P is the transition matrix from S_2 to S_1.

Note: P^{-1} is the transition matrix from S_1 to S_2.

$$P^{-1} = \left[[\mathbf{v}_1]_{S_2} \middle| [\mathbf{v}_2]_{S_2} \middle| \ldots \middle| [\mathbf{v}_n]_{S_2}\right]$$

Example 1: Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V and let $T: V \to V$ be a linear operator such that

$$[T]_{S_1} = \begin{bmatrix} -3 & 4 & 7 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

Find $[T]_{S_2}$ where $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is the basis for V defined by $\mathbf{w}_1 = \mathbf{v}_1$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$

Solution:

$$[T]_{S_1} = \begin{bmatrix} -3 & 4 & 7 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are expressed as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as,

$$\mathbf{w}_1 = \mathbf{v}_1, \quad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$$

$$[\mathbf{w}_1]_{S_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{w}_2]_{S_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{w}_3]_{S_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the transition matrix from S_2 to S_1 is

$$P = \left[[\mathbf{w}_1]_{S_1} \,\middle|\, [\mathbf{w}_2]_{S_1} \,\middle|\, [\mathbf{w}_3]_{S_1} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of T w.r.t the basis S_2 is

$$[T]_{S_2} = P^{-1}[T]_{S_1}P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 & 7 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 9 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Example 2: $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the bases for a vector space V and let

$$P = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

be the transition matrix from S_2 to S_1.

(i) Express $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(ii) Express $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as linear combinations of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

Solution: (i) Since P represents the transition matrix from the basis S_2 to S_1,

$$P = \left[[\mathbf{w}_1]_{S_1} \middle| [\mathbf{w}_2]_{S_1} \middle| [\mathbf{w}_3]_{S_1} \right] = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\mathbf{w}_1]_{S_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{w}_2]_{S_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, [\mathbf{w}_3]_{S_1} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

Hence,

$$\begin{aligned} \mathbf{w}_1 &= 2\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{w}_2 &= -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{w}_3 &= 3\mathbf{v}_1 + 4\mathbf{v}_2 + 2\mathbf{v}_3 \end{aligned}$$

(ii)

$$P^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 5 & -7 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$$

Since P^{-1} represents the transition matrix from the basis S_1 to S_2,

$$P^{-1} = \left[[\mathbf{v}_1]_{S_2} \middle| [\mathbf{v}_2]_{S_2} \middle| [\mathbf{v}_3]_{S_2} \right] = \begin{bmatrix} -2 & 5 & -7 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$[\mathbf{v}_1]_{S_2} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_2]_{S_2} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}, \quad [\mathbf{v}_3]_{S_2} = \begin{bmatrix} -7 \\ -5 \\ 3 \end{bmatrix}$$

Hence,

$$\begin{aligned} \mathbf{v}_1 &= -2\mathbf{w}_1 - 2\mathbf{w}_2 + \mathbf{w}_3 \\ \mathbf{v}_2 &= 5\mathbf{w}_1 + 4\mathbf{w}_2 - 2\mathbf{w}_3 \\ \mathbf{v}_3 &= -7\mathbf{w}_1 - 5\mathbf{w}_2 + 3\mathbf{w}_3 \end{aligned}$$

Example 3: $T : R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_2 \end{bmatrix}$$

$S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

(ii) Find the matrix of T w.r.t the basis S_1.
(ii) Find the matrix of T w.r.t the basis S_2.

Solution: (i) The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

Since S_1 is the standard basis for R^2,

$$[T]_{S_1} = [T] = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

(ii) Since S_1 is the standard basis for R^2,

$$[\mathbf{w}_1]_{S_1} = [\mathbf{w}_1] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[\mathbf{w}_2]_{S_1} = [\mathbf{w}_2] = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

The transition matrix from S_2 to S_1 is

$$P = \left[[\mathbf{w}_1]_{S_1} \; \left[|\mathbf{w}_2|_{S_1}\right]\right]$$
$$= \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

The matrix of T w.r.t the basis S_2 is

$$[T]_{S_2} = P^{-1}[T]_{S_1}P = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{11} & -\frac{56}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Example 4: Let $T: R^2 \rightarrow R^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$

where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(i) Find the matrix of T w.r.t the basis S_1.
(ii) Find the matrix of T w.r.t the basis S_2.

Solution: (i) $$T(\mathbf{v}_1) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(1)+2 \\ 1-3(2) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Expressing $T(\mathbf{v}_1)$ as linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$T(\mathbf{v}_1) = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$

$$\begin{bmatrix} 4 \\ -5 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ 2k_1 + 3k_2 \end{bmatrix}$$

Equating corresponding components,

$$k_1 + k_2 = 4$$
$$2k_1 + 3k_2 = -5$$

Solving these equations,

$$k_1 = 17$$
$$k_2 = -13$$
$$T(\mathbf{v}_1) = 17\mathbf{v}_1 - 13\mathbf{v}_2$$

$$[T(\mathbf{v}_1)]_{S_1} = \begin{bmatrix} 17 \\ -13 \end{bmatrix}$$

Similarly, $$T(\mathbf{v}_2) = T\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}2(1)+3\\1-3(3)\end{bmatrix} = \begin{bmatrix}5\\-8\end{bmatrix}$$

Expressing $T(\mathbf{v}_2)$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$T(\mathbf{v}_2) = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$

$$\begin{bmatrix}5\\-8\end{bmatrix} = k_1\begin{bmatrix}1\\2\end{bmatrix} + k_2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}k_1+k_2\\2k_1+3k_2\end{bmatrix}$$

Equating corresponding components,

$$k_1 + k_2 = 5$$
$$2k_1 + 3k_2 = -8$$

Solving these equations,

$$k_1 = 23$$
$$k_2 = -18$$
$$T(\mathbf{v}_2) = 23\,\mathbf{v}_1 - 18\,\mathbf{v}_2$$

$$[T(\mathbf{v}_2)]_{S_1} = \begin{bmatrix}23\\-18\end{bmatrix}$$

$$[T]_{S_1} = \left[[T(\mathbf{v}_1)]_{S_1} \,\middle|\, T(\mathbf{v}_2)|_{S_1}\right]$$
$$= \begin{bmatrix}17 & 23\\-13 & -18\end{bmatrix}$$

(ii) Expressing $\mathbf{w}_1$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{w}_1 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$

$$\begin{bmatrix}-1\\1\end{bmatrix} = k_1\begin{bmatrix}1\\2\end{bmatrix} + k_2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}k_1+k_2\\2k_1+3k_2\end{bmatrix}$$

Equating corresponding components,

$$k_1 + k_2 = -1$$
$$2k_1 + 3k_2 = 1$$

Solving these equations,

$$k_1 = -4$$
$$k_2 = 3$$
$$\mathbf{w}_1 = -4\mathbf{v}_1 + 3\mathbf{v}_2$$

$$[T(\mathbf{w}_1)]_{S_1} = \begin{bmatrix}-4\\3\end{bmatrix}$$

Expressing $\mathbf{w}_2$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{w}_2 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$$
$$\begin{bmatrix}0\\1\end{bmatrix} = k_1\begin{bmatrix}1\\2\end{bmatrix} + k_2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}k_1+k_2\\2k_1+3k_2\end{bmatrix}$$

Equating corresponding components,

$$k_1 + k_2 = 0$$
$$2k_1 + 3k_2 = 1$$

Solving these equations,

$$k_1 = -1$$
$$k_2 = 1$$
$$\mathbf{w}_2 = -\mathbf{v}_1 + \mathbf{v}_2$$
$$[T(\mathbf{w}_2)]_{S_1} = \begin{bmatrix}-1\\1\end{bmatrix}$$

The transition matrix from S_2 to S_1 is

$$P = \left[[T(\mathbf{w}_1)]_{S_1} \,\middle|\, [T(\mathbf{w}_2)]_{S_1}\right]$$
$$= \begin{bmatrix}-4 & -1\\3 & 1\end{bmatrix}$$
$$P^{-1} = \begin{bmatrix}-1 & -1\\3 & 4\end{bmatrix}$$

The matrix of T w.r.t the basis S_2 is

$$[T]_{S_2} = P^{-1}[T]_{S_1}P = \begin{bmatrix}-1 & -1\\3 & 4\end{bmatrix}\begin{bmatrix}17 & 23\\-13 & -18\end{bmatrix}\begin{bmatrix}-4 & -1\\3 & 1\end{bmatrix}$$
$$= \begin{bmatrix}1 & -1\\-5 & -2\end{bmatrix}$$

Example 5: Let $T: P_1 \rightarrow P_1$ is defined by $T(a_0 + a_1x) = a_0 + a_1\,(x + 1)$;
$S_1 = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $S_2 = \{\mathbf{q}_1, \mathbf{q}_2\}$ where $\mathbf{p}_1 = 6 + 3x$, $\mathbf{p}_2 = 10 + 2x$, $\mathbf{q}_1 = 2$, $\mathbf{q}_2 = 3 + 2x$
Find the matrix of T w.r.t the basis S_1 and matrix of T w.r.t the basis S_2.

Solution: $T(\mathbf{p}_1) = T(6 + 3x) = 6 + 3(x + 1) = 9 + 3x$

Expressing $T(\mathbf{p}_1)$ as linear combinations of $\mathbf{p}_1$ and $\mathbf{p}_2$

$$T(\mathbf{p}_1) = k_1\mathbf{p}_1 + k_2\mathbf{p}_2$$
$$9 + 3x = k_1(6 + 3x) + k_2(10 + 2x)$$
$$= (6k_1 + 10k_2) + (3k_1 + 2k_2)x$$

Equating corresponding coefficients,

$$6k_1 + 10k_2 = 9$$
$$3k_1 + \ \ 2k_2 = 3$$

Solving these equations,

$$k_1 = \frac{2}{3}, k_2 = \frac{1}{2}$$

$$\therefore T(\mathbf{p}_1) = \frac{2}{3}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2$$

$$[T(\mathbf{p}_1)]_{S_1} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}$$

Similarly, $T(\mathbf{p}_2) = T(10 + 2x) = 10 + 2\,(x + 1) = 12 + 2x$

Expressing $T(\mathbf{p}_2)$ as linear combinations of $\mathbf{p}_1$ and $\mathbf{p}_2$

$$\begin{aligned} T(\mathbf{p}_2) &= k_1\mathbf{p}_1 + k_2\mathbf{p}_2 \\ 12 + 2x &= k_1(6 + 3x) + k_2(10 + 2x) \\ &= (6k_1 + 10k_2) + (3k_1 + 2k_2)x \end{aligned}$$

Equating corresponding the coefficients,

$$6k_1 + 10k_2 = 12$$
$$3k_1 + \ \ 2k_2 = \ 2$$

Solving these equations,

$$k_1 = -\frac{2}{9}, k_2 = \frac{4}{3}$$

$$\therefore T(\mathbf{p}_2) = -\frac{2}{9}\mathbf{p}_1 + \frac{4}{3}\mathbf{p}_2$$

$$[T(\mathbf{p}_2)]_{S_1} = \begin{bmatrix} -\frac{2}{9} \\ \frac{4}{3} \end{bmatrix}$$

$$[T]_{S_1} = \left[T(\mathbf{p}_1)_{S_1} \,\middle|\, T(\mathbf{p}_2)_{S_1} \right] = \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix}$$

Expressing $\mathbf{q}_1$ as linear combinations of $\mathbf{p}_1$ and $\mathbf{p}_2$,

$$\begin{aligned}\mathbf{q}_1 &= k_1\mathbf{p}_1 + k_2\mathbf{p}_2\\ 2 &= k_1(6+3x)+k_2(10+2x)\\ &= (6k_1+10k_2)+(3k_1+2k_2)x\end{aligned}$$

Equating corresponding coefficients,

$$6k_1 + 10k_2 = 2$$
$$3k_1 + \ \ 2k_2 = 0$$

Solving these equations,

$$k_1 = -\frac{2}{9},\ k_2 = \frac{1}{3}$$
$$\therefore \mathbf{q}_1 = -\frac{2}{9}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$$

$$[\mathbf{q}_1]_{S_1} = \begin{bmatrix} -\frac{2}{9} \\ \frac{1}{3} \end{bmatrix}$$

Expressing $\mathbf{q}_2$ as linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$,

$$\begin{aligned}\mathbf{q}_2 &= k_1\mathbf{p}_1 + k_2\mathbf{p}_2\\ 3+2x &= k_1(6+3x)+k_2(10+2x)\\ &= (6k_1+10k_2)+(3k_1+2k_2)x\end{aligned}$$

Equating corresponding coefficients,

$$6k_1 + 10k_2 = 3$$
$$3k_1 + \ \ 2k_2 = 2$$

Solving these equations,

$$k_1 = \frac{7}{9},\ k_2 = -\frac{1}{6}$$
$$\therefore \mathbf{q}_2 = \frac{7}{9}\mathbf{p}_1 - \frac{1}{6}\mathbf{p}_2$$

$$[\mathbf{q}_2]_{S_1} = \begin{bmatrix} \frac{7}{9} \\ -\frac{1}{6} \end{bmatrix}$$

Hence, transition matrix from S_2 to S_1 is

$$P = \left[[\mathbf{q}_1]_{S_1} \middle| [\mathbf{q}_2]_{S_1} \right]$$

$$= \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

Thus

$$P^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$[T]_{S_2} = P^{-1}[T]_{S_1}P = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

3.11 SIMILARITY OF MATRICES

If A and B are two square matrices then B is said to be similar to A, if there exists a non-singular matrix P such that $B = P^{-1}AP$

Properties of Similar Matrices

(i) Similar matrices have the same determinant.
(ii) Similar matrices have the same rank.
(iii) Similar matrices have the same nullity.
(iv) Similar matrices have the same trace.
(v) Similar matrices have the same characteristic polynomial.
(vi) Similar matrices have the same eigenvalues.
(vii) If λ is an eignvalue of two similar matrices, the eigenspace of both the similar matrices corresponding to λ have the same dimension.

Two matrices representing the same linear operator $T : V \rightarrow V$ with respect to different bases are similar. If S_1 and S_2 are two different bases for a vector space V then matrices $[T]_{S_1}$ and $[T]_{S_2}$ are similar.

Hence,
$$\det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right)$$

The value of the determinant depends on T, but not on any basis that is used to obtain the matrix for T. Thus, if V is a finite-dimensional vector space then

$$\det(T) = \det\left([\mathrm{T}]_{S_1}\right)$$

where S_1 is any basis for V.

Example 1: Show that the matrices $\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ are similar but that $\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ are not.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} = 5$$

$$\det(B) = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

$$\det(C) = \begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = 0$$

$$\det(D) = \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

Since $\det(A) = \det(B)$, matrices A and B are similar.
Since $\det(C) \neq \det(D)$, matrices C and D are not similar.

Example 2: Let $T: R^3 \to R^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_2 \\ x_1 + 7x_3 \end{bmatrix}$$

S_1 is the standard basis for R^3 and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Verify that
$$\det(T) = \det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right).$$

Solution: The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

Since S_1 is the standard basis for R^3,

$$[T]_{S_1} = [T] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

$$\det(T) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{vmatrix} = -8$$

Since S_1 is the standard basis for R^3,

$$[\mathbf{w}_1]_{S_1} = [\mathbf{w}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[\mathbf{w}_2]_{S_1} = [\mathbf{w}_2] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[\mathbf{w}_3]_{S_1} = [\mathbf{w}_3] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Transition matrix from S_2 to S_1 is

$$P = \left[[\mathbf{w}_1]_{S_1} \,\middle|\, [\mathbf{w}_2]_{S_1} \,\middle|\, [\mathbf{w}_3]_{S_1} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus
$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of T w.r.t the basis S_2 is

$$[T]_{S_2} = P^{-1}[T]_{S_1} P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{bmatrix}$$

$$\det([T]_{S_2}) = \begin{vmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{vmatrix} = -8$$

Hence, $\det(T) = \det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right)$

Exercise 3.4

1. Find the transition matrix from S_2 to S_1 where

(i) $S_2 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}; S_1 = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

(ii) $S_2 = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}; S_1 = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

(iii) $S_2 = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right\};$

$S_1 = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

(iv) $S_2 = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right\},$

$S_1 = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

$$\left[\textbf{Ans.: } (i)\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix} \quad (ii)\begin{bmatrix}-1 & -1\\ 3 & 4\end{bmatrix} \quad (iii)\begin{bmatrix}0 & -1 & -2\\ 1 & 0 & 2\\ 0 & 1 & 1\end{bmatrix} \quad (iv)\begin{bmatrix}1 & 0 & \frac{3}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & -\frac{1}{2}\end{bmatrix}\right]$$

2. Let $T : R^2 \to R^2$ be defined by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 + x_2\\ -2x_1 + 4x_2\end{bmatrix}$$

(i) Find the matrix of T w.r.t. the standard basis $S_1 = \{e_1, e_2\}$ for R^2.

(ii) Find the matrix of T w.r.t. the basis $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ where

$$[\mathbf{w}_1] = \begin{bmatrix}1\\1\end{bmatrix} \text{ and } [\mathbf{w}_2] = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\left[\textbf{Ans.: } (i)\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix} \quad (ii)\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\right]$$

3. Let $T : R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}8x_1 - 3x_2\\ 6x_1 - x_2\end{bmatrix}$$

$S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$

where

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{w}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}1\\2\end{bmatrix}$$

(i) Find the matrix of T w.r.t. the basis S_1.

(ii) Find the matrix of T w.r.t. the basis S_2.

$$\left[\textbf{Ans.: } (i)\begin{bmatrix}2 & 0\\ 6 & 5\end{bmatrix} \quad (ii)\begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}\right]$$

4. Let I: $R^3 \to R^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

(i) Find the matrix of T w.r.t. the standard basis S_1 for R^3.

(ii) Find the transition matrix from S_2 to S_1 where $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, $\mathbf{w}_1 = \{1, 0, 1\}$, $\mathbf{w}_2 = \{-1, 2, 1\}$, $\mathbf{w}_3 = \{2, 1, 1\}$.

(iii) Find the matrix of T w.r.t. the basis S_2.

$$\left[\textbf{Ans.:} \text{ (i) } \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}\right]$$

$$\left[\begin{array}{l} \text{(ii) } \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \text{(iii) } \begin{bmatrix} \frac{17}{4} & \frac{35}{4} & \frac{11}{2} \\ -\frac{3}{4} & \frac{15}{4} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{7}{2} & 0 \end{bmatrix} \end{array}\right]$$

5. Show directly that $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ are not similar.

6. Show directly that there does not exist an invertible matrix P that satisfies equation $A = P^{-1}BP$ for

$$A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$$

7. Prove that if A is similar to B and B is similar to C, then A is similar to C.

8. Prove that if A is similar to B, then A^2 is similar to B^2.

9. Prove that if A is similar to B, then A^T is similar to B^T.

10. Prove that every square matrix is similar to itself.

Inner Product Spaces

Chapter 4

4.1 INTRODUCTION

Inner product space is a vector space with an inner product on it. It associates each pair of vectors in the space with a scalar quantity known as the inner product of vectors. It helps defining the orthogonality between vectors. They generalize Euclidean spaces to the vector spaces of any dimension.

4.2 INNER PRODUCT SPACES

Let V be a real vector space. An inner product on V denoted by $\langle \cdot, \cdot \rangle$ is a function from $V \times V \to R$ that assigns a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ to each ordered pair of vectors **u** and **v** in V in such a way that for all **u**, **v**, **w** in V and all scalars k, the following axioms are satisfied.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (Symmetry)

(b) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (Additivity)

(c) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ (Homogeneity)

(d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ (non-negativity)

if and only if $\mathbf{u} = 0$

If the given product satisfies all the above 4 axioms then V is called a real inner product space with respect to the given product.

Note: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n then Euclidean inner product (dot product)

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

satisfies all the four axioms of inner product space. Hence, any vector space with respect to Euclidean inner product is an inner product space.

Weighted Euclidean Inner Product If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, and $w_1, w_2, \ldots, w_n$ are positive real numbers called weights then

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

is called a weighted Euclidean inner product with weights $w_1, w_2, \ldots, w_n$.

4.2.1 *Properties of Inner Products*

If **u**, **v**, and **w** are vectors in an inner product space V and k is any scalar then

(i) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(ii) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(iii) $\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$

(iv) $\langle \mathbf{u}-\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$

(v) $\langle \mathbf{u}, \mathbf{v}-\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$

Example 1: Let $\mathbf{u} = (3, -1)$, $\mathbf{v} = (2, -2)$, $\mathbf{w} = (-1, 6)$ and $k = -2$. Verify the following using Euclidean inner product.

(i) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(ii) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(iii) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$

(iv) $\langle \mathbf{0}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$

Solution: (i) $\mathbf{u}+\mathbf{v} = (3, -1) + (2, -2) = (5, -3)$

$$\begin{aligned}
\text{L.H.S.} &= \langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle \\
&= (\mathbf{u}+\mathbf{v})\cdot\mathbf{w} \\
&= (5, -3)\cdot(-1, 6) \\
&= -5-18 \\
&= -23 \\
\text{R.H.S.} &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \\
&= \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w} \\
&= (3, -1)\cdot(-1, 6) + (2, -2)\cdot(-1, 6) \\
&= (-3-6) + (-2-12) \\
&= -23
\end{aligned}$$

L.H.S. = R.H.S.

(ii) $\mathbf{v}+\mathbf{w} = (2, -2) + (-1, 6) = (1, 4)$

$$\begin{aligned}
\text{L.H.S.} &= \langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle \\
&= \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) \\
&= (3, -1)\cdot(1, 4) \\
&= 3-4 = -1 \\
\text{R.H.S.} &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \\
&= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} \\
&= (3, -1)\cdot(2, -2) + (3, -1)\cdot(-1, 6) \\
&= (6+2) + (-3-6) \\
&= -1
\end{aligned}$$

L.H.S. = R.H.S.

(iii) $k\mathbf{u} = -2\,(3, -1) = (-6, 2),\ k\mathbf{v} = -2(2, -2) = (-4, 4)$

$$\begin{aligned}\langle k\mathbf{u}, \mathbf{v}\rangle &= (k\mathbf{u})\cdot\mathbf{v}\\ &= (-6, 2)\cdot(2, -2)\\ &= (-12-4)\\ &= -16\end{aligned}$$

$$\begin{aligned}k\langle \mathbf{u}, \mathbf{v}\rangle &= k(\mathbf{u}\cdot\mathbf{v})\\ &= -2[(3, -1)\cdot(2, -2)]\\ &= -2(6+2)\\ &= -16\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}, k\mathbf{v}\rangle &= \mathbf{u}\cdot(k\mathbf{v})\\ &= (3, -1)\cdot(-4, 4)\\ &= -12-4\\ &= -16\end{aligned}$$

$$\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, k\mathbf{v}\rangle$$

(iv)
$$\begin{aligned}\langle \mathbf{0}, \mathbf{u}\rangle &= \mathbf{0}\cdot\mathbf{u}\\ &= (0, 0)\cdot(3,\ -1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}, \mathbf{0}\rangle &= \mathbf{u}\cdot\mathbf{0}\\ &= (3, -1)\cdot(0, 0)\\ &= 0\end{aligned}$$

$$\langle \mathbf{0}, \mathbf{u}\rangle = \langle \mathbf{u}, \mathbf{0}\rangle = 0$$

Example 2: Determine which of the following are inner products on R^3 if, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$

(i) $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + u_2v_2 + 4u_3v_3$

(ii) $\langle \mathbf{u}, \mathbf{v}\rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$

(iii) $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 - u_2v_2 + u_3v_3$

Solution: (i) (a)
$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= 2u_1v_1 + u_2v_2 + 4u_3v_3\\ &= 2v_1u_1 + v_2u_2 + 4v_3u_3\\ &= \langle \mathbf{v}, \mathbf{u}\rangle\end{aligned}$$

Symmetry axiom is satisfied.

(b) Let $w = (w_1, w_2, w_3)$ be also in R^3.

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3)\\ &= (u_1+v_1, u_2+v_2, u_3+v_3)\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= 2(u_1+v_1)w_1+(u_2+v_2)w_2+4(u_3+v_3)w_3\\ &= (2u_1w_1+u_2w_2+4u_3w_3)+(2v_1w_1+v_2w_2+4v_3w_3)\\ &= \langle \mathbf{u}, \mathbf{w}\rangle+\langle \mathbf{v}, \mathbf{w}\rangle\end{aligned}$$

Additivity axiom is satisfied.

(c) Let k be any scalar.

$$\begin{aligned}k\mathbf{u} &= k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3)\\ \langle k\mathbf{u}, \mathbf{v}\rangle &= 2(ku_1)v_1+(ku_2)v_2+4(ku_3)v_3\\ &= k(2u_1v_1+u_2v_2+4u_3v_3)\\ &= k\langle \mathbf{u}, \mathbf{v}\rangle\end{aligned}$$

Homogeneity axiom is satisfied.

(d) $$\begin{aligned}\langle \mathbf{u}, \mathbf{u}\rangle &= 2u_1u_1+u_2u_2+4u_3u_3\\ &= 2u_1^2+u_2^2+4u_3^2 \geq 0\end{aligned}$$

Also, $\langle \mathbf{u}, \mathbf{u}\rangle = 2u_1^2+u_2^2+4u_3^2 = 0$

if and only if $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ i.e. $\mathbf{u} = 0$

Non-negativity axiom is satisfied.

Hence, the given product is an inner product in R^3.

(ii) (a) $$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2\\ &= v_1^2u_1^2+v_2^2u_2^2+v_3^2u_3^2\\ &= \langle \mathbf{v}, \mathbf{u}\rangle\end{aligned}$$

Symmetry axiom is satisfied.

(b) Let $\mathbf{w} = (w_1, w_2, w_3)$ be also in R^3.

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (u_1, u_2, u_3)+(v_1, v_2, v_3)\\ &= (u_1+v_1, u_2+v_2, u_3+v_3)\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= (u_1+v_1)^2 w_1^2+(u_2+v_2)^2 w_2^2+(u_3+v_3)^2 w_3^2\\ &= (u_1^2w_1^2+u_2^2w_2^2+u_3^2w_3^2)+(v_1^2w_1^2+v_2^2w_2^2+v_3^2w_3^2)+2(u_1v_1w_1^2+u_2v_2w_2^2+u_3v_3w_3^2)\\ &= \langle \mathbf{u}, \mathbf{w}\rangle+\langle \mathbf{v}, \mathbf{w}\rangle+2(u_1v_1w_1^2+u_2v_2w_2^2+u_3v_3w_3^2)\\ &\neq \langle \mathbf{u}, \mathbf{w}\rangle+\langle \mathbf{v}, \mathbf{w}\rangle\end{aligned}$$

Additivity axiom is failed.

Hence, the given product is not an inner product in R^3.

(iii) (a) $$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= u_1v_1-u_2v_2+u_3v_3\\ &= v_1u_1-v_2u_2+v_3u_3\\ &= \langle \mathbf{v}, \mathbf{u}\rangle\end{aligned}$$

Symmetry axiom is satisfied.

(b) Let $\mathbf{w} = (w_1, w_2, w_3)$ be also in R^3.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3)\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= (u_1 + v_1)w_1 - (u_2 + v_2)w_2 + (u_3 + v_3)w_3\\ &= (u_1 w_1 - u_2 w_2 + u_3 w_3) + (v_1 w_1 - v_2 w_2 + v_3 w_3)\\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

Additivity axiom is satisfied.

(c) Let k be any scalar.

$$\begin{aligned}k\mathbf{u} &= k(u_1, u_2, u_3)\\ \langle k\mathbf{u}, \mathbf{v} \rangle &= (ku_1)v_1 - (ku_2)v_2 + (ku_3)v_3\\ &= k(u_1 v_1 - u_2 v_2 + u_3 v_3)\\ &= k\langle \mathbf{u}, \mathbf{v} \rangle\end{aligned}$$

Homogeneity axiom is satisfied.

(d) $\langle \mathbf{u}, \mathbf{u} \rangle = u_1 u_1 - u_2 u_2 + u_3 u_3$

$$= u_1^2 - u_2^2 + u_3^2$$

which is not necessarily positive because one term is with a negative sign.
Non-negativity axiom failed.
Hence, the given product is not an inner product.

Example 3: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in R^2 then verify that the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$

$$\begin{aligned}&= 3v_1 u_1 + 2v_2 u_2\\ &= \langle \mathbf{v}, \mathbf{u} \rangle\end{aligned}$$

Symmetry axiom is satisfied.

(b) Let $w = (w_1, w_2)$ be also in R^2.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)\\ \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2\\ &= 3u_1 w_1 + 2u_2 w_2 + 3v_1 w_1 + 2v_2 w_2\\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

Additivity axiom is satisfied.

(c) Let k be any scalar.

$$k\mathbf{u} = (ku_1, ku_2)$$
$$\langle k\mathbf{u}, \mathbf{v}\rangle = 3(ku_1)v_1 + 2(ku_2)v_2$$
$$= k(3u_1v_1 + 2u_2v_2)$$
$$= k\langle \mathbf{u}, \mathbf{v}\rangle$$

Homogeneity axiom is satisfied.

(d) $$\langle \mathbf{u}, \mathbf{u}\rangle = 3u_1u_1 + 2u_2u_2$$
$$= 3u_1^2 + 2u_2^2 \geq 0$$

Also, $\langle \mathbf{u}, \mathbf{u}\rangle = 3u_1^2 + 2u_2^2 = 0$.

if and only if $u_1 = 0$, $u_2 = 0$ i.e. $\mathbf{u} = 0$
Non-negativity axiom is satisfied.
Hence, given product satisfies all the four inner product axioms.

Example 4: If $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ are vectors in R^2 then prove that R^2 is an inner product space with respect to the inner product defined as

$$\langle \mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + u_2v_1 + 4u_1v_2 + 4u_2v_2.$$

Solution: R^2 will be an inner product space with respect to the given product if it satisfies all the four inner product axiom.

(a) $$\langle \mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + u_2v_1 + 4u_1v_2 + 4u_2v_2$$
$$= 4v_1u_1 + v_1u_2 + 4v_2u_1 + 4v_2u_2$$
$$= \langle \mathbf{v}, \mathbf{u}\rangle$$

Symmetry axiom is satisfied.

(b) Let $\mathbf{w} = (w_1, w_2)$ be also in R^2.

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2)$$
$$= (u_1 + v_1, u_2 + v_2)$$
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = 4(u_1 + v_1)w_1 + (u_2 + v_2)w_1 + 4(u_1 + v_1)w_2 + 4(u_2 + v_2)w_2$$
$$= (4u_1w_1 + u_2w_1 + 4u_1w_2 + 4u_2w_2) + (4v_1w_1 + v_2w_1 + 4v_1w_2 + 4v_2w_2)$$
$$= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$$

Additivity axiom is satisfied.

(c) Let k be any scalar.

$$k\mathbf{u} = k(u_1, u_2) = (ku_1, ku_2)$$
$$\langle k\mathbf{u}, \mathbf{v}\rangle = 4(ku_1)v_1 + (ku_2)v_1 + 4(ku_1)v_2 + 4(ku_2)v_2$$
$$= k(4u_1v_1 + u_2v_1 + 4u_1v_2 + 4u_2v_2)$$
$$= k\langle \mathbf{u}, \mathbf{v}\rangle$$

Homogeneity axiom is satisfied.

(d) $\langle \mathbf{u}, \mathbf{u} \rangle = 4u_1u_1 + u_2u_1 + 4u_1u_2 + 4u_2u_2$

$$= 4u_1^2 + 5u_1u_2 + 4u_2^2$$

$$= \frac{5}{2}u_1^2 + 5u_1u_2 + \frac{5}{2}u_2^2 + \frac{3}{2}\left(u_1^2 + u_2^2\right)$$

$$= \frac{5}{2}\left(u_1 + u_2\right)^2 + \frac{3}{2}\left(u_1^2 + u_2^2\right) \geq 0$$

Also, $\langle \mathbf{u}, \mathbf{u} \rangle = \frac{5}{2}\left(u_1 + u_2\right)^2 + \frac{3}{2}\left(u_1^2 + u_2^2\right) = 0$

if and only if $u_1 = 0$, $u_2 = 0$ i.e. $\mathbf{u} = 0$
Non-negativity axiom is satisfied.
Hence, R^2 is an inner product space.

4.2.2 *Inner Products Generated by Matrices*

Let **u** and **v** be vectors in R^n expressed as $n \times 1$ matrices and A be an $n \times n$ invertible matrix. If $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n then

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} \qquad \text{...(4.1)}$$

represents the inner product on R^n generated by matrix A.

If **u** and **v** are in matrix form, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

then, $$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$= (\mathbf{v})^T \mathbf{u}$$

Applying this formula in Eq. (4.1),

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

$$= (A\mathbf{v})^T A\mathbf{u}$$

$$= (\mathbf{v}^T A^T)\, A\mathbf{u}$$

$$= \mathbf{v}^T A^T A\mathbf{u}$$

Note: (i) If $A = I$ (identity matrix) then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= I\mathbf{u} \cdot I\mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v}\end{aligned}$$

Thus, inner product on R^n generated by identity matrix is the Euclidean inner product (dot product).

(ii) The weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$ is the inner product in R^n generated by the matrix $A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{w_1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{w_1} \end{bmatrix}$

Example 1: Show that $\langle \mathbf{u}, \mathbf{v} \rangle = 9u_1v_1 + 4u_2v_2$ is the inner product on R^2 generated by the matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution: Inner product generated by A is

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= A\mathbf{u} \cdot A\mathbf{v}, \text{ where } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 3u_1 \\ 2u_2 \end{bmatrix} \cdot \begin{bmatrix} 3v_1 \\ 2v_2 \end{bmatrix} \\ &= 3u_1 3v_1 + 2u_2 2v_2 \\ &= 9u_1v_1 + 4u_2v_2\end{aligned}$$

Example 2: Show that $\langle \mathbf{u}, \mathbf{v} \rangle = 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$ is the inner product on R^2 generated by the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

Solution: Inner product generated by A is

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= A\mathbf{u} \cdot A\mathbf{v}, \text{ where } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 2u_1 + u_2 \\ -u_1 + 3u_2 \end{bmatrix} \cdot \begin{bmatrix} 2v_1 + v_2 \\ -v_1 + 3v_2 \end{bmatrix}\end{aligned}$$

$$= (2u_1 + u_2)(2v_1 + v_2) + (-u_1 + 3u_2)(-v_1 + 3v_2)$$
$$= 4u_1v_1 + 2u_1v_2 + 2u_2v_1 + u_2v_2 + u_1v_1 - 3u_1v_2 - 3u_2v_1 + 9u_2v_2$$
$$= 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$$

Example 3: Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$. Find a matrix that generates the following inner products.
(i) $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$ (ii) $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 6u_2v_2$.

Solution: (i) $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$

In this weighted Euclidean inner product

$$w_1 = 3, \quad w_2 = 5$$

The matrix that generates it is

$$A = \begin{bmatrix} \sqrt{w_1} & 0 \\ 0 & \sqrt{w_2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

(ii) $\langle \mathbf{u}, \mathbf{v} \rangle = 4\, u_1v_1 + 6\, u_2v_2$

In this weighted Euclidean inner product

$$w_1 = 4, \quad w_2 = 6$$

The matrix that generates it is

$$A = \begin{bmatrix} \sqrt{w_1} & 0 \\ 0 & \sqrt{w_2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}.$$

4.2.3 *Norm or Length in Inner Product Spaces*

The norm or length of a vector $\mathbf{u}$ in an inner product space V is denoted by $\|\mathbf{u}\|$ and is defined by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}}$$

Unit Vector: Let $\mathbf{u}$ be a vector in an inner product space V. If $\|\mathbf{u}\| = 1$ then $\mathbf{u}$ is called a unit vector in V.

Properties of Length

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V and k is any scalar then

1. $\|\mathbf{u}\| \geq 0$
2. $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = 0$

3. $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$
4. $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$, Triangle inequality.

4.2.4 *Distance in Inner Product Spaces*

The distance between two vectors **u** and **v** in an inner product space V is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Properties of Distance

If **u**, **v** and **w** are vectors in an inner product space V then

1. $d(\mathbf{u}, \mathbf{v}) \ge 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$, Triangle inequality

Example 1: Find $\|\mathbf{u}\|$ if $\mathbf{u} = (3, 4)$ and weighted Euclidean inner product is $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_1v_2 - u_2v_1 + 3u_2v_2$ where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$.

Solution:

$$\begin{aligned}\|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \\ &= \left(u_1u_1 - u_1u_2 - u_2u_1 + 3u_2u_2\right)^{\frac{1}{2}} \\ &= \left(3^2 - 3\cdot 4 - 4\cdot 3 + 3(4)^2\right)^{\frac{1}{2}} \\ &= \sqrt{33}.\end{aligned}$$

Example 2: Find $d(\mathbf{u}, \mathbf{v})$ if $\mathbf{u} = (5, 4)$, $\mathbf{v} = (2, -6)$ and weighted Euclidean inner product is $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$.

Solution:

$$\mathbf{u} - \mathbf{v} = (5, 4) - (2, -6) = (3, 10)$$

$$\begin{aligned}d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{\frac{1}{2}} \\ &= \langle (3, 10), (3, 10) \rangle^{\frac{1}{2}} \\ &= (3\cdot 3\cdot 3 + 2\cdot 10\cdot 10)^{\frac{1}{2}} \\ &= \sqrt{227}\end{aligned}$$

Example 3: Find $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$ using the following inner products.

(i) the Euclidean inner product

(ii) the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$

(iii) the inner product generated by the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

Solution: $\mathbf{u} - \mathbf{v} = (-1, 2) - (2, 5) = (-1-2, 2-5) = (-3, -3)$

(i)

$$\begin{aligned}
\|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \\
&= (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \\
&= \left(u_1^2 + u_2^2\right)^{\frac{1}{2}} \\
&= (1+4)^{\frac{1}{2}} \\
&= \sqrt{5}
\end{aligned}$$

$$\begin{aligned}
d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\
&= \|(-3, -3)\| \\
&= [(-3, -3) \cdot (-3, -3)]^{\frac{1}{2}} \\
&= \left[(-3)^2 + (-3)^2\right]^{\frac{1}{2}} \\
&= \sqrt{18} = 3\sqrt{2}
\end{aligned}$$

(ii)

$$\begin{aligned}
\|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \\
&= \left(3u_1u_1 + 2u_2u_2\right)^{\frac{1}{2}} \\
&= \left[3(-1)^2 + 2(2)^2\right]^{\frac{1}{2}} \\
&= \sqrt{11}
\end{aligned}$$

$$\begin{aligned}
d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\
&= \|(-3, -3)\| \\
&= \langle (-3, -3), (-3, -3) \rangle^{\frac{1}{2}} \\
&= \left[3(-3)^2 + 2(-3)^2\right]^{\frac{1}{2}} \\
&= \sqrt{45} = 3\sqrt{5}
\end{aligned}$$

(iii) Inner product generated by the matrix A is

$$\langle \mathbf{u}, \mathbf{v}\rangle = A\mathbf{u}\cdot A\mathbf{v}, \text{ where } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\cdot\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + 2u_2 \\ -u_1 + 3u_2 \end{bmatrix}\cdot\begin{bmatrix} v_1 + 2v_2 \\ -v_1 + 3v_2 \end{bmatrix}$$

$$= (u_1 + 2u_2)(v_1 + 2v_2) + (-u_1 + 3u_2)(-v_1 + 3v_2)$$

$$= u_1v_1 + 2u_1v_2 + 2u_2v_1 + 4u_2v_2 + u_1v_1 - 3u_1v_2 - 3u_2v_1 + 9u_2v_2$$

$$= 2u_1v_1 - u_1v_2 - u_2v_1 + 13u_2v_2$$

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{\frac{1}{2}}$$

$$= (2u_1u_1 - u_1u_2 - u_2u_1 + 13u_2u_2)^{\frac{1}{2}}$$

$$= \left[2(-1)^2 - (-1)(2) - (2)(-1) + 13(2)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{58}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

$$= \|(-3, -3)\|$$

$$= \langle(-3, -3), (-3, -3)\rangle^{\frac{1}{2}}$$

$$= \left[2(-3)^2 - (-3)(-3) - (-3)(-3) + 13(-3)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{117}$$

$$= 3\sqrt{13}.$$

Example 4: Find $\|\mathbf{p}_1\|$ and $d(\mathbf{p}_1, \mathbf{p}_2)$ if $\mathbf{p}_1 = 3 - x + x^2$, $\mathbf{p}_2 = 2 + 5x^2$ and weighted inner product $\langle \mathbf{p}_1, \mathbf{p}_2\rangle = a_0b_0 + a_1b_1 + a_2b_2$ where $\mathbf{p}_1 = a_0 + a_1x + a_2x^2$, $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$

Solution:

$$\|\mathbf{p}\| = \langle \mathbf{p}_1, \mathbf{p}_1\rangle^{\frac{1}{2}}$$

$$= \left[3^2 + (-1)^2 + (1)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{11}$$

$$\mathbf{p}_1 - \mathbf{p}_2 = (3 - x + x^2) - (2 + 5x^2)$$

$$= 1 - x - 4x^2$$

$$d(\mathbf{p}_1, \mathbf{p}_2) = \|\mathbf{p}_1 - \mathbf{p}_2\|$$

$$= \left[1^2 + (-1)^2 + (-4)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

Example 5: Find $\|B\|$ and $d(A, B)$ if $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$ and weighted inner product $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

Solution:

$$\begin{aligned}
\|B\| &= \langle B, B\rangle^{\frac{1}{2}} \\
&= \left[(-4)^2 + (7)^2 + (1)^2 + (6)^2\right]^{\frac{1}{2}} \\
&= \sqrt{102}
\end{aligned}$$

$$A - B = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix}$$

$$\begin{aligned}
d\langle A, B\rangle &= \|A - B\| \\
&= \langle A - B, A - B\rangle^{\frac{1}{2}} \\
&= \left[(6)^2 + (-1)^2 + (8)^2 + (-2)^2\right]^{\frac{1}{2}} \\
&= \sqrt{105}
\end{aligned}$$

Example 6: Find inner product $\langle A, B\rangle = \text{tr}(B^T A)$ if

$$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution:

$$\begin{aligned}
\langle A, B\rangle &= \text{tr}(B^T A) \\
&= \text{tr}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}\right\} \\
&= [1 \quad 4]\begin{bmatrix} 9 \\ 6 \end{bmatrix} + [2 \quad 5]\begin{bmatrix} 8 \\ 5 \end{bmatrix} + [3 \quad 6]\begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \left[\begin{matrix}\text{Sum of diagonal} \\ \text{elements}\end{matrix}\right] \\
&= (9 + 24) + (16 + 25) + (21 + 24) \\
&= 119
\end{aligned}$$

Note: If can be observed that the second last step gives the sum of product of corresponding elements of A and B.

Hence, $\langle A, B\rangle = (9)(1) + (6)(4) + (8)(2) + (5)(5) + (7)(3) + (4)(6) = 119.$

Example 7: Find $\langle \mathbf{f}, \mathbf{g}\rangle$ if $\mathbf{f} = f(x) = 1 - x + x^2 + 5x^3$, $\mathbf{g} = g(x) = x - 3x^2$ and the inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Solution: $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{-1}^{1} (1 - x + x^2 + 5x^3)(x - 3x^2)\,dx$

$$= \int_{-1}^{1} (x - 3x^2 - x^2 + 3x^3 + x^3 - 3x^4 + 5x^4 - 15x^5)\, dx$$

$$= \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)\, dx$$

$$= 2\int_{0}^{1} (-4x^2 + 2x^4)\, dx \quad \left[\because \int_{-a}^{a} f(x)\, dx = 0, \text{ if } f(x) \text{ is odd} \quad = 2\int_{0}^{a} f(x)\, dx, \text{ if } f(x) \text{ is even}\right]$$

$$= 2\left| -4\frac{x^3}{3} + 2\frac{x^5}{5} \right|_0^1$$

$$= 2\left(-\frac{4}{3} + \frac{2}{5} \right)$$

$$= -\frac{28}{15}.$$

Example 8: Find $d(\mathbf{f}, \mathbf{g})$ if $\mathbf{f} = f(x) = \cos 2\pi x$ and $\mathrm{g} = g(x) = \sin 2\pi x$ and the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x) g(x)\, dx$

Solution: $d(\mathbf{f}, \mathbf{g}) = \|\mathbf{f} - \mathbf{g}\|$

$$= \langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle^{\frac{1}{2}}$$

$$\langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle = \int_0^1 [f(x) - g(x)][f(x) - g(x)]\, dx$$

$$= \int_0^1 (\cos 2\pi x - \sin 2\pi x)^2\, dx$$

$$= \int_0^1 \left(\cos^2 2\pi x + \sin^2 2\pi x - 2\cos 2\pi x \sin 2\pi x\right) dx$$

$$= \int_0^1 (1 - \sin 4\pi x)\, dx$$

$$= \left| x + \frac{\cos 4\pi x}{4\pi} \right|_0^1$$

$$= 1 + \frac{\cos 4\pi - \cos 0}{4\pi}$$

$$= 1$$

$$d(\mathbf{f}, \mathbf{g}) = \sqrt{1} = 1.$$

Exercise 4.1

1. Determine which of the following are inner products on R^2 if $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$

(i) $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}_1\mathbf{v}_1 - \mathbf{u}_1\mathbf{v}_2 - \mathbf{u}_2\mathbf{v}_1 + 3\mathbf{u}_2\mathbf{v}_2$

(ii) $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}_1\mathbf{v}_1\ \mathbf{u}_2\mathbf{v}_2$

(iii) $\langle \mathbf{u}, \mathbf{v}\rangle = 3\mathbf{u}_1\mathbf{v}_1 + 5\mathbf{u}_2\mathbf{v}_2$

[**Ans.:** (i), (iii)]

2. Show that $\langle A, B\rangle = a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4$ is not an inner product on M_{22} where

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

3. Find $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ if $\mathbf{u} = (2, -1)$, $\mathbf{v} = (-1, 1)$ and weighted inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 - u_1v_2 - u_2v_1 + u_2v_2$ where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$

$\left[\textbf{Ans.:} \sqrt{13}, \sqrt{34}\right]$

4. Find $\|\mathbf{p}_2\|$ and $d(\mathbf{p}_1, \mathbf{p}_2)$ if $\mathbf{p}_1 = 2x - x^2$, $\mathbf{p}_2 = -1 + x + 2x^2$ and weighted inner product $\langle \mathbf{p}_1, \mathbf{p}_2\rangle = a_0b_0 + a_1b_1 + a_2b_2$ where $\mathbf{p}_1 = a_0 + a_1x + a_2x^2$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$.

$\left[\textbf{Ans.:} \sqrt{6}, \sqrt{11}\right]$

5. Find $\|B\|$ and $d(A, B)$ if

$$A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -5 & 1 \\ 6 & 2 \end{bmatrix} \text{ and}$$

weighted inner product $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ where

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

$\left[\textbf{Ans.:} \sqrt{66}, \sqrt{47}\right]$

6. Find $\langle \mathbf{f}, \mathbf{g}\rangle$ if $\mathbf{f} = f(x) = x - 5x^3$, $\mathbf{g} = g(x) = 2 + 8x^2$ and the inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

[**Ans.:** 0]

7. Find $d(\mathbf{f}, \mathbf{g})$ if $\mathbf{f} = f(x) = x$, $\mathbf{g} = g(x) = e^x$ and the inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{0}^{1} f(x)g(x)\,dx$

$\left[\textbf{Ans.:} \frac{e^2}{2} - \frac{13}{6}\right]$

4.2.5 *Angle between Vectors*

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors in an inner product space V and if θ is the angle between them then

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

4.2.6 *Orthogonality*

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V are called orthogonal if

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0 \quad \text{i.e.,} \quad \theta = \frac{\pi}{2}$$

4.2.7 *Pythagorean Theorem*

If **u** and **v** are orthogonal vectors in an inner product space V then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof: Since **u** and **v** are orthogonal, $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle = 0$

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad [\because \langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle = 0]\end{aligned}$$

4.2.8 *Cauchy–Schwarz Inequality*

If **u** and **v** are vectors in an inner product space V then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

Example 1: Find the cosine of the angle between **u** and **v** if R^2, R^3 and R^4 have the Euclidean inner product.

(i) $\mathbf{u} = (1, -3)$, $\mathbf{v} = (2, 4)$

(ii) $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = (2, 4, -9)$

(iii) $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{v} = (-3, -3, -3, -3)$

Solution: $\cos\theta = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$, where θ is angle between **u** & **v**.

$$= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

(i) $$\cos\theta = \frac{(1,-3)\cdot(2,4)}{\sqrt{1+9}\sqrt{4+16}} = \frac{2-12}{\sqrt{10}\sqrt{20}} = \frac{-10}{\sqrt{10}\sqrt{20}} = -\frac{1}{\sqrt{2}}$$

(ii) $$\cos\theta = \frac{(-1,5,2)\cdot(2,4,-9)}{\sqrt{1+25+4}\sqrt{4+16+81}} = \frac{-2+20-18}{\sqrt{30}\sqrt{101}} = 0$$

(iii) $\cos\theta = \dfrac{(1,0,1,0)\cdot(-3,-3,-3,-3)}{\sqrt{1+1}\sqrt{9+9+9+9}}$

$= \dfrac{-3-3}{\sqrt{2}\sqrt{36}}$

$= -\dfrac{1}{\sqrt{2}}$

Example 2: Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

(i) $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$
(ii) $\mathbf{u} = (-4, 6, -10, 1)$, $\mathbf{v} = (2, 1, -2, 9)$

Solution: (i) $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v}$

$$\begin{aligned}&= (-1, 3, 2)\cdot(4, 2, -1)\\ &= (-1)(4)+(3)(2)+(2)(-1)\\ &= -4+6-2\\ &= 0\end{aligned}$$

Hence, **u** and **v** are orthogonal.

(ii) $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v}$

$$\begin{aligned}&= (-4, 6, -10, 1)\cdot(2, 1, -2, 9)\\ &= (-4)(2)+(6)(1)+(-10)(-2)+(1)(9)\\ &= -8+6+20+9\\ &= 27 \neq 0\end{aligned}$$

Hence, **u** and **v** are not orthogonal.

Example 3: Determine whether there exists scalars k and l such that the vectors $\mathbf{u} = (2, k, 6)$, $\mathbf{v} = (l, 5, 3)$ and $\mathbf{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product.

Solution: Let **u**, **v** and **w** be mutually orthogonal.

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = 0$$

$$(2, k, 6)\cdot(l, 5, 3) = 0$$

$$2l+5k+18 = 0$$

$$2l+5k = -18 \qquad \text{...(1)}$$

$$\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}\cdot\mathbf{w} = 0$$

$$(l, 5, 3)\cdot(1, 2, 3) = 0$$

$$l+10+9 = 0$$

$$l = -19 \qquad \text{...(2)}$$

$$\langle \mathbf{w}, \mathbf{u} \rangle = \mathbf{w} \cdot \mathbf{u} = 0$$
$$(1, 2, 3) \cdot (2, k, 6) = 0$$
$$2 + 2k + 18 = 0$$
$$k = -10 \qquad \text{...(3)}$$

Substituting l and k in equation (1),

$$2(-19) + 5(-10) = -18$$
$$-88 = -18$$

This shows that l and k does not satisfy equation (1).
Hence, there do not exist k and l such that **u**, **v** and **w** are orthogonal.

Example 4: Find cosine of the angle between $\mathbf{p}_1 = x - x^2$, and $\mathbf{p}_2 = 7 + 3x + 3x^2$ if the inner product $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_0b_0 + a_1b_1 + a_2b_2$, where

$\mathbf{p}_1 = a_0 + a_1x + a_2x^2$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$.

Solution: Let θ be the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$.

$$\cos\theta = \frac{\langle \mathbf{p}_1, \mathbf{p}_2 \rangle}{\|\mathbf{p}_1\| \|\mathbf{p}_2\|}$$
$$= \frac{\langle x - x^2, 7 + 3x + 3x^2 \rangle}{\sqrt{(1)^2 + (-1)^2}\sqrt{(7)^2 + (3)^2 + (3)^2}}$$
$$= \frac{(0)(7) + (1)(3) + (-1)(3)}{\sqrt{2}\sqrt{67}}$$
$$= 0$$

Example 5: Show that $\mathbf{p}_1 = 1 - x + 2x^2$ and $\mathbf{p}_2 = 2x + x^2$ are orthogonal with respect to the inner product $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_0b_0 + a_1b_1 + a_2b_2$, where

$\mathbf{p}_1 = a_0 + a_1x + a_2x^2$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$.

Solution:
$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle 1 - x + 2x^2, 2x + x^2 \rangle$$
$$= (1)(0) + (-1)(2) + (2)(1)$$
$$= 0$$

Hence, $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal.

Example 6: Show that the matrices $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ are orthogonal with respect to the inner product $\langle A, B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$.

Solution:
$$\langle A, B \rangle = (2)(1) + (1)(1) + (-1)(0) + (3)(-1)$$
$$= 0$$

Hence, A and B are orthogonal.

Example 7: For the matrices $A=\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, verify Cauchy–Schwarz inequality and find the cosine of the angle between them if M_{22} have the inner product as defined in Example 6.

Solution:

$$\begin{aligned}
\langle A, B\rangle &= (2)(3)+(6)(2)+(1)(1)+(-3)(0) \\
&= 19 \\
|\langle A, B\rangle| &= 19 \\
\|A\| &= \langle A, A\rangle^{\frac{1}{2}} \\
&= \sqrt{(2)^2+(6)^2+(1)^2+(-3)^2} \\
&= \sqrt{50} \\
&= 5\sqrt{2} \\
\|B\| &= \langle B, B\rangle^{\frac{1}{2}} \\
&= \sqrt{(3)^2+(2)^2+(1)^2+(0)^2} \\
&= \sqrt{14} \\
\|A\|\|B\| &= 5\sqrt{2}\cdot\sqrt{14} \\
&= 10\sqrt{7} \\
&= 26.45
\end{aligned}$$

Since $|\langle A, B\rangle| < \|A\|\|B\|$, Cauchy–Schwarz inequality is verified.

Let θ be the angle between A and B.

$$\begin{aligned}
\cos\theta &= \frac{\langle A, B\rangle}{\|A\|\|B\|} \\
&= \frac{19}{10\sqrt{7}}.
\end{aligned}$$

Example 8: Verify that the Cauchy–Schwarz inequality holds for the following vectors.

(i) $\mathbf{u}=(-2, 1)$ and $\mathbf{v}=(1, 0)$ where $\langle \mathbf{u}, \mathbf{v}\rangle = 3u_1v_1 + 2u_2v_2$.

(ii) $A=\begin{bmatrix} -1 & 2 \\ 6 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}$ using the inner product $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ where $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$.

(iii) $\mathbf{p}_1 = -1 + 2x + x^2$ and $\mathbf{p}_2 = 2 - 4x^2$ using the inner product $\langle \mathbf{p}_1, \mathbf{p}_2\rangle = a_0b_0 + a_1b_1 + a_2b_2$ where $\mathbf{p}_1 = a_0 + a_1x + a_2x^2$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2$.

Solution: (i)

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= \langle(-2, 1), (1, 0)\rangle \\ &= 3(-2)(1) + 2(1)(0) \\ &= -6\end{aligned}$$

$$|\langle \mathbf{u}, \mathbf{v}\rangle| = |-6| = 6$$

$$\begin{aligned}\|\mathbf{u}\| &= \langle \mathbf{u}, \mathbf{u}\rangle^{\frac{1}{2}} \\ &= \left(3u_1^2 + 2u_2^2\right)^{\frac{1}{2}} \\ &= \left[3(-2)^2 + 2(1)^2\right]^{\frac{1}{2}} \\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}\| &= \langle \mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}} \\ &= \left(3v_1^2 + 2v_2^2\right)^{\frac{1}{2}} \\ &= \left[3(1)^2 + 2(0)^2\right]^{\frac{1}{2}} \\ &= \sqrt{3}\end{aligned}$$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{42} = 6.48$$

Since, $|\langle \mathbf{u}, \mathbf{v}\rangle| < \|\mathbf{u}\|\|\mathbf{v}\|$, Cauchy–Schwarz's inequality is verified.

(ii)

$$\begin{aligned}\langle A, B\rangle &= (-1)(1) + (2)(0) + (6)(3) + (1)(3) \\ &= -1 + 18 + 3 \\ &= 20\end{aligned}$$

$$|\langle A, B\rangle| = 20$$

$$\|A\| = \sqrt{(-1)^2 + (2)^2 + (6)^2 + (1)^2} = \sqrt{42}$$

$$\|B\| = \sqrt{(1)^2 + (0)^2 + (3)^2 + (3)^2} = \sqrt{19}$$

$$\|A\|\,\|B\| = \sqrt{798} = 28.25$$

Since, $|\langle A, B\rangle| < \|A\|\|B\|$, Cauchy–Schwarz's inequality is verified.

(iii)

$$\begin{aligned}\langle \mathbf{p}_1, \mathbf{p}_2\rangle &= (-1)(2) + (2)(0) + (1)(-4) \\ &= -6\end{aligned}$$

$$|\langle \mathbf{p}_1, \mathbf{p}_2\rangle| = |-6| = 6$$

$$\|\mathbf{p}_1\| = \langle \mathbf{p}_1, \mathbf{p}_1 \rangle^{\frac{1}{2}}$$
$$= \sqrt{(-1)^2 + (2)^2 + (1)^2}$$
$$= \sqrt{6}$$
$$\|\mathbf{p}_2\| = \langle \mathbf{p}_2, \mathbf{p}_2 \rangle^{\frac{1}{2}}$$
$$= \sqrt{(2)^2 + (0)^2 + (-4)^2}$$
$$= \sqrt{20}$$
$$\|\mathbf{p}_1\|\|\mathbf{p}_2\| = \sqrt{120} = 10.95$$

Since, $|\langle \mathbf{p}_1, \mathbf{p}_2 \rangle| < \|\mathbf{p}_1\|\|\mathbf{p}_2\|$, Cauchy–Schwarz's inequality is verified.

4.2.9 *Orthogonal and Orthonormal Set*

A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ of vectors in an inner product space V is called an orthogonal set if each pair of distinct vectors in S are orthogonal, i.e. $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $i \neq j$.

An orthogonal set of unit vectors (norm is 1) is called orthonormal, i.e. $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $i \neq j$ and $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = 1$ for $i = 1, 2, \ldots, p$.

The process of dividing a non-zero vector **u** by its norm is called normalizing **u**.

$$\text{normalized } \mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Example 1: Show that the vectors $\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $\mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ and $\mathbf{u}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ are orthonormal with respect to the Euclidean inner product on R^3.

Solution:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$
$$= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}\right)$$
$$= 0$$
$$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$
$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right)$$
$$= 0$$
$$\langle \mathbf{u}_3, \mathbf{u}_1 \rangle = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \cdot \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$
$$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$$
$$= 0$$

$$\|\mathbf{u}_1\| = \langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{\frac{1}{2}} = \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]^{\frac{1}{2}} = 1$$

$$\|\mathbf{u}_2\| = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle^{\frac{1}{2}} = \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2\right]^{\frac{1}{2}} = 1$$

$$\|\mathbf{u}_3\| = \langle \mathbf{u}_3, \mathbf{u}_3 \rangle^{\frac{1}{2}} = \left[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right]^{\frac{1}{2}} = 1$$

Since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_3, \mathbf{u}_1 \rangle = 0$ and $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$, the vectors are orthonormal.

Example 2: Show that the set of vectors $\mathbf{u}_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\mathbf{u}_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, $\mathbf{u}_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is orthogonal with respect to the Euclidean inner product on R^3 and then convert it to an orthonormal set by normalizing the vectors.

Solution:

$$\begin{aligned}\langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\ &= \left(\frac{1}{5}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{5}\right)(0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \cdot \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \\ &= \left(-\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(-\frac{2}{3}\right) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}_3, \mathbf{u}_1 \rangle &= \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \cdot \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{5}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{5}\right) \\ &= 0\end{aligned}$$

Hence, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are orthogonal.

$$\|\mathbf{u}_1\| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \sqrt{\frac{3}{25}} = \frac{\sqrt{3}}{5}$$

$$\|\mathbf{u}_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$$

$$\|\mathbf{u}_3\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{2}{3}}$$

Normalising the vectors,

$$\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\mathbf{w}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

Orthonormal set = $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

Exercise 4.2

1. Verify that the Cauchy–Schwarz in-equality holds for the following vectors with respect to the Euclidean inner product.

(i) $\mathbf{u} = (-4, 2, 1), \mathbf{v} = (8, -4, -2)$

(ii) $\mathbf{u} = (0, -2, 2, 1), \mathbf{v} = (-1, -1, 1, 1)$

[**Ans.:** (i), (ii)]

2. Find the cosine of the angle between **u** and **v** if R^3 and R^4 have Euclidean inner product:

(i) $\mathbf{u} = (4, 1, 8), \mathbf{v} = (1, 0, -3)$

(ii) $\mathbf{u} = (2, 1, 7, -1), \mathbf{v} = (4, 0, 0, 0)$

$\left[\textbf{Ans.: (i)} -\frac{2\sqrt{10}}{9} \quad \textbf{(ii)} \frac{2}{\sqrt{55}}\right]$

3. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product:

(i) $(1, -1, 2), (0, 2, -1), (-1, 1, 1)$

(ii) $(0, 1, 0, -1)\ (1, 0, 1, 0), (-1, 1, 1, 1)$

[**Ans.:** (ii)]

4. Let $\mathbf{u} = (1, 1, -2)$ and $\mathbf{v} = (a, -1, 2)$. For what values of a are **u** and **v** orthogonal with respect to the Euclidean inner product?

[**Ans.:** $a = 5$]

5. For what values of a and b is the set $\{\mathbf{u}, \mathbf{v}\}$ orthonormal with respect to the Euclidean inner product where

$$\mathbf{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \text{ and } \mathbf{v} = \left(a, \frac{1}{\sqrt{2}}, -b\right)$$

$\left[\textbf{Ans.: } a = \pm\frac{1}{2}, b = \pm\frac{1}{2}\right]$

4.3 ORTHOGONAL AND ORTHONORMAL BASIS

In an inner product space, a basis consisting of orthogonal vectors is called an orthogonal basis.
In an inner product space, a basis consisting of orthonormal vectors is called an orthonormal basis.

Theorem 4.1: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space V then S is linearly independent.

Theorem 4.2: Any orthogonal set of n non-zero vectors in R^n is a basis for R^n.

Theorem 4.3: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V then it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Here, $\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle$ are the coordinate vectors of $\mathbf{u}$ with respect to the orthonormal basis S,

i.e.,
$$[\mathbf{u}]_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle)$$

Corollary: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and $\mathbf{u}$ is any vector in V then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

Theorem 4.4: If S is an orthonormal basis for an n-dimensional inner product space, and if coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ with respect to S are $[\mathbf{u}]_S = (a_1, a_2, \ldots, a_n)$ and $[\mathbf{v}]_S = (b_1, b_2, \ldots, b_n)$ then,

(i) $\|\mathbf{u}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

(ii) $d(\mathbf{u}, \mathbf{v}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$

(iii) $\langle \mathbf{u}, \mathbf{v} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = [\mathbf{u}]_S \cdot [\mathbf{v}]_S$

Theorem 4.5: Every non-zero finite dimensional inner product space has an orthonormal basis.

Constructing an Orthogonal Basis from an Arbitrary Basis

An orthogonal basis for a non-zero finite dimensional inner product space V can be constructed from an arbitrary basis of V using Gram–Schmidt process.

4.4 GRAM–SCHMIDT PROCESS

Let V be any non-zero n-dimensional inner product space and $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an arbitrary basis for V. The process of constructing an orthogonal basis $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ from S_1 is as follows.

Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1$.
Step 2: Find the vectors $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ successively using the formula

$$\mathbf{v}_i = \mathbf{u}_i - \frac{\langle \mathbf{u}_i, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_i, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{u}_i, \mathbf{v}_{i-1} \rangle}{\|\mathbf{v}_{i-1}\|^2}\mathbf{v}_{i-1}$$

The set S_2 of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set. Since every orthogonal set is linearly independent, S_2 is linearly independent and also has n vectors (dim V), thus the set S_2 is an orthogonal basis for V.

Note: The orthogonal basis S_2 can be transformed to orthonormal basis by normalizing all the vectors of S_2.

Example 1: Find an orthonormal basis for R^3 containing the vectors $\mathbf{v}_1 = (3, 5, 1)$, $\mathbf{v}_2 = (2, -2, 4)$ using the Euclidean inner product.

Solution:

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_2 \\ &= (3, 5, 1)\cdot(2, -2, 4) \\ &= 6 - 10 + 4 \\ &= 0 \end{aligned}$$

Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.
Basis for R^3 will have 3 non-zero vectors.
Let $\mathbf{v}_3 = (b_1, b_2, b_3)$ be the third vector of the basis such that

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 \quad \text{and} \quad \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$$

$$(3, 5, 1) \cdot (b_1, b_2, b_3) = 0$$

$$3b_1 + 5b_2 + b_3 = 0 \qquad \text{...(1)}$$

and

$$(2, -2, 4) \cdot (b_1, b_2, b_3) = 0$$

$$2b_1 - 2b_2 + 4b_3 = 0 \qquad \text{...(2)}$$

The augmented matrix of the system of equations (1) and (2) is

$$\left[\begin{array}{ccc|c} 3 & 5 & 1 & 0 \\ 2 & -2 & 4 & 0 \end{array}\right]$$

Reducing the augment matrix to row echelon form,

$$R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -3 & 0 \\ 2 & -2 & 4 & 0 \end{array}\right]$$

$$R_2 - 2R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 7 & -3 & 0 \\ 0 & -16 & 10 & 0 \end{array}\right]$$

$$\left(-\frac{1}{16}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 7 & -3 & 0 \\ 0 & 1 & -\frac{5}{8} & 0 \end{array}\right]$$

The corresponding system of equations is

$$b_1 + 7b_2 - 3b_3 = 0$$

$$b_2 - \frac{5}{8}b_3 = 0$$

Solving for leading variables b_1 and b_2,

$$b_1 = -7b_2 + 3b_3, \quad b_2 = \frac{5}{8}b_3$$

Let $\qquad b_3 = 8$

$$b_1 = -11, b_2 = 5$$

$$\mathbf{v}_3 = (-11, 5, 8)$$

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form an orthogonal basis for R^3.
Normalizing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$,

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(3, 5, 1)}{\sqrt{9+25+1}} = \left(\frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}\right)$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(2, -2, 4)}{\sqrt{4+4+16}} = \left(\frac{2}{\sqrt{24}}, -\frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}}\right)$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{(-11, 5, 8)}{\sqrt{121+25+64}} = \left(-\frac{\sqrt{11}}{\sqrt{210}}, \frac{5}{\sqrt{210}}, \frac{8}{\sqrt{210}}\right)$$

The vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ form an orthonormal basis for R^3.

Example 2: Verify that the basis vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $\mathbf{v}_3 = (0, 0, 1)$ form an orthonormal basis S for R^3 with the Euclidean inner product. Express the vector $\mathbf{u} = (1, -1, 2)$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and find coordinate vector $[\mathbf{u}]_S$.

Solution:

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)\cdot\left(\frac{4}{5}, \frac{3}{5}, 0\right)$$

$$= \left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) + 0$$

$$= 0$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(\frac{4}{5}, \frac{3}{5}, 0\right) \cdot (0, 0, 1)$$

$$= \left(\frac{4}{5}\right)(0) + \left(\frac{3}{5}\right)(0) + (0)(1)$$

$$= 0$$

$$\langle \mathbf{v}_3, \mathbf{v}_1 \rangle = (0, 0, 1) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$= (0)\left(-\frac{3}{5}\right) + (0)\left(\frac{4}{5}\right) + (1)(0)$$

$$= 0$$

$$\|\mathbf{v}_1\| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{(1)^2} = 1$$

Hence, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form an orthonormal basis for R^3.

Since, $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis in R^3, any vector $\mathbf{u}$ in R^3 can be expressed as

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3$$

$$(1,-1,2) = \left[(1,-1,2)\cdot\left(-\frac{3}{5},\frac{4}{5},0\right)\right]\mathbf{v}_1 + \left[(1,-1,2)\cdot\left(\frac{4}{5},\frac{3}{5},0\right)\right]\mathbf{v}_2 + \left[(1,-1,2)\cdot(0,0,1)\right]\mathbf{v}_3$$

$$= -\frac{7}{5}\mathbf{v}_1 + \frac{1}{5}\mathbf{v}_2 + 2\mathbf{v}_3.$$

$$[\mathbf{u}]_S = \left(-\frac{7}{5}, \frac{1}{5}, 2\right)$$

Example 3: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{v}_1 = (1, -1, 2, -1)$, $\mathbf{v}_2 = (-2, 2, 3, 2)$, $\mathbf{v}_3 = (1, 2, 0, -1)$, $\mathbf{v}_4 = (1, 0, 0, 1)$, is an orthogonal basis for R^4 with Euclidean inner product. Express the vector $\mathbf{u} = (1, 1, 1, 1)$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and find the coordinate vector $[\mathbf{u}]_S$.

Solution: Since S is an orthogonal basis for R^4, any vector $\mathbf{u}$ in R^4 can be expressed as

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 + \frac{\langle \mathbf{u}, \mathbf{v}_4 \rangle}{\|\mathbf{v}_4\|^2}\mathbf{v}_4$$

$$(1, 1, 1, 1) = \frac{(1, 1, 1, 1)\cdot(1, -1, 2, -1)}{\sqrt{1+1+4+1}}\mathbf{v}_1 + \frac{(1, 1, 1, 1)\cdot(-2, 2, 3, 2)}{\sqrt{4+4+9+4}}\mathbf{v}_2$$

$$+ \frac{(1, 1, 1, 1)\cdot(1, 2, 0, -1)}{\sqrt{1+4+0+1}}\mathbf{v}_3 + \frac{(1, 1, 1, 1)\cdot(1, 0, 0, 1)}{\sqrt{1+0+0+1}}\mathbf{v}_4$$

$$= \frac{1}{\sqrt{7}}\mathbf{v}_1 + \frac{5}{\sqrt{21}}\mathbf{v}_2 + \frac{2}{\sqrt{6}}\mathbf{v}_3 + \frac{2}{\sqrt{2}}\mathbf{v}_4$$

$$[\mathbf{u}]_S = \left(\frac{1}{\sqrt{7}}, \frac{5}{\sqrt{21}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{2}}\right)$$

Example 4: Let R^3 have the Euclidean inner product and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthonormal basis with $\mathbf{v}_1 = \left(0, -\frac{3}{5}, \frac{4}{5}\right)$, $\mathbf{v}_2 = (1, 0, 0)$ and $\mathbf{v}_3 = \left(0, \frac{4}{5}, \frac{3}{5}\right)$.

(i) Find the vectors **u**, **v** and **w** that have the coordinate vectors $[\mathbf{u}]_S = (-2, 1, 2)$, $[\mathbf{v}]_S = (3, 0, -2)$ and $[\mathbf{w}]_S = (5, -4, 1)$.

(ii) Find $\|\mathbf{v}\|$, $d(\mathbf{u}, \mathbf{w})$ and $\langle \mathbf{w}, \mathbf{v} \rangle$ using coordinate vectors.

Solution: (i)

$$\mathbf{u} = -2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$$

$$= -2\left(0, -\frac{3}{5}, \frac{4}{5}\right) + (1, 0, 0) + 2\left(0, \frac{4}{5}, \frac{3}{5}\right)$$

$$= \left(0+1+0, \frac{6}{5}+0+\frac{8}{5}, -\frac{8}{5}+0+\frac{6}{5}\right)$$

$$= \left(1, \frac{14}{5}, -\frac{2}{5}\right)$$

$$\mathbf{v} = 3\mathbf{v}_1 + 0\mathbf{v}_2 - 2\mathbf{v}_3$$

$$= 3\left(0, -\frac{3}{5}, \frac{4}{5}\right) - 2\left(0, \frac{4}{5}, \frac{3}{5}\right)$$

$$= \left(0, -\frac{9}{5} - \frac{8}{5}, \frac{12}{5} - \frac{6}{5}\right)$$

$$= \left(0, -\frac{17}{5}, \frac{6}{5}\right)$$

$$\mathbf{w} = 5\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3$$

$$= 5\left(0, -\frac{3}{5}, \frac{4}{5}\right) - 4(1, 0, 0) + \left(0, \frac{4}{5}, \frac{3}{5}\right)$$

$$= \left(-4, -\frac{15}{5} + \frac{4}{5}, \frac{20}{5} + \frac{3}{5}\right)$$

$$= \left(-4, -\frac{11}{5}, \frac{23}{5}\right)$$

(ii)

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (0)^2 + (-2)^2}$$

$$= \sqrt{13}$$

$$
\begin{aligned}
d(\mathbf{u}, \mathbf{w}) &= \sqrt{(-2-5)^2 + (1+4)^2 + (2-1)^2} \\
&= \sqrt{75} \\
&= 5\sqrt{3} \\
\langle \mathbf{w}, \mathbf{v} \rangle &= [\mathbf{w}]_s \cdot [\mathbf{v}]_s \\
&= (5, -4, 1) \cdot (3, 0, -2) \\
&= 15 - 2 \\
&= 13
\end{aligned}
$$

Example 5: Let R^3 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (3, 7, -2)$, $\mathbf{u}_3 = (0, 4, 1)$ into an orthonormal basis.

Solution: Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1 = (1, 0, 0)$

Step 2:
$$
\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \\
&= (3, 7, -2) - \frac{(3, 7, -2)\cdot(1, 0, 0)}{1}(1, 0, 0) \\
&= (3, 7, -2) - 3(1, 0, 0) \\
&= (0, 7, -2) \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\
&= (0, 4, 1) - \frac{(0, 4, 1)\cdot(1, 0, 0)}{1}(1, 0, 0) - \frac{(0, 4, 1)\cdot(0, 7, -2)}{(49+4)}(0, 7, -2) \\
&= (0, 4, 1) - 0 - \frac{28-2}{53}(0, 7, -2) \\
&= (0, 4, 1) - \left(0, \frac{182}{53}, -\frac{52}{53}\right) \\
&= \left(0, \frac{30}{53}, \frac{105}{53}\right) \\
&= \frac{15}{53}(0, 2, 7)
\end{aligned}
$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for R^3. Normalizing $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$,

$$
\begin{aligned}
\mathbf{w}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0, 0)}{1} = (1, 0, 0) \\
\mathbf{w}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(0, 7, -2)}{\sqrt{49+4}} = \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right) \\
\mathbf{w}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\frac{15}{53}(0, 2, 7)}{\frac{15}{53}\sqrt{4+49}} = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)
\end{aligned}
$$

The vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ form an orthonormal basis for R^3.

Example 6: Let R^3 have the inner product $\langle (x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$. Use the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

Solution: Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

Step 2: $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$= (1, 1, 0) - \frac{\langle (1, 1, 0), (1, 1, 1)\rangle}{(1^2 + 2\cdot 1^2 + 3\cdot 1^2)}(1, 1, 1)$$

$$= (1, 1, 0) - \frac{(1)(1) + 2(1)(1) + 3(0)(1)}{6}(1, 1, 1)$$

$$= (1, 1, 0) - \frac{1}{2}(1, 1, 1)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= (1, 0, 0) - \frac{\langle (1, 0, 0), (1, 1, 1)\rangle}{(1^2 + 2\cdot 1^2 + 3\cdot 1^2)}(1, 1, 1) - \frac{\left\langle (1, 0, 0), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right\rangle}{\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right)}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) - \frac{1}{6}(1, 1, 1) - \frac{2}{6}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= \left(1 - \frac{1}{6} - \frac{1}{6}, -\frac{1}{6} - \frac{1}{6}, -\frac{1}{6} + \frac{1}{6}\right)$$

$$= \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for R^3. Normalizing $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$,

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 1, 1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}{\sqrt{\frac{6}{4}}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3}, -\frac{1}{3}, 0\right)}{\sqrt{\frac{6}{9}}} = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)$$

The vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ form an orthonormal basis for R^3.

Example 7: Use the Gram–Schmidt process to transform the basis $\{1, x, x^2\}$ of P_2 into an orthonormal basis if

(i) $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\,\mathbf{q}(0) + \mathbf{p}(1)\,\mathbf{q}(1) + \mathbf{p}(2)\,\mathbf{q}(2)$

(ii) $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^2 p(x)q(x)\,dx$

Solution: Let $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$

(i) **Step 1:** Let $\mathbf{q}_1 = \mathbf{p}_1 = 1$

Step 2:

$$\begin{aligned}
\mathbf{q}_2 &= \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 \\
&= x - \frac{(0)(1)+(1)(1)+(2)(1)}{(1+1+1)}(1) \\
&= (x-1) \\
\mathbf{q}_3 &= \mathbf{p}_3 - \frac{\langle \mathbf{p}_3, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 - \frac{\langle \mathbf{p}_3, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 \\
&= x^2 - \frac{(0)(1)+(1)(1)+(4)(1)}{(1+1+1)}(1) - \frac{(0)(-1)+(1)(0)+(4)(1)}{(1+0+1)}(x-1) \\
&= x^2 - \frac{5}{3} - 2(x-1) \\
&= x^2 - 2x + \frac{1}{3}
\end{aligned}$$

The polynomials $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ form an orthogonal basis for P_2. Normalizing $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$,

$$\mathbf{r}_1 = \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \frac{1}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}$$

$$\mathbf{r}_2 = \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \frac{x-1}{\sqrt{(-1)^2+0^2+(1)^2}} = \frac{x-1}{\sqrt{2}}$$

$$\mathbf{r}_3 = \frac{\mathbf{r}_3}{\|\mathbf{r}_3\|} = \frac{x^2-2x+\frac{1}{3}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2}} = \sqrt{\frac{3}{2}}\left(x^2-2x+\frac{1}{3}\right)$$

The polynomials $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis for P_2.

(ii) **Step 1:** Let $\mathbf{q}_1 = \mathbf{p}_1 = 1$

Step 2:

$$\mathbf{q}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1$$

$$= x - \frac{\int_0^2 (x)(1)\,dx}{\int_0^2 1\cdot dx}(1)$$

$$= x - \frac{2}{2}$$

$$= x - 1$$

Step 3:

$$\mathbf{q}_3 = \mathbf{p}_3 - \frac{\langle \mathbf{p}_3, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 - \frac{\langle \mathbf{p}_3, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2$$

$$= x^2 - \frac{\int_0^2 (x^2)(1)}{2}(1) - \frac{\int_0^2 x^2(x-1)\,dx}{\int_0^2 (x-1)^2\,dx}(x-1)$$

$$= x^2 - \frac{4}{3} - \frac{\left(\frac{4}{3}\right)}{\left(\frac{2}{3}\right)}(x-1)$$

$$= x^2 - 2x + \frac{2}{3}$$

The polynomials $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ form an orthonormal basis for P_2.
Normalizing $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$,

$$\mathbf{r}_1 = \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \frac{1}{\sqrt{2}}$$

$$\mathbf{r}_2 = \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \frac{x-1}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}(x-1)$$

$$\mathbf{r}_3 = \frac{\mathbf{q}_3}{\|\mathbf{q}_3\|} = \frac{x^2 - 2x + \frac{2}{3}}{\sqrt{\int_0^2 \left(x^2 - 2x + \frac{2}{3}\right)^2 dx}}$$

$$= \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - 2x + \frac{2}{3}\right)$$

The polynomials $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form an orthonormal basis for P_2.

Example 8: Use Gram–Schmidt method to transform the basis $\left\{\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}$ of M_{22} into an orthogonal basis if

$$\langle A, B\rangle = \operatorname{tr}(AB^T).$$

Solution: Let $A_1 = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, A_2 = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, A_3 = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, A_4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Step 1: Let $B_1 = A_1 = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

Step2:

$$B_2 = A_2 - \frac{\langle A_2, B_1\rangle}{\|B_1\|^2} B_1$$

$$= A_2 - \frac{\operatorname{tr}(A_2 B_1^T)}{\operatorname{tr}(B_1 B_1^T)} B_1$$

$$= A_2 - \frac{(1)(1)+(0)(1)+(1)(0)+(0)(0)}{1+1+0+0} B_1$$

$$= \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\2 & 0\end{bmatrix}$$

$$B_3 = A_3 - \frac{\langle A_3, B_1\rangle}{\|B_1\|^2} B_1 - \frac{\langle A_3, B_2\rangle}{\|B_2\|^2} B_2$$

$$= A_3 - \frac{\operatorname{tr}(A_3 B_1^T)}{\operatorname{tr}(B_1 B_1^T)} B_1 - \frac{\operatorname{tr}(A_3 B_2^T)}{\operatorname{tr}(B_2 B_2^T)} B_2$$

$$= A_3 - \frac{1}{2} B_1 - \frac{\left(-\frac{1}{2}\right)}{3} B_2$$

$$= \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1 & -1\\2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{6} & \frac{1}{3}\\ \frac{1}{3} & \frac{5}{6}\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix}-1 & 2\\2 & 5\end{bmatrix}$$

$$B_4 = A_4 - \frac{\langle A_4, B_1\rangle}{\|B_1\|^2}B_1 - \frac{\langle A_4, B_2\rangle}{\|B_2\|^2}B_2 - \frac{\langle A_4, B_3\rangle}{\|B_3\|^2}B_3$$

$$= A_4 - \frac{\text{tr}\,(A_4B_1^T)}{(B_1B_1^T)}B_1 - \frac{\text{tr}\,(A_4B_2^T)}{(B_2B_2^T)}B_2 - \frac{\text{tr}\,(A_4B_3^T)}{(B_3B_3^T)}B_3$$

$$= A_4 - \frac{1}{2}B_1 - \frac{\left(\frac{1}{2}\right)}{3}B_2 - \frac{\left(\frac{2}{3}\right)}{\left(\frac{17}{18}\right)}B_3$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{6}\cdot\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \frac{12}{17}\cdot\frac{1}{6}\begin{bmatrix} -1 & 2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{109}{204} & -\frac{133}{204} \\ -\frac{41}{102} & \frac{7}{17} \end{bmatrix}$$

The matrices B_1, B_2, B_3 and B_4 form an orthogonal basis for M_{22}.

Example 9: Let R^4 have the Euclidean inner product. Find an orthonormal basis for the subspace of R^4 consisting of all the vectors (a, b, c, d) such that

$$a - b - 2c + d = 0$$

Solution: Using $a - b - 2c + d = 0$, the vector (a, b, c, d) can be written as

$$= a(1, 0, 0, -1) + b(0, 1, 0, 1) + c(0, 0, 1, 2)$$

$$= a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

Basis for the vector $(a, b, c, d) = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

where $\mathbf{u}_1 = (1, 0, 0, -1)$, $\mathbf{u}_2 = (0, 1, 0, 1)$, $\mathbf{u}_3 = (0, 0, 1, 2)$

To convert the above basis to an orthonormal basis, apply the Gram–Schmidt process using Euclidean inner product.

Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1 = (1, 0, 0, -1)$

Step 2: $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$= (0, 1, 0, 1) - \frac{(0, 1, 0, 1) \cdot (1, 0, 0, -1)}{(1+1)}(1, 0, 0, -1)$$

$$= (0, 1, 0, 1) + \frac{1}{2}(1, 0, 0, -1)$$

$$= \left(\frac{1}{2}, 1, 0, \frac{1}{2}\right)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= (0, 0, 1, 2) - \frac{(0, 0, 1, 2) \cdot (1, 0, 0, -1)}{(1+1)}(1, 0, 0, -1)$$

$$- \frac{(0, 0, 1, 2) \cdot \left(\frac{1}{2}, 1, 0, \frac{1}{2}\right)}{\left(\frac{1}{4} + 1 + \frac{1}{4}\right)}\left(\frac{1}{2}, 1, 0, \frac{1}{2}\right)$$

$$= (0, 0, 1, 2) + (1, 0, 0, -1) - \left(\frac{1}{3}, \frac{2}{3}, 0, \frac{1}{3}\right)$$

$$= \left(\frac{2}{3}, -\frac{2}{3}, 1, \frac{2}{3}\right)$$

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form an orthogonal basis for the subspace of R^4. Normalizing $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$,

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0, 0, -1)}{\sqrt{1+1}} = \left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2}, 1, 0, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + 1 + \frac{1}{4}}} = \left(\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{6}}\right)$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3}, -\frac{2}{3}, 1, \frac{2}{3}\right)}{\sqrt{\frac{4}{9} + \frac{4}{9} + 1 + \frac{4}{9}}} = \left(\frac{2}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \sqrt{\frac{3}{7}}, \frac{2}{\sqrt{21}}\right)$$

The vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ form an orthonormal basis for the subspace of R^4.

Example 10: Find an orthonormal basis for the solution space of the homogeneous system

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 + x_2 + 2x_3 = 0$$

Solution: The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]$$

$$(-1)\ R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + x_2 - \ x_3 = 0$$
$$x_2 - 4x_3 = 0$$

Solving for leading variables x_1 and x_2,

$$x_1 = -x_2 + x_3$$
$$x_2 = 4x_3$$

Let $x_3 = t,$

$$x_1 = -3t, \quad x_2 = 4t,$$

The solution space of system consists vectors of the form $\mathbf{x} = \begin{bmatrix} -3t \\ 4t \\ t \end{bmatrix} = t\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

Hence, basis for the solution space $= \left\{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right\}$

Normalizing the basis vector,

Orthonormal basis $= \left\{\begin{bmatrix} -\frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{bmatrix}\right\}$

Exercise 4.3

1. Find an orthonormal basis for R^3 containing the vectors (2, –2, 1) and (2, 1, –2), using Euclidean inner product.

$$\left[\textbf{Ans.:} \left\{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\}\right]$$

2. Consider the orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3 with the Euclidean inner product where $\mathbf{v}_1 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$ $\mathbf{v}_2 = \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$, $\mathbf{v}_3 = (0, 1, 0)$ Express the vector $\mathbf{u} = (2, -3, 1)$ as a linear combination of the vectors in S and find the coordinate vector $[\mathbf{u}]_s$.

$$\left[\textbf{Ans.: } \mathbf{u} = \frac{4}{\sqrt{5}}\mathbf{v}_1 - \frac{3}{\sqrt{5}}\mathbf{v}_2 - 3\mathbf{v}_3, \left(\frac{4}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, -3\right)\right]$$

3. Verify that the basis vectors $\mathbf{v}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $\mathbf{v}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $\mathbf{v}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ form an orthonormal basis S for R^3 with the Euclidean inner product. Express the vector $\mathbf{u} = (3, 4, 5)$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and find the coordinate vector $[\mathbf{u}]_s$.

$$[\textbf{Ans.: } \mathbf{u} = \mathbf{v}_1 + 0\mathbf{v}_2 + 7\mathbf{v}_3, (1, 0, 7)]$$

4. Verify that the vectors $\mathbf{v}_1 = (1, -2, 3, -4)$, $\mathbf{v}_2 = (2, 1, -4, -3)$, $\mathbf{v}_3 = (-3, 4, 1, -2)$ and $\mathbf{v}_4 = (4, 3, 2, 1)$ form an orthogonal basis for R^4 with the Euclidean inner product. Express the vector $\mathbf{u} = (-1, 2, 3, 7)$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and find the coordinate vector $[\mathbf{u}]_s$.

$$\left[\textbf{Ans.: } \mathbf{u} = -\frac{4}{5}\mathbf{v}_1 - \frac{11}{10}\mathbf{v}_2 - 0\mathbf{v}_3 + \frac{1}{2}\mathbf{v}_4, \left(-\frac{4}{5}, -\frac{11}{10}, 0, \frac{1}{2}\right)\right]$$

5. Find the coordinate vector of $\mathbf{u} = (-1, 0, 2)$ with respect to the orthonormal basis $\left\{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\}$ using Euclidean inner product.

$$[\textbf{Ans.: } (0, -2, 1)]$$

6. Let R^2 have the Euclidean inner product and let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is the orthonormal basis with $\mathbf{v}_1 = \left(\frac{3}{5}, -\frac{4}{5}\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}\right)$.

(i) Find the vectors $\mathbf{u}$ and $\mathbf{v}$ that have coordinate vectors $[\mathbf{u}]_s = (1, 1)$ and $[\mathbf{v}]_s = (-1, 4)$.

(ii) Find $\|\mathbf{u}\|$, $d(\mathbf{u}, \mathbf{v})$ and $\langle \mathbf{u}, \mathbf{v} \rangle$ using coordinate vectors.

$$\left[\textbf{Ans.: } \text{(i) } \mathbf{u} = \left(\frac{7}{5}, -\frac{1}{5}\right), \mathbf{v} = \left(\frac{13}{5}, \frac{8}{5}\right) \quad \text{(ii) } \sqrt{2}, \sqrt{13}, 3\right]$$

7. Let R^3 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 2, 1)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (3, 1, 0)$ into an orthogonal basis.

$$\left[\textbf{Ans.: } \left\{(1, 2, 1), \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right), \left(\frac{3}{2}, 0, -\frac{3}{2}\right)\right\}\right]$$

8. Let R^3 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 0, 3)$, $\mathbf{u}_2 = (2, 2, 0)$, $\mathbf{u}_3 = (3, 1, 2)$ into an orthonormal basis.

$$\left[\textbf{Ans.: } \left\{\left(\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}\right), \left(\frac{9}{\sqrt{190}}, \frac{10}{\sqrt{190}}, -\frac{3}{\sqrt{190}}\right), \left(\frac{3}{\sqrt{19}}, -\frac{3}{\sqrt{19}}, -\frac{1}{\sqrt{19}}\right)\right\}\right]$$

9. Let R^3 have the inner product $\langle(x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3$ Use the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, -1, 1)$, $\mathbf{u}_3 = (1, 1, 0)$ into an orthogonal basis.

[**Ans.:** (1, 1, 1), (1, –5, 1), (3, 0, –2)]

10. Use the Gram–Schmidt process to transform the basis $\{1, x, x^2, x^3\}$ of P_3 into an orthogonal basis if

$$\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

$$\left[\textbf{Ans.:} \left\{1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x^2)\right\}\right]$$

11. Let M_{22} have the inner product $\langle A, B\rangle = \text{tr}(AB^T)$. Use the Gram–Schmidt process to transform the basis

vectors $A_1 = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, A_2 = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, A_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, A_4 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

into an orthogonal basis.

$$\left[\textbf{Ans.:} \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \frac{1}{\sqrt{15}}\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right]$$

12. Find an orthonormal basis for the subspace of R^4 consisting of all vectors of the form $(a, a+b, c, b+c)$.

$$\left[\textbf{Ans.:} \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)\right\}\right]$$

13. Use Gram–Schmidt process to construct an orthonormal basis for the subspace W of R^4 spanned by the vectors $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (2, -1, 0, 1)$, $\mathbf{v}_3 = (3, -3, 0, -2)$, $\mathbf{v}_4 = (1, -2, 0, -3)$ using Euclidean inner product.

$$\left[\textbf{Ans.:} \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \left(\frac{3}{\sqrt{22}}, -\frac{3}{\sqrt{22}}, 0, \frac{2}{\sqrt{22}}\right), \left(\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, 0, -\frac{3}{\sqrt{11}}\right)\right\}\right]$$

14. Find an orthonormal basis for the solution space of the homogeneous system

$$\begin{bmatrix}1 & 1 & -1\\2 & 1 & 3\\1 & 2 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\textbf{Ans.:} \left\{\begin{bmatrix}-\frac{4}{\sqrt{42}}\\ \frac{5}{\sqrt{42}}\\ \frac{1}{\sqrt{42}}\end{bmatrix}\right\}\right]$$

4.5 ORTHOGONAL COMPLEMENTS

Let W be a subspace of an inner product space V. A vector **u** in V is called orthogonal to W if it is orthogonal to every vector in W. The set of all vectors in V that are orthogonal to W is called the orthogonal complement of W and is denoted by $W^{\perp}$ (read as "W perpendicular" or "W perp.")

Properties of Orthogonal Complements

If W is a subspace of inner product space V then

(i) A vector **u** is in $W^{\perp}$ if and only if **u** is orthogonal to every vector in a set that spans W.

(ii) The only vector common to W and $W^{\perp}$ is **0**.

(iii) $W^{\perp}$ is a subspace of V.

(iv) $\left(W^{\perp}\right)^{\perp} = W$.

Theorem 4.6: If A is an $m \times n$ matrix then the following hold:

(i) The null space of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.

(Row space of $A)^{\perp}$ = Null space of A and (Null space of $A)^{\perp}$ = Row space of A

(ii) The null space of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product.

(Column space of $A)^{\perp}$ = Null space of A^T and (Null space of $A^T)^{\perp}$ = Column space of A

Note: This theorem can be used to find a basis for the orthogonal complement of a subspace of Euclidean n-space.

Example 1: Find a basis for the orthogonal complement of the subspace W of the corresponding space R^n spanned by the vectors

(i) $\mathbf{u}_1 = (2, 0, -1)$, $\mathbf{u}_2 = (4, 0, -2)$ in R^3.

(ii) $\mathbf{u}_1 = (2, -1, 1, 3, 0)$, $\mathbf{u}_2 = (1, 2, 0, 1, -2)$, $\mathbf{u}_3 = (4, 3, 1, 5, -4)$, $\mathbf{u}_4 = (3, 1, 2, -1, 1)$, $\mathbf{u}_5 = (2, -1, 2, -2, 3)$ in R_5.

Solution: (i) The subspace W spanned by these vectors is the row space of the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

Since, (Row space of $A)^{\perp}$ = Null space of A

Basis for (Row space $A)^{\perp}$ = Basis for the null space of A

i.e., Basis for $W^{\perp}$ = Basis for the null space of A

The null space of A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 4 & 0 & -2 & 0 \end{array}\right]$$

$$R_2 - 4R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + 0x_2 - \frac{1}{2}x_3 = 0$$

Solving for the leading variables,

$$x_1 = -0x_2 + \frac{1}{2}x_3$$

Assigning the free variables x_2 and x_3 arbitrary values t_1 and t_2 respectively,

$$x_1 = \frac{1}{2}t_2,$$
$$x_2 = t_1,$$
$$x_3 = t_2$$

Null space consists vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Basis for the null space of $A = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right\}$ which is also the basis for $W^{\perp}$.

(ii) The space W spanned by these vectors is the row space of the matrix.

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 1 & -2 \\ 4 & 3 & 1 & 5 & -4 \\ 3 & 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & -2 & 3 \end{bmatrix}$$

Since, (Row space of A)$^\perp$ = Null space of A
Basis for (Row space of A)$^\perp$ = Basis for the null space A
i.e., Basis for $W^\perp$ = Basis for the null space of A
The null space of A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 2 & -1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 1 & -2 \\ 4 & 3 & 1 & 5 & -4 \\ 3 & 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 1 & -2 & 0 \\ 4 & 3 & 1 & 5 & -4 & 0 \\ 3 & 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 2 & -2 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 4 & 3 & 1 & 5 & -4 & 0 \\ 3 & 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 2 & -2 & 3 & 0 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 4R_1,\ R_4 - 3R_1,\ R_5 - 2R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & -5 & 2 & -4 & 7 & 0 \\ 0 & -5 & 2 & -4 & 7 & 0 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - R_2,\ R_5 - R_2$$
$$\sim \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \end{array}\right]$$

$$R_5 - R_4$$
$$\sim \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_{34}$$
$$\sim \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(-\frac{1}{5}\right)R_2$$
$$\sim \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 - 2x_5 &= 0 \\ x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4 - \frac{4}{5}x_5 &= 0 \\ x_3 - 5x_4 + 3x_5 &= 0 \end{aligned}$$

Solving for the leading variables,

$$x_1 = -2x_2 - 0x_3 - x_4 + 2x_5$$
$$x_2 = \frac{1}{5}x_3 + \frac{1}{5}x_4 + \frac{4}{5}x_5$$
$$x_3 = 5x_4 - 3x_5$$

Assigning the free variable x_4 and x_5 arbitrary values t_1 and t_2 respectively,

$$x_1 = -\frac{17}{5}t_1 + \frac{8}{5}t_2,$$
$$x_2 = \frac{6}{5}t_1 + \frac{1}{5}t_2,$$
$$x_3 = 5t_1 - 3t_2,$$
$$x_4 = t_1,$$
$$x_5 = t_2$$

Null space consists vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{17}{5}t_1 + \frac{8}{5}t_2 \\ \frac{6}{5}t_1 + \frac{1}{5}t_2 \\ 5t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -\frac{17}{5} \\ \frac{6}{5} \\ 5 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the null space of $A = \left\{ \begin{bmatrix} -\frac{17}{5} \\ \frac{6}{5} \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ which is also the basis for $W^{\perp}$.

4.6 ORTHOGONAL PROJECTION

4.6.1 *Orthogonal Projection on a Vector*

If **u** and **v** are two vectors in an inner product space V such that $\mathbf{v} \neq 0$ then orthogonal projection of **u** on **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}$$

4.6.2 *Orthogonal Projection on a Subspace*

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthonormal basis for subspace W of an inner product space V and W is any vector in V then orthogonal projection of $\mathbf{w}$ on W is given by

$$\text{proj}_W \mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_r \rangle \mathbf{v}_r$$

Note: If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthogonal basis for W and $\mathbf{w}$ is any vector in V then

$$\text{proj}_W \mathbf{w} = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r$$

(1) *Approximation Theorem*

Theorem 4.7: If W is a finite dimensional subspace of an inner product space V and $\mathbf{u}$ is any vector in V then $\text{proj}_W \mathbf{u}$ is the vector in W that is closest to $\mathbf{u}$, i.e.

$$\|\mathbf{u} - \text{proj}_W \mathbf{u}\| < \|\mathbf{u} - \mathbf{v}\| \text{ for all } \mathbf{v} \text{ in } W.$$

(2) *Projection Theorem*

Theorem 4.8: If W is a subspace of a finite dimensional inner product space V and $\mathbf{w}$ is any vector in V then

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1 = \text{proj}_W \mathbf{w}$ is in W

and $\mathbf{w}_2 = \text{proj}_{W^\perp} \mathbf{w}$ is in $W^\perp$.

Example 1: Find the orthogonal projection of $\mathbf{u} = (1, -2, 3)$ along $\mathbf{v} = (1, 2, 1)$ in R^3 with respect to the Euclidean inner product.

Solution: Let W be the subspace spanned by the vector $\mathbf{v} = (1, 2, 1)$.

$$\begin{aligned}\text{proj}_W \mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \\ &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \\ &= \frac{(1, -2, 3) \cdot (1, 2, 1)}{(1^2 + 2^2 + 1^2)} (1, 2, 1) \\ &= \frac{1 - 4 + 3}{6} (1, 2, 1) \\ &= 0\end{aligned}$$

Example 2: Find the orthogonal projection of $\mathbf{u} = (4, 0, -1)$ along $\mathbf{v} = (3, 1, -5)$ in R^3 with respect to the Euclidean inner product. Also find the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

Solution: Let W be the subspace spanned by the vector $\mathbf{v} = (3, 1, -5)$.

$$\begin{aligned}\text{proj}_W\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{v}\|^2}\mathbf{v}\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\\ &= \frac{(4, 0, -1)\cdot(3, 1, -5)}{(9+1+25)}(3, 1, -5)\\ &= \frac{17}{35}(3, 1, -5)\\ &= \left(\frac{51}{35}, \frac{17}{35}, -\frac{17}{7}\right)\end{aligned}$$

The component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is

$$\begin{aligned}\text{proj}_{W^\perp}\mathbf{u} &= \mathbf{u} - \text{proj}_W\mathbf{u}\\ &= (4, 0, -1) - \left(\frac{51}{35}, \frac{17}{35}, -\frac{17}{7}\right)\\ &= \left(\frac{89}{35}, -\frac{17}{35}, \frac{10}{7}\right)\end{aligned}$$

Example 3: Let W be the subspace of R^3 with orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right), \mathbf{v}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. Using Euclidean inner product find the distance from the vector $\text{u} = (1, 1, 0)$ to W, where u is in V.

Solution: From the approximation theorem,
Distance from $\mathbf{u}$ to $W = \|\mathbf{u} - \text{proj}_W\mathbf{u}\|$

$$\begin{aligned}\text{proj}_W\mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2\\ &= \left\{(1, 1, 0)\cdot\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\right\}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\\ &\quad + \left\{(1, 1, 0)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{3}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\\ &= \left(\frac{13}{18}, -\frac{1}{9}, \frac{5}{18}\right)\end{aligned}$$

$$\begin{aligned}\|\mathbf{u}-\text{proj}_W\mathbf{u}\| &= \left\|(1, 1, 0)-\left(\frac{13}{18}, -\frac{1}{9}, \frac{5}{18}\right)\right\| \\ &= \left\|\left(\frac{5}{18}, -\frac{10}{9}, -\frac{5}{18}\right)\right\| \\ &= \sqrt{\left(\frac{25}{324}+\frac{100}{81}+\frac{25}{324}\right)} \\ &= \frac{5\sqrt{2}}{6}.\end{aligned}$$

Example 4: Let W be the subspace spanned by the orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = \left(\frac{4}{5}, 0, -\frac{3}{5}\right)$.

Find the orthogonal projection of $\mathbf{w} = (1, 2, 3)$ on W with respect to Euclidean inner product. Also find the component of $\mathbf{u}$ orthogonal to W and express $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

Solution: Since an orthonormal set is also linearly independent, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for W.

$$\begin{aligned}\text{proj}_W\mathbf{w} &= \langle \mathbf{w}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2\rangle \mathbf{v}_2 \\ &= (\mathbf{w}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w}\cdot\mathbf{v}_2)\mathbf{v}_2 \\ &= [(1, 2, 3)\cdot(0, 1, 0)](0, 1, 0) + \left[(1, 2, 3)\cdot\left(\frac{4}{5}, 0, -\frac{3}{5}\right)\right]\left(\frac{4}{5}, 0, -\frac{3}{5}\right) \\ &= (2)(0, 1, 0) + (-1)\left(\frac{4}{5}, 0, -\frac{3}{5}\right) \\ &= \left(-\frac{4}{5}, 2, \frac{3}{5}\right)\end{aligned}$$

The component of w orthogonal to W is

$$\begin{aligned}\text{proj}_{W^{\perp}}\mathbf{w} &= \mathbf{w} - \text{proj}_W\mathbf{w} \\ &= (1, 2, 3) - \left(-\frac{4}{5}, 2, \frac{3}{5}\right) \\ &= \left(\frac{9}{5}, 0, \frac{12}{5}\right)\end{aligned}$$

Using Projection theorem $\mathbf{w}$ can be expressed as

$$\begin{aligned}\mathbf{w} &= \mathbf{w}_1 + \mathbf{w}_2 \\ &= \text{proj}_W\mathbf{w} + \text{proj}_{W^{\perp}}\mathbf{w} \\ &= \left(-\frac{4}{5}, 2, \frac{3}{5}\right) + \left(\frac{9}{5}, 0, \frac{12}{5}\right)\end{aligned}$$

Example 5: Let $W = \text{span}\{(2, 5, -1), (-2, 1, 1)\}$. Find the orthogonal projection of $\mathbf{w} = (1, 2, 3)$ on W with respect to Euclidean inner product. Also find the component of w orthogonal to W.

Solution: Let $\mathbf{v}_1 = (2, 5, -1)$, $\mathbf{v}_2 = (-2, 1, 1)$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1 \cdot \mathbf{v}_2 = (2, 5, -1)\cdot(-2, 1, 1)$$
$$= 0$$

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Since an orthogonal set is linearly independent, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ form an orthogonal basis for W.

$$\begin{aligned}
\text{proj}_W \mathbf{w} &= \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= \frac{(\mathbf{w}\cdot\mathbf{v}_1)}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{(\mathbf{w}\cdot\mathbf{v}_2)}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= \frac{(1, 2, 3)\cdot(2, 5, -1)}{[2^2 + 5^2 + (-1)^2]}(2, 5, -1) + \frac{(1, 2, 3)\cdot(-2, 1, 1)}{[(-2)^2 + 1^2 + 1^2]}(-2, 1, 1) \\
&= \frac{9}{30}(2, 5, -1) + \frac{3}{6}(-2, 1, 1) \\
&= \left(\frac{18}{30}, \frac{45}{30}, -\frac{9}{30}\right) + \left(-1, \frac{3}{6}, \frac{3}{6}\right) \\
&= \left(-\frac{12}{30}, \frac{60}{30}, \frac{6}{30}\right) = \left(-\frac{2}{5}, 2, \frac{1}{5}\right)
\end{aligned}$$

The component of $\mathbf{w}$ orthogonal to W is

$$\begin{aligned}
\text{proj}_{W^\perp} \mathbf{w} &= \mathbf{w} - \text{proj}_W \mathbf{w} \\
&= (1, 2, 3) - \left(-\frac{2}{5}, 2, \frac{1}{5}\right) \\
&= \left(\frac{7}{5}, 0, \frac{14}{5}\right)
\end{aligned}$$

4.7 LEAST SQUARES APPROXIMATION

In Chapter 6 we have discussed that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if it has a solution and is inconsistent if it has no solution. In many situations an inconsistent system requires a solution. In such situations we find a value of $\mathbf{x}$ that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$, i.e. it minimizes the value of $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product. Here, $\mathbf{x}$ is regarded as an approximate solution of the linear system $A\mathbf{x} = \mathbf{b}$.

The general least squares problem is to find a vector $\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product. The vector $\mathbf{x}$ is called a least squares solution of $A\mathbf{x} = \mathbf{b}$.

If W is the column space of A then the vector in W that is closet to $\mathbf{b}$ is $\text{proj}_W\,\mathbf{b}$. Thus, if we find $\mathbf{x}$ such that

$$A\mathbf{x} = \text{proj}_W\,\mathbf{b}$$

then $\|A\mathbf{x} - \mathbf{b}\|$ is minimum and therefore $\mathbf{x}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$.

Since $\mathbf{b} - A\mathbf{x} = \mathbf{b} - \text{proj}_W\,\mathbf{b}$ is in $W^{\perp}$, it is orthogonal to every vector in W. Therefore, $\mathbf{b} - A\mathbf{x}$ is orthogonal to each column of A.

Thus,

$$A^T(\mathbf{b} - A\mathbf{x}) = 0$$

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

This is called the normal system of equation associated with $A\mathbf{x} = \mathbf{b}$.

The least squares solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ can be found as follows.

1. Find the matrices A^TA and $A^T\mathbf{b}$.
2. Solve the normal system

$$A^T A\mathbf{x} = A^T\mathbf{b} \text{ for } \mathbf{x} \text{ using Gauss-elimination method.}$$

Theorem 4.9: A vector $\mathbf{x}$ is the least squares solution to $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}$ is a solution to the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

Theorem 4.10: If $\mathbf{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$ and W is the column space of A then

$$\text{proj}_W\,\mathbf{b} = A\mathbf{x}.$$

Example 1: Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} x_1 + x_2 &= 7 \\ -x_1 + x_2 &= 0 \\ -x_1 + 2x_2 &= -7 \end{aligned}$$

and find the orthogonal projection of $\mathbf{b}$ on the column space of A.

Solution: The matrix form of the linear system is

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

The normal system $A^T A\,\mathbf{x} = A^T\mathbf{b}$ is

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & -2 & 14 \\ -2 & 6 & -7 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{cc|c} -2 & 6 & -7 \\ 3 & -2 & 14 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -3 & \frac{7}{2} \\ 3 & -2 & 14 \end{array}\right]$$

$$R_2 - 3R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -3 & \frac{7}{2} \\ 0 & 7 & \frac{7}{2} \end{array}\right]$$

The corresponding system of equations is

$$x_1 - 3x_2 = \frac{7}{2}$$

$$7x_2 = \frac{7}{2}$$

The least squares solution of the system is $x_1 = 5,\ x_2 = \dfrac{1}{2}$

The orthogonal projection of **b** on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{2} \\ -\frac{9}{2} \\ -4 \end{bmatrix}$$

Example 2: Find the orthogonal projection of $\mathbf{u} = (2, 1, 3)$ on the subspace of R^4 spanned by the vectors $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (1, 2, 1)$.

Solution: The subspace of R^4 spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\text{proj}_W \mathbf{u} = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the least squares solution of the system $A\mathbf{x} = \mathbf{u}$.

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

$$A^T \mathbf{u} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

The normal system $A^T A\mathbf{x} = A^T \mathbf{u}$ is

$$\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc:c} 2 & 3 & 3 \\ 3 & 6 & 7 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right) R_1$$

$$\sim \left[\begin{array}{cc:c} 1 & \frac{3}{2} & \frac{3}{2} \\ 3 & 6 & 7 \end{array}\right]$$

$$R_2 - 3R_1$$

$$\sim \left[\begin{array}{cc:c} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{5}{2} \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{3}{2} x_2 = \frac{3}{2}$$

$$\frac{3}{2} x_2 = \frac{5}{2}$$

The least squares solution of the system is

$$x_1 = -1, x_2 = \frac{5}{3}$$

The orthogonal projection of **u** on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \\ \frac{7}{3} \\ \frac{5}{3} \end{bmatrix}$$

Exercise 4.4

1. Find a basis for the orthogonal complement of the subspace W of the corresponding space R^n spanned by the following vectors:

(i) $\mathbf{v}_1 = (5, -2, -1)$, $\mathbf{v}_2 = (2, -3, 15)$ in R^3

(ii) $\mathbf{v}_1 = (1, -1, 3)$, $\mathbf{v}_2 = (5, -4, -4)$, $\mathbf{v}_3 = (7, -6, 2)$ in R^3

(iii) $\mathbf{v}_1 = (1, -1, 2, 0)$, $\mathbf{v}_2 = (1, 0, -2, 3)$ in R^4.

(iv) $\mathbf{v}_1 = (1, 4, 5, 6, 9)$, $\mathbf{v}_2 = (3, -2, 1, 4, -1)$, $\mathbf{v}_3 = (-1, 0, -1, -2, -1)$, $\mathbf{v}_4 = (2, 3, 5, 7, 8)$ in R^5.

$$\left[\textbf{Ans.:} \text{ (i) } \left\{\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}\right\} \quad \text{(ii) } \left\{\begin{matrix} 16 \\ 9 \\ 1 \end{matrix}\right\} \quad \text{(iii) } \left\{\begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -1 \end{bmatrix}\right\} \quad \text{(iv) } \left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right]$$

2. Find the orthogonal projection of $\mathbf{u} = (1, 2, 3, 4)$ along $\mathbf{v} = (1, -3, 4, -2)$ in R^4 with respect to the Euclidean inner product.

$$\left[\textbf{Ans.:} \left(-\frac{1}{30}, \frac{1}{10}, -\frac{2}{15}, \frac{1}{15}\right)\right]$$

3. Let W be the subspace of R^4 with basis $\{(1, 1, 0, 1), (0, 1, 1, 0), (-1, 0, 0, 1)\}$. Find the orthogonal projection of $\mathbf{w} = (2, 1, 3, 0)$ on W with respect to the Euclidean inner product.

$$\left[\textbf{Ans.:} \left(\frac{7}{5}, \frac{11}{5}, \frac{9}{5}, -\frac{3}{5}\right)\right]$$

4. Let W be the subspace of R^3 with orthonormal basis

$$\left\{\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\}.$$

Using Euclidean inner product, find the component of $\mathbf{w} = (2, 1, 3)$ orthogonal to W and express $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$.

$$\left[\textbf{Ans.:} \left(\frac{1}{6}, \frac{2}{3}, -\frac{1}{6}\right), \mathbf{w}_1 = \left(\frac{11}{6}, \frac{1}{3}, \frac{19}{6}\right), \mathbf{w}_2 = \left(\frac{1}{6}, \frac{2}{3}, -\frac{1}{6}\right)\right]$$

5. Let W be the subspace of R^4 with orthonormal basis

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right), (0, 0, 1, 0), \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)\right\}.$$

Using Euclidean inner product, find the component of $\mathbf{w} = (1, 0, 2, 3)$ orthogonal to W and express $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$.

[**Ans. :** $(0, 0, 0, 0)$, $\mathbf{w}_1 = (1, 0, 2, 3)$, $\mathbf{w}_2 = (0, 0, 0, 0)$]

6. Let W be the subspace of R^3 with orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \mathbf{v}_2 = \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right).$$

Using Euclidean inner product find the distance from the vector $\mathbf{u} = (3, 4, -1)$ to W, where $\mathbf{u}$ is in V.

$$\left[\textbf{Ans.:} \sqrt{10}\right]$$

7. Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ and find orthogonal projection of $\mathbf{b}$ onto the column space of A.

(i) $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

Ans.:

(i) $x_1 = \dfrac{17}{95}, x_2 = \dfrac{143}{285}; \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$

(ii) $x_1 = 12, x_2 = -3, x_3 = 9; \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$

8. Find the orthogonal projection of $\mathbf{u} = (6, 3, 9, 6)$ subspace of R^4 spanned by the vectors $\mathbf{v}_1 = (2, 1, 1, 1)$, $\mathbf{v}_2 = (1, 0, 1, 1)$, $\mathbf{v}_3 = (-2, -1, 0, -1)$.

$$\left[\textbf{Ans.:} \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}\right]$$

Eigenvalues and Eigenvectors

Chapter 5

5.1 INTRODUCTION

Eigenvalues and eigenvectors are important concepts in linear algebra. They are derived from the German word '*eigen*' which means proper or characteristic. Eigenvectors are non-zero vectors that get mapped into scalar multiples of themselves under a linear operator. These are useful in solving systems of differential equations, analyzing population growth models and are also useful in quantum mechanics and economics.

5.2 EIGENVALUES AND EIGENVECTORS

Any non-zero vector **x** is said to be a characteristic vector or eigenvector of a square matrix A, if there exists a number λ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $A = [a_{ij}]_{n\times n}$ is a n-rowed square matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

Also, λ is said to be characteristic root or characteristic value or eigenvalue of the matrix A.

Depending on the sign and the magnitude of the eigenvalue λ corresponding to **x**, the linear operator $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches eigenvector **x** by a factor λ. If λ is negative, direction of eigenvector reverses.

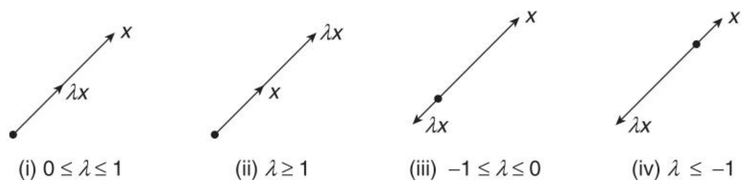


Fig. 5.1

Now
$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x}$$
$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The matrix $A - \lambda I$ is called the characteristic matrix of A where I is the unit matrix of order n.

The determinant

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A.

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A and the roots of this equation are called the eigenvalues of the matrix A. The set of all eigenvectors is called the eigenspace of A corresponding to λ. The set of all eigenvalues of A is called the spectrum of A.

Note: (1) The characteristic equation of the matrix A of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of principal diagonal elements and
S_2 = Determinant A

(2) The characteristic equation of the matrix A of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of principal diagonal elements,
S_2 = Sum of minors of principal diagonal elements and
S_3 = Determinant A

(3) The sum of the eigenvalues of a matrix is the sum of its principal diagonal elements.
(4) The product of the eigenvalues of a matrix is the determinant of the matrix.

5.2.1 *Nature of Eigenvalues of Special Types of Matrices*

Theorem 5.1: The eigenvalues of a triangular matrix are the diagonal elements of the matrix.

Theorem 5.2: The eigenvalues of a real symmetric matrix are real.

Theorem 5.3: The eigenvalues of a skew real symmetric matrix are either purely imaginary or zero.

Theorem 5.4: The eigenvalues of a Hermitian matrix are real.

Theorem 5.5: The eigenvalues of a skew Hermitian matrix are either purely imaginary or zero.

Theorem 5.6: The eigenvalues of a unitary matrix are of unit modulus.

Theorem 5.7: The eigenvalues of an orthogonal matrix are of unit modulus.

5.2.2 *Relations between Eigenvalues and Eigenvectors*

Theorem 5.8: If $\mathbf{x}$ is an eigenvector of a matrix A corresponding to the eigenvalue λ then $k\mathbf{x}$ is also an eigenvector of A corresponding to same eigenvalue λ, where k is any nonzero scalar.

Theorem 5.9: If $\mathbf{x}$ is an eigenvector of a matrix A then $\mathbf{x}$ can not correspond to more than one eigenvalue of A.

Theorem 5.10: The eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.

Theorem 5.11: If two or more eigenvalues are equal then the corresponding eigenvectors may or may not be linearly independent.

Theorem 5.12: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Theorem 5.13: Any two eigenvectors corresponding to two distinct eigenvalues of a unitary matrix are orthogonal.

Theorem 5.14: If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

Theorem 5.15: If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Theorem 5.16: If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $\lambda \pm k$ is an eigenvalue of $A \pm kI$ and $\mathbf{x}$ is a corresponding eigenvector.

Theorem 5.17: If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector then $k\lambda$ is an eigenvalue of matrix kA and $\mathbf{x}$ is a corresponding eigenvector.

Theorem 5.18: If λ is an eigenvalue of a matrix A then λ is also an eigenvalue of matrix A^T. Matrix A and A^T need not have the same eigenvectors.

Example 1: If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find eigenvalues for the following matrices:

(i) A (ii) A^T (iii) A^{-1} (iv) $4A^{-1}$ (v) A^2 (vi) $A^2 - 2A + I$ (vii) $A^3 + 2I$

Solution:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix}+\begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}+\begin{vmatrix}3 & -1\\ -1 & 5\end{vmatrix}$$

$$=(15-1)+(9-1)+(15-1)$$

$$=14+8+14$$

$$=36$$

$$S_3=\det(A)=\begin{vmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{vmatrix}$$

$$=3(15-1)+1(-3+1)+1(1-5)$$

$$=42-2-4$$

$$=36$$

Hence, the characteristic equation is

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$\lambda = 2,\ 3,\ 6$$

(i) Eigenvalues of $A = \lambda$: 2, 3, 6

(ii) Eigenvalues of $A^T = \lambda^T$: 2, 3, 6

(iii) Eigenvalues of $A^{-1} = \lambda^{-1}$: $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(iv) Eigenvalues of $4A^{-1} = 4\lambda^{-1}$: $2, \frac{4}{3}, \frac{2}{3}$

(v) Eigenvalues of $A^2 = \lambda^2$: 4, 9, 36

(vi) Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$: 1, 4, 25

(vii) Eigenvalues of $A^3 + 2I = \lambda^3 + 2$: 10, 29, 218

Example 2: Find the eigenvalues, eigenvectors and bases for eigenspaces for the following matrices.

(i) $\begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$ (iii) $\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

(iv) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

Solution: (i) $$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 4 + 3 - 3 = 4$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-9+8) + (-12+6) + (12-6)$$

$$= -1 - 6 + 6$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix}$$

$$= 4(-9+8) - 6(-3+2) + 6(-4+3)$$

$$= -4 + 6 - 6$$

$$= -4$$

Hence, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda = -1, 1, 4$$

(a) For $\lambda = -1$, $$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x+6y+6z=0$$
$$x+4y+2z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}6 & 6\\4 & 2\end{vmatrix}} = -\frac{y}{\begin{vmatrix}5 & 6\\1 & 2\end{vmatrix}} = \frac{z}{\begin{vmatrix}5 & 6\\1 & 4\end{vmatrix}}$$

$$\frac{x}{-12} = \frac{y}{-4} = \frac{z}{14}$$

$$\frac{x}{-6} = \frac{y}{-2} = \frac{z}{7} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = -1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix}-6t\\-2t\\7t\end{bmatrix} = t\begin{bmatrix}-6\\-2\\7\end{bmatrix} = \text{t}\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = -1$.

(b) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix}3 & 6 & 6\\1 & 2 & 2\\-1 & -4 & -4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x+2y+2z=0$$
$$-x-4y-4z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}2 & 2\\-4 & -4\end{vmatrix}} = -\frac{y}{\begin{vmatrix}1 & 2\\-1 & -4\end{vmatrix}} = \frac{z}{\begin{vmatrix}1 & 2\\-1 & -4\end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{-2}$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding $\lambda = 1$ are the non-zero vectors of the form

$\mathbf{x} = \begin{bmatrix}0\\t\\-t\end{bmatrix} = t\begin{bmatrix}0\\1\\-1\end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = 1$.

(c) For $\lambda = 4$,
$$[A - \lambda I]\,\mathbf{x} = 0$$
$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$0x + 6y + 6z = 0$$
$$x - \ y + 2z = 0$$
$$-x - 4y - 7z = 0$$

By Cramer's rule,
$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}}$$
$$\frac{x}{18} = \frac{y}{6} = \frac{z}{-6}$$
$$\frac{x}{3} = \frac{y}{1} = \frac{z}{-1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 4$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ forms a basis for the eigenspace corresponding to $\lambda = 4$.

(ii)
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

The characteristic equation is
$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 3 = 6$

S_2 = Sum of the minors of principal diagonal elements of A
$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= (6-2) + (3+2) + (2-0)$$
$$= 4 + 5 + 2$$
$$= 11$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-2) - 0 - 1(2-4)$$

$$= 6$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

(a) For $\lambda = 1$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 0y - z = 0$$
$$x + y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 1$.

(b) For $\lambda = 2$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 0y + z = 0$$
$$2x + 2y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -2t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = 2$.

(c) For $\lambda = 3$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + 0y - z = 0$$

$$x - y + z = 0$$

$$2x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ forms a basis for the eigenspace corresponding to $\lambda = 3$.

(iii)

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 8 + 7 + 3 = 18$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$
$$= (21-16)+(24-4)+(56-36)$$
$$= 5+20+20$$
$$= 45$$

$$S_3 = \det(A) = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$
$$= 8(21-16)+6(-18+8)+2(24-14)$$
$$= 40-60+20$$
$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$
$$\lambda = 0, 3, 15$$

(a) For $\lambda = 0$, $$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 0$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 0$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 5x - 6y + 2z &= 0 \\ -6x + 4y - 4z &= 0 \\ 2x - 4y + 0z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = 3$.

(c) For $\lambda = 15$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -7x - 6y + 2z &= 0 \\ -6x - 8y - 4z &= 0 \\ 2x - 4y - 12z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{40} = -\frac{y}{40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 15$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ forms a basis for the eigenspace corresponding to $\lambda = 15$.

Note: The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal which can be verified with this example.

$$\mathbf{x}_1^T\mathbf{x}_2 = [1 \quad 2 \quad 2]\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 0$$

$$\mathbf{x}_2^T\mathbf{x}_3 = [2 \quad 1 \quad -2]\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\mathbf{x}_3^T\mathbf{x}_1 = [2 \quad -2 \quad 1]\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$$

Thus, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are orthogonal to each other.

(iv)
$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = -2 + 1 + 0 = -1$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$
$$= (0-12) + (0-3) + (-2-4)$$
$$= -12 - 3 - 6$$
$$= -21$$

$$S_3 = \det(A) = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$
$$= (-2)(0-12) - 2(0-6) - 3(-4+1)$$
$$= 24 + 12 + 9$$
$$= 45$$

Hence, the characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$
$$\lambda = 5, -3, -3$$

(a) For $\lambda = 5$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x + 2y - 3z = 0$$
$$2x - 4y - 6z = 0$$
$$-x - 2y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 5$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 5$.

(b) For $\lambda = -3$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+2y-3z=0$$

Let $$y = t_1 \text{ and } z = t_2$$

$$x=-2t_1+3t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = -3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -2t_1+3t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ form a basis for the eigenspace corresponding to $\lambda = -3$.

(v) $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0+0+0=0$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1)+(0-1)+(0-1)$$

$$= -3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0 - 1(0-1) + 1(1-0)$$

$$= 2$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda - 2 = 0$$

$$\lambda = 2, -1, -1$$

(a) For $\lambda = 2$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 2$.

(b) For $\lambda = -1$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$

Let $y = t_1$ and $z = t_2$

$$x = -t_1 - t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = -1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ form a basis for the eigenspace corresponding to $\lambda = -1$.

(vi) $$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$
$$= (4-2) + (2+2) + (2-0)$$
$$= 2 + 4 + 2$$
$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$
$$= 1(4-2) - 2(0+1) + 2(0+2)$$
$$= 2 - 2 + 4$$
$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$, $$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + y + z = 0$$
$$-x + 2y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y + 2z = 0$$
$$0x + 0y + z = 0$$
$$-x + 2y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{0} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = 2$.

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 2$.

(vii)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0+0+3=3$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= (0+3)+(0)+(0)$$

$$= 3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix}$$

$$= 0-1(0-1)+0$$

$$= 1$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\lambda = 1, 1, 1$$

For $\lambda = 1$,

$$[A-\lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + 0z = 0$$

$$0x - y + z = 0$$

$$x - 3y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = 1$.

Hence, there is only one eigenvector corresponding to repeated root $\lambda = 1$.

Example 3: Find the values of μ which satisfy the equation $A^{100}\,\mathrm{x} = \mu\mathrm{x}$, where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 - 2 + 0 = 0$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= (0+2) + (0+1) + (-4-0)$$

$$= -1$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2(0+2) - 1(0+2) - 1(0+2)$$

$$= 4 - 2 - 2$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda = 0$$
$$\lambda = 0, 1, -1$$

If λ is an eigenvalue of A, it satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$.

For equation $A^{100}\ \mathbf{x} = \mu\mathbf{x}$, μ represents eigenvalues of A^{100}. Eigenvalues of $A^{100} = \lambda^{100}$, i.e., 0, 1, 1.

Hence, values of μ are 0 and 1.

Example 4: Find the characteristic root and characteristic vectors of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that characteristic roots are of unit modulus and characteristic vectors are orthogonal.

Solution:
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$
$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$
$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$
$$\cos\theta - \lambda = \pm i\sin\theta$$
$$\lambda = \cos\theta \pm i\sin\theta$$
$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Hence, characteristic roots are of unit modulus.

(a) $\lambda = \cos\,\theta + i\,\sin\,\theta$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$(-i\sin\theta)x - (\sin\theta)y = 0$$

Let
$$y = t$$
$$x = it$$

Thus, the eigenvectors of A corresponding to $\lambda = \cos\,\theta + i\,\sin\,\theta$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where the $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = \cos\,\theta + i\,\sin\,\theta$.

(b) For $\lambda = \cos\theta - i\sin\theta$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(i\sin\theta)x - (\sin\theta)y = 0$$

Let
$$y = t$$
$$x = -it$$

Thus, the eigenvectors of A corresponding to $\lambda = \cos\theta - i\sin\theta$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -it \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = \cos\theta - i\sin\theta$.

For orthogonality of complex matrix,

$$\mathbf{x}_1^\theta \mathbf{x}_2 = [-i \quad 1]\begin{bmatrix} -i \\ 1 \end{bmatrix} = [i^2 + 1] = [0] = \mathbf{0}$$

Similarly,
$$\mathbf{x}_1 \mathbf{x}_2^\theta = \mathbf{0}$$

Hence, characteristic vectors are orthogonal.

Example 5: If $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$ verify whether eigenvectors are mutually orthogonal.

Solution:
$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 = 0$$

$$\lambda = -3, 3$$

(a) For $\lambda = -3$

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x + (1-2i)y = 0$$

Let
$$y = t$$
$$x = -\frac{1-2i}{5}t$$

Thus, the eigenvectors of A corresponding to $\lambda = -3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -\frac{1-2i}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1-2i}{5} \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ forms a basis for the eigenspace corresponding to $\lambda = -3$.

(b) For $\lambda = 3$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & 1-2i \\ 1+2i & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + (1-2i)y = 0$$

Let
$$y = t$$
$$x = (1-2i)t$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} (1-2i)t \\ t \end{bmatrix} = t\begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ forms a basis for the eigenspace corresponding to $\lambda = 3$.

For orthogonality of complex matrix,

$$\mathbf{x}_1^{\theta}\mathbf{x}_2 = \begin{bmatrix} -\frac{1+2i}{5} & 1 \end{bmatrix}\begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1+2i}{5}(1-2i) + 1 \end{bmatrix} = [0] = \mathbf{0}$$

Similarly, $\mathbf{x}_1\mathbf{x}_2^{\theta} = \mathbf{0}$

Hence, eigenvectors are mutually orthogonal.

5.2.3 *Algebraic and Geometric Multiplicity of an Eigenvalue*

If the eigenvalue λ of the equation det $(A - \lambda I) = 0$ is repeated n times then n is called the algebraic multiplicity of λ. The number of linearly independent eigenvectors is the difference between the number of unknowns and the rank of the corresponding matrix $A - \lambda I$ and is known as geometric multiplicity of eigenvalue λ.

Example 1: Determine algebraic and geometric multiplicity of each eigenvalue of the following matrices:

$$\text{(i)} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution: (i) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Since A is upper triangular matrix, its diagonal elements are the eigenvalues of A.

$$\lambda = 2, 2, 2$$

Since eigenvalue $\lambda = 2$ is repeated thrice, its algebraic multiplicity is 3.

For $\lambda = 2$

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2

Number of unknowns = 3

Number of linearly independent eigenvectors = 3 − 2 = 1

Hence, geometric multiplicity is 1.

(ii) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= (4-2) + (2+2) + (2-0)$$

$$= 2 + 4 + 2$$

$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$
$$= 1(4-2) - 2(0+1) + 2(0+2)$$
$$= 2 - 2 + 4$$
$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$\lambda = 1, 2, 2$$

(a) Since eigenvalue $\lambda = 1$ is non-repeated, its algebraic multiplicity is 1.
For $\lambda = 1$

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}
$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$
$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent eigenvectors = 3 − 2 = 1
Hence, geometric multiplicity is 1.

(b) Since eigenvalue $\lambda = 2$ is repeated twice, its algebraic multiplicity is two.
For $\lambda = 2$

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2

Number of unknowns = 3

Number of linearly independent eigenvectors = 3 − 2 = 1

Hence, geometric multiplicity is 1.

Exercise 5.1

1. Find the eigenvalues and eigenvectors for the following matrices:

(i) $\begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$

(x) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

(xi) $\begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$

(xii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(xiii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(xiv) $\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$

(xv) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(xvi) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(xvii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(xviii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

(xix) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

(xx) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Ans.: (i) $-1, 0, 2; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

(ii) $-1, 2, 1; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(iii) $-1, -2, -3; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

(iv) $1, 2, 3; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(v) $1, 1, 7; \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(vi) $5, 1, 1; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(vii) $5, -3, -3; \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(viii) $-1, 1, 1; \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ix) $-2, 9, -18; \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

(x) $3, 6, 9; \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}$

(xi) $-3, 3, 9; \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}$

(xii) $2, 3, 6; \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}$

(xiii) $8, 2, 2; \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$

(xiv) $12, 6, 6; \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

(xv) $4, 1, 1; \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}$

(xvi) $1, 3, 3; \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}$

(xvii) $1, 2, 2; \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$

(xviii) $3, 2, 2; \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}5\\2\\-5\end{bmatrix}$

(xix) $1, 1, 1; \begin{bmatrix}-3\\1\\1\end{bmatrix}$

(xx) $2, 2, 2; \begin{bmatrix}1\\0\\0\end{bmatrix}$

2. Determine algebraic and geometric multiplicity of the following matrices:

(i) $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & -3 & 3\end{bmatrix}$ (ii) $\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\0 & 0 & 1\end{bmatrix}$

Ans.:
(i) For $\lambda = 1$, $AM = 3, GM = 1$
(ii) For $\lambda = 1$, $AM = 2, GM = 2$
For $\lambda = 3$, $AM = 1$, $GM = 1$

3. If $A = \begin{bmatrix}2 & 2 & 1\\1 & 3 & 1\\1 & 2 & 2\end{bmatrix}$, find eigenvalues of the following matrices:

(i) $A^3 + I$ (ii) A^{-1} (iii) $A^2 - 2A + I$
(iv) adj A (v) $A^3 - 3A^2 + A$

Ans.: (i) 2, 2, 126 (ii) $1, 1, \frac{1}{5}$
(iii) 0, 0, 16 (iv) 5, 5, 1
(v) −1, −1, 55

4. Verify that $\mathbf{x} = [2, 3, -2, -3]^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$ of the matrix

$$A = \begin{bmatrix}1 & -4 & -1 & -4\\2 & 0 & 5 & -4\\-1 & 1 & -2 & 3\\-1 & 4 & -1 & 6\end{bmatrix}$$

5. If $A = \begin{bmatrix}3 & -1 & 1\\-1 & 3 & -1\\1 & -1 & 3\end{bmatrix}$ then check whether eigenvectors of A are orthogonal.

6. If $A = \begin{bmatrix}3 & 10 & 5\\-2 & -3 & -4\\3 & 5 & 7\end{bmatrix}$ then verify whether eigenvectors of A are linearly independent or not.

5.3 CAYLEY–HAMILTON THEOREM

Theorem 5.19: Every square matrix satisfies its own characteristic equation.

Proof: Let A be an n-rowed square matrix. Its characteristic equation is

$$|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$$

$$(A-\lambda I)\ \text{adj}\,(A-\lambda I) = |A-\lambda I|\,I \qquad \ldots(5.1)$$

$$[\because A\ \text{adj}(A) = |A|\,I]$$

Since adj $(A - \lambda I)$ has element as cofactors of elements of $|A - \lambda I|$, the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $n - 1$ or less. Hence, adj $(A - \lambda I)$ can be written as a matrix polynomial in λ.

adj $(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}$
where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order n.

$$(A-\lambda I)\ \text{adj}(A-\lambda I) = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$|A-\lambda I|\,I = (A-\lambda I)[B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}]$$

$$(-1)^n[I\lambda^n + a_1 I\lambda^{n-1} + a_2 I\lambda^{n-2} + \cdots + a_{n-1}I\lambda + a_n I]$$
$$= (-IB_0)\lambda^n + (AB_0 - IB_1)\lambda^{n-1} + (AB_1 - IB_2)\lambda^{n-2} + \cdots + (AB_{n-2} - IB_{n-1})\lambda + AB_{n-1}$$

Equating corresponding coefficients,

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$
$$\vdots \qquad \vdots$$
$$AB_{n-2} - IB_{n-1} = (-1)^n a_{n-1} I$$
$$AB_{n-1} = (-1)^n a_n I$$

Premultiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots I$ and adding,

$$(-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI] = \mathbf{0}$$

Hence,
$$A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = \mathbf{0} \qquad \ldots(5.2)$$

Corollary: If A is a non-singular matrix, i.e. det $(A) \neq 0$ then premultiplying Eq. (5.2) by A^{-1}, we get

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_nA^{-1} = \mathbf{0}$$

$$A^{-1} = -\frac{1}{a^n}[A^{n-1} + a_1A^{n-2} + \ldots + a_{n-1}I]$$

Example 1: Verify Cayley–Hamilton theorem for the following matrix and hence, find A^{-1} and A^4.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-1) + (4-1)$$

$$= 9$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4I$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\mathbf{0} \qquad \ldots(1)$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^3-6A^2+9A-4I)=\mathbf{0}$$

$$A^2-6A+9I-4A^{-1}=\mathbf{0}$$

$$4A^{-1}=A^2-6A+9I$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Multiplying Eq. (1) by A,

$$A(A^3-6A^2+9A-4I)=\mathbf{0}$$

$$A^4-6A^3+9A^2-4A=\mathbf{0}$$

$$A^4=6A^3-9A^2+4A$$

$$=\begin{bmatrix}132 & -126 & 126\\ -126 & 132 & -126\\ 126 & -126 & 132\end{bmatrix}-\begin{bmatrix}54 & -45 & 45\\ -45 & 54 & -45\\ 45 & -45 & 54\end{bmatrix}+\begin{bmatrix}8 & -4 & 4\\ -4 & 8 & -4\\ 4 & -4 & 8\end{bmatrix}$$

$$=\begin{bmatrix}86 & -85 & 85\\ -85 & 86 & -85\\ 85 & -85 & 86\end{bmatrix}$$

Example 2: Show that the matrix $A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$ satisfies Cayley–Hamilton theorem and hence find A^{-1}, if it exists.

Solution:

$$A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}-\lambda & c & -b\\ -c & -\lambda & a\\ b & -a & -\lambda\end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where S_1 = Sum of the principal diagonal elements of $A=0$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}0 & a\\ -a & 0\end{vmatrix}+\begin{vmatrix}0 & -b\\ b & 0\end{vmatrix}+\begin{vmatrix}0 & c\\ -c & 0\end{vmatrix}$$

$$=(0+a^2)+(0+b^2)+(0+c^2)$$

$$=a^2+b^2+c^2$$

$$S_3=\det(A)=\begin{vmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{vmatrix}$$

$$=0-c(0-ab)-b(ac-0)$$

$$=abc-abc$$

$$=0$$

Hence, the characteristic equation is

$$\lambda^3+(a^2+b^2+c^2)\lambda=0$$

$$A^2=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}=\begin{bmatrix}-c^2-b^2 & ab & ac\\ ab & -c^2-a^2 & bc\\ ac & bc & -b^2-a^2\end{bmatrix}$$

$$A^3=\begin{bmatrix}-c^2-b^2 & ab & ac\\ ab & -c^2-a^2 & bc\\ ac & bc & -b^2-a^2\end{bmatrix}\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & -c^3-cb^2-ca^2 & b^3+bc^2+ba^2\\ c^3+ca^2+cb^2 & 0 & -ab^2-ac^2-a^3\\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & 0\end{bmatrix}$$

$$= -(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = -(a^2+b^2+c^2)A$$

$$A^3 + (a^2+b^2+c^2)A = \mathbf{0}$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

$$\det(A) = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0-ab) - b(ac-0) = abc - abc = 0$$

Hence, A^{-1} does not exist.

Example 3: Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley–Hamilton theorem for this matrix. Find A^{-1} and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

Solution:
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 3 = 4$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3 - 8$$

$$= -5$$

Hence, the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = -1, 5$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \qquad \ldots(1)$$

The matrix A satisfies its own characteristic equation. Hence, Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by A^{-1},

$$A^{-1}(A^2 - 4A - 5I) = \mathbf{0}$$
$$A - 4I - 5A^{-1} = \mathbf{0}$$
$$A^{-1} = \frac{1}{5}(A - 4I)$$
$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I)$
$+ 3(A^2 - 4A - 5I) + A + 5I$
$= (A^2 - 4A - 5I)(A^3 - 2A + 3) + A + 5I$
$= A + 5I$ [using Eq. (1)]

which is a linear polynomial in A.

Example 4: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 1 + 2 = 5$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0)+(4-1)+(2-0)$$

$$= 7$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0)-1(0-0)+1(0-1)$$

$$= 4-0-1$$

$$= 3$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley–Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = \mathbf{0} \qquad \text{...(1)}$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

$$= \mathbf{0} + (A^2 + A + I)$$

$$= A^2 + A + I \qquad \text{[using Eq. (1)]}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 5: If $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$, prove by induction that for every integer $n \ge 3$, $A^n = A^{n-2} + A^2 - I$. Hence, find A^{50}.

Solution:

$$A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}1-\lambda&0&0\\1&-\lambda&1\\0&1&-\lambda\end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1+0+0=1$

S_2 = Sum of the minors of principal diagonal elements of A

$$=\begin{vmatrix}0&1\\1&0\end{vmatrix}+\begin{vmatrix}1&0\\0&0\end{vmatrix}+\begin{vmatrix}1&0\\1&0\end{vmatrix}$$

$$=(0-1)+(0-0)+(0-0)$$

$$=-1$$

$$S_3=\det(A)=\begin{vmatrix}1&0&0\\1&0&1\\0&1&0\end{vmatrix}$$

$$=1(0-1)+0+0$$

$$=-1$$

Hence, the characteristic equation is

$$\lambda^3-\lambda^2-\lambda+1=0$$

By Cayley–Hamilton theorem,

$$A^3-A^2-A+I=\mathbf{0}$$

$$A^3 = A^2 + A - I = A^1 + A^2 - I$$

$$= A^{3-2} + A^2 - I \qquad \ldots(1)$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Assuming that the Eq. (1) is true for $n = k$,

$$A^k = A^{k-2} + A^2 - I = A^l + A^2 - I$$

Pre multiplying both the sides by A,

$$A^{k+1} = A^{k-1} + A^3 - A$$

Substituting the value of A^3,

$$A^{k+1} = A^{k-1} + (A^2 + A - I) - A$$
$$= A^{(k+1)-2} + A^2 - I$$

Hence, $A^n = A^{n-2} + A^2 - I$ is true for $n = k + 1$.
Thus, by mathematical induction, it is true for every integer $n \geq 3$.

We have,
$$A^n = A^{n-2} + A^2 - I = (A^{n-4} + A^2 - I) + A^2 - I$$
$$= A^{n-4} + 2(A^2 - I) = (A^{n-6} + A^2 - I) + 2(A^2 - I)$$
$$= A^{n-6} + 3(A^2 - I)$$
$$\vdots$$
$$A^n = A^{n-2r} + r(A^2 - I)$$

Putting $n = 50$ and $r = 24$,

$$A^{50} = A^{50-2(24)} + 24(A^2 - I)$$
$$= A^2 + 24A^2 - 24I = 25A^2 - 24I$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Exercise 5.2

1. Verify Cayley–Hamilton theorem for the matrix A and hence, find A^{-1} and A^4.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \text{(i)}\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix},\begin{bmatrix}-55&104&24\\-20&-15&32\\32&-42&13\end{bmatrix}\\ \text{(ii)}\ \frac{1}{40}\begin{bmatrix}-3&5&11\\14&-10&2\\5&5&-5\end{bmatrix},\\ \qquad\begin{bmatrix}248&101&218\\272&109&50\\104&98&204\end{bmatrix}\end{array}\right]$$

$$\left[\begin{array}{l}\text{(iii)}\ \frac{1}{6}\begin{bmatrix}4&0&2\\0&3&0\\2&0&4\end{bmatrix},\begin{bmatrix}41&0&-40\\0&16&0\\-40&0&41\end{bmatrix}\\ \text{(iv)}\ \begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix},\begin{bmatrix}-55&104&24\\-20&-15&32\\32&-40&-23\end{bmatrix}\end{array}\right]$$

2. Verify that the matrix $A=\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}$ satisfies the characteristic equation and hence, find A^{-2}.

$$\left[\begin{array}{l}\textbf{Ans.:}\ A^3+A^2-5A-5I=\mathbf{0},\\ \qquad A^{-2}=\frac{1}{5}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{array}\right]$$

3. Use Cayley–Hamilton theorem to find $2A^5-3A^4+A^2-4I$, where $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$.

$$\left[\textbf{Ans.:}\ 138A-403I=\begin{bmatrix}11&138\\-138&127\end{bmatrix}\right]$$

4. If $A=\begin{bmatrix}1&4\\1&1\end{bmatrix}$, find A^7-9A^2+I.

[**Ans. :** $609A+640I$]

5. Verify Cayley–Hamilton theorem for

(i) $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (ii) $A=\begin{bmatrix}2&4\\1&2\end{bmatrix}$ and

hence, find A^{-1} and A^3-5A^2.

$$\left[\begin{array}{l}\textbf{Ans.:}\ \text{(i)}\ \begin{bmatrix}-2&1\\1.5&-0.5\end{bmatrix},2A\\ \qquad\text{(ii)}\ A^{-1}\text{ does not exist, }A^2\end{array}\right]$$

6. Compute $A^9-6A^8+10A^7-3A^6+A+I$, where $A=\begin{bmatrix}1&2&3\\-1&3&1\\1&0&2\end{bmatrix}$.

$$\left[\textbf{Ans.:}\begin{bmatrix}2&2&3\\-1&4&1\\1&0&3\end{bmatrix}\right]$$

7. Verify Cayley–Hamilton theorem for $A=\begin{bmatrix}1&2\\2&2\end{bmatrix}$ and evaluate $2A^4-5A^3-7A+6I$.

$$\left[\textbf{Ans.:}\begin{bmatrix}36&32\\32&52\end{bmatrix}\right]$$

5.4 SIMILARITY OF MATRICES

If A and B are two square matrices of order n then B is said to be similar to A, if there exists a non-singular matrix P such that

$$B = P^{-1} AP$$

Theorem 5.20: Similarity of matrices is an equivalence relation.

Theorem 5.21: Similar matrices have the same determinant.

Theorem 5.22: Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to the eigenvalue λ where $B = P^{-1} AP$.

5.5 DIAGONALIZATION

A matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1} AP = D$ where D is a diagonal matrix, also known as spectral matrix. The matrix P is then said to diagonalize A or transform A to diagonal form and is known as modal matrix.

Theorem 5.23: If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

Theorem 5.24: An $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

Theorem 5.25: The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues is equal to the algebraic multiplicity.

Corollary: If A is similar to a diagonal matrix D, the diagonal elements of D are the eigenvalues of A.

5.5.1 *Orthogonally Similar Matrices*

If A and B are two square matrices of order n then B is said to be orthogonally similar to A, if there exists an orthogonal matrix P such that

$$B = P^{-1} AP$$

Since P is orthogonal, $\quad P^{-1} = P^T$

$$B = P^{-1} AP = P^T AP$$

Theorem 5.26: Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

Corollary 1: A real symmetric matrix of order n has n mutually orthogonal real eigenvectors.

Corollary 2: Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

Note: To find the orthogonal matrix P, each element of the eigenvector is divided by its norm (or length).

Example 1: Show that the following matrices are not diagonalizable.

(i) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solution: (i)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$
$$= (4-2) + (2+2) + (2-0)$$
$$= 2 + 4 + 2$$
$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$
$$= 1(4-2) - 2(0+1) + 2(0+2)$$
$$= 2 - 2 + 4$$
$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$, number of linearly independent eigenvectors = 1

(b) For $\lambda = 2$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$
$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$
$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2
Number of unknowns = 3
Number of linearly independent eigenvectors = 3 – 2 = 1
Since the matrix A has a total of 2 linearly independent eigenvectors which is less than its order 3, the matrix A is not diagonalizable.

(ii)
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 1 & 2-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 + 3 = 6$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$$
$$= (6-4) + (3-0) + (2+2)$$
$$= 9$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 1(6-4) + 2(3-2) + 0$$

$$= 4$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\lambda = 4, 1, 1$$

(a) For $\lambda = 4$, number of linearly independent eigenvectors = 1

(b) For $\lambda = 1$, $[A - \lambda I]\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2

Number of unknowns = 3

Number of linearly independent eigenvectors = 3 – 2 = 1

Since the matrix A has a total of 2 linearly independent eigenvectors which is less than its order 3, the matrix A is not diagonalizable.

Example 2: Show that the following matrices are similar to diagonal matrices. Find the diagonal and modal matrix in each case.

(i) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Solution: (i) $$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 4 + 3 + 1 = 8$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix}$$

$$= (3-8) + (4-4) + (12+10)$$

$$= -5 + 0 + 22$$

$$= 17$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= 4(3-8) - 2(-5+4) - 2(-20+6)$$

$$= -20 + 2 + 28$$

$$= 10$$

Hence, the characteristic equation is

$$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$$

$$\lambda = 1, 2, 5$$

Since all the eigenvalues are distinct, the matrix A is diagonalizable.

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + 2y - 2z = 0$$
$$-5x + 2y + 2z = 0$$
$$-2x + 4y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 2 \\ -5 & 2 \end{vmatrix}}$$

$$\frac{x}{8} = \frac{y}{4} = \frac{z}{16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{4} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 4t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 2y - 2z = 0$$
$$-5x + y + 2z = 0$$
$$-2x + 4y - z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}$$

$$\frac{x}{6} = \frac{y}{6} = \frac{z}{12}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 2$.

(c) For $\lambda = 5$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 2y - 2z = 0$$
$$-5x - 2y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{y}{12} = \frac{z}{12}$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 5$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 5$.

Modal matrix P has eigenvectors as its column vectors.

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(ii)

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix}=0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = -9+3+7=1$
S_2 = Sum of the minors of principal diagonal elements of A

$$\begin{aligned} &= \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix} \\ &= (21-32)+(-63+64)+(-27+32) \\ &= -11+1+5 \\ &= -5 \end{aligned}$$

$$S_3 = \det(A) = \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix}$$

$$\begin{aligned} &= -9(21-32)-4(-56+64)+4(-64+48) \\ &= 99-32-64 \\ &= 3 \end{aligned}$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$\lambda = -1, -1, 3$$

(a) For $\lambda = -1$, $\quad [A-\lambda I]\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8x+4y+4z=0$$

Let $\quad y = t_1 \quad \text{and} \quad z = t_2$

$$x = \frac{1}{2}t_1 + \frac{1}{2}t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = -1$ are the non-zero vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}t_1 + \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors corresponding to $\lambda = -1$.

(b) For $\lambda = 3$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x + 4y + 4z = 0$$
$$-8x + 0y + 4z = 0$$
$$-16x + 8y + 4z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{16} = \frac{z}{32}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form

$\mathbf{x} = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

Since the matrix A has a total of 3 linearly independent eigenvectors which is same as its order, matrix A is diagonalizable.

Modal matrix P has eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(iii)
$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 4 - 3 = 2$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}$$

$$= (-12+12) + (-3+0) + (4+0)$$

$$= 1$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= 1(-12+12) - 6(0-0) - 4(0-0)$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda = 0, 1, 1$$

(a) For $\lambda = 0$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 6y - 4z = 0$$
$$0x + 4y + 2z = 0$$
$$0x - 6y - 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & -4 \\ 4 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{y}{-2} = \frac{z}{4}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 0$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -t \\ 2t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 3y + 2z = 0$$

Let

$$x = t_1 \quad \text{and} \quad z = t_2$$

$$y = -\frac{2}{3}t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t_1 \\ -\frac{2}{3}t_2 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 1$.

Since matrix A has total 3 linearly independent eigenvectors which is same as its order, the matrix A is diagonalizable.

Modal matrix P has eigenvectors as its column vectors.

$$P = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & -\frac{2}{3} \\ 2 & 0 & 1 \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3: Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also find modal matrices.

(i) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: (i)

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x - y + z &= 0 \\ -x + 3y - z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2}$$

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 0x - y + z &= 0 \\ -x + 2y - z &= 0 \\ x - y + 0z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 6$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -3x - y + z &= 0 \\ -x - y - z &= 0 \\ x - y - 3z &= 0 \end{aligned}$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 6$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 6$.

Length of the eigenvector $\mathbf{x}_1 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

Length of the eigenvector $\mathbf{x}_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Length of the eigenvector $\mathbf{x}_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

(ii)
$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 8 + 7 + 3 = 18$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 5 + 20 + 20$$

$$= 45$$

$$S_3 = \det(A) = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 60 + 20$$

$$= 0$$

Hence, the characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$
$$\lambda = 0, 3, 15$$

(a) For $\lambda = 0$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$
$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 0$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 0$.

(b) For $\lambda = 3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y + 0z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 15$,

$$[A - \lambda I]\mathbf{x} = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x - 6y + 2z = 0$$

$$-6x - 8y - 4z = 0$$

$$2x - 4y - 12z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{40} = \frac{y}{-40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 15$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 15$.

Length of the eigenvector $\mathbf{x}_1 = \sqrt{1^2 + 2^2 + 2^2} = 3$

Length of the eigenvector $\mathbf{x}_2 = \sqrt{2^2 + 1^2 + (-2)^2} = 3$

Length of the eigenvector $\mathbf{x}_3 = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

(iii)
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 6 + 3 + 3 = 12$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$
$$= (9-1) + (18-4) + (18-4)$$
$$= 8 + 14 + 14$$
$$= 36$$

$$S_3 = \det(A) = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1)+2(-6+2)+2(2-6)$$

$$= 48-8-8$$

$$= 32$$

Hence, the characteristic equation is

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 2, 2, 8$$

(a) For $\lambda = 8$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + 2z = 0$$

$$-2x - 5y - z = 0$$

$$2x - y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 8$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 8$.

(b) For $\lambda = 2$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 2y + 2z = 0$$

Let $y = t_1$ and $z = t_2$

$$x = \frac{1}{2}t_1 - \frac{1}{2}t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} \frac{1}{2}t_1 - \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 2$.

The orthogonal matrix P has mutually orthogonal eigenvectors. Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, we must choose $\mathbf{x}_3$ such that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are orthogonal.

Let $\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

For orthogonality of eigenvectors,

$$\mathbf{x}_1^T\mathbf{x}_3 = 0 \text{ and } \mathbf{x}_2^T\mathbf{x}_3 = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$2l - m + n = 0 \quad \text{and} \quad \frac{1}{2}l + m = 0$$

By Cramer's rule,

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 2 & 1 \\ \frac{1}{2} & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}}$$

$$\frac{l}{-1} = \frac{m}{\frac{1}{2}} = \frac{n}{\frac{5}{2}}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{5} = t, \text{ say}$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 5t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = t\mathbf{x}_3$$

where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 2$.

$$\text{Length of eigenvector } \mathbf{x}_1 = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$\text{Length of eigenvector } \mathbf{x}_2 = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$$

$$\text{Length of eigenvector } \mathbf{x}_3 = \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{30}$$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{30} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$$

The modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.

5.5.2 *Powers of a Matrix*

If A is an $n \times n$ matrix and P is an invertible matrix then

$$(P^{-1}AP)^k = P^{-1}A^k P$$

If the matrix A is diagonalizable and $D = P^{-1}AP$ is a diagonal matrix then

$$D^k = (P^{-1}AP)^k = P^{-1}A^k P$$

Premultiplying D^k by P and post-multiplying by P^{-1},

$$PD^k P^{-1} = P(P^{-1}A^k P)P^{-1} = (PP^{-1})\, A^k\, (PP^{-1}) = IA^k I = A^k$$

$$\therefore A^k = PD^k P^{-1}$$

Example 1: Find A^{10} where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 1 + 2 = 3$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 - 0 = 2$$

Hence, the characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = 1, 2$$

Since all the eigenvalues are distinct, the matrix A is diagonalizable.

(a) For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + y = 0$$
$$x = y$$

Let
$$y = t$$
$$x = t$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + 0y = 0$$

Let
$$y = t$$
$$x = 0$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 2$.

Modal matrix P has eigenvectors as its column vectors.

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$$

Example 2: Find a matrix P that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Hence, find A^{13}.

Solution: $$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 0 + 2 + 3 = 5$

S_2 = Sum of the minors of principal diagonal elements of A.

$$= \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-0) + (0+2) + (0-0)$$

$$= 8$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 0 + 0 - 2(0-2) = 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + 0y - 2z = 0$$
$$x + y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-1} = t, \text{ say}$$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 0y + z = 0$$

Let

$$y = t_1 \quad \text{and} \quad z = t_2$$
$$x = -t_2$$

Thus, the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 2$.

Modal matrix P has eigenvectors as its column vectors.

$$P = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 2^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

Exercise 5.3

1. Show that the following matrices are not similar to diagonal matrices.

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

2. Show that the following matrices are similar to diagonal matrices. Find the diagonal and modal matrix in each case.

(i) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Ans.:

(i) $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(ii) $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(iii) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -2 & 1 \\ 3 & 3 & -2 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

3. Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also, find the modal matrix in each case.

(i) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

Ans.:

(i) $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$, $P = \begin{bmatrix} \frac{4}{\sqrt{18}} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix}$

(ii) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

4. Find A^{11}, where $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$

Ans.: $\begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$

5.6 QUADRATIC FORM

A homogeneous polynomial of second degree in n variables is called a quadratic form. An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$ where $a_{ij} = a_{ji}$ are all real, is called a quadratic form in n variables $x_1, x_2, \ldots x_n$.

Matrix of a Quadratic Form

The quadratic form corresponding to a symmetric matrix A can be written as

$$Q = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \qquad \text{...(1)}$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Coefficient of $x_i x_j$ in equation (1) $= a_{ij} + a_{ji}$

$$= 2a_{ij}$$

$$= 2a_{ji}$$

Coefficient of x_i^2 in equation (1) = a_{ii}

5.6.1 *Linear Transformation*

Let $Q = \mathbf{x}^T A\mathbf{x}$ be a quadratic form and $\mathbf{x} = P\mathbf{y}$ be a non-singular linear transformation.

$$Q = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y})$$

$$= \mathbf{y}^T P^T AP\mathbf{y}$$

$$= \mathbf{y}^T B\mathbf{y} \qquad \text{where } B = P^T AP$$

The form $\mathbf{y}^T B\mathbf{y}$ is called linear transformation of the quadratic form $\mathbf{x}^T A\mathbf{x}$ under a non-singular transformation $\mathbf{x} = P\mathbf{y}$ and P is called the matrix of the transformation.

Further, $$B^T = (P^T AP)^T = P^T A^T (P^T)^T = P^T AP \quad [\because A \text{ is symmetric}]$$
$$= B$$

Hence, matrix B is also symmetric.

5.6.2 *Rank of Quadratic Form*

The rank of the coefficient matrix A is called the rank of the quadratic form $\mathbf{x}^T A\mathbf{x}$. The number of non-zero eigen values of A also gives the rank of the quadratic form of A.

If $\rho(A) < n$ (order of A), i.e. det $(A) = 0$ then the quadratic form is singular, otherwise it is non-singular.

5.6.3 *Canonical or Normal Form*

Let $Q = \mathbf{x}^T A\mathbf{x}$ be a quadratic form of rank r. An orthogonal transformation $\mathbf{x} = P\mathbf{y}$ which diagonalises A, i.e., $P^T AP = D$, transforms the quadratic form Q to $\sum_{i=1}^{r} \lambda_i y_i^2$ (i.e., sum of r squares) or in matrix form $\mathbf{y}^T D\mathbf{y}$ in new variables. This new quadratic form containing only the squares of y_i is called the canonical form or normal form or sum of squares form of the given quadratic form.

(1) *Index*

The number of positive terms in the canonical form is called the index of the quadratic form and is denoted by p.

(2) *Signature*

The difference between the number of positive and negative terms in the canonical form is called the signature of the quadratic form and is denoted by s.

If index is p and total terms are r then

$$\text{signature } s = p - (r - p)$$
$$= 2p - r$$

The signature of a quadratic form is invariant for all normal reductions.

5.6.4 *Value Class or Nature of Quadratic Form*

Let $Q = \mathbf{x}^T A\mathbf{x}$ be the quadratic form in n variables $x_1, x_2 \ldots x_n$. Let r be the rank and p be the number of positive terms in the canonical form of Q. Then we have the following criteria for the definiteness of value class of Q.

Value Class	Criteria	Canonical Form
1. Positive definite	$r = p = n$	$\sum_{i=1}^{n} y_i^2$ (only positive terms)
2. Positive semidefinite	$r = p, p < n$	$\sum_{i=1}^{r} y_i^2$ (only positive terms)
3. Negative definite	$r = n, p = 0$	$-\sum_{i=1}^{n} y_i^2$ (only negative terms)
4. Negative semidefinite	$r < n, p = 0$	$-\sum_{i=1}^{r} y_i^2$ (only negative terms)
5. Indefinite	Otherwise	both positive and negative terms

(1) *Criteria for the Value Class of a Quadratic Form in Terms of the Nature of Eigen Values*

Value Class	Nature of Eigen Values
1. Positive definite	positive eigenvalues
2. Positive semidefinite	positive eigenvalues and at least one is zero
3. Negative definite	negative eigenvalues
4. Negative semidefinite	negative eigenvalues and at least one is zero
5. Indefinite	positive as well as negative eigenvalues

(2) *Criteria for the Value Class of a Quadratic Form in Terms of Leading Principal Minors*

For the matrix $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The leading principal minors of matrix A are those determinants starting with a_{11} of orders 1, 2, … n, i.e.

$$|a_{11}|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \cdots \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Value Class	Nature of Leading Principal Minors
1. Positive definite	positive leading principal minors
2. Positive semidefinite	positive leading principal minors and at least one is zero
3. Negative definite	negative leading principal minors
4. Negative semidefinite	negative leading principal minors and at least one is zero
5. Indefinite	positive as well as negative leading principal minors

5.6.5 *Maximum and Minimum Value of Quadratic Form*

Maximum and minimum values of a quadratic form $\mathbf{x}^T A\mathbf{x}$ are λ_1 and λ_n respectively if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ subject to the constraint

$$\|x\| = (x_1^2 + x_2^2 \cdots x_n^2)^{\frac{1}{2}} = 1$$

The quadratic form $\mathbf{x}^T A\mathbf{x} = \lambda_1$ if $\mathbf{x}$ is a normalized eigenvector of A corresponding to λ_1 and $\mathbf{x}^T A\mathbf{x} = \lambda_n$ if $\mathbf{x}$ is a normalized eigenvector of A corresponding to λ_n.

5.6.6 *Methods to Reduce Quadratic Form to Canonical Form*

(1) *Orthogonal Transformation*

If $Q = \mathbf{x}^T A\mathbf{x}$ is a quadratic form, then there exists a real orthogonal transformation $\mathbf{x} = P\mathbf{y}$ (where P is an orthogonal matrix) which transforms the given quadratic form $\mathbf{x}^T A\mathbf{x}$ to

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_r y_r^2$$

where $\lambda_1, \lambda_2 \ldots \lambda_r$ are the r non-zero eigen values of matrix A.

(2) *Congruent Transformation*

Congruent transformation consist of a pair of elementary transformations, one row and one similar column such that pre and post matrices are transpose of each other.

If $Q = \mathbf{x}^T A\mathbf{x}$ is a quadratic form then there exists a non-singluar linear transformation $\mathbf{x} = P\mathbf{y}$ which transforms the given quadratic form $\mathbf{x}^T A\mathbf{x}$ to a sum of square terms.

$$b_1 y_1^2 + b_2 y_2^2 + \cdots + b_r y_r^2$$

Example 1: Express the following quadratic forms in matrix notation:

(i) $x^2 - 6xy + y^2$

(ii) $2x^2 + 3y^2 - 5z^2 - 2xy + 6xz - 10yz$

(iii) $x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 - 8x_3x_4$

Solution: (i) $\mathbf{x}^T A\mathbf{x} = [x \quad y]\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $\mathbf{x}^T A\mathbf{x} = [x \quad y \quad z]\begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -5 \\ 3 & -5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(iii) $\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]\begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & 3 & -4 \\ -1 & -3 & -4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Example 2: Write down the quadratic forms corresponding to the following matrices:

(i) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

Solution: (i) $Q = 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 + 10x_1x_3 - 4x_2x_3$

(ii) $Q = 2x_2^2 + 4x_3^2 + 6x_4^2 + 2x_1x_2 + 4x_1x_3 + 6x_1x_4 + 6x_2x_3 + 8x_2x_4 + 10x_3x_4$

Example 3: Determine the nature (value class), index and signature of the following quadratic forms:

(i) $x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

(ii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

(iii) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$

(iv) $-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

Solution: (i) $$Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\lambda = -2, 3, 6.$$

Since there are positive as well as negative eigenvalues, value class of quadratic form is indefinite.

Index p = Number of positive eigenvalues = 2

Signature s = Difference between the number of positive and negative eigenvalues
= 2 − 1 = 1

(ii) $$Q = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 8, 2, 2$$

Since all the eigenvalues of A are positive, the value class of the quadratic form is positive definite.

Index p = Number of positive eigenvalues = 3

Signature s = Difference between the number of positive and negative eigenvalues
$= 3 - 0 = 3$

(iii)
$$Q = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 = 0$$

$$\lambda = 0, 0, 6$$

Since the eigenvalues of A are positive and two eigenvalues are zero, the value class of the quadratic form is positive semidefinite.

Index p = Number of positive eigenvalues = 1.

Signature s = Difference between the number of positive and negative eigenvalues
$= 1 - 0 = 1$

(iv)
$$Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix}=0$$

$$\lambda^3+9\lambda^2+24\lambda+16=0$$

$$\lambda=-1,-4,-4$$

Since all the eigenvalues of A are negative, the quadratic form is negative definite.

Index p = Number of positive eigenvalues = 0

Signature s = Difference between the number of positive and negative eigenvalues
$= 0 - 3 = -3$

Example 4: Find the value of k so that the value class of the quadratic form $k\left(x_1^2+x_2^2+x_3^2\right)+2x_1x_2-2x_2x_3+2x_3x_1$ is positive definite.

Solution:

$$Q=k\left(x_1^2+x_2^2+x_3^2\right)+2x_1x_2-2x_2x_3+2x_3x_1$$

$$Q=\mathbf{x}^T A\mathbf{x}=[x_1 \quad x_2 \quad x_3]\begin{bmatrix} k & 1 & 1 \\ 1 & k & -1 \\ 1 & -1 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A=\begin{bmatrix} k & 1 & 1 \\ 1 & k & -1 \\ 1 & -1 & k \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} k-\lambda & 1 & 1 \\ 1 & k-\lambda & -1 \\ 1 & -1 & k-\lambda \end{vmatrix}=0$$

$$(k-\lambda)[(k-\lambda)^2-1]-1(k-\lambda+1)+1[-1-(k-\lambda)]=0$$

$$(k-\lambda)(k-\lambda+1)(k-\lambda-1)-(k-\lambda+1)-(k-\lambda+1)=0$$

$$(k-\lambda+1)[(k-\lambda)(k-\lambda-1)-2]=0$$

$$(k-\lambda+1)[(k-\lambda)^2-(k-\lambda)-2]=0$$

$$(k-\lambda+1)(k-\lambda+1)(k-\lambda-2)=0$$

$$\lambda=(k+1),(k+1),(k-2)$$

For value class of quadratic form to be positive definite, all the eigenvalues should be greater than zero, i.e.

$$k > -1 \quad \text{and} \quad k > 2$$

Hence, value class of quadratic form is positive definite if $k > 2$.

Example 5: Find the maximum and minimum values of the quadratic form $x_1^2 + x_2^2 + 4x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$, and determine values of x_1 and x_2 at which the maximum and minimum occur.

Soltuion:

$$Q = x_1^2 + x_2^2 + 4x_1x_2$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1$$

Thus, the eigenvalues of A are $\lambda = 3$, and $\lambda = -1$ which are the maximum and minimum values, respectively of the quadratic form subject to the constraint.

(a) For $\lambda = 3$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 2x_2 = 0$$

$$-x_1 + x_2 = 0$$

$$x_1 = x_2$$

Let

$$x_2 = t$$

$$x_1 = t$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

(b) For $\lambda = -1$

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

$$x_1 = -x_2$$

Let

$$x_2 = t$$

$$x_1 = -t$$

Thus, the eigenvectors of A corresponding to $\lambda = -1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = -1$.

$$\text{Length of the eigenvector } \mathbf{x}_1 = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\text{Length of the eigenvector } \mathbf{x}_2 = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, subject to the constraint $x_1^2 + x_2^2 = 1$, the maximum value of the quadratic form is $\lambda = 3$, which occurs if $x_1 = \frac{1}{\sqrt{2}}, x_2 = \frac{1}{\sqrt{2}}$ and the minimum value is $\lambda = -1$, which occurs if $x_1 = -\frac{1}{\sqrt{2}},\ x_2 = \frac{1}{\sqrt{2}}$.

Example 6: Reduce the following quadratic forms to canonical forms by orthogonal transformation. Also find the rank, index, signature and value class (nature) of the quadratic forms.

(i) $Q = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_3$

(ii) $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$

(iii) $Q = 2x^2 + 2y^2 - z^2 - 4yz + 4xz - 8xy$

Solution: (i) $\quad Q = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_3$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 2 + 2 + 2 = 6$

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= (4-0) + (4-1) + (4-0)$$

$$= 11$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 2(4-0) + 0 + 1(0-2)$$

$$= 8 - 2$$

$$= 6$$

Hence, the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

(a) For $\lambda = 1$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 1$.

(b) For $\lambda = 2$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 0x_2 + x_3 = 0$$

$$x_1 + 0x_2 + 0x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 2$.

(c) For $\lambda = 3$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 0x_2 + x_3 = 0$$

$$0x_1 - x_2 + 0x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

$$\text{Length of the eigenvector } \mathbf{x}_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\text{Length of the eigenvector } \mathbf{x}_2 = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$\text{Length of the eigenvector } \mathbf{x}_3 = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to canonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_3$$

$$x_2 = y_2$$

$$x_3 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_3$$

The canonical form is

$$Q = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$= [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 3y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3
Index p = Number of positive terms in canonical form = 3
Signature s = Difference between the number of positive and negative terms in canonical form = 3 – 0 = 3

Since only positive terms occur in the canonical form, the value class of the quadratic form is positive definite.

(ii) $$Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of $A = 3 + 5 + 3 = 11$
S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 14 + 8 + 14$$

$$= 36$$

$$S_1 = \det(A) = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$= 36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

(a) For $\lambda = 2$,
$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 2$.

(b) For $\lambda = 3$,
$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 0x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 2 & -1\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}0 & 1\\ -1 & -1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}0 & -1\\ -1 & 2\end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 3$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t\\ t\\ t\end{bmatrix} = t\begin{bmatrix}1\\1\\1\end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 3$.

(c) For $\lambda = 6$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix}-3 & -1 & 1\\ -1 & -1 & -1\\ 1 & -1 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - 3x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-3 & 1\\ -1 & -1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-3 & -1\\ -1 & -1\end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 6$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} t\\ -2t\\ t\end{bmatrix} = t\begin{bmatrix}1\\-2\\1\end{bmatrix} = t\mathbf{x}_3$ where $\mathbf{x}_3$ is a linearly independent eigenvector corresponding to $\lambda = 6$.

$$\text{Length of the eigenvector } \mathbf{x}_1 = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\text{Length of the eigenvector } \mathbf{x}_2 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{Length of the eigenvector } \mathbf{x}_3 = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to canonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_2 - \frac{2}{\sqrt{6}} y_3$$

$$x_3 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{1}{\sqrt{6}} y_3$$

The canonical form is

$$Q = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$= [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index p = Number of positive terms in canonical form = 3

Signature s = Difference between the number of positive and negative terms in canonical form = 3 – 0 = 3

Since only positive terms occur in the canonical form, the value class of the quadratic form is positive definite.

(iii)
$$Q = 2x^2 + 2y^2 - z^2 - 8xy + 4xz - 4yz$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x \quad y \quad z]\begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -4 & 2 \\ -4 & 2-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of the principal diagonal elements of A = 2 + 2 – 1 = 3

S_2 = Sum of the minors of principal diagonal elements of A

$$= \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix}$$

$$= (-2-4) + (-2-4) + (4-16)$$

$$= -6 - 6 - 12$$

$$= -24$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{vmatrix}$$

$$= 2(-2-4) + 4(4+4) + 2(8-4)$$

$$= -12 + 32 + 8$$

$$= 28$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 - 24\lambda - 28 = 0$$

$$\lambda = -2, -2, 7$$

(a) For $\lambda = 7$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x - 4y + 2z = 0$$
$$-4x - 5y - 2z = 0$$
$$2x - 2y - 8z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ -5 & -2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -5 & 2 \\ -4 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix}}$$

$$\frac{x}{18} = \frac{y}{-18} = \frac{z}{9}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t, \text{ say}$$

Thus, the eigenvectors of A corresponding to $\lambda = 7$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 7$.

(b) For $\lambda = -2$, $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x - 2y + z = 0$$

Let $y = t_1$ and $z = t_2$

$$x = t_1 - \frac{1}{2}t_2$$

Thus, the eigenvectors of A corresponding to $\lambda = -2$ are the non-zero vectors of the form

$$\mathbf{x} = \begin{bmatrix} t_1 - \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = -2$.

The orthogonal matrix P has mutually orthogonal eigenvectors. Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, we must choose $\mathbf{x}_3$ such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are orthogonal.

Let
$$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For orthogonality of eigenvectors,

$$\mathbf{x}_1^T\mathbf{x}_3 = 0 \quad \text{and} \quad \mathbf{x}_2^T\mathbf{x}_3 = 0$$

$$[2 \quad -2 \quad 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{and} \quad [1 \quad 1 \quad 0]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$2l - 2m + n = 0 \text{ and } \quad l + m = 0$$

By Cramer's rule,

$$\frac{l}{\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{l}{-1} = -\frac{m}{1} = \frac{n}{4} = t, \text{ say}$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 4t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = t\mathbf{x}_3$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = -2$.

Length of eigenvector $\mathbf{x}_1 = \sqrt{2^2 + (-2) + 1^2} = 3$

Length of eigenvector $\mathbf{x}_2 = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

Length of eigenvector $\mathbf{x}_3 = \sqrt{(-1)^2 + 1^2 + 4^2} = \sqrt{18}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Hence, the diagonal matrix D has eigenvalues as its diagonal elements.
Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to cannonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{2}{3} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{18}} y_3$$

$$x_2 = -\frac{2}{3} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{18}} y_3$$

$$x_3 = \frac{1}{3} y_1 + \frac{4}{\sqrt{18}} y_3$$

The cannonical form is

$$Q = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$= [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 7y_1^2 - 2y_2^2 - 2y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index p = Number of positive terms in canonical form = 1

Signature s = Difference between the number of positive and negative terms in canonical form = 1 – 2 = –1

Since both positive and negative terms occur in canonical form, the value class of quadratic form is indefinite.

Example 7: Reduce the following quadratic forms to canonical form by congruent transformation. Also find the rank, index, signature and value class nature of the quadratic forms.

(i) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

(ii) $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_1x_3 + 12x_1x_2$

(iii) $x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$

Solution: (i) $Q = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Let $$A = I_3 AI_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1, R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1, C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\left(\frac{1}{\sqrt{2}}\right)R_3, \qquad \left(\frac{1}{\sqrt{2}}\right)C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{3}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Comparing with $D = P^T AP$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to the canonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{3}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = y_1 - y_2 + \frac{3}{\sqrt{2}} y_3$$
$$x_2 = y_2 - \sqrt{2} y_3$$
$$x_3 = \frac{1}{\sqrt{2}} y_3$$

The canonical form is

$$Q = y^T (P^T AP) y = y^T D y$$
$$= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
$$= y_1^2 + y_2^2 - y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index p = Number of positive terms in canonical form = 2

Signature s = Difference between the number of positive and negative terms in canonical form = 2 – 1 = 1

Since both positive and negative terms occur in the canonical form, the value class of the quadratic form is indefinite.

(ii) $$Q = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_1x_3 + 12x_1x_2$$

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Let $$A = I_3 A I_3$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1, R_3 + R_1$

$$\begin{bmatrix} 2 & 6 & -2 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 3C_1, C_3 + C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + \frac{2}{17}R_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 0 & -\frac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{11}{17} & \frac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + \frac{2}{17}C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -\frac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{11}{17} & \frac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & \frac{11}{17} \\ 0 & 1 & \frac{2}{17} \\ 0 & 0 & 1 \end{bmatrix}$$

$\left(\frac{1}{\sqrt{2}}\right)R_1, \left(\frac{1}{\sqrt{2}}\right)C_1$

$\left(\frac{1}{\sqrt{17}}\right)R_2, \left(\frac{1}{\sqrt{17}}\right)C_2$

$\left(\sqrt{\frac{17}{81}}\right)R_3, \left(\sqrt{\frac{17}{81}}\right)C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ \frac{11}{9\sqrt{17}} & \frac{2}{9\sqrt{17}} & \frac{\sqrt{17}}{9} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Comparing with $$D = P^T AP$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to canonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 - \frac{3}{\sqrt{17}} y_2 + \frac{11}{9\sqrt{17}} y_3$$

$$x_2 = \frac{1}{\sqrt{17}} y_2 + \frac{2}{9\sqrt{17}} y_3$$

$$x_3 = \frac{\sqrt{17}}{9} y_3$$

The canonical form is

$$Q = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 - y_2^2 - y_3^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index p = Number of positive terms in canonical form = 1

Signature s = Difference between the number of positive and negative terms in canonical form = 1 – 2 = –1

Since both positive and negative terms occur in the canonical form, the value class of the quadratic form is indefinite.

(iii) $$Q = x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$$

$$Q = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$$

Let $$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 + R_1, R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 + C_1, C_3 - \frac{1}{2}C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 + \frac{1}{2}C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\sqrt{\frac{2}{3}}\right)R_3, \left(\sqrt{\frac{2}{3}}\right)C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

Comparing with $$D = P^T AP$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to canonical form.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$x = u + v$$
$$y = v + \frac{1}{\sqrt{6}} w$$
$$z = \sqrt{\frac{2}{3}} w$$

The canonical form is

$$Q = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$
$$= \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
$$= u^2 + v^2 + w^2$$

Rank r = Number of non-zero terms in canonical form = 3

Index p = Number of positive terms in canonical form = 3

Signature s = Difference between the number of positive and negative terms in canonical form = 3 – 0 = 3

Since only positive terms occur in the canonical form, the value class of the quadratic form is positive definite.

Example 8: Show that $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_1x_3 + 6x_1x_2$ is positive semi-definite and find a non-zero set of values of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ which makes the form zero.

Solution: $Q = 5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_1x_3 + 6x_1x_2$

$$Q = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let $A = I_3 A I_3$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - \frac{3}{5}R_1,\ R_3 - \frac{7}{5}R_1$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{7}{5} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - \frac{3}{5}C_1,\ C_3 - \frac{7}{5}C_1$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{7}{5} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{7}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + \frac{1}{11}R_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{16}{11} & \frac{1}{11} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{7}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 + \frac{1}{11}C_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{16}{11} & \frac{1}{11} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing with $D = P^T AP$,

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\mathbf{x} = P\mathbf{y}$ be the linear transformation which transforms the given quadratic form to the canonical form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = y_1 - \frac{3}{5}y_2 - \frac{16}{11}y_3$$

$$x_2 = y_2 + \frac{1}{11}y_3$$

$$x_3 = y_3$$

The canonical form is

$$Q = \mathbf{y}^T(P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{121}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 5y_1^2 + \frac{121}{5}y_2^2$$

Rank r = Number of non-zero terms in canonical form = 2

Index p = Number of positive terms in canonical form = 2

Signature s = Difference between the number of positive and negative terms in canonical form = 2 – 0 = 2

Since all the terms in canonical form are positive and one term is zero, the value class of the quadratic form is positive semi definite.

The set of values $y_1 = 0$, $y_2 = 0$, $y_3 = 1$ will reduce the quadratic form to zero. For this set of value,

$$x_3 = 1,\ x_2 = \frac{1}{11},\ x_1 = -\frac{16}{11}$$

This is a non-zero set of values of x_1, x_2, x_3 which makes the quadratic form zero.

Exercise 5.4

1. Express the following quadratic forms in matrix notation.

(i) $2x^2 + 3y^2 + 6xy$

(ii) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

(iii) $x_1^2 + 2x_2^2 - 7x_3^2 + x_4^2 - 4x_1x_2 + 8x_1x_3 - 6x_3x_4$

(iv) $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

(v) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 4x_1x_3 - 6x_1x_4 - 4x_2x_3 - 8x_2x_4 + 12x_3x_4$

Ans.: (i) $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 & 4 & 0 \\ -2 & 2 & 0 & 0 \\ 4 & 0 & -7 & -3 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

$$\begin{bmatrix} \text{(iv)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ \text{(v)} \begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ 2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix} \end{bmatrix}$$

2. Write down the quadratic forms corresponding to following matrices:

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -1 & \frac{3}{2} & -2 \\ -1 & -3 & -\frac{5}{2} & 3 \\ \frac{3}{2} & -\frac{5}{2} & 4 & \frac{1}{2} \\ -2 & 3 & \frac{1}{2} & 1 \end{bmatrix}$

[**Ans.:** (i) $x_1^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$
(ii) $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2$
$-4x_1x_3 - 6x_3x_4$
(iii) $2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1x_2$
$+3x_1x_3 - 4x_1x_4 - 5x_2x_3 + 6x_2x_4 + x_3x_4$]

3. Reduce the following quadratic forms to canonical forms by orthogonal transformation. Also find rank, index and signature.

(i) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$

(ii) $2x_1^2 + 2y_1^2 + 2z_1^2 - 2x_1x_2$
$+ 2x_1x_3 - 2x_2x_3$

(iii) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$

[**Ans.:** (i) $2y_1^2 + 2y_2^2 + 6y_3^2; r = 3,$
$p = 3, s = 3$
(ii) $4y_1^2 + y_2^2 + y_3^2; r = 3,$
$p = 3, s = 3$
(iii) $3y_1^2 + 6y_2^2 - 9y_3^2; r = 3,$
$p = 2, s = 1$]

4. Reduce the following quadratic forms to canonical forms by congruent transformation. Also find rank, index and signature.

(i) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$

(ii) $2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx$

(iii) $x^2 + 3y^2 + 8z^2 + 4w^2 + 4xy + 6xz$
$- 4xw + 12yz - 8yw - 12zw$

[**Ans.:** (i) $y_1^2 - y_2^2 - y_3^2; r = 3,$
$p = 1, s = -1$
(ii) $y_1^2 - y_2^2 - y_3^2; r = 3,$
$p = 1, s = -1$
(iii) $y_1^2 - y_2^2 - y_3^2; r = 3,$
$p = 1, s = -1$]

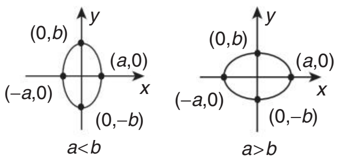
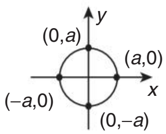
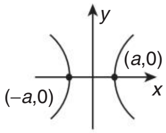
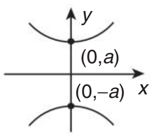
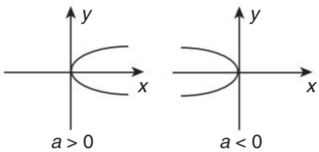
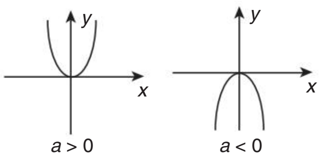
5.7 CONIC SECTIONS

Consider quadratic equations of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

where $a, b, \ldots, f$ are real numbers and at least one of the numbers a, b, c is not zero. In this equation $ax^2 + 2bxy + cy^2$ is called the associated quadratic form. Graphs of quadratic equations in x and y are called conics or conic sections. The most important conics are ellipses, circles, hyperbolas and parabolas. *A* conic is said to be in standard position relative to the coordinate axes if its equation can be expressed in one of the forms given in the table.

A conic in standard position does not contain an xy-term in its equation. The presence of an xy-term in the equation indicates that the conic is rotated out of standard position. Also, a conic in standard position does not contain both an x^2 and an x term or both a y^2 and a y term. If there is no xy-term, the occurence of either of these pairs in the equation indicates that the conic is translated out of standard position. The occurrence of either of these pairs and an xy-term usually indicates that the conic is both rotated and translated out of standard position.

Conic	Equation	Standard Position
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\ a,\ b>0$	
Circle	$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1;\ a>0$	
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;\ a,\ b>0$	
Hyperbola	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1;\ a,\ b>0$	
Parabola	$y^2=ax;\ a\neq 0$	
Parabola	$x^2=ay;\ a\neq 0$	

Eliminating xy-term from Quadratic Form

Let $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ be the equation of a conic c. Writing the equation in the matrix form,

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + f = \mathbf{0}$$

$$\mathbf{x}^T A\mathbf{x} + k\mathbf{x} + f = \mathbf{0}$$

where
$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, k = \begin{bmatrix} d & e \end{bmatrix}$$

The coordinate axes can be rotated so that the conic equation in the new $x'\ y'$ coordinate system becomes

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d'x' + e'y' + f = 0$$

where λ_1 and λ_2 are the eigenvalues of A. The rotation can be accomplished by the transformation $\mathbf{x} = P\mathbf{x}'$ where P orthogonally diagonalizes A and det $(P) = 1$.

Example 1: Describe the conic whose equation is

$$9x^2 + 4y^2 - 36x - 24y + 36 = 0.$$

Give its equation in the translated coordinate system.

Solution: Since the quadratic equation

$$9x^2 + 4y^2 - 36x - 24y + 36 = 0$$

contains x^2, x, y^2, and y-terms but no cross-product term, its graph is a conic that is translated out of standard position. This conic can be brought into the standard position by suitably translating the co-ordinate axes.

Collecting x-terms and y-terms,

$$(9x^2 - 36x) + (4y^2 - 24y) + 36 = 0$$
$$9(x^2 - 4x) + 4(y^2 - 6y) + 36 = 0$$

Completing the squares,

$$9(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 36$$
$$9(x-2)^2 + 4(y-3)^2 = 36$$

Translating the coordinate axes by translation equations $x' = x - 2,\ y' = y - 3$,

$$9x'^2 + 4y'^2 = 36$$
$$\frac{x'^2}{4} + \frac{y'^2}{9} = 1$$

This is the equation of the ellipse in the standard position in the $x'y'$ system.

Example 2: Describe the conic whose equation is

$$5x^2 - 4xy + 8y^2 - 36 = 0.$$

Give its equation in the rotated coordinate system.

Solution: Let $5x^2 - 4xy + 8y^2 - 36 = 0$ be the equation of a conic and let the associated quadratic form be

$$Q = 5x^2 - 4xy + 8y^2$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 4, 9$$

(a) For $\lambda=4$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - 2y = 0$$

Let

$$y = t$$
$$x = 2t$$

Thus, the eigenvectors of A corresponding to $\lambda = 4$, are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 4$.

(b) $\lambda = 9$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-2x - y = 0$$

Let

$$y = t$$
$$x = -\frac{1}{2}t$$

Thus, the eigenvectors of A corresponding to $\lambda = 9$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 9$.

Length of the eigenvector $\quad \mathbf{x}_1 = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$

Length of the eigenvector $\quad \mathbf{x}_2 = \sqrt{\left(-\frac{1}{2}\right)^2 + (1)^2} = \frac{\sqrt{5}}{2}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Thus, matrix P orthogonally diagonalizes A.

$$\det(P) = \begin{vmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{vmatrix} = \frac{4}{5} + \frac{1}{5} = 1$$

Thus, the rotation can be accomplished by the transformation $\mathbf{x} = P\mathbf{x}'$.
The matrix form of the conic equation is

$$\mathbf{x}^T A\mathbf{x} - 36 = 0$$

$$(P\mathbf{x}')^T A(P\mathbf{x}') - 36 = 0$$

$$\mathbf{x}'^T (P^T AP)\mathbf{x}' - 36 = 0$$

$$\mathbf{x}'^T D\mathbf{x}' - 36 = 0$$

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - 36 = 0$$

$$4x'^2 + 9y'^2 - 36 = 0$$

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

This is the equation of the ellipse.

Example 3: Translate and rotate the coordinate axes, if necessary, to put the conic $9x^2 - 4xy + 6y^2 - 10x - 20y = 5$ in the standard position. Find the equation of the conic in the final coordinate system.

Solution: Let $9x^2 - 4xy + 6y^2 - 10x - 20y = 5$ be the equation of a conic and let the associated quadratic form be

$$Q = 9x^2 - 4xy + 6y^2$$
$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$$
$$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 15\lambda + 50 = 0$$
$$\lambda = 5, 10$$

(a) For $\lambda = 5$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-2x + y = 0$$

Let
$$y = t$$
$$x = \frac{1}{2}t$$

Thus, the eigenvectors of A corresponding to $\lambda = 5$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = t\mathbf{x}_1$ where $\mathbf{x}_1$ is a linearly independent eigenvector corresponding to $\lambda = 5$.

(b) $\lambda = 10$,
$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$-x - 2y = 0$$

Let
$$y = t$$
$$x = -2t$$

Thus, the eigenvectors of A corresponding to $\lambda = 10$ are the non-zero vectors of the form $\mathbf{x} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix} = t\mathbf{x}_2$ where $\mathbf{x}_2$ is a linearly independent eigenvector corresponding to $\lambda = 10$.

Length of the eigenvector $\quad \|\mathbf{x}_1\| = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2} = \frac{\sqrt{5}}{2}$

Length of the eigenvector $\quad \|\mathbf{x}_2\| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

Modal matrix P has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Diagonal matrix D has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

Thus, matrix P orthogonally diagonalizes A.

$$\det(P) = \begin{vmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{vmatrix} = \frac{1}{5} + \frac{4}{5} = 1$$

Thus, the rotation can be accomplished by the transformation $\mathbf{x} = P\mathbf{x}'$.
The matrix form of the conic equation is

$$\left.\begin{aligned} \mathbf{x}^T A\mathbf{x} + k\mathbf{x} &= 5 \\ (P\mathbf{x}')^T A(P\mathbf{x}') + k(P\mathbf{x}') &= 5 \\ \mathbf{x}'^T (P^T AP)\mathbf{x}' + (kP)\mathbf{x}' &= 5 \\ \mathbf{x}'^T D\mathbf{x}' + (kP)\mathbf{x}' &= 5 \end{aligned}\right\}, \qquad \text{where } k = [-10 \quad -20]$$

$$\begin{bmatrix} x' & y' \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}+\begin{bmatrix} -10 & -20 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}=5$$

$$5x'^2+10y'^2+\begin{bmatrix} -10\sqrt{5} & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}=5$$

$$5x'^2+10y'^2-10\sqrt{5}x'=5$$

Collecting x'-terms and y'-terms,

$$(5x'^2-10\sqrt{5}x')+10y'^2=5$$

$$5(x'^2-2\sqrt{5}x')+10y'^2=5$$

Completing the squares,

$$5(x'^2-2\sqrt{5}x'+5)+10y'^2=5+25$$

$$5(x'-\sqrt{5})^2+10y'^2=30$$

Translating the coordinate axes by translation equations, $x''=x'-\sqrt{5}$, $y''=y'$,

$$5x''^2+10y''^2=30$$

$$\frac{x''^2}{6}+\frac{y''^2}{3}=1$$

This is the equation of the ellipse.

Exercise 5.6

1. In each case, a translation will put the conic in standard position. Name the conic and give its equation in the translated coordinate system.

(i) $x^2-16y^2+8x+128y=256$

(ii) $y^2-8x-14y+49=0$

(iii) $x^2+10x+7y=-32$

(iv) $x^2+y^2+6x-10y+18=0$

Ans.: (i) $x'^2-16y'^2=16$, hyperbola
(ii) $y'^2=8x'$, parabola
(iii) $y'=-\frac{1}{7}x'^2$, parabola
(iv) $x'^2+y'^2=16$, circle

2. In each case, rotate axes to identify the graph of the equation and write the equation in standard form.

(i) $x^2+xy+y^2=6$

(ii) $9x^2+y^2+6xy=4$

(iii) $4x^2+4y^2-10xy=0$

Ans.: (i) $\frac{x'^2}{12}+\frac{y'^2}{4}=1$, ellipse
(ii) $y'=\frac{2}{\sqrt{10}}$ and $y'=-\frac{2}{\sqrt{10}}$;
$y'^2=\frac{4}{10}$, pair of parallel lines
(iii) $y'=3x'$ and $y'=-3x'$;
$9x'^2-y'^2=0$, two intersecting lines

3. In each case, translate and rotate the coordinate axes, if necessary, to put the conic in standard position. Find the equation of the conic in final coordinate system.

(i) $2x^2 - 4xy - y^2 - 4x - 8y = -14$

(ii) $5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0$

(iii) $9x^2 + y^2 + 6xy - 10\sqrt{10}x + 10\sqrt{10}y + 90 = 0$

Ans.: (i) $2x''^2 - 3y''^2 = 24$, hypherbola

(ii) $\dfrac{x''^2}{9} + \dfrac{y''^2}{4} = 1$, ellipse

(iii) $y''^2 = -4x''$, parabola

Vector Functions

Chapter 6

6.1 INTRODUCTION

A vector field or a scalar field can be differentiated w.r.t. position in three ways to produce another vector field or scalar field. This chapter details the three derivatives, i.e., (i) the gradient of a scalar field, (ii) the divergence of a vector field, and (iii) the curl of a vector field.

6.2 VECTOR FUNCTION OF A SINGLE SCALAR VARIABLE

If, in some interval (a, b) or $[a, b]$, for every value of a scalar variable t, there corresponds a value of $\bar{r}$, then $\bar{r}$ is called a vector function of the scalar variable 't' and is denoted by $\bar{r} = \bar{f}(t)$.

6.2.1 *Decomposition of a Vector Function*

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along the three mutually perpendicular fixed directions (x, y, and z axes), then $\bar{r} = \bar{f}(t)$ can be decomposed as

$$\bar{r} = \bar{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

where, $f_1(t), f_2(t)$ and $f_3(t)$ are scalar functions of t. This relation can also be denoted by $\bar{f} = (f_1, f_2, f_3)$

$$|\bar{f}(t)| = \sqrt{[f_1(t)]^2 + [f_2(t)]^2 + [f_3(t)]^2}$$

6.2.2 *Derivative of a Vector Function*

Derivative of a vector function $\bar{f}(t)$ with respect to a scalar variable t is defined as

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$$

where, δt is the change in t.

If $\bar{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$ where $f_1(t), f_2(t)$ and $f_3(t)$ are the components of $\overline{f}(t)$ in the direction of x, y, z-axes, then derivative in the component form is $\frac{d\overline{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$.

6.2.3 *Some Standard Results*

Most of the basic rules of differentiation that are true for a scalar function of scalar variable hold good for vector function of a scalar variable, provided the order of factors in vector products is maintained.

Let $\overline{a}, \overline{b}, \overline{c}$ are differentiable vector functions of a scalar variable t.

1. $\frac{d\overline{k}}{dt} = 0, \overline{k}$ is a constant vector
2. $\frac{d}{dt}(\overline{a} \pm \overline{b}) = \frac{d\overline{a}}{dt} \pm \frac{d\overline{b}}{dt}$
3. $\frac{d}{dt}(\phi\overline{a}) = \phi\frac{d\overline{a}}{dt} + \overline{a}\frac{d\phi}{dt}$, ϕ is a scalar function of t.
4. $\frac{d}{dt}(\overline{a} \cdot \overline{b}) = \frac{d\overline{a}}{dt} \cdot \overline{b} + \overline{a} \cdot \frac{d\overline{b}}{dt}$
5. $\frac{d}{dt}(\overline{a} \times \overline{b}) = \frac{d\overline{a}}{dt} \times \overline{b} + \overline{a} \times \frac{d\overline{b}}{dt}$
6. $\frac{d}{dt}\left[\overline{a}\,\overline{b}\,\overline{c}\right] = \left[\frac{d\overline{a}}{dt}\,\overline{b}\,\overline{c}\right] + \left[\overline{a}\,\frac{d\overline{b}}{dt}\,\overline{c}\right] + \left[\overline{a}\,\overline{b}\,\frac{d\overline{c}}{dt}\right]$
7. $\frac{d}{dt}\left[\overline{a} \times (\overline{b} \times \overline{c})\right] = \frac{d\overline{a}}{dt} \times (\overline{b} \times \overline{c}) + \overline{a} \times \left(\frac{d\overline{b}}{dt} \times \overline{c}\right) + \overline{a} \times \left(\overline{b} \times \frac{d\overline{c}}{dt}\right)$

6.3 TANGENT, NORMAL AND BINORMAL VECTORS

(1) *Tangent Vector*

Let $P(t)$ and $Q(t + \delta t)$ be the two points on the curve $\overline{r} = \overline{f}(t)$. The tangent vector at P is the limiting position of the chord PQ when $Q \to P$, i.e., $\delta t \to 0$.

$$\lim_{\delta t \to 0} \frac{\delta\overline{r}}{\delta t} = \frac{d\overline{r}}{dt}$$

Hence, tangent vector is

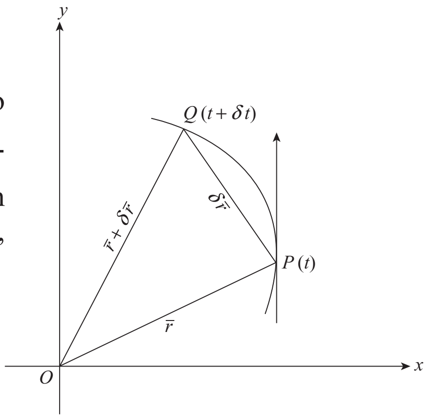


Fig. 6.1

$$\overline{T} = \frac{d\overline{r}}{dt}$$

The line containing the tangent vector is known as *tangent line*.

(2) *Osculating Plane*

The limiting position of a plane passing through three points P, Q, R on a curve as Q and R approaches P is known as osculating plane.

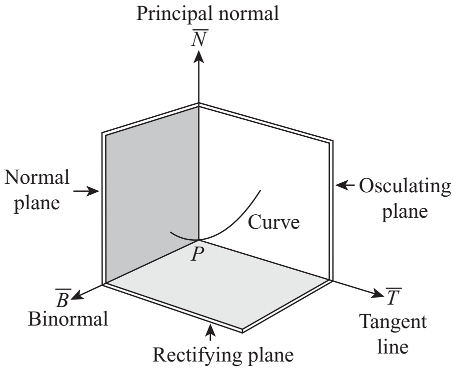


Fig. 6.2

(3) *Normal Plane*

A plane containing all the normals to the tangent line at P is known as normal plane.

(4) *Principal Normal*

The line perpendicular to the tangent line and lying in the osculating plane is known as principal normal.

(5) *Binormal*

The line perpendicular to the osculating plane and passing through P is known as binormal.

(6) *Rectifying Plane*

The plane containing the tangent line and the binormal is known as rectifying plane.

(7) *Unit Tangent Vector*

A unit vector along the tangent line is known as the unit tangent vector and is denoted by $\hat{T}$.

$$\hat{T} = \frac{\overline{r}'}{|\overline{r}'|}$$

(8) *Unit Normal Vector*

A unit vector along the principal normal is known as the unit normal vector and is denoted by $\hat{N}$.

$$\hat{N} = \frac{\overline{r}''}{|\overline{r}''|}$$

(9) *Unit Binormal Vector*

A unit vector along the binormal is known as the unit binormal vector and is denoted by $\hat{B}$.

$$\hat{B} = \hat{T} \times \hat{N}$$

Example 1: Write down the formula for $\frac{d}{dt}\left(\overline{A} \times \overline{B}\right)$ and verify the same for $\overline{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$, and $\overline{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$.

Solution: $\frac{d}{dt}\left(\overline{A}\times\overline{B}\right) = \frac{d\overline{A}}{dt} \times \overline{B} + \overline{A} \times \frac{d\overline{B}}{dt}$

Given,
$$\overline{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k},$$
$$\overline{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$$
$$\overline{A}\times\overline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$
$$= \hat{i}\,(0 - t^3\cos t) - \hat{j}\,(0 + t^3\sin t) + \hat{k}\,(-5t^2\cos t - t\sin t)$$
$$= (-t^3\cos t)\,\hat{i} - (t^3\sin t)\,\hat{j} - (5t^2\cos t + t\sin t)\,\hat{k}$$
$$\frac{d}{dt}\left(\overline{A}\times\overline{B}\right) = (-3t^2\cos t + t^3\sin t)\,\hat{i} - (3t^2\sin t + t^3\cos t)\,\hat{j}$$
$$- (10t\cos t - 5t^2\sin t + \sin t + t\cos t)\,\hat{k} \qquad \ldots (1)$$

Now,
$$\frac{d\overline{A}}{dt} = 10t\,\hat{i} + \hat{j} - 3t^2\,\hat{k},$$
$$\frac{d\overline{B}}{dt} = \cos t\,\hat{i} + \sin t\,\hat{j}$$
$$\frac{d\overline{A}}{dt}\times\overline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$
$$= \hat{i}(0 - 3t^2\cos t) - \hat{j}(0 + 3t^2\sin t) + \hat{k}\,(-10\,t\cos t - \sin t)$$
$$\overline{A}\times\frac{d\overline{B}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix}$$
$$= \hat{i}(0 + t^3\sin t) - \hat{j}(0 + t^3\cos t) + \hat{k}\,(5t^2\sin t - t\cos t)$$
$$\frac{d\overline{A}}{dt}\times\overline{B} + \overline{A}\times\frac{d\overline{B}}{dt} = (-3t^2\cos t + t^3\sin t)\,\hat{i} - (3t^2\sin t + t^3\cos t)\,\hat{j}$$
$$- (10t\cos t + \sin t - 5t^2\sin t + t\cos t)\,\hat{k} \qquad \ldots (2)$$

Comparing Eqs. (1) and (2),
$$\frac{d}{dt}\left(\overline{A}\times\overline{B}\right) = \frac{d\overline{A}}{dt}\times\overline{B} + \overline{A}\times\frac{d\overline{B}}{dt}$$

Example 2: If $\frac{d\bar{u}}{dt} = \bar{w} \times \bar{u}$ and $\frac{d\bar{v}}{dt} = \bar{w} \times \bar{v}$, then prove that $\frac{d}{dt}(\bar{u}\times\bar{v}) = \bar{w} \times (\bar{u}\times\bar{v})$.

Solution: We know that, $\frac{d}{dt}(\bar{u}\times\bar{v}) = \frac{d\bar{u}}{dt}\times\bar{v}+\bar{u}\times\frac{d\bar{v}}{dt}$

But $$\frac{d\bar{u}}{dt} = \bar{w} \times \bar{u},\quad \frac{d\bar{v}}{dt} = \bar{w} \times \bar{v}$$

$$\begin{aligned}\frac{d}{dt}(\bar{u}\times\bar{v}) &= (\bar{w}\times\bar{u})\times\bar{v}+\bar{u}\times(\bar{w}\times\bar{v})\\ &= (\bar{v}\cdot\bar{w})\bar{u}-(\bar{v}\cdot\bar{u})\bar{w}+(\bar{u}\cdot\bar{v})\bar{w}-(\bar{u}\cdot\bar{w})\bar{v}\\ &= (\bar{v}\cdot\bar{w})\bar{u}-(\bar{u}\cdot\bar{w})\bar{v} = (\bar{w}\cdot\bar{v})\bar{u}-(\bar{w}\cdot\bar{u})\bar{v}\\ &= \bar{w}\times(\bar{u}\times\bar{v})\end{aligned}$$

Example 3: If $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$, then show that $\bar{r} \times \frac{d\bar{r}}{dt} = \hat{k}$.

Solution: $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$

$$\frac{d\bar{r}}{dt} = 3t^2\,\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}$$

$$\begin{aligned}\bar{r}\times\frac{d\bar{r}}{dt} &= \left[t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}\right]\times\left[3t^2\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}\right]\\ &= 3t^5(\hat{i}\times\hat{i}) + \left(6t^5 + \frac{2}{5}\right)(\hat{i}\times\hat{j}) + \left(6t^5 - \frac{3}{5}\right)(\hat{j}\times\hat{i})\\ &\quad + \left(2t^3 - \frac{1}{5t^2}\right)\left(6t^2 + \frac{2}{5t^3}\right)(\hat{j}\times\hat{j})\\ &= 0 + \left(6t^5 + \frac{2}{5}\right)\hat{k} + \left(6t^5 - \frac{3}{5}\right)(-\hat{k}) + 0 \qquad [\because\ \hat{i}\times\hat{i} = 0 = \hat{j}\times\hat{j}\,]\\ &= \hat{k}\end{aligned}$$

Example 4: If $\bar{a}$ and $\bar{b}$ are constant vectors and ω is constant and $\bar{r} = \bar{a}\sin\omega t + \bar{b}\cos\omega t$, prove that $\bar{r} \times \frac{d\bar{r}}{dt} + \omega(\bar{a}\times\bar{b}) = 0$.

Solution: $\bar{r} = \bar{a}\sin\omega t + \bar{b}\cos\omega t$

$$\frac{d\bar{r}}{dt} = \bar{a}\,\omega\cos\omega t + \bar{b}\,\omega(-\sin\omega t)$$

$$\bar{r}\times\frac{d\bar{r}}{dt}=\left(\bar{a}\sin\omega t+\bar{b}\cos\omega t\right)\times\left(\bar{a}\,\omega\cos\omega t-\bar{b}\,\omega\sin\omega t\right)$$
$$=\left(\bar{a}\times\bar{a}\right)\omega\sin\omega t\cos\omega t-\left(\bar{a}\times\bar{b}\right)\omega\sin^2\omega t+\left(\bar{b}\times\bar{a}\right)\omega\cos^2\omega t$$
$$-\left(\bar{b}\times\bar{b}\right)\omega\cos\omega t\sin\omega t$$
$$=0-\left(\bar{a}\times\bar{b}\right)\omega\sin^2\omega t-\left(\bar{a}\times\bar{b}\right)\omega\cos^2\omega t-0\quad\left[\because\bar{a}\times\bar{a}=0=\bar{b}\times\bar{b}\right]$$
$$=-\left(\bar{a}\times\bar{b}\right)\omega\left(\sin^2\omega t+\cos^2\omega t\right)=-\left(\bar{a}\times\bar{b}\right)\omega$$

Hence, $\bar{r}\times\dfrac{d\bar{r}}{dt}+(\bar{a}\times\bar{b})\,\omega=0.$

Example 5: If $\bar{r}=\bar{a}\,\sinh t+\bar{b}\cosh t$, where $\bar{a}$ and $\bar{b}$ are constant, then show that

(i) $\dfrac{d^2\bar{r}}{dt^2}=\bar{r}$ (ii) $\dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}=$ constant.

Solution: $\bar{r}=\bar{a}\,\sinh t+\bar{b}\,\cosh t,$

(i) $$\frac{d\bar{r}}{dt}=\bar{a}\,\cosh t+\bar{b}\,\sinh t\qquad[\because\bar{a}\text{ and }\bar{b}\text{ are constant}]$$
$$\frac{d^2\bar{r}}{dt^2}=\bar{a}\,\sinh t+\bar{b}\,\cosh t=\bar{r}$$

Hence, $\dfrac{d^2\bar{r}}{dt^2}=\bar{r}$

(ii) $$\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}=\left(\bar{a}\cosh t+\bar{b}\sinh t\right)\times\left(\bar{a}\sinh t+\bar{b}\cosh t\right)$$
$$=\left(\bar{a}\times\bar{a}\right)\cosh t\sinh t+\left(\bar{a}\times\bar{b}\right)\cosh^2 t+\left(\bar{b}\times\bar{a}\right)\sinh^2 t+\left(\bar{b}\times\bar{b}\right)\sinh t\cosh t$$
$$=0+\left(\bar{a}\times\bar{b}\right)\cosh^2 t-\left(\bar{a}\times\bar{b}\right)\sinh^2 t+0$$
$$=\left(\bar{a}\times\bar{b}\right)\left(\cosh^2 t-\sinh^2 t\right)$$
$$=\left(\bar{a}\times\bar{b}\right)\qquad[\because\cosh^2 t-\sinh^2 t=1]$$

Hence, $\dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}=$ constant.

Example 6: If $\bar{r}=a\,(\sin\omega t)\,\hat{i}+b\,(\sin\omega t)\,\hat{j}+\dfrac{ct}{\omega^2}(\sin\omega t)\,\hat{k}$, prove that

$\dfrac{d^2\bar{r}}{dt^2}+\omega^2\bar{r}=\dfrac{2c}{\omega}(\cos\,\omega t)\,\hat{k}.$

Solution: $\bar{r} = a(\sin \omega t)\,\hat{i} + b(\sin \omega t)\,\hat{j} + \dfrac{ct}{\omega^2}(\sin \omega t)\,\hat{k}$

$$\frac{d\bar{r}}{dt} = a\omega(\cos \omega t)\,\hat{i} + b\omega(\cos \omega t)\,\hat{j} + \frac{c}{\omega^2}(\sin \omega t + t\,\omega\cos \omega t)\,\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = a\omega(-\omega\sin \omega t)\,\hat{i} + b\omega(-\omega\sin \omega t)\,\hat{j} + \frac{c}{\omega^2}[\omega(\cos \omega t) + \omega(\cos \omega t) + t\omega(-\omega\sin \omega t)]\,\hat{k}$$

$$= -a\omega^2(\sin \omega t)\,\hat{i} - b\omega^2(\sin \omega t)\,\hat{j} + \frac{c}{\omega^2}(2\omega\cos \omega t - t\omega^2\sin \omega t)\,\hat{k}$$

$$= -\omega^2\left[a(\sin \omega t)\,\hat{i} + b(\sin \omega t)\,\hat{j} + \frac{ct}{\omega^2}(\sin \omega t)\,\hat{k}\right] + \frac{2c}{\omega}(\cos \omega t)\,\hat{k}$$

$$= -\omega^2\,\bar{r} + \frac{2c}{\omega}(\cos \omega t)\,\hat{k}$$

Example 7: If $\bar{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + (at\tan \alpha)\,\hat{k}$, prove that

(i) $\left|\dfrac{d\bar{r}}{dt} \times \dfrac{d^2\bar{r}}{dt^2}\right| = a^2 \sec\alpha$ (ii) $\left[\dfrac{d\bar{r}}{dt}\ \dfrac{d^2\bar{r}}{dt^2}\ \dfrac{d^3\bar{r}}{dt^3}\right] = a^3\tan\alpha.$

Solution: $\bar{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + (at\tan \alpha)\,\hat{k}$,

$$\frac{d\bar{r}}{dt} = (-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + (a\tan\alpha)\,\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = (-a\cos t)\,\hat{i} + (-a\sin t)\,\hat{j} + 0\,\hat{k}$$

$$\frac{d^3\bar{r}}{dt^3} = (a\sin t)\,\hat{i} + (-a\cos t)\,\hat{j} + 0\cdot\hat{k}$$

(i) $$\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 + a^2\sin t\tan\alpha) - \hat{j}\,(0 + a^2\cos t\tan\alpha) + \hat{k}\,(a^2\sin^2 t + a^2\cos^2 t)$$

$$= a^2(\sin t\tan\alpha)\,\hat{i} - a^2(\cos t\tan\alpha)\,\hat{j} + a^2\,\hat{k}$$

$$\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\cdot\tan^2\alpha + a^4\cos^2 t\cdot\tan^2\alpha + a^4} = a^2\sqrt{\tan^2\alpha + 1} = a^2\sec\alpha$$

(ii) $$\left(\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right)\cdot\frac{d^3\bar{r}}{dt^3} = [a^2(\sin t\tan\alpha)\,\hat{i} - a^2(\cos t\tan\alpha)\,\hat{j} + a^2\,\hat{k}].\ [(a\sin t)\,\hat{i} + (-a\cos t)\,\hat{j} + 0\,\hat{k}]$$

$$= a^3\sin^2 t\tan\alpha + a^3\cos^2 t\tan\alpha$$

$$[\because\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0]$$

$$= a^3\tan\alpha$$

Hence, $\left[\frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] = a^3 \tan\alpha.$

Example 8: If $\bar{A} = (\sin t)\,\hat{i} + (\cos t)\,\hat{j} + t\,\hat{k}$, $\bar{B} = (\cos t)\,\hat{i} - (\sin t)\,\hat{j} - 3\,\hat{k}$, $\bar{C} = 2\,\hat{i} + 3\,\hat{j} - \hat{k}$, find $\frac{d}{dt}\left[\bar{A}\times(\bar{B}\times\bar{C})\right]$ at $t = 0$.

Solution: $(\bar{B}\times\bar{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & -\sin t & -3 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}\,(\sin t + 9) - \hat{j}(-\cos t + 6) + \hat{k}\,(3\cos t + 2\sin t)$

$$\bar{A}\times(\bar{B}\times\bar{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin t & \cos t & t \\ \sin t + 9 & \cos t - 6 & 3\cos t + 2\sin t \end{vmatrix}$$

$$= \hat{i}\,(3\cos^2 t + 2\sin t\cos t - t\cos t + 6t) - \hat{j}\,(3\cos t\sin t + 2\sin^2 t - t\sin t - 9t) + \hat{k}\,(\sin t\cos t - 6\sin t - \cos t\sin t - 9\cos t)$$

$$= (3\cos^2 t + \sin 2t - t\cos t + 6t)\hat{i} - \left(\frac{3}{2}\sin 2t + 2\sin^2 t - t\sin t - 9t\right)\hat{j} + (-6\sin t - 9\cos t)\,\hat{k}$$

$$\frac{d}{dt}\left[\bar{A}\times(\bar{B}\times\bar{C})\right] = [6\cos t\,(-\sin t) + 2\cos 2t - \cos t + t\sin t + 6]\,\hat{i} - (3\cos 2t + 4\sin t\cos t - \sin t - t\cos t - 9)\,\hat{j} - (6\cos t - 9\sin t)\,\hat{k}$$

Putting $t = 0$,

$$\frac{d}{dt}\left[\bar{A}\times(\bar{B}\times\bar{C})\right] = 7\,\hat{i} + 6\,\hat{j} - 6\,\hat{k}.$$

Example 9: Find the derivative of $\bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)$ with respect to 't'.

Solution:

$$\frac{d}{dt}\left[\bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)\right] = \frac{d\bar{r}}{dt}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right)$$

$$= \frac{d\bar{r}}{dt}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right) + \bar{r}\times\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right) \qquad \left[\because \frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2} = 0\right]$$

Example 10: Find $\frac{d}{dt}\left(\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right)$, where $\bar{r}$ is a vector function of scalar variable t and $\bar{r}$ is a constant vector.

Solution: $\frac{d}{dt}\left(\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right) = \frac{\left[\frac{d}{dt}(\bar{r}\times\bar{a})\right](\bar{r}\cdot\bar{a}) - (\bar{r}\times\bar{a})\frac{d}{dt}(\bar{r}\cdot\bar{a})}{(\bar{r}\cdot\bar{a})^2}$

$$= \frac{\left(\frac{d\bar{r}}{dt} \times \bar{a} + \bar{r} \times \frac{d\bar{a}}{dt}\right)(\bar{r} \cdot \bar{a}) - (\bar{r} \times \bar{a})\left(\frac{d\bar{r}}{dt} \cdot \bar{a} + \bar{r} \cdot \frac{d\bar{a}}{dt}\right)}{(\bar{r} \cdot \bar{a})^2}$$

But, $\frac{d\bar{a}}{dt} = 0$, as $\bar{a}$ is constant.

Hence,
$$\frac{d}{dt}\left(\frac{\bar{r} \times \bar{a}}{\bar{r} \cdot \bar{a}}\right) = \frac{\left(\frac{d\bar{r}}{dt} \times \bar{a}\right)(\bar{r} \cdot \bar{a}) - (\bar{r} \times \bar{a})\left(\frac{d\bar{r}}{dt} \cdot \bar{a}\right)}{(\bar{r} \cdot \bar{a})^2}.$$

Example 11: Find $\frac{d\bar{f}}{dt}$ if $\bar{f} = r^2\,\bar{r} + (\bar{a} \cdot \bar{r})\,\bar{b}$ where $\bar{r}$ is a function of t and $\bar{a}, \bar{b}$ are constant vectors.

Solution: $\bar{f} = r^2\bar{r} + (\bar{a} \cdot \bar{r})\,\bar{b}$

$$\frac{d\bar{f}}{dt} = \frac{d}{dt}(r^2\bar{r}) + \frac{d}{dt}(\bar{a} \cdot \bar{r})\bar{b}$$

$$= \left(\frac{dr^2}{dt}\right)(\bar{r}) + r^2\frac{d\bar{r}}{dt} + \left(\frac{d\bar{a}}{dt} \cdot \bar{r} + \bar{a} \cdot \frac{d\bar{r}}{dt}\right)\bar{b} + (\bar{a} \cdot \bar{r})\frac{d\bar{b}}{dt}$$

$$= \left(2r\frac{dr}{dt}\right)(\bar{r}) + r^2\frac{d\bar{r}}{dt} + \left(\bar{a} \cdot \frac{d\bar{r}}{dt}\right)\bar{b} \qquad \left[\because \frac{d\bar{a}}{dt} = \frac{d\bar{b}}{dt} = 0\right]$$

Hence,
$$\frac{d\bar{f}}{dt} = 2r\bar{r}\frac{dr}{dt} + r^2\frac{d\bar{r}}{dt} + \bar{b}\left(\bar{a} \cdot \frac{d\bar{r}}{dt}\right).$$

Example 12: If $\bar{f}(t)$ is a unit vector, prove that $\left|\bar{f}(t) \times \frac{d\bar{f}(t)}{dt}\right| = \left|\frac{d\bar{f}(t)}{dt}\right|$.

Solution: Since $\bar{f}$ is a unit vector,

$$\bar{f} \cdot \bar{f} = 1$$

Differentiating w.r.t. t,

$$\frac{d\bar{f}}{dt} \cdot \bar{f} + \bar{f} \cdot \frac{d\bar{f}}{dt} = 0$$

$$2\bar{f} \cdot \frac{d\bar{f}}{dt} = 0$$

$$\bar{f} \cdot \frac{d\bar{f}}{dt} = 0$$

This shows that $\bar{f}$ and $\frac{d\bar{f}}{dt}$ are perpendicular to each other.

Now, $$\bar{f} \times \frac{d\bar{f}}{dt} = |\bar{f}|\left|\frac{d\bar{f}}{dt}\right| \sin\theta \hat{n}$$

where, θ is the angle between $\bar{f}$ and $\frac{d\bar{f}}{dt}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\bar{f}$ and $\frac{d\bar{f}}{dt}$.

Since $\bar{f}$ and $\frac{d\bar{f}}{dt}$ are perpendicular, $\theta = \frac{\pi}{2}$.

$$\bar{f} \times \frac{d\bar{f}}{dt} = |\bar{f}|\left|\frac{d\bar{f}}{dt}\right| \sin\frac{\pi}{2} \hat{n}$$

$$\left|\bar{f} \times \frac{d\bar{f}}{dt}\right| = \left|\frac{d\bar{f}}{dt}\right| |\hat{n}| \qquad [\because \bar{f} \text{ is a unit vector}]$$

Hence, $$\left|\bar{f} \times \frac{d\bar{f}}{dt}\right| = \left|\frac{d\bar{f}}{dt}\right|. \qquad [\because |\hat{n}| = 1]$$

Example 13: Find the magnitude of the velocity and acceleration of a particle which moves along the curve $x = 2 \sin 3t$, $y = 2 \cos 3t$, $z = 8t$ at any time $t > 0$. Find unit tangent vector to the curve.

Solution: The position vector $\bar{r}$ of the particle is

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = (2 \sin 3t)\hat{i} + (2 \cos 3t)\hat{j} + (8t)\hat{k}$$

Velocity, $$\bar{v} = \frac{d\bar{r}}{dt} = (6 \cos 3t)\,\hat{i} + (-6 \sin 3t)\,\hat{j} + 8\hat{k}$$

$$|\bar{v}| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64} = \sqrt{36 + 64} = 10$$

Acceleration, $$\bar{a} = \frac{d^2\bar{r}}{dt^2} = (-18 \sin 3t)\,\hat{i} + (-18 \cos 3t)\,\hat{j} + (0)\hat{k}$$

$$|\bar{a}| = \sqrt{(18)^2 \sin^2 3t + (18)^2 \cos^2 3t} = 18.$$

$$\text{Unit tangent vector} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{1}{10}\left[(6\cos 3t)\hat{i} - (6\sin 3t)\,\hat{j} + 8\hat{k}\right].$$

Example 14: A particle moves along a plane curve such that its linear velocity is perpendicular to the radius vector. Show that the path of the particle is a circle.

Solution: Let position vector $\bar{r}$ of the particle is

$$\bar{r} = x\,\hat{i} + y\,\hat{j}$$

Velocity, $$\bar{v} = \frac{d\bar{r}}{dt}$$

To find path of the particle, we have to develop a relation in x and y. Velocity is perpendicular to the radius vector.

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

$$2\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

$$\bar{r} \cdot \frac{d\bar{r}}{dt} + \frac{d\bar{r}}{dt} \cdot \bar{r} = 0$$

$$\frac{d}{dt}\left(\bar{r} \cdot \bar{r}\right) = 0$$

$$\bar{r} \cdot \bar{r} = c^2, \text{ constant}$$

$$x^2 + y^2 = c^2$$

which is a circle with center at the origin and radius c.

Example 15: Find the magnitude of tangential components of acceleration at any time t of a particle whose position at any time t is given by $x = \cos t + t \sin t$, $y = \sin t - t \cos t$.

Solution: Position vector $\bar{r}$ of the particle is

$$\bar{r} = (\cos t + t \sin t)\,\hat{i} + (\sin t - t \cos t)\,\hat{j}$$

Velocity,
$$\bar{v} = \frac{d\bar{r}}{dt}$$
$$= (-\sin t + \sin t + t \cos t)\,\hat{i} + (\cos t - \cos t + t \sin t)\,\hat{j}$$
$$= (t \cos t)\,\hat{i} + (t \sin t)\,\hat{j}$$

Acceleration, $$\bar{a} = \frac{d^2\bar{r}}{dt^2} = (\cos t - t \sin t)\,\hat{i} + (\sin t + t \cos t)\,\hat{j}$$

Unit vector in the direction of the tangent is

$$\hat{t} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{(t\cos t)\hat{i} + (t\sin t)\hat{j}}{\sqrt{t^2\cos^2 t + t^2\sin^2 t}} = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$$

Magnitude of tangential component of acceleration

$$= \bar{a} \cdot \hat{t}$$
$$= [(\cos t - t \sin t)\,\hat{i} + (\sin t + t \cos t)\,\hat{j}] \cdot [(\cos t)\,\hat{i} + (\sin t)\,\hat{j}]$$
$$= \cos^2 t - t \sin t \cos t + \sin^2 t + t \cos t \sin t$$
$$= 1$$

Example 16: Show that a particle whose position vector $\bar{r}$ at any time t is given by $\bar{r} = (a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}$ moves in an ellipse whose center is at the origin and that its acceleration varies directly as its distance from the center and is directed towards it.

Solution: $\bar{r} = (a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}$

$$x = a \cos nt,\ y = b \sin nt$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 nt + \sin^2 nt = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse with center at origin.

Now, $$\frac{d\bar{r}}{dt} = (-a\,n \sin nt)\,\hat{i} + (b\,n \cos nt)\,\hat{j}$$

Acceleration, $$\bar{a} = \frac{d^2\bar{r}}{dt^2}$$

$$= (-a\,n^2 \cos nt)\,\hat{i} + (-b\,n^2 \sin nt)\,\hat{j}$$

$$= -n^2\,[(a \cos nt)\,\hat{i} + (b \sin nt)\,\hat{j}]$$

$$= -n^2\,\bar{r}$$

This shows that acceleration of the particle varies directly as its distance $\bar{r}$ from the origin (center of the ellipse) and negative sign shows that acceleration is directed towards the origin.

Example 17: Find unit tangent, unit normal and unit binormal vectors for the curve $x = t,\ y = 3 \sin t\ ,\ z = 3 \cos t$.

Solution:

$$\bar{r} = t\,\hat{i} + 3\sin t\,\hat{j} + 3\cos t\,\hat{k}$$

$$\bar{r}' = \hat{i} + 3\cos t\,\hat{j} - 3\sin t\,\hat{k}$$

$$\bar{r}'' = -3\sin t\,\hat{j} - 3\cos t\,\hat{k}$$

$$|\bar{r}'| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

$$|\bar{r}''| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$$

$$\hat{T} = \frac{\bar{r}'}{|\bar{r}'|} = \frac{\hat{i} + 3\cos t\,\hat{j} - 3\sin t\,\hat{k}}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}\hat{i} + \frac{3}{\sqrt{10}}\cos t\,\hat{j} - \frac{3}{\sqrt{10}}\sin t\,\hat{k}$$

$$\hat{N} = \frac{\bar{r}''}{|\bar{r}''|} = \frac{-3\sin t\,\hat{j} - 3\cos t\,\hat{k}}{3}$$

$$= -\sin t\,\hat{j} - \cos t\,\hat{k}$$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix}$$

$$= \hat{i}\left(-\frac{3}{\sqrt{10}}\cos^2 t - \frac{3}{\sqrt{10}}\sin^2 t\right) - \hat{j}\left(-\frac{1}{\sqrt{10}}\cos t\right) + \hat{k}\left(-\frac{1}{\sqrt{10}}\sin t\right)$$

$$= -\frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\cos t\,\hat{j} - \frac{1}{\sqrt{10}}\sin t\hat{k}$$

Example 18: Find unit tangent, unit normal and unit binormal vectors to the curve $x = a\cos\,\theta, y = a\sin\,\theta, z = b\theta.$

Solution:

$$\bar{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + b\theta\hat{k}$$

$$\bar{r}' = -a\sin\theta\hat{i} + a\cos\theta\hat{j} + b\hat{k}$$

$$\bar{r}'' = -a\cos\theta\hat{i} - a\sin\theta\hat{j}$$

$$|\bar{r}'| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2} = \sqrt{a^2 + b^2}$$

$$|\bar{r}''| = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a$$

$$\hat{T} = \frac{\bar{r}'}{|\bar{r}'|} = \frac{-a\sin\theta\hat{i} + a\cos\theta\hat{j} + b\hat{k}}{\sqrt{a^2 + b^2}}$$

$$= -\frac{a}{\sqrt{a^2 + b^2}}\sin\theta\hat{i} + \frac{a}{\sqrt{a^2 + b^2}}\cos\theta\hat{j} + \frac{b}{\sqrt{a^2 + b^2}}\hat{k}$$

$$\hat{N} = \frac{\bar{r}''}{|\bar{r}''|} = \frac{-a\cos\theta\hat{i} - a\sin\theta\hat{j}}{a}$$

$$= -\cos\theta\hat{i} - \sin\theta\hat{j}$$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$= \frac{1}{\sqrt{a^2 + b^2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & b \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2 + b^2}}[\hat{i}(0 + b\sin\theta) - \hat{j}(0 + b\cos\theta) + \hat{k}(a\sin^2\theta + a\cos^2\theta)]$$

$$= \frac{1}{\sqrt{a^2 + b^2}}(b\sin\theta\hat{i} - b\cos\theta\hat{j} + a\hat{k})$$

Example 19: Find unit tangent, unit normal and unit binormal vectors for the curve $\bar{r} = 2\cos t\hat{i} + 3\sin t\hat{j} + 4t\hat{k}$ at $t = \pi$.

Solution:

$$\bar{r} = 2\cos t\hat{i} + 3\sin t\hat{j} + 4t\hat{k}$$
$$\bar{r}' = -2\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}$$
$$\bar{r}'' = -2\cos t\hat{i} - 3\sin t\hat{j}$$

At $t = \pi$,

$$\bar{r}' = 0\hat{i} - 3\hat{j} + 4\hat{k} = -3\hat{j} + 4\hat{k}$$
$$\bar{r}'' = 2\hat{i} + 0\hat{j} + 0\hat{k} = 2\hat{i}$$
$$|\bar{r}'| = \sqrt{9+16} = 5$$
$$|\bar{r}''| = \sqrt{4} = 2$$
$$\hat{T} = \frac{\bar{r}'}{|\bar{r}'|} = \frac{-3\hat{j}+4\hat{k}}{5} = -\frac{3}{5}\hat{j} + \frac{4}{5}\hat{k}$$
$$\hat{N} = \frac{\bar{r}''}{|\bar{r}''|} = \frac{2\hat{i}}{2} = \hat{i}$$
$$\hat{B} = \hat{T} \times \hat{N}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \end{vmatrix}$$
$$= \hat{i}(0) - \hat{j}\left(-\frac{4}{5}\right) + \hat{k}\left(\frac{3}{5}\right)$$
$$= \frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}$$

Exercise 6.1

1. If $\bar{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$ and $\bar{B} = \sin t\,\hat{i} - \cos t\,\hat{j}$, find the value of

(i) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$ (ii) $\frac{d}{dt}(\bar{A}\times\bar{B})$.

$$\left[\begin{array}{l}\textbf{Ans.:} \\ \text{(i)}\,(5t^2-1)\cos t + 11t\sin t, \\ \text{(ii)}\,(t^3\sin t - 3t^2\cos t)\hat{i} \\ \quad -(t^3\cos t + 3t^2\sin t)\hat{j} \\ \quad +(5t^2\sin t - \sin t - 11\cos t)\hat{k}\end{array}\right]$$

2. If $\bar{A} = 4t^3\,\hat{i} + t^2\,\hat{j} - 6t^2\,\hat{k}$ and $\bar{B} = (\sin t)\,\hat{i} - (\cos t)\,\hat{j}$, verify the formula of $\frac{d}{dt}(\bar{A}\cdot\bar{B})$.

3. If $\bar{r} = \bar{A}\,e^{nt} + \bar{B}\,e^{-nt}$, show that $\frac{d^2\bar{r}}{dt^2} - n^2\bar{r} = 0$.

4. If $\bar{r} = t^3\,\hat{i} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{j}$, show that $\bar{r} \times \frac{d\bar{r}}{dt} = \hat{k}$.

5. Prove that

$$\frac{d}{dt}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right]=\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right].$$

6. Prove that

$$\frac{d^2}{dt^2}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right]=\left[\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right]+\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dt^4}\right].$$

7. Find the derivatives of the following:

(i) $r^3\ \bar{r}+\bar{a}\times\frac{d\bar{r}}{dt}$ (ii) $\frac{\bar{r}}{r^2}+\frac{r\bar{b}}{\bar{a}\cdot\bar{r}}$

where, $r=|\bar{r}|$, $\bar{a}$ and $\bar{b}$ are constant vectors.

[**Ans.:**

(i) $3r^2\frac{dr}{dt}\bar{r}+r^3\frac{d\bar{r}}{dt}+\bar{a}\times\frac{d^2\bar{r}}{dt^2}$

(ii) $\frac{1}{r^2}\left(\frac{d\bar{r}}{dt}\right)-2\frac{\bar{r}}{r^3}\frac{dr}{dt}+\frac{\bar{b}}{(\bar{a}\cdot\bar{r})}\frac{dr}{dt}-\frac{\bar{b}r}{(\bar{a}\cdot\bar{r})^2}\left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)$]

8. A particle moves along the curve $\bar{r}=e^{-t}(\cos t)\,\hat{i}+e^{-t}(\sin t)\,\hat{j}+e^{-t}\,\hat{k}$. Find the magnitude of velocity and acceleration at time t.

[**Ans.:** $\bar{v}=\sqrt{3}e^{-t}, \bar{a}=\sqrt{5}e^{-t}$]

9. A particle moves on the curve $x=2t^2, y=t^2-4t, z=3t-5$. Find the velocity and acceleration at $t=1$ in the direction of $\hat{i}-3\hat{j}+2\hat{k}$

[**Hint:** unit vector in the direction of $\hat{i}-3\hat{j}+2\hat{k}$ is

$$\hat{n}=\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{1+9+4}}=\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}},$$

Find $\bar{v}$ and $\bar{a}$ at $t=1$, velocity in the given direction $=\bar{v}\cdot\hat{n}$ and acceleration in the given direction $=\bar{a}\cdot\hat{n}$]

[**Ans.:** $\bar{v}=\frac{8\sqrt{2}}{\sqrt{7}}, \bar{a}=-\sqrt{\frac{2}{7}}$]

10. A particle is moving along the curve $\bar{r}=\bar{a}t^2+\bar{b}t+\bar{c}$, where $\bar{a},\bar{b},\bar{c}$ are constant vectors. Show that acceleration is constant.

11. A particle moves such that its position vector is given by $\bar{r}=(\cos\omega t)\,\hat{i}+(\sin\omega t)\,\hat{j}$. Show that velocity $\bar{v}$ is perpendicular to $\bar{r}$.

[**Hint:** Prove that $\frac{d\bar{r}}{dt}\cdot\bar{r}=0$]

6.4 ARC LENGTH

The parameterization for a curve is a set of functions depending only on a parameter t along with the bounds for the parameter. When we parameterize a curve by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as,

$$\bar{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$$

The tangent vector $\bar{r}'(t)$ *is*

$$\bar{r}'(t)=x'(t)\hat{i}+y'(t)\hat{j}+z'(t)\hat{k}$$

$$|\bar{r}'(t)|=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}$$

The length of the curve is

$$l=\int_a^b|\bar{r}'(t)|\,dt$$

The arc length function or arc length of the curve is obtained by replacing the constant limit with a variable *t*.

$$s = \int_a^t |\bar{r}'(u)|\, du$$

Example 1: Find the length of the curve $\bar{r}(t) = 2t\hat{i} + 3\sin 2t\hat{j} + 3\cos 2t\hat{k}$ on the interval $0 \le t \le 2\pi$

Solution:

$$\bar{r} = 2t\hat{i} + 3\sin 2t\hat{j} + 3\cos 2t\hat{k}$$

$$\bar{r}' = 2\hat{i} + 6\cos 2t\hat{j} - 6\sin 2t\hat{k}$$

$$|\bar{r}'| = \sqrt{4 + 36\cos^2 2t + 36\sin^2 2t}$$

$$= \sqrt{4+36}$$

$$= 2\sqrt{10}$$

$$l = \int_a^b |\bar{r}'|\,\mathrm{d}t$$

$$= \int_0^{2\pi} 2\sqrt{10}\,\mathrm{d}t$$

$$= 2\sqrt{\pi}\,[t]_0^{2\pi}$$

$$= 4\pi\sqrt{10}$$

$$s = \int_0^t |r'(u)|\,\mathrm{d}u$$

$$= \int_0^t 2\sqrt{10}\,du$$

$$= 2\sqrt{10}\,[t]_0^t$$

$$= 2\sqrt{10}t$$

Example 2: Find the length of the arc of the curve

$\bar{r}(t) = \dfrac{2\sqrt{2}}{3} t^{\frac{3}{2}}\hat{i} + \dfrac{t^2}{2}\hat{j} + (t+3)\hat{k}$ between $t = 0$ and $t = 2$

Solution:

$$\bar{r}' = \sqrt{2}t^{\frac{1}{2}}\hat{i} + t\hat{j} + \hat{k}$$

$$|\bar{r}'| = \sqrt{2t + t^2 + 1} = t + 1$$

$$l = \int_a^b |\bar{r}'(t)|\mathrm{d}t$$

$$= \int_0^2 (t+1)\mathrm{d}t$$

$$= \left[\frac{1}{2}t^2 + t\right]_0^2$$

$$= 2+2$$

$$= 4$$

6.5 CURVATURE AND TORSION

Curvature

The magnitude of rate of change of tangent vector w.r.t the arc length is known as curvature and is denoted by κ.

$$\kappa = \left|\frac{d\hat{T}}{ds}\right|$$

The reciprocal of the curvature is known as the radius of curvature of the curve and is denoted by ρ.

$$\rho = \frac{1}{\kappa}$$

If the equation of the curve is

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \text{ then}$$

$$\kappa = \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|^3}$$

Torsion

The magnitude of rate of change of binormal w.r.t. the arc length is known as torsion and is denoted by τ.

$$\tau = \left|\frac{d\hat{B}}{ds}\right|$$

The reciprocal of the torsion is known as the *radius of torsion of the curve* and is denoted by σ.

$$\sigma = \frac{1}{\tau}$$

If the equation of the curve is

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

then,

$$\tau = \frac{[\bar{r}'\bar{r}''\bar{r}''']}{|\bar{r}' \times \bar{r}''|^2}$$

Example 1: Find curvature and torsion for the curve $\bar{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$. Also prove that $2(\kappa^2 + \tau^2) = 1$

Solution:

$$
\begin{aligned}
\bar{r} &= \cos t\hat{i} + \sin t\hat{j} + t\hat{k} \\
\bar{r}' &= -\sin t\hat{i} + \cos t\hat{j} + \hat{k} \\
\bar{r}'' &= -\cos t\hat{i} - \sin t\hat{j} \\
\bar{r}''' &= \sin t\hat{i} - \cos t\hat{j} \\
\bar{r}' \times \bar{r}'' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} \\
&= \hat{i}(\sin t) - \hat{j}(\cos t) + \hat{k}\left(\sin^2 t + \cos^2 t\right) \\
&= \sin t\hat{i} - \cos t\hat{j} + \hat{k} \\
\left|\bar{r}' \times \bar{r}''\right| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \\
\left|\bar{r}'\right| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \\
\left[\bar{r}'\bar{r}''\bar{r}'''\right] &= \left(\bar{r}' \times \bar{r}''\right) \cdot \bar{r}''' = \sin^2 t + \cos^2 t + 0 = 1 \\
\kappa &= \frac{\left|\bar{r}' \times \bar{r}''\right|}{\left|\bar{r}'\right|^3} \\
&= \frac{\sqrt{2}}{\left(\sqrt{2}\right)^3} \\
&= \frac{1}{2} \\
\tau &= \frac{\left|\bar{r}'\bar{r}''\bar{r}'''\right|}{\left|\bar{r}' \times \bar{r}''\right|^2} \\
&= \frac{1}{\left(\sqrt{2}\right)^2} \\
&= \frac{1}{2} \\
\kappa^2 + \tau^2 &= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2} \\
2\left(\kappa^2 + \tau^2\right) &= 1
\end{aligned}
$$

Example 2: For the curve $\bar{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + b\theta\hat{k}$, find the radius of curvature and torsion.

Solution:

$$
\begin{aligned}
\bar{r} &= a\cos\theta\hat{i} + a\sin\theta\hat{j} + b\theta\hat{k} \\
\bar{r}' &= -a\sin\theta\hat{i} + a\cos\theta\hat{j} + b\hat{k} \\
\bar{r}'' &= -a\cos\theta\hat{i} - a\sin\theta\hat{j} + 0\hat{k} \\
\bar{r}''' &= a\sin\theta\hat{i} - a\cos\theta\hat{j} + 0\hat{k} \\
\bar{r}' \times \bar{r}'' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & b \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}
\end{aligned}
$$

$$= \hat{i}\left(0 + ab\sin\theta\right) - \hat{j}\left(0 + ab\cos\theta\right) + \hat{k}\left(a^2\sin^2\theta + a^2\cos^2\theta\right)$$
$$= ab\sin\theta\hat{i} - ab\cos\theta\hat{j} + a^2\hat{k}$$

$$\left|\bar{r}' \times \bar{r}''\right| = \sqrt{a^2b^2\sin^2\theta + a^2b^2\cos^2\theta + a^4}$$
$$= a\sqrt{b^2\sin^2\theta + b^2\cos^2\theta + a^2}$$
$$= a\sqrt{b^2 + a^2}$$
$$= a\sqrt{a^2 + b^2}$$

$$\left|\bar{r}'\right| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2} = \sqrt{a^2 + b^2}$$

$$\left[\bar{r}'\bar{r}''\bar{r}'''\right] = \left(\bar{r}' \times \bar{r}''\right)\cdot\bar{r}'''$$
$$= a^2b\sin^2\theta + a^2b\cos^2\theta + 0$$
$$= a^2b$$

$$\kappa = \frac{\left[\bar{r}' \times \bar{r}''\right]}{\left|\bar{r}'\right|^3} = \frac{a\sqrt{a^2 + b^2}}{\left(a^2 + b^2\right)^{\frac{3}{2}}} = \frac{a}{a^2 + b^2}$$

$$\rho = \frac{1}{\kappa} = \frac{a^2 + b^2}{a}$$

$$\tau = \frac{\left[\bar{r}'\bar{r}''\bar{r}'''\right]}{\left|\bar{r}' \times \bar{r}''\right|^2} = \frac{a^2b}{\left(a\sqrt{a^2 + b^2}\right)^2} = \frac{b}{a^2 + b^2}$$

Example 3: For the curve $x = a\cos\theta, y = a\sin\theta, z = a\theta\tan\alpha$, find ρ.

Solution:

$$\bar{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + a\theta\tan\alpha\hat{k}$$
$$\bar{r}' = \left(-a\sin\theta\right)\hat{i} + a\cos\theta\hat{j} + a\tan\alpha\hat{k}$$
$$\bar{r}'' = (-a\cos\theta)\hat{i} + (-a\sin\theta)\hat{j} + 0\hat{k}$$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & a\tan\alpha \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$
$$= \hat{i}\left(0 + a^2\tan\alpha\sin\theta\right) - \hat{j}\left(0 + a^2\tan\alpha\cos\theta\right) + \hat{k}\left(a^2\sin^2\theta + a^2\cos^2\theta\right)$$
$$= a^2\left\{\left(\tan\alpha\sin\theta\right)\hat{i} - \left(\tan\alpha\cos\theta\right)\hat{j} + \hat{k}\right\}$$

$$\left|\bar{r}' \times \bar{r}''\right| = a^2\sqrt{\tan^2\alpha\sin^2\theta + \tan^2\alpha\cos^2\theta + 1} = a^2\sec\alpha$$

$$\left|\bar{r}'\right| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + a^2\tan\alpha} = a\sec\alpha$$

$$\rho = \frac{\left|\bar{r}'\right|^3}{\left|\bar{r}' \times \bar{r}''\right|} = \frac{a^3\sec^3\alpha}{a^2\sec\alpha} = a\sec^2\alpha$$

Example 4: Find curvature and torsion for the curve $x = t\cos t, y = t\sin t, z = \lambda t$ at $t = 0$

Solution:

$$\bar{r} = t\cos t\hat{i} + t\sin t\hat{j} + \lambda t\hat{k}$$

$$\bar{r}' = (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j} + \lambda\hat{k}$$

$$\bar{r}'' = (-2\sin t - t\cos t)\hat{i} + (2\cos t - t\sin t)\hat{j} + 0\hat{k}$$

$$\bar{r}''' = (-3\cos t + t\sin t)\hat{i} + (-3\sin t - t\cos t)\hat{j}$$

At $t = 0$,

$$\bar{r}' = \hat{i} + 0\hat{j} + \lambda\hat{k}$$

$$\bar{r}'' = 0\hat{i} + 2\hat{j} + 0\hat{k}$$

$$\bar{r}''' = -3\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \lambda \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 2\lambda) - \hat{j}(0 - 0) + \hat{k}(2 - 0)$$
$$= -2\lambda\hat{i} - 0\hat{j} + 2\hat{k}$$

$$|\bar{r}' \times \bar{r}''| = 2\sqrt{\lambda^2 + 1}$$

$$|\bar{r}'| = \sqrt{1 + \lambda^2}$$

$$[\bar{r}'\bar{r}''\bar{r}'''] = (\bar{r}' \times \bar{r}'') \cdot \bar{r}''' = 6\lambda + 0 + 0 = 6\lambda$$

$$\kappa = \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|^3} = \frac{2\sqrt{1+\lambda^2}}{(1+\lambda^2)^{\frac{3}{2}}} = \frac{2}{1+\lambda^2}$$

$$\tau = \frac{[\bar{r}'\bar{r}''\bar{r}''']}{|\bar{r}' \times \bar{r}''|^2} = \frac{6\lambda}{4(1+\lambda^2)} = \frac{3\lambda}{2(1+\lambda^2)}$$

Example 5: Prove that radii of curvature and torsion are equal to $\frac{2x^2}{c}$ for the curve $x = c\cosh t, y = c\sinh t, z = ct$.

Solution:

$$\bar{r} = c\cosh t\hat{i} + c\sinh t\hat{j} + ct\hat{k}$$

$$\bar{r}' = c\sinh t\hat{i} + c\cosh t\hat{j} + c\hat{k}$$

$$\bar{r}'' = c\cosh t\hat{i} + c\sinh t\hat{j} + 0\hat{k}$$

$$\bar{r}'' = c\sinh t\hat{i} + c\cosh t\hat{j} + 0\hat{k}$$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c\sinh t & c\cosh t & c \\ c\cosh t & c\sinh t & 0 \end{vmatrix}$$

$$= \hat{i}\left(0 - c^2\sinh t\right) - \hat{j}\left(0 - c^2\cosh t\right) + \hat{k}\left(c^2\sinh^2 t - c^2\cosh^2 t\right)$$

$$= -c^2\sinh t\hat{i} + c^2\cosh t\hat{j} - c^2\hat{k}$$

$$\left|\bar{r}'\times\bar{r}''\right| = c^2\sqrt{\sinh^2 t+\cosh^2 t+1} = c^2\sqrt{\cosh^2 t+\cosh^2 t} = c^2\sqrt{2}\cosh t$$

$$\left|\bar{r}'\right| = \sqrt{c^2\sinh^2 t+c^2\cosh^2 t+c^2} = c\sqrt{2}\cosh t$$

$$\left[\bar{r}'\bar{r}''\bar{r}'''\right] = \left(\bar{r}'\times\bar{r}''\right)\cdot\bar{r}''' = -c^3\sinh^2 t+c^3\cosh^2 t+0 = c^3$$

$$\rho = \frac{\left|\bar{r}'\right|^3}{\left|\bar{r}'\times\bar{r}''\right|} = \frac{c^3 2\sqrt{2}\cosh^3 t}{c^2\sqrt{2}\cosh t} = \frac{2c^2\cosh^2 t}{c} = \frac{2x^2}{c}$$

$$\tau = \frac{\left[\bar{r}'\bar{r}''\bar{r}'''\right]}{\left|\bar{r}'\times\bar{r}''\right|^2} = \frac{c^3}{c^4 2\cosh^2 t} = \frac{c}{2c^2\cosh^2 t} = \frac{c}{2x^2}$$

$$\sigma = \frac{1}{\tau} = \frac{2x^2}{c}$$

$$\rho = \sigma = \frac{2x^2}{c}$$

Example 6: Find curvature and torsion for the curve $x = t^2-1, y = t^3-1, z = t^4-1$ at $t = 1$.

Solution:

$$\bar{r} = \left(t^2-1\right)\hat{i}+\left(t^3-1\right)\hat{j}+\left(t^4-1\right)\hat{k}$$

$$\bar{r}' = 2t\hat{i}+3t^2\hat{j}+4t^3\hat{k}$$

$$\bar{r}'' = 2\hat{i}+6t\hat{j}+12t^2\hat{k}$$

$$\bar{r}''' = 0\hat{i}+6\hat{j}+24t\hat{k}$$

At $t = 1$,

$$\bar{r}' = 2\hat{i}+3\hat{j}+4\hat{k}$$

$$\bar{r}'' = 2\hat{i}+6\hat{j}+12\hat{k}$$

$$\bar{r}''' = 0\hat{i}+6\hat{j}+24\hat{k}$$

$$\bar{r}'\times\bar{r}'' = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 2 & 6 & 12\end{vmatrix}$$

$$= \hat{i}(36-24)-\hat{j}(24-8)+\hat{k}(12-6)$$

$$= 12\hat{i}-16\hat{j}+6\hat{k}$$

$$\left|\bar{r}'\times\bar{r}''\right| = \sqrt{144+256+36} = \sqrt{436} = 2\sqrt{109}$$

$$\left|\bar{r}'\right| = \sqrt{4+9+16} = \sqrt{29}$$

$$\left[\bar{r}'\bar{r}''\bar{r}'''\right] = (\bar{r}'\times\bar{r}'')\cdot\bar{r}''' = 0-96+144 = 48$$

$$\kappa = \frac{\left|\bar{r}'\times\bar{r}''\right|}{\left|\bar{r}'\right|^3} = \frac{2\sqrt{109}}{29\sqrt{29}}$$

$$\tau = \frac{\left[\bar{r}'\bar{r}''\bar{r}'''\right]}{\left|\bar{r}'\times\bar{r}''\right|^2} = \frac{48}{4\times 109} = \frac{12}{109}$$

Example 7: Find the radius of curvature and the radius of torsion for the curve $\bar{r} = 3t\bar{i} + 3t^2\bar{j} + 3t^3\bar{k}$

Solution:

$$\bar{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$$
$$\bar{r}' = 3\hat{i} + 6t\hat{j} + 6t^2\hat{k}$$
$$\bar{r}'' = 0\hat{i} + 6\hat{j} + 12t\hat{k}$$
$$\bar{r}''' = 0\hat{i} + 0\hat{j} + 12t\hat{k}$$
$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix}$$
$$= \hat{i}\,(72t^2 - 36t^2) - \hat{j}\,(36t - 0) + \hat{k}(18 - 0)$$
$$= 36t^2\hat{i} - 36t\,\hat{j} + 18\hat{k}$$
$$|\bar{r}' \times \bar{r}''| = 18\sqrt{4t^4 + 4t^2 + 1} = 18\,(2t^2 + 1)$$
$$|\bar{r}'| = \sqrt{9 + 36t^2 + 36t^4} = 3(2t^2 + 1)$$
$$\left[\bar{r}'\ \bar{r}''\ \bar{r}'''\right] = (\bar{r}' \times \bar{r}'').\bar{r}''' = 0 + 0 + 216 = 216$$
$$\rho = \frac{|\bar{r}'|^3}{|\bar{r}' \times \bar{r}''|} = \frac{27(2t^2 + 1)^3}{18(2t^2 + 1)} = \frac{3}{2}(2t^2 + 1)^2$$
$$\sigma = \frac{|\bar{r}' \times \bar{r}''|^2}{|\bar{r}'\ \bar{r}''\ \bar{r}'''|} = \frac{(18)^2(2t^2 + 1)^2}{216} = \frac{3}{2}(2t^2 + 1)^2$$

Example 8: Find the curvature and torsion of the curve $\bar{r} = a(t - \sin t)\hat{i} + a(t - \cos t)\hat{j} + at\hat{k}$ at $t = \dfrac{\pi}{3}$.

Solution:

$$\bar{r} = a(t - \sin t)\hat{i} + a(t - \cos t)\hat{j} + at\,\hat{k}$$
$$\bar{r}' = a(1 - \cos t)\hat{i} + a(1 + \sin t)\hat{j} + at\,\hat{k}$$

At $t = \dfrac{\pi}{3}$,

$$\bar{r}' = a\left(1 - \frac{1}{2}\right)\hat{i} + a\left(1 + \frac{\sqrt{3}}{2}\right)\hat{j} + a\hat{k}$$
$$= \frac{a}{2}\hat{i} + \frac{a\left(2 + \sqrt{3}\right)}{2}\hat{j} + a\hat{k}$$

$$\bar{r}'' = a(\sin t)\hat{i} + a(\cos t)\hat{j} + a\hat{k}$$

At $t = \dfrac{\pi}{3}$, $$\bar{r}'' = \frac{a\sqrt{3}}{2}\hat{i} + \frac{a}{2}\hat{j} + 0\hat{k}$$

$$\bar{r}''' = a(\cos t)\hat{i} + a(-\sin t)\hat{j} + 0\hat{k}$$

At $t = \dfrac{\pi}{3}$, $$\bar{r}''' = \frac{a}{2}\hat{i} - \frac{a\sqrt{3}}{2}\hat{j} + 0\hat{k}$$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{a}{2} & \dfrac{a(2+\sqrt{3})}{2} & a \\ \dfrac{a\sqrt{3}}{2} & \dfrac{a}{2} & 0 \end{vmatrix}$$

$$= \hat{i}\left(0 - \frac{a^2}{2}\right) - \hat{j}\left(0 - \frac{a^2\sqrt{3}}{2}\right) + \hat{k}\left\{\frac{a^2}{4} - \frac{a^2\sqrt{3}\left(2+\sqrt{3}\right)}{4}\right\}$$

$$= -\frac{a^2}{2}\hat{i} + \frac{a^2\sqrt{3}}{2}\hat{j} - \frac{a^2}{2}\left(\sqrt{3}+1\right)$$

$$\left|\bar{r}' \times \bar{r}''\right| = \frac{a^2}{2}\sqrt{1+3+3+1+2\sqrt{3}} = \frac{a^2}{2}\sqrt{8+2\sqrt{3}}$$

$$\left|\bar{r}'\right| = a\sqrt{\frac{1}{4} + \frac{\left(4+3+4\sqrt{3}\right)}{4} + 1} = a\sqrt{3+\sqrt{3}}$$

$$\left[\bar{r}'\,\bar{r}''\,\bar{r}'''\right] = \left(\bar{r}' \times \bar{r}''\right).\bar{r}''' = \frac{-a^3}{4} - \frac{3a^3}{4} = -a^3$$

$$\kappa = \frac{\left|\bar{r}' \times \bar{r}''\right|}{\left|\bar{r}'\right|^3} = \frac{\dfrac{a^2}{2}\sqrt{8+2\sqrt{3}}}{a^3\left(3+\sqrt{3}\right)^{\frac{3}{2}}} = \frac{\sqrt{8+2\sqrt{3}}}{2a\,(3+\sqrt{3})^{\frac{3}{2}}}$$

$$\tau = \frac{\left[\bar{r}'\,\bar{r}''\,\bar{r}'''\right]}{\left|\bar{r}' \times \bar{r}''\right|^2} = \frac{-a^3}{\dfrac{a^4}{4}\left(8+2\sqrt{3}\right)} = -\frac{4}{a\left(8+2\sqrt{3}\right)}$$

Exercise 6.2

1. Find the arc length of the following curves:

(a) $\overline{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ for $0 \le t \le 6\pi$

(b) $\overline{r} = (1+3t^2)\hat{i} + (4+2t^3)\hat{j}$ for $0 \le t \le 1$

$$\left[\begin{array}{l}\textbf{Ans.:}\ \text{(a) } 6\sqrt{2}\pi \\ \qquad\quad \text{(b) } 2\left(2\sqrt{2}-1\right)\end{array}\right]$$

2. Find the arc length of the curve $\overline{r} = t^2\hat{i} + t^3\hat{j}$ between $(1,1)$ and $(4,8)$

$$\left[\textbf{Ans.:}\ \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)\right]$$

3. For the curve in space $x = a\cos 2t$, $y = \sin 2t$, $z = 2a\sin t$, show that

$$\tau = \frac{3\cos t}{a\,(5 + 3\cos^2 t)}$$

4. For the space curve $\overline{r} = a(3t - t^3)\hat{i} + 3at^2\hat{j} + a(3t + t^3)\hat{k}$,

show that $\kappa = \tau = \dfrac{1}{3a(1+t^2)^2}$

5. For the space curve $x = e^t\cos t, y = e^t\sin t, z = e^t$, find the radius of curvature and radius of torsion.

$$\left[\textbf{Ans.:}\ \rho = \frac{3}{\sqrt{2}}e^t,\ \ \sigma = 3e^t\right]$$

6.6 SCALAR AND VECTOR POINT FUNCTION

6.6.1 *Field*

If a function is defined in any region of space, for every point of the region, then this region is known as field.

6.6.2 *Scalar Point Function*

A function ϕ (x, y, z) is called scalar point function defined in the region R, if it associates a scalar quantity with every point in the region R of space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

6.6.3 *Vector Point Function*

A function $\overline{F}\,(x, y, z)$ is called vector point function defined in the region R, if it associates a vector quantity with every point in the region R of space. The velocity of a moving fluid, gravitational force are the examples of vector point function.

6.6.4 *Vector Differential Operator Del* (∇)

The vector differential operator Del (or nabla) is denoted by ∇ and is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

6.7 GRADIENT

The gradient of a scalar point function ϕ is written as $\nabla\phi$ or grad ϕ and is defined as

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

grad ϕ is a vector quantity.
$\phi\,(x, y, z)$ is a function of three independent variables and its total differential $\mathrm{d}\phi$ is given as

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z\right)$$

$$= \nabla\phi\cdot\mathrm{d}\bar{r} \qquad \left[\because \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} + \therefore\ \mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{j}\mathrm{d}z\right]$$

$$= |\nabla\phi||\mathrm{d}\bar{r}|\cos\theta \qquad \text{...(6.1)}$$

where, θ is the angle between the vectors $\nabla\phi$ and $\mathrm{d}\bar{r}$. If $\mathrm{d}\bar{r}$ and $\nabla\phi$ are in the same direction, then $\theta = 0$,

$$\mathrm{d}\phi = |\nabla\phi||\mathrm{d}\bar{r}|$$

cos $\theta = 1$ is the maximum value of cos θ. Hence, $\mathrm{d}\phi$ is maximum at $\theta = 0$.

6.7.1 *Normal*

Let $\phi\,(x, y, z) = c$ represents a family of surfaces for different values of the constant c. Such a surface for which the value of the function is constant is called **level surface**.

Now differentiating ϕ, we get

$$\mathrm{d}\phi = 0$$

But from Eq. (6.1) of Section 6.7,

$$\mathrm{d}\phi = \nabla\phi\cdot\mathrm{d}\bar{r}$$

$$\nabla\phi\cdot\mathrm{d}\bar{r} = 0$$

Hence, $\nabla\phi$ and $\mathrm{d}\bar{r}$ are perpendicular to each other. Since vector $\mathrm{d}\bar{r}$ is in the direction of the tangent to the given surface, vector $\nabla\phi$ is perpendicular to the tangent to the surface and hence $\nabla\phi$ is in the direction of normal to the surface.
Thus geometrically $\nabla\phi$ represents a vector normal to the surface $\phi\,(x, y, z) = c$.

6.7.2 *Directional Derivative*

(i) Let $\phi\,(x, y, z)$ be a scalar point function. Then $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are the directional derivative of ϕ in the **direction of the coordinate axes**.

Similarly, if $\overline{f}\,(x, y, z)$ be a vector point function, then $\frac{\partial \overline{f}}{\partial x}, \frac{\partial \overline{f}}{\partial y}, \frac{\partial \overline{f}}{\partial z}$ are the directional derivative of $\overline{f}$ in the **direction of the coordinate axes.**

(ii) The directional derivative of a scalar point function $\phi\,(x, y, z)$ in the direction **of a line** whose direction cosines are l, m, n,

$$= l\frac{\partial \phi}{\partial x} + m\frac{\partial \phi}{\partial y} + n\frac{\partial \phi}{\partial z}$$

(iii) The directional derivative of scalar point function $\phi\,(x, y, z)$ in the **direction of vector $\overline{a}$**, is the component of $\nabla\phi$ in the direction of $\overline{a}$. If $\hat{a}$ is the unit vector in the direction of $\overline{a}$, then directional derivatives of ϕ in the direction of $\overline{a}$

$$= \nabla\phi \cdot \hat{a} = \frac{\nabla\phi \cdot \overline{a}}{|\overline{a}|}$$

6.7.3 *Maximum Directional Derivative*

Since the component of a vector is maximum in its own direction, [$\because$ cos θ is maximum when $\theta = 0$], the directional derivative is maximum in the direction of $\nabla\phi$. Since $\nabla\phi$ is normal to the surface, directional derivative is maximum in the direction of normal.

Maximum directional derivative $= |\nabla\phi|\cos\theta$

$$= |\nabla\phi|\cos 0$$

$$= |\nabla\phi|$$

Standard Results:

(i) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$

(ii) $\nabla(\phi\psi) = \phi(\nabla\psi) + (\nabla\phi)\psi$

(iii) $\nabla f(u) = \hat{i}\frac{\partial f(u)}{\partial x} + \hat{j}\frac{\partial f(u)}{\partial y} + \hat{k}\frac{\partial f(u)}{\partial z} = f'(u)\nabla u.$

Example 1: Find $\nabla\phi$ at (1, – 2, 1), if $\phi = 3x^2y - y^3 z^2$.

Solution: $\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$= \hat{i}\,(6xy - 0) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(0 - 2y^3z)$$

At $x = 1, y = -2, z = 1$

$$\nabla\phi = \hat{i}\,(-12) + \hat{j}\,(3 - 12) + \hat{k}\,(16)$$

$\nabla\phi$ at $(1, -2, 1) = -12\,\hat{i} - 9\,\hat{j} + 16\,\hat{k}$

Example 2: Evaluate ∇e^{r^2}, where $r^2 = x^2 + y^2 + z^2$.

Solution: $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x, y and z,

$$2r\frac{\partial r}{\partial x} = 2x, \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z}$$

$$= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}(e^{r^2}\cdot 2r)\frac{x}{r} + \hat{j}(e^{r^2}\cdot 2r)\frac{y}{r} + \hat{k}(e^{r^2}\cdot 2r)\frac{z}{r} = 2e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Example 3: If $f(x, y) = \log\sqrt{x^2 + y^2}$ and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$\text{grad } f = \frac{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}}{\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]\cdot\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]}.$$

Solution: $f(x, y) = \log\sqrt{x^2 + y^2}$

$$= \frac{1}{2}\log(x^2 + y^2)$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{k}\frac{\partial}{\partial z}\left[\frac{1}{2}\log(x^2 + y^2)\right]$$

$$= \frac{\hat{i}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2x + \frac{\hat{j}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2y + 0$$

$$= \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$$

$$= \frac{x\hat{i} + y\hat{j}}{\left(x\hat{i} + y\hat{j}\right)\cdot\left(x\hat{i} + y\hat{j}\right)}$$

Now, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{k}\cdot\bar{r} = z$ $\qquad [\because \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0, \hat{k}\cdot\hat{k} = 1]$

$\bar{r} = x\hat{i} + y\hat{j} + (\hat{k}\cdot\bar{r})\,\hat{k}$

$\bar{r} - (\hat{k}\cdot\bar{r})\hat{k} = x\hat{i} + y\hat{j}$

Substituting $x\hat{i} + y\hat{j}$ in ∇f,

$$\nabla f = \frac{\bar{r} - (\hat{k} \cdot \bar{r})\hat{k}}{[\bar{r} - (\hat{k} \cdot \bar{r})\hat{k}] \cdot [\bar{r} - (\hat{k} \cdot \bar{r})\hat{k}]}.$$

Example 4: Prove that $\nabla r^n = nr^{n-2}\bar{r}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$.

Solution: $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z} = i\frac{\partial r^n}{\partial r} \cdot \frac{\partial r}{\partial x} + \hat{j}\frac{\partial r^n}{\partial r} \cdot \frac{\partial r}{\partial y} + \hat{k}\frac{\partial r^n}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$= \hat{i}\, nr^{n-1} \cdot \frac{x}{r} + \hat{j}\, nr^{n-1} \cdot \frac{y}{r} + \hat{k}\, nr^{n-1} \cdot \frac{z}{r}$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= nr^{n-2}\,\bar{r}.$$

Example 5: Show that $\nabla\left(\frac{\bar{a} \cdot \bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} - \frac{n(\bar{a} \cdot \bar{r})}{r^{n+2}}(\bar{r})$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$, $\bar{a}$ is constant vector.

Solution: Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, and $\frac{\bar{a} \cdot \bar{r}}{r^n} = \phi$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\phi = \left(\frac{\bar{a} \cdot \bar{r}}{r^n}\right) = \left[\frac{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})}{r^n}\right]$$

$$= \left(\frac{a_1x + a_2y + a_3z}{r^n}\right)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{a_1x + a_2y + a_3z}{r^n}\right)$$

$$= \frac{\left[\frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\right]r^n - (a_1x + a_2y + a_3z)\frac{\partial r^n}{\partial x}}{r^{2n}}$$

$$= \frac{a_1r^n - (a_1x + a_2y + a_3z)nr^{n-1}\frac{\partial r}{\partial x}}{r^{2n}}$$

But, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\frac{\partial \phi}{\partial x}=\frac{a_1 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{x}{r}\right)}{r^{2n}}$$

Similarly, $\dfrac{\partial \phi}{\partial y}=\dfrac{a_2 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{y}{r}\right)}{r^{2n}}$

and $\quad \dfrac{\partial \phi}{\partial z}=\dfrac{a_3 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{z}{r}\right)}{r^{2n}}$

$$\nabla\phi=\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}$$

$$=\frac{\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)r^n-(a_1x+a_2y+a_3z)nr^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{r^{2n}}$$

$$=\frac{\bar{a}r^n-\left(\bar{a}\cdot\bar{r}\right)nr^{n-2}\bar{r}}{r^{2n}}$$

$$\left[\because a_1x+a_2y+a_3z=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\bar{a}\cdot\bar{r}\right]$$

Hence, $\nabla\left(\dfrac{\bar{a}\cdot\bar{r}}{r^n}\right)=\dfrac{\bar{a}}{r^n}-\dfrac{n\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^{n+2}}.$

Example 6: If $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $\bar{a},\ \bar{b}$ are constant vectors, prove that $\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\dfrac{1}{r}\right)=\dfrac{3\left(\bar{a}\cdot\bar{r}\right)\left(\bar{b}\cdot\bar{r}\right)}{r^5}-\dfrac{\bar{a}\cdot\bar{b}}{r^3}.$

Solution: Let $\quad \bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\quad \bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$=\hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

But, $\quad \bar{r}=x\hat{i}+y\hat{j}+z\hat{k},\ r^2=x^2+y^2+z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)+\hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right)+\hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)=-\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=-\frac{\bar{r}}{r^3}.$$

$$\bar{b}\cdot\nabla\left(\frac{1}{r}\right) = (b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\cdot\left(-\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3}\right)$$

$$= -\left(\frac{b_1x+b_2y+b_3z}{r^3}\right)$$

$$= \phi, \text{ say}$$

$$\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{b_1x+b_2y+b_3z}{r^3}\right)$$

$$= -\left[\frac{b_1r^3-(b_1x+b_2y+b_3z)\frac{\partial}{\partial x}r^3}{r^6}\right]$$

$$= -\left[\frac{b_1r^3-(b_1x+b_2y+b_3z)3r^2\frac{\partial r}{\partial x}}{r^6}\right]$$

$$= -\left[\frac{b_1r^3-(\bar{b}\cdot\bar{r})3r^2\frac{x}{r}}{r^6}\right]$$

$$= \frac{-b_1r^2+3(\bar{b}\cdot\bar{r})x}{r^5}$$

Similarly,

$$\frac{\partial\phi}{\partial y} = \frac{-b_2r^2+3(\bar{b}\cdot\bar{r})y}{r^5}$$

$$\frac{\partial\phi}{\partial z} = \frac{-b_3r^2+3(\bar{b}\cdot\bar{r})y}{r^5}$$

and

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

$$= -\frac{(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})}{r^3}+\frac{3(\bar{b}\cdot\bar{r})(x\hat{i}+y\hat{j}+z\hat{k})}{r^5}$$

$$= -\frac{\bar{b}}{r^3}+\frac{3(\bar{b}\cdot\bar{r})\bar{r}}{r^5}$$

$$\bar{a}\cdot\nabla\phi = \bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{a}\cdot\bar{b}}{r^3}+\frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5}$$

Hence,

$$\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5}-\frac{\bar{a}\cdot\bar{b}}{r^3}.$$

Example 7: Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Solution: $\nabla\phi$ is the vector which is normal to the surface $\phi(x, y, z) = c$

Given surface is $x^2 + y^2 + z^2 = a^2$

$$\phi(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

At the point $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$,

$$\nabla\phi = \frac{2a}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

Unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$$= \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2a}{\sqrt{3}} \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{\frac{4a^2}{3} + \frac{4a^2}{3} + \frac{4a^2}{3}}}$$

$$= \frac{2a(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3} \cdot \frac{2a\sqrt{3}}{\sqrt{3}}}$$

$$= \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}.$$

Example 8: Find unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

Solution: Given surface is $x^2y + 2xz^2 = 8$

$$\phi(x, y, z) = x^2y + 2xz^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2y + 2xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2y + 2xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2y + 2xz^2)$$

$$= \hat{i}(2xy + 2z^2) + \hat{j}(x^2) + \hat{k}(4xz)$$

At the point (1, 0, 2), $\nabla\phi = 8\hat{i} + \hat{j} + 8\hat{k}$

Unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2)

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\hat{i} + \hat{j} + 8\hat{k}}{\sqrt{64 + 1 + 64}} = \frac{8\hat{i} + \hat{j} + 8\hat{k}}{\sqrt{129}}.$$

Example 9: Find the directional derivatives of $\phi = xy^2 + yz^2$ at the point (2, −1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution: $\nabla\phi = \hat{i}\dfrac{\partial}{\partial x}(xy^2 + yz^2) + \hat{j}\dfrac{\partial}{\partial y}(xy^2 + yz^2) + \hat{k}\dfrac{\partial}{\partial z}(xy^2 + yz^2)$

$$= \hat{i}y^2 + \hat{j}\,(2xy + z^2) + \hat{k}\,(2yz)$$

At the point (2, –1, 1),

$$\nabla\phi = \hat{i} + \hat{j}\,(-4+1) + \hat{k}\,(-2) = \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$= (\nabla\phi)\cdot\frac{\bar{a}}{|\bar{a}|}$$

$$= \left(\hat{i} - 3\hat{j} - 2\hat{k}\right)\cdot\frac{\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)}{\sqrt{1+4+4}}$$

$$= \frac{(1-6-4)}{3}$$

$$= -3.$$

Example 10: Find the directional derivative of $\phi = \dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ at the point $P\,(1, -1, 1)$ in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{j}\frac{\partial}{\partial y}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{k}\frac{\partial}{\partial z}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{i} + \hat{j}\left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{k}$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, –1, 1),

$$\nabla\phi = \frac{-(i - j + k)}{(3)^{\frac{3}{2}}}$$

Directional derivative in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$

$$= \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$$

$$= \frac{-(i - j + k)\cdot(i + j + k)}{(3)^{\frac{3}{2}}\sqrt{1+1+1}}$$

$$= \frac{-1+1-1}{3^2}$$

$$= -\frac{1}{9}.$$

Example 11: Find the directional derivative of $\phi = xy^2 + yz^3$ at (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1).

Solution: Let $\psi = x \log z - y^2$

$\nabla\psi$ is normal to the surface $x \log z - y^2 = -4$

$$\nabla\psi = \hat{i}\frac{\partial}{\partial x}(x\log z - y^2) + \hat{j}\frac{\partial}{\partial y}(x\log z - y^2) + \hat{k}\frac{\partial}{\partial z}(x\log z - y^2)$$

$$= \hat{i}(\log z) + \hat{j}(-2y) + \hat{k}\left(\frac{x}{z}\right)$$

At the point (–1, 2, 1),

$$\nabla\psi = \hat{i}\ (\log 1) - 4\hat{j} - \hat{k}$$
$$= -4\hat{j} - \hat{k}$$

$-4\hat{j} - \hat{k}$ is a vector normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1).

Now, $\phi = xy^2 + yz^3$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= \hat{i}\ (y^2) + \hat{j}\ (2xy + z^3) + \hat{k}\ (3yz^2)$$

At the point (2, –1, 1),

$$\nabla\phi = \hat{i} + \hat{j}\ (-4 + 1) + \hat{k}\ (-3) = \hat{i} - 3\hat{j} - 3\hat{k}$$

Directional derivative of ϕ in the direction of the vector $-4\hat{j} - \hat{k}$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{(-4\hat{j} - \hat{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Example 12: Find directional derivative of the function $\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1, 1, 1).

Solution: Tangent to the curve is

$$\overline{T} = \frac{d\overline{r}}{dt}$$

$$= \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^3\hat{k})$$

$$= (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

If $x = 1, y = 1, z = 1$, then $t = 1$

At the point (1, 1, 1), $t = 1$

$$\overline{T} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$$

$$= \hat{i}\ (y^2 + 2xz) + \hat{j}\ (2xy + z^2) + \hat{k}\ (2yz + x^2)$$

At the point (1, 1, 1),

$$\nabla\phi = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Directional derivative of ϕ in the direction of the tangent $\overline{T} = \hat{i} + 2\hat{j} + 3\hat{k}$ at the point (1, 1, 1)

$$= \nabla\phi \cdot \frac{\overline{T}}{|\overline{T}|} = (3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

Example 13: Find the directional derivative of $\phi = e^{2x} \cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = a\,t$ at $t = \frac{\pi}{4}$.

Solution: Tangent to the curve is

$$\overline{T} = \frac{d\overline{r}}{dt} = \frac{d}{dt}\left[(a\sin t)\hat{i} + (a\cos t)\hat{j} + (at)\hat{k}\right]$$

$$= (a \cos t)\,\hat{i} + (-a \sin t)\,\hat{j} + (a)\,\hat{k}$$

At the point $t = \frac{\pi}{4}, \overline{T} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$$\phi = e^{2x} \cos yz$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(e^{2x}\cos yz) + \hat{j}\frac{\partial}{\partial y}(e^{2x}\cos yz) + \hat{k}\frac{\partial}{\partial z}(e^{2x}\cos yz)$$

$$= \hat{i}\,(2e^{2x}\cos yz) + \hat{j}\,(-e^{2x} z \sin yz) + \hat{k}\,(-e^{2x} y \sin yz)$$

At the origin, $\nabla\phi = 2i$

Directional derivative in the direction of the tangent to the given curve

$$= \nabla\phi\frac{\overline{T}}{|\overline{T}|} = 2\hat{i} \cdot \frac{\left(\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}\right)}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{2a}{2a} = 1.$$

Example 14: Find the directional derivative of v^2, where $\overline{v} = xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}$ at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

Solution: $v^2 = \overline{v} \cdot \overline{v}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}) \cdot (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$= x^2y^4 + z^2y^4 + x^2z^4$$

Let $v^2 = \phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (2xy^4 + 2xz^4)\,\hat{i} + (4x^2y^3 + 4z^2y^3)\,\hat{j} + (2zy^4 + 4x^2z^3)\,\hat{k}$$

At the point (2, 0, 3),

$$\nabla\phi = (0 + 324)\,\hat{i} + (0 + 0)\,\hat{j} + (0 + 432)\,\hat{k} = 324\,\hat{i} + 432\,\hat{k}$$

Given sphere is $x^2 + y^2 + z^2 = 14$.
Let $\psi = x^2 + y^2 + z^2$

Normal to the sphere $= \nabla\psi = \hat{i}\dfrac{\partial\psi}{\partial x} + \hat{j}\dfrac{\partial\psi}{\partial y} + \hat{k}\dfrac{\partial\psi}{\partial z} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

At the point (3, 2, 1),

$$\nabla\psi = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

Directional derivative in the direction of normal to the sphere

$$= \nabla\phi \cdot \frac{\nabla\psi}{|\nabla\psi|}$$

$$= \left(324\hat{i} + 432\hat{k}\right) \cdot \frac{\left(6\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{36+16+4}}$$

$$= \frac{1404}{\sqrt{14}}.$$

Example 15: Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4). In what direction it will be maximum? Find the maximum value of it.

Solution: Position vector of the point P

$$\overline{OP} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Position vector of the point Q

$$\overline{OQ} = 5\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$= (2x)\,\hat{i} + (-2y)\,\hat{j} + (4z)\,\hat{k}$$

At the point, (1, 2, 3),

$$\nabla\phi = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Directional derivative at the point (1, 2, 3) in the direction of the line PQ

$$= \left(2\hat{i} - 4\hat{j} + 12\hat{k}\right) \cdot \frac{\left(4\hat{i} - 2\hat{j} + \hat{k}\right)}{\sqrt{16+4+1}}$$

$$= \frac{8+8+12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{7}\sqrt{3}}$$

$$= \frac{4\sqrt{7}}{\sqrt{3}}$$

Directional derivative is maximum in the direction of $\nabla\phi$ i.e. $2\hat{i} - 4\hat{j} + 12\hat{k}$
Maximum value of directional derivative

$$= |\nabla\phi| = \sqrt{4+16+144} = \sqrt{164} = 2\sqrt{41}$$

Example 16: Find the directional derivative of $\phi = 6x^2y + 24y^2z - 8z^2x$ at (1, 1, 1) in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$. Hence, find its maximum value.

Solution: $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(6x^2y + 24y^2z - 8z^2x) + \hat{j}\frac{\partial}{\partial y}(6x^2y + 24y^2z - 8z^2x) + \hat{k}\frac{\partial}{\partial z}(6x^2y + 24y^2z - 8z^2x)$$

$$= (12xy - 8z^2)\,\hat{i} + (6x^2 + 48yz)\,\hat{j} + (24y^2 - 16zx)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 4\hat{i} + 54\,\hat{j} + 8\hat{k}$$

Given line is $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$.

Direction ratios of the line are 2, –2, 1.

Direction of the line $= 2\hat{i} - 2\,\hat{j} + \hat{k}$

Directional derivative in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$ at the point (1, 1, 1)

$$= (4\hat{i} + 54\hat{j} + 8\hat{k}) = \frac{(2\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{4+4+1}} = \frac{8 - 108 + 8}{3} = \frac{-92}{3}.$$

Maximum value of directional derivative

$$= \left|4\hat{i} + 54\hat{j} + 8\hat{k}\right| = \sqrt{16 + 2916 + 64} = \sqrt{2996}.$$

Example 17: Find the values of a, b, c if the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, –1) has maximum magnitude 64 in the direction parallel to the z-axis.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) + \hat{j}\frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) + \hat{k}\frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

At the point (1, 2, –1),

$$\nabla\phi = (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \qquad \dots (1)$$

The directional derivative is maximum in the direction of $\nabla\phi$ i.e. in the direction of $(4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$. But it is given that directional derivative is maximum in the direction of z-axis i.e., in the direction of $0\,\hat{i} + 0\,\hat{j} + \hat{k}$. Therefore, $\nabla\phi$ and z-axis are parallel.

$$\frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1} = l\text{, say}$$

$$4a + 3c = 0 \qquad \dots (2)$$

$$4a - b = 0 \qquad \dots (3)$$

Substituting in Eq. (1),

$$\nabla\phi = (2b - 2c)\,\hat{k}$$

Maximum value of directional derivative is $|\nabla\phi|$. But it is given as 64.

$$|\nabla\phi| = 64$$

$$|(2b - 2c)\hat{k}| = 64$$

$$2b - 2c = 64, \qquad b - c = 32$$

From Eqs. (2) and (3),

$$4a + 3c = 0, \qquad 4a - b = 0,$$

Solving, $\qquad b = -3c$

Substituting in $b - c = 32$, $-4c = 32$,

$$c = -8,\ b = 24,\ a = 6$$

Hence, $\qquad a = 6,\ b = 24,\ c = -8.$

Example 18: For the function $\phi(x, y) = \dfrac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle $30°$ with the positive x-axis at (0, 2).

Solution: $\nabla\phi = \hat{i}\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) + \hat{j}\dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right) + \hat{k}\dfrac{\partial}{\partial z}\left(\dfrac{x}{x^2+y^2}\right)$

$$= \left[\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}\right]\hat{i} + \left[-\frac{x(2y)}{(x^2+y^2)^2}\right]\hat{j} + 0$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}\hat{i} - \frac{2xy}{(x^2+y^2)^2}\hat{j}$$

At the point (0, 2),

$$\nabla\phi = \frac{4-0}{(0+4)^2}\hat{i} - \frac{0}{(0+4)^2}\hat{j} = \frac{\hat{i}}{4}$$

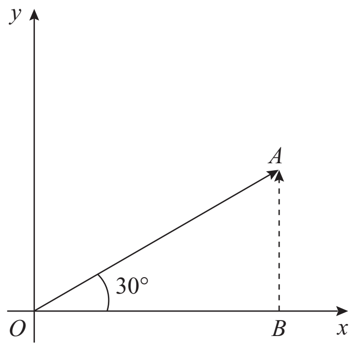


Fig. 6.3

Line OA makes an angle $30°$ with positive x-axis.

$$\overline{OA} = \overline{OB} + \overline{BA}$$

Unit vector in the direction of $\overline{OA}$

$$= \hat{i}\cos 30° + \hat{j}\sin 30°$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Directional derivative in the direction of $\dfrac{\sqrt{3}}{2}\hat{i} + \dfrac{1}{2}\hat{j}$ at (0, 2)

$$= \frac{\hat{i}}{4}\cdot\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = \frac{\sqrt{3}}{8}$$

Example 19: Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Solution: Rate of change of ϕ in the given direction is the directional derivative of ϕ in that direction.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz) = (yz)\,\hat{i} + (xz)\,\hat{j} + (xy)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = \hat{i} + \hat{j} + \hat{k}$$

Given surface is $x^2y + y^2x + yz^2 = 3$.

Let $\psi = x^2y + y^2x + yz^2$

Normal to the surface = $\nabla\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$

$$= (2xy + y^2)\,\hat{i} + (x^2 + 2xy + z^2)\,\hat{j} + (2yz)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\psi = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

Directional derivative in the direction of normal to the given surface

$$= \nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|} = (\hat{i} + \hat{j} + \hat{k})\cdot\frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Example 20: Find the direction in which temperature changes most rapidly with distance from the point (1, 1, 1) and determine the maximum rate of change if the temperature at any point is given by $\phi(x, y, z) = xy + yz + zx$.

Solution: Temperature is given by $\phi(x, y, z) = xy + yz + zx$. Temperature will change most rapidly i.e., rate of change of temperature, will be maximum in the direction of $\nabla\phi$.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy + yz + zx) + \hat{j}\frac{\partial}{\partial y}(xy + yz + zx) + \hat{k}\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= (y + z)\,\hat{i} + (x + z)\,\hat{j} + (y + x)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

This shows that temperature will change most rapidly in the direction of $2\hat{i} + 2\hat{j} + 2\hat{k}$ and maximum rate of change = maximum directional derivative

$$= |\nabla\phi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Example 21: Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - z$ at the point (2, –1, 2).

Solution: The angle between the surfaces at any point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2 + y^2 + z^2$, $\phi_2 = x^2 + y^2 - z$

Normal to ϕ_1,

$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z} = (2x)\,\hat{i} + (2y)\,\hat{j} + (2z)\,\hat{k}$$

Normal to ϕ_2, $$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z} = (2x)\,\hat{i} + (2y)\,\hat{j} - \hat{k}$$

At (2, –1, 2), $\nabla\phi_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$, $\nabla\phi_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

Let θ be the angle between the normals $\nabla\phi_1$ and $\nabla\phi_2$.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

$$(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} - \hat{k}) = |4\hat{i} - 2\hat{j} + 4\hat{k}||4\hat{i} - 2\hat{j} - \hat{k}|\cos\theta$$

$$(16 + 4 - 4) = \sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$= \sqrt{36}\sqrt{21}\cos\theta$$

$$16 = 6\sqrt{21}\cos\theta$$

$$\cos\theta = \frac{16}{6\sqrt{21}} = \frac{8\sqrt{21}}{63}$$

Hence, acute angle $$\theta = \cos^{-1}\frac{8\sqrt{21}}{63} = 54°25^1$$

Example 22: Find the angle between the normals to the surface $xy = z^2$ at $P(1, 1, 1)$ and Q (4, 1, 2).

Solution: Given surface is $xy = z^2$.

Let $\phi = xy - z^2$

Normal to ϕ, $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy - z^2) + \hat{j}\frac{\partial}{\partial y}(xy - z^2) + \hat{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\,\hat{i} + x\,\hat{j} - 2z\,\hat{k}$$

Normal at point P (1, 1, 1),

$$\overline{N_1} = \hat{i} + \hat{j} - 2\hat{k}$$

Normal at point Q (4, 1, 2),

$$\overline{N_2} = \hat{i} + 4\hat{j} - 4\hat{k}$$

Let θ be the angle between $\overline{N_1}$ and $\overline{N_2}$.

$$\overline{N_1}\cdot\overline{N_2} = |\overline{N_1}||\overline{N_2}|\cos\theta$$

$$\cos\theta = \frac{\overline{N_1}\cdot\overline{N_2}}{|\overline{N_1}||\overline{N_2}|} = \frac{(\hat{i} + \hat{j} - 2\hat{k})\cdot(\hat{i} + 4\hat{j} - 4\hat{k})}{\sqrt{1+1+4}\sqrt{1+16+16}} = \frac{1+4+8}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{198}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

Example 23: Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution: If surfaces cut orthogonally, then their normals will also cut orthogonally, i.e., angle between their normals will be 90°.

Given surfaces are $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$.
Let $\phi_1 = 5x^2 - 2yz - 9x$ and $\phi_2 = ax^2y + bz^3$

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\dfrac{\partial}{\partial x}(5x^2 - 2yz - 9x) + \hat{j}\dfrac{\partial}{\partial y}(5x^2 - 2yz - 9x) + \hat{k}\dfrac{\partial}{\partial z}(5x^2 - 2yz - 9x)$

$$= (10x - 9)\,\hat{i} + (-2z)\,\hat{j} + (-2y)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\dfrac{\partial}{\partial x}(ax^2y + bz^3) + \hat{j}\dfrac{\partial}{\partial y}(ax^2y + bz^3) + \hat{k}\dfrac{\partial}{\partial z}(ax^2y + bz^3)$

$$= (2axy)\,\hat{i} + (ax^2)\,\hat{j} + (3bz^2)\,\hat{k}$$

At the point $(1, -1, 2)$,

$$\nabla\phi_1 = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\nabla\phi_2 = -2a\hat{i} + a\hat{j} + 12b\hat{k}$$

$\nabla\phi_1$ and $\nabla\phi_2$ are orthogonal.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2|\,\cos\frac{\pi}{2}$$

$$(\hat{i} - 4\hat{j} + 2\hat{k})\cdot(-2a\hat{i} + a\hat{j} + 12b\hat{k}) = 0$$

$$-2a - 4a + 24b = 0$$

$$-6a + 24b = 0$$

$$a - 4b = 0 \quad \ldots (1)$$

The point $(1, -1, 2)$ lies on the surface $ax^2y + bz^3 = 4$.

$$a(1)^2(-1) + b(2)^3 = 4$$

$$-a + 8b = 4 \quad \ldots (2)$$

Solving Eqs. (1) and (2), we get

$$a = 4,\ b = 1$$

Example 24: Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$.

Solution: Let $\phi_1 = ax^2 + y^2 + z^2 - xy$

$$\phi_2 = bx^2y + y^2z + z$$

The point $(1, 1, 0)$ lies on both the surfaces.

$$a(1)^2 + (1)^2 + 0 - (1)(1) = 1$$

$$a = 1$$

and

$$b(1)^2 + 0 + 0 = 1$$

$$b = 1$$

Angle between the given surface is the angle between their normals.

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\,\dfrac{\partial}{\partial x}(x^2 + y^2 + z^2 - xy) + \hat{j}\,\dfrac{\partial}{\partial y}(x^2 + y^2 + z^2 - xy)$

$$+ \hat{k}\,\frac{\partial}{\partial z}(x^2 + y^2 + z^2 - xy)$$

$$= (2x - y)\,\hat{i} + (2y - x)\,\hat{j} + (2z)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\,\dfrac{\partial}{\partial x}(x^2y + y^2z + z) + \hat{j}\,\dfrac{\partial}{\partial y}(x^2y + y^2z + z) + \hat{k}\,\dfrac{\partial}{\partial z}(x^2y + y^2z + z)$

$$= (2xy)\,\hat{i} + (x^2 + 2yz)\,\hat{j} + (y^2 + 1)\,\hat{k}$$

At the point (1, 1, 0),

$$\nabla\phi_1 = \hat{i} + \hat{j} + 0\hat{k}$$

$$\nabla\phi_2 = 2\,\hat{i} + \hat{j} + 2\hat{k}$$

Let the angle between $\overline{N_1}$ and $\overline{N_2}$ is θ.

$$\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|}\cdot\frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{(\hat{i} + \hat{j})\cdot(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{1+1}\,\sqrt{4+1+4}} = \frac{2+1}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

Hence, angle between the surfaces is $\dfrac{\pi}{4}$.

Example 25: Find the constants a, b if the directional derivative of $\underline{\phi} = ay^2 + 2bxy + xz$ at $P(1, 2, -1)$ is maximum in the direction of the tangent to the curve, $r = (t^3 - 1)\,\hat{i} + (3t - 1)\,\hat{j} + (t^2 - 1)\,\hat{k}$ at point (0, 2, 0).

Solution: $\phi = ay^2 + 2bxy + xz$

$$\nabla\phi_1 = \hat{i}\,\frac{\partial}{\partial x}(ay^2 + 2bxy + xz) + \hat{j}\,\frac{\partial}{\partial y}(ay^2 + 2bxy + xz) + \hat{k}\,\frac{\partial}{\partial z}(ay^2 + 2bxy + xz)$$

$$= (2by + z)\,\hat{i} + (2ay + 2bx)\,\hat{j} + (x)\,\hat{k}$$

At the point (1, 2, –1),

$$\nabla\phi = (4b - 1)\,\hat{i} + (4a + 2b)\,\hat{j} + \hat{k}$$

Tangent to the curve $\overline{r} = (t^3 - 1)\,\hat{i} + (3t - 1)\,\hat{j} + (t^2 - 1)\,\hat{k}$ is

$$\frac{d\overline{r}}{dt} = (3t^2)\,\hat{i} + 3\,\hat{j} + (2t)\,\hat{k}$$

At the point (0, 2, 0), i.e., at $t = 1$

$$\frac{d\overline{r}}{dt} = 3\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}$$

Directional derivative is maximum in the direction of $\nabla\phi$ but it is given that directional derivative is maximum in the direction of the tangent.

Hence, $\nabla\phi$ and $\frac{d\bar{r}}{dt}$ are parallel.

$$\frac{4b-1}{3}=\frac{4a+2b}{3}=\frac{1}{2}$$

$$\frac{4b-1}{3}=\frac{1}{2} \text{ and } \frac{4a+2b}{3}=\frac{1}{2},\ 8a+4b=3$$

$$b=\frac{5}{8} \text{ and } 8a=3-4b=3-\frac{5}{2}=\frac{1}{2}$$

$$a=\frac{1}{16}$$

Hence, $$a=\frac{1}{16},\ b=\frac{5}{8}.$$

Example 26: The temperature of the points in space is given by $\phi = x^2 + y^2 - z$. A mosquito located at point (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

Solution: Temperature is given by $\phi = x^2 + y^2 - z$

Rate of change (increase) in temperature = $\nabla\phi$

$$=\hat{i}\frac{\partial}{\partial x}(x^2+y^2-z)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2-z)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$=(2x)\,\hat{i}+(2y)\,\hat{j}-\hat{k}$$

At the point (1, 1, 2),

$$\nabla\phi = 2\hat{i}+2\hat{j}-\hat{k}$$

Mosquito will get warm as soon as possible if it moves in the direction in which rate of increase in temperature is maximum, i.e., $\nabla\phi$ is maximum. Now, $\nabla\phi$ is maximum in its own direction, i.e., in the direction of $\nabla\phi$.

Unit vector in the direction of $\nabla\phi = \frac{\nabla\phi}{|\nabla\phi|}$

$$=\frac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{4+4+1}}$$

$$=\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$$

Hence, mosquito should move in the direction of $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

Example 27: Find the direction in which the directional derivative of $\phi = \dfrac{(x^2 - y^2)}{xy}$ at (1, 1) is zero.

Solution: $\phi(x, y) = \dfrac{x}{y} - \dfrac{y}{x}$,

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(\frac{x}{y}-\frac{y}{x}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x}{y}-\frac{y}{x}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{x}{y}-\frac{y}{x}\right)$$

$$= \left(\frac{1}{y}+\frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2}-\frac{1}{x}\right)\hat{j},$$

At the point (1, 1) $\nabla\phi = 2\hat{i} - 2\hat{j}$.

Let the direction in which directional derivative is zero is $\bar{r} = x\hat{i} + y\hat{j}\cdot$

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{i}}{\sqrt{x^2+y^2}} = 0$$

$$(2\hat{i} - 2\hat{j})\cdot(xi + y\hat{j}) = 0$$

$$2x - 2y = 0,\ x = y$$

$$\bar{r} = x\hat{i} + x\hat{j}$$

Unit vector in this direction $= \dfrac{x(\hat{i}+\hat{j})}{x\sqrt{1+1}} = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$

Hence, directional derivative is zero in the direction of $\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$.

Exercise 6.3

1. Find $\nabla\phi$ if
   (i) $\phi = \log(x^2 + y^2 + z^2)$
   (ii) $\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$.

   $\left[\textbf{Ans.:}\ \text{(i)}\ \dfrac{2\bar{r}}{r^2}\ \text{(ii)}\ (2-r)e^{-r}\,\bar{r}\quad \text{where } \bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\bar{r}|\right]$

2. Find $\nabla\phi$ and $|\nabla\phi|$ if
   (i) $\phi = 2xz^4 - x^2y$ at (2, –2, –1)
   (ii) $\phi = 2xz^2 - 3xy - 4x$ at (1, –1, 2).

   $\left[\textbf{Ans.:}\ \text{(i)}\ 10\hat{i} - 4\hat{j} - 16\hat{k},\ 2\sqrt{93}\quad \text{(ii)}\ 7\hat{i} - 3\hat{j} + 8\hat{k},\ 2\sqrt{29}\right]$

3. If $\bar{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$ find
   (i) $\bar{A}\cdot\nabla\phi$
   (ii) $\bar{A}\times\nabla\phi$ at (1, –1, 1).

   [**Ans. :** (i) 5 (ii) $7\hat{i} - \hat{j} - 11\hat{k}$]

4. If $\phi = 3x^2y$, $\psi = xz^2 - 2y$, find $\nabla(\nabla\phi\cdot\nabla\psi)$.

   $\left[\textbf{Ans.:}\ (6yz^2 - 12x)\,\hat{i} + 6xz^2\,\hat{j} + 12xyz\hat{k}\right]$

5. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$, prove that

(i) $\nabla (\log r) = \dfrac{\bar{r}}{r^2}$

(ii) $\nabla |\bar{r}|^3 = 3r\,\bar{r}$

(iii) $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$.

6. Prove that $\nabla\left(\dfrac{\bar{a}\cdot\bar{r}}{r^n}\right) = \dfrac{\bar{a}}{r^n} - \dfrac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$,

where $\bar{a}$ is a constant vector.

7. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point (2, –2, 3).

$\left[\textbf{Ans. :} \dfrac{1}{3}(\hat{i} - 2\hat{j} - 2\hat{k})\right]$

8. Find unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point (3, 1, –4).

$\left[\textbf{Ans. :} \dfrac{(2\hat{i} + \hat{j} - 2\hat{k})}{3}\right]$

9. Find a unit vector normal to the surface $xy^3 z^2 = 4$ at the point (–1, –1, 2).

$\left[\textbf{Ans. :} \dfrac{\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{11}}\right]$

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, –2, –1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

$\left[\textbf{Ans.:} \dfrac{37}{\sqrt{3}}\right]$

11. Find the directional derivative of $\phi = xy + yz + zx$ at (1, 2, 0) in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

$\left[\textbf{Ans. :} \dfrac{10}{3}\right]$

12. Find the maximal directional derivative of x^3y^2z at (1, –2, 3).

$\left[\textbf{Ans.:} 4\sqrt{91}\right]$

13. In what direction from the point (2, 1, –1) is the directional derivative of $\phi = x^2yz^3$ a maximum? Find its maximum value of magnitude.

$\left[\textbf{Ans.:} \text{ maximum in the direction of } \nabla\phi = 4\hat{i} - 4\hat{j} + 12\hat{k}, 4\sqrt{11}\right]$

14. In what direction from the point (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find its maximum value of magnitude.

$\left[\textbf{Ans.:} 96(\hat{i} + 3\hat{j} - 3\hat{k}), 96\sqrt{19}\right]$

15. In what direction from the point (1, 3, 2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? Find its maximum value of magnitude.

$\left[\textbf{Ans. :} 4\hat{i} - 6\hat{j} + 2\hat{k}, 2\sqrt{14}\right]$

16. What is the greatest rate of change of $\phi = xyz^2$ at the point (1, 0, 3)?

[**Ans. :** $\nabla\phi = 9$]

17. If the directional derivative of $\phi = ax^2 + by + 2z$ at (1, 1, 1) is maximum in the direction of $\hat{i} + \hat{j} + \hat{k}$, then find values of a and b.

[**Ans. :** $a = 1, b = 2$]

18. If the directional derivative of $\phi = ax + by + cz$ at (1, 1, 1) has maximum magnitude 4 in a direction parallel to x axis, then find values of a, b, c.

[**Ans. :** $a = 2, b = -2, c = 2$]

19. Find the directional derivative of $\phi = x^2y + y^2z + z^2x^2$ at (1, 2, 1) in the direction of the normal to the surface $x^2 + y^2 - z^2x = 1$ at (1, 1, 1).

$\left[\textbf{Ans. :} \dfrac{4}{3}\right]$

20. Find the directional derivative of $\phi = x^2y + yz^2$ at (2, –1, 1) in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at (1, 1, 1).

$\left[\textbf{Ans. :} \dfrac{-13}{\sqrt{29}}\right]$

21. Find the directional derivative of $\phi = x^2y + y^2z + z^2x$ at (2, 2, 2) in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point (2, –1, 3).

$\left[\textbf{Ans. :} \dfrac{36}{\sqrt{41}}\right]$

22. Find the rate of change of $\phi = xy + yz + zx$ at (1, –1, 2) in the direction of the normal to the surface $x^2 + y^2 = z + 4$.

$$\left[\textbf{Ans.: } \frac{14}{\sqrt{21}}\right]$$

23. Find the directional derivative of $\phi = x^2yz^2$ along the curve $x = e^{-t}$, $y = 2 \sin t + 1$, $z = t - \cos t$ at $t = 0$.

$$\left[\textbf{Ans.: } -\frac{1}{\sqrt{6}}\right]$$

24. Find the directional derivative of $\phi = x^2y^2z^2$ at (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = 2 \sin t + 1$, $z = t - \cos t$, at $t = 0$.

$$\left[\textbf{Ans.: } \frac{2\sqrt{3}}{3}\right]$$

25. Find the directional derivative of the scalar function $\phi = x^2 + xy + z^2$ at the point $P(1, -1, -1)$ in the direction of the line PQ where Q has coordinates (3, 2, 1).

$$\left[\begin{aligned}&\textbf{Hint: } \overline{PQ} = \overline{OQ} - \overline{OP}\\ &= (3\hat{i} + 2\hat{j} + \hat{k}) - (-\hat{i} - \hat{j} - \hat{k})\\ &= 2\hat{i} + 3\hat{j} + 2\hat{k}\end{aligned}\right]$$

$$\left[\textbf{Ans.: } \frac{1}{\sqrt{17}}\right]$$

26. Find the directional derivative of $\phi = 2x^3y - 3y^2z$ at the point P (1, 2, –1) in the direction towards Q (3, –1, 5). In what direction from P is the directional derivative maximum? Find the magnitude of maximum directional derivative.

$$\left[\textbf{Ans.: } -\frac{90}{7}, 12\hat{i} + 14\hat{j} - 12\hat{k}, 22\right]$$

27. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2, –1, 2) in the direction from this point towards the point (4, –4, 8).

$$\left[\textbf{Ans.: } \frac{376}{7}\right]$$

28. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 47$ at (4, –3, 2).

$$\left[\textbf{Ans.: } \cos^{-1}\left(\frac{19}{29}\right)\right]$$

29. Find the angle between the normals to the surface $2x^2 + 3y^2 = 5z$ at the point (2, –2, 4) and (–1, –1, 1).

$$\left[\textbf{Ans.: } \cos^{-1}\frac{65}{\sqrt{233}\sqrt{77}}\right]$$

30. Find the angle between the normals to the surface $xy = z^2$ at the points (1, 4, 2) and (–3, –3, 3).

$$\left[\textbf{Ans.: } \theta = \cos^{-1}\frac{1}{\sqrt{22}}\right]$$

31. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1).

$$\left[\textbf{Ans.: } \cos^{-1}\frac{\sqrt{6}}{14}\right]$$

32. Find the constant a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).

$$\left[\begin{aligned}&\textbf{Hint: } \text{condition for orthogonal-}\\ &\quad\text{ity is } \nabla\phi \cdot \nabla\psi = 0\end{aligned}\right].$$

$$\left[\textbf{Ans.: } a = \frac{5}{2}, b = 1\right]$$

33. Find the angle between the two surfaces $x^2 + y^2 + a\,z^2 = 6$ and $z = 4 - y^2 + bxy$ at P (1, 1, 2).

$$\left[\begin{aligned}&\textbf{Hint: } (1, 1, 2) \text{ lies on both}\\ &\quad\text{surfaces, } a = 1, b = -1\end{aligned}\right]$$

$$\left[\textbf{Ans.: } \cos^{-1}\frac{\sqrt{6}}{11}\right]$$

34. Find the directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3).

$$\left[\textbf{Ans. } \frac{26}{5}\sqrt{2}\right]$$

35. Find the direction in which the directional derivative of $\phi = (x + y) = \frac{(x^2 - y^2)}{xy}$ at (1, 1) is zero.

$$\left[\begin{array}{l}\textbf{Hint: } \phi(x, y) = \frac{x}{y} - \frac{y}{x}, \\ \nabla\phi = \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j}, \\ \text{At}(1, 1), \nabla\phi = 2\hat{i} - 2\hat{j}\end{array}\right.$$

Let the direction in which directional derivative is zero is $\bar{r} = x\hat{i} + y\hat{j}$

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = 0$$

$$(2\hat{i} - 2\hat{j}) \cdot (x\hat{i} + y\hat{j}) = 0$$

$$2x - 2y = 0, x = y$$

$$\bar{r} = x\hat{i} + x\hat{j}$$

unit vector in this direction

$$= \frac{x(\hat{i} + \hat{j})}{x\sqrt{1+1}}$$

$$= \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

Hence, directional derivative is zero in the direction of $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$.

6.8 DIVERGENCE

The divergence of a vector point function $\overline{F}$ is denoted by div $\overline{F}$ or $\nabla \cdot \overline{F}$ and is defined as

$$\nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \overline{F}$$

If
$$\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$$

then
$$\nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar quantity.

Note:

(i) $\nabla \cdot \overline{F} \neq \overline{F} \cdot \nabla$, because $\nabla \cdot \overline{F}$ is a scalar quantity whereas

$\overline{F} \cdot \nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar differential operator.

(ii) $\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= \hat{i} \cdot \frac{\partial \overline{F}}{\partial x} + \hat{j} \cdot \frac{\partial \overline{F}}{\partial y} + \hat{k} \cdot \frac{\partial \overline{F}}{\partial z} \qquad (\text{if } \overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

6.8.1 *Physical Interpretation of Divergence*

Consider the case of a homogeneous and incompressible fluid flow. Consider a small rectangular parallelepiped of dimensions δx, δy, δz parallel to x, y and z axes respectively.

Let $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be the velocity of the fluid at point A (x, y, z).

The velocity component parallel to x – axis (normal to the face $PQRS$) at any point of the face $PQRS$

$$= v_1 (x + \delta x, y, z)$$

$$= v_1 + \frac{\partial v_1}{\partial x}\delta x$$ [expanding by Taylor's series and ignoring higher powers of δx]

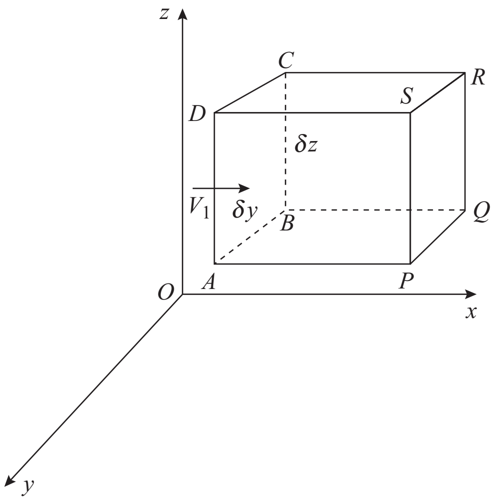


Fig. 6.4

Mass of the fluid flowing in across the face $ABCD$ per unit time

= velocity component normal to the face $ABCD$ × area of the face $ABCD$

$$= v_1 (\delta y \, \delta z)$$

Mass of the fluid flowing out across the face $PQRS$ per unit time

= velocity component normal to the face $PQRS$ × area of the face $PQRS$

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y \, \delta z$$

Gain of fluid in the parallelepiped per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y \, \delta z - v_1 \, \delta y \, \delta z$$

$$= \frac{\partial v_1}{\partial x}\delta x \, \delta y \, \delta z$$

Similarly, gain of fluid in the parallelepiped per unit time in the direction of y-axis

$$= \frac{\partial v_2}{\partial y} \delta x \delta y \delta z$$

and gain of fluid in the parallelepiped per unit time in the direction of z-axis

$$= \frac{\partial v_3}{\partial z} \delta x \delta y \delta z$$

Total gain of fluid in the parallelepiped per unit time

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x \, \delta y \, \delta z$$

But, $\delta x \; \delta y \; \delta z$ is the volume of the parallelepiped.

Hence, total gain of fluid per unit volume $= \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$

$$= \text{div } \bar{v}$$
$$= \nabla \cdot \bar{v}$$

Note: A point in a vector field $\bar{F}$ is said to be a **source** if div $\bar{F}$ is positive, i.e., $\nabla \cdot \bar{F} > 0$ and is said to be a **sink** if div $\bar{F}$ is negative, i.e, $\nabla \cdot \bar{F} < 0$.

6.8.2 *Solenoidal Function*

A vector function $\bar{F}$ is said to be **solenoidal** if div $\bar{F} = 0$ at all points of the function. For such a vector, there is no loss or gain of fluid.

6.9 CURL

The curl of a vector point function $\bar{F}$ is denoted by curl $\bar{F}$ or $\nabla \times \bar{F}$ and is defined as

$$\nabla \times \bar{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

which is a vector quantity.

6.9.1 *Physical Interpretation of Curl*

Let $\bar{\omega}$ be the angular velocity of a rigid body moving about a fixed point. The linear velocity $\bar{v}$ of any particle of the body with position vector $\bar{r}$ w.r.t. to the fixed point is given by,

$$\bar{v} = \bar{\omega} \times \bar{r}$$

Let $\bar{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k},\ \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{v} = \bar{\omega} \times \bar{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(\omega_2 z - \omega_3 y) - \hat{j}\,(\omega_1 z - \omega_3 x) + \hat{k}\,(\omega_1 y - \omega_2 x)$$

Curl $\bar{v} = \nabla \times \bar{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) - \hat{j}\,(-\omega_2 - \omega_2) + \hat{k}\,(\omega_3 + \omega_3)$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$$= 2\bar{\omega}$$

Curl $\bar{v} = 2\bar{\omega}$

Thus, the curl of the linear velocity of any particle of a rigid body is equal to twice the angular velocity of the body.

This shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

6.9.2 *Irrotational Field*

A vector point function $\bar{F}$ is said to be **irrotational**, if curl $\bar{F} = 0$ at all points of the function, otherwise it is said to be rotational.

Note: If $\bar{F} = \nabla\phi$, then curl $\bar{F} = \nabla \times \bar{F} = \nabla \times \nabla\phi = 0$.

Thus, if $\nabla \times \bar{F} = 0$, then we can find a scalar function ϕ so that $\bar{F} = \nabla\phi$. A vector field $\bar{F}$ which can be derived from a scalar field ϕ so that $\bar{F} = \nabla\phi$ is called a **conservative vector field** and ϕ is called the **scalar potential.**

Example 1: If $\overline{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla\cdot\overline{A}$ at the point (1, –1, 1).

Solution: $\nabla\cdot\overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$, where $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla\cdot\overline{A} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

At the point (1, –1, 1),

$$\nabla\cdot\overline{A} = 2(1)(1) - 6(-1)^2(1)^2 + 1(-1)^2$$

$$= 2 - 6 + 1$$

$$= -3$$

Example 2: If $\overline{\boldsymbol{r}} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that div (grad r^n) = $n(n+1)r^{n-2}$.

Solution:

$$\text{grad}\, r^n = \hat{i}\,\frac{\partial r^n}{\partial x} + \hat{j}\,\frac{\partial r^n}{\partial y} + \hat{k}\,\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

But $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$r^2 = |\overline{r}|^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad}\, r^n = nr^{n-1}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= nr^{n-1}\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{r}$$

$$= nr^{n-2}\,\overline{r}$$

$$\text{div}(\text{grad}\, r^n) = \nabla\cdot(nr^{n-2}\,\overline{r})$$

$$= n\,\nabla\cdot r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n\left[\frac{\partial}{\partial x}(r^{n-2}x) + \frac{\partial}{\partial y}(r^{n-2}y) + \frac{\partial}{\partial z}(r^{n-2}z)\right]$$

$$= n\left(x\frac{\partial}{\partial x}r^{n-2} + r^{n-2} + y\frac{\partial}{\partial y}r^{n-2} + r^{n-2} + z\frac{\partial}{\partial z}r^{n-2} + r^{n-2}\right)$$

$$= n\left[3r^{n-2} + x(n-2)r^{n-3}\frac{\partial r}{\partial x} + y(n-2)r^{n-3}\frac{\partial r}{\partial y} + z(n-2)r^{n-3}\frac{\partial r}{\partial z}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{(x^2+y^2+z^2)}{r}\right] \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{r^2}{r}\right]$$

$$= nr^{n-2}(3 + n - 2)$$

$$= n\,(n+1)\,r^{n-2}$$

Example 3: Prove that for vector function $\overline{A}$, $\nabla \times (\nabla \times \overline{A}) = \nabla\,(\nabla \cdot \overline{A}) - \nabla^2\,\overline{A}$.

Solution: Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \times \left(\nabla \times \overline{A}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x \partial z}\right)\right]$$

$$- \hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_1}{\partial x \partial y}\right) - \left(\frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_2}{\partial z^2}\right)\right]$$

$$+ \hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2}\right) - \left(\frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_2}{\partial y \partial z}\right)\right]$$

Consider $\hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x \partial z}\right)\right]$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}\right) - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)+\left(\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial x^2}\right)\right]$$

$$\left[\text{Adding and subtracting } \frac{\partial^2 A_1}{\partial x^2}\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \hat{i}\,\frac{\partial}{\partial x}(\nabla\cdot\overline{A})-\hat{i}\;\nabla^2 A_1$$

Similarly,

$$-\hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_1}{\partial x\partial y}\right)-\left(\frac{\partial^2 A_3}{\partial y\partial z}-\frac{\partial^2 A_2}{\partial z^2}\right)\right]=\hat{j}\,\frac{\partial}{\partial y}(\nabla\cdot\overline{A})-\hat{j}\,\nabla^2 A_2$$

$$\text{and } \hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x\partial z}-\frac{\partial^2 A_3}{\partial x^2}\right)-\left(\frac{\partial^2 A_3}{\partial y^2}-\frac{\partial^2 A_2}{\partial y\partial z}\right)\right]=\hat{k}\,\frac{\partial}{\partial z}(\nabla\cdot\overline{A})-\hat{k}\,\nabla^2 A_3$$

Hence, $\nabla\times(\nabla\times\overline{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\nabla\cdot\overline{A}) - \nabla^2(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$

$$= \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$$

Example 4: If $\overline{\boldsymbol{A}} = \nabla\,(xy+yz+zx)$, find $\nabla\cdot\overline{\boldsymbol{A}}$ and $\nabla\times\overline{\boldsymbol{A}}$.

Solution: $\overline{A} = \nabla\,(xy+yz+zx)$

$$= \hat{i}\,\frac{\partial}{\partial x}(xy+yz+zx)+\hat{j}\,\frac{\partial}{\partial y}(xy+yz+zx)+\hat{k}\,\frac{\partial}{\partial z}(xy+yz+zx)$$

$$= (y+z)\,\hat{i}+(x+z)\,\hat{j}+(y+x)\,\hat{k}$$

$$\nabla\cdot\overline{A} = \nabla\cdot[(y+z)\,\hat{i}+(z+x)\,\hat{j}+(x+y)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial y}(z+x)+\frac{\partial}{\partial z}(x+y)$$

$$= 0$$

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y)-\frac{\partial}{\partial z}(z+x)\right]-\hat{j}\left[\frac{\partial}{\partial x}(x+y)-\frac{\partial}{\partial z}(y+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(z+x)-\frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}\ (1-1)-\hat{j}\ (1-1)+\hat{k}\,(1-1)$$

$$= 0$$

Example 5: Verify $\nabla\left(\nabla\times\overline{A}\right)=\nabla\left(\nabla\cdot\overline{A}\right)-\nabla^2\overline{A}$ for $\overline{A}=x^2y\hat{i}+x^3y^2\hat{j}-3\,x^2z^2\hat{k}$.

Solution:

$$\nabla\times\overline{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & x^3y^2 & -3x^2z^2\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(-3x^2z^2)-\frac{\partial}{\partial z}(x^3y^2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-3x^2z^2)-\frac{\partial}{\partial z}(x^2y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^3y^2)-\frac{\partial}{\partial y}(x^2y)\right]$$

$$= 0\cdot\hat{i}-(-6xz^2)\hat{j}+(3x^2y^2-x^2)\hat{k}$$

$$\nabla\times\left(\nabla\times\overline{A}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 6xz^2 & (3x^2y^2-x^2)\end{vmatrix}$$

$$=\hat{i}\ (6x^2y-12xz)-\hat{j}\ (6xy^2-2x-0)+\hat{k}\ (6z^2-0)$$

$$=(6x^2y-12xz)\ \hat{i}-(6xy^2-2x)\ \hat{j}+(6z^2)\ \hat{k}$$

$$\nabla\cdot\overline{A}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2y\hat{i}+x^3y^2\hat{j}-3x^2z^2\hat{k})$$

$$=\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(x^3y^2)+\frac{\partial}{\partial z}(-3x^2z^2)$$

$$=2xy+2\,x^3y-6x^2z$$

$$\nabla\left(\nabla\cdot\overline{A}\right)=\hat{i}\,\frac{\partial}{\partial x}\ (2xy+2x^3y-6x^2z)+\hat{j}\,\frac{\partial}{\partial y}\ (2xy+2x^3y-6x^2z)$$

$$+\hat{k}\,\frac{\partial}{\partial z}\,(2xy+2x^3y-6x^2z)$$

$$= (2y + 6x^2y - 12xz)\,\hat{i} + (2x + 2x^3 - 0)\,\hat{j} + (-6x^2)\,\hat{k}$$

$$\nabla^2 \overline{A} = \frac{\partial^2}{\partial x^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k}) + \frac{\partial^2}{\partial y^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k}) + \frac{\partial^2}{\partial z^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(2xy\hat{i} + 3x^2y^2\hat{j} - 6xz^2\hat{k}) + \frac{\partial}{\partial y}(x^2\hat{i} + 2x^3y\,\hat{j}) + \frac{\partial}{\partial z}(-6x^2z\,\hat{k})$$

$$= (2y\hat{i} + 6xy^2\hat{j} - 6z^2\hat{k}) + 2x^3\hat{j} - 6x^2\hat{k}$$

$$= 2y\hat{i} + (6xy^2 + 2x^3)\hat{j} - 6\,(z^2 + x^2)\hat{k}$$

$$\nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2 \overline{A} = (6x^2y - 12xz)\,\hat{i} + (2x - 6xy^2)\,\hat{j} + (6z^2)\,\hat{k}$$

Hence, $\nabla \times \left(\nabla \times \overline{A}\right) = \nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2 \overline{A}$

Example 6: Show that $\overline{A} = 3\,y^4z^2\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution: $\overline{A} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$

$$\nabla \cdot \overline{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) = 0$$

Since $\nabla \cdot \overline{A} = 0$, $\overline{A}$ is solenoidal.

Example 7: Determine the constant b such that $\overline{A} = (bx + 4y^2z)\,\hat{i} + (x^3\sin z - 3y)\,\hat{j} - (e^x + 4\cos x^2y)\,\hat{k}$ is solenoidal.

Solution: If $\overline{A}$ is solenoidal, then

$$\nabla \cdot \overline{A} = 0$$

$$\frac{\partial}{\partial x}(bx + 4y^2z) + \frac{\partial}{\partial y}(x^3\sin z - 3y) + \frac{\partial}{\partial z}(-e^x - 4\cos x^2y) = 0.$$

$$b - 3 = 0$$

$$b = 3$$

Example 8: Show that the vector field $\overline{A} = \dfrac{a(x\hat{i} + y\hat{j})}{\sqrt{x^2 + y^2}}$ is a source field or sink field according as $a > 0$ or $a < 0$.

Solution: Vector field $\overline{A}$ is a source field if $\nabla \cdot \overline{A} > 0$ and vector field $\overline{A}$ is a sink field if $\nabla \cdot \overline{A} < 0$.

$$\nabla\cdot\overline{A}=\nabla\cdot\left(\frac{ax}{\sqrt{x^2+y^2}}\hat{i}+\frac{ay}{\sqrt{x^2+y^2}}\hat{j}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{ax}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{ay}{\sqrt{x^2+y^2}}\right)$$

$$=a\left[\frac{1}{\sqrt{x^2+y^2}}-\frac{x\cdot 2x}{2(x^2+y^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{x^2+y^2}}-\frac{y\cdot 2y}{2(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=a\left[\frac{2}{\sqrt{x^2+y^2}}-\frac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=\frac{a}{\sqrt{x^2+y^2}}$$

Since $\sqrt{x^2+y^2}$ is always positive, $\nabla\cdot\overline{A}>0$ if $a>0$, and $\nabla\cdot\overline{A}<0$ if $a<0$. Hence, $\overline{A}$ is a source field if $a>0$ and sink field if $a<0$.

Example 9: If $\overline{A}=(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}$ is solenoidal, find the constant a.

Solution: If $\overline{A}$ is solenoidal, then $\nabla\cdot\overline{A}=0$,

$$\nabla\cdot[(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}]=0$$

$$\frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)+\frac{\partial}{\partial z}(2xyz-2x^2y^2)=0$$

$$2axy+2xy+2xy=0$$

$$2a=-4$$

$$a=-2$$

Example 10: Find the curl of $\overline{\boldsymbol{A}}=e^{xyz}(\hat{i}+\hat{j}+\hat{k})$ at the point (1, 2, 3).

Solution: Curl of $\overline{A}=\nabla\times\overline{A}$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & e^{xyz} & e^{xyz}\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial}{\partial y}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)-\hat{j}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)+\hat{k}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial y}e^{xyz}\right)$$

$$=(e^{xyz}\cdot xz-e^{xyz}\cdot xy)\,\hat{i}-(e^{xyz}\cdot yz-e^{xyz}\cdot xy)\,\hat{j}+(e^{xyz}\cdot yz-e^{xyz}\cdot xz)\,\hat{k}$$

At the point (1, 2, 3),

$$\text{Curl } \overline{A} = e^6 [\hat{i}\,(3-2) - \hat{j}\,(6-2) + \hat{k}\,(6-3)]$$

$$= e^6 (\hat{i} - 4\hat{j} + 3\hat{k})$$

Example 11: Find curl $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\hat{k}$ at the point (1, 0, 2).

Solution: Curl $\overline{A} = \nabla \times \overline{A}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= (2z + 2x)\,\hat{i} - (0 - 0)\,\hat{j} + (-2z - x^2)\,\hat{k}$$

$$\text{Curl}\left(\text{Curl } \overline{A}\right) = \nabla \times (\nabla \times \overline{A})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(z+x) & 0 & -(x^2+2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2 - 2z) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}2(z+x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}2(z+x)\right]$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(-2x - 2) + \hat{k}\,(0 - 0)$$

$$= (2x + 2)\,\hat{j}$$

At the point (1, 0, 2),

$$\text{Curl}\left(\text{Curl } \overline{A}\right) = (2 + 2)\,\hat{j}$$

$$= 4\,\hat{j}$$

Example 12: Prove that $\overline{F} = 2xyz^2\hat{i} + [x^2z^2 + z\cos(yz)]\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}$ is a conservative vector field.

Solution: Vector field $\overline{F}$ is conservative if $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + zy\sin yz)\,\hat{i}$$

$$- (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$$

$$= 0$$

Hence, $\overline{F}$ is conservative vector field.

Example 13: Determine the constants a and b such that curl of $(2xy + 3yz)\,\hat{i} + (x^2 + axz - 4\,z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$ is zero.

Solution: Let $\overline{F} = (2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$

Curl $\overline{F} = \nabla \times \overline{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & 3xy + 2byz \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(3xy + 2byz) - \frac{\partial}{\partial z}(x^2 + axz - 4z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xy + 2byz) - \frac{\partial}{\partial z}(2xy + 3yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2 + axz - 4z^2) - \frac{\partial}{\partial y}(2xy + 3yz)\right] = 0$$

$$(3x + 2bz - ax + 8z)\,\hat{i} - (3y - 3y)\,\hat{j} + (2x + az - 2x - 3z)\,\hat{k} = 0$$

$$[(3 - a)x + 2z\,(b + 4)]\,\hat{i} - 0\,\hat{j} + z\,(a - 3)\,\hat{k} = 0$$

Comparing coefficients of $\hat{i}$ and $\hat{k}$,

$$(3 - a)\,x + 2\,(b + 4)z = 0$$
$$(a - 3)z = 0$$

Solving both the equations $\quad a = 3,\ b = -4$

Example 14: Show that $\overline{F} = (y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational .

Solution: If $\overline{F}$ is solenoidal, $\nabla \cdot \overline{F} = 0$

$$\begin{aligned}\nabla \cdot \overline{F} &= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)\\ &= -2 + 2x - 2x + 2\\ &= 0\end{aligned}$$

Hence, $\overline{F}$ is solenoidal.

If $\overline{F}$ is irrotational, $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$\begin{aligned}&= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]\\ &\quad - \hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]\\ &\quad + \hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]\\ &= (3x - 3x)\,\hat{i} - (3y - 2z + 2z - 3y)\,\hat{j} + (3z + 2y - 2y - 3z)\,\hat{k}\\ &= 0\end{aligned}$$

Hence, $\overline{F}$ is irrotational.

Example 15: Find the directional derivative of the divergence of $\overline{F}(x, y, z) = xy\,\hat{i} + xy^2\hat{j} + z^2\hat{k}$ at the point (2, 1, 2) in the direction of the outer normal to the sphere $x^2 + y^2 + z^2 = 9$.

Solution: $\overline{F}\,(x, y, z) = xy\,\hat{i} + xy^2\,\hat{j} + z^2\,\hat{k}$

Divergence of $\overline{F}\,(x, y, z) = \nabla \cdot \overline{F}$

$$\begin{aligned}&= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(z^2)\\ &= y + 2xy + 2z\end{aligned}$$

Gradient of divergence of $\overline{F} = \nabla(\nabla \cdot \overline{F})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y + 2xy + 2z)$$

$$= 2y\,\hat{i} + (1 + 2x)\,\hat{j} + 2\hat{k}$$

At the point (2, 1, 2),

$$\nabla(\nabla \cdot \overline{F}) = 2\,\hat{i} + 5\,\hat{j} + 2\hat{k}$$

Normal to sphere $= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$

$$= 2\,(x\hat{i} + y\hat{j} + z\hat{k})$$

Normal at (2, 1, 2) $= 2\,(2\hat{i} + \hat{j} + 2\hat{k})$

Directional derivative in the direction of the outer normal to the sphere $x^2 + y^2 + z^2 = 9$

$$= (2\hat{i} + 5\,\hat{j} + 2\hat{k}) \cdot \frac{4\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{16 + 4 + 16}}$$

$$= \frac{1}{6}(8 + 10 + 8)$$

$$= \frac{13}{3}$$

Exercise 6.4

1. Find divergence and curl of $x^2 \cos z\,\hat{i} + y \log x\,\hat{j} - yz\,\hat{k}$.

   $\left[\textbf{Ans. :}\ 2x\cos z + \log x - y,\ \hat{i}z - \hat{j}x^2 \sin z + \hat{k}\frac{y}{x}\right]$

2. If $\phi = 2x^3y^2z^4$, prove that div (grad ϕ) $= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$.

3. Find curl (curl $\overline{A}$), if $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$.

   [**Ans. :** $(2x + 2)\,\hat{j}$]

4. If $\overline{A} = 2yz\hat{i} - x^2y\hat{j} + x\,z^2\hat{k}$, $\overline{B} = x^2\hat{i} + yz\hat{j} - xy\hat{k}$ and $\phi = 2x^2yz^3$, find

   (i) $(\overline{A} \cdot \nabla)\,\phi$ (ii) $\overline{A} \cdot \nabla\,\phi$

   (iii) $(\overline{B} \cdot \nabla)\,\overline{A}$ (iv) $(\overline{A} \times \nabla)\,\phi$

   (v) $\overline{A} \times \nabla\phi$

   [**Ans. :** (i) and (ii) $8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$ (iii) $(2yz^2 - 2xy^2)\,\hat{i} - (2x^3y + x^2yz)\,\hat{j} + (x^2z^2 - 2x^2yz)\hat{k}$ (iv) and (v) $-(6x^4y^2z^2 + 2x^3z^5)\,\hat{i} + (4x^2yz^5 - 12x^2y^2z^3)\,\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$]

5. If $\overline{A} = x^2\,\hat{i} + xye^x\,\hat{j} + \sin z\,\hat{k}$, find $\nabla \cdot (\nabla \times \overline{A})$.

   [**Ans. :** 0]

6. If $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, find div (grad ϕ).

   [**Ans. :** 0]

7. If $\phi = 2x^2 - 3y^2 + 4z^2$, find curl (grad ϕ).

   [**Ans. :** 0]

8. Prove that for every field $\overline{A}$, div (curl $\overline{A}$) = 0.

9. Prove that gradient field describing a motion is irrotational.

[Hint: Prove that $\nabla \times \nabla\phi = 0$]

10. Prove that $\overline{A} = \hat{i}\, yz + \hat{j}\, xz + \hat{k}\, xy$ is irrotational and find a scalar function $\phi(x, y, z)$ such that $\overline{A}$ = grad ϕ.

[**Ans. :** $xyz + c$]

11. Prove that $\overline{A} = (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotational. Find the function ϕ such that $\overline{A} = \nabla\phi$.

[**Ans. :** $\phi = 3x^2y + xz^3 - yz$]

12. Prove that the velocity given by $\overline{A} = (y + z)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}$ is irrotational and find its scalar potential. Is the motion possible for an incompressible fluid?

[**Ans. :** $\phi = yz + zx + xy$, motion is possible because $\nabla \cdot \overline{A} = 0$]

13. Prove that $\overline{A} = (z^2 + 2xy + 3y)\,\hat{i} + (3x + 2y + z)\,\hat{j} + (y + 2zx)\,\hat{k}$ is irrotational and find scalar potential ϕ such that $\overline{A} = \nabla\phi$ and $\phi(1, 1, 0) = 4$.

[**Ans. :** $\phi = z^2x + x^2 + 3xy + y^2 + yz - 1$]

14. Prove that $\overline{A} = (z^2 + 2x + 3y)\,\hat{i} + (3x + 2y + z)\,\hat{j} + (y + 2zx)\,\hat{k}$ is conservative and find scalar potential ϕ such that $\overline{A} = \nabla\phi$.

[**Ans. :** $\phi = x^2 + y^2 + z^2x + 3xy + zy$]

15. Prove that $\overline{A} = (y^2\cos x + z^3)\,\hat{i} + (2y\sin x - 4)\,\hat{j} + (3xz^2 + 2)\,\hat{k}$ is irrotational and find its scalar potential.

[**Ans. :** $\phi = y^2 \sin x + z^3x - 4y + 2z$]

16. Prove that $a = -1$ or $b = 0$, if $(xyz)^b\,(x^a\hat{i} + y^a\hat{j} + z^a\hat{k})$ is an irrotational vector.

17. Find the constant a if $\overline{A} = (ax + 3y + 4z)\,\hat{i} + (x - 2y + 3z)\,\hat{j} + (3x + 2y - z)\,\hat{k}$ is solenoidal.

[**Ans. :** $a = 3$]

18. Find the constant a if $\overline{A} = (x + 3y^2)\,\hat{i} + (2y + 2z^2)\,\hat{j} + (x^2 + az)\,\hat{k}$ is solenoidal.

[**Ans. :** $a = -3$]

19. Find the constants a, b, c if $\overline{A} = (axy + bz^2)\,\hat{i} + (3x^2 - cz)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotational.

[**Ans. :** $a = 6, b = 1, c = 1$]

20. Find the directional derivative of $\nabla \cdot (\nabla f)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

6.10 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

6.10.1 *Sum and Difference*

The gradient, divergence and curl are distributive with respect to the sum and difference of the functions. If f, g are scalars and $\overline{A}$ and $\overline{B}$ are vectors, then

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla \cdot (\overline{A} \pm \overline{B}) = (\nabla \cdot \overline{A}) \pm (\nabla \cdot \overline{B})$

(iii) $\nabla \times (\overline{A} \pm \overline{B}) = (\nabla \times \overline{A}) \pm (\nabla \times \overline{B})$

Proof: (i) $\nabla(f \pm g) = \hat{i}\frac{\partial}{\partial x}(f \pm g) + \hat{j}\frac{\partial}{\partial y}(f \pm g) + \hat{k}\frac{\partial}{\partial z}(f \pm g)$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \pm \left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right)$$
$$= \nabla f \pm \nabla g$$

(ii) Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k},\ \overline{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$$\nabla\cdot(\overline{A} \pm \overline{B}) = \nabla\cdot\left[(A_1 \pm B_1)\hat{i} + (A_2 \pm B_2)\hat{j} + (A_3 \pm B_3)\hat{k}\right]$$
$$= \frac{\partial}{\partial x}(A_1 \pm B_1) + \frac{\partial}{\partial y}(A_2 \pm B_2) + \frac{\partial}{\partial z}(A_3 \pm B_3)$$
$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \pm \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$
$$= \nabla\cdot\overline{A} \pm \nabla\cdot\overline{B}$$

(iii) $\nabla\times(\overline{A} \pm \overline{B}) = \nabla\times(\overline{A} \pm \overline{B})$

$$= \hat{i}\times\frac{\partial}{\partial x}(\overline{A} \pm \overline{B}) + \hat{j}\times\frac{\partial}{\partial y}(\overline{A} \pm \overline{B}) + \hat{k}\times\frac{\partial}{\partial z}(\overline{A} \pm \overline{B})$$
$$= \sum \hat{i}\times\frac{\partial}{\partial x}(\overline{A} \pm \overline{B})$$
$$= \sum \hat{i}\times\left(\frac{\partial\overline{A}}{\partial x} \pm \frac{\partial\overline{B}}{\partial x}\right)$$
$$= \sum \hat{i}\times\frac{\partial\overline{A}}{\partial x} + \sum \hat{i}\times\frac{\partial\overline{B}}{\partial x}$$
$$= (\nabla\times\overline{A}) \pm (\nabla\times\overline{B})$$

6.10.2 *Products*

If f, g are scalars and $\overline{A}$ and $\overline{B}$ are vectors, then

(i) $\nabla(f\,g) = f\nabla g + g\nabla f$ or grad $(fg) = f\,(\text{grad } g) + g\,(\text{grad } f)$

(ii) $\nabla(\overline{A}\cdot\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} + (\overline{A}\cdot\nabla)\overline{B} + \overline{B}\times(\nabla\times\overline{A}) + \overline{A}\times(\nabla\times\overline{B})$

or grad $(\overline{A}\cdot\overline{B}) = (\overline{B}\cdot\nabla)\overline{A} + (\overline{A}\cdot\nabla)\overline{B} + \overline{B}\times(\text{curl } \overline{A}) + \overline{A}\times(\text{curl } \overline{B})$

(iii) $\nabla\cdot(f\overline{A}) = f(\nabla\cdot\overline{A}) + (\nabla f)\cdot\overline{A}$

or div $(f\overline{A}) = f(\text{div } \overline{A}) + (\text{grad } f)\cdot\overline{A}$

(iv) $\nabla\cdot(\overline{A}\times\overline{B}) = \overline{B}\cdot(\nabla\times\overline{A}) - \overline{A}\cdot(\nabla\times\overline{B})$

or div $(\overline{A}\times\overline{B}) = \overline{B}\cdot\text{curl } \overline{A} - \overline{A}\cdot\text{curl } \overline{B}$

(v) $\nabla\times\left(f\overline{A}\right)=f\left(\nabla\times\overline{A}\right)+(\nabla f)\times\overline{A}$

or curl $\left(f\overline{A}\right)=f\left(\text{curl }\overline{A}\right)+(\text{grad} f)\times\overline{A}$

(vi) $\nabla\times\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\overline{B}\left(\nabla\cdot\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\nabla\cdot\overline{B}\right)$

or curl $\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\overline{B}\left(\text{div }\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\text{div }\overline{B}\right).$

Proof:

(i)
$$\nabla(fg)=\hat{i}\frac{\partial}{\partial x}(fg)+\hat{j}\frac{\partial}{\partial y}(fg)+\hat{k}\frac{\partial}{\partial z}(fg)$$
$$=\sum\hat{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)$$
$$=\sum f\left(\hat{i}\frac{\partial g}{\partial x}\right)+\sum g\left(\hat{i}\frac{\partial f}{\partial x}\right)$$
$$=f\left(\hat{i}\frac{\partial g}{\partial x}+\hat{j}\frac{\partial g}{\partial y}+\hat{k}\frac{\partial g}{\partial z}\right)+g\left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right)$$
$$=f\nabla g+g\nabla f$$

(ii)
$$\nabla\left(\overline{A}\cdot\overline{B}\right)=\hat{i}\frac{\partial}{\partial x}\left(\overline{A}\cdot\overline{B}\right)+\hat{j}\frac{\partial}{\partial y}\left(\overline{A}\cdot\overline{B}\right)+\hat{k}\frac{\partial}{\partial z}\left(\overline{A}\cdot\overline{B}\right)$$
$$=\sum\hat{i}\frac{\partial}{\partial x}\left(\overline{A}\cdot\overline{B}\right)$$
$$=\sum\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}+\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}\right)$$
$$=\sum\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)+\sum\hat{i}\left(\overline{B}\cdot\frac{\partial\overline{A}}{\partial x}\right) \quad \text{... (1)}$$

Consider,

$$\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)=\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)\hat{i}-\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x} \qquad \left[\because\overline{a}\times\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}\cdot\overline{c}\right)\overline{b}-\left(\overline{a}\cdot\overline{b}\right)\overline{c}\right]$$

$$\hat{i}\left(\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}\right)=\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+\left(\overline{A}\cdot\hat{i}\right)\frac{\partial\overline{B}}{\partial x}$$

Similarly, interchanging $\overline{A}$ and $\overline{B}$,

$$\hat{i}\left(\overline{B}\cdot\frac{\partial\overline{A}}{\partial x}\right)=\overline{B}\times\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+\left(\overline{B}\cdot\hat{i}\right)\frac{\partial\overline{A}}{\partial x}$$

Substituting in Eq. (1),

$$\nabla(\overline{A}\cdot\overline{B}) = \sum\left[\overline{A}\times\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+(\overline{A}\cdot\hat{i})\frac{\partial\overline{B}}{\partial x}\right]+\sum\left[\overline{B}\times\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+(\overline{B}\cdot\hat{i})\frac{\partial\overline{A}}{\partial x}\right]$$

$$=\overline{A}\times\sum\left(\hat{i}\times\frac{\partial\overline{B}}{\partial x}\right)+\overline{B}\times\sum\left(\hat{i}\times\frac{\partial\overline{A}}{\partial x}\right)+\sum(\overline{A}\cdot\hat{i})\frac{\partial\overline{B}}{\partial x}+\sum(\overline{B}\cdot\hat{i})\frac{\partial\overline{A}}{\partial x}$$

$$=\overline{A}\times(\nabla\times\overline{B})+\overline{B}\times(\nabla\times\overline{A})+(\overline{A}\cdot\nabla)\overline{B}+(\overline{B}\cdot\nabla)\overline{A}$$

$$\left[\because \sum(\overline{A}\cdot\hat{i})\frac{\partial\overline{B}}{\partial x}=\left(A_1\hat{i}\frac{\partial\overline{B}}{\partial x}+A_2\hat{j}\frac{\partial\overline{B}}{\partial y}+A_3\hat{k}\frac{\partial\overline{B}}{\partial z}\right)\right]$$

(iii) $$\nabla\cdot(f\overline{A})=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(f\overline{A})$$

$$=\sum\hat{i}\cdot\frac{\partial}{\partial x}(f\overline{A})$$

$$=\sum\hat{i}\cdot\left(f\frac{\partial\overline{A}}{\partial x}+\overline{A}\frac{\partial f}{\partial x}\right)$$

$$=\sum f\left(\hat{i}\cdot\frac{\partial\overline{A}}{\partial x}\right)+\sum\left(\overline{A}\cdot\hat{i}\frac{\partial f}{\partial x}\right)$$

$$=f\sum\left(\hat{i}\cdot\frac{\partial\overline{A}}{\partial x}\right)+\overline{A}\cdot\sum\hat{i}\frac{\partial f}{\partial x}$$

$$=f(\nabla\cdot\overline{A})+(\overline{A}\cdot\nabla f)$$

$$=f(\nabla\cdot\overline{A})+(\nabla f\cdot\overline{A})$$

(iv) $$\nabla\times(\overline{\ddot{u}\ddot{u}\ddot{u}})=\left(\hat{\ }\frac{\partial}{\partial x}+\hat{\ }\frac{\partial}{\partial y}+\hat{\ }\frac{\partial}{\partial z}\right)\cdot(\overline{\ }\times\overline{\ })$$

$$=\sum\hat{i}\frac{\partial}{\partial x}\cdot(\overline{A}\times\overline{B})=\sum\hat{i}\cdot\frac{\partial}{\partial x}(\overline{A}\times\overline{B})$$

$$=\sum\hat{i}\cdot\left(\overline{A}\times\frac{\partial\overline{B}}{\partial x}+\frac{\partial\overline{A}}{\partial x}\times\overline{B}\right)$$

$$=\sum\hat{i}\cdot\left(\overline{A}\times\frac{\partial\overline{B}}{\partial x}\right)+\sum\hat{i}\cdot\left(\frac{\partial\overline{A}}{\partial x}\times\overline{B}\right)$$

$$=\sum\hat{i}\times\overline{A}\cdot\frac{\partial\overline{B}}{\partial x}+\sum\hat{i}\times\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}\qquad\left[\because\overline{a}\cdot\overline{b}\times\overline{c}=\overline{a}\times\overline{b}\cdot\overline{c}\right]$$

$$=-\sum\hat{i}\times\frac{\partial\overline{B}}{\partial x}\cdot\overline{A}+\sum\hat{i}\times\frac{\partial\overline{A}}{\partial x}\cdot\overline{B}$$

[Interchanging $\overline{A}$ and $\frac{\partial\overline{B}}{\partial x}$ in scalar triple product.]

$$= -(\nabla \times \overline{B}) \cdot \overline{A} + (\nabla \times \overline{A}) \cdot \overline{B}$$
$$= \overline{B} \cdot (\nabla \times \overline{A}) - \overline{A} \cdot (\nabla \times \overline{B})$$

(v) $$\nabla \times (f\overline{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times (f\overline{A})$$
$$= \sum \hat{i} \times \frac{\partial}{\partial x}(f\overline{A})$$
$$= \sum \hat{i} \times \left(f\frac{\partial \overline{A}}{\partial x} + \frac{\partial f}{\partial x}\overline{A} \right)$$
$$= \sum f\left(\hat{i} \times \frac{\partial \overline{A}}{\partial x} \right) + \sum \hat{i} \times \frac{\partial f}{\partial x}\overline{A}$$
$$= f\sum\left(\hat{i} \times \frac{\partial \overline{A}}{\partial x} \right) + \sum\left(\hat{i}\frac{\partial f}{\partial x} \right) \times \overline{A}$$
$$= f(\nabla \times \overline{A}) + (\nabla f) \times \overline{A}$$

(vi) $$\nabla \times (\overline{A} \times \overline{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times (\overline{A} \times \overline{B})$$
$$= \sum \hat{i}\frac{\partial}{\partial x} \times (\overline{A} \times \overline{B})$$
$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\overline{A} \times \overline{B})$$
$$= \sum \hat{i} \times \left(\overline{A} \times \frac{\partial \overline{B}}{\partial x} + \frac{\partial \overline{A}}{\partial x} \times \overline{B} \right)$$
$$= \sum \hat{i} \times \left(\overline{A} \times \frac{\partial \overline{B}}{\partial x} \right) + \sum \hat{i} \times \left(\frac{\partial \overline{A}}{\partial x} \times \overline{B} \right)$$
$$= \sum\left[\left(\hat{i} \cdot \frac{\partial \overline{B}}{\partial x} \right)\overline{A} - (\hat{i} \cdot \overline{A})\frac{\partial \overline{B}}{\partial x} \right] + \sum\left[(\hat{i} \cdot \overline{B})\frac{\partial \overline{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \overline{A}}{\partial x} \right)\overline{B} \right]$$
$$\left[\because \overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c} \right]$$
$$= \overline{A}\sum\left(\hat{i} \cdot \frac{\partial \overline{B}}{\partial x} \right) - \overline{B}\sum\left(\hat{i} \cdot \frac{\partial \overline{A}}{\partial x} \right) - \sum(\overline{A} \cdot \hat{i})\frac{\partial \overline{B}}{\partial x} + \sum(\overline{B} \cdot \hat{i})\frac{\partial \overline{A}}{\partial x}$$
$$= \overline{A}(\nabla \cdot \overline{B}) - \overline{B}(\nabla \cdot \overline{A}) - (\overline{A} \cdot \nabla)\overline{B} + (\overline{B} \cdot \nabla)\overline{A}$$
$$= (\overline{B} \cdot \nabla)\overline{A} - \overline{B}(\nabla \cdot \overline{A}) - (\overline{A} \cdot \nabla)\overline{B} + \overline{A}(\nabla \cdot \overline{B})$$

6.11 SECOND ORDER DIFFERENTIAL OPERATOR

It is a two fold application of the operator ∇ . Some second order differential operators are given below.

(1) *Laplacian Operator* ∇^2

Div (grad f) = $\nabla \cdot (\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$$= \nabla^2 f$$

$$= \Delta f$$

Thus, the scalar differential operator (read as "nabla squared" or "delta")

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is known as Laplacian operator.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is known as Laplacian equation.

(ii) $\nabla \times \nabla f$ = curl grad f

$$= \nabla \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)$$

$$= 0$$

Hence, curl grad $f = \nabla \times \nabla f = 0$.

(iii) $\nabla \cdot (\nabla \times \overline{A}) = \text{div curl } \overline{A}$

Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times A) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\, \partial y} - \frac{\partial^2 A_2}{\partial x\, \partial z} - \frac{\partial^2 A_3}{\partial y\, \partial x} + \frac{\partial^2 A_1}{\partial y\, \partial z} + \frac{\partial^2 A_2}{\partial z\, \partial x} - \frac{\partial^2 A_1}{\partial z\, \partial y}$$

$$= 0$$

Hence, $\nabla \cdot (\nabla \times \overline{A}) = \text{div curl } \overline{A} = 0.$

Example 1: If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(r^n \overline{r}) = (n+3)r^n$.

Solution: r^n is a scalar and $\overline{r}$ is a vector.

We know that div $(f\overline{A}) = f(\nabla \cdot \overline{A}) + (\nabla f) \cdot \overline{A}$

$$\text{div}(r^n \overline{r}) = r^n (\nabla \cdot \overline{r}) + (\nabla r^n) \cdot \overline{r}$$

$$= r^n \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})\right] + \left(\hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}\right) \cdot \overline{r}$$

$$= r^n (1+1+1) + \left[\hat{i}(nr^{n-1})\frac{\partial r}{\partial x} + \hat{j}(nr^{n-1})\frac{\partial r}{\partial y} + \hat{k}(nr^{n-1})\frac{\partial r}{\partial z}\right] \cdot \overline{r}$$

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, div $(r^n \overline{r}) = 3r^n + nr^{n-1}\left(\hat{i}\dfrac{x}{r} + \hat{j}\dfrac{y}{r} + \hat{k}\dfrac{z}{r}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$= 3r^n + nr^{n-1}\left(\frac{x^2 + y^2 + z^2}{r}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{r^2}{r}\right)$$

$$= 3r^n + nr^n$$

Hence, div $(r^n \overline{r}) = (n+3)r^n$.

Example 2: Find the value of n for which the vector $r^n \bar{r}$ is solenoidal, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: If $\bar{F} = r^n \bar{r}$ is solenoidal, then

$$\nabla \cdot r^n \bar{r} = 0 \qquad \ldots (1)$$

As proved in Ex. 1.,

$$\nabla \cdot r^n \bar{r} = (n+3)\ r^n$$

Substituting in Eq. (1),

$$(n+3)\ r^n = 0$$
$$n = -3$$

Example 3: Prove that Div (grad r^n) $= n\,(n+1)\,r^{n-2}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution:

$$\begin{aligned}
\text{Div}\,(\text{grad}\, r^n) &= \nabla \cdot (\nabla r^n) \\
&= \nabla \cdot \left(\hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z} \right) \\
&= \nabla \cdot \left(nr^{n-1}\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\frac{\partial r}{\partial z}\hat{k} \right) \\
&= \nabla \cdot \left(nr^{n-1}\frac{x}{r}\hat{i} + nr^{n-1}\frac{y}{r}\hat{j} + nr^{n-1}\frac{z}{r}\hat{k} \right) \\
&= \nabla \cdot nr^{n-1}\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r} \\
&= n\nabla \cdot r^{n-2}\bar{r} \\
&= n\left[r^{n-2}\,(\nabla \cdot \bar{r}) + (\nabla r^{n-2}) \cdot \bar{r} \right] \\
&= n\left[r^{n-2}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \right. \\
&\qquad \left. + \left(\hat{i}\frac{\partial r^{n-2}}{\partial x} + \hat{j}\frac{\partial r^{n-2}}{\partial y} + \hat{k}\frac{\partial r^{n-2}}{\partial z} \right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \right] \\
&= n\left[r^{n-2}(1+1+1) + \left\{ (n-2)\,r^{n-3}\frac{\partial r}{\partial x}\hat{i} + (n-2)\,r^{n-3}\frac{\partial r}{\partial y}\hat{j} \right. \right. \\
&\qquad \left. \left. + (n-2)\,r^{n-3}\frac{\partial r}{\partial z}\hat{k} \right\} \cdot \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \right]
\end{aligned}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r^2 = x^2 + y^2 + z^2$$
$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, $\nabla\cdot(\nabla r^n) = n\left[3r^{n-2} + (n-2)r^{n-3}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})\right]$

$$= n\,[3r^{n-2} + (n-2)\,r^{n-4}\,(x^2 + y^2 + z^2)]$$
$$= n\,[3r^{n-2} + (n-2)\,r^{n-4}\cdot r^2]$$
$$= n\,[3r^{n-2} + (n-2)\,r^{n-2}]$$
$$= n\,(n+1)\,r^{n-2}$$

Example 4: If ϕ and ψ are two scalar point functions, show that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi.\nabla\psi + \psi\nabla^2\phi$.

Solution: $\nabla^2(\phi\psi) = \frac{\partial^2}{\partial x^2}(\phi\psi) + \frac{\partial^2}{\partial y^2}(\phi\psi) + \frac{\partial^2}{\partial z^2}(\phi\psi)$ ... (1)

Consider, $\frac{\partial^2}{\partial x^2}(\phi\psi) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\phi\psi)\right]$

$$= \frac{\partial}{\partial x}\left(\psi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\psi}{\partial x}\right)$$
$$= \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \phi\frac{\partial^2\psi}{\partial x^2}$$

Similarly, $\frac{\partial^2}{\partial y^2}(\phi\psi) = \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} + \psi\frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y} + \phi\frac{\partial^2\psi}{\partial y^2}$

and $\frac{\partial^2}{\partial z^2}(\phi\psi) = \frac{\partial\psi}{\partial z}\frac{\partial\phi}{\partial z} + \psi\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial\psi}{\partial z} + \phi\frac{\partial^2\psi}{\partial z^2}$

Substituting in Eq. (1),

$$\nabla^2(\phi\psi) = 2\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right)$$
$$+\phi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \psi\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$
$$= 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi + \psi\,\nabla^2\phi$$
$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$$

Example 5: Prove that $\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = 2r^{-4}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: $\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = \nabla\cdot(r^{-2}\bar{r})$

$$= r^{-2}(\nabla\cdot\bar{r}) + (\nabla r^{-2})\cdot\bar{r}$$
$$= r^{-2}\left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})\right]$$
$$+\left(\hat{i}\frac{\partial r^{-2}}{\partial x} + \hat{j}\frac{\partial r^{-2}}{\partial y} + \hat{k}\frac{\partial r^{-2}}{\partial z}\right)\cdot\bar{r}$$

$$= r^{-2}\left[(1+1+1)+\left\{(-2r^{-3})\frac{\partial r}{\partial x}\hat{i} + (-2r^{-3})\frac{\partial r}{\partial y}\hat{j} + (-2r^{-3})\frac{\partial r}{\partial z}\hat{k}\right\}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\right]$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence,

$$\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = r^{-2}\left[3-2r^{-3}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right]$$

$$= r^{-2}\left[3-2r^{-3}\frac{(x^2+y^2+z^2)}{r}\right]$$

$$= 3r^{-2} - 2r^{-4}\,r^2$$

$$= 3r^{-2} - 2r^{-2}$$

$$= r^{-2}$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = \nabla^2(r^{-2})$$

$$= \frac{\partial^2}{\partial x^2}(r^{-2}) + \frac{\partial^2}{\partial y^2}(r^{-2}) + \frac{\partial^2}{\partial z^2}(r^{-2})$$

Now,

$$\frac{\partial^2 r^{-2}}{\partial x^2} = \frac{\partial}{\partial x}\left[(-2r^{-3})\frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[(-2r^{-3})\frac{x}{r}\right]$$

$$= -2\frac{\partial}{\partial x}(r^{-4}\cdot x)$$

$$= -2\left(-4r^{-5}\frac{\partial r}{\partial x}x + r^{-4}\right)$$

$$= -2\left(-4r^{-5}\frac{x}{r}x + r^{-4}\right)$$

$$= -2r^{-4}\left(\frac{-4x^2}{r^2}+1\right)$$

Similarly,

$$\frac{\partial^2 r^{-2}}{\partial y^2} = -2r^{-4}\left(\frac{-4y^2}{r^2}+1\right)$$

$$\frac{\partial^2 r^{-2}}{\partial z^2} = -2r^{-4}\left(\frac{-4z^2}{r^2}+1\right)$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right]=\frac{\partial^2 r^{-2}}{\partial x^2}+\frac{\partial^2 r^{-2}}{\partial y^2}+\frac{\partial^2 r^{-2}}{\partial z^2}$$

$$=-2r^{-4}\left[\frac{-4(x^2+y^2+z^2)}{r^2}+3\right]$$

$$=-2r^{-4}\left[\frac{-4r^2}{r^2}+3\right]$$

$$=2r^{-4}$$

Hence, $$\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right]=2r^{-4}$$

Example 6: Prove that $\nabla\left(\nabla\cdot\frac{\bar{r}}{r}\right)=-\frac{2}{r^3}\bar{r}.$

Solution: $$\nabla\cdot\frac{\bar{r}}{r}=\nabla\cdot\left(r^{-1}\bar{r}\right)$$

$$=r^{-1}\left(\nabla\cdot\bar{r}\right)+(\nabla r^{-1})\cdot\bar{r}$$

$$=r^{-1}\left[\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right]+\left(\hat{i}\frac{\partial r^{-1}}{\partial x}+\hat{j}\frac{\partial r^{-1}}{\partial y}+\hat{k}\frac{\partial r^{-1}}{\partial z}\right)\cdot\bar{r}$$

$$=3r^{-1}+\left(-r^{-2}\frac{\partial r}{\partial x}\hat{i}-r^{-2}\frac{\partial r}{\partial y}\hat{j}-r^{-2}\frac{\partial r}{\partial z}\hat{k}\right)\cdot\bar{r}$$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

Hence, $$\nabla\cdot\frac{\bar{r}}{r}=3r^{-1}-r^{-2}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)\cdot\bar{r}$$

$$=3r^{-1}-r^{-2}\left(\frac{\bar{r}}{r}\right)\cdot\bar{r}$$

$$=3r^{-1}-r^{-2}\frac{(\bar{r}\cdot\bar{r})}{r}$$

$$=3r^{-1}-r^{-2}\left(\frac{r^2}{r}\right)$$

$$=2r^{-1}$$

$$\nabla\left(\nabla\cdot\frac{\bar{r}}{r}\right)=\hat{i}\frac{\partial}{\partial x}(2r^{-1})+\hat{j}\frac{\partial}{\partial y}(2r^{-1})+\hat{k}\frac{\partial}{\partial z}(2r^{-1})$$

$$=-2r^{-2}\frac{\partial r}{\partial x}\hat{i}-2r^{-2}\frac{\partial r}{\partial y}\hat{j}-2r^{-2}\frac{\partial r}{\partial z}\hat{k}$$

$$=-2r^{-2}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)$$

$$=-2r^{-2}\frac{\bar{r}}{r}$$

$$=-\frac{2}{r^3}\bar{r}.$$

Example 7: Show that $\bar{\boldsymbol{E}}=\frac{\bar{\boldsymbol{r}}}{\boldsymbol{r}^2}$ is irrotational.

Solution: Curl $\bar{E}=\nabla\times\bar{E}$

$$=\nabla\times\frac{\bar{r}}{r^2}$$

$$=\nabla\times\left(r^{-2}\bar{r}\right)$$

We know that, $\nabla\times\left(f\bar{A}\right)=f\left(\nabla\times\bar{A}\right)+(\nabla f)\times\bar{A}$

$$\text{curl }\bar{E}=\nabla\times\left(r^{-2}\bar{r}\right)$$

$$=r^{-2}\left(\nabla\times\bar{r}\right)+(\nabla r^{-2})\times\bar{r}$$

$$=r^{-2}\sum\hat{i}\times\frac{\partial}{\partial x}(x\hat{i})+\left(\sum\hat{i}\,\frac{\partial r^{-2}}{\partial x}\right)\times\bar{r}$$

$$=r^{-2}\sum\left(\hat{i}\times\hat{i}\right)+\left[\sum(-2r^{-3})\frac{\partial r}{\partial x}\hat{i}\right]\times\bar{r}$$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$$

Hence, $\nabla\times\left(r^{-2}\bar{r}\right)=0-2r^{-3}\left(\sum\frac{x}{r}\hat{i}\right)\times\bar{r}$

$$=-2r^{-3}\frac{\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{r}\times\bar{r}$$

$$=-2r^{-4}\left(\bar{r}\times\bar{r}\right)$$

$$=0$$

Hence, $\bar{E}$ is irrotational.

Example 8: Prove that $\nabla\times(\bar{a}\times\bar{r})=2\bar{a}$, where a is a constant vector.

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\bar{a}\times\bar{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}$$

$$=\hat{i}\,(a_2z-a_3y)-\hat{j}\,(a_1z-a_3x)+\hat{k}\,(a_1y-a_2x)$$

$$\nabla\times(\bar{a}\times\bar{r})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ a_2z-a_3y & a_3x-a_1z & a_1y-a_2x\end{vmatrix}$$

$$=\hat{i}\,(a_1+a_1)-\hat{j}\,(-a_2-a_2)+\hat{k}\,(a_3+a_3)$$

$$=2(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$

$$=2\,\bar{a}$$

Example 9: Prove that $\nabla\times\left(\dfrac{\bar{a}\times\bar{r}}{r^n}\right)=\dfrac{(2-n)\bar{a}}{r^n}+\dfrac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$.

Solution: $\nabla\times\left(\dfrac{\bar{a}\times\bar{r}}{r^n}\right)=\nabla\times(r^{-n}\bar{A})$, where $\bar{a}\times\bar{r}=\bar{A}$, say

We know that,

$$\nabla\times(f\bar{A})=f\,(\nabla\times\bar{A})+(\nabla f)\times\bar{A}$$

$$\nabla\times(r^{-n}\bar{A})=r^{-n}\,(\nabla\times\bar{A})+(\nabla r^{-n})\times\bar{A}$$

$$=r^{-n}\left[\nabla\times(\bar{a}\times\bar{r})\right]+\left(\hat{i}\frac{\partial r^{-n}}{\partial x}+\hat{j}\frac{\partial r^{-n}}{\partial y}+\hat{k}\frac{\partial r^{-n}}{\partial z}\right)\times\bar{A}$$

As proved in Ex. 8

$$\nabla\times(\bar{a}\times\bar{r})=2\bar{a}$$

$$\nabla\times(r^{-n}\bar{A})=r^{-n}\,(2\bar{a})+(-nr^{-n-1})\left(\frac{\partial r}{\partial x}\hat{i}+\frac{\partial r}{\partial y}\hat{j}+\frac{\partial r}{\partial z}\hat{k}\right)\times\bar{A}$$

As proved earlier, $\dfrac{\partial r}{\partial x}=\dfrac{x}{r},\dfrac{\partial r}{\partial y}=\dfrac{y}{r},\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$

Hence, $\nabla\times\left(r^{-n}\bar{A}\right)=2\bar{a}r^{-n}-nr^{-n-1}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)\times\bar{A}$

$$=\frac{2\bar{a}}{r^n}-\frac{n}{r^{n+1}}\frac{\bar{r}}{r}\times(\bar{a}\times\bar{r})$$

$$=\frac{2\bar{a}}{r^n}-\frac{n}{r^{n+2}}\left[(\bar{r}\cdot\bar{r})\bar{a}-(\bar{r}\cdot\bar{a})\bar{r}\right]\quad\left[\because \bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}\right]$$

$$=\frac{2\bar{a}}{r^n}-\frac{n}{r^{n+2}}\left[r^2\bar{a}-(\bar{r}\cdot\bar{a})\bar{r}\right]$$

$$=\frac{2\bar{a}}{r^n}-\frac{n\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$$

$$=\frac{(2-n)\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}.$$

Example 10: If $\bar{a}$ is a constant vector, show that $\bar{a}\times(\nabla\times\bar{r})=\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\nabla)\bar{r}$.

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\bar{r}=x_1\hat{i}+x_2\hat{j}+x_3\hat{k}$$

$$\nabla\times\bar{r}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ r_1&r_2&r_3\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)-\hat{j}\left(\frac{\partial r_3}{\partial x}-\frac{\partial r_1}{\partial z}\right)+\hat{k}\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)$$

$$\bar{a}\times(\nabla\times\bar{r})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ a_1&a_2&a_3\\ \left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)&\left(\frac{\partial r_1}{\partial z}-\frac{\partial r_3}{\partial x}\right)&\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)\end{vmatrix}$$

$$=\hat{i}\left[a_2\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)-a_3\left(\frac{\partial r_1}{\partial z}-\frac{\partial r_3}{\partial x}\right)\right]-\hat{j}\left[a_1\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)-a_3\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)\right]$$

$$+\hat{k}\left[a_1\left(\frac{\partial r_1}{\partial z}-\frac{\partial r_3}{\partial x}\right)-a_2\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)\right]$$

$$= \hat{i}\left(a_2\frac{\partial r_2}{\partial x}+a_3\frac{\partial r_3}{\partial x}+a_1\frac{\partial r_1}{\partial x}-a_1\frac{\partial r_1}{\partial x}\right)+\hat{j}\left(a_1\frac{\partial r_1}{\partial y}+a_3\frac{\partial r_3}{\partial y}+a_2\frac{\partial r_2}{\partial y}-a_2\frac{\partial r_2}{\partial y}\right)$$

$$+\hat{k}\left(a_1\frac{\partial r_1}{\partial z}+a_2\frac{\partial r_2}{\partial z}+a_3\frac{\partial r_3}{\partial z}-a_3\frac{\partial r_3}{\partial z}\right)-\hat{i}\left(a_2\frac{\partial r_1}{\partial y}+a_3\frac{\partial r_1}{\partial z}\right)$$

$$-\hat{j}\left(a_1\frac{\partial r_2}{\partial x}+a_3\frac{\partial r_2}{\partial z}\right)-\hat{k}\left(a_1\frac{\partial r_3}{\partial x}+a_2\frac{\partial r_3}{\partial y}\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(a_1r_1+a_2r_2+a_3r_3)$$

$$-\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)(r_1\hat{i}+r_2\hat{j}+r_3\hat{k})$$

$$=\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\nabla)\bar{r}$$

Example 11: If $\bar{a}$ is a constant vector such that $|\bar{a}|=a,$ prove that $\nabla\cdot\left[(\bar{a}\cdot\bar{r})\bar{a}\right]=a^2.$

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

We know that, $\nabla\cdot(f\bar{A})=f(\nabla\cdot\bar{A})+(\nabla f)\cdot\bar{A}$

$$\nabla\cdot\left[(\bar{a}\cdot\bar{r})\bar{a}\right]=(\bar{a}\cdot\bar{r})(\nabla\cdot\bar{a})+\left[\nabla(\bar{a}\cdot\bar{r})\right]\cdot\bar{a}$$

Since $\bar{a}$ is constant, $\nabla\cdot\bar{a}=0$

$$\nabla(\bar{a}\cdot\bar{r})=\hat{i}\frac{\partial}{\partial x}(\bar{a}\cdot\bar{r})+\hat{j}\frac{\partial}{\partial y}(\bar{a}\cdot\bar{r})+\hat{k}\frac{\partial}{\partial z}(\bar{a}\cdot\bar{r})$$

$$=\hat{i}\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\hat{j}\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\hat{k}\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)$$

$$=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$

$$=\bar{a}$$

Hence, $\nabla\cdot\left[(\bar{a}\cdot\bar{r})\bar{a}\right]=0+\bar{a}\cdot\bar{a}$

$$=a^2.$$

Example 12: If $\bar{F}=(\bar{a}\cdot\bar{r})\bar{r}$ where $\bar{a}$ is a constant vector, find curl $\bar{F}$ and prove that it is perpendicular to $\bar{a}.$

Solution: Curl $\bar{F} = \nabla \times \bar{F} = \nabla \times \left[(\bar{a} \cdot \bar{r}) \bar{r} \right]$

We know that, $\nabla \times (f \bar{A}) = f (\nabla \times \bar{A}) + (\nabla f) \times \bar{A}$

$$\text{Curl } \bar{F} = \nabla \times \left[(\bar{a} \cdot \bar{r}) \bar{r} \right]$$

$$= (\bar{a} \cdot \bar{r})(\nabla \times \bar{r}) + \left[\nabla (\bar{a} \cdot \bar{r}) \right] \times \bar{r}$$

Now,
$$\nabla \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\ (0 - 0) - \hat{j}\ (0 - 0) + \hat{k}\ (0 - 0)$$
$$= 0$$

As proved in Ex. 11

$$\nabla (\bar{a} \cdot \bar{r}) = \bar{a}$$

$$\nabla \times \left[(\bar{a} \cdot \bar{r}) \bar{r} \right] = 0 + \bar{a} \times \bar{r}$$
$$= \bar{a} \times \bar{r}$$

$$\nabla \times \left[(\bar{a} \cdot \bar{r}) \bar{r} \right] \cdot \bar{a} = (\bar{a} \times \bar{r}) \cdot \bar{a} = 0$$

Hence, $\nabla \times \left[(\bar{a} \cdot \bar{r}) \bar{r} \right]$ is perpendicular to $\bar{a}$.

Example 13: Prove that $\nabla \cdot \left(\frac{\bar{a} \times \bar{r}}{r} \right) = 0$, where $\bar{a}$ is a constant vector.

Solution: $\nabla \cdot \left(\frac{\bar{a} \times \bar{r}}{r} \right) = \nabla \cdot \left[r^{-1} (\bar{a} \times \bar{r}) \right]$

We know that, $\nabla \cdot (f \bar{A}) = f (\nabla \cdot \bar{A}) + (\nabla f) \cdot \bar{A}$

$$\nabla \cdot \left[r^{-1} (\bar{a} \times \bar{r}) \right] = r^{-1} \left[\nabla \cdot (\bar{a} \times \bar{r}) \right] + (\nabla r^{-1}) \cdot (\bar{a} \times \bar{r})$$

$$= r^{-1} \left[\nabla \cdot (\bar{a} \times \bar{r}) \right] + \left(\hat{i} \frac{\partial r^{-1}}{\partial x} + \hat{j} \frac{\partial r^{-1}}{\partial y} + \hat{k} \frac{\partial r^{-1}}{\partial z} \right) \cdot (\bar{a} \times \bar{r})$$

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$
$$\nabla \cdot (\bar{a} \times \bar{r}) = \bar{r} \cdot (\nabla \times \bar{a}) - \bar{a} \cdot (\nabla \times \bar{r})$$

Since $\bar{a}$ is constant, $\nabla \times \bar{a} = 0$.

Also, $\nabla \times \bar{r} = 0$ as proved in Ex. 12.

$$\nabla \cdot (\bar{a} \times \bar{r}) = 0$$

Hence, $\nabla \cdot \left[r^{-1} (\bar{a} \times \bar{r}) \right] = 0 + \left(-r^{-2} \frac{\partial r}{\partial x} \hat{i} - r^{-2} \frac{\partial r}{\partial y} \hat{j} - r^{-2} \frac{\partial r}{\partial z} \hat{k} \right) \cdot (\bar{a} \times \bar{r})$

$$= 0 - r^{-2} \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \right) \cdot (\bar{a} \times \bar{r})$$

$$= -r^{-3} \left[\bar{r} \cdot (\bar{a} \times \bar{r}) \right]$$

$$= 0$$

Example 14: Prove that curl $\left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = \bar{b} \times \bar{a}$, where $\bar{a}$ and $\bar{b}$ are constants.

Solution: We know that, $(\bar{r} \times \bar{a}) \times \bar{b} = (\bar{r} \cdot \bar{b}) \bar{a} - (\bar{a} \cdot \bar{b}) \bar{r}$

Let $\bar{r} \cdot \bar{b} = f$, say and $\bar{a} \cdot \bar{b} = g$, say

(¢✗¢)× = ‾ –

$$\text{curl } \left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = \nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b} \right]$$

$$= \nabla \times \left(f\bar{a} - g\bar{r} \right)$$

$$= \nabla \times \left(f\bar{a} \right) - \nabla \times \left(g\bar{r} \right)$$

We know that, $\nabla \times \left(f\bar{A} \right) = f(\nabla \times \bar{A}) + (\nabla f) \times \bar{A}$

$$\nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = f(\nabla \times \bar{a}) + (\nabla f) \times \bar{a} - g(\nabla \times \bar{r}) - (\nabla g) \times \bar{r}$$

Since $\bar{a}$ is constant, $\nabla \times \bar{a} = 0$. Also $\nabla \times \bar{r} = 0$

$$\nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = \left[\nabla (\bar{r} \cdot \bar{b}) \right] \times \bar{a} - \left[\nabla (\bar{a} \cdot \bar{b}) \right] \times \bar{r} \qquad \text{[Substituting } f \text{ and } g\text{]}$$

$$= \left[\nabla (\bar{r} \cdot \bar{b}) \right] \times \bar{a} - 0 \qquad \ldots (1) \qquad \left[\because \bar{a} \text{ and } \bar{b} \text{ are constant} \right]$$

Let $\bar{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla (\bar{r} \cdot \bar{b}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (b_1 x + b_2 y + b_3 z)$$

$$= b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$= \bar{b}$$

Substituting in Eq. (1),

$$\nabla \times \left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = \bar{b} \times \bar{a}$$

Hence, curl $\left[(\bar{r} \times \bar{a}) \times \bar{b} \right] = \bar{b} \times \bar{a}$.

Example 15: Prove that $\nabla\cdot\left(r\nabla\frac{1}{r^n}\right)=\frac{n(n-2)}{r^{n+1}}$.

Solution: $\nabla\frac{1}{r^n}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^{-n}$

$$=(-nr^{-n-1})\frac{\partial r}{\partial x}\hat{i}+(-nr^{-n-1})\frac{\partial r}{\partial y}\hat{j}+(-nr^{-n-1})\frac{\partial r}{\partial z}\hat{k}$$

$$=(-nr^{-n-1})\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)$$

$$=\frac{-n}{r^{n+1}}\frac{\bar{r}}{r}=-\frac{n}{r^{n+2}}\bar{r}$$

$$\Re\Re\Re\left(\frac{1}{r^n}\right)\qquad\left[r\left(\frac{n}{r^{n+2}}\bar{r}\right)\right]$$

$$=-n\nabla\cdot\left(r^{-n-1}\bar{r}\right)$$

We know that, $\nabla\cdot\left(f\bar{A}\right)=f\left(\nabla\cdot\bar{A}\right)+(\nabla f)\cdot\bar{A}$

$$-n\nabla\cdot\left(r^{-n-1}\bar{r}\right)=-n\left[\frac{1}{r^{n+1}}(\nabla\cdot\bar{r})+(\nabla r^{-n-1})\cdot\bar{r}\right]$$

$$=-n\left[\frac{3}{r^{n+1}}+\left\{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(r^{-n-1})\right\}\cdot\bar{r}\right]\qquad[\because\nabla\cdot\bar{r}=3]$$

$$=-n\left[\frac{3}{r^{n+1}}-(n+1)r^{-n-2}\left(\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right)\cdot\bar{r}\right]$$

$$=-n\left[\frac{3}{r^{n+1}}-\frac{(n+1)}{r^{n+2}}\left(\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}\right)\cdot\bar{r}\right]$$

$$=-n\left[\frac{3}{r^{n+1}}-\frac{(n+1)}{r^{n+2}}\frac{\bar{r}\cdot\bar{r}}{r}\right]$$

$$=-n\left[\frac{3}{r^{n+1}}-\frac{(n+1)r^2}{r^{n+2}r}\right]$$

$$=-\frac{n(2-n)}{r^{n+1}}$$

$$=\frac{n(n-2)}{r^{n+1}}.$$

Example 16: Prove that $\nabla\log r=\frac{\bar{r}}{r^2}$ and hence, show that

$\nabla\times\left(\bar{a}\times\nabla\log r\right)=2\frac{\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^4}$ where $\bar{a}$ is a constant vector.

Solution: $$\nabla \log r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)\log r$$
$$= \left(\frac{1}{r}\frac{\partial r}{\partial x}\hat{i} + \frac{1}{r}\frac{\partial r}{\partial y}\hat{j} + \frac{1}{r}\frac{\partial r}{\partial z}\hat{k} \right)$$
$$= \frac{1}{r}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} \right)$$
$$= \frac{\bar{r}}{r^2}$$

$$\nabla \times \left(\bar{a} \times \nabla \log r \right) = \nabla \times \left(\bar{a} \times \frac{\bar{r}}{r^2} \right)$$
$$= \nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^2} \right)$$

Let $\bar{a} \times \bar{r} = \bar{A}$,

$$\nabla \times \left(\bar{a} \times \nabla \log r \right) = \nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^2} \right) = \nabla \times \left(r^{-2}\bar{A} \right)$$

We know that, $\nabla \times \left(f\bar{A} \right) = f\left(\nabla \times \bar{A} \right) + (\nabla f) \times \bar{A}$

$$\nabla \times \left(r^{-2}\bar{A} \right) = r^{-2}\left(\nabla \times \bar{A} \right) + (\nabla r^{-2}) \times \bar{A}$$
$$= r^{-2}\left[\nabla \times \left(\bar{a} \times \bar{r} \right) \right] + (\nabla r^{-2}) \times \left(\bar{a} \times \bar{r} \right)$$
$$= r^{-2}\left[\left(\bar{r} \cdot \nabla \right)\bar{a} - \bar{r}\left(\nabla \cdot \bar{a} \right) - \left(\bar{a} \cdot \nabla \right)\bar{r} + \bar{a}\left(\nabla \cdot \bar{r} \right) \right]$$
$$+ \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) r^{-2} \right] \times \left(\bar{a} \times \bar{r} \right)$$

Since $\bar{a}$ is a constant vector, $\nabla \cdot \bar{a} = 0, \left(\bar{r} \cdot \nabla \right)\bar{a} = 0.$

Let
$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times \left(\bar{a} \times \nabla \log r \right) = r^{-2}\left[-\left(\bar{a} \cdot \nabla \right)\bar{r} + \bar{a}\left(\nabla \cdot \bar{r} \right) \right] + (-2r^{-3})\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} \right) \times \left(\bar{a} \times \bar{r} \right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla\times\left(\bar{a}\times\nabla\log r\right)=r^{-2}\left[-\left(a_1\frac{\partial\bar{r}}{\partial x}+a_2\frac{\partial\bar{r}}{\partial y}+a_3\frac{\partial\bar{r}}{\partial z}\right)+\bar{a}(3)\right]$$

$$+(-2r^{-3})\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)\times\left(\bar{a}\times\bar{r}\right)$$

$$=r^{-2}\left[-\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)+3\bar{a}\right]+(-2r^{-3})\frac{\bar{r}}{r}\times\left(\bar{a}\times\bar{r}\right)$$

$$=r^{-2}\left(-\bar{a}+3\bar{a}\right)-\frac{2}{r^4}\left[(\bar{r}\cdot\bar{r})\bar{a}-(\bar{r}\cdot\bar{a})\bar{r}\right]$$

$$=\frac{2\bar{a}}{r^2}-\frac{2}{r^4}\left[r^2\bar{a}-(\bar{r}\cdot\bar{a})\bar{r}\right]$$

$$=\frac{2\bar{a}}{r^2}-\frac{2\bar{a}}{r^2}+\frac{2\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^4}$$

$$=\frac{2\left(\bar{a}\cdot\bar{r}\right)\bar{r}}{r^4}$$

Example 17: Calculate $\nabla^2 f$ when $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$ at the point (1, 1, 0).

Solution: $$\nabla^2 f=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(3x^2z-y^2z^3+4x^3y+2x-3y-5) \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(3x^2z-y^2z^3+4x^3y+2x-3y-5)$$

$$=6xz+12x^2y+2$$

$$\frac{\partial^2 f}{\partial x^2}=6z+24xy$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(3x^2z-y^2z^3+4x^3y+2x-3y-5)$$

$$=-2yz^3+4x^3-3$$

$$\frac{\partial^2 f}{\partial y^2}=-2z^3$$

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(3x^2z-y^2z^3+4x^3y+2x-3y-5)$$

$$=3x^2-3y^2z^2$$

$$\frac{\partial^2 f}{\partial z^2}=-6y^2z.$$

Substituting in Eq. (1),

$$\nabla^2 f=6z+24xy-2z^3-6y^2z$$

At the point (1, 1, 0), $\nabla^2 f = 24$

Example 18: Prove that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution:

$$\nabla^2 f = \nabla \cdot \nabla f$$

$$\nabla f = \hat{i}\frac{\partial f(r)}{\partial x} + \hat{j}\frac{\partial f(r)}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial z}$$

$$= \left[f'(r)\frac{\partial r}{\partial x}\right]\hat{i} + \left[f'(r)\frac{\partial r}{\partial y}\right]\hat{j} + \left[f'(r)\frac{\partial r}{\partial z}\right]\hat{k}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla f = f'(r)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= f'(r)\frac{\bar{r}}{r}$$

$$= \frac{f'(r)}{r}\bar{r}$$

$$\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left[\frac{f'(r)}{r}\bar{r}\right]$$

We know that, $\nabla \cdot (f\bar{A}) = f(\nabla \cdot \bar{A}) + (\nabla f) \cdot \bar{A}$

$$\nabla^2 f = \nabla \cdot \left[\frac{f'(r)}{r}\bar{r}\right]$$

$$= \frac{f'(r)}{r}(\nabla \cdot \bar{r}) + \left[\nabla \frac{f'(r)}{r}\right] \cdot \bar{r} \qquad \left[\because \frac{f'(r)}{r} \text{ is a scalar function}\right]$$

Now,

$$\nabla \cdot \bar{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = 3$$

$$\nabla \frac{f'(r)}{r} = \frac{\partial}{\partial x}\left[\frac{f'(r)}{r}\right]\hat{i} + \frac{\partial}{\partial y}\left[\frac{f'(r)}{r}\right]\hat{j} + \frac{\partial}{\partial z}\left[\frac{f'(r)}{r}\right]\hat{k}$$

$$= \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial x}i + \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial y}j + \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\frac{\partial r}{\partial z}\hat{k}$$

$$= \left[\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right]\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]\bar{r}$$

Hence, $\nabla^2 f = \frac{f'(r)}{r}(3) + \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]\bar{r} \cdot \bar{r}$

$$= \frac{3f'(r)}{r} + \left[\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right]r^2$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= f''(r) + \frac{2}{r} f'(r)$$

$$= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

Exercise 6.5

1. Evaluate div $(\overline{A} \times \overline{r})$ if curl $\overline{A} = 0$, $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

[**Ans. :** 0]

2. If $\overline{r_1}$ and $\overline{r_2}$ are vectors joining the $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ to a variable point P (x, y, z), show that curl $(\overline{r_1} \times \overline{r_2}) = 2(\overline{r_1} - \overline{r_2})$.

3. Prove that

$\nabla \times [(\overline{r} \times \overline{a}) \times \overline{b}] = \overline{b} \times \overline{a}$, where

$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\overline{a}$, $\overline{b}$ are constant vectors.

4. If $\overline{a}$ is a constant vector, prove that

$\nabla \times [\overline{r} \times (\overline{a} \times \overline{r})] = 3\overline{r} \times \overline{a}$.

5. Prove that

$\nabla \cdot \left[\frac{f(r)}{r}\overline{r}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$

Hence, or otherwise prove that div $(r^n \overline{r}) = (n+3)r^n$.

6. Prove that

$\nabla \cdot \left(\frac{\log r}{r}\overline{r}\right) = \frac{1}{r}(1 + 2\log r)$.

7. Prove that

$\overline{a} \cdot [\text{grad } (\overline{f} \cdot \overline{a}) - \text{curl } (\overline{f} \times \overline{a})] = \text{div } \overline{f}$

where $\overline{a}$ is a constant unit vector.

8. Find $f(r)$, so that the vector $f(r)\,\overline{r}$ is both solenoidal and irrotational.

$\left[\textbf{Ans.:}\ f(r) = \frac{c}{r^3}\right]$

9. If ϕ_1 and ϕ_2 are scalar functions, then prove that, $\nabla \times (\phi_1 \nabla \phi_2) = \nabla \phi_1 \times \nabla \phi_2$.

10. Is $\overline{A} = \frac{\overline{a} \times \overline{r}}{r^n}$ a solenoidal vector, where $\overline{a}$ is constant vector?

[**Ans. :** Yes]

11. Prove that div $(\overline{a} \cdot \overline{r})\overline{a} = a^2$.

12. If $\overline{r}$ is the positive vector of the point (x, y, z) and r is the modulus of $\overline{r}$, then prove that $r^n \overline{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$.

13. If ϕ_1 and ϕ_2 are scalar functions, then prove that

$\nabla \times (\phi_1 \nabla \phi_2) = \nabla \phi_1 \times \nabla \phi_2 = -\nabla \times (\phi_2 \nabla \phi_1)$ and deduce that $\nabla \times (f \nabla f) = 0$.

14. Prove that $\nabla \cdot (\phi_1 \nabla \phi_2 \times \phi_2 \nabla \phi_1) = 0$, where ϕ_1 and ϕ_2 are scalar functions.

15. Prove that

$\nabla^2 (fg) = f\nabla^2 g + 2\nabla g \cdot \nabla f + g\nabla^2 f$,

where f and g are scalar functions.

16. Calculate $\nabla^2 f$ where $f = 4x^2 + 9y^2 + z^2$.

[**Ans. :** 28]

Vector Calculus

Chapter 7

7.1 INTRODUCTION

Vector calculus deals with the differentiation and integration of vector functions. We learn about derivative of a vector function, gradient, divergence and curl in vector differential calculus. In vector integral calculus, we learn about line integral, surface integral, volume integral and three theorems, namely Green's theorem, divergence theorem and Stokes' theorem. It plays an important role in the differential geometry and in the study of partial differential equations. It is useful in the study of rigid dynamics, fluid dynamics, heat transfer, electromagnetism, theory of relativity, etc.

7.2 LINE INTEGRALS

The line integral is a simple generalisation of a definite integral $\int_a^b f(x)\,\mathrm{d}x$ which is integrated from $x = \mathrm{a}$ (point A) to $x = \mathrm{b}$ (point B) along the x-axis. In a line integral, the integration is done along a curve C in space.

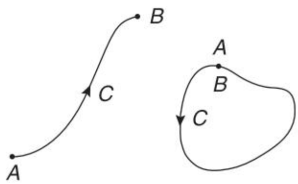


Fig. 7.1

Let $\overline{F}(\overline{r})$ be a vector function defined at every point of a curve C. If $\overline{r}$ is the position vector of a point $P\,(x, y, z)$ on the curve C, then the line integral of $\overline{F}(\overline{r})$ over a curve C is defined by

$$\int_C \overline{F}(\overline{r})\cdot \mathrm{d}\overline{r} = \int_C (F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z)$$

where $\qquad \overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If the curve C is represented by a parametric representation

$$\overline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},$$

then the line integral along the curve C from $t = a$ to $t = b$ is

$$\int_C \overline{F}(\overline{r})\cdot \mathrm{d}\overline{r} = \int_a^b \overline{F}\cdot\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\mathrm{d}t$$

$$= \int_a^b \left(F_1\frac{\mathrm{d}x}{\mathrm{d}t} + F_2\frac{\mathrm{d}y}{\mathrm{d}t} + F_3\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t$$

If C is a closed curve, then the symbol of the line integral $\int_C$ is replaced by $\oint_C$.

Note:

(1) The curve C is called the path of integration, the points $\overline{r}(a)$ and $\overline{r}(b)$ are called initial and terminal points respectively.

(2) The direction from A to B along which t increases is called positive direction on C.

7.2.1 *Circulation*

If $\overline{F}$ is the velocity of a fluid particle and C is a closed curve, then the line integral $\oint_C \overline{F} \cdot \mathrm{d}\overline{r}$ represents the circulation of $\overline{F}$ around the curve C.

Note: If the circulation of $\overline{F}$ around every closed curve C in the region R is zero, then $\overline{F}$ is irrotational, i.e. if $\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = 0$, $\overline{F}$ is irrotational.

7.2.2 *Work done by a Force*

If $\overline{F}$ is the force acting on a particle moving along the arc AB of the curve C, then the line integral $\int_A^B \overline{F} \cdot \mathrm{d}\overline{r}$ represents the work done in displacing (moving) the particle from the point A to the point B.

7.3 PATH INDEPENDENCE OF LINE INTEGRALS (CONSERVATIVE FIELD AND SCALAR POTENTIAL)

If $\overline{F}$ is conservative, i.e. $\overline{F} = \nabla\phi$ where ϕ is a scalar potential, then the line integral along the curve C from the points A to B is

$$\begin{aligned}\int_C \overline{F} \cdot \mathrm{d}\overline{r} &= \int_A^B \nabla\phi \cdot \mathrm{d}\overline{r} \\ &= \int_A^B \left(\frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z \right) \\ &= \int_A^B \mathrm{d}\phi \\ &= \phi(B) - \phi(A)\end{aligned}$$

Thus, line integral depends only on the start and end values and therefore is independent of the path.

Hence, for a conservative force field, line integral is independent of the path.

Note 1: If $\overline{F}$ is conservative and curve C is closed, then

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = \phi(A) - \phi(A) = 0$$

Fig. 7.2

Note 2: Work done in moving a particle from points A to B under a conservative force field is

$$\text{work done} = \phi(B) - \phi(A)$$

Example 1: Evaluate $\int_C \bar{F}\cdot \mathrm{d}\bar{r}$ along the parabola $y^2 = x$ between the points (0, 0) and (1, 1) where $\bar{F} = x^2\hat{i} + xy\hat{j}$.

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$$\mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y$$

(ii) $$\bar{F}\cdot\mathrm{d}\bar{r} = (x^2\hat{i} + xy\hat{j})\cdot(\hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y)$$
$$= x^2\mathrm{d}x + xy\mathrm{d}y$$

(iii) Path of integration is the parabola

$$x = y^2$$
$$\mathrm{d}x = 2y\mathrm{d}y$$

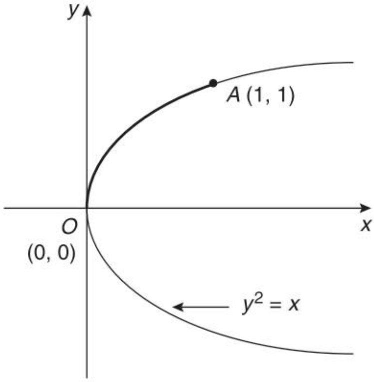


Fig. 7.3

Substituting in $\bar{F}\cdot\mathrm{d}\bar{r}$ and integrating between the limits $y = 0$ to $y = 1$,

$$\int_C \bar{F}\cdot\mathrm{d}\bar{r} = \int_0^1 (y^4\cdot 2y\,\mathrm{d}y + y^2\cdot y\,\mathrm{d}y)$$
$$= \int_0^1 (2y^5 + y^3)\,\mathrm{d}y$$
$$= \left| 2\frac{y^6}{6} + \frac{y^4}{4} \right|_0^1$$
$$= \frac{1}{3} + \frac{1}{4}$$
$$= \frac{7}{12}$$

Example 2: Prove that $\int_C \bar{F}\cdot\mathrm{d}\bar{r} = 3\pi$, where $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ and C is the arc of the curve $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

Solution : (i) $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$$x = \cos t,\ y = \sin t,\ z = t$$
$$\mathrm{d}x = -\sin t\,\mathrm{d}t,\ \mathrm{d}y = \cos t\,\mathrm{d}t,\ \mathrm{d}z = \mathrm{d}t$$

(ii) $$\bar{F}\cdot\mathrm{d}\bar{r} = (z\hat{i} + x\hat{j} + y\hat{k})\cdot(\hat{i}\,\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z)$$
$$= z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z$$
$$= t(-\sin t)\mathrm{d}t + \cos t\cdot\cos t\,\mathrm{d}t + \sin t\,\mathrm{d}t$$
$$= (-t\sin t + \cos^2 t + \sin t)\mathrm{d}t$$

(iii) Path of integration is the arc of the curve $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}(-t\sin t + \cos^2 t + \sin t)dt$$

$$= -\left|t(-\cos t) - (-\sin t)\right|_0^{2\pi} + \int_0^{2\pi}\frac{(1+\cos 2t)}{2}dt + \left|-\cos t\right|_0^{2\pi}$$

$$= -(-2\pi) + \left|\frac{t}{2} + \frac{\sin 2t}{4}\right|_0^{2\pi} - (\cos 2\pi - \cos 0)$$

$$= 2\pi + \frac{2\pi}{2}$$

$$= 3\pi$$

Example 3: If $\bar{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$, calculate the circulation of $\bar{F}$ along the circle in the *xy*-plane of 2 unit radius and centre at the origin.

Solution: Circulation = $\oint_C \bar{F}\cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}\right]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$

(iii) Path of integration is the circle in *xy*-plane of radius of 2 units and centre at the origin, i.e. $x^2 + y^2 = 4$ and in *xy*-plane $z = 0$

Parametric equation of the circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$
$$dx = -2\sin\theta\,d\theta, \qquad dy = 2\cos\theta\,d\theta$$

For the complete circle, θ varies from 0 to 2π.
Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\text{Circulation} = \int_0^{2\pi}[(2\cdot 2\cos\theta - 2\sin\theta)(-2\sin\theta\,d\theta) + (2\cos\theta + 2\sin\theta)(2\cos\theta\,d\theta)]$$

$$= 4\int_0^{2\pi}(-2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \cos\theta\sin\theta)\,d\theta$$

$$= 4\int_0^{2\pi}\left(1 - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= 4\left|\theta + \frac{\cos 2\theta}{4}\right|_0^{2\pi}$$

$$= 8\pi$$

Example 4: Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F}=(x^2+y^2)\hat{i}-2xy\,\hat{j}$ and C is the rectangle in the xy-plane bounded by $y=0, x=a, y=b, x=0$.

Solution : (i) Let $\overline{r}=x\hat{i}+y\hat{j}$

$$d\overline{r}=\hat{i}\,dx+\hat{j}\,dy$$

(ii) $\overline{F}\cdot d\overline{r}=\left[(x^2+y^2)\hat{i}-2xy\,\hat{j}\right]\cdot(\hat{i}\,dx+\hat{j}\,dy)$

$$=(x^2+y^2)dx-2xy\,dy$$

(iii) Path of integration is the rectangle $OABD$ bounded by the four lines $y=0, x=a, y=b, x=0$.

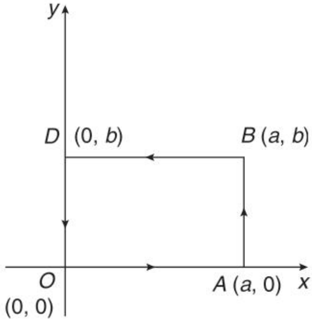


Fig. 7.4

$$\int_C \overline{F}\cdot d\overline{r}=\int_{OA}\overline{F}\cdot d\overline{r}+\int_{AB}\overline{F}\cdot d\overline{r}+\int_{BD}\overline{F}\cdot d\overline{r}+\int_{DO}\overline{F}\cdot d\overline{r} \qquad \ldots (1)$$

(a) Along $OA : y=0,\quad dy=0$
x varies from 0 to a.

$$\int_{OA}\overline{F}\cdot d\overline{r}=\int_0^a x^2dx=\left|\frac{x^3}{3}\right|_0^a=\frac{a^3}{3}$$

(b) Along $AB : x=a,\quad dx=0$
y varies from 0 to b.

$$\int_{AB}\overline{F}\cdot d\overline{r}=\int_0^b(-2ay)dy=-\left|ay^2\right|_0^b=-ab^2$$

(c) Along $BD : y=b,\quad dy=0$
x varies from a to 0.

$$\int_{BD}\overline{F}\cdot d\overline{r}=\int_a^0(x^2+b^2)dx=\left|\frac{x^3}{3}+b^2x\right|_a^0=-\left(\frac{a^3}{3}+b^2a\right)$$

(d) Along $DO : x=0,\quad dx=0$
y varies from b to 0.

$$\int_{DO}\overline{F}\cdot d\overline{r}=\int_b^0 0\,dy=0$$

Substituting in Eq. (1),

$$\int_C \overline{F}\cdot d\overline{r}=\frac{a^3}{3}-ab^2-\frac{a^3}{3}-b^2a$$
$$=-2ab^2$$

Example 5: Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F}=(3x^2+6y)\hat{i}-14yz\hat{j}+20xz^2\hat{k}$ and C is the straight line joining the points (0, 0, 0) to (1, 1, 1).

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= (3x^2+6y)dx - 14yzdy + 20xz^2dz$$

(iii) Path of integration is the straight line joining the points $A(0, 0, 0)$ to $B(1, 1, 1)$. Equation of the line AB is

$$\frac{x-0}{0-1} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$

$$x = y = z$$

$$dx = dy = dz$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $x = 0$ to $x = 1$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 \left[(3x^2+6x)dx - 14x^2\,dx + 20x^3dx\right]$$

$$= \int_0^1 (20x^3 - 11x^2 + 6x)\,dx$$

$$= \left|20\frac{x^4}{4} - \frac{11x^3}{3} + \frac{6x^2}{2}\right|_0^1$$

$$= \frac{13}{3}$$

Example 6: Evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve $x^2 + y^2 = 1$, $z = 1$ in the positive direction from (0, 1, 1) to (1, 0, 1), where $\bar{F} = (yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}$.

Solution: (i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}\right]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= (yz+2x)dx + xzdy + (xy+2z)dz$$

(iii) Path of integration is the part of the curve $x^2 + y^2 = 1$, $z = 1$ from (0, 1, 1) to (1, 0, 1).

Parametric equation of the curve is

$$x = \cos\theta, \qquad y = \sin\theta, \qquad z = 1$$

$$dx = -\sin\theta\, d\theta, \qquad dy = \cos\theta\, d\theta, \qquad dz = 0$$

At point $A : x = 0$

$$\cos\theta = 0,\ \theta = \frac{\pi}{2}$$

At point $B : x = 1$

$$\cos\theta = 1,$$

$$\theta = 2\pi$$

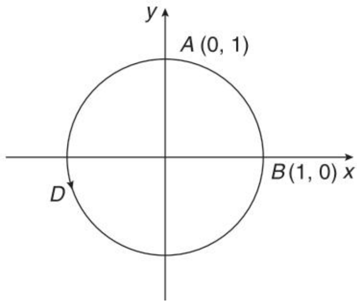


Fig. 7.5

Substituting in $\overline{F}\cdot \mathrm{d}\overline{r}$ and integrating between the limits $\theta = \dfrac{\pi}{2}$ to $\theta = 2\pi$,

$$\begin{aligned}
\int_{ADB}\overline{F}\cdot \mathrm{d}\overline{r} &= \int_{\frac{\pi}{2}}^{2\pi}\left[(\sin\theta+2\cos\theta)(-\sin\theta\,\mathrm{d}\theta)+\cos\theta(\cos\theta\,\mathrm{d}\theta)\right]\\
&= \int_{\frac{\pi}{2}}^{2\pi}(\cos^2\theta-\sin^2\theta-2\cos\theta\sin\theta)\mathrm{d}\theta\\
&= \int_{\frac{\pi}{2}}^{2\pi}(\cos 2\theta-\sin 2\theta)\mathrm{d}\theta\\
&= \left|\frac{\sin 2\theta}{2}+\frac{\cos 2\theta}{2}\right|_{\frac{\pi}{2}}^{2\pi}\\
&= \frac{1}{2}(\sin 4\pi-\sin\pi+\cos 4\pi-\cos\pi)\\
&= 1
\end{aligned}$$

Example 7: Evaluate $\int_C \overline{F}\cdot \mathrm{d}\overline{r}$ over the circular path $x^2 + y^2 = a^2$ where $\overline{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$.

Solution: (i) Let $\overline{r} = x\hat{i} + y\hat{j}$

$$\mathrm{d}\overline{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y$$

(ii)

$$\begin{aligned}
\overline{F}\cdot \mathrm{d}\overline{r} &= \left[\sin y\,\hat{i} + x(1+\cos y)\hat{j}\right]\cdot\left(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y\right)\\
&= \sin y\,\mathrm{d}x + x(1+\cos y)\mathrm{d}y\\
&= \sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y + x\,\mathrm{d}y\\
&= \mathrm{d}(x\sin y) + x\mathrm{d}y
\end{aligned}$$

(iii) Path of integration is the circle $x^2 + y^2 = a^2$.
Parametric equation of the circle is

$$x = a\cos\theta, \qquad y = a\sin\theta$$
$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = a\cos\theta\,\mathrm{d}\theta$$

For complete circle, θ varies from 0 to 2π.
Substituting in $\overline{F}\cdot \mathrm{d}\overline{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\begin{aligned}
\int_C \overline{F}\cdot \mathrm{d}\overline{r} &= \int_0^{2\pi}\left[\mathrm{d}\{a\cos\theta\sin(a\sin\theta)\} + a\cos\theta\cdot a\cos\theta\,\mathrm{d}\theta\right]\\
&= \left|a\cos\theta\sin(a\sin\theta)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\mathrm{d}\theta\\
&= 0 + \frac{a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}\\
&= \pi a^2
\end{aligned}$$

Example 8: Find work done in moving a particle in the force field $\overline{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve $x^2 = 4y$ and $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution: Work done $= \int_C \overline{F} \cdot \mathrm{d}\overline{r}$

(i) Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\mathrm{d}\overline{r} = \hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y + \hat{k}\,\mathrm{d}z$$

(ii) $\overline{F} \cdot \mathrm{d}\overline{r} = \left[3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}\right] \cdot (\hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y + \hat{k}\,\mathrm{d}z)$

$$= 3x^2\mathrm{d}x + (2xz - y)\mathrm{d}y + z\mathrm{d}z$$

(iii) Path of integration is the curve $x^2 = 4y$ and $3x^3 = 8z$.

$$y = \frac{x^2}{4}, \qquad z = \frac{3}{8}x^3$$

$$\mathrm{d}y = \frac{x}{2}\mathrm{d}x, \quad \mathrm{d}z = \frac{9x^2}{8}\mathrm{d}x$$

Substituting in $\overline{F} \cdot \mathrm{d}\overline{r}$ and integrating between the limits $x = 0$ to $x = 2$,

$$\text{Work done} = \int_0^2 \left[3x^2\mathrm{d}x + \left(2x \cdot \frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{x}{2}\mathrm{d}x + \frac{3x^3}{8} \cdot \frac{9x^2}{8}\mathrm{d}x\right]$$

$$= \int_0^2 \left(3x^2 + \frac{51x^5}{64} - \frac{x^3}{8}\right)\mathrm{d}x$$

$$= \left|\frac{3x^3}{3} + \frac{51}{64} \cdot \frac{x^6}{6} - \frac{1}{8} \cdot \frac{x^4}{4}\right|_0^2$$

$$= 8 + \frac{51}{6} - \frac{1}{2}$$

$$= 16$$

Example 9: Find the work done in moving a particle from $A(1, 0, 1)$ to $B(2, 1, 2)$ along the straight line AB in the force field $\overline{F} = x^2\hat{i} + (x - y)\hat{j} + (y + z)\hat{k}$.

Solution: Work done $= \int \overline{F} \cdot \mathrm{d}\overline{r}$

(i) Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\mathrm{d}\overline{r} = \hat{i}\,\mathrm{d}x + \hat{j}\,\mathrm{d}y + \hat{k}\,\mathrm{d}z$$

(ii) $\overline{F} \cdot \mathrm{d}\overline{r} = \left[x^2\hat{i} + (x - y)\hat{j} + (y + z)\hat{k}\right] \cdot (\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z)$

$$= x^2\mathrm{d}x + (x - y)\mathrm{d}y + (y + z)\mathrm{d}z$$

(iii) Path of integration is the straight line AB joining the points $A(1, 0, 1)$ and $B(2, 1, 2)$.
Equation of the line AB is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

$$\frac{x-1}{1-2}=\frac{y-0}{0-1}=\frac{z-1}{1-2}$$

$$x-1=y=z-1$$

$$x=1+y, \qquad z=1+y$$

$$dx=dy, \qquad dz=dy$$

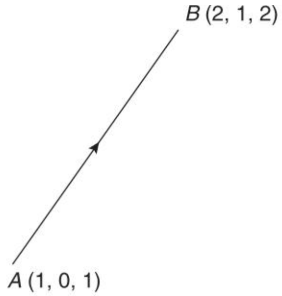


Fig. 7.6

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $y=0$ to $y=1$,

$$\text{Work done} = \int_0^1\left[(1+y)^2 dy + (1+y-y)dy + (y+1+y)dy\right]$$

$$= \int_0^1\left[(1+y)^2 + 2 + 2y\right]dy$$

$$= \left|\frac{(1+y)^3}{3} + 2y + y^2\right|_0^1$$

$$= \frac{8}{3} + 2 + 1 - \frac{1}{3}$$

$$= \frac{16}{3}$$

Example 10: Find work done in moving a particle along the straight line segments joining the points (0, 0, 0) to (1, 0, 0), then to (1, 1, 0) and finally to (1, 1, 1) under the force field $\bar{F}=(3x^2+6y)\hat{i}-14yz\hat{j}+20xz^2\hat{k}$.

Solution: Work done $=\int \bar{F}\cdot d\bar{r}$

(i) Let $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$d\bar{r}=\hat{i}dx+\hat{j}dy+\hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r}=\left[(3x^2+6y)\hat{i}-14yz\hat{j}+20xz^2\hat{k}\right]\cdot(\hat{i}dx+\hat{j}dy+\hat{k}dz)$

$$=(3x^2+6y)dx-14yz\,dy+20xz^2dz$$

(iii) Path of integration is the line segments joining the points $O(0, 0, 0)$ to $A(1, 0, 0)$, $A(1, 0, 0)$ to $B(1, 1, 0)$ and then $B(1, 1, 0)$ to $D(1, 1, 1)$.

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BD}\bar{F}\cdot d\bar{r} \qquad \text{... (1)}$$

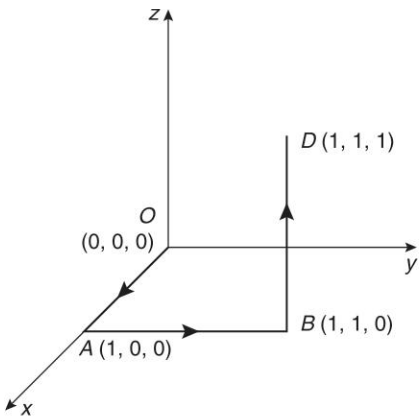


Fig. 7.7

(a) Along OA : $y = 0, \quad z = 0$

$dy = 0, \quad dz = 0$

x varies from 0 to 1.

$$\int_{OA} \overline{F} \cdot d\overline{r} = \int_0^1 3x^2 dx = \left| x^3 \right|_0^1 = 1$$

(b) Along AB : $x = 1, \quad z = 0$

$dx = 0, \quad dz = 0$

y varies from 0 to 1.

$$\int_{AB} \overline{F} \cdot d\overline{r} = \int_0^1 0\, dy = 0$$

(c) Along BD : $x = 1, \quad y = 1$

$dx = 0, \quad dy = 0$

z varies from 0 to 1.

$$\int_{BD} \overline{F} \cdot d\overline{r} = \int_0^1 20z^2 dz = 20\left| \frac{z^3}{3} \right|_0^1 = \frac{20}{3}$$

Substituting in Eq. (1),

$$\text{Work done } = 1 + 0 + \frac{20}{3}$$

$$= \frac{23}{3}$$

Example 11: Find the work done by the force $\overline{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in displacing the particle along the triangle OAB, where

$$OA : 0 \le x \le 1, \quad y = x, \quad z = 0$$
$$AB : 0 \le z \le 1, \quad x = 1, \quad y = 1$$
$$BO : 0 \le x \le 1, \quad y = z = x$$

Solution: Work done $= \int_C \bar{F} \cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

(ii) $\bar{F} \cdot d\bar{r} = (x\hat{i} - z\hat{j} + 2y\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= x\,dx - z\,dy + 2y\,dz$$

(iii) Path of integration is the triangle *OAB*.

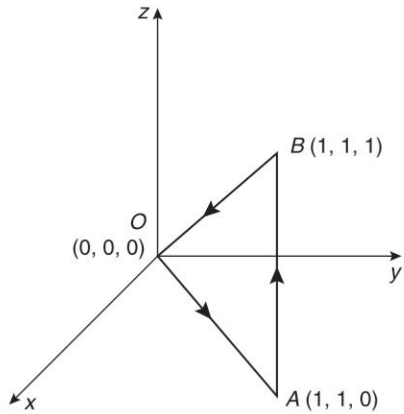


Fig. 7.8

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BO} \bar{F} \cdot d\bar{r} \quad \ldots (1)$$

(a) Along OA : $y = x$, $z = 0$

$dy = dx$, $dz = 0$

x varies from 0 to 1.

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_0^1 x\,dx = \left|\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

(b) Along AB: $x = 1$, $y = 1$

$dx = 0$, $dy = 0$

z varies from 0 to 1.

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 2\,dz = |2z|_0^1 = 2$$

(c) Along BO: $x = y = z$

$dx = dy = dz$

x varies from 1 to 0.

$$\int_{BO} \bar{F} \cdot d\bar{r} = \int_1^0 (x\,dx - x\,dx + 2x\,dx) = |x^2|_1^0 = -1$$

Substituting in Eq. (1),

$$\int_C \bar{F} \cdot d\bar{r} = \frac{1}{2} + 2 - 1 = \frac{3}{2}$$

Example 12: Find the work done by the force $\bar{F} = 16y\hat{i} + (3x^2 + 2)\hat{j}$ in moving a particle once round the right half of the ellipse $x^2 + a^2y^2 = a^2$ from (0, 1) to (0, −1).

Solution: Work done $= \int_C \bar{F} \cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy$$

(ii) $\bar{F}\cdot d\bar{r} = \left[16y\hat{i} + (3x^2+2)\hat{j}\right]\cdot(\hat{i}dx + \hat{j}dy)$

$$= 16y\,dx + (3x^2+2)\,dy$$

(iii) Path of integration is the right half of the ellipse $x^2 + a^2y^2 = a^2$ from (0, 1) to (0, – 1).

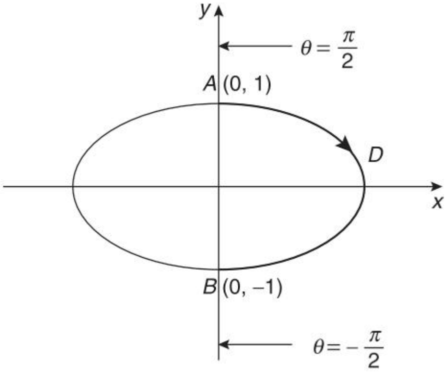


Fig. 7.9

Parametric equation of the ellipse is

$x = a\cos\theta,\qquad y = \sin\theta$

$dx = -a\sin\theta\,d\theta,\qquad dy = \cos\theta\,d\theta$

At point A : $y = 1$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

At point B : $y = -1$

$$\sin\theta = -1$$

$$\theta = -\frac{\pi}{2}$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = \frac{\pi}{2}$ to $\theta = -\frac{\pi}{2}$,

$$\text{Work done} = \int_{ADB}\bar{F}\cdot d\bar{r}$$

$$= \int_{\frac{\theta}{2}}^{-\frac{\theta}{2}}\left[16\sin\theta(-a\sin\theta d\theta) + (3a^2\cos^2\theta + 2)(\cos\theta d\theta)\right]$$

$$= -\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}}(-16a\sin^2\theta + 3a^2\cos^3\theta + 2\cos\theta)\,d\theta$$

$$= -2\int_0^{\frac{\pi}{2}}(-16a\sin^2\theta + 3a^2\cos^3\theta + 2\cos\theta)\,d\theta$$

$$= -2\left[-16a\cdot\frac{1}{2}B\left(\frac{3}{2}, \frac{1}{2}\right) + 3a^2\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right) + 2\cdot\frac{1}{2}B\left(1, \frac{1}{2}\right)\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right]$$

$$= -2\left[-8a\frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma 2} + \frac{3a^2}{2}\frac{\Gamma 2\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{2}} + \frac{\Gamma 1\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{2}}\right]$$

$$= -2\left[-8a\cdot\frac{1}{2}\pi + \frac{3a^2}{2}\cdot\frac{4}{3} + 2\right]$$

$$= 8a\pi - 4a^2 - 4$$

Example 13: If $\bar{F} = 2xyz\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$, then

(i) if $\bar{F}$ is conservative, find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (0, 1, 1) to (1, 2, 0)

Solution :

(i) Since $\bar{F}$ is conservative,

$$\bar{F} = \nabla\phi$$

$$(2xyz)\hat{i} + (x^2z + 2y)\hat{j} + (x^2y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficient of $\hat{i},\ \hat{j},\ \hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = 2xyz, \frac{\partial\phi}{\partial y} = x^2z + 2y, \frac{\partial\phi}{\partial z} = x^2y$$

But, $\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\,\mathrm{d}x + \frac{\partial\phi}{\partial y}\,\mathrm{d}y + \frac{\partial\phi}{\partial z}\,\mathrm{d}z$

$$= (2xyz)\,\mathrm{d}x + (x^2z + 2y)\,\mathrm{d}y + (x^2y)\,\mathrm{d}z$$

Integrating both the sides,

$$\int \mathrm{d}\phi = \int_{\substack{y,z\\ \text{constant}}} 2xyz\,\mathrm{d}x + \int_{\substack{x,z\\ \text{constant}}} (x^2z + 2y)\,\mathrm{d}y + \int_{\substack{x,y\\ \text{constant}}} (x^2y)\,\mathrm{d}z$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e. omitting the x^2yz term in second and third integral,

$$\phi = x^2yz + y^2 + c$$

where c is the constant of integration.

(ii) $\bar{F}$ is conservative and hence the work-done is independent of the path.

Work done $= \int_C \bar{F}\cdot\mathrm{d}\bar{r}$

$$= \int_{(0,1,1)}^{(1,2,0)} \mathrm{d}\phi = \left|\phi\right|_{(0,1,1)}^{(1,2,0)}$$

$$= \left|x^2yz + y^2 + c\right|_{(0,1,1)}^{(1,2,0)}$$

$$= 3$$

Example 14: If $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$, then

(i) If $\bar{F}$ is conservative, find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (1, 1, 0) to (2, 0, 1)

Solution:

(i) Since $\bar{F}$ is conservative,

$$\bar{F} = \nabla\phi$$

$$(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = x^2 - yz, \quad \frac{\partial\phi}{\partial y} = y^2 - zx, \quad \frac{\partial\phi}{\partial z} = z^2 - xy$$

But,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz$$

Integrating both the sides,

$$\int d\phi = \int_{\substack{y,z\\ \text{constant}}} (x^2 - yz)\,dx + \int_{\substack{x,z\\ \text{constant}}} (y^2 - zx)\,dy + \int_{\substack{x,y\\ \text{constant}}} (z^2 - xy)\,dz$$

Considering only those terms in R.H.S. integral which have not appeared in the previous integral, i.e. omitting the *xyz* term in second and third integral,

$$\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$$

where c is the constant of integration.

(ii) $\bar{F}$ is conservative and hence the work done is independent of the path.

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_{(1,1,0)}^{(2,0,1)} d\phi$$

$$= \left|\phi\right|_{(1,1,0)}^{(2,0,1)}$$

$$= \left|\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c\right|_{(1,1,0)}^{(2,0,1)}$$

$$= \frac{7}{3}$$

Exercise 7.1

1. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=(x+y)\hat{i}+(y-x)\hat{j}$ and C is

(i) the parabola $y^2=x$ between the points (1, 1) and (4, 2)

(ii) the straight line joining the points (1, 1) and (4, 2)

$\left[\textbf{Ans.:} \text{(i) } \frac{34}{3} \text{ (ii) } 11\right]$

2. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=(3x-2y)\hat{i}+(y+2z)\hat{j}-x^2\hat{k}$ and C is

(i) the curve $x=t, y=t^2, z=t^3$ between the points (0, 0, 0) to (1, 1, 1)

(ii) the straight line joining the points (0, 0, 0) to (1, 1, 1).

(iii) the straight lines from (0, 0, 0) to (0, 1, 0) then to (0, 1, 1) and then to (1, 1, 1).

$\left[\textbf{Ans.:} \text{(i) } \frac{23}{15} \text{ (ii) } \frac{5}{3} \text{ (iii) } 0\right]$

3. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=(2x+y^2)\hat{i}+(3y-4x)\hat{j}$ and C is the triangle in the xy-plane with vertices (0, 0), (2, 0) and (2, 1).

$\left[\textbf{Ans.:} -\frac{14}{3}\right]$

4. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and C is the curve $y^2=x, z=0$ from (0, 0, 0) to (1, 1, 0) followed by the straight line from (1, 1, 0) to (1, 1, 1).

$\left[\textbf{Ans.:} \frac{3}{4}\right]$

5. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=2x\hat{i}+4y\hat{j}-3z\hat{k}$ and C is the curve $\bar{r}=\cos t\,\hat{i}+\sin t\,\hat{j}+t\hat{k}$ from $t=0$ to $t=\pi$.

$\left[\textbf{Ans.:} -\frac{3\pi^2}{2}\right]$

6. Find the circulation of $\bar{F}=(x-3y)\hat{i}+(y-2x)\hat{j}$ around the ellipse in the xy-plane with the origin as centre and 2 and 3 as semi-major and semi-minor axes respectively.

$[\textbf{Ans.:} 6\pi]$

7. Find the circulation of $\bar{F}=y\hat{i}+z\hat{j}+x\hat{k}$ around the curve $x^2+y^2=1$, $z=0$.

$[\textbf{Ans.:} -\pi]$

8. Find the work done in moving a particle in a force field $\bar{F}=3xy\hat{i}-5z\hat{j}+10x\hat{k}$ along the curve $x=1+t^2, y=2t^2, z=t^3$ from $t=1$ to $t=2$.

$[\textbf{Ans.:} 303]$

9. Find the work done in moving a particle in a force field $\bar{F}=3x^2\hat{i}+(2xz-y)\hat{j}+z\hat{k}$ along the

(i) straight line joining the points (0, 0, 0) and (2, 1, 3)

(ii) curve $x=2t^2, y=t, z=4t^2-t$ from $t=0$ to $t=1$

$\left[\textbf{Ans.:} \text{(i) } 16 \text{ (ii) } \frac{71}{5}\right]$

10. Find the work done in moving a particle in a force field $\bar{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$ once around the circle in xy-plane with centre at the origin and radius of 3 units.

$[\textbf{Ans.:} 18\pi]$

11. If $\overline{F}=(2xy+z^3)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is conservative then

(i) find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (1, –2, 1) to (3, 1, 4)

$[\textbf{Ans.:}\ \text{(i)}\ \phi=x^2y+xz^3+c\ \text{(ii)}\ 202]$

12. If $\overline{F}=3x^2y\hat{i}+(x^3-2yz^2)\hat{j}+(3z^2-2y^2z)\hat{k}$, is conservative

(i) find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (2, 1, 1) to (2, 0, 1)

$\left[\textbf{Ans.:}\ \text{(i)}\ \phi=x^3y+z^3-y^2z^2+c \quad \text{(ii)}\ -7\right]$

13. If $\overline{F}=2xye^z\hat{i}+x^2e^z\hat{j}+x^2ye^z\hat{k}$ is conservative, then find

(i) the scalar potential ϕ

(ii) the work done in moving a particle under this force field from (0, 0, 0) to (1, 1, 1)

$[\textbf{Ans.:}\ \text{(i)}\ \phi=x^2ye^z+c\ \text{(ii)}\ e]$

14. Evaluate

$\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F}=\cos y\,\hat{i}-x\sin y\,\hat{j}$

and C is the curve $y=\sqrt{1-x^2}$ in the xy-plane from (1, 0) to (0, 1).

$[\textbf{Ans.:}\ -1]$

7.4 GREEN'S THEOREM IN THE PLANE

Statement: If $M(x, y)$, $N(x, y)$ and their partial derivatives $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous in some region R of xy-plane bounded by a closed curve C, then

$$\oint_C (M\,dx+N\,dy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

Proof: Let the region R be bounded by the curve C.

Let the curve C be divided into two parts, the curves EAB and BDE.

Let the equations of the curves EAB and BDE are $x=f_1(y)$, $x=f_2(y)$ respectively and are bounded between the lines $y=c$ and $y=d$.

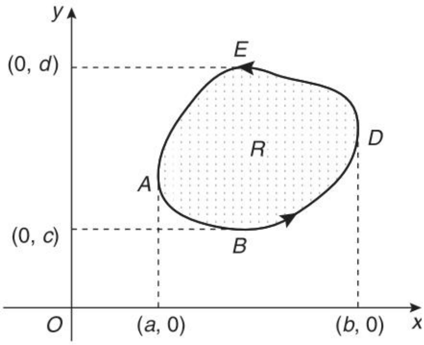


Fig. 7.10

Consider,

$$\iint_R \frac{\partial N}{\partial x}dx\,dy=\int_c^d\left[\int_{f_1(y)}^{f_2(y)}\frac{\partial N}{\partial x}dx\right]dy$$

$$=\int_c^d |N(x,y)|_{f_1(y)}^{f_2(y)}\,dy$$

$$=\int_c^d [N(f_2,y)-N(f_1,y)]\,dy$$

$$=\int_c^d N(f_2,y)\,dy+\int_d^c N(f_1,y)\,dy$$

$$=\int_{BDE} N(x,y)\,dy+\int_{EAB} N(x,y)\,dy$$

$$=\oint_C N(x,y)\,dy$$

$$\oint_C N(x, y)\,\mathrm{d}y = \iint_R \frac{\partial N}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{... (7.1)}$$

Similarly, let the curve C be divided into two parts, the curves ABD and DEA.
Let the equations of the curves ABD and DEA are $y = g_1(x)$, $y = g_2(x)$ respectively and are bounded between the lines $x = a$ and $x = b$.
Consider,

$$\begin{aligned}
\iint_R \frac{\partial M}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,\mathrm{d}y\right]\mathrm{d}x \\
&= \int_a^b \left|M(x, y)\right|_{g_1(x)}^{g_2(x)}\,\mathrm{d}x \\
&= \int_a^b \left[M(x, g_2) - M(x, g_1)\right]\mathrm{d}x \\
&= -\int_b^a M(x, g_2)\,\mathrm{d}x - \int_a^b M(x, g_1)\,\mathrm{d}x \\
&= -\left[\int_{DEA} M(x, y)\,\mathrm{d}x + \int_{ABD} M(x, y)\,\mathrm{d}x\right] \\
&= -\oint_C M(x, y)\,\mathrm{d}x
\end{aligned}$$

$$\oint_C M(x, y)\,\mathrm{d}x = -\iint_R \frac{\partial M}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{... (7.2)}$$

Adding Eqs. (7.1) and (7.2),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Note: Vector form of Green's theorem is given as

$$\oint_C \bar{F} \cdot \mathrm{d}\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

where $\bar{F} = M\hat{i} + N\hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is the unit vector along z-axis.

Area of the Plane Region Let A be the area of the plane region R bounded by a closed curve C.

Let
$$M = -y, \quad N = x$$
$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$$

Using Green's theorem,

$$\oint_C (-y\,\mathrm{d}x + x\,\mathrm{d}y) = \iint_R (1+1)\,\mathrm{d}x\,\mathrm{d}y = 2\iint_R \mathrm{d}x\,\mathrm{d}y = 2A$$

Hence, $$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Note: In polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$dx = \cos\theta dr - r\sin\theta d\theta, \qquad dy = \sin\theta dr + r\cos\theta d\theta$$

$$A = \frac{1}{2}\oint_C \left[r\cos\theta(\sin\theta\,dr + r\cos\theta\,d\theta) - r\sin\theta(\cos\theta\,dr - r\sin\theta\,d\theta)\right]$$

$$= \frac{1}{2}\oint_C r^2 d\theta$$

Example 1: Verify Green's theorem for $\oint_C \left[(x^2 - 2xy)dx + (x^2 y + 3)dy\right]$ where C is the boundary of the region bounded by the parabola $y = x^2$ and the line $y = x$.

Solution: (i) The points of intersection of the parabola $y = x^2$ and the line $y = x$ are obtained as $x = x^2$, $x = 0, 1$ and $y = 0, 1$.

Hence, $O(0, 0)$ and $B(1, 1)$ are the points of intersection.

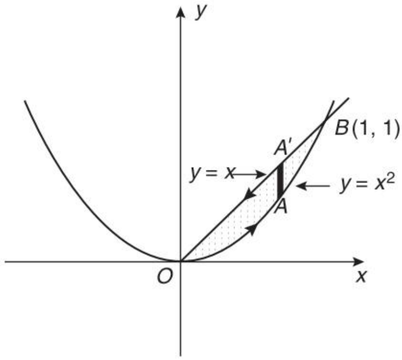


Fig. 7.11

(ii) $M = x^2 - 2xy, \qquad N = x^2 y + 3$

$$\frac{\partial M}{\partial y} = -2x, \qquad \frac{\partial N}{\partial x} = 2xy$$

(iii) $\oint_C (M\,dx + N\,dy)$

$$= \int_{OAB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy) \quad \dots (1)$$

(a) Along OAB : $y = x^2$

$$dy = 2x\,dx$$

x varies from 0 to 1.

$$\int_{OAB} (M\,dx + N\,dy) = \int_{OAB} \left[(x^2 - 2xy)dx + (x^2 y + 3)dy\right]$$

$$= \int_0^1 \left[(x^2 - 2x \cdot x^2)dx + (x^2 \cdot x^2 + 3)2x\,dx\right]$$

$$= \int_0^1 (x^2 - 2x^3 + 2x^5 + 6x)\,dx$$

$$= \left| \frac{x^3}{3} - \frac{2x^4}{4} + \frac{2x^6}{6} + \frac{6x^2}{2} \right|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + 3$$

$$= \frac{19}{6}$$

(b) Along BO : $y = x$

$dy = dx$

x varies from : $x = 1$ to $x = 0$.

$$\begin{aligned}\int_{BO}(M\,dx + N\,dy) &= \int_{BO}\left[(x^2 - 2xy)dx + (x^2y + 3)dy\right]\\ &= \int_1^0\left[(x^2 - 2x^2)dx + (x^3 + 3)dx\right]\\ &= \left|-\frac{x^3}{3} + \frac{x^4}{4} + 3x\right|_1^0\\ &= \frac{1}{3} - \frac{1}{4} - 3\\ &= -\frac{35}{12}\end{aligned}$$

Substituting in Eq. (1),

$$\oint_C(M\,dx + N\,dy) = \frac{19}{6} - \frac{35}{12} = \frac{1}{4} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the line $y = x$ and the parabola $y = x^2$.
Along the vertical strip AA', y varies from x^2 to x and in the region R, x varies from 0 to 1.

$$\begin{aligned}\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy &= \int_0^1\int_{x^2}^x(2xy + 2x)\,dy\,dx\\ &= \int_0^1\left|xy^2 + 2xy\right|_{x^2}^x dx\\ &= \int_0^1(x^3 + 2x^2 - x^5 - 2x^3)\,dx\\ &= \left|\frac{-x^4}{4} + \frac{2x^3}{3} - \frac{x^6}{6}\right|_0^1\\ &= -\frac{1}{4} + \frac{2}{3} - \frac{1}{6}\\ &= \frac{1}{4}\end{aligned} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C(M\,dx + N\,dy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \frac{1}{4}$$

Hence, Green's theorem is verified.

Example 2: Verify Green's theorem for $\oint_C [(x-y)\,dx + 3xy\,dy]$, where C is the boundary of the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Solution: (i) The points of intersection of the parabolas

$x^2 = 4y$ and $y^2 = 4x$ are obtained as

$$\left(\frac{y^2}{4}\right)^2 = 4y,\ y(y^3 - 64) = 0$$

$$y = 0, 4$$

$$x = 0, 4$$

Hence, $O(0, 0)$ and $C(4, 4)$ are the points of intersection.

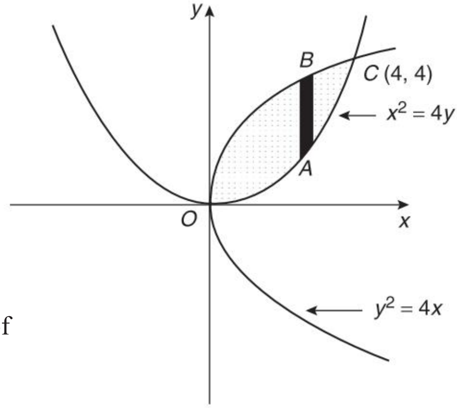


Fig. 7.12

(ii) $M = x - y, \qquad N = 3xy$

$$\frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 3y$$

(iii) $$\oint_C (M\,dx + N\,dy) = \int_{OAC} (M\,dx + N\,dy) + \int_{CBO} (M\,dx + N\,dy) \qquad \text{... (1)}$$

(a) Along OAC: $x^2 = 4y,\ y = \dfrac{x^2}{4}$

$$dy = \frac{x}{2}\,dx$$

x varies from 0 to 4.

$$\int_{OAC} (M\,dx + N\,dy) = \int_{OAC} [(x-y)dx + (3xy)dy]$$

$$= \int_0^4 \left(x - \frac{x^2}{4}\right)dx + \left(3x \cdot \frac{x^2}{4}\right)\frac{x}{2}\,dx$$

$$= \int_0^4 \left(x - \frac{x^2}{4} + \frac{3}{8}x^4\right)dx$$

$$= \left|\frac{x^2}{2} - \frac{x^3}{12} + \frac{3}{8}\cdot\frac{x^5}{5}\right|_0^4$$

$$= 8 - \frac{16}{3} + \frac{384}{5}$$

$$= \frac{1192}{15}$$

(b) Along CBO : $y^2 = 4x,\ \ x = \frac{y^2}{4}$

$$dx = \frac{y}{2}\,dy$$

y varies from 0 to 4.

$$\int_{CBO}(M\,dx + N\,dy) = \int_{CBO}\left[(x-y)dx + 3\,xy\,dy\right]$$

$$= \int_4^0 \left(\frac{y^2}{4} - y\right)\frac{y}{2}\,dy + \left(3\cdot\frac{y^2}{4}\cdot y\right)dy$$

$$= \int_4^0 \left(\frac{7y^3}{8} - \frac{y^2}{2}\right)dy = \left|\frac{7}{8}\cdot\frac{y^4}{4} - \frac{1}{2}\cdot\frac{y^3}{3}\right|_4^0$$

$$= -\frac{7}{8}\cdot 64 + \frac{1}{2}\cdot\frac{64}{3}$$

$$= -\frac{136}{3}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = \frac{1192}{15} - \frac{136}{3} = \frac{512}{15} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Along the vertical strip AB, y varies from $\frac{x^2}{4}$ to $2\sqrt{x}$ and in the region R, x varies from 0 to 4.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} (3y+1)\,dx\,dy$$

$$= \int_0^4 \left|\frac{3y^2}{2} + y\right|_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(6x + 2\sqrt{x} - \frac{3}{32}x^4 - \frac{x^2}{4}\right)dx$$

$$= \left|3x^2 + \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{32}\cdot\frac{x^5}{5} - \frac{1}{4}\cdot\frac{x^3}{3}\right|_0^4$$

$$= 48 + \frac{32}{3} - \frac{96}{5} - \frac{16}{3}$$

$$= \frac{512}{15} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{512}{15}$$

Hence, Green's theorem is verified.

Example 3: Verify Green's theorem for $\oint_C [(y - \sin x)\,dx + \cos x\,dy]$ where C is the plane triangle enclosed by the lines $y = 0,\ x = \frac{\pi}{2},\ y = \frac{2x}{\pi}$.

Solution: (i) The point of intersection of the lines $y = \frac{2x}{\pi}$ and $x = \frac{\pi}{2}$ is obtained as

$$y = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1.$$

Hence, $B\left(\frac{\pi}{2}, 1\right)$ is the point of intersection.

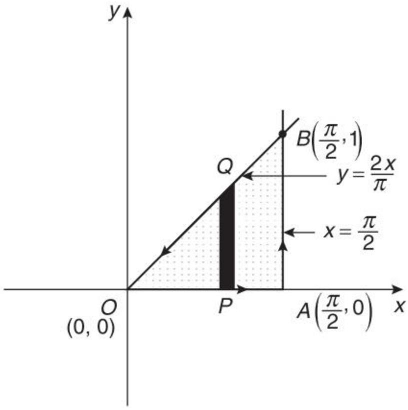


Fig. 7.13

(ii) $M = y - \sin x,\ N = \cos x$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = -\sin x$$

(iii) $\oint_C (M\,dx + N\,dy)$

$$= \int_{OA} (M\,dx + N\,dy) + \int_{AB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy) \qquad \text{... (1)}$$

(a) Along OA : $y = 0$

$dy = 0$

x varies from 0 to $\frac{\pi}{2}$.

$$\int_{OA} (M\,dx + N\,dy) = \int_{OA} [(y - \sin x)dx + \cos x\,dy]$$

$$= \int_0^{\frac{\pi}{2}} (-\sin x)\,dx$$

$$= \left|\cos x\right|_0^{\frac{\pi}{2}}$$

$$= -1$$

(b) Along AB : $x = \frac{\pi}{2}$

$dx = 0$

y varies from 0 to 1.

$$\int_{AB} (M\,dx + N\,dy) = \int_{AB} [(y - \sin x)dx + \cos x\,dy]$$

$$= \int_0^1 \cos\frac{\pi}{2}\,dy$$

$$= 0$$

(c) Along BO: $y = \frac{2x}{\pi}$

$$dy = \frac{2}{\pi}dx$$

x varies from $\frac{\pi}{2}$ to 0.

$$\begin{aligned}
\int_{BO}(M\,dx + N\,dy) &= \int_{BO}[(y-\sin x)dx + \cos x\,dy] \\
&= \int_{\frac{\pi}{2}}^{0}\left[\left(\frac{2x}{\pi} - \sin x\right)dx + \cos x \cdot \frac{2}{\pi}dx\right] \\
&= \left|\frac{2}{\pi}\cdot\frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right|_{\frac{\pi}{2}}^{0} \\
&= \cos 0 - \frac{1}{\pi}\cdot\frac{\pi^2}{4} - \cos\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2} \\
&= 1 - \frac{\pi}{4} - \frac{2}{\pi}
\end{aligned}$$

Substituting in Eq. (1),

$$\oint_C(M\,dx + N\,dy) = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left(\frac{\pi^2+8}{4\pi}\right) \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the triangle OAB.

Along the vertical strip PQ, y varies from 0 to $\frac{2x}{\pi}$ and in the region R, x varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}
\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{2x}{\pi}}(-\sin x - 1)\,dx\,dy \\
&= \int_0^{\frac{\pi}{2}}\left|-y\sin x - y\right|_0^{\frac{2x}{\pi}}dx \\
&= \int_0^{\frac{\pi}{2}}\left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)dx \\
&= -\frac{2}{\pi}\left|x(-\cos x) - (-\sin x) + \frac{x^2}{2}\right|_0^{\frac{\pi}{2}} \\
&= -\frac{2}{\pi}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \frac{\pi^2}{8} - 0\right) \\
&= -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right)
\end{aligned}$$

$$= -\left(\frac{\pi^2+8}{4\pi}\right) \quad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = -\left(\frac{\pi^2+8}{4\pi}\right)$$

Hence, Green's theorem is verified.

Example 4: Verify Green's theorem for $\int_C \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$ where C is the boundary of the region bounded by the parabola $y = \sqrt{x}$ and the lines $x = 1, x = 4, y = 1$.

Solution:

(i) The point of intersection of the

(a) parabola $y = \sqrt{x}$ and the line $x = 1$ is obtained as

$$y = \sqrt{1} = 1$$

Hence, $A(1, 1)$ is the point of intersection.

(b) parabola $y = \sqrt{x}$ and the line $x = 4$ is obtained as

$$y = \sqrt{4} = 2$$

Hence, $D(4, 2)$ is the point of intersection.

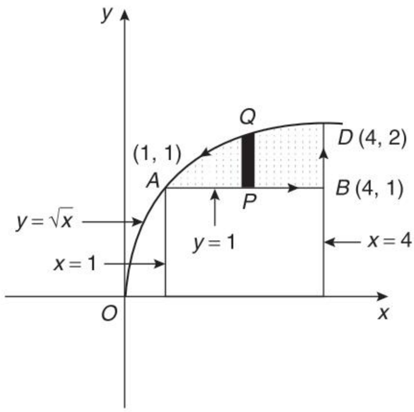


Fig. 7.14

(ii) $M = \frac{1}{y}, \qquad N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

(iii) $$\oint_C (M\,dx + N\,dy) = \int_{AB} (M\,dx + N\,dy) + \int_{BD} (M\,dx + N\,dy) + \int_{DQA} (M\,dx + N\,dy) \quad \text{... (1)}$$

(a) Along $AB : y = 1, dy = 0$
x varies from 1 to 4.

$$\int_{AB} (M\,dx + N\,dy) = \int_{AB} \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$$

$$= \int_1^4 dx$$

$$= |x|_1^4$$

$$= 3$$

(b) Along $BD: x = 4$, $dx = 0$
y varies from 1 to 2.

$$\int_{BD}(M\,dx + N\,dy) = \int_{BD}\left(\frac{1}{y}dx + \frac{1}{x}dy\right)$$
$$= \int_1^2 \frac{1}{4}dy$$
$$= \frac{1}{4}|y|_1^2$$
$$= \frac{1}{4}$$

(c) Along $DQA: \ y = \sqrt{x}, \ dy = \frac{1}{2\sqrt{x}}dx$

x varies from 4 to 1.

$$\int_{DQA}(M\,dx + N\,dy) = \int_{DQA}\left(\frac{1}{y}dx + \frac{1}{x}dy\right)$$
$$= \int_4^1\left(\frac{1}{\sqrt{x}}dx + \frac{1}{x}\cdot\frac{1}{2\sqrt{x}}dx\right)$$
$$= \left|2\sqrt{x} - \frac{1}{\sqrt{x}}\right|_4^1$$
$$= 2 - 1 - 4 + \frac{1}{2}$$
$$= -\frac{5}{2}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = 3 + \frac{1}{4} - \frac{5}{2} = \frac{3}{4} \qquad \text{... (2)}$$

(iv) Let R be the region bounded by the parabola $y = \sqrt{x}$ and the lines $x = 1, x = 4, y = 1$. Along the vertical strip, y varies from 1 to $\sqrt{x}$ and in the region R, x varies from 1 to 4.

$$\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_1^4\int_1^{\sqrt{x}}\left(-\frac{1}{x^2} + \frac{1}{y^2}\right)dx\,dy$$
$$= \int_1^4\left|-\frac{1}{x^2}\cdot y - \frac{1}{y}\right|_1^{\sqrt{x}} dx$$
$$= \int_1^4\left(-x^{-\frac{3}{2}} - x^{-\frac{1}{2}} + \frac{1}{x^2} + 1\right)dx$$
$$= \left|2x^{-\frac{1}{2}} - 2x^{\frac{1}{2}} - \frac{1}{x} + x\right|_1^4$$
$$= 1 - 4 - \frac{1}{4} + 4 - 2 + 2 + 1 - 1$$
$$= \frac{3}{4} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \frac{3}{4}$$

Hence, Green's theorem is verified.

Example 5: Verify Green's theorem for $\oint_C (2xy\,\mathrm{d}x - y^2\mathrm{d}y)$ where C is the boundary of the region bounded by the ellipse $3x^2 + 4y^2 = 12$.

Solution:

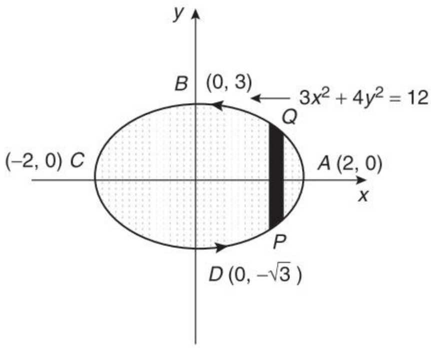


Fig. 7.15

(i) $M = 2xy, \qquad N = -y^2$

$\dfrac{\partial M}{\partial y} = 2x, \qquad \dfrac{\partial N}{\partial x} = 0$

(ii) $\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \oint_C (2xy\,\mathrm{d}x - y^2\,\mathrm{d}y),$ ... (1)

where C is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1.$

Parametric equation of the ellipse is

$$x = 2\cos\theta, \qquad y = \sqrt{3}\sin\theta$$
$$\mathrm{d}x = -2\sin\theta\,\mathrm{d}\theta, \quad \mathrm{d}y = \sqrt{3}\cos\theta\,\mathrm{d}\theta$$

For the given ellipse, θ varies from 0 to 2π.
Substituting in Eq. (1),

$$\begin{aligned}\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) &= \int_0^{2\pi} \left[(2\cdot 2\cos\theta\cdot\sqrt{3}\sin\theta)(-2\sin\theta\,\mathrm{d}\theta) - 3\sin^2\theta\cdot\sqrt{3}\cos\theta\,\mathrm{d}\theta\right] \\ &= \int_0^{2\pi} \left(-11\sqrt{3}\cos\theta\sin^2\theta\right)\mathrm{d}\theta \\ &= -11\sqrt{3}\cdot 2\int_0^{\pi} \cos\theta\sin^2\theta\,\mathrm{d}\theta \\ &= 0 \qquad \text{...(2)}\end{aligned}$$

$$\left[\begin{aligned}\because \int_0^{2a} f(x)\mathrm{d}x &= 2\int_0^{a} f(x)\mathrm{d}x, \text{ if } f(2a-x) = f(x) \\ &= 0, \text{ if } f(2a-x) = -f(x)\end{aligned}\right]$$

(iii) Let R be the region bounded by the ellipse, $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1.$

Along the vertical strip *PQ*, *y* varies from $-\sqrt{3-\dfrac{3x^2}{4}}$ to $\sqrt{3-\dfrac{3x^2}{4}}$ and in the region *R*, *x* varies from –2 to 2.

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{-2}^{2}\int_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} (0-2x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-2}^{2} -2x\left|y\right|_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} \mathrm{d}x$$

$$= -4\int_{-2}^{2} x\sqrt{3-\frac{3x^2}{4}}\,\mathrm{d}x$$

$$= 0 \quad ...(3) \qquad \left[\because \int_{-a}^{a} f(x)\mathrm{d}x = 0, \text{ if } f(-x) = -f(x)\right]$$

From Eqs. (2) and (3),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0$$

Hence, Green's theorem is verified.

Example 6: Evaluate $\oint_C \left[(x^2-\cosh y)\mathrm{d}x + (y+\sin x)\mathrm{d}y\right]$ by Green's theorem where *C* is the rectangle with vertices (0, 0), (π, 0), (π, 1), (0, 1).

Solution: By Green's theorem,

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \quad ... (1)$$

where *R* is the region bounded by the rectangle *OABC*.

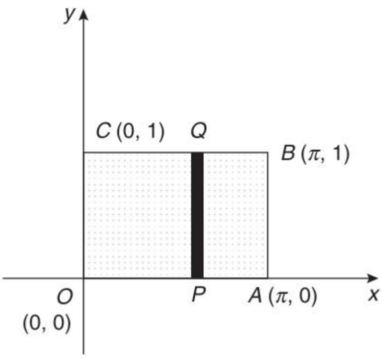


Fig. 7.16

$$M = x^2 - \cosh y, \quad N = y + \sin x$$

$$\frac{\partial M}{\partial y} = -\sinh y, \qquad \frac{\partial N}{\partial x} = \cos x$$

Along the vertical strip *PQ*, *y* varies from 0 to 1 and in the region *R*, *x* varies from 0 to π.

Substituting in Eq. (1),

$$\oint_C \left[(x^2-\cosh y)\mathrm{d}x + (y+\sin x)\mathrm{d}y\right]$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{\pi} \left|y\cos x + \cosh y\right|_0^1 \mathrm{d}x$$

$$= \int_0^{\pi} (\cos x + \cosh 1 - 0 - \cosh 0)\,\mathrm{d}x$$

$$= \int_0^{\pi} (\cos x + \cosh 1 - 1)\, dx$$

$$= \left| \sin x + x \cosh 1 - x \right|_0^{\pi}$$

$$= \sin \pi + \pi \cosh 1 - \pi - \sin 0$$

$$= \pi (\cosh 1 - 1)$$

Example 7: Evaluate by Green's theorem $\oint_C (-x^2 y\, dx + xy^2 dy)$ where C is the cardioid $r = a\,(1 + \cos\theta)$.

Solution: By Green's theroem,

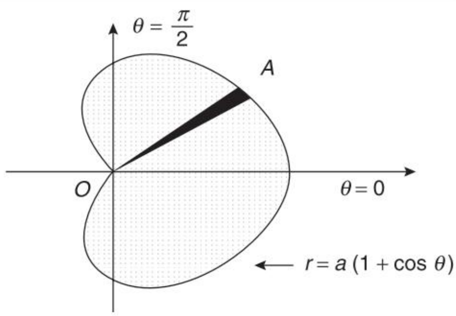


Fig. 7.17

$$\oint_C (M\, dx + N\, dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad \ldots (1)$$

where R is the region bounded by the cardioid $r = a\,(1 + \cos\theta)$.

$$M = -x^2 y, \qquad N = xy^2$$

$$\frac{\partial M}{\partial y} = -x^2, \qquad \frac{\partial N}{\partial x} = y^2$$

Putting $\qquad x = r \cos\theta, \qquad y = r \sin\theta$

$$\frac{\partial M}{\partial y} = -r^2 \cos^2 \theta, \qquad \frac{\partial N}{\partial x} = r^2 \sin^2 \theta$$

$$dx\, dy = r\, dr\, d\theta$$

Along the radius vector OA, r varies from 0 to $a\,(1 + \cos\theta)$ and in the region R, θ varies from 0 to 2π.

Substituting in Eq. (1),

$$\oint_C (-x^2 y\, dx + xy^2\, dy) = \int_0^{2\pi} \int_0^{a(1+\cos\theta)} (r^2 \sin^2 \theta + r^2 \cos^2 \theta) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} r^3 dr\, d\theta$$

$$= \int_0^{2\pi} \left| \frac{r^4}{4} \right|_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{4} \int_0^{2\pi} (1 + \cos\theta)^4\, d\theta \qquad \left[\because \int_0^{2a} f(\theta) d\theta = 2\int_0^{a} f(\theta) d\theta \text{ if } f(2a \quad \theta) = f(\theta) \right]$$

$$= \frac{a^4}{4} \cdot 2\int_0^{\pi} (1 + \cos\theta)^4 d\theta$$

$$= \frac{a^4}{2} \int_0^{\pi} \left(2\cos^2 \frac{\theta}{2} \right)^4 d\theta$$

Putting $\frac{\theta}{2} = t, \quad d\theta = 2dt$

When $\theta = 0, \quad t = 0$

$\theta = \pi, \quad t = \frac{\pi}{2}$

$$\oint_C (-x^2 y\,dx + xy^2 dy) = 8a^4 \int_0^{\frac{\pi}{2}} \cos^8 t \cdot 2\,dt$$

$$= 16a^4 \cdot \frac{1}{2} B\left(\frac{9}{2}, \frac{1}{2}\right)$$

$$= 8a^4 \cdot \frac{\overline{|\frac{9}{2}}\,\overline{|\frac{1}{2}}}{\overline{|5}}$$

$$= \frac{8a^4}{24} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \overline{|\frac{1}{2}}\,\overline{|\frac{1}{2}}$$

$$= \frac{35\pi}{16} a^4$$

Example 8: Evaluate $\oint_C \left[(x^2 + 2y)dx + (4x + y^2)dy\right]$ by Green's theorem where C is the boundary of the region bounded by $y = 0$, $y = 2x$ and $x + y = 3$.

Solution: By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad \text{... (1)}$$

where R is the region bounded by the triangle OAB.

$$M = x^2 + 2y, \qquad N = 4x + y^2$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 4$$

Substituting in Eq. (1),

$$\oint_C \left[(x^2 + 2y)dx + (4x + y^2)dy\right]$$

$$= \iint_R (4 - 2)dx\,dy$$

$$= 2\iint_R dx\,dy$$

$$= 2(\text{Area of } \Delta OAB)$$

$$= 2 \cdot \frac{1}{2} \cdot 3 \cdot 2$$

$$= 6$$

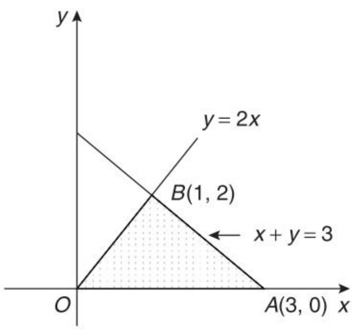


Fig. 7.18

Example 9: Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Solution: (i) The points of intersection of the parabola $y = x^2$ and the line $y = x + 2$ are obtained as

$$x + 2 = x^2, \quad x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0,$$

$$x = 2, -1 \text{ and } y = 4, 1$$

Hence, $A(-1, 1)$ and $B(2, 4)$ are the points of intersection.

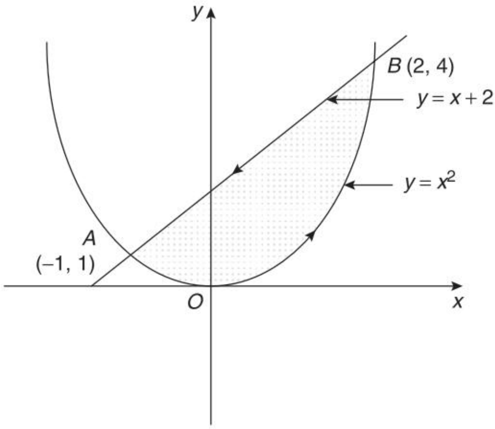


Fig. 7.19

(ii) By Green's theorem, the area of the region bounded by a closed curve C is

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\left[\int_{AOB} (x\,dy - y\,dx) + \int_{BA} (x\,dy - y\,dx)\right] \quad \ldots (1)$$

(a) Along AOB : $y = x^2$, $dy = 2x\,dx$
x varies from -1 to 2.

$$\int_{AOB} (x\,dy - y\,dx) = \int_{-1}^{2} (x \cdot 2x\,dx - x^2 dx)$$

$$= \left|\frac{x^3}{3}\right|_{-1}^{2}$$

$$= \frac{8}{3} + \frac{1}{3}$$

$$= 3$$

(b) Along BA : $y = x + 2$, $dy = dx$
x varies from 2 to -1.

$$\int_{BA} (x\,dy - y\,dx) = \int_{2}^{-1} [x\,dx - (x + 2)\,dx]$$

$$= -2|x|_{2}^{-1}$$

$$= -2(-1 - 2)$$

$$= 6$$

Substituting in Eq. (1),

$$A = \frac{1}{2}(3 + 6) = \frac{9}{2}$$

Example 10: Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

(i) By Green's theorem, the area of the region bounded by a closed curve C is

$$A=\frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x) \quad \text{... (1)}$$

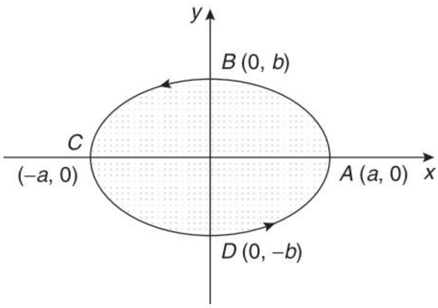


Fig. 7.20

(ii) Parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$x = a\cos\theta, \qquad y = b\sin\theta$$
$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta, \quad \mathrm{d}y = b\cos\theta\,\mathrm{d}\theta$$

For the given ellipse, θ varies from 0 to 2π.

Substituting in Eq. (1),

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} [a\cos\theta(b\cos\theta\,\mathrm{d}\theta) - b\sin\theta(-a\sin\theta\,\mathrm{d}\theta)] \\ &= \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}\theta \\ &= \frac{1}{2}ab\left|\theta\right|_0^{2\pi} \\ &= \pi\, ab \end{aligned}$$

Example 11: Find the area of the loop of the folium of descartes $x^3+y^3=3axy$.

Solution: (i) Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the curve changes to

$$r^3(\cos^3\theta+\sin^3\theta) = 3ar^2\sin\theta\cos\theta$$
$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

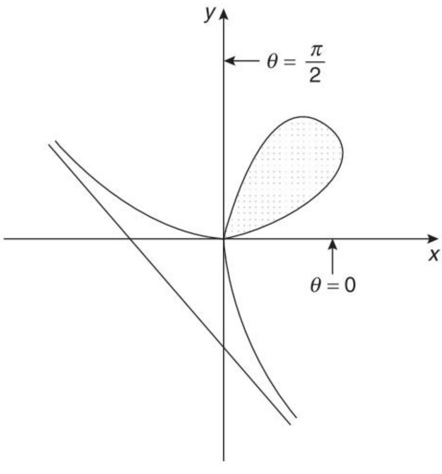


Fig. 7.21

(ii) By Green's theorem, the area of the region bounded by a closed curve C in polar form is

$$A=\frac{1}{2}\oint_C r^2\,\mathrm{d}\theta$$

For the loop of the given curve, θ varies from 0 to $\frac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} \mathrm{d}\theta$$

$$= \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta \cdot \sec^2\theta}{(1+\tan^3\theta)^2} \mathrm{d}\theta$$

Putting
$$1+\tan^3\theta = t$$
$$3\tan^2\theta \sec^2\theta \, \mathrm{d}\theta = \mathrm{d}t$$

When $\theta = 0,\ t = 1$

$\theta = \frac{\pi}{2}, \quad t \to \infty$

$$A = \frac{9a^2}{2}\int_1^{\infty} \frac{\mathrm{d}t}{3t^2}$$

$$= \frac{3a^2}{2}\left| -\frac{1}{t} \right|_1^{\infty}$$

$$= \frac{3a^2}{2}$$

Exercise 7.2

(I) Verify Green's theorem in plane for the following:

1. $\oint_C \left[(x^2 - 2xy)\,\mathrm{d}x + (x^2 y + 3)\,\mathrm{d}y\right]$, where C is the boundary of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$.

$$\left[\textbf{Ans.: } \frac{128}{5}\right]$$

2. $\oint_C \left[(xy - x^2)\,\mathrm{d}x + x^2 y\,\mathrm{d}y\right]$, where C is the boundary of the triangle formed by the lines $y = 0$, $x = 1$ and $y = x$.

$$\left[\textbf{Ans.: } -\frac{1}{12}\right]$$

3. $\oint_C \left[(3x^2 - 8y^2)\,\mathrm{d}y + (4y - 6xy)\,\mathrm{d}y\right]$, where C is the boundary of the region bounded by $y = x^2$ and $y = \sqrt{x}$.

$$\left[\textbf{Ans.: } \frac{3}{2}\right]$$

4. $\oint_C (e^{-x}\sin y\,\mathrm{d}x + e^{-x}\cos y\,\mathrm{d}y)$, where C is the boundary of the region bounded by the square with vertices $(0,0), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$.

$$\left[\textbf{Ans.: } 2\left(e^{-\frac{\pi}{2}} - 1\right)\right]$$

5. $\oint_C (xy^2 - 2xy)\,\mathrm{d}x + (x^2 y + 3)\,\mathrm{d}y$, where C is the boundary of the region bounded by the rectangle with vertices $(-1, 0)$, $(1, 0)$, $(1, 1)$ and $(-1, 1)$.

[**Ans.:** 0]

6. $\oint_C (x^3\mathrm{d}y - y^3\mathrm{d}x)$, where C is the circle $x^2 + y^2 = 4$.

[**Ans.:** 48π]

7. $\oint_C \left[(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\right]$, where C is the boundary of the region bounded by the x-axis and the circle $y = \sqrt{1-x^2}$. $\left[\textbf{Ans.: } \frac{4}{3}\right]$

(II) Evaluate the following integrals using Green's theorem:

1. $\oint_C e^{-x}(\cos y\,dx - \sin y\,dy)$, where C is the boundary of the region bounded by the rectangle with vertices $(0,0)$, $(\pi,0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

$\left[\textbf{Ans.: } 2(1-e^{-\pi})\right]$

2. $\oint_C \left[(x^2 + y^2)\,dx + (5x^2 - 3y)\,dy\right]$, where C is the boundary of the region bounded by the parabola $x^2 = 4y$ and the line $y = 4$.

$\left[\textbf{Ans.: } -\frac{512}{5}\right]$

3. $\oint_C \left[(y^3 - xy)\,dx + (xy + 3xy^2)\,dy\right]$, where C is the boundary of the region in the first quadrant bounded by the y-axis and the parabolas $y = 1 - x^2$, $y = x^2$.

$\left[\textbf{Ans.: } \left(\frac{1}{8} + \frac{\sqrt{2}}{6}\right).\right]$

4. $\oint_C (xy\,dx + x^3\,dy)$, where C is the boundary of the region bounded by the x-axis and the circle $y = \sqrt{4-x^2}$.

[**Ans.:** 6π]

5. $\oint_C e^x(\sin y\,dx + \cos y\,dy)$, where C is the boundary of the region bounded by the ellipse $4(x+1)^2 + 9(y-3)^2 = 36$.

[**Ans.:** 0]

(III) Find the area of the following regions using Green's theorem:

1. Bounded by the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$\left[\textbf{Ans.: } \frac{3\pi}{8}a^2\right]$

2. Bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the x-axis.

[**Ans.:** $3\pi a^2$]

3. In the first quadrant, bounded by the lines $y = x$, $x = 4y$ and rectangular hyperbola $xy = 1$.

[**Ans.:** $\log 2$]

4. Bounded by one loop of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

$\left[\textbf{Ans.: } \frac{a^2}{2}\right]$

7.5 SURFACE INTEGRALS

The surface integral over a curved surface S is the generalisation of a double integral over a plane region R. Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a continuous vector point function defined over a two-sided surface S. Divide S into a finite number of subsurfaces $S_1, S_2, \ldots\ldots, S_m$ with surface areas $\delta S_1, \delta S_2, \ldots\ldots, \delta S_m$. Let δS_r be the surface area of S_r and $\hat{n}_r$ be the unit vector at some point P_r (in S_r) in the direction of the outward normal to S_r.

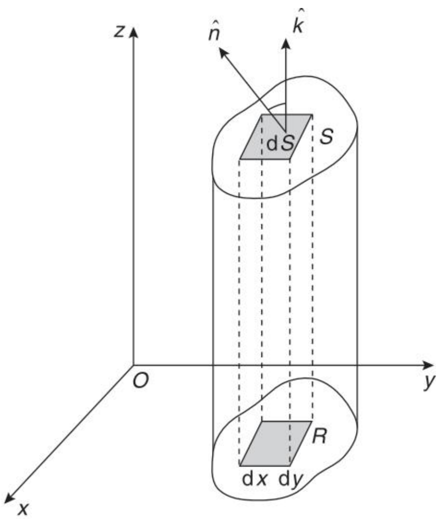


Fig. 7.22

If we increase the number of subsurfaces, then the surface area δS_r of each subsurface will decrease. Thus, as $m \to \infty, \delta S_r \to 0$

Then,

$$\lim_{m \to \infty} \sum_{r=1}^{m} \overline{F}(P_r) \cdot \hat{n}_r \, \delta S_r = \iint_S \overline{F} \cdot \hat{n} \, dS$$

This is called surface integral of $\overline{F}$ over the surface S.

The surface integral can also be written as

$$\iint_S \overline{F} \cdot \mathrm{d}\overline{S}, \text{ where } \mathrm{d}\overline{S} = \hat{n} \, \mathrm{d}S$$

If equation of the surface S is $\phi(x, y, z) = 0$, then $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

7.5.1 *Flux*

If $\overline{F}$ represents velocity of the fluid at any point P on a closed surface S, then surface integral $\iint_S \overline{F} \cdot \hat{n} \, \mathrm{d}S$ represents the flux of $\overline{F}$ over S, i.e., volume of the fluid flowing out from S per unit time.

Note: If $\iint_S \overline{F} \cdot \hat{n} \, \mathrm{d}S = 0$, then $\overline{F}$ is called a solenoidal vector point function.

7.5.2 *Evaluation of Surface Integral*

A surface integral is evaluated by expressing it as a double integral over the region R. The region R is the orthogonal projection of S on one of the coordinate planes (xy, yz or zx). Let R be the orthogonal projection of S on the xy-plane and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$. Then

$$\hat{n} = \cos\alpha \, \hat{i} + \cos\beta \, \hat{j} + \cos\gamma \, \hat{k}$$

$$\begin{aligned} dx\,dy &= \text{Projection of } dS \text{ on } xy\text{-plane} \\ &= dS\cos\gamma \\ dS &= \frac{dx\,dy}{\cos\gamma} \\ &= \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \end{aligned}$$

Hence, $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Similarly, taking projection on yz and zx-plane,

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} \quad\text{and}\quad \iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$

Component Form of Surface Integral

$$\begin{aligned} \iint_S \bar{F}\cdot\hat{n}\,dS &= \iint_S (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot(\cos\alpha\,\hat{i}+\cos\beta\,\hat{j}+\cos\gamma\,\hat{k})\,dS \\ &= \iint_S (F_1\,dS\cos\alpha + F_2\,dS\cos\beta + F_3\,dS\cos\gamma) \\ &= \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) \end{aligned}$$

Example 1: Evaluate $\iint_S \bar{F}\cdot\hat{n}\,dS,$ where $\bar{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ in the first octant.

Solution:

(i) The given surface is the plane $2x + 3y + 6z = 12$ in the first octant.
Let $\phi = 2x + 3y + 6z$

$$\begin{aligned} \hat{n} &= \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} \\ &= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7} \end{aligned}$$

(ii) Let R be the projection of the plane $2x + 3y + 6z = 12$ (in the first octant) on the xy-plane, which is a triangle OAB bounded by the lines $y = 0$, $x = 0$ and $2x + 3y = 12$.

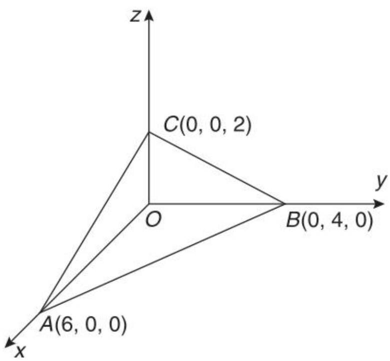


Fig. 7.23

(iii) $\mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$

$$= \frac{7}{6}\mathrm{d}x\,\mathrm{d}y$$

(iv) Along the vertical strip PQ, y varies from 0 to $\dfrac{12-2x}{3}$ and in the region R, x varies from 0 to 6.

$$\iint_S \overline{F}\cdot\hat{n}\mathrm{d}S = \iint_R (18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)\frac{7}{6}\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{6}\iint_R (36z - 36 + 18y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= 3\iint_R \left[2\left(\frac{12-2x-3y}{6}\right) - 2 + y\right]\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\,\mathrm{d}y\,\mathrm{d}x$$

$$= 2\int_0^6 (3-x)\,|y|_0^{\frac{12-2x}{3}}\,\mathrm{d}x$$

$$= 2\int_0^6 (3-x)\frac{(12-2x)}{3}\mathrm{d}x$$

$$= \frac{4}{3}\int_0^6 (x^2 - 9x + 18)\,\mathrm{d}x$$

$$= \frac{4}{3}\left|\frac{x^3}{3} - \frac{9x^2}{2} + 18x\right|_0^6$$

$$= \frac{4}{3}(72 - 162 + 108)$$

$$= 24$$

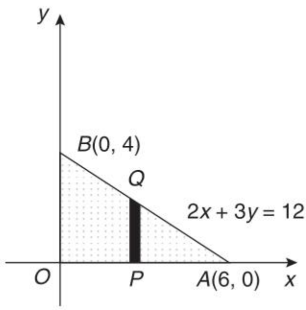


Fig. 7.24

Example 2: Evaluate $\iint_S (yz\,\mathrm{d}y\,\mathrm{d}z + xz\,\mathrm{d}z\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y)$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Solution:

(i) $\iint_S \overline{F}\cdot\hat{n}\mathrm{d}S = yz\,\mathrm{d}y\,\mathrm{d}z + xz\,\mathrm{d}z\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y$

$$\overline{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

(ii) The given surface is the sphere $x^2 + y^2 + z^2 = 1$.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= x\hat{i} + y\hat{j} + z\hat{k} \qquad [\because x^2 + y^2 + z^2 = 1]$$

(iii) Let R be the projection of the sphere $x^2 + y^2 + z^2 = 1$ (in the positive octant) on the xy-plane ($z = 0$), which is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.

(iv) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

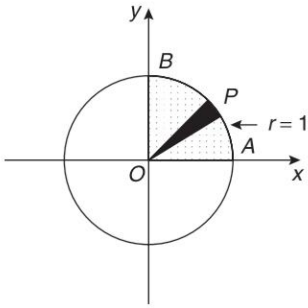


Fig. 7.25

(v) $$\iint_S (yz\,\mathrm{d}y\,\mathrm{d}z + xz\,\mathrm{d}z\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y) = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

$$= \iint_R (yz\hat{i} + xz\hat{j} + xy\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R (3xyz)\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= 3\iint_R xy\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, the equation of the circle $x^2 + y^2 = 1$ reduces to $r = 1$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Along the radius vector OP, r varies from 0 to 1 and in the first quadrant of the circle, θ varies from 0 to $\frac{\pi}{2}$.

$$\iint_S (yz\,\mathrm{d}y\,\mathrm{d}z + xz\,\mathrm{d}z\,\mathrm{d}x + xy\,\mathrm{d}x\,\mathrm{d}y) = 3\int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta\cdot r\sin\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 3\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\,\mathrm{d}\theta\cdot\int_0^1 r^3\,\mathrm{d}r$$

$$= \frac{3}{2}\left|\frac{-\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\cdot\left|\frac{r^4}{4}\right|_0^1$$

$$= \frac{3}{16}(-\cos\pi + \cos 0)$$

$$= \frac{3}{8}$$

Example 3: Find the flux of $\bar{F} = \hat{i} - \hat{j} + xyz\hat{k}$ through the circular region S obtained by cutting the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$.

Solution: Flux $= \iint_S \bar{F} \cdot \hat{n}\, dS$

(i) Surface S is the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$, which is an ellipse $2x^2 + z^2 = a^2$.

(ii) Normal to the ellipse $2x^2 + z^2 = a^2$ is also normal to the plane $y = x$.

Let $\phi = x - y$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

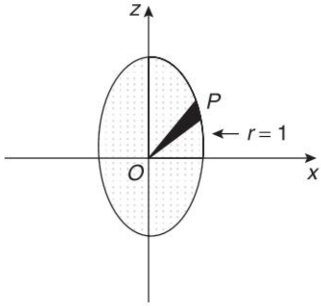


Fig. 7.26

(iii) Let R be the projection of the surface S on the xz-plane, which is an ellipse $2x^2 + z^2 = a^2$

(iv) $dS = \frac{dx\, dz}{|\hat{n} \cdot \hat{j}|}$

$$= \sqrt{2}\, dx\, dz$$

(v) $\iint_S \bar{F} \cdot \hat{n}\, dS = \iint_R (\hat{i} - \hat{j} + xyz\,\hat{k}) \cdot \left(\frac{\hat{i} - \hat{j}}{\sqrt{2}}\right) \sqrt{2}\, dx\, dz$

$$= \iint_R 2\, dx\, dz$$

Putting $x = \frac{a}{\sqrt{2}} r\cos\theta,\ z = ar\sin\theta,$ the equation of the ellipse $2x^2 + z^2 = a^2$ reduces to

$r = 1$ and $dx\, dz = \frac{a^2}{\sqrt{2}} r\, dr\, d\theta$

Along the radius vector OP, r varies from 0 to 1 and for a complete ellipse, θ varies from 0 to 2π.

$$\iint_S \bar{F} \cdot \hat{n}\, dS = 2\int_0^{2\pi} \int_0^1 \frac{a^2}{\sqrt{2}} r\, dr\, d\theta$$

$$= \frac{2a^2}{\sqrt{2}} \left|\frac{r^2}{2}\right|_0^1 |\theta|_0^{2\pi}$$

$$= \sqrt{2}\, a^2 \cdot \frac{1}{2} \cdot 2\pi$$

$$= \sqrt{2}\, \pi a^2$$

Aliter

$$\iint_S \bar{F}\cdot\hat{n}\,dS = 2\iint_R dx\,dz$$

$$= 2\left[\text{Area of the ellipse } \frac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2}+\frac{y^2}{a^2}=1\right]$$

$$= 2\cdot\pi\frac{a}{\sqrt{2}}\cdot a$$

$$= \sqrt{2}\,\pi a^2$$

Hence, $\quad$ flux $= \sqrt{2}\,\pi a^2$

Example 4: Evaluate $\iint_S \bar{F}\cdot\hat{n}\,dS$ where $\bar{F} = 3y\,\hat{i}+2z\,\hat{j}+x^2yz\,\hat{k}$ and S is the surface $y^2 = 5x$ in the positive octant bounded by the planes $x = 3$ and $z = 4$.

Solution:

(i) The given surface is $y^2 = 5x$.
Let $\phi = y^2 - 5x$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-5\hat{i}+2y\,\hat{j}}{\sqrt{25+4y^2}}$$

(ii) Let R be the projection of the surface $y^2 = 5x$ (in the positive octant) bounded by the planes $x = 3$ and $z = 4$ in the xz-plane.

(iii) $$dS = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \frac{\sqrt{25+4y^2}}{2y}\,dx\,dz$$

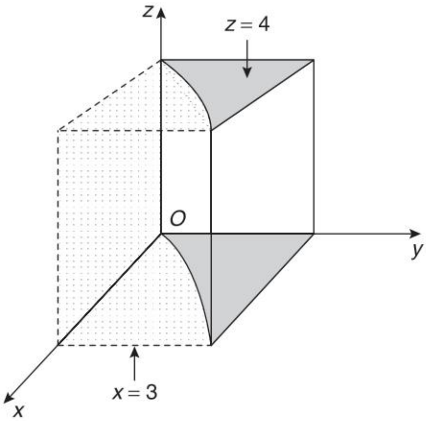


Fig. 7.27

(iv) In the region R, x varies from 0 to 3 and z varies from 0 to 4.

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R (3y\,\hat{i}+2z\,\hat{j}+x^2yz\,\hat{k})\cdot\left(\frac{-5\hat{i}+2y\,\hat{j}}{\sqrt{25+4y^2}}\right)\left(\frac{\sqrt{25+4y^2}}{2y}\right)dx\,dz$$

$$= \frac{1}{2}\iint_R (-15y+4yz)\frac{dx\,dz}{y}$$

$$= \frac{1}{2}\int_{z=0}^{4}\int_{x=0}^{3}(-15+4z)\,dx\,dz$$

$$= \frac{1}{2}\int_0^4 \left(\left|-15x\right|_0^3 + 4z\left|x\right|_0^3 \right) dz$$

$$= \frac{1}{2}\int_0^4 (-45 + 12z) dz$$

$$= \frac{1}{2}\left|-45z + 6z^2\right|_0^4$$

$$= \frac{1}{2}(-180 + 96)$$

$$= -42$$

Exercise 7.3

Evaluate the following integrals:

1. $\iint_S \overline{F} \cdot \hat{n}\, dS$, where $\overline{F} = (x + y^2)\hat{i} - 2x\,\hat{j} + 2yz\,\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

[**Ans.**: 81]

2. $\iint_S \overline{F} \cdot \hat{n}\, dS$, where $\overline{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\hat{k}$ and S is the surface of the parallelepiped $0 \le x \le 1$, $0 \le y \le 2$ and $0 \le z \le 3$.

[**Ans.**: 33]

3. $\iint_S \overline{F} \cdot \hat{n}\, dS$, where $\overline{F} = x\,\hat{i} + (z^2 - zx)\,\hat{j} - xy\,\hat{k}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).

$\left[\textbf{Ans.: } -\frac{22}{3}\right]$

4. $\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS$, where $\overline{F} = y^2\,\hat{i} + y\,\hat{j} - xz\hat{k}$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

[**Ans.**: 0]

5. Find the flux of the vector field $\overline{F}$ through the portion of the sphere $x^2 + y^2 + z^2 = 36$ lying between the planes $z = \sqrt{11}$ and $z = \sqrt{20}$ where $\overline{F} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$.

$\left[\textbf{Ans.: } 72\pi\sqrt{20} - \sqrt{11}\right]$

6. Find the flux of the vector field $\overline{F} = x\hat{i} + y\,\hat{j} + \sqrt{x^2 + y^2 - 1}\,\hat{k}$ through the outer side of the hyper-boloid $z = \sqrt{x^2 + y^2 - 1}$ bounded by the planes $z = 0$ and $z = \sqrt{3}$.

$\left[\textbf{Ans.: } 2\sqrt{3}\pi\right]$

7. Find the flux of the vector field $\overline{F} = 2y\hat{i} - z\,\hat{j} + x^2\,\hat{k}$ across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$.

[**Ans.**: 132]

7.6 VOLUME INTEGRALS

If V be a region in space bounded by a closed surface S, then the volume integral of a vector point function $\bar{F}$ is $\iiint_V \bar{F}\,\mathrm{d}V$.

Component Form of Volume Integral

If $$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\iiint_V \bar{F}\,\mathrm{d}V = \iiint_V (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \hat{i}\iiint F_1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \hat{j}\iiint F_2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \hat{k}\iiint F_3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Another type of volume integral is $\iiint_V \phi\,\mathrm{d}V$, where ϕ is a scalar function.

Example 1: Evaluate $\iiint_V \bar{F}\mathrm{d}V$ where $\bar{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$ and V is the volume enclosed by the planes $x = 0$, $y = 0$, $y = a$, $z = b^2$ and the surface $z = x^2$.

Solution:

(i) V is the volume of the cylinder in positive octant with base as OAB and bounded between the planes $y = 0$ and $y = a$. y varies from 0 to a.

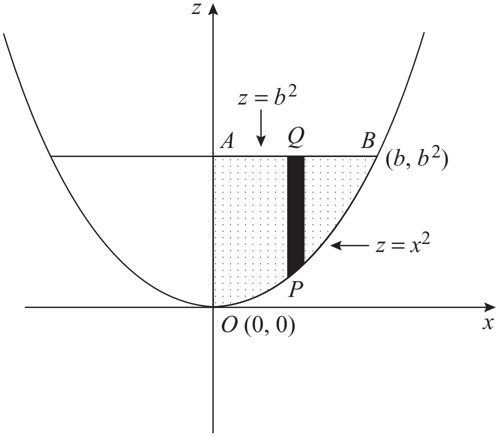


Fig. 7.28

(ii) Along the vertical strip PQ, z varies from x^2 to b^2 and in the region OAB, x varies from 0 to b.

$$\iiint_V \bar{F}\,\mathrm{d}V = \int_{x=0}^{b}\int_{z=x^2}^{b^2}\int_{y=0}^{a}(x\hat{i} + y\hat{j} + 2z\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^b\int_{x^2}^{b^2}\left(x\hat{i}\,\left|y\right|_0^a + \hat{j}\left|\frac{y^2}{2}\right|_0^a + 2z\hat{k}\,\left|y\right|_0^a\right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b\int_{x^2}^{b^2}\left(\hat{i}xa + \hat{j}\frac{a^2}{2} + \hat{k}\,2za\right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b \left(\hat{i}\, xa |z|_{x^2}^{b^2} + \hat{j} \frac{a^2}{2} |z|_{x^2}^{b^2} + \hat{k}\, a |z^2|_{x^2}^{b^2} \right) \mathrm{d}x$$

$$= \int_0^b \left[\hat{i}\, xa(b^2 - x^2) + \hat{j} \frac{a^2}{2}(b^2 - x^2) + \hat{k}\, a(b^4 - x^4) \right] \mathrm{d}x$$

$$= \left| \hat{i}\, a\left(\frac{b^2x^2}{2} - \frac{x^4}{4} \right) + \hat{j} \frac{a^2}{2}\left(b^2x - \frac{x^3}{3} \right) + \hat{k}\, a\left(b^4x - \frac{x^5}{5} \right) \right|_0^b$$

$$= \hat{i}\, a\left(\frac{b^4}{2} - \frac{b^4}{4} \right) + \hat{j} \frac{a^2}{2}\left(b^3 - \frac{b^3}{3} \right) + \hat{k}\, a\left(b^5 - \frac{b^5}{5} \right)$$

$$= \frac{ab^4}{4}\hat{i} + \frac{a^2b^3}{3}\hat{j} + \frac{4ab^5}{5}\hat{k}.$$

Example 2: Evaluate $\iiint_V (\nabla \times \overline{F})\, \mathrm{d}V$, where $\overline{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ and V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution: (i) $\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$

$$= \hat{i}(0 - 0) - \hat{j}(-4 + 3) + \hat{k}(-2y - 0)$$

$$= \hat{j} - 2y\hat{k}$$

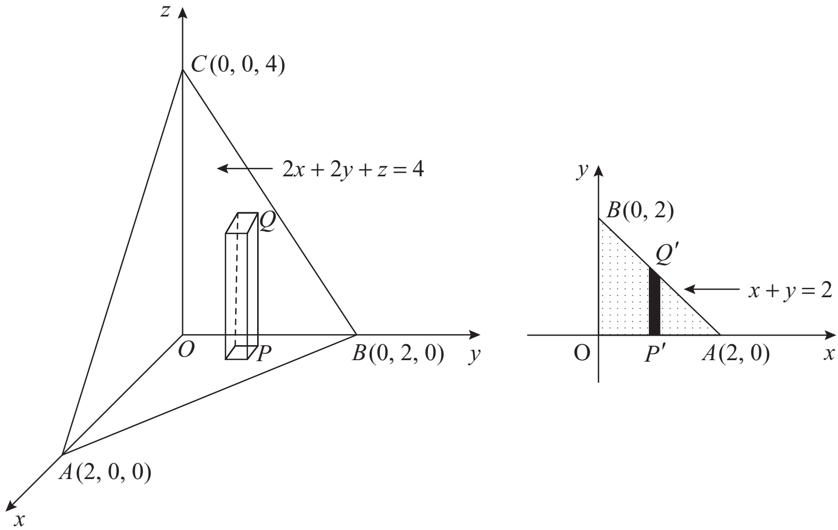


Fig. 7.29

(ii) Along the elementary volume PQ, z varies from 0 to $4-2x-2y$.
Along the vertical strip $P'Q'$, y varies from 0 to $2-x$ and in the region, x varies from 0 to 2.

$$\iiint_V (\nabla\times\overline{F})\,\mathrm{d}V = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y}(\hat{j}-2y\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^2\int_0^{2-x}(\hat{j}-2y\hat{k})\left|z\right|_0^{4-2x-2y}\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2\int_0^{2-x}(\hat{j}-2y\hat{k})(4-2x-2y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2\int_0^{2-x}\left[(4-2x-2y)\,\hat{j}-2(4-2x)\,y\hat{k}+4y^2\,\hat{k}\right]\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2\left[\left\{(4-2x)\left|y\right|_0^{2-x}-\left|y^2\right|_0^{2-x}\right\}\hat{j}-\left\{2(2-x)\left|y^2\right|_0^{2-x}-4\left|\frac{y^3}{3}\right|_0^{2-x}\right\}\hat{k}\right]\mathrm{d}x$$

$$= \int_0^2\left[\left\{2(2-x)(2-x)-(2-x)^2\right\}\hat{j}-\left\{2(2-x)(2-x)^2-\frac{4}{3}(2-x)^3\right\}\hat{k}\right]\mathrm{d}x$$

$$= \int_0^2\left[(2-x)^2\,\hat{j}-\frac{2}{3}(2-x)^3\hat{k}\right]\mathrm{d}x$$

$$= \left|\frac{(2-x)^3}{-3}\hat{j}-\frac{2}{3}\cdot\frac{(2-x)^4}{-4}\hat{k}\right|_0^2$$

$$= \frac{8}{3}\hat{j}-\frac{8}{3}\hat{k}$$

$$= \frac{8}{3}(\hat{j}-\hat{k})$$

Exercise 7.4

Evaluate the following integrals:

1. $\iiint_V (\nabla\cdot\overline{F})\,\mathrm{d}V$ where

$\overline{F} = 2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$

and V is region in the first octant bounded by the cylinder $y^2+z^2=9$ and the plane $z=2$.

[**Ans.:** 180]

2. $\iiint_V \overline{F}\,\mathrm{d}V$ where $\overline{F} = 2xz\,\hat{i}-x\hat{j}+y^2\hat{k}$

and V is the region bounded by the surfaces $x=0$, $y=0$, $y=6$, $z=x^2$, $z=4$.

[**Ans.:** $128\hat{i}-24\hat{j}+384\hat{k}$]

3. $\iiint_V f\,\mathrm{d}V$ where $f=45x^2y$ and V is the region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

[**Ans.:** 128]

4. $\iiint_V \nabla\times\overline{F}\,\mathrm{d}V$ where $\overline{F}=(x+2y)\hat{i}-3z\,\hat{j}+x\,\hat{k}$ and V is the closed region in the first octant bounded by the plane $2x+2y+z=4$.

[**Ans.:** $\frac{8}{3}(3\hat{i}-\hat{j}+2\hat{k})$]

7.7 GAUSS' DIVERGENCE THEOREM

Statement: If $\overline{F}$ be a vector point function having continuous partial derivatives in the region bounded by a closed surface S, then $\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

Proof: Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \text{... (7.3)}$$

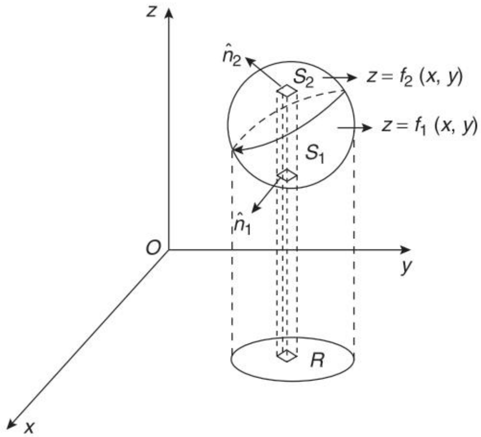


Fig. 7.30

Assume a closed surface S such that any line parallel to the coordinate axes intersects S at most at two points.

Divide the surface S into two parts: S_1, the lower and S_2, the upper part. Let $z = f_1(x, y)$ and $z = f_2(x, y)$ be the equations and $\hat{n}_1$ and $\hat{n}_2$ be the normals to the surfaces S_1 and S_2 respectively. Let R be the projection of the surface S on the xy-plane.

$$\iiint_V \frac{\partial F_3}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_R\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R F_3(x, y, z)\Big|_{f_1}^{f_2}\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R F_3(x, y, f_2)\mathrm{d}x\,\mathrm{d}y - \iint_R F_3(x, y, f_1)\mathrm{d}x\,\mathrm{d}y \qquad \text{... (7.4)}$$

$dx\,dy$ = projection of dS on xy-plane

$= \hat{n}\cdot\hat{k}\,dS$

For surface S_2: $z = f_2(x, y)$

$$dx\,dy = \hat{n}_2\cdot\hat{k}\;dS_2$$

For surface S_1: $z = f_1(x, y)$

$$dx\,dy = -\hat{n}_1\cdot\hat{k}\,dS_1$$

Substituting in Eq. (7.4),

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,dS_2 - \iint_{S_1} F_3\,(-\hat{n}_1\cdot\hat{k})\,dS_1$$

$$= \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,dS_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,dS_1$$

$$= \iint_S F_3\,\hat{n}\cdot\hat{k}\,dS \qquad \text{... (7.5)}$$

Similarly, projecting the surface S on yz and zx-planes, we get

$$\iiint_V \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \iint_S F_1\,\hat{n}\cdot\hat{i}\,dS \qquad \text{... (7.6)}$$

and
$$\iiint_V \frac{\partial F_2}{\partial y}\,dx\,dy\,dz = \iint_S F_2\,\hat{n}\cdot\hat{j}\,dS \qquad \text{... (7.7)}$$

Substituting Eqs. (7.5), (7.6) and (7.7) in Eq. (7.3),

$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S F_1\,\hat{n}\cdot\hat{i}\,dS + \iint_S F_2\,\hat{n}\cdot\hat{j}\,dS + \iint_S F_3\,\hat{n}\cdot\hat{k}\,dS$$

$$= \iint_S (F_1\,\hat{i}\cdot\hat{n} + F_2\,\hat{j}\cdot\hat{n} + F_3\,\hat{k}\cdot\hat{n})\,dS$$

$$= \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot\hat{n}\,dS$$

$$= \iint_S \overline{F}\cdot\hat{n}\,dS$$

Hence,
$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S \overline{F}\cdot\hat{n}\,dS$$

Note: Cartesian form of Gauss' divergence theorem is

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z}\right)dx\,dy\,dz = \iint_S \left(F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\right)$$

Example 1: Verify Gauss' divergence theorem for $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: By Gauss' divergence theorem,

$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S \overline{F}\cdot\hat{n}\,dS$$

(i) $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$
$$= 4z - 2y + y = 4z - y$$

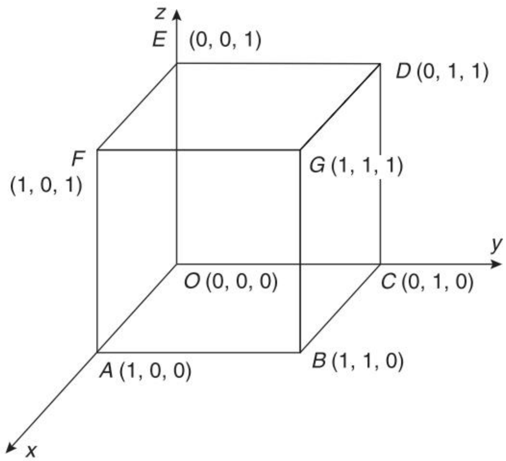


Fig. 7.31

(ii) For the cube: x varies from 0 to 1
y varies from 0 to 1
z varies from 0 to 1

$$\iiint_V \nabla \cdot \overline{F} \mathrm{d}V = \int_0^1 \int_0^1 \int_0^1 (4z - y) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_0^1 \int_0^1 \left| 2z^2 - yz \right|_0^1 \mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 \mathrm{d}x \int_0^1 (2 - y)\mathrm{d}y$$
$$= \left| x \right|_0^1 \left| 2y - \frac{y^2}{2} \right|_0^1$$
$$= 2 - \frac{1}{2}$$
$$= \frac{3}{2} \qquad \text{... (1)}$$

(iii) Surface S of the cube consists of 6 surfaces, S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \overline{F} \cdot \hat{n}\,\mathrm{d}S = \iint_{S_1} \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_2} \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_3} \overline{F} \cdot \hat{n}\,\mathrm{d}S$$
$$+ \iint_{S_4} \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_5} \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_6} \overline{F} \cdot \hat{n}\,\mathrm{d}S \qquad \text{... (2)}$$

(a) On $S_1(OABC): z = 0,\ \hat{n} = -\hat{k},\quad dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$

x and y both varies from 0 to 1.

$$\iint_{S_1} \overline{F}\cdot\hat{n}\,dS = \iint_{S_1} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-\hat{k})\,dx\,dy$$
$$= 0$$

(b) On $S_2(DEFG): z = 1,\ \hat{n} = \hat{k},\ dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$

x and y both varies from 0 to 1.

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,dS = \iint_{S_2} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot\hat{k}\,dx\,dy$$
$$= \int_0^1\int_0^1 y\,dx\,dy$$
$$= \int_0^1 \left|\frac{y^2}{2}\right|_0^1 dx$$
$$= \frac{1}{2}|x|_0^1$$
$$= \frac{1}{2}$$

(c) On $S_3(OAFE):\ y = 0,\ \hat{n} = -\hat{j},\ dS = \dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|} = dz\,dx$

x and z both varies from 0 to 1.

$$\iint_{S_3} \overline{F}\cdot\hat{n}\,dS = \iint_{S_3} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-\hat{j})\,dz\,dx$$
$$= 0$$

(d) On $S_4(BCDG):\ y = 1,\ \hat{n} = \hat{j},\ dS = \dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|} = dz\,dx$

x and z both varies from 0 to 1.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,dS = \iint_{S_4} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(\hat{j})\,dz\,dx$$
$$= \int_0^1\int_0^1 -dz\,dx$$
$$= -1$$

(e) On $S_5(OCDE):\ x = 0,\ \hat{n} = -\hat{i},\ dS = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

y and z both varies from 0 to 1.

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,dS = \iint_{S_5} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot(-\hat{i})\,dy\,dz$$
$$= 0.$$

(f) On $S_6\left(ABGF\right)$: $x=1,\ \ \hat{n}=\hat{i},\ \ \mathrm{d}S=\dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|}=\mathrm{d}y\,\mathrm{d}z$

y and z both varies from 0 to 1.

$$\iint_{S_6}\overline{F}\cdot\hat{n}\,\mathrm{d}S=\iint_{S_6}(4xz\hat{i}-y^2\hat{j}+yz\hat{k})\cdot\hat{i}\,\mathrm{d}y\,\mathrm{d}z$$
$$=\int_0^1\int_0^1 4z\,\mathrm{d}y\,\mathrm{d}z$$
$$=4\left|y\right|_0^1\left|\frac{z^2}{2}\right|_0^1$$
$$=2$$

Substituting in Eq. (2),

$$\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S=0+\frac{1}{2}+0+(-1)+0+2$$
$$=\frac{3}{2} \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S=\frac{3}{2}$$

Hence, Gauss' divergence theorem is verified.

Example 2: Verify Gauss' divergence theorem for $\overline{F}=2x^2y\hat{i}-y^2\,\hat{j}+4xz^2\,\hat{k}$ over the region bounded by the cylinder $y^2+z^2=9$ and the plane $x=2$ in the first octant.

Solution: By Gauss' divergence theorem,

$$\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iint_S\overline{F}\cdot\hat{n}\,\mathrm{d}S$$

(i) $\overline{F}=2x^2y\hat{i}-y^2\,\hat{j}+4xz^2\,\hat{k}$

$$\nabla\cdot\overline{F}=\frac{\partial}{\partial x}(2x^2y)+\frac{\partial}{\partial y}(-y^2)+\frac{\partial}{\partial z}(4xz^2)$$
$$=4xy-2y+8xz$$

(ii) $\iiint_V\nabla\cdot\overline{F}\,\mathrm{d}V=\iiint(4xy-2y+8xz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

For the given region, x varies from 0 to 2. Putting $y=r\cos\theta$, $z=r\sin\theta$, the equation of the cylinder $y^2+z^2=9$ reduces to $r=3$ and $\mathrm{d}y\,\mathrm{d}z=r\,\mathrm{d}r\,\mathrm{d}\theta$.
Along the radius vector OP, r varies from 0 to 3 and for the region in the first octant, θ varies from 0 to $\dfrac{\pi}{2}$.

$$\iiint_V \nabla\cdot\overline{F}\,dV = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{3}\int_{x=0}^{2}(4x\cdot r\cos\theta - 2\cdot r\cos\theta + 8x\cdot r\sin\theta)\,dx\cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^3\left(2r^2\cos\theta\left|x^2\right|_0^2 - 2r^2\cos\theta\left|x\right|_0^2 + 4r^2\sin\theta\left|x^2\right|_0^2\right)dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^3(4r^2\cos\theta + 16r^2\sin\theta)\,dr\,d\theta$$

$$= 4\left|\frac{r^3}{3}\right|_0^3\left|\sin\theta\right|_0^{\frac{\pi}{2}} + 16\left|\frac{r^3}{3}\right|_0^3\left|-\cos\theta\right|_0^{\frac{\pi}{2}}$$

$$= 36 + 144$$

$$= 180 \qquad \ldots (1)$$

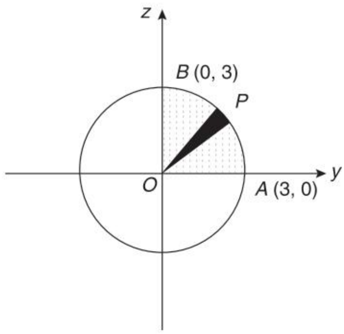


Fig. 7.32

(iii) The surface S consists of 5 surfaces, S_1, S_2, S_3, S_4, S_5.

$$\iint_S \overline{F}\cdot\hat{n}\,dS = \iint_{S_1}\overline{F}\cdot\hat{n}\,dS + \iint_{S_2}\overline{F}\cdot\hat{n}\,dS + \iint_{S_3}\overline{F}\cdot\hat{n}\,dS + \iint_{S_4}\overline{F}\cdot\hat{n}\,dS + \iint_{S_5}\overline{F}\cdot\hat{n}\,dS \qquad \ldots (2)$$

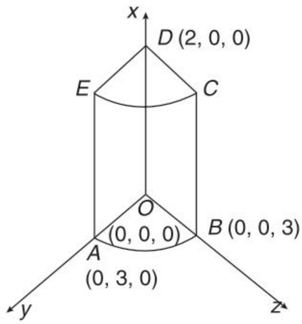


Fig. 7.33

(a) On $S_1(OAED): z = 0,\ \hat{n} = -\hat{k}$

$$dS = \frac{dx\,dy}{\left|\hat{n}\cdot\hat{k}\right|} = dx\,dy$$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_1}\overline{F}\cdot\hat{n}\,dS = \iint_{S_1}(2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= 0$$

(b) On $S_2\ (OBCD): y = 0,\ \hat{n} = -\hat{j},\ dS = \dfrac{dz\,dx}{\left|\hat{n}\cdot\hat{j}\right|} = dz\,dx$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_2}\overline{F}\cdot\hat{n}\,dS = \iint_{S_2}(2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot(-\hat{j})\,dz\,dx = 0$$

(c) On S_3 (OAB) : $x = 0,\ \hat{n} = -\hat{i},\ \mathrm{d}S = \dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|} = \mathrm{d}y\,\mathrm{d}z$

y and z varies from 0 to 3.

$$\iint_{S_3} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_3} (2x^2 y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{i})\,\mathrm{d}y\,\mathrm{d}z = 0$$

(d) On S_4 (DEC) : $x = 2,\ \hat{n} = \hat{i},\ \mathrm{d}S = \dfrac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|} = \mathrm{d}y\,\mathrm{d}z$

y and z varies from 0 to 3.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_4} (2x^2 y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\hat{i}\,\mathrm{d}y\,\mathrm{d}z = \iint 8y\,\mathrm{d}y\,\mathrm{d}z$$

Putting $y = r\cos\theta$, $z = r\sin\theta$, equation of the cylinder $y^2 + z^2 = 9$ reduces to $r = 3$ and $\mathrm{d}y\ \mathrm{d}z = r\ \mathrm{d}r\ \mathrm{d}\theta$.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S = 8\int_0^{\frac{\pi}{2}}\int_0^3 r\sin\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta\cdot\int_0^3 r^2\,\mathrm{d}r$$

$$= 8\left|-\cos\theta\right|_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^3$$

$$= 72$$

(e) On S_5 $(ABCE)$: This is the curved surface of the cylinder $y^2 + z^2 = 9$ bounded between $x = 0$ and $x = 2$.

Let $\phi = y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4y^2 + 4z^2}}$$

$$= \frac{y\hat{i} + z\hat{k}}{3} \qquad [\because y^2 + z^2 = 9]$$

$$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{3\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_5} (2x^2 y\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot\left(\frac{y\,\hat{j}+z\,\hat{k}}{3}\right)\cdot\frac{3\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}(-y^3+4xz^3)\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

The parametric equation of the cylinder $y^2+z^2=9$ is,

$$y = 3\cos\theta,\ z = 3\sin\theta$$

$$\mathrm{d}y = -3\sin\theta\,\mathrm{d}\theta = -z\,\mathrm{d}\theta,\ \frac{\mathrm{d}y}{z} = -\mathrm{d}\theta$$

When $y = 0,\ \theta = \frac{\pi}{2}$

$y = 3, \theta = 0$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,\mathrm{d}S = \int_{\frac{\pi}{2}}^{0}\int_{0}^{2}(-27\cos^3\theta + x108\sin^3\theta)(-\mathrm{d}\theta)\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{2}}\left(-27\cos^3\theta\,|x|_0^2 + 108\sin^3\theta\left|\frac{x^2}{2}\right|_0^2\right)\mathrm{d}\theta$$

$$= \int_{0}^{\frac{\pi}{2}} -54\cos^3\theta + 216\sin^3\theta$$

$$= -54\cdot\frac{1}{2}B\left(2,\frac{1}{2}\right) + 216\cdot\frac{1}{2}B\left(2,\frac{1}{2}\right) \qquad \left[\because \int\sin^p\theta\cos^q\theta\,\mathrm{d}\theta = B\left(\frac{p+1}{2},\frac{q+1}{2}\right)\right]$$

$$= \frac{\overline{|2}\,\overline{|\frac{1}{2}}}{\overline{|\frac{5}{2}}}(-27+108)$$

$$= \frac{1\cdot\overline{|\frac{1}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\overline{|\frac{1}{2}}}\cdot 81$$

$=108$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = 0+0+0+72+108 = 180 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = 180$$

Hence, Gauss' divergence theorem is verified.

Example 3: Verify Gauss' divergence theorem for $\overline{F} = 2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: By Gauss' divergence theorem,

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

(i) $\overline{F} = 2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = 2z + z + 2z = 5z$$

(ii) $\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint_V 5z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

For upper half of the sphere (hemisphere),

r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$

ϕ varies from 0 to 2π

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = 5\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r\cos\theta\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= 5\int_0^{2\pi}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,\mathrm{d}\theta\left|\frac{r^4}{4}\right|_0^a$$

$$= 5\left|\phi\right|_0^{2\pi}\cdot\frac{1}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\cdot\frac{a^4}{4}$$

$$= -\frac{5a^4}{16}\cdot 2\pi(\cos\pi - \cos 0)$$

$$= \frac{5}{4}\pi a^4 \qquad \text{... (1)}$$

(iii) Given surface is not closed. We close this surface from below by the circular surface S_2 in xy-plane.

Thus, the surface S consists of two surfaces S_1 and S_2.

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1}\overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_2}\overline{F}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (2)}$$

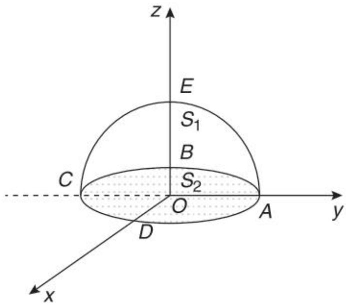


Fig. 7.34

$[\because x^2 + y^2 + z^2 = a^2]$

(a) Surface $S_1(ABCEA)$: This is the curved surface of the upper half of the sphere.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Let R be the projection of S_1 on the xy-plane, which is a circle $x^2 + y^2 = a^2$.

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{a\,dx\,dy}{z}$$

$$\iint_{S_1} \overline{F}\cdot\hat{n}\,dS = \iint_R (2xz\hat{i} + yz\hat{j} + z^2\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)\frac{a\,dx\,dy}{z}$$

$$= \iint_R (2x^2 + y^2 + z^2)\,dx\,dy$$

$$= \iint_R (2x^2 + y^2 + a^2 - x^2 - y^2)\,dx\,dy \qquad [\because z^2 = a^2 - x^2 - y^2]$$

$$= \iint_R (x^2 + a^2)\,dx\,dy$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2 + y^2 = a^2$ reduces to $r = a$ and $dx\,dy = r\,dr\,d\theta$. Along the radius vector OP, r varies from 0 to a and for the complete circle, θ varies from 0 to 2π.

$$\iint_{S_1} \nabla\times\overline{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^a (r^2\cos^2\theta + a^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\left|\frac{r^4}{4}\right|_0^a \cos^2\theta + a^2\left|\frac{r^2}{2}\right|_0^a\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{a^4}{4}\left(\frac{1+\cos 2\theta}{2}\right) + \frac{a^4}{2}\right]d\theta$$

$$= a^4\left|\frac{5}{8}\theta + \frac{1}{8}\frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= \frac{5}{4}\pi a^4$$

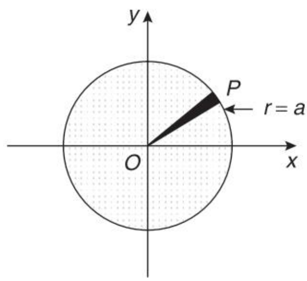


Fig. 7.35

(b) Surface S_2 $(ABCDA)$: This is the circle $x^2 + y^2 = a^2$ in xy-plane $z = 0$, $\hat{n} = -\hat{k}$

$$dS = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$
$$= dx\,dy$$

$$\iint_{S_2} \overline{F} \cdot \hat{n}\,dS = \iint_{S_2} (2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k}) \cdot (-\hat{k})\,dx\,dy$$
$$= 0 \qquad [\because z = 0]$$

Substituting in Eq. (2),

$$\iint_S \overline{F} \cdot \hat{n}\,dS = \frac{5}{4}\pi a^4 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla \cdot \overline{F}\,dV = \iint_S \overline{F} \cdot \hat{n}\,dS = \frac{5}{4}\pi a^4$$

Hence, Gauss' divergence theorem is verified.

Example 4: Evaluate $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot d\overline{S}$, where S is the surface of the sphere in the first octant.

Solution: By Gauss' divergence theorem,

$$\iint_S \overline{F} \cdot d\overline{S} = \iiint_V \nabla \cdot \overline{F}\,dV \qquad \text{... (1)}$$

$$\overline{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$
$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

From Eq. (1), $\iint_S \overline{F} \cdot d\overline{S} = 0$

Example 5: Evaluate $\iint_S (x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy)$ where S is the closed surface consisting of the circular cylinder $x^2 + y^2 = a^2$, $z = 0$ and $z = b$.

Solution: By Gauss' divergence theorem,

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \qquad \text{... (1)}$$

(i) $$F_1\,\mathrm{d}y\,\mathrm{d}z + F_2\,\mathrm{d}z\,\mathrm{d}x + F_3\,\mathrm{d}x\,\mathrm{d}y = x^3\,\mathrm{d}y\,\mathrm{d}z + x^2y\,\mathrm{d}z\,\mathrm{d}x + x^2z\,\mathrm{d}x\,\mathrm{d}y$$

$$F_1 = x^3,\ F_2 = x^2y, F_3 = x^2z$$

(ii) $$\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial z}(x^2z)$$

$$= 3x^2 + x^2 + x^2 = 5x^2$$

(iii) $$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint 5x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, circular cylinder $x^2 + y^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

Along the radius vector OA, r varies from 0 to a and for complete circle, θ varies from 0 to 2π. Along the volume of the cylinder, z varies from 0 to b.

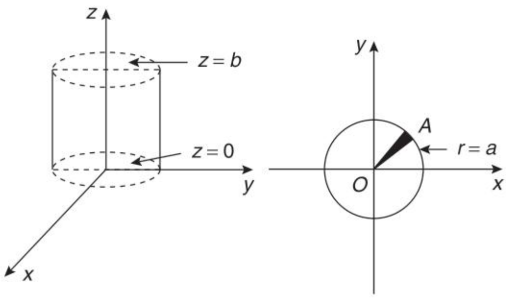


Fig. 7.36

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 5\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\cos^2\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

$$= 5\left|z\right|_0^b\left|\frac{r^4}{4}\right|_0^a\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$$= \frac{5}{4}\cdot b a^4\cdot\frac{1}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= \frac{5}{4}\cdot\frac{ba^4}{2}\cdot 2\pi$$

$$= \frac{5}{4}\pi a^4 b$$

From Eq. (1),

$$\iint_S (x^3\,\mathrm{d}y\,\mathrm{d}z + x^2y\,\mathrm{d}z\,\mathrm{d}x + x^2z\,\mathrm{d}x\,\mathrm{d}y) = \frac{5}{4}\pi a^4 b$$

Example 6: Evaluate $\iint_S (lx + my + nz)\,dS$, where l, m, n are the direction cosines of the outer normal to the surface whose radius is 2 units.

Solution: By Gauss' divergence theorem,

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V \qquad \ldots (1)$$

(i) $\overline{F}\cdot\hat{n} = lx + my + nz$

$$= (x\hat{i} + y\hat{j} + z\hat{k})\cdot(l\hat{i} + m\hat{j} + n\hat{k})$$

$$\overline{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

(ii) $\nabla\cdot\overline{F} = \dfrac{\partial}{\partial x}x + \dfrac{\partial}{\partial y}y + \dfrac{\partial}{\partial z}z$

$$= 3$$

(iii) $\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = \iiint_V 3\,\mathrm{d}V$

$= 3$ (Volume of the region bounded by the sphere of 2-unit radius)

$$= 3\cdot\frac{4}{3}\pi(2)^3$$

$$= 32\pi$$

From Eq. (1),

$$\iint_S (lx + my + nz)\,\mathrm{d}S = 32\pi.$$

Example 7: Prove that $\iint_S \dfrac{\mathrm{d}S}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \dfrac{4\pi}{\sqrt{abc}}$, where S is the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Solution: By Gauss' divergence theorem,

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V \qquad \ldots (1)$$

(ii) $\overline{F}\cdot\hat{n} = \dfrac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

where $\hat{n}$ = unit normal to the ellipsoid, $ax^2 + by^2 + cz^2 = 1$

$$= \frac{2ax\,\hat{i} + 2by\,\hat{j} + 2cz\,\hat{k}}{\sqrt{4a^2x^2 + 4b^2y^2 + 4c^2z^2}}$$

$$= \frac{ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

Now, $\overline{F}\cdot\hat{n} = \dfrac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

$$= \frac{ax^2 + by^2 + cz^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \qquad [\because ax^2 + by^2 + cz^2 = 1]$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \left(\frac{ax\hat{i} + by\hat{j} + cz\hat{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \right)$$

Hence, $\overline{F} = x\hat{i} + y\hat{j} + z\hat{k}$

(iii) $$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

(iv) $$\iiint_V \nabla \cdot \overline{F}\, \mathrm{d}V = \iiint_V 3\, \mathrm{d}V$$

$= 3$ (Volume of the region bounded by the ellipsoid)

$$= 3 \cdot \frac{4}{3}\pi \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}} \left[\because \frac{x^2}{\left(\frac{1}{\sqrt{a}}\right)^2} + \frac{y^2}{\left(\frac{1}{\sqrt{b}}\right)^2} + \frac{z^2}{\left(\frac{1}{\sqrt{c}}\right)^2} = 1 \right]$$

$$= \frac{4\pi}{\sqrt{abc}}$$

From Eq. (1),

$$\iint_S \frac{\mathrm{d}S}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \frac{4\pi}{\sqrt{abc}}$$

Example 8: Evaluate $\iint_S \overline{F} \cdot \mathrm{d}\overline{S}$ using divergence theorem where $\overline{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: By Gauss' divergence theorem,

$$\iint_S \overline{F} \cdot \mathrm{d}\overline{S} = \iiint_V \nabla \cdot \overline{F}\mathrm{d}V \qquad \text{... (1)}$$

(i) $$\overline{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3$$

$$= 3x^2 + 3y^2 + 3z^2$$

(ii) $$\iiint_V \nabla \cdot \overline{F}\, \mathrm{d}V = 3\iiint_V (x^2 + y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
For complete sphere,

r varies from 0 to a
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V = 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\cdot r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= 3\int_0^{2\pi}\mathrm{d}\phi\cdot\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\cdot\int_0^a r^4\,\mathrm{d}r$$

$$= 3\left|\phi\right|_0^{2\pi}\left|-\cos\theta\right|_0^{\pi}\left|\frac{r^5}{5}\right|_0^a$$

$$= 3\cdot 2\pi(-\cos\pi + \cos 0)\frac{a^5}{5}$$

$$= \frac{12}{5}\pi a^5$$

From Eq. (1),

$$\iint_S \overline{F}\cdot\mathrm{d}\overline{S} = \frac{12}{5}\pi a^5.$$

Example 9: Evaluate $\iint \overline{F}\cdot\mathrm{d}\overline{S}$ using Gauss' divergence theorem where $\overline{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + zx\,\hat{k}$ and S is the surface of the region bounded by $x = 0$, $y = 0$, $z = 0$, $y = 3$, $x + 2z = 6$.

Solution: By Gauss' divergence theorem,

$$\iint_S \overline{F}\cdot\mathrm{d}\overline{S} = \iiint_V \nabla\cdot\overline{F}\,\mathrm{d}V \qquad \ldots (1)$$

(i)

$$\overline{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + zx\,\hat{k}$$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx) = 2y + z^2 + x$$

(ii)

$$\iiint_V \nabla\cdot\overline{F} = \iiint_V (2y + z^2 + x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

In the given region, y varies from 0 to 3.
In xz-plane, region is bounded by the lines $x = 0$, $z = 0$, $x + 2z = 6$.

Along the vertical strip PQ, z varies from 0 to $\frac{6-x}{2}$ and in the region, x varies from 0 to 6.

$$\iiint_V \nabla\cdot\overline{F} = \int_{x=0}^{6}\int_{z=0}^{\frac{6-x}{2}}\int_{y=0}^{3}(2y+z^2+x)\,dy\,dz\,dx$$

$$= \int_0^6\int_0^{\frac{6-x}{2}}\left|y^2+z^2y+xy\right|_0^3 dz\,dx$$

$$= \int_0^6\int_0^{\frac{6-x}{2}}(9+3z^2+3x)\,dz\,dx$$

$$= \int_0^6\left|9z+z^3+3xz\right|_0^{\frac{6-x}{2}} dx$$

$$= \int_0^6\left[9\left(\frac{6-x}{2}\right)+\left(\frac{6-x}{2}\right)^3+3x\left(\frac{6-x}{2}\right)\right]dx$$

$$= \int_0^6\left[27+\frac{9x}{2}-\frac{3x^2}{2}+\left(\frac{6-x}{2}\right)^3\right]dx$$

$$= \left|27x+\frac{9}{2}\cdot\frac{x^2}{2}-\frac{x^3}{2}+\frac{1}{8}\cdot\frac{(6-x)^4}{-4}\right|_0^6$$

$$= 162+81-108+\frac{6^4}{32}$$

$$= \frac{351}{2}$$

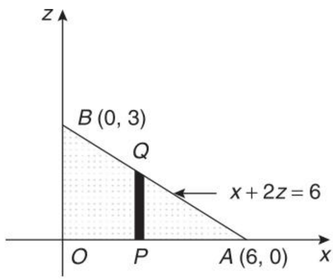


Fig. 7.37

Hence, From Eq. (1),

$$\iint \overline{F}\cdot d\overline{s} = \frac{351}{2}.$$

Example 10: Evaluate $\iint_S \overline{F}\cdot\hat{n}\,dS$ using Gauss' divergence theorem where $\overline{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$ over the region bounded by the cone $z^2 = x^2 + y^2$ and plane $z = 4$, above the xy plane.

Solution: By Gauss' divergence theorem,

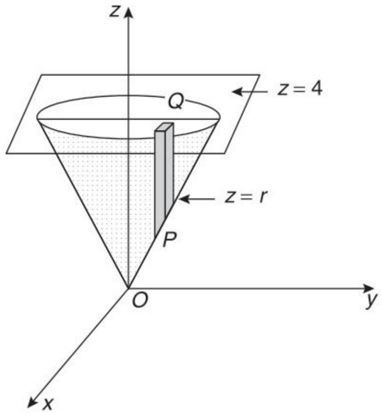
$$\iint_S \overline{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\overline{F}\,dV \quad \text{... (1)}$$

(i)
$$\overline{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)$$

$$= 4z + xz^2 + 3$$

(ii)
$$\iiint_V \nabla\cdot\overline{F}\,dV = \iiint (4z + xz^2 + 3)\,dx\,dy\,dz$$



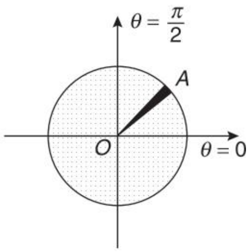


Fig. 7.38

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cone $z^2 = x^2 + y^2$ reduces to $z = r$, and $dx\,dy\,dz = r\,dr\,d\theta\,dz$. Along the elementary volume PQ, z varies from r to 4.

Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and plane $z = 4$ which is a circle $r = 4$.

Along the radius vector OA, r varies from 0 to 4 and for the complete circle, θ varies from 0 to 2π.

$$\iiint_V \nabla\cdot\overline{F}\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=r}^{4}(4z + r\cos\theta\cdot z^2 + 3)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left|2z^2 + r\cos\theta\cdot\frac{z^3}{3} + 3z\right|_r^4 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left[2r(16-r^2) + \frac{r^2\cos\theta}{3}(64-r^3) + 3r(4-r)\right]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left(44r + \frac{64}{3}r^2\cos\theta - 3r^2 - 2r^3 - \frac{r^5}{3}\cos\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left|22r^2 + \frac{64}{3}\cos\theta\cdot\frac{r^3}{3} - r^3 - \frac{r^4}{2} - \frac{1}{3}\cdot\frac{r^6}{6}\cos\theta\right|_0^4 d\theta$$

$$= \int_0^{2\pi}\left(160 + \frac{2048}{9}\cos\theta\right)d\theta$$

$$= 160\left|\theta\right|_0^{2\pi} + \frac{2048}{9}\left|\sin\theta\right|_0^{2\pi}$$

$$= 160\cdot 2\pi + 0$$

$$= 320\pi$$

From Eq. (1),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = 320\pi.$$

Example 11: Evaluate $\iint_S (x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}) \cdot \hat{n}\,\mathrm{d}S$ using Gauss' divergence theorem where S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Solution: By Gauss' divergence theorem,

$$\iint_S \bar{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \bar{F}\, dV \qquad \text{... (1)}$$

(i)

$$\bar{F} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

(ii) $\iiint_V \nabla \cdot \bar{F}\,\mathrm{d}V = \iiint_V (2x + 2y + 2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Putting $x = ar\sin\theta\cos\phi$, $y = br\sin\theta\sin\phi$, $z = cr\cos\theta$, equation of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ reduces to $r = 1$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = abc\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

For the complete ellipsoid,

r varies from 0 to 1

θ varies from 0 to π

ϕ varies from 0 to 2π

$$\iiint_V \nabla \cdot \bar{F}\,\mathrm{d}V = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} [(ar\sin\theta\cos\phi + br\sin\theta\sin\phi + cr\cos\theta)$$

$$abc\,r^2\sin\theta]\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} (a\sin^2\theta\cos\phi + b\sin^2\theta\sin\phi + c\cos\theta\sin\theta)\left|\frac{r^4}{4}\right|_0^1 abc\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{abc}{4}\left[\int_0^{\pi}\left(a\sin^2\theta\left|\sin\phi\right|_0^{2\pi} + b\sin^2\theta\left|-\cos\phi\right|_0^{2\pi} + c\cos\theta\sin\theta\left|\phi\right|_0^{2\pi}\right)\mathrm{d}\theta\right.$$

$$= \frac{abc}{4}\int_0^{\pi}(0 + 0 + c\cos\theta\sin\theta \cdot 2\pi)\,\mathrm{d}\theta$$

$$= \pi\frac{abc^2}{4}\left|-\frac{\cos 2\theta}{2}\right|_0^{\pi}$$

$$= \frac{-\pi abc^2}{8}(\cos 2\pi - \cos 0)$$

$$= 0$$

From Eq. (1),

$$\iint_S \bar{F} \cdot \hat{n}\, dS = 0$$

Exercise 7.5

(I) Verify Gauss' divergence theorem for the following:

1. $\overline{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ over the region R bounded by the parallelepiped $0\le x\le a, 0\le y\le b, 0\le z\le c$.

$\left[\textbf{Ans.: } abc(a+b+c)\right]$

2. $\overline{F}=x\hat{i}+y\hat{j}+z\hat{k}$ over the region R bounded by the sphere $x^2+y^2+z^2=16$.

$\left[\textbf{Ans.: } 256\pi\right]$

3. $\overline{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ over the region bounded by the cylinder $x^2+y^2=4$ and the planes $z=0, z=3$.

$\left[\textbf{Ans.: } 84\pi\right]$

4. $\overline{F}=2xy\hat{i}+6yz\hat{j}+3zx\hat{k}$ over the region bounded by the coordinate planes and the plane $x+y+z=2$.

$\left[\textbf{Ans.: } \frac{22}{3}\right]$

(II) Evaluate the following integrals using Gauss' divergence theorem:

1. $\iint_S (y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k})\cdot\hat{n}\,dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane.

$\left[\textbf{Ans.: } \frac{\pi}{12}\right]$

2. $\iint_S (x^2y\hat{i}+y^3\hat{j}+xz^2\hat{k})\cdot\hat{n}\,dS$, where S is the surface of the parallelepiped $0\le x\le 2, 0\le y\le 3, 0\le z\le 4$.

$\left[\textbf{Ans.: } 384\right]$

3. $\iint_S (4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\cdot\hat{n}\,dS$, where S is the surface of the region bounded by $y^2=4x, x=1, z=0, z=3$.

$\left[\text{Ans.: } 56\right]$

4. $\iint_S (x\,dy\,dz+y\,dz\,dx+z\,dx\,dy)$, where S is the part of the plane $x + 2y + 3z = 6$ which lies in the first octant.

$\left[\textbf{Ans.: } 18\right]$

5. $\iint_S (x\,dy\,dz+y\,dz\,dx+z\,dx\,dy)$, where S is the surface of the sphere $(x-2)^2+(y-2)^2+(z-2)^2=4$.

$\left[\textbf{Ans.: } 32\pi.\right]$

6. $\iint_S (2xy^2\hat{i}+x^2y\hat{j}+x^3\hat{k})$, where S is the surface of the region bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=4$.

$\left[\textbf{Ans.: } \frac{3072\pi}{5}\right]$

7. $\iint_S (x^3\hat{i}+y^3\hat{j}+z^3\hat{k})$, where S is the surface of the region bounded within $z=\sqrt{16-x^2-y^2}$ and $x^2+y^2=4$.

$\left[\textbf{Ans.: } \frac{2\pi}{5}(2188-1056\sqrt{3})\right]$

7.8 STOKES' THEOREM

Statement: If S be an open surface bounded by a closed curve C and $\overline{F}$ be a continuous and differentiable vector function, then

$$\oint_C \overline{F}\cdot d\overline{r}=\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

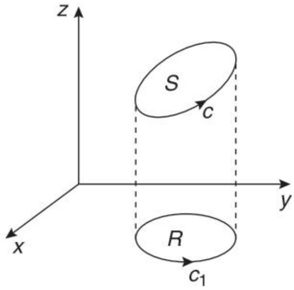


Fig. 7.39

Proof: Let $\quad \bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\,\mathrm{d}S = \iint_S \nabla \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot \hat{n}\,\mathrm{d}S$$

$$= \iint_S (\nabla \times F_1\hat{i}) \cdot \hat{n}\,\mathrm{d}S + \iint_S (\nabla \times F_2\hat{j}) \cdot \hat{n}\,\mathrm{d}S + \iint_S (\nabla \times F_3\hat{k}) \cdot \hat{n}\,\mathrm{d}S \qquad \text{... (7.8)}$$

Consider,

$$\iint_S (\nabla \times F_1\hat{i}) \cdot \hat{n}\,\mathrm{d}S = \iint_S \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times F_1\hat{i}\right] \cdot \hat{n}\,\mathrm{d}S$$

$$= \iint_S \left(-\hat{k}\frac{\partial F_1}{\partial y} + \hat{j}\frac{\partial F_1}{\partial z}\right) \cdot \hat{n}\,\mathrm{d}S$$

$$= \iint \left(\frac{\partial F_1}{\partial z}\hat{j} \cdot \hat{n} - \frac{\partial F_1}{\partial y}\hat{k} \cdot \hat{n}\right)\mathrm{d}S \qquad \text{... (7.9)}$$

Let equation of the surface S be $z = f(x, y)$,

Then, $\quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

Differentiating partially w.r.t. y,

$$\frac{\partial \bar{r}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

Taking dot product with $\hat{n}$,

$$\frac{\partial \bar{r}}{\partial y} \cdot \hat{n} = \hat{j} \cdot \hat{n} + \frac{\partial f}{\partial y}\hat{k} \cdot \hat{n} \qquad \text{... (7.10)}$$

$\dfrac{\partial \bar{r}}{\partial y}$ is tangential and $\hat{n}$ is normal to the surface S.

$$\frac{\partial \bar{r}}{\partial y} \cdot \hat{n} = 0$$

Substituting in Eq. (7.10),

$$0 = \hat{j} \cdot \hat{n} + \frac{\partial f}{\partial y}\hat{k} \cdot \hat{n}$$

$$\hat{j} \cdot \hat{n} = -\frac{\partial f}{\partial y}\hat{k} \cdot \hat{n} = -\frac{\partial z}{\partial y}\hat{k} \cdot \hat{n} \qquad [\because z = f(x, y)]$$

Substituting in Eq. (7.9),

$$\iint_S (\nabla \times F_1 \hat{i}) \cdot \hat{n}\, dS = \iint_S \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial z}{\partial y}\hat{k}\cdot\hat{n} \right) - \frac{\partial F_1}{\partial y}\hat{k}\cdot\hat{n} \right] dS$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y} \right)\hat{k}\cdot\hat{n}\, dS \qquad \text{... (7.11)}$$

Equation of the surface is $z = f(x, y)$.

$$F_1(x, y, z) = F_1[x, y, f(x, y)] = G(x, y) \text{ say}$$

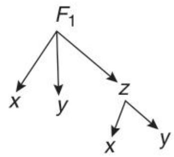


Fig. 7.40

Differentiating partially w.r.t. y,

$$\frac{\partial G}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y}$$

Substituting in Eq. (7.11),

$$\iint_S (\nabla \times F_1 \hat{i}) \cdot \hat{n}\, dS = -\iint_S \frac{\partial G}{\partial y}\hat{k}\cdot\hat{n}\, dS$$

Let R is the projection of S on the xy-plane and dxdy is the projection of dS on the xy-plane, then $\hat{k}\cdot\hat{n}\, dS = dx\, dy$

Thus, $\displaystyle\iint_S (\nabla \times F_1 \hat{i}) \cdot \hat{n}\, dS = -\iint_R \frac{\partial G}{\partial y}\, dx\, dy$

$$= \oint_{C_1} G\, dx \qquad \text{[Using Green's theorem]}$$

Since the value of G at each point (x, y) of C_1 is same as the value of F_1 at each point (x, y, z) of C and dx is same for both the curves C_1 and C, we get

$$\iint_S (\nabla \times F_1 \hat{i}) \cdot \hat{n}\, dS = \oint_C F_1\, dx \qquad \text{... (7.12)}$$

Similarly, by projecting the surface S on to yz and zx planes,

$$\iint_S (\nabla \times F_2 \hat{j}) \cdot \hat{n}\, dS = \oint_C F_2\, dy \qquad \text{... (7.13)}$$

and

$$\iint_S (\nabla \times F_3 \hat{k}) \cdot \hat{n}\, dS = \oint_C F_3\, dz \qquad \text{... (7.14)}$$

Substituting Eqs. (7.12), (7.13) and (7.14) in Eq. (7.8),

$$\iint_S \nabla \times \overline{F}\cdot\hat{n}\, dS = \oint_C (F_1\, dx + F_2\, dy + F_3\, dz) = \oint_C \left(\overline{F}\cdot d\overline{r}\right)$$

Note: If surfaces S_1 and S_2 have the same bounding curve C, then

$$\iint_{S_1} \nabla \times \overline{F}\cdot\hat{n}\, dS = \iint_{S_2} \nabla \times \overline{F}\cdot\hat{n}\, dS = \oint_C \overline{F}\cdot d\overline{r}$$

Example 1: Verify Stokes' theorem for the vector field $\overline{F} = (x^2 - y^2)\hat{i} + 2xy\,\hat{j}$ in the rectangular region in the xy-plane bounded by the lines $x = -a$, $x = a$, $y = 0$, $y = b$.

Solution: By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r}$$

(i)
$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(2y + 2y)$$

$$= 4y\hat{k}$$

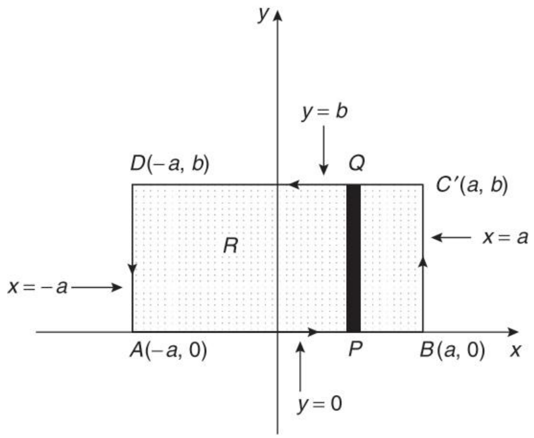


Fig. 7.41

(ii) Surface S is the rectangle $ABCD$ in xy-plane.

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\, dy}{|\hat{n} \cdot \hat{k}|} = dx\, dy$$

(iii) Let R be the region bounded by the rectangle $ABCD$ in xy-plane. Along the vertical strip PQ, y varies from 0 to b and in the region R, x varies from $-a$ to a.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_R 4y\hat{k} \cdot \hat{k}\, dx\, dy$$

$$= 4\int_{x=-a}^{a} \int_{y=0}^{b} y\, dy\, dx$$

$$= 4\int_{-a}^{a} \left| \frac{y^2}{2} \right|_0^b dx$$

$$= 2b^2 |x|_{-a}^{a}$$

$$= 4ab^2 \qquad \ldots (1)$$

(iv) Let C be the boundary of the rectangle $ABC'D$.

$$\oint_C \overline{F}\cdot d\overline{r} = \oint_{AB} \overline{F}\cdot d\overline{r} + \oint_{BC'} \overline{F}\cdot d\overline{r} + \oint_{C'D} \overline{F}\cdot d\overline{r} + \oint_{DA} \overline{F}\cdot d\overline{r} \qquad \text{... (2)}$$

(a) Along AB : $y = 0$, $dy = 0$

x varies from $-a$ to a.

$$\oint_{AB} \overline{F}\cdot d\overline{r} = \int_{AB} \left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$
$$= \int_{-a}^{a} x^2\,dx$$
$$= \left|\frac{x^3}{3}\right|_{-a}^{a}$$
$$= \frac{2a^3}{3}$$

(b) Along BC' : $x = a$, $dx = 0$

y varies from 0 to b.

$$\int_{BC'} \overline{F}\cdot d\overline{r} = \int_{BC'} \left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$
$$= \int_0^b 2ay\,dy$$
$$= 2a\left|\frac{y^2}{2}\right|_0^b$$
$$= ab^2$$

(c) Along $C'D$: $y = b$, $dy = 0$

x varies from a to $-a$.

$$\int_{C'D} \overline{F}\cdot d\overline{r} = \int_{C'D} \left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$
$$= \int_a^{-a} (x^2 - b^2)\,dx$$
$$= \left|\frac{x^3}{3} - b^2x\right|_a^{-a}$$
$$= -\frac{2a^3}{3} + 2ab^2$$

(d) Along DA : $x = -a$, $dx = 0$

y varies from b to 0.

$$\int_{DA} \overline{F}\cdot d\overline{r} = \int_{DA} \left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$
$$= \int_b^0 (-2ay)\,dy$$

$$= -2a\left|\frac{y^2}{2}\right|_b^0$$

$$= ab^2$$

Substituting in Eq. (2),

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \frac{2a^3}{3} + ab^2 - \frac{2a^3}{3} + 2ab^2 + ab^2$$

$$= 4ab^2 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\ \mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r} = 4ab^2$$

Hence, Stokes' theorem is verified.

Example 2: Verify Stokes' theorem for $\overline{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$ and S is the surface of the plane $2x + y + z = 2$ which is in the first octant.

Solution: By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\ \mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r}$$

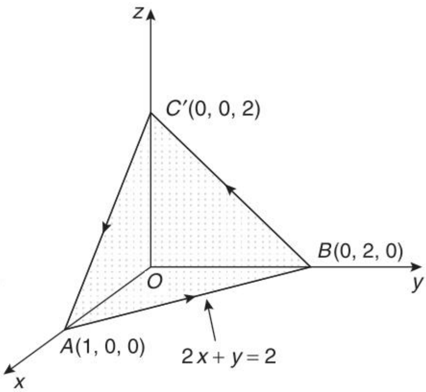


Fig. 7.42

(i) $\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$

$$= \hat{i}(0-1) - \hat{j}(-1-0) + \hat{k}(0-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

(ii) Let $\phi = 2x + y + z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{4+1+1}}$$

$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

(iii) Projection of the plane $2x + y + z = 2$ on xy-plane $(z = 0)$ is the triangle OAB bounded by the lines $x = 0$, $y = 0$, $2x + y = 2$.

(iv) $\mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \sqrt{6}\,\mathrm{d}x\,\mathrm{d}y$

(v) Let R be the region bounded by the triangle OAB in the xy-plane.

Along the vertical strip PQ, y varies from 0 to $(2-2x)$ and in the region R, x varies from 0 to 1.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R(-\hat{i}+\hat{j}-\hat{k})\cdot\frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}\sqrt{6}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1\int_0^{2-2x}(-2+1-1)\,\mathrm{d}x\,\mathrm{d}y$$

$$= -2\int_0^1 |y|_0^{2-2x}\,\mathrm{d}x$$

$$= -2\int_0^1(2-2x)\,\mathrm{d}x$$

$$= -4\left|x-\frac{x^2}{2}\right|_0^1$$

$$= -4\left(1-\frac{1}{2}\right)$$

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = -2 \qquad \text{... (1)}$$

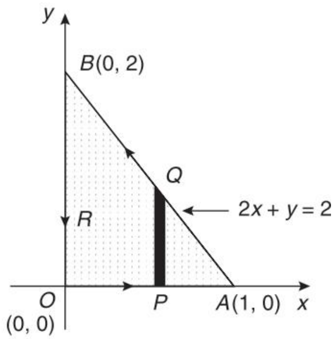


Fig. 7.43

Aliter

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = -2\iint_R \mathrm{d}x\,\mathrm{d}y$$

$$= -2(\text{Area of }\Delta OAB)$$

$$= -2\cdot\frac{1}{2}\cdot 1\cdot 2$$

$$= -2$$

(vi) Let C be the boundary of the triangle ABC'.

$$\overline{F}\cdot\mathrm{d}\overline{r} = (x+y)\,\mathrm{d}x + (y+z)\,\mathrm{d}y - x\,\mathrm{d}z$$

$$\oint\overline{F}\cdot\mathrm{d}\overline{r} = \int_{AB}\overline{F}\cdot\mathrm{d}\overline{r} + \int_{BC'}\overline{F}\cdot\mathrm{d}\overline{r} + \int_{C'A}\overline{F}\cdot\mathrm{d}\overline{r} \qquad \text{... (2)}$$

(a) Along AB: $z = 0$, $y = 2-2x$
$\mathrm{d}z = 0$, $\mathrm{d}y = -2\mathrm{d}x$
x varies from 1 to 0.

$$\int_{AB}\overline{F}\cdot\mathrm{d}\overline{r} = \int_{AB}\left[(x+y)\mathrm{d}x + (y+z)\mathrm{d}y - x\,\mathrm{d}z\right]$$

$$= \int_1^0\left[(x+2-2x)\mathrm{d}x + (2-2x)(-2\mathrm{d}x)\right]$$

$$= \int_1^0 (3x-2)\,\mathrm{d}x$$

$$= \left| 3 \cdot \frac{x^2}{2} - 2x \right|_1^0$$
$$= -\frac{3}{2} + 2$$
$$= \frac{1}{2}$$

(b) Along BC': $x = 0, \quad y + z = 2$
$dx = 0, \quad dz = -dy$
y varies from 2 to 0.

$$\int_{BC'} \overline{F} \cdot d\overline{r} = \int_{BC'} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_2^0 2\,dy$$
$$= 2|y|_2^0$$
$$= -4$$

(c) Along $C'A$: $y = 0, \quad 2x + z = 2$
$dy = 0, \quad dz = -2dx$
x varies from 0 to 1.

$$\int_{C'A} \overline{F} \cdot d\overline{r} = \int_{C'A} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_0^1 [x\,dx - x(-2\,dx)]$$
$$= \int_0^1 3x\,dx$$
$$= 3\left|\frac{x^2}{2}\right|_0^1$$
$$= \frac{3}{2}$$

Substituting in Eq. (2),

$$\oint_C \overline{F} \cdot d\overline{r} = \frac{1}{2} - 4 + \frac{3}{2}$$
$$= -2 \qquad \ldots (3)$$

From Eqs. (1) and (3),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \oint_C \overline{F} \cdot d\overline{r} = -2$$

Hence, Stokes' theorem is verified.

Example 3: Verify Stokes' theorem for $\overline{F} = xz\hat{i} + y\hat{j} + xy^2\hat{k}$ where S is the surface of the region bounded by $y = 0$, $z = 0$ and $4x + y + 2z = 4$ which is not included in the yz-plane.

Solution: By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

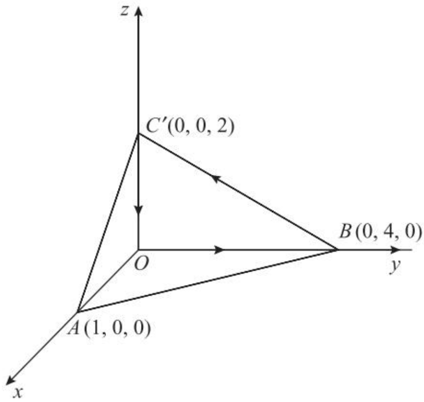


Fig. 7.44

(i)
$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y & xy^2 \end{vmatrix}$$

$$= \hat{i}(2xy - 0) - \hat{j}(y^2 - x) + \hat{k}(0 - 0)$$
$$= 2xy\,\hat{i} + (x - y^2)\,\hat{j}$$

(ii) Surface S consists of three surfaces, $y = 0$, $z = 0$ and $4x + y + 2z = 4$.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S + \iint_{S_3} \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S \quad \text{... (1)}$$

(a) Surface S_1 ($\Delta OAC'$): $y = 0$, $\hat{n} = -\hat{j}$ and $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}z$.

Let R_1 be the region bounded by the $\Delta OAC'$. Along the vertical strip P_1Q_1, z varies from 0 to $2 - 2x$ and in the region R_1, x varies from 0 to 1.

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \iint_{R_1} -(x - y^2)\,\mathrm{d}x\,\mathrm{d}z$$
$$= \int_0^1 \int_0^{2-2x} (-x)\,\mathrm{d}x\,\mathrm{d}z \quad [\because y = 0]$$
$$= -\int_0^1 x \left| z \right|_0^{2-2x} \mathrm{d}x$$

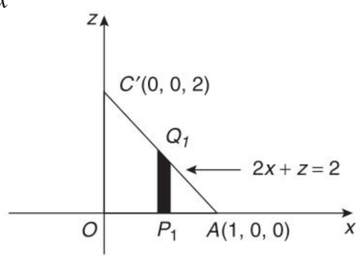


Fig. 7.45

$$= -\int_0^1 x(2-2x)\,dx$$

$$= -\left| x^2 - \frac{2x^3}{3} \right|_0^1$$

$$= -\left(1 - \frac{2}{3}\right)$$

$$= -\frac{1}{3}$$

(b) Surface S_2 (ΔOAB): $z = 0$, $\hat{n} = -\hat{k}$ and $dS = dx\,dy$.
Let R_2 be the region bounded by the ΔOAB. Along the vertical strip $P_2 Q_2$, y varies from 0 to $4 - 4x$ and in the region R_2, x varies from 0 to 1.

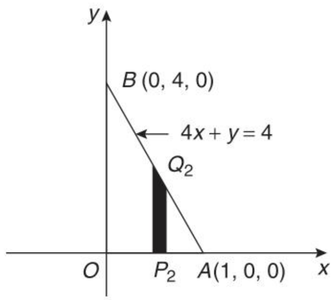


Fig. 7.46

$$\iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\,dS = \iint_{R_2} \left[2xy\,\hat{i} + (x - y^2)\,\hat{j}\right] \cdot (-\hat{k})\,dx\,dy$$

$$= 0$$

(c) Surface $S_3 (4x + y + 2z = 4)$:

Let $\phi = 4x + y + 2z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{16+1+4}}$$

$$= \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{21}}$$

Projection of the plane $4x + y + 2z = 4$ on xy-plane is the triangle OAB.

$$dS = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{\sqrt{21}}{2}\,dx\,dy$$

Let R_3 be the region bounded by the ΔOAB. Along the vertical strip P_2Q_2, y varies from 0 to $4 - 4x$ and x varies from 0 to 1.

$$\iint_{S_3} \nabla \times \overline{F} \cdot \hat{n}\;dS = \iint_{R_2} \left[2xy\,\hat{i} + (x - y^2)\,\hat{j}\right] \cdot \left(\frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{21}}\right) \frac{\sqrt{21}}{2}\,dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{4-4x}(8xy + x - y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left| 8x\frac{y^2}{2} + xy - \frac{y^3}{3} \right|_0^{4-4x} dx$$

$$= \frac{1}{2}\int_0^1 \left[4x(4-4x)^2 + x(4-4x) - \frac{(4-4x)^3}{3} \right] dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{256}{3}x^3 - 196x^2 + 132x - \frac{64}{3} \right) dx$$

$$= \frac{1}{2}\left| \frac{256}{3}\cdot\frac{x^4}{4} - 196\frac{x^3}{3} + 132\frac{x^2}{2} - \frac{64}{3}x \right|_0^1$$

$$= \frac{1}{2}\left(\frac{64}{3} - \frac{196}{3} + 66 - \frac{64}{3} \right)$$

$$= \frac{1}{3}$$

Substituting in Eq. (1),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = -\frac{1}{3} + 0 + \frac{1}{3} = 0 \qquad \text{... (2)}$$

(iii) Since the surface S does not include the yz-plane, it is open on the yz-plane. $\Delta OBC'$ is the boundary of the surface S.

Let C be the boundary of the $\Delta OBC'$ bounded by the lines $y = 0, z = 0, y + 2z = 4$.

$$\oint_C \overline{F} \cdot d\overline{r} = \int_{C'O} \overline{F} \cdot d\overline{r} + \int_{OB} \overline{F} \cdot d\overline{r} + \int_{BC'} \overline{F} \cdot d\overline{r} \qquad \text{... (3)}$$

$$\overline{F} \cdot d\overline{r} = xz\,dx + y\,dy + xy^2 dz = y\,dy \qquad [\because x = 0, dx = 0]$$

(a) Along $C'O : y = 0 \quad dy = 0$

z varies from 2 to 0.

$$\int_{C'O} \overline{F} \cdot d\overline{r} = \int_2^0 y\,dy = 0$$

(b) Along $OB : z = 0, \quad dz = 0$

y varies from 0 to 4.

$$\int_{OB} \overline{F} \cdot d\overline{r} = \int_0^4 y\,dy = \left| \frac{y^2}{2} \right|_0^4 = 8$$

(c) Along BC': $y = 4 - 2z$, $dy = -2\,dz$

z varies from 0 to 2.

$$\int_{BC'} \overline{F} \cdot d\overline{r} = \int_0^2 y\,dy$$

$$= \int_0^2 (4 - 2z)(-2dz)$$

$$= -4\left|2z - \frac{z^2}{2}\right|_0^2$$

$$= -4(4-2)$$

$$= -8$$

Substituting in Eq. (3),

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = 0 + 8 - 8 = 0 \qquad \text{... (4)}$$

From Eqs. (2) and (4),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\ \mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r} = 0$$

Hence, Stokes' theorem is verified.

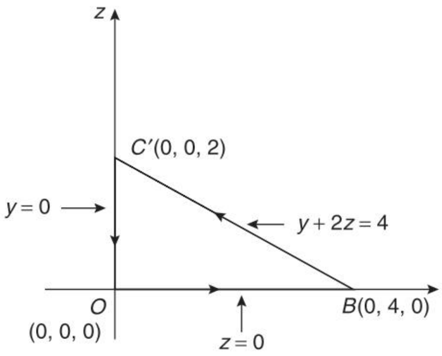


Fig. 7.47

Example 4: Verify Stokes' theorem for $\overline{F} = 4y\,\hat{i} - 4x\,\hat{j} + 3\hat{k}$, where S is a disk of 1-unit radius lying on the plane $z = 1$ and C is its boundary.

Solution: By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\ \mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r}$$

where S is the surface of the disk of 1-unit radius lying on the plane $z = 1$ and C is the circle $x^2 + y^2 = 1$.

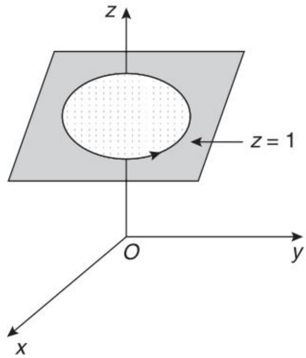


Fig. 7.48

(i) $$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & -4x & 3 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-4-4)$$

$$= -8\hat{k}$$

(ii) Since disc lies on the plane $z = 1$, parallel to the xy-plane,

$$\hat{n} = \hat{k}$$

(iii) Projection of the disc in the xy-plane is the circle $x^2 + y^2 = 1$.

(iv) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \mathrm{d}x\,\mathrm{d}y$$

(v) Let R be the region bounded by the circle $x^2 + y^2 = 1$ in the xy-plane.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\ \mathrm{d}S = \iint_R (-8\hat{k})\cdot\hat{k}\,\mathrm{d}x\,\mathrm{d}y$$

$$= -8\iint_R \mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$,
equation of the circle $x^2 + y^2 = 1$
reduces to $r = 1$ and $dx\,dy = r\,dr\,d\theta$.
Along, the radius vector OA, r varies from 0 to 1 and for a complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = -8\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= -8\left|\theta\right|_0^{2\pi}\left|\frac{r^2}{2}\right|_0^1 \qquad (1)$$

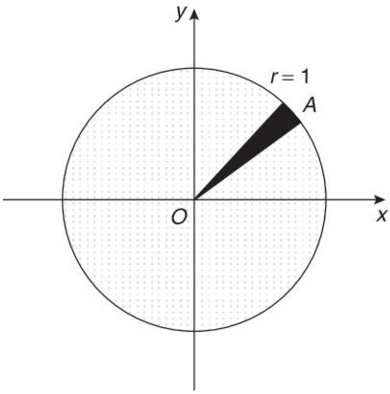


Fig. 7.49

Aliter

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = -8\iint_R dx\,dy$$

$$= -8(\text{Area of the circle})$$

$$= -8\pi(1)^2$$

$$= -8\pi$$

(vi) C is the boundary of the disc, i.e., the circle $x^2 + y^2 = 1$ lying on the plane $z = 1$.

$$\overline{F}\cdot d\overline{r} = 4y\,dx - 4x\,dy + 3dz$$

$$= 4y\,dx - 4x\,dy \qquad [\because z = 1,\ dz = 0]$$

$$\oint_C \overline{F}\cdot d\overline{r} = \oint_C (4y\,dx - 4x\,dy)$$

Parametric equation of the circle is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$dx = -\sin\theta\,d\theta, \qquad dy = \cos\theta\,d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi}\left[4\sin\theta(-\sin\theta\,d\theta) - 4\cos\theta(\cos\theta\,d\theta)\right]$$

$$= -4\int_0^{2\pi} d\theta$$

$$= -4\left|\theta\right|_0^{2\pi}$$

$$= -8\pi \qquad \text{... (2)}$$

From Eqs. (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r} = -8\pi$$

Hence, Stokes' theorem is verified.

Example 5: Verify Stokes' theorem for $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$ over the surface of the sphere $x^2 + y^2 + z^2 = 16$ above xy-plane.

Solution: By Stokes' theorem,

(i) $$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r}$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(2z-0) + \hat{k}(3y-1)$$

$$= -2z\,\hat{j} + (3y-1)\hat{k}$$

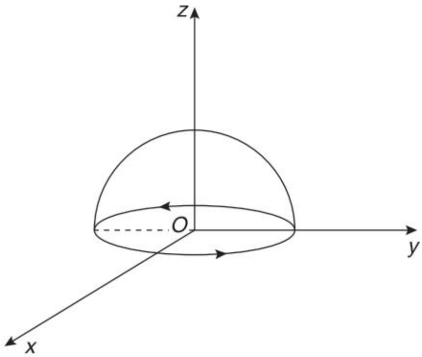


Fig. 7.50

(ii) Let $\phi = x^2+y^2+z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\,\hat{i}+y\,\hat{j}+z\,\hat{k}}{4} \qquad \left[\because x^2+y^2+z^2=16\right]$$

(iii) Let R be the projection of the hemisphere $x^2+y^2+z^2=16$ on the xy-plane ($z=0$) which is a circle, $x^2+y^2=16$.

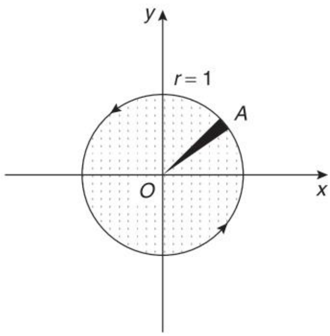


Fig. 7.51

(iv) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

(v) $$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$$

$$= \iint_R \left[-2z\,\hat{j}+(3y-1)\hat{k}\right]\cdot\left(\frac{x\,\hat{i}+y\,\hat{j}+z\,\hat{k}}{4}\right)\frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R [-2zy+(3y-1)z]\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R (y-1)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2+y^2=16$ reduces to $r=4$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Along the radius vector OA, r varies from 0 to 4 and for the complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S=\int_0^{2\pi}\int_0^4(r\sin\theta-1)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=\left|\frac{r^3}{3}\right|_0^4\left|-\cos\theta\right|_0^{2\pi}-\left|\frac{r^2}{2}\right|_0^4\left|\theta\right|_0^{2\pi}$$

$$=-\frac{4^3}{3}(\cos 2\pi-\cos 0)-\frac{16}{2}\cdot 2\pi$$

$$=-16\pi \qquad \text{... (1)}$$

(vi) The boundary C of the hemisphere S is the circle $x^2+y^2=4$ in xy-plane ($z=0$).

$$\overline{F}\cdot\mathrm{d}\overline{r}=(x^2+y-4)\mathrm{d}x+3xy\,\mathrm{d}y+(2xz+z^2)\mathrm{d}z$$

$$=(x^2+y-4)\,\mathrm{d}x+3xy\,\mathrm{d}y \qquad [\because z=0,\mathrm{d}z=0]$$

$$\oint_C\overline{F}\cdot\mathrm{d}\overline{r}=\oint_C\left[(x^2+y-4)\,\mathrm{d}x+3xy\,\mathrm{d}y\right]$$

Parametric equation of the circle $x^2+y^2=4$ is

$$x=4\cos\theta,\qquad y=4\sin\theta$$

$$\mathrm{d}x=-4\sin\theta\,\mathrm{d}\theta,\qquad \mathrm{d}y=4\cos\theta\,\mathrm{d}\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C\overline{F}\cdot\mathrm{d}\overline{r}=\left[\int_0^{2\pi}(16\cos^2\theta+4\sin\theta-4)(-4\sin\theta\,\mathrm{d}\theta)+(3\cdot 4\cos\theta\cdot 4\sin\theta)(4\cos\theta\,\mathrm{d}\theta)\right]$$

$$=\int_0^{2\pi}(-64\cos^2\theta\sin\theta-16\sin^2\theta+16\sin\theta+192\cos^2\theta\sin\theta)\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}-16\sin^2\theta\,\mathrm{d}\theta \qquad \left[\because \int_0^{2a}f(2a-x)=0,\text{ if } f(2a-x)=-f(x)\right]$$

$$=-16\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=-8\left|\theta-\frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$=-16\pi \qquad \text{... (2)}$$

From Eqs. (1) and (2),

$$\iint_S\nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S=\oint_C\overline{F}\cdot\mathrm{d}\overline{r}=-16\pi$$

Hence, Stokes' theorem is verified.

Example 6: Verify Stokes' theorem for $\overline{F}=y\hat{i}+z\hat{j}+x\hat{k}$ over the surface $x^2+y^2=1-z,\ z>0$.

Solution: By Stokes' theorem,

$$\iint_S\nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S=\oint_C\overline{F}\cdot\mathrm{d}\overline{r}$$

(i) $\nabla\times\overline{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}$

$$=\hat{i}(0-1)-\hat{j}(1-0)+\hat{k}(0-1)$$
$$=-(\hat{i}+\hat{j}+\hat{k})$$

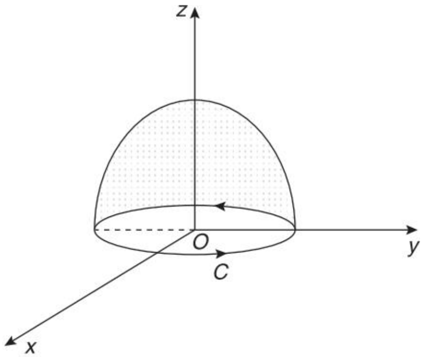


Fig. 7.52

(ii) Let $\phi=x^2+y^2+z$

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}$$
$$=\frac{2x\hat{i}+2y\hat{j}+\hat{k}}{\sqrt{4x^2+4y^2+1}}$$

(iii) Let R be the projection of the surface $x^2+y^2=1-z$ on the xy-plane ($z=0$) which is a circle $x^2+y^2=1$.

(iv) $\mathrm{d}S=\dfrac{\mathrm{d}x\,\mathrm{d}y}{\hat{n}\cdot\hat{k}}$

$$=\sqrt{4x^2+4y^2+1}\;\mathrm{d}x\,\mathrm{d}y$$

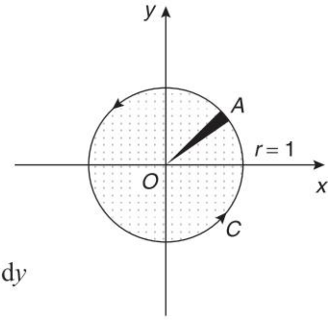


Fig. 7.53

(v) $\iint\limits_S \nabla\times\overline{F}\cdot\hat{n}\;\mathrm{d}S$

$$=\iint\limits_R -(\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2x\hat{i}+2y\hat{j}+\hat{k})}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}\,\mathrm{d}x\,\mathrm{d}y$$
$$=-\iint\limits_R(2x+2y+1)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x=r\cos\theta$, $y=r\sin\theta$, circle $x^2+y^2=1$ reduces to $r=1$ and $\mathrm{d}x\,\mathrm{d}y=r\,\mathrm{d}r\,\mathrm{d}\theta$

Along the radius vector OA, r varies from 0 to 1 and for the complete circle, θ varies from 0 to 2π.

$$\iint\limits_S\nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S=-\int_0^{2\pi}\int_0^1(2r\cos\theta+2r\sin\theta+1)r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$=-\int_0^{2\pi}\left[2(\cos+\sin\theta)\left|\frac{r^3}{3}\right|_0^1+\left|\frac{r^2}{2}\right|_0^1\right]\mathrm{d}\theta$$
$$=-\int_0^{2\pi}\left[\frac{2}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]\mathrm{d}\theta$$
$$=-\left|\frac{2}{3}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right|_0^{2\pi}$$

$$= -\frac{2}{3}(\sin 2\pi - \cos 2\pi - \sin 0 + \cos 0) - \pi$$

$$= -\pi \qquad \text{... (1)}$$

(vi) The boundary C of the surface $x^2 + y^2 = 1 - z$ is the circle $x^2 + y^2 = 1$ in the xy-plane ($z = 0$).

$$\overline{F} \cdot \mathrm{d}\overline{r} = y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$$

$$= y\,\mathrm{d}x \qquad [\because z = 0, \mathrm{d}z = 0]$$

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = \oint_C y\,\mathrm{d}x$$

Parametric equation of the circle $x^2 + y^2 = 1$ is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = \cos\theta\,\mathrm{d}\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\,\mathrm{d}\theta)$$

$$= -\int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2}\right) \mathrm{d}\theta$$

$$= -\frac{1}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= -\frac{1}{2}\left(2\pi - \frac{\sin 4\pi}{2} - 0\right) \qquad \text{... (2)}$$

$$= -\pi$$

From Eqs. (1) and (2),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r} = -\pi$$

Hence, Stokes' theorem is verified.

Example 7: Evaluate by Stokes' theorem $\oint_C (e^x \mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z)$, where C is the curve $x^2 + y^2 = 4,\ z = 2$.

Solution: By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$$

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C (e^x \mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z) \qquad \text{... (1)}$$

$$\overline{F} = e^x\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

Substituting in Eq. (1),

$$\oint_C (e^x\,\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z) = 0$$

Example 8: Evaluate $\iint_S (\nabla\times\overline{F})\cdot\hat{n}\,\mathrm{d}S$ for the vector field $\overline{F} = (2y^2 + 3z^2 - x^2)\hat{i} + (2z^2 + 3x^2 - y^2)\hat{j} + (2x^2 + 3y^2 - z^2)\hat{k}$ over the part of the sphere $x^2 + y^2 + z^2 - 2ax + az = 0$ cut off by the plane $z = 0$.

Solution: By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r} \qquad \text{... (1)}$$

(i) $\overline{F}\cdot\mathrm{d}\overline{r} = (2y^2 + 3z^2 - x^2)\mathrm{d}x + (2z^2 + 3x^2 - y^2)\mathrm{d}y + (2x^2 + 3y^2 - z^2)\mathrm{d}z$

(ii) Let C be the boundary of the part of the sphere $x^2 + y^2 + z^2 - 2ax + az = 0$ cut off by the plane $z = 0$, which is a circle, $x^2 + y^2 - 2ax = 0, (x-a)^2 + y^2 = a^2$.

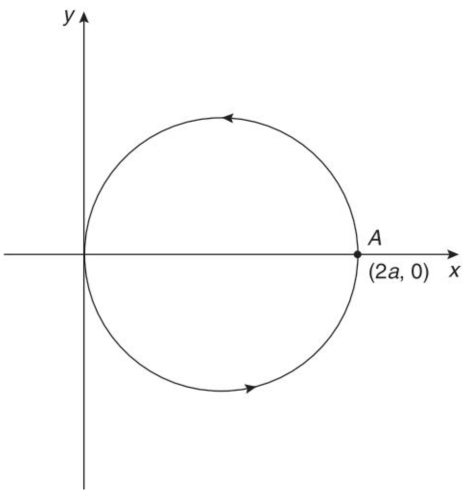


Fig. 7.54

Parametric equation of the circle

$$x - a = a\cos\theta, \qquad y = a\sin\theta$$
$$dx = -a\sin\theta\, d\theta, \quad dy = a\cos\theta\, d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\overline{r} = \oint_C \left[(2y^2 - x^2)dx + (3x^2 - y^2)dy\right] \qquad [\because z = 0, dz = 0]$$
$$= \int_0^{2\pi} \left[\left\{2a^2\sin^2\theta - (a + a\cos\theta)^2\right\}(-a\sin\theta\, d\theta) + \left\{3(a + a\cos\theta)^2 - a^2\sin^2\theta\right\}(a\cos\theta\, d\theta).\right]$$
$$= a^3 \int_0^{2\pi} (-2\sin^3\theta + \sin\theta + \sin\theta\cos^2\theta + 2\cos\theta\sin\theta + 3\cos\theta + 3\cos^3\theta + 6\cos^2\theta - \sin^2\theta\cos\theta)d\theta$$
$$= 2a^3 \int_0^{\pi} (3\cos\theta + 3\cos^3\theta + 6\cos^2\theta - \sin^2\theta\cos\theta)d\theta \qquad \left[\because \int_0^{2a} f(\theta)d\theta = 0, \text{if } f(2a-\theta) = -f(\theta) \atop = 2\int_0^{a} f(\theta)d\theta, \text{ if } f(2a-\theta) = f(\theta)\right]$$
$$= 4a^3 \int_0^{\frac{\pi}{2}} 6\cos^2\theta\, d\theta \qquad [\because \cos(\pi - \theta) = -\cos\theta]$$
$$= 24a^3 \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= 12a^3 \left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$
$$= 12a^3 \left(\frac{\pi}{2} + \frac{\sin\pi - 0}{2}\right)$$
$$= 6\pi a^3$$

From Eq. (1),

$$\iint_S \nabla \times \overline{F}\cdot\hat{n}\, ds = 6\pi a^3$$

Example 9: Evaluate by Stokes' theorem $\oint_C (4y\,dx + 2z\,dy + 6y\,dz)$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6z$ and the plane $z = x + 3$.

Solution: By Stokes' theorem,

$$\iint_S \nabla \times \overline{F}\cdot\hat{n}\, dS = \oint_C \overline{F}\cdot d\overline{r} \qquad \text{... (1)}$$

(i) $$\overline{F}\cdot d\overline{r} = 4y\,dx + 2z\,dy + 6y\,dz$$

$$= (4y\hat{i} + 2z\,\hat{j} + 6y\,\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$\therefore\ \overline{F} = 4y\hat{i} + 2z\,\hat{j} + 6y\,\hat{k}$$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix}$$

$$= \hat{i}(6-2) - \hat{j}(0-0) + \hat{k}(0-4)$$

$$= 4\hat{i} - 4\hat{k}$$

(ii) Normal to the surface which is bounded by the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6z$ and the plane $z = x + 3$ is also normal to the plane $z = x + 3$.

Let $\phi = x - z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{\hat{i} - \hat{k}}{\sqrt{2}}$$

$$dS = dx\,dz$$

(iii) Let C be the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$ which is a circle $x^2 + z^2 = 6z$ (since $y = 0$ on xz-plane).

(iv) Let R be the region bounded by the circle $x^2 + z^2 - 6z = 0$ with 3-unit radius.

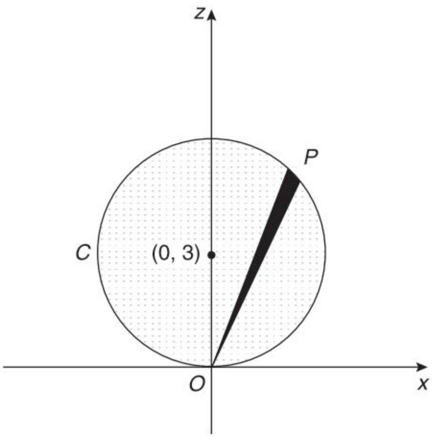


Fig. 7.55

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \iint_R \frac{4+4}{\sqrt{2}}\,dx\,dz$$

$$= 4\sqrt{2} \iint_R dx\,dz$$

Putting $x = r\cos\theta$, $z = r\sin\theta$, the equation of the circle $x^2 + z^2 = 6z$ reduces to $r = 6\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. Along the radius vector OP, r varies from 0 to 6 $\sin\theta$ and for the complete circle, θ varies from 0 to π.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = 4\sqrt{2}\int_0^{\pi}\int_0^{6\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\sqrt{2}\int_0^{\pi} \left| \frac{r^2}{2} \right|_0^{6\sin\theta} \mathrm{d}\theta$$

$$= \frac{4\sqrt{2}}{2}\int_0^{\pi} 36\sin^2\theta\,\mathrm{d}\theta$$

$$= 36\sqrt{2}\int_0^{\pi}(1-\cos 2\theta)\mathrm{d}\theta$$

$$= 36\sqrt{2}\left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi}$$

$$= 36\sqrt{2}\left(\pi - \frac{\sin 2\pi}{2} \right)$$

$$= 36\pi\sqrt{2}$$

Aliter

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S = 4\sqrt{2}\iint_R \mathrm{d}x\,\mathrm{d}z$$

$$= 4\sqrt{2}\,(\text{Area of the circle } C)$$

$$= 4\sqrt{2}(\pi \cdot 3^2)$$

$$= 36\pi\sqrt{2}$$

From Eq. (1),

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = 36\pi\sqrt{2}$$

Example 10: Using Stokes' theorem, find the work done in moving a particle once around the perimeter of the triangle with vertices at (2, 0, 0), (0, 3, 0) and (0, 0, 6) under the force field $\overline{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$.

Solution: Work done $= \oint_C \overline{F} \cdot \mathrm{d}\overline{r}$

By Stokes' theorem,

$$\oint_C \overline{F} \cdot \mathrm{d}\overline{r} = \iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S$$

Thus, work done $= \iint_S \nabla \times \overline{F} \cdot \hat{n}\,\mathrm{d}S$

where S is the surface of the ΔABC.

Equation of the ΔABC is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

$$3x+2y+z=6$$

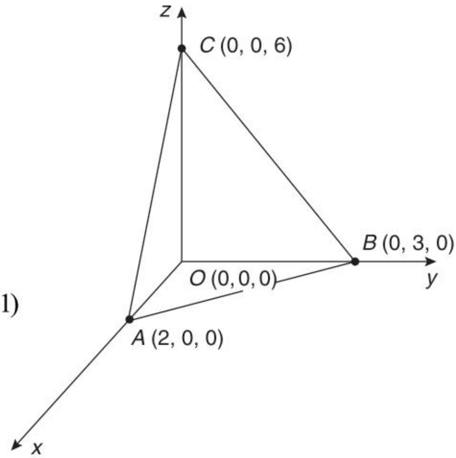


Fig. 7.56

(i) $$\nabla\times\overline{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y & 2x-z & y+z\end{vmatrix}$$

$$=\hat{i}(1+1)-\hat{j}(0-0)+\hat{k}(2-1)$$

$$=2\hat{i}+\hat{k}$$

(ii) Let $\phi=3x+2y+z$

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}$$

$$=\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}}$$

$$=\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

(iii) Projection of ΔABC on the xy-plane is the ΔOAB bounded by the lines $y=0$, $3x+2y=6$, $x=0$.

$$\mathrm{d}S=\frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$$

$$=\sqrt{14}\,\mathrm{d}x\,\mathrm{d}y$$

(iv) Let R be the region bounded by the ΔOAB. Along the vertical strip PQ, y varies from 0 to $\frac{6-3x}{2}$ and in the region R, x varies from 0 to 2.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S=\iint_R(2\hat{i}+\hat{k})\cdot\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)\sqrt{14}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^2\int_0^{\frac{6-3x}{2}}7\,\mathrm{d}y\,\mathrm{d}x$$

$$=7\int_0^2|y|_0^{\frac{6-3x}{2}}\,\mathrm{d}x$$

$$=7\int_0^2\left(\frac{6-3x}{2}\right)\mathrm{d}x$$

$$=7\left|3x-\frac{3x^2}{4}\right|_0^2$$

$$=21$$

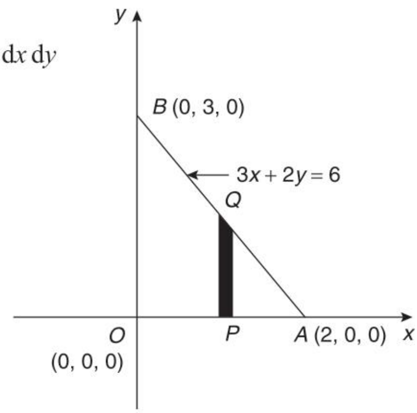


Fig. 7.57

Aliter

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = 7\iint_R dx\, dy$$
$$= 7(\text{Area of } \Delta OAB)$$
$$= 7 \cdot \frac{1}{2} \cdot 2 \cdot 3$$
$$= 21$$

Example 11: Evaluate $\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS$ using Stokes' theorem, where $\overline{F} = yz\hat{i} + (2x + y - 1)\hat{j} + (x^2 + 2z)\hat{k}$ and S is the surface of intersection of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in the positive octant.

Solution: By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r} \quad \text{... (1)}$$

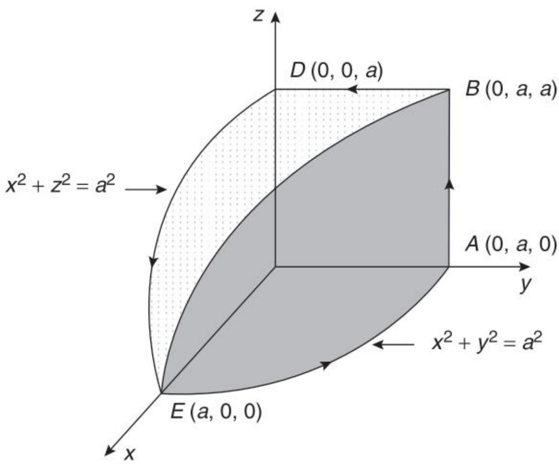


Fig. 7.58

(i) $\overline{F} \cdot d\overline{r} = yz\, dx + (2x + y - 1)dy + (x^2 + 2z)dz$

(ii) C is $EABDE$ which is the boundary of the surface of intersection of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in the positive octant.

$$\oint_C \overline{F} \cdot d\overline{r} = \int_{EA} \overline{F} \cdot d\overline{r} + \int_{AB} \overline{F} \cdot d\overline{r} + \int_{BD} \overline{F} \cdot d\overline{r} + \int_{DE} \overline{F} \cdot d\overline{r} \quad \text{... (2)}$$

(a) Along EA: $z = 0$, $\quad x^2 + y^2 = a^2$

$$dz = 0$$

Putting $\quad x = a\cos\theta, \quad y = a\sin\theta$

$$dx = -a\sin\theta\, d\theta, \quad dy = a\cos\theta\, d\theta$$

Along EA, θ varies from 0 to $\frac{\pi}{2}$.

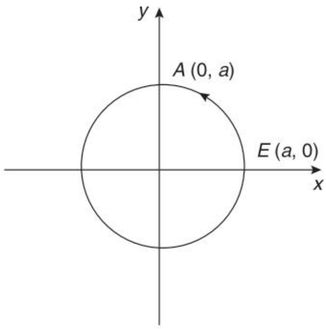


Fig. 7.59

$$\int_{EA} \overline{F}\cdot \mathrm{d}\overline{r} = \int_{EA}\left[yz\,\mathrm{d}x + (2x+y-1)\mathrm{d}y + \left(x^2+2z\right)\mathrm{d}z\right]$$

$$= \int_0^{\frac{\pi}{2}} (2a\cos\theta + a\sin\theta - 1)a\cos\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} (2a^2\cos^2\theta + a^2\sin\theta\cos\theta - a\cos\theta)\mathrm{d}\theta$$

$$= 2a^2\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right) + a^2\cdot\frac{1}{2}B(1,1) - a\left|\sin\theta\right|_0^{\frac{\pi}{2}}$$

$$= a^2\frac{\left\lceil\frac{3}{2}\right.\left\lceil\frac{1}{2}\right.}{\lceil 2} + \frac{a^2}{2}\frac{\lceil 1\lceil 1}{\lceil 2} - a\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= a^2\cdot\frac{1}{2}\pi + \frac{a^2}{2} - a$$

$$= \frac{\pi a^2}{2} + \frac{a^2}{2} - a$$

(b) Along AB: $x = 0$, $y = a$

$\mathrm{d}x = 0$, $\mathrm{d}y = 0$

z varies from 0 to a.

$$\int_{AB} \overline{F}\cdot \mathrm{d}\overline{r} = \int_{AB}\left[yz\,\mathrm{d}x + (2x+y-1)\mathrm{d}y + (x^2+2z)\mathrm{d}z\right]$$

$$= \int_0^a 2z\,\mathrm{d}z$$

$$= \left|z^2\right|_0^{a^2}$$

$$= a^2$$

(c) Along BD: $x = 0$, $z = a$

$\mathrm{d}x = 0$, $\mathrm{d}z = 0$

y varies from a to 0.

$$\int_{BD} \overline{F}\cdot \mathrm{d}\overline{r} = \int_{BD}\left[yz\,\mathrm{d}x + (2x+y-1)\mathrm{d}y + (x^2+2z)\mathrm{d}z\right]$$

$$= \int_a^0 (y-1)\mathrm{d}y$$

$$= \left|\frac{y^2}{2} - y\right|_a^0$$

$$= -\frac{a^2}{2} + a.$$

(d) Along DE: $y = 0, \quad x^2 + z^2 = a^2$

$$dy = 0$$

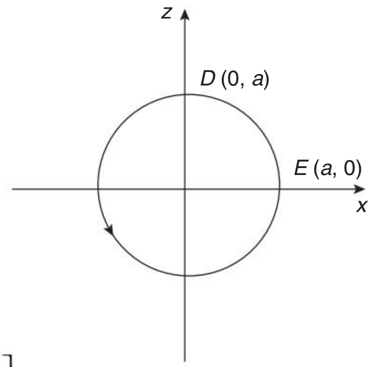


Fig. 7.60

Putting $x = a\cos\theta, \quad z = a\sin\theta$

$$dx = -a\sin\theta\, d\theta, \quad dz = a\cos\theta\, d\theta$$

Along DE, θ varies from $\frac{\pi}{2}$ to 2π.

$$\int_{DE} \overline{F}\cdot d\overline{r} = \int_{DE}\left[yz\, dx + (2x + y - 1)dy + (x^2 + 2z)dz \right]$$

$$= \int_{\frac{\pi}{2}}^{2\pi} (a^2\cos^2\theta + 2a\sin\theta)\,(a\cos\theta\, d\theta)$$

$$= a^2\int_{\frac{\pi}{2}}^{2\pi} (a\cos^3\theta + 2\sin\theta\cos\theta)d\theta$$

$$= a^2\int_{\frac{\pi}{2}}^{2\pi} \left[\frac{a}{4}(\cos 3\theta + 3\cos\theta) + \sin 2\theta\right]d\theta$$

$$= \frac{a^3}{4}\left|\frac{\sin 3\theta}{3} + 3\sin\theta\right|_{\frac{\pi}{2}}^{2\pi} + a^2\left|-\frac{\cos 2\theta}{2}\right|_{\frac{\pi}{2}}^{2\pi}$$

$$= \frac{a^3}{4}\left(\frac{\sin 6\pi}{3} + 3\sin 2\pi - \frac{1}{3}\sin\frac{3\pi}{2} - 3\sin\frac{\pi}{2}\right) - \frac{a^2}{2}(\cos 4\pi - \cos\pi)$$

$$= \frac{a^3}{4}\left(\frac{1}{3} - 3\right) - \frac{a^2}{2}(1+1)$$

$$= -\frac{2a^3}{3} - a^2$$

Substituting in Eq. (2),

$$\int_C \overline{F}\cdot d\overline{r} = \frac{\pi a^2}{2} + \frac{a^2}{2} - a + a^2 - \frac{a^2}{2} + a - \frac{2a^3}{3} - a^2$$

$$= \frac{\pi a^2}{2} - \frac{2a^3}{3}$$

From Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, dS = \frac{\pi a^2}{2} - \frac{2a^3}{3}.$$

Exercise 7.6

(I) Verify Stoke's theorem for the following vector point functions:

1. $\overline{F}=\left(x^3+\dfrac{yz^2}{2}\right)\hat{i}+\left(\dfrac{xz^2}{2}+y^2\right)\hat{j}+(x\,y\,z)\hat{k}$ over the surface S of the cube $0\le x\le 3,\ 0\le y\le 3,\ 0\le z\le 3.$

[**Ans.:** 0]

2. $\overline{F}=x\,z\,\hat{i}+y\,\hat{j}+y^2x\,\hat{k}$ over the surface S of the tetrahedron bounded by the planes $y=0$, $z=0$ and $4x+y+2z=4$ above the yz-plane.

[**Ans.:** 0]

3. $\overline{F}=\left(x^3+\dfrac{z^4}{4}\right)\hat{i}+4x\,\hat{j}+(x\,z^3+z^2)\hat{k}$ over the upper half surface S of the sphere $x^2+y^2+z^2=1$.

[**Ans.:** 4π]

4. $\overline{F}=(x^2+y+2)\hat{i}+2xy\,\hat{j}+4ze^x\,\hat{k}$ over the surface S of the paraboloid $z=9-(x^2+y^2)$ above xy-plane.

[**Ans.:** -9π]

(II) Evaluate the following integrals using Stokes' theorem:

1. $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ where $\overline{F}=(x^2+y+z)\hat{i}+2xy\,\hat{j}-(3xyz+z^3)\hat{k}$ and S is the surface of the hemisphere $x^2+y^2+z^2=9$ above the xy-plane.

[**Ans.:** -9π]

2. $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ where $\overline{F}=3y\,\hat{i}-xz\,\hat{j}+yz^2\hat{k}$ and S is the surface of the paraboloid $x^2+y^2=2z$ bounded by the plane $z=2$ and C is its boundary traversed in the clockwise direction.

[**Ans.:** 20π]

3. $\int_C (y\,\mathrm{d}x+z\,\mathrm{d}y+x\,\mathrm{d}z)$ where C is the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $x+z=a$.

$\left[\textbf{Ans.:}\ \dfrac{-\pi a^2}{\sqrt{2}}\right]$.

(III) For the vector field:

1. $\overline{F}=-\dfrac{y}{x^2+y^2}\hat{i}+\dfrac{x}{x^2+y^2}\hat{j}$ over the surface of the sphere $x^2+y^2+z^2=1$ above the xy-plane, evaluate

(i) $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$ **(ii)** $\oint_C \overline{F}\cdot\mathrm{d}\overline{r}$, where C is the boundary of S. Are the results compatible with Stokes' theorem?

[**Ans.:** (i) 0 (ii) 2π (iii) no since $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S\neq\oint_C \overline{F}\cdot\mathrm{d}\overline{r}$ Also in this case, Stokes' theorem cannot be applied since at (0, 0) which is inside C, $\overline{F}$ is neither continuous nor differentiable].

Integral Formulae

Appendix 1

1. $\int x^n \mathrm{d}x = \dfrac{x^{n+1}}{n+1}\ (n \neq -1)$

2. $\int \dfrac{1}{x}\mathrm{d}x = \log|x|$

3. $\int e^x \mathrm{d}x = e^x$

4. $\int a^x \mathrm{d}x = \dfrac{a^x}{\log a},\ a > 0,\ a \neq 1$

5. $\int \sin x\,\mathrm{d}x = -\cos x$

6. $\int \cos x\,\mathrm{d}x = \sin x$

7. $\int \tan x\,\mathrm{d}x = -\log \cos x$

8. $\int \cot x\,\mathrm{d}x = \log \sin x$

9. $\int \sec x\,\mathrm{d}x = \log(\sec x + \tan x)$

10. $\int \operatorname{cosec} x\,\mathrm{d}x = \log(\operatorname{cosec} x - \cot x)$

11. $\int \sec^2 x\,\mathrm{d}x = \tan x$

12. $\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x$

13. $\int \sec x \tan x\,\mathrm{d}x = \sec x$

14. $\int \operatorname{cosec} x \cot x\,\mathrm{d}x = -\operatorname{cosec} x$

15. $\int \dfrac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \sin^{-1}\left(\dfrac{x}{a}\right)$

16. $\int \dfrac{1}{\sqrt{x^2 - a^2}}\mathrm{d}x =$

$\log\left(x + \sqrt{x^2 - a^2}\right) = \cosh^{-1}\left(\dfrac{x}{a}\right)$

17. $\int \dfrac{1}{\sqrt{x^2 + a^2}}\mathrm{d}x =$

$\log\left(x + \sqrt{x^2 + a^2}\right) = \sinh^{-1}\left(\dfrac{x}{a}\right)$

18. $\int \dfrac{1}{a^2 - x^2}\mathrm{d}x = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)$

$= \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right),\ x^2 < a^2$

19. $\int \dfrac{1}{x^2 - a^2}\mathrm{d}x = \dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right)$

$= -\dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right),\ x^2 > a^2$

20. $\int \dfrac{1}{a^2 + x^2}\mathrm{d}x = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

21. $\int \sqrt{a^2 - x^2}\,\mathrm{d}x$

$= \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

22. $\int \sqrt{a^2 + x^2}\,\mathrm{d}x$

$= \dfrac{x}{2}\sqrt{a^2 + x^2} + \dfrac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right)$

23. $\int \sqrt{x^2 - a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2})$

24. $\int e^{ax} \sin bx\, dx$

$= \frac{e^{ax}}{a^2 + b^2}\ (a \sin bx - b \cos bx)$

25. $\int e^{ax} \cos bx\, dx$

$= \frac{e^{ax}}{a^2 + b^2}\ (a \cos bx + b \sin bx)$

26. $\int uv\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\int v\, dx\right) dx$

27. $\int [f(x)]^n f'(x)\, dx$

$= \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$

28. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)|$

29. $\int e^{f(x)} f'(x)\, dx = e^{f(x)}$

30. $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

31. $\int \sin[f(x)]\, f'(x)\, dx = -\cos f(x)$

32. $\int \cos[f(x)]\, f'(x)\, dx = \sin f(x)$

33. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

34. $\int_0^{2a} f(x)\, dx$

$= \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$

35. $\int_{-a}^a f(x)\, dx$

$= 2\int_0^a f(x)\, dx,$ if $f(x)$ is even

$= 0,$ if $f(x)$ is odd

36. $\int_0^{2a} f(x)\, dx$

$= 2\int_0^a f(x)\, dx$ if $f(x) = f(2a - x)$

$= 0,$ if $f(x) = -f(2a - x)$

GUJARAT TECHNOLOGICAL UNIVERSITY

B.E. Sem-I/II Examination Summer-2014

Subject Code: 2110015

Subject Name: Vector Calculus and Linear Algebra

Total Marks: 70

Instructions:

1. **Question No. 1 is compulsory. Attempt any four out of remaining six questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figure to the right indicate full marks.**

Q.1 (a) Objective Question **07**

1. The number of solutions of the system of equations $AX = 0$ where A is a singular matrix is

(a) 0 (b) 1 (c) 2 (d) infinite

Solution:

Here, system of equation $AX = 0$ which is a homogeneous linear system. And it has two types of solutions:

(i) Trivial solution

(ii) Infinitely many solutions

Ans: (d) Infinite

2. Let A be a unitary matrix; then A^{-1} is

(a) A (b) $\overline{A}$ (c) A^T (d) $(\overline{A})^T$

Solution:

If A be a unitary matrix then

$$A(\overline{A})^T = I = (\overline{A})^T A$$

So $$(\overline{A})^T = I\,A^{-1}$$

$$A^{-1} = (\overline{A})^T$$

Ans: (d) $(\overline{A})^T$

3. Let $W = span\ \{\cos^2 x, \sin^2 x, \cos 2x\}$ then the dimension of W is

(a) 0 (b) 1 (c) 2 (d) 3

Solution:

W can be expressed as a linear combination of function

$$W = \cos^2 x + \sin^2 x + \cos 2x$$

$$= \cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x$$

$$W = 2\cos^2 x$$

$$W = 2f$$

$\therefore$ $\dim(W) = 2$

Ans: (c) 2

4. Let P_2 be the vector space of all polynomials with degree less than or equal to two; then the dimension of P_2 is

(a) 1 (b) 2 (c) 3 (d) 4

Solution:

Here, P_2 is the polynomial with degree less than or equal to two. Therefore,

$$P_2 = a_0 + a_1x + a_2x^2$$

$$\dim(P_2) = 3$$

Ans: (c) 3

5. The column vectors of an orthogonal matrix are

(a) orthogonal (b) orthonormal

(c) dependent (d) none of these

Solution:

The column vectors of an orthogonal matrix are *orthonosmal*.

6. Let $T : R^2 \to R^2$ be a linear transformation defined by $T(x, y) = (y, x)$; then it is

(a) one to one (b) onto

(c) both (d) neither

Solution:

A linear transformation is one to one if and only if $\text{Ker}(T) = \{0\}$

let $T(x, y) = 0$

$$(y, x) = (0, 0)$$

$$y = 0 \text{ and } x = 0$$

$$\left[\frac{x}{y}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Ker}(T) = \{0\}$$

Hence, T is one to one.

A linear transformation is onto if $R(T) = W$

Let $V = (x, y)$ and $W = (a, b)$ be in R^2 where a and b are real numbers such that

$$T(v) = W \quad \Rightarrow \quad T(x, y) = (a, b)$$

$$\Rightarrow \qquad (y, x) = (a, b)$$

$$\Rightarrow \qquad y = a, x = b$$

Thus, for every $W = (a, b)$ in R^2, $\exists V = (b, a)$ in R^2.

Hence, T is onto.

Ans: (c) Both

7. Let $T : R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (y, z, 0)$; then the dimension of $R(T)$ is

(a) 0 (b) 1 (c) 2 (d) 3

Solution:

The image of T is the entire yz-plane. i.e. point of the form $(y, z, 0)$

$R(T) = Im(T) = \{(a, b, c) \mid a = 0\} = yz$-plane

$\dim [R(T)] = 2$

Q.1 (b) **07**

1. If $||u + v||^2 = ||u||^2 + ||v||^2$; then u and v are
   (a) parallel (b) perpendicular
   (c) dependent (d) none of these

Solution:

By Pythagorean theorem, if u and v are orthogonal (Perpendicular) vectors in Inner product space then $<u, v> = <v, u> = 0$

$$\begin{aligned} ||u \mid v||^2 &= <u + v, u + v> \\ &= <u, u + v> + <v, u + v> \\ &= <u, v> + <u, v> + <v, u> + <v, v> \\ &= ||u||^2 + ||v||^2 \qquad [\because <u, v> = <v, u> = 0] \end{aligned}$$

Ans: (b) Perpendicular

2. $||u + v||^2 - ||u - v||^2$ is
   (a) $<u, v>$ (b) $2<u, v>$ (c) $3<u, v>$ (d) $4<u, v>$

Solution:

$$\begin{aligned} ||u + v||^2 &= <u + v, u + v> \\ &= <u, u + v> + <v, u + v> \end{aligned}$$

and $$||u - v||^2 = <u - v, u - v> = <u, u> + <u, v> + <v, u> + <v, u> \quad (1)$$

$$\begin{aligned} ||u - v||^2 &= <u, u - v> - <v, u - v> \\ &= <u, u> - <u, v> - <v, u> + <v, v> \quad (2) \end{aligned}$$

eq. (1) – eq. (2)

$$\begin{aligned} &||u + v||^2 - ||u - v||^2 \\ &= <u, u> + <u, v> + <v, u> + <v, v> - <u, u> + <u, v> \\ &\quad + <v, u> - <v, u> \\ &= 2<u, v> + 2<v, u> \\ &= 2<u, v> + 2<u, v> \\ &= 4<u, v> \end{aligned}$$

Ans: (d) $4<u, v>$

3. Let $T : R^3 \rightarrow R^3$ be a one-to-one linear transformation; then the dimension of Ker(T) is
   (a) 0 (b) 1 (c) 2 (d) 3

Solution:

A linear transformation is one to one if and only if Ker $(T) = 0$

$\dim \{\text{Ker}(T)\} = 0$ **Ans:** (a) 0

4. Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$; then the eigen values of A^2 are

(a) 1, 2 (b) 1, 4 (c) 1, 6 (d) 1, 16

Solution:

The characteristic equation is $|A - dI| = 0$

$$\begin{bmatrix} 2-d & 1 \\ 2 & 3-d \end{bmatrix} = 0$$

$$(2 - d) - (3 - d) - 2 = 0$$
$$6 - 2d - 3d + d^2 - 2 = 0$$
$$d^2 - 5d + 4 = 0$$
$$d^2 - 4d - d + 4 = 0$$
$$d(d - 4) - 1(d - 4) = 0$$
$$(d - 4)(d - 1) = 0$$
$$d = 1, 4$$

The eigenvalues of A are $d = 1, 4$. Therefore, the eigenvalues of A^2 are 1, 16.

Ans: (d) 1, 16

5. Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$; then the eigenvalues of $A + 3I$ are

(a) 1, 2 (b) 2, 5 (c) 3, 6 (d) 4, 7

Solution:

The characteristic equation is $|A - dI| = 0$

$$\begin{bmatrix} 2-d & 1 \\ 2 & 3-d \end{bmatrix} = 0$$

$$\Rightarrow \quad (2 - d)(3 - d) - 2 = 0$$
$$\Rightarrow \quad d^2 - 5d + 4 = 0$$
$$\Rightarrow \quad d^2 - 4d - d + 4 = 0$$
$$\Rightarrow \quad d(d - 4) - 1(d - 4) = 0$$
$$\Rightarrow \quad (d - 1)(d - 4) = 0$$
$$\Rightarrow \quad d = 1, 4$$

The eigenvalues of A is $d = 1, 4$. Therefore, the eigenvalue

6. div $\overline{r}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Solution:

Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \Rightarrow \quad \text{div } \vec{r} = \nabla \cdot \vec{r}$

$$\text{div } \vec{r} = \frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}$$
$$= 1 + 1 + 1 = 3$$

7. If the value of line integral $\oint_C \bar{F} \cdot d\bar{r}$ does not depend on the path C then $\bar{F}$ is

(a) solenoidal (b) incompressible
(c) irrotational (d) none of these

Solution:

If $\vec{F}$ is the velocity of a fluid particle and c is a closed curve then the line integral $\oint_C \vec{F} \cdot d\vec{r}$ in the region R is zero. Then $\vec{F}$ is irrotational, i.e., if $\oint_C \vec{F} \cdot d\vec{r} = 0$, $\vec{F}$ is irrotational.

Q.2

(a) Solve the following system of equations using the Gauss elimination method: **05**

$2x_1 + x_2 + 2x_3 + x_4 = 6$, $6x_1 - x_2 + 6x_3 + 12x_4 = 36$
$4x_1 + 3x_2 + 3x_3 - 3x_4 = 1$, $2x_1 + 2x_2 - x_3 + x_4 = 10$

Solution:

The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -1 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 6 & -1 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & 1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

Reducing the augmented matrix to row echelon form

$$\sim \left[\begin{array}{cccc:c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 6 & -1 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & 1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right] \frac{R_1}{2}$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 3 \\ 0 & -4 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -11 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right] \begin{array}{l} R_2 - 6R_1 \\ R_3 - 4R_1 \\ R_4 - 2R_1 \end{array}$$

$R_2/4$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & \frac{9}{4} & \frac{18}{4} \\ 0 & 1 & -1 & -5 & -11 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_3 + R_2 \qquad R_4 + R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & \frac{9}{4} & \frac{18}{4} \\ 0 & 0 & -1 & -\frac{11}{4} & -\frac{26}{4} \\ 0 & 0 & -3 & \frac{9}{4} & \frac{34}{4} \end{array}\right]$$

$R_3 \to -R_3$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & \frac{9}{4} & \frac{18}{4} \\ 0 & 0 & 1 & \frac{11}{4} & \frac{26}{4} \\ 0 & 0 & -3 & \frac{9}{4} & \frac{34}{4} \end{array}\right]$$

$R_4 + 3R_3$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & \frac{9}{4} & \frac{18}{4} \\ 0 & 0 & 1 & \frac{11}{4} & \frac{26}{4} \\ 0 & 0 & 0 & \frac{42}{4} & \frac{112}{4} \end{array}\right]$$

Now, the corresponding system equation is

$$2x_1 + x_2 + 2x_3 + x_4 = 6 \qquad \text{then } \boxed{x_4 = \frac{112}{42} = \frac{8}{3}}$$

$$x_2 + \frac{9}{4}x_4 = \frac{18}{4} \qquad x_3 = \frac{26}{4} - \frac{11}{\cancel{4}}\,\frac{^2\cancel{8}}{3} = -\frac{10}{12}$$

$$x_3 + \frac{11}{4}xy = \frac{26}{4} \qquad \boxed{x_3 = -\frac{5}{6}}$$

$$\frac{42}{4}x_4 = \frac{112}{4} \qquad x_2 = \frac{9}{4}x_4 - \frac{18}{4} = \frac{9}{\cancel{4}}\,\frac{^2\cancel{8}}{3} - \frac{18}{4}$$

$$= \frac{18}{3} - \frac{18}{4} = \frac{18}{12} = \frac{3}{2}$$

$$2x_1 = 6 - x_2 - x_4 - 2x_3 \Rightarrow x_1 = 3 - \frac{x_2}{2} - \frac{x_4}{2} - x_3$$

$$x_1 = 3 - \frac{3}{4} - \frac{4}{3} + \frac{5}{6}$$

$$= \frac{36 - 9 - 16 + 10}{12} = \frac{46 - 25}{12} = \frac{21}{12} = \frac{7}{4} \qquad \boxed{x_1 = \frac{7}{4}}$$

(b) Find the inverse of $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ using the Gauss–Jordan method. **05**

Solution:

Here, $$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = I_4 A$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Now, reducing the matrix A to the reduced row echelon form

$$R_2 - R_1,\ R_3 - 2R_1,\ R_4 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix} A$$

$\frac{R_3}{-3}$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix} A$$

$R_4 + 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ -\frac{2}{3} & 1 & -\frac{2}{3} & 1 \end{bmatrix} A$$

$3R_4$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ -2 & 3 & -2 & 3 \end{bmatrix} A$$

$R_3 + \frac{1}{3}$ $\quad R_2 - R_4 \quad R_1 - R_4$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 2 & -3 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix} A$$

$R_1 - 3R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 5 & -6 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix} A$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix} A$$

$I_4 = A^{-1} A$

$$\therefore \qquad A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$$

(c) Express $\begin{bmatrix} 4+2i & 7 & 3-i \\ 0 & 3i & -2 \\ 5+3i & -7+i & 9+6i \end{bmatrix}$ as the sum of a hermitian and a skew-hermitian matrix. **04**

Solution:

$$A = \begin{bmatrix} 4+2i & 7 & 3-i \\ 0 & 3i & -2 \\ 5+3i & -7+i & 9+6i \end{bmatrix}$$

$$A^{\Theta} = (\bar{A})^T = \begin{bmatrix} 4-2i & 0 & 5-3i \\ 7 & -3i & -7-i \\ 3+i & -2 & 9-6i \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^{\Theta})$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4+2i & 7 & 3-i \\ 0 & 3i & -2 \\ 5+3i & -7+i & 9+6i \end{bmatrix} + \begin{bmatrix} 4-2i & 0 & 5-3i \\ 7 & -3i & -7-i \\ 3+i & -2 & 9-6i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 7 & 8-4i \\ 7 & 0 & -9-i \\ 8+4i & -9+i & 18 \end{bmatrix}$$

And $Q = \frac{1}{2}\,(A - A^{\theta})$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4+2i & 7 & 3-i \\ 0 & 3i & -2 \\ 5+3i & -7+i & 9+6i \end{bmatrix} - \begin{bmatrix} 4-2i & 0 & 5-3i \\ 7 & -3i & -7-i \\ 3+i & -2 & 9-6i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 4i & 7 & -2+2i \\ -7 & 6i & 5+i \\ 2+2i & -5+i & 12i \end{bmatrix}$$

We know that P is a hermitian and Q is a skew-hermition matrix.

$$A = P + Q = \begin{bmatrix} 4 & \frac{7}{2} & 4-2i \\ \frac{7}{2} & 0 & -\frac{9}{2}-\frac{i}{2} \\ 4+2i & -\frac{9}{2}+\frac{i}{2} & 9 \end{bmatrix} + \begin{bmatrix} 2i & \frac{7}{2} & -1+i \\ -\frac{7}{2} & 3i & \frac{5}{2}+\frac{i}{2} \\ 1+i & -\frac{5}{2}+\frac{i}{2} & 6i \end{bmatrix}$$

Q.3

(a) Let V be the set of all ordered pairs of real numbers with vector addition defined as $(x_2, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$. Show that the first five axioms for vector addition are satisfied. Clearly mention the zero vector and additive inverse. **05**

Solution:

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ and $w = (x_3, y_3)$ are objects in V.

(1) $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1)$

Since x_1, x_2, y_1, y_2 are real numbers, $x_1 + x_2 + 1$ and $y_1 + y_2 + 1$ are also real numbers.

Therefore, $u + v \in V$.

(2) $u + v = (x_1 + x_2 + 1, y_1 + y_2 + 1)$

$= x_2 + x_1 + 1, y_2 + y_1 + 1)$

$= v + w$

Hence, vector addition is commutative.

(3) $u + (v + w) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$= (x_1, y_1) + (x_2 + x_3 + 1, y_2 + y_3 + 1)$

$= [x_1 + (x_2 + x_3 + 1) + 1, y_1 + (y_2 + y_3 + 1) + 1]$

$= [(x_1 + x_2 + 1) + x_3 + 1, (y_1 + y_2 + 1) + y_3 + 1]$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1) + (x_3, y_3)$

$= (u + v) + w$

Hence, vector addition is associative.

(4) Let $(a, b) \in V$, such that

$$(a, b) + u = u$$
$$(a, b) + (x_1, y_1) = (x_1, y_1)$$
$$(a + x_1 + 1, b + y_1 + 1) = (x_1, y_1)$$
$$a + x_1 + 1 = x_1, b + y_1 + 1 = y_1$$
$$a = -1, b = -1$$

Also, $u + (a, b) = u$

Hence, $(-1, -1)$ is the zero vector in V.

(5) Let $(a, b) \in V$ such that

$$u + (a, b) = (-1, -1)$$
$$(x_1, y_1) + (a, b) = (-1, -1)$$
$$(x_1 + a + 1, y_1 + b + 1) = (-1, -1)$$
$$x_1 + a + 1 = -1 \qquad y_1 + b + 1 = -1$$
$$a = -x_1 - 2 \qquad b = -y_1 - 2$$

Also, $(a, b) + u = (-1, -1)$

Hence, $(-x_1, -2, -y_1, -2)$ is the inverse in V.

(b) Find a basis for the subspace of P_2 spanned by the vector $1 + x$, x^2, $-2 + 2x^2$, $-3x$ **05**

Solution:

Let $v_1 = 1 + x$, $v_2 = x^2$, $v_3 = -2 + 2x^2$, $v_4 = -3x$

Here, $\{v_1, v_2, v_3, v_4\}$ spans subspace of P_2 but it is not a basis for the subspace of P_2. Since dim (subspace of P_2) = 3 and the basis of the subspace of P_2 contains exactly three vectors. We now need to remove one vector from $\{v_1, v_2, v_3, v_4\}$ to get a basis.

We can remove that vector only which is a linear combination of some of the other vectors of the set $\{v_1, v_2, v_3, v_4\}$. Let

$$c_1(1 + x) + c_2(x^2) + c_3(-2 + 2x^2) + c_4(-3x) = 0 + 0x + 0x^2$$

Then

$$(c_1 - 2c_3) + (c_1 - 3c_4)x + (c_2 + 2c_3)x^2 = 0 + 0x + 0x^2$$

which implies

$$c_1 - 2c_3 = 0 \qquad c_1 - 3c_4 = 0 \qquad c_2 + 2c_3 = 0$$

Now, the matrix form of the system is

$$AX = 0$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$$

$R_2 - R_1$

$$\approx \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\approx \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \end{array}\right]$$

$R_3/2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

The corresponding system of equation is

$$c_1 = 2c_3 = 0$$

$$c_2 + 2c_3 = 0$$

$$c_3 = \frac{3}{2}c_4 = 0$$

Now, take $c_4 = t$

$$c_3 = \frac{3}{2}t$$

Then $\qquad c_2 = -3t \qquad c_1 = 3t$

Here, c_1, c_2, c_3, c_4, not all zero, the given vectors are linearly are dependent and the relation between them is given by

$$3tv_1 - 3tv_2 + \frac{3}{2}tv_3 + tv_4 = 0$$

Thus, we can remove any one of the vectors v_1, v_2, v_3, v_4. Let us remove v_4. Then the set $\{v_1, v_2, v_3\}$ still spans the subspace of P_2 and has exactly three vectors. So it must be a basis for P_2.

Basis = $\{1 + x, x^2, -2 + 2x^2\}$

(c) Express the matrix $\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ **04**

Solution:

Here, the matrix

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$$

Let $$A = A_1k_1 + A_2k_2 + A_3k_3$$

$$\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} = k_1\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + k_3\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 + 2k_3 & -k_1 + k_2 + 2k_3 \\ -k_3 & 3k_1 + 2k_2 + k_3 \end{bmatrix}$$

Now, equating the corresponding components,

$$\begin{aligned} k_1 + k_2 + 2k_3 &= 5 \\ -k_1 + k_2 + 2k_3 &= 1 \\ -k_3 &= -1 \\ 3k_1 + 2k_2 + k_3 &= 9 \end{aligned} \qquad (1)$$

Now, solving these equations,

$$\begin{aligned} k_3 &= 1 \\ k_1 + k_2 &= 5 - 2 = 3 \\ -k_1 + k_2 &= 1 - 2 = -1 \\ 2k_2 &= 2 \\ k_2 &= 1 \end{aligned}$$

Then $$k_1 + k_2 = 3$$

$$k_1 = 3 - 1 = 2$$

Now, $$k_1 = 2, k_2 = 1, k_3 = 1$$

Hence the linear combination of A is $\boxed{A = 2A_1 + A_2 + A_3}$

Q4.

(a) Consider the basis $S = \{v_1, v_2\}$ for R^2 where $v_1 = (1, 1)$ and $v_2 = (2, 3)$. Let $T : R^2 \to P^2$ be the linear transformation such that $T(v_1) = 2 - 3x + x^2$ and $T(v_2) = 1 - x^2$; then find the formula of $T(a, b)$ **05**

Solution:

Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$ be an arbitrary vector in R^2 expressed as a linear combination of v_1 and v_2

$$V = k_1v_1 + k_2v_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} k_1 + 2k_2 \\ k_1 + 3k_2 \end{bmatrix}$$

New, equating corresponding components,

$$\begin{aligned} k_1 + 2k_2 &= a \\ k_1 + 3k_2 &= b \end{aligned} \qquad (1)$$

Solving these equations,

$$-k_2 = a - b$$

$$\boxed{k_2 = b - a}$$

And $\qquad k_1 + 2(b - a) = a$

$\Rightarrow \qquad k_1 = a - 2b + 2a = 3a - 2b$

$$\boxed{k_1 = 3a - 2b}$$

$\therefore \qquad V = (3a - 2b)V_1 + (b - a)V_2$

$$T(V) = k_1\, T(v_1) + k_2\, T(v_2)$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = (3a - 2b)\,(2 - 3x + x^2) + (b - a)(1 - x^2)$$

$$= (6a - 4b + b - a) + (-9a + 6b)x + (3a - 2b - b + a)x^2$$

$$\boxed{T\begin{bmatrix} a \\ b \end{bmatrix} = (5a - 3b) + (-9a + 6b)x + (4a - 3b)x^2}$$

(b) Verify Rank-Nullity theorem for the linear transformation $T : R^4 \rightarrow R^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4)$$

05

Solution:

The basis for ker (T) is the basis for the solution of the homogeneous system

$$\begin{aligned} 4x_1 + x_2 - 2x_3 - 3x_4 &= 0 \\ 2x_1 + x_2 + x_3 - 4x_4 &= 0 \\ 6x_1 - 9x_3 + 9x_4 &= 0 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right]$$

Reducing in row echelon form,

$$\frac{R_1}{4}, \frac{R_3}{3}$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{2}{4} & -\frac{3}{4} & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 2 & 0 & -3 & 3 & 0 \end{array}\right]$$

$R_2 - 2R_1 \quad R_3 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 2 & -\frac{5}{2} & 0 \\ 0 & -\frac{1}{2} & -2 & \frac{9}{2} & 0 \end{array}\right]$$

$2R_2, \quad 2R_3$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & -1 & -4 & 9 & 0 \end{array}\right]$$

$R_3 + R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

$\frac{1}{4}R_4$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the corresponding system of equations is

$$x_1 + \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{3}{4}x_4 = 0$$

$$x_2 + 4x_3 - 5x_4 = 0$$

$$x_4 = 0$$

Now, take $x_3 = t$

$$x_2 = -4t$$

and $$x_1 = \frac{-1}{4}(-4t) + \frac{1}{2}t = t + \frac{t}{2} = \frac{3}{2}t$$

so $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}t \\ -4t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

Hence the basis per ker(T) = Null space

$$= \begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ dim [ker(T)] = 1

The basis for the range of T is the basis for the column space of [T].

$$[T] = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}$$

Reducing [T] to row echelon form as above,

$$\approx \begin{bmatrix} l & \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the leading entry appears in column 1, 2 and 4.

Hence, the basis of $R(T)$ = Basis for column space of [T].

$$= \left\{ \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix} \right\}$$

$$\text{dim } [R(T)] = 3$$

$$\text{Rank}(T) = \dim(R(T)) = 3$$

$$\text{nullity } (T) = \dim (\text{Ker}(T)) = 1$$

so $$\text{Rank}(T) + \text{Nullity } (T) = 3 + 1$$

$$= 4$$

$$= \dim R_4$$

Hence, the dimension theorem is verified.

(c) Find the algebraic and geometric multiplicity of each of the eigenvalues of
$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **04**

Solution:

Here, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristic equation is

$$|A - dI| = 0$$

$$\begin{vmatrix} -d & 1 & 1 \\ 1 & -d & 1 \\ 1 & 1 & -d \end{vmatrix} = 0$$

$\Rightarrow \; d^3 - s_1 d^2 + s_2 d = s_3 = 0$

s_1 = sum of the principal diagonal element

$= \phi + \phi + \phi = 0$

s_2 = sum of the minors of principal diagonal element

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 - 1 - 1$$

$$= -3$$

$$s_3 = \det(A) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= 0 - 1(-1) + 1(1 - 0)$

$= 1 + 1 = 2$

Hence, the characteristic equation is

$d^3 - 3d = 2 = 0$

$d^2(d + 1) - d(d + 1) - 2\,(d + 1) = 0$

$(d + 1)(d^2 - 2d + d - 2) = 0$

$(d + 1)(d - 2)(d + 1) = 0$

$d = 2, -1, -1$

eigen values of the matrix are 2, –1, –1.

Since the eigenvalue $d = -1$ is repeated twice. So its Algebraic multiplicity is 2.

For $d = -1$, the corresponding eigenvectors are

$$[A - dI]x = 0$$

$$\begin{bmatrix} +1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1 - R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Rank of matrix = 1

And the corresponding equation is

$$x_1 + x_2 + x_3 = 0$$
$$x_3 = t$$
$$x_2 = 3$$
$$x_1 = -t - S$$

So the eigenvector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t - S \\ S \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + S\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Number of unknowns = 3

Number of linearly independent eigenvectors

$\therefore$ Geometric multiplicity is 2.

Since eigenvalue $d = 2$ is non-repeated, so its Algebraic multiplicity is 1.

For $d = 2$, the corresponding eigenvectors are

$$[A - dI]x = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1 \quad R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{R_2}{-3} \quad \frac{R_3}{3}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the corresponding equation is

$$x_1 + x_2 - 2x_3 = 0$$

$$x_2 + x_3 = 0$$

Let $x_3 = t \qquad x_2 = t$

$$x_1 = 2x_3 - x_2$$

$$= 2t - t = t$$

So the eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$

$$= t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, the rank of matrix = 2

Number of unknowns = 3

Number of linearly independent eigenvectors = 3 – 2 = 1

Hence, geometric multiplicity is 1.

Q.5

(a) For $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, let the inner product on M_{22} be defined as $\langle A, B\rangle = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$. Let $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$; then verify Cauchy–Schwarz inequality and find the angle between A and B. **05**

Solution:

Cauchy–Schwarz Inequality:

If A and B are vectors in an inner product space M_{22} then

$$|<A, B>| \leq ||A||\ ||B||$$

Now,
$$||B|| = <B, B>^{1/2}$$
$$= [(3)^2 + (2)^2 + (1)^2 + (0)^2]^{1/2}$$
$$= \sqrt{9+4+1} = \sqrt{14}$$
$$||A|| = <A, A>^{1/2}$$
$$= [(2)^2 + (6)^2 + (1)^2 + (-3)^2]^{1/2}$$
$$= \sqrt{4+36+1+9}$$
$$= \sqrt{50} = 5\sqrt{2}$$
$$<A, B> = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$$
$$= 2.3 + 6.2 + 1.1 + (-3)\cdot 0$$
$$= 6 + 12 + 1 + 0$$
$$= 19$$

Therefore, $|19| \leq 5\sqrt{2}\cdot\sqrt{14}$

$$|19| \leq 26.45$$

so the Cauchy–Schwarz inequality is verified.

The angle between A and B is

$$\cos\theta = \frac{<A, B>}{||A||\,||B||}$$

$$\therefore \qquad \theta = \cos^{-1}\left[\frac{<A, B>}{||A||\,||B||}\right]$$
$$= \cos^{-1}\left[\frac{19}{\sqrt{50}\sqrt{14}}\right]$$
$$= \cos^{-1}\left[\frac{19}{\sqrt{700}}\right]$$
$$\boxed{\theta = \cos^{-1}\left[\frac{19}{10\sqrt{7}}\right]}$$

(b) Let R^3 have the inner product defined by $< (x_1, x_2, x_3), (y_1, y_2, y_3) > = x_1y_1 + 2x_2y_2 + 3x_3y_3$. Apply the Gram–Schmidt process to transform the vectors (1, 1, 1), (1, 1, 0) and (1, 0, 0) into orthonormal vectors. **05**

Solution:

Gram-Schmidt Process:

Step-1 Let $v_1 = u_1 = (1, 1, 1)$

Step-2

$$v_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} \cdot v_1$$

$$= (1, 1, 0) - \frac{<(1,1,0),(1,1,1)>}{(1^2 + 2\cdot 1^2 + 3\cdot 1^2)} \cdot (1, 1, 1)$$

$$= (1, 1, 0) - \frac{[(1)(1) + 2(1)(1) + 3(1)(0)]}{1+2+3} (1, 1, 1)$$

$$= (1, 1, 0) - \frac{3}{6}(1, 1, 1) = (1, 1, 0) - \frac{-1}{2}(1, 1, 1)$$

$$v_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

Step-3

$$v_3 = u_3 - \frac{<u_3, v_1>}{\|v_1\|^2} v_1 - \frac{<u_3, v_2>}{\|v_2\|^2} v_2$$

$$= (1, 0, 0) - \frac{<(1,0,0),(1,1,1)>}{(1^2 + 2\cdot 1^2 + 3\cdot 1^2)} \cdot (1, 1, 1)$$

$$- \frac{\left\langle (1,0,0), \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)\right\rangle}{\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right)} \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

$$= (1, 0, 0) - \frac{(1)}{6}(1, 1, 1) - \frac{\frac{1}{2}}{\frac{6}{4}}\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

$$= (1, 0, 0) - \frac{1}{6}(1, 1, 1) - \frac{1}{3}\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

$$= \left(1 \frac{-1}{6} - \frac{1}{6}, \frac{-1}{6} - \frac{1}{6}, \frac{-1}{6} + \frac{1}{6}\right)$$

$$= \left(\frac{4}{6}, \frac{-2}{6}, 0\right) = \left(\frac{2}{3}, \frac{-1}{3}, 0\right)$$

Thus, the vectors v_1, v_2, v_3 form an orthogonal basic for R^3.

$$\text{Orthogonal basis} = \left\{(1,1,1), \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right), \left(\frac{2}{3}, \frac{-1}{3}, 0\right)\right\}$$

Now, normalizing v_1, v_2, v_3

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1+2\cdot 1+3\cdot 1}} = \frac{(1,1,1)}{\sqrt{6}}$$

$$= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$W_2 = \frac{v_2}{\|v_2\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{2}{4}+\frac{3}{4}}}$$

$$= \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)}{\sqrt{\frac{6}{4}}} = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$W_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$W_3 = \frac{v_3}{\|v_3\|} = \frac{\left(\frac{2}{3}, -\frac{1}{3}, 0\right)}{\sqrt{\frac{4}{9}+\frac{2}{9}+3\cdot 0}}$$

$$= \frac{\left(\frac{2}{3}, -\frac{1}{3}, 0\right)}{\sqrt{\frac{6}{9}}} = \frac{\left(\frac{2}{3}, -\frac{1}{3}, 0\right)}{\sqrt{\frac{6}{3}}}$$

$$W_3 = \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, 0\right)$$

Thus, the vectors w_1, w_2, w_3 form an orthonormal basis for R^3.

$$\text{Orthonormal basis} = \left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, 0\right)\right\}$$

(c) Find a basis for the orthogonal complement of the subspace spanned by the vectors (2, –1, 1, 3, 0), (1, 2, 0, 1, –2), (4, 3, 1, 5, –4), (3, 1, 2, –1, 1) and (2, –1, 2, –2, 3). **04**

Solution:

Let the W subspace spanned by these vectors be the row space of the matrix.

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 1 & -2 \\ 4 & 3 & 1 & 5 & -4 \\ 3 & 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & -2 & 3 \end{bmatrix}$$

Since (Row space)^T = Null space

Basis for (Row space)^T = Basis for the null space

$\therefore$ the null space of A is the solution space of the homogeneous system $AX = 0$

$$\begin{bmatrix} 2 & -1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 1 & -2 \\ 4 & 3 & 1 & 5 & -4 \\ 3 & 1 & 2 & -1 & 1 \\ 2 & -1 & 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 1 & -2 & 0 \\ 4 & 3 & 1 & 5 & -4 & 0 \\ 3 & 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 2 & -2 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix into row echelon form,

$R_1 \leftrightarrow R_2$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 4 & 3 & 1 & 5 & -4 & 0 \\ 3 & 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 2 & -2 & 3 & 0 \end{array}\right]$$

$R_2 - 2R_1 \quad R_3 - 4R_1 \quad R_4 - 3R_1 \quad R_5 - 2R_1$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & -5 & 2 & -4 & 7 & 0 \\ 0 & -5 & 2 & -4 & 7 & 0 \end{array}\right]$$

$R_3 - R_2 \quad R_4 - R_2 \quad R_5 - R_2$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \end{array}\right]$$

$R_5 - R_4$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & -5 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\dfrac{R_2}{-5}$

$$\approx \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the corresponding system of equation is

$$x_1 + 2x_2 + 0x_3 + x_4 - 2x_5 = 0$$

$$x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4 - \frac{4}{5}x_5 = 0$$

$$x_3 - 5x_4 + 3x_5 = 0$$

Now, let $x_4 = S \quad x_5 = t$

$$x_3 = 5S - 3t$$

$$x_2 = \frac{1}{5}(5S - 3t) + \frac{1}{5}(S) + \frac{4}{5}(t)$$

$$= S - \frac{3}{5}t + \frac{S}{5} + \frac{4t}{5}$$

$$= \frac{6}{5}S + \frac{t}{5}$$

$$x_1 = 2x_5 - x_4 - 2x_2$$

$$= 2t - S - 2\left(\frac{6}{5}S + \frac{t}{5}\right)$$

$$= 2t - S - \frac{12}{5}S - \frac{2t}{5}$$

$$= \frac{-17}{5}S + \frac{8}{5}t$$

Null space vectors of the form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{17}{5}S + \frac{8}{5}t \\ \frac{6}{5}S + \frac{1}{5}t \\ 5S - 3t \\ S \\ t \end{bmatrix}$$

$$= S\begin{bmatrix} -\frac{17}{5} \\ \frac{6}{5} \\ 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the null space of A

$$\left\{\begin{bmatrix} -\frac{17}{5} \\ \frac{6}{5} \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

which is also the basis for the subspace w.

Q.6

(a) Verify Cayley–Hamilton theorem for $A = \begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix}$ and, hence, find A^4. **05**

Solution:

Here, $A = \begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix}$

The characteristic equation is

$$|A - dI| = 0$$

$$\begin{vmatrix} 6-d & -1 & 1 \\ -2 & 5-d & -1 \\ 2 & 1 & 7-d \end{vmatrix} = 0$$

$$d^3 = S_1 d^2 + S_2 d - S_3 = 0$$

where $S_1 = 6 + 5 + 7 = 18$

$$S_2 = 36 + 40 + 28 = -104$$

$$S_3 = \det(A) = \begin{vmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{vmatrix}$$

$$= 6(36) + 1(-12) + 1(-12)$$

$$= 216 - 24 = 192$$

Hence, the characteristic equation is

$$d^3 - 18d^2 + 104d - 192d = 0$$

Now, the Cayley–Hamilton theorem is every square matrix satisfies its own characteristic equation. Therefore,

$$A^3 - 18A^2 + 104A - 192I = 0 \tag{1}$$

Now, $A^2 = \begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 40 & -10 & 14 \\ -24 & 26 & -14 \\ 24 & 10 & 50 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 40 & -10 & 14 \\ -24 & 26 & -14 \\ -24 & 10 & 50 \end{bmatrix}\begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 288 & -76 & 148 \\ -224 & 140 & -148 \\ 224 & 76 & 364 \end{bmatrix}$$

$$A^3 - 18A^2 + 104A - 192I$$

$$= \begin{bmatrix} 288 & -76 & 148 \\ -224 & 140 & -148 \\ 224 & 76 & 364 \end{bmatrix} - \begin{bmatrix} 720 & -180 & 252 \\ -432 & 468 & -252 \\ 432 & \pm 80 & 900 \end{bmatrix}$$

$$+ \begin{bmatrix} 624 & -104 & 104 \\ -208 & 520 & -104 \\ 208 & 104 & 728 \end{bmatrix} - \begin{bmatrix} 192 & 0 & 0 \\ 0 & 192 & 0 \\ 0 & 0 & 192 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, the Cayley–Hamilton theorem is verified.

Now, multiplying Eq. (1) by A,

$A(A^3 - 18A^2 + 104A - 192I) = 0$

$A^4 - 18A^3 + 104A^2 - 192A = 0$

$A^4 = 18A^3 - 104A^2 + 192A$

$$= 18\begin{bmatrix} 288 & -76 & 148 \\ -224 & 140 & -148 \\ 224 & 76 & 364 \end{bmatrix} - 104\begin{bmatrix} 40 & -10 & 14 \\ -24 & 26 & -14 \\ 24 & 10 & 50 \end{bmatrix} + 192\begin{bmatrix} 6 & -1 & 1 \\ -2 & 5 & -1 \\ 2 & 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5184 & -1368 & 2664 \\ -4032 & 2520 & -2664 \\ 4032 & 1368 & 6552 \end{bmatrix} - \begin{bmatrix} 4160 & -1040 & 1456 \\ -2496 & 2704 & -1456 \\ 2496 & 1040 & 5200 \end{bmatrix}$$

$$+ \begin{bmatrix} 1152 & -192 & 192 \\ -384 & 960 & -192 \\ 384 & 192 & 1344 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2176 & -520 & 1400 \\ -1920 & 776 & -1400 \\ 1920 & 520 & 2696 \end{bmatrix}$$

(b) Show that the vector field $\overline{F} = (y \sin z - \sin x)i + (x \sin z + 2yz)j + (xy \cos z + y^2)\, k$ is conservative and find the corresponding scalar potential. **05**

Solution:

Since $\vec{F}$ is conservative then

curl $\vec{F} = 0$

$$\nabla \times F = \begin{vmatrix} \dfrac{\hat{i}}{\dfrac{\partial}{\partial x}} & \dfrac{\hat{j}}{\dfrac{\partial}{\partial y}} & \dfrac{\hat{k}}{\dfrac{\partial}{\partial z}} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy \cos z + y^2) - \frac{\partial}{\partial z}(x \sin z + 2yz)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(xy \cos z + y^2) - \frac{\partial}{\partial z}(y \sin z - \sin x)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x \sin z + 2yz) - \frac{\partial}{\partial y}(y \sin z - \sin x)\right]$$

$$= \hat{i}\,[x \cos z + 2y - x \cos z - 2y]$$

$$-\hat{j}\,[y \cos z - y \cos z] + \hat{k}\,[\sin z - \sin z]$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = 0$$

So $\vec{F}$ is a conservative field.

Now, since $\vec{F}$ is conservative,

$$\vec{F} = \nabla\phi$$

$$(y \sin z - \sin x)\hat{i} + (x \sin z + 2yz)\hat{j} + (xy \cos z + y^2)\hat{k}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing the coefficient of $\hat{i}, \hat{j}, \hat{k}$ on both sides,

$$\frac{\partial\phi}{\partial x} = y \sin z - \sin x \quad \frac{\partial\phi}{\partial y} = x \sin z + 2yz \quad \frac{\partial\phi}{\partial z} = xy \cos z + y^2$$

But $\quad d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$

$$d\phi = (y \sin z - \sin x)dx + (x \sin z + 2yz)\,dy + (xy \cos z + y^2)\,dz$$

Integrating both the sides, we get

$$\int d\phi = \int_{y,z\to\cos wt}(y \sin z - \sin x)dx + \int_{x,z\to\cos wt}(x \sin z + 2yz)dy$$
$$+ \int_{x,y\to\cos wt}(xy \cos z + y^2)dz$$

$$\therefore \qquad \phi = -(-\cos x) + 2 \cdot \frac{y^2}{2} \cdot z + xy \sin z + c$$

$$\phi = \cos x + y^2 z + cy \sin z + c$$

It is scalar potential.

(c) Find the directional derivative of $x^2y^2z^2$ at $(1, 1, -1)$ along a direction equally inclined with coordinate axes. **04**

Solution:

Here, $\phi = x^2y^2z^2$ point $(1, 1, -1)$

In the direction equally inclined with coordinate axes, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Now, $\nabla\phi = \hat{i}\dfrac{\partial}{\partial x}(x^2y^2z^2) + \hat{j}\dfrac{\partial}{\partial y}(x^2y^2z^2) + \hat{k}\dfrac{\partial}{\partial z}(x^2y^2z^2)$

$$= y^2z^2(2x)\hat{i} + x^2z^2(2y)\hat{j} + x^2y^2(2z)\hat{k}$$

At the point $(1, 1, -1)$,

$$\nabla\phi = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

Therefore, the directional derivative in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$$= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\hat{i} + 2\hat{j} - 2\hat{k})\frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{1+1+1}}$$

$$= \frac{2+2-2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Q.7

(a) Verify Green's theorem for $\oint_C (3x - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the triangle with vertices (0, 0), (1, 0) and (0, 1). **05**

Solution:

The region bounded by the triangle vertices (0, 0), (1, 0) and (0, 1).

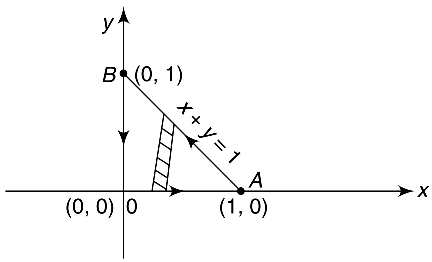


Fig. 1

Here, $M = 3x - 8y^2 \quad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \oint_C (Mdx + Ndy) = \int_{OA}(Mdx + Ndy) + \int_{AB}(Mdx + Ndy) + \int_{OO}(Mdx + Ndy) \quad (1)$$

Along the path OA: $y = 0$

$dy = 0$

$$\int_{OA}(Mdx + Ndy) = \int_0^1 (3x - 8y^2)\, dx$$

$$= \int_0^1 3xdx = 3 \cdot \left|\frac{x^2}{2}\right|_0^1 = \frac{3}{2}$$

Along the path ABC: $x + y = 1$

$$y = 1 - x \quad \Rightarrow \quad dy = (-dx)$$

$$\int_{AB}(Mdx + Ndy) = \int_1^0 (3x - 8y^2)dx + (4y - 6xy)(-dx)$$

$$= \int_1^0 [3x - 8y^2 - 4y + 6xy]dx$$

$$= \int_1^0 [3x - 8(1 - x)^2 - 4(1 - x) + 6x(1 - x)]dx$$

$$= \int_0^1 [4(1-x) - 6x(1-x) + 8(1-2x+x^2) - 3x]dx$$

$$= \int_0^1 (4 - 4x - 6x + 6x^2 + 8 - 16x + 8x^2 - 3x)\, dx$$

$$= \int_0^1 (14x^2 - 29x + 12)dx$$

$$= \left[14\frac{x^3}{3} - 29\frac{x^2}{2} + 12x\right]_0^1 = \frac{14}{3} - \frac{29}{2} + 12$$

$$= \frac{28 - 87 + 72}{6}$$

$$= \frac{100 - 87}{6} = \frac{13}{6}$$

Along the path BO: $x = 0$

$dx = 0$

$$\int_{BO}(Mdx + Ndy) = \int_1^0 (3x - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 4ydy = 4\left|\frac{y^2}{2}\right|_1^0$$

$$= 4\left(\frac{-1}{2}\right) = -2$$

From Eq. (1),

$$\int_c(Mdx + Wdy) = \frac{3}{2} + \frac{13}{6} - 2 = \frac{9 + 13 - 12}{6}$$

$$= \frac{22 - 12}{6} = \frac{10}{6} = \frac{5}{3} \qquad (2)$$

Let R be the region bounded by the triangle. Along the vertical strip.
y varies from: $y \to 0$ to $y \to 1 - x$
x varies from: $x \to 0$ to $x \to 1$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} (-6y + 16y)dxdy$$

$$= \int_0^1 \int_0^{1-x} 10y\, dy\, dx$$

$$= \int_0^1 \left[10\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 5(1-x)^2\, dx$$

$$= 5\left|\frac{(1-x)^3}{3}\right|_0^1$$

$$= \frac{5}{3} \qquad (3)$$

From Eq. (2) and Eq. (3)

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \frac{5}{3}$$

Hence, Green's theorem is verified.

(b) Verify Stokes' theorem for $\overline{F} = (x + y)i + (y + z)j - xk$ and S is the surface of the plane $2x + y + z = 2$ which is in the first octant. **05**

Solution:

By Stokes' theorem,

$$\iint_S (\nabla \times F) \cdot \hat{n} ds = \oint_C \vec{F} \cdot d\vec{r}$$

The given surface is the plane $2x + y + z = 2$ in the first octant. Let $\phi = 2x + y + z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{4+1+1}}$$

$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

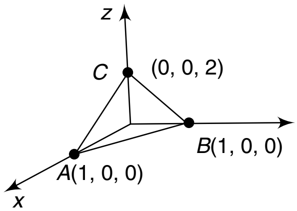


Fig. 1

Let R be the projection of the plane $2x + y + z = 2$ (in the first octant) on the xy-plane which is the triangle OAB bounded by the lines $x = 0$, $y = 0$, $2x + y = 2$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & y + z & -x \end{vmatrix}$$

$$= \hat{i}\,(0 - 1) - \hat{j}\,(-1 - 0) + \hat{k}\,(0 - 1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dxdy}{\frac{1}{\sqrt{6}}} = \sqrt{6}\,dxdy$$

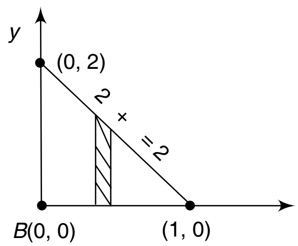


Fig. 2

Let R be the region bounded by the triangle OAB in the xy-plane

Along the vertical strip:

y varies from: $y = 0$, $y = 2 - 2x$

x varies from: $x = 0$, $x = 1$

$$\iint_S (\nabla \times F)\hat{n}\, ds = \iint_R (-1 + \hat{j} - k) \cdot \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}} \cdot \sqrt{6}\,dxdy$$

$$= \int_0^1 \int_0^{2-2x} (-2 + 1 - 1) dxdy$$

$$= -2\int_0^1 |y|_0^{2-2x}$$

$$= -2\int_0^1 (2-2x)\,dx = -2\left|\left(2x - \frac{2x^2}{2}\right)\right|_0^1 = -2 \qquad (1)$$

Let C be the boundary of the triangle ABC.

$$\vec{F} \cdot dr \quad \cdot dr = (x+y)dx + (y+z)dy - xdz$$

$$\oint \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CA} \vec{F} \cdot d\vec{r} \qquad (2)$$

Along the path AB: $z = 0 \quad y = 2 - 2x$

$dz = 0 \quad dy = -2dx$

x varies from 1 to 0

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_1^0 [(x+y)dx + (y+z)dy - xdz]$$

$$= \int_1^0 (x + 2 - 2x)dx + (2-2x)(-2\,dx)] = \int_1^0 (3x-2)dx$$

$$= \left|3 \cdot \frac{x^2}{2} - 2x\right|_1^0 = -\frac{3}{2} + 2 = \frac{1}{2}$$

Along the path BC: $x = 0 \quad y + z = 2$

$dx = 0 \quad dz = -dy$

y varies from 2 to 0

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 [(x+y)dx + (y+2)dy - xdz]$$

$$= \int_2^0 2dy = 2|y|_2^0 = -4$$

Along the path CA: $y = 0 \quad 2x + z = 2$

$dy = 0 \quad dz = -2dx$

z varies from 0 to 1

$$\int_{CA} \vec{F} \cdot d\vec{r} = \int_0^1 [(x+y)dx + (y+z)dy - xdz] = \int_0^1 [xdx - x(-2dx)]$$

$$= \int_0^1 3xdx = 3\left|\frac{x^2}{2}\right|_0^1 = \frac{3}{2}$$

From Eq. (2),

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{2} - 4 + \frac{3}{2} = -2 \qquad (3)$$

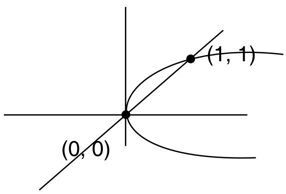
From Eq. (1) and (3),

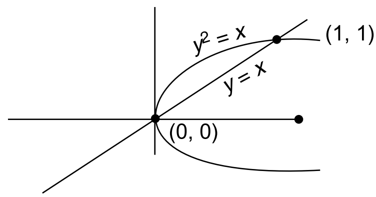
$$\iint_S (\nabla \times F)\hat{n}\,ds = \oint_C \vec{F} \cdot d\vec{r} = -2$$

Hence, Stokes' theorem is verified.

(c) Find the work done when a force $\overline{F} = (x^2 - y^2 + x)i - (2xy + y)j$ moves a particle in the XY-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. **04**

Solution:





Work done $\vec{F} \cdot d\vec{r}$

Let $\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Path of the integration along the ______ parabola

$$y^2 = x \Rightarrow dx = 2y\, dy$$

and y varies from 0 to 1.

$\therefore$ Work done $W = \int_C \vec{F} \cdot d\vec{r}$

$$= \int_0^1 [(x^2 - y^2 + x)dx - (2xy + y)dy]$$

$$= \int_0^1 [(y^4 - \cancel{y}^3 + \cancel{y}^2) \cdot (2y dy) - (2y^3 + y)dy]$$

$$= \int_0^1 [(2y^5) - 2y^3 - y]\, dy$$

$$= \left| 2 \cdot \frac{y^6}{6} - \frac{2y^4}{4} - \frac{y^2}{2} \right|_0^1$$

$$= \frac{2}{6} - \frac{2}{4} - \frac{1}{2}$$

$$= \frac{4 - 6 - 6}{12} = -\frac{8}{12} = -\frac{2}{3}$$

Index

E

F

G

H

I

K

L

M

N

O

P

Q

R

S

T

U

V

W

Z