Linear Algebra and Partial Differential Equations

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Preface

Linear Algebra and Partial Differential Equations, has been designed specifically to cater to the needs of third semester B Tech students. The current edition aims at preparing the students for examination alongside strengthening the fundamental concepts related to Partial Differential Equations. Lucidity of the text, ample worked examples and notes highlighted within the text help students navigate through complex topics seamlessly. Stepwise explanation, use of multiple methods of problem solving, and additional information presented by the means of appendices are few other notable features of the content.

Salient Features

- Strict adherence to the syllabus
- Stepwise solutions of solved problems which will enable students to score marks

Chapter Organization

The book is organised into 5 units. *Unit 1* deals with Vector Spaces. *Unit 2* explains in detail about Linear Transformation. *Unit 3* discusses the Inner Product Spaces. *Unit 4* focuses on Partial Differential Equations while *Chapter 5* elaborates on Fourier Series Solutions of Partial Differential Equations.

Acknowledgements

I have great pleasure in dedicating this book to the students and teachers. I hope that both the faculty and the students will receive the present edition as willingly as the earlier editions and my other books.

A number of reviewers took pains to provide valuable feedback for the book. We are grateful to all of them.

T Veerarajan

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Unit 1

Vector Spaces

1.1 VECTOR SPACES – DEFINITION

A vector space is a non-empty set of objects (called vectors) for which rules of addition and scalar multiplication are defined as follows and for which the following axioms hold good:

Addition means a rule that assigns to each pair of vectors u and v in the vector space V, a vector $(u + v)$ in V.

Scalar multiplication means a rule that assigns to each scalar c in a field F (viz., a set of real or complex scalars which obey the elementary rules of algebra) and each vector in V, a vector cu in V.

Axioms:

1. Addition is commutative. viz., for any two vectors, $u, v, \in V$, $u + v = v + u$.
2. Addition is associative. viz., for any vectors $u, v, \in V$, $(u + v) + w = u + (v + w)$
3. There is a unique vector 0 in V (called zero vector) such that $u + 0 = 0 + u = u$ for any vector u in V.
4. For each vector u in V. there is a unique vector $-u$ in V, such that $u + (-u) = 0$
5. For any scalar c in F and any vector u, v in V, $c(u + v) = Cu + Cv$.
6. For any two scalars C_1 and C_2 in F and any vector $u \in V$, $(C_1 + C_2)u = C_1u + C_2u$.
7. For the unit scalars $1 \in F$, $1u = u$ for any $u \in V$.
8. For any two scalars C_1 and C_2 in F and any vector $u \in V$, $(C_1C_2)\, u = C_1\, (C_2u)$.

Note ✍
The Vector space is also referred to as the vector space over the field F or linear space.

Examples of Vector Spaces

1. The set of all n-triples of scalars in any field F with addition and scalar multiplication defined by:
   $(a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) = a_1 + b_1, a_2 + b_2, ..., a_n + b_n$ and $c\,(a_1, a_2, ..., a_n) = (ca_1, ca_2, ..., ca_n)$, where $a_i, b_i, c \in F$. This vector space is denoted by F^n. Particular cases are R^n and C^n.

Note ✍

The zero vector of F^n is 0 = (0, 0, ..., 0)

2. The set of all $(m \times n)$ matrices with entries from any field F is a vector space over F w.r.t. the operations of matrix addition and scalar multiplication is denoted by $F^{m \times n}$

Note ✍

$F^{1 \times n} = F^n$

3. The set of all polynomials $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, with the coefficients c_i from any field F with respect to additions of polynomials and multiplication by a constant.
4. The set V of all function from a non-empty set X into any arbitrary field F for which addition and scalar multiplication are defined as follows is a vector space.
   The sum of any two functions f and $g \in V$ is the function $(f + g)(x) = f(x) + g(x)$
   The product of a scalar $c \in F$ and a function $f \in V$ in the function $cf \in V$, defined by $(cf)(x) = cf(x)$.

Subspaces

If W is a subset of a vector space V over a field F, such that W is itself a vector space over F w.r.t. vector addition and scalar multiplication [viz., (1) W is non-empty, (2) $v, w \in W$ implies $v + w \in W$ and (3) $v \in W$ implies $c\,v \in W$ for every $c \in F$], then W is called a ***sub-space of V***.

Examples of Subspaces

1. If V is R^3, then the set W consisting of those vectors whose first component is zero. i.e., $W = \{(0, a, b): a, b \in R\}$ is a sub-space of V.
2. If V is the space of all $n \times n$ matrices, then the set of all symmetric matrices of order n is a sub-space of V.
3. If V is any space, then the set $\{0\}$ consisting of the zero vector alone and the entire space V are sub-spaces V.

Span

If S is a non-empty sub set of a vector space V, the set of all linear combinations of vectors in S is a subspace of V and is called *the span of S* and denoted by $L(S)$. The subspace $L(S)$ is said to be generated by S.

If $L(S) = V$, then V is said to be finitely generated by S.

Examples

1. The vector $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and span the vector space R, for, any vector (a, b, c) in R^3 can be expressed as a linear combination of e_1, e_2, e_3 as $(a, b, c) = ae_1 + be_2 + ce_3$
2. The polynomial $1, t, t^2, \ldots$ generate the vector space of all polynomials in t, as any polynomial can be expressed as a linear combination of $1, t, t^2, \ldots$.

Linear Dependence and Independence of Vectors

The vectors $u_1, u_2, ..., u_m$ are said to be *linear by dependent* if scalars, $c_1, c_2, ... c_m$ (not all zero simultaneously) can be found such that

$$c_1u_1 + c_2 u_2 + \cdots + c_m u_m = 0 \tag{1}$$

where the symbol 0 on the right denotes the null vector.

Otherwise the m vectors are said to be *linearly independent.* In this case the equation (1) will be satisfied only if $c_1 = c_2 = ... = c_m = 0$.

In (1), suppose $c_k \neq 0$, then

$$c_k u_k = -c_1u_1 - c_2u_2 - \cdots - c_{k-1}u_{k-1} - c_{k+1}u_{k+1} - \cdots - c_m u_m$$

or equivalently $u_k = d_1u_1 + d_2u_2 + \cdots + d_m u_m$. In the case, the vector u_k is said to be *a linear combination* of all the others.

1.2 BASIS AND DIMENSION

A vector space V is said to be *finite-dimensional* (n-dimensional or dim $V = n$), if there exists a linearly independent set of vectors $\{e_1, e_2, ..., e_n)$ in V which spans the space V. The set $\{e_1, e_2, ..., e_n\}$ is called a *basis of* V and the number of elements in a basis is called *the dimension of* V

Examples

1. The vectors e_1, (1, 0, 0, ..., 0), $e_2 = (0, 1, 0, ..., 0)$, $e_3 = (0, 0, 1, ..., 0)$, $e_n = (0, 0, 0, ..., 1)$ form a basis of R^n, called *the standard basis* and dim $(R^n) = n$.
2. If V is the vector space of all ($m \times n$) matrices over F, then dim $V = mn$.
   In particular, if V is the vector space of all (2×2) matrices over R, then dim $V = 4$.
   The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ from the basis of V.
3. If V is the vector space of polynomials in t of degree n, then dim $(V) = n + 1$, for the linearly independent set $\{1, t, t^2, ..., t^n\}$ is a basis of V.

Worked Examples (1)

Example 1

Determine whether the vector $v = (3, 9, -4, -2)$ belongs to the space spanned by $u_1 = (1, -2, 0, 3)$, $u_2 = (2, 3, 0, -1)$ and $u_3 = (2, -1, 2, 1)$

If v belongs to the space spanned by u_1, u_2 and u_3, then constants k_1, k_2, k_3 should exist such that $v = k_1u_1 + k_2u_2 + k_3u_3$.

viz., $$(3, 9, -4, -2) = k_1(1, -2, 0, 3) + k_2(2, 3, 0, -1) + k_2(2, -1, 2, 1)$$

viz.,
$$k_1 + 2k_2 + 2k_3 = 3 \tag{1}$$
$$-2k_1 + 3k_2 - k_3 = 9 \tag{2}$$
$$2k_3 = -4 \tag{3}$$

and $$3k_1 - k_2 + k_3 = -2 \tag{4}$$

Equations (1), (2), (3) and (4) are satisfied by $k_1 = 1$, $k_2 = 3$ and $k_3 = -2$

$\therefore$ The vector v belongs to the space spanned by the vectors u_1, u_2, u_3.

Example 2

Find whether the vector (2, 4, 6, 7, 8) is in the subspace of R^5 spanned by (1, 2, 0, 3, 0), (0, 0, 1, 4, 0) and (0, 0, 0, 0, 1)

If possible, let $(2, 4, 6, 7, 8) = k_1(1, 2, 0, 3, 0) + k_2(0, 0, 1, 4, 0) + k_3(0, 0, 0, 0, 1)$.

Then $k_1 = 2$, $2k, = 4$, $k_2 = 6$, $3k_1 + 4k_2 = 7$, $k_3 = 8$

There equations are not satisfied by the same set of values of k_1, k_2 and k_3,

$\therefore$ The given vector does not belong to the subspace of R^5.

Example 3

Examine the linear dependence or independence of the following vectors:

$$u_1 = (1, -2, 3, 4),\ u_2 = (-2, 4, -1, -3) \text{ and } u_3 = (-1, 2, 7, 6)$$

Writing the vectors as row vectors, one below the other, we have

$$\begin{pmatrix} 1, & -2, & 3, & 4 \\ -2, & 4, & 1, & -3 \\ -1, & 2, & 7, & 6 \end{pmatrix} \sim \begin{pmatrix} 1, & -2, & 3, & 4 \\ 0, & 0, & 5, & 5 \\ 0, & 0, & 10, & 10 \end{pmatrix} (u_1, u_2 + 2u_1, u_3 + u_1)$$

$$\sim \begin{pmatrix} 1, & -2, & 3, & 4 \\ 0, & 0, & 5, & 5 \\ 0, & 0, & 0, & 0 \end{pmatrix} [u_1, u_2 + 2u_1, u_3 + u_1 - 2\,(u_2 + 2u_1)$$

We see that $u_3 - 3u_1 - 2u_2 = 0$

$\therefore$ The 3 vectors are linearly dependent.

Example 4

Find the maximum number of linearly independent vectors among the following and express each of the remaining vectors as a linear combination of these.

$$u_1 = (1, 2, 1);\ u_2 = (4, 1, 2);\ \ u_3 = (6, 5, 4) \text{ and } u_4 = (-3, 8, 1).$$

Writing the vectors as row vectors one below the other, we have

$$\begin{pmatrix} 1, & 2, & 1 \\ 4, & 1, & 2 \\ 6, & 5, & 4 \\ -3, & 8, & 1 \end{pmatrix} \sim \begin{pmatrix} 1, & 2, & 1 \\ 0, & 7, & -2 \\ 0, & -7, & -2 \\ 0, & 14, & 4 \end{pmatrix} \begin{matrix} [u_1, u_2, -4u_1, u_3 - 6u_1, u_4 + 3u_1] \\ [u_1', u_2', u_3', u_2' \text{ say}] \end{matrix}$$

$$\sim \begin{pmatrix} 1, & 2, & 1 \\ 0, & -7, & -2 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix} (u_1', u_2', u_3' - u_2', u_4' + 2n_2')$$

Maximum number of linearly independent vectors = 2

Also $$u_3' - u_2' = 0$$

viz., $$u_3 - 6u_1 - (u_2 - 4u_1) = 0$$

viz., $$u_3 = 2u_1 + u_2$$

and $$u_4' + 2u_2' = 0$$

viz., $$u_4 = 3u_1 + 2(u_2 - 4u_1) = 0$$

viz., $$u_4 = 5u_1 - 2u_2.$$

Example 5

Determine whether the set of vectors (4, 1, 2, 0), (1, 2, –1, 0), (1, 3, 1, 2) and (6, 1, 0, 1) is linearly independent.

Let $k_1(4, 1, 2, 0) + k_2(1, 2, -1, 0) + k_3(1, 3, 1, 2) + k_4(6, 1, 0, 1) = 0$ (A)

Then $$4k_1 + k_2 + k_3 + 6k_4 = 0 \quad (1)$$

$$k_1 + 2k_2 + 3k_3 + k_4 = 0 \quad (2)$$

$$2k_1 - k_2 + k_3 = 0 \quad (3)$$

and $$2k_3 + k_4 = 0 \quad (4)$$

using (4) in (1); $$4k_1 + k_2 - 11k_3 = 0 \quad (5)$$

using (4) in (2); $$k_1 + 2k_2 + k_3 = 0 \quad (6)$$

Eliminating k_3 from (3) and (5); $26k_1 - 10k_2 = 0$ or $13k_1 - 5k_2 = 0$ (7)

Eliminating k_3 from (3) and (6); $k_1 - 3k_2 = 0$ (8)

Solving (7) and (8), we get $k_1 = 0$ and $k_2 = 0$

From (3), $k_3 = 0$ and from (4), $k_4 = 0$

viz., the only values satisfying (A) are $k_1 = k_2 = k_3 = k_4 = 0$.

$\therefore$ The given system in linearly independent.

Example 6

Show that the vectors $u = (1, 2, 3)$, $v = (0, 1, 2)$ and $w = (0, 0, 1)$ generate R^3.

If u, v, w generate R^3, a general vectors (a, b, c) in R^3 should expressed as a linear combination of u, v, w.

Let $(a, b, c) = k_1(1, 2, 3) + k_2(0, 1, 2) + k_3(0, 0, 1)$

$\therefore$ $k_1 = a$; $2k_1 + k_2 = b$ $\therefore$ $k_2 = b - 2a$

and $3k_1 + 2k_2 + k_3 = c$ $\therefore$ $k_3 = c - 3a - 2(b - 2a) = c - 2b + a$

Hence the three given vectors generate R^3.

Example 7

Find the condition on a, b, c so that $(a, b, c) \in R^3$ belong to the space generated by $u = (2, 1, 0)$, $v = (1, -1, 2)$ and $w = (0, 3, -4)$

Let $$(a, b, c) = k_1(2, 1, 0) + k_2(1, -1, 2) + k_3(0, 3, -4)$$

Then
$$2k_1 + k_2 = a \quad (1)$$
$$k_1 - k_2 + 3k_3 = b \quad (2)$$
$$2k_2 - 4k_3 = c \quad (3)$$

Eliminating k_3 from (2) and (3), $4k_1 + 2k_2 = 4b + 3c$

viz., $2a = 4b + 3c$ from (1)

Example 8

Show that the vectors $u = (1, 0, -1)$, $v = (1, 2, 1)$ and $w = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of u, v, w.

Writing u, v, w as row vectors one below the other and row reducing, we get

$$\begin{pmatrix} 1, & 0, & -1 \\ 1, & 2, & 1 \\ 0, & -3, & 2 \end{pmatrix} \sim \begin{pmatrix} 1, & 0, & -1 \\ 0, & 2, & 2 \\ 0, & -3, & 2 \end{pmatrix} (R_1, R_2 - R_1, R_3)$$

$$\sim \begin{pmatrix} 1, & 0, & -1 \\ 0, & 2, & 2 \\ 0, & 0, & 5 \end{pmatrix} (R_1', R_2', R_3' + \frac{3}{2} R_2')$$

$\therefore$ The given vectors are linearly independent.

Let $(a, b, c) = k_1(1, 0, -1) + k_2(1, 2, 1)$ and $k_3(0, -3, 2)$

Then
$$k_1 + k_2 = a$$
$$2k_2 - 3k_3 = b$$
$$-k_1 + k_2 + 2k_3 = c$$

Solving, $k_1 = \frac{1}{10}(7a - 2b - 3c)$; $k_2 = \frac{1}{10}(3a + 2b + 3c)$ and $k_3 = \frac{1}{10}(2a - 2b + 2c)$

The standard basis vectors and given by

$$e_1 = (1, 0, 0) = \frac{7}{10}u + \frac{3}{10}v + \frac{2}{10}w$$

$$e_2 = (0, 1, 0) = -\frac{2}{10}u + \frac{2}{10}v - \frac{2}{10}w$$

and
$$e_3 = (0, 0, 1) = -\frac{3}{10}u + \frac{3}{10}v + \frac{2}{10}w$$

Example 9

Find a basis and the dimension of the sub space W of R^4, generated by the vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$ and $(3, 8, -3, -5)$.

We form the matrix with the given vectors as rows and then row-reduce to echelon form as given below:

$$\begin{pmatrix} 1, & -2, & 5, & -3 \\ 2, & 3, & 1, & -4 \\ 3, & 9, & -3, & -5 \end{pmatrix} \sim \begin{pmatrix} 1, & -2, & 5, & -3 \\ 0, & 7, & -9, & 2 \\ 0, & 7, & -9, & 2 \end{pmatrix} (R_1, R_2 - 2R_1, R_3 - R_1 - R_2)$$

$$\sim \begin{pmatrix} 1, & -2, & 5, & 3 \\ 0, & 7, & -9, & 2 \\ 0, & 0, & 0, & 0 \end{pmatrix} (R_1, R_2, R_3 - R_2)$$

The non zero row vectors in the echelon form, namely, (1, –2, 5, –3) and (0, 7, –9, 2) form a basis of W and dim (W) = 2

Example 10

If W is the space spanned by the polynomial $v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = t^3 + 6t - 5$, $v_3 = 2t^3 - 3t^2 + 9t - 1$ and $v_4 = 2t^3 - 5t^2 + 7t + 5$, find a basis and dimension of W.

The coefficient vectors relative to the basis (t^3, t^2, t, 1) are (1, –2, 4, 1), (1, 0, 6, –5), (2, –3, 9, –1) and (2, –5, 7, 5)

We form the matrix with these coefficient vectors as rows and row-reduce to the echelon form as given below:

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 1 & 0 & 6 & -5 \\ 2 & -3 & 9 & -1 \\ 2 & -5 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & 3 \end{pmatrix} (R_1, R_2 - R_1, R_3 - 2R_1, R_4 - 2R_1)$$

$$\sim \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & 3 \end{pmatrix} (R_1, R_2 \div 2, R_3, R_4)$$

$$\sim \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (R_1, R_2, R_3 - R_2, R_4 + R_2)$$

$\therefore$ (1, –2, 4, 1) and (0, 1, 1, –3) form a basis for the space generated by the coefficient vectors

viz., $t^3 - 2t^2 + 4t + 1$ and $t^2 + t - 3$ form a basis for the space W and dim (W) = 2.

Example 11

Find the dimension and a basis for the solution space W of the system of homogeneous equation given below.

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0$$

$$3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0$$

Row-reducing the given system of equation, we get

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0 \quad (1)$$

$$x_3 + 2x_4 - 2x_5 = 0 \quad (2)$$

$$\dim (W) = \text{No. of unknowns} - \text{No. of non-zero equations}$$

$$= 5 - 2 = 3.$$

The free variables are taken as x_2, x_4 and x_5 ($\because$ x_2 is not present in (2))
Taking $x_2 = 1, x_4 = 0, x_5 = 0$; $v_1 = (x_1, x_2, x_3, x_4, x_5) = (-2, 1, 0, 0, 0)$, using (1) and (2)
Taking $x_2 = 0$, $x_4 = 1$, $x_5 = 0$; $v_2 = (5, 0, -2, 1, 0)$, using (1) and (2)
Taking $x_2 = 0$, $x_4 = 0$, $x_5 = 1$; $v_3 = (-7, 0, 2, 0, 1)$, using (1) and (2)
v_1, v_2, v_3 form a basis for the solution space W.

Example 12

Find a homogenous system of equations whose solution set W is spanned by (1, –2, 0, 3 –1), (2, –3, 2, 5, –3) and (1, –2, 1, 2, –2).
$v = (x_1, x_2, x_3, x_4, x_5) \in W$, if and only if v is a linear combination of the given vectors.
$\therefore$ $(x_1, x_2, x_3, x_4, x_5) = k_1(1, -2, 0, 3, -1) + k_2 (2, -3, 2, 5, -3) + k_3(1, -2, 1, 2, -2)$

viz.,

$$k_1 + 2k_2 + k_3 = x_1 \quad (1)$$

$$-2k_1 - 3k_2 - 2k_3 = x_2 \quad (2)$$

$$2k_2 + k_3 = x_3 \quad (3)$$

$$3k_1 + 5k_2 + 2k_3 = x_4 \quad (4)$$

$$-k_1 - 3k_2 - 2k_3 = x_5 \quad (5)$$

Row-reducing the above equation, we get

$$k_1 + 2k_2 + k_3 = x_1 \quad (1')$$

$$k_2 = 2x_1 + x_2 \quad (2')$$

$$2k_2 + k_3 = x_3 \quad (3')$$

$$-k_2 - k_3 = x_4 - 3x_1 \quad (4')$$

$$-k_2 - k_3 = x_1 + x_5 \quad (5')$$

(3′) + (4′) gives $\quad k_2 = -3x_1 + x_3 + x_4$

Also $\quad k_2 = 2x_1 + x_2$

$\therefore \quad 2x_1 + x_2 = -3x_1 + x_3 + x_4$

viz; $\quad 5x_1 + x_2 - x_3 - x_4 = 0 \quad (6)$

Equating (4′) and (5′), we also get

$$x_1 + x_5 = x_4 - 3x_1$$

viz; $4x_1 - x_4 + x_5 = 0$

(OR) $4x_1 - (5x_1 + x_2 - x_3) + x_5 = 0$ viz., $x_1 + x_2 - x_3 - x_5 = 0$ (7)

$v \in W$, if and only if the above system has a solution.

viz., if $5x_1 + x_2 - x_3 - x_4 = 0$ (6)

and $x_1 + x_2 - x_3 - x_5 = 0$ (7′)

Equations (6) and (7′) form the required homogeneous system of equations.

Exercise 1

Part A (Short-Answer Questions)

1. Define vector space with two examples.
2. Define subspace with two examples.
3. Define span of a vector space.
4. Define standard vectors in R^3 and prove that they span the vector space R^3
5. Define linear dependence and independence of vectors.
6. Define basis and dimension of a vector space.
7. If V is the vector space of all (2 × 2) matrices over R, give a basis of V and dimension of V.
8. Determine whether the vectors (1, 1, 1) and (1, –1, 5) form a basis for the vector space R^3.
9. Find whether the vectors (1, 1, 2), (1, 2, 5) and (5, 3, 4) form a basis for the vector space R^3.
10. Find whether the vector (1, 1, 1), (1, 2, 3) and (2, –1, 1) form a basis for the vector space R^3.

Part B

11. Is the vector (3, –1, 0, –1) in the sub-space of R^4 spanned by the vectors (2, –1, 3, 2), (–1, 1, 1, –3) and (1, 1, 9, –5)?
12. Find whether the vector (–3, –6, 1, –5, 2) is in the sub space of R^5 spanned by (1, 2, 0, 3, 0), (0, 0, 1, 4, 0) and (0, 0, 0, 0, 1).
13. Examine the linear dependence or independence of the following vectors:
    (i) $u_1 = (2, -1, 3, 2)$, $u_2 = (1, 3, 4, 2)$ and $u_3 = (3, -5, 2, 2)$.
    (ii) $u_1 = (1, -1, 0, 1)$, $u_2 = (-1, -1, -1, 2)$ and $u_3 = (2, 0, 1, -1)$
14. Find the maximum number of linearly independent vectors among the following and express each of the remaining vectors as a linear combination of these:
    $u_1 = (3, 1, -4)$; $u_2 = (2, 2, -3)$; $u_3 = (0, -4, 1)$ and $u_4 = (-4, -4, 6)$
15. Show that the vectors u_1 (2, 3, –1, –1); $u_2 = (1, -1, -2, -4)$; $u_3 = (3, 1, 3, -2)$ and $u_4 = (6, 3, 0, -7)$ form a linearly dependent system, also express u4 as a linear combination of other.
16. Determine whether the vector (4, 2, 1, 0) is a linear combination of the vectors $u_1 = (6, -1, 2, 1)$, $u_2 = (1, 7, -3, -2)$, $u_3 = (3, 1, 0, 0)$ and $u_4 = (3, 3, -2, -1)$.
17. Show that the vector $(a, b, 0)$ in R^3 is generated by
    (i) $u_1 = (1, 2, 0)$ and $u_2 = (0, 1, 0)$

(ii) $u_1 = (2, -1, 0)$ and $u_2 = (1, 3, 0)$

18. Show that the vector $(a, b, 0)$ in R^3 is generated by
(i) $u_1 = (0, 1, 1)$ and $u_2 = (0, 2, -1)$
(ii) $u_1 = (0, 1, 2)$ and $u_2 = (0, 2, 3)$
19. Show that the vectors (1, 1, 1), (1, 2, 3) and (2, 3, 8) form a basis for R^3. Express each of the standard basis vectors as a linear combination of these vectors.
20. Show that the vectors $u = (1, 2, 2)$, $v = (2, 1, -2)$ and $w = (2, -2, 1)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of u, v and w.
21. Find a basis and dimension for the subspace of R^4 spanned by the four vectors $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 9)$, $v_3 = (1, -1, -4, 0)$ and $v_4 = (2, 1, 1, 6)$
22. Find a basis and dimension of the subspace of R^4 spanned by
(i) (1, 4, –1, 3); (2, 1, –3, –1) and (0, 2, 1, –5)
(ii) (1, –4, –2, 1); (1, –3, –1, 2) and (3, –8, –2, 7)
23. Find a basis and dimension of the solution space W of the homogeneous system $x + 3y + 2z = 0$, $x + 5y + z = 0$ and $3x + y + 8z = 0$.
24. Find a basis and dimension of the solution space W of the homogeneous system: $x_1 + 2x_2 - x_3 + x_4 = 0$ and $x_1 - 2x_2 + x_3 + 2x_4 = 0$
25. Find a homogeneous system of equations whose solution set W is spanned by (1, –2, 0, 3), (1, –1, –1, 4) and (1, 0, –2, 5)

Answers

Exercise 1

8. No, since dim $(R^3) = 3$, but there are only 2 elements.
9. No, since the vectors are linearly dependent.
10. Yes, since the vectors are linearly independent.
11. No, the given vectors are linearly dependent.
12. Yes, since $(-3, -6, 1, -5, 2) = -3u_1 + u_2 + 2u_3$
13. (i) Linearly dependent, since $u_3 = 2u_1 - u_2$
(ii) Linearly dependent since $u_1 = u_2 + u_3$
14. 2, let them be u_1 and u_2. Then $u_3 = 2u_1 - 3u_2$ and $u_4 = 0\,u_1 - 2u_2$
15. $u_4 = u_1 + u_2 + u_3$
16. Yes since $u = 2u_1 + u_2 - 3u_3 + 0.u_4$
17. (i) $(a, b, 0) = a(1, 2, 0) + (b - 2a)(0, 1, 0)$
(ii) $(a, b, 0) = \frac{1}{7}(3a - b)(2, -1, 0) + \frac{1}{7}(a + 2b)(1, 3, 0)$
18. (i) $(0, b, c) = \frac{1}{3}(b + 2c)(0, 1, 1) + \frac{1}{3}(b - c)(0, 2, -1)$
(ii) $(0, b, c) = (-3b + 2c)(0, 1, 2) + (2b - c)(0, 2, 3)$

19. $e_1 = \frac{7}{4}u_1 - \frac{5}{4}u_2 + \frac{1}{4}u_3$; $e_2 = -\frac{1}{2}u_1 + \frac{3}{2}u_2 - \frac{1}{2}u_3$

and $e_3 = -\frac{1}{4}u_1 - \frac{1}{4}u_2 + \frac{1}{4}u_3$

20. $e_1 = \frac{1}{9}u + \frac{2}{9}v + \frac{2}{9}w$; $e_2 = \frac{2}{9}u + \frac{1}{9}v - \frac{2}{9}w$; $e_3 = \frac{2}{9}u - \frac{2}{9}v + \frac{1}{9}w$

21. (1, 1, 2, 4), (0, –3, –11, 1) and (0, –2, –6, –4) form a basis and dim = 3.
22. (i) dim (W) = 3; Basis ≡ [(1, 4, –1, 3), (0, –7, –1, –7) and (0, 2, 1, 5)
    (ii) dim (W) = 2; Basis ≡ [(1, –4, –2, 1) and (0, 1, 1, 1)]
23. Basis ≡ (7, –1, 2) and dim (W) = 1
24. dim (W) = 2; Basis ≡ [(–5, 1, 0, 3) and (3, 0, 1 –2)]
25. $2x_1 + x_2 + x_3 = 0$ and $5x_1 + x_2 - x_4 = 0$

Unit 2

Linear Transformation

2.1 LINEAR TRANSFORMATION–DEFINITION

If V and W are vector spaces over the same field F, a function T from V into W is called a *linear transformation* or *linear mapping*, provided it preserves the two basic operations of a vector space and denoted by $T : V \to W$.

viz., (i) $T(v + w) = T(v) + T(w)$ for any $v, w \in V$

(ii) $T(cv) = cT(v)$, for any $v \in V$ and $c \in$ F.

Note ✍

$T(0) = T(0v) = 0$. $T(v) = 0$.

Examples of Linear Transformation

1. *Zero transformation:* Let $N : V \to W$ be a transformation such that $N(v) = 0 \in W$, for every $v \in V$.

   Now $N(v + w) = 0 = 0 + 0 = N(v) + N(w)$

   and $N(cv) = 0 = c \times 0 = cN(v)$

   Hence W is a linear transformation, usually denoted by 0.
2. *Identity transformation:* Let $I: V \to V$ be a transformation such that $I(v) = v$, for every $v \in V$.

   Now $I(v + w) = v + w = I(v) + I(w)$

   and $I(cv) = cv = cI(v)$

   Hence I is a linear transformation.
3. If V is *the vector of polynomials* in the variable x over the real field R and if

   $D(f) = \dfrac{df}{dx}$ and $I(x) = \int_0^1 f(x)\,dx$, then

   $D : V \to V$ and $: V \to R$ are linear transformation, for

   $D(c_1 v + c_2 w) = c_1\, D(v) + c_2\, D(w)$ and,

   $$I(c_1 v + c_2 w) = \int_0^1 (c_1 v + c_2 w)\,dx = c_1 \int_0^1 v\,dx + c_2 \int_0^1 w dx = c_1 I(v) + c_2 I(w)$$
4. Let P be a fixed $(m \times m)$ matrix with entries over F and Q be a fixed $(n \times n)$ matrix over F.

If T is a transformation over the space $F^{m \times n}$ is defined as $T(A) = PAQ$, then T is a linear transformation, for

$$T(c_1A + c_2B) = P(c_1A + c_2B)Q = [c_1(PA) + c_2(PB)]Q = c_1(PAQ) + c_2(PBQ)$$
$$= c_1T(A) + c_2T(B)$$

Note ✍

T: V → W can be uniquely determined by arbitrarily assigning elements of W to the elements of a basis of V as per the following theorem which is stated without proof.

Theorem: If V is a finite dimensional vector space over a field F, with $\{v_1, v_2, ..., v_n\}$ as a basis and W is another vector space over the same field containing the arbitrary vector $\{w_1, w_2, ..., w_n\}$ (which may be linearly dependent or equal to each other), there exists a unique linear transformation $T : V \to W$ such that $T(v_j) = w_j$ $ij = 1, 2, ..., n$.

For example, let us find $T : R^2 \to R^3$ defined by $T((1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$

Since $(1, 2)$ and $(0, 1)$ are linearly independent, they form a basis of R^2. The vectors $(3, -1, 5)$ and $(2, 1, -1)$ have been arbitrarily chosen in R^3

Now $(a, b) = c_1(1, 2) + c_2(0, 1)$ $\quad \therefore c_1 = a$ and $c_2 = b - 2a$

$\therefore \quad T(a, b) = a(3, -1, 5) + (b - 2a)(2, 1, -1)$

$= (2b - a, b - 3a, 7a - b)$, which is the unique linear transformation required.

2.2 NULL SPACE AND RANGE SPACE

Definitions: If V and W are vector spaces over the field F and if $T : V \to W$ is a linear transformation, the set of all vectors v in V such that $T(v) = 0$ is called *the null space of T* or *kernel of T* and denoted by N_T.

The set of all vector w is W such that $T(v) = w$, $v \in V$ is called *the range space* or *the image space of T* and denoted by R_T.

Note ✍

Null space of T is a subspace of V and range space of T is a subspace of W.)

If V is finite dimensional, the dimension of the range of T is called *the rank of T* and that of the null space of T is called *the nullity of T*

Dimension Theorem

The sum of the dimension of the range space and null space of a linear transformation is equal to the dimension of its domain viz., if V and W are vector spaces over the field F and if $T : V \to W$ is a linear transformation and if V is finite dimensional, then rank (T) + nullity (T) = dim (V).

Proof: Let $\{v_1, v_2, ..., v_k\}$ be a basis for N_T, so that dim $(N_T) = k$

We can find vectors $v_{k+1}, v_{k+2}, \ldots v_n$ such that $(v_1, v_2, \ldots v_n)$ is a basis of V, so that $\dim(V) = n$.

The theorem is proved, if we can prove that $(Tv_{k+1}, Tv_{k+2}, \ldots Tv_n)$ is basis for R_T

Clearly the vectors $Tv_1, Tv_2, \ldots Tv_n$ span R_T

But $\quad Tv_1 = Tv_2 = \ldots = Tv_k = 0$

$\therefore \quad Tv_{k+1}, Tv_{k+2}, \ldots, Tv_n$ span R_T

These vector Tv_{k+1}, Tv_{k+2}, Tv_n will be a basis of R_T, provided they are linearly independent.

Let these be scalars c_i such that $\sum\limits_{i=k+1}^{n} c_i(Tv_i) = 0$

viz., $T \sum\limits_{i=k+1}^{n} c_i\, v_i = 0 \quad (\because T \text{ is linear})$

This means that $v = \sum\limits_{i=k+1}^{n} c_i v_i$ is (in N_T) viz., $T \sum\limits_{i=k+1}^{n} c_i v_i = 0 \quad [\because T \text{ is linear}]$ (1)

Since $\{v_1, v_2, \ldots, v_k\}$ is a basis of N_T, there exist scalars $b_1, b_2, \ldots, b_k$ such that

$$v = \sum_{i=1}^{k} b_i v_i \tag{2}$$

From (1) and (2), we get $\sum\limits_{i=1}^{k} b_i v_i - \sum\limits_{i=K+1}^{n} c_i v_i = 0$

Since $v_1, v_2, .., v_n$ form a basis of V, they are linearly independent.

$\therefore \quad b_1 = b_2 = \ldots = b_k = c_{k+1} = c_{k+2} = \ldots = c_n = 0$

$\therefore \quad Tv_{k+1}, Tv_{k+2}, \ldots, Tv_n$ form a basis for R_T

$\therefore \quad \dim(R_T) = n - k$ and $\dim(N_T) = k$

$\therefore \quad$ Rank (T) + nullity $(T) = \dim(V)$

Worked Examples 2(a)

Example 1

Show that the transformation $T : R^3 \to R^2$ defined by $T(x, y, z) = (z, x + y)$ is linear.

Let $\quad v = (x, y, z)$ and $w = (x', y', z')$

Then $\quad c_1v + c_2w = (c_1x + c_2x', c_1y + c_2y', c_1z + c_2z')$

$\therefore \quad T(c_1v + c_2w) = \{c_1z + c_2z', c_1(x + y) + c_2(x' + y')\}$, by definition of T

$\quad = c_1(z, x + y) + c_2(z', x' + y')$

$$= c_1T(x, y, z) + c_2T(x', y', z')$$

$$= c_1T(v) + c_2T(w)$$

$\therefore$ T is linear.

Example 2

Show that the transformation $T : R^2 \to R^2$ defined by $T(x, y) = (\sin x, y)$ is not linear.

Let $v = (x, y)$ and $w = (x', y')$

Then $\quad c_1v + c_2w = (c_1x + c_2x', c_1y + c_2y')$

$$\therefore \quad T(c_1v + c_2w) = \{\sin(c_1x + c_2x'), c_1y + c_2y'\} \tag{1}$$

But $c_1T(v) + c_2T(w) = c_1 (\sin x, y) + c_2 (\sin x', y')$

$$= \{c_1 \sin x + c_2 \sin x', c_1y + c_2y'\} \tag{2}$$

From (1) and (2), we see that $T(c_1v + c_2w) \neq c_1T(v) + c_2T(w)$

$\therefore$ T is not linear.

Example 3

Find whether the transformation $T : R^2 \to R^3$ defined by

$$T(x, y) = (x + 1, 2y, x + y) \text{ is linear.}$$

Let $\quad v\,(x, y)$ and $w = (x', y')$

Then $\quad c_1v + c_2w = (c_1x + c_2x', c_1y + c_2y')$

$$\therefore \quad T(c_1v + c_2w) = \{c_1x + c_2x' + 1, 2(c_1y + c_2y'), c_1(x + y) + c'_2(x' + y')$$

$$\neq c_1T(v) + c_2T(w)$$

$\therefore$ T is not linear.

Example 4

If V is the vector space of all $n \times n$ matrices over F and if B is an arbitrary matrix in V, show that the transformation $T : V \to V$ defined by $T(A) = AB - BA$, where $A \in V$ is linear. Show also that $T(A) = A + B$ is not linear, unless $B = 0$.

$$T(c_1A + c_2A') = (c_1A + c_2A')B - B(c_1A + c_2A')$$

$$= c_1(AB - BA) + c_2(A'B - BA') \tag{1}$$

$$c_1T(A) + c_2T(A') = c_1(AB - BA) + c_2(A'B - BA') \tag{2}$$

From (1) and (2), we see that T is linear.

Now $\quad T(c_1A + c_2A') = c_1A + c_2A' + B \quad (3)$

and $c_1T(A) + c_2T(A') = c_1(A + B) + c_2(A' + B)$

$$= c_1A + c_2A' + 2B \tag{4}$$

From (3) and (4); we see that T is not linear, but linear when $B = 0$

Example 5

Find $T(1, 0)$, where $T : R^2 \to R^3$ is defined by $T(1, 2) = (3, 2, 1)$ and $T(3, 4) = (6, 5, 4)$

Since (1, 2) and (3, 4) are linearly independent, they form a basis of R^2.(3. 2, 1) and (6, 5, 4) are arbitrary vectors in R^3.

$\therefore$ $T : R^2 \to R^3$ can be uniquely determined.

Let $(1, 0) = c_1(1, 2) + c_2(3, 4)$

Then $c_1 + 3c_2 = 1$ and $2c_1 + 4c_2 = 0$

Solving there equations, $c_1 = -2$ and $c_2 = 1$

$$\therefore \quad T(1, 0) = -2T(1, 2) + T(3, 4)$$
$$= -2(3, 2, 1) + (6, 5, 4)$$
$$= (0, 1, 2)$$

Example 6

Find a basis and dimension of R_T and N_T for the linear transformation $T : R^3 \to R^3$, defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3 = 2x_1 + x_2, -x_1 -2x_2 + 2x_3)$

The images of standard basis vectors of R^3, viz: (1, 0, 0), (0, 1, 0) and (0, 0, 1) generate R_T

$$T(1, 0, 0) = (1, 2, -1);\ T(0, 1, 0) = (-1, 1, -2) \text{ and } T(0, 0, 1) = (2, 0, 2)$$

We have to test wether three images can form the basis of R_T

viz., to find the number of independent vectors from the images.

$$\text{Now} \quad \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ {(1, 2, 1) and (0, 1, 1)} is a basis of R_T and rank of $T = 2$.

Let (x_1, x_2, x_3) be an element of N_T

Then $T(x_1, x_2, x_3) = (0, 0, 0)$

viz., $x_1 - x_2 + 2x_3 = 0,$ (1)

$2x_1 + x_2 = 0$ (2)

and $-x_1 - 2x_2 + 2x_3 = 0$ (3)

viz., $x_1 - x_2 + 2x_3 = 0$ (1)

$3x_2 - 4\,x_3 = 0$ [(2) –3 × (1)]

$-3x_2 + 4x_3 = 0$ [(1) + (3)]

viz., $x_1 - x_2 + 2x_3 = 0$

and $3x_2 - 4x_3 = 0$

$\therefore$ dim (N_T) = 1 (No. of unknowns–No. of non-zero equations)

Taking x_3 as the free variable and putting $x_3 = 3$, we get $x_2 = 4$ and $x_1 = -2$

$\therefore$ (–2, 4, 3) is a basis of N_T.

Example 7

Find a basis and dimension of R_T and N_T for the linear transformation $T : R^4 \to R^3$ defined by $T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4)$.

The images of the standard basis vectors of R^4 generate R_T.

viz., $T(1, 0, 0, 0) = (1, 1, 1)$, $T(0, 1, 0, 0) = (-1, 0, 1)$, $T(0, 0, 1, 0) = (1, 2, 3)$ and $T(0, 0, 0, 1) = (1, -1, -3)$

Now let us find the number of independent vectors from the images

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ {(1, 1, 1,) and (0, 1, 2) from a basis of R_T and dim (R_T) = 2

Let $(x_1, x_2, x_3, x_4) \in N_T$

Then $T(x_1, x_2, x_3, x_4) = (0, 0, 0)$

$$\therefore \quad \begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_3 - x_4 &= 0 \\ x_1 + x_2 + 3x_3 - 3x_4 &= 0 \end{aligned} \quad \Bigg| \quad \text{viz.,} \begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_2 + x_3 - 2x_4 &= 0 \\ 2x_2 + 2x_3 - 4x_4 &= 0 \end{aligned} \quad \Bigg| \quad \begin{aligned} \text{viz.,}\ x_1 - x_2 + x_3 + x_4 &= 0 \\ \text{and} \quad x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

Since x_3 and x_4 can be taken as free variables,

$x_2 = -1, x_1 = 0$ (corresponding to $x_3 = 1$ and $x_4 = 0$)

and $x_2 = 2$, $x_1 = 1$ (corresponding to $x_3 = 0$ and $x_4 = 1$)

$\therefore$ dm (N_T) = 2 and {(0, –1, 1,0), (1, 2, 0, 1)} is a basis of N_T.

Example 8

If V is the vector space of 2 × 2 matrices, if $M = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ and if $T : V \to V$ be the linear transformation defined by $T(A) = MA$, find a basis and dimension of (i) R_T and (ii) N_T.

The images of the standard basis element of V are given by

$$\begin{pmatrix} 1 & -1 \\ -2, & 2 \end{pmatrix}\begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}; \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}; \begin{pmatrix} 1 & -1 \\ -2, & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 2, & 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$$

These images span R_T. To find the basis of R_T, we have to find the number of independent vectors among these images. Writing these images as row vector, we have

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ dim $(R_T) = 2$ and a basis is $\left\{ \begin{bmatrix} 1, & 0 \\ -2, & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$

Let $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be an element of N_T.

Then $T\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viz.,

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0 \\ -2x_1 + 2x_3 &= 0 \\ -2x_2 + 2x_4 &= 0 \end{aligned}$$

$\therefore$ The non- zero eqations are

$x_1 - x_3 = 0$ and $x_2 - x_4 = 0$

$\therefore \dim(N_T) = 2$ and x_3 and x_4 can be treeted as free variable

Taking $x_3 = 1$ and $x_4 = 0$, we get $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Taking $x_3 = 0$ and $x_4 = 1$, we get $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ A basis of N_T is $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

Example 9

Find a linear transformation $R^3 \to R^3$ whose range space R_T is generated by (1, 2, 3) and (4, 5, 6)

Consider the standard basis of R^3, namely $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

The images of e_1, e_2, e_3, which are the elements of the basis of R_{T_1} are given by $T(1, 0, 0) = (1, 2, 3)$; $T(0, 1, 0) = (4, 5, 6)$; $T(0, 0, 1) = (0, 0, 0)$.

Now $xe_1 + ye_2 + ze_3 = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = (x, y, z)$

$$\begin{aligned} \therefore \quad T(x, y, z) &= T(xe_1 + ye_2 + xe_3) \\ &= xT(e_1) + yT(e_2) + zT(e_3) \\ &= x(1, 2, 3) + y(4, 5, 6) + z(0, 0, 0) \\ &= (x + 4y, 2x + 5y, 3x + 6y) \end{aligned}$$

Example 10

Find a linear transformation $T : R^4 \to R^3$ whose null space N_T is generated by (1, 2, 3, 4) and (0, 1, 1, 1)

Let $T(x, y, z, w) = [(a_1x + b_1y + c_{1z} + d_1w, a_2x + b_2y + c_2z + d_2w),$

$$(a_3x + b_3y + c_3z + d_3w)] \text{ (since } T \text{ is linear)}$$

Since (1, 2, 3, 4) and (0, 1, 1, 1) generate the null space,

$$T(1, 2, 3, 4) = (0, 0, 0) \text{ and } T(0, 1, 1, 1) = (0, 0, 0)$$

viz., $a_1 + 2b_1 + 3c_1 + 4d_1 = 0$ (1) | viz., $b_1 + c_1 + d_1 = 0$ (4)

$a_2 + 2b_2 + 3c_2 + 4d_2 = 0$ (2) | $b_2 + c_2 + d_2 = 0$ (5)

$a_3 + 2b_3 + 3c_3 + 4d_3 = 0$ (3) | $b_3 + c_3 + d_3 = 0$ (6)

Taking $d_1 = 0$ and solving (1) and (4), we have

viz., $(a_1, b_1, c_1,) = (–1, –1, 1)$

Solving taking $c_2 = 0$ and solving (2) and (5), we get

$$(a_2, b_2, d_2) = (–2, –1, 1)$$

Taking $b_3 = 0$ and solving (3) and (6), we get

$$(a_3, c_3, d_3) = (–1, –1, + 1)$$

$\therefore$ $T(x, y, z, w) = \{–x –y + z, –2x – y + w, –x – z + w\}$

or $(x + y, – z, 2x + y – w, x + z – w)$

Exercise 2(A)

Part A (Short-Answer Questions)

1. Define linear transformation with an example
2. How can we find the unique linear transformation $T : V \to W$?
3. Show that the transformation $T : R \to R^2$ defined by $T(x) = (2x, 3x)$ is linear.
4. Show that the transformation $T : R^2 \to R$ defined by $T(x, y) = |x – y|$ not linear.
5. Find the linear transformation $T : R^2 \to R$ defined by $T(1, 1) = 3$ and $T(0, 1) = –2$.
6. Define the null space of a linear transformation.
7. Define the range space of a linear transformation.
8. Define rank and nullity of a linear transformation. How are they related?

Part B

9. Show that the transformation $T : R^2 \to R^2$ defined by $T(x, y) = (ax + by, cx + dy)$; where $a, b, c, d \in R$, is linear.
10. Show that the transformation $T : R^3 \to R^2$ defined by $T(x, y, z) = (x + 1, y + z)$ is not linear.
11. Show that the transformation $T : R^2 \to R^2$, defined by $T(x, y) = (x^2, y^2)$ is not linear.
12. Show that the transformation $T : R^2 \to R$, defined by $T(x, y) = xy$ is not linear.

13. If V is the vector space of all $n \times n$ matrices over F and if B is an arbitrary matrix in V, show that the transformation $T : V \to V$, defined by $T(A) = AB + BA$, where $A \in V$ is linear.
14. If V is the vector space of polynomials in t over F, show that $T : V \to V$, defined by (i) $T(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) = a_0t + a_1t^2 + \cdots + a_nt^{n+1}$ and (ii) $T(a_0 + a_1t + a_2t^2 + \cdots + a_nt_n) = 0 + a_1 + a_2t + \cdots + a_nt^{n-1}$ are linear.
15. Find $T(a, b, c)$, where $T : R^3 \to R$ is defined by $T(1, 1, 1) = 3$, $T(0, 1, -2) = 1$ and $T(0, 0, 1) = -2$.
16. Is there a linear transformation $T : R^2 \to R^2$ such that $T(1, -1) = (1, 0)$, $T(2, -1) = (0, 1)$ and $T(-3, 2) = (1, 1)$?
17. Find a basis and dimension for R_T and N_T for the linear transformation $T : R^3 \to R^3$, given by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$
18. Find a basis and dimension for R_T and N_T, for the linear transformation $T : R^3 \to R^3$, defined by $T(x, y, z) = (x + 2y, y - z, x + 2z)$
19. Find a basis and dimension for R_T and N_T for the linear transformation $T : R^2 \to R^2$, defined by $T(x, y) = (x + y, x + y)$.
20. Find a basis and dimension for R_T and N_T for the linear transformation $T : R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y + z)$.
21. If V is the vector space of 2×2 matrices over R and $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and if $T : V \to V$ is the linear transformation, defined by $T(A) = AM - MA$, find a basis and dimension of N_T.
22. Find a linear transformation $T : R^3 \to R^4$ whose range space is generated by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$
23. If T is the linear operator on R^3 [viz; $T : R^3 - R^3$], the matrix of which in the usual basis is $[A]_e^A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$, find a basis for R_T and N_T.

    [Hint: $T(x, y, z) = (x + 2y + z, y + z, -x + 3y + 4z)$
24. Find a basis and dimension for R_T and N_T of the linear transformation $T : R^4 \to R^3$ determined by $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{pmatrix}$
25. Find a basis and dimension for R_T and N_T of the linear transformation $T : R^4 \to R_3$ determined by $B = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & -1 & 1 \\ -2 & 0 & -5 & 3 \end{pmatrix}$

2.3 MATRIX REPRESENTATION OF LINEAR TRANSFORMATION

Definition: Let V and W be vector spaces over the field F, of dimensions n and m respectively. Let $e = \{e_1, e_2, \cdots, e_n\}$ and $f = \{f_1, f_2, \cdots, f_m\}$ be ordered (arbitrary but fixed) bases for V and W respectively. If $v \in V$, then $v = c_1e_1 + c_2e_2 + \; + c_ne_n$.

The column vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is called *the co-ordinate* vector of v relative to $\{e_j\}$ and denoted as $[v]_e$.

Let $T : V \to W$ be a linear transformation. Since $T(e_1), T(e_2), \ldots, T(e_n)$ are vectors in W, each can be expressed as a linear combination of $f_1, f_2, \ldots, f_m$ uniquely as

$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i$

viz., $T(e_j) = a_{1j}f_1 + a_{2j}f_2 + \cdots + a_{mj}f_m \ (j = 1, 2, \cdots n)$

viz.,
$$T(e_1) = a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m$$
$$T(e_2) = a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m$$

$$T(e_n) = a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m$$

The co-ordinate vector of $T(e_j)$ relative to $\{f_i\}$ is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

The matrix of dimension $(m \times n)$ for by the co-ordinate vectors $T(e_j)$ $(j = 1, 2, \cdots, n)$ which determine T is called the *matrix of T relative to the bases e and f* and denoted by A_T or A.

Working rule: To get the matrix representation of T w.r.t. e and f, we have to express each $T(e_j)$ as a linear combination of $f_i'\,s$. The transpose of the matrix of coefficients in the above equations is the required matrix.

Definition: If $T : V \to V$, where V is a vectors space over the field F, the linear transformation T is called a *linear operator on V*

Note ✍

If T is a linear operator on a vector space V over the field F and if $e = \{e_1, e_2, \cdots, e_n\}$ is a basis of V, then the matrix of T relative to e is the $(n \times n)$ square matrix A whose elements of a_{ij} are defined by the equation $T(e_j) = \sum_{i=1}^{n} a_{ij}\, e_i \,(j = 1, 2, \cdots n)$(since $e = f$)

Also $[T(v)]_e = A[v)_e$ (1)

Since A depends on the basis e used, it is usually denoted by $[T]_e$.

Thus $[T(v)]_e = [T]_e[v]_e$ (2)

Eigenvalue and Eigenvectors

In equation (1) above, we shall take $[v]_e = X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $[T(v)]_e = Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, it takes the form $Y = AX$, where A is the square matrix $[a_{ij}]$ $(i, j = 1, 2, \cdots, n)$.

There are situation where certain column vectors are transformed into scalar multiples of themselves.

viz., $Y = AX = \lambda X$ (1), where λ is a scalar,

From (1), we get $AX = \lambda IX$ or $(A - \lambda I)X = 0$ (2), where I is the unit matrix of order n.

If X is a non-zero column vector, then λ is called an *eigenvalue* of A and X is the Eigenvalues of A and the corresponding eigenvectors are as follows:

If λ is an eigenvalue of A and X is the corresponding eighenvector, then $(A - \lambda I)$ $X = 0$

viz.,
$$\left.\begin{array}{l} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n = 0 \\ ------------------- \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{n2} - \lambda)x_n = 0 \end{array}\right\} \quad (3)$$

Equations (3) are a system of homogeneous linear equations in the unknowns x_1, x_2, $\cdots$, x_n (which are the element of the non-zero vector X). The condition for the system (3) to have a non-zero solution is

$$\begin{vmatrix} a_{11} - \lambda \; a_{12} - a_{1n} \\ a_{21} \; a_{22} - \lambda \cdots a_{2n} \\ ---------------- \\ a_{n1} \; a_{n2} \cdots a_{nn} - \lambda \end{vmatrix} = 0, \text{ viz., } |A - \lambda I| = 0 \quad (4)$$

The equation (3) is called *the characteristic equation* of A

The n roots of the characteristic equation are called the *eigenvaluses of* A.

Note ✍

(1) Corresponding to each value of λ, equation (2) possess a non-trivial solution which will be a one-parameter family of solutions. Hence the eigenvector corresponding to an eigenvalue is not unique.

(2) If all the eighenvalues of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.

(3) If two or more eigenvalues are equal, then the eigenvectors may be linearly independent or linearly dependent.

Properties of Eigenvalues

We state certain properties of eignevalues without proof: They may be verified in individual problems

1. A square matrix A and it transpose A^T have the same eighenvlaues.
2. The sum of the eigenvalues of A is equal to the trace of the matrix, viz., to the sum of the principal diagonal elements of A.

3. The product of the eigenvalues of a matrix A is equal to $|A|$. If $|A| = 0$, viz. If A is a singular matrix, at least one of the eigenvalues of A is zero and conversely.
4. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A, then
   (a) $k\lambda_1, k\lambda_2, \cdots k\lambda_n$ are the eigenvalues of kA, where k is a non-zero scalar.
   (b) $\lambda_1^p, \lambda_2^p, \cdots \lambda_n^p$ are the eigenvalues of A^p, where p is a $+ve$ integer.
   (c) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots \frac{1}{\lambda_n}$ are the eigen values of A^{-1}, viz., the inverse of A, provided $\lambda_r \neq 0$, viz., $A$is non-singular.
5. The eigenvalues of a real symmetric matrix, viz., symmetric matrix with real elements are real.
6. The eigen vectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

2.2 SIMILARITY TRANSFORMATION AND DIAGONALISATION

Definition: If A and B are $(n \times n)$ square matrices over F for which there exists an invertible (non-singular) $(n \times n)$ matrix P over F such that $B = P^{-1}AP$, B is said to be *similar* to A or B is said to be obtained from A by a *similarity transformation*.

Note ✍

(1) When B is similar to A, Ais similar B, for

$B = P^{-1}AP$; *viz.,* $PBP^{-1} = PP^{-1}APP^{-1} = IAI = A$

Assuming $P^{-1} = Q$, *this means that* $A = Q^{-1}BQ$

viz., A is similar to B.

(2) Since $A = [T]_e$ *and* $B = [T]_f$, *A and B represent the same linear operator T, if and only if they are similar to each other.*

Definition: A linear operator T is said to be *diagonalisable* if for some basis $\{e_i\}$ it is represented by a diagonal matrix.

viz., T is diagonalisable if and only if its matrix representation can be diagonalised by a similarity transformation.

Property (1) (Proof omitted)

If A is a square matrix with distinct eigenvalues and P is the matrix whose columns are the eighenvectors of A, then A can be diagonalised by the similarity transformation $P^{-1}AP = D$, where D is the diagonal matrix whose diagonal element are the eighenvalues of A.

Property (2)

If A is a real symmetric matrix, then the eigen vectors will be linearly independent and pairwise orthogonal. If we normalise each eigenvector X_r, viz., divide each element of X_r by the square root of the sum of the squares of all the elements of X_r and use the normalised eigenvector of A as columns of Q, then Q will be an orthogonal matrix such that $Q^{-1} = Q^T$.

The the similarity transformation $P^{-1}AP = D$ takes the form $Q^TAQ = D$. In this case, A is said to be diagonalised by an *orthogonal transformation*.

Worked Examples 2(b)

Example 1

If $T : R^3 \to R^2$ is the linear transformation defined by $T(x, y, z) = (2x + y - z, 3x - 2y + 4z)$, find the matrix of T relative to the bases e and f where $e \equiv \{e_1 = (1, 1, 1), e_2 = (1, 1, 0)$ and $e_3 = (1, 0, 0)$ and $f \equiv \{f_1 = (1, 3); f_2 = (1, 4)\}$. Also verify that $[T(v)]_f = [T]_e^f[v]_e$.

Let $(a, b) = k_1f_1 + k_2f_2 = k_1(1, 3) + k_2(1, 4)$

$$\therefore \quad k_1 + k_2 = a \tag{1}$$

$$3k_1 + 4k_2 = b \tag{2}$$

Solving (1) and (2), we get $k_1 = 4a - b$ and $k_2 = b - 3a$ $\therefore$ $(a + b) = (4a - b)f_1 + (b - 3a)f_2$

$$(a, b) = (4a - b)\, f_1 +)b - 3a)\, f_2$$

Using the definition of $T(x, y, z)$ and step (1),

$$T(e_1) = T(1, 1, 1) = (2, 5) = 3f_1 - f_2,$$

$$T(e_2) = T(1, 1, 0) = (3, 1) = 11f_1 - 8f_2,$$

$$T(e_3) = T(1, 0, 0) = (2, 3) = 5f_1 - 3f_2$$

$$[T]_e^t = \begin{pmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{pmatrix}$$

Let $(a, b, c) = k_1e_1 + k_2b_2 + k_3e_3 = k_1(1, 1, 1) + k_2(1, 1, 0) + k_3(1, 0, 0)$

$$\therefore \quad k_1 + k_2 + k_3 = a \tag{3}$$

$$k_1 + k_2 = b \tag{4}$$

$$k_1 = c \tag{5}$$

Solving (3), (4) and (5), we get $k_1 = c$, $k_2 = b - c$, $k_3 = a - b$

If $(a, b, c) = v \in R^3$, then $v = ce_1 + (b - c)e_2 + (a - b)\, e_3$

$$\therefore \quad [v]_e = \begin{pmatrix} c \\ b - c \\ a - b \end{pmatrix}$$

$$\therefore \quad T(v) = T(a, b, c) = [(2a + b - c), (3a - 2b + 4c)], \text{ using } T(x, y, z)$$

$$= [\{4(2a + b - c) - (3a - 2b + 4c)\}f_1 + (3a - 2b + 4c) - 3(2a + b - c)f_2, \text{ using (1)}$$

$$= (5a + 6b - 8c)f_1 + (-3a - 5b + 7c)f_2$$

$$\therefore \quad [T(v)]_f = \begin{pmatrix} 5a + 6b - 8c \\ -3a - 5b + 7c \end{pmatrix}$$

Now $[T]_e^f\,[v]_e = \begin{pmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{pmatrix}\begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix} = \begin{pmatrix} 5a+6b-8c \\ -3a-5b+7c \end{pmatrix} = [T(v)]_f$, which completes the verification.

Example 2

If T is the linear operator on R^2 defined by $T(x_1, x_2) = (-x_2, x_1)$, find the matrix of T in the basis $e = \{e_1 = (1, 2), e_2 = (1, -1)$

Since T is the linear operator, $T : R^2 \to R^2$ and hence $f = \{f_1 = (1, 2); f_2 = (1, -1)\}$.

Let $(a, b) = k_1(1, 2) + k_2(1, -1)$ we have $k_1 + k_2 = a$ and $2k_1 - k_2 = b$.

Solving these equation, we get $k_1 = \dfrac{a+b}{3}$ and $k_2 = \dfrac{2a-b}{3}$

$$\therefore \qquad (a, b) = \left[\left(\frac{a+b}{3}\right)(1, 2) + \left(\frac{2a-b}{3}\right)(1, -1)\right] = \frac{1}{3}(a+b)e_1 + \frac{1}{3}(2a-b)e_2$$

$$\therefore \qquad T(e_1) = T(1, 2) = (-2, 1) = -\frac{1}{3}e_1 - \frac{5}{3}e_2$$

$$T(e_2) = T(1, -1) = (1, 1) = \frac{2}{3}e_1 + \frac{1}{3}e_2$$

$$\therefore \qquad [T]_e = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$$

Example 3

The matrix $A = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}$ determines a linear transformation $T : R^3 \to R^2$, defined by $T(v) = Av$, where v is a column vector.

(i) Show that the matrix representation of T relative to the usual bases of R^3 and R^2 is A itself.

(ii) Find the matrix representation of T relative to the following bases of R^3 and R^2

$$e = \{e_1 = (1, 1, 1), e_2 = (1, 1, 0), e_3 = (1, 0, 0) \text{ and}$$

$$f = \{f_1 = (1, 3), f_2 = (2, 5)$$

(i) $$T(1, 0, 0) = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2f_1 + 2f_2$$

$$T(0, 1, 0) = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} = 5f_1 - 4f_2$$

$$T(0,0,1) = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix} = 3f_1 + 7f_2$$

$$\therefore \quad [T]_e^f = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix} = A$$

(ii) Let $(a, b) = k_1 f_1 + k_2 f_2 = k_1(1, 3) + k_2(2, 5)$ (1)

$\therefore\ k_1 + 2k_2 = a$ and $3k_1 + 5k_2 = b$

Solving these equation, we get $k_1 = 2b - 5a$ and $k_2 = 3a - b$

Since $[T]_e^f = A$, $T(e_1) = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = -12f_1 + 8f_2$, by (1)

$$T(e_2) = \begin{pmatrix} 2 & 5 & 3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} = -41f_1 + 24f_1,\ \text{by (1)}$$

and $$T(e_3) = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -8f_1 + 5f_2,\ \text{by (1)}$$

$$\therefore \quad [T]_e^f = \begin{pmatrix} -12 & -41 & -8 \\ 8 & 24 & 8 \end{pmatrix}$$

Example 4

If $T(x, y) = (2x - 3y, x + y)$, find $(T]_e$ where $e \equiv f\{e_1 = (1, 2), e_2(2, 3)\}$.
Verify also that $[T]_e[v]e = [T(v)]_e$ for any $v \in R^2$.

Let $(a, b) = k_1 e_1 + k_2 e_2 = k_1(1, 2) + k_2(2, 3)$

$\therefore$ $k_1 + 2k_2 = a$ and $2k_1 + 3k_2 = b$

Solving these equation, we get $k_1 = -3a + 2b$ and $k_2 = 2a - b$

$\therefore$ $(a, b) = (-3a + 2b)e_1 + (2a - b)e_2$ (1)

$$\therefore \quad \left.\begin{aligned} T(e_1) &= T(1, 2) = (-4, 3) = 18e_1 - 11e_2 \\ T(e_2) &= T(2, 3) = (-5, 5) = 25e_1 - 15e_2 \end{aligned}\right\}\text{using (1)}$$

$$\therefore \quad [T]_e = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix}$$

Let $(a, b) = v = (-3a + 2b)e_1 + (2a - b)e_2$

$$\therefore \quad [v]_e = \begin{bmatrix} -3a + 2b \\ 2a - b \end{bmatrix}$$

$$\therefore \qquad [T]_e[v]_e = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix} \begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$$

$$= [-54a + 36b + 50a - 25b, 33a - 22b - 30a + 15b]^T$$

$$= (-4a + 11b, 3a - 7b)^T \qquad (2)$$

Let $T(v) = (2a - 3b, a + b)$, by the definition of $T(x, y)$

$$= k_1e_1 + k_2e_2 = k_1(1, 2) + k_2(2,3)$$

$$\therefore \qquad k_1 + 2k_2 = 2a - 3b$$

$$2k_1 + 3k_2 = a + b$$

Solving these equation, we get $k_1 = -4a + 11b$ and $k_2 = 3a - 7b$

$$\therefore \qquad [T(v)]e = \begin{bmatrix} -4a+11b \\ 3a-7b \end{bmatrix} \qquad (3)$$

From (2) and (3), the result $[T(v)]_e = [T]_e \cdot [v]_e$ has been verified

Example 5

If V is the vector space of 2×2 matrices over R and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find the matrix of the linear operator on V in the usual basis when (i) $T(A) = MA$, (ii) $T(A) = AM$ and (iii) $T(A) = MA - AM$.

The usual basis is $E \equiv \left\{ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(i) $T(E_1) = ME_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = aE_1 + 0 \cdot E_2 + cE_3 + 0 \cdot E_4$

Similarly $T(E_2) = ME_2 = 0 \cdot E_1 + aE_2 + 0 \cdot E_3 + cE_4$

$T(E_3) = ME_3 = b \cdot E_1 + 0 \cdot E_2 + dE_3 + 0 \cdot E_4$

and $T(E_4) = ME_4 = 0 \cdot E_1 + bE_2 + 0 \cdot E_3 + dE_4$

$$\therefore \qquad [T]_E = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

(ii) When $T(A) = AM$, $[T]_E = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$

(iii) When $T(A) = MA - AM$, $[T]_E = \begin{pmatrix} 0 & -c & b & 0 \\ -b & (a-d) & 0 & b \\ c & 0 & (d-a) & -c \\ 0 & c & -b & 0 \end{pmatrix}$

Example 6

In the vector space of polynomials in x of degree ≤ 3 over R if $D : V \to V$ is the differential operator defined by $Df(x) = \frac{d}{dx} f(x)$, find the matrix of D in the basis $(1, x, x^2, x^3)$. Verify that $[D]_e \cdot [f(x)]_e = [Df(x)]_e$, where $f(x) = a + bx + cx_2 + dx_3$.

$$D(e_1) = D(1) = 0 = 0e_1 + 0e_2 + 0e_3 + 0e_4$$

$$D(e_2) = D(x) = 1 = e_1 + 0e_2 + 0e_3 + 0e_4$$

$$D(e_3) = D(x^2) = 2x = 0e_1 + 2e_2 + 0e_3 + 0e_4$$

$$D(e_4) = D(x^3) = 3x^2 = 0e_1 + 0e_2 + 3e_3 + 0e_4$$

$$\therefore \qquad [D]_e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As $f(x) = a + bx + cx^2 + dx^3$ viz., $f(x) = ae_1 + be_2 + ce_3 + de_4$, $[f(_x)]_2 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

As $Df(x) = b + 2cx + 3dx^2$ viz., $Df(x) = be_1 + 2ce_2 + 3de_3$, $[Df(_x)] = \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix}$

Now $[D]_e \cdot [F(x)_e = \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix} = [Df(x)]_e$. Verification is completes.

Example 7

If V is a two dimensional vector space over R and if T is a linear operator on V such that its matrix representation in the usual basis is $[T]_e = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that $T^2 - (a + d)T + (ad - bc)I = 0$.

$$T(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy)$$

$$T^2(x, y) = T\{T(x, y\} = T(ax + by, cx + dy)$$
$$= \{a(ax + by + b(cx + dy), c(ax + by) + d(cx + dy)\}$$
$$= \{(a^2 + bc)x + (ab + bd)y, (ac + dc)x + (bc + d^2)y\} \qquad (1)$$
$$-(a + d)T(x, y) = \{-(a + d)(ax + by), -(a + d)(cx + dy)\} \qquad (2)$$
$$(ad - bc)\, I(x, y) = \{(ad - bc)x, (ad - bc)y\} \qquad (3)$$

Adding (1), (2) and (3), we get $T^2 - (a + d)T + (ad - bc)I = 0$.

Example 8

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Verify that their sum and product are equal to the trace of A and $|A|$ respectively.

The characteric equation of A is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

viz., $(1 - \lambda)(\lambda^2 - 6\lambda + 8) = 0$. $\therefore$ The eigenvalues are 1, 2, 4.
When $\lambda = 1$, the eigenvector is given by

$$\left.\begin{aligned} 0\cdot x_1 + 2x_2 - x_3 &= 0 \\ \text{and } x_1 - x_2 + 2x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{4-1} = \frac{x_2}{-1-0} = \frac{x_3}{0-2}$$

$\therefore$ $$X_1 = (3, -1, -2)^T$$

When $\lambda = 2$, the eigenvector is given by

$$\left.\begin{aligned} -x_1 + 0\cdot x_2 + 0\cdot x_3 &= 0 \\ 0x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$\therefore$ $$X_2 = (0, 1, 1)^T$$

When $\lambda = 4$, the eigenvector is given by

$$\left.\begin{aligned} -3x_1 + 0x_2 + 0\cdot x_3 &= 0 \\ 0x_1 - x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0}$$

$\therefore$ $$X_3 = (0, -1, 1)^T$$

Sum of the eigenvalues = 1 + 2 + 4 = 7

= Trace of the matrix = 1 + 3 + 3

Product of the eigenvalue $= 8 = |A|$

Example 9

Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

viz., $(3 - \lambda)(2-\lambda)(5-\lambda) = 0$

$\therefore$ Eigenvalues of A are 2, 3, 5.

Now $$A^2 = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{pmatrix}$$

The characteristic equation of A^2 is $\begin{vmatrix} 9-\lambda & 5 & 38 \\ 0 & 4-\lambda & 42 \\ 0 & 0 & 25-\lambda \end{vmatrix} = 0$

viz., $(9-\lambda)(4-\lambda)(25-\lambda) = 0$

$\therefore$ The eigenvalues of A^2 are 4, 9, 25, which are the square of eigenvalue of A.

Let $$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{11} = cofactor of $a_{11} = 10$; $A_{12} = 0$; $A_{13} = 0$; $A_{21} = -5$; $A_{22} = 15$;
$A_{23} = 0$; $A_{31} = -2$; $A_{32} = -18$; $A_{33} = 0$ and $|A| = 30$

$\therefore$ $$A^{-1} = \frac{1}{30}\begin{bmatrix} 10 & -5 & -2 \\ 0 & 15 & -18 \\ 0 & 0 & 6 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2} & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

Characteristic equation of A^{-1} is $\begin{vmatrix} \frac{1}{3}-\lambda & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2}-\lambda & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5}-\lambda \end{vmatrix}$

viz., $\left(\frac{1}{3}-\lambda\right)\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{5}-\lambda\right)=0$

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$, which are the reciprocals of the eigenvalues of A.

Hence the two properties have been verified.

Example 10

Verify that the eigenvectors of the following real symmetric matrix are orthogonal in pairs.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The caracteristic equation of A is $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

viz., $\lambda^3 - 11\lambda^2 + 30\lambda - 36 = 0$

viz., $(\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore$ The eigenvalues of A are 2, 3, 6.

When $\lambda = 2$, the eigenvector is given by

$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \quad \therefore X_1 = (-1, 0, 1)^T$$

When $\lambda = 3$, the eigenvector is given by

$$\left.\begin{aligned} 0\cdot x_1 - x_2 + x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 0\cdot x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \quad \therefore X_2 = (1, 1, 1)^T$$

When $\lambda = 6$, the eigenvector is given by

$$\left.\begin{aligned} 3x_1 - x_2 + x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \end{aligned}\right| \therefore \frac{x_1}{2} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \quad \therefore X_3 = (1, -2, 1)^T$$

Now $X_1^T X_2 = [-1, 0 + 1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0;\ X_2^T X_3 = [1, 1, 1]\begin{bmatrix}1\\-2\\1\end{bmatrix} = 0$ and

$$X_3^T X_1 = [1, -2, 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = 0.\ \therefore \text{ Eigenvectors are orthogonal in pairs;}$$

Example 11

If λ is an eigenvalues of the matrix $A = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\-2 & 1 & 2\\1 & -2 & 2\end{bmatrix}$, verify that $\frac{1}{\lambda}$ is also an eigenvalue of A. Also verify that the eigenvalues are of unit modulus.

The characterstic equation of the matrix $3A$ is $0 = \begin{vmatrix}2-\lambda & 2 & 1\\-2 & 1-\lambda & 2\\1 & -2 & 2-\lambda\end{vmatrix}$

viz., $(2 - \lambda)(\lambda^2 - 3\lambda + 6) - 2(2\lambda - 4 - 2) + (4 - 1 + \lambda) = 0$

viz., $\lambda^3 - 5\lambda^2 + 15\lambda - 27 = 0$

viz., $(\lambda - 3)(\lambda^2 - 2\lambda + 9) = 0$

$\therefore\ \lambda = 3$ and $\lambda = \dfrac{2 \pm \sqrt{4 - 36}}{2}$ or $1 \pm i2\sqrt{2}$

$\therefore$ Eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = \dfrac{1 + i2\sqrt{2}}{3}$ and $\lambda_3 = \dfrac{i - i2\sqrt{2}}{3}$

Now $\dfrac{1}{\lambda_1} = 1 = \lambda_1;\ \dfrac{1}{\lambda_2} = \dfrac{3}{1 + i2\sqrt{2}} = \dfrac{3(1 - i2\sqrt{2})}{1 + 8} = \dfrac{i - i2\sqrt{2}}{3} = \lambda_3$

Similarly $\dfrac{1}{\lambda_3} = \lambda_2$.

Thus, when λ is an eigenvalue of A, $\dfrac{1}{\lambda}$ is also an eigenvalue of A.

Now $|\lambda_1| = 1$ and $|\lambda_2| = \left|\dfrac{1 \pm i2\sqrt{2}}{3}\right| = \sqrt{\dfrac{1}{9} + \dfrac{8}{9}} = 1$

Hence the eigenvalues of A are of unit modulus.

Note ✍

The two results verified above are properties of an orthogonal matrix. In fact, the matrix A is an orthogonal matrix as it satisfies the definition of an orthogonal matrix, namely $A \cdot A^T = A^T A = I$.

Example 12

Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ by similarity transformation and hence find A^4.

The characteristic equation of A is $\begin{bmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{bmatrix} = 0$

i.e., $(2-\lambda)(\lambda^2 + 2\lambda - 5) - 2(-6 - 2\lambda + 7) = 0$

i.e., $\lambda^3 - 13\lambda + 12 = 0$

i.e., $(\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$. i.e., $\lambda = -4, 1, 3$

When $\lambda = -4$, the eigenvector is given by

$$\left.\begin{aligned} 6x_1 + 2x_2 - 7x_3 &= 0 \\ 2x_1 + 5x_2 + 2x_3 &= 0 \\ 0 \cdot x_1 + x_2 + x_3 &= 0 \end{aligned}\right| \begin{aligned} &\therefore \frac{x_1}{4+35} = \frac{x_2}{-14-12} = \frac{x_3}{30-4} \\ &\text{i.e., } \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}. \text{ i.e., } X_1 = (3, -2, 2)^T \end{aligned}$$

When $\lambda = 1$, the eigenvector is given by

$$\left.\begin{aligned} x_1 + 2x_2 - 7x_3 &= 0 \\ 2x_1 + 0 \cdot x_2 + 2x_3 &= 0 \\ 0x_1 + x_2 - 4x_3 &= 0 \end{aligned}\right| \begin{aligned} &\therefore \frac{x_1}{4-0} = \frac{x_2}{-14-2} = \frac{x_3}{0-4} \\ &\text{i.e., } \frac{x_1}{1} = \frac{x_2}{-4} = \frac{x_3}{-1}. \text{ i.e., } X_1 = (1, -4, -1)^T \end{aligned}$$

When $\lambda = 3$, the eigenvector is given by

$$\left.\begin{aligned} -x_1 + 2x_2 - 7x_3 &= 0 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \\ 0x_1 + x_2 - 6x_3 &= 0 \end{aligned}\right| \begin{aligned} &\therefore \frac{x_1}{4-14} = \frac{x_2}{-14+2} = \frac{x_3}{2-4} \\ &\text{i.e., } \frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2}. \text{ i.e., } X_3 = (5, 6, 1)^T \end{aligned}$$

$\therefore$ The modal (diagonalising) matrix P is given by

$$P = \begin{bmatrix} 3 & 1 & 5 \\ -2 & -4 & 6 \\ 2 & -1 & 1 \end{bmatrix}; \text{Let } P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $A_{11} = 2$, $A_{12} = 14$, $A_{13} = 10$, $A_{21} = -6$, $A_{22} = -7$, $A_{23} = 5$

$$A_{31} = 26, A_{32} = -28, A_{33} = -10 \text{ and } |P| = 70$$

$$\therefore \qquad P^{-1} = \frac{1}{70}\begin{bmatrix} 2 & -6 & 26 \\ 14 & -7 & -28 \\ 10 & 5 & -10 \end{bmatrix}$$

$\therefore$ The required similarity transformation is

$$P^{-1}AP = D(-4, 1, 3)$$

$$\therefore \qquad A = PDP^{-1}$$

$$\therefore \qquad A^4 = PD^4P^{-1}$$

Now $$D^4P^{-1} = \begin{bmatrix} 256 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 2 & -6 & 26 \\ 14 & -7 & -28 \\ 10 & 5 & -10 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 512 & -1536 & 6656 \\ 14 & -7 & -28 \\ 810 & 405 & -810 \end{bmatrix}$$

and $$PD^4P^{-1} = \frac{1}{70}\begin{bmatrix} 3 & 1 & 5 \\ -2 & -4 & 6 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 512 & -1536 & 6656 \\ 14 & -7 & -28 \\ 810 & 405 & -810 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 5600 & -2590 & 15890 \\ 3780 & 5530 & -18060 \\ 1820 & -2660 & 12530 \end{bmatrix}$$

i.e., $$A^4 = \begin{bmatrix} 80 & -37 & 227 \\ 54 & 79 & -258 \\ 26 & -38 & -179 \end{bmatrix}$$

Example 13

Find the matrix P that diagonalises the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ by means of similarity transformation. Verify your answer.

The characteristic equation A is $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

i.e., $(2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + (\lambda - 1) = 0$

i.e., $$A^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

i.e., $$(\lambda - 1)^2 (\lambda - 5) = 0$$

$\therefore$ The eigenvalue are 5, 1, 1.

When $\lambda = 5$, the eigenvector is given by

$$\left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 - 3x_3 &= 0 \end{aligned}\right| \begin{aligned} &\therefore \ \frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2} \\ &\therefore \quad X_1 = (1, 1, 1)^T \end{aligned}$$

When $\lambda = 1$, the eigenvector is given by the same single equation, namely, $x_1 + 2x_2 + x_3 = 0$

Treating x_2 and x_3 as free variables and putting $x_2 = -1$ and $x_3 = 0$, we get $x_1 = 2$.

Putting $x_2 = 0$ and $x_3 = -1$, we $x_1 = 1$

$\therefore \quad X_2 = [2, -1, 0]^T$ and $X_3 = (1, 0, -1)^T$

$$\therefore \text{ the modal matrix } P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the cofactos are given by $A_{11} = 1$, $A_{12}, = 1$, $A_{13} = 1$, $A_{21} = 2$, $A_{22} = -2$, $A_{23} = 2$, $A_{31} = 1$, $A_{32} = 1$ and $A_{33} = -3$

$$|P| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4$$

$$\therefore \qquad P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$\therefore$ The required similarity transformation is

Verification:

$$AP = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 5 & -1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$$

$$\therefore \qquad P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 1 \\ 5 & -1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 14

Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation.

Verify your answer.

The characteristic equation of A is $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$

i.e., $(2-\lambda)(\lambda^2 - 2\lambda - 3) - (\lambda - 1) - (\lambda - 1) = 0$

i.e., $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$; i.e., $(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

$\therefore$ The eigenvalues of A are $-1, 1, 4$.

When $\lambda = -1$, the eigenvector is given by

$$\left.\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right| \quad \begin{aligned} &\therefore \frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1} \\ &\therefore \quad X_1 = (0, 1, 1)^T \end{aligned}$$

When $\lambda = 1$, the eigenvector is given by

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \\ -x_1 - 2x_1 + 0.x_3 &= 0 \end{aligned}\right| \quad \begin{aligned} &\therefore \frac{x_1}{-2+10} = \frac{x_2}{-1+2} = \frac{x_3}{0-1} \\ &\therefore \quad X_2 = (2, -1, 1)^T \end{aligned}$$

When $\lambda = 4$, the eigenvector is given by

$$\left.\begin{aligned} -2x_1 + x_2 - x_3 &= 0 \\ x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 - 3x_3 &= 0 \end{aligned}\right| \quad \begin{aligned} &\therefore \frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1} \\ &\therefore \quad X_3 = (1, 1, -1)^T \end{aligned}$$

Hence the modal matrix $P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

The normalised modal matrix Q is got by normalising each column vector of P, viz., by dividing the elements of each column vector by its norm.

Thus
$$Q=\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is $Q^T AQ = D(-1, 1, 4)$

Verification:

$$AQ=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}=\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

and
$$Q^T AQ=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}=D(-1,1,4)$$

Example 15

Diagonalise the matrix $A=\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation, Verify your answer.

The characteristic equation of A is $\begin{bmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{bmatrix}=0$

i.e., $(2-\lambda)(6-\lambda)(2-\lambda) - 16(6-\lambda) = 0$

i.e., $(6-\lambda)(\lambda^2 - 4\lambda - 12) = 0$

i.e., $(6-\lambda)(-6+\lambda)(\lambda+2) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -2, 6, 6$.

When $\lambda = -2$, the eigenvector is given by

$$\left.\begin{aligned} 4x_1 + 0x_2 + 4x_3 &= 0 \\ 0x_1 + 8x_2 + 0\cdot x_3 &= 0 \\ 4x_1 + 0x_2 + 4x_3 &= 0 \end{aligned}\right| \quad \begin{aligned} &\therefore \ \frac{x_1}{0-32} = \frac{x_2}{0-0} = \frac{x_3}{32-0} \\ &\therefore \quad X_1 = (1, 0, -1)^T \end{aligned}$$

When $\lambda = 6$, the eigenvector is given by

$$-4x_1 + 0\cdot x_2 + 4x_3 = 0$$

$$4x_1 + 0\cdot x_2 - 4x_3 = 0$$

or $$x_1 - x_3 = 0$$

$\therefore$ x_2 is free variable, but it must be so chosen that X_2 and X_3 are orthogonal among themselves and also each orthogonal with X (by the property or orthogonal matrix)

Taking $x_2 = 0$ arbitrarily, we choose $X_2 = (1, 0, 1)^T$

This choice makes X_1 and X_2 orthogonal.

To find X_3, we assume it as X_3 $(a, b, c)^T$

Since X_3 is orthogonal to X_1, $a - c = 0$ (1)

Since X_3 is orthogonal to X_2, $a + c = 0$ (2)

Solving (1) and (2), we get $a = c = 0$ and b arbitrary.

$\therefore$ Let $X_3 = (0, 1, 0)^T$

$\therefore$ The modal matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised modal matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

The required orthogonal transformation that diagonalises A is $Q^T AQ = D(-2, 6, 6)$

Verification

$$AQ = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

and $$Q^TAQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D(-2, 6, 6)$$

Note ✍

Had we assumed another values for x_2, say, 2, we would have got a different X_2. For example, if $X_2 = [1, 2, 1]^T$, X_3 would have [1, –1, 1]. In this case the modal matrix Q will be $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Exercise 2(B)

Part A (Short-Answer Questions)

1. Define matrix of a linear transformation relative to the bases e and f.
2. Define the matrix representation of a linear operator T on a vector space V.

Find the matrix representation of the following linear transformations relative to the usual basis of R^n:

3. $T : R^2 \to R^3$ defined by $T(x, y) = (3x - y, 2x + 4y, 5x - 6y)$
4. $T : R^4 \to R^2$ defined by $T(x, y, s, t) = (3x - 4y + 2s - 5t, 5x + 7y - s - 2t)$
5. $T : R^3 \to R^4$ defined by $T(x, y, z) = (2x + 3y - 8z, x + y + z, 4x - 5z, 6y)$
6. $T : R^3 \to R^2$ defined by $T(x, y, z) = (2x - 4y + 9z, 5x + 3y - 2z)$
7. $T : R^2 \to R^4$ defined by $T(x, y, z) = (3x + 4y, 5x - 2y, x + 7y, 4x)$
8. $T : R^3 \to R^3$ defined by $T(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z)$
9. Define eigenvalues and eigenvaectors of a square matrix.
10. State five properties of eigenvalues.
11. Define similarity transformation.
12. When are two matrices said to be similar?
13. What is meant by diagonalising a linear operator?
14. Define orthogonal transformation.
15. How will you derive an orthogonal matrix from a real symmetric matrix?

Part B

16. If $T : R^3 \to R^2$ is the linear transformation defined by $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$, find the matrix of T relative to the bases e and f, where $e \equiv \{e_1 = (1, 1, 1), e_2 = (1, 1, 0), e_3 = (1, 0, 0)\}$ and $f \equiv \{f_1 = (1, 3), f_2 = (2, 5)$. Also verify that $[T(v)]_f = [T]_e^f [v]_e$.
17. If $T : R^3 \to R^2$ is the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$, find the matrix of T relative to the bases e and f, where (i) $e \equiv \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $f \equiv \{(1, 0), (0, 1)\}$ (ii) $e = \{1, 0, -1), (1, 1, 1), (1, 0, 0)$ and $f \equiv \{(0, 1), (1, 0)\}$

18. If T is the linear operator on R^3, defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$, find the matrix of T in the basis $e \equiv \{e_1 = (1, 0, 1), e_2 = (-1, 2, 1)$ and $e_3 = (2, 1, 1)$. Also verify that $[T]_e[v]_e = [T(v)]_e$.
19. If T is the linear operator on R^3, defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$ Find $[T]_e$, where $e \equiv \{e_1 = (1, 1, 1), e_2 = (1, 1, 0)$ and $e_3 = (1, 0, 0)$
20. If $T(x, y) = (2y, 3x - y)$, find $[T]_e$, where $e \equiv \{e_1 = (1, 2)$ and $e_2 = (2, 5)$
21. If $T(x, y) = (5x + y, 3x - 2y)$, find $[T]_e$, where $e \equiv \{e_1 = (1, 3)$ and $e_2 = (1, 4)\}$
22. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and T is the linear operator on R^2, defined by $T(v) = Av$, where v is written as column, find the matrix of T in the (i) usual basis and (ii) basis $e \equiv \{(1, 3), (2, 5)$
23. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$
24. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$
25. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

    What can you infer about the matrix A from the eigenvalues? Verify your answer.
26. Verify that the sum and product of the eigenvalues of A are equal to the trace of A and $|A|$ respectively, given that $A = \begin{bmatrix} -15 & 4 & 1 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$
27. Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ by similarity transformation.
28. Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 1 & -5 & 3 \\ 0 & -6 & 4 \end{bmatrix}$ by similarity transformation.
29. Diagonalise the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ by orthogonal transformation.

30. Diagonalise the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ by orthogonal transformation

Answers

Exercise 2(A)

5. $T(a, b) = 5a - 2b$
15. $T(a, b, c) = 8a - 3b - 2c$
16. No
17. $\dim(R_T) = 2$ Basis $\equiv \{(1, 0, 1), (0, 1, -1)\}$; $\dim(N_T) = 1$, Basis $\equiv (3, -1, 1)$
18. $\dim(R_T) = 2$, Basis $\equiv \{(1, 0, 1), (0, 1, -2)$; $\dim(N_T) = 1$, Basis $\equiv (2, -1, 1)$
19. $\dim(R_T) = 1$, Basis $\equiv (1, 1)$; $\dim(N_T) = 1$, Basis $\equiv (1, -1)$
20. $\dim(R_T) = 2$, Basis .$\equiv \{(1, 0), (0, 1)\}$; $\dim(N_T) = 1$, Basis $\equiv (1, -1, 1)$
21. $\dim(N_T) = 2$, Basis $\equiv \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
22. $T(x, y, z) = (x + 2y, 2x, -4x - 3y)$
23. $\dim(R_T) = 2$, Basis $\equiv \{(1, 0, -1), (0, 1, 5)\}$; $\dim(N_T) = 1$, Basis $\equiv (1, 1, -1)$
24. $\dim(R_T) = 2$, Basis $\equiv \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$; $\dim(N_T) = 1$, Basis $\equiv \left\{ \begin{pmatrix} -4 \\ 2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ -5 \end{pmatrix} \right\}$
25. $\dim(R_T) = 3$ and $R_T = R^3$, Basis $\equiv \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$; $\dim(N_T) = 1$ Basis $\equiv \begin{pmatrix} -1 \\ \frac{2}{3} \\ 1 \\ 1 \end{pmatrix}$

Exercise 2(B)

3. $[T] = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 & 2 & -5 \\ 5 & 7 & -1 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 & -8 \\ 1 & 1 & 1 \\ 4 & 0 & -5 \\ 0 & 6 & 0 \end{bmatrix}$

6. $[T] = \begin{bmatrix} 2 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$

7 $[T] = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$

8. $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

16. $\begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

17. (i) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

18. $[T]_e = \dfrac{1}{4}\begin{bmatrix} 17 & 35 & 22 \\ -3 & 15 & -6 \\ -2 & -14 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

20. $\begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$

21. $\begin{bmatrix} 35 & 41 \\ -27 & -32 \end{bmatrix}$

22. (i) $[T]_e = A$

(ii) $[T]_e = \begin{bmatrix} 5 & -8 \\ 6 & 10 \end{bmatrix}$

23. 1, 3, –4; $(-2, 1, 4)^T$, $(2, 1, -2)^T$, $(1, -3, 13)^T$
24. 5, 1, 1; $(1, 1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
25. 0, 3, 15; $(1, 2, 2)^T$, $(2, 1, -2)^T$, $(2, -2, 1)^T$; A singular.
26. Eigenvalues are 5, –10, –20; Trace = –25; $|A| = 1000$
27. $D(1, 3, -4);\ P = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{bmatrix}$
28. $D(4, -2, -2);\ P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$
29. $D(1, 3, 4);\ Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$
30. $D(4, 1, 1);\ Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

Unit **3**

Inner Product Spaces

3.1 INNER PRODUCT – DEFINITION

If V is a vector space over F, viz., the field of real or complex numbers and there exists a function (or mapping), which assigns a scalar $(u,v) \in F$ corresponding to each ordered pair of vectors $u,v \in Y$, then (u,v) is called an *inner product in V,* provided it satisfies the following axioms:

(i) $(au_1 + bu_2, v) = a(u_1, v) + b(u_2, v)$

(ii) $(u, v) = \overline{(v, u)}$. If F is the field of real numbers, then $\overline{(v, u)} = (v,u)$

(iii) $(u, u) \geq 0$ and equality holds if $u = 0$ [$\because$ $(u,u) = \overline{(u,u)}$ and so real]

The vector space with an inner product is called *an inner product space*.

A finite dimensional real inner product space is called an *Euclidean space* and complex inner product space is called *a unitary space*.

Note ✍

(1) In axiom (i), if $v = av_1 + bv_2$, then $(u, av_1 + bv_2) = \bar{a}(u, v_1) + \bar{b}(u, v_2)$, for

$$(u, av_1 + bv_2) = \overline{(av_1 + bv_2, u)}, \text{ by axiom (ii)}$$
$$= \overline{a(v_1, u)} + \overline{b(v_2, u)}, \text{ by axiom (i)}$$
$$= \bar{a}\,\overline{(v_1, u)} + b\,\overline{(v_2, u)}$$
$$= \bar{a}(u, v_1) + \bar{b}(u, v_2), \text{ by axiom (ii)}$$

(2) If $u \equiv (a_1, a_2, \dots a_n)$ and $v \equiv (b_1, b_2, \dots b_n)$ over R^n, then, $(u, v) = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)$ is called the standard inner product in R^n. It is also called the scalar product or the dot product and denoted by u.v

(3) If $u = (z_1, z_2, \dots z_n)$ and $v = (w_1, w_2, \dots w_n)$ are in C^n, then $(u, v) = (z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n)$ is called the standard inner product in C^n, It is also denoted as (u/v).

(4) *The non-negative real number* $\sqrt{(u, u)}$ *is called the norm or length of u and denoted by* $\|u\|$. If $\|u\| = 1, u$ *is called a unit vector. To normalize u* $\in$ *V, we have to divide u by* $\|u\|$ and $\cos\theta = \dfrac{(u, v)}{\|u\| \cdot \|v\|}$ *where q is the angle between u and v. The non-negative real number* $d(u, v) = \|v - u\|$ *is called the distance between u and v. It can be verified that (i) d(u, v) ≥ 0 (equality holds good when u = v), (ii) d(u, v) = d (v, u) and (iii) d(u, v) ≤ d (u, w) + d(w, v).*

Theorem

If V is an inner product space, then for any vectors u, v in V and any scalar C,

(i) $\|cu\| = |c| \cdot \|u\|$

(ii) $\|u\| > 0,$ for $u \neq 0$

(iii) **Cauchy-schwraz inequality:** $|(u, v)| \leq \|u\|.\|v\|$

(iv) **Triangle Inequality:** $\|u + v\| \leq \|u\| + \|v\|$.

Proof

(i) $\|cu\|^2 = (cu, cu) = |c\bar{c}|(u, u)$

$$= |c|^2 \; \|u\|^2$$

$\therefore \quad \|cu\| = |c| \; \|u\|.$

(ii) By axiom (iii), $(u, u) \geq 0$, viz., $\|u\|^2 \geq 0 \quad \therefore \; \|u\| \geq 0.$

Also $\|u\| = 0$, only when $u = 0$

$\therefore \quad \|u\| > 0$, only when $u \neq 0$.

(iii) The inequality is valid when $u = 0$ (viz., $0 \leq 0$)

When $u \neq 0$, consider $\|u - (u, v)\, t\, v\|^2 \geq 0$, where t is real

$$= (u - (u, v)\, tv, u - (u, v)\, tv \geq 0$$

viz., $(u, v) - \overline{(u, u)}\; t\; (u, v) - t\; (u, v)\; (v, u) + (u, v)\; \overline{(u, v)}\, t^2\; (v, v) \geq 0$

viz., $\|u\|^2 - 2t\,|u, v)|^2 + |u, v)|^2\; t^2\; \|v\|^2 \geq 0 \qquad [\because\; (v, u) = \overline{(u, v)}$ and $z\bar{z} = |\bar{z}|^2]$

Now putting $t = \dfrac{1}{\|v\|^2}$ in (1), we get ...(1)

$$\|u\|^2 - \frac{|(u, v)|^2}{\|v\|^2} \geq 0$$

viz., $\quad \left|(u, v)^2\right| \leq \|u\|^2 \cdot \|v\|^2$

viz., $|(u, v)| \le \|u\| \cdot \|v\|$.

(iv) Consider $\|(u+v)\|^2 = (u+v, u+v)$

$= (u, u) + (u, v) + \overline{(u, v)} + (v, v)$

$= \|u\|^2 + 2R(u, v) + \|v\|^2$

$\le \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2$, since $R(u,v) \le |u, v|$ as $R(z)$ $\le |z|$ and so $R(u, v) \le \|u\| \cdot \|v\|$ by (iii)

$\le \{\|u\| + \|v\|\}^2$

$\therefore \quad \|u+v\| \le \|u\| + \|v\|$

Note ✍

(1) In the Cauchy-Schwarz inequality, the equality holds when u = (u, v)t v

i.e., $u = \dfrac{(u, v)v}{\|v\|^2}$

i.e., when u and v are linearly dependent.

(2) If u = (a_1, a_2, … a_n) and v = (b_1, b_2, … b_n), then (iii) becomes

$$(a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n)^2 \le \{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\}\{|b_1|^2 + |b_2|^2 + \cdots |b_n|^2\}$$

(3) If f and g are real continuous functions over $0 \le t \le 1$, then (iii) gives

$$\left[\int_0^1 f(t)g(t)dt\right]^2 \le \int_0^1 f^2(t)dt + \int_0^1 g^2(t)dt.$$

Orthogonality

Definition: The vectors $u, v \in$ an inner product space V are said to be *orthogonal*, if $(u, v) = 0$.

Note ✍

(1) If u is orthogonal to v, then $(u, v) = 0$

Now $(v, u) = \overline{(u, v)} = \bar{0} = 0$

$\therefore$ v is orthogonal to u.

(2) $0 \in V$ is orthogonal to every $v \in V$ for

$(0, v) = (0v, v) = 0\,(v, v) = 0$

(3) If u is orthogonal to ever $v \in V$, then $u = 0$,

for $(u, u) = 0 \;\therefore\; u = 0$, by axiom (iii)

Definition: A set (u_i) of vector in V is said to be *an orthogonal set*, if all pairs of distinct vectors are orthogonal, i.e., $(u_i, u_j) = 0$ when $i \ne j$.

Definition: The set $\{u_i\}$ of vectors in V is said to be an orthonormal set, if it is orthogonal and if $\|u_i\| = 1$ for each u_i.

For example, the standard basis (e_1, e_2, e_3) is an orthonormal set with respect to the standard inner product, for $\|e_i\| = 1$ and $(e_i, e_j) = 0$, when $i \ne j$.

Theorem

An orthogonal set of non-zero vectors (or an orthonormal set of vectors) is linearly independent.

Proof

Let $(u_1, u_2, \ldots, u_n)$ be an orthogonal set of vectors in a given inner product space.

Let $v = c_1u_1 + c_2u_2 + \ldots + c_nu_n$

Now $$(v, u_j) = \left(\sum_i c_i\ u_i, u_j\right)$$

$$= \sum_i c_i\ (u_i, u_j), \text{ by axiom (1)}$$

$$= c_j\ (u_j, u_j)$$

$\therefore$ $$c_j = \frac{(v, u_j)}{\|u_j\|^2};\ j = 1, 2, \cdots, m \text{ since } (u_j, u_j) \neq 0, \text{ as } u_j \text{ is a non-zero vector}$$

$\therefore$ when $v = 0$ c_j $(j = 1, 2, \ldots, m) = 0$

This means that $(u_1, u_2, \ldots, u_m)$ is linearly independent.

Corollary: If $v \in V$ then $v - \frac{(v, u_1)}{\|u_1\|^2} u_1 - \frac{(v, u_2)}{\|u_2\|^2} u_2 - \cdots - \frac{(v, u_m)}{\|u_m\|^2} u_m$ is orthogonal to each u_i.

for $$\left(v - \frac{(v, u_1)}{\|u_1\|^2} u_1 - \frac{(v, u_2)}{\|u_2\|^2} u_2 \cdots - \frac{(v - u_m)}{\|u_m\|^2} u_m, u_i\right) = (v, u_i) - \frac{(v, u_i)}{\|u_i\|^2}(u_i, u_i) = 0$$

3.2 GRAM–SCHMIDT ORTHOGONALISATION PROCESS

Constructions of an orthogonal basis for an inner product space can be done according to the following theorem which is stated below without proof.

If $(v_1, v_2, \ldots, v_n)$ is a basis of an inner product space V, then an orthogonal basis $(u_1, u_2, \ ,u_n)$ can be found out using the rules $u_1 = v_1$ and

$$u_{m+1} = v_{m+1} - \sum_{i=1}^{m} \frac{(v_{m+1}, u_i)}{\|u_i\|^2} u_i$$

Working rule for the construction of an orthogonal basis.

(1) $u_1 = v_1$

(2) $u_2 = v_2 - \dfrac{(v_2, u_1)}{\|u_1\|^2}, u_1$

(3) $u_3 = v_3 - \dfrac{(v_3, u_1)}{\|u_1\|^2} u_1 - \dfrac{(v_3, u_2)}{\|u_2\|^2} u_2$

(4) $u_4 = v_4 - \frac{(v_4, u_1)}{\|u_1\|^2}u_1 - \frac{(v_4, u_2)}{\|u_2\|^2}u_2 - \frac{(v_4, u_3)}{\|u_3\|^2}u_3$ and so on.

Adjoint of Linear Operations

Definition: A linear operator T on an inner product space V is said to have *an adjoint-operator T^* on V*, if $(T(u), v) = (u, T^*(v))$ for all $u, v \in V$.

Note ✍

(1) If V is finite dimensional, T exists for every T. This is not true, if V is infinite dimensional.*

(2) $[T^*]_e = \overline{[T]}_e^T$. *This will not be true, if e is an artillery basis of V.*

Theorem

If V is a finite dimensional inner product space and if T and S are linear operators on V and c is scalar, then

(i) $(T + S)^* = T^* + S^*$; (ii) $(cT)^* = \bar{c}\,T^*$,

(iii) $(TS)^* = S^*T^*$ (iv) $(T^*)^* = T$ and

(v) If T is invertible, T^* is also invertible such that $(T^{-1})^* = (T^*)^{-1}$

Proof

(i) If $u, v \in v$, then $((T + S)(u), v) = (T(u) + S(u), v)$

$= (T(u), v) + (S(u), v)$

$= (u, T^*(v)) + (u, S^*(v))$

i.e., $(u, (T + S)^*v) = (u, T^*(v) + S^*(v))$

$\therefore$ $(T + S)^* = T^* + S^*$, (since adjoint operator is unique)

(ii) $((cT)u, v) = (c\,T(u), v) = c(T(u),v) = c(u, T^*v) = (u, \bar{c}\,T^*(v))$

$\therefore$ $(cT)^* = \bar{c}\,T^*$

(iii) $(TS(u), v) = (T(S(u),v) = (S(u), T^*(v)$

$= (u, S^*(T^*(v)))$

$= (u, S^*T^*(v))$

$\therefore$ $(TS)^* = S^*T^*$

(iv) $(T^*(u), v) = \overline{(v, T^*(u))} = \overline{(T(v), u)} = \overline{(u, T(v))}$ $\therefore$ $(T^*)^* = T)$

(v) [**Note:** The linear operator T is said to be invertible, if $[T]_e$ is invertible, Equivalently $[T^{-1}]e\,[T]_e = [T]_e\,[T^{-1}]_e = I$

viz., $[T]_e^{-1} = [T^{-1}]_e]$

For every $u, v \in V$, $(I(u), v) = (u, v) = (u, I(v)$ $\therefore$ $I^* = I$

$\therefore$ $I = I^* = (TT^{-1})^* = (T^{-1})^*\,T^*$, by (iii)

$\therefore$ $(T^{-1})^* = (T^*)^{-1}$

Definitions

A linear operator T is called *self-adjoint*, if $T^* = T$.

A self-adjoint operator is called *Hermitian* in the complex case and *symmetric* in the real case.

A linear operator T is called *skew-adjoint*, if $T^* = -T$.

Proof: Any operator T can be expressed as the sum of a self-adjoint operator and a skew adjoint operator.

Property: Let S and U be any linear operators

Let $S = \frac{1}{2}(T + T^*)$ and $U = \frac{1}{2}(T - T^*)$, so that $T = S + U$

Now $S^* = \left(\frac{1}{2}(T + T^*\right)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T + T^*) = S$

$\therefore$ S is self-adjoint

$$U^* = \left(\frac{1}{2}(T - T^*)\right)^* = \frac{1}{2}(T^* - T) = \frac{-1}{2}(T - T^*) = -U$$

$\therefore$ S is skew-adjoint.

Worked Examples (3)

Example 1

If $u = (x_1, x_2)$ and $v = (y_1, y_2)$, prove that $(u, v) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$ is an inner product in R^2

Let $u' = (x'_1, x'_2)$, Then $au + bu' = (ax_1 + bx'_1, ax_2 + bx'_2)$

$\therefore$ $(au + bu'_1, v) = (ax_1 + bx'_1)y_1 - (ax_1 + bx'_1)y_2 - (ax_2 + bx_2')y_1 + (ax_2 + bx_2')y_2$

$$= a(x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2) + b(x'_1y_1 - x'_1y_2 - x_2'y_1 + 4x_2'y_2)$$

$$= a(u, v) + b(u', v)$$

$\therefore$ Axiom (1) of I.P. is satisfied

$$(v, u) = y_1x_1 - y_1x_2 - y_2x_1 + 4y_2x_2$$
$$= x_1y_1 - x_2y_1 - x_2y_2 + 4x_2y_2$$

$\therefore$ Axiom (2) is satisfied

$$(u, u) = x_1^2 - x_1x_2 - x_2x_1 + 4x_2^2$$

$$= (x_1 - x_2)^2 + 3x_2^2 \geq 0$$

Equality holds only when $x_1 = x_2 = 0$

$\therefore$ Axiom (3) is satisfied.

$\therefore$ (u, v) is an inner product in R^2.

Example 2

For what values of a, b, c, d, for which $f(u, v) = ax_1y_1 + bx_1y_2 + cx_2y_1 = dx_2y_2$ is an inner product on R^2?

Axioms (1) and (2) will hold good for all real values of a, b, c, d.

$$f(u,u) = ax_1^2 + b\,x_1x_2 + c\,x_1\,x_2 + d\,x_2^2.$$

$$= a\left[x_1^2 + \frac{(b+c)}{a}\,x_1x_2 + \frac{d}{a}\,x_2^2\right]$$

$$= a\left[\left\{x_1 + \left(\frac{(b+c)}{2a}\right)x_2\right\}^2 + \left\{\frac{d}{a} - \frac{(b+c)^2}{4a^2}\right\}x_2^2\right]$$

$$= a\left[\left\{x_1 + \left(\frac{(b+c)}{2a}\right)x_2\right\}^2 + \left\{\frac{4ad-(b+c)^2}{4a^2}\right\}x_2^2\right]$$

$$\geq 0, \text{ if } a > 0 \text{ and } 4ad > (b+c)^2$$

i.e., $ad > \dfrac{(b+c)^2}{4} > bc$, since $(b-c) > 0$ i.e., $b^2 + c^2 > 2bc$ i.e., $(b+c)^2 > 4bc$

$\therefore$ $f(u, v)$ is an inner product on R^2, if $a > 0$ and $ad - bc > 0$

Example 3

If V is a vector space of $m \times n$ matrices over R, prove that $(A, B) = Tr\,(B^TA)$ is an inner product in V.

$$(c_1A_1 + c_2A_2, B) = Tr\,\{B^T(c_1A_1 + c_2A_2)\}$$

$$= c_1Tr\,(B^TA_1) + c_2Tr\,(B^TA_2)$$

$$= c_1\,(A_1, B) + c_2\,(A_2, B)$$

i.e., Axiom (1) holds good

$(A, B) = Tr\,(B^TA)$ can be proved to be equal to $(B, A) = Tr\,(A^TB)$

$(A, A) = Tr\,(A^TA) = Tr\,(I) = n \geq 0$. Equality holds only when A is a null matrix

$\therefore$ $(A, B) = Tr\,(B^TA)$ is an inner product in R.

Example 4

If V is the vector space of real continuous functions on the real interval $a \leq t \leq b$, prove that $(f, g) = \int_0^1 f(t)g(t)\,dt$ is an inner product on V.

$$(c_1f_1 + c_2f_2, g) = \int_a^b (c_1f_1 + c_2f_2)g\;dt$$

$$= c_1\int_a^b f_1\,g\;dt + c_2\int_a^b f_2 g\;dt$$

$$= c_1(f_1, g) + c_2(f_2, g)t$$

$\therefore$ Axiom (1) holds good.

$$\int_a^b f(t)\, g(t)\, dt = \int_a^b g(t)\, f(t)\, dt.\ viz., (f, g) = (g, f)$$

$\therefore$ Axiom (2) holds good.

$$(f, f) = \int_a^b [f(t)]^2\, dt \geq 0, \text{since } a \leq f(t) \leq b)$$

$\therefore$ Axiom (3) holds good.

$\therefore$ (f, g) is an inner product

Example 5

If V is the vector space of $m \times n$ matrices over R, find the norm of $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ w.r.t. the inner product $(A, B) = Tr(B^+A)$

$$\begin{aligned} \|A\|^2 &= (A, A) = Tr(A^T A) \\ &= Tr\left\{\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\right\}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \\ &= Tr\begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix} \\ &= 10 + 20 = 30 \end{aligned}$$

$\therefore$ $\|A\| = \sqrt{30}$

Example 6

Find the vectors which form an orthogonal basis with the vectors (1, –2, 2,–3) and (2, –3, 2, 4) in R^4.

Let the required vector be $w = (x, y, z, t)$

w is orthogonal with $u = (1, -2, 2, -3)$

$\therefore$ $(u, w) = 0$. viz., $x - 2y + 2z - 3t = 0$ …(1)

w is orthogonal with $v = (2, -3, 2, 4)$

$\therefore$ $(v, w) = 0$. viz., $2x - 3y + 2z + 4t = 0$ …(2)

(2) – 2 ×(1) gives $y - 2z + 10t = 0$ …(3)

$\therefore$ z and t are free variables.

Putting $t = 0$, $z = 1$, we get $y = 2$ from (3); $x = 2$ from (2).

Putting $t = 1$, $z = 0$, we get $y = -10$ from (3); $x = -17$ from (2)

$\therefore$ The required vectors are (2, 2, 1, 0) and (–17, –10, 1, 0)

Example 7

Find the vectors which form an orthonormal basis with $u = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $v = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ in R^4.

Let the required vector be $w = (x, y, z, t)$

Then $(u, w) = 0$. viz., $\frac{x}{2} + \frac{y}{2} + \frac{z}{2} + \frac{t}{2} = 0$ …(1)

and $\quad (v, w) = 0$. viz., $\dfrac{x}{2} + \dfrac{y}{2} - \dfrac{z}{2} - \dfrac{t}{2} = 0$ ...(2)

(1) + (2) gives $x + y = 0$ and $z + t = 0$ from (1) or (2)

viz., $x = -y$ and $z = -t$

Taking $y = -1$ $x = 1$ $x = -1$ or $y = 1$: Taking $t = -1$, $z = 1$ or if $t = 1$, $z = -1$

$\therefore$ The orthogonal basis is given by (1, –1, –1, 1)and (1, –1, 1, –1)

The corresponding orthonormal basis is given by $\left(\dfrac{1}{\sqrt{4}}, -\dfrac{1}{\sqrt{4}}, -\dfrac{1}{\sqrt{4}}, \dfrac{1}{\sqrt{4}}\right)$

and $\left(\dfrac{1}{\sqrt{4}}, -\dfrac{1}{\sqrt{4}}, \dfrac{1}{\sqrt{4}}, -\dfrac{1}{\sqrt{4}}\right)$; viz., $\left(\dfrac{1}{2}, -\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{1}{2}\right)$

Example 8

Find an orthonormal basis of R^3, given that an arbitrary basis of R^3 is $\{v_1$ (1, 1,1), $v_2 = (0, 1, 1)$ and $v_3 = (0, 0, 1)\}$ using Gram-Schmidt process.

Let (u_1, u_2, u_3) be the required orthogonal basis.

Then by Gram-Schmidt process,

$$u_1 = v_1 = (1, 1, 1)$$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 = (0, 1, 1) - \frac{2}{(\sqrt{3})^2}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_1\|^2} u_2$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/2}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

The corresponding orthonormal basis (u_1', u_2', u_3') by normalizing each of (u_1, u_2, u_3)

$\therefore$ The orthonormal basis required is $\left\{u_1' = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right); u_2' = \dfrac{1}{\sqrt{6}}(-2, 1, 1);\right.$

$u_3' = \left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

Example 9

Find an orthogonal basis of the subspace of R^4, given than an arbitrary basis $\{v_1 = (2, 1, 3, -1)$, $v_2 = (7, 4, 3, -3)$, and $v_1 = (5, 7, 7, 8)$

Let (u_1, u_2, u_3, u_4) be the required orthogonal basis. Then, by G.S.O process, $u_1 = v_1 = (2, 1, 3, -1)$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 = (7, 4, 3, -3) - \frac{30}{15}(2, 1, 3, -1) = (3, 2, -3, -1)$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2^2\|^2} u_2$$

$$= (5, 7, 7, 8) - \frac{30}{15}(2, 1, 3, -1) - \frac{0}{23}(3, 2, -3, -1)$$

$$= (5, 7, 7, 8) - 2, (2, 1, 3, -1) = (1, 5, 1, 10)$$

$\therefore$ The required orthogonal basis is $\{u_1 = (2, 1, 3, -1); u_2 = (3, 2, -3, -1); u_3 = (1, 5, 1, 10)\}$

Example 10

Find an orthonormal basis of R^3, given that an arbitrary basis is $\{v_1\ (3, 0, 4)$, $v_2 = (-1, 0, 7)$, $v_3 = (2, 9, 11)$. Express (x, y, z) as a linear combination of the orthogonal basis vectors.

Let (u_1, u_2, u_3) be an orthogonal basis of R^3.

Then by G.S.O process, $u_1 = v_1 = (3, 0, 4)$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1^2\|} u_1 = (-1, 0, 7) - \frac{25}{25}(3, 0, 4) = (-4, 0, 3)$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1^2\|} u_1 - \frac{(v_3, u_2)}{\|u_2^2\|} u_2$$

$$= (2, 9, 11) - \frac{50}{25}(3, 0, 4) - \frac{25}{25}(-4, 0, 3)$$

$$= (2, 9, 11) - (6, 0, 8) - (-4, 0, 3)$$

$$= (0, 9, 0)$$

$\therefore$ The orthogonal basis is $\{u_1 = (3, 0, 4); u_2 = (-4, 0, 3); u_3 = (0, 9, 0)\}$

$\therefore$ The corresponding orthonormal basis is

$$\left\{u_1{}' = \left(\frac{3}{5}, 0, \frac{4}{5}\right); u_2{}' = \left(-\frac{4}{5}, 0, \frac{3}{5}\right); u_3{}' = (0, 1, 0\right\}$$

Let $u = k_1 u_1 + k_2 u_2 + k_3 u_3$ ...(1)

$\therefore$ $(u, u_1) = k_1 \|u_1{}^2\|$ [$\because$ $(u_2, u_1) = 0, (u_3, u_1) = 0$, by orthogonality]

$\therefore$ $k_1 = \dfrac{(u, u_1)}{\|u_1^2\|}$. Similarly, $k_2 = \dfrac{(u, u_1)}{\|u_2^2\|}$ and $k_3\ \dfrac{(u, u_3)}{\|u_0^2\|}$

If we take $u = (x, y, z\ )$ in (1), we get

$$(x, y, z) = \left(\frac{3x + 4z}{25}\right)(3, 0, 4) + \left(\frac{-4x + 3z}{25}\right)(-4, 0, 3) + \frac{9y}{81}(0,9,0) \qquad \text{...(2)}$$

(2) is the required linear combination of u_1, u_2, and u_3 equivalent to (x, y, z)

Example 11

Find an orthonormal basis of the subspace w of C^3, spanned by $v_1 = (1, i, 0)$ and v_2 $(1, 2, 1-i)$

Let u_1 and u_2 be the orthogonal basis of the subspace.

Then by GSO process, $u_1 = v_1 = (1, i, 0)$

and $$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 = (1, 2, 1-i) - \frac{(1-2i)}{2}(1, i, 0)$$

$$= \frac{1}{2}\{2(1, 2, 1-i) - (1-2i)(1, i, 0)$$

$$= \frac{1}{2}\{1+2i, 2-i, 2-i\}$$

The corresponding orthonormal basis given by $\left\{\frac{u_1}{\sqrt{2}}, \frac{u_2}{\frac{1}{2}\sqrt{18}}\right\}$

$$\text{viz., } \left\{u_1^1\left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0\right); u_2^1 = \frac{1}{\sqrt{18}}(1+2i, 2-i, 2-2i)\right\}$$

Example 12

If V is the vector space of polynomials over R of degree ≤ 2 with inner product $(f, g) = \int_0^1 f(t)\, g(t) dt$, (i) find a basis of the subspace W orthogonal to $f(t) = 2t + 1$ and (ii) apply G.S.O. process to the basis $(1, t, t^2)$ to find an orthonormal basis $\{u_1(t), u_2(t), u_3(t)\}$

(i) Let $g(t) = at^2 + bt + c$ be orthogonal to $f(t) = 2t + 1$.

Then $$\int_0^1 f(b)g(t)\, dt = \int_0^1 (at^2 + bt + c)\ (2t+1)dt = 0$$

$$\text{viz., } \int_0^1 [2at^3 + (a+2b)t^2 + (b+2c)t + c]\, dt = 0$$

$$\text{viz., } \frac{a}{2} + \frac{1}{3}(a+2b) + \frac{1}{2}(b+2c) + c = 0 \text{ or } 5a + 7b + 12c = 0 \qquad \ldots(1)$$

b and c are free variables in (1)

Taking $b = -5$ and $c = 0$, we get, from (1), $a = 7$

Taking $b = 0$ and $c = -5$, we get from (1), $a = 12$.

$\therefore$ $g_1(t) = 7t^2 - 5t$ and $g_2(t)$ $12t^2$ -5 form a basis of W, orthogonal to $f(t) = 2t + 1$.

(ii) By G.S.O process, $u_1(t) = v_1(t) = 1$

$$u_2(t) = v_2(t) - \frac{(v_2, u)}{\|u_1\|^2} u_1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1dt} \times 1 = t - \frac{1}{2}$$

$$u_3(t) = v_3(t) - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} u_2$$

$$= t^2 - \frac{\int_0^1 t^2\,dt}{\int_0^1 1dt}.1 - \frac{\int_0^1 t^2\left(t - \frac{1}{2}\right)}{\int_0^1 \left(t - \frac{1}{2}\right)^2} \cdot \left(t - \frac{1}{2}\right)$$

$$= t^2 - \frac{1}{3} - \frac{\left(\frac{1}{4} - \frac{1}{6}\right)}{\frac{1}{24} + \frac{1}{24}} \cdot \left(t - \frac{1}{2}\right) = t^2 - \frac{1}{3} - \left(t - \frac{1}{2}\right) = t^2 - t + \frac{1}{6}$$

Normalizing the vectors $u_1(t)$, $u_2(t)$, $u_3(t)$, we get the orthonormal basis .

$$u_1'(t) = 1; u_2'(t) = \frac{t - \frac{1}{2}}{\sqrt{\int_0^1 \left(t - \frac{1}{2}\right)^2 dt}} = \frac{t - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$= 2\sqrt{3}\left(t - \frac{1}{2}\right) = \sqrt{3}\,(2t - 1)$$

$$u_3'(t) = \left(t^2 - t + \frac{1}{6}\right) \div \sqrt{\int_0^1 \left(t^2 - t + \frac{1}{6}\right)^2 dt}$$

$$= \left(t^2 - t + \frac{1}{6}\right) \div \sqrt{\int_0^1 \left(t^4 + t^2 + \frac{1}{36} - 2t^3 - \frac{t}{3} + \frac{t^2}{3}\right) dt}$$

$$= \left(t^2 - t + \frac{1}{6}\right) \div \sqrt{\frac{1}{5} + \frac{4}{3} \cdot \frac{1}{3} - \frac{2}{4} - \frac{1}{6} + \frac{1}{36}}$$

$$= \left(t^2 - t + \frac{1}{6}\right) \div \frac{1}{6\sqrt{5}} = \sqrt{5}\,(6t^2 - 6t + 1)$$

Required orthonormal basis is $\{1, \sqrt{3}(2t - 1), \sqrt{5}\,(6t^2 - 6t + 1)\}$

Example 13

Define orthogonal projection of the vector $v \in V$ on W, which is a subspace of V. Find the orthogonal projection $v = (-10, 2, 8)$ on the subspace W spanned by $\{w_1 = (-1, 1, 1)$ and $w_2 = (1, -2, 2)\}$

Definition

If V is an inner product space and W a subspace of V and if $v \in V$ and $(w_1, w_2, \cdots w_r$ span W, then $\sum_{i=1}^{r} \frac{(v, w_i)}{\|w_i\|} w_i$ is called the orthogonal projection of v on W.

For the given problem, the orthogonal projection of $v = (-10, 2, 8)$ on

$$w = \frac{(v, w_1)}{\|w_1^2\|} w_1 + \frac{(v, w_2)}{\|w_2^2\|} w_2$$

$$= \frac{(10+2+8)}{3}(-1, 1, 1) + \frac{(-10-4+16)}{9}(1, -2, 2)$$

$$= \frac{(20)}{3}(-1, 1, 1) + \frac{2}{9}(1, -2, 2)$$

$$= \left\{\left(-\frac{20}{3} + \frac{2}{9}\right), \left(\frac{20}{3} - \frac{4}{9}\right), \left(\frac{20}{3} + \frac{4}{9}\right)\right\}$$

$$= \left(-\frac{58}{9}, \frac{56}{9}, \frac{64}{9}\right)$$

Example 14

Find the angle between $u = f(t) = 2t - 1$ and $v = g(t) = t^2$ in the space in which the inner product is defined by $(f, g) = \int_0^1 f(t)\, g(t)\, dt$.

If θ is the angle between the vector $u, v \in V$, then $\cos\theta = \frac{(u, v)}{\|u\| \cdot \|v\|}$

$$(u, v) = \{f(t), g(t)\} = \int_0^1 (2t-1)t^2\, dt = \frac{2}{4} - \frac{1}{3} = \frac{1}{6}$$

$$\|u\|^2 = \int_0^1 [f(t)]^2\, dt = \int_0^1 (4t^2 - 4t + 1)dt = \frac{4}{3} - 2 + 1 = \frac{1}{3}$$

$$\|v\|^2 = \int_0^1 [g(t)]^2\, dt = \int_0^1 t^4\, dt = \frac{1}{5}$$

$$\therefore \quad \cos\theta = \frac{\frac{1}{6}}{\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{5}}} = \frac{1}{6}\sqrt{15}$$

Example 15

If V is the vector space C^2 with standard inner product and if T is an adjoint operator on C^2 defined by $T(e_1) = (1, -2)$, $T(e_2) = (i, -1)$, find T^* assuming that $v = (x_1, x_2)$.

$$(x_1, x_2) = x_1(1, 0) + x_2(0, 1) = x_1 e_1 + x_2 e_2$$

$\therefore \quad T(x_1, x_2) = x_1(1, -2) + x_2\ (i, -1)$

$$= (x_1 + i\,x_2, -2x_1 - x_2)$$

$$[T]_e = \begin{pmatrix} 1 & i \\ -2 & -1 \end{pmatrix}$$

Since e is an orthonormal basis for c^2, $[T^*]_e = \overline{[T]_e}^T$

Now $\overline{[T]_e} = \begin{pmatrix} 1 & -i \\ -2 & -1 \end{pmatrix} \quad \therefore\ \overline{[T]_e}^T = \begin{pmatrix} 1 & -2 \\ -i & -1 \end{pmatrix} = [T^*]_e$

$\therefore \quad T^*(x_1, x_2) = (x_1 - 2x_2, -ix_1 - x_2$

Example 16

If $T : C^3 \to C^3$, defined by $T(x, y, z) = \{ix + (2 + 3i)y, 3x + (3 - i)z, (2 - 5i)\ y + iz\}$, find $T^*\ (x, y, z)$

$$[T]_e = \begin{pmatrix} i & (2+3i) & 0 \\ 3 & 0 & (3-i) \\ 0 & (2-5i) & i \end{pmatrix}$$

Since e is the stand and orthonormal basis for c^3, $[T^*]_e = \overline{[T]_e}^T$

Now $\overline{[T]_e} = \begin{pmatrix} -i & 2-3i & 0 \\ 3 & 0 & 3+i \\ 0 & 2+5i & -i \end{pmatrix}$

$$[T^*]_e = \overline{[T]}_e^T = \begin{pmatrix} -i & 3 & 0 \\ 2-3i & 0 & 2+5i \\ 0 & 3+i & -i \end{pmatrix}$$

$\therefore \quad T^*\ (x, y, z) = \{-ix + 3y, (2 - 3i)x + (2 + 5i)z, (3 + i)y - iz\}$

Example 17

If T is the linear operator on C^3 with the standard inner product whose matrix in the standard basis is defined by $A_{jk} = i^{j+k}$, where $i = \sqrt{-1}$ find a basis for the null space of T^*

$$[T]_e = \begin{pmatrix} i^2 & i^3 & i^4 \\ i^3 & i^4 & i^5 \\ i^4 & i^5 & i^6 \end{pmatrix} = \begin{pmatrix} -1 & -i & 1 \\ -i & 1 & i \\ 1 & i & -1 \end{pmatrix}$$

$\therefore \quad [T^*]_e = \overline{[T]_e}^T = \begin{pmatrix} -1 & i & 1 \\ i & 1 & -i \\ 1 & -i & -1 \end{pmatrix}$

$\therefore \quad T^*(x, y, z) = (-x + iy + z, ix + y - iz, x - iy - z)$

Let $(x, y, z) \in N_{T^*}$. Then $T^*(x, y, z) = (0, 0, 0)$

viz.,
$$-x + iy + z = 0$$
$$ix + y - iz = 0$$
$$x - iy - z = 0$$

viz., $-x + iy + z = 0 \quad \therefore \quad \dim(N_{T^*}) = 2$ and y and z are free variable.

$\therefore$ Basis of $N_{T^*} = \{(i, 1, 0); (1, 0, 1)\}$

Example 18

Express the linear operator $T(x, y, z) = (x + 2y, 3x - 4z, y)$ as the sum of a self-adjoint operator and a skew adjoint operator.

$$[T]_e = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & -4 & 0 \end{pmatrix} \quad \therefore \ \overline{[T]_e}^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -4 \\ 0 & 1 & 0 \end{pmatrix} = [T^*]_e$$

$\therefore \quad T^*(x, y, z) = \{x + 3y, 2x + z, -4y\}$

Let $S = \frac{1}{2}(T + T^*)$ and $U = \frac{1}{2}(T - T^*)$

$$S(x, y, z) = \frac{1}{2}\{T(x, y, z) + T^*(x, y, z)\}$$
$$= \left(\frac{2x + 5y}{2}, \frac{5x - 3z}{2}, -\frac{3y}{2}\right)$$

and
$$U(x, y, z) = \frac{1}{2}\{T(x, y, z) - T^*(x, y, z)\}$$
$$= (-y, x - 5z, 5y)$$

$$[S]_e = \begin{pmatrix} 1 & \frac{5}{2} & 0 \\ \frac{5}{2} & 0 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 0 \end{pmatrix} = \overline{[S]_e}^T = [S^*]_e \quad \therefore \ S \text{ is self-adjoint}$$

$$[U]_e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 5 \\ 0 & -5 & 0 \end{pmatrix} = -\overline{[U]_e}^T = -[U^*]_e \quad \therefore \ U \text{ is self-adjoint}$$

Hence the result.

Exercise 3

Part A (Short-Answer Questions)

1. Define inner product.
2. When is a vector space called inner product space.
3. Define standard inner product in R^n and C^n.
4. Show that, for any vector α in R^2, $\alpha = (\alpha, e_1)\, e_1 + (\alpha, e_2)\, e$.
5. Define the norm of a vector and find it when $u = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right) \in R^4$.
6. State Cauchy-Shwary inequality and use it to prove that
$$\left(\sum a_r \bar{b}_r\right)^2 \le \sum |a_r|^2 \times \sum |b_r|^2.$$
7. If f and g are real continuous functions over $0 \le t \le 1$, prove that
$$\left[\int_0^1 f(t)\, g(t)\, dt\right]^2 \le \int_0^1 f^2(t) dt.\ \int_0^1 g^2(t)\, dt.$$
8. Define orthonormal set of vectors and give an example.
9. State the working rule for construction of orthogonal basis from a given basis
10. Find a unit vector orthogonal to $u = (1, 1, 2)$ and $v = (0, 1, 3)$ in R^3.
11. Find the orthogonal projection of $v = (1, -1, 2)$ on $w = (0, 1, 1)$
12. Find the orthogonal projection of $v = (-10, 2, 8)$ on the subspace spanned by $w = (3, 12, -1)$
13. Define adjoint operator of a linear operator on an inner product space.
14. State the relation between a linear operator T and its adjoint T^*
15. If $T(x, y, z) = x + 2y + 3z$, prove that T is a self-adjoint operator.

Part B

16. If $u = (x_1, x_2)$ and $v = (y_1, y_2)$, prove that $f(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$ is an inner product.
17. For what value of k, is $f(u, v) = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$, where $u = (x_1, x_2)$ and $v = (y_1, y_2)$, an inner product?
18. If $\alpha = (1, 2)$ and $\beta = (-1, 1)$ are vector in R^2 such that $(\alpha, r) = -1$ and $(\beta, r) = 3$, find r where (α, r) and (β, r) are standard inner products in R^2.
19. Find the norm of $u = (1, 2) \in R^2$ w.r.t. (i) the standard inner product and (ii) the inner product defined by $(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$.
20. If V is the vector space of polynomials over R, find the norm of the vector $f(t) = t^2 - 2t + 3$ w.r.t. the inner product $\int_0^1 f(t)\, g(t)\, dt$.
21. If V is the vector space of real continuous functions in the interval $-\pi \le t \le \pi$, prove that $\{1, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots\}$ is an orthogonal set.
22. Find the vector which forms an orthonormal basis with $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ in R^3.

23. Find the vectors which form an orthogonal basis with (1, 1, 1, 2) and (1, 2, 3, –3) in R^4.
24. Find the vectors which form an orthogonal basis with $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ in R^4.
25. Find an orthonormal basis of R^3 with the standard inner product, given that an arbitrary basis is $\{v_1 = (1, 0, 1), v_2 = (1, 0, -1), v_3 = (0, 3, 4)\}$
26. Find a basis of the subspace W of R^4, orthogonal to $u_1 = (1, -2, 3, 4)$ and $u_2 = (3\ -5, 7, 8)$.
27. Find a basis of the subspace W of R^4, orthogonal to $u_1 = (1, 0, -1, 1)$ and $u_2 = (2, 3, -1, 2)$.
28. Construct an orthogonal basis of the subspace of R^4, spanned by (1, 2, 2, –1), (1, 1, – 5, 3) and (3, 9, 3, –7).
29. Construct an orthogonal basis of the subspace of R^4, spanned by (1, 1, –1, –2), (5, 8, – 2, –3) and (3, 9, 3, 8).
30. Find an orthonormal basis of the subspace W of C^3, spanned by $v_1 = (1, i, 1)$ and $v_2\ (1 + i, 0, 2)$.
31. Find an orthonormal basis of the subspace W of C^3, spanned by $(1, 0, i)$ and $v^2 = (2, 1, 1 + i)$.
32. Find the angle between the vector $u = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ in the space in which the inner product is defined as $(A, B) = Tr\ (B^T A)$.
33. If T is the linear operator on C^2 with standard inner product defined by $T(e_1) = (1 + i, 2)$ and $T(e_2) = (i, i)$ find $T^*\ (x_1, x_2)$.
34. If T is the linear operator on C^3 with standard inner product defined by $T(x, y, z) = \{2x + iy, y - 5\ iz, \underline{x} + (1-i)\ y + 3z\}$, find $T^*\ (x, y, \underline{z})$
35. Express the linear operator $T(x, y, z) = \{x - 2y, 2y - 3z, 3z - 4x\}$ as the sum of a self adjoint operator and a skew-adjoint operator.

Answers

Exercise 3

5. $\dfrac{\sqrt{65}}{12}$;

10. $\left(\dfrac{1}{\sqrt{11}}, -\dfrac{3}{11}, \dfrac{1}{\sqrt{11}}\right)$;

11. $\dfrac{1}{2}(0, 1, 1)$;

12. $-\dfrac{1}{11}(3, 12 - 1)$;

17. $k > 9$

18. $\left(-\frac{7}{3}, \frac{2}{3}\right)$;

19. $\sqrt{5}, \sqrt{13}$;

20. $\sqrt{\frac{83}{15}}$;

22. $\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$;

23. (1, –2, 1, 0 and (25, 4, –17, –6);

24. $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} - \frac{1}{2}\right)$ and $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

25. $\left(\frac{1}{\sqrt{2}}, 0\,\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0 - \frac{1}{\sqrt{2}}\right)$, (0, 1, 0) ;

26. $v_1 = (1, 2, 1, 0)$, $v_2 = (4, 4, 0, 1)$;

27. (–1, 0, 0, 1) and (– 3, 1, –3, 0);

28. $u_1 = (1, 2, 2, -1)$, $u_2 = (2, 3, -3, 2)$, $u_3 = (2, -1, -1, -2)$;

29. u_1 (1, 1,–1, –2), $u_2 = (2, 5, 1, 3)$, and $u_3 = (0, 0, 0, 0)$;

30. $u_1 = \frac{1}{\sqrt{3}}(1, i, 1)$ and $u_2 = \frac{1}{\sqrt{24}}(2i, 1-3i, 3-i)$;

31. $u_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$ and $u_2 = \frac{1}{2\sqrt{2}}(1+i, 2, 1-i)$;

32. $\cos\theta = \frac{2}{\sqrt{210}}$;(33) $T^* = \begin{pmatrix} 1-i & 2 \\ -i & -i \end{pmatrix}$

34. $T^* = \{2x + z, -ix + y + (1+i)z, 5iy + 3z\}$

35. $S(x, y, z) = \left\{x - y - 2z, -x + 2y - \frac{3}{2}z, -2x - \frac{3}{2}y + 3z\right\}$

$U(x, y, z) = \left\{-y + 2z, x - \frac{3}{2}z, -2x + \frac{3}{2}y\right\}$

Chapter **4**

Partial Differential Equations

4.1 INTRODUCTION

Partial differential equations are found in problems involving wave phenomena, heat conduction in homogeneous solids and potential theory. As an equation containing ordinary differential coefficients is called an ordinary differential equation, an equation containing partial differential coefficients is called a partial differential equation. Partial derivatives come into being only when there is a dependent variable which is a function of two or more independent variables. Hence in a partial differential equation, there will be one dependent variable and two or more independent variables. However we will mostly deal with partial differential equations containing only two independent variables. In what follows, z ill be taken as the dependent variable and z and y the independent variables so that $z = f(x, y)$. We will use the following standard notations to denote the partial derivatives:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} - r, \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

The *order* of a partial differential equation is that of the highest order derivative occurring in it.

4.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Thought our main interest is to solve partial differential equations, it will be advantageous if we know how partial differential equations are formed. Knowledge of the formation of partial differential equations will help us to distinguish between two kinds of solutions of the equation. Partial differential equations can be formed by eliminating either arbitrary constants or arbitrary functions from functional relations satisfied by the dependent and independent variables. When we form partial differential equations the following points may be considered for proper procedure and checking.

1. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the process of elimination results in a partial differential equation of the first order.

Note ✍

4. In the formation of ordinary differential equations, the order of the equation is equal to the number of constants eliminated.

2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to partial differential equation of second or higher order.

3. If the partial differential equation is formed by eliminating arbitrary functions, the order of the equation will be, in general, equal to the number of arbitrary functions eliminated.

4.3 ELIMINATION OF ARBITRARY CONSTANTS

By way of verifying point 3 of Section 4.2, let us consider the functional relation among

$$x, y, z, \text{ i.e. } f(x, y, z, a, b) = 0 \tag{1}$$

where a and b are arbitrary constants to be eliminated.

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0, \text{ i.e. } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \tag{2}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0, \text{ i.e. } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \tag{3}$$

Equations (2) and (3) will contain a and b.

If we eliminate a and b from equations (1), (2) and (3), we get partial differential equation (involving p and q) of the first order. This justifies point 1 of Section 4.2.

4.4 ELIMINATION OF ARBITRARY FUNCTIONS

By way of verifying point 3 of Section 4.2 above, let us consider the relation

$$f(u, v) = 0 \tag{1}$$

where u and v are functions of x, y, z and f is an arbitrary function to be eliminated. Differentiating (1) partially with respect to x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \tag{2}$$

[since u and v are functions of x, y, z and z is z is in turn, a function of x, y]
Differentiating (2) partially with respect to y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0 \tag{3}$$

Instead of eliminating, f, let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3).

From (2) and (3), we get

$$\frac{u_x + u_z p}{u_y + u_z q} = \frac{v_x + v_z p}{v_y + v_z q}, \text{ where } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

i.e. $$u_x v_y + u_x v_z q + u_z v_y p = u_y v_x + u_y v_z p + u_z v_x q$$

i.e. $$(u_y v_z - u_z v_y)p + (u_z v_z - u_x v_z)q = (u_x v_y - u_y v_x) \quad (4)$$

i.e. $Pp + Qq = R$, say, where P, Q and R are functions of x, y, z.

Now equation (4) is a partial differential equation of order 1.

This justifies point 3 of Section 4.2.

Note ✍

1. *To verify point 3 of Section 4.2, we could have taken a functional relation containing a function of one argument, but we have shown that the order of the partial differential equation formed depends only on the number of arbitrary functions eliminated and not on the number of arguments of the function.*
2. *The equation (4) is called Lagrange's linear equation, whose solution will be discussed later.*

Worked Examples 4(a)

Example 1

Form the partial differential equation by eliminating the arbitrary constants a and b from the following.

(i) $\log z = a \log x + \sqrt{1-a^2} \log y + b$

(ii) $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$

(i) $$\log z = a \log x + \sqrt{1-a^2} \log y + b \quad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$\frac{1}{z} p = \frac{a}{x} \quad (2)$$

and $$\frac{1}{z} q = \frac{\sqrt{1-a^2}}{y} \quad (3)$$

If we ignore (1), b is eliminated.

Form (2), $a = \dfrac{px}{z}$ and using this in (3), we get

$$\frac{1}{z^2}q^2 = \frac{1}{y^2}\left\{1 - \frac{p^2x^2}{z^2}\right\}$$

i.e. $$\frac{p^2x^2}{z^2} + \frac{q^2y^2}{z^2} = 1$$

or $$p^2x^2 + q^2y^2 = z^2$$

(ii) $$(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha \quad (1)$$

Differentiating (1) partially with respect to x and then respect to y, we get

$$2(x-a) = 2zp \cot^2 \alpha \quad (2)$$

and $$2(y-b) = 2zq \cot^2 \alpha \quad (3)$$

Using (2) and (3) in (1), we have

$$z^2(p^2+q^2)\cot^4 \alpha = z^2 \cot^2 \alpha$$

i.e. $$p^2 + q^2 = \tan^2 \alpha$$

Example 2

Form the partial differential equation by eliminating the arbitrary constants a and b from the following.

(i) $\sqrt{1+a^2}\,\log(z+\sqrt{z^2-1}) = x + ay + b$

(ii) $z = \dfrac{1}{2}x\sqrt{x^2+a^2} + \dfrac{1}{2}y\sqrt{y^2-a^2} + \dfrac{a^2}{2}\log\left\{\dfrac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}\right\} + b$

(i) $$\sqrt{1+a^2}\,\log(z+\sqrt{z^2-1}) = x + ay + b \quad (1)$$

Differentiating (1) partially with respect to z and then with respect to y, we get

$$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1+\frac{z}{\sqrt{z^2-1}}\right\}p = 1$$

i.e. $$\sqrt{1+a^2}\cdot p/\sqrt{z^2-1} = 1 \quad (2)$$

and $$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1+\frac{z}{\sqrt{z^2-1}}\right\}q = a$$

i.e. $$\sqrt{1+a^2}\cdot\frac{q}{\sqrt{z^2-1}} = a \quad (3)$$

From (2) and (3), we get $\frac{p}{q} = \frac{1}{a}$ (4)

Using (4) in (2), we get

$$\sqrt{1 + \frac{q^2}{p^2}} \cdot p = \sqrt{z^2 - 1}$$

i.e. $$\sqrt{p^2 + q^2} = \sqrt{z^2 - 1} \quad \text{or} \quad p^2 + q^2 + 1 = z^2$$

(ii) $$z = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} y\sqrt{y^2 - a^2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2})$$

$$-\frac{a^2}{2} \log(y + \sqrt{y^2 - a^2}) + b \qquad (1)$$

Differentiating (1) partially with respect to x,

$$p = \frac{1}{2}\left\{x \cdot \frac{x}{\sqrt{x^2 + a^2}} + \sqrt{x^2 + a^2}\right\} + \frac{a^2}{2} \cdot \frac{1}{x + \sqrt{x^2 + a^2}}\left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$

$$= \frac{1}{2}\left[\frac{2x^2 + a^2}{\sqrt{x^2 + a^2}} + \frac{a^2}{\sqrt{x^2 + a^2}}\right] = \sqrt{x^2 + a^2} \qquad (2)$$

Similarly, differentiating (1) partially with respect to y, we get

$$q = \sqrt{y^2 - a^2}$$

From (2) and (3),

$$p^2 - x^2 = y^2 - q^2$$

i.e. $$p^2 + q^2 = x^2 + y^2$$

Example 3

Form the partial differential equation by eliminating the arbitrary constants a, b and c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

We note that the number of constants is more than the number of independent variables. Hence the order of the resulting equation will be more than 1.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} p = 0 \qquad (2)$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2} q = 0 \qquad (3)$$

Differentiating (2) partially with respect to x,

$$\frac{1}{a^2}+\frac{1}{c^2}(zr+p^2) \tag{4}$$

where $$r = \frac{\partial^2 z}{\partial x^2}$$

From (2), $$-\frac{c^2}{a^2} = \frac{zp}{x} \tag{5}$$

From (4), $$\frac{-c^2}{a^2} = zr+p^2 \tag{6}$$

From (5) and (6), we get

$xz\dfrac{\partial^2 z}{\partial x^2}+x\left(\dfrac{\partial z}{\partial x}\right)^2 = z\dfrac{\partial z}{\partial x}$ which is the required partial differential equation. This is not the only way of eliminating a, b and c. Had we differentiated (2) partially with respect to y, we would have got

$$\frac{2}{c^2}\{zs+pq\} = 0, \text{ where } s = \frac{\partial^2 z}{\partial x \partial y}$$

i.e. $$z\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = 0$$

which is also a partial equation corresponding to (1).

If we differentiate (3) partially with respect to y and eliminate b and c, we will get yet another partial differential equation, namely

$$yz\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0$$

Example 4

Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is unity.

The equation of a plane which cuts off intercepts a, b, c on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \tag{1}$$

If sum of the intercepts is unity, $a + b + c = 1$ or

$$c = 1 - a - b \tag{2}$$

Using (2) in (1), we get the equation of a plane, the sum of whose x, y, z-intercepts is unity as

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b} = 1$$

or $$b(1 - a - b)x + a(1 - a - b)y + abz = ab(1 - a - b) \tag{3}$$

If a and b are treated as arbitrary constants, (3) represents the family of planes having the given property. Differentiating (3) partially with respect to x and then with respect to y, we have

$$b(1 - a - b) + abp = 0 \quad \text{or} \quad 1 - a - b = -ap \tag{4}$$

and $$a(1 - a - b) + abq = 0 \quad \text{or} \quad 1 - a - b = -bq \tag{5}$$

From (4) and (5), we get

$$ap = bq \quad \text{or} \quad \frac{a}{q} = \frac{b}{p} = k \tag{5}$$

Using (6) in (4), $1 - k(p + q) = -kpq$

i.e. $$k = \frac{1}{p+q-pq}$$

$\therefore$ $$a = \frac{q}{p+q-pq},\ b = \frac{p}{p+q-pq} \text{ and } 1-a-b = \frac{-pq}{p+q-pq}$$

Using these values in (3), we have

$$-k^2p^2qx - k^2pq^2y + k^2pqz = -k^3p^2q^2$$

i.e. $$-px - qy + z = -kpq$$

or $z = px + qy - \dfrac{pq}{p+q-pq}$, which is the required partial differential equation.

Example 5

Find differential equation of all planes which are at a constant distance k from the origin.

The equation of a plane which is at a distance k from the origin is

$$x \cos a + y \cos \beta + z \cos \nu = k$$

where $\cos \alpha$, $\cos \beta$, $\cos \nu$ are the direction cosines of a normal to the plane.

Taking $\cos \alpha = a$, $\cos \beta = b$ and $\cos \nu = c$ and noting that $a^2 + b^2 + c^2 = 1$, the equation of the plane can be assumed as

$$ax + by + \sqrt{1 - a^2 - b^2 z} = k \tag{1}$$

If a and b are treated as arbitrary constants, equation (1) represents all planes having the given property.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$a + \sqrt{1 - a^2 - b^2 p} = 0 \tag{2}$$

and $$b + \sqrt{1 - a^2 - b^2 q} = 0 \tag{3}$$

From (2) and (3), $\frac{a}{p} = \frac{b}{q} = -\sqrt{1-a^2-b^2} = \lambda$ say

$\therefore$ $a = \lambda p$, $b = \lambda q$ and $\sqrt{1-\lambda^2(p^2+q^2)} = -\lambda$

i.e. $$1 - \lambda^2(p^2 + q^2) = \lambda^2$$

$\therefore$ $$\lambda^2 = \frac{1}{1+p^2+q^2} \text{ or } \lambda = -\frac{1}{\sqrt{1+p^2+q^2}}$$

($\because$ λ is negative, as $\lambda = -\sqrt{1-a^2-b^2}$)

Using these values in (1), we get

$$\lambda px + \lambda qy - \lambda z = k$$

i.e. $$z = px + qy - \frac{k}{\lambda} \text{ or}$$

$z = px + qy + k\sqrt{1+p^2+q^2}$, which is the required partial differential equation.

Example 6

Find the differential equation of all spheres of the same radius c having their centres on the yoz-plane.

The equation of a sphere having its centre at (0, a, b), that lies on the yoz-plane and having its radius equal to c is

$$x^2 + (y - a)^2 + (z - b)^2 = c^2 \tag{1}$$

If a and b are treated as arbitrary constants, (1) represents the family of spheres having the given property.

$$2x + 2(z - b)q = 0 \tag{2}$$

and $$2(y - a) + 2(z - b)q = 0 \tag{3}$$

From (2), $$z - b = -\frac{x}{p} \tag{4}$$

Using (4) in (3), $$y - a = \frac{qx}{p} \tag{5}$$

Using (4) and (5) in (1), we get

$$x^2 + \frac{q^2x^2}{p^2} + \frac{x^2}{p^2} = c^2$$

i.e. $(1 + p^2 + q^2)x^2 = c^2 p^2$, which is the required partial differential equation.

Example 7

Find the differential equation of all spheres whose centres lie on the x-axis.

The equation of any sphere whose centre is $(a, 0, 0)$ (that lies on the x-axis) and whose radius is b is

$$(x - a)^2 + y^2 + z^2 = b^2 \tag{1}$$

If a and b are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to z and then with respect to y, we have

$$2(x - a) + 2zp = 0 \tag{2}$$

$$2y + 2zq = 0 \tag{3}$$

The required equation is provided by (3).

i.e. it is $z\dfrac{\partial z}{\partial y} + y = 0$

Example 8

Find the differential equation of all spheres whose radii are the same.

The equation of all spheres with equal radius can be taken as

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2 \tag{1}$$

where a, b, c are arbitrary constants and R is a given constant.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$(x - a) + (z - c)p = 0 \tag{2}$$

$$(y - b) + (z - c)q = 0 \tag{3}$$

Differentiating (2) and (3) with respect to x and y respectively, we get

$$1 + (z - c)\,r + p^2 = 0 \tag{4}$$

and

$$1 + (z - c)\,t + q^2 = 0 \tag{5}$$

Eliminating $(z - c)$ from (4) and (5), we have

$$\frac{r}{t} = \frac{1+p^2}{1+q^2}$$

i.e. $r(1 + q^2) = t(1 + p^2)$, where $r = \dfrac{\partial^2 z}{\partial x^2}$ and $t = \dfrac{\partial^2 z}{\partial y^2}$.

Note ✍

The answer is not unique. We can get partial differential equations.

Example 9

Form the partial differential equation by eliminating the arbitrary function 'f' from

(i) $z = e^{ay} f(x + by)$; and

(ii) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

(i)
$$z = e^{ay} . f(x + by)$$

i.e.
$$e^{-ay} z = f(x + by) \tag{1}$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$e^{-ay} p = f'(u) \cdot 1 \tag{2}$$

$$e^{-ay} q - ae^{-ay} z = f'(u) b \tag{3}$$

where $u = x + by$

Eliminating $f'(u)$ from (2) and (3), we get

$$\frac{q - az}{p} = b$$

i.e.
$$q = az + bp$$

(ii)
$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

i.e.
$$z - y^2 = 2f\left(\frac{1}{x} + \log y\right)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$p = 2f'(u) \cdot \left(\frac{-1}{x^2}\right) \tag{2}$$

and
$$q - 2y = 2f'(u) \cdot \left(\frac{1}{y}\right) \tag{3}$$

where $u = \frac{1}{x} + \log y$

Dividing (2) by (3), we have

$$\frac{p}{q - 2y} = \frac{-y}{x^2}$$

i.e.
$$px^2 + qy = 2y^2$$

which is the required partial differential equation.

Example 10

From the partial differential equation by eliminating function 'f' from

(i) $xy + yz + zx = f\left(\dfrac{z}{x+y}\right)$ and

(ii) $f(z - xy, x^2 + y^2) = 0$

(i) $$xy + yz + zx = f\left(\frac{z}{x+y}\right) \quad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we have

$$y + yp + xp + z = f'(u)\left\{\frac{(x+y)p - z}{(x+y)^2}\right\} \quad (2)$$

and $$y + yp + z + xq = f'(u)\left\{\frac{(x+y)q - z}{(x+y)^2}\right\} \quad (3)$$

Dividing (2) by (3), we have

$$\frac{(y+z)+(x+y)p}{(z+x)+(x+y)q} = \frac{(x+y)p - z}{(x+y)\,q - z}$$

i.e. $$(x + y)(z + x)\,p - z(z + x) - z(x + y)q$$
$$= (x + y)\,(y + z)q - z(y + z) - z(x + y)p$$

i.e. $$(x + y)(x + 2z)\,p - (x + y)\,(y + 2z)q = z(x - y)$$

which is a Lagrange linear equation.

(ii) $f(z - xy, x^2 + y^2) = 0$ (1)

$$f(u \cdot v) = 0$$

If we assume that u can be expressed as a single-valued function of v, (1) can be rewritten as

$$z - xy = \phi(x^2 + y^2) \quad (2)$$

where ϕ is an arbitrary function.

Differentiating (2) partially with respect to x and then with respect to y, we have

$$p - y = \phi'(u)\cdot 2x \quad (3)$$

and $$q - x = f'(u)\cdot 2y \quad (4)$$

Eliminating $\phi'(u)$ from (3) and (4), we get

$$\frac{p-y}{q-x} = \frac{x}{y} \text{ or } yp - xq = y^2 - x^2$$

Note ✍

Without assuming that u = ϕ(v), we can eliminate 'f' and form the equation alternatively as given in the following example.

Example 11

Form the differential equation by eliminating 'f' from

(i) $f(z-xy, x^2+y^2)=0$ and

(ii) $f(x^2+y^2+z^2, ax+by+cz)=0$

(i) $\qquad f(z-xy, x^2+y^2)=0 \qquad (1)$

By putting $z-xy=u$ and $x^2+y^2=v$, (1) becomes

$$f(u, v)=0 \tag{2}$$

Differentiating (2) partially with respect to x and then with respect to y, we have

$$\frac{\partial f}{\partial u}\cdot(p-y)+\frac{\partial f}{\partial v}(2x)=0 \tag{3}$$

and

$$\frac{\partial f}{\partial u}(q-x)+\frac{\partial f}{\partial v}(2y)=0 \tag{4}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} p-y & 2x \\ q-x & 2y \end{vmatrix}=0$$

i.e. $\qquad 2y(p-y)-2x(q-x)=0$

or $\qquad yp-xq=y^2-x^2$

(ii) $\qquad f(x^2+y^2+z^2, ax+by+cz)=0 \qquad (1)$

Putting $u=x^2+y^2+z^2$ and $v=ax+by+cz$, (1) becomes

$$f(u, v)=0 \tag{2}$$

Differentiating (2) partially with respect to x and then with respect to y, we have

$$\frac{\partial f}{\partial u}(2x+2zp)+\frac{\partial f}{\partial v}(a+cp)=0 \tag{3}$$

and

$$\frac{\partial f}{\partial u}(2y+2zq)+\frac{\partial f}{\partial v}(b+cq)=0 \tag{4}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} x+zp & a+cp \\ y+zq & b+cq \end{vmatrix}=0$$

i.e. $\qquad (x+zp)(b+cq)=(y+zq)(a+cp)$

i.e. $\qquad (cy-bz)p+(az-cx)q=b-ay$

Example 12

Form the partial differential equation by eliminating the arbitrary functions f and g from $z = f(2x + y) + g(3x - y)$

$$z = f(2x + y) + g(3x - y) \tag{1}$$

Differentiating (1) partially with respect to x,

$$p = f'(u)\cdot 2 + g'(v)\cdot 3 \tag{2}$$

where $u = 2x + y$ and $v = 3x - y$

Differentiating (1) partially with respect to y,

$$q = f'(u)\cdot 1 + g'(v)\,(-1) \tag{3}$$

Differentiating (2) partially with respect to x and then with respect to y,

$$r = f''(u)\cdot 4 + g''(v)\cdot 9 \tag{4}$$

and

$$s = f''(u)\cdot 2 + g''(v)\cdot(-3) \tag{5}$$

Differentiating (3) partially with respect to y,

$$t = f''(u)\cdot 1 + g''(v)\cdot 1 \tag{6}$$

Eliminating $f''(u)$ and $g''(v)$ from (4), (5) and (6) using determinants, we have

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0$$

i.e.

$$5r + 5s - 30t = 0$$

or

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

Example 13

Form the differential equation by eliminating the arbitrary function f and ϕ from $z = f(ax + by) + \phi(cx + dy)$.

$$z = f(u) + \phi(v) \tag{1}$$

where $u = ax + by$ and $v = cx + dy$

Differentiating partially with respect to x and y,

$$p = f'(u)\cdot a + \phi'(v)\cdot c \tag{2}$$

$$q = f'(u)\cdot b + \phi'(v)d \tag{3}$$

$$r = f''(u)\cdot a^2 + \phi''(v)\cdot c^2 \tag{4}$$

$$s = f''(u)\cdot ab + \phi''(v)\cdot cd \tag{5}$$

$$t = f''(u)\cdot b^2 + \phi''(v)\cdot d^2 \tag{6}$$

Eliminating $f''(u)$ and $\phi''(v)$ from (4), (5), (6), we have

$$\begin{vmatrix} r & a^2 & c^2 \\ s & ab & cd \\ t & b^2 & d^2 \end{vmatrix} = 0$$

i.e. $$(abd^2 - b^2cd)r - (a^2d^2 - b^2c^2)s + (a^2cd - abc^2)t = 0$$

i.e. $$bd(ad - bc)r - (ad + bc)\,(ad - bc)s + ac(ad - bc\,)t = 0$$

i.e. $$bd\frac{\partial^2 z}{\partial x^2} - (ad + bc)\frac{\partial^2 z}{\partial x \partial y} + ac\frac{\partial^2 z}{\partial y^2} = 0.$$

Example 14

Form the differential equation by eliminating f and g from $z = xf(ax + by) + g(ax + by)$.

$$z = x\cdot f(u) + g(u) \tag{1}$$

where $u = ax + by$.

Differentiating partially with respect to x and y,

$$p = xf'(u)\cdot a + f(u) + g'(u)\cdot a \tag{2}$$

$$q = xf'(u)\cdot b + g'(u)\cdot b \tag{3}$$

$$r = x\cdot f''(u)a^2 + f'(u)\cdot 2a + g''(u)\cdot a^2 \tag{4}$$

$$s = xf''(u)ab + f'(u)b + g''(u)ab \tag{5}$$

$$t = xf''(u)b^2 + g''(u)\cdot b^2 \tag{6}$$

[(4) × b – (5) × $2a$] gives

$$br - 2as = -a^2b[xf''(u) + g''(u)] \tag{7}$$

$$= -a^2b\times\frac{1}{b^2}t, \text{ from (6)}$$

i.e. $$b^2\frac{\partial^2 z}{\partial x^2} - 2ab\frac{\partial^2 z}{\partial x \partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0$$

Example 15

Form the differential equation by eliminating the arbitrary functions f and g from

$z = f(x + iy) + (x + iy)g\,(x - iy)$, where $i = \sqrt{-1}$ and $x + iy \neq z$

$$z = f(u) + (x + iy)\,g(v) \tag{1}$$

where $u = x + iy$ and $v = x - iy$.

Differentiating partially with respect to x and y,

$$p = f'(u) \cdot 1 + (x + iy)g'(v) \cdot 1 + g(v) \tag{2}$$

$$q = f'(u) \cdot i + (x + iy)g'(v)\,(-i) + g(v) \cdot i \tag{3}$$

$$r = f''(u) \cdot 1 + (x + iy)g''(v) \cdot 1 + 2g'(v) \cdot 1 \tag{4}$$

$$s = f''(u) \cdot i + (x + iy)g''(v)\,(-i) \tag{5}$$

$$t = f''(u)\,(-1) + (x + iy)g''(v) \cdot (-1) + 2g'(v) \tag{6}$$

Adding (4) and (6), we get

$$r + t = 4g'(v) \tag{7}$$

From (2) and (3), we get

$$p + iq = 2(x + iy)g'(v) \tag{8}$$

Eliminating $g'(v)$ from (7) and (8), we get

$$r + t = 2\frac{(p+iq)}{x+iy}$$

i.e.

$$(x + iy)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 2\left(\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right)$$

Note ✍

Equation (5), giving the value of s, is not all used.

Example 16

If $u = f(x^2 + y) + \phi(x^2 - y)$, show that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial u}{\partial x} - 4x^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

$$u = f(v) + f(w) \tag{1}$$

where $v = x^2 + y$ and $w = x^2 - y$.

Differentiating partially with respect to x and y,

$$\frac{\partial u}{\partial x} = f'(v)\cdot 2x + \phi'(w)\cdot 2x \qquad (2)$$

$$\frac{\partial u}{\partial y} = f'(v)\cdot 1 + \phi'(w)\cdot(-1) \qquad (3)$$

$$\frac{\partial^2 u}{dx^2} = f'(v)\cdot 2 + f''(v)\cdot 4x^2 + \phi'(w)\cdot 2 + \phi''(w)\cdot 4x^2 \qquad (4)$$

$$\frac{\partial^2 u}{\partial x\partial y} = f''(v)\cdot 2x + \phi''(w)\cdot(-2x) \qquad (5)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(v)\cdot 1 + \phi''(w)\cdot 1 \qquad (6)$$

Eq. (4) can be rewritten as

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= 2\{f'(v)+\phi'(w)\} + 4x^2\{f''(v)+\phi''(w)\}\\ &= 2\times\frac{1}{2x}\frac{\partial u}{\partial x} + 4x^2\cdot\frac{\partial^2 u}{\partial y^2}, \qquad \text{from (2) and (6)}\end{aligned}$$

i.e. $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial y^2} = 0$$

Example 17

Form the differential equation by eliminating f and ϕ from $z = f(x + y)\cdot\phi(x - y)$.

$$z = f(u)\cdot\phi(v) \qquad (1)$$

where $u = x + y$ and $v = x - y$.

Differentiating partially with respect to x and y, we get

$$p = f(u)\cdot\phi'(v) + f'(u)\cdot\phi(v) \qquad (2)$$

$$q = f(u)\phi'(v)\,(-1) + f'(u)\phi(v) \qquad (3)$$

$$r = f(u)\phi''(v) + 2f'(u)\phi'(v) + f''(u)\cdot\phi(v) \qquad (4)$$

$$s = f(u)\phi''(v)\,(-1) + f''(u)\cdot\phi(v) \qquad (5)$$

$$t = f(u)\cdot\phi''(v) - 2f'(u)\phi'(v) + f''(u)\phi(v) \qquad (6)$$

Subtracting (5) from (3), we get

$$r - t = 4f'(u)\cdot\phi'(v) \qquad (7)$$

From (1) and (2), we get

$$p^2 - q^2 = 4f(u) \cdot \phi(u) \cdot f'(u) \cdot \phi'(v)$$

$$= z(r - t) \text{ from (1) and (7)}$$

i.e.
$$z\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Example 18

Form the differential equation by eliminating f and ϕ from $z = xf(y/x) + y\phi(x)$.

$$z = xf(u) + y\phi(x) \qquad (1)$$

where $u = \dfrac{y}{x}$

Differentiating partially with respect to x and y, we get

$$p = xf'(u) \cdot \left(-\frac{y}{x^2}\right) + f(u) + y\phi'(x)$$

i.e.
$$p = -\frac{y}{x} \cdot f'(u) + f(u) + y\phi'(x) \qquad (2)$$

$$q = x \cdot f'(u) \cdot \frac{1}{x} + \phi(x)$$

i.e.
$$q = f'(u) + \phi'(x) \qquad (3)$$

$$r = -\frac{y}{x} \cdot f''(u)\left(-\frac{y}{x^2}\right) + y\phi''(x)$$

i.e.
$$r = \frac{y^2}{x^3} f''(u) + y\phi''(x) \qquad (4)$$

$$s = -\frac{y}{x^2} f''(u) + \phi'(x) \qquad (5)$$

$$t = \frac{1}{x} f''(u) \qquad (6)$$

Eliminating $f''(u)$ from (5) and (6), we get

$$s + \frac{y}{x} t = \phi'(x) \qquad (7)$$

From (2) and (3), we get

$$px + qy = \{xf(u) + y\phi(x)\} + dy\phi'(x)$$

i.e.
$$px + qy = z + xy\phi'(x) \qquad (8)$$

Eliminating $\phi'(x)$ from (7) and (8), we get

$$xys + y^2 t = px + qy - z$$

i.e.
$$xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

Example 19

Form differential equation by eliminating f and ϕ from $z = f(y) + \phi(x + y + z)$

$$z = f(y) + \phi(u) \tag{1}$$

where $u = x + y + z$.

Differentiating partially with respect to x and y, we get

$$p = \phi'(u)\,(1 + p) \tag{2}$$

$$q = f'(y) + \phi'(u)\,(1 + q) \tag{3}$$

$$r = \phi'(u) \cdot r = \phi''(u) \cdot (1 + p)^2 \tag{4}$$

$$s = \phi'(u) \cdot s + \phi''(u)\,(1 + p)\,(1 + q) \tag{5}$$

$$t = f''(y) + \phi'(u)t + \phi'(u)\,(1 + q)^2 \tag{6}$$

From (4), $$r\{1 - \phi'(u)\} = (1 + p)^2\phi''(u) \tag{7}$$

From (5), $$s\{1 - \phi'(u)\} = (1 + p)\,(1 + q)\phi''(u) \tag{8}$$

Dividing (7) by (8), we get

$$\frac{r}{s} = \frac{1+p}{1+p}$$

i.e. $$\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial x^2} = \left(1+\frac{\partial z}{\partial x}\right)\frac{\partial^2 z}{\partial x^2}$$

Example 20

Form the differential equation by eliminating the arbitrary function ϕ from

$$z = \frac{1}{x}\phi(y-x)+\phi'(y-x).$$

Note ✍

Though ϕ' is the derivative of ϕ, we should not assume that only one function is to be eliminated. We have to eliminate two functions ϕ and ϕ' and hence the resulting partial differential equation will be of order 2.

$$z = \frac{1}{x}\phi(u)+\phi'(u) \tag{1}$$

where $u = y - x$

Differentiating partially with respect to x and y, we get

$$p = \frac{1}{x}\phi'(u)\cdot(-1)-\frac{1}{x^2}\phi(u)+\phi''(u)(-1) \tag{2}$$

$$q = \frac{1}{x}\phi'(u)\cdot 1+\phi''(u)\cdot 1 \tag{3}$$

$$r = \frac{1}{x}\phi''(u)\cdot 1+\frac{2}{x^2}\phi'(u)+\frac{2}{x^3}\phi(u) + \phi'''(u)\cdot 1 \tag{4}$$

$$s = -\frac{1}{x^2}\phi'(u) - \frac{1}{x}\phi''(u) + \phi'''(u)(-1) \tag{5}$$

$$t = \frac{1}{x}\phi''(u)\cdot 1 + \phi'''(u)\cdot 1 \tag{6}$$

From (4) and (6), we get

$$r - t = \frac{2}{x^2}\phi'(u) + \frac{2}{x^3}\phi(u)$$

$$= \frac{2}{x^2}\left\{\frac{1}{x}\phi(u) + \phi'(u)\right\}$$

$$= \frac{2}{x^2} z$$

i.e. $$x^2\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = 2z$$

Exercise 4(a)

Part A (Short-Answer Questions)

1. Write down the form of the P.D.E. (partial differential equation), obtained by eliminating 'f' from $f(u, v) = 0$.

Form the P.D.E.s by eliminating the arbitrary constants a and b from the following relations:

2. $z = (x + a)(y + b)$
3. $z = (x^2 + a^2)(y^2 + b^2)$
4. $z = ax + by + ab$
5. $z = ax + by + a^2 + b^2$
6. $z = ax^3 + by^3$
7. $z = a(x + y) + b$
8. $ax^2 + by^2 + z^2 = 1$
9. $(x - a)^2 + (y - b)^2 = z^2$.

Form the P.D.E.s by eliminating the arbitrary functions from the following relations.

10. $z = f(x^2 + y^2)$
11. $z = \phi(x^3 - y^3)$
12. $z = f(bx - ay)$

13. $z = \phi(xy)$
14. $z = f\left(\frac{y}{x}\right)$
15. $z = f(x) + \phi(y)$
16. $z = f(x) + \phi(y) + axy$
17. $z = f(y) + x\phi(y)$
18. $z = yf(x) + \phi(x)$
19. $z = xf(y) + \phi(y) - \sin x$
20. $z = yf(x) + \phi(x) - \cos y$

Part B

21. Form the P.D.E. by eliminating a and b from $z = xy + y\sqrt{x^2 - a^2} + b$.
22. From the P.D.E. by eliminating a and b from $z = ax - \frac{a}{a+1}y + b$.
23. Form the P.D.E. by eliminating a and b from $4z(1 + a^2) = (x + ay + b)^2$.
24. Form the P.D.E. by eliminating a and b from $z^2 + \left\{ z\sqrt{z^2 - 4a^2} - 4a^2 \log(z + \sqrt{z^2 - 4a^2}) \right\} = 4(x + ay + b)$.
25. Form the P.D.E. by eliminating a and b from $3z = ax^3 + 2\sqrt{a - 1y^{3/2}} + b$.
26. Form the P.D.E. of all planes which cut off equal intercepts on the x and y axes.
27. Form the P.D.E. of all planes passing through the origin.
28. Form the P.D.E. of all spheres whose centres lie on the z-axis.
29 Form the P.D.E. of all spheres of radius c having their centres on the xoy-plane.
30. Final the P.D.E. of all spheres of radius c having their centres on the zox-plane.
31. Form the P.D.E. by eliminating the arbitrary function 'f' from (a)$z = f\left(\frac{xy}{z}\right)$; (b) $z = f(x^2 + y^2 + z^2)$
32. Form the P.D.E. by eliminating the arbitrary function f from (a) $xyz = f(x + y + z)$; (b) $\frac{xy}{z} = f(x^2 - y + z)$
33. Form the P.D.E. by eliminating 'ϕ' from $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$; (b) $\phi(x^3 - y^3, x^2 - z^2) = 0$
34. Form the P.D.E. by eliminating 'ϕ' from (a) $\phi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$; (b) $\phi\left(x^2 - y^2 - 2z, \frac{y}{zx}\right) = 0$

35. Form the P.D.E. by eliminating 'ϕ' from

(a) $\phi\left(\dfrac{x-y}{y-z}, xy+yz+zx\right)=0$; (b) $\phi\left(\dfrac{x+y+z}{z}, x^2-y^2\right)=0$

From the P.D.E.s by eliminating the arbitrary functions from the following relations.

36. $z = f(x + iy) + g(x - iy)$, where $i = \sqrt{-1}$ and $x + iy \neq z$.

37. $z = f(2y + 3x) + g(y + x)$.

38. $z = f_1(y - x) + f_2(y + x) + f_3(y + 2x)$.

39. $z = xf(2x + 3y) + g(2x + 3y)$

40. $z = f(x + y) + yg(x + y)$

41. $z = (x - iy)\, f(x + iy) + g(x - iy)$, where $i = \sqrt{-1}$ and $x + iy \neq z$.

42. $z = f(\sqrt{x}+y)+g(\sqrt{x}-y)$

43. $z = f(x) \cdot \phi(y)$

44. $z = yf(x) + x\phi(y)$

45. $z = f(x + y + z)+ \phi(x - y)$

4.5 SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

The relation between the independent variables and the dependent variable (containing arbitrary constants or functions) from which a partial differential equation is formed is called the *primitive* or *solution* of the P.D.E.

In other words, a *solution* of a P.D.E. is a relation between the independent and the dependent variables, which satisfies the P.D.E. solution of a P.D.E. is also called *integral* of the P.D.E.

As was seen in Section 4.2, the primitive of a P.D.E. may contain arbitrary constants or arbitrary function. Accordingly, we have two types of solutions for a P.D.E.

A solution of a P.D.E. which contains as many arbitrary constants as the number of independent variables is called *the complete solution* or *complete integral* of the equation.

A solution of a P.D.E. which contains as many arbitrary functions as the order of the equation is called the *general solution* or *general integral* of the equation.

Both these types of solutions can be obtained for the same P.D.E. For example, the equation $z = px + qy$ is obtained when we eliminate the arbitrary constants a and b from $z = ax + by$ or the arbitrary functions 'f' from $z = x \cdot f\left(\dfrac{y}{x}\right)$.

Thus $z = ax + by$ is the complete solution and $z = x \cdot f\left(\dfrac{y}{x}\right)$ is the general solution of the P.D.E. $z = px + qy$.

The complete solution $z = ax + by$ can be rewritten as $z = x\left\{a + b\left(\frac{y}{x}\right)\right\}$.

Comparing this with the general solution $z = xf\left(\frac{y}{x}\right)$, we note that $a + b\left(\frac{y}{x}\right)$ is a particular case of $f(y/x)$. Hence the general solution of a P.D.E. is more general than the complete solution. Thus when the solution of a P.D.E. is required, we should try to give the general solution. However there are certain. P.D.E.s for which methods are not available for finding the general solutions directly, but methods are available for finding the complete solutions only in other cases. In such cases, we indicate the procedure for finding the general solution from the complete solution as explained in Section 4.6.

4.6 PROCEDURE TO FIND GENERAL SOLUTION

Let $$F(x, y, z, p, q) = 0 \tag{1}$$

be a first order P.D.E. Let its complete solution be

$$\phi(x, y, z, a, b) = 0 \tag{2}$$

where a and b are arbitrary constants.

Let $b = f(a)$ [or $a = g(b)$], where 'f' is an arbitrary function.

Then (2) becomes

$$\phi[x, y, z, a, f(a)] = 0 \tag{3}$$

Differentiating (2) partially with respect to a, we get

$$\frac{d\phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \tag{4}$$

Theoretically, it is possible to eliminate 'a' between (3) and (4).

This eliminant, which contains the arbitrary function 'f', is general solution of (1).

A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary functions in the general solution is called *a particular solution* or *particular integral* of the P.d.E.

Thus for the P.D.E. $z = px + qy$, for which the complete solution is $z = ax + by$ and the general solution is $z = x \cdot f\left(\frac{y}{x}\right)$, the following are particular solution.

(i) $z = 2x + 3y$

(ii) $z = 3x - 4y$

(iii) $z = x \cdot e^{\frac{y}{x}}$

(iv) $z = x \sin\left(\frac{y}{x}\right)$

There is yet another type of solution of a P.D.E., called the *singular solution* or *singular integral*. Geometrically the singular solution of a P.D.E. represents the envelope of the family of surfaces represented by the complete solution of that P.d.E. the singular solution will neither contain arbitrary constants nor arbitrary functions but at the same time cannot be obtained as particular case of the complete or general solution.

4.7 PROCEDURE TO FIND SINGULAR SOLUTION

Let $$F(x, y, z, p, q) = 0 \quad (1)$$

be a first order P.D.E.

Let its complete solution be

$$\phi(x, y, z, a, b) = 0 \quad (2)$$

Differentiating (2) partially with respect to a and then b, we have

$$\frac{\partial \phi}{\partial a} = 0 \quad (3)$$

and $$\frac{\partial \phi}{\partial b} = 0 \quad (4)$$

The eliminant of a and b from equations (2), (3) and (4), if it exists, is the singular solution of the P.D.E. (1).

As pointed out earlier, P.D.E.s can be divided into two categories — one for which methods are readily available only for finding complete solutions and the other for which methods are available for finding general solutions. first order non-linear equations that belong to the first category will be discussed in Section 4.8.

4.8 COMPLETE SOLUTIONS OF FIRST ORDER NONLINEAR P.D.E.s

A P.D.E., the partial derivatives occurring in which are of the first degree, is said to be *linear*; otherwise it is said to be *non-linear*.

First order non-linear P.D.E.s, for which complete solution can be found out, are divided into four standard types. Some first-order non-linear P.D.E.s, which do not fall under any of the four standard types, can be transformed into one or the other of the standard types by suitable changes of variables. We shall discuss below the special methods of finding the complete solutions for these types of equations.

Type I

Equations of the form $f(p, q) = 0$, i.e. the P.D.E.s that contain p and q only explicitly.

For equations of this type, it is known that a solution will be of the following form,

$$z = ax + by + c \quad (1)$$

But this solution contains three arbitrary constants, whereas the number of independent variables is two. Hence if we can reduce the number of arbitrary constants in (1) by one, it becomes the complete solution of the equation $f(p, q) = 0$. Now from (1), $p = a$ and $q = b$. If (1) is to be a solution of $f(p, q) = 0$, the values of p and q obtained from (1) should satisfy the given equation.

i.e. $$f(a, b) = 0$$

Solving this, we can get $b = \phi(a)$, where ϕ is a known function. Using this value of b in (1), the complete solution of the given P.D.E. is

$$z = ax + \phi(a)y + c \tag{2}$$

The general solution can be obtained from (2) by the method given earlier.

To find the singular solution, we have to eliminate a and c from

$$z = ax + \phi(a)y + c, \qquad x + \phi'(a)y = 0 \text{ and } 1 = 0$$

of which the last equation is absurd. Hence there is no singular solution for equations of type I.

Type II

Clairaut's type, the P.D.E.s of the form

$$z = px + qy + f(p, q) \tag{1}$$

For equations of this type also, it is known that a solution will be on the form

$$z = ax + by + c \tag{2}$$

If we can reduce the number of arbitrary constants in (2) by one, it becomes the complete solution of (1).

From (2) we get $p = a$ and $q = b$.

As before, $$z = ax + by + f(a, b) \tag{3}$$

From (2) and (3), we get $c = f(a, b)$

Thus the complete solution of (1) is given by (3).

Note ✍

Without going through all these formalities, we can quickly write down the complete solution of a clairaut's type of P.D.E. by simply replacing p and q by a and b in it respectively.

The general and singular solution of (1) can be found out by the usual methods.

For clairaut's type of equations, singular solutions will normally exist.

Type III

Equations not containing x and y explicitly, i.e. equations of the form

$$f(z, p, q) = 0 \tag{1}$$

For equations of this type, it is known that a solution will be of the form

$$z = \phi(x + ay) \tag{2}$$

where 'a' is an arbitrary constant and ϕ is a specific function to be found out.

Putting $x + ay = u$, (2) becomes $z = \phi(u)$ or $z(u)$

$$\therefore \qquad p = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$$

and
$$q = \frac{dz}{du}\cdot\frac{\partial u}{\partial u} = a\frac{dz}{du}$$

If (2) is to be a solution of (1), the values of p and q obtained should satisfy (1).

i.e.
$$f\left(z, \frac{\partial z}{du}, a\frac{dz}{du}\right) = 0 \qquad (3)$$

From (3), we can get
$$\frac{dz}{du} = \psi(z, a) \qquad (4)$$

Now (4) is an ordinary differential equation, which can be solved by the variable separable method.

The solution of (4), which will be of the form $g(z, a) = u + b$ or $g(z, a) = x + ay + b$, is the complete solution of (1).

The general and singular solutions of (1) can be found out by the usual methods.

Type IV

Equations of the form
$$f(x, p) = g(y, q) \qquad (1)$$

that is equations which do not contain z explicitly and in which terms containing p and x can be separated from those containing q and y.

To find the complete solution of (1), we assume that $f(x, p) = g(y, q) = a$, where 'a' is an arbitrary constant.

Solving $f(x, p) = a$, we can get $p = \phi(x, a)$ and solving $g(y, q) = a$, we can get $q = \psi(y, a)$.

Now
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{or} \quad p\,dx + q\,dy$$

i.e.
$$dz = \phi(x, a)dx + \psi(y, a)dy$$

Integrating with respect to the concerned variables, we get
$$z = \int \phi(x, a)\,dx + \int \psi(y, a)\,dy + b \qquad (2)$$

The complete solution of (1) is given by (2), which contains two arbitrary constants a and b.

The general and singular solutions of (1) are found out by the usual methods.

4.9 EQUATIONS REDUCIBLE TO STANDARD TYPES TRANSFORMATION

Type A

Equations of the form $f(x^m p, y^n q) = 0$ or $f(x^m p, y^n q, z) = 0$,
where m and n are constants, each not equal to 1.

We make the transformations $x^{1-m} = X$ and $y^{1-n} = Y$.

Then
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = (1-m)x^{-m}P, \text{ where } P \equiv \frac{\partial z}{\partial X} \text{ and}$$
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = (1-n)y^{-n}Q, \text{ where } Q \equiv \frac{\partial z}{\partial Y}$$

Therefore the equation $f(x^m p, y^n q) = 0$ reduces to $f\{(1-m)P, (1-n)Q\} = 0$, which is a type I equation.

The equation $f(x^m p, y^n q, z) = 0$ reduces to $f\{(1-m)P, (1-n)Q, z\} = 0$, which is a type III equation.

Type B

Equations of the form $f(px, qy) = 0$ or $f(px, qy, z) = 0$

Note ✍

These equations correspond to m = 1 and n = 1 of the type A equations.

The required transformations are
$$\log x = X \text{ and } \log y = Y$$

In this case, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{1}{x}$ or $px = P$ and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} =$
$\frac{\partial z}{\partial Y}\cdot\frac{1}{y}$ or $qy = Q$, where $P \equiv \frac{\partial z}{\partial X}$ and $Q \equiv \frac{\partial z}{\partial Y}$.

Therefore the equation $f(px, qy) = 0$ reduces to $f(P, Q) = 0$, which is a type I equation.

The equation $f(px, qy, z) = 0$ reduces to $f(P, Q, z) = 0$, which is a type III equation.

Type C

Equations of the form $f(z^k p, z^k q) = 0$ or $f(z^k p, z^k q, x, y) = 0$, where k is a constant $\neq -1$.

We make the transformation $Z = z^{k+1}$

Then
$$P = \frac{\partial Z}{\partial x} = (k+1)z^k p \text{ and}$$
$$Q = \frac{\partial Z}{\partial y} = (k+1)z^k q$$

Therefore the equation $f(zk\ p, zk\ q) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}\right) = 0$, which is a type I equation and the equation $f(z^k p, z^k q, x, y) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}, x, y\right) = 0$, which may be a type IV equation.

Type D

Equations of the form $f\left(\frac{p}{z}, \frac{q}{z}\right) = 0$ or $f(p/z, q/z, x, y) = 0$, which correspond to $k = -1$ of type C equations.

The required transformation is $Z = \log z$

Then $P = \frac{\partial Z}{\partial x} = \frac{1}{z} p$ and $Q = \frac{\partial Z}{\partial y} = \frac{1}{z} q$

Therefore the equations $f(p/z, q/z) = 0$ and $f(p/z, q/z, x, y) = 0$ reduce respectively to type I and type IV equations.

Type E

Equations of the from $f(x^m z^k p, y^n z^k q) = 0$ where $m, n \neq 1$; $k \neq -1$

We make the transformations

$$X = x^{1-m},\ Y = y^{1-n} \text{ and } Z = z^{k+1}$$

Then

$$P = \frac{\partial Z}{\partial X} = \frac{\mathrm{d}Z}{\mathrm{d}z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}X}$$

$$= (k+1) z^k p \cdot \frac{x^m}{1-m}$$

and

$$Q = (k+1) z^k q \frac{y^n}{1-n}$$

$\therefore$ The given equation reduces to

$$f\left\{\left(\frac{1-m}{k+1}\right) P, \left(\frac{1-n}{k+1}\right) Q\right\} = 0$$

which is a type I equation.

Type F

Equations of the form $f\left(\frac{px}{z}, \frac{qy}{z}\right) = 0$

By putting $X = \log x$, $Y = \log y$ and $Z = \log z$ the equation reduces to $f(P, Q) = 0$ where $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$.

Worked Examples 4(b)

Example 1

Solve the equation $pq + p + q = 0$.

This equation contains only p and q explicitly.

$\therefore$ Let a solution of the equation be

$$z = ax + by + c \tag{1}$$

From (1), we get $p = a$ and $q = b$.

Since (1) is a solution of the given equation,

$$ab + a + b = 0 \tag{1}$$

$\therefore$
$$b = -\frac{a}{a+1} \tag{2}$$

Using (2) in (1), the required complete solution of the equation

$$z = ax - \frac{a}{a+1}y + c \tag{3}$$

To find the general solution, we put $c = f(a)$ in (3), where 'f' is an arbitrary function.

i.e.
$$z = ax - \frac{a}{a+1}y + f(a) \tag{4}$$

Differentiating (4) partially with respect to a, we get

$$x - \frac{1}{(a+1)^2}y + f'(a) = 0 \tag{5}$$

Eliminating a between (4) and (5), we get the required general solution.

To find the singular solution, we have to differentiate (3) partially with respect to a and c.

When we differentiate (3) partially with respect to c, we get $0 = 1$, which is absurd.

Hence, no singular solution exists for the given equation.

Example 2

Solve the equation $p^2 + q^2 = 4pq$.

$$p^2 + q^2 - 4pq = 0 \tag{1}$$

As (1) contains only p and q, a solution of (1) will be of the form

$$z = ax + by + c \tag{2}$$

From (2), we get $p = a$ and $q = b$.

Since (2) is a solution of (1),

$$a^2 + b^2 - 4ab = 0$$

Solving for b, we get

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2}$$

$$= (2 \pm \sqrt{3})a$$

Using in (2), the complete solution of (1) is

$$z = ax + (2 \pm \sqrt{3})ay + c \tag{3}$$

There is no singular solution for (1), as in Example 1.

To get the general solution, we put $c = f(a)$ in (3), which becomes

$$z = ax + (2 \pm \sqrt{3})ay + f(a) \tag{4}$$

where f is an arbitrary function.

Differentiating (4) partially with respect to a, we get

$$0 = x + (2 \pm \sqrt{3})y + f'(a) \tag{5}$$

The eliminant of 'a' between (4) and (5) gives the general solution of (1).

Example 3

Solve the equation $x^4 p^2 - yzq - z^2 = 0$

As it is, the equation

$$x^4 p^2 - yzq - z^2 = 0 \tag{1}$$

does not belong to any of the four standard types.

Rewriting Eq. (1), we get

$$\left(\frac{x^2 p}{z}\right)^2 - \left(\frac{yq}{z}\right) = 1 \tag{2}$$

As L.H.S. of (2) is a function of $\frac{x^2 p}{z}$ and $\frac{yq}{z}$, we make the transformations

$X = x^{-1}$, $Y = \log y$ and $Z = \log z$

(by the transformation rules for type A and type F equations)

Then $$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = zP\left(-\frac{1}{x^2}\right)$$

$\therefore$ $$\frac{x^2 p}{x} = -P$$

and $$q = \frac{\partial z}{\partial y} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial Y} \cdot \frac{dY}{dy} = zQ \cdot \frac{1}{y}$$

$\therefore$ $$\frac{yq}{z} = Q$$

Equation (2) becomes

$$P^2 - Q = 1 \tag{3}$$

Equation 3 contains only P and Q explicitly.

Therefore a solution of (3) will be of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from (4), satisfy Eq. 3.

$$\therefore \quad a^2 - b = 1$$

$$\therefore \quad b = a^2 - 1$$

$\therefore$ The complete solution of (3) is

$$Z = aX + (a^2 - 1)Y + c$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{a}{x} + (a^2 - 1)\log y + c$$

Singular solution does not exist and general solution is found out as usual.

Example 4

Solve the equation $z^2\left(\dfrac{p^2}{x^2} + \dfrac{q^2}{y^2}\right) = 1.$

The given equation does not belong to any of the four standard types.

It can be rewritten as

$$(x^{-1}zp)^2 + (y^{-1}zq)^2 = 1 \tag{1}$$

which of the form $(x^m z^k p)^2 + (y^n z^k q)^2 = 1$ [Refer to type E equations]

$\therefore$ We make the transformations

$$X = z^{1-m},\ Y = y^{1-n} \text{ and } Z = z^{k+1}$$

i.e. $$X = x^2,\ Y = y^2 \text{ and } Z = z^2$$

Then

$$p = \frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}Z}\cdot\frac{\partial Z}{\partial X}\cdot\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{2z}\cdot P\cdot 2x$$

$$\therefore \quad P = x^{-1}zp$$

Similarly, $Q = y^{-1}zq$.

Using these in (1), it becomes

$$P^2 + Q^2 = 1 \tag{2}$$

As (2) contains only P and Q explicitly, a solution of the equation will be of the form

$$Z = aX + bY + c \tag{3}$$

$\therefore$ $P = a$ and $Q = b$, obtained form (3), satisfy Eq. 2.

i.e. $$a^2 + b^2 = 1.$$

$\therefore$ The complete solution of (2) is

$$Z = aX \pm \sqrt{1-a^2}\, Y + c$$

$\therefore$ The complete solution of (1) is

$$z^2 = ax^2 \pm \sqrt{1-a^2}\, y^2 + c$$

Singular solution does not exist and general solution is found out as usual.

Example 5

Solve the equation $pq\, xy = z^2$.

The equation

$$pq\, xy = z^2 \tag{1}$$

does not belong to any of the four standard types.

Rewriting (1),

$$\left(\frac{px}{z}\right)\left(\frac{qy}{z}\right) = 1 \tag{2}$$

As (2) contains $\frac{px}{z}$ and $\frac{qy}{z}$, we make the substitutions $X = \log x$, $Y = \log y$ and $Z = \log z$ [Refer to type F equations]

Then $$P = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = z \cdot P \cdot \frac{1}{x}$$

i.e. $$\frac{px}{x} = P$$

Similarly $$\frac{qy}{z} = Q$$

Using these in (2), it becomes

$$PQ = 1 \tag{3}$$

which contains only P and Q explicitly. A solution of (3) is of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from 4, satisfy (3)

i.e. $$ab = 1 \text{ or } b = \frac{1}{a}$$

$\therefore$ The complete solution of (3) is $Z = aX + \frac{1}{a}Y + c$

$\therefore$ The complete solution of (1) is

$$\log z = a \log x + \frac{1}{a} \log y + c \tag{5}$$

General solution of (1) is obtained as usual.

Note ✍

To find the singular solution of (1), we should not use the complete solution of (3). We should use only that of (1) given in (5).

If we put $c = \log k$, (5) becomes

$$\log z = \log (x^a y^{1/a} k)$$

i.e. $$z = x^a y^{1/a} k \tag{6}$$

Differentiating (6) partially with respect to a,

$$\log x - \frac{1}{a^2} \log y = 0 \tag{7}$$

Differentiating (6) partially with respect to k,

$$0 = x^a y^{1/a} \tag{8}$$

Eliminating a and k form (6), (7) and (8), that is using (8) in (6), the singular solution of equation (1) is $z = 0$.

Example 6

Solve the equation $z^4 q^2 - z^2 p = 1$.

The equation can be solved directly, as it contains p, q and z only explicitly.

However we shall transform it into a simpler equation and solve it.

The equation can be rewritten as

$$(z^2 q)^2 - (z^2 p) = 1 \tag{1}$$

which contains $z^2 p$ and $z^2 q$.

Hence we make the transformation $Z = z^3$ [Refer to type C equations]

$$\therefore \qquad P = \frac{\partial Z}{\partial x} = 3z^2 p$$

i.e.
$$z^2 p = \frac{P}{3}$$

Similarly $z^2 q = \dfrac{Q}{3}$

Using these values in (1), we get

$$Q^2 - 3P = 0 \tag{2}$$

As (2) is an equation containing P and Q only a solution of (2) will be of the form

$$Z = ax + by + c \tag{3}$$

Now $P = a$ and $Q = b$, obtained from (3) satisfy Eq. 2.

$$\therefore \qquad b^2 - 3a = 9$$

i.e.
$$b = \pm\sqrt{3a+9}$$

$\therefore$ Complete solution of (2) is $Z = ax \pm \sqrt{3a+9y} + c$, i.e. complete solution of (1) is $z^3 = ax \pm \sqrt{3a+9y} + c$. Singular solution does not exist. General solution is found out as usual.

Example 7

Solve the equation $z = px + qy + p^2 + pq + q^2$

The given equation

$$z = px + qy + (p^2 + pq + q^2) \tag{1}$$

is a Clairaut's type equation.

$\therefore$ The complete solution of (1) is

$$z = ax + by + a^2 + ab + b^2 \tag{2}$$

[got by replacing p and q in (1) by a and b]

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to a and then b, we get

$$x + 2a + b = 0 \tag{3}$$

and
$$y + a + 2b = 0 \tag{4}$$

The eliminant of a and b from (2), (3) and (4) is the required singular solution.

Solving (3) and (4) for a and b, we get

$$a = \frac{1}{3}(y-2x) \quad \text{and} \quad b = \frac{1}{3}(x-2y)$$

Using these values in (2), the singular solution is

$$z = \frac{x}{2}(y-2x)+\frac{y}{3}(x-2y)+\frac{1}{9}(y-2x)^2$$

$$+\frac{1}{9}(y-2x)(x-2y)+\frac{1}{9}(x-2y)^2$$

i.e.
$$9z = 3x(y-2x) + 3y(x-2y)$$
$$+ (y-2x)^2 + (y-2x)(x-2y) + (x-2y)^2$$

i.e.
$$3z + x^2 - xy + y^2 = 0$$

General solution of (1) is found out as usual.

Example 8

Solve the equation $z = px+qy+\left(\frac{q}{p}-p\right)$.

The given equation

$$z = px+qy+\left(\frac{q}{p}-p\right) \qquad (1)$$

is a Clairaut's type equation.

∴ The complete solution of (1) is

$$z = ax+by+\frac{b}{a}-a \qquad (2)$$

The general solution of (1) is found out as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to a and then b.

We get

$$0 = x - b/a^2 - 1 \qquad (3)$$

and
$$0 = y + 1/a \qquad (4)$$

Using $a = -\frac{1}{y}$ got from (4) in (3), we get

$$x - by^2 - 1 = 0$$

i.e.
$$b = \frac{x-1}{y^2}$$

Using thee values of a and b in (2), we get

$$z = -x/y+\frac{x-1}{y}-\left(\frac{x-1}{y}\right)+\frac{1}{y}$$

i.e. $yz = 1 - x$, which is the singular solution of (1).

Example 9

Solve the equation $Z = px + qy + c\sqrt{1 + p^2 + q^2}$.

The given equation

$$z = px + qy + c\sqrt{1 + p^2 + q^2} \tag{1}$$

is a Clairaut's type equation.

$\therefore$ Its complete solution is

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \tag{2}$$

where a and b are arbitrary constants and c is a given constant.

The general solution of (1) is found out from (2) as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to a and then b.

$$0 = x + \frac{ca}{\sqrt{1 + a^2 + b^2}} \tag{3}$$

and

$$0 = y + \frac{cb}{\sqrt{1 + a^2 + b^2}} \tag{4}$$

From (3) and (4), we get $\frac{a}{b} = \frac{x}{y}$ or $\frac{a}{x} = \frac{b}{y} = k$, say

$\therefore$ $a = kx$ and $b = ky$

Using these values in (3), we have

$$\frac{kc}{\sqrt{1 + k^2(x^2 + y^2)}} = -1$$

since k is negative,

i.e. $$1 + k^2(x^2 + y^2) = k^2c^2$$

or $$k^2(c^2 - x^2 - y^2) = 1$$

i.e. $$k = -\frac{1}{\sqrt{c^2 - x^2 - y^2}}$$

$\therefore$ $$a = -\frac{x}{\sqrt{c^2 - x^2 - y^2}}, \quad b = -\frac{y}{\sqrt{c^2 - x^2 - y^2}}$$

and $$\sqrt{1 + a^2 + b^2} = \frac{c}{\sqrt{c^2 - x^2 - y^2}}$$

Using these values in (2), the singular solution of (1) is got as

$$z = -\frac{x^2}{\sqrt{c^2 - x^2 - y^2}} - \frac{y^2}{\sqrt{c^2 - x^2 - y^2}} + \frac{c^2}{\sqrt{c^2 - x^2 - y^2}}$$

i.e. $$z = \sqrt{c^2 - x^2 - y^2} \text{ or}$$

$$x^2 + y^2 + z^2 = c^2$$

Example 10

Solve the equation $(pq - p - q)(z - px - qy) = pq$.

Rewriting the given equation as

$$z = px + qy + \frac{pq}{pq - p - q} \tag{1}$$

we identify it as Clairaut's type equation.

Hence its complete

$$z = ax + by + \frac{ab}{ab - a - b} \tag{2}$$

The general solution of (1) is found out as usual from (2).

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to a and then b, we get

$$0 = x + \frac{(ab - a - b)b - ab(b - 1)}{(ab - a - b)^2}$$

i.e.

$$0 = x - \frac{b^2}{(ab - a - b)^2} \tag{3}$$

and similarly

$$0 = y - \frac{a^2}{(ab - a - b)^2} \tag{4}$$

From (3) and (4), we get $\dfrac{a^2}{b^2} = y/x$ or

$$\frac{a}{\sqrt{y}} = \frac{b}{\sqrt{x}} = k, \text{ say}$$

$\therefore$

$$a = k\sqrt{y} \quad \text{and} \quad b = k\sqrt{x}$$

Using thee values in (3), we get

$$k^2 x - (k^2\sqrt{xy} - k\sqrt{y} - k\sqrt{x})^2 x = 0$$

i.e.

$$(k\sqrt{xy} - \sqrt{x} - \sqrt{y}) = 1$$

$\therefore$

$$k = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{xy}}$$

Hence $a = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{x}}$ and $b = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{y}}$

Also

$$\frac{ab}{ab - a - b} = \frac{1}{1 - 1/b - 1/a} = \frac{1}{1 - \dfrac{\sqrt{y}}{1 + \sqrt{x} + \sqrt{y}} - \dfrac{\sqrt{x}}{1 + \sqrt{x} + \sqrt{y}}}$$

$$= 1 + \sqrt{x} + \sqrt{y}$$

Using these values in (2), the singular solution of (1) is

$$z = \sqrt{x}(1 + \sqrt{x} + \sqrt{y}) + \sqrt{y}(1 + \sqrt{x} + \sqrt{y}) + (1 + \sqrt{x} + \sqrt{y})$$

i.e.

$$z = (1 + \sqrt{x} + \sqrt{y})^2.$$

Example 11

Transform the equation $4xyz = pq + 2px^2y + 2qxy^2$ by means of the substitutions $X = x^2$ and $Y = y^2$ and hence solve it.

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{p}{2x}$$

and similarly $$Q = \frac{q}{2y}$$

Rewriting the given equation, we have

$$1z = \frac{pq}{xy} + 2px + 2qy \tag{1}$$

Using the transformations in (1), it becomes

$$4z = 4P\,Q + 4P\,X + 4Q\,Y$$

i.e. $$z = P\,X + Q\,Y + P\,Q \tag{2}$$

which is a Clairaut' type of equation.

The complete solution of (2) is

$$z = aX + bY + ab \tag{3}$$

Therefore the complete solution (1) is

$$z = ax^2 + by^2 + ab \tag{4}$$

The general solution of (1) is obtained form (4) as usual.

The singular solution of (1) is obtained as follows.

Differentiating (4) partially with, respect to a and then, b, we get

$$0 = x^2 + b \tag{5}$$

and $$0 = y^2 + a \tag{6}$$

From (5) and (6), $a = -y^2$ and $b = -x^2$. Using these values in (4), the singular solution of (1) is

$$z = -x^2y^2 - x^2y^2 + x^2y^2$$

i.e. $$z + x^2y^2 = 0$$

Example 12

Solve the equation $z^2(p^2 + q^2 + 1) = c^2$, where c is a constant.

The given equation

$$z^2(p^2 + q^2 + 1) = c^2 \tag{1}$$

does not contain x and y explicity.

Therefore (1) has a solution of the form

$$z = y(u) = z(x + ay)$$

where $z(u) = z(x + ay)$ is a function of $(x + ay)$, where a is an arbitrary constant.

From (2) we have $p = \dfrac{dz}{du}$ and $q = \dfrac{dz}{du} \cdot a$

Since (2) is a solution of (1), we get

$$z^2\left\{\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right\} = c^2$$

i.e.
$$(1+a^2)\left(\frac{dz}{du}\right)^2 = \frac{c^2}{z^2} - 1$$

i.e.
$$\sqrt{1+a^2}\,\frac{dz}{du} = \frac{\sqrt{c^2 - z^2}}{z}$$

i.e.
$$-\sqrt{1+a^2}\,\frac{z\,dz}{\sqrt{c^2 - z^2}} = du \tag{3}$$

Integrating (3), the complete solution of (1) is

$$-\frac{1}{2}\sqrt{1+a^2}\int \frac{-2z\,dz}{\sqrt{c^2 - z^2}} = u + b$$

i.e.
$$-\sqrt{1+a^2}\sqrt{c^2 - z^2} = x + ay + b \quad \text{or}$$

$$(1 + a^2)(c^2 - z^2) = (x + ay + b)^2 \tag{4}$$

The general and singular solutions of (1) are found out from (4) as usual.

Example 13

Solve the equation $p(1 - q^2) = q(1 - z)$

The given equation

$$p(1 - q^2) = q(1 - z) \tag{1}$$

does not contain x and y explicitly.

Therefore (1) has a solution of the form

$$z = z(u) = z(x + ay) \tag{2}$$

where a is an arbitrary constant.

From (2), $p = \dfrac{dz}{dz}$ and $q = a\dfrac{dz}{du}$

Since (2) is a solution of (1), we get

$$\frac{dz}{du}\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}(1 - z)$$

i.e.
$$\frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2 - a + az\right] = 0$$

As z is not constant, $\dfrac{dz}{du} \neq 0$

$$\therefore \qquad 1 - a^2\left(\frac{dz}{du}\right)^2 - a + az = 0$$

i.e.
$$a^2\left(\frac{dz}{du}\right)^2 = az + 1 - a$$

or
$$a\frac{dz}{du} = \sqrt{az+1-a} \tag{3}$$

Solving (3), we get

$$a\int\frac{dz}{\sqrt{az+1-a}} = u + b$$

i.e.
$$2\sqrt{az+1-a} = x + ay + b \quad \text{or}$$
$$4(az + 1 - a) = (x + ay + b)^2 \tag{4}$$

which is the complete solution of (1).

The general and singular solution of (1) are found out from (4) as usual.

Example 14

Solve the equation $\quad 9pqz^2 = 4(1 + z^3)$.

The given equation

$$9pqz^4 = 4(1 + z^3) \tag{1}$$

does not contain x and y explicitly.

Therefore (1) has got a solution of the form

$$z = z(u) = z(x + ay) \tag{2}$$

where a is an arbitrary constant.

From (2), $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$.

Since (2) is solution of (1), we get

$$9a\left(\frac{dz}{du}\right)^2 z^4 = 4(1 + z^3)$$

i.e.
$$3\sqrt{az^2}\,\frac{dz}{du} = 2\sqrt{1+z^3} \tag{3}$$

Solving (3), we get

$$\frac{\sqrt{a}}{2}\int\frac{3z^2dz}{\sqrt{1+z^3}} = u + b$$

i.e.
$$\sqrt{a}\cdot\sqrt{1+z^3} = x + ay + b \qquad \text{or}$$
$$a(1 + z^3) = (x + ay + b)^2 \tag{4}$$

which is the complete solution of (1).

The general and singular of (1) are found out form (4) as usual.

Example 15

Solve the equation $\dfrac{x^2}{p}+\dfrac{y^2}{q}=z.$

The given equation does not belong to any of the standard types.

It can be rewritten as

$$\frac{1}{px^{-2}}+\frac{1}{qy^{-2}} = z \tag{1}$$

As equation (1) contains px^{-2} and qy^{-2}, we make the substitution $X = x^3$ and $Y = y^2$. [Refer type A equations]

Then $P=\dfrac{\partial z}{\partial X}=p\cdot\dfrac{1}{3x^2}$ or $px^{-2}=3P$ and similarly $qy^{-2} = 3Q$.

Then (1) becomes

$$\frac{1}{P}+\frac{1}{Q} = 3Z \tag{2}$$

As (2) does not contain X and Y explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

Form (3), $P=\dfrac{\mathrm{d}z}{\mathrm{d}u}$ and $Q=a\dfrac{\mathrm{d}z}{\mathrm{d}u}$

Since (3) is a solution of (2), we get

$$\frac{\mathrm{d}z}{\mathrm{d}u}(1+a) = 3az\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2$$

$$\frac{\mathrm{d}z}{\mathrm{d}u}\left(3az\frac{\mathrm{d}z}{\mathrm{d}u}-a-1\right) = 0$$

$$\text{As } \frac{\mathrm{d}z}{\mathrm{d}u}\neq 0,\ 3az\frac{\mathrm{d}z}{\mathrm{d}u}=a+1 \tag{4}$$

Solving (4), $\int 3az\,dz=(a+1)u+b$

i.e. $$\frac{3}{2}az^2 = (a + 1)(X + aY) + b$$

which is the complete solution of equation (2).

$\therefore$ The complete solution of equation (1) is

$$\frac{3}{2}az^2 = (a + 1)(x^3 + ay^3) + b$$

where a and b are arbitrary constants.

The general and singular solutions are found out as usual.

Example 16

Solve the equation

$$p^2 + x^2 y^2 q^2 = x^2 z^2$$

The given equation does not belong to any of the standard types.

Rewriting, it, we have

$$(x^{-1}p)^2 + (yq)^2 = z^2 \tag{1}$$

As equation (1) contains x^{-1} p and yq, we make the transformations $X = x^2$ and $Y = \log y$ [Refer to type A and type B equations]

$$\therefore \qquad \frac{\partial z}{\partial X} = p \cdot \frac{1}{2x} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = qy$$

i.e. $\qquad x^{-1} p = 2P \qquad$ and $\qquad yq = Q$

Using these values in (1), it becomes

$$4P^2 + Q^2 = z^2 \tag{2}$$

As (2) does not contain X and Y explicitly, it has got a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), we have

$$P = \frac{\mathrm{d}z}{\mathrm{d}u} \quad \text{and} \quad Q = a\frac{\mathrm{d}z}{\mathrm{d}u}$$

Using these values in (2), we get

$$\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 (4 + a^2) = z^2$$

i.e.

$$\sqrt{a^2 + 4}\,\frac{\mathrm{d}z}{\mathrm{d}u} = z \tag{4}$$

Solving (4), we get

$$\sqrt{a^2 + 4}\log z = X + aY + b$$

which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{a^2 + 4}\log z = x^2 + a \log y + b$$

where a and b are arbitrary constants.

The general and singular solutions are found out as usual.

Example 17

Solve the equation

$$x^2 p^2 + xpq = z^2$$

The given equation can be rewritten as

$$(xp)^2 + (xp)q = z^2 \tag{1}$$

Putting $X = \log x$, we get $P = \dfrac{\partial z}{\partial X} = px$

Using this in (1), it becomes

$$P^2 + Pq = z^2 \tag{2}$$

As Eq. 2 does not contain X and y explicitly, it has a solution of the form

$$z = z(u) = z(X + ay) \tag{3}$$

From (3),

$$P = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Using these values in (2), we have

$$\left(\frac{dz}{du}\right)^2 + a\left(\frac{dz}{du}\right)^2 = z^2$$

i.e.

$$\sqrt{1+a}\,\frac{dz}{du} = z \tag{4}$$

Solving (4), we get $\sqrt{1+a}\ \log z = X + ay + b,$ which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{1+a}\,\log z = \log x + ay + b$$

The general and singular solution are found out as usual.

Example 18

Solve the equation

$$q^2 y^2 = z(z - px)$$

As the given equation contains px and qy, we make the following substitutions.

$$X = \log x \qquad \text{and} \qquad Y = \log y$$

$\therefore$

$$P = \frac{\partial z}{\partial X} = px \qquad \text{and} \qquad Q = \frac{\partial z}{\partial Y} = qy$$

Using these in the given equation, it becomes

$$Q^2 = z(z - P) \qquad \text{or} \qquad Pz + Q^2 = z^2 \tag{1}$$

As Eq. (1) does not contain X and Y explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{2}$$

From (2),

$$P = \frac{\mathrm{d}z}{\mathrm{d}u} \quad \text{and } Q = a\frac{\mathrm{d}z}{\mathrm{d}u}$$

Using these values in (1), it becomes

$$z\frac{\mathrm{d}z}{\mathrm{d}u} + a^2\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 = z^2$$

or
$$a^2\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 + z\frac{\mathrm{d}z}{\mathrm{d}u} - z^2 = 0 \tag{3}$$

Solving (3) for $\frac{\mathrm{d}z}{\mathrm{d}u}$, we get

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2}$$

$$= \frac{(-1 \pm \sqrt{1+4a^2})z}{2a^2}$$

Solving this equation, we get

$$2a^2\int\frac{\mathrm{d}z}{z} = (-1 \pm \sqrt{1+4a^2})u + b$$

i.e.
$$2a^2 \log z = (-1 \pm \sqrt{1+4a^2})(X + aY) + b$$

which is the complete solution of (1).

$\therefore$ The complete solution of the given equation is

$$2a \log z = (-1 \pm \sqrt{1+4a^2})(\log x + a\log y) + b$$

The general and singular solution are found out as usual.

Example 19

Solve the equation

$$\sqrt{p} + \sqrt{q} = x + y$$

The given equation does not contain z explicitly and is variable separable.

That is the equation can be rewritten as

$$\sqrt{p}-x = y-\sqrt{q} = a, \quad \text{say} \tag{1}$$

$\therefore$ $$p = (x+a)^2 \quad \text{and} \quad q = (y-a)^2$$

No $$dz = p\,dx + q\,dy$$

$$= (x+a)^2 dx + (y-a)^2\, dy \tag{2}$$

Integrating both sides with respect to the concerned variables, we get

$$z = \frac{(x+a)^3}{3}+\frac{(y-a)^3}{3}+b \tag{3}$$

where a and b are arbitrary constants. Equation (3) is the complete solution of the given equation.

General solution is found out as usual. Singular solution does not exist.

Example 20

Solve the equation

$$yp = 2xy + \log q$$

The given equation, which does not contain, z, can be rewritten as

$$p - 2x = \frac{1}{y}\log q = a, \text{ say} \tag{1}$$

$$p = 2x + a \quad \text{and} \quad q = e^{ay}$$

$\therefore$ $$dz = p\,dx + q\,dy$$

i.e. $$dz = (2x+a)dx + e^{ay}dy \tag{2}$$

Integrating (2), we get

$$z = x^2 + ax + \frac{1}{a}e^{ay} + b \tag{3}$$

where a and b are arbitrary constant.

Equation (3) is the complete solution of the given equation.

General solution is found out as usual.

Singular solution does not exist.

Example 21

Solve the equation

$$p^2(1+x^2)y = qx^2$$

The given equation, which does not contain z, can be rewritten as

$$p^2\frac{(1+x^2)}{x^2} = \frac{q}{y} = a, \text{ say} \tag{1}$$

$$p = \frac{\sqrt{a} \cdot x}{\sqrt{1+x^2}} \qquad \text{and} \qquad q = ay$$

$$dz = p dx + q dy$$

$$= \sqrt{a} \cdot \frac{x}{\sqrt{1+x^2}} dx + ay dy \tag{2}$$

Integrating (2), we get the complete solution of the given equation as

$$z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + b \tag{3}$$

where a and b are arbitrary constants.

From (3), we get the general solution as usual. Singular solution does not exist.

Example 22

Solve the equation $z^2(p^2 + q^2) = x + y$

The given equation

$$z^2(p^2 + q^2) = x + y \tag{1}$$

does not belong to any of the standard types.

Equation (1) can be rewritten as

$$(zp)^2 + (zp)^2 = x + y$$

Since the equation contains zp and zq, we make the substitution $Z = z^2$

$\therefore$
$$P = \frac{\partial Z}{\partial x} = 2zp \quad \text{and} \quad Q = \frac{\partial Q}{\partial y} = 2zq$$

Using these in (1), it becomes

$$P^2 + Q^2 = 4x + 4y \tag{2}$$

which does not contain Z explicitly.

Rewriting (2), we get

$$P^2 - 4x = 4y - Q^2 = 4a, \text{ say} \tag{3}$$

$\therefore$
$$P = 2\sqrt{x+a} \quad \text{and} \quad Q = 2\sqrt{y-a}$$

$$dZ = P dx + Q dy$$

$$= 2\sqrt{x+a}\, dx + 2\sqrt{y-a}\, dy$$

Integrating, we get

$$Z = \frac{4}{3}(x+a)^{3/2} + \frac{4}{3}(y-a)^{3/2} + b$$

i.e.
$$z^2 = \frac{4}{3} \cdot (x+a)^{3/2} + \frac{4}{3} \cdot (y-a)^{3/2} + b$$

which is the complete solution of (1).
General solution is found out as usual.
Singular solution does not exist.

Example 23

Solve the equation

$$p^2 + q^2 = z^2(x^2 + y^2)$$

The given equation does not belong to any of the standard types.

It can be rewritten as

$$(z^{-1} p)^2 + (z^{-1} q)^2 = x^2 + y^2 \tag{1}$$

As the Eq. (1) contains $z^{-1}p$ and $z^{-1}q$, we make the substitution $Z = \log z$

$$\therefore \qquad P = \frac{p}{z} \quad \text{and} \quad Q = \frac{q}{z}$$

Using these values in (1), it becomes

$$P^2 + Q^2 = x^2 + y^2 \tag{2}$$

As Eq. 2 doe not contain Z explicitly, we rewrite it as

$$P^2 - x^2 = y^2 - Q^2 = a^2, \text{ say} \tag{3}$$

From (3),

$$P = \sqrt{x^2 + a^2} \quad \text{and} \quad Q = \sqrt{y^2 - a^2}$$

$$dZ = P dx + Q dy$$

$$= \sqrt{x^2 + a^2}\, dx + \sqrt{y^2 - a^2}\, dy$$

Integrating, we get

$$Z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

where a and b are arbitrary constants.
General solution is found out as usual.
Singular solution does not exist.

Example 24

Solve the equation $(x + pz)^2 + (y + qz)^2 = 1$.

The given equation does not belong to any of the standard types.

But the equation contains pz and qz.

Therefore we make the substitution $Z = z^2$.

Then $P = \frac{\partial Z}{\partial x} = 2zp$ and $Q = 2zq$.

Using these values in the given equation, it becomes

$$\left(x + \frac{P}{2}\right)^2 + \left(y + \frac{Q}{2}\right)^2 = 1 \tag{1}$$

Equation (1) does not contain Z explicitly. Rewriting (1), we have

$$\left(x + \frac{P}{2}\right)^2 = 1 - \left(y + \frac{Q}{2}\right)^2 = a^2, \text{ say} \tag{2}$$

From (2), $x + \frac{P}{2} = a$ or $P = 2(a - x)$ and $y + \frac{Q}{2} = \sqrt{1 - a^2}$ or $Q = 2(\sqrt{1 - a^2} - y)$

Now $dZ = P dx + Q dy$

$$= 2(a - x)\,dx + 2(\sqrt{1 - a^2 - y})\,dy \tag{3}$$

Integrating (3) and replacing Z by z^2, the complete solution of the given equation is

$$z^2 = -(a - x)^2 + 2\sqrt{1 - a^2}\,y - y^2 + b$$

General solution is found out as usual. Singular solution does not exist.

Example 25

Solve the equation $pz^2 \sin^2 x + qz^2 \cos^2 y = 1$. The given equation does not belong to any of the standard types.

The given equation contains $(z^2 p)$ and $(z^2 q)$.

Therefore we make the substitution $Z = z^3$

$$\therefore \quad P = \frac{\partial Z}{\partial x} = 3z^2 p \quad \text{and} \quad Q = 3z^2 q$$

Using these values in the given equation, it becomes

$$\frac{P}{3}\sin^2 x + \frac{Q}{3}\cos^2 y = 1 \tag{1}$$

Equation (1) does not contain Z explicitly. Rewriting (1), we have

$$\frac{P}{3}\sin^2 x = 1 - \frac{Q}{3}\cos^2 y = a, \text{ say} \tag{2}$$

From (2), $P = 3a \operatorname{cosec}^2 x$ and $Q = 3(1 - a)\sec^2 y$

Now $dZ = P dx + Q dy$

$$= 3a \operatorname{cosec}^2 x dx + 3(1-a) \sec^2 y dy \tag{3}$$

Integrating (3) and replacing Z by z^3, the complete solution of the given equation is

$$z^3 = -3a \cot x + 3(1-a) \tan y + b$$

General solution is found out as usual. Singular solution does not exist.

Exercise 4(b)

Part A (Short-Answer Questions)

1. Define complete solution and general solution of a P.D.E.
2. How will you find the general solution of a P.D.E. from its complete solution?
3. What is the geometrical significance of the singular solution of a P.D.E.?
4. How will you find the singular solution of a P.D.E. from its complete solution?
5. Find the complete solution of the P.D.E. $q = f(p)$
6. Find the complete solution of the P.D.E. $z = px + qy + f(p, q)$.

Find the complete solution of the following P.D.E.s.

7. $pq = k$
8. $p = e^q$
9. $p^2 + q^2 = 2$
10. $p + q = z$
11. $p^2 = qz$
12. $pq = z$
13. $pq = xy$
14. $px = qy$
15. $pe^y = qe^x$
16. Rewrite the equation $pqz = p^2(qx + p^2) + q^2(py + q^2)$ as a Clairaut's equation and hence write down its complete solution.

Part B

17. Solve the equation (a) $\sqrt{p} + \sqrt{q} = 1$; (b) $p^2 + q^2 = k^2$. Find the singular solution, if they exist.
18. Solve the equation $3p^2 - 2q^2 = 4pq$. Find the singular solution, if it exist.

19. Solve the equation $p^2 - 2pq + 3q = 5$. Find the singular solution, if it exists.

Convert the following equations into equations of the form $f(p, q) = 0$ and hence solve them.

20. $p^2 x^2 + q^2 y^2 = z^2$
21. $p^2 x + q^2 y = z$
22. $px^2 + qy^2 = z^2$
23. $z^2(p^2 - q^2) = 1$
24. $2x^4 p^2 - yzq - 3z^2 = 0$
25. $(y - x)(qy - px) = (p - q)^2$ [**Hint:** Put $x + y = X$ and $xy = Y$]

Find the singular solution os the following partial differential equations.

26. $z = px + qy - 2\sqrt{pq}$
27. $\dfrac{z}{pq} = \dfrac{x}{q} + \dfrac{y}{p} - \sqrt{pq}$
28. $z = px + qy + p^2 q^2$
29. $(p + q)(z - px - qy) = 1$
30. $z = px + qy + p^2 - q^2$
31. $z = px + qy + \sqrt{p^2 + q^2}$
32. $(1 - x) - p(2 - y)q = 3 - z$

Solve the following equations.

33. $p^2 + q^2 = z$
34. $1 + p^2 + q^2 = z^2$
35. (a) $pz = 1 + q^2$; (b) $qz = 1 + p^2$
36. $p(1 + q^2) = q(z - a)$
37. $9(p^2 z + q^2) = 4$

Convert the following equations into equations of the form $f(p, q, z) = 0$ and hence solve them.

38. $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z$
38. $(p^2 x^2 + q^2)z^2 = 1$
40. $p^2 x^4 + y^2 zq = 2z^2$

Solve the following equations

41. $q = px + p^2$

42. $yp + xq + pq = 0$
43. $yp - x^2 q^2 = x^2 y$
44. $q(p - \sin x) = \cos y$

Convert the following equations into equations of the form $f(p, q, x, y) = 0$ and hence solve them.

45. $(p^2 - q^2)z = x - y$
46. $(p^2 + q^2)z^2 = x^2 + y^2$
47. $p^2 + x^2 y^2 q^2 = x^2 z^2$
48. $4z^2 q^2 = y - x + 2zp$
49. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$ [**Hint:** Put $x + y = X$ and $x - y = Y$]
50. $(p^2 + q^2)(x^2 + y^2) = 1$ [**Hint:** Put $x = r \cos \theta$ and $y = r \sin \theta$]

4.10 GENERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations, for which the general solution can be obtained directly, can be divided into the following three categories.

1. Equations that can be solved by direct (partial) integration. For example, consider the equation.

$$\frac{\partial z}{\partial x} = a \tag{1}$$

If z were a function of x only, direct integration with respect to x will give the solution as

$$z = ax + b \tag{2}$$

If (2) is to be the general solution of (1), b need not be a constant, but it may be an arbitrary function of y, say $f(y)$. Then (2) becomes

$$z = ax + f(y) \tag{3}$$

When we differentiate (3) partially with respect to x, we get Eq. (1). As (3) contains an arbitrary function, it is the general solution.
Thus when we get the solution of an equation by partial integration with respect to x [or y], we should take an arbitrary function of y [or x] in the place of arbitrary constants taken when ordinary integration is performed.

Equations, in which the dependent variable occurs only in the partial derivatives, can be solved by this partial integration method.

2. Lagrange's linear equation of the first order, which will be discussed in Section 4.11.
3. Linear partial differential of higher order with constant coefficients, which will be discussed in Section 4.12.

4.11 LAGRANGE'S LINEAR EQUATION

A linear partial differential equation of the first order, which is of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z, is called *Lagrange's linear equation*.
We have already shown that the elimination of the arbitrary function 'f' from $f(u, v) = 0$ leads to Lagrange's linear equation.

General solution of Lagrange's linear equation

The general solution of the equation $Pp + Qq = R$ is $f(u, v) = 0$, where 'f' is an arbitrary function and $u(x, y, z) = a$ and $v(x, y, z) = b$ are independent solutions of the simultaneous differential equations $\dfrac{dx}{p} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

Proof

$$f(u, v) = 0 \tag{1}$$

Differentiating (1) partially with respect to x and then y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \tag{2}$$

and
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \tag{3}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (2) and (3), we get

$$\frac{u_x + u_z p}{u_y + u_z q} = \frac{v_x + v_z p}{v_y + v_z q}$$

i.e.
$$(u_y v_z - u_z v_y)p + (u_z v_x - u_x v_z)q = u_x v_y - u_y v_x \tag{4}$$

Taking , $P = u_y v_z - u_z v_y$, $Q = u_z v_x - u_x v_z$ and $R = u_x v_y - u_y v_x$, Eq. (4) takes the form

$$Pp + Qq = R \tag{5}$$

Since the primitive of equation (5) is equation (1), that contains an arbitrary function 'f', we conclude that $f(u, v) = 0$ is the general solution of the Lagrange's linear equation (5).

Now consider $u = a$ and $v = b$

$\therefore \quad du = 0$ and $dv = 0$

i.e.
$$u_x dx + u_y dy + u_z dz = 0 \tag{6}$$

and
$$v_x dx + v_y dy + v_z dz = 0 \tag{7}$$

Solving (6) and (7) for dx, dy, dz, we get

$$\frac{\mathrm{d}x}{u_y v_z - u_z v_y} = \frac{\mathrm{d}y}{u_z v_x - u_x v_z} = \frac{\mathrm{d}z}{u_x v_y - u_y v_x} \tag{8}$$

When we eliminate a and b from $u = a$ and $v = b$, we get the simultaneous equations (8). In other words, the solutions of equations (8) are $u = a$ and $v = b$.

Therefore the general solution of $Pp + Qq = R$ is $f(u, v) = 0$, where $u = a$ and $v = b$ are independent solutions of $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

Working rule to solve Pp + Qq = R

(i) To solve $Pp + Qq = R$, we form the corresponding subsidiary simultaneous equations $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

(ii) Solving these equations, we get two independent solution $u = a$ and $v = b$.

(iii) Then the required general is $f(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$.

4.12 SOLUTION OF THE SIMULTANEOUS EQUATIONS $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$

Method of grouping

By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, even though P; Q; R are, in general, functions of x, y, z. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings.

Method of multipliers

If we can find a set of three quantities l, m, n, which may be constants or functions of the variables x, y, z, such that lP + mQ + nR = 0, then a solution of the simultaneous equations is found out as follows.

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R} = \frac{l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z}{lP + mQ + nR}$$

Since $l\mathrm{P} + m\mathrm{Q} + n\mathrm{R} = 0$, $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0$. If $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z$ is an exact differential of some function $u(x, y, z)$, then we get d$u = 0$. Integrating this, we get $u = a$, which is a solution of $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

Similarly, if we can find another set of independent multipliers l', m', n', we can get another independent solution $v = b$.

Note ✍

1. *We may use the method of grouping to get one solution and the method of multipliers to get the other of* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. *The subsidiary equations are called Lagrange's subsidiary simultaneous equations.*
3. *The multipliers l, m, n are called Lagrange multipliers.*

Worked Examples 4(c)

Example 1

Solve the equations (i) $\frac{\partial^2 z}{\partial x^2} = xy$; (ii) $\frac{\partial^2 z}{\partial y^2} = \sin xy$

(i)
$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrating both sides of (1) partially with respect to x (i.e. treating y as a constant),

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + \phi(y) \tag{2}$$

Integrating (2) partially with respect to x,

$$z = \frac{x^3}{6}y + f(y) + x \cdot \phi(y) \tag{3}$$

where $f(y)$ and $\phi(y)$ are arbitrary functions. Equation (3) is the required general solution of (1).

(ii)
$$\frac{\partial^2 z}{\partial y^2} = \sin xy \tag{4}$$

Integrating (4) partially with respect to y,

$$\frac{\partial z}{\partial y} = -\frac{1}{x}\cos xy + \phi(x) \tag{5}$$

Integrating (5) partially with respect to y,

$$z = -\frac{1}{x^2}\sin xy + f(x) + y \cdot \phi(x) \tag{6}$$

where $f(x)$ and $\phi(x)$ are arbitrary functions. Equations (6) is the required general solution of (4)

Example 2

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, if $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$

Also show that $u \to \sin x$, when $t \to \infty$.

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \tag{1}$$

Integrating (1) partially with respect to x,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \tag{2}$$

When $x = 0$, $\dfrac{\partial u}{\partial t} = 0$. (given)

Using this in (2), we get $f(t) = 0$.

$\therefore$ Equation (2) becomes $\dfrac{\partial u}{\partial t} = e^{-t} \sin x$ (3)

Integrating (3) partially with respect to t, we get

$$u = -e^{-t} \sin x + g(x) \tag{4}$$

Using the given condition, namely, $u = 0$ when $t = 0$, in (4), we get

$$0 = -\sin x + g(x) \text{ or } g(x) = \sin x$$

Using the value in (4), the required particular solution of (1) is $u = \sin x\,(1 - e^{-t})$.

$$\text{Now} \quad \lim_{t\to\infty}(u) = \sin x \left[\lim_{t\to\infty}(1 - e^{-t})\right]$$

$$= \sin x$$

That is when $t \to \infty$ $\quad u \to \sin x$.

Example 2

Solve the equation $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, given that $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$ when $x = 0$.

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \tag{1}$$

If z were a function of x alone, the equation (1) would have been the ordinary differential equation

$$\frac{d^2 z}{dx^2} + z = 0, \text{ i.e., } (D^2, + 1)z = 0 \tag{2}$$

The auxiliary equation of (2) is $m^2 + 1 = 0$. Its roots are $\pm i$. Hence the solution of (2) is

$$z = A \cos x + B \sin x \tag{3}$$

Solution (3) can be assumed to be obtained by integrating (2) ordinarily with respect to x.

If we replace A and B in (3) by arbitrary functions of y, the solution can be assumed to have been obtained by integrating (1) partially with respect to x.

Thus the general solution of (1) is

$$z = f(y) \cdot \cos x + g(y) \cdot \sin x \tag{4}$$

From (4),
$$\frac{\partial z}{\partial x} = -f(y) \sin x + g(y) \cos x \tag{5}$$

Using the condition that $z = e^y$ when $x = 0$ in (4), we get

$$f(y) = e^y \tag{6}$$

Using the condition that $\frac{\partial z}{\partial x} = 1$ when $x = 0$ in (5),

$$g(y) = 1 \tag{7}$$

Using (6) and (7) in (4), the required solution of (1) is $z = e^y \cos x + \sin x$.

Example 4

Solve the equations $\frac{\partial z}{\partial x} = 3x - y$ and $\frac{\partial z}{\partial y} = -x + \cos y$ simultaneously.

$$\frac{\partial z}{\partial x} = 3x - y \tag{1}$$

$$\frac{\partial z}{\partial y} = -x + \cos y \tag{2}$$

Integrating (1) partially with respect to x,

$$z = \frac{3x^2}{2} - yx + f(y) \tag{3}$$

Differentiating (3) partially with respect to y,

$$\frac{\partial z}{\partial y} = -x + f'(y) \tag{4}$$

Comparing (2) and (4), we get $f'(y) = \cos y$

$\therefore$
$$f(y) = \sin y + c \tag{5}$$

$\therefore$ The required solution is

$z = \frac{3}{2}x^2 - xy + \sin y + c,$ where c is an arbitrary constant.

Example 5

By changing the independent variables by the transformations $u = x - y$ and $v = x + y$, show that the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ can be transformed as $\frac{\partial^2 z}{\partial v^2} = 0$ and hence solve it.

$$u = x - y \text{ and } v = x + y$$

$$\therefore \qquad x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{v-u}{2}$$

If we express x and y in z in terms of u and v, z becomes a function of u and v.

$$z_x = \frac{\partial z}{\partial x} = z_u \cdot u_x + z_v \cdot v_x \text{ where } z_u = \frac{\partial z}{\partial u} \text{ and } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

$$= z_u + z_v.$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = -z_u + z_v$$

$$z_{xx} = (z_{uu} + z_{uv}) + (z_{vu} + z_{vv}) = z_{uu} + 2z_{uv} + z_{vv}$$

$$z_{xy} = (-z_{uu} + z_{uv}) + (-z_{vu} + z_{vv}) = -z_{uu} + z_{vv}$$

$$z_{yy} = z_{uu} - z_{uv} + (-z_{vu} + z_{vv}) = z_{uu} - 2z_{uv} + z_{vv}$$

Using these values in the given equation $z_{xx} + 2z_{xy} + z_{yy} = 0$, it becomes $4z_{vv} = 0$.

i.e.
$$\frac{\partial^2 z}{\partial v^2} = 0 \tag{1}$$

Integrating (1) partially with respect to v,

$$\frac{\partial z}{\partial v} = g(u) \tag{2}$$

Integrating (2) partially with respect to v,

$$z = v \cdot g(u) + f(u) \tag{3}$$

$\therefore$ The solution of the given equation is

$$z = f(x - y) + (x + y)g\,(x - y)$$

Example 6

By changing the independent variables by the transformations $u = x$ and $v = \dfrac{y}{x}$, transformation the equation $x^2 \dfrac{\partial^2 x}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$ and hence solve it.

When $u = x$ and $v = y/x$, $\quad x = u$ and $y = uv$.

$\therefore$ z, which is a function of x and y, can also treated as a function of u and v.

$$z_x = z_u \cdot u_x + z_v \cdot v_x = z_u - \frac{y}{x^2} z_v$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = \frac{1}{x} \cdot z_v$$

$$z_{xx} = z_{uu} + z_{uv}\left(-\frac{y}{x^2}\right) + \frac{2y}{x^3} z_v - \frac{y}{x^2}\left[z_{vu} + z_{vv}\left(\frac{-y}{x^2}\right)\right]$$

$$z_{xy} = z_v \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x}\left[z_{vu} + z_{vv}\left(-\frac{y}{x^2}\right)\right]; z_{yy} = \frac{1}{x}\left[z_{vv} \cdot \frac{1}{x}\right]$$

Using these values in the given equation, it becomes,

$$\left(x^2 z_{uu} - yz_{uv} + \frac{2y}{x} z_v - yz_{uv} + \frac{y^2}{x^2} z_{vv}\right)$$

$$+\left(-\frac{2y}{x} z_v + 2yz_{uv} - \frac{2y^2}{x^2} z_{vv}\right) + \left(\frac{y^2}{x^2} z_{vv}\right) = 0$$

i.e. $$x^2 z_{uu} = 0 \quad \text{or} \quad z_{uu} = 0 \tag{1}$$

Integrating (1) partially with respect to u,

$$z_u = \phi(v) \tag{2}$$

Integrating (2) partially with respect to u,

$$z = f(v) + u \cdot \phi(v) \tag{3}$$

$\therefore$ Solution of the given equation is

$$z = f(y/x) + x \cdot \phi(y/x)$$

Example 7

Transform the partial differential equation $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$ to the form $\frac{\partial^2 z}{\partial u \partial v} = 0$ by using the substitutions $u = x + \alpha y$ and $v = x + \beta y$, where α and β are appropriate constants and hence solve the given equation.

Clearly z, which is a function of x and y, can also be treated as a function of u and v.

$$z_x = z_u + z_v; \qquad z_y = \alpha z_u + \beta z_v$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}; \qquad z_{xy} = z_{uu} \cdot \alpha + z_{uv} \cdot \beta$$

$$+ z_{vu} \cdot \alpha + z_{vv} \cdot \beta \text{ or } \alpha z_{uu} + (\alpha + \beta) z_{uv} + \beta z_{vv}$$

$$z_{yy} = \alpha(z_{uu} \cdot \alpha + z_{uv} \cdot \beta) + \beta(z_{vu} \cdot \alpha + z_{vv} \cdot \beta)$$

$$= \alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}.$$

Using these values in the given equation, it becomes

$$(z_{uu} + 2z_{uv} + z_{vv}) - 5[\alpha z_{uu} + (\alpha + \beta) z_{uv} + \beta z_{vv}]$$

$$+ 6[\alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}] = 0$$

i.e. $$(6\alpha^2 - 5\alpha + 1) z_{uu} + [2 - 5(\alpha + \beta) + 12\alpha\beta] z_{uv} + (6\beta^2 - 5\beta + 1) z_{vv} = 0 \tag{1}$$

Since (1) has to reduce to the form $z_{uv} = 0$, coefficient of $z_{uu} = 0$ coefficient of z_{vv}.

i.e. $$6\alpha^2 - 5\alpha + 1 = 0 \text{ and } 6\beta^2 - 5\beta + 1 = 0$$

i.e. $$\alpha = \frac{1}{2}, \frac{1}{3} \quad \text{and} \quad \beta = \frac{1}{2}, \frac{1}{3}$$

If we choose equal values for α and β, coefficient of z_{uv} also becomes zero. Hence we choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.

For these value of α and β, equation (1) becomes

$$-\frac{1}{6}z_{uv} = 0 \text{ or } \frac{\partial^2 z}{\partial u \partial v} = 0 \tag{2}$$

Integrating (2) partially with respect to u,

$$\frac{\partial z}{\partial v} = \phi(v) \tag{3}$$

Integrating (3) partially with respect to v,

$$z = \int \phi(v)\,\mathrm{d}v + f(u)$$

i.e. $$z = f(u) + g(v)$$

$\therefore$ The solution of the given equation is

$$z = f\left(x + \frac{1}{2}y\right) + g\left(x + \frac{1}{3}y\right)$$

$$\text{or } z = f(y + 2x) + g(y + 3x)$$

Example 8

Solve the equation $x^2p + y^2q + z^2 = 0$

The given equation

$$x^2 p + y^2 q = -z^2 \tag{1}$$

is a Lagrange's linear equation with $P = x^2$, $Q = y^2$ and $R = -z^2$

The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}z}{-z^2}$$

Taking the first two ratios, we get an ordinary differential equation in x and y, namely, $\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2}$.

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} - a$

i.e. $$\frac{1}{x} - \frac{1}{y} = a \qquad (2)$$

Taking the last two raties, we get the equation $\frac{dy}{y^2} = \frac{-dz}{z^2}$

$$\frac{dy}{y^2} = \frac{-dz}{z^2}$$

Integrating, we get $\frac{-1}{y} = \frac{1}{z} - b$

Solving,

$$\frac{1}{y} + \frac{1}{z} = b \qquad (3)$$

∴ The general solution of the given equation is $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0,$ where 'f' is an arbitrary function.

Example 9

Solve the equation $y^2p - xyq = x(z - 2y)$.

The given equation is a Lagrange's linear equation with $P = y^2$, $Q = -xy$, $R = x(z - 2y)$. The subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking the first two ratios, we get

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad -x\,dx = y\,dy$$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} = \frac{a}{2}$ or $x^2 + y^2 = a$ (1)

From the subsidiary equations, we have

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} = \frac{z\,dy + y\,dz}{-2xy^2}$$

From the first and last ratios, we get

$$\frac{dx}{1} = \frac{d(yz)}{-2x} \quad \text{or} \quad -2x\,dx = d(yz)$$

Integrating, we get $x^2 + yz = b$ (2)

From (1) and (2) the general solution of the given equation is $f(x^2 + y^2, x^2 + yz) = 0$.

Example 10

Solve the equation $(p - q)z = z^2 + (x + y)$. This is a Lagrange's linear equation with $P = z$, $Q = -z$ and $R = z^2 + (x + y)$.

The subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)}$$

From the first two ratios, we get $dx = -dy$

Integrating, we get $\qquad x + y = a^2 \qquad (1)$

Note ✍

Neither the method of grouping nor the method of multipliers can be used to get the second solution.

We make use of solution (1), i.e. we put $x + y = a^2$ in the third ratio.

From the first and third ratios, we get

$$\frac{dx}{z} = \frac{dz}{z^2 + a^2} \quad \text{or} \quad 2dx = \frac{2z dz}{z^2 + a^2}$$

Integrating, we get $2x = \log (z^2 + a^2) + b$. Now using the value of a^2 from (1), the second solution is

$$2x - \log (z^2 + x + y) = b \qquad (2)$$

From (1) and (2), the general solution of the given equation is

$$f[x + y, 2x - \log (x + y + z^2)] = 0$$

Example 11

Solve the equation $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

This is a Lagrange's linear equation with $P = (z^2 - 2yz - y^2)$, $Q = xy + zx$ and $R = xy - zx$.

The subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

From the last two ratio, we have

$$(y - z)dy = (y = z)dz$$

i.e. $\qquad y dy - (z dy + y dz) - z dz = 0$

i.e. $\qquad y dy - d(yz) - z dz = 0$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{a}{2} \text{ or}$$

$$y^2 - 2yz - z^2 = a \tag{1}$$

Using the multipliers, x, y, z, each of the above ratios = $\dfrac{x\mathrm{d}x + y\mathrm{d}y + a\mathrm{d}z}{0}$

$\therefore$ $x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$

Integrating, we get $x^2 + y^2 + z^2 = b$ (2)

Therefore the general solution of the given equation is $f(y^2 - 2yz - z^2, x^2, + y^2 + z^2) = 0$

Example 12

Solve the equation $(x - 2z)p + (2z - y)q = y - x$. This is a Lagrange's linear equation with $P = x - 2z$, $Q = 2z - y$ and $R = y - x$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x-2z} = \frac{\mathrm{d}y}{2z-y} = \frac{\mathrm{d}z}{y-x} \tag{1}$$

Using the multipliers 1, 1, 1, each ratio in (1) = $\dfrac{\mathrm{d}x + \mathrm{d}y + \mathrm{d}z}{0}$

$\therefore$ $\mathrm{d}x + \mathrm{d}y + \mathrm{d}z = 0$

Integrating, we get, $x + y + z = a$ (2)

Using the multipliers y, x, $2z$, each ratio in (1) = $\dfrac{y\mathrm{d}x + x\mathrm{d}y + 2z\mathrm{d}z}{0}$

$\therefore$ $\mathrm{d}(xy) + 2z\mathrm{d}z = 0$

Integrating, we get $xy + z^2 = b$ (3)

Therefore the general solution of the given equation is $f(x + y + z, xy + z^2) = 0$

Example 13

Solve the equation $(x^2 - y^2 - z^2)p + 2xyq = 2zx$. This is a Lagrang's linear equation with $P = x^2 - y^2 - z^2$, $Q = 2xy$, $R = 2zx$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2 - y^2 - z^2} = \frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2zx} \tag{1}$$

Taking the last two ratios, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log a$

i.e. $$\frac{y}{z} = a \tag{2}$$

Using the multipliers, x, y, z, each of the ratios in (1) = $\dfrac{r\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$ (3)

Taking the last ratio in (1) and the ratio in (3),

$$\frac{dz}{2zx} = \frac{\frac{1}{2}d(x^2+y^2+z^2)}{x(x^2+y^2+z^2)}$$

i.e. $$\frac{dz}{z} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

Integrating, we get $\log b + \log z = \log (x^2 + y^2 + z^2)$

i.e. $$\frac{x^2+y^2+z^2}{z} = b \tag{4}$$

Therefore the general solution of the given equation is $f\left(\dfrac{y}{z}, \dfrac{x^2+y^2+z^2}{z}\right) = 0.$

Example 14

Solve the equations $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

This is a Lagrange's linear equation with $P = x^2(y-z)$, $Q = y^2(z-x)$, $R = z^2(x-y)$.

The subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \tag{1}$$

Using the multipliers $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$, each of the ratios in (1) = $\dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$\therefore$ $$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get $-\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -a$

or $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \tag{2}$$

Using the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$, each of the ratios in (1) = $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \qquad \frac{1}{x}\mathrm{d}x+\frac{1}{y}\mathrm{d}y+\frac{1}{z}\mathrm{d}z = 0$$

Integrating, we get log x + log + log z = log b

or $$xyz = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0$.

Example 15

Solve the equation $(mz - ny)p + (nx - lz)q = ly - mx$. Hence down the solution of the equation $(2z - y)p + (x + z)q + 2x + y = 0$.

The equation $(mz - ny)p + (nx - lz)q = ly - mx$

is a Lagrang's linear equation with $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$.
The subsidiary equations are

$$\frac{\mathrm{d}x}{mz-ny} = \frac{\mathrm{d}y}{nx-lz} = \frac{\mathrm{d}z}{ly-mx} \tag{1}$$

Using the two set of multipliers l, m, n and x, y, z, each of the above ratios in (1)

$$= \frac{l\mathrm{d}x+m\mathrm{d}y+n\mathrm{d}z}{0} \text{ and also } = \frac{x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z}{0}$$

$$\therefore \qquad l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0 \quad \text{and} \quad x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$$

Integrating both the equations, we get

$$lx + my + nz = a \text{ and } x^2 + y^2 + z^2 = b$$

Therefore the general solution of the given equation is $f(lx + my + nz, x^2 + y^2 + z^2) = 0$.

Comparing the equation

$$(2z - y)p + (x + z)q = -2x - y \tag{2}$$

with the previous equations (1), we get $l = -1$, $m = 2$, $n = 1$.

Therefore the solution of equation (2) is

$$f(-x + 2y = z, x^2 + y^2 + z^2) = 0$$

Example 16

Solve the equation $(y + z)p + (z + x)q = x + y$.

This is a Lagrange's equation with $P = y + z$, $Q = z + x$ and $R = x + y$.

The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \tag{1}$$

Each of the ratios in (1) is equal to

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(z-x)}{-(z-x)} \tag{2}$$

Taking the first two ratios in (2), we get

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get log $(x-y)$ = log $(y-z)$ + log a

i.e.
$$\frac{x-y}{y-z} = a \tag{3}$$

Note ✍

Taking the last two ratios in (2) and integrating, we get another solution, namely

$$\frac{z-x}{y-z} = b \tag{4}$$

But solution (4) is not independent of solution (3), since $-\left(1+\frac{x-y}{y-z}\right)$ = $-(1 + a)$, *i.e.* $\frac{z-x}{y-z} = b$.

Hence we should use solution, (3) or (4) only to write down the general solution of the give equation.

Now each of the ratios in (1) is also equal to

$$\frac{d(x+y+z)}{2(x+y+z)} \tag{5}$$

Taking the first ratio in (2) and the ratio (5), we have $\frac{d(x+y+z)}{(x+y+z)} = -\frac{2d(x-y)}{x-y}$

Integrating, we get log $(x + y + z) = -2$ log $(x - y)$ + log c

i.e.
$$(x-y)^2 (x+y+z) = c \tag{6}$$

Therefore the general solution of the $f\left\{\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right\} = 0$

Example 17

Solve the equation $x(y^2 + z^2)p + y(z^2 + x^2)q = z(y^2 - x^2)$.

This is a Lagrange's linear equation with $P = x(y^2 + z^2)$, $Q = y(z^2 + x^2)$ and $R = z(y^2 - x^2)$.

The subsidiary equations are

$$\frac{dx}{x(y^2+z^2)} = \frac{dy}{y(z^2+x^2)} = \frac{dz}{z(y^2-x^2)} \tag{1}$$

Using the multipliers $-\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) = $\dfrac{-\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

Integrating, we get – log x + log y + log z = log a

i.e.
$$\frac{yz}{x} = a \tag{2}$$

Using the multipliers x, $-y$, z, each of the ratios in (1) = $\dfrac{xdx-ydy+zdz}{0}$

$\therefore$
$$xdx - ydy + zdx = 0$$

Integrating, we get
$$x^2 - y^2 + z^2 = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{yz}{x}, x^2-y^2+z^2\right) = 0$

Example 18

Find the integral surface of the equation $px + qy = x$, passing through $x + y = 1$ and $x^2 + y^2 + z^2 = 4$.

The general solution or integral of the Lagrange's linear equation

$$px + qy = z \tag{1}$$

represent a surface. This surface is called the integral surface of the equation.

Now the particular integral passing through the circle given by (2) and (3) is required.

$$x + y = 1 \tag{2}$$

$$x^2 + y^2 + z^2 = 4 \tag{3}$$

First let us find the general integral surface of equation (1).
The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy}{x} \tag{4}$$

Two independent solution of (4) are easily found as

$$\frac{x}{y} = a \tag{5}$$

and
$$\frac{y}{z} = b \tag{5$'$}$$

Therefore the general integral surface of (1) is

$$f\left(\frac{x}{y},\frac{y}{z}\right)=0 \tag{6}$$

Instead of finding the particular value of 'f' that satisfies (2) and (3), we proceed alternatively as follow.

We eliminate x, y, z from (2), (3), (5) and (5)′ and get a relation satisfied by a and b, which are then replaced by their equivalents, namely, $\frac{x}{y}$ and $\frac{y}{z}$ respectively.

Using (5)′ in (3),
$$x^2+y^2+\frac{y^2}{b^2}=4 \tag{7}$$

Using (5) in (2) and (7), we have

$$x\left(1+\frac{1}{a}\right)=1 \tag{8}$$

and
$$x^2\left(1+\frac{1}{a^2}+\frac{1}{a^2b^2}\right)=4 \tag{9}$$

Eliminating x between (8) and (9), we get

$$\frac{(a^2b^2+b^2+1)}{b^2(a+1)^2}=4 \tag{10}$$

Substituting for a and b from (5) and (6) in (10), we get

$$\frac{x^2}{z^2}+\frac{y^2}{z^2}+1=4\frac{y^2}{z^2}\left(\frac{x+y}{y}\right)^2$$

viz., $x^2+y^2+z^2=4(x+y)^2$, which is the equation of the required integral surface.

Example 19

Show that the integral surface of the equation $2y(z-3)p+(2x-z)q=y(2x-3)$ that passes through the circle $x^2+y^2=2x$, $z=0$ is $x^2+y^2-z^2-2x+4z=0$.

The subsidiary equations of the given Lagrange's equation are

$$\frac{dx}{2y(z-3)}=\frac{dy}{2x-z}=\frac{dz}{y(2x-3)} \tag{1}$$

Taking the first and lat ratios in (1), we have

$$\frac{dx}{2z-6}=\frac{dz}{2x-3}\quad\text{or } (2x-3)dx=(2z-6)dz$$

Integrating, we get
$$x^2-z^2-3x+6z=a \tag{2}$$

Using the multipliers 1, $2y$, -2, each ratio in (1) $=\dfrac{dx+2ydy-2dz}{0}$

$\therefore \qquad dx + 2ydy - 2dz = 0$

Integrating, we get $\qquad x + y^2 - 2z = b \qquad (3)$

The required surface has to pass through

$$x^2 + y^2 = 2x \quad \text{and} \qquad (4)$$

$$z = 0 \qquad (5)$$

Using (5) in (2) and (3), we get

$$x^2 - 3x = a \qquad (6)$$

and $\qquad x + y^2 = b \qquad (7)$

From (6) and (7), we get

$$x^2 + y^2 - 2x = a + b \qquad (8)$$

Using (4) in (8), we have

$$a + b = 0 \qquad (9)$$

Substituting for a and b from (2) and (3) in (9), we get $x^2 + y^2 - z^2 - 2x + 4z = 0$, which is the equation of the required integral surface.

Example 20

Show that the integral surface of the partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$ is $x^2 + y^2 + 2xyz - 2z + 2 = 0$.

The subsidiary equations of the given Lagrange's equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \qquad (1)$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) = $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$xyz = a \qquad (2)$$

Using the multipliers $x, y, -1$, each of the ratios in (1) = $\dfrac{xdx + ydy - dz}{0}$

$\therefore \qquad xdx + ydy - dz = 0$

Integrating, we get

$$x^2 + y - 2z = b \tag{3}$$

The required surface has to pass through

$$x + y = 0 \tag{4}$$

and
$$z = 1 \tag{5}$$

Using (4) and (5) in (2) and (3), we have

$$-x^2 = a \tag{6}$$

and
$$2x^2 - 2 = b \tag{7}$$

Eliminating x between (6) and (7), we get

$$2a + b + 2 = 0 \tag{8}$$

Substituting for a and b from (2) and (3) in (8), we get $2xyz + x^2 + y^2 - 2z + 2 = 0$ or $x^2 + y^2 + 2xyz - 2z + 2 = 0$, which is the equation of the required surface.

Exercise 4(c)

Part A (Short-Answer Questions)

Solve the following equations.

1. $\dfrac{\partial^2 z}{\partial x^2} = 0$

2. $\dfrac{\partial^2 z}{\partial y^2} = 0$

3. $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

4. $\dfrac{\partial^2 z}{\partial x^2} = e^{x+y}$

5. $\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + 3y)$

6. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}$

7. $\dfrac{\partial^2 z}{\partial x^2} = \sin y$

8. $\dfrac{\partial^2 z}{\partial y^2} = \cos y$

9. $\dfrac{\partial^2 z}{\partial x \partial y} = k$

10. $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

11. Give the working rule to solve the Lagrange's linear equation.

Find the general solutions of the following Lagrange's equations.

12. $pyx + qzx = xy$
13. $yq - xp = z$
14. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$
15. $p \tan x + q \tan y = \tan z$
16. $px^2 + qy^2 = z^2$

Part B

17. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\dfrac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\dfrac{\pi}{2}$.

18. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, given that $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$.

19. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} = z$, given that $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$ when $y = 0$.

20. Solve the equation $p = 6x + 3y$, $q = 3x - 4y$ simultaneously.

21. Solve the equation $x\dfrac{\partial z}{\partial y} = 2x + y + 3z$.

22. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} + 18xy^2 + \sin(2x - y) = 0$.

23. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} - 5\dfrac{\partial z}{\partial y} + 6z = 12y$.

24. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = 0$, $\dfrac{\partial^2 z}{\partial y^2} = 0$ simultaneously.

25. By changing the independent variables by the transformations $u = x + at$, $v = x - at$, show that the equation $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ get transformed into the equation $\dfrac{\partial^2 z}{\partial u \partial v} = 0$. Hence obtain the general solution of the equation.

26. By changing the independent variable by the transformations $z = x + iy$, $z^* = x - iy$, where $i = \sqrt{-1}$, show that the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ gets transformed into the equation $\frac{\partial^2 u}{\partial z \partial z^*} = 0$. Hence obtain the general solution of the equation.

27. Use the transformations $x = u + v$, $y = u - v$ to change the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ as $\frac{\partial^2 z}{\partial u \partial v} = 0$ and hence solve it.

28. Find the solution of the equation $y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$, by transforming it to a simpler form using the substitutions $u = x^2 + y^2$, $v = x^2 - y^2$.

29. Reduce the equation $4y^3 z_{xx} - yz_{yy} + z_y - 0$ to a simpler form by using the transformations $u = y^2 + x$ and $v = y^2 - x$ and hence solve it.

Find the general solutions of following linear partial differential equations.

30. (i) $p \cot x + q \cot y = \cot z$
(ii) $(a - x)p + (b - y)q = (c - z)$

31. $\frac{y^2 z}{x} p + xzq = y^2$

32. (i) $x^2 p + y^2 q = (x + y)z$; (ii) $x^2 p - y^2 q = (x - y)z$

33. $(y^2 + z^2)p - xyq + xz = 0$

34. $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$

35. $p - q = \log (x + y)$

36. $x(xp - yq) = y^2 - x^2$

37. (i) $(y - z)p + (z - x)q = x - y$; (ii) $(y - z)p + (x - y)q = z - x$

38. (i) $x(y - z)p + y(z - x)q = z(x - y)$

(ii) $\frac{y - z}{yz} p + \frac{z - x}{zx} \cdot q = \frac{x - y}{xy}$

39. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

40. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. [See example (16)]

41. (i) $(y + z)p - (x + z)q = x - y$ (ii) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

42. $(y^3 x - 2x^4)p + (2y^4 - x^3 y)q = (x^3 - y^3)z$.

43. Find the integral surface of the equation $px + qy = z$, that passes through the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 2$.

44. Find the integral surface of the equation $yp + xq + 1 = z$, that passes through the curve $z = x^2 + y + 1$ and $y = 2x$.

45. Show that the integral surface of the equation $(x^2 - a^2)p + (xy - az \tan \alpha) q = xz - ay \cot \alpha$, that passes through the curve $x^2 + y^2 = a^2$, $z = 0$ is $x^2 + y^2 - a^2 = z^2 \tan^2 \alpha$.

4.13 LINEAR P.D.E.'s OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

Linear partial differential equations of higher order with constant coefficient may be divided into two categories as given below.

(i) Equations in which the partial derivatives occurring are all of the same order (of course, with degree 1 each) and the coefficients are constants. Such equations are called *homogeneous linear* P.D.E.s with constant coefficients.

(ii) Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called *non-homogeneous linear* P.D.E.s with constant coefficients.

For example,

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y} \text{ and}$$

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} - 4\frac{\partial^3 z}{\partial x \partial y^2} + 12\frac{\partial^3 z}{\partial y^3} = x + 2y$$

are equation of the first category.

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = x^2 + y^2 \text{ and}$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = \cos(x + 2y)$$

The standard form of a homogeneous linear partial differential equation of n^{th} order with constant coefficients is

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots a_n \frac{\partial^n z}{\partial y^n} = R(x, y) \quad (1)$$

where a' are constants.

If we use operators $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$, we can symbolically write equation (1) as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots + a_n D'^n)z = R(x, y) \quad (2)$$

i.e. $$f(D, D')z = R(x, y) \quad (3)$$

where $f(D, D')$ is a homogeneous polynomial of the n^{th} degree in D and D'.

The method of solving (3) is similar to that of solving ordinary linear differential equations with constant coefficients.

The general solution of (3) is of the form z = (Complementary function) + (Particular integral), where the complementary function (C.F.) is the R.H.S. of the general solution of $f(D, D')z = 0$ and the particular integral (P.I.) is given symbolically by $\frac{1}{f(D,D')}R(x,y)$.

Complementary function of $f(D, D')z = R(x, y)$

C.F. of the solution of $f(D, D')z = R(x, y)$ is the R.H.S. of the solution of

$$f(D, D')z = 0 \tag{1}$$

Let us assume that

$$z = \phi(y + mx) \tag{2}$$

is a solution of equation (1), where ϕ is an arbitrary function.

Differentiating (2) partially with respect to x and then with respect to y, we have

$$Dz = \frac{\partial z}{\partial x} = m\phi'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2\phi''(y+mx)$$

$$\vdots$$

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n\phi^{(n)}(y+mx)$$

Similarly, $$D_z^{r^n}\frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + mx) \text{ and}$$

$$D^{n-r}D_y^{r} = \frac{\partial^n z}{\partial x^{n-r}\partial y^r} = m^{n-r}\phi^{(n)}(y+mx)$$

Since (2) is solution of (1), we have

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n)\, \phi^{(n)} (y + mx) = 0 \tag{3}$$

Since ϕ is arbitrary, $\phi^{(n)}(y+mx) \not\equiv 0$

$\therefore$ (3) reduces to $a_0 m^n + a_1 m^{n-1} + \ldots + a_n = 0$ or $f(m, 1) = 0$ (4)

Thus $z = \phi(y + mx)$ will be a solution of (1), if m satisfies the algebraic equation (4) or m is a root of equation (4), which we get by replacing D by m and D' by 1 in the equation $f(D, D')z = 0$ and by dropping z from it.

The equation $f(m, 1) = 0$ is called the auxiliary equation, which is an algebraic equation of the n^{th} degree in m and hence will have n roots.

Case (i)

The roots of (4) are distinet (real or complex).

Let them be $m_1, m_2, \ldots m_n$.

The solutions of (1) corresponding to these roots are $z = \phi_1(y + m_1x)$, $z = \phi_2(y + m_2x)\ldots$, $z = \phi_n(y + m_nx)$. The general solution of (1) is given by a linear combination of these solutions.

That is the general solution of (1) is given by

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is $\phi_1(y + m_1x) + \phi_2(y + m_2x) + \cdots + \phi_n(y + m_nx)$, where ϕ_r's are arbitrary functions.

Case (ii)

Two of the roots of (4) are equal and other are distinct.

Let them be $m_1, m_3, m_4, \ldots, m_n$.

Note ✍

If we apply the rule arrived at in Case (i), the solution of (1) will be $z = [\phi_1(y + m_1x) + \phi_2(y + m_1x)] + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$, i.e. $z = \phi(y + m_1x) + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$, which contains only $(n - 1)$ arbitrary functions. Hence it cannot be the general solution of Equation (1).

Then $$f(m, 1) \equiv a_0(m - m_1)^2 (m - m_3) \cdots (m - m_n)$$

$\therefore$ $$f(D, D') = a_0(D - m_1D')^2 (D - m_3D') \cdots (D - m_nD')$$

Hence solution of (1) will be a combination of the solutions of the component equations

$$(D - m_1D')^2z = 0, (D - m_3D')z = 0, \cdots, (D - m_nD')z = 0$$

Consider $(D - m_rD')z = 0$, i.e $p - m_r q = 0$ which is a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$$

Solving, we get $y + m_r x = a$ and $z = b$.

$\therefore$ General solution of $(D - m_rD')z = 0$ is $f_r(y + m_r x, z) = 0$ or $z = \phi_r(y + m_r x)$.

Now consider $$(D - m_1D')^2z = 0 \tag{5}$$

Let $$(D - m_1D')z = u \tag{6}$$

$\therefore$ becomes $$(D - m_1D')u = 0 \tag{7}$$

The solution of (7) is $u = \phi_1(y + m_1x)$. Using this value of u in (6), it becomes

$$(D - m_1D')z = \phi_1(y + m_1x)$$

or $$p - m_1q = \phi_1(y + m_1x) \tag{8}$$

which is a Lagrange's equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi_1(y+m_1x)}$$

Solving, we get $y + m_1x = a$ and $z - x\phi_1(y + m_1x) = b$

$\therefore$ The solution of Eq. (8) and hence Eq. (5) is

$$f[y + m_1x, z - x \cdot \phi_1(y + m_1x)] = 0$$

or $$z - x \cdot \phi_1(y + m_1x) = \phi_2(y + m_1x)$$

or $$z = x \cdot \phi_1(y + m_1x) + \phi_2(y + m_1x)$$

$\therefore$ General solution of equation (1) is

$$z = x\phi_1(y + m_1x) + \phi_2(y + m_1x) + \phi_3(y + m_3x) + \cdots + \phi_n(y + m_nx)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$x\phi_1(y + m_1x) + \phi_2(y + m_1x) + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$$

Case (iii)

'r' of the roots of Eq. (4) are equal and others distinct.

i.e. $$m_1 = m_2 = m_3 = \ldots = m_r$$

Proceeding as in Case (ii), we can show that the part of the C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$\phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + \ldots + x^{r-1}\phi_r(y + m_1x)$$

The Particular Integral of the solution of $f(D, D')z = R(x, y)$.

As in the case of ordinary differential equations, there are formulas/methods for finding particular integrals (P.I.) of the solution of homogeneous (and also nonhomogeneous) linear P.D.E.s with constant coefficients. The formulas/methods are given below without proof.

1. $$\frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a,b)}e^{ax+by}, \text{ if } f(a,b) \neq 0$$

1(a).

If $f(a, b) = 0$, $\left(D - \frac{a}{b}D'\right)$ or its power will be a factor of $f(D, D')$. In this case we factorise $f(D, D')$ and proceed as in ordinary differential equations and use the following results.

$$\frac{1}{\left(D-\frac{a}{b}D'\right)}e^{ax+by} = xe^{ax+by}; \frac{1}{\left(D-\frac{a}{b}D'\right)^2}e^{ax+by} =$$

$$\frac{x^2}{2!}e^{ax+by}, \cdots, \frac{1}{\left(D-\frac{a}{b}D'\right)^r}e^{ax+by} = \frac{x^r}{r!}e^{ax+by}$$

The above results can be derived by using Lagrange's linear equation method.

For example, let $\frac{1}{\left(D-\frac{a}{b}D'\right)^r}e^{ax+by} = z.$

i.e., $p - \frac{a}{b}q = e^{ax+by}$

The subsidiary equations are

$$\frac{dx}{1} = \frac{b\,dy}{-a} = \frac{dz}{e^{ax+by}}$$

The solutions of these equations are $ax + by = c_1$ and $z = xe^{c_1}$ or $z = xe^{ax+by}$.

2. $\frac{1}{f(D^2, DD', D'^2)} \begin{matrix}\sin\\\cos\end{matrix}(ax+by)$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \begin{matrix}\sin\\\cos\end{matrix}(ax+by)$$

provided $f(a^2, -ab, -b^2) \neq 0$.

2(a).

If $f(-a^2, -ab, -b^2) = 0$, then will be $\left(D^2 - \frac{a^2}{b^2}D'^2\right)$ will be a factor of $f(D^2, DD', D'^2)$. In this case, we proceed as in ordinary differential equations and use the results.

$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\sin(ax+by) = -\frac{x}{2a}\cos(ax+by) \text{ and}$$

$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax+by) = \frac{x}{2a}\sin(ax+by)$$

which may be verified by the reader.

3. $\frac{1}{f(D, D')}x^m y^n = [f(D, D')]^{-1} x^m y^n$ where $[f(D, D')]^{-1}$ is to be expanded in series of power of D and D'.

4. $\dfrac{1}{f(D,D')}e^{ax+by}F(x,y)=e^{ax+by}\cdot\dfrac{1}{f(D+a,D'+b)}F(x,y).$

5. $\dfrac{1}{D-mD'}F(x,y)=\left[\int F(x,a-x)\mathrm{d}x\right]_{a\to y+mx}$

This result can be derived by assuming that $\dfrac{1}{D-mD'}F(x,y)=z$ and solving for z by using Lagrange's linear equation method.

4.14 COMPLEMENTARY FUNCTION FOR A NON-HOMOGENEOUS LINEAR EQUATION

Let the non-homogeneous linear equation be $f(D, D') = 0$.

We resolve $f(D, D')$ into linear factor of form $(D - aD' - b)$.

The C.F. is the linear combination or simply the sum of (the R.H.S. functions of) the solution of the component equations $(D - a_rD' - b_r)z = 0$.

Now let us consider the equation $(D - aD' - b)z = 0$.

i.e. $p - aq = bz$, which is a Lagrang's linear equation

The subsidiary equations are

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}y}{-a}=\frac{\mathrm{d}z}{bz}$$

One solution of these equations $y + ax = c_1$. The other solution is log $z = bx +$ log c_2

or $$z = c_2e^{bx}$$

$\therefore$ The general solution of the equation is

$$\phi\left(\frac{z}{e^{bx}}, y+ax\right)=0 \text{ or } z=e^{bx}f(y+ax)$$

Note ✍

The rules/methods for finding P.I.s are the same as those for homogeneous linear equations.

4.15 SOLUTION OF P.D.E.s BY THE METHOD OF SEPARATION OF VARIABLES

In the next few chapters on applications of partial differential equations, we will have to solve *boundary value problems*, i.e. partial differential equations that satisfy certain given conditions called boundary conditions.

When solving a boundary value problem, if we first find the general solution of the concerned partial differential equation, it will be very difficult to find particular values of the arbitrary functions involved in the general solution that satisfy the boundary conditions. Hence in such situations, we try to find particular solutions of the partial differential equation that satisfy the boundary conditions and then combine them to get the solution of the boundary value problem.

A simple but powerful method of obtaining such particular solution is the *method of separation of variables*. In this method of solving a P.D.E. with z as the dependent variable and x and y as independent variables, the solution is assumed to be of the form $z = f(x) \cdot g(y)$, where f is a function of x alone and g is a function of y alone.

This assumption makes the solution of the P.D.E. depend on solutions of ordinary differential equations.

This variable separable solution of a P.D.E. is called a particular solution, as it can be verified to be a particular form of the general solution of the P.D.E.

For example, consider the equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \tag{1}$$

A variable separable solution of (1) can be obtained as

$$z = (ae^{px} + be^{-px})\,(ce^{\text{pat}} + de^{-\text{pat}}) \tag{2}$$

where a, b, c, d, p are constants.

(2) can be rewritten as

$$z = \{ac\, e^{p(x + at)} + bd\, e^{-p(x + at)}\} + \{ad\, e^{p(x - at)} + bc\, e^{-p(x - at)}\} \tag{3}$$

(3) is a particular case of

$$z = f(x + at) + \phi(x - at)$$

which is the general solution of (1) [see Problem 25 in Exercise 1(c)].

Worked Examples 4(d)

Example 1

Solve the equation

$$(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = 0$$

The auxiliary equation (got by replacing D by m and D' by 1 in the given P.D.E.) is

$$m^3 + 2m^2 - m - 2 = 0$$

i.e. $$m^3 + (m^2 + 2) - (m + 2) = 0$$

i.e. $$(m - 1)(m + 1)(m + 2) = 0$$

$\therefore$ $$m = 1, -1, -2$$

$\therefore$ General solution of the given equation is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x)$$

Note ✍

There is no particular integral in the general solution, since the R.H.S. member of the given P.D.E. is zero.

Example 2

Solve the equation

$$(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$$

The auxiliary equation is $m^3 - m^2 - 8m + 12 = 0$

$m = 2$ is a root of the auxiliary equation.

It is $(m - 2)(m^2 + m - 6) = 0$ or $(m - 2)(m - 2)(m + 3) = 0$

$\therefore \quad m = 2, 2, -3$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y + 2x) + f_2(y + 2x) + f_3(y - 3x)$$

Example 3

Solve the equation $\quad (D^2 - 3DD' + 2D'^2)z = 2\cosh(3x + 4y)$

The auxiliary equation is $\quad m^2 - 3m + 2 = 0$

i.e. $\quad (m - 1)(m - 2) = 0$

$$m = 1, 2$$

$\therefore$ The C.F. of the given P.D.E. $= f_1(y\ \ x) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2} 2\cosh(3x + 4y)$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}[e^{3x+4y} + e^{-(3x+4y)}]$$

$$= \frac{1}{3^2 - 3.34 + 2.4^2} e^{3x+4y} + \frac{1}{(-3)^2 - 3(-3)(-4) + 2(-4)^2} e^{-(3x+4y)}$$

$$= \frac{1}{5}[e^{3x+4y} + e^{-(3x+4y)}]$$

$$= \frac{2}{5}\cosh(3x + 4y)$$

$\therefore$ The general solution of the given equation is $z = f_1(y + x) + f_2(y + 2x) + \frac{2}{3}$ cosh $(3x + 4y)$.

Example 4

Solve the equtation

$$(9D^2 + 6DD' + D'^2)z = (e^x + e^{-2y})^2$$

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \text{ i.e. } (3m + 1)^2 = 0$$

$$\therefore \quad m = -1/3, -1/3$$

$$\therefore \quad \text{C.F.} = x f_1\left(y - \frac{1}{3}\cdot x\right) + f_2\left(y - \frac{1}{3}\cdot x\right) \text{ or}$$

$$x f_1(3y - x) + f_2(3y - x)$$

$$\text{P.I.} = \frac{1}{9D^2 + 6DD' + D'^2}(e^x + e^{2y})^2$$

$$= \frac{1}{9D^2 + 6DD' + D'^2}(e^{2x} + e^{-4y} + 2e^{x-2y})$$

$$= \frac{1}{(3D + D')^2}e^{2x} + \frac{1}{(3D + D')}e^{-4y} + 2\cdot\frac{1}{(3D + D')^2}e^{x-2y}$$

$$= \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

$\therefore$ The general solution of the given equation is

$$z = x f_1(3y - x) + f_2(3y - x) + \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

Example 5

Solve the equation

$$(D^3 - 3DD'^2 + D'^2)z = e^{2x-y} + e^{x+y}$$

The auxiliary equation is $m^3 - 3m + 2 = 0$

i.e. $\quad (m - 1)(m^2 + m - 2) = 0$

i.e. $\quad (m - 1)^2(m + 2) = 0$

$$\therefore \quad m = 1, 1, -2$$

$$\therefore \quad \text{C.F.} = x f_1(y + x) + f_2(y + x) + f_3(y - 2x)$$

$$\text{P.I.} = \frac{1}{D^3 + 3DD'^2 + 2D'^3}(e^{2x-y} + e^{x+y})$$

$$= \frac{1}{(D+2D')(D-D')^2}e^{2x-y} + \frac{1}{(D-D')^2(D+2D')}e^{x+y}$$

$$= \frac{1}{9}\cdot\frac{1}{D+2D'}e^{2x-y} + \frac{1}{9}\cdot\frac{1}{(D-D')^2}e^{x+y}$$

$$= \frac{1}{9}\left[xe^{2x-y} + \frac{x^2}{2}e^{x+y}\right]$$

$\therefore$ The general solution of the given equation is

$$z = x f_1(y+x) + f_2(y+x) + f_3(y-2x) + \frac{x}{9}e^{2x-y} + \frac{x^2}{18}e^{x+y}$$

Example 6

Solve the equation

$$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = (1 + e^{2x+y})^2$$

The auxiliary equation is $m^3 - 6m^2 + 12m - 8 = 0$

i.e. $(m-2)^3 = 0$

$\therefore$ $m = 2, 2, 2$

$\therefore$ $\text{C.F.} = x^2f_1(y + 2x) + x \cdot f_2(y + 2x) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{(D-2D')^3}(1+e^{2x+y})^2$$

$$= \frac{1}{(D-2D')^3}(1) + 2\cdot\frac{1}{(D-2D')^3}e^{2x+y} + \frac{1}{(D-2D')^3}e^{4x+2y}$$

$$= \frac{x^3}{3!} + 2\frac{x^3}{3!}e^{2x+y} + \frac{x^3}{3!}e^{4x+2y}$$

$$\left[\text{since } \frac{1}{(D-2D')^3}(1) = \frac{1}{(D-2D')^3}e^{0\cdot x+0\cdot y} \text{ and } \frac{1}{\left(D-\frac{a}{b}D'\right)^3}e^{ax+by} = \frac{x^3}{3!}e^{ax+by}\right]$$

$$= \frac{x^3}{6}(1+e^{2x+y})^2$$

$\therefore$ The general solution of the given equation is

$$z = x^2 f_1(y + 2x) + x f_2(y + 2x) + f_3(y + 2x) + \frac{x^3}{6}(1 + e^{2x+y})^2$$

Example 7

Solve the equation $(D^2 + 2DD' + D'^2)z = x^2y + e^{x-y}$

The auxiliary equation is $m^2 + 2m + 1 = 0$ or $(m + 1)^2 = 0$ $\therefore$ $m = -1, -1$

$\therefore$ $\text{C.F.} = xf_1(y - x) + f_2(y - x)$

$$(\text{P.I.})_1 = \frac{1}{(D+D')^2} x^2 y$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2} x^2 y$$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D}\right)^{-2} (x^2 y)$$

$$= \frac{1}{D^2}\left(1-\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2 y)$$

$$= \frac{1}{D^2}\left\{x^2 y-\frac{2}{D}x^2\right\}$$

$$= y\cdot\frac{1}{D^2}(x^2)-2\cdot\frac{1}{D^3}(x^2)$$

$$= y\cdot\frac{x^4}{3.4}-2\cdot\frac{x^5}{3.4.5}$$

$$= \frac{x^4 y}{12}-\frac{x^5}{30}$$

$$(\text{P.I.})_2 = \frac{1}{(D+D')^2} e^{x-y} = \frac{x^2}{2!} e^{x-y}$$

$\therefore$ The general solution is

$$z = x f_1(y-x)+f_2(y-x)+\frac{x^4 y}{12}-\frac{x^5}{30}+\frac{x^2}{2}e^{x-y}$$

Example 8

Solve the equation

$$(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3$$

The auxiliary equation is

$$m^3 - 7m - 6 = 0, \quad \text{i.e. } (m + 1)(m^2 - m - 6) = 0$$

i.e. $(m + 1)(m + 2)(m - 3) = 0$

$\therefore$ $m = -1, -2, 3$

$\therefore$ $\text{C.F.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

$$\text{P.I.} = \frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3)$$

$$= \frac{1}{D^3}\left\{1-\frac{(7DD'^2+6D'^3}{D^3}\right\}^{-1}(x^2+xy^2+y^3)$$

$$= \frac{1}{D^3}\left[1+\frac{D'^2}{D^3}(7D+6D')+\cdots\right](x^2+xy^2+y^3)$$

$$= \left[\frac{1}{D^3} + \frac{1}{D^6}(7DD'^2 + 6D'^3)\right](x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}\{7D \cdot (2x + 6y) + 36\}$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}(50)$$

$$= \frac{x^5}{3.4.5} + y^2 \cdot \frac{x^4}{2.3.4} + y^3 \cdot \frac{x^3}{1.2.3} + 50 \cdot \frac{x^3}{1.2.3}$$

$$= \frac{1}{60}x^5 + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

$\therefore$ The general solution is

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x^5}{60} + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

Example 9

Solve the equation

$$(D^2 + 4DD' - 5D'^2)z = xy + \sin(2x + 3y)$$

The auxiliary equation is $m^2 + 4m - 5 = 0$

i.e. $(m + 5)(m - 1) = 0$

$\therefore$ $m = -5.1$

$\therefore$ C.F. $= \phi_1(y - 5x) + \phi_2(y + x)$

$$(\text{P.I.})_1 = \frac{1}{D^2 + 4DD' - 5D'^2}(x, y)$$

$$= \frac{1}{D^2\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}}(xy)$$

$$= \frac{1}{D^2}\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}^{-1}(xy)$$

$$= \frac{1}{D^2}\left\{1 - \frac{D'}{D^2}(4D - 5D') + \cdots\right\}(xy)$$

$$= \frac{1}{D^2}(xy) - \frac{1}{D^4} \cdot 4D'(xy)$$

$$= \frac{x^3y}{6} - \frac{1}{D^4}(4x)$$

$$= \frac{1}{6}x^3y - \frac{1}{30}x^5$$

$$(\text{P.I.})_2 = \frac{1}{D^2+4DD'+5D'^2}\sin(2x+3y)$$

$$= \frac{1}{-2^2+4\cdot(-2.3)-5(-3^2)}\sin(2x+3y)$$

$$= \frac{1}{17}\sin(2x+3y)$$

$\therefore$ General solution is

$$z = \phi_1(y-5x)+\phi_2(y+x)+\frac{1}{6}x^3y-\frac{1}{30}x^5+\frac{1}{17}\sin(2x+3y)$$

Example 10

Solve the equation

$$(D^2 + D'^2)z = \sin 2x \sin 3y + 2\sin^2(x+y)$$

The auxiliary equation is $m^2 + 1 = 0$

i.e. $m = \pm i$

$\therefore$ $\text{C.F.} = \phi_1(y+ix) + \phi_2(y-ix)$

$$(\text{P.I.})_1 = \frac{1}{D^2+D'^2}\sin 2x \sin 3y$$

$$= \frac{1}{D^2+D'^2}\cdot\frac{1}{2}\{\cos(2x-3y)-\cos(2x+3y)\}$$

$$= \frac{1}{2}\left[\frac{1}{-4-9}\cos(2x-3y)-\frac{1}{-4-9}\cos(2x+3y)\right]$$

$$= -\frac{1}{13}\cdot\frac{1}{2}[\{\cos(2x-3y)-\cos(2x+3y)\}]$$

$$= -\frac{1}{13}\sin 2x \sin 3y$$

$$(\text{P.I.})_2 = \frac{1}{D^2+D'^2}2\sin^2(x+y)$$

$$= \frac{1}{D^2+D'^2}\{1-\cos(2x+2y)\}$$

$$= \frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(1)-\frac{1}{D^2+D'^2}\cos(2x+2y)$$

$$= \frac{1}{D^2}(1)-\frac{1}{-4-4}\cos(2x+2y)$$

$$= \frac{x^2}{2}+\frac{1}{8}\cos(2x+2y)$$

$\therefore$ General solution is

$$z = \phi_1(y+ix)+\phi_2(y-ix)-\frac{1}{13}\sin 2x \sin 3y+\frac{x^2}{2}+\frac{1}{8}\cos(2x+2y)$$

Example 11

Solve the equation

$$(16D^4 - D'^4)z = \cos(x + 2y)$$

The auxiliary equation is $16\,m^4 - 1 = 0$

i.e. $(m^2 - 1/4)(m^2 + 1/4) = 0$

$\therefore$ $m = \pm 1/2, \pm i/2$

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + f_2\left(y - \frac{1}{2}x\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right)$$

$$\text{P.I.} = \frac{1}{16D^4 - D'^4}\cos(x+2y)$$

$$= \frac{1}{(4D^2 - D'^2)(4D^2 + D'^2)}\cos(x+2y)$$

$$= \frac{1}{(4D^2 - D'^2)} \cdot \frac{1}{4(-1)+(-4)}\cos(x+2y)$$

$$= -\frac{1}{8} \cdot \frac{1}{4D^2 - D'^2}\cos(x+2y)$$

$$= -\frac{1}{32} \cdot \frac{1}{D^2 - \frac{1}{4}D'^2}\cos(x+2y)$$

$$= -\frac{1}{32} \cdot \frac{x}{2}\sin(x+2y) \left[\because \quad \frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax+by) = \frac{x}{2a}\sin(ax+by)\right]$$

$$= -\frac{1}{64}x\sin(x+2y)$$

$\therefore$ General solution is

$$z = f_1\left(y + \frac{x}{2}\right) + f_2\left(y - \frac{x}{2}\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right) - \frac{1}{64}x\sin(x+2y)$$

Example 12

Solve the equation

$$(D^3 + D^2D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$$

The auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$

i.e. $m^2(m + 1) - 4(m + 1) = 0$

i.e. $(m + 1)(m + 2)(m - 2) = 0$

$\therefore$ $m = -1, -2, 2$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x)$

$$\text{P.I.} = \frac{1}{(D^2-4D'^2)(D+D')}\cos(2x+y)$$

$$= \frac{1}{(D^2-4D'^2)}\cdot\frac{(D-D')}{D^2-D'^2}\cos(2x-y)$$

$$= \frac{1}{(D^2-4D'^2)}\cdot\frac{1}{-4-(-1)}(D-D')\cos(2x-y)$$

$$= \frac{1}{3}\cdot\frac{1}{D^2-4D'^2}\{-2\sin(2x-y)-\sin(2x-y)\}$$

$$= \frac{1}{D^2-4D'^2}\sin(2x-y)$$

$$= -\frac{x}{4}\cos(2x-y)\left[\because \quad \frac{1}{D^2-\dfrac{a^2}{b^2}D'^2}\sin(ax+by) = -\frac{x}{2a}\cos(ax+by)\right]$$

$\therefore$ General solution is

$$z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+2x)-\frac{x}{4}\cos(2x-y)$$

Example 13

Solve the equation

$$(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y}$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\therefore$ $m = 1, 1$

$\therefore$ C.F. $= xf_1(y+x) + f_2(y+x)$

$$\text{P.I.} = \frac{1}{(D-D')^2}e^{x+y}(x^2y^2)$$

$$= e^{x+y}\frac{1}{\{(D+1)-(D'+1)\}^2}x^2y^2$$

$$= e^{x+y}\frac{1}{(D-D')^2}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}(x^2y^2)$$

$$= e^{x+y}\frac{1}{D^2}\left(1+\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y^2)$$

$$= e^{x+y}\frac{1}{D^2}\left\{x^2y^2+\frac{2}{D}(2x^2y+\frac{3}{D^2}(2x^2)\right\}$$

$$= e^{x+y}\left[y^2 \cdot \frac{1}{D^2}(x^2) + 4y \cdot \frac{1}{D^3}(x^2) + 6 \cdot \frac{1}{D^4}(x^2)\right]$$

$$= \left(\frac{1}{12}x^4y^2 + \frac{1}{15}x^5y + \frac{1}{60}x^6\right)e^{x+y}$$

$\therefore$ General solution is

$$z = x\,f_1(y+x) + f_2(y+x) + \left(\frac{1}{12}y^2 + \frac{1}{15}xy + \frac{1}{60}x^2\right)x^4e^{x+y}$$

Example 14

Solve the equation

$$(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$$

The auxiliary equation is $m^2 - 1 = 0$

$\therefore$ $m = \pm 1$

$\therefore$ $\text{C.F.} = f_1(y+x) + f_2(y-x)$

$$\text{P.I.} = \frac{1}{D^2 - D'^2}e^{x-y}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2 - D'^2 + 2(D+D')}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{2(D+D')+5}\sin(2x+3y)$$

$$= e^{x-y}\frac{2(D+D')-5}{4(D+D')^2-25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\} \cdot \frac{1}{4(D^2+2DD'+D'^2)-25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\} \cdot \left(-\frac{1}{125}\right)\sin(2x+3y)$$

$$= -\frac{1}{125}e^{x-y}\{4\cos(2x+3y) + 6\cos(2x+3y) - 5\sin(2x+3y)\}$$

$$= \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

$\therefore$ General solution is

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

Example 15

Solve the equation

$$(D^2 - 5DD' + 6D'^2)z = y\sin x$$

The auxiliary equation is $m^2 - 5m \;\; 6 = 0$

i.e. $(m-2)(m-3)=0$

$\therefore$ $m = 2, 3$

$\therefore$ $\text{C.F.} = \phi_1(y+2x) + \phi_2(y+3x)$

$$\text{P.I.} = \frac{1}{(D+2D')(D-3D')} y \sin x$$

$$= \frac{1}{D-2D'}\left[\int (a-3x)\sin x \, dx\right]_{a \to y+3x}$$

$$= \frac{1}{D-2D'}[(a-3x)(-\cos x) + 3(-\sin x)]_{a \to y+3x}$$

$$= \frac{1}{D-2D'}[-y\cos x - 3\sin x]$$

$$= \left\{\int [(a-2x)\cos x + 3\sin x] dx\right\}_{a \to y+2x}$$

$$= -[(a-2x)\sin x + 2(-\cos x) - 3\cos x]_{a \to y+2x}$$

$$= 5\cos x - y\sin x$$

$\therefore$ General solution is

$$z = \phi_1(y+2x) + \phi_2(y+3x) + 5\cos x - y\sin x$$

Example 16

Solve the equation

$$(4D^2 - 4DD' + D'^2)z = 16 \log(x+2y)$$

The auxiliary equation is $4m^2 - 4m + 1 = 0$

i.e. $(2m-1)^2 = 0$

$\therefore$ $m = 1/2, 1/2$

$\therefore$ $\text{C.F.} = x f_1\left(y + \frac{1}{2}x\right) + f_2\left(y + \frac{1}{2}x\right)$ or

$x f_1(2y+x) + f_2(2y+x)$

$$\text{P.I.} = \frac{1}{(2D-D')^2} 16 \log(x+2y)$$

$$= 4 \cdot \frac{1}{(D-1/2D')} \cdot \frac{1}{D-1/2D'} \log(x+2y)$$

$$= 4 \cdot \frac{1}{D-1/2D'}\left\{\int \log\left[x + 2\left(a - \frac{1}{2}x\right)\right] dx\right\}_{a \to y+\frac{1}{2}x}$$

$$= 4\cdot\frac{1}{D-1/2D'}\left[\int \log(2a)\,dx\right]_{a\to y+\frac{1}{2}x}$$

$$= 4\cdot\frac{1}{D-1/2D'}\{x\log(x+2y)\}$$

$$= 4\left[\int x\log\left\{x+2\left(a-\frac{1}{2}x\right)\right\}dx\right]_{a\to y+\frac{1}{2}x}$$

$$= 4\left[\int x\log(2a)\,dx\right]_{a\to y+\frac{1}{2}x}$$

$$= 2x^2(\log 2a)_{a\to y+\frac{1}{2}x}$$

$$= 2x^2\log(x+2y)$$

∴ General solution is

$$z = xf_1(x+2y) + f_2(x+2y) + 2x^2\log(x+2y)$$

Example 17

Solve the equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \cosh(x-y)$$

The given equation is a non-homogeneous linear equation

$$D^2 + 2DD' + D'^2 - 2D - 2D' \equiv (D+D')^2 - 2(D+D')$$
$$= (D+D')(D+D'-2)$$

∴ The given equation

$$(D+D')(D+D'-2)z = \cosh(x-y)$$

∴ C.F. $= f_1(y-x) + e^{2x}\cdot f_2(y-x)$ [∵ the part of C.E. corresponding to $(D-aD'-b)z = 0$ is $e^{bx}f(y+ax)$]

$$\text{P.I.} = \frac{1}{(D+D')(D+D'-2)}\frac{1}{2}\{e^{x-y}+e^{-x+y}\}$$

$$= \frac{1}{2}\cdot\frac{1}{D+D'}\cdot\frac{-1}{2}(e^{x-y}+e^{-x+y})$$

$$= -\frac{1}{4}\cdot(xe^{x-y}+xe^{-x+y})$$

$$= -\frac{x}{2}\cosh(x-y)$$

∴ General solution is

$$z = f_1(y-x)+e^{2x}f_2(y-x)-\frac{x}{2}\cosh(x-y)$$

Example 18

Solve the equation $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y} + xy$

$$D^2 - D'^2 - 3D + 3D' \equiv (D + D')(D - D') - 3(D - D')$$
$$= (D - D')(D + D' - 3)$$

$\therefore$ The given equation is

$$(D - D')(D + D' - 3)z = e^{x+2y} + xy$$

$\therefore$ C.F. $= f_1(y + x) + e^{3x} f_2(y - x)$

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} e^{x+2y}$$

$$= \frac{1}{(D+D'-3)} \cdot (-1) e^{x+2y}$$

$$= -xe^{x+2y}$$

$$(\text{P.I.})_2 = \frac{1}{(D-D')(D+D'-3)} xy$$

$$= -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left\{1-\frac{D+D'}{3}\right\}^{-1} xy$$

$$= \frac{1}{3D}\left(1+\frac{D'}{D}\right)\left\{1+\frac{1}{3}(D+D')+\frac{1}{9}(D+D')^2+\frac{1}{27}(D+D')^3+\cdots\right\}xy$$

$$= -\frac{1}{3}\left(\frac{1}{D}+\frac{D'}{D2}\right)\left\{1+\frac{1}{3}D+\frac{1}{3}D'+\frac{1}{9}D^2+\frac{2}{9}DD'+\frac{1}{27}D^3+\frac{1}{9}D^2D'\right\}(xy)$$

$$= -\frac{1}{3}\left[\frac{1}{D}+\frac{1}{3}+\frac{1}{3}\frac{D'}{D}+\frac{1}{9}D+\frac{2}{9}D'+\frac{1}{9}DD'+\frac{D'}{D^2}+\frac{1}{3}\frac{D'}{D}+\frac{1}{9}D'+\frac{1}{27}DD'\right](xy)$$

$$= -\frac{1}{3}\left[\frac{D'}{D^2}+\frac{2}{3}\frac{D'}{D}+\frac{1}{D}+\frac{1}{3}+\frac{1}{9}D+\frac{1}{3}D'+\frac{4}{27}DD'\right]xy$$

$$= -\frac{1}{3}\left[\frac{x^3}{6}+\frac{x^2}{3}+\frac{x^2y}{2}+\frac{1}{3}xy+\frac{1}{9}y+\frac{1}{3}x+\frac{4}{27}\right]$$

$\therefore$ General solution is

$$z = \text{C.F.} + (\text{P.I.})_1 + (\text{P.I.})_2$$

Example 19

Solve the equation $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = x + y + \sin(2x + y)$

$$D^2 - 3DD' + 2D'^2 + 2D - 2D' \equiv (D - D')(D - 2D') + 2(D - D')$$
$$= (D - D')(D - 2D' + 2)$$

$\therefore$ The given equation is

$$(D - D')(D - 2D' + 2)z = (x + y) + \sin(2x + y)$$

$$\text{C.F.} = f_1(y + x) + e^{-2x} f_2(y + 2x)$$

$$(\text{P.I.})_1 = \frac{1}{(D-D')(D-2D'+2)}(x+y)$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D^2}\right)^{-1}\left(1+\frac{D-2D'}{2}\right)^{-1}(x+y)$$

$$= \frac{1}{2}\left(\frac{1}{D}+\frac{D'}{D^2}\right)\left\{1-\frac{1}{2}(D-2D')+\frac{1}{4}(D-2D')^2+\cdots\right\}(x+y)$$

$$= \frac{1}{2}\left[\frac{1}{D}+\frac{D'}{D^2}\right]\left[1-\frac{1}{2}D+D'+\frac{1}{4}D^2-DD'\right](x+y)$$

$$= \frac{1}{2}\left[\frac{1}{D}-\frac{1}{2}+\frac{D'}{D}+\frac{1}{4}D-D'+\frac{D'}{D^2}-\frac{1}{2}\frac{D'}{D}+\frac{1}{4}D'\right](x+y)$$

$$= \frac{1}{2}\left(\frac{D'}{D^2}+\frac{1}{2}\cdot\frac{D'}{D}+\frac{1}{D}-\frac{1}{2}+\frac{1}{4}D-\frac{3}{4}D'\right)(x+y)$$

$$= \frac{1}{2}\left[\frac{x^2}{2}+\frac{x}{2}+\frac{x^2}{2}+xy-\frac{1}{2}y-\frac{1}{2}x+1/4-3/4\right]$$

$$= \frac{1}{2}x^2+\frac{1}{2}xy-\frac{1}{4}y-1/4$$

$$(\text{P.I.})_2 = \frac{1}{D^2-3DD'+2D'^2+2D-2D'}\sin(2x+y)$$

$$= \frac{1}{-4+6-2+2(D-D')}\sin(2x+y)$$

$$= \frac{(D+D')}{2(D^2-D'^2)}\sin(2x+y)$$

$$= \frac{-1}{6}\{2\cos(2x+y)+\cos(2x+y)\}$$

$$= \frac{-1}{2}\cos(2x+y)$$

$\therefore$ General solution is

$$z = f_1(y+x)+e^{-2x}f_2(y+2x)+\frac{1}{2}x^2+\frac{1}{2}xy-\frac{1}{4}y-\frac{1}{4}-\frac{1}{2}\cos(2x+y)$$

Example 20

Solve the equation $(D^2 - DD' + D' - 1)z = e^{2x+3y} + \cos^2(x + 2y)$

$$D^2 - DD' + D' - 1 \equiv (D^2 - 1) - D'(D - 1)$$

$$= (D - 1)(D - D' + 1)$$

$\therefore$ The given equation is

$$(D-1)(D-D'+1)z = e^{2x+3y} + \cos^2(x+2y)$$

$$\text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x)$$

$$(\text{P.I.})_1 = \frac{1}{(D-1)(D-D'+1)} e^{2x+3y}$$

$$= \frac{1}{(2-1)(D-D'+1)} e^{2x+3y}$$

$$= xe^{2x+3y}$$

$$(\text{P.I.})_2 = \frac{1}{(D-1)(D-D'+1)} \frac{1}{2}\{1+\cos(2x+4y)\}$$

$$= \frac{1}{2}(-1) + \frac{1}{2} \cdot \frac{1}{(D^2 - DD' + D' - 1)} \cos(2x+4y)$$

$$= -1/2 + \frac{1}{2} \cdot \frac{1}{-4+8+D'-1} \cos(2x+4y)$$

$$= -1/2 + 1/2 \cdot \frac{D'-3}{(D'^2-9)} \cos(2x+4y)$$

$$= \frac{-1}{2} - \frac{1}{50}\{-4\sin(2x+4y) - 3\cos(2x+4y)\}$$

$$= \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\} - 1/2$$

$\therefore$ General solution is

$$z = e^x f_1(y) + e^{-x} f_2(y+x) + xe^{2x+3y} - 1/2 + \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\}$$

Example 21

Solve the equation $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y + ye^x$

$$2D^2 - DD' - D'^2 + 6D + 3D' \equiv (2D+D')(D-D') + 3(2D+D')$$

$$= (2D+D')(D-D'+3)$$

$\therefore$ The given equation is

$$(2D+D')(D-D'+3)z = xe^y + ye^x$$

$\therefore$
$$\text{C.F.} = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y+x)$$

or
$$f_1(2y-x) + e^{-3x} \cdot f_2(y+x)$$

$$(\text{P.I.})_1 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(xe^y)$$

$$= e^y \cdot \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)}(x)$$

$$= e^y \cdot \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2}(x)$$

$$= \frac{e^y}{2} \cdot \left\{1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)\right\}^{-1}(x)$$

$$= \frac{e^y}{2}\left\{1 - \frac{5}{2} \cdot D\right\}(x)$$

$$= \frac{1}{4}(2x - 5)e^y$$

$$(\text{P.I.})_2 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(ye^x)$$

$$= e^x \cdot \frac{1}{2(D+1)^2 - (D+1)D' - D'^2 + 6(D+1) + 3D'}(y)$$

$$= e^x \cdot \frac{1}{8 + 10D + 2D' + 2D^2 - DD' - D'^2}(y)$$

$$= \frac{e^x}{8}\left\{1 + \frac{1}{8}(10D + 2D' + 2D^2 - DD' - D'^2)\right\}^{-1}(y)$$

$$= \frac{e^x}{8}\left\{1 - \frac{1}{4}D'\right\}(y)$$

$$= \frac{1}{32}(4y - 1)e^x$$

$\therefore$ General solution is

$$z = f_1(2y - x) + e^{-3x} f_2(y + x) + \frac{1}{4}(2x - 5)e^y + \frac{1}{32}(4y - 1)e^x$$

Example 22

Solve the equation $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $z = X(x) \cdot Y(y)$ be a solution of

$$2xz_x - 3yz_y = 0 \tag{1}$$

Then $z_x = X'Y$ and $z_y = XY'$, where $X' = \dfrac{dX}{dx}$ and $Y' = \dfrac{dY}{dy}$ satisfy Eq. (1).

i.e. $\quad 2xX'Y - 3yXY' = 0$

i.e. $\quad 2x\dfrac{X'}{X} = 3y\dfrac{Y'}{Y}$

L.H.S. is a function of x alone and R.H.S. is a function of y alone. They are equal for all values of x and y. This is possible only if each is a constant.

$\therefore$ $$2x\frac{X'}{X} = 3y\frac{Y'}{Y} = k$$

i.e. $$2\frac{X'}{X} = \frac{k}{x} \tag{2}$$

and $$\frac{3Y'}{Y} = \frac{k}{y} \tag{3}$$

Integrating both sides of (2) with respect to x,

$$2\log X = k \log x + \log A$$

i.e. $$X^2 = Ax^k \text{ or } X = ax^{k/2} \tag{4}$$

Similarly, from (3), $Y = by^{k/3}$

$\therefore$ Required solution of (1) is

$$z = abx^{k/2}\,y^{k/3} \text{ or } z = cx^{k/2}\,y^{k/3}$$

Example 23

Solve the equation $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z$, by the method of separation of variables, given that

$$z(x, 0) = 6e^{-3x}$$

Let

$$z = X(x)\cdot Y(y) \tag{1}$$

be a solution of

$$z_x = 2z_y + z \tag{2}$$

Then $z_x = X'Y$ and $z_y = XY'$ satisfy equation (2).

i.e. $$X'Y = 2XY' + XY$$

Dividing throughout by X, Y, we get

$$\frac{X'}{X} = 2\frac{Y'}{Y} + 1 = k$$

[$\because$ the L.H.S. is a function of a x alone and the R.H.S. is a function of y alone]

$$\frac{X'}{X} = k \tag{3}$$

and $$\frac{Y'}{Y} = \frac{k-1}{2} \tag{4}$$

Integrating (3) and (4) with respect to x and y respectively, we get

$$\log X = kx + \log A \text{ and } \log Y = \left(\frac{k-1}{2}\right)y + \log B$$

i.e $$X = Ae^{kx} \text{ and } Y = Be^{\left(\frac{k-1}{2}\right)y}$$

$\therefore$ Required solution is

$$z = ce^{kx} \cdot e^{\left(\frac{k-1}{2}\right)y} \qquad (5)$$

Given that $z(x, 0) = 6e^{-3x}$

$\therefore$ $$ce^{kx} = 6e^{-3x}$$

$\therefore$ $$c = 6 \text{ and } k = -3$$

Using these values in (5), the required solution is $z = 6e^{-(3x + 2y)}$.

Example 24

Solve the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $$z = X(x).Y(y) \qquad (1)$$

be a solution of the equation

$$z_{xx} - 2z_x + z_y = 0 \qquad (2)$$

Then $z_x = X'Y$, $z_{xx} = X''Y$ and $z_y = XY'$ satisfy (2).

i.e. $$X''Y - 2X'Y + XY' = 0$$

Dividing throughout by XY, we get

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0$$

i.e. $$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k$$

i.e. $$X'' - 2X' - kX = 0 \qquad (3)$$

and $$Y' + kY = 0 \qquad (4)$$

i.e. $$(D^2 - 2D - k)X = 0$$

where $D \equiv \dfrac{d}{dx}$ and

$$\frac{Y'}{Y} = -k \tag{6}$$

A.E. of (5) is $m^2 - 2m - k = 0$

$$\therefore \quad m = \frac{2 \pm \sqrt{4+4k}}{2} \text{ or } 1 \pm \sqrt{k+1}$$

$\therefore$ Solution of (5) is

$$X = Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}$$

Solution of (6) is

$$Y = ce^{-ky}$$

Using these values in (1), the required solution is

$$z = \{Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}\}ce^{-ky}$$

or $$z = \{c_1e^{(1+\sqrt{k+1})x} + c_2e^{(1-\sqrt{k+1})x}\}e^{-ky}$$

Example 25

Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 5u$, by the method of separation of variables, given that $u = 0$ and $\dfrac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ and for all values of y.

Let $$u(x, y) = X(x) \cdot Y(y) \tag{1}$$

be a solution of

$$u_{xx} = u_y + 5u \tag{2}$$

Then $u_{xx} = X''Y$ and $u_y = XY'$ satisfy (2)

i.e. $$X''Y = XY' + 5XY$$

Dividing throughout by XY, we get

$$\frac{X''}{X} = \frac{Y'}{Y} + 5 = k$$

$$X'' - kX = 0 \tag{3}$$

and $$\frac{Y'}{Y} = k - 5 \tag{4}$$

Assuming that k is positive, the solutions of (3) and (4) are

$$X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

and $$Y = ce^{(k-5)y}$$

Using these values in (1), the required solution is

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{5}$$

Given: $u = 0$ when $x = 0$ and for all y

$\therefore$ $$(C_1 + C_2)e^{(k-5)y} = 0$$

i.e. $$C_1 + C_2 = 0 \tag{6}$$

Differentiating (5) partially with respect to x, we have

$$\frac{\partial u}{\partial x} = \sqrt{k}(C_1 e^{\sqrt{k}x} - C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{7}$$

Given: $\dfrac{\partial u}{\partial x} = e^{-3y}$, when $x = 0$ and for all y.

$\therefore$ $$\sqrt{k}(C_1 - C_2)e^{(k-5)y} = e^{-3y}$$

$\therefore$ $$\sqrt{k}(C_1 - C_2) = 1 \tag{8}$$

and $$k - 5 = -3 \tag{9}$$

Solving (6), (8) and (9), we get

$$k = 2,\ C_1 = \frac{1}{2\sqrt{2}} \text{ and } C_2 == \frac{1}{2\sqrt{2}}$$

Using these values in (5), the required solution is

$$u(x, y) = \frac{1}{\sqrt{2}}\sinh x\sqrt{2} \cdot e^{-3y}$$

Exercise 4(d)

Part A (Short-Answer Questions)

Solve the following equations:

1. $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = 0$
2. $(D - D')^3 z = 0$

3. $(D^2 + D'^2)^2 z = 0$
4. $(D^3 + 4D^2D' - 5DD'^2)z = 0$
5. $(2D^2D' - 5DD'^2 - 3D'^3)z = 0$
6. $(D + D' - 1)(D - D' + 1)z = 0$
7. $D(D - 2D' + 3)z = 0$
8. $D'(D + 3D' - 2)z = 0$
9. $(D + D')(D - D' - 1)z = 0$
10. $(D - D')(D + D' + 1)z = 0$

Find the particular integrals of the following equations:

11. $(D^2 + 2DD' + D'^2)z = e^{x-y}$
12. $(D^2 - DD'^2 - 2D'^2)z = \sin(3x + 4y)$
13. $(D^2 - 4D'^2)z = \sin(2x + y)$
14. $\{(D - 1)^2 - D'^2)z = e^{x+y}$
15. $(D^2 - D'^2 + D)z = \cos(x + y)$

Solve the following partial differential equations by the method of separation of variables.

16. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, given that $u(x, 0) = 4e^{-x}$
17. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$
18. $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given that $z(x, 0) = 4e^{-3x}$
19. $x^2\frac{\partial z}{\partial y} + y^3\frac{\partial z}{\partial x} = 0$
20. $\frac{\partial u}{\partial y} = 2\frac{\partial^2 u}{\partial x^2}$

Part B

Solving the following partial differential equations:

21. $(D^2 + 3DD' - 4D'^2)z = (e^{2x} - e^{-y})^3$
22. $(D^3 - 7DD'^2 - 6D'^3)z = \sinh(2x - 3y)$
23. $(D^2 - 7DD' + 12D'^2)z = (e^{3x} + e^{4x})e^{y}$
24. $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$
25. $(D^3 + 2D^2D')z = e^{2x} - 3x^2y$
26. $(D^2 - 3DD' + 2D'^2)z = (e^{2x+3y}) + \sin(x - 2y)$

27. $(D^2 - 6DD' + 9D'^2)z = x^2y^2 + \cos(3x + y)$
28. $(D^2 - DD')z = \cos x \cos 2y$
29. $(8D^3 - 4D^2D' - 18DD'^2 + 9D'^3)z = \sin(3x + 2y)$
30. $(D^2 - 3DD' + 2D'^2)z = (2 + 4x)e^{x + 2y}$
31. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$
32. $(D^2 + DD' - 6D'^2)z = y \cos x$
33. $(D^2 + D'^2)z = \dfrac{8}{x^2 + y^2}$
34. $D(D^2 + 4DD' + 3D'^2 - 3D - 5D' + 2)z = e^x + e^y$
35. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = \cosh(2x + y)$
36. $(D^2 - DD' + D)z = x^2 + y^2$
37. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
38. $(D^2 - D'^2 - 2D + 1)z = xy + e^{2x + 3y}$
39. $(D^2 + DD' + D' - 1)z = \sinh(3x - 2y)$
40. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \cos 2x \cos y$
41. $(2DD' + D'^2 - 3D')z = 4\sin^3(x + 2y)$
42. $(D^2 - D'^2 + D + 3D' - 2)z = xe^x + ye^y$
43. Solve equation $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, by the method of separation of variables, given that $u(0, y) = 3e^{-y} - e^{-5y}$ [**Hint:** Assume the R.H.S. of the solution as the sum of two terms of the form $Ce^{\frac{kx}{4}+(3-k)y}$ with different values of c and k]
44. Solve equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 4e^{-3y} + 6e^{-8y}$ when $x = 0$.
45. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y} + 2z$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 1 + e^{-3y}$ when $x = 0$.

Answers

Exercise 4(a)

2. $pq = z$

3. $pq = 4xyz$

4. $z = px + qy + pq$

5. $z = px + qy + p^2 + q^2$

6. $px + qy = 3z$

7. $p = q$

8. $px + qy = z - \dfrac{1}{z}$

9. $p^2 + q^2 = 1$

10. $py = qx$

11. $py^2 + qx^2 = 0$

12. $ap + bq = 0$

13. $px = qy$

14. $px + qy = 0$

15. $s = 0$

16. $s = a$

17. $r = 0$

18. $t = 0$

19. $r = \sin x$

20. $t = \cos y$

21. $px + qy = pq$

22. $pq = p + q$

23. $p^2 + q^2 = z$

24. $pz = 1 + q^2$

25. $yp - x^2q^2 = x^2y$

26. $p = q$

27. $z = px + qy$

28. $py = qx$

29. $z^2(p^2 + q^2 + 1) = c^2$

30. $(p^2 + q^2 + 1)y^2 = c^2q^2$

31. (a) $px = qy$; (b) $py = qx$

32. (a) $x(y - z)p + y(z - x)q = z(x - y)$;

    (b) $x(y - z)p + y(z + 2x^2)q = z(x + 2x^2)$

33. (a) $px^2 + qy^2 = z^2$

    (b) $y^2zp + x^2zq = xy^2$

34. (a) $(y^2 + z^2)p - xyq + xz = 0$;

    (b) $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$

35. (a) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$;

    (b) $yp + xq = z$

36. $r + t = 0$

37. $2r + 3s - 9t = 0$

38. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{(\partial x)^2 \partial y} - \dfrac{\partial^3 z}{\partial x(\partial y)^2} + 2\dfrac{\partial^3 z}{\partial y^3} = 0$

39. $9r - 12s + 4t = 0$

40. $r - 2s + t = 0$

41. $(x - iy)(r - t) = 2(p - iq)$

42. $4xr - t + 2p = 0$

43. $zs = pq$

44. $xys = px + py - z$

45. $(1 + q)r + (q - p)s - (1 + p)t = 0$

Exercise 4(b)

7. $z = ax + \dfrac{k}{a}y + b$

8. $z = ax + y \log a + b$

9. $z = ax \pm \sqrt{2 - a^2}\ y + b$

10. $(1 + a)\log z = x + ay + b$

11. $\log z = a(x + ay) + b$

12. $4az = (x + ay + b)^2$

13. $z = a\dfrac{x^2}{2} + \dfrac{y^2}{2a} + b$

14. $z = a \log(xy) + b$

15. $z = a(e^x + e^y) + b$

16. $z = ax + by + \dfrac{a^3}{b} + \dfrac{b^3}{a}$

17. (a) C.S. is $z = ax \pm (1-\sqrt{a})^2 y$; No singular solution (S.S.).

(b) $z = ax \pm \sqrt{k^2 - a^2}\ y +$; No S.S.

18. C.S. is $z = ax + \frac{1}{2}(-2 \pm \sqrt{10})ay + b$; No S.S.

19. C.S. is $z = ax + \left(\frac{5-a^2}{3-2a}\right)y + b$; No S.S.

20. $\log z = a \log x \pm \sqrt{1-a^2}\ \log y + b$.

21. $\sqrt{z} = a\sqrt{x} \pm \sqrt{1-a^2}\ \sqrt{y} + b$.

22. $\frac{1}{z} = \frac{a}{x} + \frac{(1-a)}{y} + b$.

23. $z^2 = ax \pm \sqrt{a^2 - 4} \cdot y + b$.

24. $\log z = \frac{a}{x} + (2a^2 - 3)\log y + b$

25. $z = a^2(x + y) + axy + b$.

26. $xy = 1$.

27. $729z^2 = 1024\ xy$.

28. $16z^3 + 27x^2y^2 = 0$.

29. $z^4 = 16\ xy$.

30. $4z = y^2 - x^2$.

31. $x^2 + y^2 = 1$.

32. $z = 3$.

33. $4(1 + a^2)z = (x + ay + b)^2$.

34. $\sqrt{1+a^2}\ \log(z + \sqrt{z^2 - 1}) = x + ay + b$

35. (a) $z^2 \pm z\sqrt{z^2-4a^2} - 4a^2 \log(z+\sqrt{z^2-4a^2}) = 4(x+ay+b)$

(b) $az^2 \mp z\sqrt{a^2z^2-4} \pm \dfrac{4}{a}\log(az+\sqrt{a^2z^2-4}) = 4(x+ay+b)$

36. $4(bz - ab - 1) = (x + by + c)^2$.

37. $(z + a^2)^3 = (x + ay + b)^2$.

38. $3(1 + a)\log z = x^3 + ay^3 + b$.

39. $\sqrt{a^2+1z^2} = 2(\log x + ay + b)$.

40. $2\log z = (a \pm \sqrt{a^2+8})\left(\dfrac{1}{x}+\dfrac{a}{y}+b\right)$.

41. $4z = -x^2 \pm \{x\sqrt{x^2+4a^2} + 4a^2 \log(x+\sqrt{x^2+4a^2}) + 4(a^2y+b)\}$.

42. $2z = ax^2 - \dfrac{a}{a+1}y^2 + b$.

43. $3z = ax^3 + 2\sqrt{a-1}\, y^{3/2} + b$.

44. $z = ax - \cos x + \dfrac{1}{a}\sin y + b$.

45. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$.

46. $z^2 = x\sqrt{x^2+a^2} + a^2 \sinh^{-1}\dfrac{x}{a} + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\dfrac{y}{a} + b$.

47. $\log z = \dfrac{\sqrt{a}x^2}{2} + \sqrt{1-a}\log y + b$.

48. $z^2 = x^2 + ax + \dfrac{2}{3}(y+a)^{3/2} + b$.

49. $z = \sqrt{a(x+y)} + \sqrt{(1-a)(x-y)} + b$.

50. $z = \dfrac{a}{2}\log(x^2y^2) + \sqrt{1-a^2}\tan^{-1}\left(\dfrac{y}{x}\right) + b$.

Exercise 4(c)

1. $z = x f(y) + \phi(y)$.
2. $z = y f(x) + \phi(x)$.

3. $z = f(x) + \phi(y)$.

4. $z = xf(y) + \phi(y) + e^{x+y}$.

5. $z = y\,f(x) + \phi(x) - \dfrac{1}{9}\cos(2x + 3y)$.

6. $z = f(x) + \phi(y) + \log x \cdot \log y$.

7. $z = x\,f(y) + \phi(y) + \dfrac{x^2}{2}\sin y$.

8. $z = yf(x) + \phi(x) - \cos y$.

9. $z = f(x) + \phi(y) + kxy$.

10. $z = f(x) + \phi(y) + \dfrac{xy}{3}(x^2 + y^2)$.

12. $f(x^2 - y^2, y^2 - z^2) = 0$.

13. $f\left(xy, \dfrac{y}{z}\right) = 0$.

14. $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$.

15. $f\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$.

16. $f\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{1}{y} - \dfrac{1}{z}\right) = 0$.

17. $z = (1 + \cos x)\cos y$.

18. $z = c \cosh ax + \sinh ax \sin y$.

19. $z = e^y \cosh x + e^{-y} \sinh x$.

20. $z = 3x^2 + 3xy - 2y^2 + c$.

21. $z = x^3 f(y) - x - y/3$.

22. $z = f(x) + \phi(y) - 3x^2 y^3 - \dfrac{1}{2}\sin(2x - y)$.

23. $z = e^{2y} f(x) + e^{3y} g(x) + 2y + 5/3.$

24. $z = Axy + Bx + Cy + D.$

25. $z = f(x + at) + \phi(x - at).$

26. $z = f(x + iy) + \phi(x - iy).$

27. $z = f(x + y) + \phi(x - y).$

28. $z = (x^2 - y^2) f(x^2 + y^2) + \phi(x^2 + y^2).$

29. $z = f(y^2 + x) + \phi(y^2 - x).$

30. (i) $f\left(\dfrac{\sec x}{\sec y}, \dfrac{\sec y}{\sec z}\right) = 0;$

(ii) $f\left(\dfrac{a-x}{b-y}, \dfrac{b-y}{c-z}\right) = 0.$

31. $f(x^3 - y^3, x^2 - z^2) = 0.$

32. (i) $f\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{x-y}{z}\right) = 0;$

(ii) $f\left(\dfrac{1}{x} + \dfrac{1}{y}, \dfrac{x+y}{z}\right) = 0.$

33. $f(y/z, x^2 + y^2 + z^2) = 0.$

34. $f\left(\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right) = 0.$

35. $f[x \log(x + y) - z, x + y] = 0.$

36. $f(xy, x^2 + y^2 + z^2) = 0.$

37. (i) $f(x + y + z, x^2 + y^2 + z^2) = 0;$

(ii) $f(x + y + z, x^2 + 2yz) = 0.$

38. (i) $f(x + y + z, xyz) = 0;$

(ii) $f(x + y + z, xyz) = 0.$

39. $f(xyz, x^2 + y^2 + z^2) = 0.$

40. $f\left(\dfrac{x-y}{y-z}, xy + yz + zx\right) = 0 \cdot$

41. (i) $f(x + y + z, x^2 + y^2 - z^2) = 0;$

(ii) $f(2x + 3y + 4z, x^2 + y^2 + z^2) = 0.$

42. $f\left(x^3 y^3 z, \dfrac{x}{y^2} + \dfrac{y}{x^2}\right) = 0 \cdot$

43. $xy + yz + zx = 0.$

44. $3(2x + 2y - 3y + 3)^2 = (y - x)(x + y)^3.$

Exercise 4(d)

1. $z = f_1(y - 2x) + f_2(y + 2x) + f_3(y + 3x).$

2. $z = f_1(y + x) + xf_2(y + x) + x^2 f_3(y + x).$

3. $z = f_1(y + ix) + xf_2(y + ix) + f_3(y - ix) + xf_4(y - ix).$

4. $z = f_1(y) + f_2(y - 5x) + f_3(y + x).$

5. $z = f_1(x) + f_2\left(y - \dfrac{x}{2}\right) + f_3(y + 3x) .$

6. $z = e^x f_1(y - x) + e^{-x} f_2(y + x).$

7. $z = f_1(y) + e^{-3x} f_2(y + 2x).$

8. $z = f_1(x) + e^{2x} f_2(y - 3x).$

9. $z = f_1(y - x) + e^x f_2(y + x).$

10. $z = f_1(y + x) + e^{-x} f_2(y - x).$

11. $\dfrac{x^2}{2} e^{x-y} .$

12. $\dfrac{1}{35} \sin(3x + 4y) .$

13. $-\frac{x}{4}\cos(2x+y)$.

14. $-e^{x+y}$.

15. $\sin(x+y)$.

16. $u=4e^{-x+\frac{3}{2}y}$.

17. $u=8e^{-12x-3y}$.

18. $z=4e^{-3x+t}$.

19. $z=ce^{k(3y^4-4x^3)}$.

20. $u=e^{2ky}(Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x})$.

21. $z=f_1(y+x)+f_2(y-4x)+\frac{1}{36}e^{6x}-\frac{3}{5}xe^{4x-y}-\frac{1}{8}e^{2x-2y}-\frac{1}{36}e^{-3y}$.

22. $z=f_1(y-x)+f_2(y-2x)+f_3(y+3x)+\frac{1}{44}\cosh(2x-3y)$.

23. $z=f_1(y+3x)+f_2(y+4x)+x(e^{4x+y}-e^{3x+y})$.

24. $z=f_1(y-x)+x\,f_2(y-x)+\frac{1}{4}\cdot(x^4-2x^3y+2x^2y^2)$.

25. $z=f_1(y)+x\,f_2(y)+f_3(y+2x)+\frac{1}{4}xe^{2x}+\frac{x^5}{60}\left(y+\frac{x}{3}\right)$.

26. $z=f_1(y+x)+f_2(y+2x)+\frac{1}{4}e^{2x+3y}-\frac{1}{15}\sin(x-2y)$.

27. $z=f_1(y+3x)+x\,f_2(y+3x)+\frac{x^4}{60}(9x^2+12xy+5y^2)+\frac{x^2}{2}\cos(3x+y)$.

28. $z=f_1(y)+f_2(y+x)+\frac{1}{2}\cos(x+2y)-\frac{1}{6}\cos(x-2y)$.

29. $z=f_1(2y+x)+f_2(2y+3x)+f_3(2y-3x)-\frac{x}{96}\sin(3x+2y)$.

30. $z=f_1(y+x)+f_2(y+2x)+\frac{2}{9}e^{x+2y}(11+6x)$.

31. $z = f_1(y-x) + x f_2(y-x) + f_3(y+x) + \frac{e^x}{25}(2\sin 2y + \cos 2y)$.

32. $z = f_1(y - 3x) + f_2(y + 2x) - y \cos x + \sin x$.

33. $z = f_1(y+ix) + f_2(y-ix) + \frac{1}{2}[\log(x^2+y^2)]^2 + 2\left(\tan^{-1}\frac{y}{x}\right)^2$.

34. $z = f_1(y) + e^x f_2(y - x) + e^{2x} f_3(y - 3x) - xe^x + xye^y$.

35. $z = e^x f_1(y+x) + e^{2x} f_2(y+x) - \frac{x}{2}e^{2x+y} + \frac{1}{12}e^{-2x-y}$.

36. $z = f_1(y) + e^{-x} f_2(y+x) + \frac{x^3}{3} + xy^2 - x^2 + 2xy + 4x$.

37. $z = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + (x + 2y + 6)$.

38. $z = e^x f_1(y+x) + e^x f_2(y-x) + (x+2)y - \frac{1}{8}e^{2x+3y}$.

39. $z = e^{-x} f_1(y) + e^x f_2(y-x) + \frac{1}{8}xe^{3x-2y} - \frac{1}{8}e^{2y-3x}$.

40. $z = f_1(y-x) + e^{-2x} f_2(y+2x) + \frac{1}{12}\sin(2x+y) + \frac{1}{4}\sin(2x-y) - \frac{1}{2}\cos(2x-y$.

41. $z = f_1(x) + e^{3x/2} \cdot f_2\left(y - \frac{x}{2}\right) + \frac{3}{50}\{3\cos(x+2y) - 4\sin(x+2y)\}$

$+ \frac{1}{306}\{4\sin(3x+6y) - \cos(3x+6y)\}$.

42. $z = e^x f_1(y-x) + e^{-2x} f_2(y+x) + \frac{e^x}{54}(9x^2 - 6x + 2) + e^y\left(xy - \frac{x^2}{2} - y - 3\right)$.

43. $u = 3e^{x-y} - e^{2x-5y}$.

44. $z = (e^{3x} - e^{-x})e^{-3y} + (e^{4x} - e^{-2x})e^{-8y}$.

45. $z = \frac{1}{\sqrt{2}}\sinh x\sqrt{2} + e^{-3y}\sin x$.

Unit **5**

Fourier Series Solutions of Partial Differential Equations

Part A

Fourier Series

5A.1 INTRODUCTION

Periodic functions appear in a variety of physical problems, such as those containing vibrating springs and membranes, planetary motion, a swinging pendulum and musical sounds. In some of these problems, the periodic function may be quite complicated and hence in order to understand its basic nature batter, it may be convenient to represent it in a series of simple periodic functions. Since trigonometric functions are the simplest examples of periodic functions, we usually look for series representation in terms of sines and cosines.

Originally Fourier series was applied in the study of vibration and heat diffusion. There are numerous problems in Science and Engineering in which sinusoidal signals and hence Fourier series play an important role. For example, sinusoidal signals are useful in describing the periodic behaviour of the earth's climate. Alternting current sources generate sinusoidal voltages and currents. Fourier analysis enables us to analyse the response of a Linear Time Invariant system, such as a circuit, to such sinusoidal inputs. Waves in the ocean consist of the linear combination of sinusoidal waves with different wavelengths. Signals transmitted by radio and television stations are sinusoidal in nature.

Many of the ordinary functions that occur frequently in Science and Engineering can be expressed in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

or more generally in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \tag{2}$$

Now $\cos n\,(2\pi + x) = \cos(2n\pi + nx) = \cos nx$, for $n = 1, 2, 3, \ldots$; and

$\sin n(2\pi + x) = \sin(2n\pi + nx) = \sin nx$, for $1, 2, 3, \ldots$

Thus all the trigonometric functions in (1) are periodic with period 2π. The constant $\left(\frac{a_0}{2}\right)$ may be regarded as periodic with period 2π. Hence the infinite trigonometric series (1) is periodic with period 2π.

If a function $f(x)$ is to be expressed (or expanded) in the form of the series (1), as a prerequisite, $f(x)$ should be defined in an interval of length 2π and should satisfy certain conditions, known as *Dirichlet's conditions*, which are stated below.

The infinite trigonometric series (2) is periodic with period $2l$, since

$\cos\frac{n\pi}{1}(2l+x) = \cos\frac{n\pi x}{l}$; and

$\sin\frac{n\pi}{1}(2l+x) = \sin\frac{n\pi x}{l}$ and $\frac{a_0}{2}$ may be regarded as periodic with period $2l$.

If a function $f(x)$ is to be expressed (or expanded) in the form of the series (2), as a prerequisite, it should be defined in an interval of length $2l$ and should satisfy Dirichlet's conditions.

Note ✍

Since series (1) is only a particular case of series (2) when $l = \pi$, we shall develop the theory of Fourier series in the form (2) and obtain the derivations with reference to series (2). Whenever results are required relating to series (1), we simply replace l by π and obtain the required results.

5A.2 DIRICHLET'S CONDITIONS

A function $f(x)$ defined $c \le x \le c + 2l$ can be expanded as an infinite trigonometric series of the form $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$, provided

1. $f(x)$ is single-values and infinite in $(c, c + 2l)$.
2. $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c + 2l)$.
3. $f(x)$ has no or finite number of maxima or minima in $(c, c + 2l)$.

Note ✍

All functions that we deal with will satisfy the above Dirichlet's conditions and hence can be expanded in the form of the infinite trigonometric series given above.

5A.3 EULER'S FORMULAS

If a function $f(x)$ defined in $(c, c + 2l)$ can be expanded as the infinite trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ then

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx,\ n \ge 0 \text{ and}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx,\ n \ge 1 \text{ and}$$

[Formulas given above for a_n and b_n are called *Euler's formulas for Fourier coefficients*]

Proof

Before we proceed to find the values of a_n and b_n, we shall obtain the values of certain definite integrals, which are required in the evaluation of a_n and b_n.

$$\int_c^{c+2l} \cos\frac{n\pi x}{l}\,dx = \frac{l}{n\pi}\left(\sin\frac{n\pi x}{l}\right)_c^{c+2l}$$

$$= \frac{1}{n\pi}\left\{\sin\frac{n\pi}{l}(c+2l) - \sin\frac{n\pi c}{l}\right\}$$

$$= \frac{1}{n\pi}\left\{\sin\frac{n\pi c}{l} - \sin\frac{n\pi c}{l}\right\} = 0 \tag{1}$$

$$\int_c^{c+2l} \sin\frac{n\pi x}{l}\,dx = -\frac{1}{n\pi}\left(\cos\frac{n\pi x}{l}\right)_c^{c+2l}$$

$$= -\frac{1}{n\pi}\left\{\cos\frac{n\pi}{l}(c+2l) - \cos\frac{n\pi x}{l}\right\}$$

$$= -\frac{1}{n\pi}\left\{\cos\frac{n\pi x}{l} - \cos\frac{n\pi x}{l}\right\} = 0 \tag{2}$$

$$\int_c^{c+2l} \cos\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{l}{2}\int_c^{c+2l}\left[\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\right]dx$$

$$= 0, \text{ if } m \neq n \text{ [by (1)]} \tag{3}$$

$$\int_c^{c+2l} \cos^2\frac{n\pi x}{l}\,dx = \frac{l}{2}\int_c^{c+2l}\left(1+\cos\frac{2n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\times 2l \text{ [}\because\text{ the second term vanishes as in (1)]}$$

$$= l \tag{4}$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_c^{c+2l}\left[\cos\frac{(m-n)\pi x}{l} - \cos\frac{(m+n)\pi x}{l}\right]dx$$

$$= 0, \text{ if } m \neq n \text{ [by (1)]} \tag{5}$$

$$\int_c^{c+2l} \sin^2\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_c^{c+2l}\left(1-\cos\frac{2n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\times 2l \text{ [}\because\text{ the second term vanishes as in (1)]}$$

$$= l \tag{6}$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_c^{c+2l}\left[\sin\frac{(m+n)\pi x}{l}+\sin\frac{(m-n)\pi x}{l}\right]dx$$

$$= 0, \text{ when } m \neq n \text{ and also when } m = n \tag{7}$$

[by (2)].

Now
$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{8}$$

Integrating both sides of (8) with respect to x between the limits c and $c + 2l$, we get

$$\int_c^{c+2l} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2l} dx+\sum_{n=1}^{\infty} a_n \int_c^{c+2l} \cos\frac{n\pi x}{l}\,dx+\sum_{n=1}^{\infty} b_n \int_c^{c+2l} \sin\frac{n\pi x}{l}\,dx,$$

assuming that the term by term integration is possible.

$$= \frac{a_0}{2}[x]_c^{c+2l}+\sum_{n=1}^{\infty} a_n \times 0+\sum_{n=1}^{\infty} b_n \times 0$$

[by (1) and (2)]

$$= a_0 \cdot l$$

$$\therefore \qquad a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \tag{9}$$

Multiplying both sides of (8) by $\frac{m\pi x}{l}$ where m is a fixed opposite integer and integrating term by term with respect to x between c and $c + 2l$, we get

$$\int_c^{c+2l} f(x)\cos\frac{m\pi x}{l}\,dx = \frac{a_0}{2}\int_c^{c+2l} \cos\frac{m\pi x}{l}\,dx+\sum_{n=1}^{\infty} a_n \int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$+\sum_{n=1}^{\infty} b_n \int_c^{c+2l} \sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx$$

$$= a_m \int_c^{c+2l} \cos^2\frac{m\pi x}{l}\,dx, \text{ [by (1), (3) and (7)]}$$

$$= a_m \cdot l, \text{ [by (4)]}$$

$$\therefore \qquad a_m = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{m\pi x}{l}\,dx, \text{ where } m = 1, 2, 3, \ldots \tag{10}$$

Combining (9) and (10), we have,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx\text{ , for } n \ge 0 \tag{11}$$

Note ✍

Only if the constant term is taken as $\frac{a_0}{2}$*, formula (11) is true for n = 0*

Similarly, multiplying both sides of (8) by $\sin\frac{mpx}{l}$, integrating term by term with respect to x between c and $c + 2l$ and using (2), (5), (6) and (7), we get

$$b_m = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{m\pi x}{l}\,dx \text{ or}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx \text{ for } n \ge 1 \tag{12}$$

5A.4 DEFINITION OF FOURIER SERIES

The infinite trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ is called *the Fourier series of* $f(x)$ in the interval $c \le x \le c + 2l$, provided the coefficients are given by the Euler's formulas. Very often, the Fourier series expansions of $f(x)$ are required in the intervals $(-l, l)$ and $(0, 2l)$ which are obtained by taking $c = -l$ and $c = 0$ respectively in the above discussions.

When we require Fourier series expansions of $f(x)$ in $(-\pi, \pi)$ and $(0, 2\pi)$ we simply put $l = \pi$ in all the assumptions and the results derived.

5A.5 IMPORTANT CONCEPTS

1. We have already observed that if a function $f(x)$ is to be expanded in Fourier series of the form $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ which is of *period* $2l$, $f(x)$ should be defined in an interval of length $2l$ and should satisfy Dirichlet's Conditions in that interval. *Conversely*, if a function $f(x)$ is defined and satisfied Dirichlet's conditions in an interval of length $2l$, it can be expanded in Fourier series of period $2l$.
2. Since the Fourier series of $f(x)$ in $(0, 2l)$ [or $(-l, l)$], i.e. $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ is periodic with period $2l$, we may expect $f(x)$ also to be periodic with period $2l$. In fact, $f(x)$ is periodic with period $2l$, in the sense that the Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ represents (or converges to) $f(x)$ in $(0, 2l)$ and its periodic extensions outside $(0, 2l)$.

3. We recall that the function $f(x)$ is said to be *periodic* with period $2l$, if the graphs of $y = f(x)$ in the intervals $(c - 4l, c - 2l)$, $(c - 2l, c)$, $(c + 2l, c + 4l)$, $(c + 4l, c + 6l)$ etc. are periodic repetitions of the graph of $y = f(x)$ in $(c, c + 2l)$ as given in Figs 5A.1, 5A.2 and 5A.3.

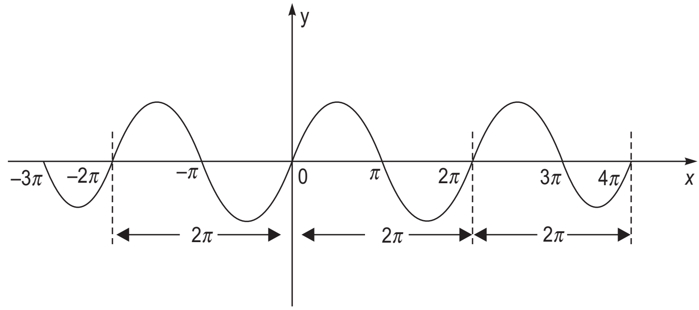


Fig 5A.1

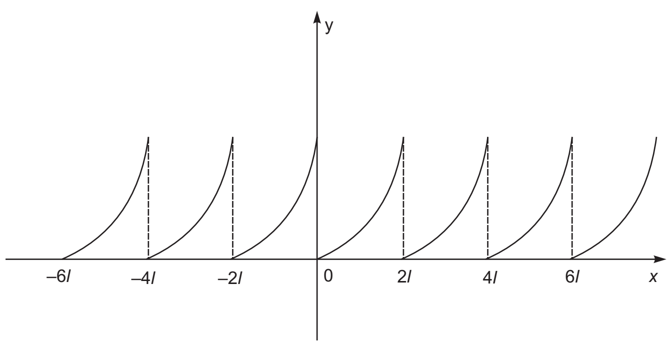


Fig 5A.2

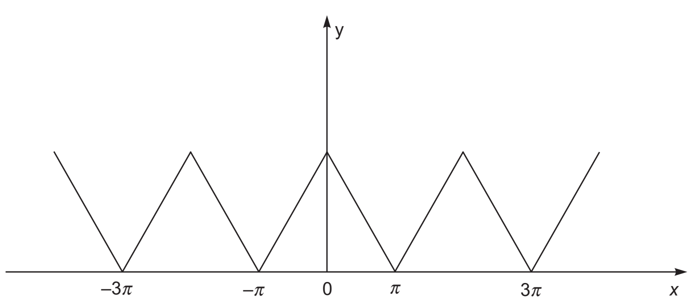


Fig 5A.3

The functions represented by the graph in Fig 5A.1 and 5A.3 are periodic with period 2π, whereas the function represented by the graph in Fig 5A.2 is periodic with period $2l$. The function represented by Fig. 5A.1 take the same value $f(x) = \sin x$ in $(-\infty, \infty)$. The function represented by Fig. 5A.1 assumes different values in $(-4l, -2l)$, $(-2l, 0)$, $(0, 2l)$, $(2l, 4l)$ etc. namely.

$$f(x)=\begin{cases}(x+4l)^2 & \text{in } (-4l,-2l)\\ (x+2l)^2 & \text{in } (-2l,0)\\ x^2 & \text{in } (0,2l)\\ (x-2l)^2 & \text{in } (2l,4l)\\ (x-4l)^2 & \text{in } (4l,6l)\end{cases}$$

The function represented by Fig. 5A.3 assumes different values in $(-3\pi, -\pi)$, $(-\pi, \pi)$, $(\pi, 3\pi)$, etc. namely,

$$f(x)=\begin{cases}x+3\pi, & \text{in } (-3\pi,-2\pi)\\ -x-\pi, & \text{in } (-2\pi,-\pi)\end{cases}$$

$$f(x)=\begin{cases}x+\pi, & \text{in } (-\pi,0)\\ -x+\pi, & \text{in } (0,\pi)\end{cases}$$

$$f(x)=\begin{cases}x-\pi, & \text{in } (\pi,2\pi)\\ 3\pi-x, & \text{in } (2\pi,3\pi), \text{etc.}\end{cases}$$

4. From the examples given above, a periodic function can be defined analytically as follows.
   (a) If $f(x) = \phi(x)$ in $(-\infty, \infty)$, i.e. $f(x)$ assumes the same value in $(-\infty, \infty)$, then $f(x)$ is said to be periodic with period $2l$, if

   $$\phi(x + 2l) = \phi(x), \text{ for } -\infty < x < \infty$$

   The Fourier series of $f(x)$ of period $2l$, in this case, will represent $\phi(x)$ everywhere.
   (b) If $f(x)$ assumes different values in different intervals of length $2l$, i.e. if

   $$f(x)=\begin{cases}\ldots\ldots\\ \phi_{-2}(x) & \text{in } (c-4l,c-2l)\\ \phi_{-1}(x) & \text{in } (c-2l,c)\\ \phi_1(x) & \text{in } (c,c+2l)\\ \phi_2(x) & \text{in } (c+2l,c+4l)\\ \phi_3(x) & \text{in } (c+4l,c+6l)\\ \ldots\ldots\end{cases}$$

   then $f(x)$ is said to be periodic with period $2l$, if

   $$\phi_{-2}(x) = \phi_1(x + 4l),\ \phi_{-1}(x) = \phi_1(x + 2l),$$

   $$\phi_2(x) = \phi_1(x - 2l),\ \phi_3(x) = \phi_1(x - 4l), \text{ etc.}$$

   In this case, the Fourier series of $f(x)$ of period $2l$ will represent $\phi_1(x)$ in $(c, c + 2l)$, $\phi_2(x)$ in $(c, 2l, c + 4l)$, etc.

   In other words, the Fourier series of $\phi_{-1}(x)$ in $(c - 2l, c)$, that of $\phi_1(x)$ in $(c, c + 2l)$, that of $\phi_2(x)$ in $(c, 2l, c + 4l)$, etc. will be identical.

5. Examples

(a) The Fourier series of $f(x) = \sin^4 x \cdot \cos^3 x$ in $(0, 2\pi)$ or $(-2\pi, 0)$ or in $(2\pi, 4\pi)$ etc. will be $\frac{3}{64}\cos x - \frac{3}{64}\cos 3x - \frac{1}{64}\cos 5x + \frac{1}{64}\cos 7x$.

In other words, the Fourier series $\frac{3}{64}\cos x - \frac{3}{64}\cos 3x - \frac{1}{64}\cos 5x + \frac{1}{64}\cos 7x$ will represent $\sin^4 x \cos^3 x$ in $(-2\pi, 0)$ and in $(2\pi, 4\pi)$ etc., since $\sin^4(2\pi + x)\cdot\cos^3(2\pi + x) = \sin^4 x \cdot \cos^3 x$ for all x and $f(x)$ assumes the same value $\sin^4 x \cdot \cos^3 x$ for all x in $(-\infty, \infty)$.

(b) The Fourier series of $f(x) = x^2$ in $(0, 2l)$ is

$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} - \frac{4l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

The same will be the Fourier series of $f(x) = (x + 2l)^2$ in $(-2l, 0)$ and $f(x) = (x - 2l)^2$ in $(2l, 4l)$, etc.

In other words, the above Fourier series represent x^2 in $(0, 2l)$, $(x + 2l)^2$ in $(-2l, 0)$, $(x - 2l)^2$ in $(2l, 4l)$, etc. This is because $(x + 2l)^2$ and $(x - 2l)^2$ are periodic extensions in $(-2l, 0)$ and $(2l, 4l)$ respectively of x^2 in $(0, 2l)$.

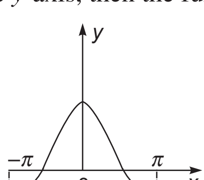
(c) The Fourier series of $f(x) = \phi_0(x) = \begin{cases} x + \pi, & \text{in } (-\pi, 0) \\ -x + \pi, & \text{in } (0, \pi) \end{cases}$ is $\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n^2}\cos nx$. The same will be the Fourier series of its periodic extensions in $(-3\pi, -\pi)$ and $(\pi, 3\pi)$ and $(\pi, 3\pi)$, etc., i.e., the above Fourier series will represent $f(x) = \phi_{-1}(x) = \begin{cases} x + 3\pi, & \text{in } (-3\pi, -2\pi) \\ -x - \pi, & \text{in } (-2\pi, -\pi) \end{cases}$

and

$$f(x) = \phi_1(x) = \begin{cases} x - \pi, & \text{in } (\pi, 2\pi) \\ 3\pi - x, & \text{in } (2\pi, 3\pi) \end{cases}, \text{ etc.}$$

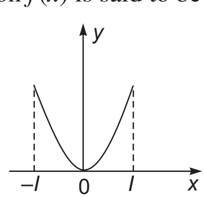
5A.6 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

Certain functions defined in symmetric ranges of the form $(-l, l)$, $(-\pi, \pi)$ or $(-\infty, \infty)$ can be classified as even and odd functions. If the graph of $y = f(x)$ in $(-l, l)$ is symmetric about the y-axis, then the function $f(x)$ is said to be an even function in $(-l, l)$.



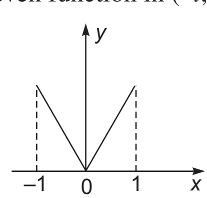
Graph of $y = \cos x$

Fig. 5A.4



Graph of $y = \cos x^2$

Fig. 5A.5



Graph of $y = |x|$

Fig. 5A.6

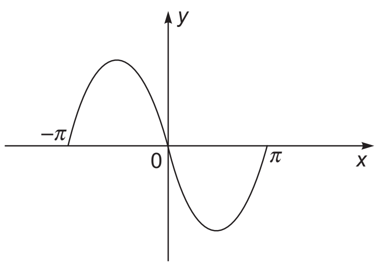
Analytically, an even function can be defined as follows.

If $f(x) = \phi(x)$ in $(-l, l)$ such that $\phi(-x) = \phi(x)$, then $f(x)$ is said to be an even function of x in $(-l, l)$. [Refer to Fig. 5A.4 and 5A.5]

If $$f(x) = \begin{cases} \phi_1(x) & \text{in } (-l, 0) \\ \phi_2(x) & \text{in } (0, l) \end{cases}$$

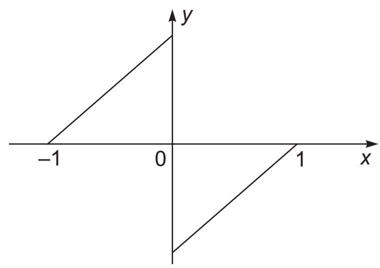
such that $\phi_1(-x) = \phi_2(x)$ or $\phi_2(-x) = \phi_1(x)$, then $f(x)$ is said to be an even function of x in $(-l, l)$. [Refer to Fig. 5A.6]

If the graph of $y = f(x)$ in $(-l, l)$ is symmetric about the origin, then the function $f(x)$ is said to be an odd function of x in $(-l, l)$.



Graph of $y = \sin x$

Fig. 5A.7



Graph of $y = \begin{cases} x + 1, & \text{in } (-1, 0) \\ x - 1, & \text{in } (0, 1) \end{cases}$

Fig. 5A.8

Analytically, an odd function can be defined as follows:

If $f(x) = \phi(x)$ in $(-l, l)$ such that $\phi(-x) = -\phi(x)$, then $f(x)$ is said to be an odd function of x in $(-l, l)$ [Refer to Fig. 5A.7]

If $$f(x) = \begin{cases} \phi_1(x) & \text{in } (-l, 0) \\ \phi_2(x) & \text{in } (0, l) \end{cases}$$

such that $\phi_1(-x) = -\phi_2(x)$ or $\phi_2(-x) = -\phi_1(x)$, then $f(x)$ is said to be an odd function of x in $(-l, l)$. [Refer to Fig. 5A.8].

Note ✍

1. *Function defined in $(-l, l)$ may be neither even nor odd.*
2. *The question of a function, defined in a non-symmetric range like $(0, 2l)$, being even or odd does not arise at all.*

5A.7 THEOREM

(i) The Fourier series of an even function $f(x)$ in $(-l, l)$ contains only cosine terms (constant term included), i.e. the Fourier series of an even function $f(x)$ in $(-l, l)$ is given by $f(x) = \dfrac{a_0}{2} + \sum a_n \cos\dfrac{n\pi x}{l}$

where $a_n = \dfrac{2}{l}\int_0^l f(x)\cos\dfrac{n\pi x}{l}\,dx$.

(ii) The Fourier series of an odd function $f(x)$ in $(-l, l)$ contains only sine terms, i.e. the Fourier series of an odd function $f(x)$ in $(-l, l)$ is given by

$$f(x) = \sum b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Proof

Since $f(x)$ is defined in an interval of length $2l$, it can be expanded as a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Case (i) $f(x)$ is even in $(-l, l)$.

Since $f(x)$ is even and $\sin\frac{n\pi x}{l}$ is odd in $(-l, l)$, $f(x)\cdot\sin\frac{n\pi x}{l}$ is an odd function of x in $(-l, l)$.

$$\therefore \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

= 0, by the property of the definite integral of an odd function in a symmetric range.

Since $\frac{n\pi x}{l}$ is even in $(-l, l)$, $f(x)\cos\frac{n\pi x}{l}$ is an even function of x in $(-l, l)$

$\therefore$ By the property of the definite integral of an even function in a symmetric range,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \; n \geq 0$$

Case (ii) $f(x)$ is odd in $(-l, l)$

$\therefore$ $f(x)\cos\frac{n\pi x}{l}$ is an odd function of x and $f(x)\sin\frac{n\pi x}{l}$ is an even function of x in $(-l, l)$

$$\therefore \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 \text{ and}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \; n \geq 1$$

by the properties mentioned above. Hence the results.

5A.8 CONVERGENCE OF FOURIER SERIES AT SPECIFIC POINTS

When $f(x)$ is expandable as a Fourier series of the form $\frac{a_0}{2}+\sum a_n \cos\frac{n\pi x}{l}$ + $\sum b_n \sin\frac{n\pi x}{l}$ in $(c, c + 2l)$, $f(x)$ is either continuous in $(c, c + 2l)$ or discontinuous with a finite number of finite discontinuities in $(c, c + 2l)$ [by Dirichlet's conditional]. In both the cases, we say that the Fourier series represents or converges to $f(x)$ in $(c, c, + 2l)$. Let us now consider a specific point $x = \alpha$ in $(c, c + 2l)$.

(i) If $x = \alpha$ is a point of continuity of $f(x)$ in $(c, c + 2l)$, then the Fourier series of $f(x)$ at $x = \alpha$ converges to $f(\alpha)$, since $f(\alpha)$ assumes a unique value.
i.e. [the sum of the Fourier series $f(x)]_{x = \alpha} = f(\alpha)$ (1)

Note ✍

If x = α is a point of discontinuity of f(x) in (c, c + 2l), the above result does not hold good, since f(α) is not uniquely defined.]

(ii) If $x = \alpha$ is a point of discontinuity of $f(x)$ in $(c, c + 2l)$, i.e., $c < \alpha < c + 2l$, then the Fourier series of $f(x)$ at $x = \alpha$ converges to $\frac{1}{2}\lim\limits_{h\to 0}$ $[f(\alpha - h) + f(\alpha + h)]$. (Proof assumed),
i.e. [Sum of the Fourier series of $f(x)]_{x=\alpha} = \frac{1}{2}\lim\limits_{h\to 0}[f(\alpha-h)+f(\alpha+h)$ (2)

(iii) If α coincides with the left extremity c of the interval $(c, c + 2l)$, $(\alpha + h)$ lies within $(c, c + 2l)$, but $(\alpha - h)$ lies within $(c - 2l, c)$. We have already observed that the Fourier series of $f(x)$ in $(c, c + 2l)$ represents $f(x)$ in this interval but it represents $f(x + 2l)$ in $(c - 2l, c)$.
∴ Formula (ii) gets modified as follows:
[Sum of the Fourier series of $f(x)]_{x = \alpha = c}$

$$= \frac{1}{2}\lim_{h\to 0}[f(\alpha-h+2l)+f(\alpha+h)] \tag{3}$$

(iv) If α coincides with the right extremity $(c + 2l)$ of the interval $(c, c + 2l)$, $(\alpha - h)$ lies within $(c, c + 2l)$, but $(\alpha + h)$ lies within $(c + 2l, c + 4l)$. As observed already, the Fourier series of $f(x)$ in $(c, c + 2l)$ represents $f(x)$ in this interval, but it represents $f(x - 2l)$ in $(c + 2l, c + 4l)$.
∴ Formula (ii) gets modified as follows:
[Sum of the Fourier series of $f(x)_{x = \alpha = c + 2l}$

$$= \frac{1}{2}\lim_{h\to 0}[f(\alpha-h)+f(\alpha+h-2l)] \tag{4}$$

Worked Examples 5A(a)

Example 1

Find the Fourier series of period $2l$ for the function $f(x) = x(2l - x)$ in $(0, 2l)$. Deduce the sum of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ in } (0, 2l) \quad (1)$$

$$a_n = \frac{1}{l}\int_0^{2l} x(2l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi x}{l}}\right) - (2l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. +(-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}, \text{ using Bernoulli's formula}$$

$$= \frac{l}{n^2\pi^2}[-2l\cos 2n\pi - 2l] = -\frac{4l^2}{n^2\pi^2}$$

Note ✍

Though Euler's formula for a_0 is a particular case of that of a_n, corresponding to $n = 0$, the value of a_0 cannot be deduced from that of a_n by putting $n = 0$ in this example. In some problems, a_0 can be deduced from a_n. Hence in all problem we shall first find a_n and if possible deduce the value of a_0 from it.

$$a_0 = \frac{1}{l}\int_0^{2l} x(2l-x)\,dx = \frac{1}{l}\left[lx^2 - \frac{x^3}{3}\right]_0^{2l} = \frac{4}{3}l^2$$

$$b_n = \frac{1}{l}\int_0^{2l} x(2l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= 0$$

Using these values in (1), we have

$$x(2l-x)=\frac{2}{3}l^2-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\text{ in }(0,2l) \quad (2)$$

The required series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty$ can be obtained by putting $x = l$ in the Fourier series in (2).

$x = l$ lies in $(0, 2l)$ and is a point of continuity of the function $f(x) = x(2l - x)$.

$\therefore$ [Sum the Fourier series in (2)]$_{x=l} = f(l)$

i.e.
$$\frac{2}{3}l^2-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = l(2l-l)$$

i.e.
$$-\frac{4l^2}{\pi^2}\left\{-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\infty\right\}=\frac{l^2}{3}$$

$\therefore$
$$\frac{1}{l^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty=\frac{\pi^2}{12}$$

Example 2

Find the Fourier series expansion of the function $f(x)=\begin{cases}0, & \text{in } -\pi\le x\le 0\\ \sin x, & \text{in } 0\le x\le\pi\end{cases}$

Hence find the values of

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots\infty$
2. $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\infty$

Since $f(x)$ is defined in a range of length 2π, it can be expanded as a Fourier series of period 2π

Let
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\text{ in}(-\pi,\pi) \quad (1)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0.\cos nx\,dx+\int_{0}^{\pi}\sin x\cos nx\,dx\right]$$

$$=\frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{2\pi}\left[\frac{-\cos(n+1)}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}, \text{ if } n\neq 1$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right\}-\left\{-\frac{1}{n+1}+\frac{1}{n-1}\right\}\right],$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\{(-1)^{n-1}-1\right],$$

if $n \neq 1$ [$\because$ $(-1)^{n+1} = (-1)^{n-1}$]

$$= \frac{1}{\pi(n^2-1)}\{(-1)^{n-1}-1\}$$

$$= \begin{cases} -\dfrac{2}{\pi(n^2-1)}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd, but} \neq 1 \end{cases}$$

Putting $n = 0$ in the value of a_n, we get $a_0 = \dfrac{2}{\pi}$.

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos x\,dx + \int_0^{\pi}\sin x\cos\,dx\right], \text{ by Euler's formula}$$

$$= \frac{1}{2\pi}(\sin^2 x)_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]_0^{\pi}, \text{ if } n \neq 1$$

$= 0$, if $n \neq 1$

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin x\,dx + \int_0^{\pi}\sin^2 x\,dx\right], \text{ by Euler's formula}$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left(x-\frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{2}$$

Using these values in (1),

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x \text{ in } (-\pi,\pi) \quad (2)$$

Putting $x = 0$ in the Fourier series in (2), we get the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots\infty$.

The value of $f(x)$ at $x = 0$ is uniquely found as 0, both from the value of $f(x)$ in $-\pi \le x \le 0$ and from the value of $f(x)$ in $0 \le x \le \pi$.

$\therefore$ $x = 0$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = f(0)$.

i.e. $$\frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots\infty\right] + \frac{1}{2}\times 0 = 0.$$

$\therefore$ $$\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots\infty = \frac{1}{2}$$

Now putting $x = \frac{\pi}{2}$, which is a point of continuity of $f(x)$, in the Fourier series in (2) we get

$$\frac{1}{\pi} - \frac{2}{\pi}\left\{-\frac{1}{1\cdot3}+\frac{1}{3\cdot5}-\frac{1}{5\cdot7}+\cdots\infty\right\} + \frac{1}{2} = 1$$

i.e. $$\frac{2}{\pi}\left[\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\cdots\infty\right] = \frac{1}{2}-\frac{1}{\pi}$$

$$\frac{1}{1\cdot3}-\frac{1}{1\cdot3}+\frac{1}{5\cdot7}\cdots\infty = \frac{\pi-2}{4}$$

Example 3

Find the Fourier series of period 2 for the function

$$f(x) = \begin{cases} k, & \text{in } -1 < x < 0 \\ x, & \text{in } 0 < x < 1 \end{cases}$$

Hence find the sum of

(i) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty$

(ii) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\infty$

Let $$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x \text{ in } (-1,1)$$

[$\because$ $2l = 2$ and hence $l = 1$] (1)

$$a_n = \frac{1}{1}\int_0^1 f(x)\cos n\pi x\, dx$$

$$= \int_{-1}^{0} k\cos n\pi x\,dx + \int_{0}^{1} x\cos n\pi x\,dx$$

$$= k = \left(\frac{\sin n\pi x}{n\pi}\right)_{-1}^{0} + \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - 1\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{1}{n^2\pi^2}\{(-1)^n - 1\},\ \text{if } n \neq 0$$

$$= \begin{cases} \dfrac{-2}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{1}{1}\left[\int_{-1}^{0} k\,dx + \int_{0}^{1} x\,dx\right] = k(x)_{-1}^{0} + \frac{1}{2}(x^2)_0^1 = k + \frac{1}{2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx$$

$$= \int_{-1}^{0} k\sin n\pi x\,dx + \int_{0}^{1} x\sin n\pi\,dx$$

$$= k\left(-\frac{\cos n\pi x}{n\pi}\right)_{-1}^{0} + \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\cdot\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= -\frac{k}{n\pi}\{1 - (-1)^n\} - \frac{1}{n\pi}(-1)^n$$

Using these values in (1), we have

$$f(x) = \left(\frac{k}{2} + \frac{1}{4}\right) - \frac{2}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos n\pi x - \frac{1}{\pi}\sum [k\{1 - (-1)^n\}$$

$$+(-1)^n]\frac{1}{n}\sin n\pi x \text{ in } (-1, 1) \tag{2}$$

By putting $x = 0$ in (2), we get the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty$

$\therefore$ We require the sum of the Fourier series of $f(x)$ at $x = 0$.
Since $f(0-) = k$ and $f(0+) = 0$, as per the definition of $f(x)$.
$\therefore$ $x = 0$ is a point of discontinuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = \frac{1}{2}\lim_{h\to 0}[f(0-h) + f(0+h)]$

$$= \frac{1}{2}\lim_{h\to 0}[k + h] = \frac{k}{2}$$

i.e. $$\frac{k}{2}+\frac{1}{4}-\frac{2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{k}{2}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$$

By putting $x=\frac{1}{2}$ in (2), we get the series $1-\frac{1}{3}+\frac{1}{5}+\cdots$

$x=\frac{1}{2}$ is a point of continuity for $f(x)$.

∴ $$[\text{Sum of the Fourier series of } f(x)]_{x=\frac{1}{2}}=f\left(\frac{1}{2}\right)$$

i.e. $$\left(\frac{k}{2}+\frac{1}{4}\right)-\frac{1}{\pi}(2k-1)\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}=\frac{1}{2}$$

i.e. $$\frac{(2k-1)}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty\right)=\frac{k}{2}-\frac{1}{4}\text{ or }\frac{(2k-1)}{4}$$

∴ $$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty=\frac{\pi}{4}$$

Example 4

Find the Fourier series of $f(x)=x^2$ in $(0, 2l)$. Hence deduce that

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty=\frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty=\frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$

Since $f(x)$ is defined in a range of length $2l$, it can be expanded as a Fourier series of period $2l$.

Let $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\text{ in }(0, 2l) \tag{1}$$

$$a_n=\frac{1}{l}\int_0^{2l}x^2\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{l}\left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-2x\left(-\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$=\frac{4l^2}{n^2\pi^2},\text{ if } n\neq 0.$$

$$a_0=\frac{1}{l}\int_0^{2l}x^2dx=\frac{1}{l}\left(\frac{x^3}{3}\right)_0^{2l}=\frac{8l^2}{3}$$

$$b_n = \frac{1}{l}\int_0^{2l} x^2 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{-4l^2}{n\pi}$$

Using these values in (1), we have

$$x^2 = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right) \text{ in } (0, 2l) \qquad (2)$$

Putting $x = 0$ in the R.H.S. of (2), we get the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty$.

$x = 0$ is the left extremity of the range $(0, 2l)$. Since the Fourier series in (2) represents x^2 in $(0, 2l)$ and $(x + 2l)^2$ in $(-2l, 0)$, $x = 0$ is a point discontinuity.

$\therefore$ [Sum of the Fourier series of $f(x)_{x=0} = \frac{1}{2}\lim_{h\to 0}[f(0-h) + f(0+h)]$

$= \frac{1}{2}\lim_{h\to 0}[(-h+2l)^2 + h^2]$ [$\because$ $x = -h$ lies in $(-2l, 0)$ and $x = h$ lies in $(0, 2l)$]

$= 2l^2$

i.e.
$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 2l^2$$

$\therefore$
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty = \frac{\pi^2}{6} \qquad (3)$$

Putting $x = l$ in the R.H.S. of (2), we get the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty$

$x = l$ is a point of continuity of the function $f(x) = x^2$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=l} = f(l)$

i.e.
$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n = l^2$$

i.e.
$$\frac{4l^2}{\pi^2}\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\infty\right) = -\frac{l^2}{3}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\infty = \frac{\pi^2}{12} \qquad (4)$$

Adding (3) and (4), we get

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}$$

$\therefore$
$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$$

Example 5

Find the Fourier series expansion of $f(x) = x^2 + x$ in (–2, 2). Hence find the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$.

Since $f(x)$ is defined in a range of length 4, it can be expanded as a Fourier series of period 4.

Let
$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \text{ [since } 2l = 4] \tag{1}$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(x^2+x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}x^2\cos\frac{n\pi x}{2}\,dx+\frac{1}{2}\int_{-2}^{2}x\cos\frac{n\pi x}{2}\,dx$$

$$= \int_{0}^{2}x^2\cos\frac{n\pi x}{2}\,dx+0, \left[\because\ x^2\cos\frac{n\pi x}{2} \text{ is an even function and } x\cos\frac{n\pi x}{2} \text{ is an odd function of } x\right]$$

$$= \left[x^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)-2x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)+2\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16}{n^2\pi^2}(-1)^n, \text{ if } n \neq 0$$

$$a_0 = \frac{1}{2}\int_{-2}^{2}(x^2+x)\,dx=\int_{0}^{2}x^2\,dx=\frac{8}{3}$$

($\because$ x^2 is an even function and x is an odd function of x)

$$b_n = \frac{1}{2}\int_{-2}^{2}(x^2+x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}x^2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{2}x\sin\frac{n\pi x}{2}\,dx$$

$$= \int_{0}^{2}x\sin\frac{n\pi x}{2}\,dx$$

$$\left[\because\ x^2\sin\frac{n\pi x}{2} \text{ is an odd function and } x\sin\frac{n\pi x}{2} \text{ is an even function of } x \text{ in } (-2, 2)\right]$$

$$= \left[x\cdot\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}(-1)^n$$

Using these values in (1), we have

$$x^2 + x = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{2} \text{ in } (-2, 2) \qquad (2)$$

Putting $x = -2$ or 2 in the R.H.S. of (2), we get the required series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$.

Let us consider $x = 2$, which is the right extremity of the range $(-2, 2)$.

The fourier series of $f(x)$ represents $f(x)$ in $(-2, 2)$ and $f(x-4)$ in the next period (2, 6), i.e. The Fourier series in the R.H.S. of (2) represents $x^2 + x$ in $(-2, 2)$ and $\{(x-4)^2 + (x-4)\}$ in (2, 6).

Evidently $x = 2$ is a point of discontinuity of $f(x)$.

$$\therefore\ [\text{Sum of the Fourier series of } f(x)]_{x=2} = \frac{1}{2}\lim_{h\to 0}[\{(2-h)^2+(2-h)\} + \{(2+h-4)^2 + (2+h-4)\}] = 4$$

i.e.
$$\frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot(-1)^n = 4$$

i.e.
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty = \frac{\pi^2}{6}.$$

Example 6

Find the Fourier series expansion of $f(x) = x(1-x)(2-x)$ in (0, 2). Deduce the sum of the series $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}-\cdots\infty$.

Since the function $f(x)$ (1) is defined in a range of length 2, it can be expanded as a Fourier series of period 2.

$$\therefore \quad \text{Let} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi\, x \; [\text{since } 2l = 2]$$

$$a_n = \frac{1}{1}\int_0^2 x(1-x)(2-x)\cos n\pi\, x\, \mathrm{d}x$$

$$= \left[(2x - 3x^2 + x^3)\left(\frac{\sin n\pi x}{n\pi}\right) - (2 - 6x + 3x^2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. +(-6+6x)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right) - 6\cdot\left(\frac{\cos n\pi x}{n^4\pi^4}\right)\right]_0^2$$

$$a_0 = \frac{1}{1}\int_0^2 (2x - 3x^2 + x^3)\,\mathrm{d}x = \left(x^2 - x^3 + \frac{x^4}{4}\right)_0^2 = 0$$

$$b_n = \frac{1}{1}\int_0^2 (2x - 3x^2 + x^3)\sin n\pi\, \mathrm{d}x$$

$$= \left[(2x - 3x^2 + x^3)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2 - 6x + 3x^2)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. +(-6+6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) - 6\cdot\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right]_0^2$$

$$= \frac{12}{n^3\pi^3}$$

Using these values in (1), we have

$$x(1-x)(2-x) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin n\pi x \qquad (2)$$

Putting $x = \frac{1}{2}$ in the R.H.S. of (2), we get the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\infty$.

$x = \frac{1}{2}$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=\frac{1}{2}} = f\left(\frac{1}{2}\right)$

i.e.
$$\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}$$

i.e.
$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

Example 7

Find the Fourier series of period 2π for the function $f(x) = x \cos x$ in $0 < x < 2\pi$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n+1)x + \cos(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\left\{x\cdot\frac{\sin(n+1)x}{n-1} + \frac{\cos(n-1)x}{(n+1)^2}\right\}_0^{2\pi}\right.$$

$$\left. + \left\{x\cdot\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right\}_0^{2\pi}\right], \text{ if } n \neq 1$$

$$= 0, \text{ if } n \neq 1$$

$$a_0 = 0$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1+\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi} = \pi\,.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}\right\} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$+ \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n-1)x}{n-1}\right\} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}, \text{ if } n \neq 1$$

$$= -\frac{1}{n+1} - \frac{1}{n-1} = --\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = -\frac{2n}{n^2-1}, \text{ if } n \neq 1.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)+\frac{\sin 2x}{4}\right]_0^{2\pi} = -\frac{1}{2}$$

Using these values in (1), we get

$$f(x) = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=2,3,\ldots}^{\infty}\frac{n}{n^2-1}\sin nx$$

Example 8

Find the Fourier series of period 2π for the function $f(x) = \sqrt{1-\cos x}$ in $-\pi < x < \pi$.

$$f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$$

$\therefore$ $f(x) = \sqrt{1-\cos x}$ is an even function of x in $-\pi < x < \pi$.

Note ✍

Since $\sqrt{1-\cos x} = \pm\sqrt{2}\sin\frac{x}{2}$, we should not conclude that $\sqrt{1-\cos x}$ is an odd function of x in $-\pi < x < \pi$. If we note the values of $\sqrt{1-\cos x}$ and $\sqrt{2}\sin\frac{x}{2}$, we can find that

$$\sqrt{1-\cos x} = \begin{cases} -\sqrt{2}\sin\dfrac{x}{2}, & \text{in } (-\pi, \pi) \\ \sqrt{2}\sin\dfrac{x}{2}, & \text{in } (0, \pi) \end{cases} \tag{1}$$

From (1) also, it is evident that $\sqrt{1-\cos x}$ is an even function of x in $(-\pi, \pi)$.

$\therefore$ Fourier series of $f(x)$ will not contain sine terms.

Let
$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \text{ in } (-\pi, \pi)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sqrt{1-\cos x}\,\cos nx\,\mathrm{d}x$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi}\sin\frac{x}{2}\cos nx\,\mathrm{d}x$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]\mathrm{d}x$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}}+\frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$$

$$\left[\because \ \cos\left(n \pm \frac{1}{2}\right)\pi = \cos n\pi \cos\frac{\pi}{2} \mp \sin n\pi \sin\frac{\pi}{2} = 0\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)} \tag{2}$$

and $$a_0 = -\frac{4\sqrt{2}}{(-\pi)} \text{ [by putting } n = 0 \text{ in (2)]}$$

$$= \frac{4\sqrt{2}}{\pi}$$

Using these values in (1), we have

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \text{ in } (-\pi,\pi)$$

Example 9

Find the Fourier series of period 2π for the function $f(x) = |\cos x|$ in $-\pi \le x \le \pi$ $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore$ $f(x)$ is an even function of x in $-\pi \le x \le \pi$.

$\therefore$ Fourier series of $f(x)$ will not contain sine terms.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ in } -\pi \le x \le \pi \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx \, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx \, dx\right]$$

$$\left[\because \ \cos x > 0 \text{ in } \left(0, \frac{\pi}{2}\right) \text{ and } < 0 \text{ in } \left(\frac{\pi}{2}, \pi\right) \text{ and } |\cos x| \text{ is positive}\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} [\cos(n+1)x + \cos(n-1)x]\,dx - \int_{\frac{\pi}{2}}^{\pi} [\cos(n+1)x + \cos(n-1)x]\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\sin(n+1)}{n+1}+\frac{\sin(n-1)x}{n-1}\right\}_0^{\frac{\pi}{2}}$$

$$-\frac{1}{\pi}\left\{\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right\}_{\frac{\pi}{2}}^{\pi}, \text{ if } n \neq 1$$

$$= \frac{1}{\pi}\left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\}+\frac{1}{\pi}\left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\}$$

$$= \frac{2}{\pi}\left\{\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\cos\frac{n\pi}{2}\right\},$$

$$\left[\text{Since}\sin(n\pm 1)\frac{\pi}{2} = \sin\frac{n\pi}{2}\cos\frac{\pi}{2}\pm\cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2} = \pm\cos\frac{n\pi}{2}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \text{ if } n \neq 1$$

$$a_0 = \frac{4}{\pi};\ a_1 = \frac{2}{\pi}\int_0^{\pi/2} |\cos x|\cos x\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos^2 x\, dx - \int_{\frac{\pi}{2}}^{\pi}\cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\left[\int_0^{\frac{\pi}{2}}(1+\cos 2x)\, dx - \int_{\frac{\pi}{2}}^{\pi}(1+\cos 2x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x+\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} - \left(x+\frac{\sin 2x}{2}\right)_{\frac{\pi}{2}}^{\pi}\right] = \frac{1}{\pi}\left[\frac{\pi}{2}-\left(\pi-\frac{\pi}{2}\right)\right] = 0$$

Using these values in (1), we get

$$|\cos x| = \frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$= \frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$\left[\because\ \cos\frac{n\pi}{2} = 0 \text{ when } n \text{ is odd}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos n\pi \cos 2\, nx$$

$$= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{4n^2 - 1} \cos 2n\, x \text{ in } (-\pi, \pi)$$

Example 10

Find the Fourier series expansion of $f(x) = \sin ax$ in $(-l, l)$.

Since $f(x)$ is defined in a range of length $2l$, we can expand $f(x)$ in Fourier series of period $2l$.

Also $f(-x) \sin[a(-x)] = -\sin ax = -f(x)$

$\therefore$ $f(x)$ is an odd function of x in $(-l, l)$.

Hence Fourier series of $f(x)$ will not contain cosine terms.

Let
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l \sin ax \cdot \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \int_0^l \left[\cos\left(\frac{n\pi}{l} - a\right)x - \cos\left(\frac{n\pi}{l} + a\right)x \right] dx$$

$$= \frac{1}{l} \left[\frac{\sin\left(\frac{n\pi}{l} - a\right)x}{\frac{n\pi}{l} - a} - \frac{\sin\left(\frac{n\pi}{l} + a\right)x}{\frac{n\pi}{l} + a} \right]_0^l$$

$$= \frac{1}{n\pi - la} \sin\left(\frac{n\pi}{l} - a\right)l - \frac{1}{n\pi + la} \sin\left(\frac{n\pi}{l} + a\right)l$$

$$= \frac{1}{n\pi - la} \sin(n\pi - al) - \frac{1}{n\pi + la} \sin(n\pi + al)$$

$$= \frac{1}{n\pi - al} \{-(-1)^n \sin al\} - \frac{1}{n\pi + al} \{(-1)^n \sin al\}$$

$$= (-1)^{n+1} \sin al \left\{ \frac{1}{n\pi - al} + \frac{1}{n\pi + al} \right\}$$

$$= \frac{(-1)^{n+1} 2n\pi \sin al}{n^2\pi^2 - a^2 l^2}$$

Using this value in (1), we get

$$\sin ax = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - a^2 l^2} \sin \frac{n\pi x}{l}$$

Example 11

Find the Fourier series expansion of $f(x) = e^{-x}$ in $(-\pi, \pi)$. Hence obtain a series for cosec π.

Though the range $(-\pi, \pi)$ is symmetric about the origin, e^{-x} is neither an even function nor an odd function.

$\therefore$ Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

in $(-\pi, \pi)$ [$\because$ the length of the range is 2π]

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx \, dx$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-x}}{n^2+1} (-\cos nx + n \sin nx \right\}_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi(n^2+1)} \{e^{-\pi}(-1)^n - e^{\pi}(-1)^n\}$$

$$= \frac{2(-1)^n}{\pi(n^2+1)} \sinh \pi$$

and
$$a_0 = \frac{2 \sinh \pi}{\pi}$$

$$b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} e^{-x} \sin nx \, dx$$

$$= \frac{1}{\pi} \left\{ \frac{e^{-x}}{n^2+1} (-\sin nx - n \cos nx) \right\}_{-\pi}^{\pi}$$

$$= -\frac{n}{\pi(n^2+1)} \{e^{-\pi}(-1)^n - e^{\pi}(-1)^n\}$$

$$= \frac{2n(-1)^n}{\pi(n^2+1)} \sinh \pi$$

Using these values in (1), we get

$$e^{-x} = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \cos nx$$
$$+ \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} \sin nx \text{ in } (-\pi, \pi) \tag{1}$$

[Sum of the Fourier series of $f(x)]_{x=0} = f(0)$,

[Since $x = 0$ is a point of continuity of $f(x)$]

i.e. $$\frac{\sinh \pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right] = e^{-0} = 1$$

i.e. $$\pi \operatorname{cosech} \pi = 1+2\times\left(\frac{-1}{2}\right)+2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

i.e. $$\operatorname{cosech} \pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 12

Find the Fourier series of period 2π for the function $f(x) = \sinh \alpha x$ in $(-\pi, \pi)$.

$$f(-x) = \sinh(-\alpha x) = -\sinh \alpha x = -f(x) \text{ in } (-\pi, \pi)$$

$\therefore$ $\sinh \alpha x$ is an odd function of x in $(-\pi, \pi)$.

$\therefore$ Fourier series of $\sinh \alpha x$ in $(-\pi, \pi)$ will not contain the constant term and the cosine terms.

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ in } (-\pi,\pi) \tag{1}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sinh ax \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(e^{\alpha x}-e^{-\alpha x})\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{\alpha x}}{n^2+\alpha^2}(\alpha \sin nx - n\cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-\alpha x}}{n^2+\alpha^2}(-\alpha \sin nx - n\cos nx)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi(n^2+\alpha^2)}[-n(-1)^n e^{\alpha\pi}+n+ne^{\alpha\pi}(-1)^n-n]$$

$$= \frac{-n(-1)^n}{\pi(n^2+\alpha^2)}(e^{\alpha\pi}-e^{-\alpha\pi}) = \frac{2n(-1)^{n-1}\sinh \alpha\pi}{\pi(n^2+\alpha^2)}$$

Using this value of b_n in (1), we get

$$\sinh ax = \frac{2\sinh \alpha\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+\alpha^2}\sin nx$$

Example 13

Find the Fourier series expansion of period 2 for the function

$$f(x) = \begin{cases} \pi x, & \text{in } 0 \le x \le 1 \\ \pi(2-x), & \text{in } 1 \le x \le 2 \end{cases}$$

Deduce the sum of $\sum\limits_{n=1,3,...}^{\infty} \frac{1}{n^2}$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \tag{1}$$

$$a_n = \frac{1}{1}\int_0^2 f(x) \cos n\pi x \, dx \quad [\because \quad 2l = 2 \text{ or } l = l]$$

$$= \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx$$

$$= \pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - 1 \cdot \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ \pi\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) + 1\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \frac{1}{n^2\pi}\{(-1)^n - 1\} + \frac{1}{n^2\pi}\{(-1)^n - 1\}, \text{ if } n \ne 0$$

$$= \begin{cases} 0, & \text{if } n \text{ is even}, \ne 0 \\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{1}{1}\int_0^2 f(x) \, dx = \int_0^1 \pi x \, dx + \int_1^2 \pi(2-x) \, dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[\frac{(2-x)^2}{-2}\right]_1^2$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$b_n = \frac{1}{1}\int_0^2 f(x) \sin n\pi\, x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi\, x \, dx$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - 1\cdot\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) + 1\cdot\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{n}(-1)^n = 0$$

Using these values in (1), we get

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n^2}\cos n\pi x \quad \text{in } (0 \le x \le 2) \qquad (2)$$

$x = 1$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of Fourier series of $f(x)]_{x=1} = f(1)$

i.e.
$$\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right) = \pi$$

i.e.
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty = \frac{\pi^2}{8}$$

Example 14

Find the Fourier series of period $\frac{\pi}{2}$ for the function $f(x) = \begin{cases} \sin x, & \text{in } 0 \le x \le \frac{\pi}{4} \\ \cos x, & \text{in } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$

Here $2l = \frac{\pi}{2}$ $\quad \therefore \quad l = \frac{\pi}{4}$

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 4nx + \sum_{n=1}^{\infty} b_n \sin 4nx \text{ in } \left(0, \frac{\pi}{2}\right) \qquad (1)$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos 4nx\,dx$$

$$= \frac{4}{\pi}\left[\int_0^{\frac{\pi}{4}} \sin x\cos 4nx\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\cos 4nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} \sin(4n+1)x - \sin(4n-1)x\}\,dx\right.$$

$$\left.+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \{\cos 4n+1)x + \cos(4n-1)x\}\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\left\{\frac{-\cos(4n-1)x}{4n+1}+\frac{\cos(4n-1)x}{4n-1}\right\}\right]^{\frac{\pi}{4}} + \left\{\frac{\sin(4n+1)x}{4n+1}+\frac{\sin(4n-1)x}{4n-1}\right\}_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos\left(n\pi+\frac{\pi}{4}\right)}{4n+1}+\frac{\cos\left(n\pi-\frac{\pi}{4}\right)}{4n-1}+\frac{1}{4n+1}-\frac{1}{4n-1}+\frac{\sin\left(2n\pi+\frac{\pi}{2}\right)}{4n+1}\right.$$

$$\left.+\frac{\sin\left(2n\pi-\frac{\pi}{2}\right)}{4n-1}-\frac{\sin\left(n\pi+\frac{\pi}{4}\right)}{4n+1}-\frac{\sin\left(n\pi-\frac{\pi}{4}\right)}{4n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{(4n+1)\sqrt{2}}+\frac{(-1)^n}{(4n-1)\sqrt{2}}+\frac{1}{4n+1}-\frac{1}{4n-1}+\frac{1}{4n+1}-\frac{1}{4n-1}\right.$$

$$\left.-\frac{(-1)^n}{(4n+1)\sqrt{2}}+\frac{(-1)^n}{(4n-1)\sqrt{2}}\right]$$

$$= \frac{4}{\pi}\left[\frac{(-1)^n}{\sqrt{2}}\left(\frac{1}{4n-1}-\frac{1}{4n+1}\right)+\left(\frac{1}{4n+1}-\frac{1}{4n-1}\right)\right]$$

$$= \frac{8}{\pi(16n^2-1)}\left[\frac{(-1)^n}{\sqrt{2}}-1\right]$$

$$a_0 = \frac{8}{\pi}\left\{1-\frac{1}{\sqrt{2}}\right\}$$

$$b_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin 4nx\,dx$$

$$= \frac{4}{\pi}\left[\int_0^{\frac{\pi}{4}}\sin x\sin 4nx\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos x\sin 4nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}}\{\cos(4n-1)x-\cos(4n+1)\}\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\{\sin(4n+1)x+\sin(4n-1)x\}\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\sin(4n-1)x}{4n-1}-\frac{\sin(4n+1)}{4n+1}\right\}_0^{\frac{\pi}{4}}-\left\{\frac{\cos(4n+1)x}{4n+1}+\frac{\cos(4n-1)x}{4n-1}\right\}_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n\pi - \frac{\pi}{4}\right)}{4n-1} - \frac{\sin\left(n\pi + \frac{\pi}{4}\right)}{4n+1} - \frac{\cos\left(2n\pi + \frac{\pi}{2}\right)}{4n+1} - \frac{\cos\left(2n\pi - \frac{\pi}{2}\right)}{4n-1}\right.$$

$$\left. + \frac{\cos\left(n\pi + \frac{\pi}{4}\right)}{4n+1} + \frac{\cos\left(n\pi - \frac{\pi}{4}\right)}{4n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{(4n-1)\sqrt{2}} - \frac{(-1)^n}{(4n+1)\sqrt{2}} + \frac{(-1)^n}{(4n+1)\sqrt{2}} + \frac{(-1)^n}{(4n-1)\sqrt{2}}\right] = 0$$

Using these values in (1), we get

$$f(x) = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{16n^2-1}\left\{\frac{(-1)^n}{\sqrt{2}} - 1\right\}\cos 4nx, \text{in}\left(0, \frac{\pi}{2}\right)$$

Example 15

Find the Foureir series expansion of $f(x)$ given by $f(x) = \begin{cases} x, & \text{in } 0 < x < 2 \\ 0, & \text{in } 2 < x < 4 \end{cases}$

Since $f(x)$ is defined in a range of length 4, we can expand it as a Fourier series of period 4.

i.e. $\quad 2l = 4$

$\therefore \quad l = 2$

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \text{ in } (0,4) \qquad (1)$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2, \text{ if } n \neq 0$$

$$= \frac{2}{n^2\pi^2}\{(-1)^n - 1\}$$

$$= \begin{cases} -\frac{4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{4}(x^2)_0^2 = 1$$

$$b_n = \frac{1}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)+\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}(-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2} \text{ in } (0, 4)$$

Example 16

Find the Fourier series expansion of $f(x)$ given that $f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 2, & \text{for } 1 < x < 3 \end{cases}$

Since the function is defined in a range of length 3, it can be expanded as a Fourier series period 3.

$\therefore \quad l = \frac{3}{2}$

Let $$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{3}+\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi x}{3} \text{ in } (0, 3) \quad (1)$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,\mathrm{d}x$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot\cos\frac{2n\pi x}{3}\,\mathrm{d}x+\int_1^3 2\cos\frac{2n\pi x}{3}\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}\left(\sin\frac{2n\pi x}{3}\right)_0^1+2\cdot\frac{3}{2n\pi}\left(\sin\frac{2n\pi x}{3}\right)_1^3\right]$$

$$= \frac{1}{n\pi}\left\{\sin\frac{2n\pi}{3}-2\sin\frac{2n\pi}{3}\right\} = -\frac{1}{n\pi}\sin\frac{2n\pi}{3},\ n\neq 0$$

$$a_0 = \frac{2}{3}\left[\int_0^1 1\,\mathrm{d}x+\int_1^3 2\,\mathrm{d}x\right]$$

$$= \frac{2}{3}[1+4] = \frac{10}{3}$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot\sin\frac{2n\pi x}{3}\,dx + \int_1^3 2\cdot\sin\frac{2n\pi x}{3}\,dx\right]$$

$$\frac{2}{3}\left[\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)_0^1 - 2\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)_1^3\right]$$

$$= -\frac{1}{n\pi}\left[\left\{\cos\frac{2n\pi}{3} - 1\right\} + 2\left\{1 - \cos\frac{2n\pi}{3}\right\}\right]$$

$$= -\frac{1}{n\pi}\left(1 - \cos\frac{2n\pi}{3}\right)$$

Using these values in (1), we have

$$f(x) = \frac{5}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{3}\cos\frac{2n\pi x}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\frac{2n\pi}{3}\right)\sin\frac{2n\pi x}{3} \text{ in } (0,3)$$

Example 17

Find the Fourier series expansion of period 2π for the function

$$f(x) = \begin{cases} x(\pi - x), & \text{in } -\pi \le x \le 0 \\ x(\pi + x), & \text{in } \quad 0 \le x \le \pi \end{cases}$$

Since the range $(-\pi, \pi)$ is symmetrically divided into two subranges and $f(x)$ assumes the values $\phi_1(x) = x(\pi - x)$ in $(-\pi, 0)$ and $\phi_2(x) = x(\pi + x)$ in $(0, \pi)$, the function $f(x)$ may be odd or even. Let us first test for the oddness or evenness of $f(x)$.

$$\phi_1(-x) = -x(\pi + x)$$
$$= -\phi_2(x)$$

$\therefore$ $f(x)$ is an odd function in $(-\pi, \pi)$.

$\therefore$ The Fourier series of $f(x)$ will contain only since terms.

Let
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ in } (-\pi, \pi) \qquad (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi + x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x + x^2)\left(\frac{-\cos nx}{n}\right) - (\pi + 2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{-2}{\pi n}\cdot 2\pi^2(-1)^n + \frac{2\cdot 2}{\pi\cdot n^3}\{(-1)^n - 1\}$$

$$= -\frac{4\pi}{n}(-1)^n + \frac{4}{\pi n^3}\{(-1)^n - 1\}$$

Using this value in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{4\pi}{n}(-1)^n + \frac{4}{\pi n^3}\{(-1)^n - 1\}\right]\sin nx \text{ in } (-\pi.\pi)$$

Example 18

Obtain the Fourier series for the function given by $f(x) = \begin{cases} 1 + \dfrac{2x}{l}, & \text{in } -l \le x \le 0 \\ 1 - \dfrac{2x}{l}, & \text{in } \quad 0 \le x \le l \end{cases}$

Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots\infty = \dfrac{\pi^2}{8}$:

The range is symmetrically divided into two subranges and

$$f(x) = \phi_1(x) = 1 + \frac{2x}{l} \text{ in } -l \le x \le 0$$

$$= \phi_2(x) = 1 - \frac{2x}{l} \text{ in } 0 \le x \le l$$

$$\phi_1(-x) = 1 - \frac{2x}{l} = \phi_2(x)$$

$\therefore$ $f(x)$ is an even function of x in $(-l, l)$. $\therefore$ The Fourier will not contain sine terms and will be of period $2l$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \text{ in } (-l, l)$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{1}\left(1 - \frac{2x}{l}\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(1-\frac{2x}{l}\right)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+\frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\{1-(-1)^n\}, \text{ if } n \neq 0$$

$$= \begin{cases} 0, & \text{if } n \text{ is even and} \neq 0 \\ \frac{8}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l \left(1-\frac{2x}{l}\right)\mathrm{d}x = \frac{2}{l}\left[x-\frac{x^2}{l}\right]_0^l = 0$$

Using these values in (1), we get

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \tag{1}$$

$x = 0$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = f(0)$

i.e.
$$\frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right]=1$$

$\therefore$
$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty = \frac{\pi^2}{8}$$

Example 19

Find the Fourier series expansion of $f(x)$ in (–2, 2) which is defined as follows:

$$f(x) = \begin{cases} 0, & \text{in } (-2,-1) \\ x+x^2, & \text{in } (-1,0) \\ x-x^2, & \text{in } (0,1) \\ 0, & \text{in } (1,2) \end{cases}$$

The symmetric range (–2, 2) is symmetrically divided into 4 subranges.

Let
$$f(x) = \begin{cases} \phi_1(x) = 0, & \text{in } (-2,-1) \\ \phi_2(x) = x+x^2, & \text{in } (-1, 0) \\ \phi_3(x) = x-x^2, & \text{in } (0, 1) \\ \phi_4(x) = 0, & \text{in } (1, 2) \end{cases}$$

We note that $\phi_1(x) = -\phi_4(x)$

and $\phi_2(-x) = -\phi_3(x)$

$\therefore$ $f(x)$ is an odd function in (–2, 2)

This can also be graphically verified shown in Fig. 5A.9.

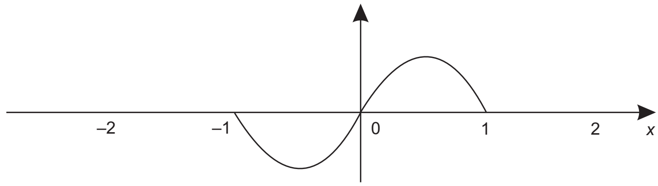


Fig. 5A.9

The Fourier series of $f(x)$ will be of period 4 and will contain only the sine terms

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \text{ in } (-2, 2) \tag{1}$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 (x-x^2)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)-(1-2x)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{16}{n^3\pi^3}\left\{1-\cos\frac{n\pi}{2}\right\}$$

Using this value in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{-4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{16}{n^3\pi^3}\left\{1-\cos\frac{n\pi}{2}\right\}\right]\sin\frac{n\pi x}{2} \text{ in } (-2, 2)$$

Example 20

Find the Fourier series expansion of $f(x)$ in $(-\pi, \pi)$, when $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \pi + x, & \text{in } -\pi \le x \le -\frac{\pi}{2} \\ -x, & \text{in } -\frac{\pi}{2} \le x \le 0 \\ x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases} \tag{1}$$

The symmetric range $(-\pi, \pi)$ is symmetrically divided into 4 subranges.

$$f(x) = \begin{cases} \phi_1(x) = \pi + x, & \text{in } (-\pi, -\pi/2) \\ \phi_2(x) = -x, & \text{in } (-\pi/2, 0) \\ \phi_3(x) = x, & \text{in } (0, \pi/2) \\ \phi_4(x) = \pi - x, & \text{in } (\pi/2), \pi) \end{cases} \tag{2}$$

We note that $\phi_1(-x) = \phi_4(x)$ and $\phi_2(-x) = \phi_3(x)$.

$\therefore$ $f(x)$ is an even function of x in $(-\pi, \pi)$. This is verified graphically also as shown in Fig. 5A.10.

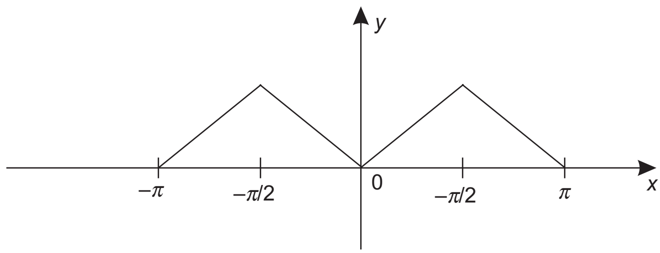


Fig. 5A.10

The Fourier series of $f(x)$ will be of period 2π and will not contain sine terms.

Let $$f(x) = \frac{a_0}{2} + \sum a_n \cos nx \; in \; (-\pi, \pi) \tag{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi/2} + \left\{(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right\}\right.$$

$$\left. + \left\{-\frac{1}{n^2}(-1)^n - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\{1 + (-1)^n\}\right], \text{ if } n \neq 0$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{2}{\pi}\left[\dfrac{1}{2m^2}(\cos m\pi - 1)\right], & \text{if } n \text{ is even and} = 2m \end{cases}$$

$$= \begin{cases} 0, & \text{if } m \text{ is even} \\ -\dfrac{2}{\pi m^2}, & \text{if } m \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right] = \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left\{\frac{(\pi - x^2}{-2}\right\}_{\pi/2}^{\pi}\right] = \frac{\pi}{2}$$

Using these values in (1), we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^2} \cos 2mx, \text{ in } (-\pi, \pi)$$

Exercise 5A(a)

Part A (Short-Answer Questions)

1. State the Dirichlet's conditions that a function $f(x)$ should satisfy so that it may be expanded in the form $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$ in $(c, c+2\pi)$.
2. State Euler's formulas for the Fourier coefficients.
3. Define Fourier series of $f(x)$ in $(c, c + 2l)$.
4. If $f(x)$ is to be expanded as a Fourier series of the form $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, in what range is $f(x)$ to be defined?
5. If the Fourier series of $f(x)$ in $(0, 2\pi)$ is $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$, what are the functions represented by the same series in $(-2\pi, 0)$ and $(2\pi, 4\pi)$?
6. If the Fourier series of $f(x)$ in $(-l, l)$ is $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, what is the Fourier series of $f(x - 2l)$ in $(l, 3l)$?
7. If the Fourier series of $f(x)$ in $(-\pi, \pi)$ is $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$, what is the Fourier series of $f(x + 2\pi)$ in $(-3\pi, -\pi)$?
8. Give the complete definition of a periodic function.
9. The Fourier series of $\sin^3 x \cos^4 x$ in $(-\pi, \pi)$, that in $(-3\pi, -\pi)$ and that in $(\pi, 3\pi)$ are identical. Support or refute this statement with reason.
10. The Fourier series of x^2 in $(0, 2)$, that of $(x + 2)^2$ in $(-2, 0)$ and that of $(x - 2)^2$ in $(2, 4)$ are identical. Support or refute this statement with reason.
11. Only if $f(x + 2l) = f(x)$ can be expanded as a Fourier series of period $2l$. Support or refute the above statement with reason.
12. Define even and odd functions graphically.
13. Since $x^2 = (-x)^2$ in $(0, 2)$, x^2 is an even function of x in $(0, 2)$. Support or refute the above statement with reason.
14. Since $-x^3 = (-x)^3$ in $(0, 2\pi)$, x^3 is an odd function of x in $(0, 2\pi)$. Support or refute the above statement with reason.
15. Write down the form of the Fourier series of an even function in $(-\pi, \pi)$ and the associated Euler's formulas for the Fourier coefficients.
16. Write down the form of the Fourier series of an odd function in $(-l, l)$ and the associated Euler's formulas for the Fourier coefficients.
17. Write down the formula for the sum of the Fourier series of $f(x)$ at the point $x = \alpha$, if

(i) $x = \alpha$ is a point of continuity of $f(x)$

(ii) $x = \alpha$ is an interior point of discontinuity of $f(x)$

18. Write down the formula for the sum of the Fourier series of $f(x)$ in $(c, c + 2\pi)$ at the point of discontinuity $x = \alpha$, if

(i) it coincides with the left end c

(ii) it coincides with the right end $c + 2\pi$

19. Find the Fourier series of $f(x) = \sin^3 x + \cos^3 x$ in $(-\pi, \pi)$.
20. Find the Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$.

Part B

21. Find the Fourier series of period 2π for the function $f(x) = x(2\pi - x)$ in $(0, 2\pi)$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty$.
22. Find the Fourier series of period $2l$ for the function $f(x) = (l - x)^2$ in $(0, 2l)$. Deduce the sum of the series $\sum \frac{1}{n^2}$.
23. Find the Fourier series expansion of $f(x) = \pi^2 - \pi^2$ in $-\pi < x < \pi$.
24. Obtain the Fourier expansion of $f(x) = 1 - x$ in $-1 < x < 1$. Deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty$.
25. Obtain the Fourier series of period $2l$ for the function

$$f(x) = l - x, \text{ in } 0 < x \le l = 0, \text{ in } l \le x < 2l$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$ and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

26. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} 0, & \text{in } (-\pi, 0) \\ \dfrac{\pi x}{4}, & \text{in } (0, \pi) \end{cases}$$. Deduce the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

27. Find the Fourier series expansion of the function $f(x) = \begin{cases} \cos \pi x, & \text{in } (-1, 0) \\ 0, & \text{in } (0, 1) \end{cases}$
28. Find the Fourier series expansion of the function $f(x) = \begin{cases} x, & \text{in } (0, \pi) \\ 2\pi - x, & \text{in } (\pi, 2\pi) \end{cases}$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$.

29. Find the Fourier series expansion of the function

$$f(x) = x, \text{ when } -l < x < 0 = k, \text{ when } 0 < x < l$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

30. Find the Fourier series of $f(x) = x^2$ in $-\pi \le x \le \pi$ and hence prove that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$, (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots \infty = \frac{\pi^2}{12}$; and

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots \infty = \frac{\pi^2}{8}$.

31. Find the Fourier series expansion of $f(x) = x^2 - x$ in $(-l, l)$. Deduce the values of (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \infty$; and (ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \cdots \infty$.
32. Find the Fourier series expansion of period 2π for the function $f(x) = x \sin x$ in $0 < x < 2\pi$. Deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$.
33. Find the Fourier series of period 2π for the function $f(x) = x \cos x$ in $-\pi < x < \pi$.
34. Find the Fourier series of period 2 for the function $f(x) = x \sin \pi x$ in $-1 \le x \le 1$. Deduce the value of $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$.
35. Find the Fourier series of period 2π for the function $f(x) = \sqrt{1 + \cos x}$ in $-\pi < x < \pi$.
36. Find the Fourier series of period 2π for the function $f(x) = \frac{1}{12} x (\pi - \pi)$ $(2\pi - x)$ in $(0, 2\pi)$. Deduce the sum of the series $1^{-3} - 3^{-3} + 5^{-3} - 7^{-3} + \cdots$.
37. Find the Fourier series of period $2l$ for the function $f(x) = |x|$ in $(-l, l)$. Hence find the value of $1^{-2} + 3^{-2} + 5^{-2} + \cdots \infty$.
38. Find the Fourier series of period 2π for the function $f(x) = |\sin x|$ in $(-\pi, \pi)$.
39. Find the Fourier series of period 2π for the function $f(x) = \cos ax$ in $-\pi \le x \le \pi$, when 'a' is not an integer. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{9n^2 - 1}$.
40. Find the Fourier series of period $2l$ for the function $f(x) = e^{ax}$ in $(0, 2l)$.
41. Find the Fourier series expansion for the function $f(x) = \cosh \alpha x$ in $(-\pi, \pi)$.
42. Find the Fourier series of period 4 for the function $f(x)$ defined as follows in $(-2, 2)$:

$$f(x) = \begin{cases} -2, & \text{in } -2 < x < 1 \\ -1, & \text{in } -1 < x < 0 \\ 1, & \text{in } 0 < x < 1 \\ 2, & \text{in } 1 < x < 2 \end{cases}$$

43. Find the fourier series of period 2π for the function

$$f(x) = \begin{cases} \cos x - \sin x, & \text{in } (-\pi, 0) \\ \cos x + \sin x, & \text{in } (0, \pi) \end{cases}$$

Hence deduce the sum of the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \cdots$

44. Find the Fourier series of period 6 for the function

$$f(x) = \begin{cases} 2x + x^2, & \text{in } (-3, 0) \\ 2x - x^2, & \text{in } (0, 3) \end{cases}$$

45. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} -\pi x - x^2, & \text{in } (-\pi, 0) \\ \pi x - x^2, & \text{in } (0, \pi) \end{cases}$$

Deduce the sum of the series (i) $\sum \frac{1}{n^2}$ and (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

46. Find the Fourier series of period 4 for the function

$$f(x) = \begin{cases} 0, & \text{in } (-2, -1) \\ x + x^3, & \text{in } (-1, 1) \\ 0, & \text{in } (1, 2) \end{cases}$$

47. Find the Fourier series of period 2l for the function

$$f(x) = \begin{cases} l + x, & \text{in } (-l, -l/2) \\ 0, & \text{in } (-l/2, l/2) \\ l - x, & \text{in } (l/2, l) \end{cases}$$

48. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} x - 1, & \text{in } -\pi < x < 0 \\ x + 1, & \text{in } 0 < x < \pi \end{cases}$$

49. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} -(\pi + x), & \text{in } (-\pi, -\pi/2) \\ x, & \text{in } (-\pi/2, \pi/2) \\ \pi - x, & \text{in } (\pi/2, \pi) \end{cases}$$

50. Find the Fourier series of period 6 for the function

$$f(x) = \begin{cases} 0, & \text{in } -3 < x < -1 \\ 1 + \cos \pi x, & \text{in } -1 < x < 1 \\ 0, & \text{in } 1 < x < 3 \end{cases}$$

5A.9 HALF-RANGE FOURIER SERIES AND PARSEVAL'S THEOREM

Introduction

If a function $f(x)$ is to be expanded as a Fourier series of period $2l$, $f(x)$ should be defined in a range of length $2l$, in particular, in the range $(-l, l)$ or $(0, 2l)$. But in some situations, the value of $f(x)$ will be available only in a range of length l, in particular in the range $(0, l)$. Without knowing the value of $f(x)$ in the full range, i.e., either in $(-l, l)$ or in $(0, 2l)$, we cannot expand $f(x)$ as a Fourier series of period $2l$, since the Fourier coefficients cannot be found out.

In such situations, i.e., when the value of $f(x)$ is given in $(0, l)$, we assign some value for $f(x)$ in $(-l, 0)$ [or in $(l, 2l)$], so that $f(x)$ is defined completely in the full range $(-l, l)$ [or in $(0, 2l)$]. If we assign arbitrary value of $f(x)$ in $(-l, 0)$, the Fourier series of $f(x)$ will contain both cosine and sine terms. This kind of Fourier series of period $2l$, resulting from arbitrary assignment of value for $f(x)$ in $(-l, 0)$ is not of interest.

If we assign a suitable value for $f(x)$ in $(-l, 0)$ so that the given value of $f(x)$ in $(0, l)$ and the assigned value of $f(x)$ in $(-l, 0)$ together make $f(x)$ even or odd in $(-l, l)$, then the Fourier series of $f(x)$ will be of period $2l$ and will contain only cosine terms or sine terms respectively. Such series are called *Fourier half-range cosine series* or *sine series* respectively and will represent the given value of $f(x)$ in $(0, l)$.

Note ✍

The term 'half-range series' is used because the Fourier series is of period 2l, even though the function is defined in a range of length l.

Theorem

A function $f(x)$ defined in $(0, l)$ can be expanded as a Fourier series of period $2l$ containing (i) only cosine terms and (ii) only sine terms, by extending $f(x)$ suitably in $(-l, 0)$.

Proof

Let $$f(x) = \phi(x) \text{ in } (0, l)$$

(i) Let us assign the value $f(x) = \phi(-x)$ in $(-l, 0)$. By the definition of an even function given in Section 5A.6, $f(x)$ is even in $(-l, l)$.

∴ Fourier series of $f(x)$ will be of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \text{ where}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \; n \geq 0$$

(ii) Let us assign the value $f(x) = -\phi(-x)$ in $(-l, 0)$. By the definition of an odd function given in the previous section, $f(x)$ is odd in $(-l, l)$.

∴ Fourier series of $f(x)$ will be of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Note ✍

1. *The values $\phi(-x)$ and $-\phi(-x)$ assigned to $f(x)$ in $(-l, 0)$ in order to make $f(x)$ even and odd respectively in $(-l, l)$ are called the even and odd extensions of $f(x)$ in $(-l, 0)$.*

2. *The evaluation of a_n and b_n by the modified Euler's formulas requires only the given value of $f(x)$ in $(0, l)$.*

Theorem

A function $f(x)$ defined in $(0, l)$ can be expanded as a Fourier series of period $2l$ containing (i) only cosine terms and (ii) only sine terms, by extending $f(x)$ suitabley in $(l, 2l)$.

Proof

Let $$f(x) = \phi(x) \text{ in } (0, l)$$

(i) Let us assign the value $f(x) = \phi(2l - x)$ in $(l, 2l)$. Let the Fourier series of $f(x)$ be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Now $$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx + \int_l^{2l} \phi(2l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx + \int_l^{0} \phi(y)\sin\frac{n\pi}{l}(2l-y)(-dy)\right]$$

on putting $2l - x = y$ in the second integral.

$$= \frac{1}{l}\left[\int_0^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx - \int_0^{l} \phi(y)\sin\frac{n\pi y}{l}\,dy\right] = 0$$

This means that the Fourier series will contain only cosine terms.

i.e. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Now $$a_n = \frac{1}{l}\left[\int_0^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} \phi(2l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{0} \phi(y)\cos\frac{n\pi}{l}(2l-y)(-dy)\right].$$

on putting $2l - x = y$ in the second integral.

$$= \frac{1}{l}\left[\int_0^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_0^{l} \phi(y)\cos\frac{n\pi y}{l}\,dy\right]$$

i.e. $$a_n = \frac{2}{l}\int_0^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx, \quad n \geq 0$$

(ii) Let us assign the value $f(x) = -\phi(2l - x)$ in $(l, 2l)$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$. Proceeding as in (i), we an prove that

$$a_n = 0, \quad h \geq 0$$

$\therefore$ The Fourier series of $f(x)$ will contain only sine terms.

i.e.
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Proceeding as in (i), we can prove that

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Note ✍

1. *The extended values of $f(x)$ in $(l, 2l)$, namely $\phi(2l - x)$ and $-\phi(2l - x)$ are only the periodic extensions in $(l, 2l)$ of $\phi(-x)$ and $-\phi(-x)$, that are the even and odd extensions of $f(x)$ in $(-l, 0)$.*

2. *Even in this case, the evaluation of a_n and b_n requires only the given value of $f(x)$ in $(0, 1)$.*

5A.10 ROOT-MEAN SQUARE VALUE OF A FUNCTION

Definition

If a function $y = f(x)$ is defined in $(c, c + 2l)$, then $\sqrt{\frac{l}{2l}\int_c^{c+2l} y^2\,dx}$ is called *the root mean-square (R.M.S.) value of* y in $(c, c + 2l)$ and is denoted by $\bar{y}$.

Thus
$$\bar{y}^2 = \frac{1}{2l}\int_c^{c+2l} y^2\,dx$$

If $y = f(x)$ can be expanded as a Fourier series in $(c, c + 2l)$, then $\bar{y}^2$ can be expressed in terms of Fourier coefficients a_0, a_n and b_n. The formula that expresses $\bar{y}^2$ in terms of a_0, a_n and b_n is known as *Parseval's formula* which is stated as a theorem.

Parseval's theorem

If $y = f(x)$ can be expanded as Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ in $(c, c + 2l)$, then the root-mean square value $\bar{y}$ of $y = f(x)$ in $(c, c + 2l)$ is given by

$$\bar{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

Proof

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ in } (c, c+2l) \tag{1}$$

$\therefore$ By Euler's formulas for the Fourier coefficients,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x,\ n \geq 0 \tag{2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x,\ n \geq 1 \tag{3}$$

Now, by definition,

$$\begin{aligned}
\overline{y}^2 &= \frac{1}{2l}\int_c^{c+2l} y^2\mathrm{d}x = \frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x \\
&= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right]\mathrm{d}x, \qquad \text{using (1)} \\
&= \frac{a_0}{4}\left[\frac{1}{l}\int_c^{c+2l} f(x)\,dx\right] + \sum_{n=1}^{\infty}\frac{a_n}{2}\left\{\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x\right\} \\
&\quad + \sum_{n=1}^{\infty}\frac{b_n}{2}\left\{\frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x\right\} \\
&= \frac{a_0}{4}\cdot a_0 + \sum_{n=1}^{\infty}\frac{a_n}{2}\cdot a_n + \sum_{n=1}^{\infty}\frac{b_n}{2}\cdot b_n, \qquad \text{by using (2) and (3)} \\
&= \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2
\end{aligned}$$

Corollary 1

If the Fourier half-range cosine series of $y = f(x)$ in $(0, l)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, then

$$\overline{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2, \text{ where } \overline{y}^2 = \frac{1}{l}\int_0^l y^2\,\mathrm{d}x$$

Corollary 2

If the Fourier half-range sine series of $y = f(x)$ in (0, 1) is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, then

$$\overline{y}^2 = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2, \text{ where } \overline{y}^2 = \frac{1}{l}\int_0^l y^2\,\mathrm{d}x$$

Worked Examples 5A(b)

Example 1

Find the half-range (i) cosine series and (ii) sine series for $f(x) = x^2$ in $(0, \pi)$

(i) To get the half-range cosine series for $f(x)$ in $(0, \pi)$, we should given an even extension for $f(x)$ in $(-\pi, 0)$.

i.e. put $f(x) = (-x)^2 = x^2$ in $(-\pi, 0)$

Now $f(x)$ is even in $(-\pi, \pi)$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2}\cdot\pi(-1)^n = \frac{4(-1)^n}{n^2},\ n \neq 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2\,\mathrm{d}x = \frac{2}{3}\pi^2$$

$\therefore$ The Fourier half-range cosine series of x^2 is given by

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \text{ in } (0, \pi)$$

(ii) To get the half-range sine series of $f(x)$ in $(0, \pi)$, we should give an odd extension for $f(x)$ in $(-\pi, 0)$.

i.e. put $f(x) = -(-x)^2$ in $(-\pi, 0)$

$= -x^2$ in $(-\pi, 0)$

Now $f(x)$ is odd in $(-\pi, \pi)$.

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\{(-1)^n - 1\}\right]$$

$$= \begin{cases} \dfrac{2}{\pi}\left[\dfrac{\pi^2}{n} - \dfrac{4}{n^3}\right], & \text{if } n \text{ is odd} \\ -\dfrac{2\pi}{n}, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (2), we get the half-range sine series of x^2 in $(0, \pi)$.

Example 2

Find (i) the Fourier half-range cosine series and (ii) the Fourier half-range series of

$$f(x) = \begin{cases} x, & \text{in } 0 < x < 1 \\ 2 - x, & \text{in } 1 < x < 2 \end{cases}$$

(i) To get the half-range cosine series, we give an even extension for $f(x)$ in $(-2, 0)$.

i.e. we put
$$f(x) = \begin{cases} 2 + x, & \text{in } -2 < x < -1 \\ -x, & \text{in } -1 < x < 0 \end{cases}$$

Now $f(x)$ has been made an even function in $(-2, 2)$. Here $2l = 4$.

Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \tag{1}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \int_0^1 x\cos\frac{n\pi x}{2}\,\mathrm{d}x + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left[\left\{x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2}}\right\}_0^1 + \left\{(2-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_1^2\right]$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi$$

$$- \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\{1+(-1)^n\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{2}{m^2\pi^2}[(-1)^m - 1], & \text{if } n \text{ is even and} = 2m \end{cases}$$

$$= \begin{cases} 0, & \text{if } m = \dfrac{n}{2} \text{ is even} \\ \dfrac{-4}{m^2\pi^2}, & \text{if } m = \dfrac{n}{2} \text{ is odd} \end{cases}$$

$$= \begin{cases} 0, & \text{if } n \text{ is a multiple of 4} \\ \dfrac{-16}{n^2\pi^2}, & \text{if } n \text{ is even, but not a multiple of 4} \end{cases}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left[\frac{(2-x)^2}{-2}\right]_1^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1\,.$$

Using these values in (1), the required cosine series is given by

$$f(x) = \frac{1}{2} - \frac{16}{\pi^2}\left[\frac{1}{2^2}\cos\pi x + \frac{1}{6^2}\cos 3\pi x + \frac{1}{10^2}\cos 5\pi x + \cdots \infty\right]$$

(ii) To get the half-range sine series of $f(x)$, we give an odd extension for $f(x)$ in $(-2, 0)$.

i.e. we put
$$f(x) = \begin{cases} -(2+x), & \text{in } -2 < x < -1 \\ x, & \text{in } -1 < x < 0 \end{cases}$$

Now $f(x)$ has been made an odd function in $(-2, 2)$. Here $2l = 4$.
Let the half-range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\left\{x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_0^1\right.$$

$$\left. + \left\{(2-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_1^2\right]$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (2), the required sine series is given by

$$f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{n\pi}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \cdots\infty\right]$$

Note ✍

From the above two examples, it is clear that any function defined in (0, l) can be expanded as a cosine series and also as a sine series. Depending on the nature of the Fourier series required, we give corresponding extension for the function in (–l, 0)

Example 3

Find the Fourier half-range cosine series of the function $f(x) = (x + 1)^2$ in (–1, 0). Hence find the value of $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\infty$.

To get the half-range cosine series of $f(x)$, we give an even extension to $f(x)$ in (0, 1), i.e. we put $f(x) = (-x + 1)^2$ in (0, 1)

Now $f(x)$ is even in (–1, 1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \text{ since } 2l = 2$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 (1-x)^2 \cos n\pi x\,dx$$

Note ✍

We do not use the given value of $f(x)$ in (–1, 0) for evaluating a_n, but use the assigned value of $f(x)$ in (0, 1). Hence extra care should be taken while assigning the value of $f(x)$ in (0, 1). However, a_n can also be found out by using the formula $a_n = \frac{2}{1}\int_{-1}^{0} (x+1)^2 \cos n\pi x\,dx$

$$= 2\left[(1-x)^2\left(\frac{\sin n\pi x}{n\pi}\right) - \{-2(1-x)\}\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}, \text{ if } n \neq 0$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 (1-x)^2\,dx = 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$= 2/3$

∴ The required half-range cosine series is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

This series represents $(x + 1)^2$ in (–1, 0) and $(1 - x)^2$ in (0, 1).

$x = 0$ is a point of continuity for $f(x)$.

∴ $[\text{Sum of the Fourier series of } f(x)]_{x=0} = f(0)$

i.e $$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1$$

∴ $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 4

Find the half-range sine series of the function $f(x) = \pi - x$ in $(\pi, 2\pi)$,by suitably extending $f(x)$ in $(0, \pi)$. Deduce the sum of the series $1 - 1/3 + 1/5 - 1/7 + \cdots \infty$. If $f(x) = \phi(x)$ in $(0, l)$, we should assign $f(x) = -\phi(2l - x)$ in $(l, 2l)$ in order to get a sine series.

Hence if $f(x) = \psi(x)$ in $(l, 2l)$, we should assign $f(x) = -\psi(2l - x)$ in $(0, l)$ in order to get a sine series. This is obtained by putting $-\phi(2l - x) = \psi(x)$ and by making the transformation $2l - x = u$.

Since $f(x) = \pi - x$ in $(\pi, 2\pi)$, we put $f(x) = -\{\pi - (2\pi - x)\}$ in $(0, \pi)$ i.e. we put $f(x) = \pi - x$ in $(0, \pi)$ to get sine series.

Let the Fourier sine series of $f(x)$ be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n}$$

Hence

$$f(x) = 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx \tag{1}$$

Putting $x = \pi/2$, we can get the series, whose sum is required.

$x = \pi/2$ is a point of continuity for $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x = \pi/2} = f(\pi/2)$

i.e.

$$2\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty\right] = \pi - \pi/2$$

$\therefore$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$$

Note ✍

If the specific value of the extension of $f(x)$ in $(0, \pi)$ is not required, we can also evaluate b_n by using the formula $b_n = \frac{2}{\pi}\int_{\pi}^{2\pi} (\pi - x)\sin nx\,dx$

Example 5

Find the half-range cosine series of $f(x) = x(l - x)$ in $(0, l)$. How should $f(x)$ be extended in order to get this cosine series (i) in the range $(-l, 0)$ and (ii) in the range $(l, 2l)$?

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, since the length of the given half-range $= l$.

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}[-l\cos n\pi - l]$$

$$= -\frac{2l^2}{n^2\pi^2}\{(-1)^n + 1\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\dfrac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$$

$\therefore$ Required half-range cosine series is given by

$$f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^2}\cos nx \quad \text{or}$$

$$= \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nx$$

To get this half-range cosine series, we should assign $f(x) = -(lx + x^2)$ in $(-l, 0)$ and assign $f(x) = (2l - x)(x - l)$ in $(l, 2l)$.

Example 6

Find the half-range sine series of $f(x)$ in $(0, \lambda)$, given that

$$f(x) = \begin{cases} (\lambda - c)x, & \text{in } (0, c) \\ (\lambda - x)c, & \text{in } (c, \lambda) \end{cases}$$

We give an odd extension to $f(x)$ in $(-\lambda, 0)$.

i.e. we put
$$f(x) = \begin{cases} -(\lambda + x)c, & \text{in } (-\lambda, -c) \\ (\lambda - c)x, & \text{in } (c, 0) \end{cases}$$

Now $f(x)$ is odd in $(-\lambda, \lambda)$.

Let
$$f(x) = \sum b_n \sin\frac{n\pi x}{\lambda} \qquad (1)$$

$$b_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin\frac{n\pi x}{\lambda}\,dx$$

$$= \frac{2}{\lambda}\left[\int_0^{c}(\lambda-c)x\sin\frac{n\pi x}{\lambda}\,dx + \int_c^{\lambda}(\lambda-x)c\sin\frac{n\pi x}{\lambda}\,dx\right]$$

$$= \frac{2}{\lambda}(\lambda-c)\left[x\left(\frac{-\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right)+\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right]_0^c$$

$$+\frac{2c}{\lambda}\left[(\lambda-x)\left(\frac{-\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right)-\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right]_c^{\lambda}$$

$$= \frac{2(\lambda-c)}{\lambda}\left[-\frac{c\lambda}{n\pi}\cos\frac{n\pi c}{\lambda}+\frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}\right]$$

$$+\frac{2c}{\lambda}\left[\frac{\lambda(\lambda-c)}{n\pi}\cos\frac{n\pi c}{\lambda}+\frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}\right]$$

$$= \frac{2\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}$$

Using in (1), we get the required half-range sine series as

$$f(x) = \frac{2\lambda^2}{\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi c}{\lambda}\sin\frac{n\pi x}{\lambda}$$

Example 7

Find the half-range cosine series of $f(x) = \sin x$ in $(0, \pi)$.
We give an even extension for $f(x)$ in $(-\pi, 0)$.
i.e. we put $f(x) = -\sin x$ in $(-\pi, 0)$.
Now $f(x)$ is even in $(-\pi, \pi)$.

Let
$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left\{1-(-1)^{n-1}\right\}\right]$$

$$= -\frac{2}{\pi(n^2-1)}\{1-(-1)^{n-1}\}$$

$$= \begin{cases} -\dfrac{4}{\pi(n^2-1)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd and } \neq 1 \end{cases}$$

$$a_0 = \frac{4}{\pi}, \text{ on putting } n = 0 \text{ in } a_n$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx = \frac{1}{\pi}\int_0^\pi \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^\pi = 0$$

Using these values in (1), the required half-range cosine series is obtained as

$$\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n^2-1}\cdot \cos nx\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx\right]$$

Example 8

Find the half-range sine series of $f(x) = \sin ax$ in $(0, l)$.

We give an odd extension for $f(x)$ in $(-l, 0)$.

i.e. we put $f(x) = -\sin[a(-x)] = \sin ax$ in $(-l, 0)$

$\therefore$ $f(x)$ is odd in $(-l, l)$

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l \sin ax \cdot \sin\frac{n\pi x}{l} dx$$

$$= \frac{1}{l}\int_0^l \left[\cos\left(\frac{n\pi}{l} - a\right)x - \cos\left(\frac{n\pi}{l} + a\right)x\right] dx$$

$$= \frac{1}{l}\left[\frac{\sin\left(\frac{n\pi}{l}-a\right)x}{\left(\frac{n\pi}{l}-a\right)} - \frac{\sin\left(\frac{n\pi}{l}+a\right)x}{\left(\frac{n\pi}{l}+a\right)}\right]_0^l$$

$$= \frac{1}{n\pi - al}\sin(n\pi - al) - \frac{1}{n\pi + al}\sin(n\pi + al)$$

$$= \frac{1}{n\pi - al}(-1)^{n+1}\sin al + \frac{1}{n\pi + al}(-1)^{n+1}\sin al$$

$$= (-1)^{n+1}\sin al \cdot \frac{2n\pi}{n^2\pi^2 - a^2l^2}$$

Using this value in (1), we get the half-range sine series as

$$\sin ax = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot n}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}$$

Example 9

Find the half-range cosine series of $f(x) = x \sin x$ in $(0, \pi)$. Deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \cdots \infty$.

We give an even extension for $f(x)$ in $(-\pi, 0)$

i.e. we put $f(x) = -x \sin(-x)$

$= x \sin x$ in $(-\pi, 0)$

Now $f(x)$ is even in $(-\pi, \pi)$.

$\therefore$ Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}\right\} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi}$$

$$-\frac{1}{\pi}\left[x\left\{\frac{-\cos(n-1)x}{n-1}\right\} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= -\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}$$

$$= (-1)^{n-1}\left\{\frac{1}{n-1} - \frac{1}{n+1}\right\} = \frac{2(-1)^{n-1}}{n^2-1}, \text{ if } n \neq 1$$

$$a_0 = 2$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{-1}{2}$$

Using these values, we get the required cosine series as

$$x \sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2-1}\cos nx \text{ in } (0, \pi) \tag{2}$$

$$x = \frac{\pi}{2} \text{ is a point of continuity of } x \sin x$$

$\therefore$ $[\text{Sum of the Fourier series of } f(x)]_{x=\frac{\pi}{2}} = f\left(\frac{\pi}{2}\right)$

i.e.
$$1 + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2-1}\cos\frac{n\pi}{2} = \frac{\pi}{2}$$

i.e.
$$1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty\right\} = \frac{\pi}{2}$$

$\therefore$
$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty = \frac{\pi - 2}{4}$$

Example 10

Find the half-range sine series of $f(x) = \dfrac{\sinh ax}{\sinh a\pi}$ in $(0, \pi)$

We give an odd extension for $f(x)$ in $(-\pi, 0)$.

i.e. we put
$$f(x) = \frac{-\sinh(-x)}{\sinh a\pi} = \frac{\sinh ax}{\sin ax} \text{ in } (-\pi, 0)$$

Now $f(x)$ is odd in $(-\pi, \pi)$.

$\therefore$ Let
$$f(x) = \sum b_n \sin nx \tag{1}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\sinh ax}{\sinh a\pi}\sin nx \, dx$$

$$= \frac{1}{\pi \sinh a\pi}\left[\int_0^\pi e^{ax}\sin nx\,dx - \int_0^\pi e^{-ax}\sin nx\,dx\right]$$

$$= \frac{1}{\pi \sinh a\pi}\left[\left\{\frac{e^{ax}}{n^2+a^2}(a\sin nx - n\cos nx\right\}_0^\pi\right.$$

$$\left. - \left\{\frac{e^{-ax}}{n^2+a^2}(-a\sin nx - n\cos nx\right\}_0^\pi\right]$$

$$= \frac{1}{\pi \sinh a\pi}\left[\frac{-n(-1)^n e^{a\pi}}{n^2+a^2} + \frac{n(-1)^n e^{-a\pi}}{n^2+a^2}\right]$$

$$= \frac{1}{\pi \sinh a\pi}\cdot\frac{2(-1)^{n-1} n \sinh a\pi}{n^2+a^2}$$

$$= \frac{2}{\pi}\cdot\frac{(-1)^{n-1}\cdot n}{n^2+a^2}$$

Using this value in (1), we get the required half-range sine series as

$$\frac{\sinh ax}{\sinh a\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n}{n^2+a^2}\sin nx \text{ in } (0, \pi)$$

Example 11

Find the Fourier series of period 2π for the function $f(x) = x^2 - x$ in $(-\pi, \pi)$. Hence deduce the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\infty$, assuming that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Let $$x^2 - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ in } (-\pi, \pi) \quad (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx \; [\because \; x\cos nx \text{ is odd in } (-\pi, \pi)$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^2}(-1)^n, \text{ if } n \neq 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \; [\because \quad x^2 \sin nx \text{ is odd in } (-\pi, \pi)]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{-2}{n}(-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \text{ in } (-\pi, \pi)$$

Now the terms of the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^4}$, whose sum is required, are the square of the Fourier coefficients a_n multiplied by a constant. Whenever this situation arises, we apply Parseval's theorem, which states that

$\frac{1}{4}a_0^2 + \frac{1}{2}\sum\limits_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum\limits_{n=1}^{\infty} b_n^2 = \overline{y}^2$, the square of the R.M.S. value of $y = f(x)$ in $(-\pi, \pi)$

Thus
$$\frac{1}{4}\cdot\frac{4}{9}\pi^4 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16\cdot(-1)^{2n}}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4\cdot(-1)^{2n+2}}{n^2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2 - x)^2\,dx$$

i.e.
$$\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\cdot 2\int_0^{\pi}(x^4 + x^2)\,dx$$

$$= \frac{1}{5}\pi^4 + \frac{1}{3}\pi^2$$

i.e.
$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{1}{5} - \frac{1}{9}\right)\pi^4 + \frac{1}{3}\pi^2 - 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

i.e.
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}\left(\because \quad \sum\frac{1}{n^2} = \frac{\pi^2}{6}\right)$$

Example 12

Find the Fourier series expansion of period $2l$ for the function

$f(x) = \begin{cases} x, & \text{in } (0, l) \\ 0, & \text{in } (l, 2l) \end{cases}$ Hence deduce the sum of the series $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\infty$,

assuming that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty = \frac{\pi^2}{6}$.

Let
$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ in } (0, 2l)$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\cdot\int_0^{l} x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l}}\right)\right]_0^l$$

$$= \frac{l}{n^2\pi^2}\{(-1)^n-1\}$$

$$= \begin{cases} \frac{-2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{1}{l}\int_0^l x\,dx = \frac{l}{2}$$

$$b_n = \frac{1}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l}}\right)\right]_0^l = \frac{-l}{n\pi}(-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{l}{4}-\frac{2l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} \text{ in } (0, 2l)$$

Now the series to be summed up contains constant multiples of squares of a_n. Hence we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{l}{2l}\int_0^{2l}[f(x)]^2\,\mathrm{d}x$$

i.e
$$\frac{l^2}{16} + \frac{1}{2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{4l^2}{n^4\pi^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{l^2}{n^2\pi^2} = \frac{1}{2l}\int_0^{l}x^2\,\mathrm{d}x$$

i.e.
$$\frac{l^2}{16} + \frac{2l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} + \frac{l^2}{2\pi^2}\cdot\frac{\pi^2}{6} = \frac{l^2}{6}$$

$$\left(\because\ \sum\frac{1}{n^2} = \frac{\pi^2}{6}\right)$$

i.e.
$$\frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{1}{6} - \frac{1}{16} - \frac{1}{12} = \frac{1}{48}$$

$\therefore$
$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 13

Find the Fourier series expansion of period l for the function

$$f(x) = \begin{cases} x, & \text{in } \left(0, \frac{l}{2}\right) \\ l - x, & \text{in } \left(\frac{l}{2}, l\right) \end{cases}$$ Hence deduce the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Here the length of the full range = period of the Fourier series required = l.

$\therefore$ The Fourier series of $f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{2n\pi x}{l} + \sum b_n \sin\frac{2n\pi x}{l} \text{ in } (0, l) \quad (1)$$

$$a_n = \frac{1}{l/2}\int_0^{l} f(x)\cos\frac{2n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\cos\frac{2n\pi x}{l}\,\mathrm{d}x + \int_{l/2}^{l}(l-x)\cos\frac{2n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) + \frac{\cos\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right\}_0^{\frac{l}{2}} + \left\{(l-x)\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - \left(\frac{\cos\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right)\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{l}{n^2\pi^2}\{(-1)^n - 1\} = \begin{cases} -\dfrac{2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{1}{\frac{l}{2}}\int_0^l f(x)\,\mathrm{d}x = \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\,\mathrm{d}x + \int_{\frac{l}{2}}^{l}(l-x)\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left(\frac{x^2}{2}\right)_0^{\frac{l}{2}} + \left\{\frac{(l-x)^2}{-2}\right\}_{\frac{l}{2}}^{l}\right] = \frac{l}{2}$$

$$b_n = \frac{1}{\frac{l}{2}}\int_0^l f(x)\sin\frac{2n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{2n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l}(l-x)\sin\frac{2n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) + \frac{\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right\}_0^{\frac{l}{2}}\right.$$

$$\left. + \left\{(1-x)\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - \frac{\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l}}\right\}_{\frac{l}{2}}^{l}\right] = 0$$

Using these values in (1), we get

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l} \text{ in } (0, l)$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{l}\int_0^l [f(x)]^2\,\mathrm{d}x$$

i.e.

$$\frac{l^2}{16} + \frac{1}{2}\cdot\frac{4l^2}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4} = \frac{l}{l}\left[\int_0^{\frac{l}{2}} x^2\mathrm{d}x + \int_{\frac{l}{2}}^{l}(l-x)^2\,\mathrm{d}x\right]$$

i.e. $$\frac{l^2}{16}+\frac{2l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{1}{l}\left[\frac{l^3}{24}+\frac{l^3}{24}\right]$$

$$= \frac{l^2}{12}$$

$$\frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{1}{48}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 14

Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} 1, & \text{in } (0, \pi) \\ 2, & \text{in } (\pi, 2\pi) \end{cases}$$

Hence find the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty$.

Let $$f(x) = \frac{a_0}{2}+\sum a_n \cos nx+\sum b_n \sin nx \text{ in } (0, 2\pi) \quad (1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\cdot\cos nx\,\mathrm{d}x+\int_{\pi}^{2\pi} 2\cdot\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi}+2\left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right], \text{ if } n \neq 0$$

$$= 0, \text{ if } n \neq 0$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx+\int_{\pi}^{2\pi} 2\,\mathrm{d}x\right] = 3$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\cdot\sin nx\,\mathrm{d}x+\int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\cos nx}{n}\right)_0^{\pi} - 2\left(\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= -\frac{1}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{-2}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using these values in (1), we get

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin nx \text{ in } (0, 2\pi)$$

Since the series to be summed up contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2\,dx$$

i.e. $$\frac{9}{4} + \frac{1}{2}\cdot\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{2\pi}\left[\int_0^{\pi} 1^2\cdot dx + \int_{\pi}^{2\pi} 2^2\cdot dx\right]$$

i.e. $$\frac{9}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{5}{2}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{2}\left(\frac{5}{2}-\frac{9}{4}\right) = \frac{\pi^2}{8}$$

Example 15

Find the half-range sine series of $f(x) = a$ in $(0, l)$. Deduce the sum of $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots\infty$.

Giving an odd extension for $f(x)$ in $(-l, 0)$, $f(x)$ is made an odd function in $(-l, l)$.

$\therefore$ Let $$f(x) = \sum b_n \sin\frac{n\pi x}{l} \tag{1}$$

$$b_n = \frac{2}{l}\int_0^l a\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2a}{l}\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l = \frac{2a}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{4a}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (1), we get

$$a = \frac{4a}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \text{ in } (0, l)$$

Since the series whose sum is required contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2} \sum b_n^2 = \frac{1}{l} \int_0^l [f(x)]^2 \, dx$$

i.e.
$$\frac{1}{2} \cdot \frac{16a^2}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{(2n-1)^2} = a^2$$

i.e.
$$\frac{8a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = a^2$$

$\therefore$
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example 16

Find the half-range cosine series of $f(x) = x$ in (0, 1). Deduce the sum of the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty$.

Giving an even extension for $f(x)$ in (–1, 0), $f(x)$ is made an even function in (–1, 1).

$\therefore$ Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \qquad (1)$$

$$a_n = \frac{2}{1} \int_0^1 f(x) \cos n\pi x \, dx$$

$$= 2 \left[x \left(\frac{\sin n\pi x}{n\pi} \right) + \left(\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= \frac{2}{n^2\pi^2} \{(-1)^n - 1\}$$

$$= \begin{cases} \dfrac{-4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and } \neq 0 \end{cases}$$

$$a_0 = \frac{2}{1} \int_0^1 x \, dx = 1$$

Using these values in (1), we get

$$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\pi x \text{ in } (0, 1)$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{1}\int_0^1 x^2\,dx$$

i.e.
$$\frac{1}{4} + \frac{1}{2}\cdot\frac{16}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4} = \frac{1}{3}$$

i.e.
$$\frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{1}{12}$$

$\therefore$
$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 17

Find the half-range sine series of $f(x) = l - x$ in $(0, l)$. Hence prove that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty = \frac{\pi^2}{6}$.

Giving an odd extension for $f(x)$ in $(-l, 0)$, $f(x)$ is made an odd function in $(-l, l)$.

$\therefore$ Let
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1}$$

$$b_n = \frac{2}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2l}{n\pi}$$

Using this value in (1), we get

$$l - x = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l} \text{ in } (0, l)$$

Since the series to be summed up contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2}\sum b_n^2 = \frac{1}{l}\int_0^l (l-x)^2\,dx$$

i.e $$\frac{1}{2}\cdot\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^l = \frac{l^3}{3}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 18

Find the half-range cosine series of $f(x) = (\pi - x)^2$ in $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\infty$.

Giving an even extension for $f(x)$ in $(-\pi, 0)$, the function $f(x)$ is made an even function in $(-\pi, \pi)$.

$\therefore$ Let $$f(x) = \frac{a_0}{2}+\sum a_n \cos nx \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) = \{-2(\pi - x)\}\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^2}, \text{ if } n \neq 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)^2\, dx = \frac{2}{\pi}\left\{\frac{(\pi - x)^3}{-3}\right\}_0^{\pi} = \frac{2}{3}\pi^2$$

Using these values in (1), we get $(\pi - x)^2 = \frac{\pi^2}{3}+16\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$ in $(0, \pi)$. Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{a_0^2}{4}+\frac{1}{2}\sum a_n^2 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)^4\, dx$$

i.e. $$\frac{\pi^4}{9}+\frac{1}{2}\cdot 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{\pi}\left\{\frac{(\pi - x)^5}{-5}\right\}_0^{\pi} = \frac{\pi^4}{5}$$

i.e $$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{4}{45}\pi^4$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Example 19

Find the half-range sine series of

$$f(x)=\begin{cases} x, & \text{in } \left(0, \dfrac{\pi}{2}\right) \\ \pi - x, & \text{in } \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$$

in $(0, \pi)$. Hence find the sum of the series $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\cdots\infty$

Giving an odd extension for $f(x)$ in $(-\pi, 0)$, the function $f(x)$ is made an odd function in $(-\pi, \pi)$.

$\therefore$ Let $f(x) = \sum b_n \sin nx$ (1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right)-\frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$, which becomes zero for even values of n.

Using this value in (1), we get

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx \text{ in } (0, \pi)$$

Since the series whose sum is required contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2}\sum b_n^2 = \frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx$$

i.e. $$\frac{1}{2}\cdot\frac{16}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2} = \frac{1}{\pi}\left[\int_0^{\pi/2}x^2\,dx + \int_{\pi/2}^{\pi}(\pi - x)^2\,dx\right]$$

i.e. $$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^2}{12}$$

$$\left(\because \quad \sin\frac{n\pi}{2} = \pm 1, \text{ when } n \text{ is odd and } \sin^2\frac{n\pi}{2} = 1\right)$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 20

Find the half-range cosine series of $f(x) = x\,(\pi - x)$ in $(0, \pi)$. Hence find the sum of the series $1/1^4 + 1/2^4 + 1/3^4 + \cdots \infty$.

Giving an even extension for $f(x)$ in $(-\pi, 0)$, the function $f(x)$ is made an even function in $(-\pi, \pi)$

$\therefore$ Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + 2\frac{\sin nx}{n^3}\right]_{\theta}^{\pi}$$

$$= -\frac{2}{n^2}\{(-1)^n + 1\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\dfrac{4}{n^2}, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,\mathrm{d}x = \frac{2}{\pi}\left\{\pi\frac{x^2}{2} - \frac{x^3}{3}\right\}_0^{\pi} = \frac{\pi^2}{3}$$

Using these values in (1), we get

$$x(\pi - x) = \frac{\pi^2}{6} - 4\cdot\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^2}\cos nx \text{ in } (0, \pi)$$

or $$x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nx \text{ in } (0, \pi)$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,\mathrm{d}x$$

i.e. $$\frac{\pi^4}{36} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{\pi}\int_0^{\pi}(\pi^2 x^2 - 2\pi x^3 + x^4)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left(\pi^2 \frac{x^3}{3} - 2\pi \frac{x^4}{4} + \frac{x^5}{5}\right)_0^{\pi} = \frac{\pi^4}{30}$$

i.e. $$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{180}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Exercise 5A(b)

Part A (Short-Answer Questions)

1. Why is Fourier half-range series called so?
2. When a function is defined in $(0, 2l)$, is it possible to expand it as a Fourier half-range series? How?
3. If $f(x)$ is defined in $(0, l)$, how should $f(x)$ be defined in $(-l, 0)$, so that the Fourier half-range series of $f(x)$ may contain (i) only cosine terms and (ii) only sine terms?
4. If $f(x)$ is defined in $(0, l)$, how should $f(x)$ be defined in $(l, 2l)$, so that the Fourier half-range series of $f(x)$ may contain (i) only cosine terms and (ii) only sine terms?
5. When $f(x)$, defined in $(-\pi, 0)$, is expanded as a Fourier half-range cosine series, write down the formula for the Fourier coefficients.
6. When $f(x)$ defined in $(-l, 0)$ is expanded as a Fourier half-range sine series, write down the formula for the Fourier coefficients.
7. Write down the even and odd extensions of $f(x)$ in $(-l, 0)$, if $f(x) = x^2 + x$ in $(0, l)$.
8. Write down the extension of $f(x)$ in $(l, 2l)$, if $f(x) = x(l - x)$ in $(0, l)$ so as to get cosine and sine series.
9. Define the root-mean square value of a function $f(x)$ in $(0, 2\pi)$.
10. State Parseval's theorem.
11. If the impressed voltage E at time t is given by the series $E = \sum_{n=1,3,5,\ldots}^{\infty} E_n \sin(n\omega t + \alpha_n)$, find the effective value of E.

Note ✍

The R.M.S. value is also called the effective value. Rewrite E as $E = \sum_{n=1,3,5,\ldots}^{\infty} (E_n \sin\alpha_n)\cos n\omega t + \sum_{n=1,3,5,\ldots}^{\infty} (E_n \cos\alpha_n)\sin n\omega t$ *and use Parseval's theorem.*

12. If an alternating current I is represented by the series $I = \sum_{n=1,3,5,\ldots}^{\infty} I_n \sin(n\omega t + \alpha_n)$, find the effective value of I.
13. If the half-range series of $f(x) = 1$ in $(0, l)$ is given by $1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}$, find the value of $1^{-2} + 3^{-2} + 5^{-2} + \cdots \infty$.

14. If the half-range cosine series of $(x^2 - x + 1/6)$ in $0 \le x \le 1$ is $\frac{a_0}{2} + \sum a_n \cos n\pi x$, find the value of $a_0^2 + 2\sum a_n^2$.
15. If the half-range sine series of $x(\pi - x)$ in $0 \le x \le \pi$ is $\Sigma b_n \sin nx$, find the value of Σb_n^2.

Part B

16. Obtain the half-range cosine series of $f(x) = \pi^2 - x^2$ in $(0, \pi)$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots \infty$.
17. Find the half-range sine series of $f(x)$ in $(0, 2l)$, given that

$$f(x) = \begin{cases} kx, & \text{in } (0, l) \\ k(2l - x), & \text{in } (l, 2l) \end{cases}$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$.

18. Find the half-range cosine series of the function $f(x) = (x + 2)^2$ in $(-2, 0)$. Hence find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
19. Find the half-range sine series of the function $f(x) = l - x$ in $(l, 2l)$. Deduce the sum of the series $1 - 1/3 + 1/5 - 1/7 + \cdots \infty$.
20. Find the half-range sine series of $f(x) = x(\pi - x)$ in $(0, \pi)$. How should $f(x)$ be extended in order to get this sine series in $(-\pi, 0)$ and in $(\pi, 2\pi)$? Also find the sum of the series $1 - 1/3^3 + 1/5^3 \cdots \infty$.
21. Find the half-range sine series of $f(x)$ in $(0, l)$, given that

$$f(x) = \begin{cases} \frac{b}{a}x, & \text{in } (0, a) \\ \frac{b}{l-a}(l - x), & \text{in } (a, l) \end{cases}$$

22. Find the half-range sine series of $f(x) = \cos x$ in $0 < x < \pi$. How should $f(x)$ be defined at $x = 0$ and $x = \pi$, so that the series converges to $f(x)$ in $0 \le x \le \pi$?
23. Find the half-range cosine series of $f(x) = \cos ax$ in $(0, \pi)$, where a is neither zero nor an integer.
24. Find the half-range sine series of $f(x) = \begin{cases} \sin x, & \text{in } 0 \le x \le \pi/4 \\ \cos x, & \text{in } \pi/4 \le x \le \pi/2 \end{cases}$
25. Find the half-range sine series of $f(x) = x \cos \pi x$ in $(0, 1)$. Deduce the sum of the series $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} \cdots \infty$.
26. Find the half-range sine series of $f(x) = x \sin x$ in $(0, \pi)$.
27. Find the half-range cosine series of $f(x) = 6x^2 - 6x + 1$ in $(0, 1)$. Deduce the sum of the the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty$.
28. Find the Fourier sine series of $f(x) = e^{ax}$ in $(0, \pi)$.
29. Find the Fourier series of period 2π for the function $f(x) = x^2$ in $(-\pi, \pi)$. Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

30. Find the Fourier series of period 2 for the function $f(x) = x^2$ in (0, 2). Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

31. Find the Fourier series of period 2 for the function $f(x) = x^2 + x$ in (–1, 1). Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

32. Find the Fourier series of period 3 for the function $f(x) = 2x - x^2$ in (0, 3). Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

33. Find the Fourier series of period π for the function

$$f(x) = \begin{cases} x, & \text{in } (0, \pi/2) \\ \frac{\pi}{2} - x, & \text{in } (\pi/2, \pi) \end{cases}$$

Hence find the sum of $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

34. Find the Fourier series of period 4 for the function $f(x) = \begin{cases} 2, & \text{in } (-2, 0) \\ x, & \text{in } (0, 2) \end{cases}$.

Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$, assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

35. Find the Fourier series of period 2π for the function $f(x) = \begin{cases} 0, & \text{in } (0, \pi) \\ a, & \text{in } (\pi, 2\pi) \end{cases}$

Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

36. Find the half-range cosine series of $f(x) = \begin{cases} 1, & \text{in } (0, 1) \\ 2, & \text{in } (1, 2) \end{cases}$ in (0, 2). Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

37. Find the half-range sine series of $f(x) = \frac{\pi}{2} - x$ in $(0, \pi)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

38. Find the half-range cosine series of $f(x) = 1 + x$ in (0, 1). Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

39. Find the half-range sine series of $f(x) = \begin{cases} 2x, & \text{in } (0, 1) \\ 4 - 2x, & \text{in } (1, 2) \end{cases}$

Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

40. Find the half-range cosine series of $f(x) = \begin{cases} x, & \text{in } (0, \pi/2) \\ \pi - x, & \text{in } (\pi/2, \pi) \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

5A.11 HARMONIC ANALYSIS

Introduction

We know that the Fourier series of $f(x)$ in $(0, 2l)$ or $(-l, l)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left\{\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos\frac{n\pi x}{l} + \frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin\frac{n\pi x}{l}\right\} \quad (1)$$

Let $$A_n = \sqrt{a_n^2+b_n^2} \text{ and } \alpha_n = \tan^{-1}\frac{b_n}{a_n}, \text{ so that}$$

$$\cos\alpha_n = \frac{a_n}{\sqrt{a_n^2+b_n^2}} \text{ and } \sin\alpha_n = \frac{b_n}{\sqrt{a_n^2+b_n^2}}$$

Using these in (1), we get the Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\left(\cos\frac{n\pi x}{l}\cos\alpha_n + \sin\frac{n\pi x}{l}\sin\alpha_n\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right) \quad (2)$$

If we assume $A_n = \sqrt{a_n^2+b_n^2}$ and $\beta_n = \tan^{-1}\frac{a_n}{b_n}$,

(1) will take the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right) \quad (3)$$

$A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ or $A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$ is called the n^{th} *harmonic* in the Fourier expansion of $f(x)$.

The first harmonic $A_1 \cos\left(\frac{n\pi}{l} - \alpha_1\right)$ or $A_1 \sin\left(\frac{n\pi}{l} + \beta_1\right)$ is also called *the fundamental term* in the Fourier expansion of $f(x)$.

The second harmonic $A_2 \cos\left(\frac{2n\pi}{l} - \alpha_2\right)$ or $A_2 \sin\left(\frac{2n\pi}{l} + \beta_2\right)$ is also called *the octave* in the Fourier expansion of $f(x)$.

It is clear that we require the values of a_n and b_n to calculate the n^{th} harmonic. When $f(x)$ is defined by one or more mathematical expressions, the Fourier coefficients a_n and b_n are found out by integration using Euler's formulas. But in some practical problems, $f(x)$ will be defined by means of its values at equally spaced values of x in the given interval. In such problems, $f(x)$ will be defined in $(0, 2l)$ in a tabular form as given below:

x	x_0	x_1	x_2	–	x_{k-1}
$y = f(x)$	y_0	y_1	y_2	–	y_{k-1}

Here $x_1 - x_0 = x_2 - x_1 = \cdots = x_k - x_{k-1} = \dfrac{2l}{k}$ and $x_0 = 0$ and $x_k = 2l$.

When $y = f(x)$ is defined in a tabular form as given above, a_n and b_n cannot be evaluated exactly by mathematical integration, but are evaluated approximately by numerical integration as explained below:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 2\times\left[\frac{1}{2l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= 2 \times \text{Mean value of } f(x)\ \cos\frac{n\pi x}{l} \text{ over } (0, 2l)$$

Note ✍

We recall that the mean square value of $y = f(x)$ over $(0, 2l)$ was defined as

$$\overline{y}^2 = \frac{1}{2l}\int_0^{2l} y^2 dx \text{ or } \frac{1}{2l}\int_0^{2l} [f(x)]^2\, dx$$

$a_n \simeq 2 \times$ statistical average value of $f(x)$ $\cos\dfrac{n\pi x}{l}$ or $y\cos\dfrac{n\pi x}{l}$ over $(0, 2l)$

$\simeq 2 \times \dfrac{1}{k}\sum\limits_{r=0}^{k-1} y_r \cos\dfrac{n\pi x}{l}$, $n = 0, 1, 2,\ldots$

In particular, $$a_0 \simeq 2\times\frac{1}{k}\sum_{r=0}^{k-1} y_r$$

Similarly $$b_n \simeq 2\times\frac{1}{k}\sum_{r=0}^{k-1} y_r \sin\frac{n\pi x_r}{l},\ n = 1, 2,\ldots$$

Note ✍

1. *When the interval $(0, 2l)$ is divided into k equal sub-intervals, each of length $\dfrac{2l}{k}$, only k values of $y = f(x)$ are taken into consideration for numerical computation of a_n and b_n.*

   i.e. either the values $y_0, y_1, \ldots, y_{k-1}$ corresponding to the left ends of the various sub-intervals, namely $x_0, x_1, \ldots, x_{k-1}$ are considered or the values

$y_1, y_2, ..., y_k$ corresponding to the right ends of the various sub-intervals, namely $x_1, x_2, ..., x_k$ are considered, where $x_0 = 0$ and $x_k = 2l$.

2. *The process of finding the harmonics in the Fourier expansion of a function numerically is known as harmonic analysis.*
3. *In most situations, the amplitudes of the successive harmonics A_1, A_2, A_3, ... will decrease very rapidly. Hence inmost harmonic analysis problems, we may have to find the first new harmonics only.*
4. *Though $A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ or $A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$ is called the n^{th} harmonic, it need not be put in either of these forms. It is enough if we give the n^{th} harmonic in the form $\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$.*

5A.12 COMPLEX FORM OF FOURIER SERIES

The Fourier series of $f(x)$ in $(c, c + 2l)$ can also be put in the exponential form with complex coefficients as explained below:
The trigonometric form of the Fourier series of $f(x)$ defined in $(c, c + 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1}$$

Using the exponential values of $\cos\frac{n\pi x}{l}$ and $\sin\frac{n\pi x}{l}$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/l} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{\frac{in\pi x}{l}} \tag{2}$$

If we put $\frac{a_0}{2} = c_0, \frac{a_n - ib_n}{2} = c_n$ and $\frac{a_n + ib_n}{2} = c_{-n}$, then (2) can be put as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/l} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/l}$$

i.e.

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/l} + \sum_{n=-\infty}^{-1} c_n e^{in\pi x/l}$$

i.e.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{3}$$

Equation (3) is called the *complex form* or *exponential form* of the Fourier series of $f(x)$ in $(c, c + 2l)$. The coefficient c_n in (3) is given by

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left[\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right],$$

by Euler's formulas for Fourier Coefficients.

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right]dx$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{in\pi x/l}\,dx \tag{4}$$

This formula for c_n holds good for positive and negative integral values of n and for $n = 0$.

Note ✍

When $l = \pi$, the complex form of Fourier series of $f(x)$ in $(c, c + 2\pi)$ takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where}$$

$$c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{-inx}\,dx$$

Worked Examples 5A(c)

Example 1

Obtain the first three harmonics in the Fourier series expansion in (0, 12) for the function $y = f(x)$ defined by the table given below:

x:	0	1	2	3	4	5	6	7	8	9	10	11
y:	1.8	1.1	0.3	0.16	0.5	1.5	2.16	1.88	1.25	1.30	1.76	2.00

The length of the interval $= 2l = 12 \quad \therefore \quad l = 6$.

$\therefore$ The Fourier series of $y = f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{6} + \sum b_n \sin\frac{n\pi x}{6}$$

The interval (0, 12) is divided into 12 subintervals, each of length 1.

The values of y at the left end-points of the 12 sub-intervals, namely at $x = 0, 1, 2, \ldots 11$, are given.

$$\therefore \qquad a_0 = 2\times\frac{1}{12}\sum_{r=0}^{11} y_r$$

$$a_n = 2\times\frac{1}{12}\sum_{r=0}^{11} y_r \cos\frac{n\pi x_r}{6};\ b_n = 2\times\frac{1}{12}\sum y_r \sin\frac{n\pi x_r}{6}$$

To compute a_0, a_1, a_2, a_3, b_1, b_2, b_3, we tabulate the values of x_r, y_r, $\cos\frac{n\pi x_r}{6}$ $(n = 1, 2, 3)$ and $\sin\frac{n\pi x_r}{6}$ $(n = 1, 2, 3)$ as shown below:

x_r	y_r	$\cos\frac{\pi x_r}{6}$	$\sin\frac{\pi x_r}{6}$	$\cos\frac{\pi x_r}{3}$	$\sin\frac{\pi x_r}{3}$	$\cos\frac{\pi x_r}{2}$	$\sin\frac{\pi x_r}{2}$
0	1.8	1	0	1	0	1	0
1	1.1	0.866	0.5	0.5	0.866	0	1
2	0.3	0.5	0.866	–0.5	0.866	–1	0
3	0.16	0	1	–1	0	0	–1
4	0.5	–0.5	0.866	–0.5	–0.866	1	0
5	0.15	–0.866	0.5	0.5	–0.866	0	1
6	2.16	–1	0	1	0	–1	0
7	1.88	–0.866	–0.5	0.5	0.866	0	–1
8	1.25	–0.5	–0.866	–0.5	0.866	1	0
9	1.30	0	–1	–1	0	0	1
10	1.76	0.5	–0.866	–0.5	–0.866	–1	0
11	2.00	0.866	0.5	0.5	–0.866	0	–1

$$a_0 = \frac{1}{6}\sum y_r = \frac{1}{6}\times 14.36 = 2.393$$

$$a_1 = \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{6}$$

$$= \frac{1}{6}\ [(1.8 - 2.16) + (1.1 + 2.00 - 0.15 - 1.88) \times 0.866 + (0.3 + 1.76 - 0.5 - 1.25) \times 0.5]$$

$$= 0.120$$

$$b_1 = \frac{1}{6}\sum y_r \sin\frac{\pi x_6}{6}$$

$$= \frac{1}{6}\ [(0.16 - 1.30) + (0.3 + 0.5 - 1.25 - 1.76) \times 0.866 + (1.1 + 0.15 - 1.88 - 2.00) \times 0.5]$$

$$= -0.728$$

$$a_2 = \frac{1}{6}\sum y_r \cos\frac{2\pi x_r}{6} \text{ or } \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{3}$$
$$= \frac{1}{6}\ [(1.8 - 0.16 + 2.16 - 1.30) + (1.1 - 0.3 - 0.5$$
$$+ 0.15 + 1.88 - 1.25 - 1.76 + 2.00) \times 0.5]$$
$$= 0.527$$
$$b_2 = \frac{1}{6}\sum y_r \sin\frac{\pi x_r}{3}$$
$$= \frac{1}{6}\ [(1.1 + 0.3 - 0.5 - 0.15 + 1.88 + 1.25 - 1.76$$
$$- 2.00) \times .866]$$
$$= 0.104$$
$$a_3 = \frac{1}{6}\sum y_r \cos\frac{3\pi x_r}{6} \text{ or } \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{2}$$
$$= \frac{1}{6}\ (1.8 - 0.3 + 0.5 - 2.16 + 1.25 - 1.76)$$
$$= -0.112$$
$$b_3 = \frac{1}{6}\sum y_r \sin\frac{\pi x_r}{2}$$
$$= \frac{1}{6}\ [1.1 - 0.16 + 0.15 - 1.88\ \ 1.30 - 2.00]$$
$$= -0.248$$

∴ The Fourier series of $f(x)$ in (0, 12) upto the third harmonic is

$$f(x) = 1.197 + \left(0.120\cos\frac{\pi x}{6} - 0.728\sin\frac{\pi x}{6}\right)$$
$$+\left(0.527\cos\frac{\pi x}{3} + 0.104\sin\frac{\pi x}{3}\right) + \left(-0.112\cos\frac{\pi x}{2} - 0.248\sin\frac{\pi x}{2}\right)$$

Example 2

The following are 12 values of y corresponding to equidistant values of the angles x° in the range 0° to 360°. Find the first three harmonics in the Fourier series expansion of y in $(0, 2\pi)$

x°:	0	30	60	90	120	150	180	210	240
y:	10.5	20.5	26.4	29.3	27.0	21.5	12.8	1.6	–11.2
x°:	270	300	330						
y:	–18.0	–15.8	–3.5						

Since $f(x)$ is defined in$(0, 2\pi)$, the Fourier series is of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

$$a_0 = 2 \times \frac{1}{12} \sum_{r=0}^{11} y_r = \frac{1}{6} \times 100.8 = 16.8$$

To compute $a_1, a_2, a_3, b_1, b_2, b_3$, we may use the following graphical method, known as *Harrison's method*, instead of the tabulation method.

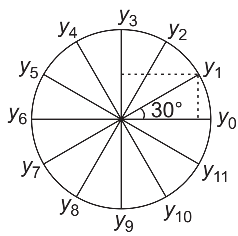


Fig. 5A.11

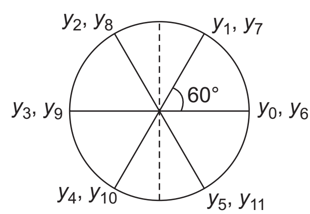


Fig. 5A.12

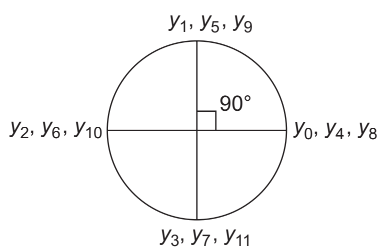


Fig. 5A.13

We draw a circle of convenient radius and divide the central angle into 12 equal parts each of magnitude 30º by means of radii vectors. The radii vectors which measure the angles 0º, 30º, 60º, 90º, ..., 330º are supposed to be of lengths y_0, y_1, y_2, ..., y_{11} (not geometrically) and this is indicated near the ends of corresponding radii vectors. [Fig. (5A.11)]

Now $$a_1 = \frac{1}{6} \sum y_r \cos x_r$$

$$= \frac{1}{6} \left[y_0 \cos 0^\circ + y_1 \cos 30^\circ + y_2 \cos 60^\circ + \cdots + y_{11} \cos 330^\circ \right]$$

$$= \frac{1}{6} \times \text{sum of the horizontal projections of the various radii vectors in the Harrison's circle for } 30^\circ.$$

While computing the sum, those horizontal projections that lie on the right of the vertical are taken to be positive and those on the left are taken to be negative. Also those horizontal projections that contain cos 30º are grouped separately and so are those that contain cos 60º.

Thus

$$a_1 = \frac{1}{6}\ [y_0 - y_6) + (y_1 + y_{11} - y_5 - y_7) \cos 30^\circ + (y_2 + y_{10} - y_4 - y_8)\cos 60^\circ]$$

Note ✍

The horizontal projections of y_3 and y_9 are zero each.

$$a_1 = \frac{1}{6}\ [(10.5 - 12.8) + (20.2 - 3.5 - 21.5 - 1.6) \times 0.866 + (26.4 - 15.8 - 27.0 + 11.2) \times 0.5]$$

$$= -1.740$$

$$b_1 = \frac{1}{6}\sum y_r \sin x_r$$

$$= [y_0 \sin 0^\circ + y_1 \sin 30^\circ + y_2 \sin 60^\circ + \ldots + y_{11} \sin 330^\circ]$$

$= \frac{1}{6}\ \times$ Sum of the vertical projections of the various radii vectors in the Harrison's circle for 30°

While computing the sum, those vertical projections that lie above the horizontal line are taken to be positive and those below the horizontal line are taken to be negative. As before the term with sin 30° are grouped together and those with sin 60° are grouped separately. The vertical projections of the horizontal radii vectors (i.e. y_0 and y_6) are taken as zero each.
Thus

$$b_1 = \frac{1}{6}\ [(y_3 - y_9) + (y_1 + y_5 - y_7 - y_{11}) \sin 30^\circ + (y_2 + y_4 - y_8 - y_{10}) \sin 60^\circ]$$

$$= \frac{1}{6}\ [(29.3 + 18.0) + (20.2 + 21.5 - 1.6 + 3.5) \times 0.5 + (26.4 + 27.0 + 11.2 + 15.8) \times 0.866]$$

$$= 23.121$$

To compute a_2 and b_2, we use Harrison's circle for 60° [Fig. (5A.12)]

$$a_2 = \frac{1}{6}\sum y_r \cos 2x_r$$

$$= \frac{1}{6}\ [(y_0 + y_6 - y_3 - y_9) + (y_1 + y_7 + y_5 + y_{11} - y_2 - y_8 - y_4 - y_{10}) \times 0.5]$$

$$= 3.117$$

$$b_2 = \frac{1}{6}\sum y_r \sin 2x_r$$

$$= \frac{1}{6}\ [y_1 + y_7 + y_2 + y_8 - y_4 - y_{10} - y_5 - y_{11}) \times 0.866$$

0.866]

$$= 1.126$$

To compute a_3 and b_3, we use Harrison's circle for 90º [Fig. (5A.13)]

$$a_3 = \frac{1}{6}\sum y_r \cos 3x_r$$

$$= \frac{1}{6}\ (y_0 + y_4 + y_8 - y_2 - y_6 - y_{10}) = 0.483$$

$$b_3 = \frac{1}{6}\sum y_r \sin 3x_r = \frac{1}{6}\ (y_1 + y_5 + y_9 - y_3 - y_7 - y_{11}) = -0.617$$

∴ The required Fourier series is

$$f(x) = 8.4 + (-1.740 \cos x + 23.121 \sin x) + (3.117 \cos 2x + 1.126 \sin 2x) + (0.483 \cos 3x - 0.167 \sin 3x) + \cdots$$

Example 3

A function $y = f(x)$ is given by the following table of values. Make a harmonic analysis of the function upto the third harmonic.

x^o:	45	90	135	180	225	270	315	360	405
y:	1.5	1.0	0.5	0	0.5	1.0	1.5	2.0	1.5
x^o:	450	495	540	585	630	675	720		
y:	1.0	0.5	0	0.5	1.0	1.5	1.0		

We note that $f(2\pi + x) = f(x)$

∴ $f(x)$ is periodic with period 2π

∴ It is enough we consider the values of $f(x)$ in one period, say $(0, 2\pi)$. We also note that

$$f(360^o - 45^o) = 1.5 = f(45^o)$$
$$f(360^o - 90^o) = 1.0 = f(90^o), \text{ etc.}$$

i.e. $$f(2\pi - x) = f(x)$$

Hence the Fourier series of $f(x)$ will contain only cosine terms, i.e. $b_1 = b_2 = b_3 = 0$. The interval $(0, 2\pi)$ is divided into sub-intervals, each of length $\frac{\pi}{4}$, i.e. it is divided into 8 sub-intervals.

Hence we should consider only 8 values of $y = f(x)$ for harmonic analysis, i.e. the values of $y = f(x)$ at the right ends of various sub-intervals, namely, 45º, 90º, 135º, ..., 360º. We shall call the values of y as $y_1, y_2, y_3, \ldots, y_8$.

$$a_0 = 2\times\frac{1}{8}\sum_{r=1}^{8} y_r = \frac{1}{4}\times 8.0 = 2.0$$

The values of a_1, a_2 and a_3 are found out using Harrison's circles for 45º, 90º and 135º as shown in Figs 5A.14, 5A.15 and 5A.16 respectively.

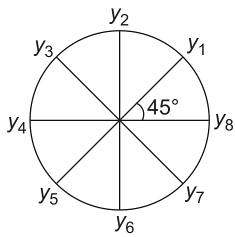


Fig. 5A.14

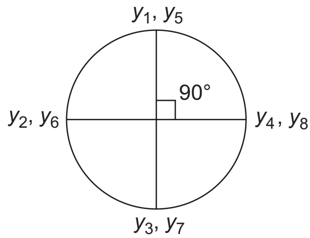


Fig. 5A.15

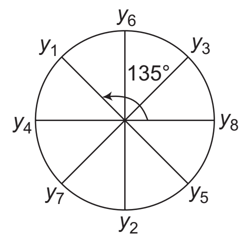


Fig. 5A.16

$$a_1 = \frac{1}{4}\,[(y_8 - y_4) + (y_1 + y_7 - y_3 - y_5)\cos 45º] = 0.854$$

$$a_2 = \frac{1}{4}\,(y_4 + y_8 - y_2 - y_6) = 0$$

$$a_3 = \frac{1}{4}\,[(y_8 - y_4) + (y_3 + y_5 - y_1 - y_7)\cos 45º] = 0.147$$

∴ The required Fourier series is

$$f(x) = 1.0 + 0.854 \cos x + 0.147 \cos 3\,x$$

Example 4

A function $y = f(x)$ is given by the following table of values. Make a harmonic analysis of the function in (0, T) upto the second harmonic.

x:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
y:	0	9.2	14.4	17.8	17.3	11.7	0

The interval (0, T) is divided into sub-intervals each of length $T/6$, i.e., it is divided into 6 sub-intervals.

Hence we consider only 6 values of $y = f(x)$ i.e., $y_0, y_1, \ldots, y_5$ corresponding to $x = 0, T/6, \ldots, \dfrac{5T}{6}$. Since $2l = T$, the Fourier series is of the form $y = \dfrac{a_0}{2} + \cos\sum\limits_{n=1}^{\infty} a_n$ $\cos\dfrac{2n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin\dfrac{2n\pi x}{l}$

$$a_0 = 2\times\frac{1}{6}\sum y_r = \frac{1}{3}\times 70.4 = 23.47$$

Since $a_1 = \dfrac{1}{3}\sum y_r \cos\dfrac{2\pi x}{T}$ *and* $b_1 = \dfrac{1}{3}\sum y_r \sin\dfrac{2\pi}{T}x_r$ and hence the arguments of cosine and sine functions increase by $\pi/3$, we use a Harrison circle for 60º [Fig. 5A.17]

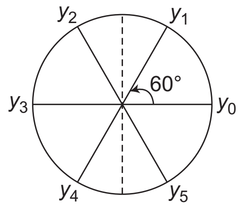


Fig. 5A.17

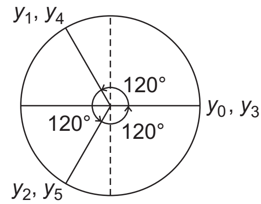


Fig. 5A.18

$$a_1 = \frac{1}{3}[(y_0 - y_3) + y_1 + y_5 - y_2 - y_2 - y_4)\cos 60^\circ]$$

$$= \frac{1}{3}[-17.8 + (-10.8) \times 0.5]$$

$$= -7.33$$

$$b_1 = \frac{1}{3}(y_1 + y_2 - y_4 - y_5)\sin 60^\circ$$

$$= \frac{1}{3} \times (-5.4) \times 0.866 = -1.599$$

As the arguments of the cosine and sine functions in the functions in the formula $a_2 = \frac{1}{3}\sum y_r \cos\frac{4\pi}{T}x_r$ and $b_2 = \frac{1}{3}\sum y_r \sin\frac{4\pi}{T}x_r$ increase by $\frac{2\pi}{3}$, we use a Harrison's circle for 120º [Fig. 5A.18].

$$a_2 = \frac{1}{3}[y_0 + y_3) - (y_1 + y_4 + y_2 + y_5)\cos 60^\circ]$$

$$= \frac{1}{3}[17.8 - 52.6 \times 0.5] = -2.833$$

$$b_2 = \frac{1}{3}[(y_1 + y_4 - y_2 - y_5)\sin 60^\circ]$$

$$= \frac{1}{3} \times 0.4 \times 0.866 = 0.115$$

$\therefore$ The Fourier series upto the second harmonic is

$$f(x) = 11.735 - 7.733\cos\frac{2\pi x}{T} - 1.559\sin\frac{2\pi x}{T}$$

$$-2.833\cos\frac{4\pi x}{T} + 0.115\sin\frac{4\pi x}{T}$$

Example 5

The turning moment T units of the crank shaft of a steam engine is given for a series of values of the crank-angle θ in degrees in the following table:

θ:	0	30	60	90	120	150	180
T:	0	5224	8097	7850	5499	2626	0

Find the first three terms in a series of sines to represent T. Also find T when $\theta = 75°$. The half-range sine series of $T = f(\theta)$ in $(0, \pi)$ is required. Let it be

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin\theta$$

Then
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta$$

$$= 2 \times \text{Mean value of } f(\theta) \sin n\theta \text{ over } (0, \pi)$$

$$= 2\times\frac{1}{k}\sum_{r=0}^{k-1} T_r \sin n\theta_r$$

Since the interval $(0, \pi)$ is divided into sub-intervals, each of length $\frac{\pi}{6}$, we consider only 6 values of T, namely $T_0, T_1, T_2,..., T_5$, corresponding to $\theta = 0, 30°, 60°, ..., 150°$

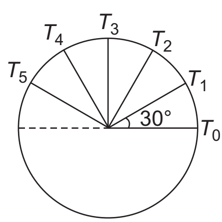


Fig. 5A.19

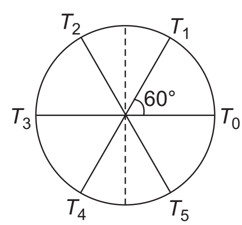


Fig. 5A.20

$$b_1 = 2\times\frac{1}{6}\sum_{r=0}^{5} T_r \sin\theta_r$$

$$= \frac{1}{3}[T_3 + (T_1 + T_5)\sin 30° + (T_2 + T_4)\sin 60°]$$

(from the Harrison's circle for 30°, Fig. 5A.19)

$$= \frac{1}{3}[7850 + 3925 + 11774] = 7850$$

$$b_2 = 2\times\frac{1}{6}\sum_{r=0}^{5} T_r \sin 2\theta_r$$

$$= \frac{1}{3}[(T_1 + T_2 - T_4 - T_5)\sin 60°]$$

(from the Harrison's circle for 60°, Fig. 5A.20)

$$= \frac{1}{3}\times 5196\times 0.866 = 1500$$

$$b_3 = 2\times\frac{1}{6}\sum_{r=0}^{5} T_r \sin 3\theta_r$$

$$= \frac{1}{3}(T_1 + T_5 - T_3)\text{, (from the Harrison's circle for } 90^\circ\text{, Fig. 5A.21)}$$

$$= \frac{1}{3}(7850 - 7850) = 0$$

$\therefore$ The required Fourier sine series upto the third harmonic is

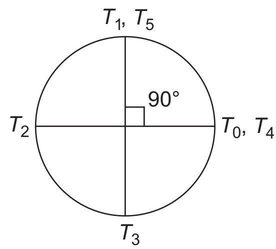


Fig. 5A.21

$$T = 7850 \sin \theta + 1500 \sin 2\theta + 0.\sin 3\theta\text{, in } (0, \pi)$$

$$[T]_{\theta = 75^\circ} = 7850 \sin 75^\circ + 1500 \sin 150^\circ$$

$$= 8332.5 \text{ units}$$

Example 6

Obtain the constant term and the first three harmonics in the Fourier cosine series of $y = f(x)$ in (0, 6) using the following table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

The interval (0, 6) is divided into 6 sub-intervals each of length l. Hence, we consider the 6 values of y, namely y_0, y_1, ..., y_5, corresponding to $x = 0, 1, ..., 5$ for harmonic analysis. As the half-range cosine series is required (0, 6), $l = 6$.

$\therefore$ Fourier cosine series of $f(x)$ in (0, 6) is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \text{ in } (0, 6)$$

$$a_0 = \frac{2}{6}\int_0^6 f(x)\,dx = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r = 14$$

$$a_1 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r \cos\frac{\pi x_r}{3}$$

$$= \frac{1}{3}[y_0 + (y_1 - y_5)\cos 30^\circ + (y_2 - y_4)\cos 60^\circ]$$

(from the Harrison's circle for 30º, Fig. 5A.22)

$= 4.565$

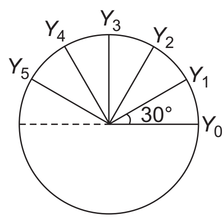


Fig. 5A.22

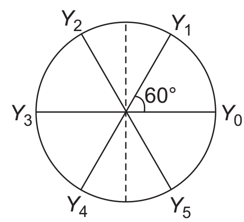


Fig. 5A.23

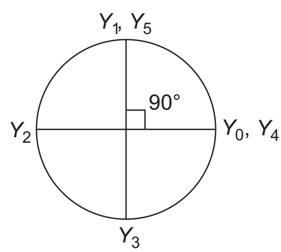


Fig. 5A.24

$$a_4 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r \cos\frac{\pi x_r}{3}$$

$$= \frac{1}{3}[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos 60^\circ]$$

(from the Harrison's circle for 60º, Fig. 5A.23)

$= -2.833$

$$a_3 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r \cos\frac{\pi x_r}{2}$$

$= \dfrac{1}{3}(y_0 + y_4 - y_2)$ (from the Harrison's circle 90º, Fig. 5A.24)

$= -1.667$

$\therefore$ The required half-range cosine series is

$$f(x) = 7 + 4.565\cos\frac{\pi x}{6} - 2.833\cos\frac{\pi x}{3} - 1.667\cos\frac{\pi x}{2} \text{ in } (0, 6)$$

Example 7

Find the complex form of the Fourier series of $f(x) = e^x$ in (0, 2).

Since $2l = 2$ or $l = 1$, the complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \tag{1}$$

$$c_n = \frac{1}{2}\int_0^2 f(x) e^{in\pi x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^2 e^x e^{-in\pi x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-in\pi}\right]_0^2$$

$$= \frac{1}{2(1-in\pi)}\{e^{2(1-in\pi)} - 1\}$$

$$= \frac{(1+in\pi)}{2(1+n^2\pi^2)}\{e^2(\cos 2n\pi - i\sin 2n\pi) - 1\}$$

$$= \frac{(e^2-1)(1+in\pi)}{2(1+n^2\pi^2)}$$

Using this value in (1), we get

$$e^x = \left(\frac{e^2-1}{2}\right)\sum_{n=-\infty}^{\infty}\frac{(1+in\pi)}{(1+n^2\pi^2)}e^{in\pi x}$$

Example 8

Find the complex form of the Fourier series of $f(x) = e^{-ax}$ in $(-l, l)$.

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{1}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} e^{-ax} e^{in\pi x/l}\,\mathrm{d}x$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{-(al+in\pi)x/l}\,\mathrm{d}x$$

$$= \frac{1}{2l}\left[\frac{e^{-(al+in\pi)x/l}}{-(al+in\pi)/l}\right]_{-l}^{l}$$

$$= -\frac{1}{2(al+in\pi)}[e^{-(al+in\pi)} - e^{(al+in\pi)}]$$

$$= \frac{1}{2(al+in\pi)}[e^{al}(-1)^n - e^{-al}(-1)^n]$$

$$[\because \quad e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n]$$

$$= \frac{\sinh al(-1)^n}{al+in\pi}$$

$$= \frac{\sinh al \cdot (al-in\pi)(-1)^n}{a^2l^2+n^2\pi^2}$$

Using this value in (1), we have

$$e^{-ax} = \sinh al \sum_{n=-\infty}^{\infty} \frac{(-1)^n (al-in\pi)}{a^2l^2+n^2\pi^2} e^{in\pi x/l} \text{ in } (-l, l)$$

Example 9

Find the complex form of the Fourier series of $f(x) = \sin x$ in $(0, \pi)$.
Here $2l = \pi$ or $l = \pi/2$.

$\therefore$ The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2nx} \tag{1}$$

$$c_n = \frac{1}{\pi}\int_0^{\pi} \sin x e^{-i2nx}\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2nx}}{1-4n^2}\{-i2n \sin x - \cos x\}\right]_0^{\pi}$$

$$= \frac{1}{\pi(4n^2-1)}[-e^{i2nx} - 1] = -\frac{2}{\pi(4n^2-1)}$$

Using this value in (1), we get

$$\sin x = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{4n^2-1} \cdot e^{i2nx} \text{ in } (0, \pi)$$

Example 10

Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is neither zero nor an integer.

Here $2l = 2\pi$ or $l = \pi$.

$\therefore$ The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax \cdot e^{inx}\, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{inx}}{a^2 - n^2} \{-in \cos ax + a \sin ax\} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2 - n^2)} \begin{array}{l} [e^{-inx} (-in \cos a\pi + a \sin a\pi) \\ -e^{-inx} (-in \cos a\pi - a \sin a\pi)] \end{array}$$

$$= \frac{1}{2\pi(a^2 - n^2)} (-1)^n 2a \sin a\pi$$

Using this value in (1), we get

$$\cos ax = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} e^{inx} \text{ in } (-\pi, \pi)$$

Exercise 5A(c)

Part A (Short-Answer Questions)

1. What do you mean by harmonics and harmonic analysis in Fourier series?
2. Give the formula used for computing a_n numerically in the Fourier half-range cosine series of $f(x)$ in $(0, l)$.
3. Give the formula used for computing b_n numerically in the Fourier half-range sine series of $f(x)$ in $(0, \pi)$.
4. Write down the complex form of the Fourier series of $f(x)$ in $(0, 2l)$ and the Euler's formula for the associated Fourier coefficient.
5. If the trigonometric and complex forms of Fourier series of $f(x)$ in $(0, 2\pi)$ are respectively $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ and $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, how are a_0, a_n, b_n and c_n related?

Part B

6. Find the Fourier series of period 2π as far as the third harmonic to represent the function $y = f(x)$ defined by the following table.

x°:	0	30	60	90	120	150	180	210	240
y:	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88

x^o:	270	300	330	360
y:	1.09	1.19	1.64	2.34

7. Obtain the first three harmonics in the Fourier series of $y = f(x)$ which is denied by means of the table given below in (0, 12).

x:	0	1	2	3	4	5	6	7
y:	6.824	7.976	8.026	7.204	5.676	3.674	1.764	0.552
x:	8	9	10	11				
y:	0.262	0.904	2.492	4.736				

8. Obtain the first three harmonics in the Fourier series of $y = f(x)$ which is defined by means of the following table in $(0, 2\pi)$.

x^o:	0	45	90	135	180	225	270	315
y:	6.824	8.001	7.204	4.675	1.764	0.407	0.904	3.614

9. Find the first three harmonics in the Fourier series of period 8 for the function $y = f(x)$ which is defined by means of the following table.

x:	1	2	3	4	5	6	7	8
y:	365	337	205	80	56	93	184	298

10. Find the Fourier series of $y = f(x)$ in $(0, 2\pi)$ upto the third harmonic, using the definition of y given by the following table:

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y:	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

11. Find the first three harmonics in the Fourier series of $y = f(x)$, which is defined in the following table, in (0, 6).

x:	0	1	2	3	4	5	6
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

12. Find the first three harmonics in the Fourier series of $y = f(x)$ in $(0, 2\pi)$, using the following table of values of x and y.

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7/6\pi$	$4/3\pi$
y:	0	0.26	0.52	0.79	1.05	1.31	0	–1.31	–1.05
x:	$3\pi/2$	$5\pi/3$	$11\pi/6$						
y:	–0.79	–0.52	–0.26						

[**Hint:** $f(2\pi - x) = -f(x)$. Hence the Fourier series of $f(x)$ in $(0, 2\pi)$ will not contain cosine terms]

13. Analyse the current i given by the following table into its constituent harmonics as far as the third harmonic.

θ^o:	0	30	60	90	120	150	180	210
i(amp):	0	24.0	32.5	27.5	18.2	13.0	0	–24.0
θ^o:	240	270	300	330				
i(amp):	–32.5	–27.5	–18.2	–13.0				

[**Hint:** $f(\pi + x) = -f(x)$. Hence $a_0, a_2, a_4, \ldots, b_2, b_4, \ldots$ are all zero. It is enough to compute a_1, a_3, b_1 and b_3]

14. Find the constant term and the first three harmonics in the Fourier cosine series of $y = f(x)$ in $(0, \pi)$ using the following table.

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$
y:	10	12	15	20	17	11

15. Find the first three harmonics in the Fourier sine series of $y = f(x)$ in $(0, 180^\circ)$ using the following table.

x°:	0	15	30	45	60	75	90	105	120	135
y:	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1
x:	150	165	180							
y:	2.6	1.2	0							

16. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $(-\pi, \pi)$.
17. Find the complex form of the Fourier series of $f(x) = e^{ax}$ in $(0, 2l)$.
18. Find the complex form of the Fourier series of $f(x) = \cos x$ in $(0, \pi)$.
19. Find the complex form of the Fourier series of $f(x) = \sin 2x$ in $(0, 1)$.
20. Find the complex form of the Fourier series of $f(x) = \sin ax$ in $(-\pi, \pi)$.

Answers

Exercise 5A(a)

19. $f(x) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x + \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

20. $f(x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x$.

21. $f(x) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx;\ \frac{\pi^2}{12}$.

22. $f(x) = \frac{l^3}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l};\ \frac{\pi^2}{6}$.

23. $f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$.

24. $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x;\ \frac{\pi}{4}$.

25. $f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l} + \frac{1}{\pi}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}$.

26. $f(x) = \frac{\pi^2}{16} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx;\ \frac{\pi^2}{8}$.

27. $f(x)=\frac{1}{2}\cos\pi x-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x$.

28. $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x;\ \frac{\pi^2}{8}$.

29. $f(x)=\left(\frac{k}{2}-\frac{l}{4}\right)+\frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}$.

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{l}{n}(-1)^{n+1}+\frac{k}{n}\{1-(-1)^n\}\right]\sin\frac{n\pi x}{l};\ \frac{\pi^2}{8}.$$

30. $f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$.

31. $f(x)=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}+\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l};\ \frac{\pi^2}{12};\frac{\pi^2}{6}$.

32. $f(x)=-1+\pi\sin x-\frac{1}{2}\cos x+2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx;\ \pi/4-1/2$.

33. $f(x)=-\frac{1}{2}\sin x+2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-1}\sin nx$.

34. $x\sin\pi x=\frac{1}{\pi}-\frac{1}{2\pi}\cos\pi x-\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos n\pi x;\ \frac{\pi}{4}-\frac{1}{2}$.

35. $\sqrt{1+\cos x}=\frac{2\sqrt{2}}{\pi}+\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos nx$.

36. $\frac{1}{12}x(\pi-x)(2\pi-x)=\sum_{n=1}^{\infty}\frac{1}{n^3}\sin nx;\ \pi^3/32$.

37. $|x|=\frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l};\ \frac{\pi^2}{8}$.

38. $|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cdot\cos 2nx$.

39. $\cos ax=\frac{\sin a\pi}{a\pi}+\frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^2-a^2};\ \frac{1}{2}\left(1-\frac{\pi}{3\sqrt{3}}\right)$.

40. $e^{ax}=\frac{e^{2al}-1}{2al}+al(e^{2al}-1)\sum_{n=1}^{\infty}\frac{1}{l^2a^2+n^2\pi^2}\cos\frac{n\pi x}{l}$.

$$-\pi(e^{2al}-1)\sum_{n=1}^{\infty}\frac{n}{l^2a^2+n^2\pi^2}\sin\frac{n\pi x}{l}.$$

41. $\cosh \alpha x = \dfrac{2\alpha^2}{\pi}\sinh \alpha\pi \left[\dfrac{1}{2\alpha^2} + \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2+\alpha^2}\cos nx\right].$

42. $f(x) = \sum_{n=1}^{\infty}\left\{\dfrac{2}{n\pi}\left(1+\cos\dfrac{n\pi}{2}\right) - \dfrac{4}{n\pi}(-1)^n\right\}\sin\dfrac{n\pi x}{2}.$

43. $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}\cos 2nx;\ \dfrac{\pi}{4}-\dfrac{1}{2}.$

44. $f(x) = \sum_{n=1}^{\infty}\left[\dfrac{6}{n\pi}(-1)^n - \dfrac{36}{n^3\pi^3}\{(-1)^n-1\}\right]\sin\dfrac{n\pi x}{3}.$

45. $f(x) = \dfrac{\pi^2}{6} - \sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos 2nx;\ \dfrac{\pi^2}{6};\dfrac{\pi^2}{12}.$

46. $f(x) = \sum_{n=1}^{\infty}\left\{-\dfrac{4}{n\pi}\cos\dfrac{n\pi}{2} + \dfrac{16}{n^2\pi^2}\sin\dfrac{n\pi}{2} + \dfrac{48}{n^3\pi^3}\cos\dfrac{n\pi}{2}\right.$
$\left. - \dfrac{96}{n^4\pi^4}\sin\dfrac{n\pi}{2}\right\}\sin\dfrac{n\pi x}{2}.$

47. $f(x) = \dfrac{l}{8} + 2l\sum_{n=1}^{\infty}\left\{-\dfrac{1}{n^2\pi^2}\cos n\pi - \dfrac{1}{2n\pi}\sin\dfrac{n\pi}{2} + \dfrac{1}{n^2\pi^2}\cos\dfrac{n\pi}{2}\right\}\cos\dfrac{n\pi x}{l}.$

48. $f(x) = \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\{1-(\pi+1)(-1)^n\}\sin nx.$

49. $f(x) = \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin nx.$

50. $f(x) = \dfrac{1}{3} + \sum_{\substack{n=1\\(n\neq 3)}}^{\infty}\left\{\dfrac{2}{n\pi}\sin\dfrac{n\pi}{3} - \dfrac{2n}{\pi(n^2-9)}\sin\dfrac{n\pi}{3}\right\}\cos\dfrac{n\pi x}{3} + \dfrac{1}{3}\cos\pi x.$

Exercise 5A(b)

7. $x^2 - x;\ x - x^2.$
8. $-x(l+x);\ x(l+x).$

11. $\dfrac{1}{2}\sum_{n=1}^{\infty}E_{2n-2}^2.$

12. $\dfrac{1}{2}\sum_{n=1}^{\infty}I_{2n-1}^2.$

13. $\dfrac{\pi^2}{8}.$

14. $\dfrac{1}{45}$.

15. $\dfrac{\pi^4}{15}$.

16. $\pi^2 - x^2 = \dfrac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^2}\cos nx;\ \dfrac{\pi^2}{12}$.

17. $f(x) = \dfrac{8kl}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi x}{2l};\ \dfrac{\pi^2}{8}$.

18. $(x+2)^2 = \dfrac{4}{3} + \dfrac{16}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{n\pi x}{2};\ \dfrac{\pi^2}{6}$.

19. $l - x = \dfrac{2l}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{l}\ in\ (l, 2l);\ \dfrac{\pi}{4}$.

20. $x(\pi - x) = \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\sin(2n-1)x$ in $(0, \pi)$;

$x(\pi + x); (\pi - x)(2\pi - x); \dfrac{\pi^3}{32}$.

21. $f(x) = \dfrac{2bl^2}{a(l-a)\pi^2}\cdot\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$.

22. $\cos x = \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{n}{4n^2-1}\sin 2nx$; $f(0)$ and $f(\pi)$ must be defined as 0 each.

23. $\cos ax = \dfrac{\sin a\pi}{a\pi} + \dfrac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2-a^2}\cos nx$.

24. $f(x) = \dfrac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}\sin\dfrac{n\pi}{2}\sin 2nx$.

25. $x\cos\pi x = -\dfrac{1}{2\pi}\sin\pi x + \dfrac{2}{\pi}\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2-1}\sin n\pi x;\ 1$.

26. $f(x) = (\pi/2)\sin x + \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}\cdot n}{(n^2-1)^2}\sin nx$.

27. $6x^2 - 6x + 1 = \dfrac{6}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos 2n\pi x;\ \dfrac{\pi^2}{12}$.

28. $e^{ax} = \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{n}{n^2+a^2}\{1+(-1)^{n-1}e^{a\pi}\}\sin nx$.

29. $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx;\ \dfrac{\pi^4}{90}$.

30. $x^2 = \dfrac{4}{3} + \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos n\pi x - \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin n\pi x;\ \dfrac{\pi^4}{90}$.

31. $x^2 + x = \dfrac{1}{3} + \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos n\pi x - \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin n\pi x;\ \dfrac{\pi^4}{90}$.

32. $2x - x^2 = -\dfrac{9}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos \dfrac{2n\pi x}{3} + \dfrac{3}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{2n\pi x}{3};\ \dfrac{\pi^4}{90}$.

33. $f(x) = -\dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)} \cos 2(2n-1)x + \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)} \sin 2(2n-1)x;\ \dfrac{\pi^4}{96}$.

34. $f(x) = \dfrac{3}{2} - \dfrac{4}{\pi^2}\sum_{n=1,3,5,...}^{\infty} \dfrac{1}{n^2} \cos \dfrac{n\pi x}{2} - \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi x}{2};\ \dfrac{\pi^4}{96}$.

35. $f(x) = \dfrac{a}{2} - \dfrac{2a}{\pi}\sum_{n=1,3,5,...}^{\infty} \dfrac{1}{n} \sin nx;\ \dfrac{\pi^2}{8}$.

36. $f(x) = \dfrac{3}{2} - \dfrac{2}{\pi}\sum_{n=1,3,5,...}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi}{2} \cos \dfrac{n\pi x}{2};\ \dfrac{\pi^2}{8}$.

37. $\dfrac{\pi}{2} - x = \sum_{n=1}^{\infty} \dfrac{1}{n} \sin 2nx;\ \dfrac{\pi^2}{6}$.

38. $1 + x = \dfrac{3}{2} - \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos(2n-1)\pi x$.

39. $f(x) = \dfrac{16}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \dfrac{n\pi}{2} \sin \dfrac{n\pi x}{2};\ \dfrac{\pi^4}{96}$.

40. $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\sum_{n=1,3,5,...}^{\infty} \cos 2nx;\ \dfrac{\pi^4}{96}$.

Exercise 5A(c)

6. $f(x) = 2.102 + 0.559 \cos x + 1.535 \sin x - 0.519 \cos 2x - 0.091 \sin 2x + 0.20 \cos 3x + 0 \sin 3x$.

7. $f(x) = 4.174 + 2.450 \cos \dfrac{\pi x}{6} + 3.160 \sin \dfrac{\pi x}{6} + 0.120 \cos \dfrac{\pi x}{3} + 0.034 \sin \dfrac{\pi x}{3} + 0.080 \cos \dfrac{\pi x}{2} + 0.010 \sin \dfrac{\pi x}{2}$

8. $f(x) = 4.174 + 2.420 \cos x + 3.105 \sin x + 0.12 \cos 2x + 0.03 \sin 2x + 0.110 \cos 3x - 0.045 \sin 3x$.

9. $$f(x) = 202 + \left(159 \cos \frac{\pi x}{4} + 10 \sin \frac{\pi x}{4}\right) + \left(-21 \cos \frac{\pi x}{2} + 13 \sin \frac{\pi x}{2}\right) + \left(-4 \cos \frac{3\pi x}{4} - \sin \frac{3\pi x}{4}\right)$$

10. $f(x) = 0.75 + 0.373 \cos x + 1.005 \sin x + 0.890 \cos 2x - 0.110 \sin 2x - 0.067 \cos 3x$.

11. $$f(x) = 1.45 - 0.367 \cos \frac{\pi x}{3} + 0.173 \sin \frac{\pi x}{3} - 0.1 \cos \frac{2\pi x}{3} - 0.05 \sin \frac{2\pi x}{3} + 0.033 \cos \pi x.$$

12. $f(x) = 0.978 \sin x - 0.456 \sin 2x + 0.26 \sin 3x$.

13. $i = 5.559 \cos \theta + 29.969 \sin \theta - 4.767 \cos 3\theta + 3.167 \sin 3\theta$.

14. $f(x) = 14.167 + 3.289 \cos x - 4.833 \cos 2x + 4 \cos 3x$.

15. $f(x) = 7.837 \sin x + 1.484 \sin 2x - 0.028 \sin 3x$.

16. $$e^{-x} = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in)}{1 + n^2} e^{inx}$$

17. $$e^{ax} = \left(\frac{e^{2al} - 1}{2}\right) \sum_{n=-\infty}^{\infty} \frac{(al + in\pi)}{a^2 l^2 + n^2 \pi^2} e^{in\pi x/l}$$

18. $$\cos x = \frac{4i}{\pi} \sum_{n=-\infty}^{\infty} \frac{n}{1 - 4n^2} e^{i2nx}$$

19. $$\sin 2x = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 \pi^2 - 1} (\cos 2 - 1 + in\pi \sin 2) e^{i2n\pi x}$$

20. $$\sin ax = \frac{i \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 - a^2} n e^{inx}$$

Part B

One-Dimensional Heat Flow

5B.1 INTRODUCTION

The partial differential equation $\frac{\partial u}{\partial t} = \alpha^2 \nabla^2 u$ governs the distribution of temperature u in homogeneous solids. As a consequence of Maxwell's electromagnetic equations, the current density J satisfies the equation $\nabla^2 J = \mu\sigma \frac{\partial J}{\partial t}$. If U is the concentration of a certain material in gms/cc in a certain homogeneous medium of diffusivity constant k measured in sq cm/sec, U satisfies the equation $\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t}$.

In the theory of consolidation of soil, it is shown that, if U is the excess hydrostatic pressure at any point, at any time t and C_v is the coefficient of consolidation, U satisfies the equation $\nabla^2 U = \frac{1}{C_v}\frac{\partial U}{\partial t}$. All these equations are of the heat flow equation form.

In this chapter, we shall derive and discuss the equation of heat flow in one dimension.

5B.2 EQUATION OF VARIABLE HEAT FLOW IN ONE DIMENSION

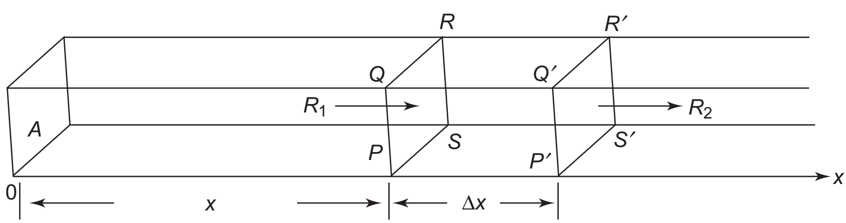


Fig. 5B.1

Consider a homogeneous bar or rod of constant cross-sectional area A made up of conducting material of density ρ, specific heat c and thermal conductivity k. It is assumed that the surface of the bar is insulated in order to make heat flow along parallel lines perpendicular to the area A.

Take one end of the bar as the origin and the direction of heat flow as the positive x-axis.

Let us now consider the heat flow in an element of the bar contained between two parallel sections $PQRS$ and $P'Q'R'S'$ which are at distances x and $x + \Delta x$ from the origin as shown in Fig. 5B.1.

Let u and $u + \Delta u$ be the temperatures of this element at times t and $t + \Delta t$ respectively.

$\therefore$ Increase in temperature in the element in Δt time = Δu

$\therefore$ Increase of heat in the element in Δt time

= (specific heat) · (mass of the element) · (increase in temperature) [by a law of thermodynamics]

$= c(\rho A \Delta x)\Delta u$

$\therefore$ Rate of increase of heat in the element at time t

$$= c\rho A\Delta x \cdot \frac{\partial u}{\partial t} \qquad (1)$$

Let R_1 and R_2 be the rate of inflow through the section $PQRS$ and rate of outflow through the section $P'Q'R'S'$ of the element.

Now $$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \text{ and } R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} \qquad (2)$$

Since the rate of flow of heat across any area A is proportional to A and the temperature gradient normal to the area, that is, $\frac{\partial u}{\partial x}$, by a law of thermodynamics. The constant of proportionality is the thermal conductivity.

Note ✍

The negative sign is taken in (2), since R_1 and R_2 are positive but $\frac{\partial u}{\partial x}$ is negative. $\frac{\partial u}{\partial x}$ is negative, since u is a decreasing function of x, as heat flows from a higher to lower temperature.

$\therefore$ Rate of increase of heat in the element at time t

$= R_1 - R_2$

$$= kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \qquad (3)$$

From (1) and (3), we have

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right] \qquad (4)$$

Equation (4) gives the temperature distribution at time t in the element of the bar.

Taking limits of Eq. (4) as $\Delta x \to 0$, we get the equation of one dimensional heat flow as

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2} \tag{5}$$

This equation gives the temperature $u(x, t)$ at any point of the bar at a distance x from one end of the bar at time t.

Let $\frac{k}{c\rho}$, a positive constant depending on the material of the bar, be denoted as α^2 or K. α^2 is called the *diffusivity* of the material of the bar.

Thus the equation takes the form

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

Note ✍

Equation (6) is also called one dimensional diffusion equation.

5B.3 VARIABLE SEPARABLE SOLUTIONS OF THE HEAT EQUATION

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let $$u(x, t) = X(x)\,.\,T(t) \tag{2}$$

be a solution of Eq. (1), where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$, where $T' = \frac{\mathrm{d}T}{\mathrm{d}t}$ and $X'' = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}$, satisfy Eq. (1).

i.e. $$XT' = \alpha^2 X'' T$$

i.e. $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone.

They are equal for all values of independent variables x and t. This is possible only if each is a constant.

$\therefore$ $$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k, \text{ where } k \text{ is a constant}$$

$\therefore$ $$X'' - kX = 0 \tag{4}$$

and $$T' - k\alpha^2 T = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of the values of k. Hence the following three cases come into being.

Case 1 k is positive. Let $k = p^2$

Then equations (4) and (5) become

$$(D^2 - p^2)X = 0 \text{ and } (D' - p^2\alpha^2)T = 0, \text{ where}$$

$$D = \frac{d}{dx} \text{ and } D' \equiv \frac{d}{dt}$$

The solutions of these equations are

$$X = C_1 e^{px} + C_2 e^{-px} \text{ and } T = C_3 e^{p^2\alpha^2 t}$$

Case 2 k is negative. Let $k = -p^2$.

Then equations (4) and (5) become

$$(D^2 + p^2)X = 0 \text{ and } (D' + p^2\alpha^2)T = 0$$

The solution of these equations are

$$X = C_1 \cos px + C_2 \sin px \text{ and } T = C_3 e^{-p^2\alpha^2 t}$$

Case 3 $k = 0$

Then equations (4) and (5) become

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

The solutions of these equations are

$$X = C_1 x + C_2 \quad \text{and} \quad T = C_3$$

Since $u(x, t) = X \cdot T$ is the solution of Eq. (1), the three mathematically possible solutions of Eq. (1) are

$$u(x, t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t} \tag{6}$$

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{7}$$

and

$$u(x, t) = Ax + B \tag{8}$$

where C_1C_3 and C_2C_3 have been taken as A and B.

Choice of Proper Solution

Out of the three mathematically possible solutions derived, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. As we are dealing with heat conduction, $u(x, t)$, representing the temperature at any point at time t, must decrease when t increases. In other words, $u(x, t)$ cannot be infinite as $t \to \infty$. Hence solution (7) is the proper solution in all variable (transient) heat flow problems.

When heat flow is under steadystate conditions, the temperature at any point does not vary with time, that it is independent of time. Hence the proper solution in steadystate heat flow problems is solution (8).

In problems, we may directly assume that (7) or (8) is the proper solution, according to whether the temperature distribution in the bar is under transient or steadystate conditions. Of course, the arbitrary constants in the suitable solution are to be found out by using the boundary conditions of the problem.

Worked Examples 5B

PROBLEMS WITH ZERO BOUNDARY VALUES (TEMPERATURES OR TEMPERATURE GRADIENTS)

Example 1

A uniform bar of length l through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by (i) $k\sin^3\dfrac{n\pi}{l}$, (ii) $k(lx - x^2)$, for $0 < x < l$, find the temperature distribution in the bar after time t.

The temperature $u(x, t)$ at a point of the bar, which is at a distance x from one end, at time t, is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

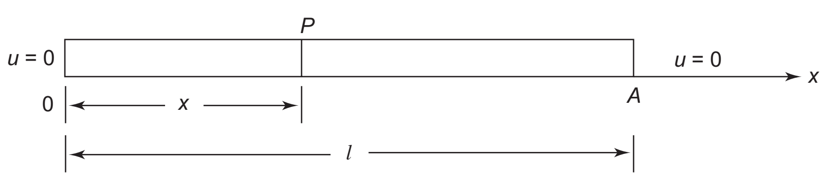


Fig. 5B.2

Since the ends $x = 0$ and $x = l$ are kept at zero temperature, that is, the ends are maintained at zero temperature at all times (Fig. 5B.2) we have

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

Since the initial temperature at the interior points of the bar is $f(x)$, we have

$$u(x, 0) = f(x), \quad \text{for } 0 < x < l \tag{4}$$

where $f(x) = k\sin^3\dfrac{n\pi}{l}$ in (i) and $= k(lx - x^2)$ in (ii).

We have to get the solution of Eq. (1) that satisfies the boundary conditions (2), (3) and (4).

Of the three mathematically possible solutions of Eq. (1), the appropriate solution that satisfies the condition $u \neq \infty$ as $t \to \infty$ is

$$u(x, t) = (A \cos px + B \sin px)\, e^{-p^2\alpha^2 t} \tag{5}$$

where A, B and p are arbitrary constants that are to be found out by using the boundary conditions.

Using boundary condition (2) in (5), we have

$$\therefore \quad Ae^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (5), we have

$$B \sin pl\, e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad B \sin pl = 0$$

i.e. $\quad$ either $B = 0$ or $\sin pl = 0$

If we assume that $B = 0$, the solution becomes $u(x, t) = 0$, which is meaningless.

$$\therefore \quad \sin pl = 0$$

$$\therefore \quad pl = n\pi$$

or $\quad p = \dfrac{n\pi}{l}$, where $n = 0, 1, 2, ..., \infty$

Using these values of A and p in (5), the solution reduces to

$$u(x, t) = B \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} \tag{6}$$

where $n = 1, 2, ..., \infty$.

Note ✍

$n = 0$ is omitted, since the solution corresponding to $n = 0$ is meaningless.

Superposing the infinitely many solution contained in Step (6), we get the most general solution of Eq. (1) as

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-n^2\pi^2\alpha^2 t/l^2} \tag{7}$$

Using the boundary condition (4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x), \quad \text{for } 0 < x < l \tag{8}$$

If we can express $f(x)$ in a series comparable with the L.H.S. series of (8), we can get the values of B_n. Since $\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$ is of form of Fourier half-range sine series of a function, in most situations we may have to expand $f(x)$ as a Fourier half-range sine series.

(i) $f(x) = k\sin^3\left(\frac{\pi x}{l}\right)$

$$= \frac{k}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Using this form of $f(x)$ in (8) and comparing like terms, we get

$$B_1 = \frac{3k}{4},\ B_3 = -\frac{k}{4},\ B_2 = B_4 = B_5 = \dots = 0$$

Using these values in (7), the required solution is

$$u(x, t) = \frac{3k}{4}\sin\frac{\pi x}{l}e^{-\pi^2\alpha^2 t/t^2} - \frac{k}{4}\sin\frac{3\pi x}{l}e^{-9\pi^2\alpha^2 t/t^2}$$

(ii) $f(x) = k(lx - x^2)$ in $0 < x < l$

Let the Fourier half-range sine series of $f(x)$ in $(0, l)$ be $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Using this form of $f(x)$ in (8) and comparing like terms, we get

$$B_n = b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l}}\right)\right]_0^d$$

$$= \frac{4kl^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$= \begin{cases} \frac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of B_n in (7), the required solution is

$$u(x, t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{l^2}}$$

Example 2

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, satisfying the following conditions.

(i) u remains finite as $t \to \infty$

(ii) $u = 0$, when $x = \pm a$, for all $t > 0$

(iii) $u = x$, when $t = 0$ and $-a < x < a$

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the following boundary conditions.

$$u(-a, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(a, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = x, \quad \text{for } -a < x < a \tag{4}$$

We have observed in Example 1 that the arbitrary constant A in the proper solution of Eq. (1) was easily calculated, when the left boundary condition was of the form $u(0, t) = 0$, for all $t \geq 0$. Using the boundary condition (2), namely, $u(-a, t) = 0$, for all $t \geq 0$ in the proper solution, the constant A cannot be immediately calculated.

Hence, to bring the left boundary condition to the required form, we shift the origin to the point $-a$, so that we have $x = X - a$, where X is the coordinate of the point x with reference to the new origin.

With reference to the new origin, Eq. (1) becomes

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial X^2} \tag{1$'$}$$

and the boundary conditions become

$$u(0, t) = 0, \qquad \text{for all } t \geq 0 \tag{2$'$}$$

$$u(2a, t) = 0, \qquad \text{for all } t \geq 0 \tag{3$'$}$$

$$u(X, 0) = X - a, \quad \text{for all } 0 < X < 2a \tag{4$'$}$$

The appropriate solution of Eq. (1′), that satisfies the condition $u \neq \infty$ as $t \to \infty$ is

$$u(X, t) = (A \cos pX + B \sin pX)e^{-p^2\alpha^2 t} \tag{5}$$

Using boundary condition (2)′ in (5), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0 \text{ for all } t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3)′ in (5), we have

$$B \sin 2ap\, e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$\therefore$ $$B \sin 2ap = 0$$

$$\text{Either } B = 0 \text{ or } \sin 2ap = 0$$

But $B = 0$ leads to a trivial solution

$\therefore$ $$\sin 2ap = 0$$

$$2ap = n\pi$$

or $$p = \frac{n\pi}{2a}, \text{ where } n = 0, 1, 2, ..., \infty$$

Using these values of A and p in (5), it reduces to

$$u(X, t) = B \sin \frac{n\pi X}{2a} \cdot e^{-n^2\pi^2\alpha^2 t/4a^2} \tag{6}$$

when $n = 1, 2, ..., \infty$

Therefore the most general solution of Eq. (1′)

$$u(X, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi X}{2a} e^{-n^2\pi^2\alpha^2 t/4a^2} \tag{7}$$

Using boundary condition (4′) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi X}{2a} = X - a \text{ in } 0 < X < 2a$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{2a}$$

which is a Fourier half-range sine series of $(X - a)$ in $(0, 2a)$.

Comparing like terms, we get

$$B_n = b_n = \frac{2}{2a} \int_0^{2a} (X - a) \sin \frac{n\pi X}{2a} \mathrm{d}X$$

$$= \frac{1}{a} \left[(X - a) \left(\frac{-\cos \frac{n\pi X}{2a}}{\frac{n\pi}{2a}} \right) - \left(-\frac{\sin \frac{n\pi X}{2a}}{\frac{n^2\pi^2}{4a^2}} \right) \right]_0^{2a}$$

$$= -\frac{2a}{n\pi} \{(-1)^n + 1\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{-4a}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

Using this value of B_n in (7), we have

$$u(X, t) = \frac{-4a}{\pi} \sum_{n=2,4,6,...}^{\infty} \frac{1}{n} \sin \frac{n\pi X}{2a} e^{-n^2\pi^2\alpha^2 t/4a^2}$$

Noting that $u(x, t) \equiv u(X, t)$, the required solution of Eq. (1), with reference to the old origin, is

$$u(x, t) = \frac{-4a}{\pi} \sum_{n=2,4,6,...}^{\infty} \left\{ \frac{1}{2n} \sin \frac{2n\pi(x+a)}{2a} \right\} e^{-4n^2\pi^2\alpha^2 t/4a^2}$$

i.e. $$u(x, t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} \sin \frac{n\pi x}{l} \cdot e^{-n^2\pi^2\alpha^2 t/a^2}$$

Example 3

Find the temperature distribution in a homogeneous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is k at the centre of the bar and falls uniformly to zero at its ends.

Figure 5B.3 represents the graph of the initial temperature in the bar.

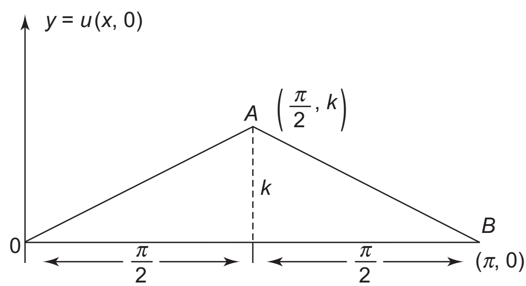


Fig. 5B.3

Equation of OA is $y = \frac{2k}{\pi} x$ and the equation of AB is $\frac{y-0}{k-0} = \frac{x-\pi}{\frac{\pi}{2}-\pi}$

i.e. $$y = \frac{2k}{\pi}(\pi - x)$$

Hence $$u(x, 0) = \begin{cases} \frac{2k}{\pi} x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases}$$

The temperature distribution $u(x, t)$ in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2} \quad (1)$$

We have to solve Eq. (1) satisfy the following boundary conditions.

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(\pi, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = \begin{cases} \dfrac{2k}{\pi} x, & \text{in } 0 \leq x \leq \dfrac{\pi}{2} \\ \dfrac{2k}{\pi}(\pi - x), & \text{in } \dfrac{\pi}{2} \leq x \leq \pi \end{cases} \tag{4}$$

As $u(x, t)$ has to remain finite when $t \to \infty$, the proper solution of Eq. (1) is

$$u(x, t) = (A \cos px + B \sin px) e^{-p^2\alpha^2 t} \tag{5}$$

Using boundary condition (2) in (5), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (5), we have

$$B \sin p\pi \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad B = 0 \quad \text{or} \quad \sin p\pi = 0$$

$B = 0$ leads to a trivial solution.

$$\therefore \quad \sin p\pi = 0$$

$$\therefore \quad p\pi = n\pi \text{ or } p = n, \text{ where } n = 0, 1, 2, \ldots\infty$$

Using these values of A and p in (5),it reduces to

$$u(x, t) = B \sin nx\, e^{-n^2\alpha^2 t} \tag{6}$$

where $n = 1, 2, 3, \ldots\infty$

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin nx\, e^{-n^2\alpha^2 t} \tag{7}$$

Using boundary condition (4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x) \text{ in } (0, \pi), \text{ where}$$

$$f(x) = \begin{cases} \dfrac{2k}{\pi} x, & \text{in } 0 \leq x \leq \pi/2 \\ \dfrac{2k}{\pi}(\pi - x), & \text{in } \pi/2 \leq x \leq \pi \end{cases}$$

If the Fourier half-range sine series of $f(x)$ in $(0, \pi)$ is $\sum_{n=1}^{\infty} b_n \sin nx$, it is comparable with $\sum_{n=1}^{\infty} B_n \sin nx$.

Hence
$$B_n = b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2k}{\pi} x \sin nx \,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \frac{2k}{\pi}(\pi - x)\sin nx \,\mathrm{d}x\right]$$

$$= \frac{4k}{\pi^2}\left[\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\frac{\pi}{2}}\right.$$

$$\left. + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of B_n in (7),the required solution is

$$u(x, t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx e^{-n^2\alpha^2 t}$$

or
$$u(x, t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x e^{-(2n-1)^2\alpha^2 t}$$

Example 4

A rod of length 20 cm has its ends A and B kept at 30ºC respectively, until steadystate conditions prevail. If the temperature at each end is then suddenly reduced to 0°C and maintained so, find the temperature $u(x, t)$ at a distance x from A at time t.

When steadystate conditions prevail, the temperature at any point of the bar does not depend on t, but only on x. Hence when steadystate conditions prevail in the bar, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

$$\left[\because \frac{\partial u}{\partial t} = 0 \text{ and } \frac{\partial^2 u}{\partial x^2} \text{ becomes } \frac{\partial^2 u}{\partial x^2}\right]$$

We have to solve (1) satisfying the following boundary conditions

$$u(0) = 30 \tag{2}$$

and
$$u(20) = 90 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = C_1 x + C_2 \tag{4}$$

Using (2) in (4), we get $C_2 = 30$

Using (3) in (5), we get $C_1 = 3$

Using these values in (4), the solution of Eq. (1) is

$$u(x) = 3x + 30 \tag{5}$$

That is, as long as the steadystate conditions prevail in the bar, the temperature distribution in it is given by (5).

Once we alter the end temperatures (or the end conditions), the heat flow or the temperature distribution in the bar will not be under steadystate conditions and hence will depend on time also. However the temperature distribution at the interior points of the bar in the steadystate will be initial temperature distribution in the transient state.

In the transient state, the temperature distribution in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \qquad \text{for all } t \geq 0 \tag{7}$$

$$u(20, t) = 0, \qquad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 3x + 30, \quad \text{for } 0 < x < 20 \tag{9}$$

As $u \neq \infty$ when $t \to \infty$, the proper solution of Eq. (6) is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{10}$$

Using boundary condition (7) in (10), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (8) in (10), we have

$$B \sin 20\, p \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \qquad B = 0 \quad \text{or } \sin 20\, p = 0$$

$B = 0$ leads to a trivial solution.

$$\therefore \qquad \sin 20\, p = 0$$

$$\therefore \qquad 20\, p = n\pi \text{ or } \frac{n\pi}{20}, \text{ where } n = 0, 1, 2, \quad , \infty$$

Using these values of A and p in (10), it reduces to

$$u(x,\, t) = B\sin\frac{n\pi x}{20}e^{-n^2\pi^2\alpha^2 t/20^2} \tag{11}$$

where $n = 1, 2, 3, ..., \infty$.

Therefore the most general solution Eq. (6) is

$$u(x,\, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20}e^{-n^2\pi^2\alpha^2 t/400} \tag{12}$$

Using boundary condition (9) in (12), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20} = 3x + 30 \text{ in } (0, 20)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}$$

which is Fourier half-range sine series of $(3x + 30)$ in $(0, 20)$.
Comparing like terms,

$$B_n = b_n = \frac{2}{20}\int_0^{20}(3x+30)\sin\frac{n\pi x}{20}\,\mathrm{d}x$$

$$= \frac{3}{10}\left[(x+10)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - \left(-\frac{\sin\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right]_0^{20}$$

$$= -\frac{6}{n\pi}\{30(-1)^n - 10\} = \frac{60}{n\pi}\{1-3(-1)^n\}$$

Using this value of B_n in (12), the required solution is

$$u(x,\, t) = \frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{1-3(-1)^n\}\sin\frac{n\pi x}{20}\cdot e^{-n^2\pi^2\alpha^2 t/400}$$

Example 5

Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions.

(i) $\dfrac{\partial u}{\partial x}(0, t) = 0$, for all $t \geq 0$

(ii) $\dfrac{\partial u}{\partial x}(\pi, t) = 0$, for all $t \geq 0$; and

(iii) $u(x, 0) = \cos^2 x$, $0 < x < \pi$

Note ✍

When conditions (i) and (ii) are satisfied, it means that the ends x = 0 and x = π of the bar are thermally insulated, so that heat cannot flow in or out through these ends.

The appropriate solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the condition that $u \neq \infty$ when $t \to \infty$ is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{2}$$

Differentiating (2) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px)e^{-p^2\alpha^2 t} \tag{3}$$

Using boundary condition (i) in (3), we have

$$p \cdot B \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$\therefore$ $B = 0$ [$\because$ if $p = 0$, $u(x, t) = A$, which is meaningless]

Note ✍

When the zero left end temperature condition was used in the proper solution, we got A = 0 in all the earlier examples. When the zero left end temperature gradient condition is used, we get B = 0.

Using boundary condition (ii) in (3), we have

$$-pA \sin p\pi \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$\therefore$ Either $A = 0$ or $\sin p\pi = 0$

$A = 0$ leads to a trivial solution.

$\therefore$ $\sin p\pi = 0$

$\therefore$ $p\pi = n\pi$ or $p = n$, where $n = 0, 1, 2, ..., \infty$

Using these values of B and p in (2), it reduces to

$$u(x, t) = A \cos nx \cdot e^{-n^2\alpha^2 t} \tag{4}$$

where $n = 0, 1, 2, ..., \infty$.

Note ✍

n = 0 gives u(x, t) = A, which cannot be omitted.

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos nx \, e^{-n^2\alpha^2 t} \tag{5}$$

Using boundary condition (iii) in (5), we have

$$\sum_{n=0}^{\infty} A_n \cos nx = \cos^2 x \text{ in } (0, \pi) \tag{6}$$

In general, we have to expand the function in the R.H.S. as a Fourier half-range cosine series in $(0, \pi)$ so that it may be compared with L.H.S. series.

In this problem, it is not necessary. We can rewrite $\cos^2 x$ as $\frac{1}{2}(1+\cos 2x)$, so that comparison is possible.

Thus $$\sum_{n=0}^{\infty} A_n \cos nx = \frac{1}{2}+\frac{1}{2}\cos 2x$$

Comparing like terms, we have

$$A_0 = \frac{1}{2}, A_2 = 1/2, A_1 = A_3 = A_4 = \ldots = 0$$

Using these values of $A_n' s$ in (5), the required solution is

$$u(x, t) = \frac{1}{2}+\frac{1}{2}\cos 2x \, e^{-4\alpha^2 t}$$

Example 6

The temperature at one end of a bar 20 cm long and with insulated sides is kept at 0°C and that the other end is kept at 60°C until steadystate conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution in the bar.

Show also that the sum of the temperature at any two points equidistant from the centre of the bar is 60°C.

When steady state conditions prevail in the bar, the temperature distribution is given by

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0) = 0 \tag{2}$$

and $$u(20) = 60 \tag{3}$$

Solving the Eq. (1), we get

$$u(x) = C_1 x + C_2 \tag{4}$$

Using (2) and (3) in (4), we get

$$C_1 = 3 \text{ and } C_2 = 0$$

$$\therefore \quad u(x) = 3x \tag{5}$$

Once the ends are insulated, the heat flow is under transient state and the subsequent temperature distribution is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$\frac{\partial u}{\partial x}(20, t) = 0, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 3x, \quad \text{for } 0 < x < 20 \tag{9}$$

As $u \neq \infty$ when $t \to \infty$, the appropriate solution of Eq. (6) is

$$u(x, t = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{10}$$

Differentiating (10) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{11}$$

Using boundary condition (7) in (11), we have

$$p \cdot B \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad \text{Either } = p = 0 \text{ or } B = 0$$

But $p = 0$ makes $u(x, t) = A$, which is meaningless.

$$\therefore \quad B = 0$$

Using boundary condition (8) in (11), we have

$$-pA \sin 20 \cdot e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad \text{Either } A = 0 \text{ or } \sin 20\, p = 0$$

$A = 0$ leads to a trivial solution

$$\therefore \quad \sin 20\, p = 0$$

$$\therefore \quad 20\, p = n\pi \text{ or } p = \frac{n\pi}{20}, \text{ where } n = 0, 1, 2, ..., \infty$$

Using these values of B and p in (10), it reduces to

$$u(x, t) = A\cos\frac{n\pi x}{20}\cdot e^{-n^2\pi^2\alpha^2 t/20^2}$$

where $n = 0, 1, 2, ..., \infty$

Therefore the most general solution of Eq. (6) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/400} \tag{12}$$

Using boundary condition (9) in (12), we have

$$\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{20} = 3x \text{ in } 0 < x < 20$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{20}$$

which is the Fourier half-range cosine series of $3x$ in (0, 20).
Comparing like terms, we get

$$A_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{2}{20}\int_0^{20} 3x\,dx$$

$$= \frac{3}{20}\left(\frac{x^2}{2}\right)_0^{20} = 30$$

and

$$A_n = a_n = \frac{2}{20}\int_0^{20} 3x\cos\frac{n\pi x}{20}\,dx$$

$$= \frac{3}{10}\left[x\left(\frac{\sin\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - \left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right]_0^{20}$$

$$= \frac{120}{n^2\pi^2}\{(-1)^n - 1\}$$

$$= \begin{cases} -\dfrac{240}{n^2\pi^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Using these values of A_0 and A_n in (12), the required solution is

$$u(x, t) = 30 - \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{20}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{400}} \tag{13}$$

Points P and Q which are equidistant from the centre of the bar can be assumed to have the x coordinates x and $20 - x$ [Fig. 5B.4]

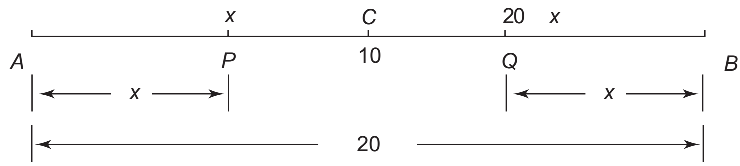


Fig. 5B.4

Temperature at P is given by (13).

Temperature at Q is given by

$$u(20-x,\, t) = 30 - \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi(20-x)}{20}\cdot e^{-\frac{(2n-1)^2\pi^2\alpha^2 t}{400}}$$

Now $$\cos\frac{(2n-1)\pi(20-x)}{20} = \cos\left\{(2n-1)\pi - \frac{(2n-1)\pi x}{20}\right\}$$

$$= (-1)^{2n-1}\cos\frac{(2n-1)\pi x}{20}$$

$$= -\cos\frac{(2n-1)\pi x}{20}$$

$\therefore$ $$u(20-x,\, t) = 30 + \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{20}\, e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{400}} \quad (14)$$

Adding (13) and (14), we get

$$u_p + u_Q = u(x,\, t) + u(20-x,\, t) = 60$$

Example 7

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$ satisfying the following conditions.

(i) u is finite when $t \to \infty$.

(ii) $\dfrac{\partial u}{\partial x} = 0$ when $x = 0$, for all values of t

(iii) $u = 0$ when $x = l$, for all values of t

(iv) $u = u_0$ when $t = 0$, for $0 < x < l$.

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \quad (1)$$

satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = u_0, \quad \text{for } 0 < x < l \tag{4}$$

Since u is finite as $t \to \infty$, the proper solution of Eq. (1) is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{5}$$

Differentiating (5) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px)e^{-p^2\alpha^2 t} \tag{6}$$

Using boundary condition (2) in (6), we have

$$pBe^{-p^2\alpha^2 t} = 0, \text{ for all values of } t \geq 0$$

$\therefore$ Either $p = 0$ or $B = 0$

$p = 0$ makes $u(x, t) = A$, which is meaningless.

$\therefore$ $B = 0$

Using boundary condition (3) in (5), we have

$$A \cos pl \cdot e^{-p^2\alpha^2 t} = 0 \text{ for all } t \geq 0$$

$\therefore$ Either $A = 0$ or $\cos pl = 0$

$A = 0$ leads to a trivial solution.

$\therefore$ $\cos pl = 0$

$\therefore$ $pl =$ an odd multiple of $\frac{\pi}{2}$ or $(2n-1)\frac{\pi}{2}$

$\therefore$ $p = \frac{(2n-1)\pi}{2l}$, where $n = 1, 2, 3, ..., \infty$.

Note ✍

In all the problems considered so far, we had $p = \frac{n\pi}{l}$*, on using the second boundary condition; but in this problem, we have* $p = \frac{(2n-1)\pi}{2l}$.

Using these values of B and p in (5), it reduces to

$$u(x, t) = A \cos \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2 \pi^2 \alpha^2 t / 4t^2} \tag{7}$$

where $n = 1, 2, 3, ..., \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2 \pi^2 t / 4l^2} \tag{8}$$

Note ✍

While superposing the solutions in (7), the unknown constants have been assumed as A_{2n-1} instead of the usual A_n, just to have one-to-one correspondence between the suffix of A and the arguments of the cosine and exponential functions in all the terms of the solution (8).

Using boundary condition (4) in (8), we have

$$\sum_{n=1}^{\infty} A_{2n-1} \cos\frac{(2n-1)\pi x}{2l} = u_0 \text{ in } (0, l) \tag{9}$$

The series in the L.H.S. of (9) is not in the form of the Fourier half-range cosine series of any function in (0, l), that is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$. Hence, to find A_{2n-1}, we proceed as in the derivation of Euler's formula for the Foureir coefficients.

Multiplying both sides of (9) by $\cos\frac{(2n-1)\pi x}{2l}$ and integrating with respect to x between 0 and l, we get

$$A_{2n-1}\int_0^l \cos^2\frac{(2n-1)\pi x}{2l}\,dx = u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx$$

[$\because$ All other integrals in the L.H.S. vanish]

i.e.
$$A_{2n-1}\cdot\frac{1}{2}\left[x + \frac{\sin\frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}}\right]_0^l = u_0\left[\frac{\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l$$

i.e
$$A_{2n-1}\cdot\frac{l}{2} = u_0\cdot\frac{2l}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2}$$

$\therefore$
$$A_{2n-1} = \frac{4u_0}{(2n-1)\pi}(-1)^{n+1}$$

Using this value of A_{2n-1} in (8), the required solution is

$$u(x, t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{(2n-1)\pi x}{2l}\cdot e^{-(2n-1)^2\pi^2\alpha^2 t/4t^2}$$

Example 8

An insulated metal rod of length 100 cm has one end A kept at 0°C and the other end B at 100°C until steady state conditions prevail. At time $t = 0$, the end B is suddenly insulated while the temperature at A is maintained at 0°C. Find the temperature at any point of the rod at any subsequent time.

When steady state conditions prevail in the rod, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0) = 0 \tag{2}$$

and $$u(100) = 100 \tag{3}$$

Solving the Eq. (1), we get

$$u(x) = c_1 x + c_2 \tag{4}$$

Using (2) and (3) in (4), we get $c_1 = 1$ and $c_2 = 0$

$$\therefore \quad u(x) = x \tag{5}$$

Once end B is insulated, though the temperature at A is not altered, the heat flow is under transient conditions and the subsequent temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \text{ for all } t \geq 0 \tag{7}$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \text{ for all } t \geq 0 \tag{8}$$

$$u(x, 0) = x, \text{ for } 0 < x < l \tag{9}$$

where $l = 100$.

As $u \neq \infty$ when $t \to \infty$, the appropriate solution of Eq. (6) is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{10}$$

Using boundary condition (7) in (10), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Differentiating (10) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = Bp \cos px \cdot e^{-p^2\alpha^2 t} \tag{11}$$

Using boundary condition (8) in (11), we have

$$Bp \cos pl \, e^{-p^2\alpha^2 t} = 0$$

$$\therefore \quad \text{Either } B = 0, p = 0 \text{ or } \cos pl = 0$$

$B = 0$ and $p = 0$ lead to meaningless solutions.

$$\therefore \quad \cos pl = 0$$

$$\therefore \quad pl = \frac{(2n-1)\pi}{2}$$

or $$p = \frac{(2n-1)\pi}{2l}, \text{ where } n = 1, 2, 3, ..., \infty$$

Using these values of A and p in (10), it reduces to

$$u(x, t) = B \sin \frac{(2n-1)\pi x}{2l} e^{-(2n-1)^2 \pi^2 \alpha^2 t/4l^2} \tag{12}$$

where $n = 1, 2, 3, ..., \infty$

Therefore the most general solution of Eq.(6) is

$$u(x, t) = \sum_{n=1}^{\infty} B_{2n-1} \sin \frac{(2n-1)\pi x}{2l} e^{-(2n-1)^2 \pi^2 \alpha^2 t/4l^2} \tag{13}$$

Using boundary condition (9) in (13),we have

$$\sum_{n=1}^{\infty} B_{2n-1} \sin \frac{(2n-1)\pi x}{2l} = x \text{ in } (0, l)$$

Proceeding as in Example 7, we get

$$B_{2n-1} = \frac{2}{l}\int_0^l x \sin \frac{(2n-1)\pi x}{2l} \mathrm{d}x$$

$$= \frac{2}{l}\left[x\left\{ \frac{-\cos \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}} \right\} - \left\{ \frac{-\sin \frac{(2n-1)\pi x}{2l}}{\frac{(2l-1)^2 \pi^2}{4l^2}} \right\} \right]_0^l$$

$$= \frac{8l(-1)}{(2n-1)^2 \pi^2} \sin \frac{(2n-1)\pi}{2}$$

$$= \frac{8l(-1)^{n+1}}{(2n-1)^2 \pi^2}$$

Using this value of B_{2n-1} in (13), the required solution is

$$u(x, t) = \frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2 \pi^2 \alpha^2 t/4t^2}$$

where $l = 100$

PROBLEMS ON TEMPERATURE IN A SLAB WITH FACES WITH ZERO TEMPERATURE

Example 9

Faces of a slab of width c are kept at temperature zero. If the initial temperature in the slab is $f(x)$, determine the temperature formula. If $f(x) = u_0$, a constant, find the flux $-k\frac{\partial u}{\partial x}(x_0, t)$ across any plane $x = x_0 (0 \le x_0 \le c)$ and show that no heat flows across the central plane $x_0 = \frac{c}{2}$, where k^2 is the diffusivity of the substance.

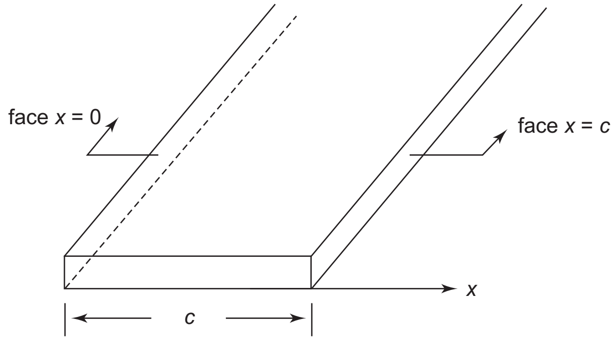


Fig. 5B.5

Though the slab is a three dimensional solid (Fig. 5B.5). It is assumed that the temperature in it at a given time t depends only on and varies with respect to x, the distance measured from one face along the width of the slab. Hence, the temperature function $u(x, t)$ at any interior point of the slab is given by

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

Note ✍

The problem of temperature distribution in a slab is exactly similar to that in a homogeneous bar.

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 0, \qquad \text{for all } t \geq 0 \tag{2}$$

$$u(c, t) = 0, \qquad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = f(x), \quad \text{for } 0 < x < c \tag{4}$$

Proceeding as in Example 1, the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} e^{-n^2\pi^2k^2t/c^2} \tag{5}$$

Using boundary condition (4) in (5), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} = f(x) \text{ in } (0, c) = \sum b_n \sin \frac{n\pi x}{c}$$

which is the Fourier half-range sine sere is of $f(x)$ in $(0, c)$.

Comparing like terms, we get

$$B_n = b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx \tag{6}$$

Using this value of B_n given by (6) in (5), the required solution is

$$u(x, t) = \frac{2}{c}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{c}\exp\left(\frac{-n^2\pi^2k^2t}{c^2}\right)\int_0^c f(\theta)\sin\frac{n\pi\theta}{c}\,d\theta \tag{7}$$

When $f(x) = u_0$, from (6), we get

$$B_n = \frac{2}{c}\int_0^c u_0 \sin\frac{n\pi x}{c}\,dx$$

$$= \frac{2u_0}{c}\left(-\frac{\cos\frac{n\pi x}{c}}{\frac{n\pi}{c}}\right)_0^c = \frac{2u_0}{n\pi}\{1-\cos n\pi\}$$

$$= \begin{cases} \dfrac{4u_0}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Therefore the required solution in this case is

$$u(x, t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{c}\cdot\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\} \tag{8}$$

Differentiating (8) partially with respect to x,

$$\frac{\partial u}{\partial x}(x, t) = \frac{4u_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi x}{c}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

Therefore the flux across the plane $x = x_0$ is given by

$$-k\frac{\partial u}{\partial x}(x_0, t) = -\frac{4ku_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi x_0}{c}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

Therefore the flux across the central plane $x = \dfrac{c}{2}$ is given by

$$-k\frac{\partial u}{\partial x}\left(\frac{c}{2}, t\right) = -\frac{4ku_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi}{2}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

$$= 0, \text{ since } \cos\frac{(2n-1)\pi}{2} = 0$$

That is no heat flows across the central plane of the slab.

Example 10

Two slabs of the same material, one 60 cm thick and the other 30 cm thick are placed face to face in perfect contact. The thicker slab is initially at temperature 100°C, the thinner one initially at zero. The outer faces are kept at zero temperature for $t > 0$. Find the temperature at the centre of the thicker slab (Fig. 5B.6)

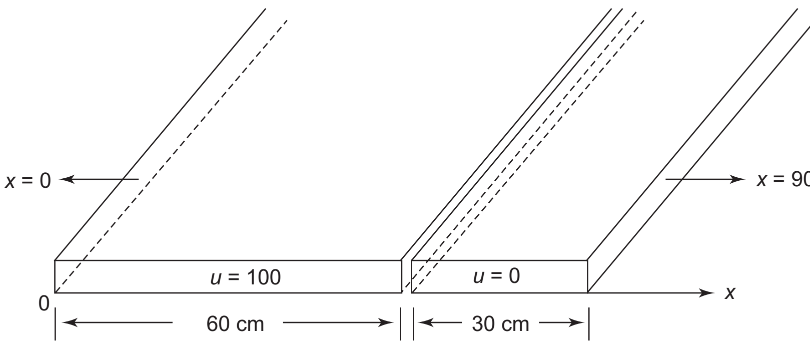


Fig: 5B.6

$u(x, t)$, the temperature function at any point of the slab at time t is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

The corresponding boundary conditions are the following:

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(90, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = \begin{cases} 100, & \text{in } 0 < x < 60 \\ 0, & \text{in } 60 < x < 90 \end{cases} \tag{4}$$

Proceeding as in Example 1, the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{90} e^{-n^2\pi^2\alpha^2 t/90^2} \tag{5}$$

Using boundary condition (4) in (5), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{90} = f(x) \text{ in } (0, 90), \text{ where}$$

$$f(x) = \begin{cases} 100, & \text{in } 0 < x < 60 \\ 0, & \text{in } 60 < x < 90 \end{cases}$$

If the Fourier half-range sine series of $f(x)$ in (0, 90) is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{90}$, then comparison of like terms gives

$$B_n = b_n = \frac{2}{90}\int_0^{60} 100 \sin\frac{n\pi x}{90}\,dx$$

$$= \frac{20}{9}\left(\frac{-\cos\frac{n\pi x}{90}}{\frac{n\pi}{90}}\right)_0^{60}$$

$$= \frac{200}{n\pi}\cdot\left\{1-\cos\frac{2n\pi}{3}\right\} = \frac{400}{n\pi}\sin^2\left(\frac{n\pi}{3}\right)$$

Using this value of B_n in (5), the required solution is

$$u(x, t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin^2\left(\frac{n\pi}{3}\right)\sin\frac{n\pi x}{90}\exp\left\{\frac{-n^2\pi^2\alpha^2 t}{90^2}\right\}$$

Therefore the temperature at the centre ($x = 30$) of the slab is given by

$$u(30, t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin^3\left(\frac{n\pi}{3}\right)\exp\left\{\frac{-n^2\pi^2\alpha^2 t}{90^2}\right\}$$

PROBLEMS WITH NON-ZERO BOUNDARY VALUES (TEMPERATURES OR TEMPERATURE GRADIENTS)

Example 11

A bar 10 cm long has originally a temperature of 0°C throughout its length. At time $t = 0$ sec, the temperature at the end $x = 0$ is raised to 20°C, while that at the end $x = 10$ is raised to 40°C. Determine the resulting temperature distribution in the bar. The temperature distribution $u(x, t)$ in the bar is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 20, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(10, t) = 40, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = 0, \quad \text{for } 0 < x < 10 \tag{4}$$

In all the earlier problems, the boundary values in (2) and (3) were zero each and hence we were able to get the values of two of the unknown constants in the proper solution easily. The usual procedure will not give the values of unknown constants in the proper solution in this example, since we have non-zero values in the boundary conditions (2) an (3). Hence we adopt a slightly different procedure, similar to the one used in Example 16 of Chapter 3(A).

Let $$u(x, t) = u_1(x) + u_2(x, t) \tag{5}$$

Using (5) in (1), we get

$$\frac{\partial}{\partial t}(u_1 + u_2) = \alpha^2 \frac{\partial^2}{\partial x^2}(u_1 + u_2)$$

This gives rise to the two equations

$$\frac{\partial u_1(x)}{\partial t} = \alpha^2 \frac{\partial^2}{\partial x^2} u_1(x) \text{ or } \frac{d^2 u_1}{dx^2} = 0 \tag{6}$$

[$u_1(x)$ is a function of x only]

and $$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{7}$$

Since $u_1(x)$ is independent of t and the end values at $x = 0$ and $x = 10$ do not change with t, we assume that $u_1(x)$ corresponds to the end points and $u_2(x, t)$ corresponds to the interior points $0 < x < 10$.

Note ✍

$u_1(x)$ is referred to as the steadystate part and $u_2(x, t)$ as the transient part of $u(x, t)$.

Thus we have to solve Eq. (6) satisfying the end conditions

$$u_1(0) = 20 \tag{8}$$

and $$u_1(10) = 40 \tag{9}$$

Solving Eq. (6), we get

$$u_1(x) = c_1 x + c_2 \tag{10}$$

Using boundary conditions (8) and (9) in (10), we get $c_1 = 2$ and $c_2 = 20$.

∴ $$u_1(x) = 2x + 20 \tag{11}$$

Now we have to solve Eq. (7), satisfying the following boundary conditions which are obtained by using (5) and the boundary conditions (2), (3), (4), (8), (9) and Step (11).

$$u_2(0, t) = u(0, t) - u_1(0) = 0, \quad \text{for all } t \geq 0 \tag{12}$$

$$u_2(10, t) = u(10, t) - u_1(10) = 0, \quad \text{for all } t \geq 0 \tag{13}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = -(2x + 20), \quad \text{for } 0 < x < 10 \tag{14}$$

Note ✍

Equation 7 is readily solvable, as the boundary conditions (12) and (13) have zero values in the R.H.S.

Proceeding as in Example 1, we get the most general solution of Equation (7) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/10^2} \tag{15}$$

Using boundary condition (14) in (15), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = -(2x + 20) \text{ in } (0, 10)$$

which is the Fourier half-range sineseries of $-(2x + 20)$ in (0, 10).

Comparing like terms, we have

$$B_n = b_n = \frac{2}{10}\int_0^{10} \{-(2x+20)\}\sin\frac{n\pi x}{10}\,\mathrm{d}x$$

$$= -\frac{2}{5}\left[(x+10)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - \left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right]_0^{10}$$

$$= \frac{4}{n\pi}\{20(-1)^n - 10\} \text{ or } \frac{40}{n\pi}\{2(-1)^n - 1\}$$

Using this value of B_n in (15), we get

$$u_2(x, t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{2(-1)^n - 1\}\sin\frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/10^2} \tag{16}$$

Using (11) and (16) in (5), the required solution is

$$u(x, t) = (2x+20) + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{2(-1)^n - 1\}\sin\frac{n\pi x}{10}\exp\left\{\frac{-n^2\pi^2\alpha^2 t}{100}\right\}$$

Example 12

The ends A and B of a rod 40 cm long have their temperatures kept at 0°C and 80°C respectively, until steadystate conditions prevail. The temperature of the end B is then suddenly reduced to 40°C and kept so, while that of the end A is kept at 0°C. Find the subsequent temperature distribution $u(x, t)$ in the rod.

When steadystate conditions prevail, the temperature distribution is given by

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the boundary conditions

$$u(0) = 0 \tag{2}$$

$$u(40) = 80 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using the boundary conditions (2) and (3) in (4), we get $a = 2$ and $b = 0$

Therefore the solution of Eq. (1) is

$$u(x) = 2x \tag{5}$$

In the transient state, the temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(40, t) = 40, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 2x, \quad \text{for } 0 < x < 40 \tag{9}$$

Since one of the end values is non-zero, we adopt the modified procedure explained in Example 11.

Let
$$u(x, t) = u_1(x) + u_2(x, t) \tag{10}$$

where $u_1(x)$ is given by

$$\frac{d^2 u_1}{dx^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$u_1(0) = 0 \tag{13}$$

and
$$u_1(40) = 40 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1 x + c_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $c_1 = 1$ and $c_2 = 0$.

$$u_1(x) = x \tag{16}$$

The boundary conditions for Eq. (12) are

$$u_2(0, t) = u(0, t) - u_1(0) = 0, \quad \text{for all } t \ge 0 \tag{17}$$

$$u_2(40, t) = u(40, t) - u_1(40) = 0, \quad \text{for all } t \ge 0 \tag{18}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = x, \quad \text{for } 0 < x < 40 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Equation (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40} e^{-n^2\pi^2\alpha^2 t/40^2} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40} = x \text{ in } (0, 40)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{40}$$

which is the Fourier half-range sine series of x in (0, 40). Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{40}\int_0^{40} x \sin\frac{n\pi x}{40}\,\mathrm{d}x$$

$$= \frac{1}{20}\left[x\left(-\frac{\cos\frac{n\pi x}{40}}{\frac{n\pi}{40}} \right) - \left(-\frac{\sin\frac{n\pi x}{40}}{\frac{n^2\pi^2}{40^2}} \right) \right]_0^{40}$$

$$= \frac{-2}{n\pi} \times 40 \cos n\pi = \frac{80}{n\pi}(-1)^{n+1}$$

Using this value of B_n in (20), we get

$$u_2(x, t) = \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \sin\frac{n\pi x}{40} \cdot e^{-n^2\pi^2\alpha^2 t/40^2} \tag{21}$$

Using (16) and (21) in (10), the required solution is

$$u(x, t) = x + \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \sin\frac{n\pi x}{40} \cdot e^{-n^2\pi^2\alpha^2 t/40^2}$$

Example 13

A bar of length 10 cm has its ends A and B kept at 50°C and 100°C until steadystate conditions prevail. The temperature at A is then suddenly raised to 90°C and at the same instant, that at B is lowered to 60°C and the end temperature are maintained thereafter. Find the temperature at distance x from the end A at time t.

When steadystate conditions prevail, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0) = 50 \tag{2}$$

and
$$u(10) = 100 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using the boundary conditions (2) and (3) in (4), we get $a = 5$ and $b = 50$.

Therefore the solution of Eq. (1) is

$$u(x) = 5x + 50 \tag{5}$$

In the transient state, the temperature distribution in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 90, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(10, t) = 60, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 5x + 50, \quad \text{for } 0 < x < 10 \tag{9}$$

Since the end values (7) and (8) are non-zero each, we adopt the modified procedure as in Examples 11 and 12.

Let
$$u(x, t) = u_1(x) + u_2(x, t) \qquad (10$$

where $u_1(x)$ is given by

$$\frac{\partial^2 u_1}{\partial x^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$u_1(0) = 90 \tag{13}$$

and $$u_1(10) = 60 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1 x + c_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $c_1 = -3$ and $c_2 = 90$

$\therefore$ $$u_1(x) = 90 - 3x \tag{16}$$

The boundary conditions for Eq. (12) are

$$u_2(0, t) = u(0, t) - u_1(0) = 0, \quad \text{for all } t \geq 0 \tag{17}$$

$$u_2(10, t) = u(10, t) - u_1(10) = 0, \quad \text{for all } t \geq 0 \tag{18}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = 8x - 40, \quad \text{for } 0 < x < 10 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Eq. (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \cdot e^{-n^2\pi^2\alpha^2 t/10^2} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 8x - 40 \text{ in } (0, 10)$$

$$= \sum b_n \sin\frac{n\pi x}{10}$$

which is the Fourier half-range sine series of $(8x - 40)$ in $(0, 10)$.
Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{10}\int_0^{10} (8x - 40)\sin\frac{n\pi x}{10}\,\mathrm{d}x$$

$$= \frac{8}{5}\left[(x-5)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - \left(\frac{\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^{10}$$

$$= \frac{-16}{n\pi}\{5\cos n\pi + 5\}$$

$$= -\frac{8-}{n\pi}\{(-1)^n + 1\}$$

$$= \begin{cases} -\dfrac{160}{n\pi}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Using this value of B_n in (20) and then using (16) and (20) in (10), the required solution is

$$u(x, t) = 90 - 3x - \frac{160}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{10} \exp\left\{\frac{-n^2\pi^2\alpha^2 t}{100}\right\}$$

i.e.
$$u(x, t) = 90 - 3x - \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{5} \exp\left(\frac{-n^2\pi^2\alpha^2 t}{25}\right)$$

Example 14

A bar AB with insulated sides is initially at temperature 0°C throughout. Heat is suddenly applied at the end $x = l$ at a constant rate A, so that $\frac{\partial u}{\partial x} = A$ for $x = l$, while the end A is not disturbed. Find the subsequent temperature distribution in the bar.

The temperature distribution $u(x, t)$ in the bar is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(l, t) = A, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = 0, \quad \text{for all } 0 < x < l \tag{4}$$

Since condition (3) has a non-zero value on the right side, we adopt the modified procedure.

Let
$$u(x, t) = u_1(x) + u_2(x, t) \tag{5}$$

where $u_1(x)$ is given by

$$\frac{d^2 u_1}{dx^2} = 0 \tag{6}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$u_1(0) = 0 \tag{8}$$

and
$$\frac{du_1}{dx}(l) = A \tag{9}$$

Solving Eq. (6), we get

$$u_1(x) = C_1(x) + C_2 \tag{10}$$

Using boundary condition (8) in (10), we get $C_2 = 0$

From (10), we have

$$\frac{du_1}{dx}(x) = C_1 \tag{11}$$

Using boundary condition (9) in (11), we get $C_1 = A$.

$$\therefore \qquad u_1(x) = Ax \tag{12}$$

The boundary conditions for Eq. (7) are

$$u_2(x, t) = u(0, t) - u_1(0) = 0, \qquad \text{for all } t \geq 0 \tag{13}$$

$$\frac{\partial u_2}{\partial x}(l, t) = \frac{\partial u}{\partial x}(l, t) - \frac{du_1}{dx}(l) = 0\,, \quad \text{for all } t \geq 0 \tag{14}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = -Ax, \quad \text{for } 0 < x < l \tag{15}$$

Proceeding as in Example 8, we get the most general solution of Eq. (7) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_{2n-1} \sin\frac{(2n-1)\pi x}{2l} \exp\left\{\frac{-(2n-1)^2\pi^2\alpha^2 t}{4l^2}\right\} \tag{16}$$

Using boundary condition (15) in (16), we have

$$\sum_{n=1}^{\infty} B_{2n-1} \sin\frac{(2n-1)\pi x}{2l} = -Ax \text{ in } (0, l)$$

$$\therefore \qquad B_{2n-1} = \frac{2}{l}\int_0^l -Ax \sin\frac{(2n-1)\pi x}{2l}\,dx$$

$$= -\frac{2A}{l}\left[x\left\{\frac{-\cos\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right\} - \left\{\frac{-\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)^2\pi^2}{4l^2}}\right\}\right]_0^l$$

$$= -\frac{8Al}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}$$

$$= \frac{8Al\cdot(-1)^n}{(2n-1)^2\pi^2}$$

Using this value of B_{2n-1} in (16) and then using (12) and (16) in (5),the required solution is

$$u(x, t) = Ax + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\sin\frac{(2n-1)\pi x}{2l}\exp\left\{\frac{-(2n-1)^2\pi^2\alpha^2 t}{4l^2}\right\}$$

Example 15

An insulated metal rod of length 100 cm has one end A kept at 0°C and the other end B at 100°C until steady state conditions prevail. At time $t = 0$, the temperature at B is suddenly reduced to 50°C and thereafter maintained, while at the same time $t = 0$, the end A is insulated. Find the temperature at any point of the rod at any subsequent time.

When steadystate conditions prevail, the temperature distribution in the rod is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the boundary conditions

$$u(0) = 0 \tag{2}$$

and

$$u(100) = 100 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using boundary conditions (2) in (3) in (4), we gat $a = 1$ and $b = 0$.

Therefore the solution of Eq. (1) is

$$u(x) = x \tag{5}$$

In the transient state, the temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(100, t) = 50, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = x, \quad \text{for } 0 < x < 100 \tag{9}$$

Since the boundary value in (8) is non-zero, we adopt the modified procedure.

Let

$$u(x, t) = u_1(x) + u_2(x, t) \tag{10}$$

where $u_1(x)$ is given by

$$\frac{d^2u_1}{dx^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$\frac{du_1}{dx}(0) = 0 \tag{13}$$

$$u_1(100) = 50 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1 x + c_2 \tag{15}$$

From (15), we have

$$\frac{du_1}{dx}(x) = c_1 \tag{16}$$

Using boundary condition (13) in (16), we get $c_1 = 0$.

Using boundary condition (14) in (16), we get $c_2 = 50$.

$$\therefore \quad u_1(x) = 50 \tag{17}$$

The boundary conditions for Eq. 12 are

$$\frac{\partial u_2}{\partial x}(0, t) = \frac{\partial u}{\partial x}(0, t) - \frac{du_1}{dx}(0) = 0, \quad \text{for all } t \ge 0 \tag{18}$$

$$u_2(100, t) = u(100, t) - u_1(100) = 0, \quad \text{for all } t \ge 0 \tag{19}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = x - 50, \quad \text{for } 0 < x < 100 \tag{20}$$

Proceeding as in Example 7, we get the most general solution of Eq. (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos\frac{(2n-1)\pi x}{200} e^{-(2n-1)^2 \pi^2 \alpha^2 t/4\times 100^2} \tag{21}$$

Using boundary condition (20) in (21), we get

$$\sum_{n=1}^{\infty} A_{2n-1} \cos\frac{(2n-1)\pi x}{200} = x - 50 \text{ in } (0, 100)$$

$$\therefore \quad A_{2n-1} = \frac{2}{100}\int_0^{100} (x-50)\cos\frac{(2n-1)\pi x}{200}\,dx$$

$$= \frac{1}{50}\left[(x-50)\left\{\frac{\sin\frac{(2n-1)\pi x}{200}}{\frac{(2n-1)\pi}{200}}\right\} - \left\{\frac{-\cos\frac{(2n-1)\pi x}{200}}{\frac{(2n-1)^2\pi^2}{200}}\right\}\right]_0^{100}$$

$$= \frac{200}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} - \frac{200^2}{(2n-1)^2\pi^2}$$

$$= \frac{200(-1)^{n+1}}{(2n-1)\pi} - \frac{200^2}{(2n-1)^2\pi^2}$$

Using this value of A_{2n-1} in (21) and then using (17) and (21) in (10), the required solution is

$$u(x, t) = 50 + \frac{200}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{2n-1} - \frac{200}{(2n-1)^2\pi^2}\right\}\cos\frac{(2n-1)\pi x}{200}$$

$$\exp\left\{\frac{(2n-1)^2\pi^2\alpha^2 t}{40000}\right\}$$

Example 16

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$ with the boundary conditions

$$u = u_0 e^{-\omega t}(\omega > 0) \text{ at } x = 0 \text{ and } u = 0 \text{ at } x = l$$

using the method of separation of variables. Show that the temperature at the mid-point of the rod is $\dfrac{1}{2}u_0 e^{-\omega t}\sec\dfrac{l}{2\alpha}\sqrt{\omega}$.

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Satisfying the boundary conditions

$$u(0, t) = u_0 e^{-\omega t}, \quad \text{for all } t \geq 0 \qquad (2)$$

and

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \qquad (3)$$

As $u \neq \infty$, when $t \to \infty$, the appropriate solution of Eq. (1) is

$$u(x, t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \qquad (4)$$

Using boundary condition (2) in (4), we have

$$Ae^{-p^2\alpha^2 t} = u_0{}^{-\omega t}$$

$$\therefore \quad A = u_0 \text{ and } p^2 = \frac{\omega}{\alpha^2} \text{ or } p = \frac{\sqrt{\omega}}{\alpha}$$

Using boundary condition (3) in (4), we have

$$\left(u_0\cos\frac{\sqrt{\omega}}{\alpha}l + B\sin\frac{\sqrt{\omega}}{\alpha}t\right)e^{-\omega t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad u_0\cos\frac{\sqrt{\omega}}{\alpha}l + B\sin\frac{\sqrt{\omega}}{\alpha}l = 0$$

$$\therefore \quad B = -\frac{u_0\cos\frac{\sqrt{\omega}}{\alpha}l}{\sin\frac{\sqrt{\omega}}{\alpha}l}$$

Using the values of A, B and p in (4), the required solution is

$$u(x, t) = u_0 \left\{ \cos\frac{\sqrt{\omega}}{\alpha} x - \frac{\cos\dfrac{\sqrt{\omega}}{\alpha} l}{\sin\dfrac{\sqrt{\omega}}{\alpha} l} \sin\frac{\sqrt{\omega}}{\alpha} x \right\} e^{-\omega t}$$

i.e. $$u(x, t) = u_0 \frac{\sin\dfrac{\sqrt{\omega}}{\alpha}(l - x)}{\sin\dfrac{\sqrt{\omega}}{\alpha}} e^{-\omega t} \tag{5}$$

The temperature at the mid point of the rod is given by $u\left(\dfrac{l}{2}, t\right)$.

From (5), on putting $x = \dfrac{l}{2}$, we get

$$u\left(\frac{l}{2}, t\right) = u_0 \frac{\sin\dfrac{\sqrt{\omega}}{2\alpha} l}{\sin\dfrac{\sqrt{\omega}}{\alpha} l} e^{-\omega t}$$

$$= \frac{u_0}{2} \sec\frac{\sqrt{\omega t}}{2\alpha} \cdot e^{-\omega t}$$

Example 17

The end $x = 0$ of a very long homogeneous rod is maintained at a temperature $u = u_0 \sin \omega t$. If $u \to 0$ as $x \to \infty$, find an expression giving u at any time, at any point of the bar.

The temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the boundary conditions

$$u(0, t) = u_0 \sin \omega t, \qquad \text{for all } t \geq 0 \tag{2}$$

$$u(x, t) \to 0, \text{ as } x \to \infty, \quad \text{for all } t \geq 0 \tag{3}$$

The variable separable solution of (1) will not give the required solution.

In order that the solution of Eq. (1) may satisfy the boundary conditions (2) and (3), let us assume the solution of (1) as

$$u(x, t) = Ae^{-Bx} \sin (Ct - Dx) \tag{4}$$

From (4), $\dfrac{\partial u}{\partial t} = ACe^{-Bx}\cos(Ct - Dx)$

$$\frac{\partial u}{\partial x} = -ABe^{-Bx}\sin(Ct - Dx) - ADe^{-Bx}\cos(Ct - Dx)$$

$$\frac{\partial^2 u}{\partial x^2} = AB^2e^{-Bx}\sin(Ct - Dx) + ABDe^{-Bx}\cos(Ct - Dx) + ABDe^{-Bx}\cos(Ct - Dx) - AD^2e^{-Bx}\sin(Ct - Dx)$$

Since (4) is a solution of (1), we have

$$C\cos\theta = \alpha^2(B^2 - D^2)\sin\theta + 2BD\alpha^2\cos\theta$$

where $\theta = Ct - Dx$.

Equating like terms, we get

$$\alpha^2(B^2 - D^2) = 0 \tag{5}$$

and

$$2BD\alpha^2 = C \tag{6}$$

From (5),

$$D = B > 0 \tag{7}$$

[$\because$ e^{-Bx} and hence $u(x, t) \to 0$ as $x \to \infty$]

Using (7) in (6), we get

$$2B^2\alpha^2 = C \tag{8}$$

Using boundary condition (2) in (4), we have

$$A\sin Ct = u_0\sin\omega t$$

$\therefore$ $A = u_0$ and $C = \omega$

From (8), $B^2 = \dfrac{\omega}{2\alpha^2}$ or $B = \dfrac{1}{\alpha}\sqrt{\dfrac{\omega}{2}}$

From (7), $D = \dfrac{1}{\alpha}\sqrt{\dfrac{\omega}{2}}$

Note ✍

Had we assumed the solution as $u(x, t) = Ae^{-Bx}\sin(Ct + Dx)$, (6) would have been $-2BD\alpha^2 = C$ and hence (8) would have been $-2B^2\alpha^2 = \omega$ or $B^2 = -\dfrac{\omega}{2\alpha^2}$ which is absurd. Hence the assumption of the solution in the form (4) is justified.

Using the values of A, B, C and D in (4), the required solution is

$$u(x, t) = u_0 e^{-\frac{x}{\alpha}\sqrt{\frac{\omega}{2}}}\cdot\sin\left(\omega t - \frac{x}{\alpha}\sqrt{\frac{\omega}{2}}\right)$$

PROBLEMS ON TRANSMISSION LINE EQUATIONS

Example 18

A telegraph is a km long. Initially the line is uncharged so that $V(x, 0) = 0$. If, at $t = 0$, the end $x = a$ is connected to a constant e.m.f. E, find $V(x, t)$ and $i(x, t)$. In particular, show that the current at the end $x = 0$ is given by

$$-\frac{E}{aR}+\frac{2E}{aR}\sum_{aR}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 t}{a^2 RC}\right)$$

(Refer to the discussion on transmission line equations in Chapter 3(A)

The potential at any point at time t in a telegraph cable is given by

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}$$

or
$$\frac{\partial V}{\partial t} = \alpha^2 \frac{\partial^2 V}{\partial x^2} \tag{1}$$

where $\alpha^2 = \dfrac{1}{RC}$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$V(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$V(a, t) = E, \quad \text{for all } t \geq 0 \tag{3}$$

$$V(x, 0) = 0, \quad \text{for all } 0 < x < a \tag{4}$$

Since condition (3) contains a non-zero boundary value, we adopt the modified procedure.

Let
$$V(x, t) = V_1(x) + V_2(x, t) \tag{5}$$

where $V_1(x)$ is given by

$$\frac{d^2 V_1}{dx^2} = 0$$

and $V_2(x, t)$ is given by

$$\frac{\partial V_2}{\partial t} = \alpha^2 \frac{\partial^2 V_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$V_1(0) = 0 \tag{8}$$

$$V_1(a) = E \tag{9}$$

Solving Eq. (6), we get

$$V_1(x) = C_1 x + C_2 \tag{10}$$

Using boundary conditions (8) and (9) in (10), we get $C_1 = \frac{E}{a}$ and $C_2 = 0$

$$V_1(x) = \frac{Ex}{a} \tag{11}$$

The boundary conditions for Eq. (7) are

$$V_2(0, t) = V(0, t) - V_1(0) = 0, \qquad \text{for all } t \geq 0 \tag{12}$$

$$V_2(a, t) = V(a, t) - V_1(a) = 0, \qquad \text{for all } t \geq 0 \tag{13}$$

$$V_2(x, 0) = V(x, 0) - V_1(x) = -\frac{Ex}{a}, \qquad \text{for } 0 < x < a \tag{14}$$

Proceeding as in Example 1, we get the most general solution of Eq. (7) as

$$V_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} e^{-n^2\pi^2\alpha^2 t/a^2} \tag{15}$$

Using boundary condition (14) in (15), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} = -\frac{Ex}{a} \text{ in } (0, a)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$$

which is the Fourier half-range sine series of $\left(-\frac{Ex}{a}\right)$ in $(0, a)$.

Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{a}\int_0^a -\frac{Ex}{a}\sin\frac{n\pi x}{a}\,dx$$

$$= -\frac{2E}{a^2}\left[x\left(-\frac{\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) - \left(-\frac{\sin\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)\right]_0^a$$

$$= \frac{2E}{n\pi}(-1)^n$$

Using this value of B_n in (15) and using (11) and (15) in (5), the required solution is

$$V(x, t) = \frac{Ex}{a} + \frac{2E}{\pi}\sum_{n=1}^{\infty}(-1)^n \sin\frac{n\pi x}{a} e^{-n^2\pi^2\alpha^2 t/a^2} \tag{16}$$

For the telegraph equation ($L = G = 0$),

$$Ri = \frac{\partial V}{\partial x} \quad \text{or} \quad i(x, t) = -\frac{1}{R}\frac{\partial V}{\partial x}(x, t)$$

Differentiating (16) partially with respect to x, we have

$$i(x, t) = -\frac{E}{aR} + \frac{2E}{aR}\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{a}e^{-n^2\pi^2\alpha^2 t/a^2}$$

$$\therefore \qquad i(0, t) = -\frac{E}{aR} + \frac{2E}{aR}\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 t}{a^2 RC}\right)$$

since $\alpha^2 = \dfrac{1}{RC}$.

Example 19

A transmission line 1000 km long is initially under steadystate conditions with potential 1200 volts at the sending end and 1100 volts at the load (x = 1000). The terminal end of the line is suddenly grounded, reducing its potential to zero, but the potential at the sending end is kept at 1200 volts. Find the potential function $e(x, t)$. Assume that $L = G = 0$.

When $L = G = 0$, the potential function $e(x, t)$ in the transmission line is given by

$$\frac{\partial e}{\partial t} = \alpha^2 \frac{\partial^2 e}{\partial x^2} \tag{1}$$

where $\alpha^2 = \dfrac{1}{RC}$.

When steadystate conditions prevail, the potential function is given by

$$\frac{d^2 e}{dx^2} = 0 \tag{2}$$

The boundary conditions for Eq. (2) are

$$e(0) = 1200 \tag{3}$$

and

$$e(1000) = 1100 \tag{4}$$

Solving Eq. (2), we get

$$e(x) = ax + b \tag{5}$$

Using boundary conditions (3) and (4) in (5), we have $a = -0.1$ and $b = 1200$.

Therefore the solution of Eq. (2) is

$$e(x) = 1200 - 0.1x \tag{6}$$

In the transient state, the potential function is given by Eq. (1).
The corresponding boundary conditions are

$$e(0, t) = 1200, \qquad \text{for all } t \geq 0 \tag{7}$$

$$e(1000, t) = 0, \qquad \text{for all } t \geq 0 \tag{8}$$

$$e(x, 0) = 1200 - 0.1x, \text{ for } 0 < x < 1000 \tag{9}$$

Since the boundary value in (7) is non-zero, we adopt the modified procedure.

Let $$e(x, t) = e_1(x) + e_2(x, t) \tag{10}$$

where $e_1(x)$ is given by

$$\frac{d^2 e_1}{dx^2} = 0 \tag{11}$$

and $e_2(x, t)$ is given by

$$\frac{\partial e_2}{\partial t} = \alpha^2 \frac{\partial^2 e_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$e_1(0) = 1200 \tag{13}$$

and $$e_1(1000) = 0 \tag{14}$$

Solving Eq. (11), we get

$$e_1(x) = C_1 x + C_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $C_1 = -1.2$ and $C_2 = 1200$

$\therefore$ $$e_1(x) = 1200 - 1.2x \tag{16}$$

The boundary conditions for Eq. (12) are

$$e_2(0, t) = e(0, t) - e_1(0) = 0, \quad \text{for all } t \geq 0 \tag{17}$$

$$e_2(1000, t) = e(1000, t) - e_1(1000) = 0, \quad \text{for all } t \geq 0 \tag{18}$$

$$e_2(x, 0) = e(x, 0) - e_1(x) = 1.1x, \quad \text{for } 0 < x < 1000 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Eq. (12) as

$$e_2(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{1000} \exp\left\{\frac{-n^2\pi^2\alpha^2 t}{1000^2}\right\} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{1000} = 1.1x \text{ in } (0, 1000)$$

$$= \sum b_n \sin \frac{n\pi x}{1000}$$

which is Fourier half-range sine series of $1.1x$ in (0, 1000).

Comparing like terms, we get

$$B_n = b_n = \frac{2}{1000}\int_0^{1000} 1.1x \sin\frac{n\pi x}{1000}\,dx$$

$$= \frac{2.2}{1000}\left[x\left(\frac{-\cos\frac{n\pi x}{1000}}{\frac{n\pi}{1000}}\right) - \left(\frac{-\sin\frac{n\pi x}{1000}}{\frac{n^2\pi^2}{1000^2}}\right)\right]_0^{1000}$$

$$= \frac{2200}{n\pi}(-1)^{n+1}$$

Using this value of B_n in (20) and then using (16) and (20) in (10), the required solution is

$$e(x, t) = 1200 - 1.2x + \frac{2200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{100}\right)\cdot\exp\left\{\frac{-n^2\pi^2 t}{10^6 RC}\right\}$$

Example 20

A submarine cable $\left(L = G = 0 \text{ and } a = \frac{1}{RC}\right)$ of length l has zero initial current and charge. The end $x = 0$ is insulated and a constant voltage E is applied at $x = l$. Show that the voltage at any point is given by

$$v(x, t) = E + \frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-a(2n-1)^2\pi^2 t/4t^2}\cos\left\{\frac{(2n-1)\pi x}{2l}\right\}$$

The voltage function function $v(x, t)$ is given by the equation

$$\frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 V}{\partial x^2} \quad\text{or}\quad \frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$\frac{\partial v}{\partial x}(0, t) = 0, \quad \text{for all } t \ge 0 \tag{2}$$

$$v(l, t) = E, \quad \text{for all } t \ge 0 \tag{3}$$

$$v(x, 0) = 0, \quad \text{for } 0 < x < l \tag{4}$$

Since the boundary value in (3) is non-zero, we adopt the modified procedure.

Let $$v(x, t) = v_1(x) + v_2(x, t) \tag{5}$$

where $v_1(x)$ is given by

$$\frac{d^2 v_1}{dx^2} = 0 \tag{6}$$

and $v_2(x, t)$ is given by

$$\frac{\partial v_2}{\partial t} = \alpha^2 \frac{\partial^2 v_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$\frac{\partial v_1}{\partial x}(0) = 0 \tag{8}$$

and
$$v_1(l) = E \tag{9}$$

Solving Eq. (6), we get

$$v_1(x) = C_1 x + C_2 \tag{10}$$

From (10), we have

$$\frac{\partial v_1(x)}{\partial x} = C_1 \tag{11}$$

Using boundary condition (8) in (11),we get $C_1 = 0$

Using boundary condition (9) in (10), we get $C_2 = E$.

$\therefore$
$$v_1(x) = E \tag{12}$$

The boundary conditions for Eq. (7) are

$$\frac{\partial v_2}{\partial x}(0, t) = \frac{\partial v}{\partial x}(0, t) - \frac{\mathrm{d}v_1}{\mathrm{d}x}(0) = 0, \text{ for all } t \geq 0 \tag{13}$$

$$v_2(l, t) = v(l, t) - v_1(l) = 0, \qquad \text{for all } t \geq 0 \tag{14}$$

$$v_2(x, 0) = v(x, 0) - v_1(x) = -E, \qquad \text{for } 0 < x < l \tag{15}$$

Proceeding as in Example 7, with u_0 replaced by $-E$, the solution of Eq. (7) is

$$v_2(x, t) = -\frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}\cos\frac{(2n-1)\pi x}{2l}\exp\left\{\frac{-(2n-1)^2\pi^2 at}{4l^2}\right\} \tag{16}$$

Using (12) and (16) in (5), the required solution is

$$v(x, t) = E + \frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\frac{\cos(2n-1)\pi x}{2l}\cdot\exp\left\{\frac{(-(2n-1)^2\pi^2 at}{4l^2}\right\}$$

Exercise 5B(b)

Part A (Short-Answer Questions)

1. State the two laws of thermodynamics used in the derivation of one dimensional heat flow equation.
2. What does α^2 represent in the equation?

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

3. Write down the three mathematically possible solutions of one dimensional heat flow equation.
4. Write down the appropriate solution of the one dimensional heat flow equation. How is it chosen?
5. Write down the form of the general solution of one dimensional heat flow equation, when both the ends of the bar are kept at zero temperature.
6. Write down the form of the general solution of one dimensional heat flow equation, when both the ends of the rod are insulated?
7. In what type of one dimensional heat flow problems, will neither the Fourier sine series nor cosine series be useful?
8. Write down the form of the temperature function, when heat flow in a bar is under steadystate conditions.
9. Explain briefly the procedure used to solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ satisfying, the conditions

$$u(0, t) = A,\ u(l, t) = B \ \text{ and } \ u(x, 0) = f(x)$$

Part B

10. A uniform bar of length 10 cm through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by

(i) $3\sin\frac{\pi x}{5} + 2\sin\frac{2\pi x}{5}$

(ii) $2\sin\frac{\pi x}{5}\cos\frac{2\pi x}{5}$ find the temperature distribution in the bar

11. Obtain the solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following conditions: (i) $u \neq \infty$, as $t \to \infty$; (ii) $u = 0$ for $x = 0$ and $x = \pi$ for any value of t; (iii) $u = \pi x - x^2$, when $t = 0$ in the range $(0, \pi)$.

12. Find the solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

that satisfies the condition $u(0, t) = 0$ and $u(l, t) = 0$ for $t \geq 0$ and

$$u(x, 0) = \begin{cases} x, & \text{for } 0 < x < \dfrac{l}{2} \\ l - x, & \text{for } \dfrac{l}{2} < x < l \end{cases}$$

13. A rod of length l has its ends A and B kept at 0°C and T°C respectively, until steadystate conditions prevail. If the temperature at B is reduced suddenly to 0°C and kept so, while that of A is maintained, find the temperature $u(x, t)$ at a distance x from the end A at time t.

14. Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions.

(i) $\dfrac{\partial u}{\partial x}(0, t) = 0$, for all $t \geq 0$

(ii) $\dfrac{\partial u}{\partial x}(l, t) = 0$, for all $t \geq 0$

(iii) (a) $u(x, 0) = 2\cos\dfrac{3\pi x}{l}\cos\dfrac{2\pi x}{l}$, for $0 < x < l$

(b) $u(x, 0) = \cos^4 \dfrac{\pi x}{l}$ in $(0, l)$;

(c) $u(x, 0) = lx - x^2$ in $(0, l)$

15. The temperature at one end of a bar, 50 cm long and with insulated sides, is kept at 0°C and that at the other end is kept at 100°C until steadystate conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution. Show also that the sum of the temperature at any two points equidistant from the centre of the bar is always 100°C.

16. A uniform rod of length a whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to temperature $\theta = 0$ and subsequently maintained at this temperature. The other end remains thermally insulated. Show that the temperature at this end at time t is given by

$$\theta = \frac{4\theta_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\{-(2n+1)^2 \alpha^2 \pi^2 t / 4a^2\}$$

17. An insulated metal rod of length 20 cm has one end A kept at 0°C and the other end B at 60°C until steadystate conditions prevail. At time $t = 0$, the end B is suddenly insulated while the temperature at A is maintained at 0°C. Find the temperature at any point of the rod at any subsequent time.

18. Two slabs of iron each 20 cm thick, one at temperature 100°C and the other at temperature 0°C throughout, are placed face to face in perfect contact and their outer faces are kept at 0°C. Find the temperature 10 minutes after contact was made, at a point on their common face.

19. Find the temperature in a flat slab of unit width such that (i) its initial temperature varies uniformly from zero at one face to u_0 at the other, (ii) the temperature of the face initially at zero remains at zero for $t > 0$ and (iii) the face initially at temperature u_0 is insulated for $t > 0$.

20. Find the temperature $\theta(x, t)$ in an infinite slab of thickness l, if the faces $x = 0$ and $x = l$ are kept at a constant temperature $T°$, the initial temperature of the slab being 0°.

21. A bar 40 cm long has originally a temperature of 0°C along all its length. At time $t = 0$ sec, the temperature at the end $x = 0$ is raised to 50°C, while that at the end $x = 40$ is raised to 100°C. Determine the resulting temperature distribution.

22. The ends A and B of a rod 10 cm long have their temperatures kept at 0°C and 20°C respectively, until steadystate conditions prevail. The temperature of the end B is then suddenly raised to 60°C and kept so while that of the end A is kept at 0°C. Find the temperature $u(x, t)$.

23. A rod l cm long with insulated lateral surface is initially at the temperature 100°C throughout. If the temperatures at the ends are suddenly reduced to 25°C and 75°C respectively, find the temperature distribution in the rod at any subsequent time.

24. The ends A and B of a bar 50 cm long are kept at 0°C and 100°C respectively, until steadystate conditions prevail. The temperatures at A and B are then suddenly raised to 50°C and 150°C respectively and they are maintained thereafter. Find an expression for the temperature at a distance x from A at any time t subsequent to the changes in the end temperatures.

25. A rod AB of length 10 cm has the ends A and B kept at temperature 40°C and 100°C respectively, until the steadystate is reached. At some time thereafter the temperatures at A and B are lowered to 10°C and 50°C and they are maintained thereafter. Find the subsequent temperature distribution.

26. The ends A and B of a rod 20 cm long have the temperatures at 30°C and 80°C until steadystate prevails. The temperatures of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.

27. A bar 25 long with its sides impervious to heat, has its ends A and B kept at 100°C and 200°C respectively. After the temperature distribution becomes steady, the end A is suddenly cooled to 50°C and at the same instant, the end B is warmed to 300°C. Find an expression for the temperature at a distance x from A at any time t subsequent to the changes in the end temperatures.

28. A bar with insulated sides is initially at temperature 0°C throughout. Heat is suddenly applied at the end $x = 0$ at a constant rate A, so that $\dfrac{\partial u}{\partial x} = A$ for

$x = 0$, while the end $x = l$ is maintained at 0°C temperature. Find the temperature in the bar at a subsequent time.

29. An insulated metal rod of length 60 cm has one end A kept at 0°C and the other end B at 60°C until steadystate conditions prevail. At time $t = 0$, the temperature at A is suddenly increased to 30°C and thereafter maintained, while at the same time $t = 0$ the end B is insulated. Find the subsequent temperature distribution in the rod.

30. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, satisfying the conditions: (i) $u(0, t) = 0$; (ii) $u(1, t) = t$ and (iii) $u(x, 0) = 0$.
[**Hint:** Assume $u(x, t) = x\{a(x^2 - 1) + t\} + u_2(x, t)$. When $u(x,t)$ satisfies the given equation and boundary conditions, $a = \dfrac{1}{6}$ and $u_2(x, t)$ satisfies $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$ such that $u_2(0, t) = 0$, $u_2(1, t) = 0$ and $u_2(x, 0) = \dfrac{1}{6}x(1 - x^2)$].

31. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, satisfying the conditions: (i) $u(0, t) = t$, (ii) $u(1, t) = 0$ and (iii) $u(x, 0) = 0$. [**Hint:** Assume $u(x, t) = \left\{ax(x^2 - 1) + (x - 1)\left(\dfrac{x}{2} - t\right)\right\} + u_2(x, t)$. When $u(x, t)$ satisfies the given equation and the boundary conditions, $a = -1/6$ and $u_2(x, t) =$ satisfies $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$ such that $u_2(0, t) = 0$, $u_2(1, t) = 0$ and $u_2(x, 0) = \dfrac{1}{6}x(x^2 - 3x + 2)$].

32. A transmission line 1000 km long is initially under steadystate conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakage to be negligible, find the potential $e(x, t)$.

33. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t secs. later. Neglect leakage and inductance.

34. In a telegraph wire, the sending end of the line is at potential e_0, the far end being earthed until steadystate conditions prevail. The sending end is suddenly earthed. Show that the potential at a point distant x from the sending end at time t is given by $e(x, t) = \dfrac{2e_0}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\dfrac{n\pi x}{l}\exp\{-n^2\pi^2 t/CRl^2\}$, where l is the length of the wire and C, R have their usual meanings.

35. A submarine cable of length l has the end $x = l$ grounded and constant voltage E is applied at the end $x = 0$ with zero initial conditions. Find the expression for the current at $x = 0$.

Answers

Exercise 5B(b)

10. (i) $u(x, t) = 3\sin\frac{\pi x}{5} e^{-4\pi^2\alpha^2 t/100} + 2\sin\frac{2\pi x}{5} e^{-16\pi^2\alpha^2 t/100}$

(ii) $u(x, t) = -\sin\frac{\pi x}{5} e^{-4\pi^2\alpha^2 t/100} + \sin\frac{3\pi x}{5} e^{-36\pi^2\alpha^2 t/100}$.

11. $u(x, t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x \exp\{-(2n-1)^2\alpha^2 t\}$.

12. $u(x, t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\exp\{-(2n-1)^2\pi^2\alpha^2 t/l^2\}$.

13. $u(x, t) = \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\exp(-n^2\pi^2\alpha^2 t/l^2)$.

14. (a) $u(x, t) = \cos\frac{\pi x}{l}\exp(-\pi^2\alpha^2 t/l^2) + \cos\frac{5\pi x}{l}\exp(-25\pi^2\alpha^2 t/l^2)$

(b) $u(x, t) = \frac{3}{8} + \frac{1}{2}\cos\frac{2\pi x}{l}\exp(-4\pi^2\alpha^2 t/l^2) + \frac{1}{8}\cos\frac{4\pi x}{l}\exp(-16\pi^2\alpha^2 t/l^2$

(c) $u(x, t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l}\exp(-4n^2\pi^2\alpha^2 t/l^2)$.

15. $u(x, t) = 50 - \frac{400}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{50}\cdot\exp\{-(2n-1)^2\pi^2\alpha^2 t/2500\}$.

17. $u(x, t) = \frac{480}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{40}\exp\{-(2n-1)^2\pi^2\alpha^2 t/1600\}$.

18. $u(20, 600) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin^2\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{2}\right)\cdot\exp(-3n^2\pi^2\alpha^2/8)$.

19. $u(x, t) = \frac{4u_0}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}\cdot\exp\left\{\frac{-(2n-1)^2\pi^2\alpha^2 t}{4}\right\}$.

20. $u(x, t) = T - \frac{4T}{\pi}\sum_{\pi}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}\cdot\exp\{-(2n-1)^2\pi^2\alpha^2 t/l^2\}$.

21. $$u(x,t) = \frac{5x}{4} + 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(2\cos n\pi - 1)\cdot\sin\frac{n\pi x}{40}\cdot\exp\{-n^2\pi^2\alpha^2 t/1600\}.$$

22. $$u(x,t) = 6x + \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\sin\frac{n\pi x}{10}\cdot\exp(-n^2\pi^2\alpha^2 t/100).$$

23. $$u(x,t) = \frac{50}{l}x + 25 + \frac{50}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{3 - \cos n\pi\}\sin\frac{n\pi x}{l}\cdot\exp\{-n^2\pi^2\alpha^2 t/l^2\}.$$

24. $$u(x,t) = (2+50) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{50}\exp\{-(2n-1)^2\pi^2\alpha^2 t/2500\}.$$

25. $$u(x,t) = (4x+10) + \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(3 - 5\cos n\pi)\sin\frac{n\pi x}{10}\cdot\exp(-n^2\pi^2\alpha^2 t/100).$$

26. $$u(x,t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(2\cos n\pi + 1)\sin\frac{n\pi x}{20}\cdot\exp(-n^2\pi^2\alpha^2 t/400).$$

27. $$u(x,t) = 10x + 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(2\cos n\pi + 1)\sin\frac{n\pi x}{25}\cdot\exp(-n^2\pi^2\alpha^2 t/625).$$

28. $$u(x,t) = A(x-l) + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2l}\exp\{-(2n-1)^2\pi^2\alpha^2 t/4l^2\}$$

29. $$u(x,t) = 30 + \frac{120}{\pi}\sum_{n=1}^{\infty}\left\{-\frac{1}{2n-1} + \frac{4(-1)^{n+1}}{(2n-1)^2\pi}\right\}\sin\frac{(2n-1)\pi x}{120}$$
$$\exp\{-(2n-1)^2\pi^2\alpha^2 t/120^2\}$$

30. $$u(x,t) = \frac{1}{6}(x^3 - x + 6xt) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\sin n\pi x\cdot\exp(-n^2\pi^2 t).$$

31. $$u(x,t) = -\frac{1}{6}(x^3 - 3x^2 + 2x + 6xt - 6t) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x\cdot\exp(-n^2\pi^2 t).$$

32. $$e(x,t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{1000}\exp\{-n^2\pi^2 t/1000^2 RC\}.$$

33. $$e(x, t) = \frac{20}{l}(l-x) - \frac{24}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\cdot\exp\left(\frac{-n^2\pi^2 t}{l^2 RC}\right)$$

$$i\,(x, t) = \frac{20}{lR} + \frac{24}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l}\exp(-n^2\pi^2 t/l^2 RC).$$

35. $$i = \frac{E}{lR}\left\{-1 + 2\sum_{n=1}^{\infty}\exp(-n^2\pi^2 t/l^2 RC)\right\}.$$

Part C

Steady State Heat Flow in Two Dimensions [Cartesian Coordinates]

3C.1 INTRODUCTION

When the heat flow is along plane curves, lying in the same or parallel planes, instead of along straight lines, then the heat flow is said to be two dimensional. When we consider heat flow or temperature distribution in this uniform plate or sheet made of conducting material, the heat flow is assumed to be two dimensional.

When all the edges of the plate are straight lines, that is, when the plate is in the form of a rectangle or square, cartesian coordinates will be used to discuss the temperature distribution in the plate, as the straight edges can be easily represented in the cartesian system. When one or more edges of the plate are circular arcs, that is, when the plate is in the form of a circle, semicircle, sector of a circle or circular ring, polar coordinates will be used to discuss the temperature distribution in the plate, as the circular edges can be easily represented in the polar system.

In this chapter, we shall fist derive the partial differential equation of variable heat flow in two dimensional cartesians and then deduce the equation of steadystate heat flow.

3C.2 EQUATION OF VARIABLE HEAT FLOW IN TWO DIMENSIONS IN CARTESIAN COORDINATES

Let us consider heat flow in a thin plate or sheet, of thickness h, which is made up of conducting material of density ρ, thermal conductivity k and specific heat c. Let the xoy-plane be taken in one face of plate. Let us assume that the surfaces of the plate are insulated, so that heat flow takes place only in the xoy-plane and not along the normal to xoy-plane.

Let us now consider the heat flow in an element of the plate in the form of a small rectangle $ABCD$, the coordinates of the vertices of which are shown in Fig. 3C.1. Let u and $u + \Delta u$ be the temperatures of this element at times t and $t + \Delta t$ respectively.

Therefore increase in temperature in the element in Δt time = Δu.

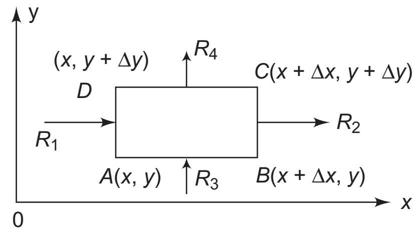


Fig. 3C.1

Therefore increase of heat in the element in Δt time =(specific heat) (mass of the element) (increase in temperature) [by a law of thermodynamics] = $c(\rho h\,\Delta x\,\Delta y)\,\Delta u$.

Therefore rate of increase of heat in the element at time t is

$$= h\rho c\Delta x\Delta y\cdot\frac{\partial u}{\partial t} \tag{1}$$

Let R_1 and R_3 be the rates of inflow of heat into the element through the sides AD and AB respectively at time t.

Let R_2 and R_4 be the rates of outflow of heat from the element through the sides BC and DC respectively at time t.

Therefore rate of increase of heat in the element at time t is

$$= R_1 - R_2 + R_3 - R_4$$

$$= \left[-k(h\Delta y)\left(\frac{\partial u}{\partial x}\right)_x\right] - \left[-k(h\Delta y)\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}\right]$$

$$+ \left[-k(h\Delta y)\left(\frac{\partial u}{\partial y}\right)_y\right] - \left[-k(h\Delta x)\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}\right]$$

by a law of thermodynamics. (For explanation, sec the derivation of one dimensional heat flow equation in Chapter 3(B)

$$= hk\Delta x\Delta y\left[\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right\} + \left\{\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\}\right] \tag{2}$$

Equating (1) and (2), we get

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\left[\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right\} + \left\{\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\}\right] \tag{3}$$

Equation (3) gives the temperature distribution at time t in the element $ABCD$ of the plate.

Taking limits as $\Delta x \to 0$ and $\Delta y \to 0$ in (3), we get the equation that gives the temperature at the point $A(x, y)$ at time t.

Thus the partial differential equation, representing variable temperature distribution in a two dimensional plate or variable heat flow in two dimensions is

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Since $\dfrac{k}{\rho c}$ depends on the material of the plate and positive, we denote it by α^2, which is called *the diffusivity* of the material of the plate.

Thus the equation of variable heat flow in two dimensional cartesians is

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4}$$

Deduction

When steadystate conditions prevail in the plate, the temperature at any point of the plate does not depend on t, but depends on x and y only.

i.e. $$\frac{\partial u}{\partial t} = 0 \text{ in Eq. (4)}$$

Thus steadystate temperature distribution in a two plate is given by $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which is the familiar *Laplace equation* in two dimensional cartesians.

Note ✍

1. *If the surfaces of the plate are not insulated, heat flow will be along non-planar curves, so that heat flow is three dimensional. In this case, the equation of heat flow will take the form*

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

2. *If heat flows along straight lines all parallel to x-axis, then $R_3 = 0 = R_4$. In this case, heat flow is one dimensional and Eq. (4) reduced to $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, which has been directly derived in Chapter 3(B).*
3. *The following functions which occur in various branches of Applied Mathematics and Engineering satisfy Laplace equation $\Delta^2 u = 0$.*
   (i) the temperature in the theory of thermal equilibrium of solids.
   (ii) the gravitational potential in regions not occupied by attracting matter.

(iii) the electrostatic potential in a uniform dielectric, in the theory of electrostatics.

(iv) the magnetic potential in free space, in the theory of magnetostatics.

(v) the electric potential, in the theory of the steady flow of electric current in solid conductors.

(vi) the velocity potential at points of a homogeneous liquid moving irrotationally in hydrodynamic problems.

3C.3 VARIABLE SEPARABLE SOLUTIONS OF LAPLACE EQUATION

Laplace equation in two dimensional cartesians is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

Let $$u(x, y) = X(x) \cdot Y(y) \tag{2}$$

be a solution of Eq. (1), where $X(x)$ is a function of x alone and $Y(y)$ is a function o y alone.

Then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$, where $X'' = \frac{d^2 X}{dx^2}$ and $Y'' = \frac{d^2 X}{dy^2}$, satisfy Eq. (1).

i.e. $$X''Y + XY'' = 0$$

i.e. $$\frac{X''}{X} = -\frac{Y''}{Y} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of y alone. They are equal for all values of the independent variables x and y. This is possible only if each is a constant.

$\therefore$ $$\frac{X''}{X} = -\frac{Y''}{Y} = k \text{, where } k \text{ is a constant.}$$

$\therefore$ $$X'' - kX = 0 \tag{4}$$

and $$Y'' + ky = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of values of k. Hence the following three cases arise:

Case (1)

k is positive. Let $k = p^2$.

Then Eq. (4) and (5) become

$$(D^2 - p^2)X = 0 \text{ and } (D_1^2 + p^2)Y = 0$$

where $$D \equiv \frac{d}{dx} \text{ and } D_1 \equiv \frac{d}{dy}$$

The solutions of these equations are $X = Ae^{px} + Be^{-px}$ and $Y = C \cos py + D \sin py$.

Case (2)

k is negative. Let $k = -p^2$.

Then Eq. (4) and (5) become

$$(D^2 + p^2)X = 0 \quad \text{and} \quad (D_1^2 - p^2)Y = 0$$

The solutions of these equations are $X = A \cos px + B \sin px$ and $Y = ce^{py} + De^{-py}$.

Case (3)

$k = 0$.

Then Eq. (4) and (5) become

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = 0 .$$

The solutions of these equations are

$$X = Ax + B \quad \text{and} \quad Y = Cy + D$$

Since $u(x, y) = X(x) \cdot Y(y)$ is solution of Eq. (1), the three mathematically possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + B^{-px})(C \cos py + D \sin py) \tag{6}$$

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{7}$$

and $$u(x, y) = (Ax + B)(Cy + D) \tag{8}$$

3C.4 CHOICE OF PROPER SOLUTION

Out of the three mathematically possible solutions derived, we have to choose that solution which is consistent with the given boundary conditions. We have already observed that Laplace equation represents steadystate heat flow in two dimensional plates in the form of rectangles or squares whose sides are parallel to the coordinate axes, that is, whose sides are $x = 0$, $x = a$, $y = 0$ and $y = b$.

Laplace Equation is readily solvable, that is, the arbitrary constants in the solutions can be easily found out, if three of the boundary values (either temperatures or gradients) prescribed on any three sides of the rectangle are zero each and the fourth boundary value is non-zero.

If the non-zero boundary value is prescribed either on $x = 0$ or on $x = a$ (in which y is varying), that solution in which periodic functions in y occur will be the proper solution. That is, (6) will be the proper solution. It can be verified in individual problems that solutions (7) and (8) become trivial in such situations.

If the non-zero boundary value if prescribed either on $y = 0$ or $y = b$ (in which x is varying), that solution in which periodic functions in x occur will be the proper solution. That is, (7) will be the proper solution. It can be verified in individual problems that solutions (6) and (8) become trivial in such situations.

Thus we cannot choose a single solution as the appropriate solution in all situations. Invariably, solution (8) need not be considered, as it will result in a trivial solution. Solution (6) or (7) will be the suitable solution, according as non-zero boundary value is prescribed on the side $x = k$ or $y = k$.

Worked Examples 3C

PROBLEMS ON TEMPERATURE DISTRIBUTION IN VERY LONG PLATES

Example 1

A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ and the short edge at infinity are kept at temperature 0°C, while the other short edge $y = 0$ is kept at temperature (i) $u_0 \sin\dfrac{n\pi}{a}$ and (ii) T (constant). Find the steadystate temperature at any point (x, y) of the plate (Fig. 3C.2)

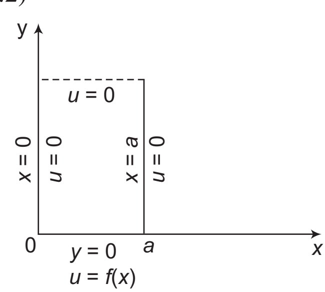


Fig. 3C.2

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for all } y > 0 \tag{2}$$

$$u(a, y) = 0, \quad \text{for all } y > 0 \tag{3}$$

$$u(x, \infty) = 0, \quad \text{for } 0 \le x \le a \tag{4}$$

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le a \tag{5}$$

where $f(x) = u_0 \sin^3 \dfrac{n\pi}{a}$ for (i) and $f(x) = T$ for (ii).

The three possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + B^{-px})(C \cos py + D \sin py) \tag{6}$$

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{7}$$

and $$u(x, y) = (Ax + B)(Cy + D) \tag{8}$$

By boundary condition (4), $u \to 0$ when $y \to \infty$. Of the three possible solutions, only Solution (7) can satisfy this condition. Hence we reject the other two solutions.

Rewriting (7), we have

$$u(x, y)e^{-py} = (A \cos px + B \sin px)(C + De^{-2py}) \tag{7'}$$

Using boundary condition (4) in (7)′, we have $(A \cos px + B \sin px)C = 0$, for $0 \le x \le a$.

$$\therefore \quad C = 0$$

Using boundary condition (2) in (7), we have

$$A \cdot D \cdot e^{-py} = 0, \text{ for all } y > 0$$

$$\therefore \quad \text{Either } A = 0 \text{ or } D = 0$$

If we assume that $D = 0$, we get a trivial solution.

$$\therefore \quad A = 0$$

Using boundary condition (3) in (7), we have

$$B \sin pa \cdot De^{-py} = 0, \quad \text{for all } y > 0$$

The assumption that $B = 0$ leads to a trivial solution.

$$\therefore \quad \sin pa = 0$$

$$\therefore \quad pa = n\pi \text{ or } p = \frac{n\pi}{a}$$

where $n = 0, 1, 2, ..., \infty$.

Using these values of A, C and p in (7), it reduces to

$$u(x, y) = \lambda \sin \frac{n\pi x}{a} \cdot e^{-n\pi y/a} \tag{9}$$

where $n = 0, 1, 2, ..., \infty$.

The most general solution of Eq. (1) [got by superposing all the solutions in (9) except the one corresponding to $n = 0$] is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}} \tag{10}$$

Using boundary condition (5) in (10), we have

$$\sum \lambda_n \sin\frac{n\pi x}{a} = f(x) \text{ in } (0, a) \tag{11}$$

(i) $f(x) = u_0 \sin^3 \dfrac{\pi x}{a}$

$$= \frac{u_0}{4}\left(3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\right)$$

Using this form of $f(x)$ in (11) and comparing like terms, we get

$$\lambda_1 = \frac{3u_0}{4}, \lambda_3 = -\frac{u_0}{4} \quad \text{and} \quad \lambda_2 = 0 = \lambda_4 = \lambda_5 = \ldots$$

Using these values of λ_n in (10), the required solution is

$$u(x, y) = \frac{3u_0}{4}\sin\frac{\pi x}{a}e^{-\pi y/a} - \frac{u_0}{4}\sin\frac{3\pi x}{a}e^{-3\pi y/a}$$

(ii) $f(x) = T$ in $(0, a)$

Let the fourier half-range sines series of $f(x)$ in $(0, a)$ be $\sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{a}$.

Using this form of $f(x)$ in (11) and comparing like terms, we get

$$\lambda_n = b_n = \frac{2}{a}\int_0^a T\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2T}{a}\left(-\frac{\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$= \frac{2T}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{4T}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (10), the required solution is

$$u(x, y) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{a}\exp\{-(2n-1)\pi y/a\}$$

Example 2

An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0$ and $y = \pi$ for positive values of x. The temperature is zero along the edges $y = 0$, $y = \pi$ and the edge at infinity. If the edge $x = 0$ is kept at temperature ky, find the steadystate temperature distribution in the plate (Fig. 3C.3)

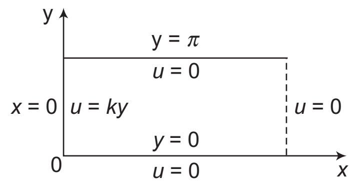


Fig. 3C.3

The steadystate temperature $u(x, y)$ at any point (x, y) of the plane is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$u(x, \pi) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty, \pi) = 0, \quad \text{for } 0 \le y \le \pi \tag{4}$$

$$u(0, y) = ky, \text{ for } 0 \le y \le \pi \tag{5}$$

Of the three possible solutions of Eq. (1), the solution

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

can satisfy the boundary condition (4). Rewriting (6), we have

$$u(x, y)e^{-py} = (A + Be^{-2px})(C \cos py + D \sin py) \tag{6$'$}$$

Using boundary condition (4) in (6′), we have

$$A(C \cos py + D \sin py) = 0, \text{ for } 0 \le y \le \pi$$

$$\therefore \quad A = 0$$

Using boundary condition (2) in (6), we have

$$B.Ce^{-px} = 0, \text{ for all } x > 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } C = 0$$

If we assume that $B = 0$, we get a trivial solution.

$$\therefore \quad C = 0$$

Using boundary condition (3) in (6), we have

$$Be^{-px} \cdot D \sin p\pi = 0, \text{ for all } x > 0$$

$\therefore$ $$B = 0, D = 0 \text{ or } \sin p\pi = 0$$

The values $B = 0$ and $D = 0$ lead to trivial solution.

$\therefore$ $$\sin p\pi = 0$$

$\therefore$ $$p = n$$

where $$n = 0, 1, 2, 3, ..., \infty.$$

Using these values of A, C and p in (6), it reduces to

$$u(x,y) = \lambda e^{-nx} \cdot \sin\ ny$$

where $$n = 0, 1, 2, ..., \infty.$$

Therefore the most general solution of Eq. 1 is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n e^{-nx} \sin ny \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin ny = ky \text{ in } (0, \pi)$$

$$= \sum b_n \sin ny$$

which is the Fourier half-range sine series of ky in $(0, \pi)$. Computing like terms in the two series, we get

$$\lambda_n = b_n = \frac{2}{\pi}\int_0^{\pi} ky \sin ny \, dy$$

$$= \frac{2k}{\pi}\left[y\left(\frac{-\cos ny}{n}\right) - \left(\frac{-\sin ny}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2k}{n}(-1)^{n+1}$$

Using this value of λ_n (7), the required solution is

$$u(x, y) = 2k\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-nx} \sin ny$$

Example 3

A long rectangular plate with insulated surface is 1 cm wide. If the temperature along one short edge ($y = 0$) is $u(x, 0) = k(lx - x^2)$ degrees, for $0 < x < l$, while the two long edge $x = 0$ and $x = l$ as well as the other short edge are kept of 0°C, find the steadystate temperature function $u(x, y)$.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \qquad \text{for all } y > 0 \tag{2}$$

$$u(l, y) = 0, \qquad \text{for all } y > 0 \tag{3}$$

$$u(x, \infty) = 0, \qquad \text{for } 0 \le x \le 1 \tag{4}$$

$$u(x, 0) = k(lx - x^2), \quad \text{for } 0 \le x \le l \tag{5}$$

Proceeding as in Example 1, most general solution of Eq. (1) can be obtained as

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cdot e^{-n\pi y/l} \tag{6}$$

Using boundary condition (5) in (6), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = k(lx - x^2) \text{ in } (0, l)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is the Fourier half-range sine series of $k(lx - x^2)$ in $(0, l)$.

Comparing like terms, we have

$$\lambda_n = b_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2k}{l} \left[(lx - x^2) \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l - 2x) \left(-\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n (6), the required solution is

$$u(x, y) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\exp\{-(2n-1)\pi y/l\}$$

Example 4

A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by

$$\begin{aligned} u &= 10\,y, && \text{for } 0 \le y \le 10 \\ &= 10(20 - y), && \text{for } 10 \le y \le 20 \end{aligned}$$

and the two long edges as well as the other short edge are kept at 0°C, find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$u(x, 20) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty , y) = 0, \quad \text{for } 0 \le y \le 20 \tag{4}$$

$$u(0, y) = f(y), \quad \text{for } 0 \le y \le 20 \tag{5}$$

where $$f(y) = \begin{cases} 10y, & \text{in } 0 \le y \le 10 \\ 10(20-y), & \text{in } 10 \le y \le 20 \end{cases}$$

Proceeding as in Example 2, the most general solution of Eq. (1) can be obtained as

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n e^{-n\pi x/20}\sin\frac{n\pi y}{20} \tag{6}$$

Using boundary condition (5) in (6), we have in $(0, 20) = \sum b_n \sin\dfrac{n\pi y}{20}$

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi y}{20} = f(y)$$

which is the Fourier half-range since series of $f(y)$ in (0, 20).

Comparing like terms, we get

$$\lambda_n = b_n = \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}\,dy$$

$$= \int_0^{10} y\sin\frac{n\pi y}{20}\,dy + \int_{10}^{20}(20-y)\sin\frac{n\pi y}{20}\,dy$$

$$= \left[\left\{ y\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - \left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_0^{10}\right.$$

$$\left. + \left\{(20-y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - 1\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_{10}^{20}\right]$$

$$= \left[\left\{-\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of λ_n in (6), the required solution is

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\exp(-n\pi x/20)\sin\frac{n\pi y}{20} \quad \text{or}$$

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\exp\{-(2n-1)\pi x/20)\sin\frac{(2n-1)\pi y}{20}$$

Example 5

A plate is in the form of the semi-infinite strip $0 \le x \le l$, $0 \le y \le \infty$. The edges $x = 0$ and $x = l$ are insulated. The edge $y = 0$ is kept at temperature

(i) $2\cos\frac{3\pi x}{l} + 3\cos\frac{4\pi x}{l}$ and

(ii) kx, $0 \le x \le l$.

Find the steadystate temperature distribution in the plate (Fig. 3C.4).

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

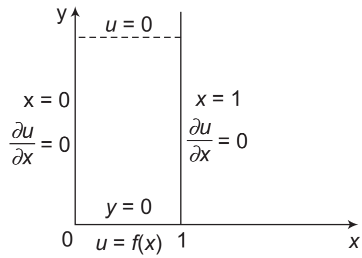


Fig. 3C.4

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad \text{for all } y > 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(l, y) = 0, \qquad \text{for all } y > 0 \tag{3}$$

$$u(x, \infty) = 0, \qquad \text{for } 0 \le x \le l \tag{4}$$

$$u(x, 0) = f(x), \qquad \text{for } 0 \le x \le l \tag{5}$$

where $$f(x) = 2\cos\frac{3\pi x}{l} + 3\cos\frac{4\pi x}{l} \text{ for (i) and}$$

$$f(x) = kx \text{ for (ii)}$$

Note ✍

When an edge is insulated, the temperature gradient at all points on that edge is zero, that is, the derivative of u with respect to the variable along the perpendicular to that edge is zero.

Though the boundary condition in the edge at infinity is not specified, we assume that the temperature in that edge is kept at zero.

Of the three mathematically possible solutions of Eq. (1), the solution

$$u(x, y) = (A \cos px + B \sin px)\,(Ce^{py} + De^{-py}) \tag{6}$$

is the proper solution, as it alone can satisfy the boundary condition (4).

Rewriting (6), we have

$$u(x, y)e^{-py} = (A \cos px + B \sin px)\,(C + De^{-2py}) \tag{6$'$}$$

Using boundary condition (4) in (6)′, we have

$$c = 0$$

Using this value of C in (6), it reduces to

$$u(x, y) = (A \cos px + B \sin px)De^{-py} \tag{7}$$

Differentiating (7) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(-A \sin px + B \cos px)De^{-py} \tag{8}$$

Using boundary condition (2) in (8), we have $p.B.D\, e^{-py} = 0$, for all $y > 0$.

$$\therefore \qquad p = 0 \text{ or } B = 0 \text{ or } D = 0$$

If we assume that $p = 0$ and $D = 0$, we get trivial solutions.

$$\therefore \qquad B = 0$$

Using boundary condition (3) in (8), we have

$$-p.A \sin pl.De^{-py} = 0, \text{ for all } y > 0$$

The values $p = 0$, $A = 0$ and $D = 0$ lead to trivial solutions.

$$\therefore \qquad \sin pl = 0$$

$$\therefore \qquad pl = n\pi \text{ or } p = \frac{n\pi}{l}, \text{ where } n = 0, 1, 2, ..., \infty.$$

Using the values of B, C and p in (6), it reduces to

$$u(x, y) = \lambda \cos\frac{n\pi x}{l} e^{n\pi y/l}, \text{ where } \lambda = AD \text{ and } n = 0, 1, 2, ..., \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x,y) = \sum_{n=0}^{\infty} \lambda_n \cos\frac{n\pi x}{l} e^{-n\pi y/l} \tag{9}$$

Note ✍

The solution corresponding to n = 0 is non-trivial and hence it is to be included in the general solution.

Using boundary condition (5) in (9), we have

$$\sum_{n=0}^{\infty} \lambda_n \cos\frac{n\pi x}{l} = f(x) \text{ in } 0 \le x \le l \tag{10}$$

(i) $f(x) = 2\cos\frac{3\pi x}{l} + 3\cos\frac{4\pi x}{l}$

Using this value of $f(x)$ in (10) and comparing like terms, we get

$$\lambda_4 = 2, \lambda_4 = 3 \text{ and } \lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = ...$$

Using these values of λ_n in (9), the required solution is

$$u(x, y) = 2\cos\frac{3\pi x}{l} e^{-3\pi y/l} + 3\cos\frac{4\pi x}{l} e^{-4\pi y/l}$$

(ii) $f(x) = kx$ in $(0, l)$

$= \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$, which is the Fourier half-range cosine series of kx in $(0, l)$.

Using this form of $f(x)$ in (10) and comparing like terms, we get

$$\lambda_n = a_n = \frac{2}{l}\int_0^l kx\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2kl}{n^2\pi^2}\{(-1)^n - 1\}$$

$$= \begin{cases} -\dfrac{4kl}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Also $\lambda_0 = \dfrac{a_0}{2} = \dfrac{1}{2}\times\dfrac{2}{l}\displaystyle\int_0^l kx\,dx = \dfrac{kl}{2}$.

Using these values of λ_0 and λ_n in(9), the required solution is

$$u(x, y) = \frac{kl}{2} - \frac{4kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}\cdot\exp\{-(2n-1)\pi y/l\}$$

Example 6

A plate is in the form of the semi-infinite strip $0 \le x \le \infty$, $0 \le y \le l$. The surface of the plate and the edge $y = l$ are insulated. If the temperatures along the edge $y = 0$ and the short edge at infinity are kept at temperature 0°C, while the temperature along the other short edge is kept at temperature T°C, find the steady temperature distribution in the plate (Fig. 3C.5).

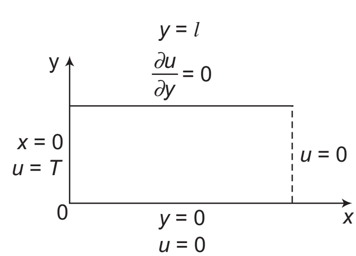


Fig. 3C.5

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$\frac{\partial u}{\partial y}(x, l) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty, y) = 0, \quad \text{for } 0 \le y \le l \tag{4}$$

$$u(0, y) = T, \quad \text{for } 0 \le y \le l \tag{5}$$

As $u(x, y) = 0$ when $x \to \infty$, as per boundary condition (4), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Rewriting (6), we have

$$u(x, y)\cdot e^{-px} = (A + Be^{-2px})(C \cos py + D \sin py) \tag{6$'$}$$

Using boundary condition (4) in (6)′, we have

$$A\cdot(C \cos py + D \sin py) = 0, \text{ for } 0 \le y \le l$$

$$\therefore \quad A = 0$$

Using this value of A in (6), it reduces to

$$u(x, y) = Be^{-px}(C \cos py + D \sin py) \tag{7}$$

Using boundary condition (2) in (7), we have

$$Be^{-px}\cdot C = 0, \text{ for all } x > 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } C = 0$$

If we assume that $B = 0$,we get a trivial solution.

$$\therefore \quad C = 0$$

Using this value of C in (7), it reduces to

$$u(x, y) = B\,De^{-px} \sin py \tag{8}$$

Differentiating (8) partially with respect to y, we get

$$\frac{\partial u}{\partial y}(x, y) = B\,Dp\cdot e^{-px} \cos py \tag{8$'$}$$

Using boundary condition (3) in (8)′, we have

$$B\,Dpe^{-px}\cos pl = 0$$

As $B \neq 0$, $D \neq 0$, $p \neq 0$, $\cos pl = 0$

$$\therefore \qquad pl = (2n-1)\frac{\pi}{2}$$

$$\therefore \qquad p = \frac{(2n-1)\pi}{2l}$$

where $\qquad n = 1, 2, 3, ..., \infty.$

Using this value of p in (8), it becomes

$$u(x, y) = \lambda e^{-(2n-1)\pi x/2l}\cdot\sin\frac{(2n-1)\pi y}{2l}$$

where $n = 1, 2, 3, ..., \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_{2n-1}e^{-(2n-1)\pi x/2l}\cdot\sin\frac{(2n-1)\pi y}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty}\lambda_{2n-1}\sin\frac{(2n-1)\pi y}{2l} = T \text{ in } (0, l)$$

$$\lambda_{2n-1} = \frac{2}{l}\int_0^l T\sin\frac{(2n-1)\pi y}{2l}\,dy$$

$$= \frac{2T}{l}\left[-\frac{\cos\frac{(2n-1)\pi y}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l = \frac{4T}{(2n-1)\pi}$$

Using this value of λ_{2n-1} in (9), the required solution is

$$u(x, y) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\exp\{-2n-1)\pi x/2l\}\sin\frac{(2n-1)\pi y}{2l}$$

PROBLEMS ON TEMPERATURE DISTRIBUTION IN FINITE PLATES

Example 7

Find the steadystate temperature distribution in a rectangular plate of sides a and b, which is insulated on the lateral surface and three of whose edges $x = 0$, $x = a$, $y = b$ are kept at zero temperature, if the temperature in the edge $y = 0$ is (i) $3\sin\frac{2\pi x}{a}+2\sin\frac{3\pi x}{a}$ and (ii) $kx(a - x)$(Fig. 3C.6).

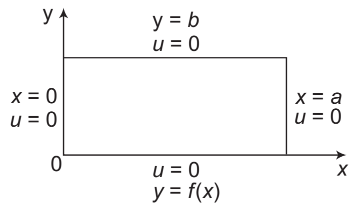


Fig. 3C.6

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{2}$$

$$u(a, y) = 0, \quad \text{for } 0 \le y \le b \tag{3}$$

$$u(x, b) = 0, \quad \text{for } 0 \le y \le a \tag{4}$$

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le a \tag{5}$$

where $f(x) = 3\sin\dfrac{2\pi x}{a}+\sin\dfrac{3\pi x}{a}$ for (i) and $f(x) = kx(a - x)$ for (ii)

The three mathematically possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

$$u(x, y) = (A \cos px + B \sin px)\,(Ce^{py} + De^{-py}) \tag{7}$$

$$u(x, y) = (Ax + B)(Cy + D) \tag{8}$$

Using boundary conditions (2) and (3) in solution (6), we get

$$A + B = 0$$

and $$Ae^{pa} + Be^{-pa} = 0$$

Solving these equations we get $A = 0 = B$, which lead to a trivial solution. Similarly, we will get a trivial solution if we use the boundary conditions in (8). Hence the suitable solution for the present problem is solution (7).

Note ✍

This conclusion is in accordance with the discussion on the choice of proper solution seen already.

Using boundary condition (2) in (7), we have

$$A(Ce^{py} + De^{-py}) = 0, \text{ for } 0 \le y \le b$$

$$A = 0$$

Using boundary condition (3) in (7), we have

$$B \sin pa(Ce^{py} + De^{-py}) = 0, \text{ for } 0 \le y \le b$$

$\therefore$ Either $B = 0$ or $\sin pa = 0$

If B is taken as zero, we get a trivial solution

$\therefore$ $\sin pa = 0$

$\therefore$ $pa = n\pi$ or $p = \dfrac{n\pi}{a}$

where $n = 0, 1, 2, ..., \infty$

Using boundary condition (4) in (7), we have

$$B \sin px(Ce^{pb} + De^{-pb}) = 0, \quad \text{for } 0 \le x \le a$$

As $B \ne 0$, $Ce^{pb} + De^{-pb} = 0$

or $D = -Ce^{2pb}$

Using these values of A, D and p in (7), it reduces to

$$u(x, y) = BC \sin\frac{n\pi x}{a}\left\{e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \cdot e^{-\frac{n\pi y}{a}}\right\}$$

$$= (BCe^{n\pi b/a})\sin\frac{n\pi x}{a}\left\{e^{\frac{n\pi}{a}(y-b)} - e^{\frac{n\pi}{a}(y-b)}\right\}$$

$$= \lambda_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b) \text{ where } n = 0, 1, 2, ...,\infty \text{ and}$$

$$\lambda_n = 2\ BCe^{n\pi b/a}$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b) \tag{9}$$

Using boundary conditions (5) in (9), we have

$$\sum_{n=1}^{\infty}\left(-\lambda_n \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = f(x) \tag{10}$$

(i) $f(x) = 3\sin\dfrac{2\pi x}{a} + 2\sin\dfrac{3\pi x}{a}$

Using this value of $f(x)$ in (10) and comparing like terms, we get

$$-\lambda_2 \sinh\frac{2\pi b}{a} = 3, -\lambda_3 \sinh\frac{3\pi b}{a} = 2 \text{ and } \lambda_1 = \lambda_4 = \lambda_5 = ... = 0$$

Using these values of λ_n in (9), the required solution is

$$u(x, y) = -3 \operatorname{cosech} \frac{2\pi b}{a} \sin \frac{2\pi x}{a} \sinh \frac{2\pi}{a}(y-b)$$

$$-2 \operatorname{cosech} \frac{3\pi b}{a} \sin \frac{3\pi x}{a} \sinh \frac{3\pi}{a}(y-b)$$

or

$$u(x, y) = 3 \operatorname{cosech} \frac{2\pi b}{a} \sin \frac{2\pi x}{a} \sinh \frac{2\pi}{a}(b-y)$$

$$+ 2 \operatorname{cosech} \frac{3\pi b}{a} \sin \frac{3\pi x}{a} \sinh \frac{3\pi}{a}(b-y)$$

(ii) $f(x) = kx(a-x)$ in $(0, a)$

Let the Fourier half-range sine series of

$$f(x) \text{ in } (0, a) \text{ be } \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$$

Using this form of $f(x)$ in (10) and comparing like terms, we get

$$-\lambda_n \sinh \frac{n\pi b}{a} = b_n = \frac{2}{a}\int_0^a kx(a-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{a}\left[(ax-x^2)\left(-\frac{\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) - (a-2x)\left(-\frac{\sin \frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)\right.$$

$$\left. + (-2)\left(\frac{\cos \frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}}\right)\right] = \frac{4ka^2}{n^3\pi^3}\{1-(-1)^n\}$$

$$\lambda_n = \begin{cases} -\dfrac{8ka^2}{n^3\pi^3} \operatorname{cosech} \dfrac{n\pi b}{a}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (9), the required solution is

$$u(x, y) = \frac{8ka^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \operatorname{cosech} \frac{(2n-1)\pi b}{a} \cdot \sin \frac{(2n-1)\pi x}{a}$$

$$\sinh \frac{(2n-1)\pi(b-y)}{a}$$

Example 8

A square plate of length 20 cm has its faces insulated and its edges along $x = 0$, $x = 20$, $y = 0$ and $y = 20$. If the temperature along the edge $x = 20$ is given by

$$u = \frac{T}{10}y, \qquad \text{for } 0 \le y \le 10$$

$$= \frac{T}{10}(20 - y), \quad \text{for } 10 \le y \le 20$$

while the other three edges are kept at 0°C, find the steadystate temperature distribution in the place (Fig. 3C.7).

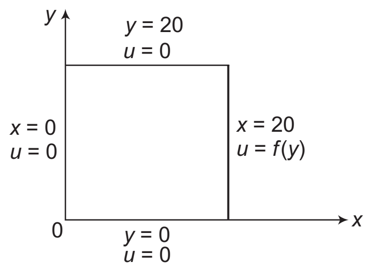


Fig. 3C.7

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfy the following boundary conditions.

$$u(x, 0) = 0, \qquad \text{for } 0 \le x \le 20 \tag{2}$$

$$u(x, 20) = 0, \qquad \text{for } 0 \le x \le 20 \tag{3}$$

$$u(0, y) = 0, \qquad \text{for } 0 \le y \le 20 \tag{4}$$

$$u(20, y) = f(y), \qquad \text{for } 0 \le y \le 20 \tag{5}$$

Since non-zero temperature is prescribed on the edge $x = 20$ in which y is varying, the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})\,(C\cos py + D\sin py) \tag{6}$$

Using boundary condition (2) in (6), we have

$$(Ae^{px} + Be^{-px})\,C = 0, \text{ for } 0 \le x \le 20$$

$$\therefore \qquad C = 0$$

Using boundary condition (3) in (6), we have

$$(Ae^{px} + Be^{-px}) \cdot D \sin 20\,p = 0, \text{ for } 0 \le x \le 20$$

$$\therefore \qquad \text{Either } D = 0 \text{ or } \sin 20\,p = 0$$

If $D = 0$, we get a trivial solution.

$\therefore$ $\sin 20\,p = 0$

$\therefore$ $20\,p = n\pi$ or $p = \dfrac{n\pi}{20}$

where $n = 0, 1, 2, ..., \infty.$

Using boundary condition (4) in (6), we have

$$(A + B)D \sin py = 0, \text{ for } 0 \le y \le 20$$

As $D \ne 0$, $A + B = 0$ or $B = -A$

Using these values of B, C and p in (6), it reduces to

$$u(x, y) = AD\left(e^{\frac{n\pi x}{20}} - e^{-\frac{n\pi y}{20}}\right)\sin\frac{n\pi y}{20}$$

or $$u(x,y) = \lambda \sinh\frac{n\pi x}{20}\sin\frac{n\pi y}{20}, \text{ where } n = 0, 1, 2, ..., \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n \sinh\frac{n\pi x}{20}\sin\frac{n\pi y}{20} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty}(\lambda_n \sinh n\pi)\sin\frac{n\pi y}{20} = f(y) \text{ in } (0, 20) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20}$$

which is the Fourier half-range sine series of $f(y)$ in (0, 20).

Comparing like terms, we get

$$\lambda_n \sinh n\pi = b_n = \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}\,dy$$

$$= \frac{T}{100}\left[\int_0^{10} \cdot y\sin\frac{n\pi y}{20}\,dy + \int_{10}^{20}(20 - y)\sin\frac{n\pi y}{20}\,dy\right]$$

$$= \frac{T}{100}\left[\left\{y\left(-\frac{\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - \left(-\frac{\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_0^{10}\right.$$

$$\left. + \left\{(20 - y)\left(-\frac{\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - (-1)\left(-\frac{\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_{10}^{20}\right]$$

$$= \frac{T}{100}\left[\left(-\frac{200}{n\pi}\cos\frac{n\pi}{2}+\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right.$$

$$\left.+\left(\frac{200}{n\pi}\cos\frac{n\pi}{2}+\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore \qquad \lambda_n = \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}\text{cosech } n\pi$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = \frac{8T}{\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi}{2}\text{cosech } n\pi \sinh\frac{n\pi x}{20}\cdot\sin\frac{n\pi y}{20}$$

or

$$u(x, y) = \frac{8T}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\text{cosech }(2n-1)\pi$$

$$\sinh\frac{(2n-1)\pi x}{20}\sin\frac{(2n-1)\pi y}{20}$$

Example 9

If a square plate is bounded by the lines $x = \pm a$ and $y = \pm a$ and three of its edges are kept at temperature 0°C, while the temperature along the edge $y = a$ is kept at $u = x + a$, $-a \le x \le a$, find the steadystate temperature in the plate (Fig. 3C.8).

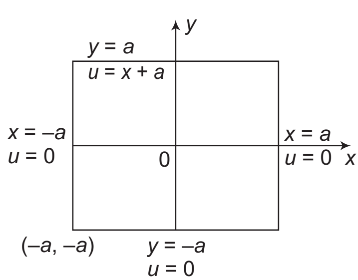


Fig. 3C.8

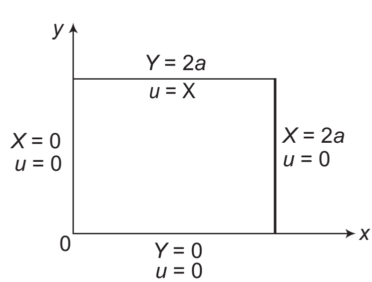


Fig. 3C.9

In Examples 7 and 8, we have observed that the arbitrary constants in the appropriate solution of the Laplace equation can be readily found out, only if two adjacent edges of the square (or rectangle) are taken as coordinate axes. As this condition is not satisfied in the present problem, we shift the origin to the point $(-a, -a)$, so that two adjacent edges may lie along the coordinate axes in the new system (Fig. 3C.9).

The transformation equations are

$$x = X - a$$

$$y = Y - a$$

The equations of the edges are $X = 0$, $X = 2a$, $Y = 0$ and $Y = 2a$ in the new system.

Let us work out the problem with reference to the new system. The steadystate temperature $u(X, Y)$ at any point (X, Y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, Y) = 0, \quad \text{for } 0 \le Y \le 2a \tag{2}$$

$$u(2a, Y) = 0, \quad \text{for } 0 \le Y \le 2a \tag{3}$$

$$u(X, 0) = 0, \quad \text{for } 0 \le X \le 2a \tag{4}$$

$$u(X, 2a) = X, \quad \text{for } 0 \le X \le 2a \tag{5}$$

Since non-zero temperature is prescribed on the edge $Y = 2a$, in which X is varying, the proper solution of equation (1) is

$$u(X, Y) = (A \cos pX + B \sin pX)\,(Ce^{pY} + De^{-pY}) \tag{6}$$

Using boundary conditions (2) in (6), we have

$$A(Ce^{pY} + De^{-pY}) = 0, \text{ for } 0 \le Y \le 2a$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin 2\,pa\,(Ce^{pY} + De^{-pY}) = 0, \text{ for } 0 \le Y \le 2a$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin 2\,pa = 0$$

If $B = 0$, we get a trivial solution and so $B \neq 0$

$$\therefore \qquad \sin 2\,pa = 0$$

$$\therefore \qquad 2pa = n\pi \text{ or } p = \frac{n\pi}{2a}, \text{ where } n = 0, 1, 2, ..., \infty.$$

Using boundary condition (4) in (6), we have

$$B \sin pX(C + D) = 0, \text{ for } 0 \le X \le 2a$$

As $B \neq 0$, we get $C + D = 0$ or $D = -C$

Using these values A, D and p in (6), it reduces to

$$u(X, Y) = BC \sin \frac{n\pi X}{2a} \left\{ e^{n\pi Y/2a} - e^{-n\pi Y/2a} \right\}$$

$$= \lambda \sin \frac{n\pi X}{2a} \sinh \frac{n\pi Y}{2a}, \text{ where } \lambda = 2BC$$

and $\qquad n = 0, 1, 2, ..., \infty.$

Therefore the most general solution of Eq. (1) is

$$u(X, Y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi X}{2a} \sinh \frac{n\pi Y}{2a} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} (\lambda_n \sinh n\pi) \sin \frac{n\pi X}{2a} = X \text{ in } (0, 2a)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{2a}$$

which is the Fourier half-range sine series of X in $(0, 2a)$.

Comparing like terms in the two series, we get

$$\lambda_n \sinh n\pi = b_n = \frac{2}{2a} \int_0^{2a} X \sin \frac{n\pi X}{2a} \mathrm{d}X$$

$$= \frac{1}{a} \left[X \left(\frac{-\cos \frac{n\pi X}{2a}}{\frac{n\pi}{2a}} \right) - \left(\frac{-\sin \frac{n\pi X}{2a}}{\frac{n^2\pi^2}{4a^2}} \right) \right]_0^{2a}$$

$$= -\frac{4a}{n\pi} \cos n\pi$$

$\therefore$
$$\lambda_n = \frac{4a}{n\pi} (-1)^{n+1} \operatorname{cosech} n\pi$$

Using this value of λ_n in (7), the required solution with reference to the new system is

$$u(X, Y) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \cdot \sin \frac{n\pi X}{2a} \sinh \frac{n\pi Y}{2a}$$

With reference to the old system, the required solution is

$$u(x, y) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \cdot \sin \frac{n\pi}{2a}(x + a) \cdot \sinh \frac{n\pi}{2a}(y + a)$$

Example 10

A rectangular place is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Its surfaces are insulated. The temperature along $x = 0$ and $y = 0$ are kept at 0°C and the others at 100°C. Find the steadystate temperature at any point of the plate (Fig. 3C.10, 3C.11 and 3C.12).

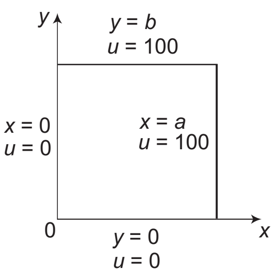


Fig. 3C.10

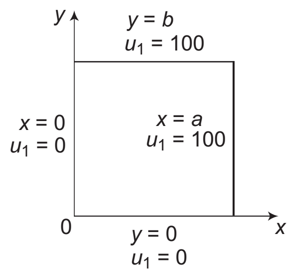


Fig. 3C.11

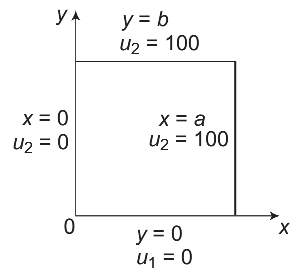


Fig. 3C.12

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{2}$$

$$u(a, y) = 100, \quad \text{for } 0 \le y \le b \tag{3}$$

$$u(x, 0) = 0, \quad \text{for } 0 < x < a \tag{4}$$

$$u(x, b) = 100, \quad \text{for } 0 < x < b \tag{5}$$

From previous examples, it is obvious that Eq. (1) is readily solvable, that is, the arbitrary constants in the proper solution of Eq. (1) can easily found out, only if three of the boundary values (temperatures along three of the edges) are zero each and the fourth boundary value (temperature along the fourth edge) in non-zero.

As two boundary values are non-zero each in this problem, we adopt a slightly modified procedure as explained below.

Let
$$u(x, y) = u_1(x, y) + u_2(x, y) \tag{6}$$

Using (6) in (1), we get

$$\frac{\partial^2}{\partial x^2}(u_1 + u_2) + \frac{\partial^2}{\partial y^2}(u_1 + u_2) = 0$$

Separating the derivatives of u_1 and those of u_2 we have

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \tag{7}$$

and
$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \tag{8}$$

We assume convenient boundary conditions for Eq. (7) [i.e. three zero and one non-zero boundary conditions] which are given below.

$$u_1(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{9}$$

$$u_1(a, y) = 0, \quad \text{for } 0 \le y \le b \tag{10}$$

$$u_1(x, 0) = 0, \quad \text{for } 0 < x < a \tag{11}$$

$$u_1(x, b) = 100, \quad \text{for } 0 < x < a \tag{12}$$

The boundary conditions for Eq. (8) are obtained by using (6) and the boundary conditions (2), (3), (4), (5) for $u(x, y)$ and the boundary conditions (9), (10), (11), (12) for $u_1(x, y)$

Thus

$$u_2(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{13}$$

$$u_2(a, y) = 100, \quad \text{for } 0 \le y \le b \tag{14}$$

$$u_2(x, 0) = 0, \quad \text{for } 0 < x < a \tag{15}$$

$$u_2(x, b) = 0, \quad \text{for } 0\ x < a \tag{16}$$

The appropriate solution of Eq. (7) consistent with the given boundary conditions for $u_1(x, y)$ is

$$u_1(x, y) = (A \cos px + B \sin px)\,(Ce^{pY} + De^{-pY}) \tag{17}$$

Using boundary conditions (9), (10) and (11) in (17) and proceeding as in Example 9, we most general solution of Eq. (7) can be obtained as

$$u_1(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{18}$$

Using boundary condition (12) in (18), we have

$$\sum_{n=1}^{\infty}\left(\lambda_n \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = 100, \text{ in } (0, a)$$

$$= \sum b_n \sin\frac{n\pi x}{a}$$

which is the Fourier half-range sine series of 100 (0, a).

Comparing like terms in the two series, we get

$$\lambda_n \sinh\frac{n\pi b}{a} = b_n = \frac{2}{a}\int_0^a 100\sin\frac{n\pi x}{a}\,\mathrm{d}x$$

$$= \frac{200}{a}\left(\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$= \frac{200}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{400}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (18), the required solution of Eq. (7) is

$$u_1(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{cosech}\frac{(2n-1)\pi b}{a}\sin\frac{(2n-1)\pi x}{a}\sinh\frac{(2n-1)\pi y}{a} \tag{19}$$

Now solving Eq. (8) subject to the boundary conditions (13), (14), (15) and (16) [proceeding as in Example 8] or by interchanging x and y and also a and b in (19), we get

$$u_2(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{cosech}\frac{(2n-1)\pi a}{b}\sinh\frac{(2n-1)\pi x}{b}\sin\frac{(2n-1)\pi y}{b} \tag{20}$$

Using (19) and (20) in (6), the required solution of Eq. (1) is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\operatorname{cosech}\frac{(2n-1)\pi b}{a}\cdot\sin\frac{(2n-1)\pi x}{a}\right.$$

$$\left.\sinh\frac{(2n-1)\pi y}{a} + \operatorname{cosech}\frac{(2n-1)\pi a}{b}\cdot\sin\frac{(2n-1)\pi y}{b}\cdot\sinh\frac{(2n-1)\pi x}{b}\right]$$

Note ✍

If non-zero temperatures are prescribed on all the four sides of the rectangle (or square), the concept used in the previous example is extended by assuming that $u(x, y) = \sum_{r=1}^{4} u_r(x, y)$. *Three of the boundary values of each of the* *equations* $\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial u_r}{\partial y^2} = 0$ *are assumed to be zero and the fourth one non-zero in such a way that we get the given boundary values of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *from those of* $\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial y^2} = 0$ *by superposition.*

Example 11

A square plate has its faces and its edge $y = 0$ insulated. Its edges $x = 0$ and $x = 10$ are kept at temperature zero and its edge $y = 10$ at temperature 100°C. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \qquad \text{for } 0 \le y \le 10 \tag{2}$$

$$u(10, y) = 0, \qquad \text{for } 0 \le y \le 10 \tag{3}$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, \qquad \text{for } 0 < x < 10 \tag{4}$$

$$u(x, 10) = 100, \qquad \text{for } 0 < x < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \sin px)\,(Ce^{pY} + De^{-pY}) \tag{6}$$

Using boundary conditions (2) and (3) in (6), we can get, as usual,

$$A = 0$$

and $$p = \frac{n\pi}{10}$$

where $$n = 0, 1, 2, ..., \infty.$$

Differentiating (6) partially with respect to y,

$$\frac{\partial u}{\partial y}(x, y) = Bp \sin px\,(Ce^{pY} - De^{-pY}) \tag{7}$$

Using boundary condition (4) in (7), we have

$$Bp \sin px(C - D) = 0, \text{ for } 0 < x < 10$$

As $B \neq 0$ and $p \neq 0$, we get $D = C$

Using these values of A, D and p in (6), it reduces to

$$u(x, y) = BC \sin\frac{n\pi x}{10}\cdot(e^{n\pi y/10} + e^{-n\pi y/10})$$

$$= \lambda \sin\frac{n\pi x}{10}\cosh\frac{n\pi y}{10}, \text{ where } \lambda = 2BC \text{ and } n = 0, 1, 2, ..., \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{10}\cosh\frac{n\pi y}{10} \qquad (8)$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty}(\lambda_n \cosh n\pi)\sin\frac{n\pi x}{10} = 100 \text{ in } (0, 10)$$

$$= \sum b_n \sin\frac{n\pi x}{10}$$

which is the Fourier half-range sine series of 100 in (0, 10).
Comparing like terms in the two series, we get

$$\lambda_n \cosh n\pi = b_n = \frac{2}{10}\int_0^{10} 100\sin\frac{n\pi x}{10}dx$$

$$= 20\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_0^{10}$$

$$= \frac{200}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases}\dfrac{400}{n\pi}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Using this value of λ_n in (8), the required solution is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\text{sech}(2n-1)\pi\cdot\sin\frac{(2n-1)\pi x}{10}\cdot\cosh\frac{(2n-1)\pi y}{10}$$

Example 12

A rectangular plate of sides 20 cm and 10 cm has its faces and the edge $x = 20$ insulated. Its edges $y = 0$ and $y = 10$ are kept at temperature zero, while the edge $x = 0$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditins.

$$u(x, 0) = 0, \quad \text{for } 0 \le x \le 20 \tag{2}$$

$$u(x, 10) = 0, \quad \text{for } 0 \le x \le 20 \tag{3}$$

$$\frac{\partial u}{\partial x}(20, y) = 0, \quad \text{for } 0 < y < 10 \tag{4}$$

$$u(0, y) = ky, \quad \text{for } 0 < y < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Using boundary conditions (2) and (3) in (6), we can get, as usual,

$$C = 0 \text{ and } p = \frac{n\pi}{10}, \text{ where } n = 0, 1, 2, ..., \infty$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(Ae^{px} - Be^{-px})\cdot D \sin py \tag{7}$$

Using boundary condition (4) in (7), we have

$$p(A^{20p} - Be^{-20p})D \sin py = 0, \text{ for } 0 < y < 10$$

As $p \neq 0$ and $D \neq 0$, $Ae^{20p} - Be^{-20p} = 0$

$$\therefore \qquad B = Ae^{40p}$$

Using these values of B, C and p in (6), it reduces to

$$u(x, y) = AD\{e^{n\pi x/10} + e^{40n\pi/10} \cdot e^{-n\pi x/10}\}\sin\frac{n\pi y}{10}$$

$$= (2ADe^{2n\pi})\cosh\frac{n\pi(x-20)}{10}\cdot\sin\frac{n\pi y}{10}$$

$$= \lambda_n \cosh\frac{n\pi(20-x)}{10}\cdot\sin\frac{n\pi y}{10}$$

[$\because$ $\cosh\theta$ is an even function]

where $n = 1, 2, 3, ..., \infty$.

$\therefore$ The most general of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \cosh \frac{n\pi(20-x)}{10} \sin \frac{n\pi y}{10} \tag{8}$$

Using boundary condition (5) in (8),we have

$$\sum_{n=1}^{\infty} (\lambda_n \cosh 2n\pi) \sin \frac{n\pi y}{10} = ky \text{ in } (0, 10)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10}$$

which is the Fourier half-range sine series of ky in (0, 10).

Comparing like terms in the two series, we have

$$\lambda_n \cosh 2n\pi = b_n = \frac{2}{10} \int_0^{10} ky \sin \frac{n\pi y}{10} dy$$

$$= \frac{2k}{10} \left[y \left(\frac{-\cos \frac{n\pi y}{10}}{\frac{n\pi}{10}} \right) - \left(\frac{-\sin \frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}} \right) \right]_0^{10}$$

$$= \frac{20k}{n\pi} (-1)^{n+1}$$

Using this value of λ_n in (8), the required solution is

$$u(x, y) = \frac{20k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \operatorname{sech} 2n\pi \cdot \cosh \frac{n\pi(20-x)}{10} \sin \frac{n\pi y}{10}$$

Example 13

A square plate has its faces and its edges $x = 0$ and $x = a$ insulated. If the edge $y = a$ is kept at temperature zero, while the edge $y = 0$ is kept at temperature $4\cos^3\left(\frac{\pi x}{a}\right)$, find the steadystate temperature distribution in the plate.

The steady state temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, x) = 0, \qquad \text{for } 0 \le y \le a \tag{2}$$

$$\frac{\partial u}{\partial x}(a, y) = 0, \qquad \text{for } 0 \le y \le a \tag{3}$$

$$u(x, a) = 0, \qquad \text{for } 0 < x < a \tag{4}$$

$$u(x, 0) = 4\cos^3\left(\frac{\pi x}{a}\right), \qquad \text{for } 0 < x < a \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \cos px)\,(Ce^{pY} + De^{-pY}) \tag{6}$$

Differentiating (6) partially with respect to x,

$$\frac{\partial u}{\partial x}(x, y) = p(-A \sin px + B \cos px)\,(Ce^{pY} + De^{-pY}) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$B = 0 \text{ and } p = \frac{n\pi}{a}, \text{ where } n = 0, 1, 2, ..., \infty$$

Using boundary conditions (4) in (6), we have

$$A \cos px(Ce^{pY} + De^{-pa}) = 0, \text{ for } 0 < x < a$$

$$\text{As } A \ne 0,\ D = -Ce^{2pa}$$

Using these values of B, D and p in (6), we get

$$u(x, y) = AC \cos \frac{n\pi x}{a}\{e^{n\pi y/a} - e^{2n\pi ya/a} \cdot e^{-n\pi y/a}\}$$

$$= (2ACe^{n\pi})\cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a}$$

or $$u(x, y) = \lambda_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a}$$

where $n = 0, 1, 2, ..., \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty}(-\lambda_n \sinh n\pi)\cos\frac{n\pi x}{a} = 4\cos^3\frac{n\pi}{a} \text{ in } (0, a)$$

$$= 3\cos\frac{n\pi}{a} + \cos\frac{3\pi x}{a}$$

Comparing like terms, we get

$$-\lambda_1 \sinh \pi = 3; -\lambda_3 \sinh 3\pi = 1; \lambda_2 = 0 = \lambda_4 = \lambda_5 = \cdots$$

$\therefore$ $$\lambda_1 = -3 \text{ cosech } \pi; \lambda_3 = \text{cosech } 3\pi; \lambda_2 = 0 = \lambda_4 = \lambda_5 = \cdots$$

Using these values in (8), the required solution is

$$u(x, y) = 3 \text{ cosech } \pi \cos\frac{n\pi}{a}\sinh\frac{\pi(a-y)}{a} + \text{cosech } 3\pi \cos\frac{3\pi x}{a}\sinh\frac{3\pi(a-y)}{a}$$

Example 14

A rectangular plate of sides a and b has its faces and the edges $y = 0$ and $y = b$ insulated. If the edge $x = 0$ is kept at temperature zero, while the edge $x = a$ is kept at temperature $k(2y - b)$, find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions:

$$\frac{\partial u}{\partial y}(x, 0) = 0, \qquad \text{for } 0 \le x \le a \tag{2}$$

$$\frac{\partial u}{\partial y}(x, b) = 0, \qquad \text{for } 0 \le x \le a \tag{3}$$

$$u(0, y) = 0, \qquad \text{for } 0 < y < b \tag{4}$$

$$u(a, y) = k(2y - b), \text{ for } 0 < y < b \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = (Ae^{px} + Be^{-px})\, p(-C \sin py + D \cos py) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$D = 0 \text{ and } p = \frac{n\pi}{b}, \text{ where } n = 0, 1, 2, ..., \infty$$

Using boundary condition (4) in (6), we have

$$(A + B)C \cos py = 0, \text{ for } 0 < y < b$$

As $C \neq 0$, we get $B = -A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = AC\left(e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}\right)\cos\frac{n\pi y}{b}$$

$$= \lambda \sinh\frac{n\pi x}{b}\cdot\cos\frac{n\pi y}{b}$$

where $\lambda = 2AC$ and $n = 0, 1, 2, ..., \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sinh\frac{n\pi x}{b}\cos\frac{n\pi y}{b} \tag{8}$$

Using boundary condition (5) in (8) we have

$$\sum_{n=1}^{\infty} \lambda_n \sinh\frac{n\pi a}{b}\cdot\cos\frac{n\pi y}{b} = k(2y - b) \text{ in } (0, b)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi y}{b},$$

which is Fourier half-range cosine series of $k(2y - b)$ in $(0, b)$.

Comparing like terms in the two series, we get

$$\lambda_n \sinh\frac{n\pi a}{b} = a_n = \frac{2}{b}\int_0^b k(2y - b)\cos\frac{n\pi y}{b}\,dy$$

$$= \frac{2k}{b}\left[(2y - b)\left(\frac{\sin\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - 2\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right)\right]_0^b$$

$$= \frac{4kb}{n^2\pi^2}\{(-1)^n - 1\}$$

$$= \begin{cases} \dfrac{-8kb}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Also
$$a_0 = \frac{2}{b}\int_0^b k(2y-b)\,dy$$
$$= \frac{2k}{b}(y^2 - by)_0^b = 0$$

We note that the constant term in the R.H.S. series is also zero.

Using this value of λ_n in (8), the required solution is

$$u(x, y) = -\frac{8kb}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,\text{cosech}\,\frac{(2n-1)\pi a}{b}$$
$$\sinh\frac{(2n-1)\pi x}{b}\cos\frac{(2n-1)\pi y}{b}$$

Example 15

Find the steadystate temperature distribution on a square plate of side a insulated along three of its sides and with the side $y = 0$ kept at temperature zero for $0 < x < \frac{a}{2}$ and at temperature T for $\frac{a}{2} < x < a$.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \text{for } 0 \le y \le a \tag{2}$$

$$\frac{\partial u}{\partial x}(a, y) = 0, \quad \text{for } 0 \le y \le a \tag{3}$$

$$\frac{\partial u}{\partial y}(x, a) = 0, \quad \text{for } 0 < x < a \tag{4}$$

$$u(x, 0) = f(x), \text{ for } 0 < x < a \tag{5}$$

where
$$f(x) = \begin{cases} 0, & \text{in } \left(0, \frac{a}{2}\right) \\ T, & \text{in } \left(\frac{a}{2}, a\right) \end{cases}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A\cos px + B\sin px)(Ce^{pY} + De^{-pY}) \tag{6}$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(-A\sin px + B\cos px)(Ce^{pY} + De^{-pY}) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$B = 0 \text{ and } p = \frac{n\pi}{a}, \text{ where } n = 0, 1, 2, ..., \infty$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = A \cos px \cdot p(Ce^{pY} + De^{-pY}) \tag{8}$$

Using boundary condition (4) in (7), we have

$$A \cdot \cos px \cdot p(Ce^{pa} - De^{-pa}) = 0, \text{ for } 0 < x < a$$

As $A \neq 0$ and $p \neq 0$, $Ce^{pa} - De^{-pa}) = 0$

$\therefore$ $$D = Ce^{2pa}$$

Using these values of B, D, and p in (6), we have

$$\begin{aligned} u(x, y) &= AC\cos\frac{n\pi x}{a}\{e^{n\pi y/a} + e^{2n\pi a/a} \cdot e^{-n\pi y/a}\} \\ &= (2ACe^{n\pi})\cos\frac{n\pi x}{a}\cosh\frac{n\pi(y-a)}{a} \\ &= \lambda_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi(a-y)}{a} \end{aligned}$$

where $n = 0, 1, 2, ..., \infty$ ($\because$ $\cosh\theta$ is even)

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=0}^{\infty} \lambda_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi(a-y)}{a} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\begin{aligned} \sum_{n=0}^{\infty}(\lambda_n \cosh n\pi)\cos\frac{n\pi x}{a} &= f(x) \text{ in } (0, a) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \end{aligned}$$

which is the Fourier half-range cosine series of $f(x)$ in $(0, a)$.

Equating like terms in the two series, we get

$$\lambda_n \cosh n\pi = a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}T\int_{a/2}^{a} \cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2T}{a}\cdot\left(\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_{a/2}^{a}$$

$$= \frac{2T}{n\pi}\left(-\sin\frac{n\pi}{2}\right)$$

$$\lambda_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{2}{a}\int_{a/2}^{a} T\,\mathrm{d}x = \frac{T}{2}$$

Using these values of λ_n in (9), the required solution is

$$u(x, y) = \frac{T}{2} - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\operatorname{sech} n\pi \cos\frac{n\pi x}{a}\cdot\cosh\frac{n\pi(a-y)}{a}$$

i.e.

$$u(x, y) = \frac{T}{2} + \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\operatorname{sech}(2n-1)\pi\cdot\cosh\frac{(2n-1)\pi x}{a}$$

$$\cosh\frac{(2n-1)\pi(a-y)}{a}$$

Example 16

Find the steadystate temperature distribution $u(x, y)$ in the uniform square $0 \le x \le \pi$; $0 \le y \le \pi$, when the edge $x = \pi$ is maintained at temperature $(2\cos 3y - 5\cos 4y)$, the other three edges being thermally insulated.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \quad \text{for} \quad 0 \le x \le \pi \tag{2}$$

$$\frac{\partial u}{\partial y}(x, \pi) = 0, \quad \text{for} \quad 0 \le x \le \pi \tag{3}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \text{for} \quad 0 < y < \pi \tag{4}$$

$$u(x, y) = 2 \cos 3y - 5 \cos 4y, \text{ for } 0 < y < \pi \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = p(Ae^{px} + Be^{-px})(-C \sin py + D \cos py) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$D = 0 \text{ and } p = n, \text{ where } n = 0, 1, 2, ..., \infty$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(Ae^{px} - Be^{-px} \cdot C \cos py \tag{8}$$

Using boundary condition (4) in (8), we have

$$p(A - B)C \cos py = 0, \text{ for } 0 < y < \pi$$

As $p \neq 0$ and $C \neq 0$, we get $B = A$.

Using these values of B, D, and p in (6), it reduces to

$$u(x, y) = AC(e^{nx} + e^{-nx}) \cos ny$$

$$= \lambda \cosh nx \cos ny$$

where $n = 0, 1, 2, ..., \infty$ and $\lambda = 2AC$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=0}^{\infty} \lambda_n \cosh nx \cos ny \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=0}^{\infty} (\lambda_n \cosh n\pi) \cos ny = 2 \cos 3y - 5 \cos 4y \text{ in } (0, \pi)$$

Comparing like terms, we get

$$\lambda_3 \cosh 3\pi = 2; \lambda_4 \cosh 4\pi = -5 \text{ and } \lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 \ldots$$

i.e. $\lambda_3 = 2 \text{ sech } 3p,\ \lambda_4 = -5 \text{ sech } 4\pi$

$$\lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 = \ldots$$

Using thse values in (9), the required solution is

$$u(x, y) = 2 \text{ sech } 3\pi \cosh 3x \cos 3y - 5 \text{ sech } 4\pi \cdot \cosh 4x \cosh 4y$$

Example 17

A square metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = a$. The edges $x = 0$ and $y = a$ are kept at zero temperature, the edge $x = a$ is insulated and the edge $y = 0$ is kept at temperature kx. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions:

$$u(0, y) = 0, \quad \text{for } 0 \le y \le a \tag{2}$$

$$\frac{\partial u}{\partial x}(a, y) = 0, \quad \text{for } 0 \le y \le a \tag{3}$$

$$u(x, a) = 0, \quad \text{for } 0 < x < a \tag{4}$$

$$u(x, 0) = kx, \text{ for } 0 < x < a \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \sin px)\,(Ce^{pY} + De^{-pY}) \tag{6}$$

Using boundary condition (2) in (6), we can get $A = 0$.

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = Bp \cos px\,(Ce^{pY} + De^{-pY}) \tag{7}$$

Using boundary conditions (3) in (7), we have

$$Bp \cos pa(Ce^{pY} + De^{-pY}), \text{ for } 0 \le y \le a$$

$$\text{Either } B = 0,\ p = 0 \text{ or } \cos pa = 0$$

If $B = 0$ and $p = 0$, we get trivial solutions

$$\therefore \qquad \cos pa = 0$$

$$\therefore \qquad pa = \frac{(2n-1)\pi}{2} \text{ or } p = \frac{(2n-1)\pi}{2a}$$

where $n = 1, 2, ..., \infty$

Using boundary condition (4) in (6), we have

$$B \sin px(Ce^{pa} + De^{-pa}) = 0$$

As $\quad B \neq 0,\ Ce^{pa} + De^{-pa} = 0$

$$D = -Ce^{2pa}$$

Using these values of A, D, and p in (6), it reduces to

$$u(x, y) = BC \sin \frac{(2n-1)\pi x}{2a} \{e^{(2n-1)\pi y/2a} - e^{2(2n-1)\pi a/2a} e^{-(2n-1)\pi y/2a}\}$$

$$= \{(2BCe^{(2n-1)\pi/2}\} \sin \frac{(2n-1)\pi x}{2a} \sinh \frac{(2n-1)\pi(y-a)}{2a}$$

$$= \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{2a} \sin \frac{(2n-1)\pi(y-a)}{2a}$$

where $n = 1, 2, ..., \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{2a} \cdot \sinh \frac{(2n-1)\pi(y-a)}{2a} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty} -\lambda_{2n-1} \sinh \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{2a} = kx \text{ in } (0, a)$$

$$\therefore \qquad -\lambda_{2n-1} \sinh \frac{(2n-1)\pi}{2} = \frac{2}{a} \int_0^a kx \sin \frac{(2n-1)}{2a} \mathrm{d}x$$

$$= \frac{2k}{a} \left[x \left\{ \frac{-\cos \frac{(2n-1)\pi x}{2a}}{\frac{(2n-1)\pi}{2a}} \right\} - \left\{ \frac{-\sin \frac{(2n-1)\pi x}{2a}}{\frac{(2n-1)^2 \pi^2}{4a^2}} \right\} \right]_0^a$$

$$= \frac{8ka}{(2n-1)^2 \pi^2} \sin \frac{(2n-1)\pi}{2}$$

$$\therefore \qquad \lambda_{2n-1} = \frac{8ka}{(2n-1)^2 \pi^2} \cdot (-1)^n \operatorname{cosech} \frac{(2n-1)\pi}{2}$$

Using these values of l_{2n-1} in (8), the required solution is

$$u(x, y) = \frac{8ka}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\text{cosech}\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2a}$$

$$\sinh\frac{(2n-1)\pi(a-y)}{2a}$$

Example 18

A rectangular plate of sides a and b is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. The edges $x = 0$ and $y = b$ are kept at zero temperature, while the edge $y = 0$ is kept insulated. If the temperature along the edge $x = a$ is kept at T°C, find the steadystate temperature distribution in the plate.

The steadystate temperature distribution in the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \quad \text{for } 0 \le x \le a \qquad (2)$$

$$u(x, b) = 0, \quad \text{for } 0 \le x \le a \qquad (3)$$

$$u(0, y) = 0, \quad \text{for } 0 < y < b \qquad (4)$$

$$u(a, y) = T, \quad \text{for } 0 < y < b \qquad (5)$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \qquad (6)$$

Differentiating (6) partially with respect to y and then using boundary condition (2), we can get $D = 0$.

Using boundary conditions (3) in (6), we can get

$$p = \frac{(2n-1)\pi}{2b}, \text{ where } n = 1, 2, 3, ..., \infty$$

Using boundary condition (4) in (6), we can get $B = -A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = \lambda \sinh\frac{(2n-1)\pi x}{2b}\cdot\cos\frac{(2n-1)\pi y}{2b}$$

where $\lambda = 2AC$ and $n = 1, 2, ..., \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sinh \frac{(2n-1)\pi x}{2b} \cos \frac{(2n-1)\pi y}{2b} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \left\{ \lambda_{2n-1} \sinh \frac{(2n-1)\pi a}{2b} \right\} \cos \frac{(2n-1)\pi y}{2b} = T \text{ in } (0, b)$$

$$\therefore \qquad \lambda_{n2-1} \sinh \frac{(2n-1)\pi a}{2b} = \frac{2}{b} \int_0^b T \cos \frac{(2n-1)}{2b} dy$$

$$= \frac{2T}{b} \left\{ \frac{\sin \frac{(2n-1)\pi y}{2b}}{\frac{(2n-1)}{2b}} \right\}_0^b$$

$$= \frac{4T}{(2n-1)} \sin \frac{(2n-1)\pi}{2}$$

$$\therefore \qquad \lambda_{2n-1} = \frac{4T}{(2n-1)\pi} \cdot \operatorname{cosech} \frac{(2n-1)\pi a}{2b} \cdot (-1)^{n+1}$$

Using this value of λ_{2-1} in (7), the required solution is

$$u(x, y) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \operatorname{cosech} \frac{(2n-1)}{2b}$$

$$\sinh \cdot \frac{(2n-1)\pi x}{2b} \cdot \cos \frac{(2n-1)\pi y}{2b}$$

Example 19

A square metal plate of side 10 cm has the edges represented by the lines $x = 10$ and $y = 10$ insulated. The edge $x = 0$ is kept at a temperature of zero degree and the edge $y = 0$ at a temperature of 100°C. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for } 0 \le y \le 10 \tag{2}$$

$$\frac{\partial u}{\partial x}(10, y) = 0, \quad \text{for } 0 \le y \le 10 \tag{3}$$

$$\frac{\partial u}{\partial y}(x, 10) = 0, \qquad \text{for } 0 < x < 10 \tag{4}$$

$$u(x, 0) = 100, \quad \text{for } 0 < x < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \sin px)\,(Ce^{pY} + De^{-pY}) \tag{6}$$

Using boundary condition (2) in (6), we can get $A = 0$.

Differentiating (6) partially with respect to x and then using the boundary condition (3), we can get

$$p = \frac{(2n-1)\pi}{20}, \text{ where } n = 1, 2, ..., \infty$$

Differentiating (6) partially with respect to y and then using the boundary condition (4), we can get $D = Ce^{20p}$.

Using these values of A, D and p in (6), it reduces to

$$u(x, y) = [2BCe^{(2n-1)\pi/2}] \sin\frac{(2n-1)\pi x}{20} \cosh\frac{(2n-1)\pi(y-10)}{20}$$

or $$u(x, y) = \lambda_{2n-1} \sin\frac{(2n-1)\pi x}{20} \cosh\frac{(2n-1)\pi(10-y)}{20}$$

where $n = 1, 2, ..., \infty$ [$\because$ $\cosh\theta$ is even]

$\therefore$ The most general solution of Eq (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sin\frac{(2n-1)\pi x}{20} = \cosh\frac{(2n-1)\pi(10-y)}{20} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \lambda_{2n-1} \cosh\frac{(2n-1)\pi}{2} \sin\frac{(2n-1)\pi x}{20} = 100 \text{ in } (0, 10)$$

$$\therefore \quad \lambda_{2n-1} \cosh\frac{(2n-1)\pi}{2} = \frac{2}{10}\int_0^{10} 100 \sin\frac{(2n-1)\pi x}{20}\, dx$$

$$= 20\left\{\frac{-\cos\frac{(2n-1)\pi x}{20}}{\frac{(2n-1)\pi}{20}}\right\}_0^{10}$$

$$= \frac{400}{(2n-1)\pi}$$

$$\lambda_{2n-1} = \frac{400}{(2n-1)\pi} \operatorname{sech}\frac{(2n-1)\pi}{2}$$

Using this value of λ_{2n-1} in (7), the required solution is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\operatorname{sech}\frac{(2n-1)}{2}\sin\frac{(2n-1)\pi x}{20}\cosh\frac{(2n-1)\pi(10-y)}{20}$$

Example 20

A square metal plate of side a has the edges $x = 0$ and $y = 0$ insulated. The edge $y = a$ is kept at temperature 0°C and the edge $x = a$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.

The steadystate temperature distribution in the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \quad \text{for } 0 \le x \le a \tag{2}$$

$$u(x, a) = 0, \quad \text{for } 0 \le x \le a \tag{3}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \text{for } 0 < y < a \tag{4}$$

$$u(a, y) = ky, \quad \text{for } 0 < y < a \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Differentiating (6) partially with respect to y and then using boundary condition (2), we can get $D = 0$.

Using boundary conditions (3), in (6), we can get

$$p = \frac{(2n-1)\pi}{2a}, \text{ where } n = 1, 2, 3, \ldots\infty.$$

Differentiating (6) partially with respect to x and then using boundary condition (4), we can get $B = A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = AC\{e^{(2n-1)\pi x/2a} + e^{-(2n-1)\pi x/2a}\}\cos\frac{(2n-1)\pi y}{2a}$$

or $$u(x, y) = \lambda\cosh\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi y}{2a} \text{ where } n = 1, 2, 3, \ldots\infty.$$

Therefore the most general solution of Eq. 1 is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \cosh\frac{(2n-1)\pi x}{2a} \cos\frac{(2n-1)\pi y}{2a} \qquad (7)$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty}\left[\lambda_{2n-1} \cosh\frac{(2n-1)\pi}{2}\right]\cos\frac{(2n-1)\pi y}{2a} = ky \text{ in } (0, a)$$

$$\therefore \qquad \lambda_{2n-1} \cosh\frac{(2n-1)\pi}{2} = \frac{2}{a}\int_0^a ky \cos\frac{(2n-1)\pi y}{2a}\,dy$$

$$= \frac{2k}{a}\left[y\left\{\frac{\sin\frac{(2n-1)\pi y}{2a}}{\frac{(2n-1)\pi}{2a}}\right\} - \left\{\frac{-\cos\frac{(2n-1)\pi y}{2a}}{\frac{(2n-1)^2\pi^2}{4a}}\right\}\right]_0^a$$

$$= \frac{4ka}{\pi^2}\left\{\frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2}\right\}$$

$$\lambda_{2n-1} = \frac{4ka}{\pi^2}\left\{\frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2}\right\} \operatorname{sech}\frac{(2n-1)\pi}{2}$$

Using this value of λ_{2n-1} in (7), the required solution is

$$u(x, y) = \frac{4ka}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2}\right\} \operatorname{sech}\frac{(2n-1)\pi\, y}{2}$$

$$\cosh\frac{(2n-1)\pi x}{2a}\cdot\cos\frac{(2n-1)\pi y}{2a}$$

Exercise 5C(c)

Part A (Short-answer Questions)

1. State the two laws of thermodynamics used in the derivation of two dimensional heat flow equation.
2. Write down the partial differential equation that represents variable heat flow in two dimensions. Deduce the equation of steadystate heat flow in two dimensions.
3. Write down the three mathematically possible solutions of Laplace equation in two dimensions.

4. Given the boundary conditions on a square or a rectangular plate, how will you identify the proper solution?
5. Explain why $u(x, y) = (Ax + B)\,(Cy + D)$ cannot be the proper solution of Laplace equation in boundary value problems, by taking an example.

Part B

6. A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ and the short edge at infinity are kept at temperature 0°C, while the other short edge $y = 0$ is kept at temperature (i) $u = 2k \sin\frac{\pi x}{2}\cos\frac{3\pi x}{a}$, (ii) $u = kx$ and (iii) $u = \begin{cases} kx, & \text{for } 0 \le x \le \frac{a}{2} \\ k(a - x), & \text{for } \frac{a}{2} \le x \le a \end{cases}$
   Find the steadystate temperature at any point (x, y) of the plate.
7. An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0$ and $y = 10$ for positive values of x. The temperature is zero along the edges $y = 0$, $y = 10$ and the edge at infinity. If the edge $x = 0$ is kept at tempera true (i) $u = 4k \sin^3\frac{\pi y}{10}$, (ii) $u = T$ and (iii) $u = ky(10 - y)$, find the steadystate temperature at any point (x, y) of the plate.
8. A plate is in the form of the semi-infinite strip $0 \le x \le \pi$, $0 \le y \le \infty$. The edges $x = 0$ and $x = \pi$ are insulated. The edge at infinity is kept at temperature 0°C, while the edge $y = 0$ is kept at temperature $u = \begin{cases} x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases}$
   Find the steadystate temperature distribution in the plate.
9. The two long edges $y = 0$ and $y = l$ of a long rectangular plate are insulated. If the temperature in the short edge at infinity is kept at 0°C, while that in the short edge $x = 0$ is kept at $ky(l - y)$, find the steadystate temperature distribution in the plate.
10. A plate is in the form of the semi-infinite strip $0 \le x \le \infty$, $0 \le y \le l$. The surface of the plate and the edge $y = 0$ are insulated. If the temperature along the edge $y = l$ and the short edge at infinity are kept temperature 0°C, while the temperature along the other short edge is kept at temperature T°C, find the steadystate temperature distribution in the plate.
11. If the temperatures along the long edge $x = 0$ and the short edge at infinity of a long plate kept at 0°C, the other long edge $x = 10$ is insulated and the other short edge $y = 0$ is kept at temperature kx, find the steadystate temperature distribution in the plate.
12. Find the steadystate temperature distribution in a square plate of side a, which is insulated on the lateral surface and three of whose edges $x = a$,

$y = 0$, $y = a$ are kept at zero temperature, if the temperature in the edge $x = 0$ is (i) $k \sin^3 \dfrac{\pi y}{a}$ and (ii) $ky(a - y)$.

13. A rectangular plate of sides a and b has its faces insulated and its edges along $x = 0$, $x = a$, $y = 0$ and $y = b$. If the temperature along the edge $y = b$ is given by (i) $u = 3\sin\dfrac{4\pi x}{a} + 5\sin\dfrac{6\pi x}{a}$ and (ii) $u = x$ in $0 \le x \le \dfrac{a}{2}$ and $u = a - x$ in $\dfrac{a}{2} \le x \le a$, while the other three edges are kept at 0°C, find the steadystate temperature in the plate.

14. If a square plate is bounded by the lines $x = \pm\pi$ and $y = \pm\pi$ and three of its edge are kept at temperature 0°C, while the edge $x = \pi$ is kept at temperature $u = y + \pi$, $-\pi \le y \le \pi$, find the steadystate temperature in the plate.

15. Find the electrostatic potential in the rectangle, $0 \le \dot{x} \le 20, 0 \le y \le 40$, whose upper edge is kept at potential 110 volts and whose other edges are grounded.

Note ✍

The electrical force of attraction or repulsion between charged particles (governed by Coulomb's law) is the gradient of a function u, called electrostatic potential and at any point, free of charges u is a solution of Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. *Hence we have to solve the Laplace equation satisfying the given boundary conditions, to get the potential distribution in the rectangle.*

16. A square plate has its face and its edge $x = 0$ insulated. Its edges $y = 0$ and $y = a$ are kept at temperature zero, while its edges $x = a$ is kept at temperature T°C. Find the steadystate temperature distribution in the plate.

17. A rectangular plate of sides a and b has its faces and the edge $y = b$ insulated. Its edges $x = 0$ and $x = a$ are kept at temperature zero, while the edge $y = 0$ kept at temperature kx. Find the steadystate temperature distribution in the plate.

18. A square plate of side 20 cm has its faces and its edges $x = 0$ and $x = 20$ insulated. If the edge $y = 0$ is kept at temperature zero, while the edge $y = 20$ is kept at temperature $u = (10 - x)$, find the steadystate temperature distribution in the plate.

19. A square plate of side π has its faces and the edges $y = 0$ and $y = \pi$ insulated. If the edge $x = \pi$ is kept at temperature zero, while the edge $x = 0$ is kept at temperature $(2 \cos 3y + 3 \cos 4y)$, find the steadystate temperature distribution in the plate.

20. Find the steadystate temperature distribution on a rectangular plate of sides a and b, insulated along three of its sides $x = 0$, $x = a$ and $y = 0$ and with side $y = b$ kept at temperature kx, $0 \le x \le a$.

21. Find the steadystate temperature distribution in a square plate of side a, insulated along three of its sides $y = 0$, $y = a$ and $x = a$ and with the side $x = 0$ kept at temperature 0° for $0 < x < \frac{a}{2}$ and 100° for $\frac{a}{2} < x < a$.
22. A square metal plate of side 10 cm is bounded by the lines $x = 0$, $x = 10$, $y = 0$ and $y = 10$. The edges $y = 0$ and $x = 10$ are kept at zero temperature, the edge $x = 0$ is kept insulated and the edge $y = 10$ is kept at temperature T°C. Find the steadystate temperature distribution in the plate.
23. A square metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = a$. The edges $x = a$ and $y = 0$ are kept at zero temperature, while the edge $y = a$ is kept insulated. If the temperature along the edge $x = 0$ is ky, find the steadystate temperature distribution in the plate.
24. A rectangular metal plate of sides a and b has the edge $x = 0$ and $y = 0$ insulated. The edge $x = a$ is kept at a temperature of 0°C and the edge $y = b$ is kept at a temperature 100°C. Find the steadystate temperature distribution in the plate.
25. A square metal plate of side 10 cm has the edges $x = 10$ and $y = 10$ insulated. The edge $y = 0$is kept at temperature zero and the edge $x = 0$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.
26. If the faces of a thin square plate of side π are perfectly insulated, the edges are kept at zero temperature and the initial temperature at any point (x, y) of the plate is $u(x, y, 0) = f(x, y)$, show that the temperature in the plate at any subsequent time is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} \sin mx \sin ny e^{-\alpha^2 (m^2+n^2)t}$$

where $\lambda_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x, y) \sin mx \sin ny \, dxdy$.

Find the temperature in the plate at time t, if $f(x, y) = xy(\pi - x)(\pi - y)$.

[**Hint:** Solve $\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ by the method of separation of variables.

Proceed as in worked Example 19 of Chapter 3(A)]

Answers

Exercise 5C(c)

6. (i) $u(x, y) = -k \sin\frac{2\pi x}{a} e^{-2\pi y/a} + k \sin\frac{4\pi x}{a} e^{-4\pi y/a}$.

(ii) $u(x, y) = \frac{2ka}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{a} \cdot \exp(-n\pi y/a)$.

(iii) $u(x, y) = \dfrac{4ka}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \sin \dfrac{(2n-1)\pi x}{a} \exp\{-(2n-1)\pi y/a\}$.

7. (i) $u(x, y) = 3k \exp(-\pi x/10) \sin \dfrac{\pi y}{10} - k \exp\left(-\dfrac{3\pi x}{10}\right) \sin \dfrac{3\pi y}{10}$.

(ii) $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{2n-1} \exp\{-(2n-1)\pi x/10\} \sin\left\{\dfrac{(2n-1)\pi y}{10}\right\}$.

(iii) $u(x, y) = \dfrac{800\,k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \exp\{-(2n-1)\pi x/10\} \sin\left\{\dfrac{(2n-1)\pi y}{10}\right\}$

8. $u(x, y) = \dfrac{\pi}{4} - \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos\{2(2n-1)\} \exp\{-2(2n-1)y\}$.

9. $u(x, y) = \dfrac{kl^2}{6} - \dfrac{kl^2}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos \dfrac{2n\pi y}{l} \cdot \exp(-2n\pi x/l)$.

10. $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)} \exp\{-(2n-1)\pi x/2l\} \cos\{(2n-1)\pi y/2l\}$.

11. $u(x, y) = \dfrac{80k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}\pi x}{(2n-1)} \sin \dfrac{(2n-1)\pi x}{20} \exp\{-(2n-1)\pi y/20\}$.

12. (i) $\dfrac{3k}{4} \operatorname{cosech} \pi \cdot \sinh \dfrac{\pi(a-x)}{a} \cdot \sin \dfrac{\pi y}{a} - \dfrac{k}{4} \operatorname{cosech} 3\pi \cdot \sinh \cdot \dfrac{3\pi(a-x)}{a} \cdot \sin \dfrac{3\pi y}{a}$

(ii) $u(x, y) = \dfrac{8ka^2}{\pi^3}$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \operatorname{cosech} (2n-1)\pi \sinh \frac{(2n-1)\pi(a-x)}{a} \sin \frac{(2n-1)\pi y}{a}.$$

13. (i) $u(x, y) = 3 \operatorname{cosech} \dfrac{4\pi b}{a} \sin \dfrac{4\pi x}{a} \sinh \dfrac{4\pi y}{a} + 5 \operatorname{cosech} \dfrac{6\pi b}{a} \cdot \sin \dfrac{6\pi x}{a} \cdot$

$\sinh \dfrac{6\pi y}{a}$

(ii) $u(x, y) = \dfrac{4a}{\pi^2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \operatorname{cosech} \frac{(2n-1)\pi b}{a} \sin \frac{(2n-1)\pi x}{a} \sinh \frac{(2n-1)\pi y}{a}.$$

14. $u(x, y) = 4 \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \sinh \dfrac{n}{2}(\pi + x) \cdot \sin \dfrac{n}{2}(\pi + y)$.

15. $u(x, y) = \dfrac{440}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\text{cosech } 2(2n-1)\pi \cdot \sin\dfrac{(2n-1)\pi x}{20} \cdot \sinh\dfrac{(2n-1)\pi y}{20}$.

16. $u(x, y) = \dfrac{4T}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\text{sech}\,(2n-1)\pi \cosh\dfrac{(2n-1)\pi x}{a} \cdot \sin\dfrac{(2n-1)\pi y}{a}$.

17. $u(x, y) = \dfrac{2ka}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n=1}\text{sech}\dfrac{n\pi b}{a} \cdot \sinh\dfrac{n\pi x}{a} \cdot \cosh\dfrac{n\pi(b-y)}{a}$.

18. $u(x, y) = \dfrac{80}{\pi^2}\dfrac{1}{(2n-1)^2}\text{cosech } (2n-1)\pi \cdot \cos\dfrac{(2n-1)\pi x}{20}\sinh\dfrac{(2n-1)\pi y}{20}$.

19. $u(x, y) = 2 \text{ cosech } 3\pi \cdot \sinh 3(\pi - x) \cdot \cos 3y + 3 \text{ cosech } 4\pi \sinh 4(\pi - x) \cdot \cos 4y$

20. $u(x, y) = \dfrac{ka}{2} - \dfrac{4k}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\text{sech}\dfrac{(2n-1)\pi b}{a} \cdot \cos\dfrac{(2n-1)\pi x}{a}\cosh\dfrac{(2n-1)\pi y}{a}$

21. $u(x,y) = 50 + \dfrac{200}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\text{sech}(2n-1)\pi\cosh\dfrac{(2n-1)\pi(a-x)}{a}\cos\dfrac{(2n-1)\pi y}{a}$

22. $u(x, y) = \dfrac{4T}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)}\text{cosech}\dfrac{(2n-1)\pi}{2} \cdot \cos\dfrac{(2n-1)\pi x}{20} \cdot \sinh\dfrac{(2n-1)\pi y}{20}$.

23. $u(x,y) = \dfrac{8ka}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\text{ cosech}\dfrac{(2n-1)\pi}{2} \cdot \sinh\dfrac{(2n-1)\pi(a-x)}{2a}$.

$\sin\dfrac{(2n-1)\pi y}{2a}$.

24. $u(x, y) = \dfrac{400}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2n-1}\text{sech}\dfrac{(2n-1)\pi b}{2a} \cdot \cos\dfrac{(2n-1)\pi x}{2a}\cosh\dfrac{(2n-1)\pi y}{2a}$.

25. $u(x, y) = \dfrac{80k}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\text{sech}\dfrac{(2n-1)}{2}\cosh\dfrac{(2n-1)\pi(10-x)}{20}$.

$\sin\dfrac{(2n-1)\pi y}{20}$.

26. $u(x, y) = \dfrac{64}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{1}{(2m-1)^3(2n-1)^3}\sin(2m-1)x \cdot \sin(2n-1)y \cdot \exp$

$[-\{(2m-1)^2 + (2n-1)^2\}\alpha^2 t]$