As per the Latest Syllabus of JNTU Kakinada

Network Analysis

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Dedicated to Our

Parents

and

Students

Contents

Preface

This book is exclusively designed for the first-year engineering students of Jawaharlal Nehru Technological University, Kakinada studying the ‘Network Analysis’ course in their second semester. The primary goal of this text is to enable the student have a firm grasp over basic principles of Network Analysis, and develop an understanding of circuits and the ability to design practical circuits that perform the desired operations. Emphasis is placed on basic laws, theorems and techniques which are used to develop a working knowledge of the methods of analysis used most frequently in further topics of electrical engineering.

Each chapter begins with principles and theorems together with illustrative and other descriptive material. A large number of solved examples showing students the step-by-step processes for applying the techniques are presented in the text. Several questions in worked examples have been selected from university question papers.

As an aid to both the instructor and the student, objective questions and tutorial problems provided at the end of each chapter progress from simple to complex. Answers to selected problems are given to instil confidence in the reader. Due care is taken to see that the reader can easily start learning the concepts of Network Analysis without prior knowledge of mathematics.

SALIENT FEATURES

- 100% coverage of JNTU Kakinada latest syllabus
- Individual topics very well supported by solved examples
- Roadmap to the syllabus provided for systematic reading of the text
- University questions incorporated at appropriate places in the text
- Excellent pedagogy:
  - Solved Examples: 490
  - Practice Problems: 214
  - Objective Type Questions: 191
  - Illustrations: 915

The book is organized in 7 chapters. All the elements with definitions, basic laws and configurations of the resistive circuits, capacitive and inductive elements are introduced in **Chapter 1.** Kirchhoff’s laws, nodal and mesh analysis with only resistive elements are explained in this chapter. **Chapter 2** includes ac fundamentals with phasor representations. Network topology has been written in an easy-to-understand manner in this unit. Steady-state analysis has been discussed in **Chapter 3.** Complex impedance, mesh and nodal analysis for ac circuits, star-delta conversion are discussed in this chapter. Coupled circuits, resonance phenomenon in series and parallel circuits are presented in **Chapter 4.** Network theorems like Thevenin’s, Norton’s, Milliman’s, Reciprocity etc. are presented in **Chapter 5. Chapter 6** deals with two-port networks, various parameters and their relations. Transient analysis has been discussed with dc and ac excitations in **Chapter 7.** Solutions using Laplace Transforms method are also presented in this chapter.

Questions that have appeared in the University Examinations are included at appropriate places which will serve to enhance understanding and build the student's confidence.

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A Sudhakar
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Roadmap to the Syllabus

UNIT I Introduction to Electrical Circuits

Network elements classification; Electric charge and current; Electric energy and potential; Resistance parameter—series and parallel combinations; Inductance parameter—series and parallel combinations; Capacitance parameter—series and parallel combinations; Energy sources: ideal, non-ideal, independent and dependent sources; Source transformation; Kirchhoff's laws; Mesh analysis and nodal analysis; Problem solving with resistances only including dependent sources also.

AC Fundamentals: Definitions of terms associated with periodic functions: time period, angular velocity and frequency, rms value, average value, form factor and peak factor—problem solving; Phase angle; Phasor representation; Addition and subtraction of phasors; Mathematical representation of sinusoidal quantities; Explanation with relevant theory; Problem solving; Principal of duality with examples.

Network Topology: Definitions of branch, node, tree, planar, non-planar graph, incidence matrix, basic tie-set schedule, basic cut-set schedule.

GO TO
Chapter 1 ***Introduction to Electrical Circuits***
Chapter 2 ***AC Fundamentals and Network Topology***

UNIT II Steady-State Analysis of AC Circuits

Response to sinusoidal excitation—pure resistance, pure inductance, pure capacitance, impedance concept, phase angle, series R-L, R-C, R-L-C circuits, problem solving; Complex impedance and phasor notation for R-L, R-C, R-L-C problem solving using mesh and nodal analysis; Star-Delta conversion, problem solving.

GO TO
Chapter 3 ***Steady-State Analysis of AC Circuits***

UNIT III Coupled Circuits

Coupled Circuits: Self-inductance; Mutual inductance; Coefficient of coupling; Analysis of coupled circuits; Natural current; Dot rule of coupled circuits; Conductively coupled equivalent circuits—problem solving.

Resonance: Introduction; Definition of Q; Series resonance; Bandwidth of series resonance; Parallel resonance; Condition for maximum impedance; Current in anti-resonance; Bandwidth of parallel resonance; General case—resistance present in both branches, anti-resonance at all frequencies.

GO TO
Chapter 4 ***Coupled Circuits and Resonance***

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Thevenin's, Norton's, Millman's, Reciprocity, Compensation, Substitution, Superposition, Maximum Power Transfer, Tellegen's—problem solving using dependent sources also.

GO TO

Chapter 5 *Network Theorems*

UNIT V Two-port Networks

Relationships of two-port networks; Z-parameters; Y-parameters; Transmission-line parameters; h-parameters; Inverse h-parameters; Inverse transmission-line parameters; Relationships between parameter sets; Parallel connection of two-port networks; Cascading of two-port networks; Series connection of two-port networks, Problem solving including dependent sources also.

GO TO

Chapter 6 *Two-port Networks*

UNIT VI Transients

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GO TO

Chapter 7 *Transients*

1

Introduction to Electrical Circuits

1.1 NETWORK ELEMENTS CLASSIFICATION *[JNTU Nov 2011]*

Simply an electric circuit consists of three parts: (1) energy source, such as battery or generator, (2) the load or sink, such as lamp or motor, and (3) connecting wires as shown in Fig. 1.1. This arrangement represents a simple circuit. A battery is connected to a lamp with two wires. The purpose of the circuit is to transfer energy from source (battery) to the load (lamp). And this is accomplished by the passage of electrons through wires around the circuit.

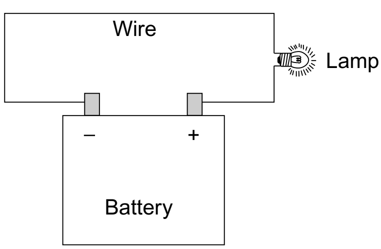


Fig. 1.1

The current flows through the filament of the lamp, causing it to emit visible light. The current flows through the battery by chemical action. A closed circuit is defined as a circuit in which the current has a complete path to flow. When the current path is broken so that current cannot flow, the circuit is called an open circuit.

More specifically, interconnection of two or more simple circuit elements (viz. voltage sources, resistors, inductors and capacitors) is called an electric network. If a network contains at least one closed path, it is called an electric circuit. By definition, a simple circuit element is the mathematical model of two terminal electrical devices, and it can be completely characterised by its voltage and current. Evidently then, a physical circuit must provide means for the transfer of energy.

Broadly, network elements may be classified into four groups, viz.

1. Active or passive
2. Unilateral or bilateral
3. Linear or nonlinear
4. Lumped or distributed

1.1.1 Active and Passive *[JNTU Jan 2010]*

Energy sources (voltage or current sources) are active elements, capable of delivering power to some external device. Passive elements are those which are capable only of receiving power. Some passive elements like inductors and capacitors are capable of storing a finite amount of energy, and return it later to an external element. More specifically, an active element is capable of delivering an average power greater than zero to some external device over an infinite time interval. For example, ideal sources are active elements. A passive element is defined as one that cannot supply average power that is greater than zero over an infinite time interval. Resistors, capacitors and inductors fall into this category.

1.1.2 Bilateral and Unilateral

In the bilateral element, the voltage-current relation is the same for current flowing in either direction. In contrast, a unilateral element has different relations between voltage and current for the two possible directions of current. Examples of bilateral elements are elements made of high conductivity materials in general. Vacuum diodes, silicon diodes, and metal rectifiers are examples of unilateral elements.

1.1.3 Linear and Nonlinear Elements

An element is said to be linear, if its voltage-current characteristic is at all times a straight line through the origin. For example, the current passing through a resistor is proportional to the voltage applied through it, and the relation is expressed as $V \propto I$ or $V = IR$. A linear element or network is one which satisfies the principle of superposition, i.e. the principle of homogeneity and additivity. An element which does not satisfy the above principle is called a nonlinear element.

1.1.4 Lumped and Distributed

Lumped elements are those elements which are very small in size and in which simultaneous actions takes place for any given cause at the same instant of time. Typical lumped elements are capacitors, resistors, inductors and transformers. Generally the elements are considered as lumped when their size is very small compared to the wave length of the applied signal. Distributed elements, on the other hand, are those which are not electrically separable for analytical purposes. For example, a transmission line which has distributed resistance, inductance and capacitance along its length may extend for hundreds of miles.

1.2 ELECTRIC CHARGE AND CURRENT

There are free electrons available in all semiconductive and conductive materials. These free electrons move at random in all directions within the structure in the absence of external pressure or voltage. If a certain amount of voltage is applied across the material, all the free electrons move in one direction depending on the polarity of the applied voltage, as shown in Fig. 1.2.

This movement of electrons from one end of the material to the other end constitutes an electric current, denoted by either I or i. The conventional direction

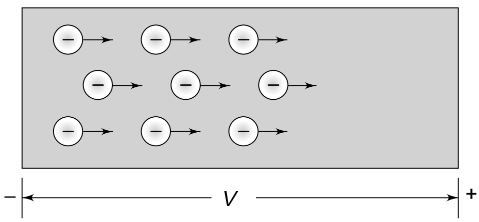


Fig. 1.2

of current flow is opposite to the flow of –ve charges, i.e. the electrons.

Current is defined as the rate of flow of electrons in a conductive or semiconductive material. It is measured by the number of electrons that flow past a point in unit time. Expressed mathematically,

$$I = \frac{Q}{t}$$

where I is the current, Q is the charge of electrons, and t is the time, or

$$i = \frac{dq}{dt}$$

where dq is the small change in charge, and dt is the small change in time.

In practice, the unit *ampere* is used to measure current, denoted by A. One ampere is equal to one coulomb per second. One coulomb is the charge carried by 6.25×10^{18} electrons. For example, an ordinary 80 W domestic ceiling fan on 230 V supply takes a current of approximately 0.35 A. This means that electricity is passing through the fan at the rate of 0.35 coulomb every second, i.e. 2.187×10^{18} electrons are passing through the fan in every second; or simply, the current is 0.35 A.

Example 1.1 *Five coulombs of charge flow past a given point in a wire in 2 s. How many amperes of current is flowing?*

Solution

$$I = \frac{Q}{t} = \frac{5}{2} = 2.5\,\text{A}$$

1.3 ELECTRIC ENERGY AND POTENTIAL

According to the structure of an atom, we know that there are two types of charges: positive and negative. A force of attraction exists between these positive and negative charges. A certain amount of energy (work) is required to overcome the force and move the charges through a specific distance. All opposite charges possess a certain amount of potential energy because of the separation between them. The difference in potential energy of the charges is called the *potential difference.*

Potential difference in electrical terminology is known as voltage, and is denoted either by V or υ. It is expressed in terms of energy (W) per unit charge (Q); i.e.

$$V = \frac{W}{Q} \quad \text{or} \quad v = \frac{dw}{dq}$$

dw is the small change in energy, and
dq is the small change in charge.

where energy (W) is expressed in joules (J), charge (Q) in coulombs (C), and voltage (V) in volts (V). One volt is the potential difference between two points when one joule of energy is used to pass one coulomb of charge from one point to the other.

Example 1.2 *If 70 J of energy is available for every 30 C of charge, what is the voltage?*

Solution $$V = \frac{W}{Q} = \frac{70}{30} = 2.33\,\text{V}$$

Example 1.3 *A resistor with a current of 3 A through it converts 500 J of electrical energy to heat energy in 12 s. What is the voltage across the resistor?*

Solution $$V = \frac{W}{Q}$$

$$Q = I \times t$$
$$= 3 \times 12 = 36\ \text{C}$$

$$V = \frac{500}{36} = 13.88\,\text{V}$$

1.4 POWER AND ENERGY

Energy is the capacity for doing work, i.e. energy is nothing but stored work. Energy may exist in many forms such as mechanical, chemical, electrical and so on. Power is the rate of change of energy, and is denoted by either P or p. If certain amount of energy is used over a certain length of time, then

$$\text{Power } (P) = \frac{\text{energy}}{\text{time}} = \frac{W}{t}, \text{ or}$$
$$p = \frac{dw}{dt}$$

where dw is the change in energy and dt is the change in time.

We can also write $$p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt}$$
$$= v \times i = vi \quad \text{W}$$

Energy is measured in joules (J), time in seconds (s), and power in watts (W).

By definition, one watt is the amount of power generated when one joule of energy is consumed in one second. Thus, the number of joules consumed in one second is always equal to the number of watts. Amounts of power less than one watt are usually expressed in fraction of watts in the field of electronics; viz.

milliwatts (mW) and microwatts (μW). In the electrical field, kilowatts (kW) and megawatts (MW) are common units. Radio and television stations also use large amounts of power to transmit signals.

Example 1.4 *What is the power in watts if energy equal to 50 J is used in 2.5 s?*

Solution

$$P = \frac{\text{energy}}{\text{time}} = \frac{50}{2.5} = 20 \text{ W}$$

Example 1.5 *A 5 Ω resistor has a voltage rating of 100 V. What is its power rating?*

Solution

$$P = VI$$

$$I = V/R$$

$$P = \frac{V^2}{R} = \frac{(100)^2}{5} = 2000 \text{ W} = 2 \text{ kW}$$

1.5 RESISTANCE PARAMETER

When a current flows in a material, the free electrons move through the material and collide with other atoms. These collisions cause the electrons to lose some of their energy. This loss of energy per unit charge is the drop in potential across the material. The amount of energy lost by the electrons is related to the physical property of the material. These collisions restrict the movement of electrons. The property of a material to restrict the flow of electrons is called resistance, denoted by R. The symbol for the resistor is shown in Fig. 1.3.

Fig. 1.3

The unit of resistance is ohm (Ω). Ohm is defined as the resistance offered by the material when a current of one ampere flows between two terminals with one volt applied across it.

According to Ohm's law, the current is directly proportional to the voltage and inversely proportional to the total resistance of the circuit, i.e.

$$I = \frac{V}{R}$$

$$\text{or} \quad i = \frac{v}{R}$$

We can write the above equation in terms of charge as follows.

$$V = R\frac{dq}{dt}, \quad \text{or} \quad i = \frac{v}{R} = Gv$$

where G is the conductance of a conductor. The units of resistance and conductance are ohm (Ω) and mho ($\mho$) respectively.

When current flows through any resistive material, heat is generated by the collision of electrons with other atomic particles. The power absorbed by the resistor is converted to heat. The power absorbed by the resistor is given by

$$P = vi = (iR)\, i = i^2 R$$

where i is the current in the resistor in amps, and v is the voltage across the resistor in volts. Energy lost in a resistance in time t is given by

$$W = \int_0^t p\,dt = pt = i^2 Rt = \frac{v^2}{R} t$$

where v is the volts
R is in ohms
t is in seconds and
W is in joules

Example 1.6 *A 10 Ω resistor is connected across a 12 V battery. How much current flows through the resistor?*

Solution $V = IR$

$$I = \frac{V}{R} = \frac{12}{10} = 1.2 \text{ A}$$

1.6 INDUCTANCE PARAMETER

[JNTU June 2009 and May/June 2008]

A wire of certain length, when twisted into a coil becomes a basic inductor. If current is made to pass through an inductor, an electromagnetic field is formed. A change in the magnitude of the current changes the electromagnetic field. Increase in current expands the fields, and decrease in current reduces it. Therefore, a change in current produces change in the electromagnetic field, which induces a voltage across the coil according to Faraday's law of electromagnetic induction.

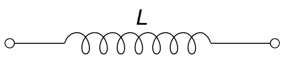


Fig. 1.4

The unit of inductance is *henry*, denoted by H. By definition, the inductance is one henry when current through the coil, changing at the rate of one ampere per second, induces one volt across the coil. The symbol for inductance is shown in Fig. 1.4.

The current-voltage relation is given by

$$v = L\frac{di}{dt}$$

where v is the voltage across inductor in volts, and i is the current through inductor in amps. We can rewrite the above equation as

$$di = \frac{1}{L}\, v\, dt$$

Integrating both sides, we get

$$\int_0^t di = \frac{1}{L}\int_0^t vdt$$

$$i(t) - i(0) = \frac{1}{L}\int_0^t vdt$$

$$i(t) = \frac{1}{L}\int_0^t vdt + i(0)$$

From the above equation we note that the current in an inductor is dependent upon the integral of the voltage across its terminals and the initial current in the coil, $i(0)$.

The power absorbed by inductor is

$$P = vi = Li\frac{di}{dt} \text{ watts}$$

The energy stored by the inductor is

$$W = \int_0^t pdt$$

$$= \int_0^t Li\frac{di}{dt}dt = \frac{Li^2}{2}$$

From the above discussion, we can conclude the following.

1. The induced voltage across an inductor is zero if the current through it is constant. That means an inductor acts as short circuit to dc.
2. A small change in current within zero time through an inductor gives an infinite voltage across the inductor, which is physically impossible. In a fixed inductor the current cannot change abruptly.
3. The inductor can store finite amount of energy, even if the voltage across the inductor is zero, and
4. A pure inductor never dissipates energy, only stores it. That is why it is also called a non-dissipative passive element. However, physical inductors dissipate power due to internal resistance.

Example 1.7 *The current in a 2 H inductor varies at a rate of 2 A/s. Find the voltage across the inductor and the energy stored in the magnetic field after 2 s.*

Solution

$$v = L\frac{di}{dt}$$
$$= 2 \times 4 = 8 \text{ V}$$
$$W = \frac{1}{2}Li^2$$
$$= \frac{1}{2} \times 2 \times (4)^2 = 16 \text{ J}$$

Example 1.8 *Find the inductance of a coil through which flows a current of 0.2 A with an energy of 0.15 J.*

Solution

$$W = \frac{1}{2}LI^2$$
$$L = \frac{2 \times W}{I^2} = \frac{2 \times 0.15}{(0.2)^2} = 7.5 \text{ H}$$

Example 1.9 *Find the inductance of a coil in which a current increases linearly from 0 to 0.2 A in 0.3 s, producing a voltage of 15 V.*

Solution

$$v = L\frac{di}{dt}$$

Current in 1 second $= \dfrac{0.2}{0.3} = 0.66 \text{ A}$

The current changes at a rate of 0.66 A/s,

$$\therefore \quad L = \frac{v}{\left(\dfrac{di}{dt}\right)}$$
$$L = \frac{15 \text{ V}}{0.66 \text{ A/s}} = 22.73 \text{ H}$$

Example 1.10 *A current of 1 A is supplied by a source to an inductor of 1 H. Calculate the energy stored in the inductor. What happens to this energy if the source is short circuited?*

Solution

$$\text{Energy stored} \frac{1}{2}LI^2 = \frac{1}{2}1 \times 1^2 = 0.5 \text{ Joules}$$

If the inductor has an internal resistance, the stored energy is dissipated in the resistance after the short circuit as per the time constant (1/r) of the coil.

If the coil is a pertect inductor. The current would circulate through the shorted coil continuously.

Example 1.11 *Derive the expression for the energy stored in an ideal inductor?*

Solution *Expression for Energy Stored in an ideal inductor*

Let 'L' be the co-efficient of self inductance and i be the current flowing through it.

Let 'dw' be the small amount of work to be expended to overcome self-induced emf.

$$\therefore \quad dw = Ei\,dt$$

$$dw = L\frac{di}{dt}idt \quad \underset{\text{from lenz law}}{\left[\because E = L\frac{di}{dt}\right]}$$

$$dw = Li\,di \tag{1}$$

Hence, total work to be done in establishing a maximum current i_0 is given by integrating (1) from 0 to i_0.

$$\therefore \quad w = \int_0^{i_0} dw = \int_0^{i_0} Li\,di = L\int_0^{i_0} i\,di$$

$$= L\left[\frac{1}{2}\frac{i_0^2}{1}\right]$$

$$w = \frac{1}{2}Li_0^2$$

$\therefore$ Energy stored in an inductor $w = \dfrac{1}{2}Li_0^2$

1.6.1 Inductance in Series

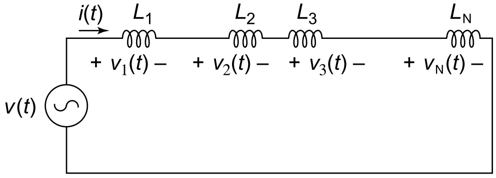


Fig. 1.5

Consider a voltage source is applied to the series combination of N inductors as shown in Fig. 1.5.

In the circuit, the current passing through each inductive element is same. Also, the source voltage applied to the circuit $v(t)$ is equal to the sum of the individual voltages.

$$\text{ie} \quad v(t) = v_1(t) + v_2(t) + v_3(t) + \ldots\ldots\ldots + v_N(t)$$

$$v(t) = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \ldots + L_N \frac{di}{dt}$$
$$= (L_1 + L_2 + L_3 + \ldots + L_N)\frac{di}{dt}$$
$$v(t) = L_{eq} \frac{di}{dt}$$

Therefore, the equivalent inductance is

$$L_{eq} = L_1 + L_2 + L_3 + \ldots\ldots\ldots\ldots + L_N$$

The equivalent inductance of any number of inductors connected in series is the sum of the individual inductances.

Example 1.12 *Three inductances are connected as shown in Fig. 1.6. What is the equivalent inductance?*

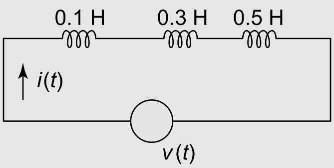


Fig. 1.6

Solution Since the current passing through each inductance is same, the three inductances are connected in series.

The equivalent inductance $L_{eq} = (0.1 + 0.3 + 0.5)$ H

$$L_{eq} = 0.9 \text{ H}$$

1.6.2 Inductance in Parallel

Consider the circuit shown in Fig. 1.7. The current source $i(t)$ is applied to circuit. Assume a voltage $v(t)$ exists across the parallel combination and let the currents in $L_1, L_2 \ldots\ldots, L_N$ be $i_1(t), i_2(t) \ldots\ldots, i_N(t)$ respectively.

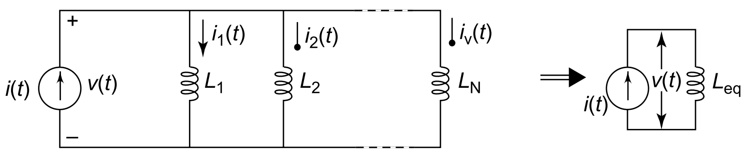


Fig. 1.7

Since the total current i_T is the sum of the branch currents.

$$i(t) = i_1(t) + i_2(t) + \ldots\ldots\ldots + i_N(t)$$

or

$$\frac{1}{L_{eq}}\int v(t)dt = \frac{1}{L_1}\int v(t)dt + \frac{1}{L_2}\int v(t)dt + \ldots\ldots + \frac{1}{L_N}\int v(t)dt$$

$$\therefore \quad \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots\ldots\ldots\ldots + \frac{1}{L_N}$$

Therefore, the reciprocal of the equivalent inductance of any number of inductors in parallel is the sum of the reciprocals of the individual inductances.

Example 1.13 *There pure inductances are connected as shown in Fig. 1.8. What equivalent inductance L_{eq} may replace this circuit?*

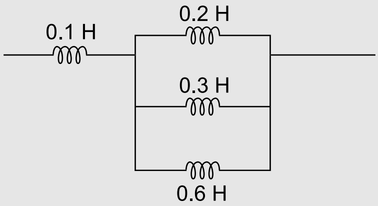


Fig. 1.8

Solution Equivalent inductance of parallel combination is

$$\frac{1}{L_{\text{eqp}}} = \frac{1}{0.2} + \frac{1}{0.3} + \frac{1}{0.6} = 10$$

$$L_{\text{eqp}} = 0.1 \text{ H}$$

The required equivalent inductance

$$L_{\text{eq}} = 0.1 \text{ H} + L_{\text{eqp}}$$

$$L_{\text{eq}} = 0.2 \text{ H}$$

1.7 CAPACITANCE PARAMETER

Any two conducting surfaces separated by an insulating medium exhibit the property of a capacitor. The conducting surfaces are called *electrodes*, and the insulating medium is called *dielectric*. A capacitor stores energy in the form of an electric field that is established by the opposite charges on the two electrodes. The electric field is represented by lines of force between the positive and negative charges, and is concentrated within the dielectric. The amount of charge per unit voltage that is capacitor can store is its capacitance, denoted by C. The unit of capacitance is *Farad* denoted by F. By definition, one Farad is the amount of capacitance when one coulomb of charge is stored with one volt across the plates. The symbol for capacitance is shown in Fig. 1.9.

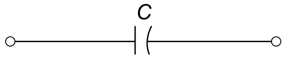


Fig. 1.9

A capacitor is said to have greater capacitance if it can store more charge per unit voltage and the capacitance is given by

$$C = \frac{Q}{V}, \quad \text{or} \quad C = \frac{q}{v}$$

(lowercase letters stress instantaneous values)

We can write the above equation in terms of current as

$$i = C\frac{dv}{dt} \qquad \left(\because\ i = \frac{dq}{dt}\right)$$

where υ is the voltage across capacitor, i is the current through it

$$dv = \frac{1}{C} i dt$$

Integrating both sides, we have

$$\int_0^t dv = \frac{1}{C}\int_0^t i dt$$

$$v(t) - v(0) = \frac{1}{C}\int_0^t i dt$$

$$v(t) = \frac{1}{C}\int_0^t i dt + v(0)$$

where $v(0)$ indicates the initial voltage across the capacitor.

From the above equation, the voltage in a capacitor is dependent upon the integral of the current through it, and the initial voltage across it.

The power absorbed by the capacitor is given by

$$p = vi = vC\frac{dv}{dt}$$

The energy stored by the capacitor is

$$W = \int_0^t p dt = \int_0^t vC\frac{dv}{dt} dt$$

$$W = \frac{1}{2} Cv^2$$

From the above discussion we can conclude the following:

1. The current in a capacitor is zero if the voltage across it is constant; that means, the capacitor acts as an open circuit to dc.
2. A small change in voltage across a capacitance within zero time gives an infinite current through the capacitor, which is physically impossible. In a fixed capacitance the voltage cannot change abruptly.
3. The capacitor can store a finite amount of energy, even if the current through it is zero, and
4. A pure capacitor never dissipates energy, but only stores it; that is why it is called *non-dissipative passive element*. However, physical capacitors dissipate power due to internal resistance.

Example 1.14 *A capacitor having a capacitance 2 μF is charged to a voltage of 1000 V. Calculate the stored energy in joules.*

Solution $W = \frac{1}{2}Cv^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (1000)^2 = 1$ J.

Example 1.15 *When a dc voltage is applied to a capacitor, the voltage across its terminals is found to build up in accordance with $V_C = 50(1 - e^{-100t})$. After a lapse of 0.01 s, the current flow is equal to 2 mA.*

(a) Find the value of capacitance in microfarads.

(b) How much energy is stored in the electric field at this time?

Solution

(a) $i = C\frac{dv_C}{dt}$

where $v_C = 50(1 - e^{-100t})$

$$i = C\frac{d}{dt}50\left(1 - e^{-100t}\right)$$

$$= C \times 50 \times 100e^{-100t}$$

At $t = 0.01$ s, $i = 2$ mA

$$C = \frac{2 \times 10^{-3}}{50 \times 100 \times e^{-100 \times 0.01}} = 1.089\ \mu F$$

(b) $W = \frac{1}{2}Cv_C^2$

At $t = 0.01$ s, $v_C = 50\,(1 - e^{-100 \times 0.01}) = 31.6$ V

$$W = \frac{1}{2} \times 1.089 \times 10^{-6} \times (31.6)^2$$

$$= 0.000543 \text{ J}$$

1.7.1 Capacitance in Series

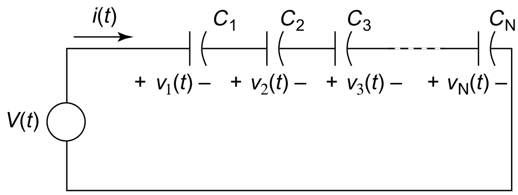


Fig. 1.10

Consider a circuit consists of N capacitors in series as shown in Fig. 1.10.

In the circuit, the total voltage applied to the circuit is equal to sum of the voltages across individual capacitive elements.

$$\therefore \quad v(t) = v_1(t) + v_2(t) + v_3(t) + \dots\dots\dots + v_N(t)$$

Assuming zero initial voltage across each capacitor,

$$v(t) = \frac{1}{C_1}\int i(t)dt + \frac{1}{C_2}\int i(t)dt + \frac{1}{C_3}\int i(t)dt + \dots\dots\dots + \frac{1}{C_N}\int i(t)dt$$

where $\quad v(t) = \dfrac{1}{C_{eq}}\displaystyle\int i(t)dt$

$$\therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots\dots + \frac{1}{C_N}$$

The reciprocal of the equivalent capacitance of any number of capacitors connected in series is the sum of the reciprocals of the individual capacitances.

Example 1.16 *The two capacitors shown in Fig. 1.11 are connected in series. Find the equivalent capacitance of the circuit.*

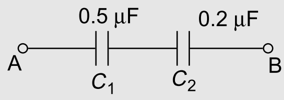


Fig. 1.11

Solution The equivalent capacitance of the circuit shown in Fig. 1.11 is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{0.5\times10^{-6}\times0.2\times10^{-6}}{0.7\times10^{-6}}$$

$$\therefore \quad C_{eq} = 0.143\ \mu\text{F}.$$

1.7.2 Capacitance in Parallel

Consider the circuit shown in Fig. 1.12 consists of N parallel capacitors. A current source is applied to the circuit. The total current applied to the circuit is the sum of the individual currents flowing in the circuit.

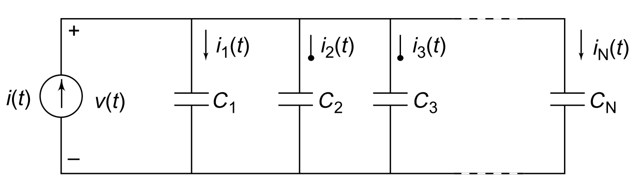


Fig. 1.12

$$i(t) = i_1(t) + i_2(t) + i_3(t) + \dots\dots\dots + i_N(t)$$

$$C_{eq}\frac{dv(t)}{dt} = C_1\frac{dv(t)}{dt} + C_2\frac{dv(t)}{dt} + C_3\frac{dv(t)}{dt} + \dots + C_N\frac{dv(t)}{dt}$$

From the above equation, we get

$$C_{eq} = C_1 + C_2 + C_3 + \ldots\ldots\ldots + C_N$$

The resultant capacitance of any number of capacitors in parallel is the sum of the individual capacitances.

Example 1.17 *Find the equivalent capacitance C_e of the combination of capacitors shown in Fig. 1.13.*

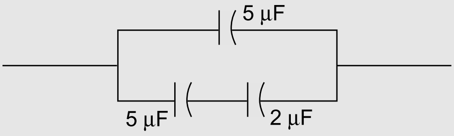


Fig. 1.13

Solution Equivalent capacitance of series branch is

$$C_s = \frac{C_1 C_2}{C_1 + C_2} = \frac{5 \times 2 \times 10^{-6}}{5+2} = 1.43\ \mu\text{F}$$

The required equivalent capacitance is

$$C_{eq} = C_s + 5\ \mu\text{F}$$

$$C_{eq} = 1.43 + 5 = 6.43\ \mu\text{F}$$

Example 1.18 *Find the total equivalent capacitance, total energy stored if the applied voltage is 100 V for the circuit as shown in Fig. 1.14. [JNTU Jan 2010]*

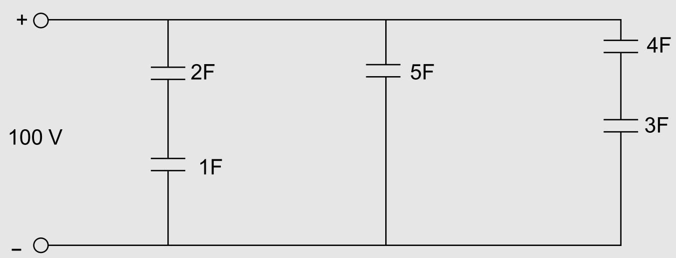


Fig. 1.14

Solution 4F and 3F in series

$$C_{eq} = \frac{4 \times 3}{4+3} = \frac{12}{4}\ \text{F}$$

$\frac{12}{7}$ F in parallel with 5 F

$$\therefore \quad C_{eq} = \frac{12}{7} + 5 = \frac{35+12}{7} = \frac{47}{7}\ \text{F}$$

$$\therefore \quad C_{eq} = 2\ \text{F and } 1\ \text{F in series}$$

$$C_{eq} = \frac{2 \times 1}{2+1} = \frac{2}{3} \text{F}$$

$\therefore \quad \frac{2}{3}$ F in parallel with $\frac{47}{7}$ F

$$\therefore \quad C_{eq} = \frac{2}{3} + \frac{47}{7} = \frac{14+141}{21} = \frac{155}{21} \text{F}$$

$$\therefore \quad E = \frac{1}{2} CV^2$$

$$\therefore \quad E = \frac{1}{2} \times \frac{155}{21} \times 100 \times 100$$

$$E = 36900 \text{ J}$$

1.7.3 *V–I* Relation of Passive Elements for Different Input Signals

In this section, we discuss about the voltage current relationship of passive elements for different input signals. Table 1.1 shows the voltage current relations of three circuit elements resistor R, inductor L and capacitor C.

Table 1.1 *V–I relation of circuit elements*

Circuit element	*Voltage (V)*	*Current (A)*	*Power (W)*
Resistor R (Ohms Ω)	$v = Ri$	$i = \frac{v}{R}$	$P = i^2 R$
Inductor L (Henry H)	$v = L\frac{di}{dt}$	$i = \frac{1}{2}\int v dt + i_0$ where i_0 is the initial current in inductor	$P = L\frac{di}{dt}$
Capacitor C (Farad F)	$v = \frac{1}{C}\int i dt + v_0$ where v_0 is the initial voltage across capacitor	$i = c\frac{dv}{dt}$	$P = cv\frac{dv}{dt}$

Resistive Element

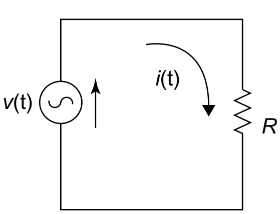


Fig. 1.15

Consider the voltage function is applied to a resistor R as shown in Fig. 1.15. The current $i(t)$ is flowing through the circuit.

The relation between $v(t)$ and $i(t)$ is

$$v(t) = R\, i(t)$$

Now, let us determine the relation between voltage and current for various input signals through following examples.

Example 1.19 *The voltage function v(t) is a repeating square wave shown in Fig. 1.16, applied to a pure resistor of 10 ohms. Plot current i(t) and power p(t).*

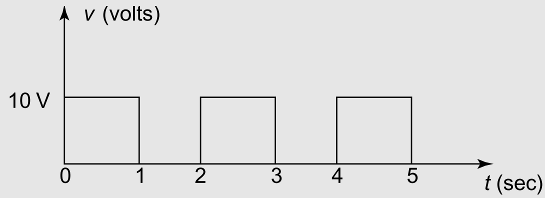


Fig. 1.16

Solution Since $v(t) = R\ i(t)$, the voltage varies directly as the current. The maximum value of current is

$$i_{\max} = \frac{v_{\max}}{R} = \frac{10}{10} = 1\text{ A}$$

Since power $P = vi$, the maximum value of power is

$$P_{\max} = v_{\max}\, i_{\max} = 10(1) = 10\text{ W}$$

The resultant current and power waveforms are shown in Figs 1.17(a) and (b) respectively.

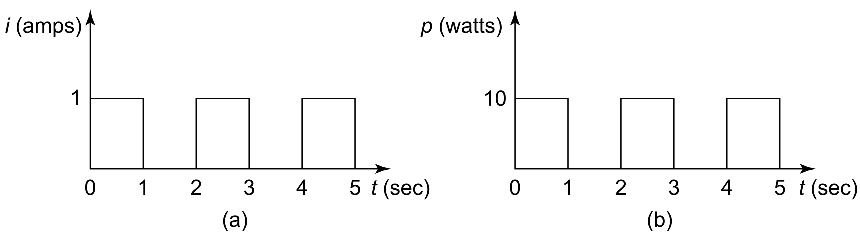


Fig. 1.17

Example 1.20 *A single pure resistance of 20 ohms passes a current of the waveform shown in Fig. 1.18. Determine and sketch the voltage V(t) and the instantaneous power p(t).*

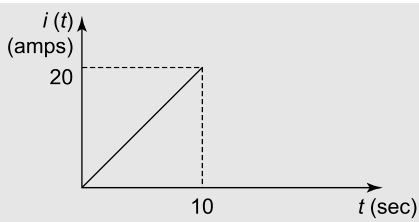


Fig. 1.18

Solution From Fig. 1.18, the instantaneous current $i(t)$ is given by $i(t) = 2t$ amperes

The corresponding voltage is

$$v(t) = R\, i(t)$$

$$= 20 \times 2t = 40\,t \text{ volts.}$$

The corresponding instantaneous power is

$$p(t) = v(t)\,i(t)$$

$$= 40\,t \times 2\,t = 80\,t^2 \text{ watts.}$$

The resultant voltage and power waveforms are shown in Figs 1.19 (a) and (b) respectively.

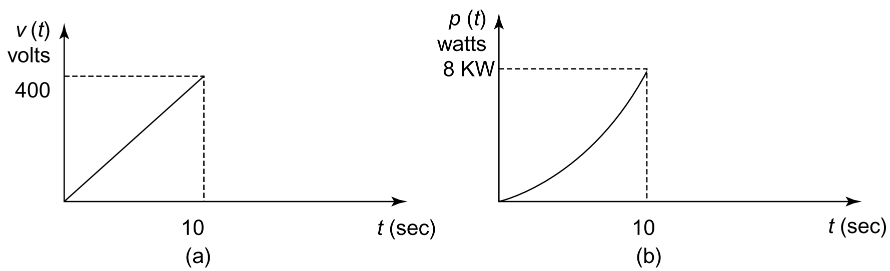


Fig. 1.19

Example 1.21 *The current function shown in Fig. 1.20 is a repeating sawtooth and exists in a pure resister of 8Ω. Find the voltage v(t) and instantaneous power p(t).*

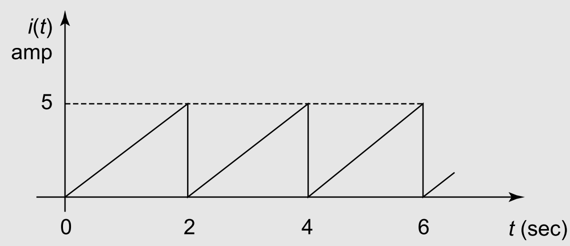


Fig. 1.20

Solution Since $v(t) = R\,i(t)$

$$v_{\text{max}} = R\,i_{\text{max}} = (8)(5) = 40 \text{ V}$$

when $0 < t < 2\text{s}$, $i(t) = \dfrac{5}{2}t = 2.5\,t$ amperes

Then, voltage $v(t) = R\,i(t)$

$$= 8(2.5\,t) = 20\,t \text{ volts}$$

Instantaneous power $p(t) = v(t)\,i(t)$

$$= 20\,t \times 2.5\,t$$

$$= 50\,t^2 \text{ watts.}$$

Therefore, the voltage $v(t)$ and power $p(t)$ waveforms are shown in Figs 1.21(a) and (b) respectively.

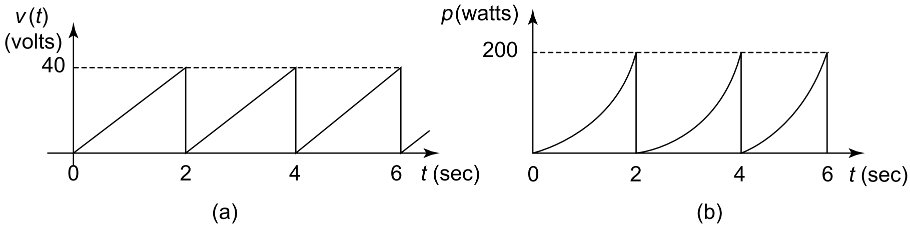


Fig. 1.21

Example 1.22 *The voltage waveform shown in Fig. 1.22 is applied to a pure resistor of 10 Ω. Sketch the current waveform and instantaneous power.*

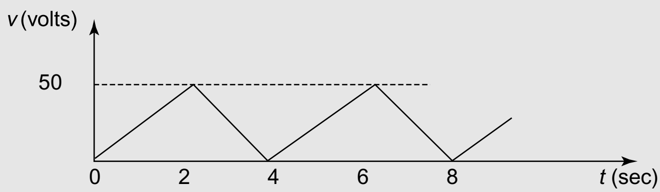


Fig. 1.22

Solution Since $v(t) = R\,i(t)$,

$$i_{max} = \frac{v_{max}}{R} = \frac{50}{10} = 5 \text{ amperes.}$$

when $0 \le t \le 2\text{s}$, $v = 25\,t$ volts,

$$\text{then} \quad i = \frac{25t}{10} = 2.5\,t \text{ amps.}$$

when $2\text{s} \le t \le 4\text{s}$, $v = -25\,t$ volts

$$\text{then} \quad i = \frac{-25t}{10} = -2.5\,t \text{ amps}$$

The instantaneous power $p(t) = v(t)\,i(t)$

when $0 < t < 2\text{s}$, $p = vi$

$$= 25\,t \times 2.5\,t$$
$$= 62.5\,t^2 \text{ watts}$$

when $2\text{s} < t < 4\text{s}$, $p = vi$

$$= -25\,t \times -2.5\,t$$
$$= 62.5\,t^2 \text{ watts}$$

Therefore, the current and power waveforms are shown in Figs 1.23(a) and (b) respectively.

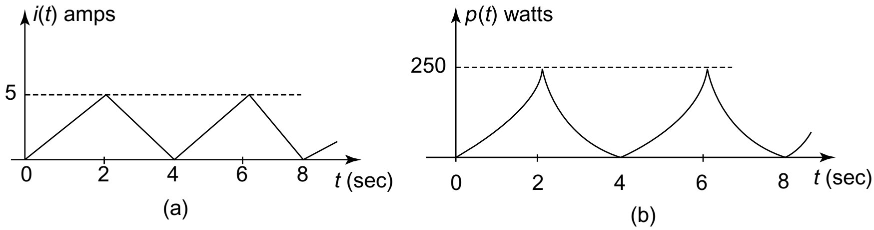


Fig. 1.23

Capacitive Element

Consider a capacitive element shown in Fig. 1.24.
The capacitance c is given by the voltage–current relationship

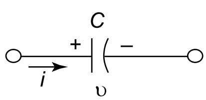


Fig. 1.24

$$i = c\frac{dv}{dt}$$

or

$$v = \frac{1}{c}\int i dt + v_0$$

where v_0 is the initial voltage across the capacitor.

Now let us determine the response of pure capacitor for various input waveforms through following examples.

Example 1.23 *The current waveform shown in Fig. 1.25 is applied to capacitor of 5 μF. Find the voltage across the capacitor.*

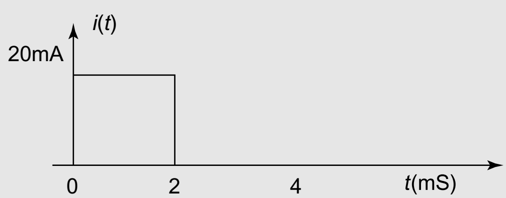


Fig. 1.25

Solution Assume initial voltage across the capacitor is zero.

The voltage across the capacitor is

$$v(t) = \frac{1}{c}\int i(t)\, dt + v_0$$

$$i(t) = 20 \text{ mA}, 0 \le t \le 2\,\text{ms}$$

$$= 0, t > 2 \text{ ms}$$

$$\therefore v(t) = \frac{1}{5\times 10^{-6}}\int 20\times 10^{-3}\, dt + v_0$$

since $v_0 = 0$

$$v(t) = 4000\,t \quad 0 \le t \le 2\,\text{ms}$$

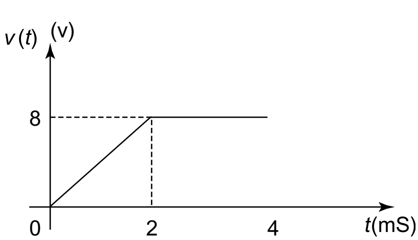


Fig. 1.26

At $t \geq 2\,\text{ms}$, the voltage across the capacitor

$$v(t) = 8 \text{ volts.}$$

The resultant waveform is shown in Fig. 1.26.

Example 1.24 *A current of the waveform shown in Fig. 1.27 flows through a capacitance $C = 100\ \mu F$. Sketch the voltage waveform and determine V_m.*

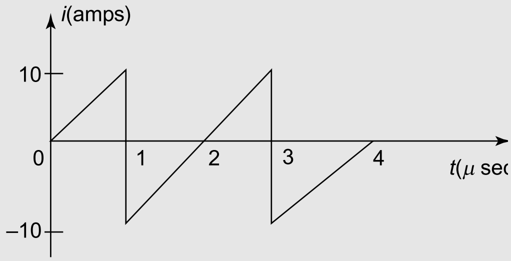


Fig. 1.27

Solution Assume initial voltage across the capacitor is zero.

The voltage across the capacitor is

$$v(t) = \frac{1}{c}\int i(t)\,dt + v_0$$

$$i(t) = 10 \times 10^6\, t\,;\ 0 \leq t \leq 1\,\mu\text{s}$$

$$= -20 + 10 \times 10^6 t;\ 1\,\mu\text{s} \leq t \leq 2\ \mu\text{s}$$

Since $v_0 = 0$

$$v(t) = \frac{1}{100 \times 10^{-6}}\int 10 \times 10^6 t\, dt;\quad 0 \leq t \leq 1\,\mu\text{s}$$

and

$$v(t) = \frac{1}{100 \times 10^{-6}}\int [-20 + 10 \times 10^6 t]dt;\quad 1\,\mu\text{s} \leq t \leq 2\ \mu\text{s}$$

Therefore

$$v(t) = \frac{10 \times 10^6}{100 \times 10^{-6}}\frac{t^2}{2};\quad 0 \leq t \leq 1\,\mu s$$

$$v(t) = \frac{1}{100 \times 10^{-6}}\left[-20t + 10 \times 10^6\,\frac{t^2}{2}\right];\quad 1\,\mu s \leq t \leq 2\ \mu\text{s}$$

The voltage waveform is shown in Fig. 1.28.

The maximum voltage $V_m = 0.05\,\text{V}$.

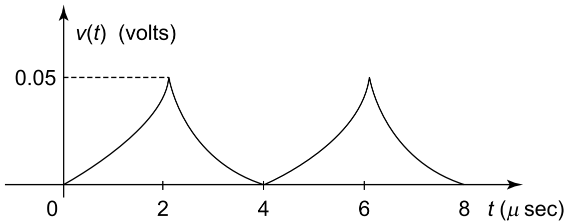


Fig. 1.28

Example 1.25 *The voltage waveform shown in Fig. 1.29 is applied to a pure capacitor of 50 μF. Sketch i(t), p(t), and determine I_m and p_m.*

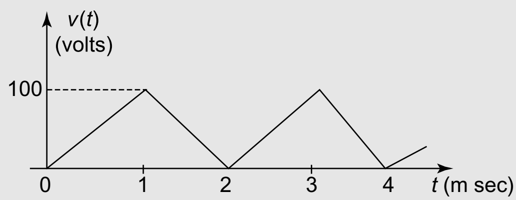


Fig. 1.29

Solution Since $i = c\dfrac{dv}{dt}$

From the voltage waveform $v(t) = 100 \times 10^3 t;\ 0 \le t \le 1\text{ms}$

$= 200 - 100 \times 10^3 t;\ 1\text{ms} \le t \le 2\text{ ms}$

Therefore, the current

$$i(t) = c\frac{dv(t)}{dt}$$

$$= 50\times10^{-6}\left[\frac{d}{dt}(100\times10^3 t)\right]$$

$$= 5\,\text{A};\qquad 0 \le t \le 1\text{ ms}$$

$$i(t) = c\frac{d}{dt}[v(t)]$$

$$= 50\times10^{-6}\left[\frac{d}{dt}(200-100\times10^3 t)\right]$$

$$= -5\,\text{A};\qquad 1\text{ ms} \le t \le 2\text{ ms}$$

The instantaneous power $p(t) = v(t)\,i(t)$

$= 100 \times 10^3 t \times 5$

$= 500 \times 10^3 t;\ 0 \le t \le 1\,\text{ms}$

and $p(t) = [\,200 - 100 \times 10^3 t][-5]$

$$= 500 \times 10^3\,t - 10^3,\ 1\text{ms} \le t \le 2\,\text{ms}.$$

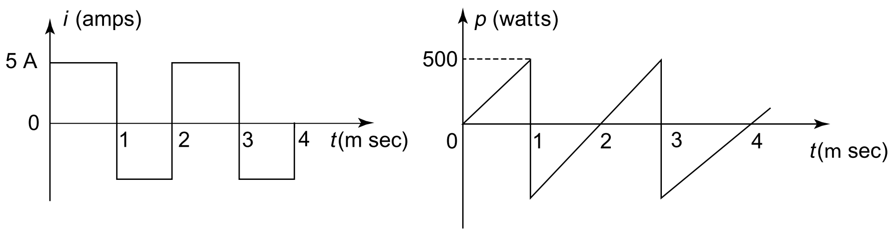
The maximum value of current

$$I_m = 5\text{ A}$$

and the maximum value of power

$$P_m = 500 \text{ watts}$$

The current and instantaneous power waveforms are shown in Figs 1.30(a) and (b) respectively.



(a) (b)

Fig. 1.30

Example 1.26 *A capacitor is charged to 1 volt at t = 0. A resistor of 1 ohm is connected across its terminals. The current is known to be of the form* $i(t) = e^{-t}$ *amperes for* $t > 0$*. At a particular time the current drops to 0.37 A at that instant determine.*

(i) At what rate is the voltage across the capacitor changing?
(ii) What is the value of the charge on the capacitor?
(iii) What is the voltage across the capacitor?
(iv) How much energy is stored in the electric field of the capacitor?
(v) What is the voltage across the resistor?

Solution

(i) The current equation is given as $i(t)$ 5 $i(0^+)\, e^{-t/RC}$; given $i(t)$ 5 $e^{-t/RC}$

$i(0^+) = 1\text{A};\ RC = 1;\ C = 1\text{F}$

When $i(t) = 0.37$ amperes

$$i(t) = 0.37 = e^{-t/1}$$

$$-t \log_e e = \log_e 0.37$$

$$t = 0.9942 \text{ sec}$$

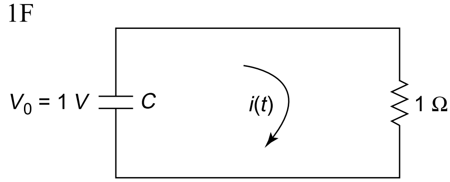


Fig. 1.31

$$i(t) = C\frac{dV(t)}{dt} \Rightarrow \frac{dV(t)}{dt} = \frac{i(t)}{C} = \frac{0.37}{1} = 0.37 \text{ V/sec}$$

or $$V_i(t) = \frac{1}{C}\int_0^t i(t)dt + V_0$$

$$= -\frac{1}{C}\int_0^t e^{-t}dt + V_0 \qquad [\therefore \ i(t) = -(t)]$$

$$= \frac{-1}{1}\frac{e^{-t}}{(-1)} + 1 = e^{-t}$$

$$= V_c(t) = e^{-t} \text{ for } t > 0$$

$$\therefore \quad \frac{dV_C(t)}{dt} = -e^{-t} = -e^{-0.9942} = -0.37 \text{ V/sec}$$

(ii) Charge on the capacitor

$$Q = C\,V_c = 1.e^{-t} = 0.37 \text{ coulombs}$$

(iii) Voltage across the capacitor

$$V_c(t) = e^{-t} = 0.37 \text{ volts}$$

(iv) Energy stored in the capacitor

$$W_C = \frac{1}{2}CV_c^2 = \frac{1}{2}1(e^{-t})^2 = \frac{e^{-2t}}{2} = 0.06845 \text{ joules}$$

(v) Voltage across the resistor at $t = 0.9942$ sec

$$V_R = i(t), R = e^{-t} = 0.37 \text{ V}$$

Inductive Element

Consider an inductive element shown in Fig. 1.32.

The inductance L is given by the voltage current relationship.

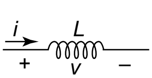


Fig. 1.32

$$v = L\frac{di}{dt}$$

or

$$i = \frac{1}{L}\int vdt + i_0$$

where i_0 is initial current flowing through inductor.

Example 1.27 *The current waveform shown in Fig. 1.33 is the current in a pure inductor L = 0.5 H. Sketch the voltage waveform.*

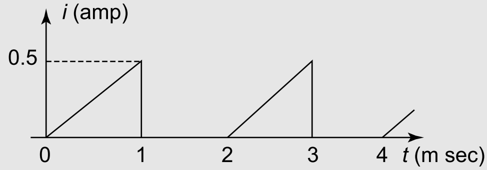


Fig. 1.33

Solution The relation between voltage and current in an inductor is given by

$$v = L\frac{di}{dt}$$

From Fig. 1.33, the current

$$i(t) = 0.5 \times 10^3 t\,;\ 0 \le t \le 1\,\text{ms}$$

$$= 0;\ 1\text{ ms} \le t \le 2\,\text{ms}$$

The voltage across inductor

$$v = 0.5 \times \frac{d}{dt}(0.5 \times 10^3 t)$$

$$= 0.5 \times 0.5 \times 10^3 = 250\,\text{V}$$

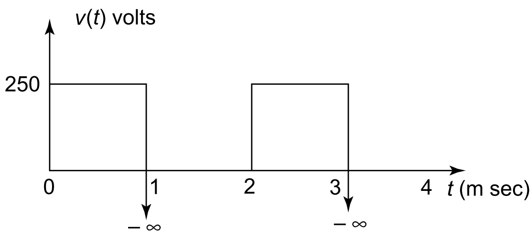


Fig. 1.34

The voltage waveform is shown in Fig. 1.34.

Practically, the current in an inductor never the discontinuous function as shown at 1 ms and 3 ms. The derivative has an infinite negative value at the points of discontinuity, there will be negative infinite spikes on the voltage waveform at these points.

Example 1.28 *An indictor element 10 mH passes a current i(t) of waveform shown in Fig. 1.35. Find the voltage across the element. Also sketch the voltage waveform.*

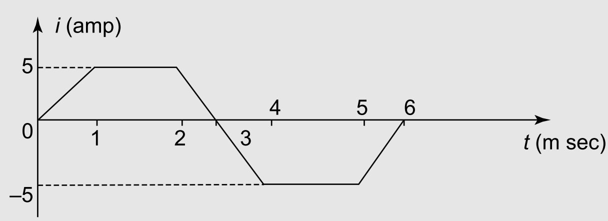


Fig. 1.35

Solution The voltage across inductor is given by

$$v = L\frac{di}{dt}$$

$$i(t) = 5 \times 10^3 t \text{ amp; } 0 \le t \le 1 \text{ m sec}$$

$$v(t) = (10 \times 10^{-3})(5 \times 10^3) = 50\text{ V}$$

$$i(t) = 5\text{A; } 1\text{ms} \le t \le 2\text{ ms}$$

$$v(t) = 10 \times 10^{-3}\ (0) = 0$$

$$i(t) = -5 \times 10^3 t + 15;\ 2\text{ ms} \le \text{t} \le 4\text{ ms}$$

$$v(t) = (10 \times 10^{-3})(-5 \times 10^3) = -50\text{ V}$$

$$i(t) = -5\text{A; } 4\text{ ms} \le \text{t} \le 6\text{ ms}$$

$$v(t) = 10 \times 10^{-3}(0) = 0$$

$$i(t) = 5 \times 10^3\ t - 30\ ;\ 5\text{ ms} \le t \le 6\text{ ms}$$

$$\therefore \quad v(t) = 10 \times 10^{-3} \times 5 \times 10^3 = 50\text{ V}$$

Therefore, the voltage waveform is shown in Fig. 1.36.

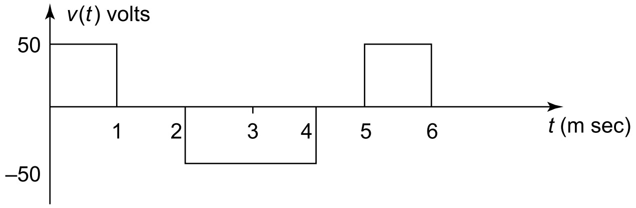


Fig. 1.36

Example 1.29 *A pure inductance of 0.01 H has an applied voltage with a wave form shown in Fig. 1.37. Sketch the corresponding current waveform and determine the expression for i in the first interval 0 < t < 1 ms.*

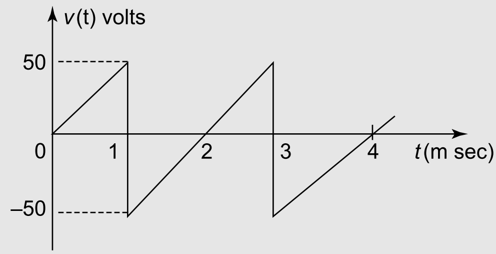


Fig. 1.37

Solution The voltage current relation in an inductor is given by

$$i(t) = \frac{1}{L}\int v(t)\,dt + i_0$$

Since $\quad i_0 = 0$

$$i = \frac{1}{0.01}\int v(t)\,dt$$

$$v(t) = 50 \times 10^3 t;\ 0 \le t \le 1\,\text{ms}$$

$$= -100 + 50 \times 10^3 t\ ;\ 1\,\text{ms} \le t \le 2\,\text{ms}.$$

Therefore, the current equation

$$i(t) = \frac{1}{0.01}\int 50\times 10^3 t\cdot dt$$

$$i(t) = 25 \times 10^5 t^2\ ;\ 0 \le t \le 1\,\text{ms}$$

and $\quad i(t) = \dfrac{1}{0.01}\displaystyle\int [-100 + 50\times 10^3 t]dt$

$$= \frac{1}{0.01}\left[-100\,t + 50\times 10^3\,\frac{t^2}{2}\right]; 1\text{ ms} \le t \le 2\text{ ms}$$

The current waveform is shown in Fig. 1.38.

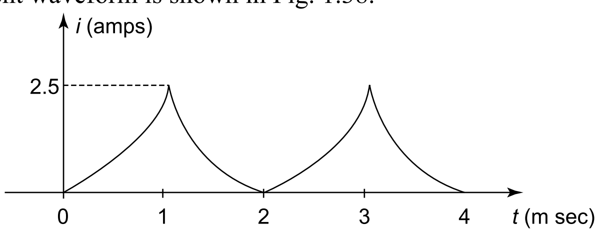


Fig. 1.38

Example 1.30 *The following current wavefrom i(t) is passed through a series RL circuit with R = 2 ; L = 2 mH. Find the voltage across each element and sketch the same. (See Fig. 1.39)* *[JNTU April/May 2003]*

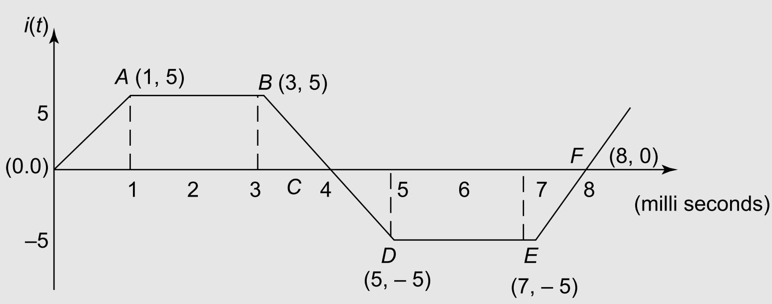


Fig. 1.39

Solution

For line $$OA, m = \frac{5}{1}$$

$$i(t) - 0 = \frac{5}{1}(t-0)$$

$$i(t) = 5t$$

For line *AB*, $i(t) = 5$

For line *BD*,

$$(i(t)-5) = \frac{-5-5}{5-3}(t-3)$$

$$i(t) - 5 = -5(t-3)$$

$$i(t) = -5t + 20$$

For line *DE*, $i(t) = -5$

For line *EF*,

$$(i(t)+5) = \frac{5}{1}(t-7)$$

$$i(t) = 5t - 40$$

Voltage induced in the inductor

Along OA

$$V_{OA} = L\frac{di}{dt} = 2\times 10^{-3}\times\frac{d(5t)}{dt} = 2\times 10^{-3}\times 5\times 10^{-3} = 10\ \mu\text{V}$$

Along AB

$$V_{AB} = L\frac{di}{dt} = 0$$

Along BD

$$V_{Bd} = L\frac{di}{dt} = 2\times 10^{-3}\times\frac{d(-5t+20)}{dt} = -10\times 10^{-6}\ \text{V} = -10\ \mu\text{V}$$

Along DE

$$V_{DE} = L\frac{di}{dt} = 0$$

Along EF

$$V_{EF} = L\frac{di}{dt} = 2\times 10^{-3}\times\frac{d(5t+40)\times 10^{-3}}{dt} = 10\ \mu\text{V}$$

The wave is shown in Fig. 1.40.

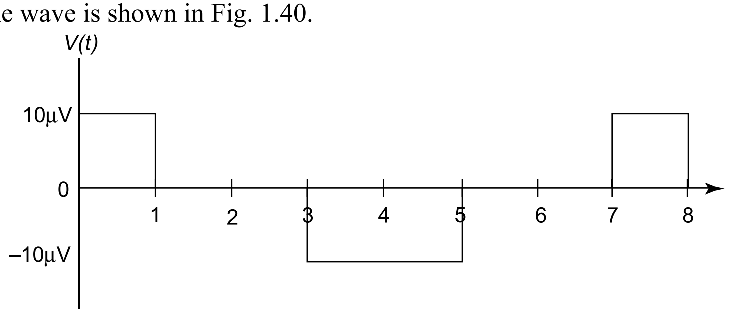


Fig. 1.40

Voltage waveform across the resistor is the same as current through the circuit as shown in Fig. 1.41.

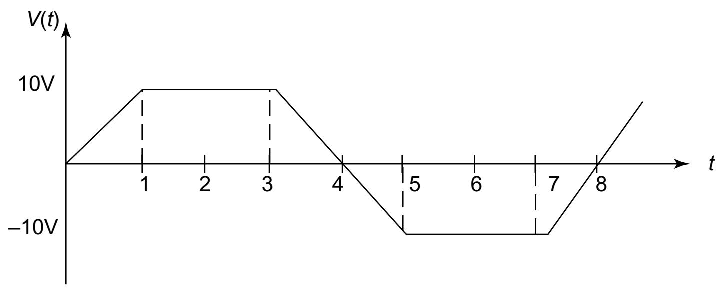


Fig. 1.41

Example 1.31 *Describe the volt-ampere relations for R, L and C parameters.*

Solution *Volt-ampere relations for R, L and C parameters*

The passive elements R, L, C are defined by the way in which the current and voltage are related for individual element.

(i) If the current 'I' and voltage 'V' are related by a constant for a single element then the element is a resistance 'R'. The Resistance 'R' represents the constant of proportionality.

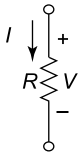


Fig. 1.42

$\therefore$ Voltage, $V = RI$ (ohms law)

Current $I = \dfrac{V}{R}$

Power, $P = VI = I^2R$

The units of resistance 'R' is ohms (Ω).

(ii) If the current and voltage are related such that the voltage is the time derivative of current, then the element is an inductance 'L'. The inductance 'L' represents the constant of proportionality.

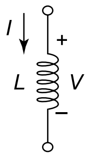


Fig. 1.43

$\therefore$ Voltage, $V = L\dfrac{dI}{dt}$

Current, $I = \dfrac{1}{L}\int V\,dt + K_1$ $[K_1 = \text{constant}]$

Power, $P = VI = LI\dfrac{dI}{dt}$

The units of inductance 'L' is Henry (H).

(iii) If the voltage and current are related such that the current is the time derivative of the voltage, then the element is a capacitance '*C*'. The capacitance '*C*' is the constant of proportionality.

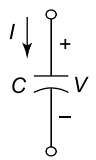


Fig. 1.44

$\therefore$ Current, $$I = C\frac{dV}{dt}$$

Voltage, $$V = \frac{1}{C}\int I\,dt + K_2 \quad [K_2 = \text{constant}]$$

Power, $$P = VI = VC\frac{dV}{dt}$$

The units of capacitance '*C*' is Farads (F).

1.8 ENERGY SOURCES: IDEAL, NON-IDEAL, INDEPENDENT AND DEPENDENT SOURCES *[JNTU Nov 2011]*

According to their terminal voltage-current characteristics, electrical energy sources are categorised into ideal voltage sources and ideal current sources. Further they can be divided into independent and dependent sources.

An ideal voltage source is a two-terminal element in which the voltage υ_s is completely independent of the current i_s through its terminals. The representation of ideal constant voltage source is shown in Fig. 1.45(a).

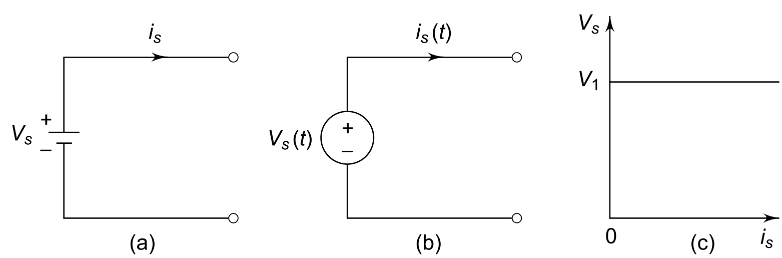


Fig. 1.45

If we observe the $\upsilon - i$ characteristics for an ideal voltage source as shown in Fig. 1.45(c) at any time, the value of the terminal voltage υ_s is constant with respect to the value of current i_s. Whenever $\upsilon_s = 0$, the voltage source is the same as that of a short circuit. Voltage sources need not have constant magnitude; in many cases the specified voltage may be time-dependent like a sinusoidal waveform. This may be represented as shown in Fig. 1.45(b). In many practical voltage sources, the internal resistance is represented in series with the source as shown in Fig. 1.46(a). In this, the voltage across the terminals falls as the current through it increases, as shown in Fig. 1.46(b).

The terminal voltage υ_t depends on the source current as shown in Fig. 1.46(b), where $\upsilon_t = \upsilon_s - i_s R$.

An ideal constant current source is a two-terminal element in which the current i_s completely independent of the voltage υ_s across its terminals. Like voltage sources we can have current sources of constant magnitude i_s or sources whose current varies with time $i_s(t)$. The representation of an ideal current source is shown in Fig. 1.47(a).

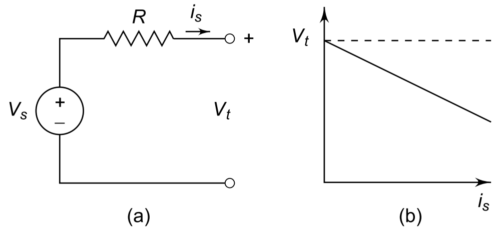


Fig. 1.46

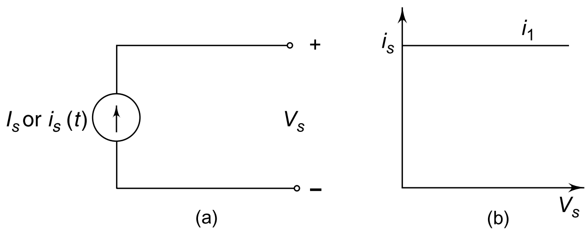


Fig. 1.47

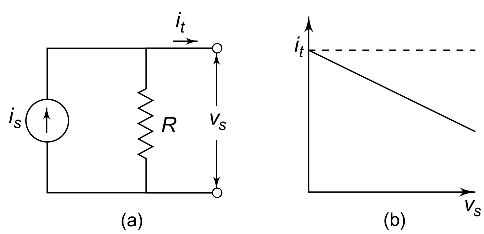


Fig. 1.48

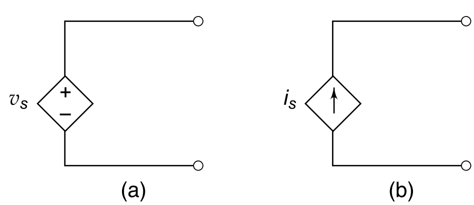


Fig. 1.49

If we observe the $\upsilon - i$ characteristics for an ideal current source as shown in Fig. 1.47(b), at any time the value of the current i_s is constant with respect to the voltage across it. In many practical current sources, the resistance is in parallel with a source as shown in Fig. 1.48(a). In this the magnitude of the current falls as the voltage across its terminals increases. Its terminal $\upsilon - i$ characteristics is shown in Fig. 1.48(b). The terminal current is given by $i_t = i_s - (\upsilon_s/R)$, where R is the internal resistance of the ideal current source.

The two types of ideal sources we have discussed are independent sources for which voltage and current are independent and are not affected by other parts of the circuit. In the case of dependent sources, the source voltage or current is not fixed, but is dependent on the voltage or current existing at some other location in the circuit.

Dependent or controlled sources are of the following types:

(i) voltage controlled voltage source (VCVS)

(ii) current controlled voltage source (CCVS)

(iii) voltage controlled current source (VCCS)
(iv) current controlled current source (CCCS)

These are represented in a circuit diagram by the symbol shown in Fig. 1.49. These types of sources mainly occur in the analysis of equivalent circuits of transistors.

1.9 SOURCE TRANSFORMATION *[JNTU Nov 2011]*

In solving networks to find solutions one may have to deal with energy sources. It has already been discussed that basically, energy sources are either voltage sources or current sources. Sometimes it is necessary to convert a voltage source to a current source and vice-versa. Any practical voltage source consists of an ideal voltage source in series with an internal resistance. Similarly, a practical current source consists of an ideal current source in parallel with an internal resistance as shown in Fig. 1.50. R_v and R_i represent the internal resistances of the voltage source V_s, and current source I_s, respectively.

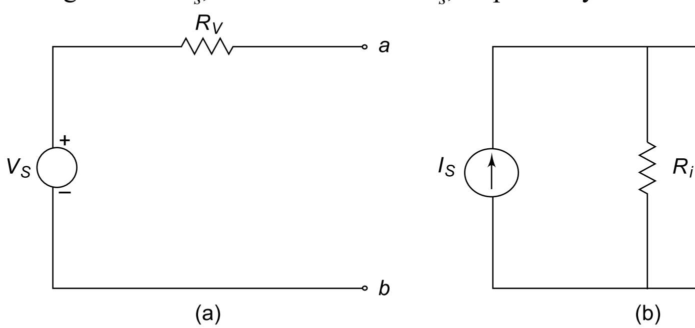


Fig. 1.50

Any source, be it a current source or a voltage source, drives current through its load resistance, and the magnitude of the current depends on the value of the load resistance. Figure 1.51 represents a practical voltage source and a practical current source connected to the same load resistance R_L.

From Fig. 1.51(a), the load voltage can be calculated by using Kirchhoff's voltage law as

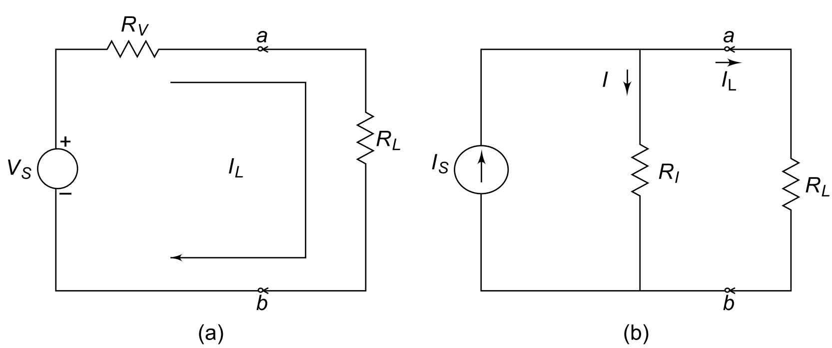


Fig. 1.51

$$V_{ab} = V_s - I_L R_v$$

The open circuit voltage $V_{OC} = V_s$

The short circuit current $I_{SC} = \dfrac{V_s}{R_v}$

From Fig. 1.51(b),

$$I_L = I_S - I = I_S - \frac{V_{ab}}{R_I}$$

The open circuit voltage $V_{OC} = I_S R_I$

The short circuit current $I_{SC} = I_S$

The above two sources are said to be equal, if they produce equal amounts of current and voltage when they are connected to identical load resistances. Therefore, by equating the open circuit voltages and short circuit currents of the above two sources we obtain

$$V_{OC} = I_s R_I = V_S$$

$$I_{SC} = I_S = \frac{V_s}{R_v}$$

It follows that $R_{\mathrm{I}} = R_V = R_s \quad \therefore \quad V_s = I_S R_S$

where R_S is the internal resistance of the voltage or current source. Therefore, any practical voltage source, having an ideal voltage V_S and internal series resistance R_S can be replaced by a current source $I_S = V_S/R_S$ in parallel with an internal resistance R_S. The reverse transformation is also possible. Thus, a practical current source in parallel with an internal resistance R_S can be replaced by a voltage source $V_S = I_s R_s$ in series with an internal resistance R_S.

Example 1.32 *Determine the equivalent voltage source for the current source shown in Fig. 1.52.*

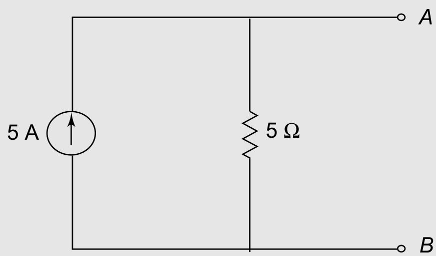


Fig. 1.52

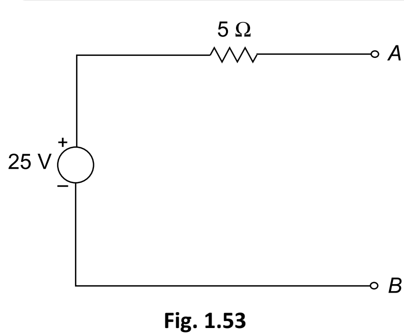


Fig. 1.53

Solution The voltage across terminals A and B is equal to 25 V. Since the internal resistance for the current source is 5 Ω, the internal resistance of the voltage source is also 5 Ω. The equivalent voltage source is shown in Fig. 1.53.

Example 1.33 *Determine the equivalent current source for the voltage source shown in Fig. 1.54.*

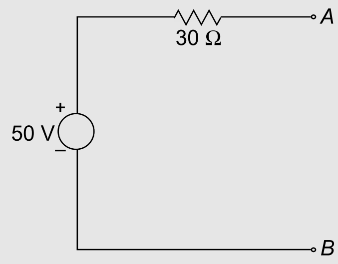


Fig. 1.54

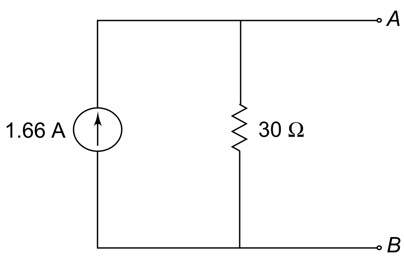


Fig. 1.55

Solution The short-circuit current at terminals A and B is equal to

$$I = \frac{50}{30} = 1.66\,\text{A}$$

Since the internal resistance for the voltage source is 30 Ω, the internal resistance of the current source is also 30 Ω. The equivalent current source is shown in Fig. 1.55.

Example 1.34 *Using source transformation, find the power delivered by the 50 V voltage source in the circuit shown in Fig. 1.56.*

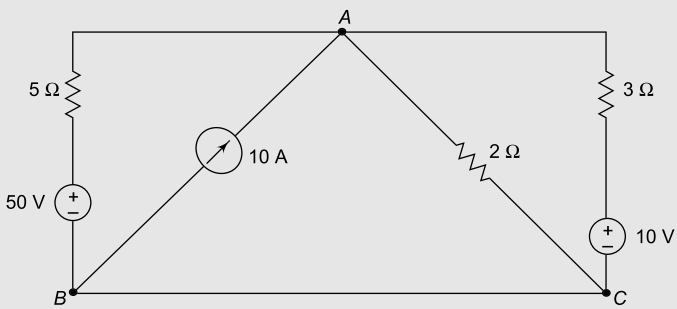


Fig. 1.56

Solution The current source in the circuit in Fig. 1.56 can be replaced by a voltage source as shown in Fig. 1.57.

$$\frac{V-50}{5} + \frac{V-20}{2} + \frac{V-10}{3} = 0$$

$$V\,[0.2 + 0.5 + 0.33] = 23.33$$

$$\text{or} \qquad V = \frac{23.33}{1.03} = 22.65\,\text{V}$$

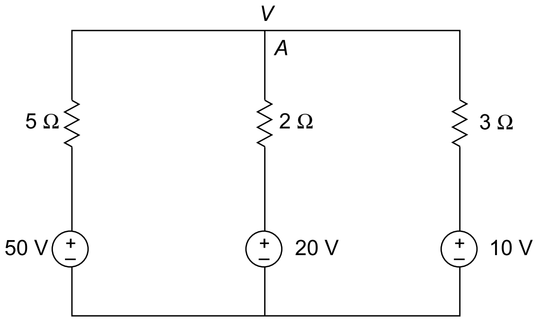


Fig. 1.57

∴ The current delivered by the 50 V voltage source is $(50 - V)/5$

$$= \frac{50 - 22.65}{5} = 5.47\text{ A}$$

Hence, the power delivered by the 50 V voltage source $= 50 \times 5.47 = 273.5$ W.

Example 1.35 *By using source transformation, source combination and resistance combination convert the circuit shown in Fig. 1.58 into a single voltage source and single resistance.*

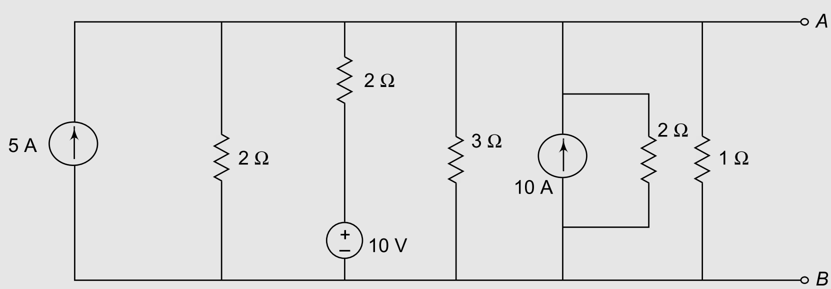
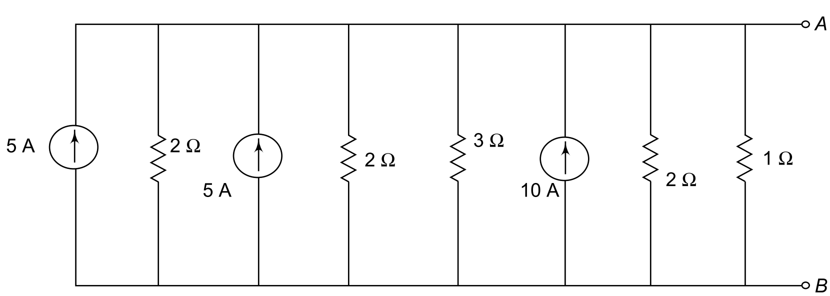


Fig. 1.58

Solution The voltage source in the circuit of Fig. 1.58 can be replaced by a current source as shown in Fig. 1.59(a).

Here the current sources can be combined into a single source. Similarly, all the resistances can be combined into a single resistance, as shown in Fig. 1.59(b).

Figure 1.59(b) can be replaced by single voltage source and a series resistance as shown in Fig. 1.59(c).



(a)

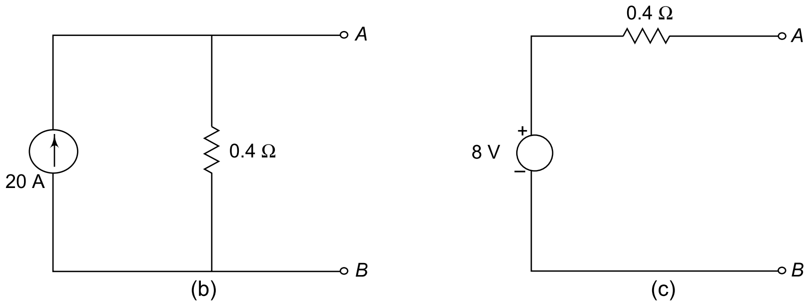


Fig. 1.59

Example 1.36 *Find the voltage and current source equivalent representation of the following network across AB, as shown in Fig. 1.60.*

[JNTU Jan 2010]

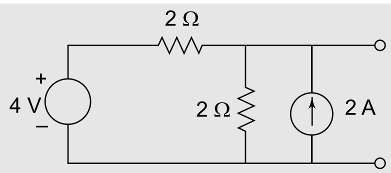


Fig. 1.60

Solution Voltage and current source equivalent representation of the following network across AB.

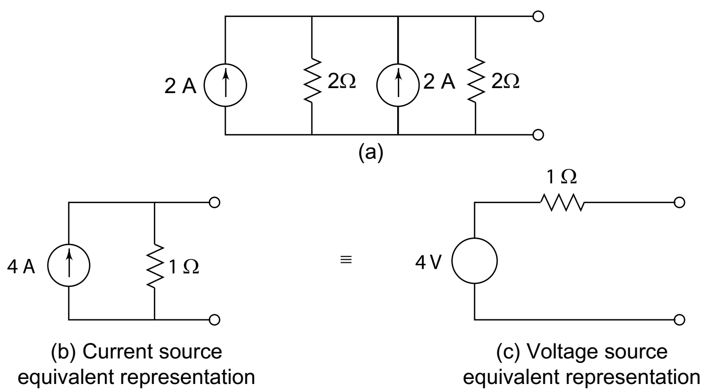


Fig. 1.61

Example 1.37 *Find the value of current I_i in Fig. 1.62.*

[JNTU April/May 2007]

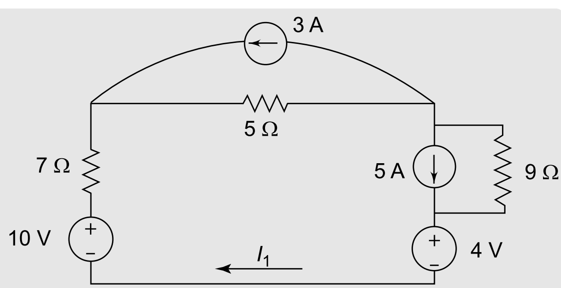


Fig. 1.62

Solution Converting current source into equivalent voltage source

By applying KVL

$$10 - 7I_1 - 15 - 5I_1 - 9I_1 + 45 - 4 = 0$$

$$36 = 21I_1$$

$$I_1 = \frac{36}{21} = 1.714 \text{ A}$$

$$I_1 = 1.714 \text{ A}$$

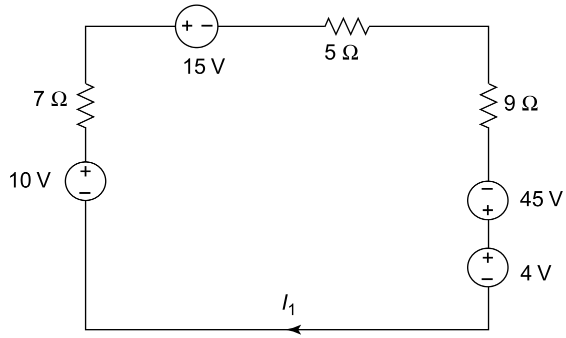


Fig. 1.63

Example 1.38 *Using source Transformation, reduce the network between A and B into an equivalent voltage source. (Fig. 1.64)*

[JNTU May/June 2006]

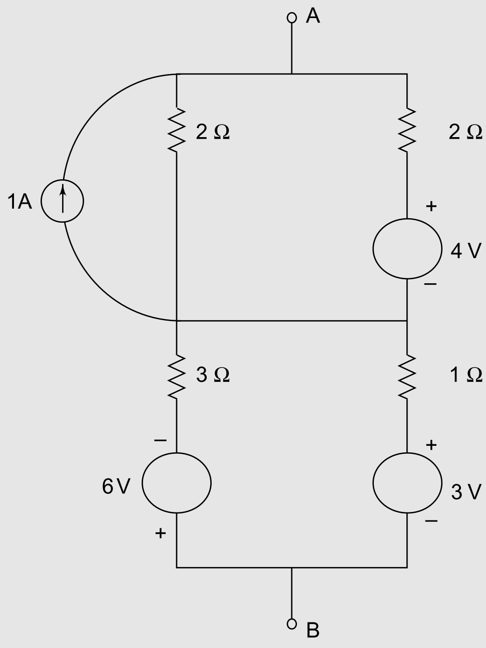


Fig. 1.64

Solution Given circuit

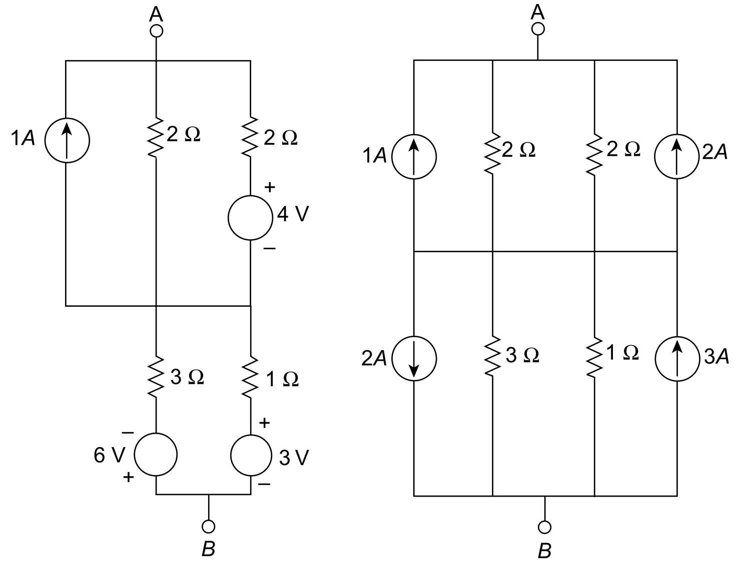


Fig. 1.65(a)

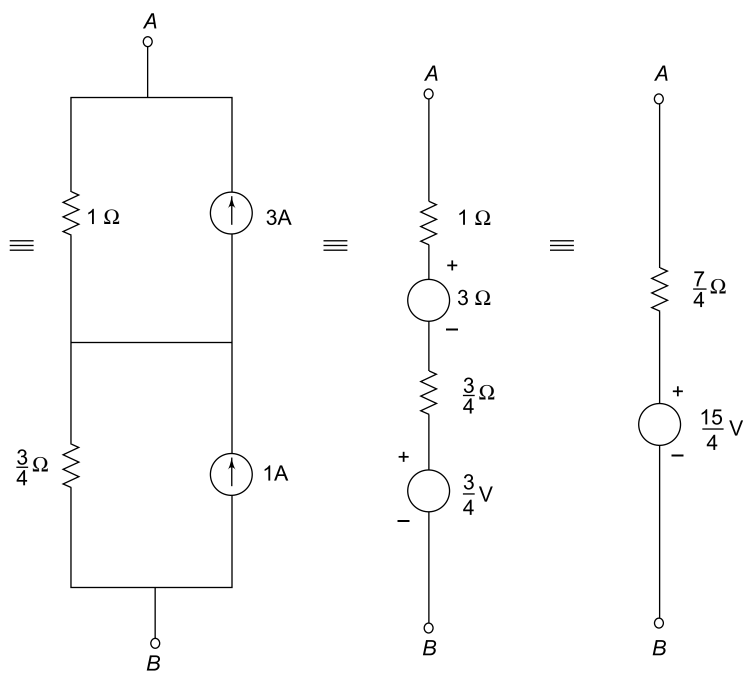


Fig. 1.65(b)

Example 1.39 *Reduce the network shown in Fig. 1.66, to a single loop network by successive source transformation, to obtain the current in the 12 Ω resistor.*

[JNTU May/June 2006]

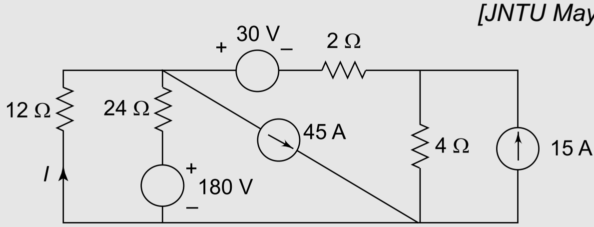


Fig. 1.66

Solution By source transformation $I = 22.5 \times \dfrac{4.8}{16.8} = 6.428$ A.

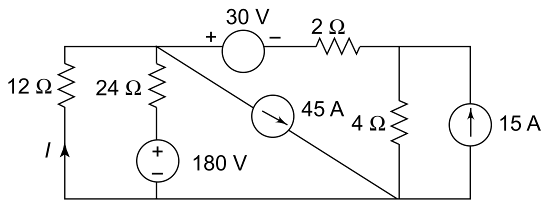


Fig. 1.67

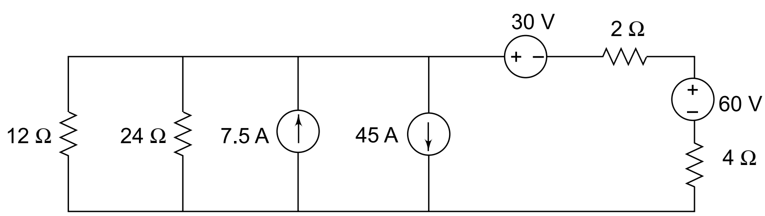


Fig. 1.68

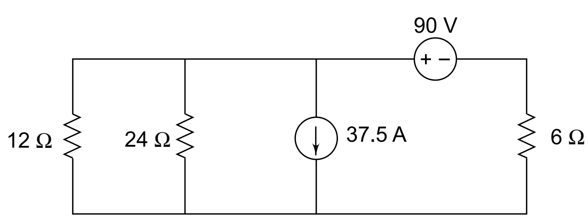


Fig. 1.69

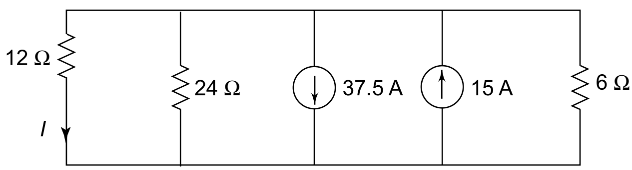


Fig. 1.70

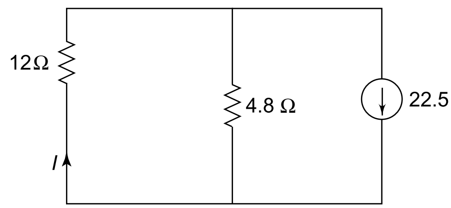


Fig. 1.71

1.10 KIRCHHOFF'S LAWS—RESISTANCE SERIES AND PARALLEL COMBINATION

1.10.1 Kirchhoff's Voltage Law *[JNTU Nov 2011]*

Kirchhoff's voltage law states that the algebraic sum of all branch voltages around any closed path in a circuit is always zero at all instants of time. When the current passes through a resistor, there is a loss of energy and, therefore, a voltage drop. In any element, the current always flows from higher potential to lower potential. Consider the circuit in Fig. 1.72. It is customary to take the direction of current *I* as indicated in the figure, i.e. it leaves the positive terminal of the voltage source and enters into the negative terminal.

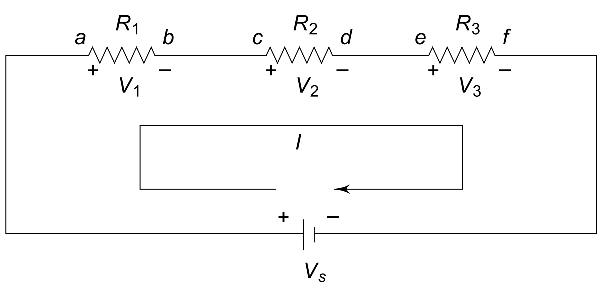


Fig. 1.72

As the current passes through the circuit, the sum of the voltage drop around the loop is equal to the total voltage in that loop. Here the polarities are attributed to the resistors to indicate that the voltages at points *a*, *c* and *e* are more than the voltages at *b*, *d* and *f*, respectively, as the current passes from *a* to *f*.

$$\therefore \quad V_s = V_1 + V_2 + V_3$$

Consider the problem of finding out the current supplied by the source *V* in the circuit shown in Fig. 1.73.

Our first step is to assume the reference current direction and to indicate the polarities for different elements. (See Fig. 1.74).

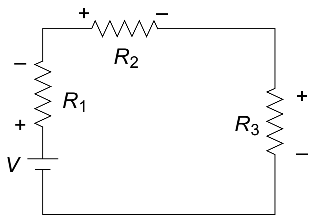


Fig. 1.73

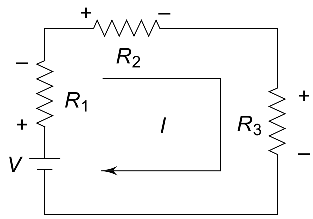


Fig. 1.74

By using Ohm's law, we find the voltage across each resistor as follows.

$$V_{R1} = IR_1,\ V_{R2} = IR_2,\ V_{R3} = IR_3$$

where V_{R1}, V_{R2} and V_{R3} are the voltages across R_1, R_2 and R_3, respectively. Finally, by applying Kirchhoff's law, we can form the equation

$$V = V_{R1} + V_{R2} + V_{R3}$$

$$V = IR_1 + IR_2 + IR_3$$

From the above equation the current delivered by the source is given by

$$I = \frac{V}{R_1 + R_2 + R_3}$$

Example 1.40 *For the circuit shown in Fig. 1.75, determine the unknown voltage drop V_1.*

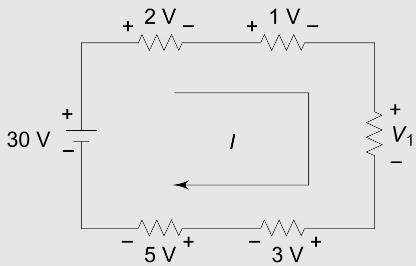


Fig. 1.75

Solution According to Kirchhoff's voltage law, the sum of the potential drops is equal to the sum of the potential rises;

Therefore, $\quad 30 = 2 + 1 + V_1 + 3 + 5$

or $\quad V_1 = 30 - 11 = 19\text{ V}$

Example 1.41 *What is the current in the circuit shown in Fig. 1.76? Determine the voltage across each resistor.*

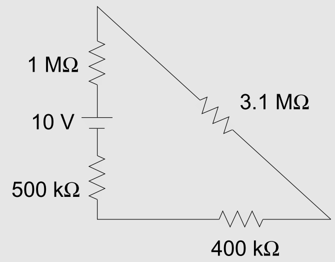


Fig. 1.76

Solution We assume current I in the clockwise direction and indicate polarities (Fig. 1.77). By using Ohm's law, we find the voltage drops across each resistor.

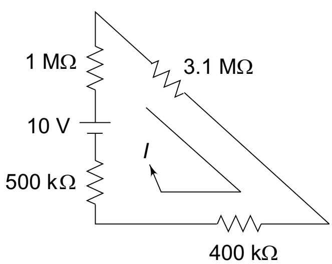


Fig. 1.77

$$V_{1M} = I, \quad V_{3.1M} = 3.1\, I$$

$$V_{500K} = 0.5\, I, \quad V_{400K} = 0.4\, I$$

Now, by applying Kirchhoff's voltage law, we form the equation.

$$10 = I + 3.1\, I + 0.5\, I + 0.4\, I$$

or $5\, I = 10$

or $I = 2\ \mu\text{A}$

$\therefore$ Voltage across each resistor is as follows

$$V_{1M} = 1 \times 2 = 2.0\ \text{V}$$

$$V_{3.1M} = 3.1 \times 2 = 6.2\ \text{V}$$

$$V_{400\ K} = 0.4 \times 2 = 0.8\ \text{V}$$

$$V_{500\ K} = 0.5 \times 2 = 1.0\ \text{V}$$

Example 1.42 *In the circuit given in Fig. 1.78, find (a) the current I, and (b) the voltage across 30 Ω.*

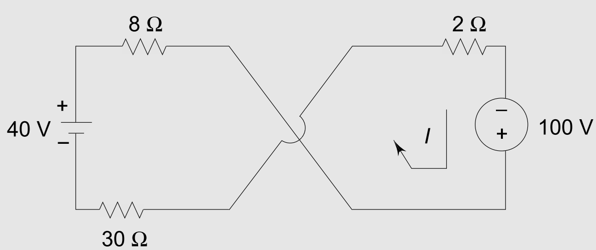


Fig. 1.78

Solution We redraw the circuit as shown in Fig. 1.79 and assume current direction and indicate the assumed polarities of resistors.

By using Ohm's law, we determine the voltage across each resistor as

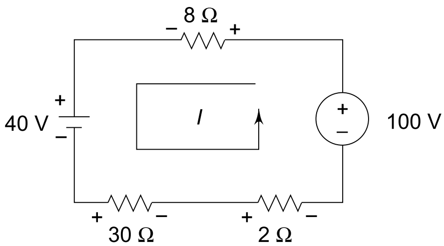


Fig. 1.79

$$V_8 = 8I,\ V_{30} = 30I,\ V_2 = 2I$$

By applying Kirchhoff's law, we get

$$100 = 8I + 40 + 30I + 2I$$

$$40\,I = 60 \text{ or } I = \frac{60}{40} = 1.5\text{ A}$$

$\therefore$ Voltage drop across 30 Ω $= V_{30} = 30 \times 1.5 = 45$ V

Example 1.43 *State Ohm's law.*

[JNTU May/June 2008]

Solution Ohm's law: Ohm's law states that the voltage across any element is proportional to current flowing through the element.

$$V \propto I$$

$$V = RI$$

R is the proportionality constant and is defined as resistance. Its unit is (Ω).

1.10.2 Voltage Division

The series circuit acts as a voltage divider. Since the same current flows through each resistor, the voltage drops are proportional to the values of resistors. Using this principle, different voltages can be obtained from a single source, called a voltage divider. For example, the voltage across a 40 Ω resistor is twice that of 20 Ω in a series circuit shown in Fig. 1.80.

In general, if the circuit consists of a number of series resistors, the total current is given by the total voltage divided by equivalent resistance. This is shown in Fig. 1.81.

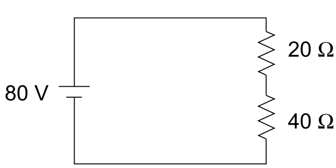


Fig. 1.80

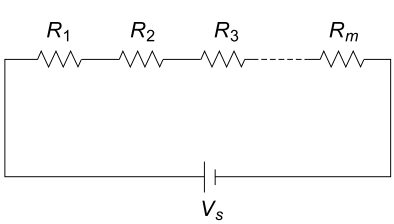


Fig. 1.81

The current in the circuit is given by $I = V_s/(R_1 + R_2 + \ldots + R_m)$. The voltage across any resistor is nothing but the current passing through it, multiplied by that particular resistor.

Therefore, $V_{R1} = IR_1$

$$V_{R2} = IR_2$$

$$V_{R3} = IR_3$$

$$\vdots$$

$$V_{Rm} = IR_m$$

or
$$V_{Rm} = \frac{V_s(R_m)}{R_1 + R_2 + \ldots + R_m}$$

From the above equation, we can say that the voltage drop across any resistor, or a combination of resistors, in a series circuit is equal to the ratio of that resistance value to the total resistance, multiplied by the source voltage, i.e.

$$V_m = \frac{R_m}{R_T} V_s$$

where V_m is the voltage across mth resistor, R_m is the resistance across which the voltage is to be determined and R_T is the total series resistance.

Example 1.44 *What is the voltage across the 10 Ω resistor in Fig. 1.82.*

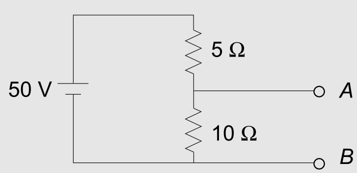


Fig. 1.82

Solution Voltage across $10\ \Omega = V_{10} = 50 \times \frac{10}{10+5} = \frac{500}{15} = 33.3\ \text{V}$

Example 1.45 *Find the voltage between A and B in a voltage divider network shown in Fig. 1.83.*

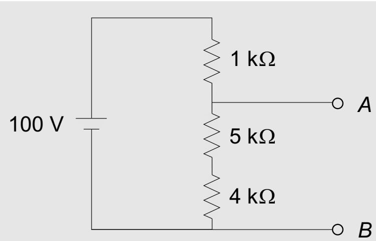


Fig. 1.83

Solution Voltage across $9\ \text{k}\Omega = V_9 = V_{AB} = 100 \times \frac{9}{10} = 90\ \text{V}$

Example 1.46 *What is the voltage across A and B in the circuit shown in Fig. 1.84.*

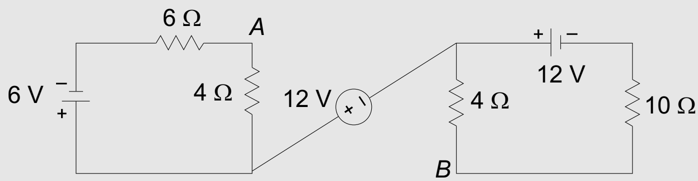


Fig. 1.84

Solution The above circuit can be redrawn as shown in Fig. 1.85.

Assume loop currents I_1 and I_2 as shown in Fig. 1.85.

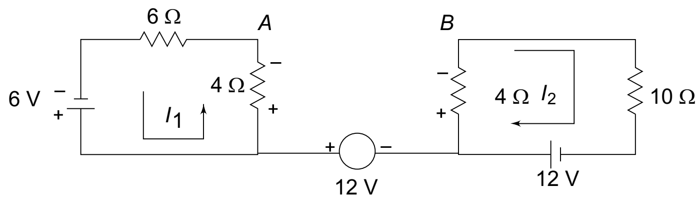


Fig. 1.85

$$I_1 = \frac{6}{10} = 0.6\,\text{A}$$

$$I_2 = \frac{12}{14} = 0.86\,\text{A}$$

V_A = Voltage drop across 4 Ω resistor = 0.6 × 4 = 2.4 V

V_B = Voltage drop across 4 Ω resistor = 0.86 × 4 = 3.44 V

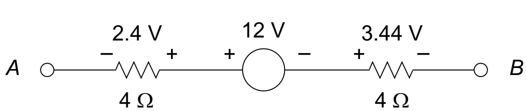


Fig. 1.86

The voltage between points A and B is the sum of voltages as shown in Fig. 1.86.

$$\therefore \quad V_{AB} = -2.4 + 12 + 3.44$$

$$= 13.04\text{ V}$$

1.10.3 Power in Series Circuit

The total power supplied by the source in any series resistive circuit is equal to the sum of the powers in each resistor in series, i.e.

$$P_S = P_1 + P_2 + P_3 + \ldots + P_m$$

where m is the number of resistors in series, P_S is the total power supplied by source and P_m is the power in the last resistor in series. The total power in the series circuit is the total voltage applied to a circuit, multiplied by the total current.

Expressed mathematically,

$$P_S = V_s I = I^2 R_T = \frac{V_s^2}{R_T}$$

where V_s is the total voltage applied, R_T is the total resistance, and I is the total current.

Example 1.47 *Determine the total amount of power in the series circuit in Fig. 1.87.*

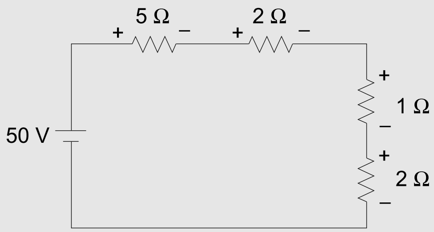


Fig. 1.87

Solution Total resistance = 5 + 2 + 1 + 2 = 10 Ω

We know $$P_S = \frac{V_s^2}{R_T} = \frac{(50)^2}{10} = 250 \text{ W}$$

Check We find the power absorbed by each resistor

$$\text{Current} = \frac{50}{10} = 5 \text{ A}$$

$$P_5 = (5)^2 \times 5 = 125 \text{ W}$$

$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$

$$P_1 = (5)^2 \times 1 = 25 \text{ W}$$

$$P_2 = (5)^2 \times 2 = 50 \text{ W}$$

The sum of these powers gives the total power supplied by the source $P_S = 250$ W.

Example 1.48 *A 20 V battery with an internal resistance of 5 ohms is connected to a resistor of x ohms. If an additional resistance of 6 Ω is connected across the battery, find the value of x, so that the external power supplied by the battery remain the same.*

Solution Power supplied to x by battery $= \left(\frac{20}{5+x}\right)^2 x = P_1$

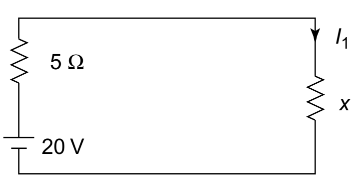


Fig. 1.88

$$I_2 = \frac{20}{5 + \frac{6x}{6+x}} = \frac{120}{30+11x}$$

$$\text{Power supplied to } x = \left(\frac{120}{30+11x}\right)^2 x = P_2$$

$$P_1 = P_2 \Rightarrow \frac{20}{5+x} = \frac{120}{11x+30}$$

$$x = 0$$

1.10.4 Kirchhoff's Current Law *[JNTU Nov 2011]*

Kirchhoff's current law states that the sum of the currents entering into any node is equal to the sum of the currents leaving that node. The node may be an interconnection of two or more branches. In any parallel circuit, the node is a junction point of two or more branches. The total current entering into a node is equal to the current leaving that node. For example, consider the circuit shown in Fig. 1.89, which contains two nodes A and B. The total current I_T entering node A is divided into I_1, I_2 and I_3. These currents flow out of node A. According to Kirchhoff's current law, the current into node A is equal to the total current out of node A: that is, $I_T = I_1 + I_2 + I_3$. If we consider node B, all three currents I_1, I_2, I_3 are entering B, and the total current I_T is leaving node B, Kirchhoff's current law formula at this node is therefore the same as at node A.

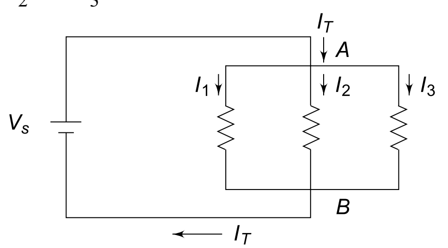


Fig. 1.89

$$I_1 + I_2 + I_3 = I_T$$

In general, sum of the currents entering any point or node or junction equal to sum of the currents leaving from that point or node or junction as shown in Fig. 1.90.

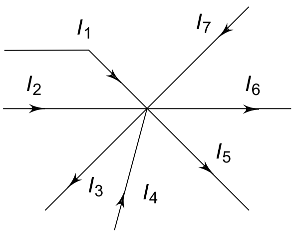


Fig. 1.90

$$I_1 + I_2 + I_4 + I_7 = I_3 + I_5 + I_6$$

If all of the terms on the right side are brought over to the left side, their signs change to negative and a zero is left on the right side, i.e.

$$I_1 + I_2 + I_4 + I_7 - I_3 - I_5 - I_6 = 0$$

This means that the algebraic sum of all the currents meeting at a junction is equal to zero.

Example 1.49 *Determine the current in all resistors in the circuit shown in Fig. 1.91.*

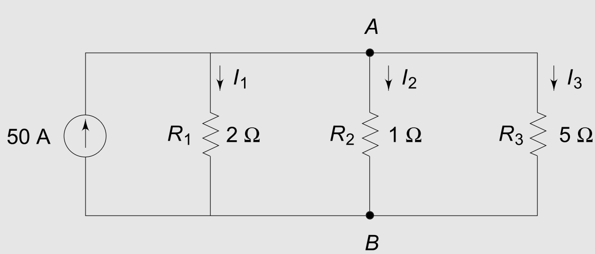


Fig. 1.91

Solution The above circuit contains a single node 'A' with reference node 'B'. Our first step is to assume the voltage V at node A. In a parallel circuit the same voltage is applied across each element. According to Ohm's law, the currents passing through each element are $I_1 = V/2, I_2 = V/1, I_3 = V/5$.

By applying Kirchhoff's current law, we have

$$I = I_1 + I_2 + I_3$$

$$I = \frac{V}{2} + \frac{V}{1} + \frac{V}{5}$$

$$50 = V\left[\frac{1}{2} + \frac{1}{1} + \frac{1}{5}\right] = V\,[0.5 + 1 + 0.2]$$

$$V = \frac{50}{1.7} = \frac{500}{17} = 29.41\text{ V}$$

Once we know the voltage V at node A, we can find the current in any element by using Ohm's law.

The current in the 2 Ω resistor is $I_1 = 29.41/2 = 14.705$ A.

Similarly, $$I_2 = \frac{V}{R_2} = \frac{V}{1} = 29.41\text{ A}$$

$$I_3 = \frac{29.41}{5} = 5.882\text{ A}$$

∴ $$I_1 = 14.7\text{ A},\ I_2 = 29.4\text{ A, and } I_3 = 5.88\text{ A}$$

Example 1.50 *For the circuit shown in Fig. 1.92, find the voltage across the 10 Ω resistor and the current passing through it.*

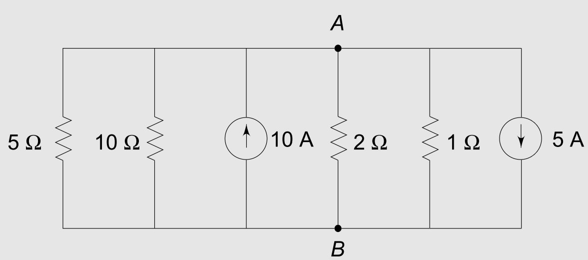


Fig. 1.92

Solution The circuit shown above is a parallel circuit, and consists of a single node A. By assuming voltage V at the node A w.r.t. B, we can find out the current in the 10 Ω branch. (See Fig. 1.93)

According to Kirchhoff's current law,

$$I_1 + I_2 + I_3 + I_4 + 5 = 10$$

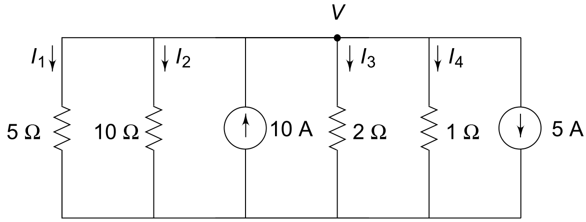


Fig. 1.93

By using Ohm's law, we have

$$I_1 = \frac{V}{5};\ I_2 = \frac{V}{10},\ I_3 = \frac{V}{2},\ I_4 = \frac{V}{1}$$

$$\frac{V}{5} + \frac{V}{10} + \frac{V}{2} + V + 5 = 10$$

$$V\left[\frac{1}{5} + \frac{1}{10} + \frac{1}{2} + 1\right] = 5$$

$$V[0.2 + 0.1 + 0.5 + 1] = 5$$

$$V = \frac{5}{1.8} = 2.78\,\text{V}$$

$\therefore$ the voltage across the 10 Ω resistor is 2.78 V and the current passing through it is

$$I_2 = \frac{V}{10} = \frac{2.78}{10} = 0.278\,\text{A}$$

Example 1.51 *Determine the current through resistance R_3 in the circuit shown in Fig. 1.94.*

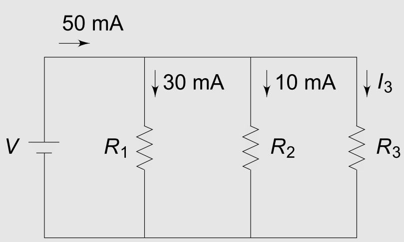


Fig. 1.94

Solution According to Kirchhoff's current law,

$$I_T = I_1 + I_2 + I_3$$

where I_T is the total current and I_1, I_2 and I_3 are the currents in resistances R_1, R_2 and R_3 respectively.

$\therefore$ $50 = 30 + 10 + I_3$

or $I_3 = 10$ mA

Example 1.52 *Using Kirchhoff's current law, find the values of the currents i_1 and i_2 in the circuit shown in Fig. 1.95.*

[JNTU May/June 2006]

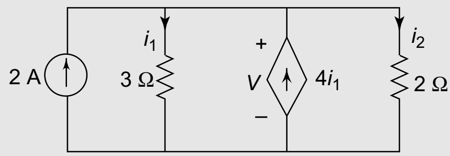


Fig. 1.95

Solution Applying KCL at node V and also $i_1 = \frac{V}{3}$

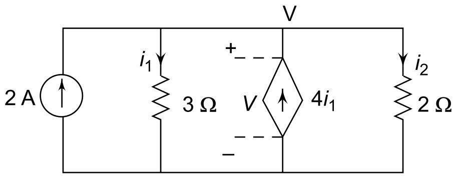


Fig. 1.96

$$\frac{V}{3} + \frac{V}{2} = 2 + 4i_1 \Rightarrow \frac{V}{3} + \frac{V}{2} = \frac{4V}{3} + 2$$

from which $V = -4$ volts

$$i_1 = -\frac{4}{3}, \quad i_2 = -2 \text{ A}$$

1.10.5 Parallel Resistance

When the circuit is connected in parallel, the total resistance of the circuit decreases as the number of resistors connected in parallel increases. If we consider m parallel branches in a circuit as shown in Fig. 1.97, the current equation is

$$I_T = I_1 + I_2 + \ldots + I_m$$

The same voltage is applied across each resistor. By applying Ohm's law, the current in each branch is given by

$$I_1 = \frac{V_s}{R_1}, I_2 = \frac{V_s}{R_2}, \ldots I_m = \frac{V_s}{R_m}$$

According to Kirchhoff's current law,

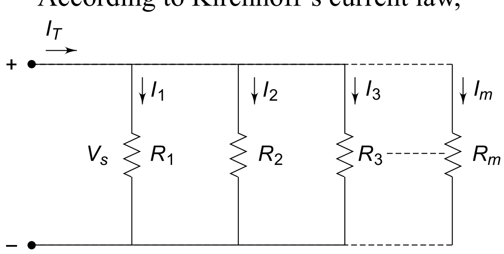


Fig. 1.97

$$I_T = I_1 + I_2 + I_3 + \ldots + I_m$$

$$\frac{V_s}{R_T} = \frac{V_s}{R_1} + \frac{V_s}{R_2} + \frac{V_s}{R_3} + \ldots + \frac{V_s}{R_m}$$

From the above equation, we have

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_m}$$

Example 1.53 *Determine the parallel resistance between points A and B of the circuit shown in Fig. 1.98.*

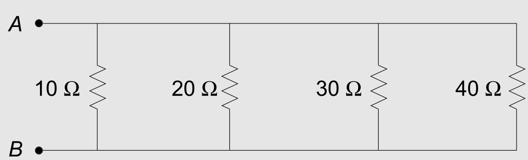


Fig. 1.98

Solution $$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

$$\frac{1}{R_T} = \frac{1}{10} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$$

$$= 0.1 + 0.05 + 0.033 + 0.025 = 0.208$$

or $$R_T = 4.8\ \Omega$$

Example 1.54 *Determine the total current in the circuit shown in Fig. 1.99.*

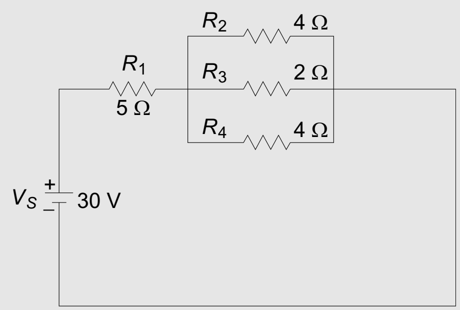


Fig. 1.99

Solution Resistances R_2, R_3 and R_4 are in parallel

$\therefore$ Equivalent resistance $R_5 = R_2 \parallel R_3 \parallel R_4$

$$= \frac{1}{1/R_2 + 1/R_3 + 1/R_4}$$

$\therefore$ $$R_5 = 1\ \Omega$$

R_1 and R_5 are in series,

$\therefore$ Equivalent resistance $R_T = R_1 + R_5 = 5 + 1 = 6\ \Omega$

And the total current $I_T = \dfrac{V_s}{R_T} = \dfrac{30}{6} = 5\text{ A}$

1.10.6 Current Division

In a parallel circuit, the current divides in all branches. Thus, a parallel circuit acts as a current divider. The total current entering into the parallel branches is divided into the branches currents according to the resistance values. The branch having higher resistance allows lesser current, and the branch with lower resistance allows more current. Let us find the current division in the parallel circuit shown in Fig. 1.100.

The voltage applied across each resistor is V_s. The current passing through each resistor is given by

$$I_1 = \frac{V_s}{R_1},\ I_2 = \frac{V_s}{R_2}$$

If R_T is the total resistance, which is given by $R_1R_2/(R_1 + R_2)$,

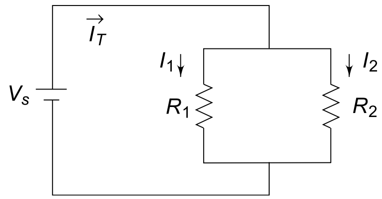


Fig. 1.100

$$\text{Total current } I_T = \frac{V_s}{R_T} = \frac{V_s}{R_1 R_2}(R_1 + R_2)$$

$$\text{or } I_T = \frac{I_1 R_1}{R_1 R_2}(R_1 + R_2) \text{ since } V_s = I_1 R_1$$

$$I_1 = I_T \cdot \frac{R_2}{R_1 + R_2}$$

Similarly,
$$I_2 = I_T \cdot \frac{R_1}{R_1 + R_2}$$

From the above equations, we can conclude that the current in any branch is equal to the ratio of the opposite branch resistance to the total resistance value, multiplied by the total current in the circuit. In general, if the circuit consists of *m* branches, the current in any branch can be determined by

$$I_i = \frac{R_T}{R_i + R_T} I_T$$

where I_i represents the current in the *i*th branch
R_i is the resistance in the *i*th branch
R_T is the total parallel resistance to the *i*th branch, and
I_T is the total current entering the circuit.

Example 1.55 *Determine the current through each resistor in the circuit shown in Fig. 1.101.*

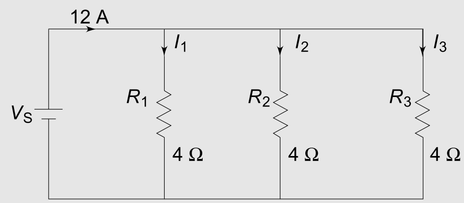


Fig. 1.101

Solution $I_1 = I_T \times \frac{R_T}{(R_1 + R_T)}$

where
$$R_T = \frac{R_2 R_3}{R_2 + R_3} = 2\ \Omega$$

$\therefore$
$$R_1 = 4\ \Omega$$
$$I_T = 12\ \text{A}$$
$$I_1 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

Similarly,
$$I_2 = 12 \times \frac{2}{2+4} = 4\ \text{A}$$

and $$I_3 = 12 \times \frac{2}{2+4} = 4\text{ A}$$

Since all parallel branches have equal values of resistance, they share current equally.

Example 1.56 *Determine the current delivered by the source in the circuit shown in Fig. 1.102.*

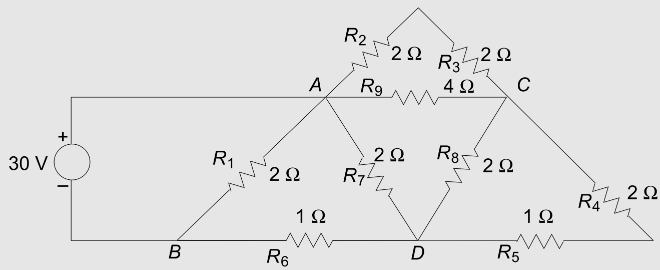


Fig. 1.102

Solution The circuit can be modified as shown in Fig. 1.103, where R_{10} is the series combination of R_2 and R_3.

$\therefore \quad R_{10} = R_2 + R_3 = 4\ \Omega$

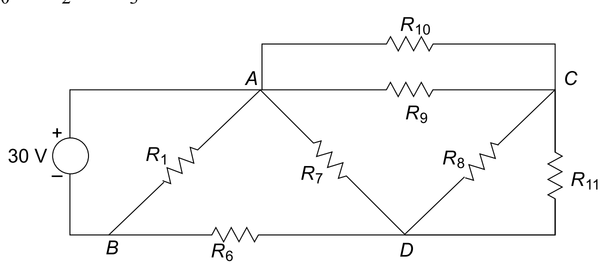


Fig. 1.103

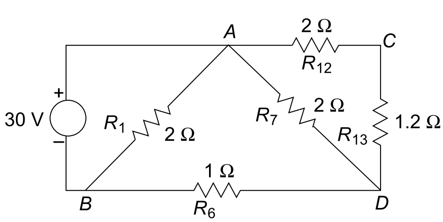


Fig. 1.104

R_{11} is the series combination of R_4 and R_5

$\therefore \quad R_{11} = R_4 + R_5 = 3\ \Omega$

Further simplification of the circuit leads to Fig. 1.104 where R_{12} is the parallel combination of R_{10} and R_9.

$\therefore \quad R_{12} = (R_{10} \parallel R_9) = (4 \parallel 4) = 2\ \Omega$

Similarly, R_{13} is the parallel combination of R_{11} and R_8

$\therefore \quad R_{13} = (R_{11} \parallel R_8) = (3 \parallel 2) = 1.2\ \Omega$

In Fig. 1.103 as shown, R_{12} and R_{13} are in series, which is in parallel with R_7 forming R_{14}. This is shown in Fig. 1.105.

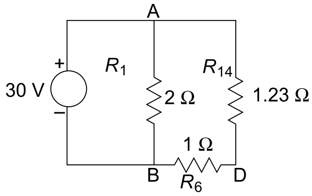


Fig. 1.105

$$\therefore \quad R_{14} = [(R_{12} + R_{13})//R_7]$$
$$= [(2 + 1.2)//2] = 1.23\ \Omega$$

Further, the resistances R_{14} and R_6 are in series, which is in parallel with R_1 and gives the total resistance

$$R_T = [(R_{14} + R_6)//R_1]$$
$$= [(1 + 1.23)//(2)] = 1.05\ \Omega$$

The current delivered by the source = 30/1.05 = 28.57 A

Example 1.57 *Determine the current in the 10 Ω resistance and find V_s in the circuit shown in Fig. 1.106.*

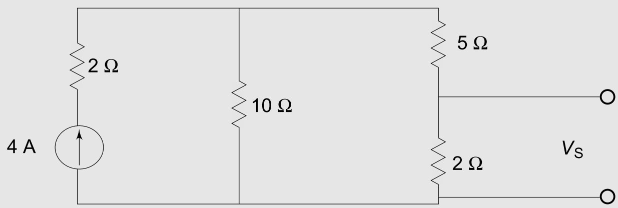


Fig. 1.106

Solution The current in 10 Ω resistance

$$I_{10} = \text{total current} \times (R_T)/(R_T + R_{10})$$

where R_T is the total parallel resistance.

$$I_{10} = 4 \times \frac{7}{17} = 1.65\ \text{A}$$

Similarly, the current in resistance R_5 is

$$I_5 = 4 \times \frac{10}{10+7} = 2.35\ \text{A}$$

or 4 – 1.65 = 2.35 A

The same current flows through the 2 Ω resistance.

$\therefore$ Voltage across 2 Ω resistance, $V_s = I_5 \times 2 = 2.35 \times 2 = 4.7$ V

Example 1.58 *Determine the value of resistance R and current in each branch when the total current taken by the circuit shown in Fig. 1.107 is 6 A.*

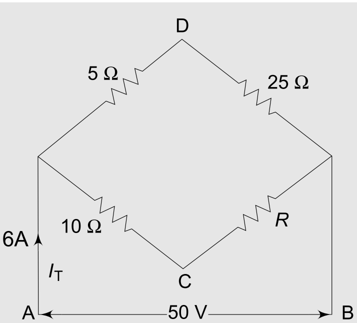


Fig. 1.107

Solution The current in branch *ADB*

$$I_{30} = 50/(25 + 5) = 1.66 \text{ A}$$

The current in branch *ACB* $I_{10} +_R = 50/(10 + R)$.

According to Kirchhoff's current law

$$I_T = I_{30} + I_{(10+R)}$$

$$6\text{A} = 1.66 \text{ A} + I_{10+R}$$

$$\therefore \quad I_{10+R} = 6 - 1.66 = 4.34 \text{ A}$$

$$\therefore \quad \frac{50}{10+R} = 4.34$$

$$10 + R = \frac{50}{4.34} = 11.52$$

$$R = 1.52\ \Omega$$

1.10.7 Power in Parallel Circuit

The total power supplied by the source in any parallel resistive circuit is equal to the sum of the powers in each resistor in parallel, i.e.

$$P_S = P_1 + P_2 + P_3 + \ldots + P_m$$

where m is the number of resistors in parallel, P_S is the total power and P_m is the power in the last resistor.

Example 1.59 *Find the current in the 10 Ω resistance, V_1, and source voltage V_s in the circuit shown in Fig. 1.108.*

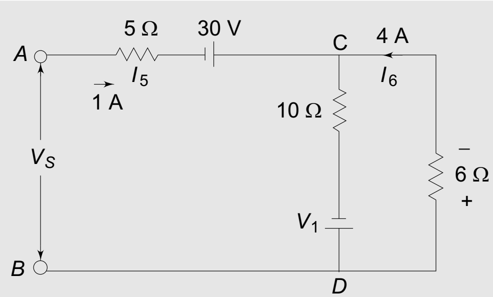


Fig. 1.108

Solution Assume voltage at node $C = V$

By applying Kirchhoff's current law, we get the current in the 10 Ω resistance

$$I_{10} = I_5 + I_6 = 4 + 1 = 5 \text{ A}$$

The voltage across the 6 Ω resistor is $V_6 = 24$ V

$\therefore$ Voltage at node C is $V_C = -24$ V.

The voltage across branch *CD* is the same as the voltage at node *C*.

Voltage across 10 Ω only = 10 × 5 = 50 V

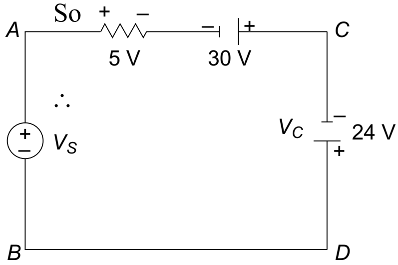


Fig. 1.109

So

$$V_C = V_{10} - V_1$$

$$-24 = 50 - V_1$$

$\therefore$

$$V_1 = 74 \text{ V}$$

Now, consider the loop CABD shown in Fig. 1.109.

If we apply Kirchhoff's voltage law we get

$$V_s = 5 - 30 - 24 = -49 \text{ V}$$

Example 1.60 *Find the power delivered by the source in the circuit shown in Fig. 1.110.*

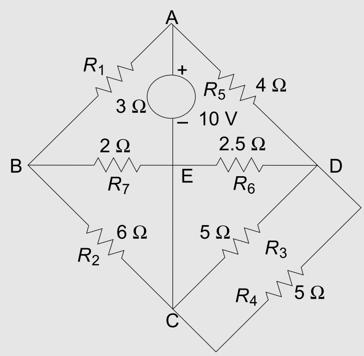


Fig. 1.110

Solution Between points $C(E)$ and D, resistances R_3 and R_4 are in parallel, which gives

$$R_8 = (R_3 // R_4) = 2.5\ \Omega$$

Between points B and $C(E)$, resistances R_2 and R_7 are in parallel, which gives

$$R_9 = (R_2 \parallel R_7) = 1.5\ \Omega$$

Between points $C(E)$ and D, resistances R_6 and R_8 are in parallel and gives

$$R_{10} = (R_6 \parallel R_8) = 1.25\ \Omega$$

The series combination of R_1 and R_9 gives

$$R_{11} = R_1 + R_9 = 3 + 1.5 = 4.5\ \Omega$$

Similarly, the series combination of R_5 and R_{10} gives

$$R_{12} = R_5 + R_{10} = 5.25\ \Omega$$

The resistances R_{11} and R_{12} are in parallel, which gives

Total resistance = $(R_{11} \parallel R_{12})$ = 2.42 ohms

These reductions are shown in Figs 1.111(a), (b), (c) and (d).

$$\text{Current delivered by the source} = \frac{10}{2.42} = 4.13 \text{ A}$$

$$\text{Power delivered by the source} = VI$$

$$= 10 \times 4.13 = 41.3 \text{ W}$$

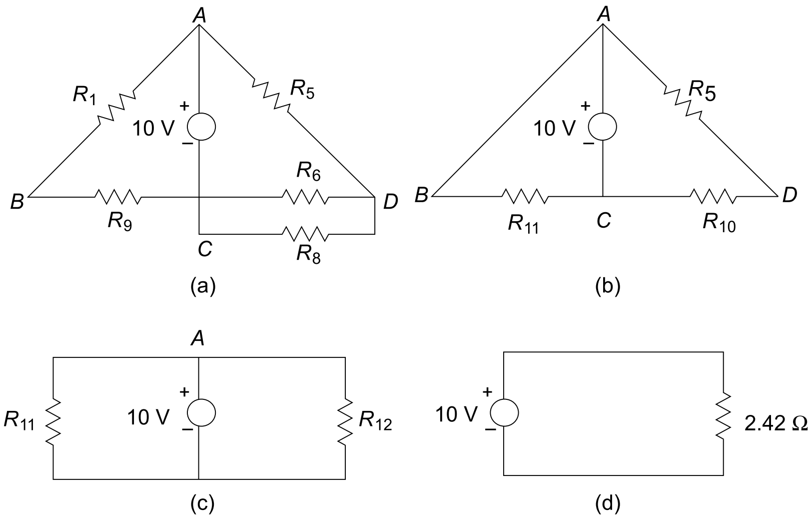


Fig. 1.111(a, b, c and d)

Example 1.61 *Determine the voltage drop across the 10 Ω resistance in the circuit as shown in Fig. 1.112.*

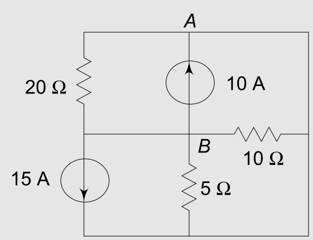


Fig. 1.112

Solution The circuit is redrawn as shown in Fig. 1.113.

This is a single node pair circuit. Assume voltage V_A at node A. By applying Kirchhoff's current law at node A, we have

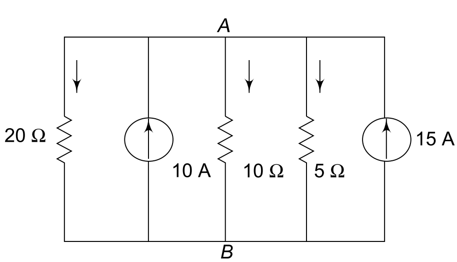


Fig. 1.113

$$\frac{V_A}{20}+\frac{V_A}{10}+\frac{V_A}{5}=10+15$$

$$V_A\left[\frac{1}{20}+\frac{1}{10}+\frac{1}{5}\right]=25\text{ A}$$

$$V_A(0.05+0.1+0.2)=25\text{ A}$$

$$V_A=\frac{25}{0.35}=71.42\text{ V}$$

The voltage across 10 Ω is nothing but the voltage at node A.

$\therefore \quad V_{10} = V_A = 71.42$ V

Example 1.62 *In the circuit shown in Fig. 1.114 what are the values of R_1 and R_2, when the current flowing through R_1 is 1 A and R_2 is 5 A? What is the value of R_2 when the current flowing through R_1 is zero?*

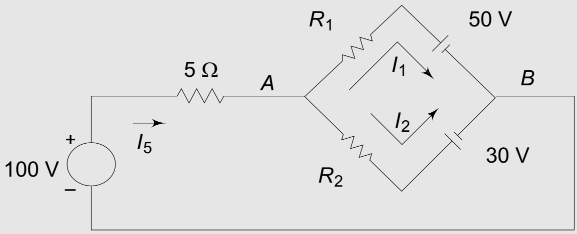


Fig. 1.114

Solution The current in the 5 Ω resistance

$$I_5 = I_1 + I_2 = 1 + 5 = 6 \text{ A}$$

Voltage across resistance 5 Ω is $V_5 = 5 \times 6 = 30$ V

The voltage at node A, $V_A = 100 - 30 = 70$ V

$$\therefore \qquad I_2 = \frac{V_A - 30}{R_2} = \frac{70 - 30}{R_2}$$

$$R_2 = \frac{70 - 30}{I_2} = \frac{40}{5} = 8\,\Omega$$

Similarly, $R_1 = \dfrac{70 - 50}{I_1} = \dfrac{20}{1} = 20\,\Omega$

When $V_A = 50$ V, the current I_1 in resistance R_1 becomes zero

$$\therefore \qquad I_2 = \frac{50 - 30}{R_2}$$

where I_2 becomes the total current

$$\therefore \qquad I_2 = \frac{100 - V_A}{5}$$

$$= \frac{100 - 50}{5} = 10 \text{ A}$$

$$\therefore \qquad R_2 = \frac{20}{I_2}$$

$$= \frac{20}{10} = 2\,\Omega$$

Example 1.63 *Determine the output voltage V_{out} in the circuit shown in Fig. 1.115.*

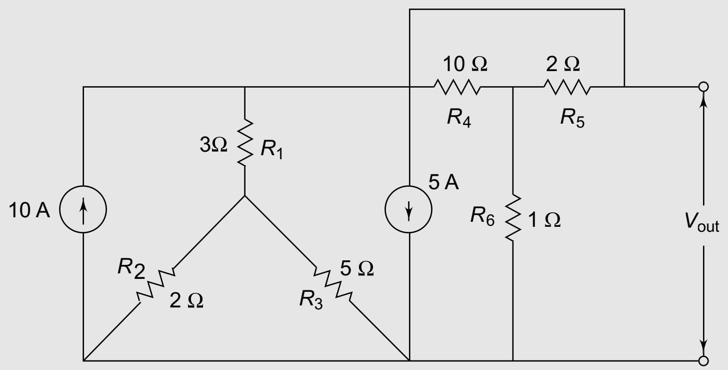


Fig. 1.115

Solution The circuit shown in Fig. 1.115 can be redrawn as shown in Fig. 1.116.

In Fig. 1.116, R_2 and R_3 are in parallel, R_4 and R_5 are in parallel. The complete circuit is a single node pair circuit. Assuming voltage V_A at node A and applying Kirchhoff's current law in the circuit, we have

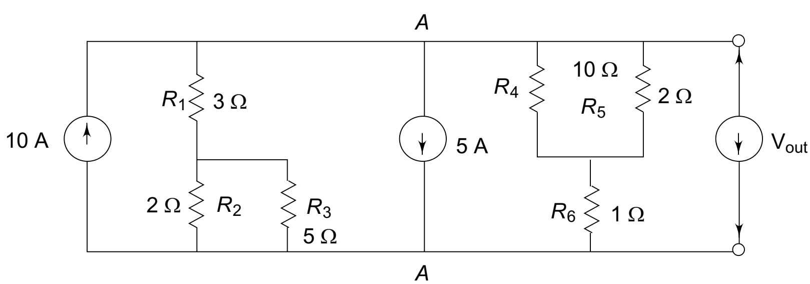


Fig. 1.116

$$10\text{A} - \frac{V_A}{4.43} - 5\text{A} - \frac{V_A}{2.67} = 0$$

$$\therefore \quad V_A\left[\frac{1}{4.43} + \frac{1}{2.67}\right] = 5\text{ A}$$

$$V_A[0.225 + 0.375] = 5$$

$$\therefore \quad V_A = \frac{5}{0.6} = 8.33\text{ V}$$

$$V_{\text{out}} = V_A = 8.33\text{ V}$$

Example 1.64 *Determine the voltage V_{AB} in the circuit shown in Fig. 1.117.*

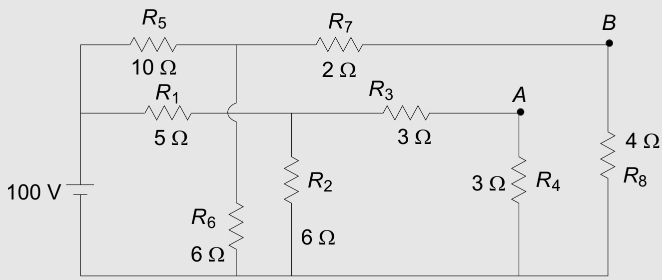


Fig. 1.117

Solution The circuit in Fig. 1.117 can be redrawn as shown in Fig. 1.118(a).

At node 3, the series combination of R_7 and R_8 are in parallel with R_6, which gives $R_9 = [(R_7 + R_8)//R_6] = 3\ \Omega$.

At node 2, the series combination of R_3 and R_4 are in parallel with R_2, which gives $R_{10} = [(R_3 + R_4)//R_2] = 3\ \Omega$.

It is further reduced and is shown in Fig. 1.118(b).

Simplifying further we draw it as shown in Fig. 1.118(c).

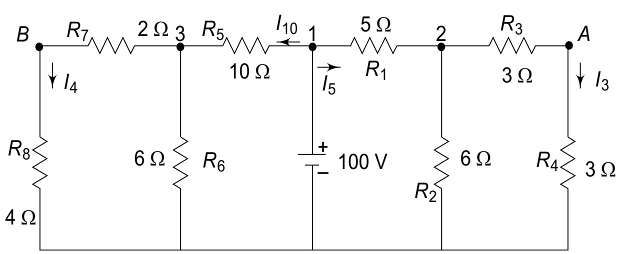


Fig. 1.118(a)

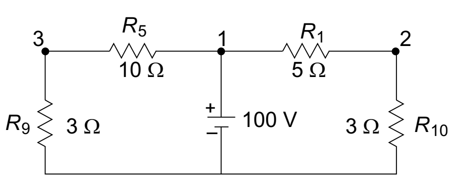


Fig. 1.118(b)

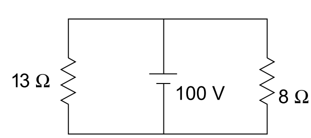


Fig. 1.118(c)

$$\text{Total current delivered by the source} = \frac{100}{R_T}$$

$$= \frac{100}{(13//8)} = 20.2\ \text{A}$$

$$\text{Current in the } 8\ \Omega \text{ resistor is } I_8 = 20.2 \times \frac{13}{13+8} = 12.5\ \text{A}$$

$$\text{Current in the } 13\ \Omega \text{ resistor is } I_{13} = 20.2 \times \frac{8}{13+8} = 7.69\ \text{A}$$

So $I_5 = 12.5$ A, and $I_{10} = 7.69$ A

Current in the 4 Ω resistance $I_4 = 3.845$ A

Current in the 3 Ω resistance $I_3 = 6.25$ A

$$V_{AB} = V_A - V_B$$

Where $\qquad V_A = I_3 \times 3\ \Omega = 6.25 \times 3 = 18.75\ \text{V}$

$$V_B = I_4 \times 4\ \Omega = 3.845 \times 4 = 15.38\ \text{V}$$

$\therefore \qquad V_{AB} = 18.75 - 15.38 = 3.37\ \text{V}$

Example 1.65 *Determine the value of R in the circuit shown in Fig. 1.119, when the current is zero in the branch CD.*

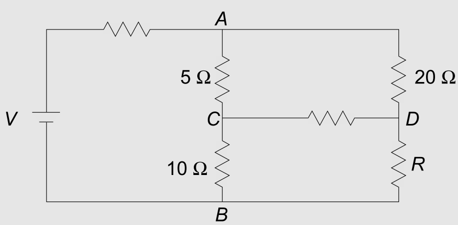


Fig. 1.119

Solution The current in the branch *CD* is zero, if the potential difference across *CD* is zero.

That means, voltage at point C = voltage at point D.

Since no current is flowing, the branch *CD* is open circuited. So the same voltage is applied across *ACB* and *ADB*

$$V_{10} = V_A \times \frac{10}{15}$$

$$V_R = V_A \times \frac{R}{20+R}$$

$\therefore \qquad V_{10} = V_R$

and $\qquad V_A \times \dfrac{10}{15} = V_A \times \dfrac{R}{20+R}$

$\therefore \qquad R = 40\ \Omega$

Example 1.66 *Find the power absorbed by each element in the circuit shown in Fig. 1.120.*

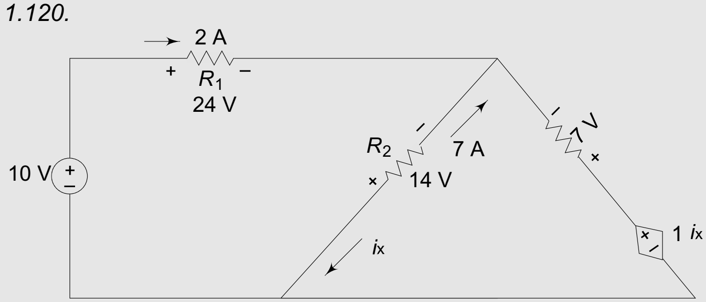


Fig. 1.120

Solution Power absorbed by any element = VI

where V is the voltage across the element and I is the current passing through that element

Here potential rises are taken as (–) sign.

Power absorbed by 10 V source = –10 × 2 = –20 W

Power absorbed by resistor R_1 = 24 × 2 = 48 W

Power absorbed by resistor R_2 = 14 × 7 = 98 W

Power absorbed by resistor R_3 = –7 × 9 = –63 W

Power absorbed by dependent voltage source = (1 × –7) × 9 = –63 W

Example 1.67 *Show that the algebraic sum of the five absorbed power values in Fig. 1.121 is zero.*

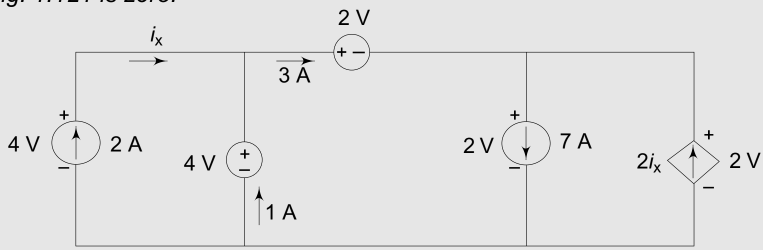


Fig. 1.121

Solution Power absorbed by 2 A current source = (–4) × 2 = –8 W

Power absorbed by 4 V voltage source = (–4) × 10 = –4 W

Power absorbed by 2 V voltage source = (2) × 3 = 6 W

Power absorbed by 7 A current source = (7) × 2 = 14 W

Power absorbed by $2i_x$ dependent current source = (–2) × 2 × 2 = –8 W

Hence, the algebraic sum of the five absorbed power values is zero.

Example 1.68 *For the circuit shown in Fig. 1.122, find the power absorbed by each of the elements.*

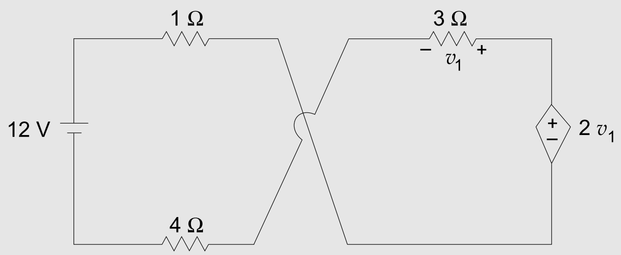


Fig. 1.122

Solution The above circuit can be redrawn as shown in Fig. 1.123.

Assume loop current I as shown in Fig. 1.223.

If we apply Kirchhoff's voltage law, we get

$$-12 + I - 2\upsilon_1 + \upsilon_1 + 4I = 0$$

The voltage across 3 Ω resistor is $\upsilon_1 = 3I$

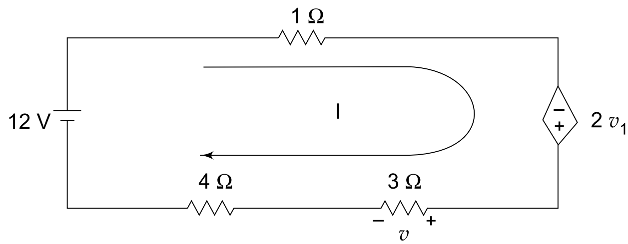


Fig. 1.123

Substituting υ_1 in the loop equation, we get $I = 6$ A

Power absorbed by the 12 V source = (–12) × 6 = –72 W

Power absorbed by the 1 Ω resistor = 6 × 6 = 36 W

Power absorbed by $2\upsilon_1$ dependent voltage source

$$= (2\upsilon_1)I = 2 \times 3 \times 6 \times 6 = -216 \text{ W}$$

Power absorbed by 3 Ω resistor = $\upsilon_1 \times I = 18 \times 6 = 108$ W

Power absorbed by 4 Ω resistor = 4 × 6 × 6 = 144 W

Example 1.69 *For the circuit shown in Fig. 1.124, find the power absorbed by each element.*

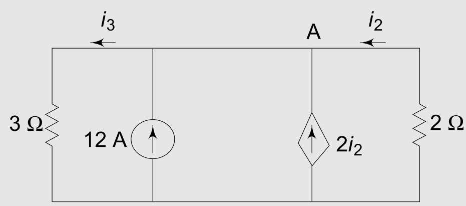


Fig. 1.124

Solution The circuit shown in Fig. 1.124 is a parallel circuit and consists of a single node A. By assuming voltage V at node A, we can find the current in each element.

According to Kirchhoff's current law

$$i_3 - 12 - 2i_2 - i_2 = 0$$

By using Ohm's law, we have

$$i_3 = \frac{V}{3},\ i_2 = \frac{-V}{2}$$

$$V\left[\frac{1}{3} + 1 + \frac{1}{2}\right] = 12$$

$$\therefore \qquad V = \frac{12}{1.83} = 6.56$$

$$i_3 = \frac{6.56}{3} = 2.187\text{A};\ i_2 = \frac{-6.56}{2} = -3.28\text{ A}$$

Power absorbed by the 3 Ω resistor = (+6.56)(2.187) = 14.35 W

Power absorbed by 12 A current source = (–6.56)12 = –78.72 W

Power absorbed by $2i_2$ dependent current source

$$= (-6.56) \times 2 \times (-3.28) = 43.03 \text{ W}$$

Power absorbed by 2 Ω resistor = (–6.52)(–3.28) = 21.51 W

Example 1.70 *Find the value of E in the network shown in Fig. 1.125.*

[JNTU April/May 2007]

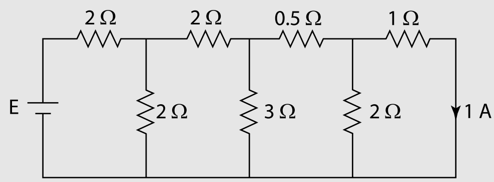


Fig. 1.125(a)

Solution Calculating current through all branches

$$E = 2 \times I + 5.916$$
$$E = 2 \times 5.04 + 5.916$$
$$E = 15.99 \text{ V}$$
$$E = 16 \text{ V}$$

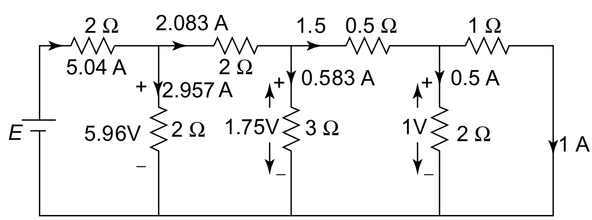


Fig. 1.125(b)

Example 1.71 *Determine the current through the 6-Ω resistor and the power supplied by the current source for the circuit shown in Fig. 1.126.*

[JNTU April/May 2006]

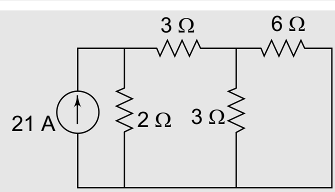


Fig. 1.126(a)

Solution Current through 6-Ω resistor and power supplied by the current source

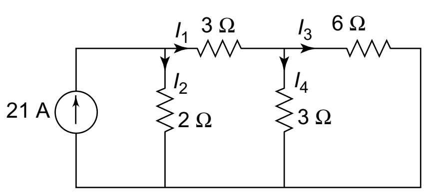


Fig. 1.126(b)

$$I_1 + I_2 = 21 \text{ A}$$
$$I_2 = \frac{21 \times 5}{7} = 15 \text{ A}$$
$$I_1 = 6 \text{ A}$$
$$I_1 = I_3 + I_4 = 6 \text{ A}$$
$$I_4 = \frac{6 \times 6}{9} = 4 \text{ A}, I_3 = 2 \text{ A}$$

Current through 6-Ω resistor is $= I_3 = 2$ A

$$I_3 = 2 \text{ A}$$

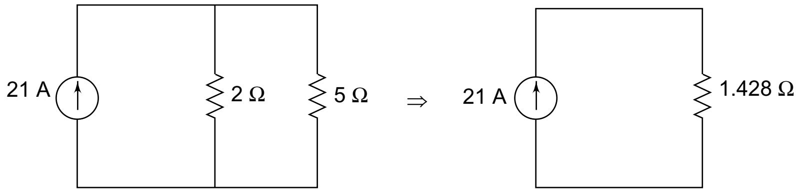


Fig. 1.127

Power supplied by the current source.

Power supplied by current source = Power consumed in the resistor.

$$= I^2R = (21)^2 \times 1.428$$

$$P = 629.748 \text{ W}$$

Exaample 1.72 *A circuit consisting of three resistances 12 Ω, 18Ω and 36 Ω respectively joined in parallel is connected in series with a fourth resistance. The whole circuit is applied with 60 V and it is found that the power dissipated in the 12 Ω resistor is 36 W. Determine the value of the fourth resistance and the total power dissipated in the circuit.* *[JNTU May/June 2008]*

Solution Given that 12 Ω, 18 Ω and 36 Ω respectively joined in parallel to each other. Let the fourth resistance be R Ω which is in series with the parallel combination as shown in the Fig. 1.128.

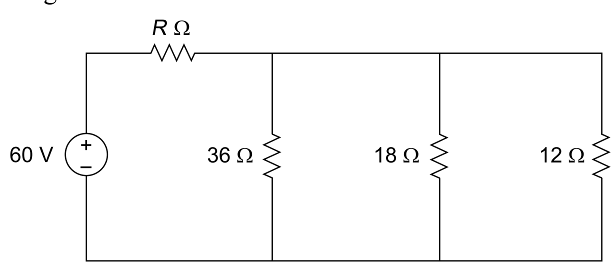


Fig. 1.128(a)

Equivalent resistance of parallel combination

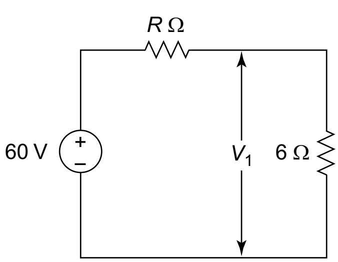


Fig. 1.128(b)

$$\frac{1}{R_{eq}} = \frac{1}{12} + \frac{1}{18} + \frac{1}{36}$$

$$R_{eq} = 6\ \Omega$$

The figure reduces to

Voltage across 6 Ω resistor , $V_1 = \dfrac{60 \times 6}{6 + R}$ [voltage division]

As the voltage across 12 Ω is also V_1 and it is given that power dissipated by 12 Ω is 36 W

$$V_1^2/R = 36 \text{ W}$$

$$\frac{(60\times 6)^2}{(6+R)^2\times 12}=36$$

$$(6 + R)^2 = 60 \times 5$$

$$R^2 + 12R + 36 = 300$$

$$R^2 + 12R - 264 = 0$$

$$R=-12\pm\sqrt{\frac{144+4(264)}{2}}=11.32\ \Omega$$

The current I flowing in the circuit is

$$I=\frac{60}{6+11.32}=3.464\text{ A}$$

Total power dissipated in the circuit $P = VI$

$$= 60 \times 3.464 = 207.852\text{ W}$$

Example 1.73 *A circuit consists of three resistors of 3 ohms, 4 ohms and 6 ohms in parallel and a fourth resistor of 4 ohms in series. A battery of 12 V emf and an internal resistance of 6 ohms is connected across the circuit. Find the total current in the circuit and terminal voltage across the battery.*

[JNTU May/June 2008]

Solution Three resistors of 3 Ω, 4 Ω and 6 Ω are in parallel and a fourth resistor of 4 Ω is in series.

The 12 V battery has a internal resistance of 6 Ω.

The circuit can be taken as

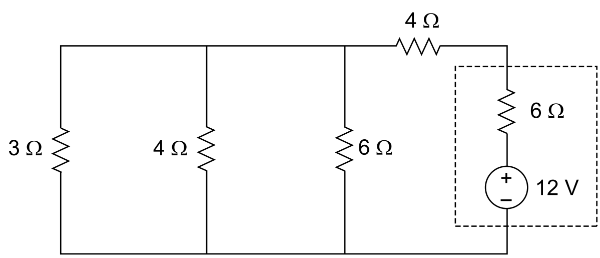


Fig. 1.129

The circuit can be reduced to as shown in Fig. 1.130

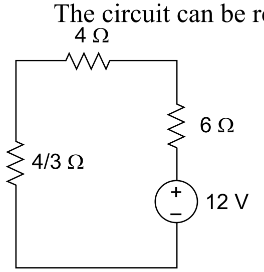


Fig. 1.130

The current I flowing in the circuit

$$=\frac{12}{10+4/3}=1.0588\text{ A}$$

Terminal voltage = battery voltage − drop due to internal resistance

$$12 - 6 \times 1.0588 = 5.647\text{ V}$$

Example 1.74 *A 50 ohm resistor is in parallel with a 100 ohm resistor. The current in a 50 ohm resistor is 7.2 A. What is the value of the third resistance to be added in parallel to make the line current as 12.1 A?.* *[JNTU May/June 2008]*

Solution A 50 Ω resistor is in parallel with 100 Ω. The current in 50 Ω is 7.2 Ω. Let the third resistance be R Ω.

The line current is 12.1 A.

The circuit is

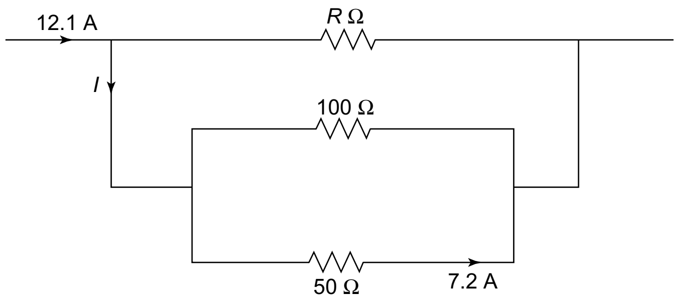


Fig. 1.131

Let I be the current flowing through parallel combination of 100 and 50 Ω. The current I flowing through 50 Ω resistor is

$$\frac{I \times 100}{150} = 7.2 \text{ [current division]}$$

$$I = 10.8\ A$$

The current through R Ω is = 12.1 − 10.8 = 1.3 A.

Thus, by current division

$$1.3 = \frac{12.1 \times 33.33}{R + 33.33}$$

$$1.3\ R + 1.3 \times 33.33 = 12.1 \times 33.33$$

$$1.3\ R = 359.99$$

$$R = 276.92\ \Omega$$

Example 1.75 *Find the current delivered by the source for the network shown in Fig. 1.132 using network reductions technique.* *[JNTU June 2009]*

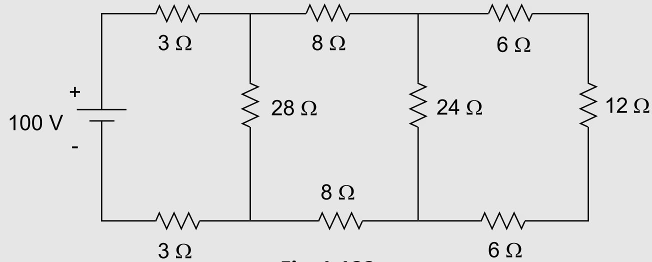


Fig. 1.132

Solution

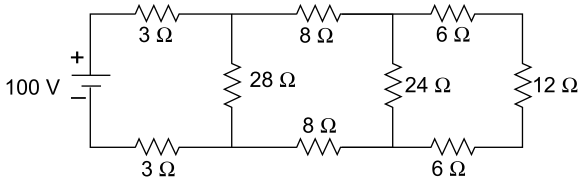


Fig. 1.133(a)

Replacing series combination of $6\,\Omega$, $12\,\Omega$ and $6\,\Omega$ by $(6 + 12 + 6)\,\Omega = 24\,\Omega$

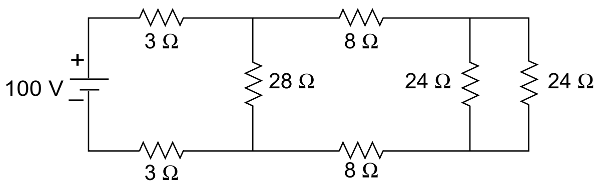


Fig. 1.133(b)

Replacing parallel combination of $24\,\Omega$ and $24\,\Omega$ by $\dfrac{24 \times 24}{24 + 24}\,\Omega = 12\,\Omega$

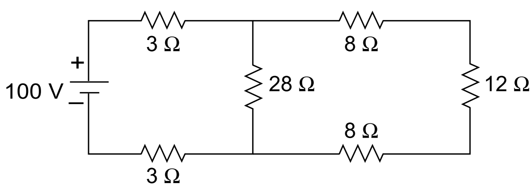


Fig. 1.133(c)

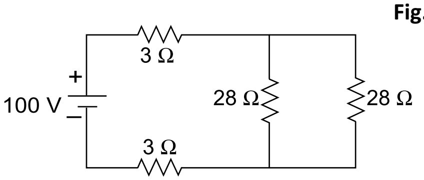


Fig. 1.133(d)

Replacing series combination of $8\,\Omega$, $12\,\Omega$ and $8\,\Omega$ by $(8 + 12 + 8)\,\Omega = 28\,\Omega$

Replacing parallel combination of $28\ \Omega$ and $28\ \Omega$ by $\dfrac{28 \times 28}{28 + 28}\,\Omega = 14\,\Omega$

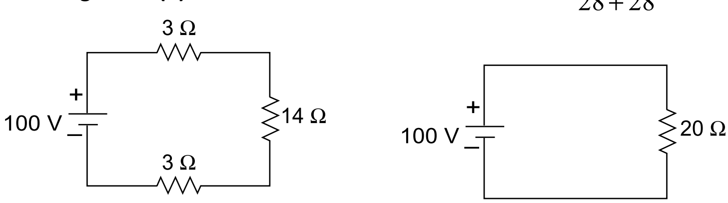


Fig. 1.133(e)

Replacing series combination of $3\,\Omega$, $14\,\Omega$ and $3\,\Omega$ by $(3 + 14 + 3)\ \Omega = 20\,\Omega$

$\therefore$ Current delivered by the source

$$= \frac{100}{20}\ \text{amp}$$

$$= 5\ \text{amp}$$

Example 1.76 *Find the value of applied d.c. voltage for the network, shown in Fig. 1.134.* *[JNTU June 2009]*

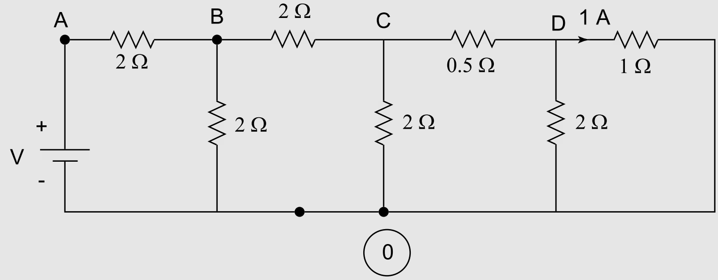


Fig. 1.134

Solution Replacing parallel combination of 1Ω and

$$2\ \Omega \text{ by } \frac{2\times 1}{2+1}\Omega = \frac{2}{3}\Omega$$

Replacing series combination of 0.5 Ω and 2/3 Ω by $\left(\frac{1}{2}+\frac{2}{3}\right)\Omega = \frac{7}{6}\Omega$

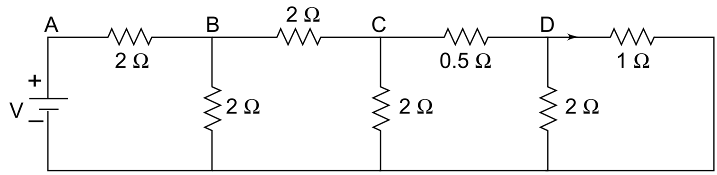


Fig. 1.135(a)

Replacing parallel combination of 2 Ω and 7/6 Ω by $\frac{2\times(7/6)}{2+(7/6)}\Omega = (14/19)\Omega$

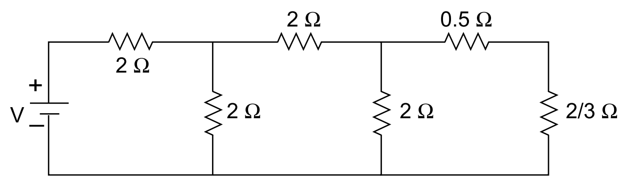


Fig. 1.135(b)

Replacing series combination of 2 Ω and 14/19 Ω by (2 + (14/19)) Ω = (52/19) Ω

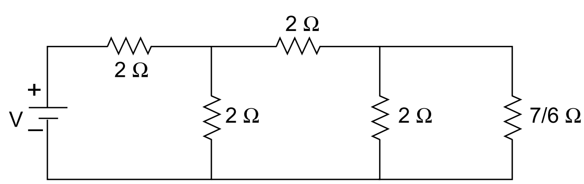


Fig. 1.135(c)

Replacing parallel combination of 2 Ω and (52/19) Ω by

$$\frac{2\times(52/19)}{2+(52/19)}\Omega=(52/45)\Omega$$

Replacing series combination of 2 Ω and (52/45) Ω by (2 + (52/45)) Ω = (142/45) Ω

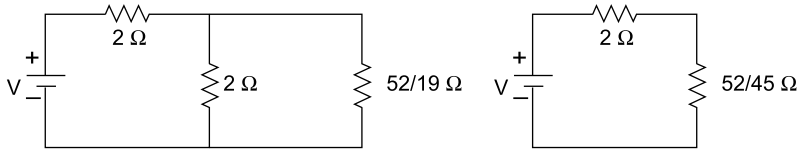


Fig. 1.135(d)

$\therefore$ Total current $=\dfrac{V}{(142/45)}$

$\therefore$ Current through 1 Ω (DO) resistance

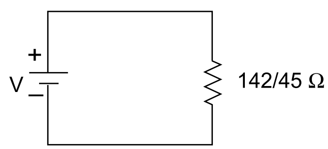


Fig. 1.135(e)

$$=\frac{V}{(142/45)}\times\frac{2}{2+(52/19)}\times\frac{2}{2+(7/6)}\times\frac{2}{2+1}$$

$$=\frac{4V}{71}=1 \quad \text{(according to the question)}$$

$$\therefore V=\frac{71}{4}\text{volt}=17.75\,\text{volt}.$$

Example 1.77 *Three resistances are connected is parallel having the ratio of 1:2:3 the total power consumed is 100 W when 10 V is applied to the combinations, find the values of the resistances.* *[JNTU June 2009]*

Solution Let, the resistances be R, $2R$ and $3R$.

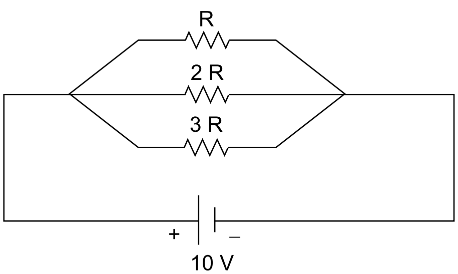


Fig. 1.136

$$\textit{Total power } VI=100$$

$$\textit{Total I current } I=\frac{100}{V}=10A$$

$$I=I_R+I_{2R}+I_{3R}$$

Then according to the question

or, $$\frac{10}{R}+\frac{10}{2R}+\frac{10}{3R}=10$$

$$\frac{1}{R}\left(1+\frac{1}{2}+\frac{1}{3}\right)=1$$

or, $$R=1+\frac{1}{2}+\frac{1}{3}=\frac{6+3+2}{6}=\frac{11}{6}\text{ ohm}$$

$\therefore$ The resistances are $\dfrac{11}{6}$ ohm, $\dfrac{11}{3}$ ohm and $\dfrac{11}{2}$ ohm.

Example 1.78 *Find the current through each element and the total power delivered by the source for the network as shown in Fig. 1.137. [JNTU Jan 2009]*

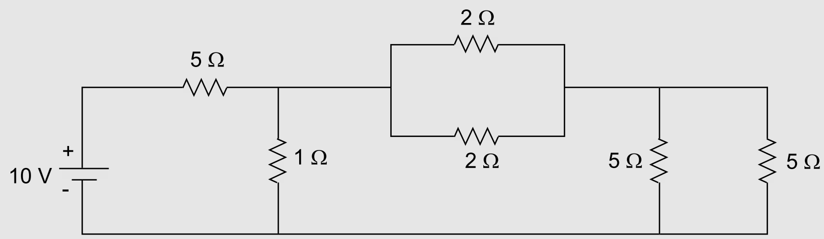


Fig. 1.137

Solution The circuit of Fig. 1.137 can be reduced to the circuit shown in Fig. 1.138.

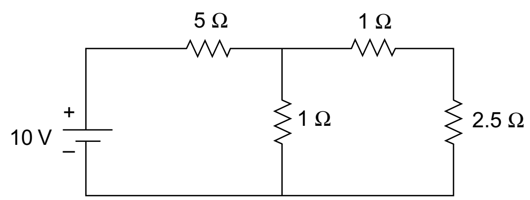


Fig. 1.138(a)

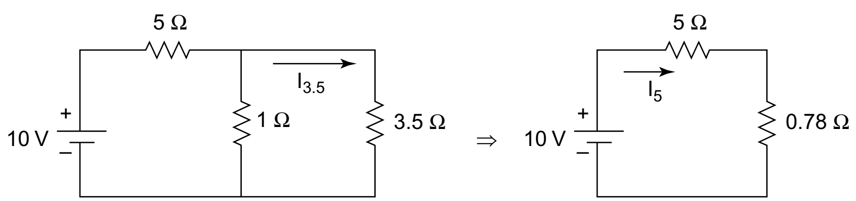


Fig. 1.138(b)

The current through 5Ω resistor

$$I_5 = \frac{10}{5.78} = 1.73\text{A}$$

Total power delivered by $10V$ source $= 1.73 \times 10$

$= 17.3$ watts

Current through 3.5Ω resistance

$$= I_5 \times \frac{1}{1+3.5} = \frac{1.73}{4.5} = 0.385\text{A}$$

Current in 1Ω is same as the current in 3.5Ω

Current in 2Ω is divided equally

$\therefore$ $I_2 = 0.1925\text{A}$

Current in 5Ω is divided equally

$\therefore$ $I_5 = 0.1925\text{A}$

Example 1.79 *Obtain the potential difference V_{AB} in the circuit shown in Fig. 1.139 using Kirchhoff's laws.* *[JNTU Jan 2010]*

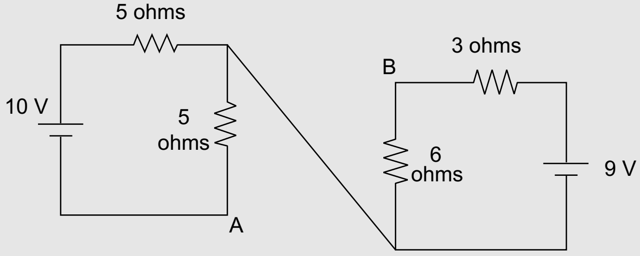


Fig. 1.139

Solution

Using KVL,

For loop-1, $10 = (5 + 5)\, i_1$ or $i_1 = 1$ amp

For loop-2, $9 = (6 + 3)\, i_2$ or $i_2 = 1$ amp

$\therefore$ Voltage drop across 6 Ω (BO) $= 6i_2 = 6$ volt

$\therefore$ Voltage drop across 5 Ω (OA) $= 5i_1 = 5$ volt

Voltage drop across $\quad V_{BA} = V_{BO} + V_{BA} = 11$ volt

$\therefore \quad V_{AB} = -11$ volt

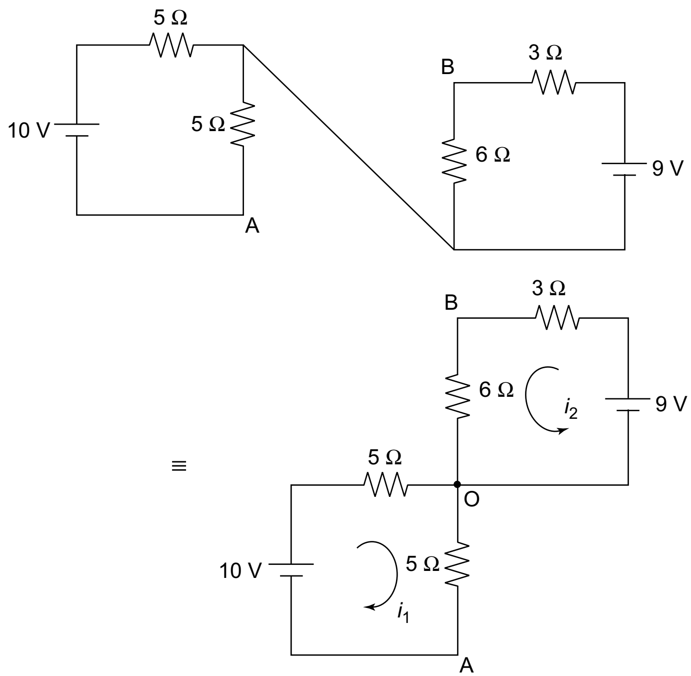


Fig. 1.140

Example 1.80 *In the circuit as shown in figure find the currents in all the resistors. Also calculate the supply voltage and power supplied by the source.*
[JNTU Jan 2010]

Solution Let current through 10 Ω = i amp

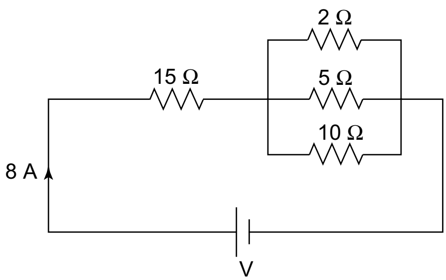


Fig. 1.141

$\therefore$ Current through 5Ω = $2i$ amp

$\therefore$ Current through 2Ω = $5i$ amp

$\therefore$ According to the question:

$$i + 2i + 5i = 8i = 8 \text{ amp}$$

$$\therefore \quad i = 1 \text{ amp}$$

$\therefore$ Current through 10 ohm = 1 amp

Current through 5 ohm = 2 amp

Current through 2 ohm = 5 amp

$$\text{Equivalent impedance} = \left(15 + 1 \Big/ \left[\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right] \text{ohm}\right)$$

$$= 16.25 \text{ ohm}$$

$\therefore$ According to the question,

$$\frac{V}{16.25} = 8$$

or, $\qquad V = 130$ volt

$\therefore$ Power supplied by the source = (130 × 8) watt

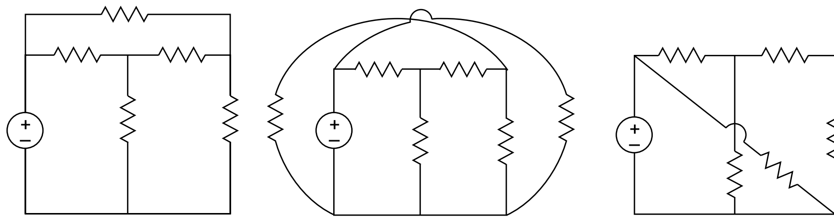
= 1040 watt

1.11 MESH ANALYSIS

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

Mesh analysis is applicable only for planar networks. For non-planar circuits mesh analysis is not applicable. A circuit is said to be planar, if it can be drawn on a plane surface without crossovers. A non-planar circuit cannot be drawn on a plane surface without a crossover.

Figure 1.142(a) is a planar circuit. Figure 1.142(b) is a non-planar circuit and Fig. 1.142(c) is a planar circuit which looks like a non-planar circuit. It has already been discussed that a loop is a closed path. A mesh is defined as a loop which does not contain any other loops within it. To apply mesh analysis, our first step is to check whether the circuit is planar or not and the second is to select mesh currents. Finally, writing Kirchhoff's voltage law equations in terms of unknowns and solving them leads to the final solution.



(a) (b) (c)

Fig. 1.142

Observation of the Fig. 1.143 indicates that there are two loops *abefa*, and *bcdeb* in the network. Let us assume loop currents I_1 and I_2 with directions as indicated in the figure. Considering the loop *abefa* alone, we observe that current I_1 is passing through R_1, and $(I_1 - I_2)$ is passing through R_2. By applying Kirchhoff's voltage law, we can write

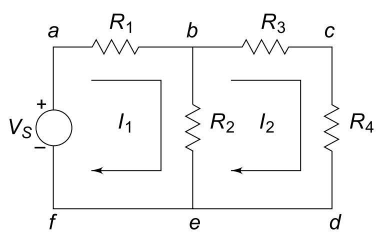


Fig. 1.143

$$V_s = I_1 R_1 + R_2 (I_1 - I_2)$$

Similarly, if we consider the second mesh *bcdeb*, the current I_2 is passing through R_3 and R_4, and $(I_2 - I_1)$ is passing through R_2. By applying Kirchhoff's voltage law around the second mesh, we have

$$R_2 (I_2 - I_1) + R_3 I_2 + R_4 I_2 = 0$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1 (R_1 + R_2) - I_2 R_2 = V_s$$

$$- I_1 R_2 + (R_2 + R_3 + R_4) I_2 = 0 \tag{1.1}$$

By solving the above equations, we can find the currents I_1 and I_2. If we observe Fig. 1.143, the circuit consists of five branches and four nodes, including the reference node. The number of mesh currents is equal to the number of mesh equations.

And the number of equations = branches – (nodes – 1). In Fig. 1.143, the required number of mesh currents would be 5 – (4 – 1) = 2.

In general, if we have B number of branches and N number of nodes including the reference node then the number of linearly independent mesh equations $M = B - (N - 1)$.

Example 1.81 *Write the mesh current equations in the circuit shown in Fig. 1.144, and determine the currents.*

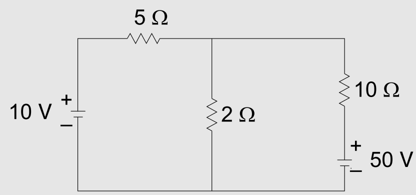


Fig. 1.144

Solution Assume two mesh currents in the direction as indicated in Fig. 1.144.

The mesh current equations are

$$5I_1 + 2(I_1 - I_2) = 10$$

$$10I_2 + 2(I_2 - I_1) + 50 = 0$$

We can rearrange the above equations as

$$7I_1 - 2I_2 = 10$$

$$-2I_1 + 12I_2 = -50$$

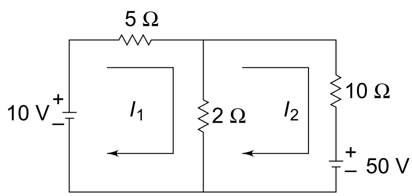


Fig. 1.145

By solving the above equations, we have

$I_1 = 0.25\ A$, and $I_2 = -4.125\ A$

Here the current in the second mesh, I_2, is negative; that is the actual current I_2 flows opposite to the assumed direction of current in the circuit of Fig. 1.145.

Example 1.82 *Determine the mesh current I_1 in the circuit shown in Fig. 1.146.*

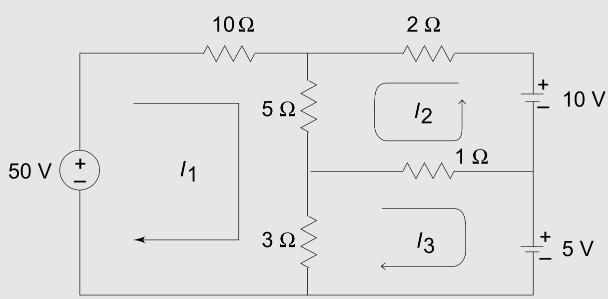


Fig. 1.146

Solution From the circuit, we can form the following three mesh equations

$$10I_1 + 5(I_1 + I_2) + 3(I_1 - I_3) = 50$$

$$2I_2 + 5(I_2 + I_1) + 1(I_2 + I_3) = 10$$

$$3(I_3 - I_1) + 1(I_3 + I_2) = -5$$

Rearranging the above equations we get

$$18I_1 + 5I_2 - 3I_3 = 50$$

$$5I_1 + 8I_2 + I_3 = 10$$

$$-3I_1 + I_2 + 4I_3 = -5$$

According to Cramer's rule

$$I_1 = \frac{\begin{vmatrix} 50 & 5 & -3 \\ 10 & 8 & 1 \\ -5 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{1175}{356}$$

or $\quad I_1 = 3.3$ A

Similarly,

$$I_2 = \frac{\begin{vmatrix} 18 & 50 & -3 \\ 5 & 10 & 1 \\ -3 & -5 & 4 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{-355}{356}$$

or $\quad I_2 = -0.997\ A$

$$I_3 = \frac{\begin{vmatrix} 18 & 5 & 50 \\ 5 & 8 & 10 \\ -3 & 1 & -5 \end{vmatrix}}{\begin{vmatrix} 18 & 5 & -3 \\ 5 & 8 & 1 \\ -3 & 1 & 4 \end{vmatrix}} = \frac{525}{356}$$

or $\quad I_3 = 1.47$ A

$\therefore \quad I_1 = 3.3$ A, $\quad I_2 = -0.997$ A, $\quad I_3 = 1.47$ A

1.11.1 Mesh Equations by Inspection Method

The mesh equations for a general planar network can be written by inspection without going through the detailed steps. Consider a three mesh networks as shown in Fig. 1.147.

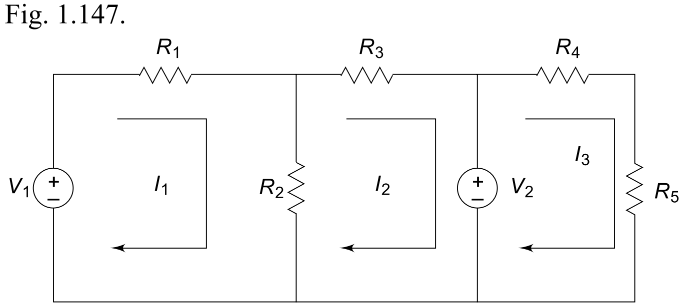


Fig. 1.147

The loop equations are

$$I_1R_1 + R_2(I_1 - I_2) = V_1 \tag{1.2}$$

$$R_2(I_2 - I_1) + I_2R_3 = -V_2 \tag{1.3}$$

$$R_4I_3 + R_5I_3 = V_2 \tag{1.4}$$

Reordering the above equations, we have

$$(R_1 + R_2)\, I_1 - R_2I_2 = V_1 \tag{1.5}$$

$$-R_2I_1 + (R_2 + R_3)I_2 = -V_2 \tag{1.6}$$

$$(R_4 + R_5)I_3 = V_2 \tag{1.7}$$

The general mesh equations for three mesh resistive network can be written as

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{1.8}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{1.9}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{1.10}$$

By comparing the Eqs 1.5, 1.6 and 1.7 with Eqs 1.8, 1.9, and 1.10 respectively, the following observations can be taken into account.

1. The self-resistance in each mesh.
2. The mutual resistances between all pairs of meshes and
3. The algebraic sum of the voltages in each mesh.

The self resistance of loop 1, $R_{11} = R_1 + R_2$, is the sum of the resistances through which I_1 passes.

The mutual resistance of loop 1, $R_{12} = -R_2$, is the sum of the resistances common to loop currents I_1 and I_2. If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign; and if the directions of the currents passing through the common resistance are opposite then the mutual resistance will have a negative sign.

$V_a = V_1$ is the voltage which drives loop one. Here, the positive sign is used if the direction of the current is the same as the direction of the source. If the current direction is opposite to the direction of the source, then the negative sign is used.

Similarly, $R_{22} = (R_2 + R_3)$ and $R_{33} = R_4 + R_5$ are the self resistances of loops two and three, respectively. The mutual resistances $R_{13} = 0$, $R_{21} = -R_2$, $R_{23} = 0$, $R_{31} = 0$, $R_{32} = 0$ are the sums of the resistances common to the mesh currents indicated in their subscripts.

$V_b = -V_2$, $V_c = V_2$ are the sum of the voltages driving their respective loops.

Example 1.83 *Write the mesh equations for the circuit shown in Fig. 1.148.*

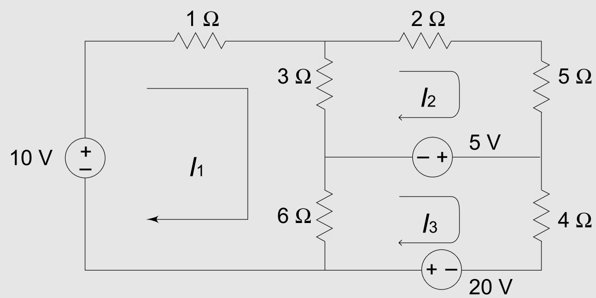


Fig. 1.148

Solution The general equations for three mesh network are

$$R_{11}I_1 \pm R_{12}I_2 \pm R_{13}I_3 = V_a \tag{1.11}$$

$$\pm R_{21}I_1 + R_{22}I_2 \pm R_{23}I_3 = V_b \tag{1.12}$$

$$\pm R_{31}I_1 \pm R_{32}I_2 + R_{33}I_3 = V_c \tag{1.13}$$

Consider Eq. (1.11)

R_{11} = self resistance of loop 1 = (1 Ω + 3 Ω + 6 Ω) = 10 Ω

R_{12} = the mutual resistance common to loop 1 and loop 2 = –3 Ω

Here, the negative sign indicates that the currents are in opposite direction

R_{13} = the mutual resistance common to loop 1 and 3 = –6 Ω

V_a = +10 V, the voltage driving the loop 1.

Here, the positive sign indicates the loop current I_1 is in the same direction as the source element.

Therefore, Eq. (1.11) can be written as

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V} \tag{1.14}$$

Consider Eq. (1.12)

R_{21} = mutual resistance common to loop 1 and loop 2 = –3 Ω

R_{22} = self resistance of loop 2 = (3 Ω + 2 Ω + 5 Ω) = 10 Ω

R_{23} = 0, there is no common resistance between loop 2 and loop 3.

V_b = –5 V, the voltage driving the loop 2.

Therefore, Eq. (1.12) can be written as

$$-3I_1 + 10I_2 = -5 \text{ V} \tag{1.15}$$

Consider Eq. (1.13)

R_{31} = mutual resistance common to loop 3 and loop 1 = –6 Ω

R_{32} = mutual resistance common to loop 3 and loop 2 = 0

R_{33} = self resistance of loop 3 = (6 Ω + 4 Ω) = 10 Ω

V_c = the algebraic sum of the voltages driving loop 3

= (5 V + 20 V) = 25 V

Therefore, Eq. (1.13) can be written as

$$-6I_1 + 10I_3 = 25 \text{ V} \tag{1.16}$$

The three mesh equation are

$$10I_1 - 3I_2 - 6I_3 = 10 \text{ V}$$
$$-3I_1 + 10I_2 = -5 \text{ V}$$
$$-6I_1 + 10I_3 = 25 \text{ V}$$

Example 1.84 *Determine the power dissipation in the 4 Ω resistor of the circuit shown in Fig. 1.149 by using mesh analysis.*

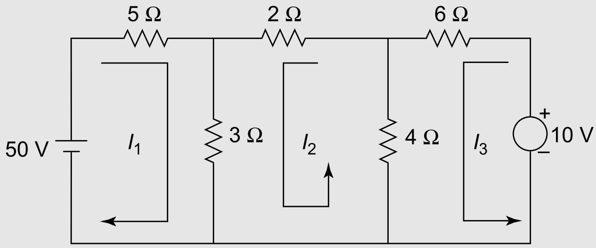


Fig. 1.149

Solution Power dissipated in the 4 Ω resistor is $P_4 = 4(I_2 - I_3)^2$

By using mesh analysis, we can find the currents I_2 and I_3.

From Fig. 1.149, we can form three equations.

From the given circuit in Fig. 1.149, we can obtain three mesh equations in terms of I_1, I_2 and I_3

$$8I_1 + 3I_2 = 50$$
$$3I_1 + 9I_2 - 4I_3 = 0$$
$$-4I_2 + 10I_3 = 10$$

By solving the above equations we can find I_1, I_2 and I_3.

$$I_2 = \frac{\begin{vmatrix} 8 & 50 & 0 \\ 3 & 0 & -4 \\ 0 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & +3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix}} = \frac{-1180}{502} = -2.35 \text{ A}$$

$$I_3 = \frac{\begin{vmatrix} 8 & 3 & 50 \\ 3 & 9 & 0 \\ 0 & -4 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & 3 & 0 \\ 3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix}} = \frac{30}{502} = 0.06 \text{ A}$$

The current in the 4 Ω resistor $= (I_2 - I_3)$

$$= (-2.35 - 0.06)\text{A} = -2.41 \text{ A}$$

Therefore, the power dissipated in the 4 Ω resistor, $P_4 = (2.41)^2 \times 4 = 23.23$ W.

Example 1.85 *Using mesh analysis, determine the voltage V_S which gives a voltage of 50 V across the 10 Ω resistor as shown in Fig. 1.150.*

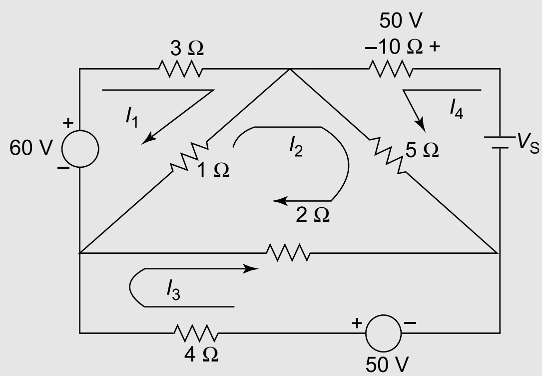


Fig. 1.150

Solution Since the voltage across the 10 Ω resistor is 50 V, the current passing through it is $I_4 = 50/10 = 5$ A.

From Fig. 1.150, we can form four equations in terms of the currents I_1, I_2, I_3 and I_4, as

$$4I_1 - I_2 = 60$$

$$-I_1 + 8I_2 - 2I_3 + 5I_4 = 0$$

$$-2I_2 + 6I_3 = 50$$

$$5I_2 + 15I_4 = VS$$

Solving the above equations, using Cramer's rule, we get

$$I_4 = \frac{\begin{vmatrix} 4 & -1 & 0 & 60 \\ -1 & 8 & -2 & 0 \\ 0 & -2 & 6 & 50 \\ 0 & 5 & 0 & V_S \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 0 & 0 \\ -1 & 8 & -2 & 5 \\ 0 & -2 & 6 & 0 \\ 0 & 5 & 0 & 15 \end{vmatrix}}$$

$$\Delta = 4\begin{vmatrix} 8 & -2 & 5 \\ -2 & 6 & 0 \\ 5 & 0 & 15 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 & 5 \\ 0 & 6 & 0 \\ 0 & 0 & 15 \end{vmatrix}$$

$$= 4\{8(90) + 2(-30) + 5(-30)\} + 1\{-1(90)\}$$

$$\Delta = 1950$$

Example 1.86 *Determine the voltage V which causes the current I_1 to be zero for the circuit shown in Fig. 1.151. Use Mesh analysis.*

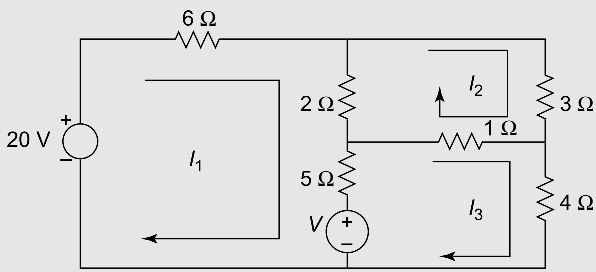


Fig. 1.151

Solution From Fig. 1.151, we can form three loop equations in terms of I_1, I_2, I_3 and V, as follows

$$13I_1 - 2I_2 - 5I_3 = 20 - V$$
$$-2I_1 + 6I_2 - I_3 = 0$$
$$-5I_1 - I_2 + 10I_3 = V$$

Using Cramer's rule, we get

$$I_1 = \frac{\begin{vmatrix} 20 - V & -2 & -5 \\ 0 & 6 & -1 \\ V & -1 & +10 \end{vmatrix}}{\begin{vmatrix} 13 & -2 & -5 \\ -2 & +6 & -1 \\ -5 & -1 & +10 \end{vmatrix}}$$

$$\Delta_1 = (20 - V)(+60 - 1) + 2(V) - 5(-6\,V)$$
$$= 1180 - 27\,V$$

we have $\Delta = 557$

$$I_1 = \frac{\Delta_1}{557}$$

$\therefore$ $\Delta_1 = 0$

$$-27\,V + 1180 = 0$$

$\therefore$ $V = 43.7$ V

$$\Delta_4 = 4\begin{vmatrix} 8 & -2 & 0 \\ -2 & 6 & 50 \\ 5 & 0 & V_S \end{vmatrix} + 1\begin{vmatrix} -1 & -2 & 0 \\ 0 & 6 & 50 \\ 0 & 0 & V_S \end{vmatrix} - 60\begin{vmatrix} -1 & 8 & -2 \\ 0 & -2 & 6 \\ 0 & 5 & 0 \end{vmatrix}$$

$$= 4\{8(6\,V_S) + 2(-2V_S - 250)\} + 1\{-1(6V_S)\} - 60\,\{-1(-30)\}$$

$$= 170\,V_S - 3800$$

$$I_4 = \frac{170V_S - 3800}{1950}$$

$$\therefore \quad V_S = \frac{1950 \times I_4 + 3800}{170} = 79.7\text{ V}$$

Example 1.87 *Write and solve the equation for Mesh Current in the network shown.*

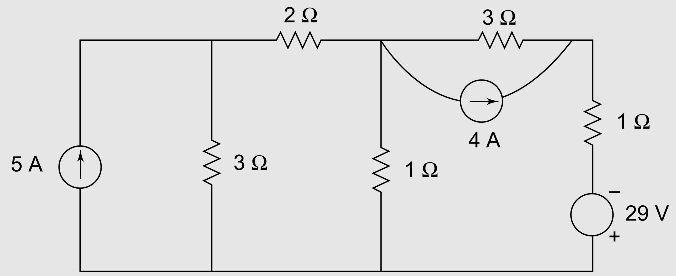


Fig. 1.152

Solution By source transformation technique transform 5 A and 4 A current sources into voltage sources.

5 A current source in parallel with 3 Ω can be transformed to 15 V in series with 3 Ω and 4 A current source in parallel with 3 Ω can be transformed to 12 volts in series with 3 Ω. The equivalent circuit is as shown below:

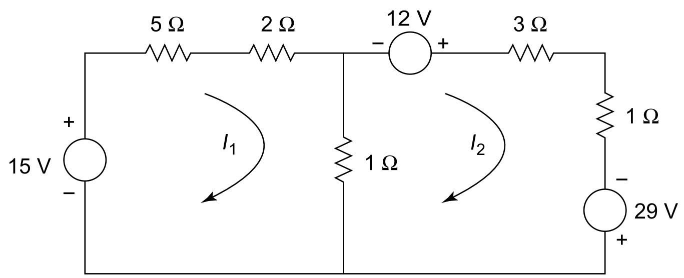


Fig. 1.153

The mesh equations are

$$2I_1 + 5I_1 + 1(I_1 - I_2) = 15$$

$$1(I_2 - I_1) + 4I_2 = 41$$

$$\Rightarrow \quad 8I_1 - I_2 = 15 \quad (1)$$

$$5I_2 - I_1 = 41 \quad (2)$$

on solving equations (1) and (2) we get

$$I_1 = 2.97\text{ Amps}$$

$$I_2 = 8.74\text{ Amps}$$

Example 1.88 *Determine the current in all branches of the following network and the voltage across for resistors using loop method.*

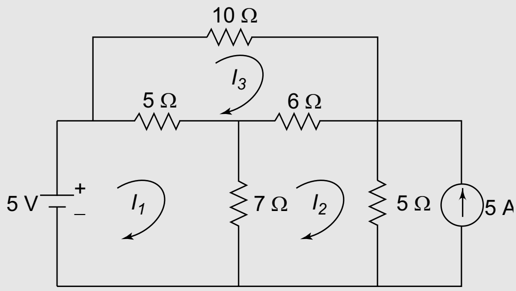


Fig. 1.154

Solution Applying mesh equation to the loops (1), (2) and (3), we get

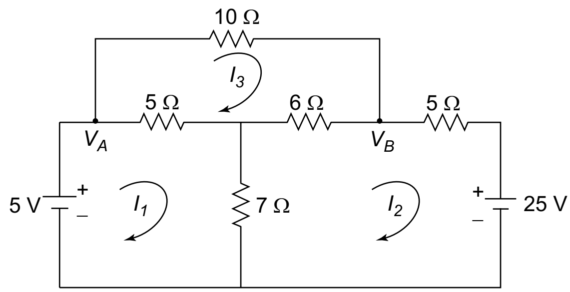


Fig. 1.155

$$5(I_1 - I_3) + 7(I_1 - I_2) = 5$$

$$12I_1 - 7I_2 - 5I_3 = 5 \tag{1}$$

$$7(I_2 - I_1) + 6(I_2 - I_3) + 5I_2 = -25$$

$$-7I_1 + 18I_2 - 6I_3 = -25 \tag{2}$$

$$10I_3 + 5(I_3 - I_1) + 6(I_3 - I_2) = 0$$

$$-5I_1 - 6I_2 + 21I_3 = 0 \tag{3}$$

By solving above three equations, we get

$$I_1 = -1.231 \text{ A}$$

$$I_2 = -2.172 \text{ A}$$

$$I_3 = -0.9138 \text{ A}$$

Current in 5 Ω resistor is −0.3172 A

7 Ω resistor is 0.941 A
6 Ω resistor is 1.2582 A
10 Ω resistor is −0.9138 A
5 Ω resistor is −2.172 A

Example 1.89 *Write the matrix loop equation for the given network and determine the loop currents, as shown in figure and find the current through each element in the network.* *[JNTU June 2009]*

Loop equations

$$4 = 10I_1 - 4I_2 - 4I_3$$
$$0 = -4I_1 + 10I_2 - 4I_3$$
$$0 = -4I_1 - 4I_2 + 10I_3$$

Matrix loop equation

$$\begin{bmatrix} 10 & -4 & -4 \\ -4 & 10 & -4 \\ -4 & -4 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Let $$\Delta = \begin{vmatrix} 10 & -4 & -4 \\ -4 & 10 & -4 \\ -4 & -4 & 10 \end{vmatrix} = 840 - 224 - 224 = 392$$

$$\Delta_1 = \begin{vmatrix} 4 & -4 & -4 \\ 0 & 10 & -4 \\ 0 & -4 & 10 \end{vmatrix} = 336$$

$$\Delta_2 = \begin{vmatrix} 10 & 4 & -4 \\ -4 & 0 & -4 \\ -4 & 0 & 10 \end{vmatrix} = 224$$

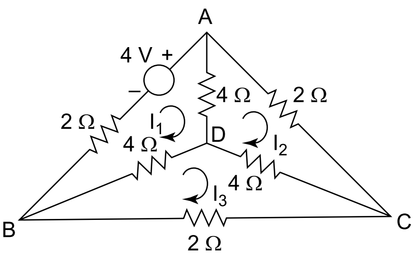


Fig. 1.156

$$\Delta_3 = \begin{vmatrix} 10 & -4 & 4 \\ -4 & 10 & 0 \\ -4 & -4 & 0 \end{vmatrix} = 224$$

$$\therefore \quad I_1 = \frac{\Delta_1}{\Delta} = \frac{336}{392} \text{ amp} = \frac{6}{7} \text{ amp}$$

$$\therefore \quad I_2 = \frac{\Delta_2}{\Delta} = \frac{224}{392} \text{ amp} = \frac{4}{7} \text{ amp}$$

$$\therefore \quad I_3 = \frac{\Delta_3}{\Delta} = \frac{224}{392} \text{ amp} = \frac{4}{7} \text{ amp}$$

$\therefore$ Current through AB $= I_1 = \frac{6}{7}$ amp

$\therefore$ Current through AC $= I_2 = \frac{4}{7}$ amp

$\therefore$ Current through BC $= I_3 = \frac{4}{7}$ amp

$$\therefore \quad \text{Current through AD} = I_1 - I_2 = \left(\frac{6}{7} - \frac{4}{7}\right) \text{amp} = \frac{2}{7} \text{ amp}$$

$$\therefore \quad \text{Current through BD} = I_1 - I_3 = \left(\frac{6}{7} - \frac{4}{7}\right) \text{amp} = \frac{2}{7} \text{ amp}$$

$$\therefore \quad \text{Current through CD} = I_2 - I_3 = \left(\frac{4}{7} - \frac{4}{7}\right) \text{amp} = 0 \text{ amp}$$

Example 1.90 *Find the power dissipated in 1 ohms resistor and voltage drop across each resistor as shown in Fig. 1.157.*

[JNTU Jan 2010]

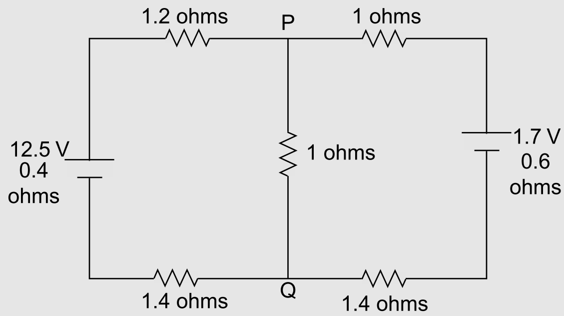


Fig. 1.157

Solution

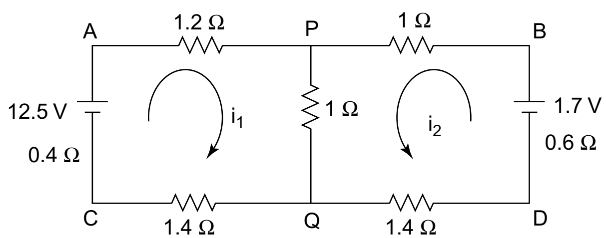


Fig. 1.158

Using KVL,

$$12.5 = 4i_1 + i_2$$

$$1.7 = 4i_2 + i_1$$

$$\therefore \quad i_1 = 3.22 \text{ amp}$$

$$\therefore \quad i_2 = -0.38 \text{ amp}$$

$$\therefore \quad \text{Current through PQ} = (3.22 - 0.38) \text{ amp} = 2.84 \text{ amp}$$

$$\therefore \quad \text{Power dissipation in PQ} = (2.84^2 \times 1) \text{ watt}$$

$$= 8.0656 \text{ watt}$$

$$\text{Voltage drop across AP} = (1.2 \times 3.22) \text{ volt} = 3.864 \text{ V}$$

$$\text{Voltage drop across PB} = -(1 \times 0.38) \text{ volt} = -0.38 \text{ V}$$

$$\text{Voltage drop across PQ} = (1 \times 2.84) \text{ volt} = 2.84 \text{ V}$$

$$\text{Voltage drop across QC} = (1.4 \times 3.22) \text{ volt} = 4.508 \text{ V}$$

$$\text{Voltage drop across QD} = -(1.4 \times 0.38) \text{ volt} = -0.532 \text{ V}$$

Example 1.91 *In the circuit shown in figure, determine the current through the 2 Ω resistor and the total current delivered by the battery. Use Kirchhoff's laws.*

[JNTU Jan 2010]

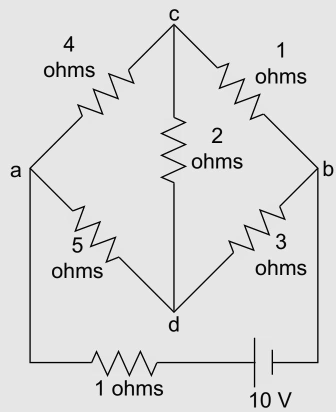


Fig. 1.159

Solution

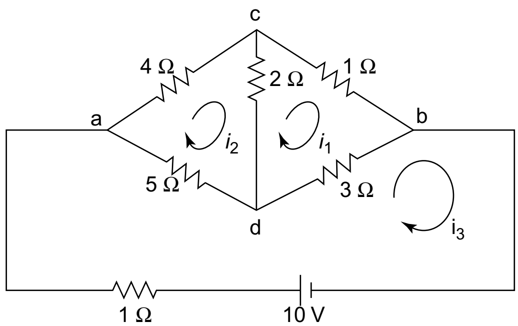


Fig. 1.160

$$11i_1 - 2i_2 - 5i_3 = 0$$

$$-2i_1 + 6i_2 - 3i_3 = 0$$

$$-5i_1 - 3i_2 + 9_3 = 10$$

$$\therefore \begin{bmatrix} 11 & -2 & -5 \\ -2 & 6 & -3 \\ -5 & -3 & 9 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\therefore\ \Delta = \begin{bmatrix} 11 & -2 & -5 \\ -2 & 6 & -3 \\ -5 & -3 & 9 \end{bmatrix} = 495 - 66 - 180 = 249$$

$$\therefore\ \Delta_1 = \begin{bmatrix} 0 & -2 & -5 \\ 0 & 6 & -3 \\ 10 & -3 & 9 \end{bmatrix} = 366$$

$$\therefore\ \Delta_2 = \begin{bmatrix} 11 & 0 & -5 \\ -2 & 0 & -3 \\ -5 & 10 & 9 \end{bmatrix} = 430$$

$$\therefore\ \Delta_3 = \begin{bmatrix} 11 & -2 & 0 \\ -2 & 6 & 0 \\ -5 & -3 & 10 \end{bmatrix} = 620$$

$$\therefore\ i_1 = \frac{\Delta_1}{\Delta} = 1.45\,\text{amp},\ i_2 = \frac{\Delta_2}{\Delta} = 1.73\,\text{amp},\ i_3 = \frac{\Delta_3}{\Delta} = 2.49\ \text{amp}$$

$\therefore$ Current through $2\Omega = \ -i_1 + i_2$ a= 0.28 amp

Total current $= i_3 = 2.49$ amp

Example 1.92 *What is the value of R such that the power supplied by both the sources are equal?* *[JNTU April/May 2003]*

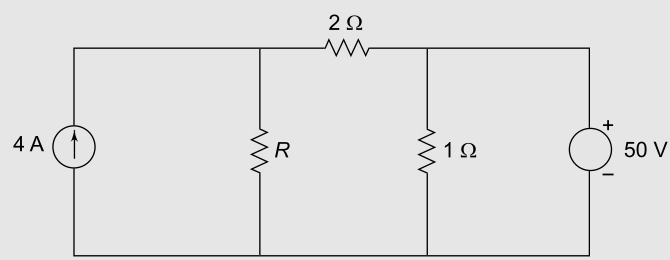


Fig. 1.161

Solution Converting current source into voltage source, we have

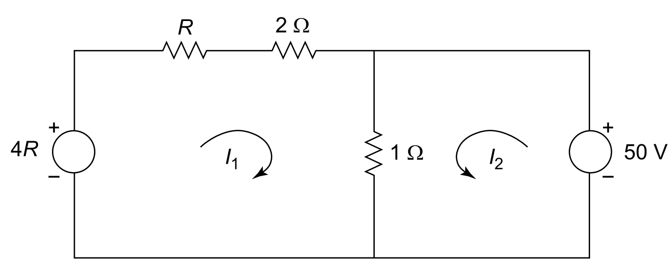


Fig. 1.162

Applying KVL for both the meshes,

$$4R = (R + 3)\ i_1 + i_2 \tag{1}$$

$$50 = i_1 + i_2 \tag{2}$$

The power supplied by both the source are equal

$$4R\ i_1 = 50 i_2$$

$$R = 12.5 \frac{i_2}{i_1} \tag{3}$$

From Eq. (1),

$$4R - i_1 R - 3i_1 - i_2 = 0$$

$$R(4 - i_1) - 3i_1 - i_2 = 0 \tag{4}$$

Substituting equation 3 in 4,

$$12.5\frac{i_2}{i_1}(4-i_1)-3i_1-i_2=0 \tag{5}$$

$$50\frac{i_2}{i_1}-13.5i_2-3i_1=0 \tag{6}$$

From equation 2, $\quad i_2 = 50 - i_1 \qquad (7)$

Substituting equation 7 in 6

$$50\left(\frac{50-i_1}{i_1}\right)-13.5(50-i_1)-3i_1=0 \tag{8}$$

$$10.5i_1^2 - 725\ i_1 + 2500 = 0 \tag{9}$$

from which $i_1 = \dfrac{725 \pm 717.72}{21} = 68.7 \text{ or } 0.347\text{ A}$

If $\quad i_1 = 68.7$ A:

from equation (2) $\quad i_2 = -15.407$ A

and $\quad R = \dfrac{12.5(-18.7)}{68.7} = -3.4\,\Omega$

If $\quad i_1 = 0.347$ A

$\quad i_2 = 46.3598$ A

and $\quad R = 12.5\times\dfrac{46.3598}{3.6402} = 1788.6\,\Omega$

Considering positive value of R = 1768.6 Ω
Power supplied by current source

$$= 4 \times 1788.6 \times 0.347 = 2482.65\text{ W}$$

Power supplied by voltage source

$$= 50 \times 49.653 = 2482.65\text{ W}$$

The value of R = 1788.6 Ω

1.11.2 Supermesh Analysis

Suppose any of the branches in the network has a current source, then it is slightly difficult to apply mesh analysis straight forward because first we should assume an unknown voltage across the current source, writing mesh equations as before, and then relate the source current to the assigned mesh currents. This is generally a difficult approach. One way to overcome this difficulty is by applying the supermesh technique. Here we have to choose the kind of supermesh. A supermesh is constituted by two adjacent loops that have a common current source. As an example, consider the network shown in Fig. 1.163.

Here, the current source I is in the common boundary for the two meshes 1 and 2. This current source creates a supermesh, which is nothing but a combination of meshes 1 and 2.

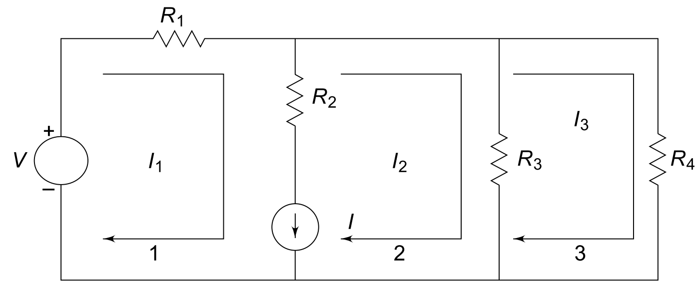


Fig. 1.163

$$R_1I_1 + R_3(I_2 - I_3) = \mathrm{V}$$

or $$R_1I_1 + R_3I_2 - R_4I_3 = \mathrm{V}$$

Considering mesh 3, we have

$$R_3(I_3 - I_2) + R_4I_3 = 0$$

Finally, the current I from current source is equal to the difference between two mesh currents, i.e.

$$I_1 - I_2 = I$$

We have, thus, formed three mesh equations which we can solve for the three unknown currents in the network.

Example 1.93 *Determine the current in the 5 Ω resistor in the network given in Fig. 1.164.*

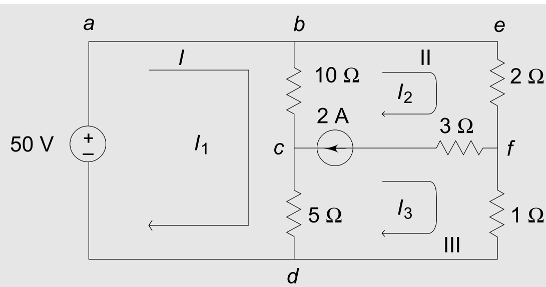


Fig. 1.164

Solution From the first mesh, i.e. *abcda*, we have

$$50 = 10(I_1 - I_2) + 5(I_1 - I_3)$$

or $$15I_1 - 10I_2 - 5I_3 = 50 \tag{1.17}$$

From the second and third meshes, we can form a supermesh

$$10(I_2 - I_1) + 2I_2 + I_3 + 5(I_3 - I_1) = 0$$

or $$-15I_1 + 12I_2 + 6I_3 = 0 \quad (1.18)$$

The current source is equal to the difference between II and III mesh currents, i.e.

$$I_2 - I_3 = 2\text{ A} \quad (1.19)$$

Solving 1.17, 1.18 and 1.19, we have

$$I_1 = 19.99\text{ A}, I_2 = 17.33\text{ A, and } I_3 = 15.33\text{ A}$$

The current in the 5 Ω resistor $= I_1 - I_3$

$$= 19.99 - 15.33 = 4.66\text{ A}$$

∴ The current in the 5 Ω resistor is 4.66 A.

Example 1.94 *Write the mesh equations for the circuit shown in Fig. 1.165 and determine the currents, I_1, I_2 and I_3.*

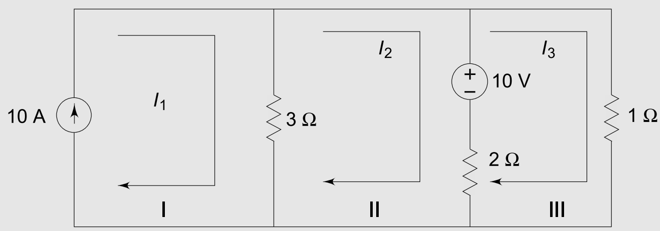


Fig. 1.165

Solution In Fig. 1.165, the current source lies on the perimeter of the circuit, and the first mesh is ignored. Kirchhoff's voltage law is applied only for second and third meshes.

From the second mesh, we have

$$3(I_2 - I_1) + 2(I_2 - I_3) + 10 = 0$$

or $$-3I_1 + 5I_2 - 2I_3 = -10 \quad (1.20)$$

From the third mesh, we have

$$I_3 + 2(I_3 - I_2) = 10$$

or $$-2I_2 + 3I_3 = 10 \quad (1.21)$$

From the first mesh,

$$I_1 = 10\text{ A} \quad (1.22)$$

From the above three equations, we get

$$I_1 = 10\text{ A}, \; I_2 = 7.27\text{ A}, I_3 = 8.18\text{ A}$$

Example 1.95 *Determine the loop currents for the circuit shown in Fig. 1.166 by using mesh analysis.*

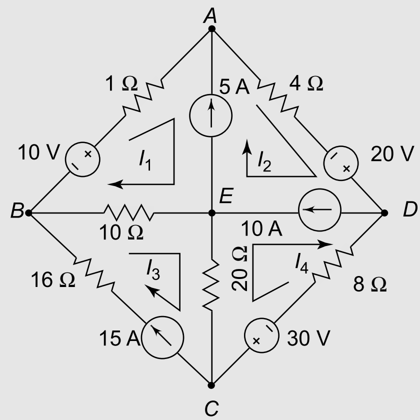


Fig. 1.166

Solution The branches AE, DE and BC consists of current sources. Here we have to apply supermesh analysis.

The combined supermesh equation is

$$10(I_1 - I_3) + I_1 - 10 + 4I_2 - 20$$

$$+ 8I_4 - 30 + 20(I_4 - I_3) = 0$$

or $$11I_1 + 4I_2 - 30I_3 + 28I_4 = 60$$

In branch AE, $\quad I_2 - I_1 = 5$ A

In branch BC, $\quad I_3 = 15$ A

In branch DE, $\quad I_2 - I_4 = 10$ A

Solving the above four equations, we can get the four currents I_1, I_2, I_3 and I_4 as

$$I_1 = 14.65 \text{ A}$$

$$I_2 = 19.65 \text{ A}, I_3 = 15 \text{ A, and } I_4 = 9.65 \text{ A}$$

Example 1.96 *Determine the power delivered by the voltage source and the current in the 10 Ω resistor for the circuit shown in Fig. 1.167.*

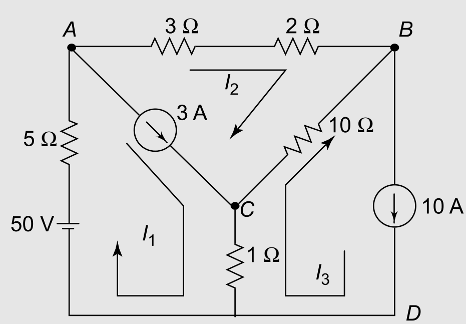


Fig. 1.167

Solution Since branches *AC* and *BD* consist of current sources, we have to use the supermesh technique.

The combined supermesh equation is

$$-50 + 5I_1 + 3I_2 + 2I_2 + 10(I_2 - I_3) + 1(I_1 - I_3) = 0$$

or $$6I_1 + 15I_2 - 11I_3 = 50$$

or $$I_1 - I_2 = 3 \text{ A and } I_3 = 10 \text{ A}$$

From the above equations we can solve for I_1, I_2 and I_3 follows

$$I_1 = 9.76 \text{ A}, I_2 = 6.76 \text{ A}, I_3 = 10 \text{ A}$$

Example 1.97 *In the circuit shown in Fig. 1.168, find the power delivered by 4 V source using mesh analysis and voltage across the 2 Ω resistor.*

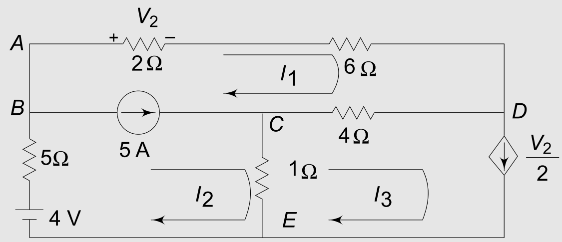


Fig. 1.168

Solution Since branches *BC* and *DE* consists of current sources, we use the supermesh technique.

The combined supermesh equation is

$$2I_1 + 6I_1 + 4(I_1 - I_3) + (I_2 - I_3) - 4 + 5I_2 = 0$$

or

$$12I_1 + 6I_2 - 5I_3 = 4$$

In branch *BC*, $I_2 - I_1 = 5$

In branch *DE*, $I_3 = \dfrac{V_2}{2}$

Solving the above equations

$$I_1 = -2 \text{ A}; I_2 = 3 \text{ A}$$

The voltage across the 2 Ω resistor $V_2 = 2I_1 = 2 \times (-2) = -4$ V

Power delivered by 4 V source $P_4 = 4I_2 = 4(3) = 12$ W

Example 1.98 *For the circuit shown in Fig. 1.169, find the current through the 10 Ω resistor by using mesh analysis.*

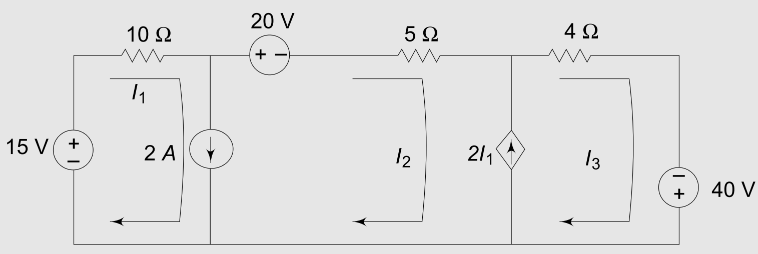


Fig. 1.169

Solution The parallel branches consist of current sources. Here we use supermesh analysis. The combined supermesh equation is.

or $-15 + 10I_1 + 20 + 5I_2 + 4I_3 - 40 = 0$

and

$$10I_1 + 5I_2 + 4I_3 = 35$$

$$I_1 - I_2 = 2$$

$$I_3 - I_2 = 2I_1$$

Solving the above equations, we get

$$I_1 = 1.96 \text{ A}$$

The current in the 10 Ω resistor is $I_1 = 1.96$ A

1.12 NODAL ANALYSIS

In Chapter 1, we discussed simple circuits containing only two nodes, including the reference node. In general, in a N node circuit, one of the nodes is chosen as reference or datum node, then it is possible to write $N - 1$ nodal equations by assuming $N - 1$ node voltages. For example, a 10 node circuit requires nine unknown voltages and nine equations. Each node in a circuit can be assigned a number or a letter. The node voltage is the voltage of a given node with respect to one particular node, called the reference node, which we assume at zero potential. In the circuit shown in Fig. 1.170, node 3 is assumed as the reference node. The voltage at node 1 is the voltage at that node with respect to node 3. Similarly, the voltage at node 2 is the voltage at that node with respect to node 3. Applying Kirchhoff's current law at node 1; the current entering is equal to the current leaving. (See Fig. 1.171).

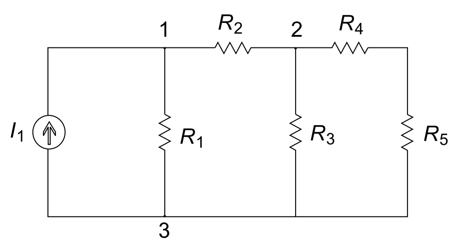


Fig. 1.170

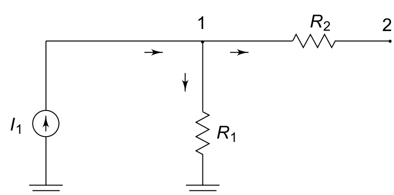


Fig. 1.171

$$I_1 = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

where V_1 and V_2 are the voltages at node 1 and 2, respectively. Similarly, at node 2, the current entering is equal to the current leaving as shown in Fig. 1.172.

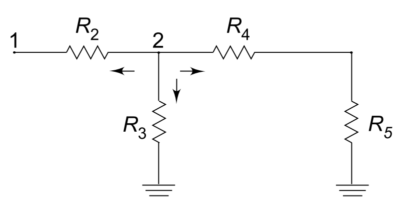


Fig. 1.172

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_2}{R_4 + R_5} = 0$$

Rearranging the above equations, we have

$$V_1\left[\frac{1}{R_1}+\frac{1}{R_2}\right]-V_2\left[\frac{1}{R_2}\right]=I_1$$

$$-V_1\left[\frac{1}{R_2}\right]+V_2\left[\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4+R_5}\right]=0$$

From the above equations, we can find the voltages at each node.

Example 1.99 *Write the node voltage equations and determine the currents in each branch for the network shown in Fig. 1.173.*

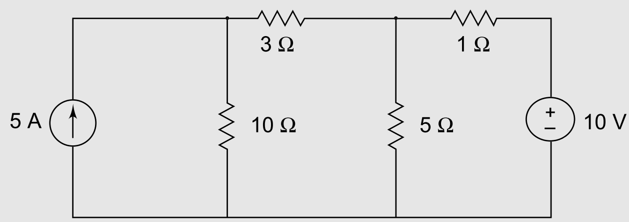


Fig. 1.173

Solution The first step is to assign voltages at each node as shown in Fig. 1.174.

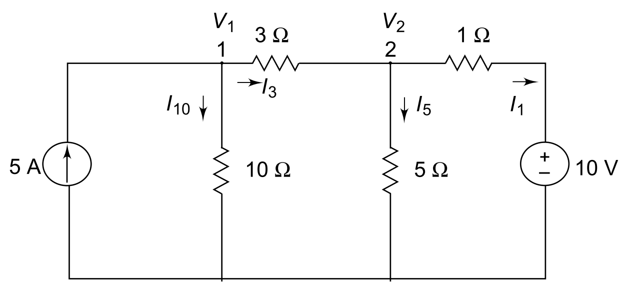


Fig. 1.174

Applying Kirchhoff's current law at node 1,

we have $5=\dfrac{V_1}{10}+\dfrac{V_1-V_2}{3}$

or $$V_1\left[\frac{1}{10}+\frac{1}{3}\right]-V_2\left[\frac{1}{3}\right]=5 \qquad (1.23)$$

Applying Kirchhoff's current law at node 2,

we have $$\frac{V_2-V_1}{3}+\frac{V_2}{5}+\frac{V_2-10}{1}=0$$

or $$-V_1\left[\frac{1}{3}\right]+V_2\left[\frac{1}{3}+\frac{1}{5}+1\right]=10 \qquad (1.24)$$

From Eqs 1.23 and 1.24, we can solve for V_1 and V_2 to get

$$V_1 = 19.85 \text{ V}, V_2 = 10.9 \text{ V}$$

$$I_{10} = \frac{V_1}{10} = 1.985\text{A},\ I_3 = \frac{V_1 - V_2}{3} = \frac{19.85 - 10.9}{3} = 2.98\text{ A}$$

$$I_5 = \frac{V_2}{5} = \frac{10.9}{5} = 2.18\text{ A},\ I_1 = \frac{V_2 - 10}{1} = 0.9\text{ A}$$

Example 1.100 *Determine the voltages at each node for the circuit shown in Fig. 1.175.*

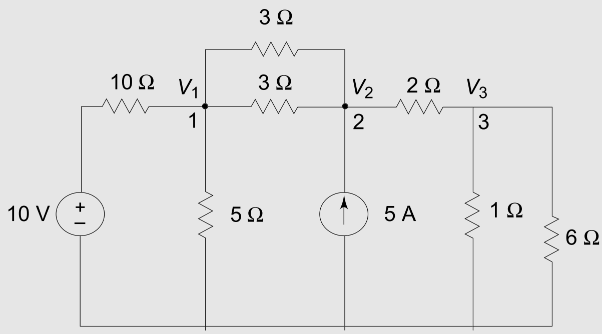


Fig. 1.175

Solution At node 1, assuming that all currents are leaving, we have

$$\frac{V_1 - 10}{10} + \frac{V_1 - V_2}{3} + \frac{V_1}{5} + \frac{V_1 - V_2}{3} = 0$$

or $$V_1\left[\frac{1}{10} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3}\right] - V_2\left[\frac{1}{3} + \frac{1}{3}\right] = 1$$

$$0.96\ V_1 - 0.66\ V_2 = 1 \tag{1.25}$$

At node 2, assuming that all currents are leaving except the current from current source, we have

$$\frac{V_2 - V_1}{3} + \frac{V_2 - V_1}{3} + \frac{V_2 - V_3}{2} = 5$$

$$-V_1\left[\frac{2}{3}\right] + V_2\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right] - V_3\left[\frac{1}{2}\right] = 5$$

$$-0.66\ V_1 + 1.16\ V_2 - 0.5\ V_3 = 5 \tag{1.26}$$

At node 3, assuming all currents are leaving, we have

$$\frac{V_3 - V_2}{2} + \frac{V_3}{1} + \frac{V_3}{6} = 0$$

$$-0.5\ V_2 + 1.66\ V_3 = 0 \tag{1.27}$$

Applying Cramer's rule, we get

$$V_1 = \frac{\begin{vmatrix} 1 & -0.66 & 0 \\ 5 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{7.154}{0.887} = 8.06\text{ V}$$

Similarly,

$$V_2 = \frac{\begin{vmatrix} 0.96 & 1 & 0 \\ -0.66 & 5 & -0.5 \\ 0 & 0 & 1.66 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{9.06}{0.887} = 10.2\text{ V}$$

$$V_3 = \frac{\begin{vmatrix} 0.96 & -0.66 & 1 \\ -0.66 & 1.16 & 5 \\ 0 & -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.96 & -0.66 & 0 \\ -0.66 & 1.16 & -0.5 \\ 0 & -0.5 & 1.66 \end{vmatrix}} = \frac{2.73}{0.887} = 3.07\text{ V}$$

1.12.1 Nodal Equations by Inspection Method

The nodal equations for a general planar network can also be written by inspection, without going through the detailed steps. Consider a three node resistive network, including the reference node, as shown in Fig. 1.176.

In Fig. 1.176, the points *a* and *b* are the actual nodes and *c* is the reference node. Now consider the nodes *a* and *b* separately as shown in Figs 1.177(a) and (b).

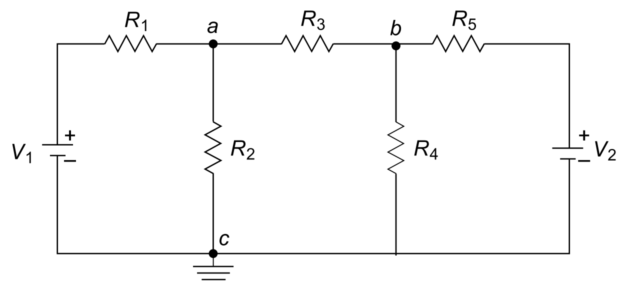


Fig. 1.176

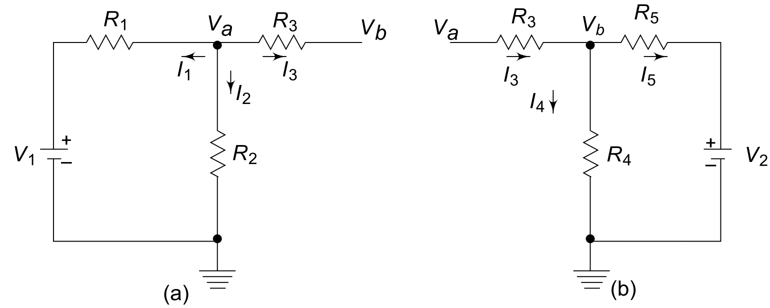


Fig. 1.177

In Fig. 1.176(a), according to Kirchhoff's current law, we have

$$I_1 + I_2 + I_3 = 0$$

$$\therefore \quad \frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_b}{R_3} = 0 \tag{1.28}$$

In Fig. 1.176(b), if we apply Kirchhoff's current law, we get

$$I_4 + I_5 = I_3$$

$$\therefore \quad \frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} + \frac{V_b - V_2}{R_5} = 0 \tag{1.29}$$

Rearranging the above equations, we get

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)V_a - \left(\frac{1}{R_3}\right)V_b = \left(\frac{1}{R_1}\right)V_1 \tag{1.30}$$

$$\left(-\frac{1}{R_3}\right)V_a + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right)V_b = \frac{V_2}{R_5} \tag{1.31}$$

In general, the above equations can be written as

$$G_{aa} V_a + G_{ab} V_b = I_1 \tag{1.32}$$

$$G_{ba} V_a + G_{bb} V_b = I_2 \tag{1.33}$$

By comparing Eqs 1.30, 1.31 and Eqs 1.32, 1.33 we have the self conductance at node a, $G_{aa} = (1/R_1 + 1/R_2 + 1/R_3)$ is the sum of the conductances connected to node a. Similarly, $G_{bb} = (1/R_3 + 1/R_4 + 1/R_5)$, is the sum of the conductances connected to node b. $G_{ab} = (-1/R_3)$, is the sum of the mutual conductances connected to node a and node b. Here all the mutual conductances have negative signs. Similarly, $G_{ba} = (-1/R_3)$ is also a mutual conductance connected between nodes b and a. I_1 and I_2 are the sum of the source currents at node a and node b, respectively. The current which drives into the node has positive sign, while the current that drives away from the node has negative sign.

Example 1.101 *For the circuit shown in Fig. 1.178, write the node equations by the inspection method.*

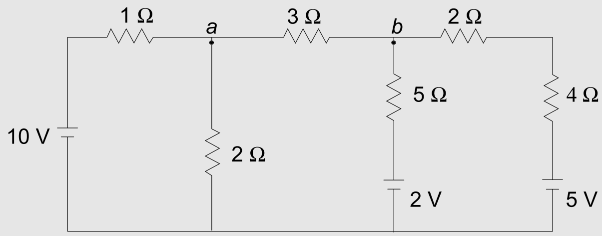


Fig. 1.178

Solution The general equations are

$$G_{aa} V_a + G_{ab} V_b = I_1 \tag{1.34}$$

$$G_{ba} V_a + G_{bb} V_b = I_2 \tag{1.35}$$

Consider Eq. 1.34.

G_{aa} = (1 + 1/2 + 1/3) mho, the self conductance at node *a* is the sum of the conductances connected to node *a*.

G_{bb} = (1/6 + 1/5 + 1/3) mho the self conductance at node *b* is the sum of the conductances connected to node *b*.

G_{ab} = –(1/3) mho, the mutual conductance between nodes *a* and *b* is the sum of the conductances connected between nodes *a* and *b*.

Similarly, G_{ba} = –(1/3), the sum of the mutual conductances between nodes *b* and *a*.

$$I_1 = \frac{10}{1} = 10\text{ A, the source current at node } a,$$

$$I_2 = \left(\frac{2}{5} + \frac{5}{6}\right) = 1.23\text{ A, the source current at node } b.$$

Therefore, the nodal equations are

$$1.83\ V_a - 0.33\ V_b = 10 \tag{1.36}$$

$$-0.33\ V_a + 0.7\ V_b = 1.23 \tag{1.37}$$

Example 1.102 *Determine the voltage ratio V_{out} / V_{in} for the circuit shown in Fig. 1.179 by using nodal analysis.*

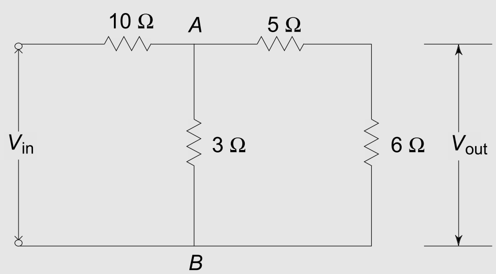


Fig. 1.179

Solution $I_{10} + I_3 + I_{11} = 0$

$$I_{10} = \frac{V_A - V_{\text{in}}}{10}$$

$$I_3 = \frac{V_A}{3}$$

$$I_{11} = \frac{V_A}{11}, \text{ or } \frac{V_{\text{out}}}{6}$$

$$\frac{V_A - V_{\text{in}}}{10} + \frac{V_A}{3} + \frac{V_A}{11} = 0$$

Also $\dfrac{V_A}{11} = \dfrac{V_{\text{out}}}{6}$

$\therefore \quad VA = V_{\text{out}} \times 1.83$

From the above equations, $V_{\text{out}}/V_{\text{in}} = 1/9.53 = 0.105$

Example 1.103 *Find the voltages V in the circuit shown in Fig. 1.180 which makes the current in the 10 Ω resistor zero by using nodal analysis.*

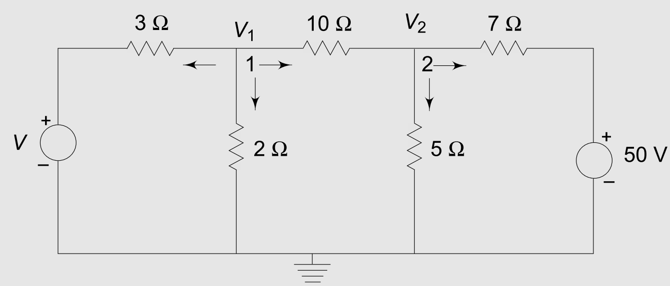


Fig. 1.180

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2. At node 1, the current equation in Fig. 1.181(a) is

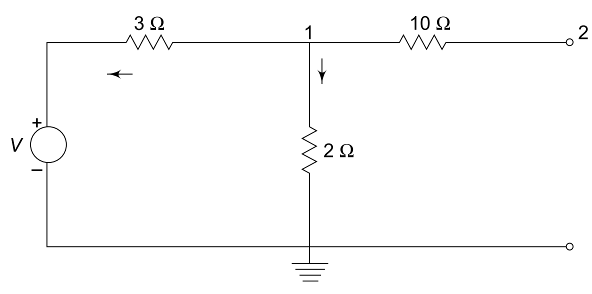


Fig. 1.181(a)

$$\frac{V_1-V}{3}+\frac{V_1}{2}+\frac{V_1-V_2}{10}=0$$

or $\quad 0.93\ V_1 - 0.1\ V_2 = V/3$

At node 2, the current equation in Fig. 1.181(b) is

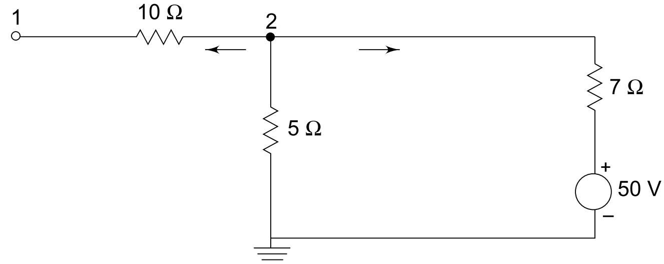


Fig. 1.181(b)

$$\frac{V_2-V_1}{10}+\frac{V_2}{5}+\frac{V_2-50}{7}=0$$

or $\quad -0.1\ V_1 + 0.443\ V_2 = 7.143$

Since the current in 10 Ω resistor is zero, the voltage at node 1 is equal to the voltage at node 2.

$\therefore \quad V_1 - V_2 = 0$

From the above three equations, we can solve for V

$V_1 = 20.83$ Volts and $V_2 = 20.83$ volts

$\therefore \quad V = 51.87$ V

Example 1.104 *Use nodal analysis to find the power dissipated in the 6 Ω resistor for the circuit shown in Fig. 1.182.*

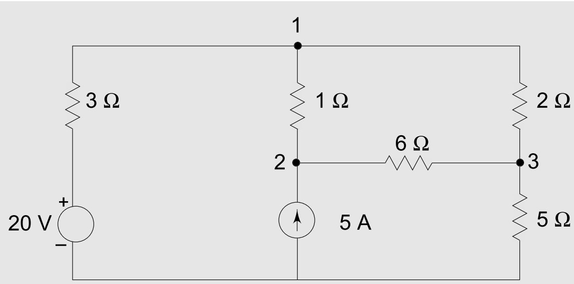


Fig. 1.182

Solution Assume voltage V_1, V_2 and V_3 at nodes 1, 2 and 3 as shown in Fig. 1.182. By applying current law at node 1, we have

$$\frac{V_1-20}{3}+\frac{V_1-V_2}{1}+\frac{V_1-V_3}{2}=0$$

or $$1.83V_1 - V_2 - 0.5V_3 = 6.67 \tag{1.38}$$

At node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{6} = 5\,\text{A}$$

or $$-V_1 + 1.167V_2 - 0.167V_3 = 5 \tag{1.39}$$

At node 3,

$$\frac{V_3 - V_1}{2} + \frac{V_3 - V_2}{6} + \frac{V_3}{5} = 0$$

or $$-0.5\ V_1 - 0.167\ V_2 + 0.867\ V_3 = 0 \tag{1.40}$$

Applying Cramer's rule to Eqs 1.38, 1.39 and 1.40, we have

$$V_2 = \frac{\Delta_2}{\Delta}$$

where $$\Delta = \begin{vmatrix} 1.83 & -1 & -0.5 \\ -1 & -1.167 & -0.167 \\ -0.5 & -0.167 & 0.867 \end{vmatrix} = -2.64$$

$$\Delta_2 = \begin{vmatrix} 1.83 & 6.67 & -0.5 \\ -1 & 5 & -0.167 \\ -0.5 & 0 & 0.867 \end{vmatrix} = 13.02$$

$$\therefore \quad V_2 = \frac{13.02}{-2.64} = -4.93\,\text{V}$$

Similarly,

$$V_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta_3 = \begin{vmatrix} 1.83 & -1 & 6.67 \\ -1 & -1.167 & 5 \\ -0.5 & -0.167 & 0 \end{vmatrix} = 1.25$$

$$\therefore \quad V_3 = \frac{1.25}{-2.64} = -0.47\,\text{V}$$

The current in the 6 Ω resistor is

$$I_6 = \frac{V_2 - V_3}{6}$$

$$= \frac{-4.93 + 0.47}{6} = -0.74\,\text{A}$$

The power absorbed or dissipated = $I_6^2 R_6$

$$= (0.74)^2 \times 6$$

$$= 3.29\ \text{W}$$

Example 1.105 *For the circuit shown in Fig. 1.183 find the voltage across the 4 Ω resistor by using nodal analysis.*

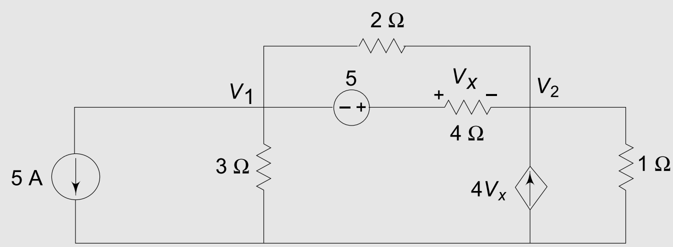


Fig. 1.183

Solution In the circuit shown, assume voltages V_1 and V_2 at nodes 1 and 2. At node 1, the current equation is

$$5+\frac{V_1}{3}+\frac{V_1+5-V_2}{4}+\frac{V_1-V_2}{2}=0$$

or $\quad 1.08\ V_1 - 0.75\ V_2 = -6.25 \qquad (1.41)$

At node 2, the current equation is

$$\frac{V_2-V_1-5}{4}+\frac{V_2-V_1}{2}-4V_x+\frac{V_2}{1}=0$$

$$V_x = V_1 + 5 - V_2$$

or $\quad -4.75\ V_1 + 5.75\ V_2 = 21.25 \qquad (1.42)$

Applying Cramer's rule to Eqs 1.41 and 1.42, we have

$$V_2=\frac{\Delta_2}{\Delta}$$

where
$$\Delta=\begin{vmatrix}1.08 & -0.75\\ -4.75 & 5.75\end{vmatrix}=2.65$$

$$\Delta_2=\begin{vmatrix}1.08 & -6.25\\ -4.75 & 21.25\end{vmatrix}=-6.74$$

$\therefore$
$$V_2=\frac{\Delta_2}{\Delta}=\frac{-6.74}{2.65}=-2.54\text{ V}$$

Similarly,
$$V_1=\frac{\Delta_1}{\Delta}$$

$$\Delta_1=\begin{vmatrix}-6.25 & -0.75\\ 21.25 & 5.75\end{vmatrix}=-20$$

$$V_1=\frac{\Delta_1}{\Delta}=\frac{-20}{2.65}=-7.55\text{ V}$$

The voltage across the 4 Ω resistor is

$$V_x = V_1 + 5 - V_2 = -7.55 + 5 - (-2.54)$$

$$V_x = -0.01 \text{ volts}$$

Example 1.106 *For the circuit shown in Fig. 1.184, find the current passing through the 5 Ω resistor by using the nodal method.*

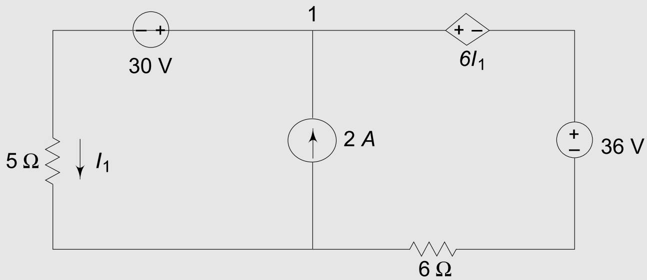


Fig. 1.184

Solution In the circuit shown, assume the voltage V at node 1.

At node 1, the current equation is

$$\frac{V-30}{5} - 2 + \frac{V-36-6I_1}{6} = 0$$

where $$I_1 = \frac{V-30}{5}$$

From the above equation

$$V = 48 \text{ V}$$

The current in the 5 Ω resistor is

$$I_1 = \frac{V-30}{5} = 3.6 \text{ A}$$

Example 1.107 *Use nodal analysis, to determine the voltage V_1 and V_2 in the circuit shown.*

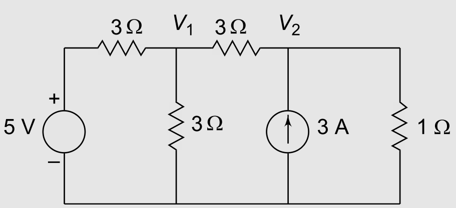


Fig. 1.185

Solution The nodal equations for the two nodes are

$$\frac{V_1 - 5}{2} + \frac{V_1}{3} + \frac{V_1 - V_2}{2} = 0 \quad (1)$$

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} = 3 \quad (2)$$

From 1 $\quad 1.333\ V_1 - 0.5\ V_2 = 2.5$

From 2 $\quad -0.5\ V_1 + 1.5\ V_2 = 3$

Solving the above equations for V_1 and V_2 yields

$$V_1 = 3\text{ V and } V_2 = 3\text{ V.}$$

Example 1.108 *For the circuit shown in Fig. 1.186, find the current through 20 Ω resistor?*

[JNTU May/June 2006]

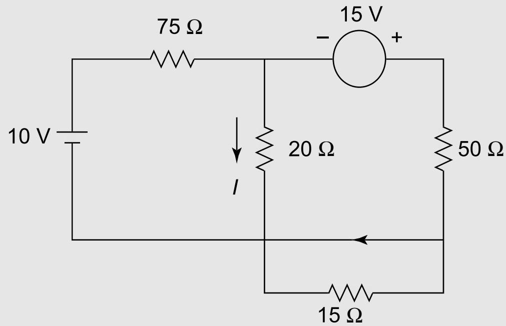


Fig. 1.186

Solution Applying nodal analysis

$$\frac{V-10}{75} + \frac{V}{20} + \frac{V+15}{50} = 0$$

$$V = -2 \text{ volts } I$$

$$I = \frac{V}{20} = -0.1 \text{ A}$$

1.12.2 Supernode Analysis

Suppose any of the branches in the network has a voltage source, then it is slightly difficult to apply nodal analysis. One way to overcome this difficulty is to apply the supernode technique. In this method, the two adjacent nodes that are connected by a voltage source are reduced to a single node and then the equations are formed by applying Kirchhoff's current law as usual. This is explained with the help of Fig. 1.186.

It is clear from Fig. 1.187, that node 4 is the reference node. Applying Kirchhoff's current law at node 1, we get

$$I = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2}$$

Due to the presence of voltage source V_x in between nodes 2 and 3, it is slightly difficult to find out the current. The supernode technique can be conveniently applied in this case.

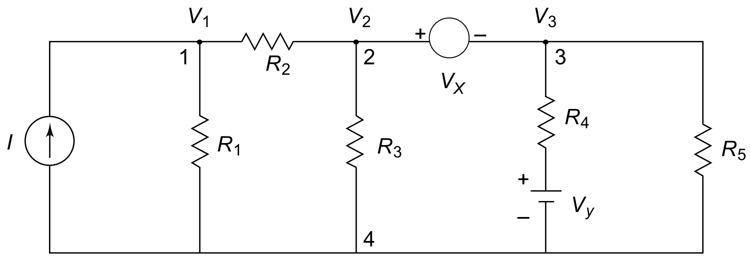


Fig. 1.187

Accordingly, we can write the combined equation for nodes 2 and 3 as under.

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3} + \frac{V_3 - V_y}{R_4} + \frac{V_3}{R_5} = 0$$

The other equation

$$V_2 - V_3 = V_x$$

From the above three equations, we can find the three unknown voltages.

Example 1.109 *Determine the current in the 5 Ω resistor for the circuit shown in Fig. 1.188.*

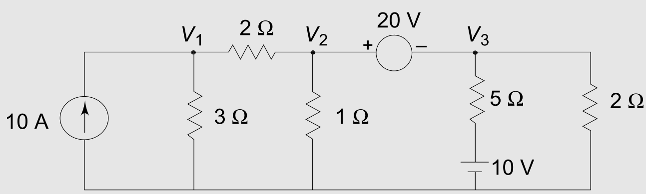


Fig. 1.188

Solution At node 1

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

or $$V_1\left[\frac{1}{3} + \frac{1}{2}\right] - \frac{V_2}{2} - 10 = 0$$

$$0.83\ V_1 - 0.5\ V_2 - 10 = 0 \tag{1.43}$$

At node 2 and 3, the supernode equation is

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

or $$\frac{-V_1}{2} + V_2\left[\frac{1}{2} + 1\right] + V_3\left[\frac{1}{5} + \frac{1}{2}\right] = 2$$

$$-0.5\ V_1 + 1.5\ V_2 + 0.7\ V_3 - 2 = 0 \tag{1.44}$$

The voltage between nodes 2 and 3 is given by

$$V_2 - V_3 = 20 \tag{1.45}$$

The current in the 5 Ω resistor $I_5 = \dfrac{V_3 - 10}{5}$

Solving Eqs 1.43, 1.44 and 1.45, we obtain

$$V_3 = -8.42\text{ V}$$

$\therefore$ Current $I_5 = \dfrac{-8.42 - 10}{5} = -3.68\text{ A}$ (current towards node 3) i.e. the current flows towards node 3.

Example 1.110 *Determine the power dissipated by 5 Ω resistor in the circuit shown in Fig. 1.189.*

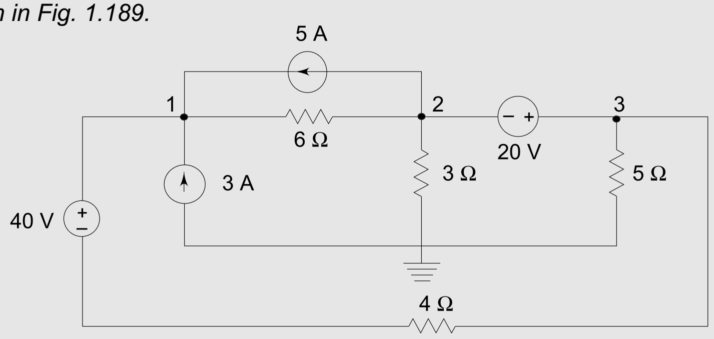


Fig. 1.189

Solution In Fig. 1.189, assume voltages V_1, V_2 and V_3 at nodes 1, 2 and 3. At node 1, the current law gives

$$\frac{V_1 - 40 - V_3}{4} + \frac{V_1 - V_2}{6} - 3 - 5 = 0$$

or $\quad 0.42\, V_1 - 0.167\, V_2 - 0.25\, V_3 = 18$

Applying the supernode technique between nodes 2 and 3, the combined equation at node 2 and 3 is

$$\frac{V_2 - V_1}{6} + 5 + \frac{V_2}{3} + \frac{V_3}{5} + \frac{V_3 + 40 - V_1}{4} = 0$$

or $\quad -0.42\, V_1 + 0.5\, V_2 + 0.45\, V_3 = -15$

Also $\quad V_3 - V_2 = 20$ V

Solving the above three equations, we get

$V_1 = 52.89$ V, $V_2 = -1.89$ V and

$V_3 = 18.11$ V

$\therefore$ The current in the 5 Ω resistor $I_5 = \dfrac{V_3}{5}$

$$= \frac{18.11}{5} = 3.62 \text{ A}$$

The power absorbed by the 5 Ω resistor $P_5 = I_5^2 R_5$

$$= (3.62)^2 \times 5$$

$$= 65.52 \text{ W}$$

Example 1.111 *Find the power delivered by the 5 A current source in the circuit shown in Fig. 1.190 by using the nodal method.*

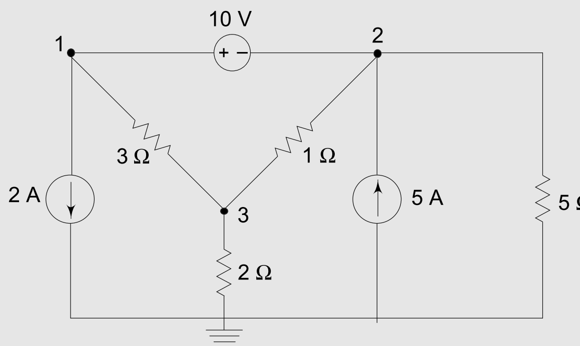


Fig. 1.190

Solution Assume the voltages V_1, V_2 and V_3 at nodes 1, 2, and 3, respectively. Here, the 10 V source is common between nodes 1 and 2. So applying the supernode technique, the combined equation at node 1 and 2 is

$$\frac{V_1 - V_3}{3} + 2 + \frac{V_2 - V_3}{1} - 5 + \frac{V_2}{5} = 0$$

or $\quad 0.34\, V_1 + 1.2\, V_2 - 1.34\, V_3 = 3$

At node 3, $\quad \dfrac{V_3 - V_1}{3} + \dfrac{V_3 - V_2}{1} + \dfrac{V_3}{2} = 0$

or $\quad -0.34\, V_1 - V_2 + 1.83\, V_3 = 0$

Also $\quad V_1 - V_2 = 10$

Solving the above equations, we get

$$V_1 = 13.72 \text{ V};\ V_2 = 3.72 \text{ V}$$

$$V_3 = 4.567 \text{ V}$$

Hence the power delivered by the source (5 A) $= V_2 \times 5$

$$= 3.72 \times 5 = 18.6 \text{ W}$$

Example 1.112 *Using nodal method, find the current through 5 Ω resistor, in the following circuit.*

[May/June 2006]

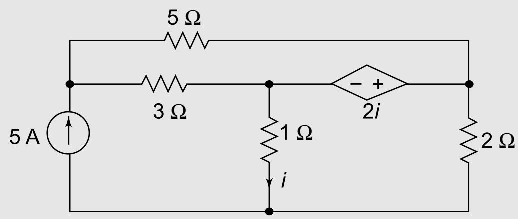


Fig. 1.191

Solution

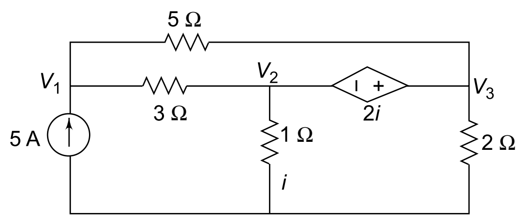


Fig. 1.192

Equation at V_1; $\dfrac{V_1 - V_3}{5} + \dfrac{V_1 - V_2}{3} = 5$

$$8V_1 - 5V_2 - 3V_3 = 75 \qquad (1)$$

Equation at supernode

$$\frac{V_2 - V_1}{3} + V_2 + \frac{V_3 - V_1}{5} + \frac{V_3}{2} = 0$$

$$-16V_1 + 40V_2 + 21V_3 = 0 \qquad (2)$$

$$V_3 - V_2 = 2i$$

$$i = V_{2/1} = V_2$$

$$V_3 - V_2 = 2V_2 \Rightarrow V_3 = 3V_2$$

Solving for V_1, V_2 and V_3

$$V_1 = 12.87;\ V_2 = 2;\ V_3 = 6 \text{ volts}$$

Current through 5 Ω from V_1 to V_3 is equal to $\dfrac{V_1 - V_3}{5} = 1.347$ amps.

Example 1.113 *Find the power supplied by 12 V source as shown in Fig. 1.193.*

[JNTU April/May 2006]

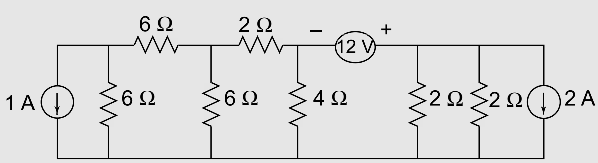


Fig. 1.193

Solution

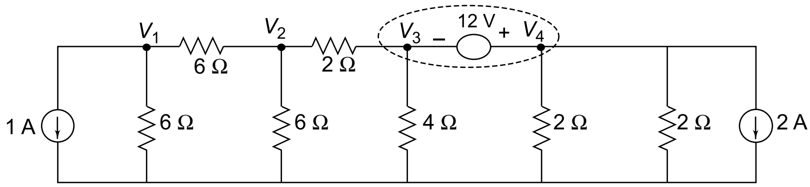


Fig. 1.194

The nodal equations are

$$1+\frac{V_1}{6}+\frac{V_1+V_2}{6}=0 \tag{1}$$

$$\frac{V_2}{6}+\frac{V_2-V_1}{6}+\frac{V_2-V_3}{2}=0 \tag{2}$$

$$\frac{V_3-V_2}{2}+\frac{V_3}{4}+\frac{V_4}{2}+\frac{V_4}{2}+2=0 \tag{3}$$

$V_4 - V_3 = 12$ is the supernode equation

(1) $\Rightarrow V_1\left[\frac{1}{3}\right]-V_2\left[\frac{1}{6}\right]+1=0$

(2) $\Rightarrow V_1\left[-\frac{1}{6}\right]+V_2\left[\frac{5}{6}\right]-V_3\left[\frac{1}{2}\right]=0$

(3) $\Rightarrow V_2\left[-\frac{1}{2}\right]+V_3\left[\frac{3}{4}\right]+V_4[1]+2=0$

$$-\frac{1}{2}V_2+\frac{7}{4}V_3+12+V_3+2=0$$

$$-\frac{1}{2}V_2+\frac{7}{4}V_3+14=0$$

$$\Rightarrow V_3=\frac{4}{7}\left[\frac{1}{2}V_2-14\right]=\frac{2}{7}V_2-8 \tag{4}$$

From (2), $-\frac{1}{6}V_1+\frac{5}{6}V_2-\frac{1}{2}\left[\frac{2}{7}V_2-8\right]=0$

$$-\frac{1}{6}V_1+\frac{5}{6}V_2-\frac{1}{7}V_2+4=0$$

$$V_1=6\left\{\frac{29}{42}V_2+4\right\}=\frac{29}{7}V_2+24 \tag{5}$$

$$\frac{1}{3}V_1-\frac{1}{6}V_2+1=0$$

Substitute for V_1

From (1), $\frac{1}{3}\left[\frac{29}{7}V_2+24\right]-\frac{1}{6}V_2+1=0$

$$\frac{29}{21}V_2+8-\frac{1}{6}V_2+1-0 \Rightarrow V_2=\frac{-126}{17}\ V$$

$$V_3=\frac{2}{7}\ V_2-8=\frac{-172}{17}\ V$$

$$V_4=V_3+12=\frac{32}{17}\ V$$

Current through 12 V source is

$$I=\frac{V_4}{2}+\frac{V_4}{2}+2=\frac{66}{17}\text{ A}$$

Power $\quad V_1=12\times\frac{66}{17}=\frac{792}{17}\text{ W}$

Example 1.114 *Find the currents I_1 and I_2 using Nodal Analysis (Fig. 1.195)*

[JNTU May/June 2006]

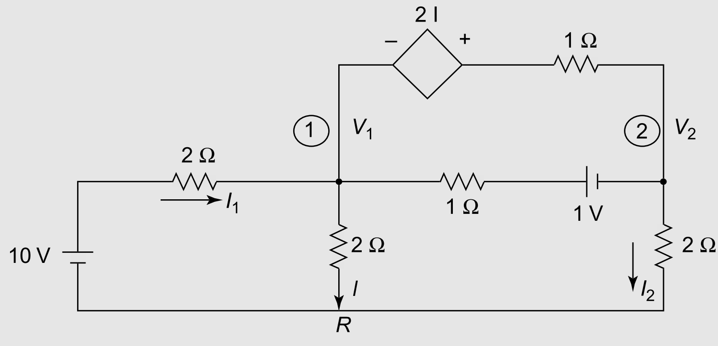


Fig. 1.195

Solution At node (1):

$$\frac{V_1-10}{2}+\frac{V_1}{2}+\frac{V_1-(1+V_2)}{1}+\frac{(2I+V_1)-V_2}{1}=0$$

$$\Rightarrow \quad V_1\left(\frac{1}{2}+\frac{1}{2}+1+1\right)+(-1-1)V_2=\frac{10}{2}+1-2I=6-2I$$

$$\Rightarrow \quad 3V_1-2V_2=6-2I \tag{1}$$

At node (2):

$$\frac{V_2}{2}+\frac{(1+V_2)-V_1}{1}+\frac{V_2-V_1-2I}{1}=0$$

$$\Rightarrow \quad (-1-1)V_1+\left(\frac{1}{2}+1+1\right)V_2=-1+2I$$

$$\Rightarrow \qquad -2V_1 + \frac{5}{2}V_2 = 2I - 1 \tag{2}$$

But
$$I = \frac{V_1}{2} \tag{3}$$

From (3) and (1) $\Rightarrow 4V_1 - 2V_2 = 6$

From (3) and (2) $\Rightarrow 3V_1 - \frac{5}{2}V_2 = 1$

Solving,

$$V_1 = 3.25 \text{ V}$$
$$V_2 = 3.5 \text{ V}$$

$$\therefore \quad I_1 = \frac{10 - V_1}{2} = 3.375 \text{ A}; \quad I_2 = \frac{V_2}{2} = 1.75 \text{ A.}$$

Example 1.115 *For the network shown (Fig. 1.196), determine the node voltages V_1 and V_2. Determine the power dissipated in each resistor.*

[JNTU May/June 2006]

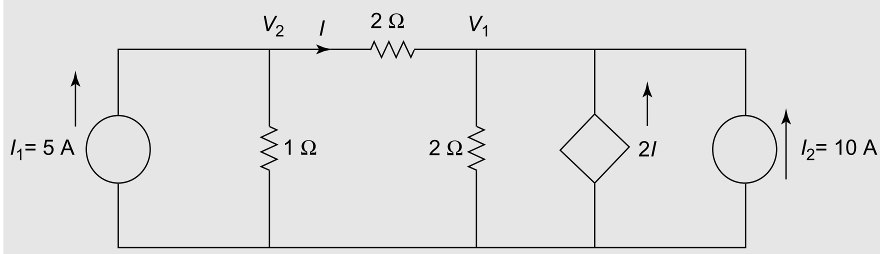


Fig. 1.196

Solution

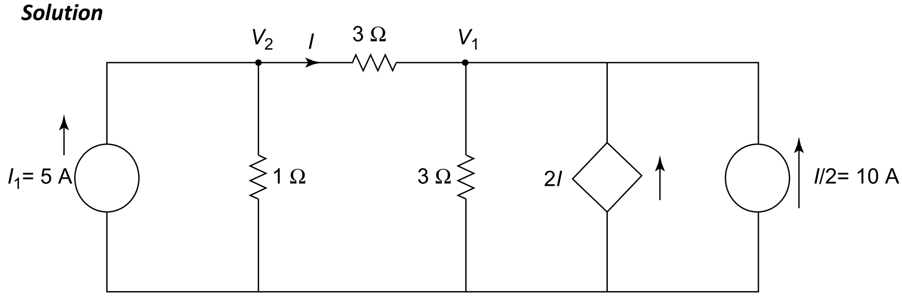


Fig. 1.197

Applying KCL,

$$5 = \frac{V_2}{1} + \frac{V_2 - V_1}{2} \quad \Rightarrow \quad V_2\left(1 + \frac{1}{2}\right) - \frac{V_1}{2} = 5$$

$$3V_2 - V_1 = 10 \tag{1}$$

$$I = \frac{V_2 - V_1}{2} \tag{2}$$

$$\frac{V_1}{3} + \frac{V_1 - V_2}{2} = 10 + 2I = 10 + 2\left(\frac{V_2 - V_1}{2}\right)$$

$$\Rightarrow \quad V_1\left(\frac{1}{2} + \frac{1}{3}\right) = 10 + V_2 - V_1$$

$$\therefore \quad 11V_1 - 9V_2 = 60 \tag{3}$$

Solving (1) and (3),

$$V_1 = 11.25 \text{ volts}$$

and $$V_2 = 7.083 \text{ volts}$$

Power dissipated in 1 Ω resistor $= VI = I^2R = \dfrac{V^2}{R} = \dfrac{V_2^2}{1} = (7.083)^2$

$= 50.17$ watts

Power dissipated in 2 Ω resistor $= \dfrac{V^2}{R} = \dfrac{(V_2 - V_1)^2}{2} = 8.682$ watts

Power dissipated in 3 Ω resistor $= \dfrac{V_1^2}{3} = \dfrac{(11.25)^2}{3} = 42.19$ watts

Example 1.116 *Using nodal analysis, determine the power supplied by 8 V voltage source. (Fig. 1.198)*
[JNTU May/June 2006]

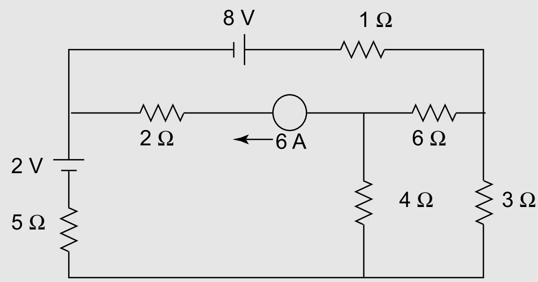


Fig. 1.198

Solution

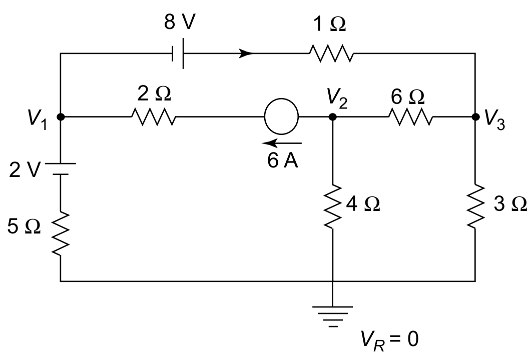


Fig. 1.199

Applying KCL at node (1);

$$\frac{V_1 - 2}{5} + \frac{V_1 - V_3 + 8}{1} = 6 \quad \Rightarrow \quad 5V_3 - 6V_1 = 8 \tag{1}$$

Applying KCL at node (2);

$$6 + \frac{V_2}{4} + \frac{V_2 - V_3}{6} = 0 \quad \Rightarrow \quad 5V_2 - 2V_3 + 72 = 0 \tag{2}$$

Applying KCL at node (3);

$$\frac{V_3 - V_2}{6} + \frac{V_3}{3} + \frac{V_3 - V_1 - 8}{1} = 0 \quad \Rightarrow \quad 9V_3 - V_2 + 6V_1 = 48 \tag{3}$$

Solving (1), (2) and (3), we get

$$V_1 = -4.593 \text{ volts}$$
$$V_2 = 11.56 \text{ volts}$$
$$V_3 = -7.11 \text{ volts}$$

From the circuit, $i = \dfrac{V_1 + 8 - V_3}{1} = 10.517 \text{ A}$

Power supplied by 8 V source is (8×10.517)

$= 84.136$ watts

Example 1.117 *Find the current through 12 Ω resistor for the given circuit by nodal method as shown in Fig. 1.200.*

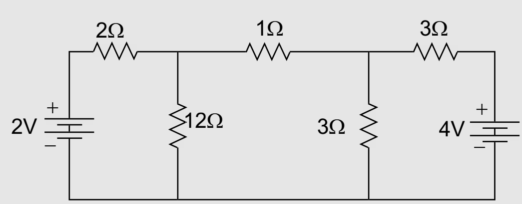


Fig. 1.200

Solution

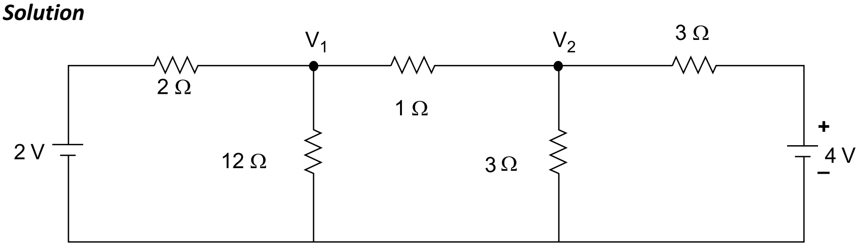


Fig. 1.201

Applying KCL at node 1

$$\frac{V_1 - 2}{2} + \frac{V_1}{12} + \frac{V_1 - V_2}{1}$$

Applying KCL at node 2

$$\therefore \quad \frac{V_2 - 4}{3} + \frac{V_2}{3} + \frac{V_2 - V_1}{1}$$

$$\therefore \quad \frac{V_1}{2} + \frac{V_1}{12} + \frac{V_1}{1} - \frac{V_2}{1} = 1 \qquad (1)$$

$$\therefore \quad \frac{V_2}{3} + \frac{V_2}{3} + \frac{V_2}{1} - \frac{V_1}{1} = \frac{4}{3} \qquad (2)$$

$$\therefore \quad V_1\left[1 + \frac{1}{2} + \frac{1}{12}\right] - V_2 = 1$$

From (1),

$$V_1\left[\frac{12+6+1}{12}\right] - V_2 = 1$$

$$\frac{19}{12}V_1 - V_2 = 1 \qquad (3)$$

From (2),

$$\therefore \quad V_2\left[\frac{1}{3} + \frac{1}{3} + 1\right] - V_1 = \frac{4}{3} \qquad (4)$$

$$\therefore \quad V_2\left[\frac{1+1+3}{3}\right] - V_1 = \frac{4}{3}$$

$$\therefore \quad \frac{5}{3}V_2 - V_1 = \frac{4}{3} \qquad (4)$$

$$\therefore \quad \frac{19}{12}V_1 - V_2 = 1 \qquad (5)$$

$$-V_1 + \frac{5}{3}V_2 = \frac{4}{3} \qquad (6)$$

Simplifying (5) and (6), we get

$$\therefore \quad V_2 = 1.89 \text{ V}$$

$$\therefore \quad V_1 = 1.82 \text{ V}$$

Practice Problems

1.1 (i) Determine the current in each of the following cases:

(a) 75 C in 1 s (b) 10 C in 0.5 s

(c) 5 C in 2 s

(ii) How long does it take 10 C to flow past a point if the current is 5 A?

1.2 A resistor of 30 Ω has a voltage rating of 500 V; what is its power rating?

1.3 A resistor with a current of 2 A through it converts 1000 J of electrical energy to heat energy in 15 s. What is the voltage across the resistor?

1.4 The filament of a light bulb in the circuit has a certain amount of resistance. If the bulb operates with 120 V and 0.8 A of current, what is the resistance of its filament?

1.5 Find the capacitance of a circuit in which an applied voltage of 20 V gives an energy store of 0.3 J.

1.6 A 6.8 kΩ resistor has burned out in a circuit. It has to be replaced with another resistor with the same ohmic value. If the resistor carries 10 mA, what should be its power rating?

1.7 If you wish to increase the amount of current in a resistor from 100 mA to 150 mA by changing the 20 V source, by how many volts should you change the source? To what new value should you set it?

1.8 A 12 V source is connected to a 10 Ω resistor.

(a) How much energy is used in two minutes?

(b) If the resistor is disconnected after one minute, does the power absorbed in resistor increase or decrease?

1.9 A capacitor is charged to 50 μC. The voltage across the capacitor is 150 V. It is then connected to another capacitor four times the capacitance of the first capacitor. Find the loss of energy.

1.10 The voltage across two parallel capacitors 5 μF and 3 μF changes uniformly from 30 to 75 V in 10 ms. Calculate the rate of change of voltage for (i) each capacitor, and (ii) the combination.

1.11 The voltage waveform shown in Fig. 1.202 is applied to a pure capacitor of 60 μF. Sketch $i(t)$, $p(t)$ and determine I_m and P_m.

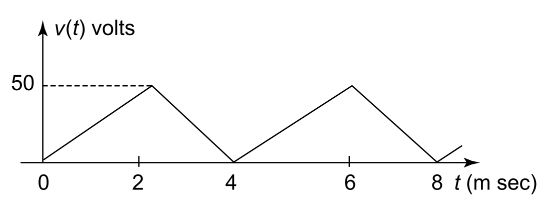


Fig. 1.202

1.12 Determine an expression for the current if the voltage across a pure capacitor is given as

$$v = V_m\left[wt - \frac{(wt)^3}{3!} + \frac{(wt)^5}{5!} - \frac{(wt)^7}{7!} + ...\right]$$

1.13 A 2 μF capacitor has a charge function $q = 100\ [1 \times e^{-5\times10^4 t}]$ μc. Determine the corresponding voltage and current functions.

1.14 A pure inductance of 0.05 H has an applied voltage with the waveform shown in Fig. 1.203. Sketch the corresponding current waveform and determine the expression for *i* in the first internal.

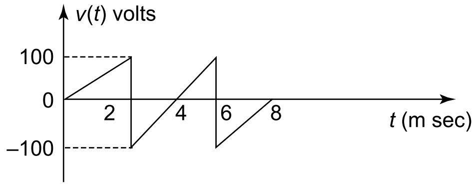


Fig. 1.203

1.15 An inductor of 0.004 H contains a current with a waveform shown in Fig. 1.204. Sketch the voltage waveform.

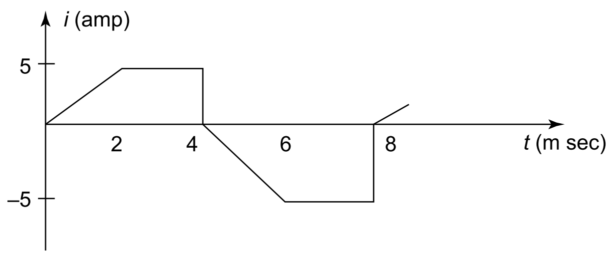


Fig. 1.204

1.16 A single pure inductance of 3 mH passes a current of the waveform shown in Fig. 1.205. Determine and sketch the voltage υ(t) and the instantaneous power *p*(*t*).

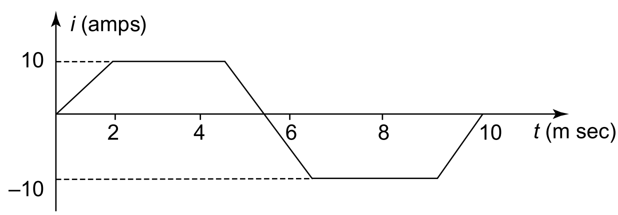


Fig. 1.205

1.17 Simplify the circuit shown in Fig. 1.206 using series parallel combinations.

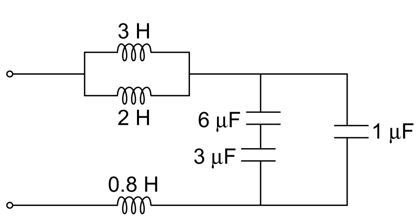


Fig. 1.206

1.18 Determine the equivalent capacitance of the circuit shown in Fig. 1.207 if all the capacitors are 10 μF.

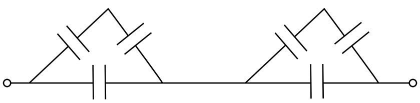


Fig. 1.207

1.19 Reduce the circuit shown in Fig. 1.208 to a single equivalent capacitance across terminals *a* and *b*.

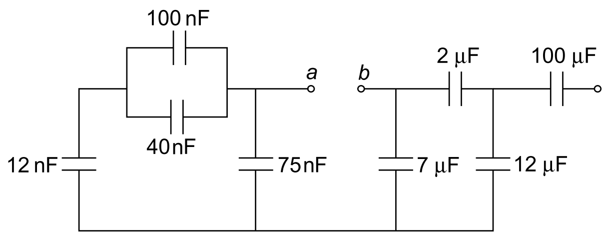


Fig. 1.208

1.20 For the circuit shown in Fig. 1.209, find the equivalent inductance.

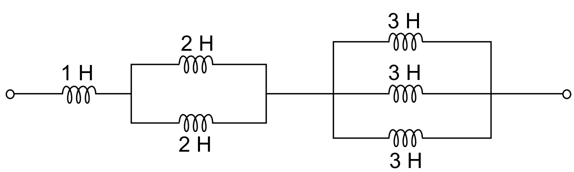


Fig. 1.209

1.21 The following voltage drops are measured across each of three resistors in series: 5.5 V, 7.2 V and 12.3 V. What is the value of the source voltage to which these resistors are connected? If a fourth resistor is added to the circuit with a source voltage of 30 V. What should be the drop across the fourth resistor?

1.22 What is the voltage V_{AB} across the resistor shown in Fig. 1.210?

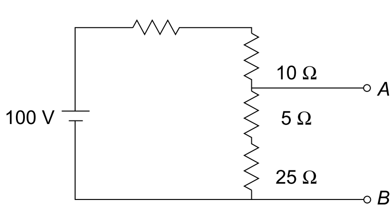


Fig. 1.210(a)

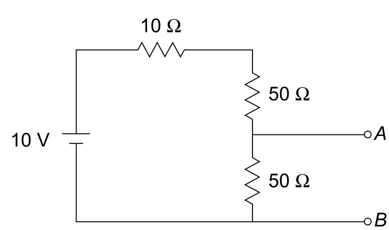


Fig. 1.210(b)

1.23 The source voltage in the circuit shown in Fig. 1.211 is 100 V. How much voltage does each metre read?

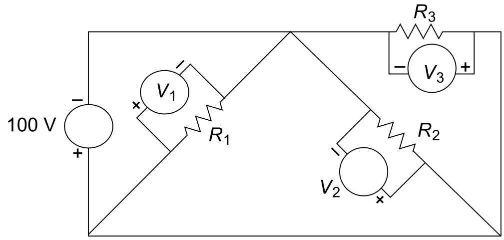


Fig. 1.211

1.24 Using the current divider formula, determine the current in each branch of the circuit shown in Fig. 1.212.

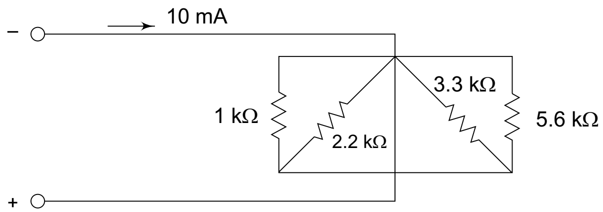


Fig. 1.212

1.25 Six light bulbs are connected in parallel across 110 V. Each bulb is rated at 75 W. How much current flows through each bulb, and what is the total current?

1.26 For the circuit shown in Fig. 1.213, find the total resistance between terminals *A* and *B*; the total current drawn from a 6 V source connected from *A* to *B*; and the current through 4.7 kΩ; voltage across 3 kΩ.

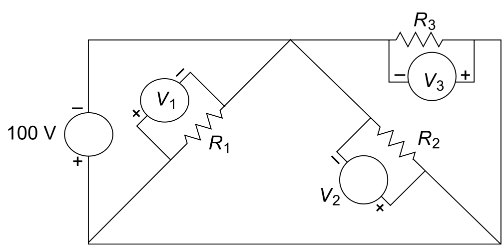


Fig. 1.213

1.27 For the circuit shown in Fig. 1.214, find the total resistance.

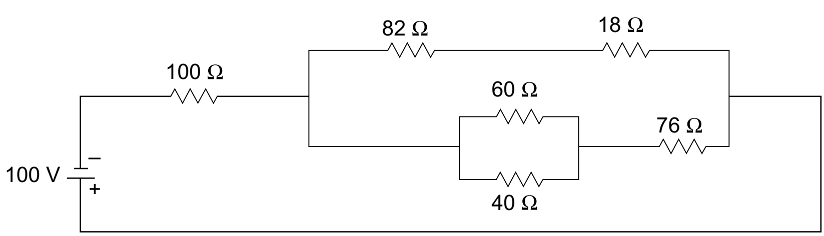


Fig. 1.214

1.28 The current in the 5 Ω resistance of the circuit shown in Fig. 1.215 is 5 A. Find the current in the 10 Ω resistor. Calculate the power consumed by the 5 Ω resistor.

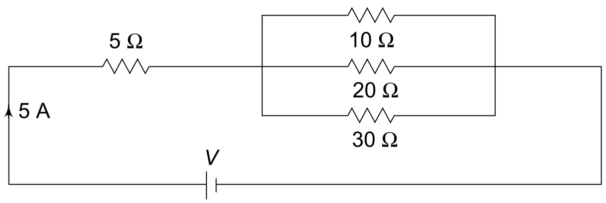


Fig. 1.215

1.29 A battery of unknown emf is connected across resistances as shown in Fig. 1.216. The voltage drop across the 8 Ω resistor is 20 V. What will be the current reading in the ammeter? What is the emf of the battery.

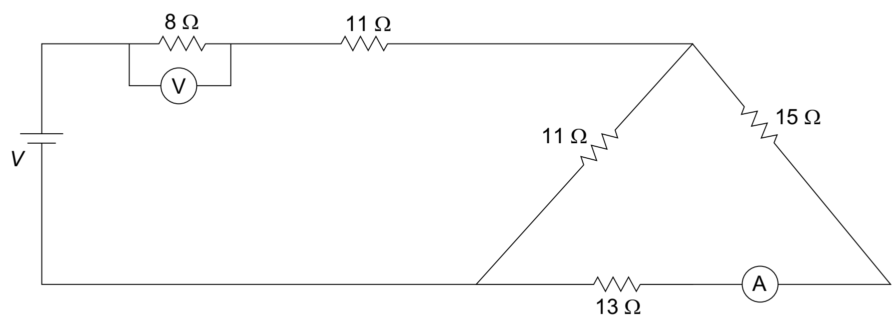


Fig. 1.216

1.30 An electric circuit has three terminals *A*, *B*, *C*. Between *A* and *B* is connected a 2 Ω resistor, between *B* and *C* are connected a 7 Ω resistor and 5 Ω resistor in parallel and between *A* and *C* is connected a 1 Ω resistor. A battery of 10 V is then connected between terminals *A* and *C*. Calculate (a) total current drawn from the battery (b) voltage across the 2 Ω resistor (c) current passing through the 5 Ω resistor.

1.31 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.217, find V_{in}, V_s and power provided by the dependent source.

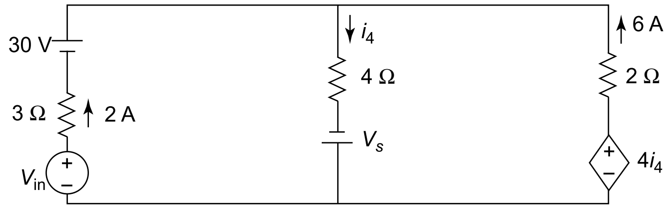


Fig. 1.217

1.32 Use Ohm's law and Kirchhoff's laws on the circuit given in Fig. 1.218, find all the voltages and currents.

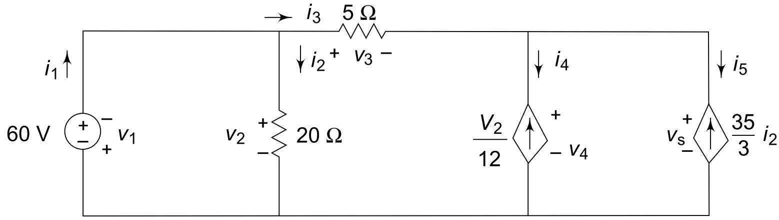


Fig. 1.218

1.33 Find the power absorbed by each element and show that the algebraic sum of powers is zero in the circuit shown in Fig. 1.219.

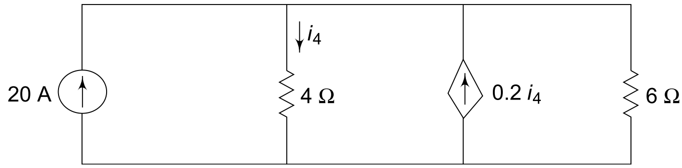


Fig. 1.219

1.34 Find the power absorbed by each element in the circuit shown in Fig. 1.220.

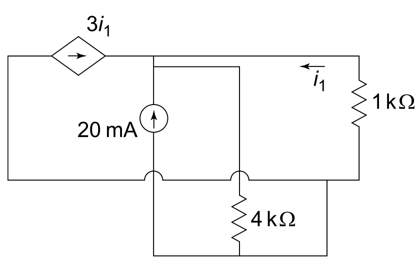


Fig. 1.220

1.35 In the circuit shown in Fig. 1.221, use mesh analysis to find out the power delivered to the 4 Ω resistor. To what voltage should the 100 V battery be changed so that no power is delivered to the 4 Ω resistor?

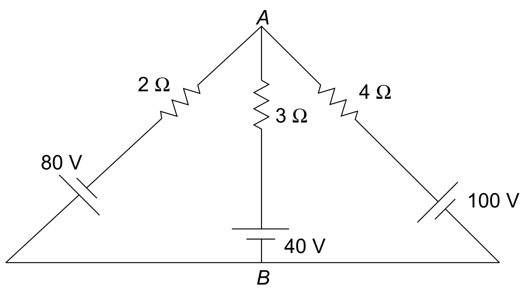


Fig. 1.221

1.36 Find the voltage between A and B of the circuit shown in Fig. 1.222 by mesh analysis.

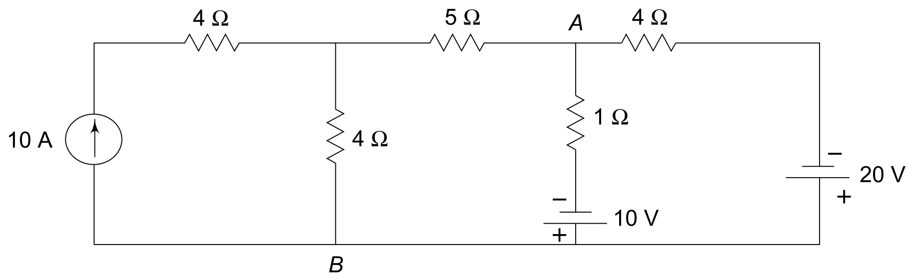


Fig. 1.222

1.37 In the circuit shown in Fig. 1.223, use nodal analysis to find out the voltage across 40 Ω and the power supplied by the 5 A source.

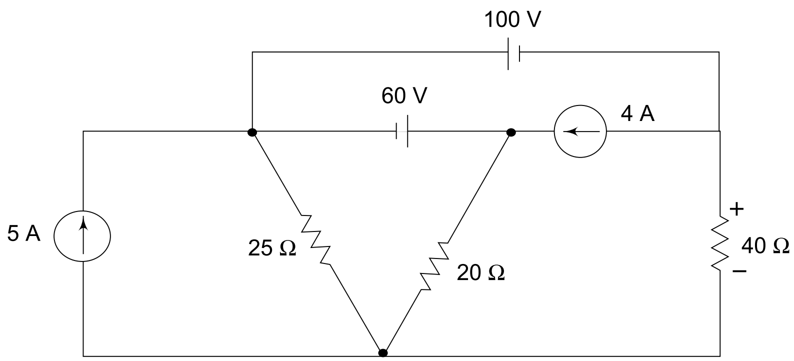


Fig. 1.223

1.38 In the network shown in Fig. 1.224, the resistance R is variable from zero to infinity. The current I through R can be expressed as $I = a + bV$, where V is the voltage across R as shown in the figure, and a and b are constants. Determine a and b.

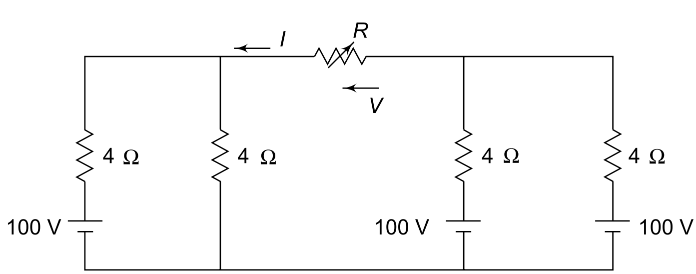


Fig. 1.224

1.39 Determine the currents in bridge circuit by using mesh analysis in Fig. 1.225.

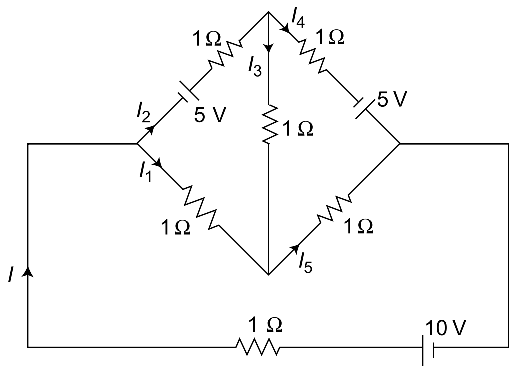


Fig. 1.225

1.40 Use nodal analysis in the circuit shown in Fig. 1.226 and determine what value of V will cause $V_{10} = 0$.

1.41 For the circuit shown in Fig. 1.227, use mesh analysis to find the values of all mesh currents.

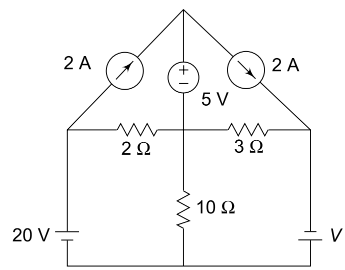


Fig. 1.226

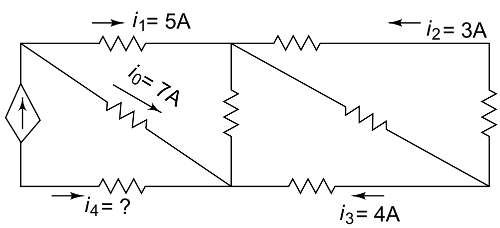


Fig. 1.227

1.42 For the circuit shown in Fig. 1.228, use node analysis to find the current delivered by the 24 V source.

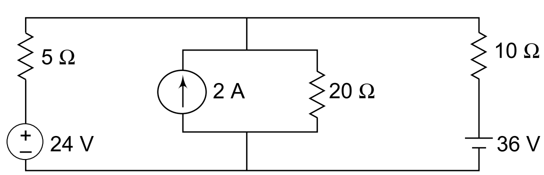


Fig. 1.228

1.43 Using mesh analysis, determine the voltage across the 10 kΩ resistor at terminals A and B of the circuit shown in Fig. 1.229.

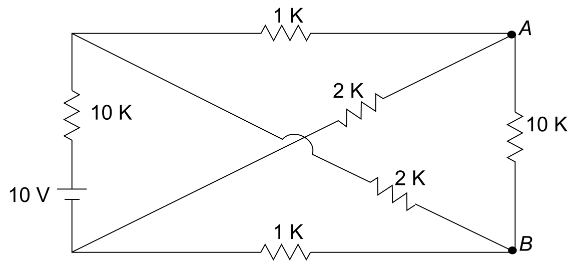


Fig. 1.229

1.44 Determine the current I in the circuit by using loop analysis in Fig. 1.230.

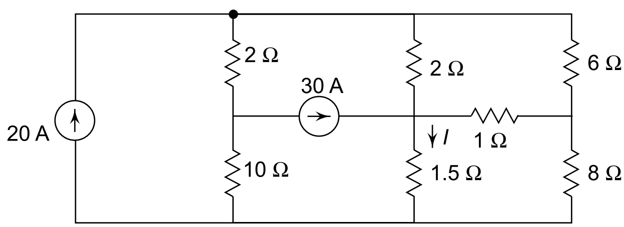


Fig. 1.230

1.45 Write nodal equations for the circuit shown in Fig. 1.231, and find the power supplied by the 10 V source.

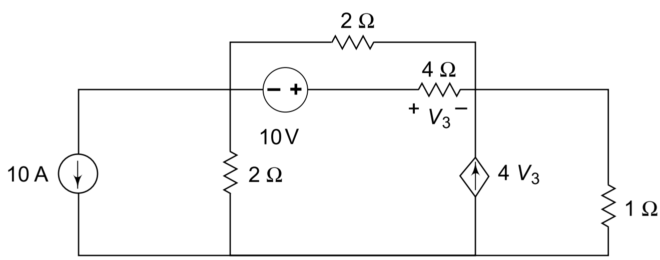


Fig. 1.231

1.46 Use nodal analysis to find V_2 in the circuit shown in Fig. 1.232.

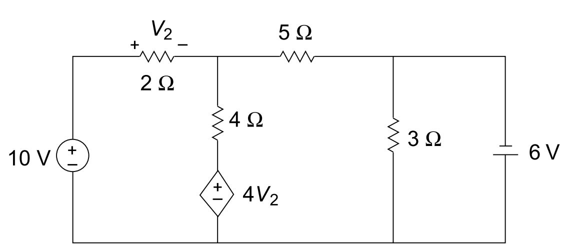


Fig. 1.232

1.47 Use mesh analysis to find V_x in the circuit shown in Fig. 1.233.

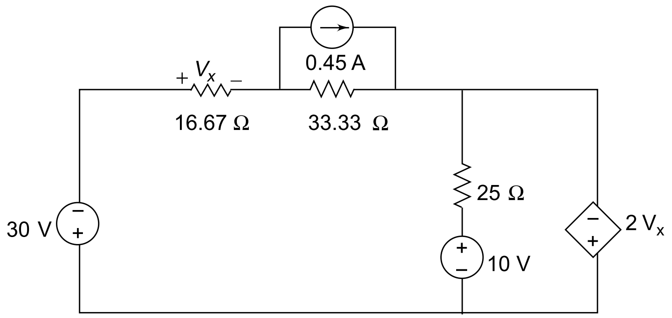


Fig. 1.233

1.48 For the circuit shown in Fig. 1.234, find the value of V_2 that will cause the voltage across 20 Ω to be zero by using mesh analysis.

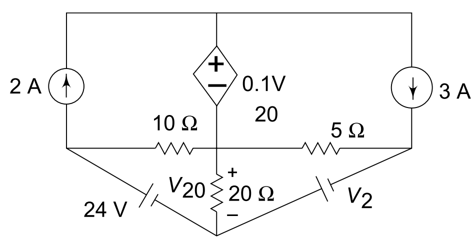


Fig. 1.234

Objective Type Questions

1.1 How many coulombs of charge do 50×10^{31} electrons possess?
(a) 80×10^{12} C (b) 50×10^{31} C
(c) 0.02×10^{-31} C (d) $1/80 \times 10^{12}$ C

1.2 Determine the voltage of 100 J/25 C.
(a) 100 V (b) 25 V
(c) 4 V (d) 0.25 V

1.3 What is the voltage of a battery that uses 800 J of energy to move 40 C of charge through a resistor?
(a) 800 V (b) 40 V
(c) 25 V (d) 20 V

1.4 Determine the current if a 10 coulomb charge passes a point in 0.5 seconds.
(a) 10 A (b) 20 A
(c) 0.5 A (d) 2 A

1.5 If a resistor has 5.5 V across it and 3 mA flowing through it, what is the power?

(a) 16.5 mW (b) 15 mW
(c) 1.83 mW (d) 16.5 W

1.6 Identify the passive element among the following.

(a) Voltage source (b) Current source
(c) Inductor (d) Transistor

1.7 If a resistor is to carry 1 A of current and handle 100 W of power, how many ohms must it be? Assume that voltage can be adjusted to any required value.

(a) 50 Ω (b) 100 Ω
(c) 1 Ω (d) 10 Ω

1.8 A 100 Ω resistor is connected across the terminals of a 2.5 V battery. What is the power dissipation in the resistor?

(a) 25 W (b) 100 W
(c) 0.4 W (d) 6.25 W

1.9 Determine total inductance of a parallel combination of 100 mH, 50 mH and 10 mH.

(a) 7.69 mH (b) 160 mH
(c) 60 mH (d) 110 mH

1.10 How much energy is stored by a 100 mH inductance with a current of 1 A?

(a) 100 J (b) 1 J
(c) 0.05 J (d) 0.01 J

1.11 Five inductors are connected in series. The lowest value is 5 μH. If the value of each inductor is twice that of the preceding one, and if the inductors are connected in order ascending values. What is the total inductance?

(a) 155 μH (b) 155 H
(c) 155 mH (d) 25 μH

1.12 Determine the charge when $C = 0.001$ μF and $\upsilon = 1$ KV.

(a) 0.001 C (b) 1 μC
(c) 1 C (d) 0.001 C

1.13 If the voltage across a given capacitor is increased, does the amount of stored charge

(a) increase (b) decrease
(c) remain constant (d) is exactly doubled

1.14 A 1 μF, a 2.2 μF and a 0.05 μF capacitors are connected in series. The total capacitance is less than

(a) 0.07 (b) 3.25
(c) 0.05 (d) 3.2

1.15 How much energy is stored by a 0.05 μF capacitor with a voltage of 100 V?

(a) 0.025 J (b) 0.05 J
(c) 5 J (d) 100 J

1.16 Which one of the following is an ideal voltage source?

(a) voltage independent of current
(b) current independent of voltage
(c) both (a) and (b)
(d) none of the above

1.17 The following voltage drops are measured across each of three resistors in series: 5.2 V, 8.5 V and 12.3 V. What is the value of the source voltage to which these resistors are connected?

(a) 8.2 V (b) 12.3 V (c) 5.2 V (d) 26 V

1.18 A certain series circuit has a 100 Ω, a 270 Ω, and a 330 Ω resistor in series. If the 270 Ω resistor is removed, the current

(a) increases (b) becomes zero
(c) decrease (d) remain constant

1.19 A series circuit consists of a 4.7 kΩ, 5.6 kΩ, 9 kΩ and 10 kΩ resistor. Which resistor has the most voltage across it?

(a) 4.7 kΩ (b) 5.6 kΩ (c) 9 kΩ (d) 10 kΩ

1.20 The total power in a series circuit is 10 W. There are five equal value resistors in the circuit. How much power does each resistor dissipate?

(a) 10 W (b) 5 W (c) 2 W (d) 1 W

1.21 When a 1.2 kΩ resistor, 100 Ω resistor, 1 kΩ resistor and 50 Ω resistor are in parallel, the total resistance is less than

(a) 100 Ω (b) 50 Ω (c) 1 kΩ (d) 1.2 kΩ

1.22 If a 10 V battery is connected across the parallel resistors of 3 Ω, 5 Ω, 10 Ω and 20 Ω, how much voltage is there across 5 Ω resistor?

(a) 10 V (b) 3 V (c) 5 V (d) 20 V

1.23 If one of the resistors in a parallel circuit is removed, what happens to the total resistance?

(a) decreases (b) increases
(c) remain constant (d) exactly doubles

1.24 The power dissipation in each of three parallel branches is 1 W. What is the total power dissipation of the circuit?

(a) 1 W (b) 4 W
(c) 3 W (d) zero

1.25 In a four branch parallel circuit, 10 mA of current flows in each branch. If one of the branch opens, the current in each of the other branches

(a) increases (b) decreases
(c) remains unaffected (d) doubles

1.26 Four equal value resistors are connected in parallel. Five volts are applied across the parallel circuit, and 2.5 mA are measured from the source. What is the value of each resistor?

(a) 4 Ω (b) 8 Ω
(c) 2.5 Ω (d) 5 Ω

1.27 Six light bulbs are connected in parallel across 110 V. Each bulb is related at 75 W. How much current flows through each bulb?

(a) 0.682 A (b) 0.7 A
(c) 75 A (d) 110 A

1.28 A 330 Ω resistor is in series with the parallel combination of four 1 kΩ resistors. A 100 V source is connected to the circuit. Which resistor has the most current through it.

(a) 330 Ω resistor
(b) parallel combination of three 1 kΩ resistors
(c) parallel combination of two 1 kΩ resistors
(d) 1 kΩ resistor

1.29 The current i_4 in the circuit shown in Fig. 1.235 is equal to

(a) 12 A (b) −12 A
(c) 4 A (d) None of the above

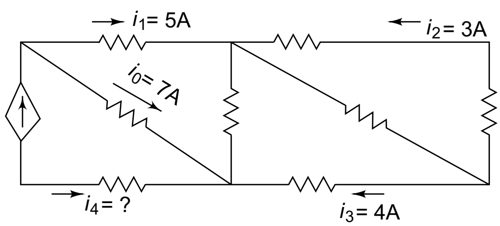


Fig. 1.235

1.30 The voltage V in Fig. 1.236 is equal to

(a) 3 V
(b) −3 V
(c) 5 V
(d) None of the above

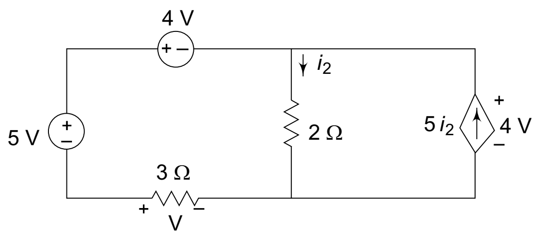


Fig. 1.236

1.31 The voltage V in Fig. 1.237 is always equal to

(a) 9 V
(b) 5 V
(c) 1 V
(d) None of the above

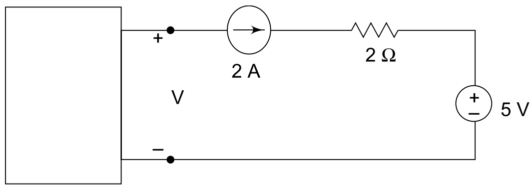


Fig. 1.237

1.32 The voltage V in Fig. 1.238 is

(a) 10 V
(b) 15 V
(c) 5 V
(d) None of the above

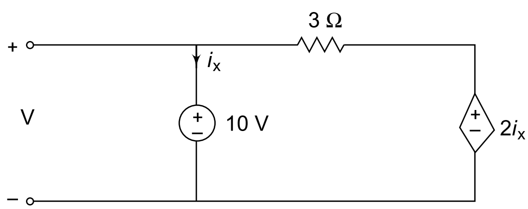


Fig. 1.238

1.33 Mesh analysis is based on

(a) Kirchhoff's current law
(b) Kirchhoff's voltage law
(c) Both
(d) None

1.34 If a network contains B branches, and N nodes, then the number of mesh current equations would be

(a) $B - (N - 1)$
(b) N 2 (B 2 1)
(c) B 2 N 2 1
(d) $(B + N) - 1$

1.35 A network has 10 nodes and 17 branches. The number of different node pair voltages would be

(a) 7
(b) 9
(c) 45
(d) 10

1.36 A circuit consists of two resistances, R_1 and R_2, in parallel. The total current passing through the circuit is I_T. The current passing through R_1 is

(a) $\dfrac{I_T R_1}{R_1 + R_2}$

(b) $\dfrac{I_T (R_1 + R_2)}{R_1}$

(c) $\dfrac{I_T R_2}{R_1 + R_2}$

(d) $\dfrac{I_T R_1 + R_2}{R_2}$

1.37 A network has seven nodes and five independent loops. The number of branches in the network is

(a) 13 (b) 12 (c) 11 (d) 10

1.38 The nodel method of circuit analysis is based on

(a) KVL and Ohm's law (b) KCL and Omp's law
(c) KCL and KVL (d) KCL, KVL and Omp's law

1.39 The number of independent loops for a network with n nodes and b branches is

(a) $n - 1$ (b) $b - n$
(c) $b - n + 1$ (d) independent of the number of nodes

2 AC Fundamentals and Network Topology

2.1 DEFINITIONS OF TERMS ASSOCIATED WITH PERIODIC FUNCTIONS

2.1.1 Time Period, Angular Velocity and Frequency *[JNTU Nov 2011]*

Sinusoidal Alternating Quantities

Many a time, alternating voltages and currents are represented by a sinusoidal wave, or simply a sinusoid. It is a very common type of alternating current (ac) and alternating voltage. The sinusoidal wave is generally referred to as a sine wave. Basically an alternating voltage (current) waveform is defined as the voltage (current) that fluctuates with time periodically, with change in polarity and direction. In general, the sine wave is more useful than other waveforms, like pulse, sawtooth, square, etc. There are a number of reasons for this. One of the reasons is that if we take any second order system, the response of this system is a sinusoid. Secondly, any periodic waveform can be written in terms of sinusoidal function according to Fourier theorem. Another reason is that its derivatives and integrals are also sinusoids. A sinusoidal function is easy to analyse. Lastly, the sinusoidal function is easy to generate, and it is more useful in the power industry. The shape of a sinusoidal waveform is shown in Fig. 2.1.

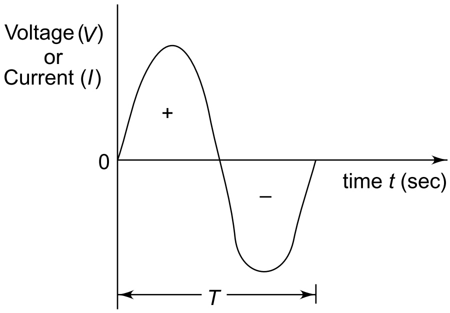


Fig. 2.1

The waveform may be either a current waveform, or a voltage waveform. As seen from the Fig. 2.1, the wave changes its magnitude and direction with time. If we start at time $t = 0$, the wave goes to a maximum value and returns to zero, and then decreases to a negative maximum value before returning to zero. The sine wave changes with time in an orderly manner. During the positive portion of voltage, the current flows in one direction; and during the negative portion of voltage, the current flows in the opposite direction. The complete positive and negative portion of the wave is one cycle of the sine wave. Time is designated by t. The time taken for any wave to complete one full cycle is called the period (T). In general, any periodic wave constitutes a number of such cycles. For example, one cycle of a sine wave repeats a number of times as shown in Fig. 2.2. Mathematically, it can be represented as $f(t) = f(t + T)$ for any t.

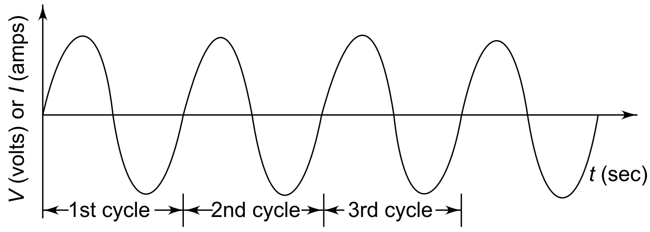


Fig. 2.2

The period can be measured in the following different ways (See Fig. 2.3).

1. From zero crossing of one cycle to zero crossing of the next cycle.
2. From positive peak of one cycle to positive peak of the next cycle, and
3. From negative peak of one cycle to negative peak of the next cycle.

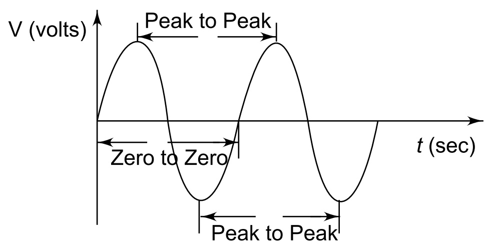


Fig. 2.3

The frequency of a wave is defined as the number of cycles that a wave completes in one second.

In Fig. 2.4, the sine wave completes three cycles in one second. Frequency is measured in hertz. One hertz is equivalent to one cycle per second, 60 hertz is 60 cycles per second and so on. In Fig. 2.4, the frequency denoted by f is 3 Hz, that is three cycles per second. The relation between time period and frequency is given by

$$f = \frac{1}{T}$$

A sine wave with a longer period consists of fewer cycles than one with a shorter period.

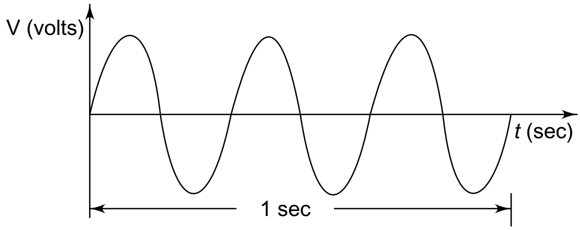


Fig. 2.4

Example 2.1 *What is the period of sine wave shown in Fig. 2.5?*

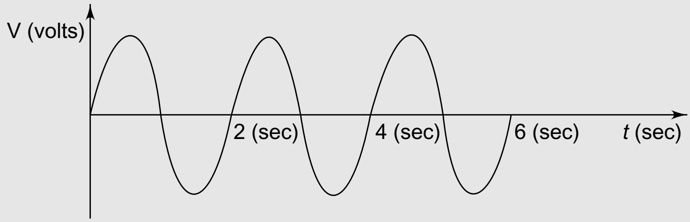


Fig. 2.5

Solution From Fig. 2.5, it can be seen the sine wave takes two seconds to complete one period in each cycle

$$T = 2\,\text{s}$$

Example 2.2 *The period of a sine wave is 20 milliseconds. What is the frequency?*

Solution

$$f = \frac{1}{T}$$

$$= \frac{1}{20\,\text{ms}} = 50\,\text{Hz}$$

Example 2.3 *The frequency of a sine wave is 30 Hz. What is its period?*

Solution

$$T = \frac{1}{f}$$

$$= \frac{1}{30} = 0.03333\ \text{s}$$

$$= 33.33\ \text{ms}$$

Example 2.4 *Calculate the frequency for each of the following values of time period.*

(a) 2 ms *(b) 100 ms* *(c) 5 ms* *(d) 5 s*

Solution The relation between frequency and period is given by

$$f = \frac{1}{T}\,\text{Hz}$$

(a) Frequency $f = \dfrac{1}{20 \times 10^{-3}} = 50\ \text{Hz}$

(b) Frequency $f = \dfrac{1}{100 \times 10^{-3}} = 10\ \text{Hz}$

(c) Frequency $f = \dfrac{1}{5\times10^{-6}} = 200\text{ kHz}$

(d) Frequency $f = \dfrac{1}{5} = 0.2\text{ Hz}$

Example 2.5 *Calculate the period for each of the following values of frequency.*

(a) 50 Hz *(b) 100 kHz* *(c) 1 Hz* *(d) 2 MHz*

Solution The relation between frequency and period is given by

$$f = \frac{1}{T}\text{ Hz}$$

(a) Time period $T = \dfrac{1}{f} = \dfrac{1}{50} = 0.02\text{ s}$

(b) Time period $T = \dfrac{1}{f} = \dfrac{1}{100\times10^{3}} = 10\ \mu\text{s}$

(c) Time period $T = \dfrac{1}{f} = \dfrac{1}{1} = 1\text{ s}$

(d) Time period $T = \dfrac{1}{f} = \dfrac{1}{2\times10^{6}} = 0.5\ \mu\text{s}$

Example 2.6 *A sine wave has a frequency of 50 kHz. How many cycles does it complete in 20 ms?*

Solution The frequency of sine wave is 50 kHz.

That means in 1 second, a sine wave goes through 50×10^3 cycles.

In 20 ms the number of cycles $= 20 \times 10^{-3} \times 50 \times 10^3$

$= 1\text{ kHz}$

That means in 20 ms the sine wave goes through 10^3 cycles.

Angular Relation of a Sinusoidal Wave

A sine wave can be measured along the X-axis on a time base which is frequency-dependent. A sine wave can also be expressed in terms of an angular measurement. This angular measurement is expressed in degrees or radians. A radian is defined as the angular distance measured along the circumference of a circle which is equal to the radius of the circle. One radian is equal to 57.3°. In a 360° revolution, there are 2π radians. The angular measurement of a sine wave is based on 360° or 2π radians for a complete cycle as shown in Figs 2.6(a) and (b).

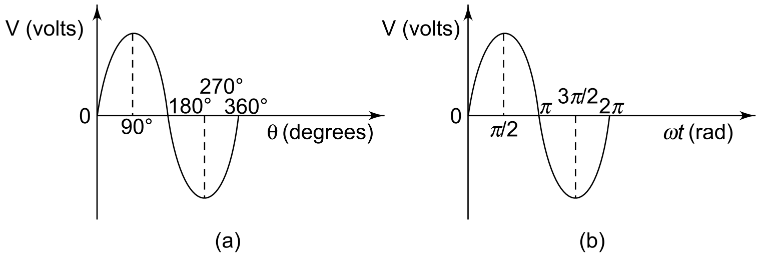


Fig. 2.6

A sine wave completes a half cycle in 180° or π radians; a quarter cycle in 90° or π/2 radians, and so on.

Phase of a Sinusoidal Wave

The phase of a sine wave is an angular measurement that specifies the position of the sine wave relative to a reference. The wave shown in Fig. 2.7 is taken as the reference wave.

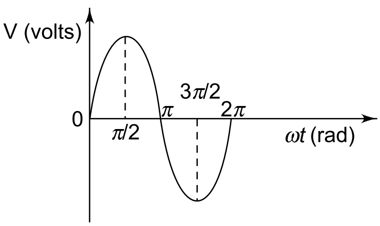


Fig. 2.7

When the sine wave is shifted left or right with reference to the wave shown in Fig. 2.7, there occurs a phase shift. Figure 2.8 shows the phase shifts of a sine wave.

In Fig. 2.8(a), the sine wave is shifted to the right by 90° (π/2 rad) shown by the dotted lines. There is a phase angle of 90° between *A* and *B*. Here the waveform *B* is lagging behind waveform *A* by 90°. In other words, the sine wave *A* is leading the waveform *B* by 90°. In Fig. 2.8(b) the sine wave *A* is lagging behind the waveform *B* by 90°. In both cases, the phase difference is 90°.

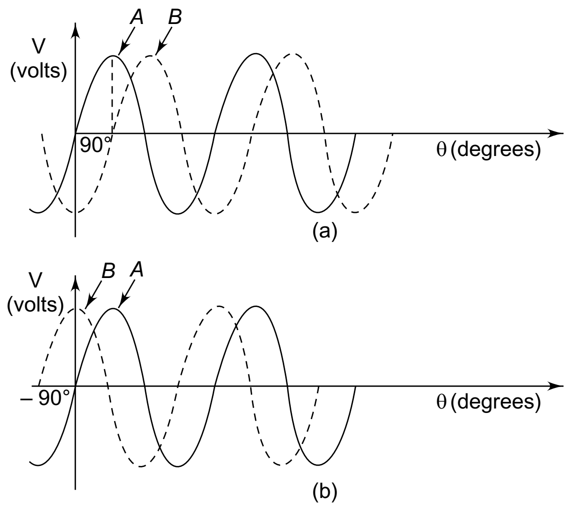


Fig. 2.8

Example 2.7 *What are the phase angles between the two sine waves shown in Figs 2.9(a) and (b)?*

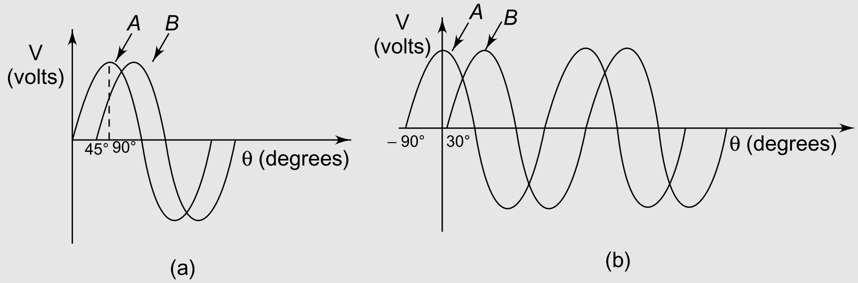


Fig. 2.9

Solution In Fig. 2.9(a), sine wave A is in phase with the reference wave; sine wave B is out of phase, which lags behind the reference wave by 45°. So we say that sine wave B lags behind sine wave A by 45°.

In Fig. 2.9(b), sine wave A leads the reference wave by 90°; sine wave B lags behind the reference wave by 30°. So the phase difference between A and B is 120°, which means that sine wave B lags behind sine wave A by 120°. In other words, sine wave A leads sine wave B by 120°.

Example 2.8 *Sine wave 'A' has a positive going zero crossing at 45°. Sine wave 'B' has a positive going zero crossing at 60°. Determine the phase angle between the signals. Which of the signal lags behind the other?*

Solution The two signals drawn are shown in Fig. 2.10.

From Fig. 2.10, the signal B lags behind signal A by 15°. In other words, signal A leads signal B by 15°.

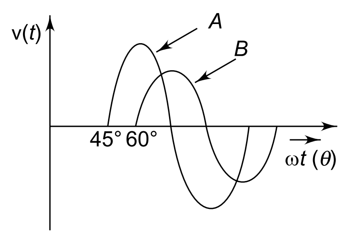


Fig. 2.10

Example 2.9 *One sine wave has a positive peak at 75°, and another has a positive peak at 100°. How much is each sine wave shifted in phase from the 0° reference? What is the phase angle between them?*

Solution The two signals are drawn as shown in Fig. 2.11.

The signal A leads the reference signal by 15°.

The signal B lags behind the reference signal by 10°.

The phase angle between these two signals is 25°.

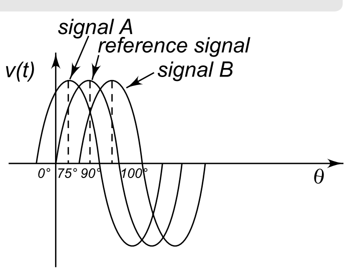


Fig. 2.11

The Mathematical Representation of Sinusoidal Quantity

A sine wave is graphically represented as shown in Fig. 2.12(a). The amplitude of a sine wave is represented on vertical axis. The angular measurement (in degrees or radians) is represented on horizontal axis. Amplitude A is the maximum value of the voltage or current on the Y-axis.

In general, the sine wave is represented by the equation

$$v(t) = V_m \sin \omega t$$

The above equation states that any point on the sine wave represented by an instantaneous value $v(t)$ is equal to the maximum value times the sine of the angular frequency at that point. For example, if a certain sine wave voltage has peak value of 20 V, the instantaneous voltage at a point $\pi/4$ radians along the horizontal axis can be calculated as

$$v(t) = V_m \sin \omega t$$
$$= 20 \sin\left(\frac{\pi}{4}\right) = 20 \times 0.707 = 14.14\,\text{V}$$

When a sine wave is shifted to the left of the reference wave by a certain angle ϕ, as shown in Fig. 2.12(b), the general expression can be written as

$$v(t) = V_m \sin(\omega t + \phi)$$

When a sine wave is shifted to the right of the reference wave by a certain angle ϕ, as shown in Fig. 2.12(c), the general expression is

$$v(t) = V_m \sin(\omega t - \phi)$$

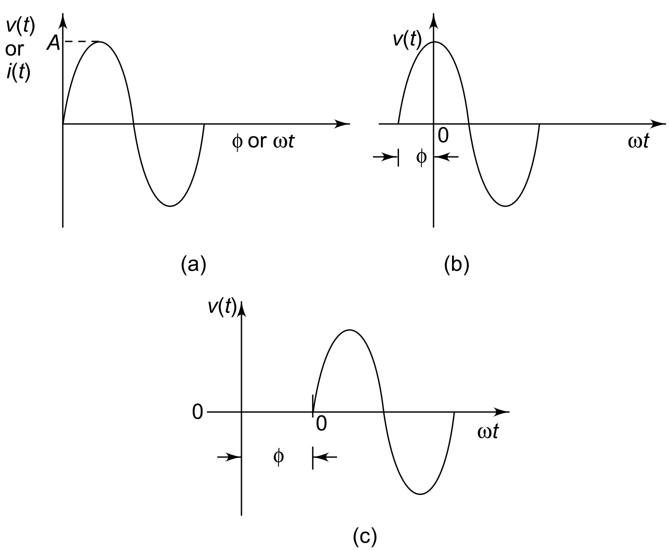


Fig. 2.12

Example 2.10 *Determine the instantaneous value at the 90° point on the X-axis for each sine wave shown in Fig. 2.13.*

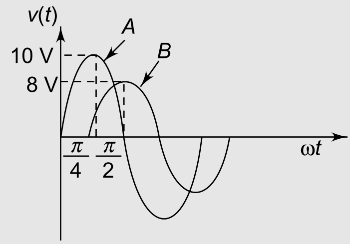


Fig. 2.13

Solution From Fig. 2.13, the equation for the sine wave *A*

$$v(t) = 10 \sin \omega t$$

The value at $\pi/2$ in this wave is

$$v(t) = 10 \sin\frac{\pi}{2} = 10\,\text{V}$$

The equation for the sine wave *B*

$$v(t) = 8 \sin(\omega t - \pi/4)$$

At $\qquad \omega t = \pi/2$

2.2 VOLTAGE AND CURRENT VALUES OF SINUSOIDAL WAVE

As the magnitude of the waveform is not constant, the waveform can be measured in different ways. These are instantaneous, peak, peak to peak, root mean square (rms) and average values.

2.2.1 Instantaneous Value

Consider the sine wave shown in Fig. 2.14. At any given time, it has some instantaneous value. This value is different at different points along the waveform.

In Fig. 2.14 during the positive cycle, the instantaneous values are positive and during the negative cycle, the instantaneous values are negative. In Fig. 2.14 shown at time 1 ms, the value is 4.2 V; the value is 10 V at 2.5 ms, –2 V at 6 ms and –10 V at 7.5 and so on.

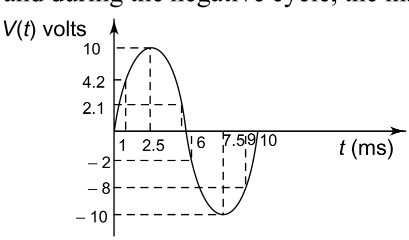


Fig. 2.14

2.2.2 Peak Value

The peak value of the sine wave is the maximum value of the wave during positive half cycle, or maximum value

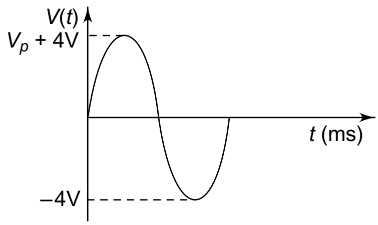


Fig. 2.15

of wave during negative half cycle. Since the value of these two are equal in magnitude, a sine wave is characterised by a single peak value. The peak value of the sine wave is shown in Fig. 2.15; here the peak value of the sine wave is 4 V.

2.2.3 Peak to Peak Value

The peak to peak value of a sine wave is the value from the positive to the negative peak as shown in Fig. 2.16. Here the peak to peak value is 8 V.

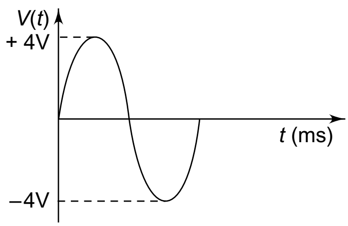


Fig. 2.16

2.2.4 Average Value

[JNTU May/June 2006, Jan 2010, Nov 2011]

In general, the average value of any function $v(t)$, with period T is given by

$$v_{\text{av}} = \frac{1}{T}\int_0^T v(t)dt$$

That means that the average value of a curve in the X-Y plane is the total area under the complete curve divided by the distance of the curve. The average value of a sine wave over one complete cycle is always zero. So the average value of a sine wave is defined over a half-cycle, and not a full cycle period.

The average value of the sine wave is the total area under the half-cycle curve divided by the distance of the curve.

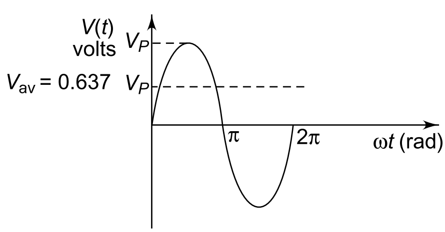


Fig. 2.17

The average value of the sine wave $v(t) = V_P \sin \omega t$ is given by

$$v_{\text{av}} = \frac{1}{\pi}\int_0^{\pi} V_P \sin \omega t \, d(\omega t)$$

$$= \frac{1}{\pi}\left[-V_P \cos \omega t\right]_0^{\pi}$$

$$= \frac{2V_P}{\pi} = 0.637\ V_P$$

The average value of a sine wave is shown by the dotted line in Fig. 2.17.

Example 2.11 *Find the average value of a cosine wave f(t) = cos ωt shown in Fig. 2.18.*

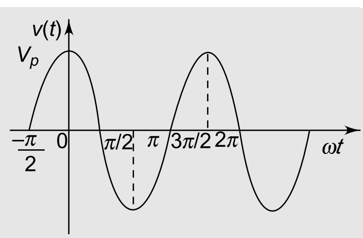


Fig. 2.18

Solution The average value of a cosine wave

$$v(t) = V_P \cos \omega t$$

$$V_{av} = \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} V_P \cos \omega t \; d(\omega t)$$

$$= \frac{1}{\pi} V_P (-\sin \omega t)_{\pi/2}^{3\pi/2}$$

$$= \frac{-V_P}{\pi}[-1-1] = \frac{2V_P}{\pi} = 0.637 \; V_P$$

2.2.5 Root Mean Square Value or Effective Value

[JNTU May/June 2006, Jan 2010, Nov 2011]

The root mean square (rms) value of a sine wave is a measure of the heating effect of the wave. When a resistor is connected across a dc voltage source as shown in Fig. 2.19(a), a certain amount of heat is produced in the resistor in a given time. A similar resistor is connected across an ac voltage source for the same time as shown in Fig. 2.19(b). The value of the ac voltage is adjusted such that the same amount of heat is produced in the resistor as in the case of the dc source. This value is called the rms value.

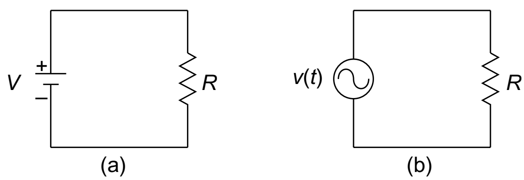


Fig. 2.19

That means the rms value of a sine wave is equal to the dc voltage that produces the same heating effect. In general, the rms value of any function with period T has an effective value given by

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T \overline{v(t)}^2 \, dt}$$

Consider a function $v(t) = V_P \sin \omega t$

The rms value, $V_{rms} = \sqrt{\frac{1}{T}\int_0^T (V_P \sin \omega t)^2 \, d(\omega t)}$

$$= \sqrt{\frac{1}{T}\int_0^{2\pi} V_P^2 \left[\frac{1-\cos 2\omega t}{2}\right] d(\omega t)}$$

$$= \frac{V_P}{\sqrt{2}} = 0.707 \; V_P$$

If the function consists of a number of sinusoidal terms, that is

$$v(t) = V_0 + (V_{c1} \cos \; \omega t + V_{c2} \cos \, 2 \, \omega t + \cdots) + (V_{s1} \sin \omega t + V_{s2} \sin 2 \, \omega t + \cdots)$$

The rms, or effective value is given by

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{1}{2}(V_{c1}^2 + V_{c2}^2 + \cdots) + \frac{1}{2}(V_{s1}^2 + V_{s2}^2 + \cdots)}$$

Example 2.12 *A wire is carrying a direct current of 20 A and a sinusoidal alternating current of peak value 20 A. Find the rms value of the resultant current in the wire.*

Solution The rms value of the combined wave

$$= \sqrt{20^2 + \frac{20^2}{2}}$$

$$= \sqrt{400 + 200} = \sqrt{600} = 24.5\text{ A}$$

Example 2.13 *Find the rms value of the voltage wave whose equation v(t) = 10 + 200 sin (wt − 30°) + 100 cos 3 wt − 50 sin (5wt + 60°).*

Solution

$$V_{\text{rms}} = \sqrt{10^2 + \frac{(200)^2}{2} + \frac{(100)^2}{2} + \frac{(50)^2}{2}}$$

$$= \sqrt{100 + 20000 + 5000 + 1250}$$

$$= 162.327\text{ V}$$

Example 2.14 *Find rms and average value of the following waveform.*

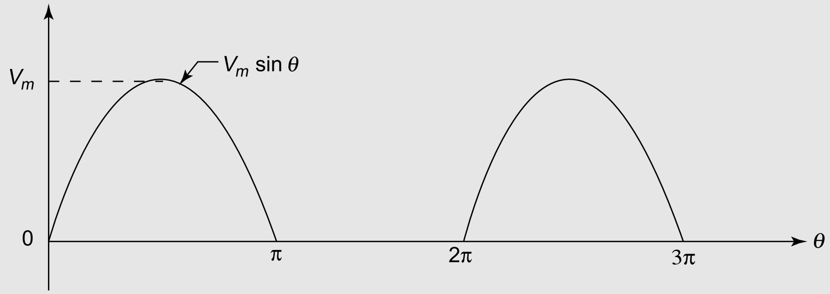


Fig. 2.20

Solution

The rms value, $V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta \, d\theta}$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi}\frac{(1-\cos 2\theta)}{2}\,d\theta}$$

$$= \sqrt{\frac{V_m^2}{4\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}}$$

$$= \frac{V_m}{2}$$

$$\text{Average value } V_{\text{av}} = \frac{1}{2\pi}\int_0^{2\pi} V_m \sin\theta\, d\theta = \frac{V_m}{2\pi}[-\cos\theta]_0^{2\pi} = \frac{V_m}{\pi}$$

Example 2.15 *For the trapezoidal waveform shown in the Fig. 2.21, determine average value.* *[JNTU Jan 2010]*

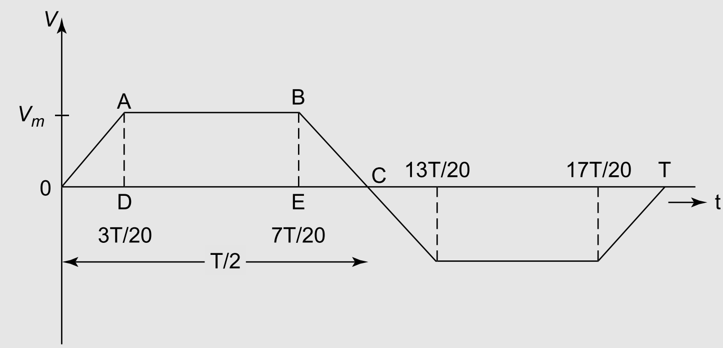


Fig. 2.21

Solution

$$V_{\text{av}} = \frac{1}{T}\int_0^{T} v(t)dt = \frac{1}{T}\left[\int_0^{3T/20}\frac{20V_m\cdot t}{3T}dt + \int_{3T/20}^{7T/20} V_m\cdot dt + \int_{7T/20}^{13T/20} V_m - \frac{20V_m}{3T}\left(t - \frac{7T}{20}\right)dt\right.$$

$$\left. - \int_{13T/20}^{17T/20} V_m\ dt + \int_{17T/20}^{T}\frac{20V_m}{3T}(t-T)dt\right]$$

$$= \frac{1}{T}\left[\left(\frac{20V_m \quad 9T^2}{3T\times 2\times 20\times 20}\right) + \cancel{\left(V_m\times\frac{4T}{20}\right)} + \cancel{\left(\frac{10V_m}{3}\times\frac{6T}{20}\right)} - \frac{20V_m}{3T\times 2}\left(\frac{13T}{20}\right)^2 - \left(\frac{7T}{20}\right)\right.$$

$$\left. - \cancel{\left(V_m\times\frac{4T}{20}\right)} - \cancel{\left(\frac{20V_m}{30}\times\frac{3T}{20}\right)} + \frac{20V_m}{3T\times 2}\left(T^2 - \left(\frac{17T}{20}\right)^2\right)\right]$$

$$= \frac{20V_m T^2}{T\times 3T\times 2}[0.0225 - 0.0225] = 0$$

2.2.6 Peak Factor *[JNTU Nov 2011]*

The peak factor of any waveform is defined as the ratio of the peak value of the wave to the rms value of the wave.

$$\text{Peak factor } = \frac{V_P}{V_{\text{rms}}}$$

$$\text{Peak factor of the sinusoidal waveform } = \frac{V_P}{V_P/\sqrt{2}} = \sqrt{2} = 1.414$$

2.2.7 Form Factor *[JNTU May/June 2006, Nov 2011]*

Form factor of a waveform is defined as the ratio of rms value to the average value of the wave.

$$\text{Form factor } = \frac{V_{\text{rms}}}{V_{\text{av}}}$$

Form factor of a sinusoidal waveform can be found from the above relation.

$$\text{For the sinusoidal wave, the form factor } = \frac{V_P/\sqrt{2}}{0.637V_P} = 1.11$$

Example 2.16 *Find the form factor for the following waveform.*

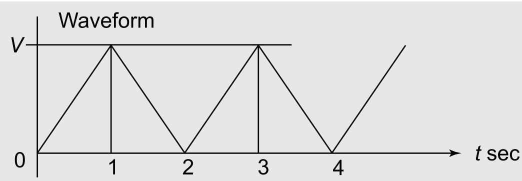


Fig. 2.22

Solution

$$\text{Form factor} = \frac{\text{rms value}}{\text{Average value}}$$

Average value of the triangular waveform 0 to 2 sec

$$V_{\text{av}} = \frac{1}{2}\left[\int_0^1 V.t\ dt + \int_1^2 -V(t-2)\,dt\right]$$

$$= \frac{1}{2}\left[V\frac{t^2}{2}\Big|_0^1 + -V\frac{t^2}{2}\Big|_1^2 + 2V.t\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\frac{V}{2} - \frac{3}{2}V + 2V\right] = \frac{V}{2}$$

$$\text{rms value } (V_{\text{rms}}) = \left[\frac{1}{2}\int_0^1 V^2t^2 dt + \int_1^2 V^2(t-2)^2\ dt\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{V^2 \frac{t^3}{3}\Big|_0^1 + V^2 \frac{t^3}{3}\Big| + 4V^2 t\Big|_1^2 - 4V^2 \frac{t^2}{2}\Big|_1^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{V^2}{3} - \frac{7V^2}{3} - 2V^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{2}\left\{\frac{8V^2 - 6V^2}{3}\right\}\right]^{1/2} = \frac{V}{\sqrt{3}}$$

$$\text{Form factor} = V/\sqrt{3}/V/2 = \frac{2}{\sqrt{3}} = 1.155$$

Example 2.17 *A sinusoidal current wave is given by i = 50 sin l00 πt. Determine*

(a) The greatest rate of change of current.
(b) Derive average and rms values of current.
(c) The time interval between a maximum value and the next zero value of current. *[JNTU Jan 2010]*

Solution (a) $i = 50 \sin 100\,\pi\, t$

$\therefore\ \sin \omega t \therefore \omega = 2\pi f$

$$\therefore \quad 2\pi f = 100\,\pi$$

$$f = \frac{100\pi}{2\pi} = 50\,\text{Hz}$$

$$\therefore \quad \frac{di}{dt} = 50 \times 100\pi\ \cos 100\pi\text{t} = 5000\pi\ \cos 100\pi\text{t}$$

$$\therefore \quad \left(\frac{di}{dt}\right)_{\max} = 5000\pi$$

(b) Average

$$I_{\text{av}} = \frac{2I_m}{\pi} = \frac{2\times 50}{3.142} = 31.826\,\text{A}$$

$$\text{rms}\ \ I = \frac{I_m}{\sqrt{2}} = \frac{50}{\sqrt{2}} = 35.35\,\text{A}$$

(c) Time interval

$$t = \frac{1}{f} = \frac{1}{50} = 0.023 = 20\ \text{ms}$$

$$\therefore\ \text{Time interval} \Rightarrow \frac{20}{4} = 5\,\text{ms}$$

Example 2.18 *A sine wave has a peak value of 25 V. Determine the following values.*

(a) rms *(b) peak to peak* *(c) average*

Solution (a) rms value of the sine wave

$$V_{\text{rms}} = 0.707\ V_P = 0.707 \times 25 = 17.68\ \text{V}$$

(b) peak to peak value of the sine wave $V_{PP} = 2V_P$

$$V_{PP} = 2 \times 25 = 50\ \text{V}$$

(c) average value of the sine wave

$$V_{\text{av}} = 0.637\ V_P = (0.637)25 = 15.93\ \text{V}$$

Example 2.19 *A sine wave has a peak value of 12 V. Determine the following values.*

(a) rms *(b) average* *(c) crest factor* *(d) form factor*

Solution (a) rms value of the given sine wave

$$= (0.707)12 = 8.48\ \text{V}$$

(b) average value of the sine wave $= (0.637)12 = 7.64$ V

(c) crest factor of the sine wave $= \dfrac{\text{Peak value}}{\text{rms value}}$

$$= \frac{12}{8.48} = 1.415$$

(d) Form factor $= \dfrac{\text{rms value}}{\text{average value}} = \dfrac{8.48}{7.64} = 1.11$

Example 2.20 *Find the form factor of the half-wave rectified sine wave shown in Fig. 2.23.*

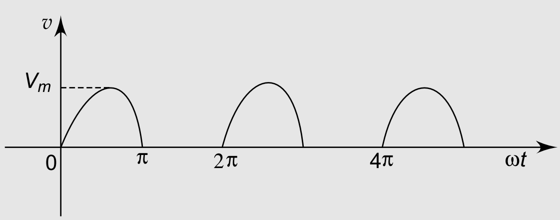


Fig. 2.23

Solution $v = V_m \sin \omega t$, for $0 < \omega t < \pi$

$= 0$, for $\pi < \omega t < 2\pi$

the period is 2π.

$$V_{\text{av}} = \frac{1}{2\pi}\left\{\int_0^{\pi} V_m \sin\omega t \, d(\omega t) + \int_{\pi}^{2\pi} 0 \, d(\omega t)\right\}$$

Average value $= 0.318\, V_m$

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi} (V_m \sin\omega t)^2 \, d(\omega t)$$

$$= \frac{1}{4} V_m^2$$

$$V_{\text{rms}} = \frac{1}{2} V_m$$

Form factor $= \dfrac{V_{\text{rms}}}{V_{\text{av}}} = \dfrac{0.5\, V_m}{0.318\, V_m} = 1.572$

Example 2.21 *Find the average and effective values of the saw tooth waveform shown in Fig. 2.24.*

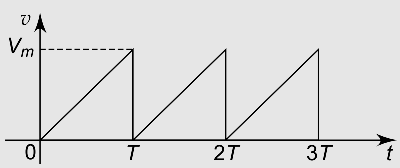


Fig. 2.24

Solution From Fig. 2.24 shown, the period is T.

$$V_{\text{av}} = \frac{1}{T}\int_0^{T} \frac{V_m}{T} t \, dt$$

$$= \frac{1}{T}\frac{V_m}{T}\int_0^{T} t \, dt$$

$$= \frac{V_m}{T^2}\frac{T^2}{2} = \frac{V_m}{2}$$

Effective value $V_{\text{rms}} = \sqrt{\dfrac{1}{T}\displaystyle\int_0^{T} v^2 \, dt}$

$$= \sqrt{\frac{1}{T}\int_0^{T}\left[\frac{V_m}{T} t\right]^2 dt}$$

$$= \frac{V_m}{\sqrt{3}}$$

Example 2.22 *Find the average and rms value of the full wave rectified sine wave shown in Fig. 2.25.*

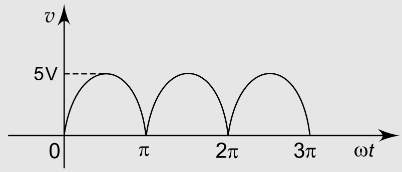


Fig. 2.25

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 5\sin\omega t\, d(\omega t)$

$$= 3.185$$

Effective value or rms value $= \sqrt{\frac{1}{\pi}\int_0^{\pi}(5\sin\omega t)^2\, d(\omega t)}$

$$= \sqrt{\frac{25}{2}} = 3.54$$

Example 2.23 *The full wave rectified sine wave shown in Fig. 2.26 has a delay angle of 60°. Calculate V_{av} and V_{rms}.*

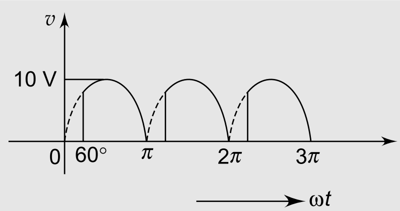


Fig. 2.26

Solution Average value $V_{av} = \frac{1}{\pi}\int_0^{\pi} 10\sin(\omega t)\, d(\omega t)$

$$= \frac{1}{\pi}\int_{60^\circ}^{\pi} 10\sin\omega t\, d(\omega t)$$

$$V_{av} = \frac{10}{\pi}(-\cos\omega t)_{60}^{\pi} = 4.78$$

Effective value $V_{rms} = \sqrt{\frac{1}{\pi}\int_{60^\circ}^{\pi}(10\sin\omega t)^2\, d(\omega t)}$

$$= \sqrt{\frac{100}{\pi}\int_{60^\circ}^{\pi}\left(\frac{1-\cos 2\omega t}{2}\right) d(\omega t)}$$

$$= 6.33$$

Example 2.24 *Find the form factor of the square wave as shown in Fig. 2.27.*

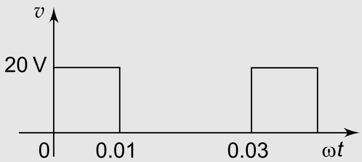


Fig. 2.27

Solution $v = 20$ for $0 < t < 0.01$

$= 0$ for $0.01 < t < 0.03$

The period is 0.03 sec.

$$\text{Average value } V_{av} = \frac{1}{0.03}\int_0^{0.01} 20\,dt$$

$$= \frac{20(0.01)}{0.03} = 6.66$$

$$\text{Effective value } V_{eff} = \sqrt{\frac{1}{0.03}\int_0^{0.01} (20)^2\,dt} = 66.6 = 0.816$$

$$\text{Form factor } = \frac{0.816}{6.66} = 0.123$$

Example 2.25 *Find the form factor of the following waveform shown in Fig. 2.28.* *[JNTU April/May 2007]*

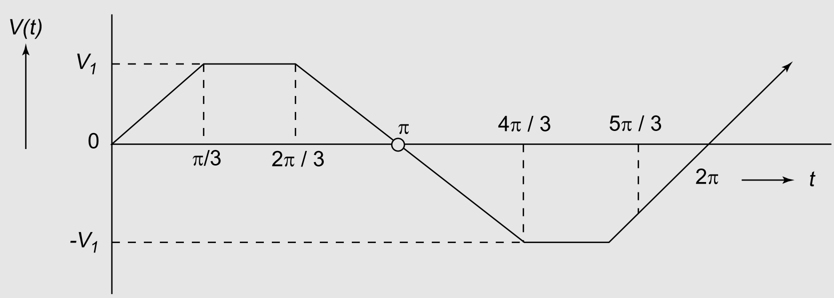


Fig. 2.28

Solution From 0 to π/3, $V = \frac{3V_1}{\pi}t$

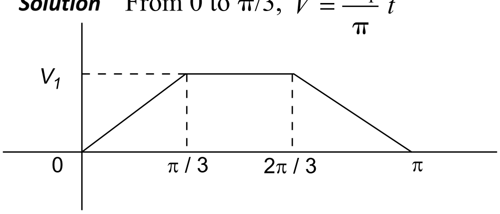


Fig. 2.29

From π/3 to 2π/3 $V = V_1$

From 2π/3 to π

$$V = 3V_1 - \frac{3V_1}{\pi}t$$

$$\text{Form factor } = \frac{V_{rms}}{V_{avg}}$$

$$V_{\text{avg}} = \frac{1}{T}\int_0^T V(t)\ dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/3} \frac{3V_1}{\pi}\ t\ dt + \int_{\pi/3}^{2\pi/3} V_1\ dt + \int_{2\pi/3}^{\pi} 3V_1 - \frac{3V_1}{\pi}\ t\ dt\right]$$

$$= \frac{1}{\pi}\left[\frac{3V_1}{\pi}\cdot\left(\frac{\pi}{3}\right)\frac{1}{2} + V_1\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) + 3V_1\left(\pi - \frac{2\pi}{3}\right) - \frac{3V_1}{\pi}\cdot\frac{1}{2}\left[\pi^2 - \frac{4\pi^2}{9}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{V_1}{6}\cdot\pi + \frac{V_1}{3}\pi + V_1\cdot\pi - \frac{5}{6}V_1\right] = \frac{2}{3}V_1$$

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T [V(t)]^2\ dt}$$

$$= \sqrt{\frac{1}{\pi}\left[\int_0^{\pi/3}\left(\frac{3V_1}{\pi}t\right)^2\ dt + \int_{\pi/3}^{2\pi/3}(V_1)^2\ dt + \int_{2\pi/3}^{\pi}\left(3V_1 - \frac{3V_1}{\pi}t\right)^2\ dt\right]}$$

$$= \sqrt{\frac{1}{\pi}\left[\int_0^{\pi/3}\frac{9V_1^2}{\pi^2}t^2\ dt + \int_{\pi/3}^{2\pi/3}V_1^2\ dt + \int_{2\pi/3}^{\pi} 9V_1^2 + \frac{9V_1^2}{\pi^2}t^2 - \frac{18V_1}{\pi}t\ dt\right]}$$

$$= \text{Sqrt}\left\{\frac{1}{\pi}\left[\frac{9V_1^2}{\pi^2}\cdot\frac{1}{3}\cdot\left(\frac{\pi}{3}\right)^3 + V_1^2\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) + 9V_1^2\left(\pi - \frac{2\pi}{3}\right)\right.\right.$$

$$\left.\left. + \frac{9V_1^2}{\pi^2}\cdot\frac{1}{3}\cdot\left(\pi^3 - \frac{8\pi^3}{27}\right) - \frac{18V_1}{\pi}\cdot\frac{1}{2}\left(\pi^2 - \frac{4\pi^2}{9}\right)\right]\right\}$$

$$= \sqrt{\frac{1}{\pi}\left[\frac{9V_1^2}{\pi^2}\cdot\frac{1}{3}\cdot\frac{\pi^3}{27} + V_1^2\cdot\frac{\pi}{3} + 9V_1^2\cdot\frac{\pi}{3} + \frac{3V_1^2}{\pi^2}\cdot\frac{19\pi^3}{27} - \frac{9V_1^2}{\pi}\cdot\frac{5\pi^2}{9}\right]}$$

$$= \sqrt{\frac{1}{\pi}\left[\frac{\pi}{9}V_1^2 + \frac{\pi}{3}V_1^2 + 3\pi V_1^2 + \frac{19}{9}\pi V_1^2 - 5\pi V_1^2\right]}$$

$$= \sqrt{\frac{5}{9}V_1^2} = \frac{\sqrt{5}}{3}V_1$$

$$\text{Form factor } = \frac{V_{\text{rms}}}{V_{\text{avg}}}$$

$$= \frac{\frac{\sqrt{5}}{3}V_1}{\frac{2}{3}V_1} = \frac{\sqrt{5}}{2} = 1.12$$

Example 2.26 *Find the average and effective values of a full wave rectified sine wave shown in Fig. 2.30.*

[JNTU May/June 2002]

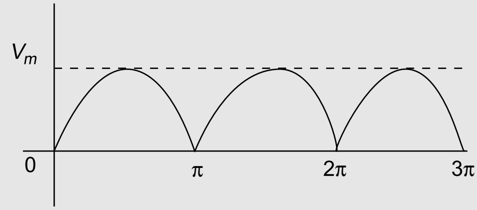


Fig. 2.30

Solution Average value

$$V_{avg} = \frac{1}{T}\int_0^T V_m \sin\theta\, d\theta = \frac{1}{\pi}\int_0^\pi V_m \sin\theta\, d\theta$$

$$= \frac{V_m}{\pi}[-\cos\theta]_0^\pi$$

$$= \frac{2V_m}{\pi}$$

Effective value

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \sin^2\theta d\theta} = \sqrt{\frac{1}{\pi}\int_0^\pi V_m^2 \sin^2\theta d\theta}$$

$$= V_m\sqrt{\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta} = V_m\sqrt{\frac{2}{\pi}\times\frac{1}{2}\times\frac{\pi}{2}}$$

$$= \frac{V_m}{\sqrt{2}}$$

Example 2.27 *Determine the RMS value of a half-wave rectified sinusoidal voltage of peak value, V_m.* *[JNTU May/June 2002]*

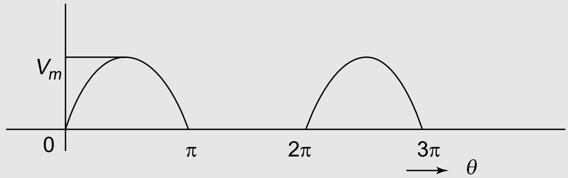


Fig. 2.31

Solution

$$\text{RMS value} = \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi} \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_0^{\pi}}$$

$$= \frac{V_m^2}{2\pi} \times \frac{\pi}{2} = \frac{V_m}{2}$$

$$\therefore \qquad V_{\text{rms}} = \frac{V_m}{2}$$

Example 2.28 *Find RMS and average value of the following waveform.*

[JNTU May/June 2004]

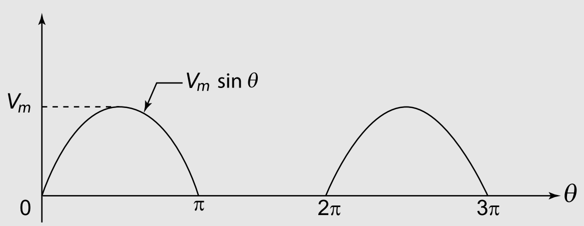


Fig. 2.32

Solution

$$\text{RMS value } V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta\; d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{4\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}}$$

$$= \frac{V_m}{2}$$

$$\text{Average value, } V_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} V_m \sin\theta \, d\theta$$

$$= \frac{V_m}{2\pi}[-\cos\theta]_0^{2\pi} = \frac{V_m}{\pi}$$

Example 2.29 *Derive expression for rms and average value of a sinusoidal alternating quantity.* *[JNTU May/June 2008]*

Solution *Average value of a sine wave:*

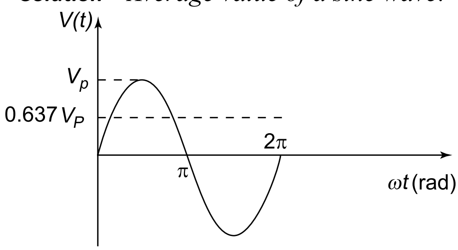


Fig. 2.33

The average value of a curve in the $X-Y$ plane is the total area under the complete curve divided by the distance of the curve. The average value of a sine wave over one complete cycle is always zero. So the average value of a sine wave is defined over a half-cycle, and not a full-cycle period.

The average value of sine wave

$$V(t) = V_P \sin \omega t$$

$$V_{\text{av}} = \frac{1}{\pi}\int_0^{\pi} V_P \sin\omega t \; d(\omega t) = \frac{1}{\pi}[-V_P \; \cos\omega t]_0^{\pi} = \frac{2V_p}{\pi} = 0.637 \; V_P$$

RMS value of a sine wave:

The root mean square (rms) value of a sine wave is a measure of the heating effect of the wave.

RMS value of any waveform is determined by using

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^{T} (V(t))^2 \; dt}$$

Let the function $V(t)$ be $V_P \sin \omega t$.

$$= \sqrt{\frac{1}{T}\int_0^{T} (V_p \sin\omega t)^2 \; d(\omega t)} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_p^2 \sin^2 \omega t \; d(\omega t)}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_p^2 \left[\frac{1-\cos 2\omega t}{2}\right] d(\omega t)} = \sqrt{\frac{1}{2\pi}\left[\tfrac{1}{2}(\omega t) - \frac{\sin 2\omega t}{4}\right]_0^{2\pi} \times V_p^2}$$

$$= \sqrt{\frac{1}{2\pi}\left[\frac{2\pi}{2} - 0\right] V_p^2}$$

$$V_{\text{rms}} = \frac{V_p}{\sqrt{2}} = 0.707 \; V_p.$$

Example 2.30 *Define*

(i) frequency,
(ii) phase,
(iii) form factor, and
(iv) peak factor. *[JNTU May/June 2008]*

Solution (i) *Frequency*: The frequency of a wave is defined as the number of cycles that a wave completes in one second. The unit of frequency is hertz.
One hertz is equivalent to one cycle per second.

(ii) *Phase*: The phase of a sine wave is an angular measurement that specifies the position of sine wave relative to reference.
When the sine wave is shifted left or right with reference to wave shown in Fig. 2.34, there occurs a phase shift.

(iii) *Form Factor*: Form factor of a wave is defined as the ratio of rms value to average value of the wave.

$$\text{Form factor} = \frac{V_{RMS}}{V_{avg}}$$

$$\text{For sine wave} = \frac{V_p/\sqrt{2}}{0.637V_p} = 1.11$$

(iv) *Peak Factor*: The peak factor of any waveform is defined as the ratio of peak value of the wave to the rms value of the wave

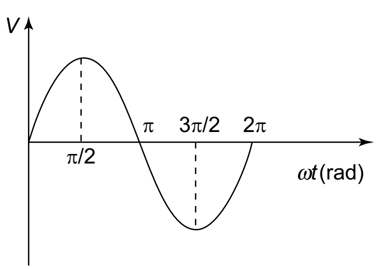


Fig. 2.34

$$\text{Peak factor} = \frac{V_p}{V_{RMS}}$$

For sine wave

$$\text{Peak factor} = \frac{V_p}{V_p/\sqrt{2}} = \sqrt{2} = 1.414$$

2.3 PHASE ANGLE AND PHASOR REPRESENTATION

A phasor diagram can be used to represent a sine wave in terms of its magnitude and angular position. Examples of phasor diagrams are shown in Fig. 2.35.

In Fig. 2.35(a), the length of the arrow represents the magnitude of the sine wave; angle θ represents the angular position of the sine wave. In Fig. 2.35(b), the magnitude of the sine wave is one and the phase angle is 30°. In Fig. 2.35(c) and (d), the magnitudes are four and three, and phase angles are 135° and 225°, respectively. The position of a phasor at any instant can be expressed as a positive or negative angle. Positive angles are measured counterclockwise from 0°, whereas negative angles are measured clockwise from 0°. For a given positive angle θ, the corresponding negative angle is θ−360°. This is shown in Fig. 2.36(a). In Fig. 2.36(b), the positive angle 135° of vector *A* can be represented by a negative angle −225°, (135°−360°).

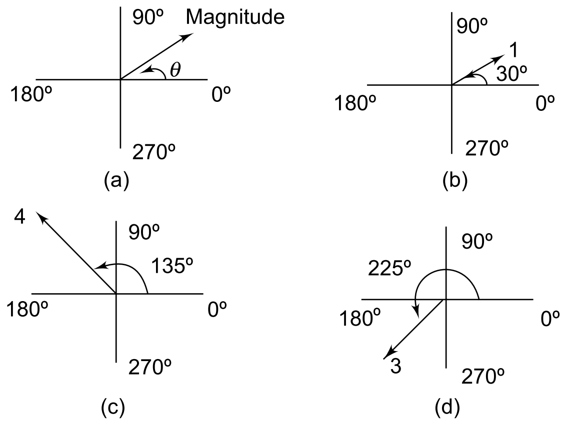


Fig. 2.35

A phasor diagram can be used to represent the relation between two or more sine waves of the same frequency. For example, the sine waves shown in Fig. 2.37(a) can be represented by the phasor diagram shown in Fig. 2.37(b).

In the above Figure, sine wave *B* lags behind sine wave *A* by 45°; sine wave *C* leads sine wave *A* by 30°. The length of the phasors can be used to represent peak, rms, or average values.

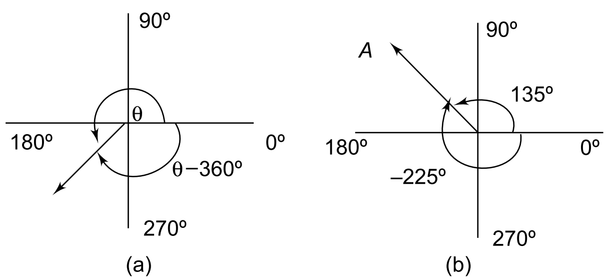


Fig. 2.36

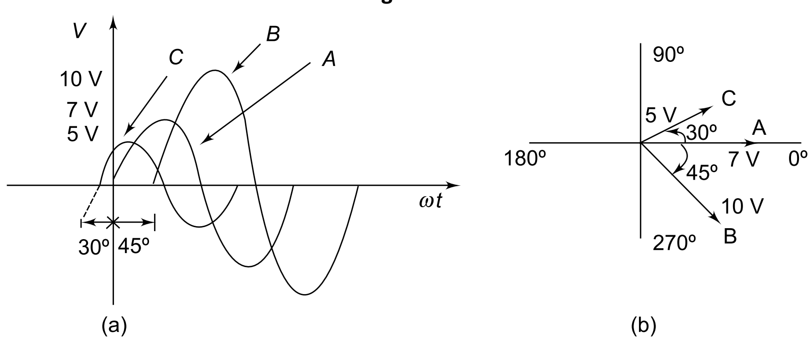


Fig. 2.37

Example 2.31 *Draw the phasor diagram to represent the two sine waves shown in Fig. 2.38.*

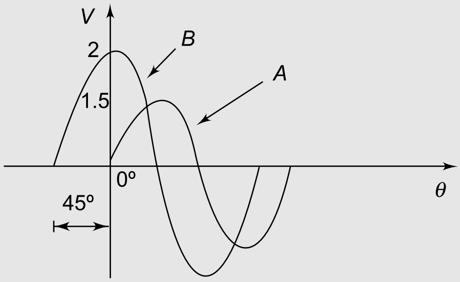


Fig. 2.38

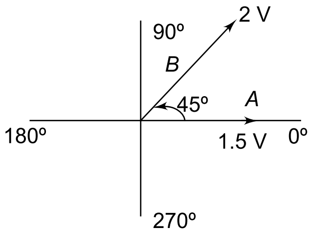


Fig. 2.39

Solution The phasor diagram representing the sine waves is shown in Fig. 2.39. The length of the each phasor represents the peak value of the sine wave.

Example 2.32 *Explain the term, phase difference.*

Solution The difference in phase between two waves is called phase difference. In the figure below the sine wave is shifted to the right by 90° shown by the dotted lines.

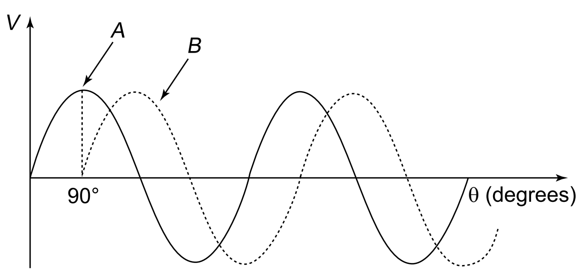


Fig. 2.40

There is a phase difference of 90° between *A* and *B*. The waveform *B* is lagging behind waveform *A* by 90° or in other words, wave *A* is leading the waveform *B* by 90°.

2.3.1 *j* Notation

j is used in all electrical circuits to denote imaginary numbers. Alternate symbol for *j* is $\sqrt{-1}$, and is known as *j* factor or *j* operator.

Thus

$$\sqrt{-1} = \sqrt{(-1)(1)} = j(1)$$
$$\sqrt{-2} = \sqrt{(-1)2} = j\sqrt{2}$$
$$\sqrt{-4} = \sqrt{(-1)4} = j2$$
$$\sqrt{-5} = \sqrt{(-1)5} = j\sqrt{5}$$

Since j is defined as $\sqrt{-1}$, it follows that $(j)(j) = j^2 = \left(\sqrt{-1}\right)\left(\sqrt{-1}\right) = -1$

$\therefore\quad (j3)(j3) = j^2 3^2$

Since $\qquad j^2 = -1$

$(j3)\,(j3) = -9$

(i.e.) the square root of -9 is $j3$

Therefore $j3$ is a square root of -9

The use of j factor provides a solution to an equation of the form $x^2 = -4$

Thus $\qquad x = \sqrt{-4} = \sqrt{(-1)4}$

$x = \left(\sqrt{-1}\right)2$

With $\qquad j = \sqrt{-1}, x = j2$

The real number 9 when multiplied three times by j becomes $-j9$.

$(j)\,(j)\,(j) = (j)^2 j = (-1)j = -j$

Finally when real number 10 is multiplied four times by j, it becomes 10

$$j = +j$$
$$j^2 = (j)\,(j) = -1$$
$$j^3 = (j^2)\,(j) = (-1)j = -j$$
$$j^4 = (j^2)\,(j)^2 = (-1)\,(-1) = +1$$

Example 2.33 *Express the following imaginary numbers using the j factor.*

(a) $\sqrt{-13}$ *(b)* $\sqrt{-9}$ *(c)* $\sqrt{-29}$ *(d)* $\sqrt{-49}$

Solution

(a) $\sqrt{-13} = \sqrt{(-1)(13)} = j\sqrt{13}$

(b) $\sqrt{-9} = \sqrt{(-1)9} = j3$

(c) $\sqrt{-29} = \sqrt{(-1)29} = j\sqrt{29}$

(d) $\sqrt{-49} = \sqrt{(-1)(49)} = j7$

2.3.2 Complex and Polar Forms

A complex number $(a + jb)$ can be represented by a point whose coordinates are (a, b). Thus, the complex number $3 + j4$ is located on the complex plane at a point having rectangular coordinates (3, 4).

This method of representing complex numbers is known as the rectangular form. In ac analysis, impedances, currents and voltages are commonly represented by complex numbers that may be either in the rectangular form or in the polar form. In Fig. 2.41 the complex number in the polar form is represented. Here R is the magnitude of the complex number and ϕ is the angle of the complex number. Thus, the polar form of the complex number is $R \angle\phi$. If the rectangular coordinates (a, b) are known, they can be converted into polar form. Similarly, if the polar coordinates (R, ϕ) are known, they can be converted into rectangular form.

In Fig. 2.41, a and b are the horizontal and vertical components of the vector R, respectively. From Fig. 2.41, R can be found as $R = \sqrt{a^2 + b^2}$.

Also from Fig. 2.41,

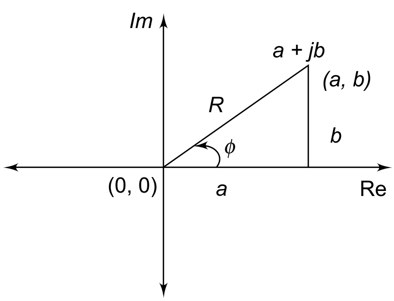


Fig. 2.41

$$\sin\phi = \frac{b}{R}$$

$$\cos\phi = \frac{a}{R}$$

$$\tan\phi = \frac{b}{a}$$

$$\phi = \tan^{-1}\frac{b}{a}$$

$$R = \sqrt{a^2 + b^2}$$

Example 2.34 *Express 10 ∠53.1° in rectangular form.*

Solution

$$a + jb = R(\cos\phi + j\sin\phi)$$

$$R = 10;\ \angle\phi = \angle 53.1°$$

$$a + jb = R\cos\phi + jR\sin\phi$$

$$R\cos\phi = 10\cos 53.1° = 6$$

$$R\sin\phi = 10\sin 53.1° = 8$$

$$a + jb = 6 + j8$$

Example 2.35 *Express 3 + j4 in polar form.*

Solution

$$R\cos\phi = 3 \qquad (1)$$

$$R\sin\phi = 4 \qquad (2)$$

Squaring and adding the above equations, we get

$$R^2 = 3^2 + 4^2$$

$$R = \sqrt{3^2 + 4^2} = 5$$

From (1) and (2), $\tan\phi = 4/3$

$$\phi = \tan^{-1}\frac{4}{3} = 53.13°$$

Hence the polar form is $5\ \angle 53.13°$

2.4 OPERATIONS WITH COMPLEX NUMBERS (ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION)

The basic operations such as addition, subtraction, multiplication and division can be performed using complex numbers.

Addition It is very easy to add two complex numbers in the rectangular form. The real parts of the two complex numbers are added and the imaginary parts of the two complex numbers are added. For example,

$$(3 + j4) + (4 + j5) = (3 + 4) + j(4 + 5) = 7 + j9$$

Subtraction Subtraction can also be performed by using the rectangular form. To subtract, the sign of the subtrahand is changed and the components are added. For example, subtract $5 + j3$ from $10 + j6$:

$$10 + j6 - 5 - j3 = 5 + j3$$

Multiplication To multiply two complex numbers, it is easy to operate in polar form. Here we multiply the magnitudes of the two numbers and add the angles algebraically. For example, when we multiply $3 \angle 30°$ with $4 \angle 20°$, it becomes $(3)(4) \angle 30° + 20° = 12 \angle 50°$.

Division To divide two complex numbers, it is easy to operate in polar form. Here we divide the magnitudes of the two numbers and subtract the angles. For example, the division of

$$9\angle 50° \text{ by } 3\angle 15° = \frac{9\angle 50°}{3\angle 15°} = 3\angle 50° - 15° = 3\angle 35°.$$

2.5 PRINCIPLE OF DUALITY

In an electrical circuit itself there are pairs of terms which can be interchanged to get new circuits. Such pair of dual terms are given below.

Current — Voltage
Open — Short
L — C
R — G
Series — Parallel
Voltage source — Current source
KCL — KVL

Consider a network containing R—L—C elements connected in series, and excited by a voltage source as shown in Fig. 2.42.

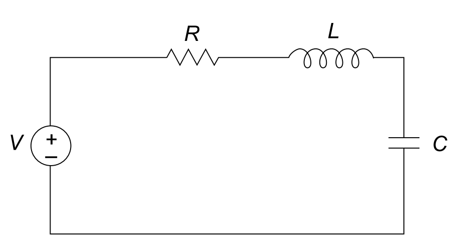


Fig. 2.42

The integrodifferential equation for the above network is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = V$$

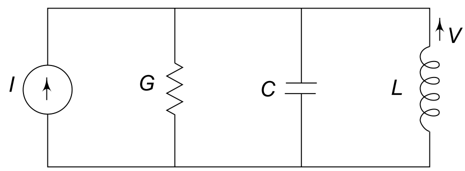


Fig. 2.43

Similarly, consider a network containing R—L—C elements connected in parallel and driven by a current source as shown in Fig. 2.43.

The integrodifferential equation for the network in Fig. 2.43 is

$$i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt$$

If we observe both the equations, the solutions of these two equations are the same. These two networks are called *duals*.

To draw the dual of any network, the following steps are to be followed.

1. In each loop of a network place a node; and place an extra node, called the *reference node*, outside the network.
2. Draw the lines connecting adjacent nodes passing through each element, and also to the reference node; by placing the dual of each element in the line passing through original elements.

For example, consider the network shown in Fig. 2.44.

Our first step is to place the nodes in each loop and a reference node outside the network.

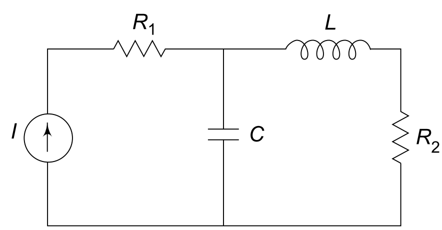


Fig. 2.44

Drawing the lines connecting the nodes passing through each element, and placing the dual of each element as shown in Fig. 2.45(a) we get a new circuit as shown in Fig. 2.45(b).

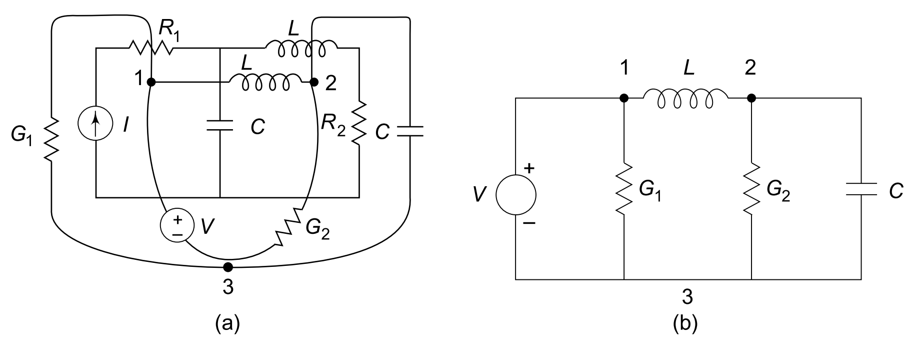


Fig. 2.45

Example 2.36 *Draw the dual network for the given network shown in Fig. 2.46.*

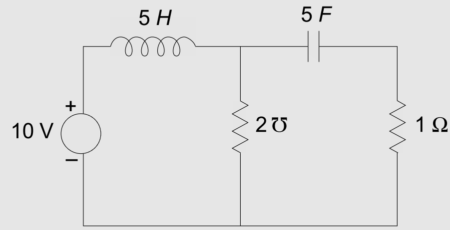


Fig. 2.46

Solution Place nodes in each loop and one reference node outside the circuit. Joining the nodes through each element, and placing the dual of each element in the line, we get the dual circuit as shown in Fig. 2.47(a).

The dual circuit is redrawn as shown in Fig. 2.47(b).

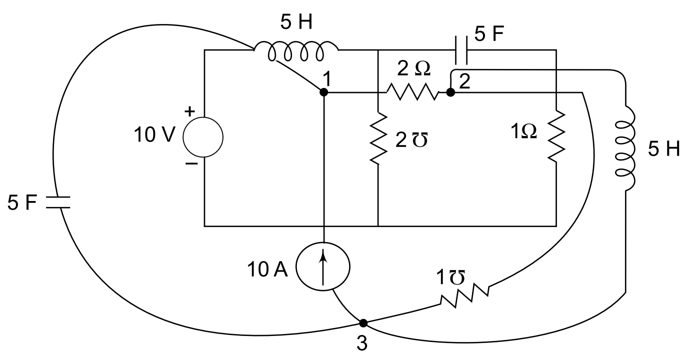


Fig. 2.47(a)

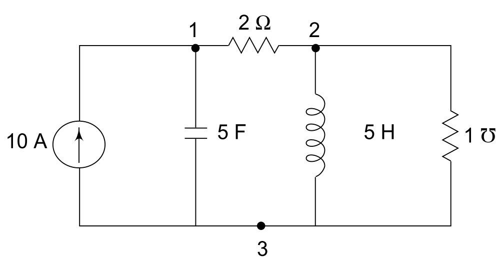


Fig. 2.47(b)

Example 2.37 *What is duality? Explain the procedure for obtaining the dual of the given planar network shown in Fig. 2.48.*

[JNTU May/June 2004]

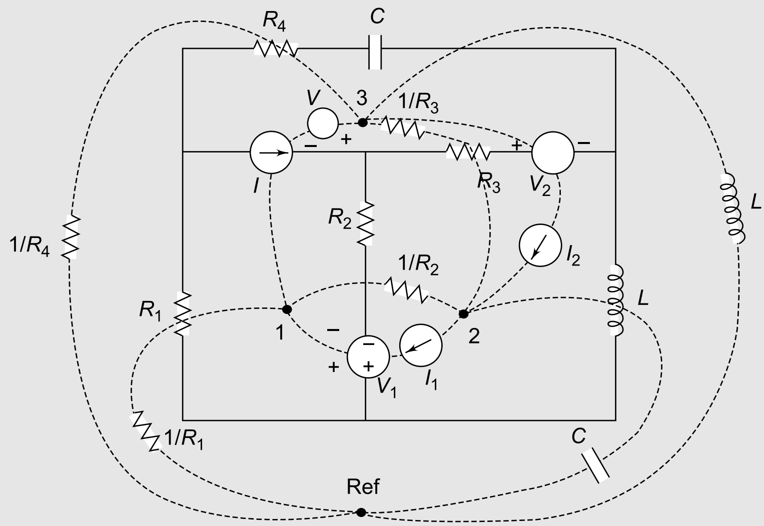


Fig. 2.48

Solution *Rule* 1: If a voltage source in the original network produces a clockwise current in the mesh, the corresponding dual element is a current source whose direction is towards the node representing the corresponding mesh.

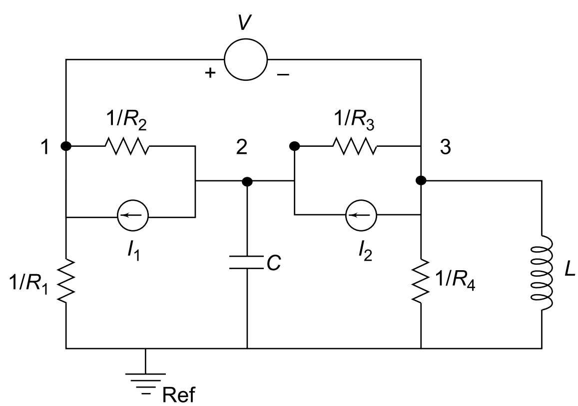


Fig. 2.49

Rule 2: If a current source in the original network produces a current in the clockwise direction in the mesh, the voltage source in the dual network will have a polarity such that the node representing the corresponding mesh is positive.

Dual of the planar circuit given in Fig. 2.49.

Example 2.38 *Draw the dual circuit for the given circuit shown in Fig. 2.50.*

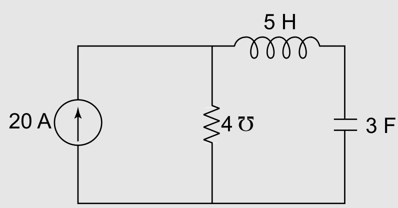
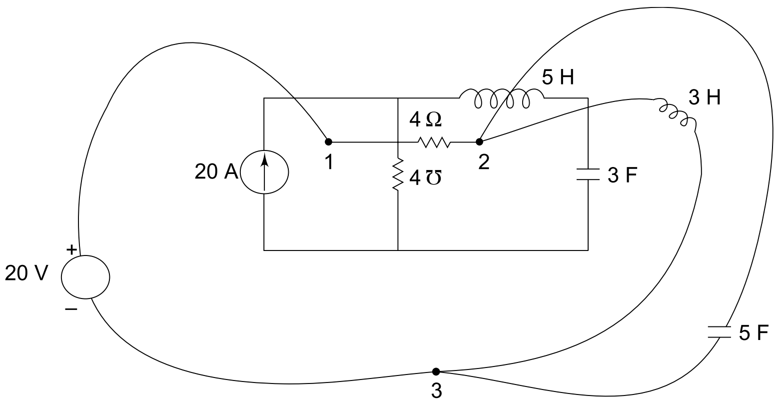


Fig. 2.50

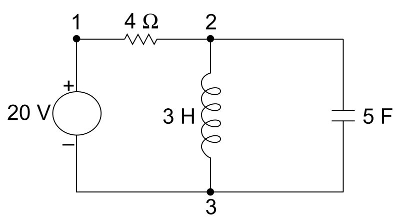
Solution Our first step is to place nodes in each loop, and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 2.51(a).

The dual circuit of the given circuit is shown in Fig. 2.51(b).



(a)



(b)

Fig. 2.51

Example 2.39 *Draw the dual circuit of Fig. 2.52.*

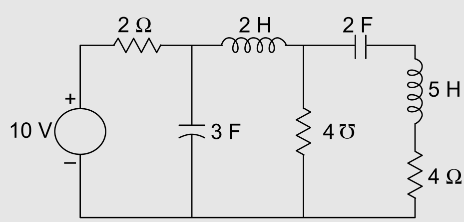
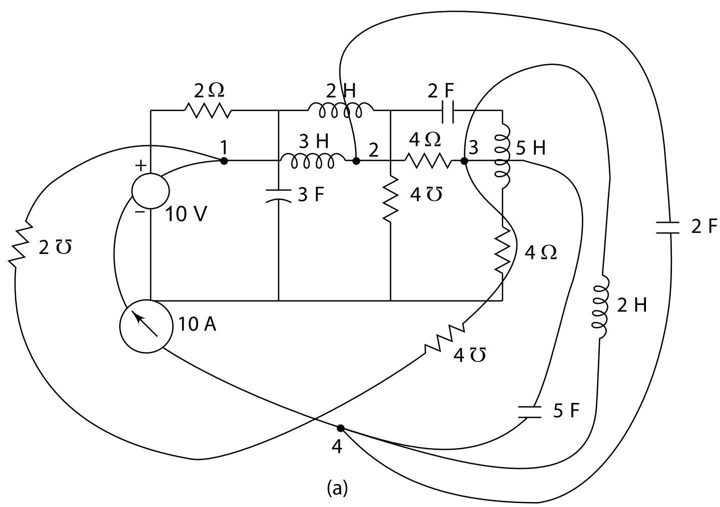


Fig. 2.52

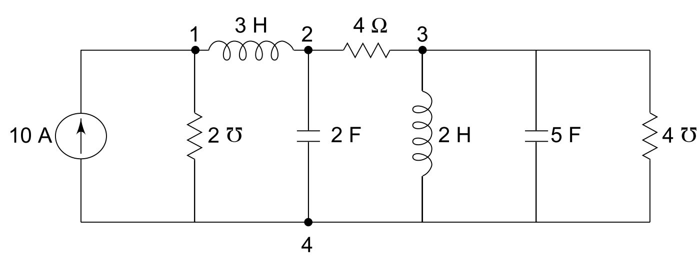
Solution Our first step is to mark nodes in each of the loop and a reference node outside the circuit.

Join the nodes with lines passing through each element and connect these lines with dual of each element as shown in Fig. 2.53(a).

The dual circuit of given circuit is shown in Fig. 2.53(b).



(a)



(b)

Fig. 2.53

Example 2.40 *Draw the dual network for the following circuit. Shown in Fig. 2.54.* *[JNTU June 2006]*

Solution

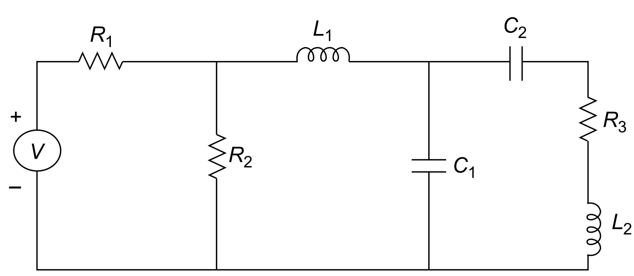


Fig. 2.54(a)

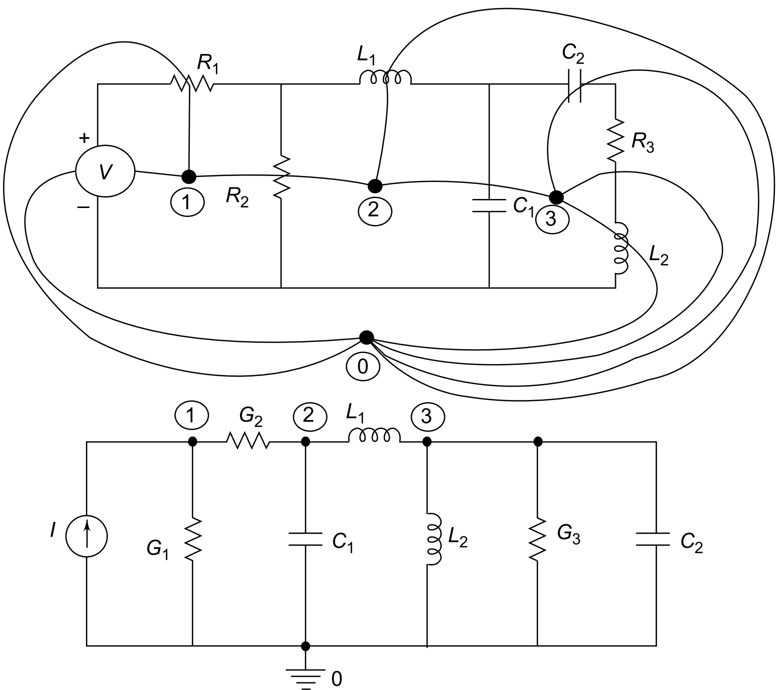


Fig. 2.54(b)

Example 2.41 *Explain clearly what you understand by "Duality" and "Dual network". Illustrate the procedure for drawing the dual of a given network.*
[JNTU June 2006]

Solution Two circuits are duals, if the mesh equations that characterise one of them have the same mathematical form as the nodal equations that characterise other.

Then they are said to duals (OH) satisfy duality of property i.e., if each mesh equation of one circuit is numerically identical with the corresponding nodal equation of other.

Network that satisfy duality property are called "Dual networks."

Dual pairs:

Resistance (R) → Conductance (G)
Inductance (L) → Capacitance (C)
Voltage (V) → Current (I)
Voltage Source → Current source
Node → Mesh
Series path → Parallel path
Open circuit → Short ckt
Thevenin → Norton

Steps to construct a dual circuit

1. Place a node at the centre of each mesh of the given circuit. Place the reference node of the dual circuit outside the given circuit.
2. Draw dotted lines between the nodes such that each line crosses a network element by its dual.

3. A voltage source that produces a positive (clockwise) mesh current has it dual or current source whose reference direction is from ground to non-reference node.

∴ Two circuits are said to be dual if they are described by the same characterising equations with dual quantities interchanged.

Example 2.42 *Draw the dual network of circuit.* *[JNTU June 2009]*

Solution

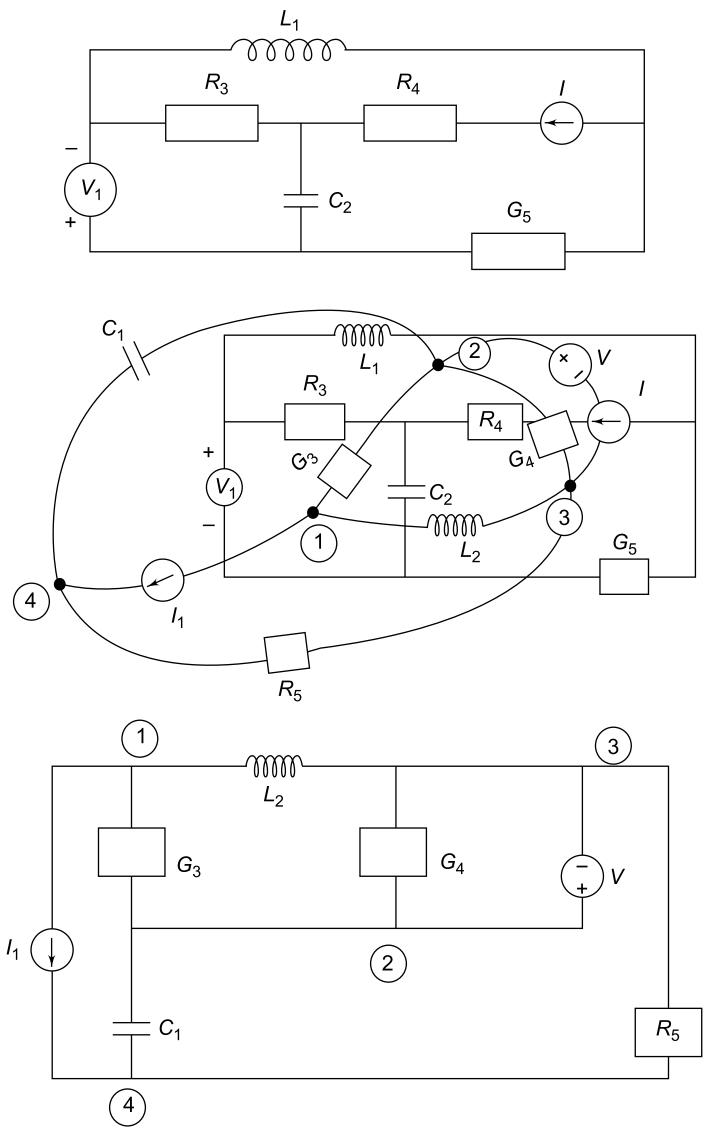


Fig. 2.55

Example 2.43 *Draw the dual network for the given network as in Fig. 2.56.*

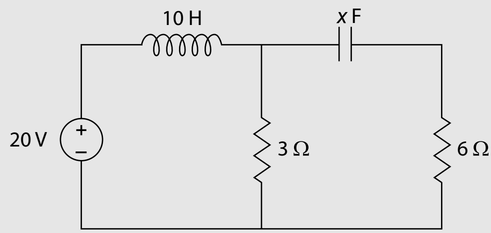


Fig. 2.56

Solution

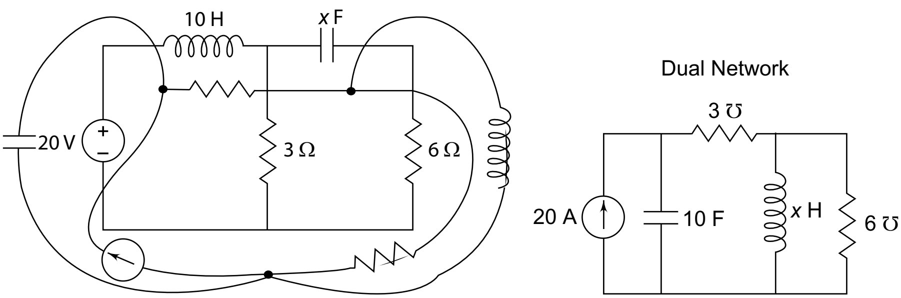


Fig. 2.57

Example 2.44 *Explain the procedure for obtaining the dual of the given planar network shown below.*

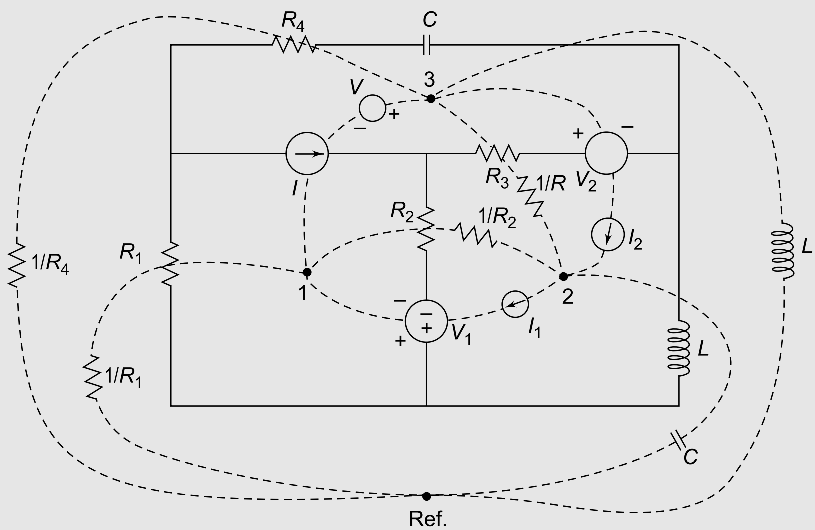


Fig. 2.58

Solution *Rule* 1: If a voltage source in the original network produces a c.w current in the mesh, the corresponding dual element is a current source whose direction is towards node representing the corresponding mesh.

Rule 2: If a current source in the original network produces a current in clockwise direction in the mesh, the voltage source in the dual network will have a polarity such that the node representing the corresponding mesh is positive.

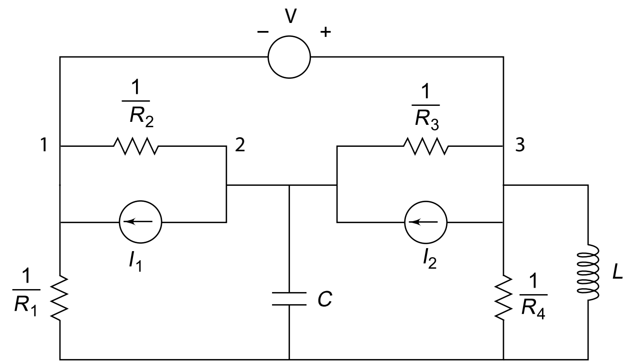


Fig. 2.59

Dual of the planar circuit given in Fig. 2.59.

2.6 DEFINITIONS OF BRANCH, NODE, PLANAR AND NON-PLANAR GRAPHS *[JNTU Nov 2011]*

A division of mathematics called topology or graph theory deals with graphs of networks and provides information that helps in the formulation of network equations. In circuit analysis, all the elements in a network must satisfy Kirchhoff's laws, besides their own characteristics. Based on these laws, we can form a number of equations. These equations can be easily written by converting the network into a graph. Certain aspects of network behaviour are brought into better perspective if a graph of the network is drawn. If each element or a branch of a network is represented on a diagram by a line irrespective of the characteristics of the elements, we get a graph. Hence, network topology is network geometry. A network is an interconnection of elements in various branches at different nodes as shown in Fig. 2.60. The corresponding graph is shown in Fig. 2.61(a).

The graphs shown in Figs 2.61(b) and (c) are also graphs of the network in Fig. 2.60.

It is interesting to note that the graphs shown in Figs 2.61(a), (b) and (c) may appear to be different but they are topologically equivalent. A branch is represented by a line segment connecting a pair of nodes in the graph of a network. A node is a terminal of a branch, which is represented by a point. Nodes are the end points of branches. All these graphs have identical relationships between branches and nodes.

The three graphs in Fig. 2.61 have six branches and four nodes. These graphs are also called undirected. If every branch of a graph has a *direction* as shown in Fig. 2.62, then the graph is called a *directed graph*.

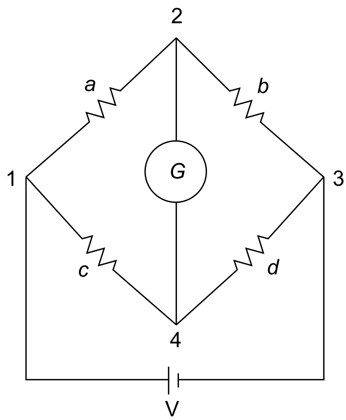


Fig. 2.60

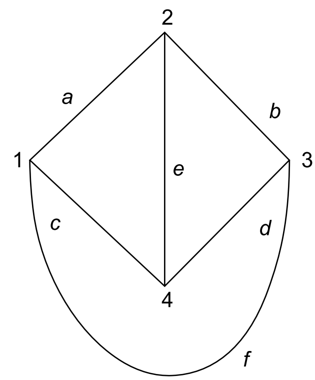


Fig. 2.61(a)

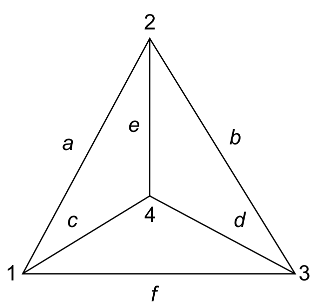


Fig. 2.61(b)

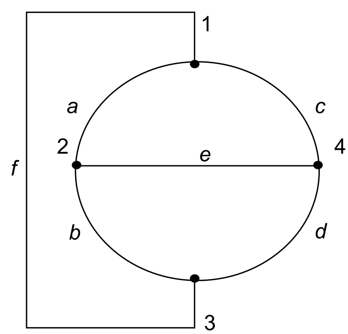


Fig. 2.61(c)

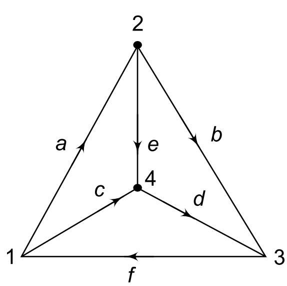


Fig. 2.62

A node and a branch are incident if the node is a terminal of the branch. Nodes can be incident to one or more elements. The number of branches incident at a node of a graph indicates the degree of the node. For example, in Fig. 2.62 the degree of node 1 is three. Similarly, the degree of node 2 is three. If each element of the connected graph is assigned a direction as shown in Fig. 2.62 it is then said to be oriented. A graph is connected if and only if there is a path between every pair of nodes. A path is said to exist between any two nodes, for example 1 and 4 of the graph in Fig. 2.62, if it is possible to reach node 4 from node 1 by traversing along any of the branches of the graph. A graph can be drawn if there exists a path between any pair of nodes. A loop exists, if there is more than one path between two nodes.

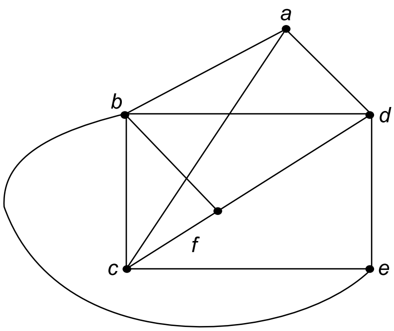


Fig. 2.63

Planar and Non-Planar Graphs

A graph is said to be planar if it can be drawn on a plane surface such that no two branches cross each other as shown in Fig 2.61. On the other hand in a non-planar graph there will be branches which are not in the same plane as others, i.e., a non-planar graph cannot be drawn on a plane surface without a crossover. Figure 2.63 illustrates a non-planar graph.

2.7 TREE AND CO-TREE [JNTU Nov 2011]

A tree is a connected subgraph of a network which consists of all the nodes of the original graph but no closed paths. The graph of a network may have a number of trees. The number of nodes in a graph is equal to the number nodes in the tree. The number of branches in a tree is less than the number of branches in a graph. A graph is a tree if there is a unique path between any pair of nodes. Consider a graph with four branches and three nodes as shown in Fig. 2.64.

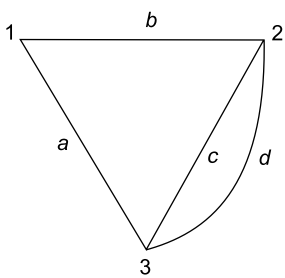


Fig. 2.64

Five open-ended graphs based on Fig. 2.64 are represented by Figs 2.65(a) to (e). Since each of these open-ended graphs satisfies all the requirements of a tree, each graph in Fig. 2.65 is a tree corresponding to Fig. 2.64.

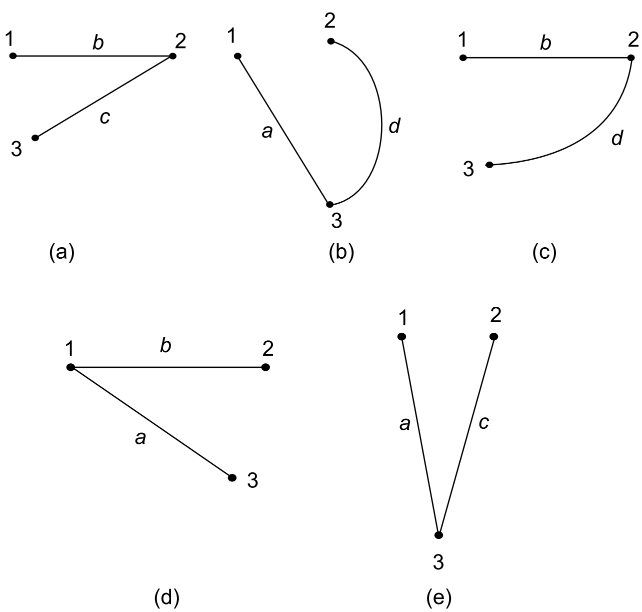


Fig. 2.65

In Fig. 2.65, there is no closed path or loop; the number of nodes $n = 3$ is the same for the graph and its tree, whereas the number of branches in the tree is only two. In general, if a tree contains n nodes, then it has $(n - 1)$ branches.

In forming a tree for a given graph, certain branches are removed or opened. The branches thus opened are called links or *link branches*. The links for Fig. 2.65(a) for example are a and d and for Fig. 2.65(b) are b and c. The set of all links of a given tree is called the co-tree of the graph. Obviously, the branches a, d are a co-tree for Fig. 2.65(a) and b, c are the co-tree. Similarly, for the tree in Fig. 2.65(b), the branches b, c are the co-tree. Thus, the link branches and the tree branches combine to form the graph of the entire network.

Example 2.45 *For the given graph shown in Fig. 2.66 draw the number of possible trees.*

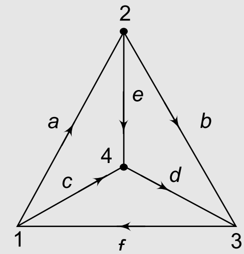


Fig. 2.66

Solution The number of possible trees for Fig. 2.66 are represented by Figs 2.67(a)–(l).

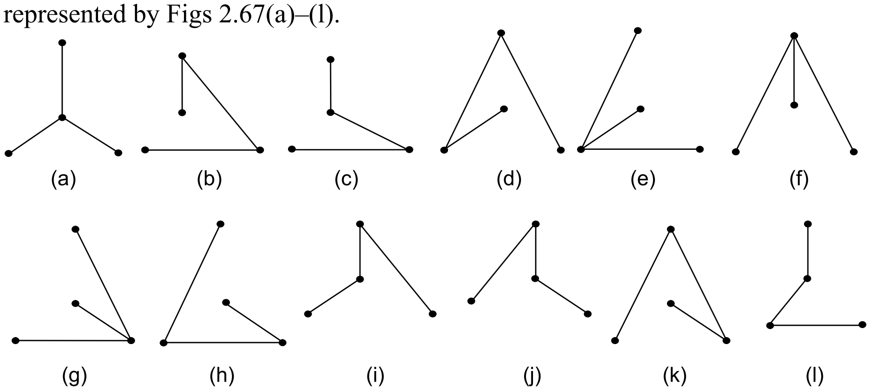


Fig. 2.67

Example 2.46 *For the network shown in Fig. 2.68 draw oriented graph and draw all possible trees.*

[JNTU Jan 2010]

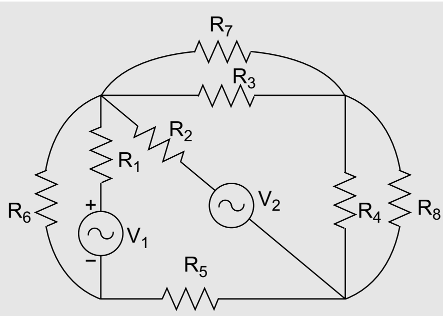
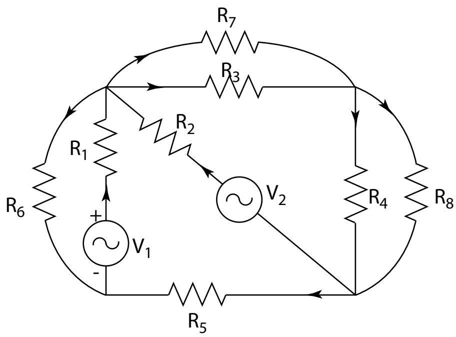


Fig. 2.68



Solution Short circuiting the voltage sources, we obtain the oriented graph as, we obtain the trees as given in Fig. 2.69.

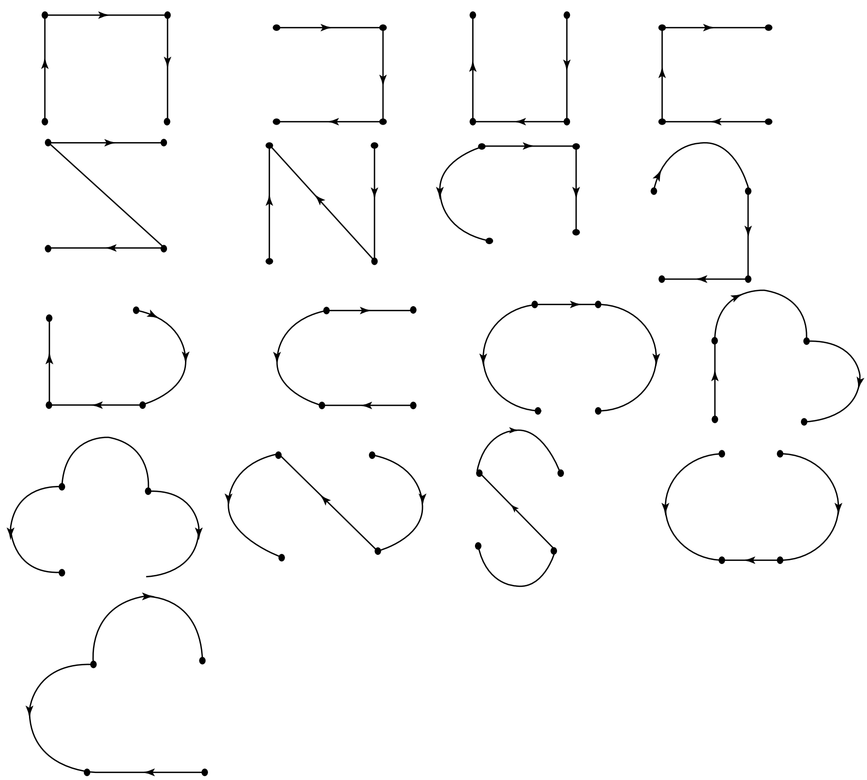


Fig. 2.69

2.8 TWIGS AND LINKS

The branches of a tree are called its 'twigs'. For a given graph, the complementary set of branches of the tree is called the co-tree of the graph. The branches of a co-tree are called links, i.e., those elements of the connected graph that are not included in the tree links and form a subgraph. For example, the set of branches (b, d, f) represented by dotted lines in Fig. 2.72 form a co-tree of the graph in Fig. 2.70 with respect to the tree in Fig. 2.71.

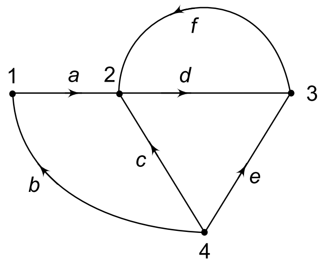


Fig. 2.70

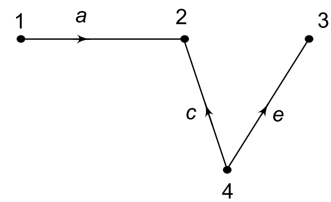


Fig. 2.71

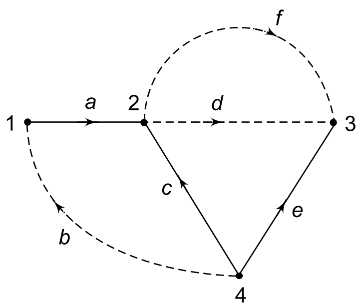


Fig. 2.72

The branches a, c and e are the twigs while the branches b, d and f are the links of this tree. It can be seen that for a network with b branches and n nodes, the number of twigs for a selected tree is $(n - 1)$ and the number of links I with respect to this tree is $(b - n + 1)$. The number of twigs $(n - 1)$ is known as the tree value of the graph. It is also called the *rank* of the tree. If a link is added to the tree, the resulting graph contains one closed path, called a loop. The addition of each subsequent link forms one or more additional loops. Loops which contain only one link are independent and are called basic loops.

2.9 INCIDENCE MATRIX (A)

The incidence of elements to nodes in a connected graph is shown by the element node incidence matrix (A). Arrows indicated in the branches of a graph result in an oriented or a directed graph. These arrows are the indication for the current flow or voltage rise in the network. It can be easily identified from an oriented graph regarding the incidence of branches to nodes. It is possible to have an analytical description of an oriented-graph in a matrix form. The dimensions of the matrix A is $n \times b$ where n is the number of nodes and b is number of branches. For a graph having n nodes and b branches, the complete incidence matrix A is a rectangular matrix of order $n \times b$.

In matrix A with n rows and b columns an entry a_{ij} in the i^{th} row and j^{th} column has the following values.

$$
\begin{aligned}
a_{ij} &= 1, \text{ if the } j^{\text{th}} \text{ branch is incident to and oriented away from the } i^{\text{th}} \text{ node.} \\
a_{ij} &= -1, \text{ if the } j^{\text{th}} \text{ branch is incident to and oriented towards the } i^{\text{th}} \text{ node.} \\
a_{ij} &= 0, \text{ if the } j^{\text{th}} \text{ branch is not incident to the } i^{\text{th}} \text{ node.}
\end{aligned} \tag{2.1}
$$

Figure 2.71 shows a directed graph.

Following the above convention its incidence matrix A is given by

$$
\begin{array}{c c}
\text{nodes} & \text{branches} \rightarrow \\
\downarrow & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{r} 1 \\ A = 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & +1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}
\end{array}
$$

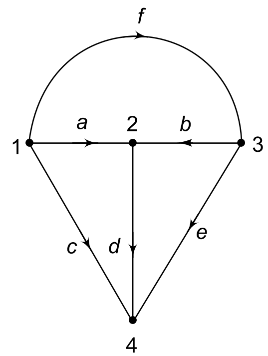


Fig. 2.73

The entries in the first row indicates that three branches a, c and f are incident to node 1 and they are oriented away from node 1 and therefore the entries a_{11}; a_{13} and a_{16} are +1. Other entries in the 1st row are zero as they are not connected to node 1. Likewise, we can complete the incidence matrix for the remaining nodes 2, 3 and 4.

Example 2.47 *Construct the incident matrix for the graph shown in Fig. 2.74.*

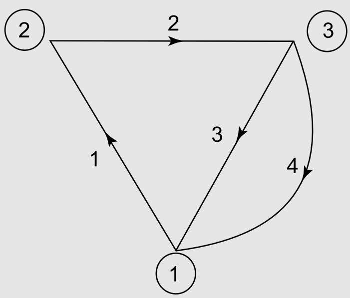


Fig. 2.74

Solution The dimensions of incidence matrix 'A' is $n \times b$ where n is number of nodes and b is number of branches, hence the dimensions of the incidence matrix for the above graph is 3×4.

Incidence matrix

n — nodes

b — branches

A =

n \ b	1	2	3	4
1	1	0	−1	−1
2	−1	1	0	0
3	0	−1	1	1

The incidence matrix is given by

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

2.9.1 Properties of Incidence Matrix (A) *[JNTU Nov 2011]*

Following properties are some of the simple conclusions from incidence matrix A.

1. Each column representing a branch contains two non-zero entries $+1$ and -1; the rest being zero. The unit entries in a column identify the nodes of the branch between which it is connected.
2. The unit entries in a row identify the branches incident at a node. Their number is called the degree of the node.
3. A degree of 1 for a row means that there is one branch incident at the node. This is commonly possible in a tree.
4. If the degree of a node is two, then it indicates that two branches are incident at the node and these are in series.
5. Columns of A with unit entries in two identical rows correspond to two branches with same end nodes and hence they are in parallel.
6. Given the incidence matrix A the corresponding graph can be easily constructed since A is a complete mathematical replica of the graph.
7. If one row of A is deleted the resulting $(n-1) \times b$ matrix is called the reduced incidence matrix A_1. Given A_1, A is easily obtained by using the first property.

It is possible to find the exact number of trees that can be generated from a given graph if the reduced incidence matrix A_1 is known and the number of possible trees is given by Det $(A_1A_1^T)$ where A_1^T is the transpose of the matrix A_1.

Example 2.48 *Draw the graph corresponding to the given incidence matrix.*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & +1 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution There are five rows and eight columns which indicate that there are five nodes and eight branches. Let us number the columns from a to h and rows as 1 to 5.

$$\begin{array}{cc} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Mark the nodes corresponding to the rows 1, 2, 3, 4 and 5 as dots as shown in Fig. 2.75(a). Examine each column of A and connect the nodes (unit entries) by a branch; label it after marking an arrow.

For example, examine the first column of A. There are two unit entries one in the first row and 2nd in the last row, hence connect branch a between node 1 and 5. The entry of A_{11} is –ve and that of A_{51} is +ve. Hence the orientation of the branch is away from node 5 and towards node 1 as per the convention. Proceeding in this manner we can complete the entire graph as shown in Fig. 2.75(b).

From the incidence matrix A, it can be verified that branches c and d are in parallel (property 5) and branches e and f are in series (property 4).

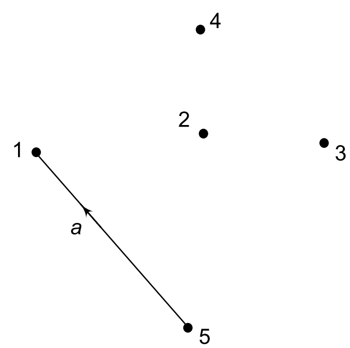


Fig. 2.75(a)

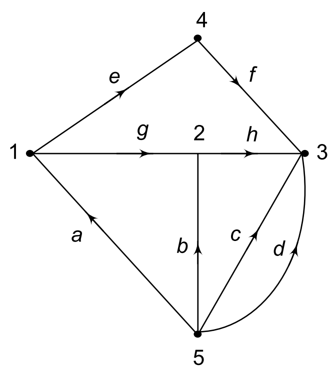


Fig. 2.75(b)

Example 2.49 *Obtain the incidence matrix A from the following reduced incidence matrix A_1 and draw its graph.*

$$[A_1] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution There are five rows and seven columns in the given reduced incidence matrix $[A_1]$. Therefore, the number of rows in the complete incidence matrix A will be $5 + 1 = 6$. There will be six nodes and seven branches in the graph. The dimensions of matrix A is 6×7. The last row in A, i.e., 6th row for the matrix A can be obtained by using the first property of the incidence matrix. It is seen that the first column of $[A_1]$ has a single non-zero element –1. Hence, the first element in the 6th row will be +1 $(-1 + 1 = 0)$. Second column of A_1 has two non-zero elements +1 and –1, hence the 2nd element in the 6th row will be 0. Proceeding in this manner we can obtain the 6th row. The complete incidence matrix can therefore be written as

$$[A] = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

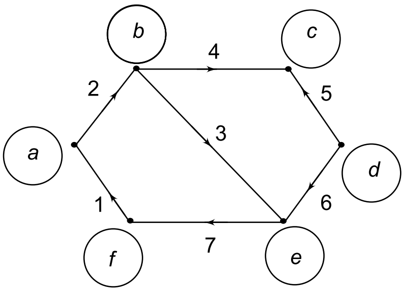


Fig. 2.76

We have seen that any one of the rows of a complete incidence matrix can be obtained from the remaining rows. Thus it is possible to delete any one row from A without loosing any information in A_1. Now the oriented graph can be constructed from the matrix A. The nodes may be placed arbitrarily. The number of nodes to be marked will be six. Taking node 6 as reference node the graph is drawn as shown in Fig. 2.76.

2.9.2 Incidence Matrix and KCL

[JNTU June 2009, Nov 2011]

Kirchhoff's current law (KCL) of a graph can be expressed in terms of the reduced incidence matrix as $A_1\, I = 0$.

A_1, I is the matrix representation of KCL, where I represents branch current vectors $I_1, I_2 \ldots I_6$.

Consider the graph shown in Fig. 2.77. It has four nodes a, b, c and d.

Let node d be taken as the reference node. The positive reference direction of the branch currents corresponds to the orientation of the graph branches. Let the branch currents be $i_1, i_2, \ldots i_6$. Applying KCL at nodes a, b and c.

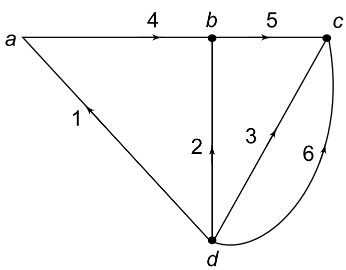


Fig. 2.77

$$-i_1 + i_4 = 0$$

$$-i_2 - i_4 + i_5 = 0$$

$$-i_3 = i_5 - i_6 = 0$$

These equations can be written in the matrix form as follows:

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1 I_b = 0 \tag{2.2}$$

Here, I_b represents column matrix or a vector of branch currents.

$$I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$$

A_1 is the reduced incidence matrix of a graph with n nodes and b branches. And it is a $(n - 1) \times b$ matrix obtained from the complete incidence matrix of A deleting one of its rows. The node corresponding to the deleted row is called the reference node or datum node. It is to be noted that $A_1 I_b = 0$ gives a set of $n - 1$ linearly independent equations in branch currents $I_1, I_2, \ldots I_6$. Here $n = 4$. Hence, there are three linearly independent equations.

Example 2.50 *For the network shown in Fig. 2.78 obtain the incidence matrix and mesh equations. Obtain the current through 20 Ω.* *[JNTU Jan 2010]*

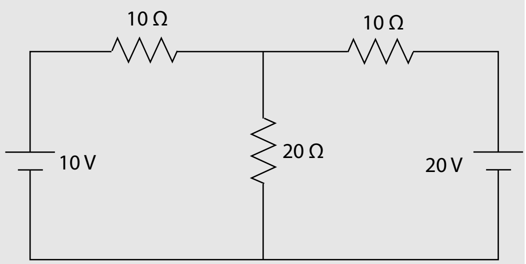


Fig. 2.78

Solution

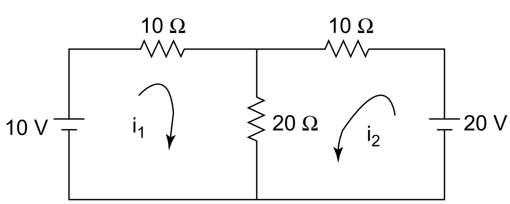


Fig. 2.79

The graph obtained

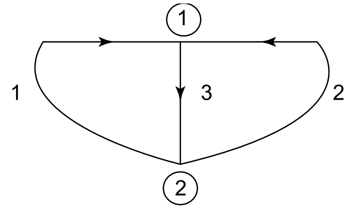


Fig. 2.80

Incidence matrix is given as

$$\begin{array}{cc} \text{nodes} & \text{branches} \rightarrow \\ \downarrow & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$$

Mesh equations are given as

$$10 = 30i_1 + 20i_2 \Rightarrow 1 = 3i_1 + 2i_2$$
$$20 = 20i_1 + 30i_2 \Rightarrow 2 = 2i_1 + 3i_2$$
$$\therefore \quad i_1 = -0.2 \text{ amp}$$
$$\therefore \quad i_2 = 0.8 \text{ amp}$$
$$\therefore \quad \text{Current through 20 V} = i_1 + i_2 = 0.6 \text{ amp}$$

2.10 BASIC TIE-SET MATRICES FOR PLANAR NETWORKS

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop there exists a closed path and a circulating current, which is called the link current. The current in any branch of a graph can be found by using link currents.

The fundamental loop formed by one link has a unique path in the tree joining the two nodes of the link. This loop is also called *f*-loop or a tie-set.

Consider a connected graph shown in Fig. 2.81(a). It has four nodes and six branches. One of its trees is arbitrarily chosen and is shown in Fig. 2.81(b). The twigs of this tree are branches 4, 5 and 6. The links corresponding to this tree are branches 1, 2 and 3. Every link defines a fundamental loop of the network.

No. of nodes $n = 4$

No. of branches $b = 6$

No. of tree branches or twigs $= n - 1 = 3$

No. of link branches $I = b - (n - 1) = 3$

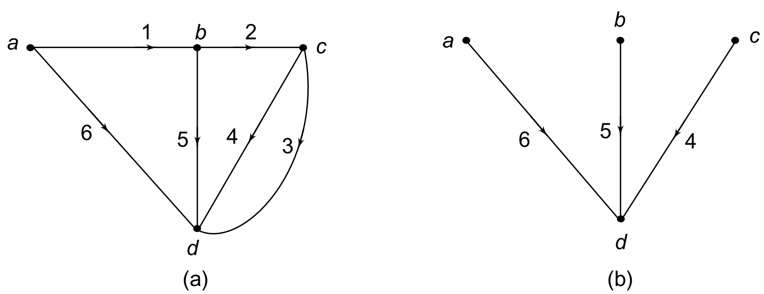


Fig. 2.81

Let $i_1, i_2, \ldots i_6$ be the branch currents with directions as shown in Fig. 2.81(a). Let us add a link in its proper place to the tree as shown in 2.81(c). It is seen that a loop I_1 is formed by the branches 1, 5 and 6. There is a formation of link current, let this current be I_1. This current passes through the branches 1, 5 and 6. By convention a fundamental loop is given the same orientation as its defining link, i.e., the link current I_1 coincides with the branch current direction i_1 in *ab*. A tie set can also be defined as the set of branches that forms a closed loop in which the link current flows. By adding the other link branches 2 and 3, we can form two more fundamental loops or *f*-loops with link currents I_2 and I_3 respectively as shown in Figs 2.81(d) and (e).

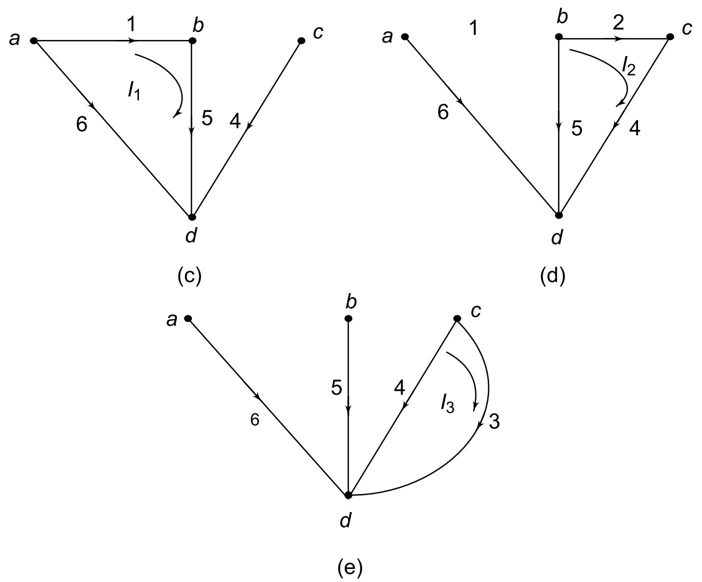


Fig. 2.81

2.10.1 Tie-Set Matrix

Kirchhoff's voltage law can be applied to the f-loops to get a set of linearly independent equations. Consider Fig. 2.82.

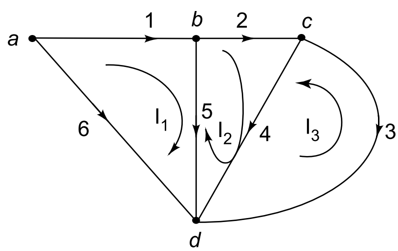


Fig. 2.82

There are three fundamental loops I_1, I_2 and I_3 corresponding to the link branches 1, 2 and 3 respectively. If V_1, V_2, … V_6 are the branch voltages the KVL equations for the three f-loops can be written as

$$\begin{aligned} V_1 + V_5 - V_6 &= 0 \\ V_2 + V_4 - V_5 &= 0 \\ V_3 - V_4 &= 0 \end{aligned} \tag{2.3}$$

In order to apply KVL to each fundamental loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equation can be written in matrix form as

$$\begin{array}{l} \text{loop branches} \rightarrow \qquad\qquad 3 \times 6 \\ \begin{array}{c|cccccc} \downarrow & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array}$$

$$\begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,V_b = 0 \tag{2.4}$$

where B is an $I \times b$ matrix called the tie-set matrix or fundamental loop matrix and V_b is a column vector of branch voltages.

The tie-set matrix B is written in a compact form as $B\ [b_{ij}]$ (2.5)

The element b_{ij} of B is defined as

$b_{ij} = 1$ when branch b_j is in the f-loop I_i (loop current) and their reference directions coincide.

$b_{ij} = -1$ when branch b_j is in the f-loop I_i (loop current) and their reference directions are opposite.

$b_{ij} = 0$ when branch b_j is not in the f-loop I_i.

2.10.2 Tie-set Matrix and Branch Currents

It is possible to express branch currents as a linear combination of link current using matrix B.

If I_B and I_I represents the branch current matrix and loop current matrix respectively and B is the tie-set matrix, then

$$[I_b] = [B^T]\,[I_L] \tag{2.6}$$

where $[B^T]$ is the transpose of the matrix $[B]$. Equation (2.6) is known as link current transformation equation.

Consider the tie-set matrix of Fig. 2.80.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

The branch current vector $[I_b]$ is a column vector.

$$[I_b] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

The loop current vector $[I_L]$ is a column vector.

$$[I_L] = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Therefore the link current transformation equation is given by $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are

$$i_1 = I_1$$
$$i_2 = I_2$$
$$i_3 = I_3$$
$$i_4 = I_2 - I_3$$
$$i_5 = I_1 - I_2$$
$$i_6 = -I_1$$

Example 2.51 *For the electrical network shown in Fig. 2.83 draw its topological graph and write its incidence matrix, tie-set matrix, link current transformation equation and branch currents.*

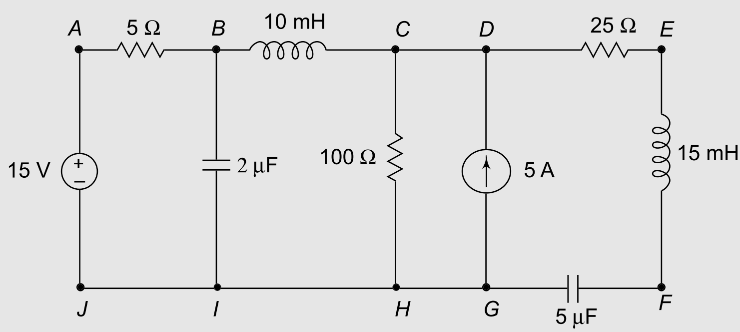


Fig. 2.83

Solution Voltage source is short circuited, current source is open circuited, the points which are electrically at same potential are combined to form a single node. The graph is shown in Fig. 2.84(a).

Combining the simple nodes and arbitrarily selecting the branch current directions the oriented graph is shown in Fig. 2.84(b). The simplified consists of three nodes. Let them be x, y and z and five branches 1, 2, 3, 4 and 5. The complete incidence matrix is given by

$$\begin{array}{cc} \text{nodes} & \text{branches} \rightarrow \\ \downarrow & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$A = \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}$$

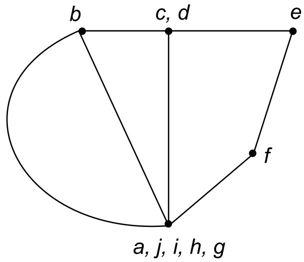


Fig. 2.84(a)

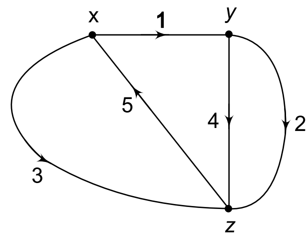


Fig. 2.84(b)

Let us choose node z as the reference or datum node for writing the reduced incidence matrix A_1 or we can obtain A_1 by deleting the last row elements in A.

$$A_1 = \begin{array}{c} \text{nodes} \\ \downarrow \\ x \\ y \end{array} \begin{array}{c} \text{branches} \rightarrow \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

For writing the tie-set matrix, consider the tree in the graph in Fig. 2.84(b).

No. of nodes $n = 3$

No. of branches $= 5$

No. of tree branches or twigs $= n - 1 = 2$

No. of link branches $I = b - (n - 1) = 5 - (3 - 1) = 3$

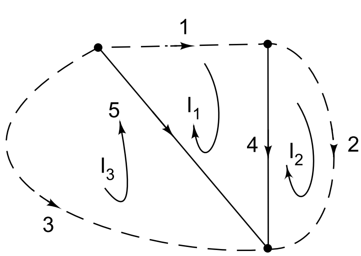


Fig. 2.84(c)

The tree shown in Fig. 2.84(c) consists of two branches 4 and 5 shown with solid lines and the link branches of the tree are 1, 2 and 3 shown with dashed lines. The tie-set matrix or fundamental loop matrix is given by

$$B = \begin{array}{c} \text{loop} \\ \downarrow \\ I_1 \\ I_2 \\ I_3 \end{array} \begin{array}{c} \text{branches} \rightarrow \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

To obtain the link current transformation equation and thereby branch currents the transpose of B should be calculated.

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The equation $[I_b] = [B^T]\,[I_L]$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The branch currents are given by

$$i_1 = I_1$$

$$i_2 = I_2$$

$$i_3 = I_3$$

$$i_4 = I_1 - I_2$$

$$i_5 = I_1 + I_3$$

Example 2.52 *Write the tie-set matrix for the graph shown in Fig. 2.85, taking the tree consisting of branches 2, 4, 5.*

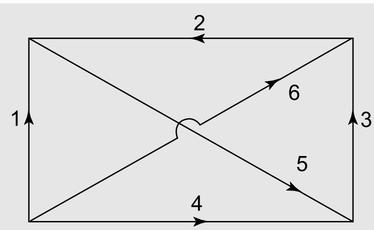


Fig. 2.85

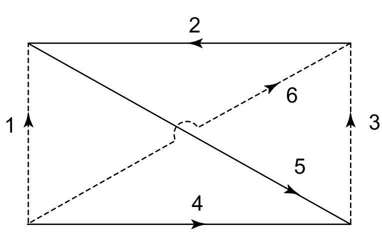


Fig. 2.86(a)

Solution The twigs of the tree are 2, 4 and 5. The links corresponding to the tree are 1, 3 and 6 as shown in the Fig. 2.86(a).

Number of nodes $n = 4$

Number of branches $b = 6$

Number of tree branches of twigs

$= n - 1 = 3$

Number of link branches $= b - (n - 1) = 3$

For writing the tie-set matrix consider the three links one at a time. The tie-set matrix of fundamental loop matrix is given by

$$B = \begin{array}{cc} \text{branches} \rightarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \text{loops } \downarrow \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The tie-sets are shown in the Figs 2.86(b), (c) and (d)

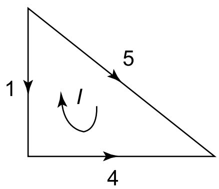


Fig. 2.86(b)

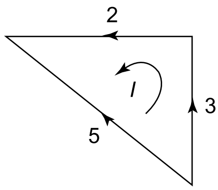


Fig. 2.86(c)

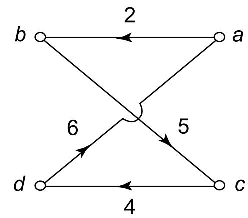


Fig. 2.86(d)

Example 2.53 *For the given graph and tree shown in Fig. 2.87, write the tie-set matrix and obtain the relationship between the branch currents and link currents.*

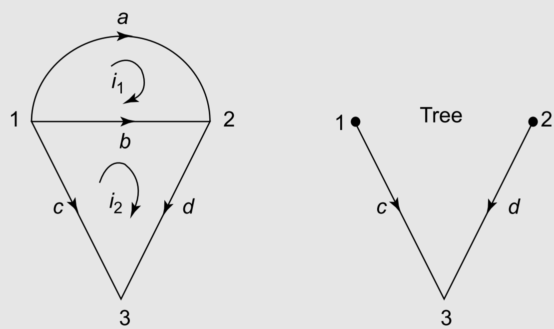


Fig. 2.87

Solution Number of link branches = $b(n-1)$

where b is number of branches and n is number of nodes

$\therefore$ Link branches = $4-(3-1)=2$

The link branches are a and b

Let the branch currents are i_a, i_b, i_c and i_d.

The two links currents are i_1 and i_2 as shown in the Fig. 2.88.

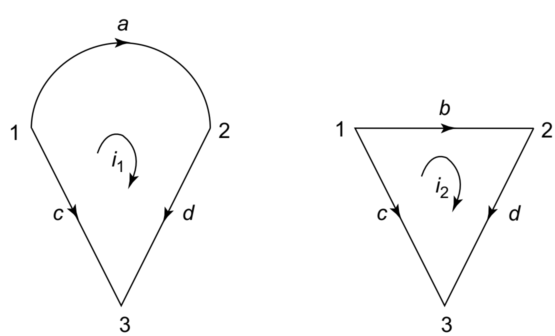


Fig. 2.88

There are two fundamental loops corresponding to the link branches a and b. If V_a and V_b are the branch voltages, the KVL equations for the two f-loops can be written as

$$V_a + V_d - V_c = 0$$

$$V_b + V_d - V_c = 0$$

The above equation can be written as

loop
currents branches →

$$\begin{array}{c} \downarrow \\ i_1 \\ i_2 \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & +1 \end{bmatrix} \end{array} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = 0$$

Example 2.54 *For the topological graph shown in Fig. 2.89, obtain the fundamental tie-set matrix choosing the tree containing the two elements 5 and 6.*

[JNTU May/June 2004]

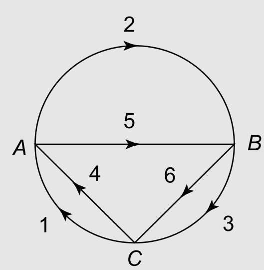


Fig. 2.89

Solution The tree of the graph is shown with solid lines (5 and 6) and the links are shown with dashed lines (1, 2, 3, 4) as in Fig. 2.90.

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop. In a loop, there exists a closed path and a circulating current, which is called the link current.

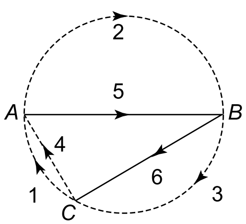


Fig. 2.90

The fundamental loop formed by one link at a time, has a unique path in the tree joining the two nodes of the link. This loop is also called f-loop or a tie-set. Every link defines a fundamental loop of the network.

No. of nodes in the graph $n = 3 = (A, B, C)$

No. of branches $b = 6 = (1, 2, 3, 4, 5, 6)$

No. of tree branches or twigs $n - 1 = 2 = (5, 6)$

No. of link branches, $l = b - (n - 1) = 4(1, 2, 3, 4)$

Tie-sets are formed as shown in Fig. 2.91.

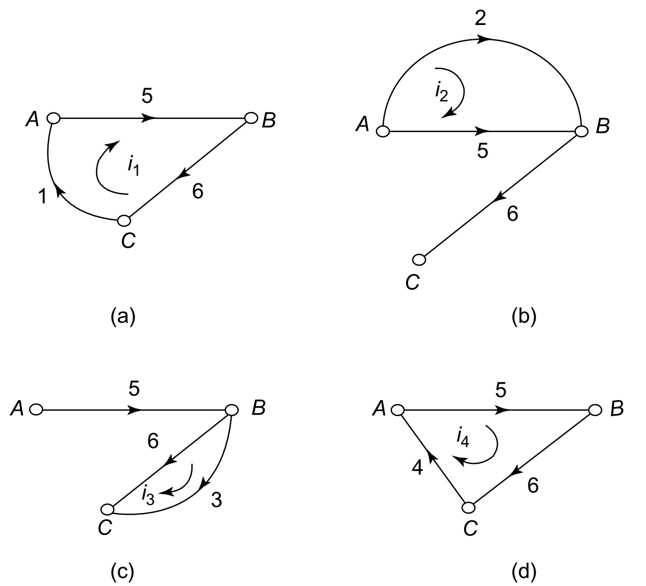


Fig. 2.91

The KVL equations for the three f-loops can be written as

$$V_1 + V_5 + V_6 = 0$$

$$V_2 - V_5 = 0$$

$$V_3 - V_6 = 0$$

$$V_4 + V_5 + V_6 = 0$$

In order to apply KVL to each loop, we take the reference direction of the loop which coincides with the reference direction of the link defining the loop.

The above equations can be written as

$[B]\,[V_b] = 0$, where B is a 4×6 tie-set matrix.

loops branches $\rightarrow$

$$\begin{array}{c} \downarrow \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}
\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Therefore, tie-set matrix, } B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Example 2.55 *Draw the oriented graph of the network shown in Fig. 2.92 and write the incidence matrix.* *[JNTU May 2007]*

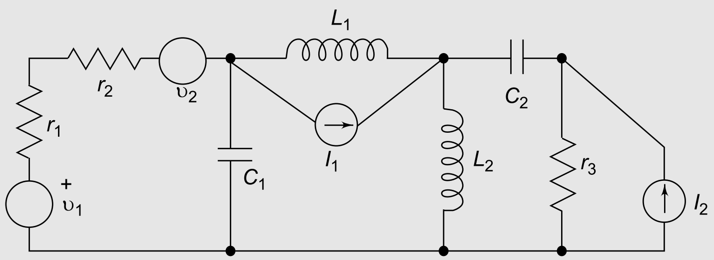


Fig. 2.92

Solution Directions of currents are arbitrarily assumed as shown in the circuit of Fig. 2.93(a).

Ideal voltage sources and current sources do not appear in the graph of a linear network. Ideal voltage source is represented by short circuit and an ideal current source is replaced by an open circuit. The nodes that appear in the graph are numbered (1) (2) (3) (4) and (5); branches as a, b, c, d, e, f and g. The graph is as shown in the Fig. 2.93(b)

For a graph with n nodes and b branches, the order of the incidence matrix is $(n - 1) \times b$. Choose node (5) as reference (or datum) node for writing incidence matrix. The required incidence matrix is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

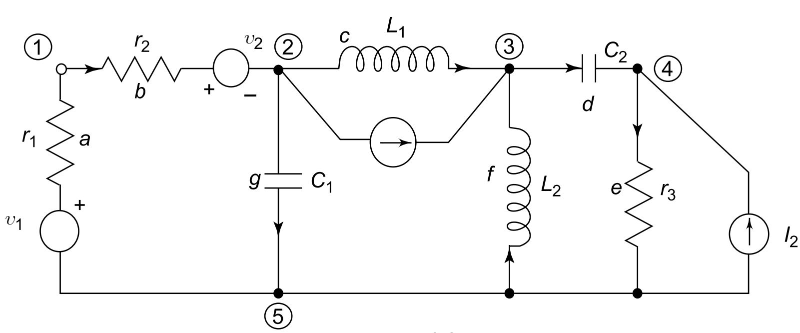


Fig. 2.93(a)

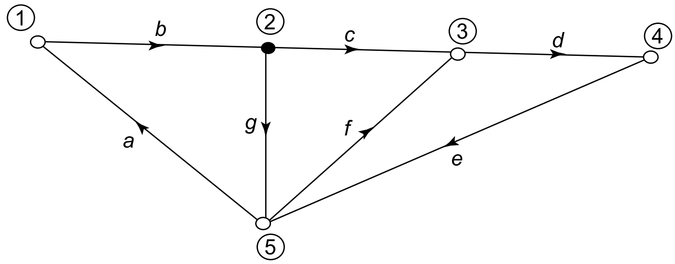


Fig. 2.93(b)

Example 2.56 *Draw the oriented graph of the network shown in Fig. 2.94.*
[JNTU May 2007]

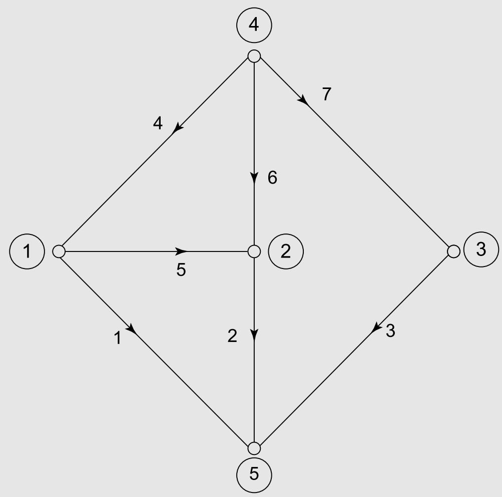


Fig. 2.94

Solution The graph represented in figure itself represents the oriented graph in which (1)–(5) are nodes and 1–7 are branches.

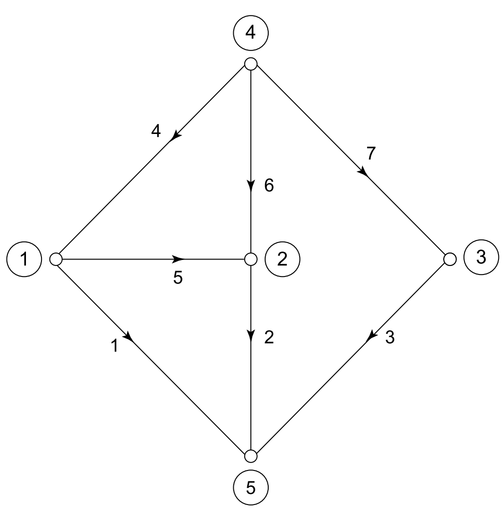


Fig. 2.95

Example 2.57 *Write the Tie-set matrix for the graph shown in Fig. 2.96, taking the tree consisting of branches.*
[JNTU May 2006]

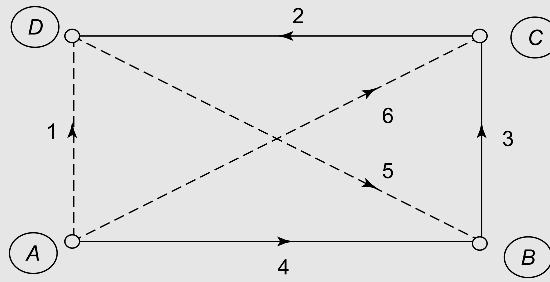


Fig. 2.96

Solution

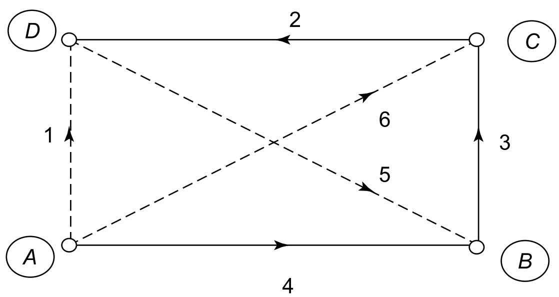


Fig. 2.97

Basic tie-sets \ *e*	*1*	*2*	*3*	*4*	*5*	*6*
(5, 3, 2)	0	−1	−1	0	1	0
(6, 3, 4)	0	0	−1	−1	0	1
(1, 2, 3, 4)	1	−1	−1	−1	0	0

Example 2.58 *For the network shown in Fig. 2.98 find the tie-set matrix loop current.* *[JNTU June 2008]*

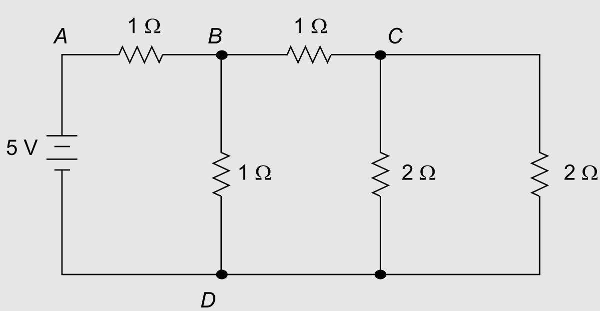


Fig. 2.98

Solution First replace the circuit with the network graph.

I_1, I_2, I_3 are loop currents corresponding to the branches.

There are three f-loops. We can apply KVL for these f-loops.

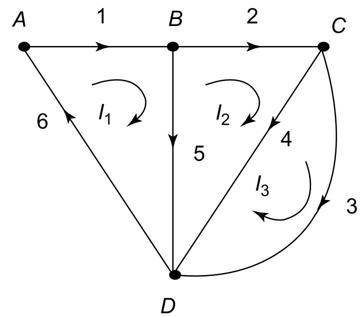


Fig. 2.99

$$V_1 + V_5 + V_6 = 0$$
$$V_2 + V_4 - V_5 = 0$$
$$V_3 - V_4 = 0$$

The above equations can be written in matrix form as

$$\begin{array}{c} \text{loop branches} \rightarrow \\ \downarrow \end{array}$$

$$\begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is possible to express branch currents as a linear combination of link currents using matrix B.

Let I_b represents branch current matrix.

I_L represents loop current matrix.

$$I_b = [B^T]\,[I_L]$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$I_b = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} \qquad I_L = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$I_1 = i_1$ Loop currents are I_1, I_2, I_3

$I_2 = i_2$

$I_3 = i_3$

Example 2.59 *Write the matrix loop equation for the network shown in Fig. 2.100 and determine the loop currents.*

[JNTU June 2008]

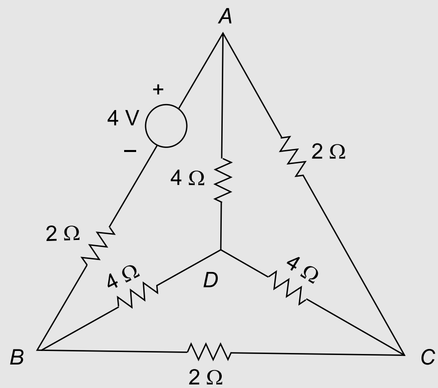


Fig. 2.100

Solution The graph for the following circuit is shown in Fig. 2.101.

This can be represented in matrix form as follows.

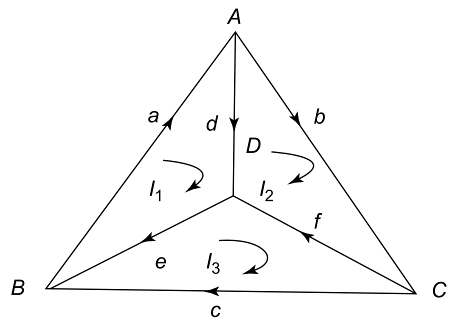


Fig. 2.101

nodes $\downarrow$ branches $\rightarrow$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

Consider the loop equations

$$V_a + V_d + V_e = 0$$
$$V_b + V_f - V_d = 0$$
$$-V_e + V_c - V_f = 0$$

In order to find loop currents, we can apply mesh analysis.

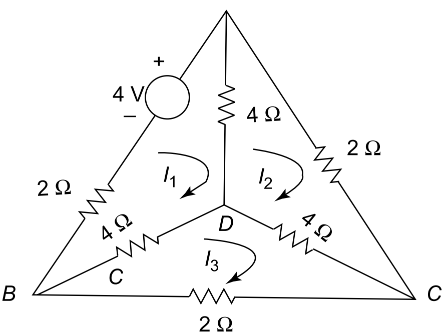


Fig. 2.102

Applying KVL to each loop,

$$10I_1 - 4I_2 - 4I_3 = 4 \quad (2.7)$$

$$10I_2 - 4I_1 - 4I_3 = 0 \quad (2.8)$$

$$10I_3 - 4I_1 - 4I_2 = 0 \quad (2.9)$$

From (2.9) we have

$$I_3 = 4/10(I_1 + I_2)$$

Substituting this in equation (2.8)

$$10I_2 - 4I_1 - \frac{16}{10}(I_1 + I_2) = 0$$

$$I_2(10 - 1.6) - 5.6I_1 = 0$$

$$-5.6I_1 + 8.4I_2 = 0 \quad (2.10)$$

The first equation reduces

$$8.4I_1 - 5.6I_2 = 4 \quad (2.11)$$

By solving I_1 and I_2 (2.10) and (2.11) we get

$$I_1 = 0.857\ A$$

$$I_2 = 0.57\ A$$

$$I_3 = 4/10(I_1 + I_2) = 0.5708\ A$$

Example 2.60 *For the resistive network as shown in Fig. 2.103 write a tie-set schedule and equilibrium equations, on current basis. Determine the branch currents and branch voltages.*

[JNTU June 2009]

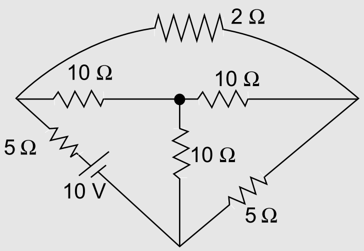


Fig. 2.103

Solution In order to determine tie-set schedule, we must draw graph of given network, and to draw the graph, we have to replace all the resistors by line segments where as the voltage source must be replaced with short circuit.

Graph

There are 4 nodes, 6 branches.

Tree contains 4 nodes, 3 branches

$d, e, f \rightarrow$ Twigs

$a, b, c \rightarrow$ Links

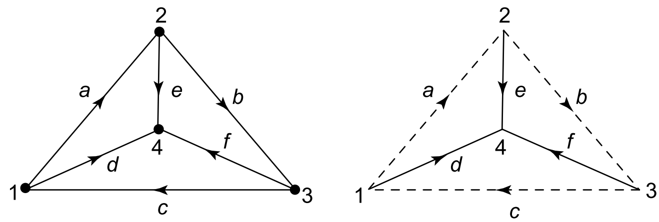


Fig. 2.104(a) *Tie-sets*

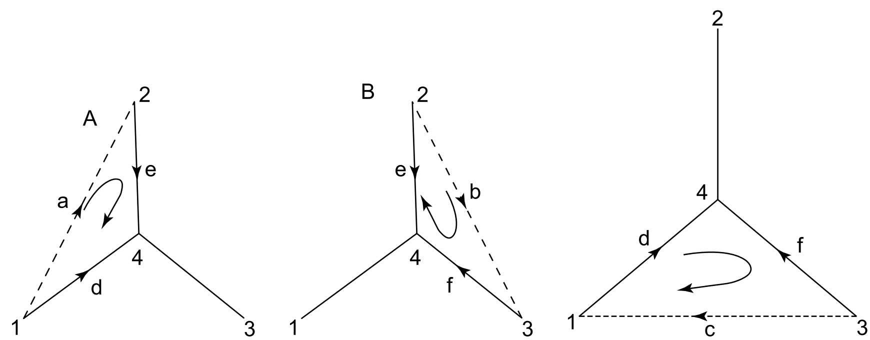


Fig. 2.104(b) *Tie-set schedule*

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

Example 2.61 *Draw the oriented network graph from the incidence matrix given below.* *[JNTU June 2009]*

Nodes	*Branches*					
	1	*2*	*3*	*4*	*5*	*6*
A	−1	0	0	1	−1	0
B	1	−1	0	0	0	−1
C	0	1	−1	0	1	0
D	0	0	+1	−1	0	+1

Solution

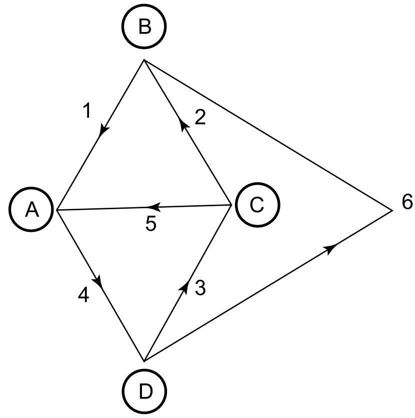


Fig. 2.105

Example 2.62 *For the network shown in Fig. 2.106, draw the oriented graph, select a tree and obtain a tie-set matrix. Write down the KVL equations from the tie-set matrix.* *[JNTU June 2009]*

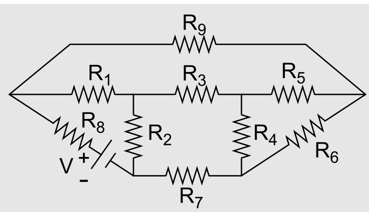


Fig. 2.106

Solution

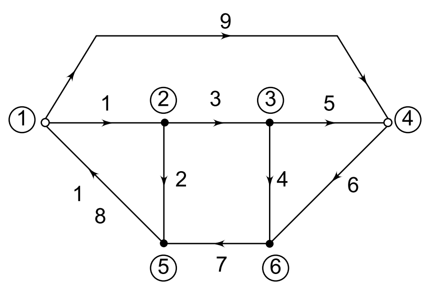


Fig. 2.107

Replacing voltage source by a short circuit, we obtain the graph as

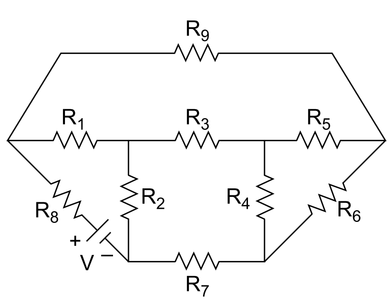


Fig. 2.108(a)

The chosen tree is shown in Fig. 2.108(b).

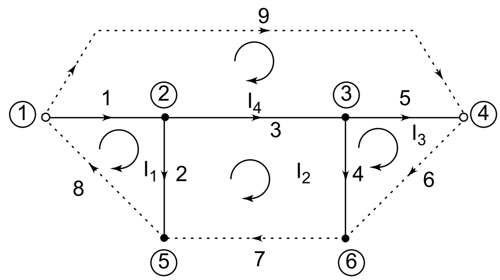


Fig. 2.108(b)

The tie-set matrix is given as

Loop current ↓ Branches →

$$\begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The KVL equations are as follows:

$$v_1 + v_2 + v_8 = 0$$
$$-v_2 + v_3 + v_4 + v_7 = 0$$
$$-v_4 + v_5 + v_6 = 0$$
$$-v_1 - v_3 - v_5 + v_9 = 0$$

Example 2.63 *For the given graph shown in Fig. 2.109 write the tie-set schedule and obtain the relation between branch currents and link currents.*

[JNTU June 2009]

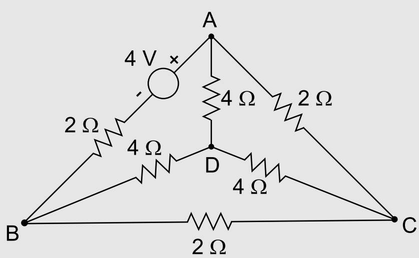


Fig. 2.109(a)

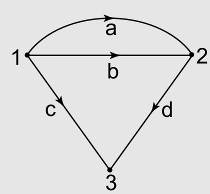


Fig. 2.109(b)

Solution

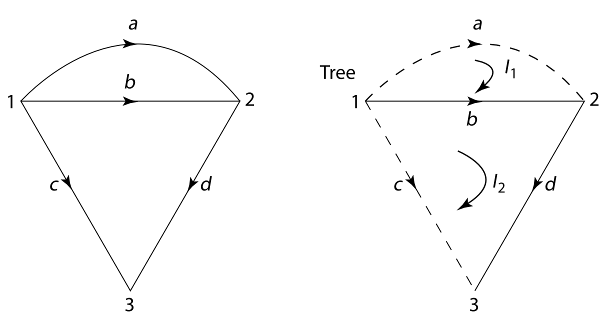


Fig. 2.110

Tie-set matrix

$$\begin{array}{c c} & \text{loop currents} \downarrow \quad \text{branches} \rightarrow \\ & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Relation between link current (I_L) and branch current (i_b)

$$I_1 = i_a$$
$$-I_1 + I_2 = i_b$$
$$-I_2 = i_c$$
$$I_2 = i_d$$

Example 2.64 *Write the tie-set schedule and write tie-set matrices also. Write the relationship between the branch current and link currents of the given Fig. 2.111.*

[JNTU Jan 2010]

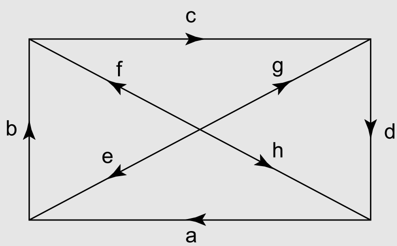


Fig. 2.111

Solution

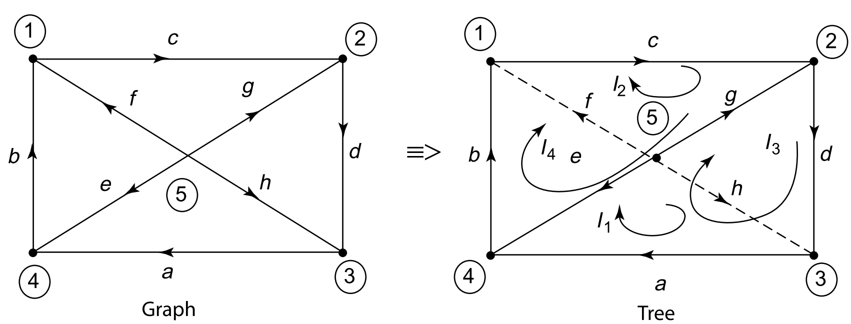


Fig. 2.112(a)

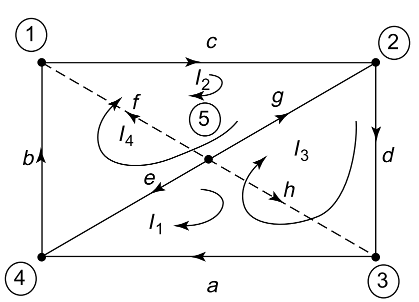


Fig. 2.112(b)

$$\begin{array}{cc} \text{loop current} & \text{branches} \rightarrow \\ \downarrow & \begin{array}{cccccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} \end{array} \\ \begin{array}{c} \text{I}_1 \\ \text{I}_2 \\ \text{I}_3 \\ \text{I}_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Tie-set matrix

Relation between
branch and loop current

$$j_a = I_1 + I_3 \qquad j_b = I_4 \qquad j_c = I_2 + I_4, \qquad j_d = I_3$$

$$j_e = -(I_1 + I_3), \qquad i_j = I_2, \qquad j_g = I_3 - (I_2 + I_4), \qquad j_h = I_1$$

($j_a, j_b, j_c \ldots$ .are the branch currents)

2.11 BASIC CUT-SET FOR PLANAR NETWORKS

A cut-set is a minimal set of branches of a connected graph such that the removal of these branches causes the graph to be cut into exactly two parts. The important property of a cut-set is that by restoring anyone of the branches of the cut-set the graph should become connected. A cut-set consists of one and only one branch of the network tree, together with any links which must be cut to divide the network into two parts.

Consider the graph shown in Fig. 2.113(a).

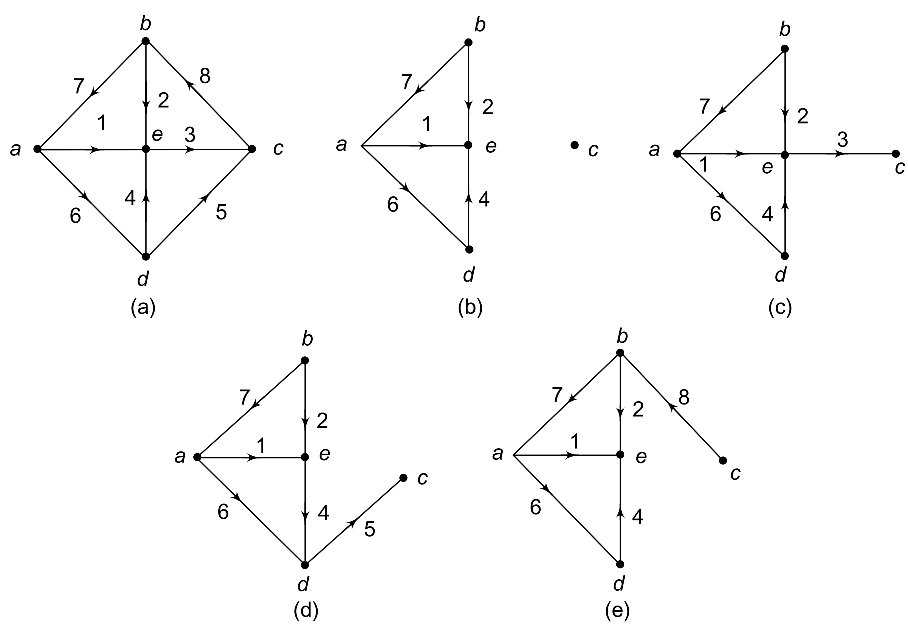


Fig. 2.113

If the branches 3, 5 and 8 are removed from the graph, we see that the connected graph of Fig. 2.113(a) is separated into two distinct parts, each of which is connected as shown in Fig. 2.113(b). One of the parts is just an isolated node. Now suppose the removed branch 3 is replaced, all others still removed. Fig. 2.113(c) shows the resultant graph. The graph is now connected. Likewise replacing the removed branches 5 and 8 of the set {3, 5, 8} one at a time, all other ones remaining removed, we obtain the resulting graphs as shown in Figs 2.113(d) and (e). The set formed by the branches 3, 5 and 8 is called the cut-set of the connected graph of Fig. 2.113(a).

2.11.1 Cut-set Orientation

A cut-set is oriented by arbitrarily selecting the direction. A cut-set divides a graph into two parts. In the graph shown in Fig. 2.114, the cut-set is {2, 3}. It is represented by a dashed line passing through branches 2 and 3. This cut-set separates the graph into two parts shown as part-1 and part-2. We may take the orientation either from part-1 to part-2 or from part-2 to part-1.

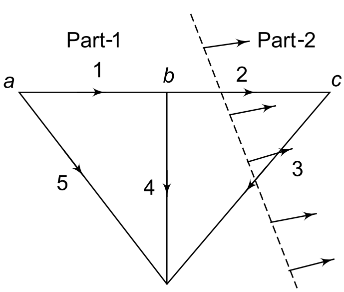


Fig. 2.114

The orientation of some branches of the cut-set may coincide with the orientation of the cut-set while some branches of the cut-set may not coincide. Suppose we choose the orientation of the cut-set {2, 3} from part-1 to part-2 as indicated in Fig. 2.114, then the orientation of branch 2 coincides with the cut-set, whereas the orientation of the branch 3 is opposite.

2.11.2 Cut-set Matrix and KCL for Cut-sets

KCL is also applicable to a cut-set of a network. For any lumped electrical network, the algebraic sum of all the cut-set branch currents is equal to zero. While writing the KCL equation for a cut-set, we assign positive sign for the current in a branch if its direction coincides with the orientation of the cut-set and a negative sign to the current in a branch whose direction is opposite to the orientation of the cut-set. Consider the graph shown in Fig. 2.115. It has five branches and four nodes. The branches have been numbered 1 through 5 and their orientations are also marked. The following six cut-sets are possible as shown in Fig. 2.116(a)–(f).

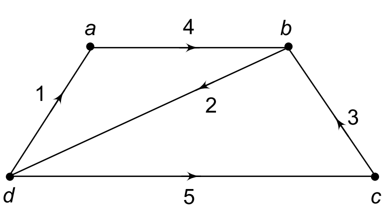


Fig. 2.115

Cut-set C_1: {1, 4}; cut-set C_2: {4, 2, 3}

Cut-set C_3: {3, 5}; cut-set C_4: {1, 2, 5}

Cut-set C_5: {4, 2, 5}; cut-set C_6: {1, 2, 3}

Applying KCL for each of the cut-set we obtain the following equations. Let $i_1, i_2 \ldots i_6$ be the branch currents.

$$\left.\begin{array}{l} C_1 : i_1 - i_4 = 0 \\ C_2 : -i_2 + i_3 + i_4 = 0 \\ C_3 : -i_3 + i_5 = 0 \\ C_4 : i_1 - i_2 + i_5 = 0 \\ C_5 : -i_2 + i_4 + i_5 = 0 \\ C_6 : i_1 - i_2 + i_3 = 0 \end{array}\right\} \quad (2.12)$$

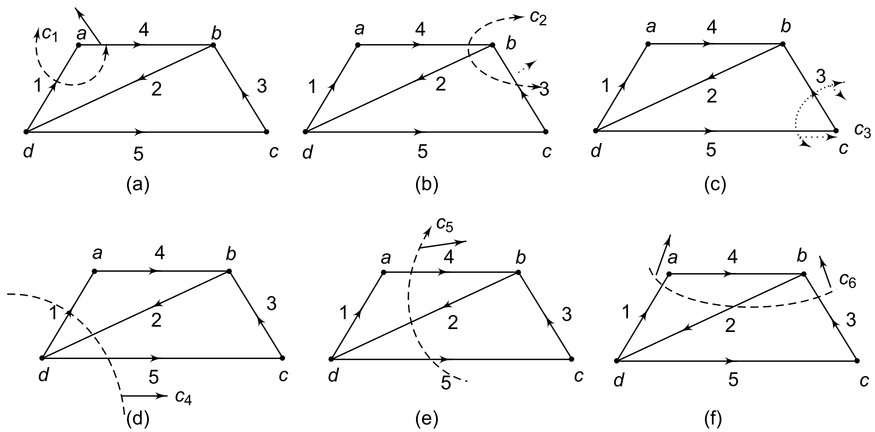


Fig. 2.116(a to f)

These equations can be put into matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$QI_b = 0 \quad (2.13)$$

where the matrix Q is called augmented cut-set matrix of the graph or all cut-set matrix of the graph. The matrix I_b is the branch-current vector.

The all cut-set matrix can be written as $Q^{\text{th}} = [q_{ij}]$.
Where q_{ij} is the element in the i^{th} row and j^{th} column. The order of Q is number of cut-sets $\times$ number of branch as in the graph.

$$\left.\begin{array}{l} q_{ij} = 1, \text{ if branch } j \text{ in the cut-set } i \text{ and the orientation} \\ \text{coincides with each other} \\ q_{ij} = -1, \text{ if branch } j \text{ is in the cut-set } i \text{ and the orientation} \\ \text{is opposite.} \\ q_{ij} = 0, \text{ if branch } j \text{ is not present in cut-set } i. \end{array}\right\} \quad (2.14)$$

Example 2.65 *For the network-graph shown in Fig. 2.117 with given orientation obtain the all cut-set (augmented cut-set) matrix.*

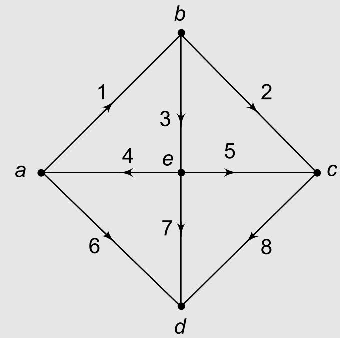


Fig. 2.117

Solution The graph has four nodes and eight branches. There are in all 12 possible cut-sets as shown with dashed lines in Figs 2.118(a) and (b). The orientation of the cut-sets has been marked arbitrarily. The cut-sets are

C_1: {1, 46}; C_2: {1, 2, 3}; C_3: {2, 5, 8}

C_4: {6, 7, 8}; C_5: {1, 3, 5, 8}; C_6: {1, 4, 7, 8}

C_7: {2, 5, 6, 7}; C_8: {2, 3, 4, 6} C_9: {1, 4, 7, 5, 2}

C_{10}: {2, 3, 4, 7, 8}; C_{11}: {6, 4, 3, 5, 8}; C_{12}: {1, 3, 5, 7, 6}

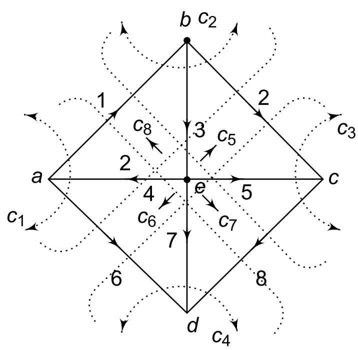


Fig. 2.118(a)

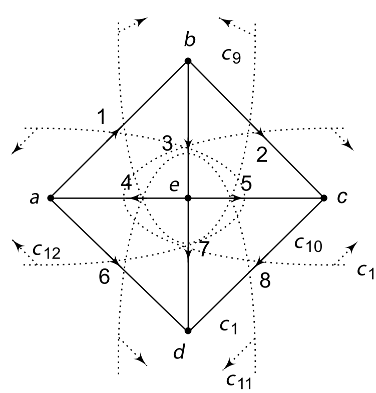


Fig. 2.118(b)

Eight cut-sets C_1 to C_8 are shown if Fig. 2.118(a) and four cut-sets C_9 to C_{11} are shown in Fig. 2.118(b) for clarity.

As explained in Section 2.11.2 with the help of Eq. 2.14, the all cut-set matrix Q is given by

$$Q = \begin{array}{c|c} \text{cut-sets} \downarrow & \text{branches} \rightarrow \\ & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{array} & \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Matrix Q is a 12×8 matrix since there are 12 cut-sets and eight branches in the graph.

Example 2.66 *Determine the basic cut-set matrix for the oriented graph given in Fig. 2.119 where the elements 1, 2 and 3 are tree branches.*

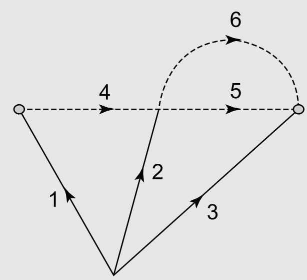


Fig. 2.119

Solution Branches 1, 2 and 3 are the twigs of the tree. The remaining branches 4, 5 and 6 are called links. Let us consider a tree as in Fig. 2.120.

For each twig, there will be a basic cut-set. Therefore, for a network graph with 'r' nodes and 'b' branches there will be $(n - 1)$ number of basic cut-sets.

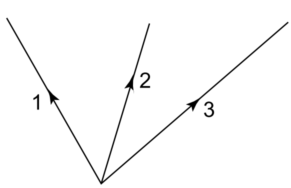


Fig. 2.120

The link that must be added to twig 1 to form a cut-set 1 is 4. Thus Corresponding to twig 1 the basic cut-set {1, 4} as shown.

As a convention the orientation of a cut-set is chosen to consider with that of its defining twig similarly, other cut-sets C_2 and C_3 corresponding to twigs 2 and 3 are also shown in the Figs 2.121(b) and (c).

$C_1 = \{1, 4\}$ Corresponding to twig 1

$C_2 = \{2, 4, 5, 6\}$ Corresponding to twig 2

$C_3 = \{3, 5, 6\}$ Corresponding to twig 3

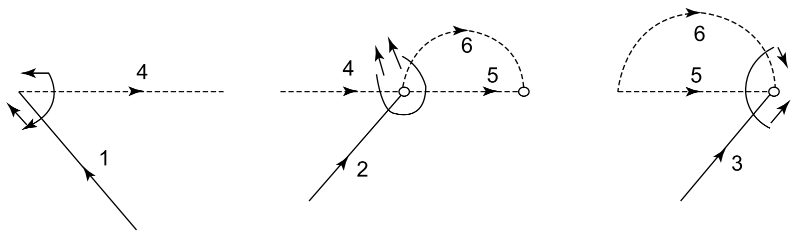


Fig. 2.121

The basic cut-set matrix Q_f of a graph with n nodes and b branches corresponds to a tree T is an $(n - 1) \times b$ matrix.

Thus the basic cut-set Matrix is given by

$$\begin{array}{ccc} & f \text{ cut-sets} & \text{branches} \rightarrow \\ & \downarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ Q_f = & \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 2.67 *For the given network Fig. 2.122, draw the oriented graph and choose one possible tree and construct the basic cutest schedule. Write down the network equations from the above matrix.*

[JNTU June 2006]

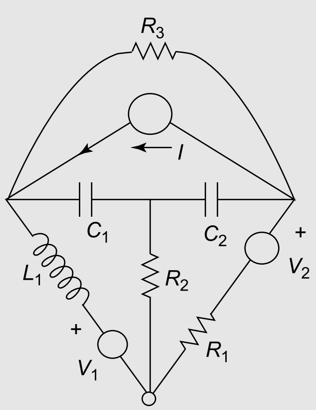


Fig. 2.122

Solution The oriented graph for the given network can be as shown in Fig. 2.123.

$$C_1: i_1 - i_5 + i_6 + i_7 = 0$$

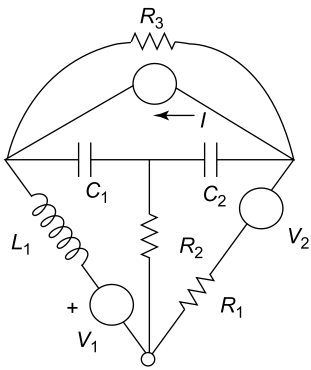


Fig. 2.123

$$C_2\colon i_2 - i_4 + i_6 + i_7 = 0$$

$$C_3\colon i_3 + i_4 - i_5 = 0$$

$$\begin{array}{lc}
 & \text{branches} \\
f\text{-cut-sets} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{lc} [7,6,\bar{2},4] & a \\ [7,6,\bar{1},4] & b \\ [3,4,\bar{5}] & c \end{array} & \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}
\end{array}$$

Example 2.68 *Draw the graph for network shown obtain a tree. What is the number of mesh currents required for network?* *[JNTU June 2009]*

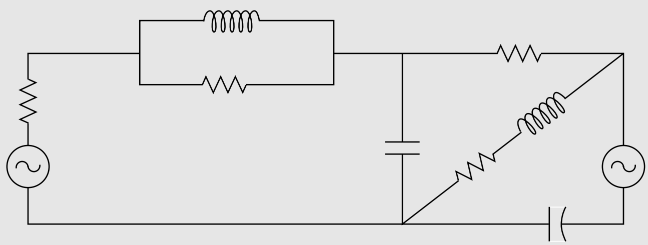
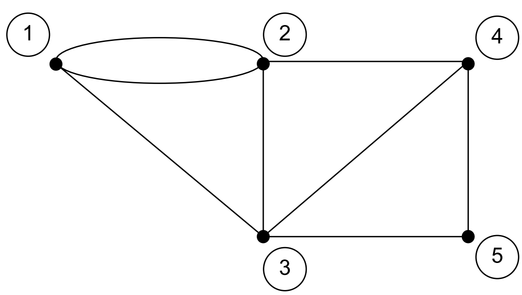


Fig. 2.124

Solution

Graph



Tree

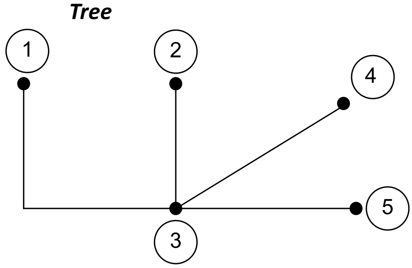


Fig. 2.125

No. of mesh currents = 4

2.11.3 Fundamental Cut-sets

Observe the set of Eq. 2.12 in Section 2.11.2 with respect to the graph in Fig. 2.116. Only first three equations are linearly independent, remaining equations can be obtained as a linear combination of the first three. The concept of fundamental cut-set (f-cut-set) can be used to obtain a set of linearly independent equations in branch current variables. The f-cut-sets are defined for a given tree of the graph. From a connected graph, first a tree is selected, and then a twig is selected. Removing this twig from the tree separates the tree into two parts. All the links which go from one part of the disconnected tree to the other, together with the twig of the selected tree will constitute a cut-set. This cut-set is called a fundamental cut-set or f-cut-set or the graph. Thus a fundamental cut-set of a graph with respect to a tree is a cut-set that is formed by one twig and a unique set of links. For each branch of the tree, i.e. for each twig, there will be a f-cut-set. So, for a connected graph having n nodes, there will be $(n - 1)$ twigs in a tree, the number of f-cut-sets is also equal to $(n - 1)$.

Fundamental cut-set matrix Q_f is one in which each row represents a cut-set with respect to a given tree of the graph. The rows of Q_1 correspond to the fundamental cut-sets and the columns correspond to the branches of the graph. The procedure for obtaining a fundamental cut-set matrix is illustrated in Example 2.54.

Example 2.69 *Obtain the fundamental cut-set matrix Q_f for the network graph shown in Fig. 2.126(a).*

Solution A selected tree of the graph is shown in Fig. 2.126(a).

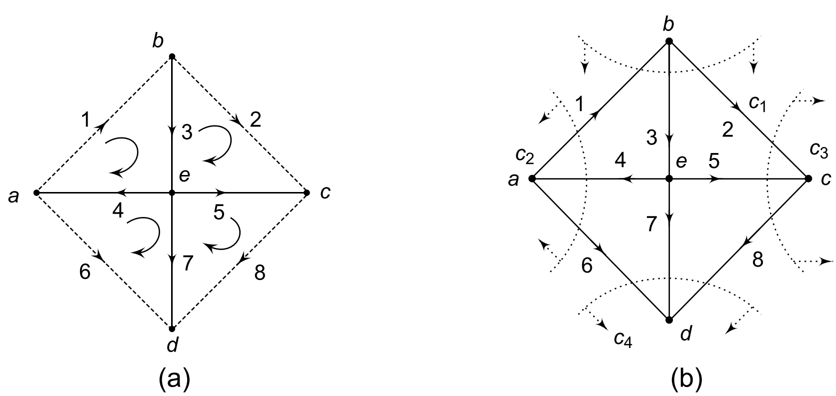


Fig. 2.126

The twigs of the tree are {3, 4, 5, 7}. The remaining branches 1, 2, 6 and 8 are the links, corresponding to the selected tree. Let us consider twig 3. The minimum number of links that must be added to twig 3 to form a cut-set C_1 is {1, 2}. This set is unique for C_1. Thus corresponding to twig 3. The f-cut-set C_1 is {1, 2, 3}. This is shown in Fig. 2.126(b). As a convention the orientation of a cut-set is chosen to coincide with that of its defining twig. Similarly, corresponding to twig 4, the f-cut-set

C_2 is {1, 4, 6} corresponding to twig 5, the *f*-cut-set C_3 is {2, 5, 8} and corresponding to twig 7, the *f*-cut-set is {6, 7, 8}. Thus the *f*-cut-set matrix is given by

$$\begin{array}{cc} f\text{-cut-sets} & \text{branches} \\ Q_f = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} & \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \tag{2.15}$$

2.11.4 Tree Branch Voltages and *f*-Cut-set Matrix

From the cut-set matrix the branch voltages can be expressed in terms of tree branch voltages. Since all tree branches are connected to all the nodes in the graph, it is possible to trace a path from one node to any other node by traversing through the tree-branches.

Let us consider Example 2.69, there are eight branches. Let the branch voltages be $V_1, V_2, \ldots V_8$. There are, four twigs, let the twig voltages be V_{t3}, V_{t4}, V_{t5} and V_{t7} for twigs 3, 4, 5 and 7 respectively.

We can express each branch voltage in terms of twig voltages as follows.

$$\begin{aligned} V_1 &= -V_3 - V_4 = -V_{t3} - V_{t4} \\ V_2 &= +V_3 + V_5 = +V_{t3} + V_{t5} \\ V_3 &= V_{t3} \\ V_4 &= V_{t4} \\ V_5 &= V_{t5} \\ V_6 &= V_7 - V_4 = V_{t7} - V_{t4} \\ V_7 &= V_{t7} \\ V_8 &= V_7 - V_5 = V_{t7} - V_{t5} \end{aligned}$$

The above equations can be written in matrix form as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} V_{t_3} \\ V_{t_4} \\ V_{t_5} \\ V_{t_7} \end{bmatrix} \tag{2.16}$$

The first matrix on the right hand side of Eq. 2.16 is the transpose of the *f*-cut-set matrix Q_f given in Eq. 2.15 in Example. 2.67. Hence, Eq. 2.11 can be written as

$$V_b = Q_f^T V_t. \tag{2.17}$$

where V_b is the column matrix of branch-voltages V_t is the column matrix of twig voltages corresponding to the selected tree and Q_f^T in the transpose of *f*-cut-set matrix.

Equation 2.17 shows that each branch voltage can be expressed as a linear combination of the tree-branch voltages. For this purpose fundamental cut-set (*f*-cut-set) matrix can be used without writing loop equations.

Example 2.70 *Find the fundamental tie-set and cut-set matrix for the graph and for the tree shown in Fig. 2.127.*

[JNTU June 2006]

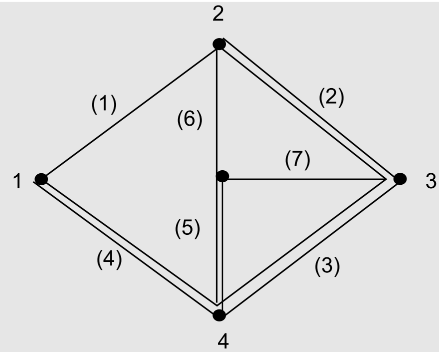


Fig. 2.127

Solution Three are 5 nodes, $n = 5$

There are 7 branches, $b = 7$

No. of twigs or tree branches

$= n - 1 = 4(2, 3, 4, 5)$

No. of link branches

$= b - (n - 1)= 3(1, 6, 7)$

The tie-sets are shown below.

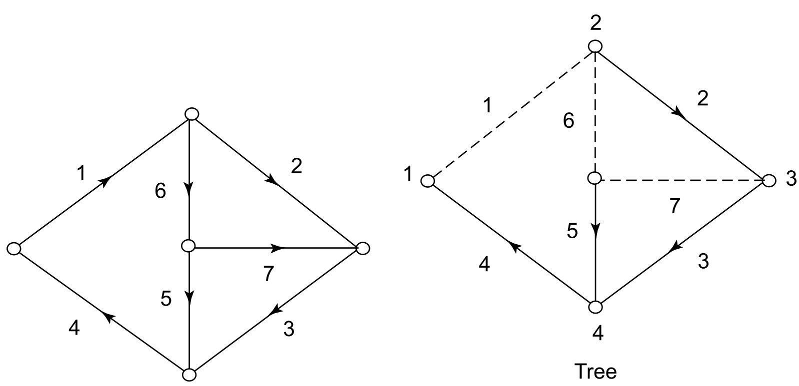


Fig. 2.128(a)

$$V_1 + V_2 + V_3 + V_4 = 0$$
$$V_2 + V_3 - V_5 - V_6 = 0$$
$$V_3 - V_5 + V_7 = 0$$

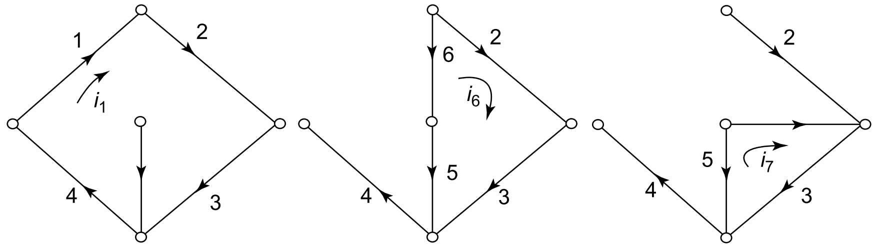


Fig. 2.128(b)

$$\begin{array}{c} \text{Tie-set matrix loop} \\ \downarrow \\ i_1 \\ i_2 \\ i_3 \end{array} \begin{array}{c} \text{branches} \rightarrow \\ \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix}$$

The required tie-set matrix is given by

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Cut-set

For the given tree, there are four fundamental cut-sets each for one twig and is given by

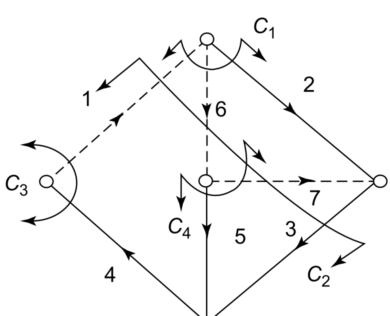


Fig. 2.128(c)

Twig 2; *f*-cut-set [1,2,6]
Twig 3; *f*-cut-set [1, 3, 6, 7]
Twig 4; *f*-cut-set [1, 4]
Twig 5; *f*-cut-set [5, 7]

The cut-sets are formed as shown

f-cut set matrix

$$\begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Example 2.71 *Draw the oriented graph of the network shown in Fig. 2.129 and write the cut-set matrix.* *[JNTU June 2006]*

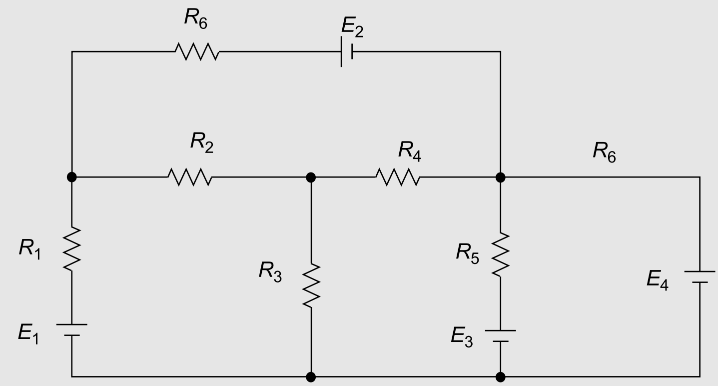


Fig. 2.129

Solution The oriented graph of the network is shown in Fig. 2.130. An arbitrary tree is selected to form fundamental cut-set (f-cut-set) matrix. The tree branches (Twigs) are shown with thick lines and the line branches are shown with dashed lines.

No. of branches = 7

No. of nodes (n) = 4

Twigs = $n - 1$ = 3(2, 3, 6)

No. of links (l) = $b - (n - 1)$ = 4(1, 4, 5, 7)

For twig 2; f-cut-set $C_1 \rightarrow (1, 2, 5)$

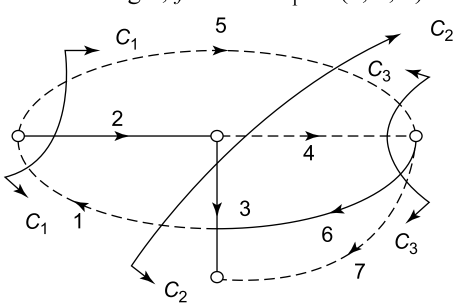


Fig. 2.130

For twig 3; f-cut-set $C_2 \rightarrow (1, 3, 4, 5)$

For twig 6; f-cut-set $C_3 \rightarrow (4, 5, 6, 7)$

Fundamental cut-set matrix

$$
\begin{array}{c}
f\text{-cut-set} \\
\\
C_1 \\
C_2 \\
C_3
\end{array}
\begin{array}{c}
\text{branches} \\
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
-1 & 1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Example 2.72 *Obtain the fundamental loop and fundamental cut-set matrices for the graph shown in Fig. 2.131.*

[JNTU May 2007]

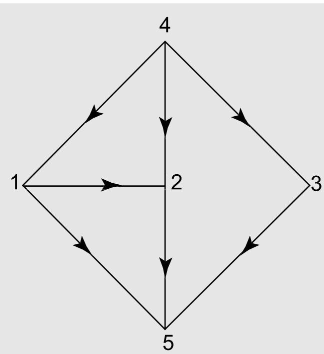
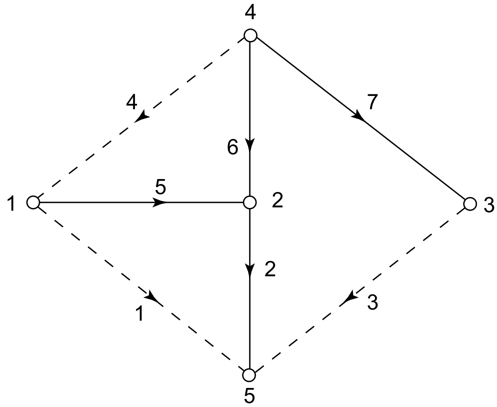


Fig. 2.131

Solution For the given graph, an arbitrary tree is chosen for which the no. of nodes $n = 5$



Tree

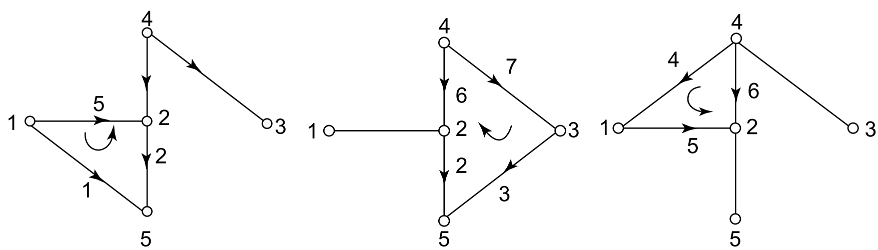
Fig. 2.132(a)

No. of branches $b = 7$

No. of tree branches of twigs $(n - 1) = 4(2, 5, 6, 7)$

No. of link branches $l = b - (n - 1)$
$= 3(1, 3, 4)$

For a given tree of a graph, addition of each link between any two nodes forms a loop called the fundamental loop, (f-loop) or a tie-set. By adding links 1, 3 and 4, we can form three fundamental loops as shown in the figure. By convention, a fundamental loop is marked with the same orientation as its defining link current.



Link 1 **Link 3** **Link 4**

Fig. 2.132(b)

Tie-sets

Tie-set schedule (Fundamental loop matrix)

Link no	*Branch No*						
	1	2	3	4	5	6	7
1	1	−1	0	0	−1	0	0
3	0	−1	1	0	0	−1	1
4	0	0	0	1	1	−1	0

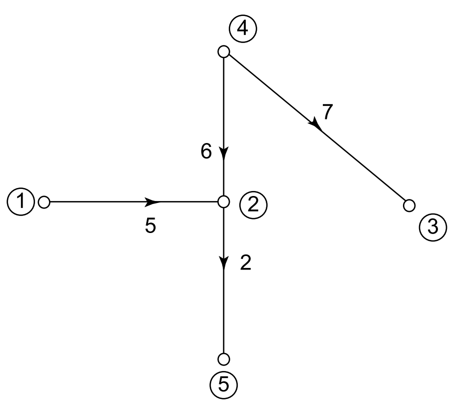


Fig. 2.132(c)

Cut-set

Consider the tree of the graph shown in figure with 5 nodes 1–5 and four tree branches.

The following are the fundamental cut-sets

f-cut-set corresponding to twig 2; $C_1 = \{1, 2, 3\}$

f-cut-set corresponding to twig 5; $C_2 = \{1, 4, 5\}$

f-cut-set corresponding to twig 6; $C_3 = \{3, 4, 6\}$

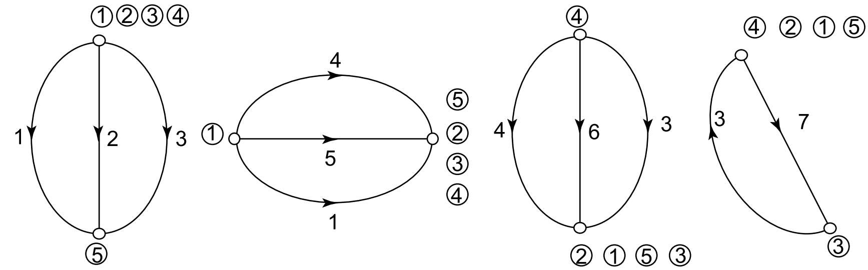


Fig. 2.133

f-cut-set corresponding to twig 7; $C_4 = \{3, 7\}$

Thus, the *f*-cut-set matrix is given by *f*-cut-sets.

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example 2.73 *Obtain the fundamental cut-set matrices for the network shown in Fig. 2.134.* *[JNTU May 2007]*

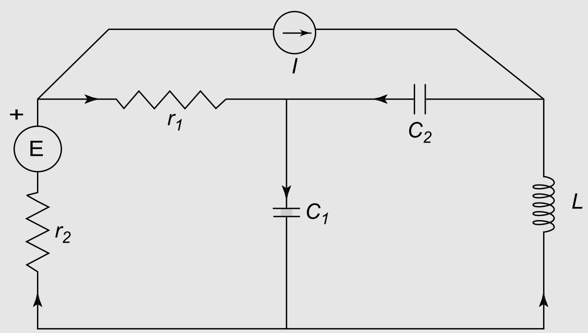


Fig. 2.134

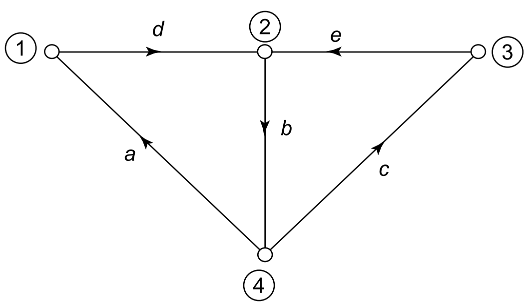
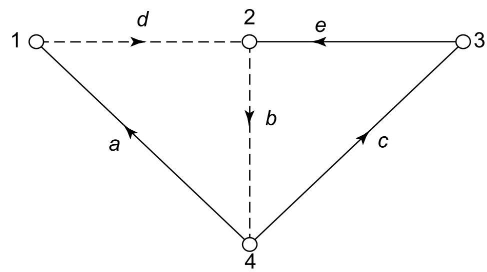


Fig. 2.135

Solution By short circuiting voltage source and open circuiting current source, the oriented graph can be drawn as shown.

The number of nodes are 4 and branches are five. An arbitrary tree is chosen as shown, with twig branches as *a*, *c*, *e* and links as *d* and *b*.



Tree
Fig. 2.136

f-loop matrix

$$\begin{array}{c} \\ \text{links} \\ b \\ d \end{array}\begin{array}{c} \text{branches} \\ \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \end{array}\right] \end{array}$$

The cut-sets are given by

$C_1 = \{a, d\}$

$C_2 = \{b, c, d\}$

$C_3 = \{b, d, e\}$

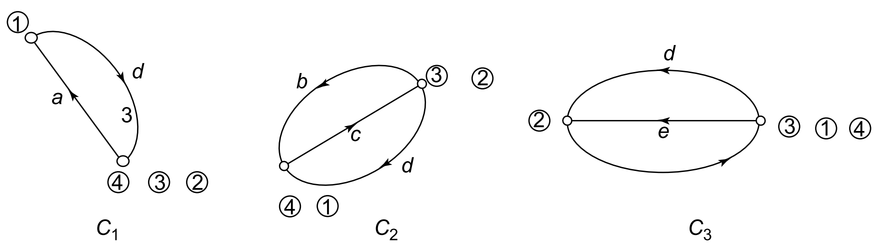


Fig. 2.137

f-cut-set matrix

$$\begin{array}{c} \\ \\ C_1 \\ C_2 \\ C_3 \end{array}\begin{array}{c} \text{branches} \\ \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Example 2.74 *For the given network Fig. 2.138 graph, construct the basic tieset incidence matrix, tracking elements 1, 6, 8, 3 as tree branches. Express the link branch voltage in terms of tree branch voltages*

[JNTU June 2006]

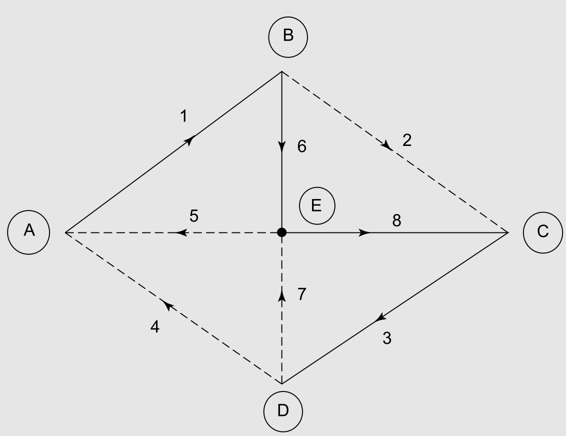


Fig. 2.138

Solution

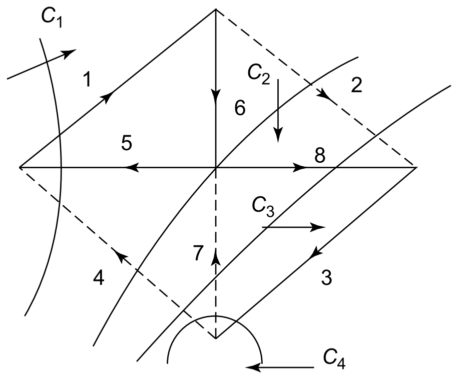


Fig. 2.139

(a) Cut-set incidence matrix is

$$Q = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

The link branch voltage in terms of tree branch voltages is given by

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

(b) Write the tie-set matrix for the graph shown in Fig. 2.140 taking the tree consisting of branches 2, 3, 4.

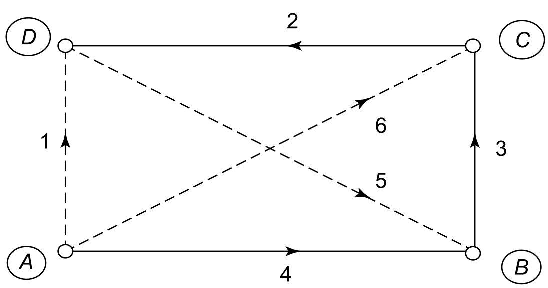


Fig. 2.140

Solution

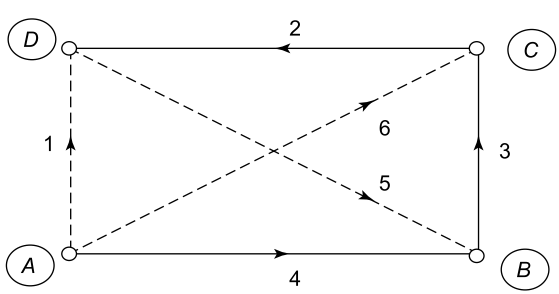


Fig. 2.141

Basic tie-sets \ e	1	2	3	4	5	6
(5, 3, 2)	0	−1	−1	0	1	0
(6, 3, 4)	0	0	−1	−1	0	1
(1, 2, 3, 4)	1	−1	−1	−1	0	0

Example 2.75 *For the given network (Fig. 2.142) graph, construct the basic cutest incidence matrix, tracking elements 1, 6, 8, 3 as tree branches. Express the link branch voltage in terms of tree branch voltages.*

[JNTU May 2007]

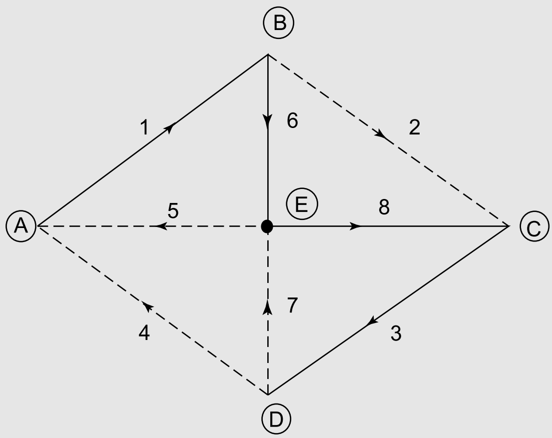


Fig. 2.142

Solution The incidence matrix is given by

Nodes \ Elements	1	2	3	4	5	6	7	8
A	1	0	0	−1	−1	0	0	0
B	−1	1	0	0	0	1	0	0
C	0	−1	1	0	0	0	0	−1
D	0	0	−1	1	0	0	1	0
E	0	0	0	0	1	−1	−1	1

Cut-set matrix is given by

Cut-set Branch \ Elements	1	2	3	4	5	6	7	8
C_1	−1	0	0	1	1	0	0	0
C_2	0	0	1	−1	0	0	−1	0
C_3	0	−1	0	1	1	−1	0	0
C_4	0	1	0	−1	0	0	−1	1

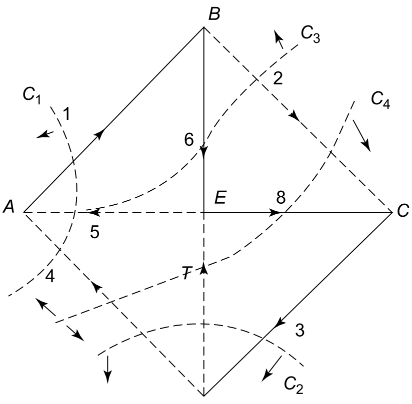


Fig. 2.143

We have $V_1 + V_6 + V_5 = 0$

$\Rightarrow \; V_5 = -(V_1 + V_6)$

$V_2 - V_8 - V_6 = 0 \Rightarrow V_2 = V_6 + V_8$

$V_8 + V_3 + V_7 = 0 \Rightarrow V_7 = -(V_3 + V_8)$

$V_5 + V_7 - V_4 = 0$

$\Rightarrow V_4 = V_5 + V_7 = -V_1 - V_6 - V_3 - V_8$

$V_4 = -(V_1 + V_3 + V_6 + V_8)$

Example 2.76 *For the network shown in Fig. 2.144, draw the oriented graph and draw all possible trees.* *[JNTU June 2008]*

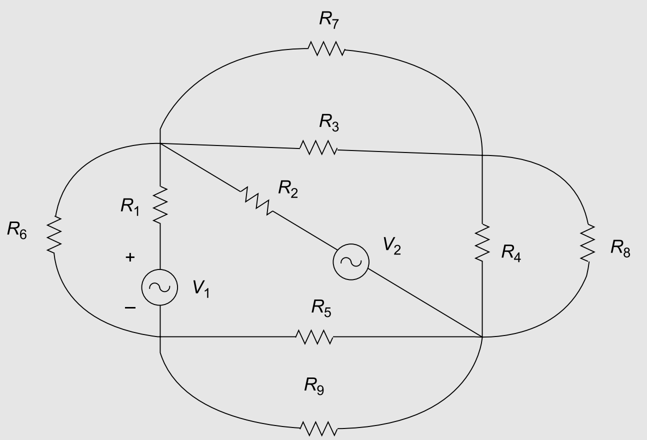


Fig. 2.144

Solution Replace the network with a graph. The voltage sources have been short-circuited.

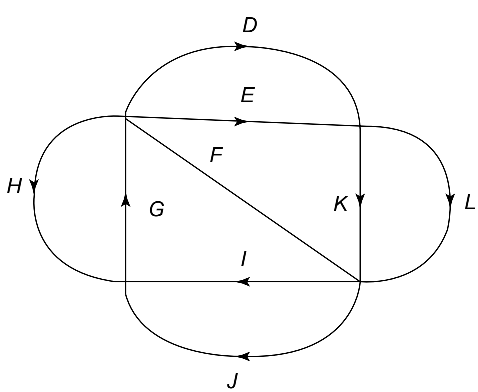


Fig. 2.145

Some possible trees are

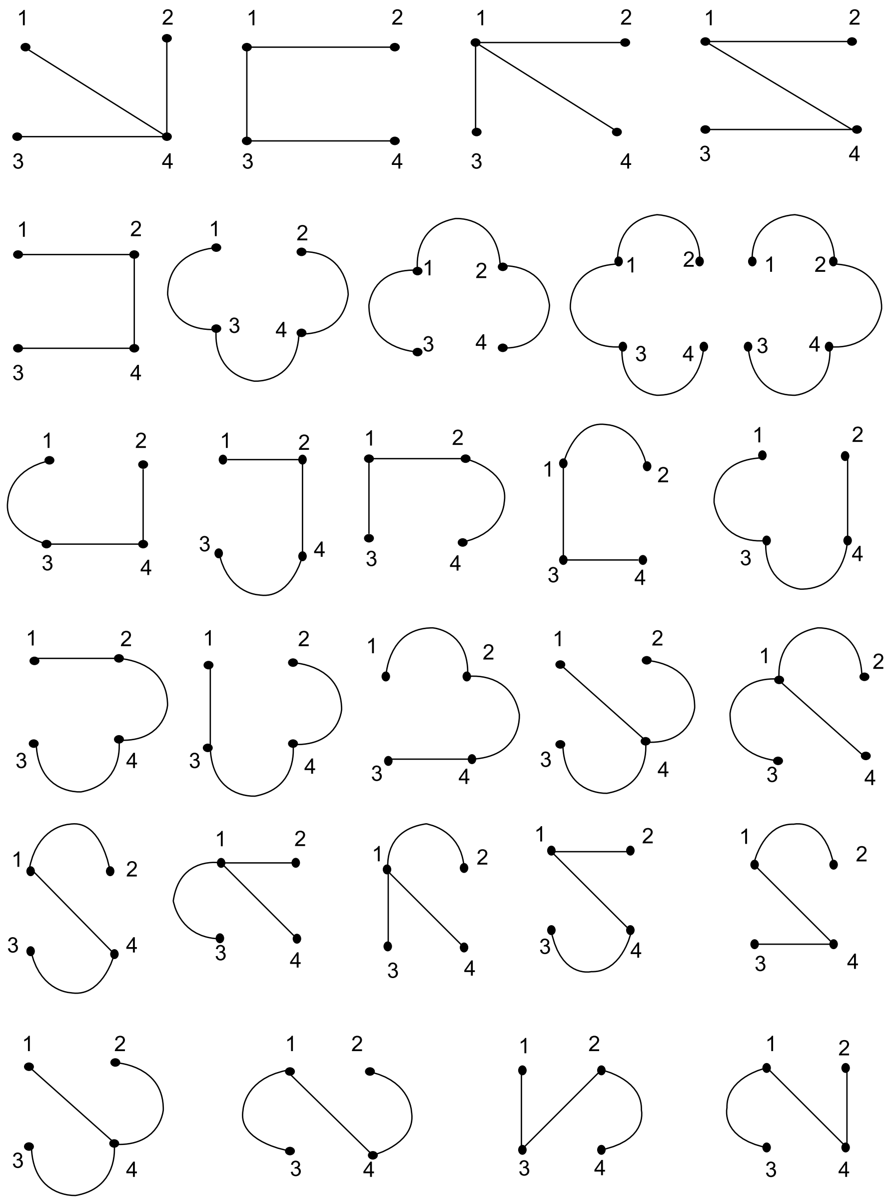


Fig. 2.146

Example 2.77 *For the graph in Fig. 2.147, write the cut-set schedule and obtain the relation between tree branch voltages and branch voltages.*

[JNTU June 2008]

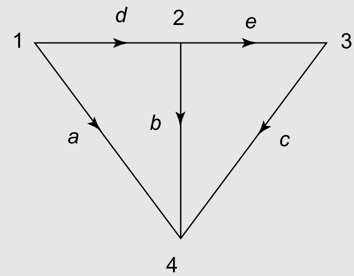


Fig. 2.147

Solution 6 cut-sets are possible for the graph.

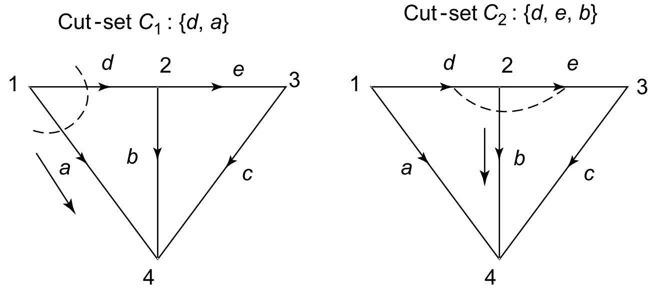


Fig. 2.148(a)

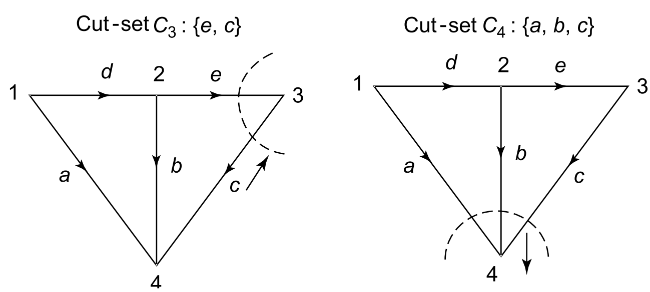


Fig. 2.148(b)

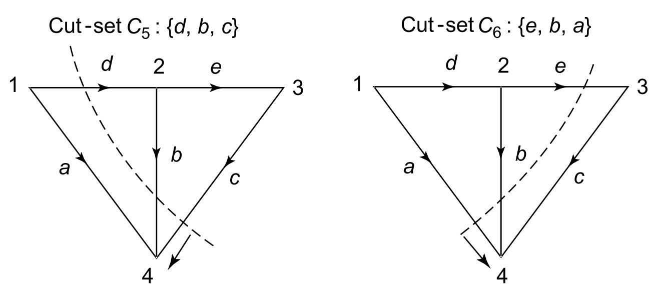


Fig. 2.148(c)

Applying KCL for each cut-set we get the following equations.

$$C_1: i_d + i_a = 0$$

$$C_2: i_b + i_e - i_d = 0$$

$$C_3: i_e - i_c = 0$$

C_4: $i_a + i_b + i_c = 0$

C_5: $i_b + i_e - i_d = 0$

C_6: $i_a + i_b + i_e = 0.$

The equations can be put into matrix form.

$$Q_f = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_d \\ i_e \\ i_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to find the relation between branch voltage and tree branch voltage. Let us consider the tree.

There are 5 branches. Let the branch voltages be V_a, V_b, V_c,V_d and V_e.

There are 3 twigs the twig–voltages be V_{td}, V_{te}, V_{tb}.

We can express branch voltages in terms of twig voltages.

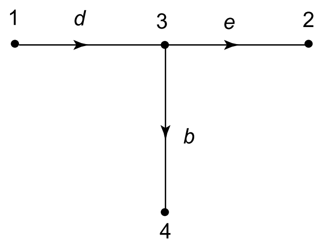


Fig. 2.149(a)

$$V_d + V_b = V_a \qquad V_a = V_{td} + V_{tb}$$

$$V_e + V_c = V_b \qquad V_c = V_{tb} - V_{te}$$

$$V_b = V_{tb}$$

$$V_d = V_{td}$$

$$V_e = V_{te}$$

The above equations can be written in matrix form as

$$V_b = Q_f^T V_t$$

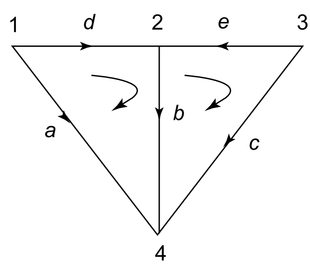


Fig. 2.149(b)

$$\begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{bmatrix} = V_b \qquad \begin{bmatrix} V_{td} \\ V_{te} \\ V_{tb} \end{bmatrix} = V_t$$

V_b is column matrix of branch–voltages and V_t is column matrix of twig voltages.

∴ The relation can be expressed as

$$\begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_{td} \\ V_{te} \\ V_{tb} \end{bmatrix}$$

Example 2.78 *For the graph shown in Fig. 2.150, find the tie-set and cut-set matrices.*

[JNTU June 2009]

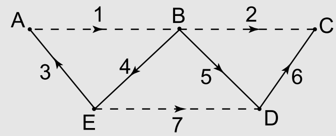


Fig. 2.150

Solution

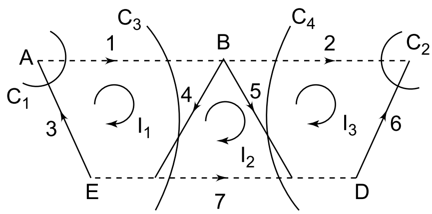


Fig. 2.151

Tie-set matrix

$$\begin{array}{c} \text{Loop currents} \\ \downarrow \\ \\ I_1 \\ I_2 \\ I_3 \end{array} \quad \begin{array}{c} \text{Branches} \rightarrow \\ \\ \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Cut-set matrix

$$\begin{array}{c} \text{Cut-sets} \\ \downarrow \\ \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \quad \begin{array}{c} \text{Branches} \rightarrow \\ \\ \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Example 2.79 *Obtain the cut-set matrix for the given network, as shown in Fig. 2.152.*

[JNTU Jan 2010]

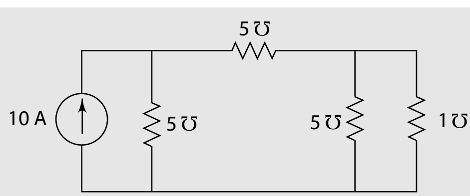


Fig. 2.152

Solution Cut-set matrix

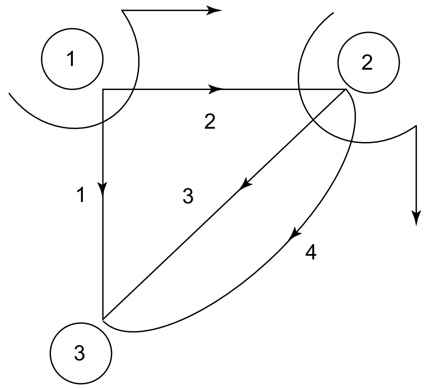


Fig. 2.153

$$
\begin{array}{cc}
\text{Cut-sets} & \text{Branches} \rightarrow \\
\downarrow & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}
\end{array}
$$

Example 2.80 *Find the cut-set matrix of the network as shown in Fig. 2.154 and obtain relationship between the branch currents.*

[JNTU Jan 2010]

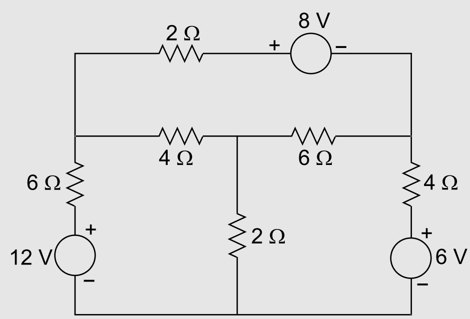


Fig. 2.154

Solution

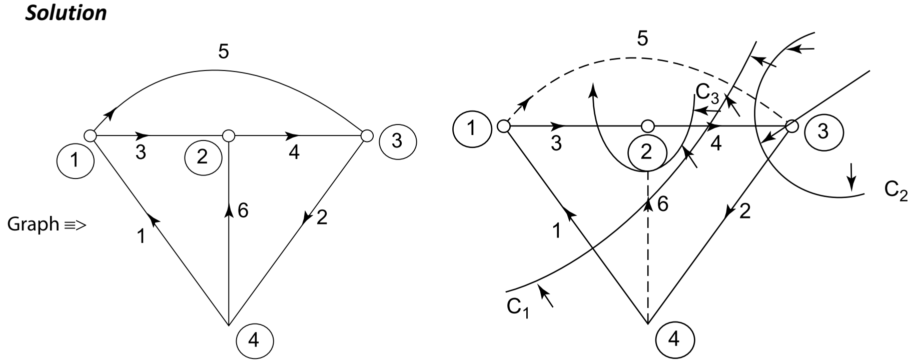


Fig. 2.155

$$\begin{array}{c|c}
\text{Cut-set} & \text{Branches} \rightarrow \\
\downarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} C_1(1,6,4) \\ C_2(2,4,5) \\ C_3(3,6,4) \end{array} & \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & +1 \end{bmatrix}
\end{array}$$

KCL equations

$$i_1 - i_4 + i_6 = 0$$
$$i_2 - i_4 - i_5 = 0$$
$$i_3 - i_4 + i_6 = 0$$

Example 2.81 *Obtain the cut-set matrix for the network as shown in Fig. 2.156.* *[JNTU Jan 2010]*

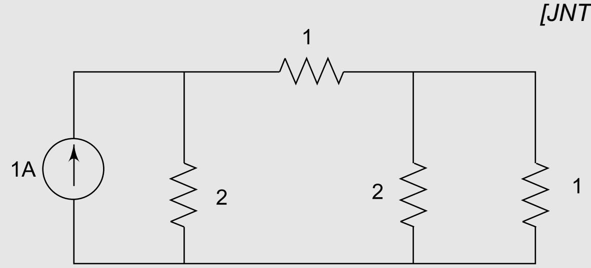


Fig. 2.156

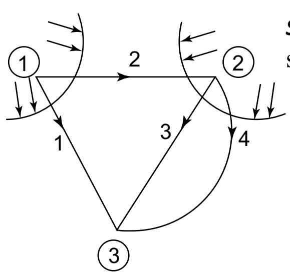


Fig. 2.157

Solution Graph obtained by open circuiting current source.

Fundamental cut-set matrix is formed as

$$\begin{array}{c|c}
\text{Cut-sets} & \text{Branches} \rightarrow \\
\downarrow & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}
\end{array}$$

Practice Problems

2.1 Calculate the frequency of the following values of period.

(a) 0.2 s (b) 50 ms
(c) 500 μs (d) 10 μs

2.2 Calculate the period for each of the values of frequency.

(a) 60 Hz (b) 500 Hz
(c) 1 kHz (d) 200 kHz
(e) 5 MHz

2.3 A certain sine wave has a positive going zero crossing at 0° and an rms value of 20 V. Calculate its instantaneous value at each of the following angles.

(a) 33° (b) 110°
(c) 145° (d) 325°

2.4 For a particular 0° reference sinusoidal current, the peak value is 200 mA; determine the instantaneous values at each of the following.

(a) 35° (b) 190°
(c) 200° (d) 360°

2.5 Sine wave *A* lags sine wave *B* by 30°. Both have peak values of 15 V. Sine wave *A* is the reference with a positive going crossing at 0°. Determine the instantaneous value of sine wave *B* at 30°, 90°, 45°, 180° and 300°.

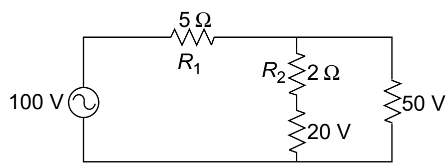


Fig. 2.158

2.6 Find the average values of the voltages across R_1 and R_2. In Fig. 2.158 values shown are rms.

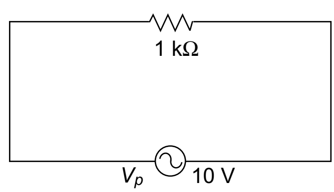


Fig. 2.159

2.7 A sinusoidal voltage is applied to the circuit shown in Fig. 2.159, determine rms current, average current, peak current, and peak to peak current.

2.8 A sinusoidal voltage of $v(t) = 50 \sin (500t)$ applied to a capacitive circuit. Determine the capacitive reactance, and the current in the circuit.

2.9 A sinusoidal voltage source in series with a dc source is shown in Fig. 2.160.

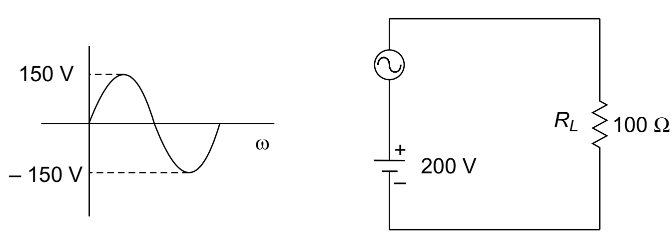


Fig. 2.160

Sketch the voltage across R_L. Determine the maximum current through R_L and the average voltage across R_L.

2.10 Find the effective value of the resultant current in a wire which carries a direct current of 10 A and a sinusoidal current with a peak value of 15 A.

2.11 An alternating current varying sinusoidally, with a frequency of 50 Hz, has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (a) 0.0025 s (b) 0.0125 s after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A?

2.12 Determine the rms value of the voltage defined by

$$v = 5 + 5 \sin(314t + \pi/6)$$

2.13 Find the effective value of the function $v = 100 + 50 \sin \omega t$.

2.14 A full wave rectified sine wave is clipped at 0.707 of its maximum value as shown in Fig. 2.161. Find the average and effective values of the function.

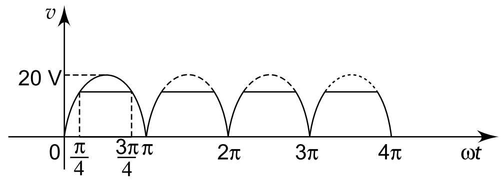


Fig. 2.161

2.15 Find the rms value of the function shown in Fig. 2.162 and described as follows

$0 < t < 0.1 \quad v = 40\,(1 - e^{-100t})$

$0.1 < t < 0.2 \quad v = 40\, e^{-50(t\,-\,0.1)}$

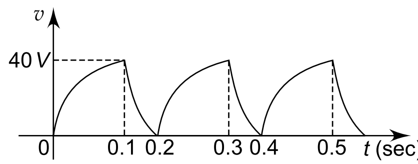


Fig. 2.162

2.16 Calculate average and effective values of the waveform shown in Fig. 2.163 and hence find from factor.

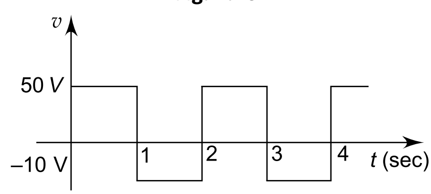


Fig. 2.163

2.17 A full wave rectified sine wave is clipped such that the effective value is 0.5 V_m as shown in Fig. 2.164. Determine the amplitude at which the wave form is clipped.

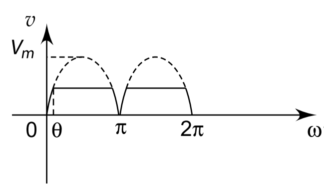


Fig. 2.164

2.18 A delayed full wave rectified sine wave has an average value of half the maximum value as shown in Fig. 2.165. Find the angle θ.

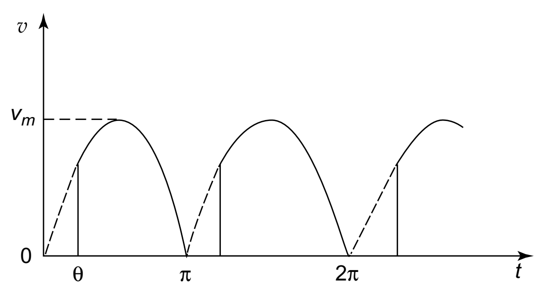


Fig. 2.165

Objective Type Questions

2.1 One sine wave has a period of 2 ms, another has a period of 5 ms, and other has a period of 10 ms. Which sine wave is changing at a faster rate?

(a) sine wave with period of 2 ms (b) sine wave with period of 5 ms
(c) all are at the same rate (d) sine wave with period of 10 ms

2.2 How many cycles does a sine wave go through in 10 s when its frequency is 60 Hz?

(a) 10 cycles (b) 60 cycles (c) 600 cycles (d) 6 cycles

2.3 If the peak value of a certain sine wave voltage is 10 V, what is the peak to peak value?

(a) 20 V (b) 10 V (c) 5 V (d) 7.07 V

2.4 If the peak value of a certain sine wave voltage is 5 V, what is the rms value?

(a) 0.707 V (b) 3.535 V (c) 5 V (d) 1.17 V

2.5 What is the average value of a sine wave over a full cycle?

(a) V_m (b) $\dfrac{V_m}{\sqrt{2}}$ (c) zero (d) $\sqrt{2}V_m$

2.6 A sinusoidal current has peak value of 12 A. What is its average value?

(a) 7.64 A (b) 24 A (c) 8.48 A (d) 12 A

2.7 Sine wave *A* has a positive going zero crossing at 30°. Sine wave *B* has a positive going zero crossing at 45°. What is the phase angle between two signals?

(a) 30° (b) 45° (c) 75° (d) 15°

2.8 A sine wave has a positive going zero crossing at 0° and an rms value of 20 V. What is its instantaneous value at 145°?

(a) 7.32 V (b) 16.22 V (c) 26.57 V (d) 21.66 V

2.9 In a pure resistor, the voltage and current are

(a) out of phase (b) in phase
(c) 90° out of phase (d) 45° out of phase

2.10 The rms current through a 10 kΩ resistor is 5 mA. What is the rms voltage drop across the resistor?

(a) 10 V (b) 5 V (c) 50 V (d) zero

2.11 In a pure capacitor, the voltage

(a) is in phase with the current (b) is out of phase with the current
(c) lags behind the current by 90° (d) leads the current by 90°

2.12 A sine wave voltage is applied across a capacitor; when the frequency of the voltage is increased, the current

(a) increases (b) decreases (c) remains the same (d) is zero

2.13 The current in a pure inductor

(a) lags behind the voltage by 90° (b) leads the voltage by 90°
(c) is in phase with the voltage (d) lags behind the voltage by 45°

2.14 A sine wave voltage is applied across an inductor; when the frequency of voltage is increased, the current

(a) increases (b) decreases (c) remains the same (d) is zero

2.15 The rms value of the voltage for a voltage function $v = 10 + 5 \cos(628t + 30°)$ volts through a circuit is

(a) 5 V (b) 10 V (c) 10.6 V (d) 15 V

2.16 For the same peak value, which is of the following wave will have the highest rms value

(a) sine wave (b) square wave
(c) triangular wave (d) half wave rectified sine wave

2.17 For 100 volts rms value triangular wave, the peak voltage will be

(a) 100 V (b) 111 V (c) 141 V (d) 173 V

2.18 The form factor of dc voltage is

(a) zero (b) infinite (c) unity (d) 0.5

2.19 For the half wave rectified sine wave shown in Fig. 2.166, the peak factor is

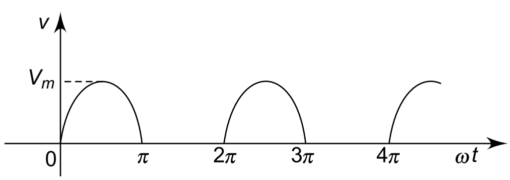


Fig. 2.166

(a) 1.41 (b) 2.0 (c) 2.82 (d) infinite

2.20 For the square wave shown in Fig. 2.167, the form factor is

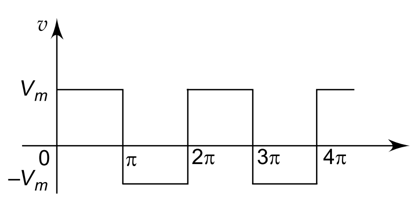


Fig. 2.167

(a) 2.0 (b) 1.0 (c) 0.5 (d) zero

2.21 The power consumed in a circuit element will be least when the phase difference between the current and voltage is

(a) 0° (b) 30° (c) 90° (d) 180°

2.22 The voltage wave consists of two components: A 50 V dc component and a sinusoidal component with a maximum value of 50 volts. The average value of the resultant will be

(a) zero (b) 86.6 V (c) 50 (d) none of the above

2.23 A tree has

(a) a closed path (b) no closed paths
(c) none

2.24 The number of branches in a tree is_______the number of branches in a graph.

(a) less than (b) more than
(c) equal to

2.25 The tie-set schedule gives the relation between

(a) branch currents and link currents
(b) branch voltages and link currents
(c) branch currents and link voltages
(d) none of the above

2.26 The cut-set schedule gives the relation between

(a) branch currents and link currents
(b) branch voltages and tree branch voltages
(c) branch voltages and link voltages
(d) branch current and tree currents

2.27 Mesh analysis is based on

(a) Kirchhoff's current law (b) Kirchhoff's voltage law
(c) Both (d) None

2.28 If a network contains B branches, and N nodes, then the number of mesh current equations would be

(a) $B - (N - 1)$ (b) $N - (B - 1)$
(c) $B - N - 1$ (d) $(B + N) - 1$

2.29 A network has seven nodes and five independent loops. The number of branches in the network is

(a) 13 (b) 12 (c) 11 (d) 10

2.30 The nodal method of circuit analysis is based on

(a) KVL and Ohm's law (b) KCL and Ohm's law
(c) KCL and KVL (d) KCL, KVL and Ohm's law

2.31 The number of independent loops for a network with n nodes and b branches is

(a) $n - 1$
(b) $b - n$

(c) $b - n + 1$
(d) independent of the number of nodes

2.32 Relative to a given fixed tree of a network

(a) link currents form an independent set
(b) branch currents form an independent set
(c) link voltages form an independent set
(d) branch voltages form an independent set

2.33 The number of independent loops for a network with 3 nodes and 6 branches is

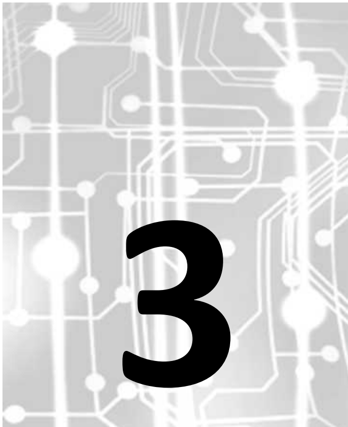
(a) 2 (b) 1 (c) 4 (d) 6

2.34 A circuit consists of two resistances, 4 Ω and 4 Ω in parallel. The total current passing through the circuit is 10 A. The current passing through R_1 is

(a) 5 A (b) 10 A (c) 4 A (d) 2 A

2.35 A network has eight nodes and five independent loops. The number of branches in the network is

(a) 13 (b) 11 (c) 12 (d) 15



3 Steady State Analysis of AC Circuits

3.1 RESPONSE TO SINUSOIDAL EXCITATION

3.1.1 Pure Resistance

When a sinusoidal voltage of certain magnitude is applied to a resistor, a certain amount of sine wave current passes through it. We know the relation between $v(t)$ and $i(t)$ in the case of a resistor. The voltage/current relation in case of a resistor is linear,

i.e. $$v(t) = i(t)R$$

Consider the function

$$i(t) = I_m \sin \omega t = IM\left[I_m e^{j\omega t}\right] \text{ or } I_m \angle 0°$$

If we substitute this in the above equation, we have

$$v(t) = I_m R \sin \omega t = V_m \sin \omega t$$

$$= IM\left[V_m e^{j\omega t}\right] \text{ or } V_m \angle 0°$$

where $$V_m = I_m R$$

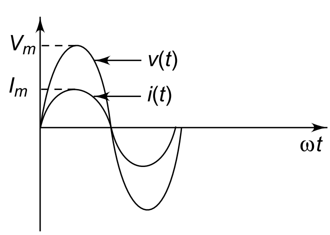


Fig. 3.1

If we draw the waveform for both voltage and current as shown in Fig. 3.1, there is no phase difference between these two waveforms. The amplitudes of the waveform may differ according to the value of resistance.

As a result, in pure resistive circuits, the voltages and currents are said to be in phase. Here the term impedance is defined as the ratio of voltage to current function. With ac voltage applied to elements, the ratio of exponential voltage to the corresponding current (impedance) consists of magnitude and phase angles. Since the phase difference is zero in case of a resistor, the phase angle is zero. The impedance in case of resistor consists only of magnitude, i.e.

$$Z = \frac{V_m \angle 0^\circ}{I_m \angle 0^\circ} = R$$

Example 3.1 *A sinusoidal voltage is applied to the resistive circuit shown in Fig. 3.2. Determine the following values.*

(a) I_{rms} *(b)* I_{av} *(c)* I_P *(d)* I_{PP}

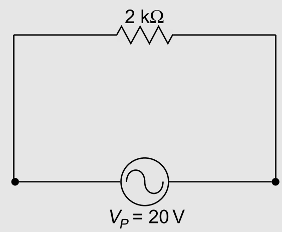


Fig. 3.2

Solution The function given to the circuit shown is

$$v(t) = V_P \sin \omega t = 20 \sin \omega t$$

The current passing through the resistor

$$i(t) = \frac{v(t)}{R}$$

$$i(t) = \frac{20}{2 \times 10^3} \sin \omega t$$

$$= 10 \times 10^{-3} \sin \omega t$$

$$I_p = 10 \times 10^{-3} \text{ A}$$

The peak value $I_P = 10$ mA

Peak to peak value $I_{PP} = 20$ mA

rms value $I_{\text{rms}} = 0.707\, I_P$

$= 0.707 \times 10 \text{ mA} = 7.07$ mA

Average value $I_{\text{av}} = (0.637)\, I_P$

$= 0.637 \times 10 \text{ mA} = 6.37$ mA

3.1.2 Pure Inductance

As discussed earlier in Chapter 1, the voltage current relation in the case of an inductor is given by

$$v(t) = L\frac{di}{dt}$$

Consider the function $i(t) = I_m \sin \omega t = IM\left[I_m e^{j\omega t}\right]$ or $I_m \angle 0°$

$$v(t) = L\frac{d}{dt}(I_m \sin \omega t)$$

$$= L\omega I_m \cos \omega t = \omega L I_m \cos \omega t$$

$$v(t) = V_m \cos \omega t, \quad \text{or} \quad V_m \sin(\omega t + 90°)$$

$$= IM\left[V_m e^{j(\omega t + 90°)}\right] \quad \text{or} \quad V_m \angle 90°$$

where $$V_m = \omega L\, I_m = X_L I_m$$

and $$e^{j90°} = j = 1\angle 90°$$

If we draw the waveforms for both, voltage and current, as shown in Fig. 3.3, we can observe the phase difference between these two waveforms.

As a result, in a pure inductor the voltage and current are out of phase. The current lags behind the voltage by 90° in a pure inductor as shown in Fig. 3.3.

The impedance which is the ratio of exponential voltage to the corresponding current, is given by

$$Z = \frac{V_m \sin(\omega t + 90°)}{I_m \sin \omega t}$$

where $V_m = \omega L\, I_m$

$$= \frac{I_m \omega L \sin(\omega t + 90°)}{I_m \sin \omega t}$$

$$= \frac{\omega L\, I_m \angle 90°}{I_m \angle 0°}$$

$\therefore$ $$Z = j\omega L = jX_L$$

where $X_L = \omega L$ and is called the inductive reactance.

Hence, a pure inductor has an impedance whose value is ωL.

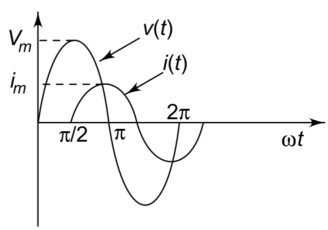


Fig. 3.3

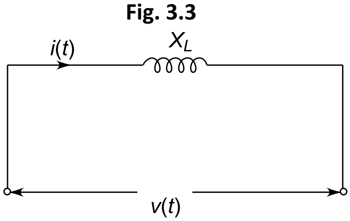


Fig. 3.4

Example 3.2 *A sinusoidal voltage is applied to the circuit shown in Fig. 3.5. The frequency is 3 kHz. Determine the inductive reactance.*

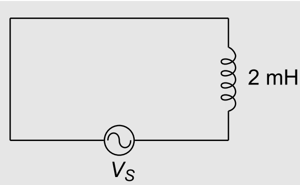


Fig. 3.5

Solution $X_L = 2\pi f L$

$$= 2\pi \times 3 \times 10^3 \times 2 \times 10^{-3}$$

$$= 37.69\ \Omega$$

Example 3.3 *Determine the rms current in the circuit shown in Fig. 3.6.*

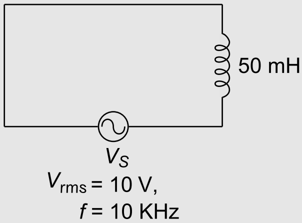


Fig. 3.6

Solution

$$X_L = 2\pi f L$$
$$= 2\pi \times 10 \times 10^3 \times 50 \times 10^{-3}$$
$$X_L = 3.141 \text{ k}\Omega$$
$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L}$$
$$= \frac{10}{3.141 \times 10^3} = 3.18 \text{ mA}$$

3.1.3 Pure Capacitance

As discussed in Chapter 1, the relation between voltage and current is given by

$$v(t) = \frac{1}{C}\int i(t)\,dt$$

Consider the function $i(t) = I_m \sin\omega t = IM\left[I_m e^{j\omega t}\right]$ or $I_m \angle 0°$

$$v(t) = \frac{1}{C}\int I_m \sin\omega t \, d(t)$$
$$= \frac{1}{\omega C} I_m[-\cos\omega t]$$
$$= \frac{I_m}{\omega C}\sin(\omega t - 90°)$$

$$\therefore \quad v(t) = V_m \sin(\omega t - 90°)$$
$$= IM\left[I_m e^{j(\omega t - 90°)}\right] \text{ or } V_m \angle -90°$$

where $$V_m = \frac{I_m}{\omega C}$$

$$\therefore \quad \frac{V_m \angle -90°}{I_m \angle 0°} = Z = \frac{-j}{\omega C}$$

Hence, the impedance is $Z = \dfrac{-j}{\omega C} = -jX_C$

where $X_C = \dfrac{1}{\omega C}$ and is called the capacitive reactance.

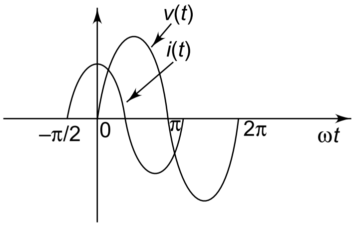


Fig. 3.7

If we draw the waveform for both, voltage and current, as shown in Fig. 3.7, there is a phase difference between these two waveforms.

As a result, in a pure capacitor, the current leads the voltage by 90°. The impedance value of a pure capacitor

$$X_C = \frac{1}{\omega C}$$

Example 3.4 *A sinusoidal voltage is applied to a capacitor as shown in Fig. 3.8. The frequency of the sine wave is 2 kHz. Determine the capacitive reactance.*

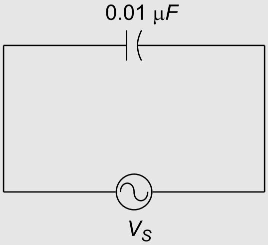


Fig. 3.8

Solution

$$X_C = \frac{1}{2\pi f C}$$

$$= \frac{1}{2\pi \times 2 \times 10^3 \times 0.01 \times 10^{-6}}$$

$$= 7.96 \text{ k}\Omega$$

Example 3.5 *Determine the rms current in the circuit shown in Fig. 3.9.*

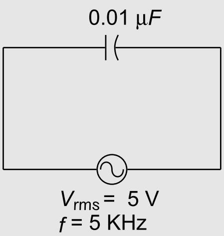


Fig. 3.9

Solution

$$X_C = \frac{1}{2\pi f C}$$

$$= \frac{1}{2\pi \times 5 \times 10^3 \times 0.01 \times 10^{-6}}$$

$$= 3.18 \text{ k}\Omega$$

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{5}{3.18 \text{ K}} = 1.57 \text{ mA}$$

3.2 IMPEDANCE CONCEPT AND PHASE ANGLE

So far our discussion has been confined to resistive circuits. Resistance restricts the flow of current by opposing free electron movement. Each element has some resistance; for example, an inductor has some resistance; a capacitance also has some resistance. In the resistive element, there is no phase difference between the voltage and the current. In the case of pure inductance, the current lags behind the voltage by 90 degrees, whereas in the case of pure capacitance, the current leads the voltage by 90 degrees. Almost all electric circuits offer impedance to the flow of current. Impedance is a complex quantity having real and imaginary parts; where the real part is the resistance and the imaginary part is the reactance of the circuit.

Consider the *RL* series circuit shown in Fig. 3.10. If we apply the real function $V_m \cos \omega t$ to the circuit, the response may be $I_m \cos \omega t$. Similarly, if we apply the imaginary function $jV_m \sin \omega t$ to the same circuit, the response is $jI_m \sin \omega t$. If we apply a complex function, which is a combination of real and imaginary functions, we will get a complex response.

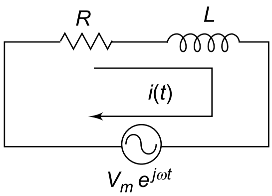


Fig. 3.10

This complex function is $V_m e^{j\omega t} = V_m (\cos \omega t + j \sin \omega t)$. Applying Kirchhoff's law to the circuit shown in Fig. 3.10,

we get $$V_m e^{j\omega t} = Ri(t) + L\frac{di}{dt}$$

The solution of this differential equation is

$$i(t) = I_m e^{j\omega t}$$

By substituting $i(t)$ in the above equation, we get

$$V_m e^{j\omega t} = RI_m e^{j\omega t} + L\frac{d}{dt}(I_m e^{j\omega t})$$

$$V_m e^{j\omega t} = RI_m e^{j\omega t} + LI_m j\omega e^{j\omega t}$$

$$V_m = (R + j\omega L)I_m$$

Impedance is defined as the ratio of the voltage to current function

$$Z = \frac{V_m e^{j\omega t}}{\frac{V_m}{R + j\omega L} e^{j\omega t}} = R + j\omega L$$

Complex impedance is the total opposition offered by the circuit elements to ac current, and can be displayed on the complex plane. The impedance is denoted by Z. Here the resistance R is the real part of the impedance, and the reactance X_L is the imaginary part of the impedance. The resistance R is located on the real axis. The inductive reactance X_L is located on the positive j axis. The resultant of R and X_L is called the complex impedance.

Figure 3.11 is called the impedance diagram for the *RL* circuit. From Fig. 3.11, the impedance $Z = \sqrt{R^2 + (\omega L)^2}$, and angle $\theta = \tan^{-1} \omega L/R$. Here, the impedance is the vector sum of the resistance and inductive reactance. The angle between impedance and resistance is the phase angle between the current and voltage applied to the circuit.

Similarly, if we consider the RC series circuit, and apply the complex function $V_m e^{j\omega t}$ to the circuit in Fig. 3.12, we get a complex response as follows.

Applying Kirchhoff's law to the above circuit, we get

$$V_m e^{j\omega t} = Ri(t) + \frac{1}{C}\int i(t)\,dt$$

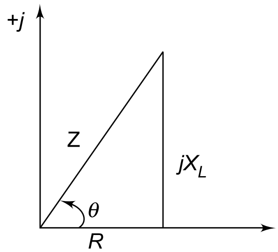


Fig. 3.11

Solving this equation we get,

$$i(t) = I_m e^{j\omega t}$$

$$V_m e^{j\omega t} = R I_m e^{j\omega t} + \frac{1}{C} I_m \left(\frac{+1}{j\omega}\right) e^{j\omega t}$$

$$= \left[RI_m - \frac{j}{\omega C} I_m\right] e^{j\omega t}$$

$$V_m = \left(R - \frac{j}{\omega C}\right) I_m$$

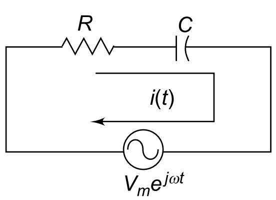


Fig. 3.12

The impedance

$$Z = \frac{V_m e^{j\omega t}}{V_m / [R - j/\omega C] e^{j\omega t}}$$

$$= [R - (j/\omega C)]$$

Here impedance Z consists of resistance (R), which is the real part, and capacitive reactance ($X_C = 1/\omega C$), which is the imaginary part of the impedance. The resistance, R, is located on the real axis, and the capacitive reactance X_C is located on the negative j axis in the impedance diagram in Fig. 3.13.

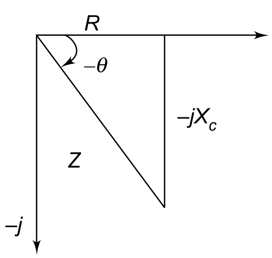


Fig. 3.13

From Fig. 3.13, impedance $Z = \sqrt{R^2 + X_C^2}$ or $\sqrt{R^2 + (1/\omega C)^2}$ and angle $\theta = \tan^{-1}(1/\omega CR)$. Here, the impedance, Z, is the vector sum of resistance and capacitive reactance. The angle between resistance and impedance is the phase angle between the applied voltage and current in the circuit.

The impedance, Z is composed of real and imaginary parts

$$Z = R + jX$$

where R is the resistance, measured in ohms
X is the reactance, measured in ohms
The admittance (Y) is the inverse of the impedance (Z).

$$Y = Z^{-1} = \frac{1}{Z}$$

where Y is the admittance, measured in Siemens.
Admittance is a measure of how easily a circuit will allow a current to flow.

$$Y = Z^{-1} = \frac{1}{R + jX} = \left(\frac{R}{R^2 + X^2}\right) + j\left(\frac{-X}{R^2 + X^2}\right)$$

Admittance is a complex number

$$Y = G + jB$$

where G (conductance) and B (susceptance) are given by

$$G = \frac{R}{R^2 + X^2}$$

$$B = \frac{-X}{R^2 + X^2}$$

The magnitude and phase of the admittance are given by

$$|Y| = \sqrt{G^2 + B^2} = \frac{1}{\sqrt{R^2 + X^2}}$$

$$\angle Y = \arctan\left(\frac{B}{G}\right) = \arctan\left(\frac{-X}{R}\right)$$

where G is the conductance, measured in Siemens.
where B is the susceptance, measured in Siemens.

3.3 SERIES RL, RC, RLC CIRCUITS *[JNTU Nov 2011]*

The impedance diagram is a useful tool for analysing series ac circuits. Basically we can divide the series circuits as RL, RC and RLC circuits. In the analysis of series ac circuits, one must draw the impedance diagram. Although the impedance diagram usually is not drawn to scale, it does represent a clear picture of the phase relationships.

If we apply a sinusoidal input to an RL circuit, the current in the circuit and all voltages across the elements are sinusoidal. In the analysis of the RL series circuit, we can find the impedance, current, phase angle and voltage drops. In Fig. 3.14(a) the resistor voltage (V_R) and current (I) are in phase with each other, but lag behind the source voltage (V_S). The inductor voltage (V_L) leads the source voltage (V_S). The phase angle between current and voltage in a pure inductor is always 90°. The amplitudes of voltages and currents in the circuit are completely dependent on the values of elements (i.e. the resistance and inductive reactance). In the circuit shown, the phase angle is somewhere between zero and 90° because of the series combination of resistance with inductive reactance, which depends on the relative values of R and X_L.

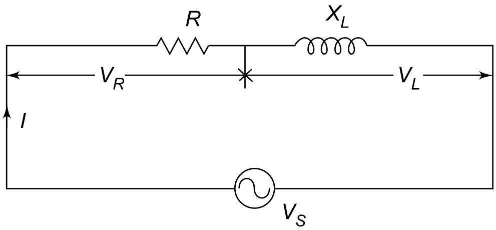


Fig. 3.14(a)

The phase relation between current and voltages in a series RL circuit is shown in Fig. 3.14(b).

Here V_R and I are in phase. The amplitudes are arbitrarily chosen. From Kirchhoff's voltage law, the sum of the voltage drops must equal the applied voltage. Therefore, the source voltage V_S is the phasor sum of V_R and V_L.

$$\therefore V_S = \sqrt{V_R^2 + V_L^2}$$

The phase angle between resistor voltage and source voltage is

$$\theta = \tan^{-1} (V_L/V_R)$$

where θ is also the phase angle between the source voltage and the current. The phasor diagram for the series RL circuit that represents the waveforms in Fig. 3.14(c).

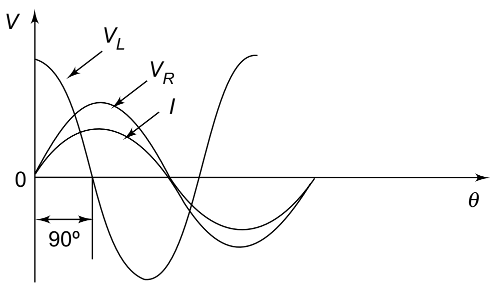


Fig. 3.14(b)

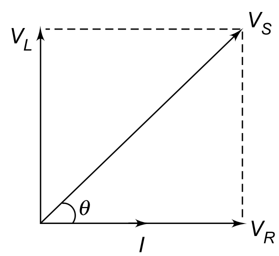


Fig. 3.14(c)

Example 3.6 *To the circuit shown in Fig. 3.15, consisting a 1 kW resistor connected in series with a 50 mH coil, a 10 V rms, 10 kHz signal is applied. Find impedance Z, current I, phase angle θ, voltage across resistance V_R, and the voltage across inductance V_L.*

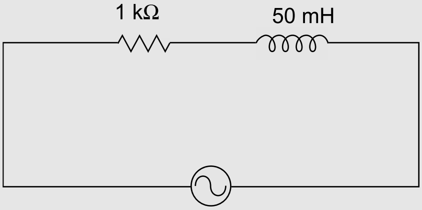


Fig. 3.15

Solution Inductive reactance $X_L = \omega L$

$$= 2\pi f L = (6.28)(10^4)(50 \times 10^{-3}) = 3140\ \Omega$$

In rectangular form,

Total impedance $Z = (1000 + j3140)\ \Omega$

$$= \sqrt{R^2 + X_L^2}$$

$$= \sqrt{(1000)^2 + (3140)^2} = 3295.4\ \Omega$$

Current $I = V_S/Z = 10/3295.4 = 3.03$ mA

Phase angle $\theta = \tan^{-1}(X_L/R) = \tan^{-1}(3140/1000) = 72.33°$

Therefore, in polar form total impedance $Z = 3295.4 \angle 72.33°$

Voltage across resistance $V_R = IR = 3.03 \times 10^{-3} \times 1000 = 3.03$ V

Voltage across inductive reactance $V_L = IX_L = 3.03 \times 10^{-3} \times 3140 = 9.51$ V

Example 3.7 *Determine the source voltage and the phase angle, if voltage across the resistance is 70 V and voltage across the inductive reactance is 20 V as shown in Fig. 3.16.*

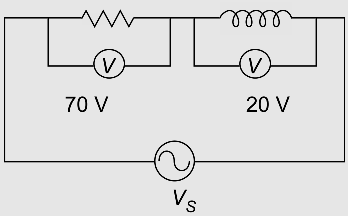


Fig. 3.16

Solution In Fig. 3.16, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_L^2}$$

$$= \sqrt{(70)^2 + (20)^2} = 72.8\ \text{V}$$

The angle between current and source voltage is

$$\theta = \tan^{-1}(V_L/V_R) = \tan^{-1}(20/70) = 15.94°$$

Example 3.8 *A signal generator supplies a 30 V, 100 Hz signal to a series circuit as shown in Fig. 3.17. Determine the impedance, the line current and phase angle in the given circuit.*

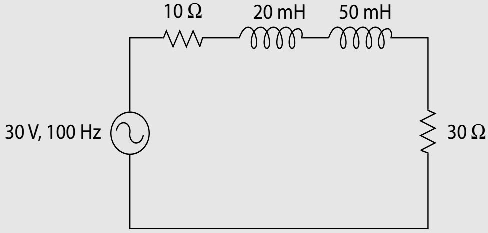


Fig. 3.17

Solution In Fig. 3.18, the resistances and inductive reactances can be combined.

Fig. 3.18

First, we find the inductive reactance

$$X_L = 2\pi fL = 2\pi \times 100 \times 70 \times 10^{-3} = 43.98\ \Omega$$

In rectangular form, the total impedance is

$$Z_T = (40 + j43.98)\ \Omega$$

Current $I = \dfrac{V_S}{Z_T} = \dfrac{30\angle 0^\circ}{40 + j43.98}$

Here we are taking source voltage as the reference voltage

$$\therefore \qquad I = \frac{30\angle 0^\circ}{59.45\angle +47.7^\circ} = 0.5\angle -47.7^\circ \text{ A}$$

The current lags behind the applied voltage by 47.7°

Hence, the phase angle between voltage and current

$$\theta = 47.7^\circ$$

Example 3.9 *For the circuit shown in Fig. 3.19, find the effective voltages across resistance and inductance, and also determine the phase angle.*

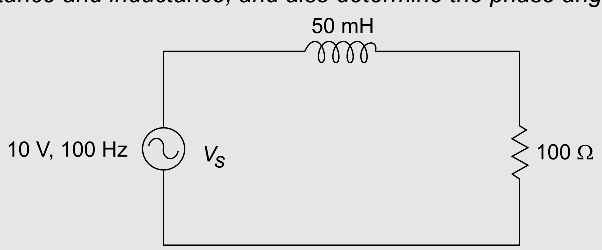


Fig. 3.19

Solution In rectangular form,

Total impedance $Z_T = R + jX_L$

where $\qquad X_L = 2\pi fL$

$$= 2\pi \times 100 \times 50 \times 10^{-3} = 31.42\ \Omega$$

$$\therefore \qquad Z_T = (100 + j31.42)\ \Omega$$

Current $$I = \frac{V_S}{Z_T} = \frac{10\angle 0^\circ}{(100 + j31.42)} = \frac{10\angle 0^\circ}{104.8\angle 17.44^\circ} = 0.095\angle -17.44^\circ$$

Therefore, the phase angle between voltage and current

$$\theta = 17.44^\circ$$

Voltage across resistance is $V_R = IR = 0.095 \times 100 = 9.5$ V

Voltage across inductive reactance is $V_L = IX_L = 0.095 \times 31.42 = 2.98$ V

When a sinusoidal voltage is applied to an RC series circuit, the current in the circuit and voltages across each of the elements are sinusoidal. The series RC circuit is shown in Fig. 3.20(a).

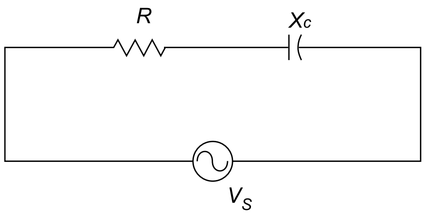


Fig. 3.20(a)

Here the resistor voltage and current are in phase with each other. The capacitor voltage lags behind the source voltage. The phase angle between the current and the capacitor voltage is always 90°. The amplitudes and the phase relations between the voltages and current depend on the ohmic values of the resistance and the capacitive reactance. The circuit is a series combination of both resistance and capacitance; and the phase angle between the applied voltage and the total current is somewhere between zero and 90°, depending on the relative values of the resistance and reactance. In a series RC circuit, the current is the same through the resistor and the capacitor. Thus, the resistor voltage is in phase with the current, and the capacitor voltage lags behind the current by 90° as shown in Fig. 3.20(b).

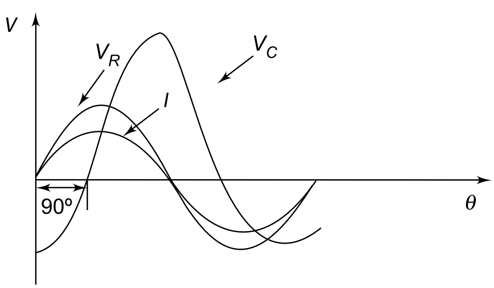


Fig. 3.20(b)

Here, I leads V_C by 90°. V_R and I are in phase. From Kirchhoff's voltage law, the sum of the voltage drops must be equal to the applied voltage. Therefore, the source voltage is given by

$$V_S = \sqrt{V_R^2 + V_C^2}$$

The phase angle between the resistor voltage and the source voltage is

$$\theta = \tan^{-1}(V_C/V_R)$$

Since the resistor voltage and the current are in phase, θ also represents the phase angle between the source voltage and current. The voltage phasor diagram for the series RC circuit, voltage and current phasor diagrams represented by the waveforms in Fig. 3.20(b) are shown in Fig. 3.20(c).

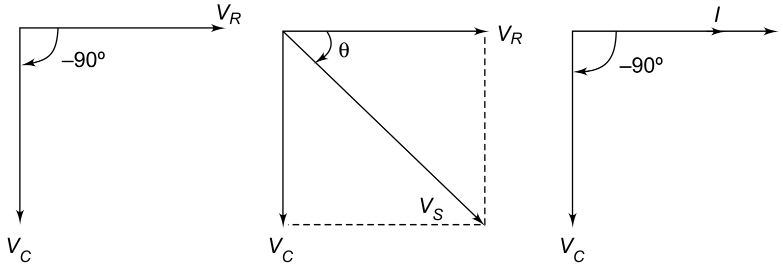


Fig. 3.20(c)

Example 3.10 *A sine wave generator supplies a 500 Hz, 10 V rms signal to a 2 kΩ resistor in series with a 0.1 μF capacitor as shown in Fig. 3.21. Determine the total impedance Z, current I, phase angle θ, capacitive voltage V_C, and resistive voltage V_R.*

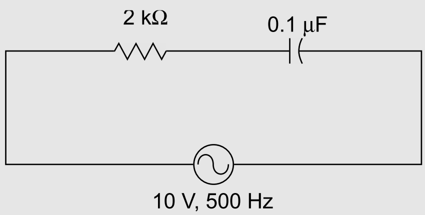


Fig. 3.21

Solution To find the impedance Z, we first solve for X_C

$$X_C = \frac{1}{2\pi f C} = \frac{1}{6.28 \times 500 \times 0.1 \times 10^{-6}}$$

$$= 3184.7\ \Omega$$

In rectangular form,

Total impedance $Z = (2000 - j3184.7)\ \Omega$

$$Z = \sqrt{(2000)^2 + (3184.7)^2}$$

$$= 3760.6\ \Omega$$

Phase angle $\theta = \tan^{-1}(-X_C/R) = \tan^{-1}(-3184.7/2000) = -57.87°$

Current $I = V_S/Z = 10/3760.6 = 2.66$ mA

Capacitive voltage $V_C = IX_C$

$$= 2.66 \times 10^{-3} \times 3184.7 = 8.47 \text{ V}$$

Resistive voltage $V_R = IR$

$$= 2.66 \times 10^{-3} \times 2000 = 5.32 \text{ V}$$

The arithmetic sum of V_C and V_R does not give the applied voltage of 10 volts. In fact, the total applied voltage is a complex quantity. In rectangular form,

Total applied voltage $V_S = 5.32 - j8.47$ V

In polar form

$$V_S = 10 \angle -57.87° \text{ V}$$

The applied voltage is complex, since it has a phase angle relative to the resistive current.

Example 3.11 *Determine the source voltage and phase angle when the voltage across the resistor is 20 V and the capacitor is 30 V as shown in Fig. 3.22.*

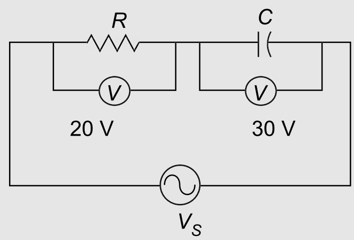


Fig. 3.22

Solution Since V_R and V_C are 90° out of phase, they cannot be added directly. The source voltage is the phasor sum of V_R and V_C.

$$\therefore\ V_S = \sqrt{V_R^2 + V_C^2} = \sqrt{(20)^2 + (30)^2} = 36\,\text{V}$$

The angle between the current and source voltage is

$$\theta = \tan^{-1}(V_C/V_R) = \tan^{-1}(30/20) = 56.3°$$

Example 3.12 *A resistor of 100 Ω is connected in series with a 50 μF capacitor. Find the effective voltage applied to the circuit at a frequency of 50 Hz. The effective voltage across the resistor is 170 V. Also determine voltage across the capacitor and phase angle. (See Fig. 3.23)*

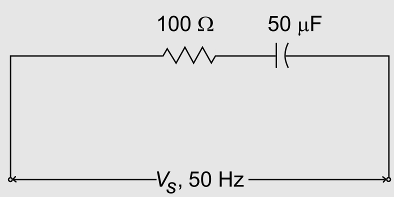


Fig. 3.23

Solution Capacitive reactance $X_C = \dfrac{1}{2\pi fC}$

$$= \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.66\,\Omega$$

Total impedance $Z_T = (100 - j63.66)\ \Omega$

Voltage across 100 Ω resistor is $V_R = 170$ V

Current in resistor, $I = \dfrac{170}{100} = 1.7$ A

Since the same current passes through capacitive reactance, the effective voltage across the capacitive reactance is

$$V_C = IX_C$$
$$= 1.7 \times 63.66 = 108.22\ \text{V}$$

The effective applied voltage to the circuit

$$V_S = \sqrt{V_R^2 + V_C^2}$$
$$= \sqrt{(170)^2 + (108.22)^2} = 201.5\,\text{V}$$

Total impedance in polar form

$$Z_T = 118.54 \angle -32.48°$$

Therefore, the current leads the applied voltage by 32.48°.

Example 3.13 *For the circuit shown in Fig. 3.24, determine the value of impedance when a voltage of (30 + j50)V is applied to the circuit and the current flowing is (−5 + j15)A. Also determine the phase angle.*

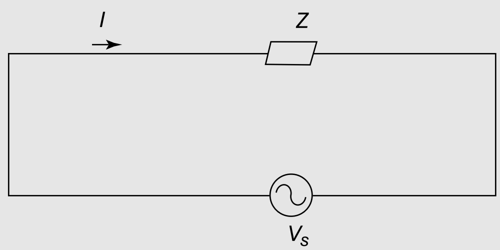


Fig. 3.24

Solution Impedance $Z = \dfrac{V_S}{I} = \dfrac{30 + j50}{-5 + j15}$

$$= \frac{58.31\angle 59°}{15.81\angle 108.43°} = 3.69\angle -49.43°$$

In rectangular form, the impedance $Z = 2.4 - j2.8$

Therefore, the circuit has a resistance of 2.4 Ω in series with capacitive reactance 2.8 Ω.

Phase angle between voltage and current is θ = 49.43°. Here, the current leads the voltage by 49.43°.

A series RLC circuit is the series combination of resistance, inductance and capacitance. If we observe the impedance diagrams of series RL and series RC circuits as shown in Fig. 3.25(a) and (b), the inductive reactance, X_L, is displayed on the $+j$ axis and the capacitive reactance, X_C, is displayed on the $-j$ axis. These reactance are 180° apart and tend to cancel each other.

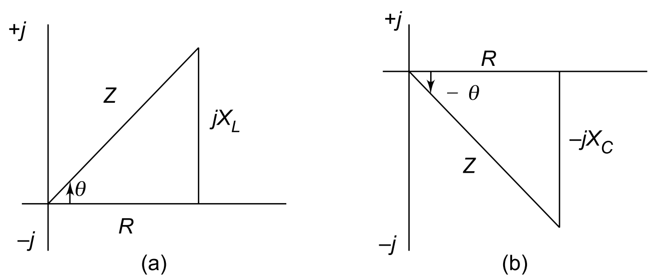


Fig. 3.25

The magnitude and type of reactance in a series RLC circuit is the difference of the two reactance. The impedance for an RLC series circuit is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. Similarly, the phase angle for an RLC circuit is

$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Example 3.14 *In the circuit shown in Fig. 3.26, determine the total impedance, current I, phase angle θ, and the voltage across each element.*

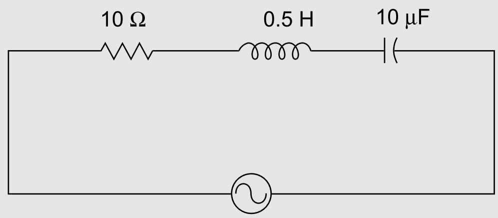


Fig. 3.26

Solution To find impedance Z, we first solve for X_C and X_L

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 50 \times 10 \times 10^{-6}}$$

$$= 318.5\ \Omega$$

$$X_L = 2\pi fL = 6.28 \times 0.5 \times 50 = 157\ \Omega$$

Total impedance in rectangular form

$$Z = (10 + j157 - j\,318.5)\ \Omega$$
$$= 10 + j(157 - 318.5)\ \Omega = 10 - j161.5\ \Omega$$

Here, the capacitive reactance dominates the inductive reactance.

$$Z = \sqrt{(10)^2 + (161.5)^2}$$

$$= \sqrt{100 + 26082.2} = 161.8\,\Omega$$

$$\text{Current } I = V_S / Z = \frac{50}{161.8} = 0.3\ \text{A}$$

Phase angle $\theta = \tan^{-1} [(X_L - X_C)/R] = \tan^{-1} (-161.5/10) = -86.45°$
Voltage across the resistor $V_R = IR = 0.3 \times 10 = 3$ V
Voltage across the capacitive reactance $= IX_C = 0.3 \times 318.5 = 95.55$ V
Voltage across the inductive reactance $= IX_L = 0.3 \times 157 = 47.1$ V

3.4 COMPLEX IMPEDANCE AND PHASOR NOTATION FOR RL, RC, RLC CIRCUITS

The complex number system simplifies the analysis of parallel ac circuits. In series circuits, the current is the same in all parts of the series circuit. In parallel ac circuits, the voltage is the same across each element.

The voltages for an RC series circuit can be expressed using complex numbers, where the resistive voltage is the real part of the complex voltage and the capacitive voltage is the imaginary part. For parallel RC circuits, the voltage is the same across each component. Here the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; the capacitive branch current is the imaginary part.

Example 3.15 *A signal generator supplies a sine wave of 20 V, 5 kHz to the circuit shown in Fig. 3.27. Determine the total current I_T, the phase angle and total impedance in the circuit.*

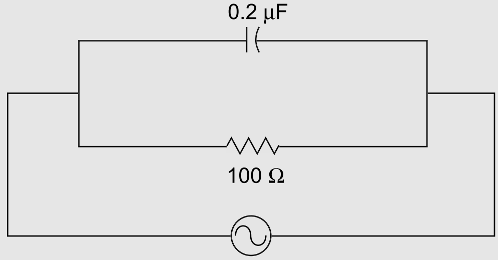


Fig. 3.27

Solution Capacitive reactance

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28 \times 5 \times 10^3 \times 0.2 \times 10^{-6}} = 159.2\ \Omega$$

Since the voltage across each element is the same as the applied voltage, we can solve for the two branch currents.

$\therefore$ Current in the resistance branch

$$I_R = \frac{V_S}{R} = \frac{20}{100} = 0.2\ \text{A}$$

and current in the capacitive branch

$$I_C = \frac{V_S}{X_C} = \frac{20}{159.2} = 0.126\ \text{A}$$

The total current is the vector sum of the two branch currents.

$\therefore$ Total current $I_T = (I_R + jI_C)$ A = (0.2 + j0.13) A

In polar form $I_T = 0.24 \angle 33°$

So the phase angle θ between applied voltage and total current is 33°. It indicates that the total line current is 0.24 A and leads the voltage by 33°. Solving for impedance, we get

$$Z = \frac{V_S}{I_T} = \frac{20\underline{|0°}}{0.24\underline{|33°}} = 83.3\underline{|-33°}\ \Omega$$

Example 3.16 *Determine the impedance and phase angle in the circuit shown in Fig. 3.28.*

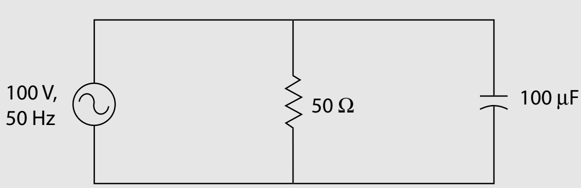


Fig. 3.28

Solution Capacitive reactance $X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\,\Omega$

Capacitive susceptance $B_C = \frac{1}{X_C}$

$$= \frac{1}{31.83} = 0.031\,\text{S}$$

Conductance G $= \frac{1}{R} = \frac{1}{50} = 0.02\,\text{S}$

Total admittance

$$Y = \sqrt{G^2 + B_C^2}$$

$$= \sqrt{(0.02)^2 + (0.031)^2}$$

$$= 0.037\,\text{S}$$

Total impedance $Z = \frac{1}{Y} = \frac{1}{0.037} = 27.02\,\Omega$

Phase angle

$$\theta = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}\left(\frac{50}{31.83}\right)$$

$$\theta = 57.52°$$

Example 3.17 *For the parallel circuit in Fig. 3.29, find the magnitude of current in each branch and the total current. What is the phase angle between the applied voltage and total current?*

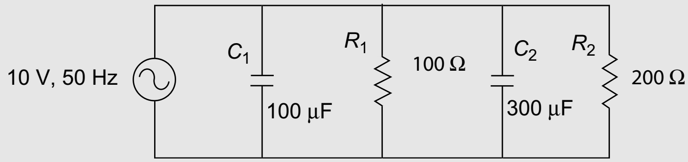


Fig. 3.29

Solution First let us find the capacitive reactances.

$$X_{C_1} = \frac{1}{2\pi f C_1}$$

$$= \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\ \Omega$$

$$X_{C_2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi \times 50 \times 300 \times 10^{-6}}$$

$$= 10.61\,\Omega$$

Here the voltage across each element is the same as the applied voltage.

Current in the 100 μF capacitor $I_C = \dfrac{V_S}{X_{C_1}}$

$$= \frac{10\angle 0^\circ}{31.83\angle -90^\circ} = 0.31\angle 90^\circ \text{ A}$$

Current in the 300 μF capacitor $I_{C_2} = \dfrac{V_S}{X_{C_2}}$

$$= \frac{10\angle 0^\circ}{10.61\angle -90^\circ} = 0.94\angle 90^\circ \text{ A}$$

Current in the 100 Ω resistor is $I_{R_1} = \dfrac{V_S}{R_1} = \dfrac{10}{100} = 0.1\text{ A}$

Current in the 200 Ω resistor is $I_{R_2} = \dfrac{V_S}{R_2} = \dfrac{10}{200} = 0.05\text{ A}$

Total current
$$\begin{aligned} I_T &= I_{R_1} + I_{R_2} + j(I_{C_1} + I_{C_2}) \\ &= 0.1 + 0.05 + j(0.31 + 0.94) \\ &= 1.26\angle 83.2^\circ \text{ A} \end{aligned}$$

The circuit shown in Fig. 3.29 can be simplified into a single parallel RC circuit as shown in Fig. 3.30.

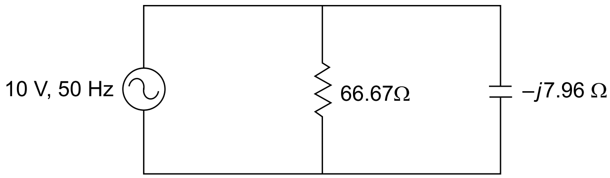


Fig. 3.30

In Fig. 3.30, the two resistances are in parallel and can be combined into a single resistance. Similarly, the two capacitive reactances are in parallel and can be combined into a single capacitive reactance.

$$R = \frac{R_1 R_2}{R_1 + R_2} = 66.67\ \Omega$$

$$X_C = \frac{X_{C_1} X_{C_2}}{X_{C_1} + X_{C_2}} = 7.96\ \Omega$$

Phase angle θ between voltage and current is

$$\theta = \tan^{-1}\left(\frac{R}{X_C}\right) = \tan^{-1}\left(\frac{66.67}{7.96}\right) = 83.19^\circ$$

Here the current leads the applied voltage by 83.19°.

In a parallel RL circuit, the inductive current is imaginary and lies on the $-j$ axis. The current angle is negative when the impedance angle is positive. Here also the total current can be represented by a complex number. The real part of the complex current expression is the resistive current; and inductive branch current is the imaginary part.

Example 3.18 *A 50 Ω resistor is connected in parallel with an inductive reactance of 30 Ω. A 20 V signal is applied to the circuit. Find the total impedance and line current in the circuit shown in Fig. 3.31.*

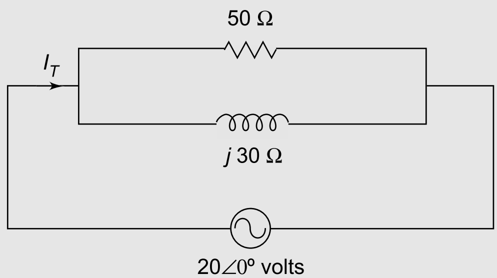


Fig. 3.31

Solution Since the voltage across each element is the same as the applied voltage, current in the resistive branch,

$$I_R = \frac{V_s}{R} = \frac{20\angle 0^\circ}{50\angle 0^\circ} = 0.4\ \text{A}$$

current in the inductive branch

$$I_L = \frac{V_s}{X_L} = \frac{20\angle 0^\circ}{30\angle 90^\circ} = 0.66\angle -90^\circ$$

Total current is $I_T = 0.4 - j0.66$

In polar form, $I_T = 0.77\ \angle -58.8^\circ$

Here the current lags behind the voltage by 58.8°

Total impedance $$Z = \frac{V_s}{I_T} = \frac{20\angle 0^\circ}{0.77\angle -58.8^\circ} = 25.97\angle 58.8^\circ\ \Omega$$

Example 3.19 *For the circuit shown in Fig. 3.32, determine the total current, impedance Z and phase angle.*

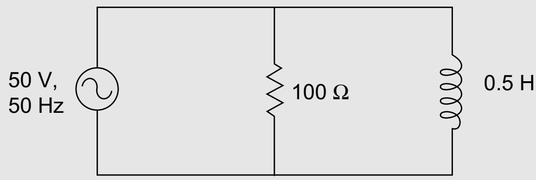


Fig. 3.32

Solution Here, the voltage across each element is the same as the applied voltage.

Current in resistive branch $$I_R = \frac{V_S}{R} = \frac{50}{100} = 0.5\ \text{A}$$

Inductive reactance $$X_L = 2\pi f L$$

$$= 2\pi \times 50 \times 0.5 = 157.06\ \Omega$$

Current in inductive branch

$$I_L = \frac{V_S}{X_L} = \frac{50}{157.06} = 0.318\ \text{A}$$

Total current $\quad I_T = \sqrt{I_R^2 + I_L^2}$

or $\quad (0.5 - j0.318)\text{A} = 0.59\ \angle{-32.5°}$

For parallel RL circuits, the inductive susceptance is

$$B_L = \frac{1}{X_L} = \frac{1}{157.06} = 0.0064\ \text{S}$$

Conductance $\quad G = \frac{1}{100} = 0.01\text{S}$

$\therefore \quad \text{Admittance} = \sqrt{G^2 + B_L^2} = \sqrt{(0.01)^2 + (0.0064)^2}$

$$= 0.0118\ \text{S}$$

Converting to impedance, we get

$$Z = \frac{1}{Y} = \frac{1}{0.012} = 83.33\,\Omega$$

Phase angle $\quad \theta = \tan^{-1}\left(\frac{R}{X_L}\right) = \tan^{-1}\left(\frac{100}{157.06}\right) = 32.48°$

3.4.1 Instantaneous Power

In a purely resistive circuit, all the energy delivered by the source is dissipated in the form of heat by the resistance. In a purely reactive (inductive or capacitive) circuit, all the energy delivered by the source is stored by the inductor or capacitor in its magnetic or electric field during a portion of the voltage cycle, and then is returned to the source during another portion of the cycle, so that no net energy is transferred. When there is complex impedance in a circuit, part of the energy is alternately stored and returned by the reactive part, and part of it is dissipated by the resistance. The amount of energy dissipated is determined by the relative values of resistance and reactance.

Consider a circuit having complex impedance. Let $v(t) = V_m \cos \omega t$ be the voltage applied to the circuit and let $i(t) = I_m \cos(\omega t + \theta)$ be the corresponding current flowing through the circuit. Then the power at any instant of time is

$$P(t) = v(t)\, i(t) = V_m \cos \omega t\, I_m \cos(\omega t + \theta) \qquad (3.1)$$

From Eq. 3.1, we get

$$P(t) = \frac{V_m I_m}{2}\left[\cos(2\omega t + \theta) + \cos\theta\right] \qquad (3.2)$$

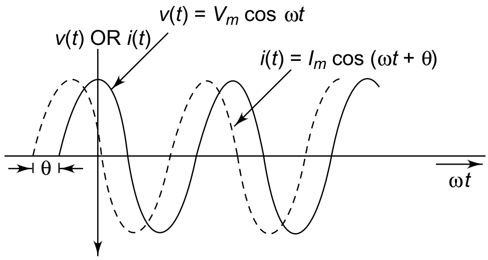


Fig. 3.33

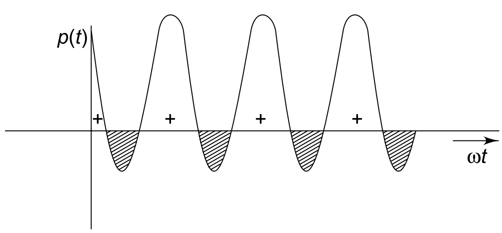


Fig. 3.34

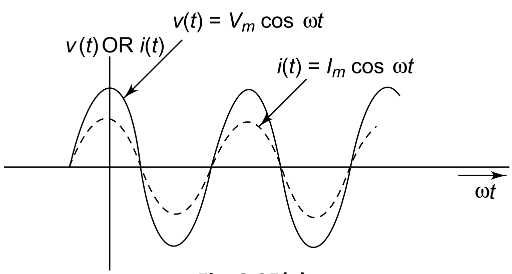


Fig. 3.35(a)

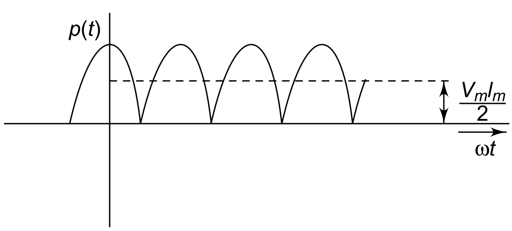


Fig. 3.35(b)

Equation 3.2 represents *instantaneous power*. It consists of two parts. One is a fixed part, and the other is time-varying which has a frequency twice that of the voltage or current waveforms. The voltage, current and power waveforms are shown in Figs 3.33 and 3.34.

Here, the negative portion (hatched) of the power cycle represents the power returned to the source. Figure 3.34 shows that the instantaneous power is negative whenever the voltage and current are of opposite sign. In Fig. 3.34, the positive portion of the power is greater than the negative portion of the power; hence the average power is always positive, which is almost equal to the constant part of the instantaneous power (Eq. 3.2). The positive portion of the power cycle varies with the phase angle between the voltage and current waveforms. If the circuit is pure resistive, the phase angle between voltage and current is zero; then there is no negative cycle in the $P(t)$ curve. Hence, all the power delivered by the source is completely dissipated in the resistance.

If θ becomes zero in Eq. 3.1, we get

$$\begin{aligned} P(t) &= v(t)\, i(t) \\ &= V_m I_m \cos^2 \omega t \\ &= \frac{V_m I_m}{2}\left(1+\cos 2\omega t\right) \qquad (3.3) \end{aligned}$$

The waveform for Eq. 3.3, is shown in Fig. 3.35, where the power wave has a frequency twice that of the voltage or current. Here the average value of power is $V_m I_m/2$.

When phase angle θ is increased, the negative portion of the power cycle increases and lesser power is dissipated. When θ becomes $\pi/2$, the positive and negative portions of the power cycle are equal. At this instant, the power dissipated in the circuit is zero, i.e. the power delivered to the load is returned to the source.

3.4.2 Average Power

To find the average value of any power function, we have to take a particular time interval from t_1 to t_2; by integrating the function from t_1 to t_2 and dividing the result by the time interval $t_2 - t_1$, we get the average power.

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P(t)\,dt \tag{3.4}$$

In general, the average value over one cycle is

$$P_{\text{av}} = \frac{1}{T} \int_0^T P(t)\,dt \tag{3.5}$$

By integrating the instantaneous power $P(t)$ in Eq. 3.5 over one cycle, we get average power

$$P_{\text{av}} = \frac{1}{T} \int_0^T \left\{ \frac{V_m I_m}{2} \left[\cos(2\omega t + \theta) + \cos\theta \right] dt \right\}$$

$$= \frac{1}{T} \int_0^T \frac{V_m I_m}{2} \left[\cos(2\omega t + \theta) \right] dt + \frac{1}{T} \int_0^T \frac{V_m I_m}{2} \cos\theta\,dt \tag{3.6}$$

In Eq. 3.6, the first term becomes zero, and the second term remains. The average power is therefore

$$P_{\text{av}} = \frac{V_m I_m}{2} \cos\theta \text{ W} \tag{3.7}$$

We can write Eq. 3.7 as

$$P_{\text{av}} = \left(\frac{V_m}{\sqrt{2}} \right) \left(\frac{I_m}{\sqrt{2}} \right) \cos\theta \tag{3.8}$$

In Eq. 3.8, $V_m / \sqrt{2}$ and $I_m / \sqrt{2}$ are the effective values of both voltage and current.

$$\therefore \qquad P_{\text{av}} = V_{eff} I_{eff} \cos\theta$$

To get average power, we have to take the product of the effective values of both voltage and current multiplied by cosine of the phase angle between voltage and the current.

If we consider a purely resistive circuit, the phase angle between voltage and current is zero. Hence, the average power is

$$P_{\text{av}} = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R$$

If we consider a purely reactive circuit (i.e. purely capacitive or purely inductive), the phase angle between voltage and current is 90°. Hence, the average power is zero or $P_{av} = 0$.

If the circuit contains complex impedance, the average power is the power dissipated in the resistive part only.

Example 3.20 *A voltage of v(t) = 100 sin ωt is applied to a circuit. The current flowing through the circuit is i(t) = 15 sin (ωt – 30°). Determine the average power delivered to the circuit.*

Solution The phase angle between voltage and current is 30°.

Effective value of the voltage $V_{eff} = \dfrac{100}{\sqrt{2}}$

Effective value of the current $I_{eff} = \dfrac{15}{\sqrt{2}}$

Average power
$$P_{av} = V_{eff} I_{eff} \cos\theta$$
$$= \frac{100}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \cos 30°$$
$$= \frac{100 \times 15}{2} \times 0.866 = 649.5 \text{ W}$$

Example 3.21 *Determine the average power delivered to the circuit consisting of an impedance Z = 5 + j8 when the current flowing through the circuit is I = 5∠30°.*

Solution The average power is the power dissipated in the resistive part only.

or $$P_{av} = \frac{I_m^2}{2} R$$

Current $I_m = 5\text{A}$

$\therefore$ $$P_{av} = \frac{5^2}{2} \times 5 = 62.5 \text{ W}$$

3.4.3 Apparent Power and Power Factor

The power factor is useful in determining useful power (true power) transferred to a load. The highest power factor is 1, which indicates that the current to a load is in phase with the voltage across it (i.e. in the case of resistive load). When the power factor is 0, the current to a load is 90° out of phase with the voltage (i.e. in case of reactive load).

Consider the following equation

$$P_{av} = \frac{V_m I_m}{2} \cos\theta \text{ W} \tag{3.9}$$

In terms of effective values

$$P_{\text{av}} = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

$$= V_{eff} I_{eff} \cos\theta \text{ W} \tag{3.10}$$

The average power is expressed in watts. It means the useful power transferred from the source to the load, which is also called true power. If we consider a dc source applied to the network, true power is given by the product of the voltage and the current. In case of sinusoidal voltage applied to the circuit, the product of voltage and current is not the true power or average power. This product is called *apparent power*. The apparent power is expressed in volt amperes, or simply VA.

$\therefore$ Apparent power $= V_{eff} I_{eff}$

In Eq. 3.10, the average power depends on the value of cos θ; this is called the *power factor* of the circuit.

$$\therefore \quad \text{Power factor}\,(pf) = \cos\theta = \frac{P_{\text{av}}}{V_{eff}\, I_{eff}}$$

Therefore, power factor is defined as the ratio of average power to the apparent power, whereas apparent power is the product of the effective values of the current and the voltage. Power factor is also defined as the factor with which the volt amperes are to be multiplied to get true power in the circuit.

In the case of sinusoidal sources, the power factor is the cosine of the phase angle between voltage and current

$$pf = \cos\theta$$

As the phase angle between voltage and total current increases, the power factor decreases. The smaller the power factor, the smaller the power dissipation. The power factor varies from 0 to 1. For purely resistive circuits, the phase angle between voltage and current is zero, and hence the power factor is unity. For purely reactive circuits, the phase angle between voltage and current is 90°, and hence the power factor is zero. In an RC circuit, the power factor is referred to as *leading* power factor because the current leads the voltage. In an RL circuit, the power factor is referred to as lagging power factor because the current lags behind the voltage.

Example 3.22 *A sinusoidal voltage $v = 50 \sin \omega t$ is applied to a series RL circuit. The current in the circuit is given by $i = 25 \sin(\omega t - 53°)$. Determine (a) apparent power (b) power factor and (c) average power.*

Solution (a) Apparent power $P = V_{eff} I_{eff}$

$$= \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

$$= \frac{50 \times 25}{2} = 625\,\text{VA}$$

(b) Power factor $= \cos\theta$

where θ is the angle between voltage and current

$$\theta = 53°$$

$\therefore$ power factor $= \cos\theta = \cos 53° = 0.6$

(c) Average power $P_{av} = V_{eff} I_{eff} \cos\theta$

$$= 625 \times 0.6 = 375 \text{ W}$$

Example 3.23 *Find the total current and the power consumed by the circuit.*

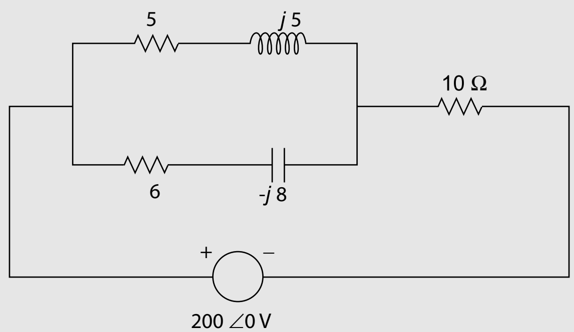


Fig. 3.36

Solution Total impedance of the circuit,

$$Z_T = (5 + j5) \,||\, (6 - j8) + 10$$

$$Z_T = 16.15 + j0.769$$

$$I = \frac{V}{Z} = \frac{200\angle 0}{16.15 + j0.769} = 12.35 - j0.588 \text{ A}$$

$$= 12.36\angle -2.72°$$

Power consumed $= I^2 R$

$$= (12.36)^2 \times 16.15 = 2467\text{W}$$

or $VI\cos\theta = 200 \times 12.36 \times \cos(-2.72) = 2467$ W.

3.4.4 Real and Reactive Power

We know that the average power dissipated is

$$P_{av} = V_{eff}[I_{eff}\cos\theta] \tag{3.11}$$

From the impedance triangle shown in Fig. 3.37

$$\cos\theta = \frac{R}{|Z|} \tag{3.12}$$

and

$$V_{eff} = I_{eff} Z \tag{3.13}$$

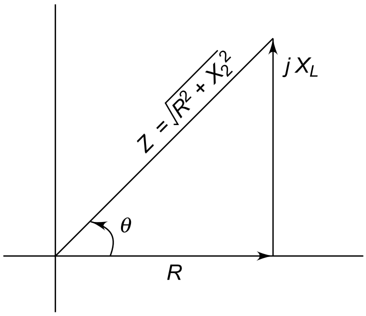


Fig. 3.37

If we substitute Eqs (3.12) and (3.13) in Eq. (3.11), we get

$$P_{av} = I_{eff}\, Z\left[I_{eff}\,\frac{R}{Z}\right]$$

$$= I_{eff}^2 R \text{ watts} \tag{3.14}$$

This gives the average power dissipated in a resistive circuit.

If we consider a circuit consisting of a pure inductor, the power in the inductor

$$P_r = iv_L \tag{3.15}$$

$$= iL\frac{di}{dt}$$

Consider $\qquad i = I_m \sin(\omega t + \theta)$

Then $\qquad P_r = I_m^2 \sin(\omega t + \theta)\, L\omega \cos(\omega t + \theta)$

$$= \frac{I_m^2}{2}(\omega L)\sin 2(\omega t + \theta)$$

$\therefore \qquad P_r = I_{eff}^2(\omega L)\sin 2(\omega t + \theta) \qquad (3.16)$

From the above equation, we can say that the average power delivered to the circuit is zero. This is called *reactive* power. It is expressed in volt-amperes reactive (VAR).

$$P_r = I_{eff}^2 X_L \text{ VAR} \tag{3.17}$$

From Fig. 3.37, we have

$$X_L = Z\sin\theta \tag{3.18}$$

Substituting Eq. 3.18 in Eq. 3.17, we get

$$P_r = I_{eff}^2 Z\sin\theta$$

$$= (I_{eff} Z) I_{eff} \sin\theta$$

$$= V_{eff} I_{eff} \sin\theta \text{ VAR}$$

3.4.5 The Power Triangle

A generalised impedance phase diagram is shown in Fig. 3.38. A phasor relation for power can also be represented by a similar diagram because of the fact that true power P_{av} and reactive power P_r differ from R and X by a factor I^2_{eff}, as shown in Fig. 3.38.

The resultant power phasor $I^2_{eff} Z$, represents the apparent power P_a.

At any instant in time, P_a is the total power that appears to be transferred between the source and reactive circuit. Part of the apparent power is true power and part of it is reactive power. Absolute value of complex power is called apparent power.

$$\therefore \qquad P_a = I^2_{eff} Z$$

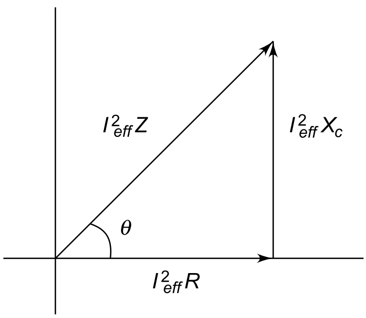


Fig. 3.38

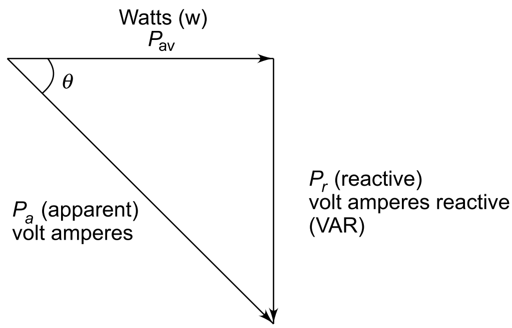


Fig. 3.39

The power triangle is shown in Fig. 3.39.

From Fig. 3.39, we can write

$$P_{\text{true}} = P_a \cos\theta$$

or average power $P_{\text{av}} = P_a \cos\theta$

and reactive power $P_r = P_a \sin\theta$

Example 3.24 *In an electrical circuit R, L and C are connected in parallel. R = 10Ω, L = 0.1 H, C = 100 μF. The circuit is energized with a supply at 230 V, 50 Hz. Calculate*

(a) Impedance

(b) Current taken from supply

(c) p.f. of the circuit

(d) Power consumed by the circuit

Solution The circuit is as shown in Fig. 3.40.

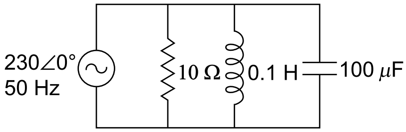


Fig. 3.40

The impedance of 3 branches are

$$Z_1 = 10\,\Omega$$

$$Z_2 = j2\pi fL = 2 \times \pi \times 50 \times 0.1 = j31.41\,\Omega$$

$$Z_3 = \frac{-j}{2\pi fc} = \frac{-j}{2\times\pi\times 50\times 100\times 10^{-6}} = -j31.84\,\Omega$$

(a) Impedance of circuit $Z = \left[\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right]^{-1}$

$$= \left[\frac{1}{10} + \frac{1}{j31.41} + \frac{1}{-j31.84}\right]^{-1}$$

$$\approx 10\ \Omega$$

(b) Current taken from supply $I = \frac{V}{Z} = \frac{230\angle 0°}{10} = 23A.$ i.e. $23\angle 0°$A

(c) p.f. of the circuit = $\cos\theta = 1$

(d) Power consumed by the circuit

Real power consumed = $I^2R = 23^2 \times 10 = 5.3$ kW

Reactive power consumed = 0 KVAR

Example 3.25 *A coil of resistance l0 V and an inductance of 0.1 H is connected in series with a capacitor of capacitance 150 μF a cross at 200 V, 50 Hz supply. Calculate (i) impedance (ii) current (iii) power and power factor of the circuit.*

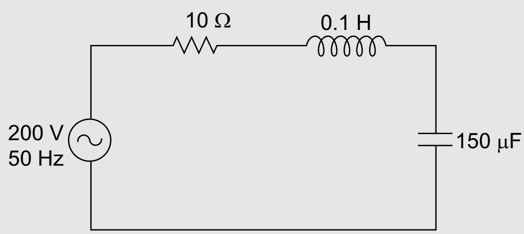


Fig. 3.41

Solution (i) Total impedance

$$Z = R + j\omega L - \frac{j}{\omega c}$$

$$= 10 + j31.45 - j21.22$$

$$= 10 + j10.194$$

$$= 14.279\underline{|45.55}$$

(ii) Current $I = \frac{V}{Z}$

$$= \frac{200\underline{|0°}}{14.279\underline{|45.55°}}$$

$$= 14\underline{|-45.55°}$$

(iii) Power factor = cos (45.55°)

= 0.7 lagging

Real power = $VI\cos\theta$

= $200 \times 14 \times 0.7$

$$= 1.9 \text{ kW}$$

$$\text{Reactive power} = VI \sin \phi$$

$$= 200 \times 14 \times \sin(-45.55)$$

$$= -1.998 \text{ KVAR}$$

"-1" Sign indicates that it absorbs the reactive power.

Example 3.26 *Two coils A and B have resistance of 12 V and 6 V and inductances of 0.02 and 0.03 H respectively. These are connected in parallel and a voltage of 200 V at 50 Hz is applied to their combination. Find*

(a) Current in the each coil.

(b) The total current and the

(c) The power factor of the circuit.

(d) Power consumed by each coil and total power. [JNTU June 2009]

Solution

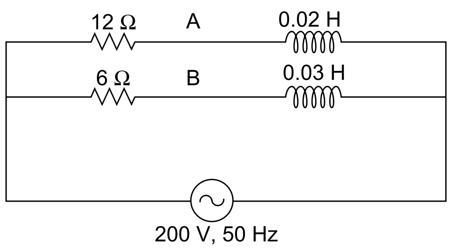


Fig. 3.42

Impedance of coil A $= (12 + j \times 50 \times 0.02 \times 2\pi)\ \Omega = (12 + j6.28)\ \Omega$.

Impedance of coil B $= (6 + j \times 50 \times 0.03 \times 2\pi)\ \Omega = (6 + j9.42)\ \Omega$.

(a) $\therefore$ Current in coil A $= \dfrac{200}{12 + j6.28}$ amp $= 14.767\angle -27.63^\circ$ amp

$$= (13.083 - j6.848) \text{ amp}$$

$\therefore$ Current in coil B $= \dfrac{200}{6 + j9.42}$ amp $= 17.907\angle -57.51^\circ$ amp

$$= (9.619 - j15.104) \text{ amp}$$

(b) $\therefore$ The total current $= [(13.083 + 9.619) - j(6.848 + 15.104)]$ amp

$$= (22.702 - j21.952) \text{ amp}$$

$$= 31.579\angle -44.04^\circ \text{ amp}$$

(c) $\therefore$ Power factor $= \cos(-44.04^\circ) = 0.719$

(d) $\therefore$ Real power consumed by coil A $= 200 \times 14.769 \times \cos(27.63°)$ watt

$= 2616.59$ watt

Real power consumed by coil B $= 200 \times 17.907 \times \cos(-57.51°)$ watt

$= 1923.76$ watt

Real power consumed by the total network

$= 200 \times 31.579 \times \cos(44.04°)$ watt

$= 4540.14$ watt

Reactive power consumed by coil A $= 200 \times 14.767 \times \sin(27.63°)$ VAR

$= 1369.67$ VAR

Reactive power consumed by coil B $= 200 \times 17.907 \times \sin(-57.51°)$ VAR

$= -3020.85$ VAR

Reactive power consumed by the total network

$= 200 \times 31.579 \times \sin(44.04°)$ VAR

$= 4390.49$ VAR

Example 3.27 *Find complex power in the following circuit.* *[JNTU May/June 2004]*

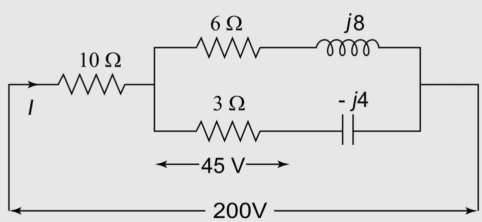


Fig. 3.43

Solution Taking the source voltage as reference

$V = 200\angle 0$ V

$$I = \frac{200\angle 0}{10 + \dfrac{(6+j8)(3-j4)}{(9+j4)}} = 13.396 + j1.886 = 13.52\angle 8°$$

Complex power $= VI^*$

$= (200\angle 0)(13.52\angle -8°)$

$S = VI^* = 2704\angle -8°$ VA

Complex power $(P + jQ) = 2704\angle -8° = (2677.68 - j376.32)$

$P = 2677.68\ W$; $Q = 376.32$ VAR leading.

Example 3.28 *In the circuit shown in Fig. 3.44, a voltage of v(t) = 50 sin (ωt + 30°) is applied. Determine the true power, reactive power and power factor.*

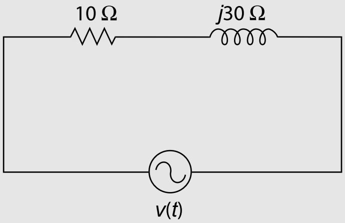


Fig. 3.44

Solution The voltage applied to the circuit is

$$v(t) = 50 \sin (\omega t + 30°)$$

The current in the circuit is

$$I = \frac{V}{Z} = \frac{50\angle 30^{\circ}}{10 + j30} = \frac{50\angle 30^{\circ}}{31.6\angle 71.56^{\circ}}$$

$$= 1.58 \angle -41.56° \text{ A}$$

The phasor diagram is shown in Fig. 3.45.

The phase angle between voltage and current θ = 71.56°

Power factor = cos θ = cos 71.56° = 0.32

True power or average power

$$P_{av} = V_{eff} I_{eff} \cos \theta$$

$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \cos 71.56°$$

$$= 12.49 \text{ W}$$

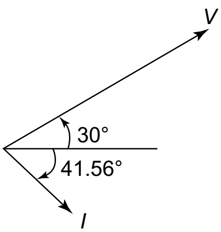


Fig. 3.45

$$\text{Reactive power} = V_{eff} I_{eff} \sin \theta$$

$$= \frac{50 \times 1.58}{\sqrt{2} \times \sqrt{2}} \sin 71.56°$$

$$= 37.47 \text{ VAR}$$

Example 3.29 *Determine the circuit constants in the circuit shown in Fig. 3.46, if the applied voltage to the circuit v(t) = 100 sin (50t + 20°). The true power in the circuit is 200 W and the power factor is 0.707 lagging.*

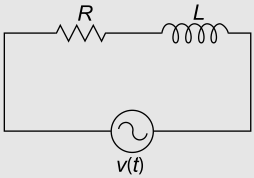


Fig. 3.46

Solution Power factor = cos θ = 0.707

∴ The phase angle between voltage and current

$$\theta = \cos^{-1} 0.707 = 45°$$

Here the current lags behind the voltage by 45°.

Hence, the current equation is $i(t) = I_m \sin(50t - 25°)$

True power $= V_{eff} I_{eff} \cos\theta = 200$ W

$$I_{eff} = \frac{200}{V_{eff}\cos\theta}$$

$$= \frac{200}{(100/\sqrt{2})\times 0.707} = 4\,\text{A}$$

$$I_m = 4\times\sqrt{2} = 5.66\,\text{A}$$

$\therefore$ The current equation is $i(t) = 5.66 \sin(50t - 25°)$

The impedance of the circuit

$$Z = \frac{V}{I} = \frac{(100/\sqrt{2})\angle 20^{\circ}}{(5.66/\sqrt{2})\angle -25^{\circ}}$$

$\therefore$ $Z = 17.67\angle 45° = 12.5 + j12.5$

Since $Z = R + jX_L = 12.5 + j12.5$

$\therefore$ $R = 12.5$ ohms, $X_L = 12.5$ ohms

$$X_L = \omega L = 12.5$$

$$L = \frac{12.5}{50} = 0.25\,\text{H}$$

Example 3.30 *A voltage v(t) = 150 sin 250t is applied to the circuit shown in Fig. 3.47. Find the power delivered to the circuit and the value of inductance in Henrys.*

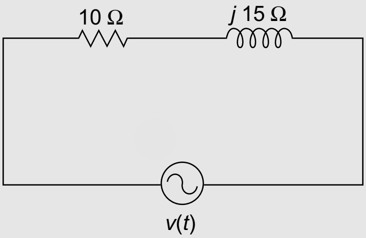


Fig. 3.47

Solution $Z = 10 + j15\ \Omega$

The impedance $Z = 18\angle 56.3°$

The impedance of the circuit $Z = \dfrac{V}{I}$

$$18\angle 56.3° = \frac{(150/\sqrt{2})\angle 0^{\circ}}{I}$$

$\therefore$ Phasor current $I = \dfrac{150/\sqrt{2}}{18\angle 56.3^{\circ}} = 5.89\angle -56.3°$

The current equation is $i(t) = 5.89\sqrt{2}\sin(250t - 56.3°)$

$$= 8.33\sin(250t - 56.3°)$$

The phase angle between the current and the voltage

$$\theta = 56.3°$$

The power delivered to the circuit

$$P_{av} = VI\cos\theta$$

$$= \frac{150}{\sqrt{2}} \times \frac{8.33}{\sqrt{2}} \cos 56.3°$$

$$= 346.6 \text{ W}$$

The inductive impedance $X_L = 15\ \Omega$

$\therefore \quad \omega L = 15$

$\therefore \quad L = \dfrac{15}{250} = 0.06\,\text{H}$

Example 3.31 *Determine the power factor, true power, reactive power and apparent power in the circuit in Fig. 3.48.*

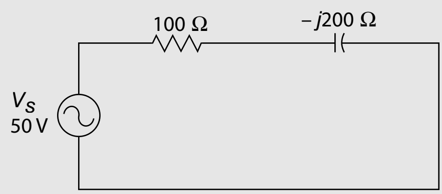


Fig. 3.48

Solution The impedance of the circuit

$$Z = \sqrt{R^2 + X_C^2}$$

$$= \sqrt{(100)^2 + (200)^2} = 223.6\ \Omega$$

The current $I = \dfrac{V_S}{Z} = \dfrac{50}{223.6} = 0.224$

The phase angle

$$\theta = \tan^{-1}\left(\frac{-X_C}{R}\right)$$

$$= \tan^{-1}\left(\frac{-200}{100}\right) = -63.4°$$

$\therefore$ The power factor $pf = \cos\theta = \cos(63.4°) = 0.448$

The true power $P_{av} = VI\cos\theta$

$$= 50 \times 0.224 \times 0.448 = 5.01 \text{ W}$$

The reactive power $P_v = I^2 X_C$

$$= (0.224)^2 \times 200 = 10.03 \text{ VAR}$$

The apparent power

$$P_a = I^2 Z = (0.224)^2 \times 223.6 = 11.21 \text{ VA}$$

Example 3.32 *In a certain RC circuit, the true power is 300 W and the reactive power is 1000 W. What is the apparent power?*

Solution The true power P_{true} or $P_{\text{av}} = VI \cos \theta$

$$= 300 \text{ W}$$

The reactive power $P_r = VI \sin \theta$

$$= 1000 \text{ W}$$

From the above results

$$\tan \theta = \frac{1000}{300} = 3.33$$

The phase angle between voltage and current, $\theta = \tan^{-1} 3.33 = 73.3°$

The apparent power $P_a = VI = \dfrac{300}{\cos 73.3^\circ} = 1043.9 \text{ VA}$

Example 3.33 *A sine wave of $v(t) = 200 \sin 50t$ is applied to a 10 Ω resistor in series with a coil. The reading of a voltmeter across the resistor is 120 V and across the coil, 75 V. Calculate the power and reactive volt-amperes in the coil and the power factor of the circuit.*

Solution The rms value of the sine wave

$$V = \frac{200}{\sqrt{2}} = 141.4 \text{ V}$$

Voltage across the resistor, $V_R = 120$ V

Voltage across the coil, $V_L = 75$ V

$\therefore$ $IR = 120$ V

The current in resistor, $I = \dfrac{120}{10} = 12 \text{ A}$

Since $IX_L = 75$ V

$\therefore$ $X_L = \dfrac{75}{12} = 6.25\ \Omega$

Power factor, $pf = \cos \theta = \dfrac{R}{Z}$

where $$Z = 10 + j6.25 = 11.8 \angle 32°$$

$\therefore$ $$\cos\theta = \frac{R}{Z} = \frac{10}{11.8} = 0.85$$

True power $$P_{\text{true}} = I^2 R = (12)^2 \times 10 = 1440 \text{ W}$$

Reactive power $$P_r = I^2 X_L = (12)^2 \times 6.25 = 900 \text{ VAR}$$

Example 3.34 *For the circuit shown in Fig. 3.49, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?*

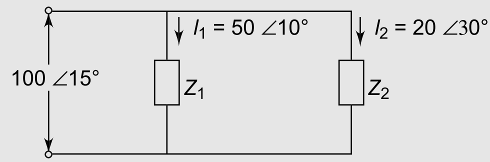


Fig. 3.49

Solution In the circuit shown in Fig. 3.49, we can calculate Z_1 and Z_2.

$$\text{Impedance } Z_1 = \frac{100\angle 15^\circ}{50\angle 10^\circ} = 2\angle 5^\circ = (1.99 + j0.174)\ \Omega$$

$$\text{Impedance } Z_2 = \frac{100\angle 15^\circ}{20\angle 30^\circ} = 5\angle -15^\circ = (4.83 - j1.29)\ \Omega$$

True power in branch Z_1 is $$P_{t_1} = I_1^2 R = (50)^2 \times 1.99 = 4975 \text{ W}$$

Reactive power in branch Z_1, $$P_{r_1} = I_1^2 X_L$$
$$= (50)^2 \times 0.174 = 435 \text{ VAR}$$

Apparent power in branch Z_1, $$P_{a_1} = I_1^2 Z_1$$
$$= (50)^2 \times 2$$
$$= 2500 \times 2 = 5000 \text{ VA}$$

True power in branch Z_2, $$P_{t_2} = I_2^2 R$$
$$= (20)^2 \times 4.83 = 1932 \text{ W}$$

Reactive power in branch Z_2, $$P_{r_2} = I_2^2 X_C$$
$$= (20)^2 \times 1.29 = 516 \text{ VAR}$$

Apparent power in branch Z_2, $$P_{a_2} = I_2^2 Z_2$$
$$= (20)^2 \times 5 = 2000 \text{ VA}$$

Total impedance of the circuit, $$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$= \frac{2\angle 5^{\circ} \times 5 \times \angle -15^{\circ}}{1.99 + j0.174 + 4.83 - j1.29}$$

$$= \frac{10\angle -10^{\circ}}{6.82 - j1.116}$$

$$= \frac{10\angle -10^{\circ}}{6.9\angle -9.29^{\circ}} = 1.45\angle -0.71^{\circ}$$

The phase angle between voltage and current, $\theta = 0.71°$

$\therefore$ Power factor $\qquad pf = \cos\theta$

$$= \cos 0.71° = 0.99 \text{ leading}$$

Example 3.35 *A voltage of $v(t) = 141.4 \sin \omega t$ is applied to the circuit shown in Fig. 3.50. The circuit dissipates 450 W at a lagging power factor, when the voltmeter and ammeter readings are 100 V and 6 A, respectively. Calculate the circuit constants.*

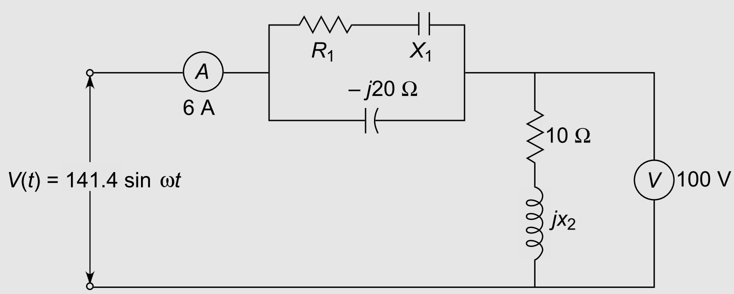


Fig. 3.50

Solution The magnitude of the current passing through $(10 + jX_2)\ \Omega$ is

$$I = 6\text{ A}$$

The magnitude of the voltage across the $(10 + jX_2)$ ohms, $V = 100$ V. The magnitude of impedance $(10 + jX_2)$ is V/I.

Hence $$\sqrt{10^2 + X_2^2} = \frac{100}{6} = 16.67\,\Omega$$

$\therefore$ $$X_2 = \sqrt{(16.67)^2 - (10)^2} = 13.33\,\Omega$$

Total power dissipated in the circuit $= VI \cos\theta = 450$ W

$\therefore$ $$V = \frac{141.4}{\sqrt{2}} = 100\text{ V}$$

$$I = 6\text{ A}$$

$$100 \times 6 \times \cos\theta = 450$$

The power factor $pf = \cos\theta = \dfrac{450}{600} = 0.75$

$$\theta = 41.4°$$

The current lags behind the voltage by 41.4°

The current passing through the circuit, $I = 6\angle{-41.4°}$

The voltage across $(10 + j13.33)\ \Omega$, $V = 6\angle{-41.4°} \times 16.66\angle 53.1°$

$$= 100\angle 11.7°$$

The voltage across parallel branch, $V_1 = 100\angle 0° - 100\angle 11.7°$

$$= 100 - 97.9 - j20.27$$

$$= (2.1 - j20.27)\text{V} = 20.38\angle{-84.08°}$$

The current in $(-j20)$ branch, $I_2 = \dfrac{20.38\angle{-84.08°}}{20\angle{-90°}} = 1.02\angle{+5.92°}$

The current in $(R_1 - jX_1)$ branch, I_1

$$= 6\angle{-41.4°} - 1.02\angle 5.92° = 4.5 - j3.97 - 1.01 - j0.1$$

$$= 3.49 - j4.07 = 5.36\angle{-49.39°}$$

The impedance $Z_1 = \dfrac{V_1}{I_1} = \dfrac{20.38\angle{-84.08°}}{5.36\angle{-49.39°}}$

$$= 3.8\angle{-34.69°} = (3.12 - j2.16)\ \Omega$$

Since $R_1 - jX_1 = (3.12 - j2.16)\ \Omega$

$$R_1 = 3.12\ \Omega$$

$$X_1 = 2.16\ \Omega$$

Example 3.36 *Determine the value of the voltage source and power factor in the following network if it delivers a power of 100 W to the circuit shown in Fig. 3.51. Find also the reactive power drawn from the source.*

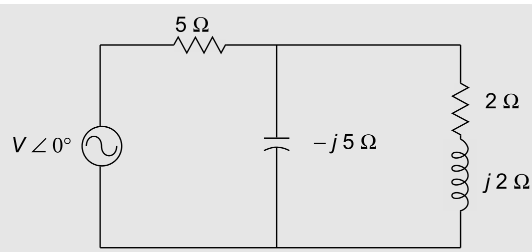


Fig. 3.51

Solution Total impedance in the circuit,

$$Z_{eq} = 5 + \frac{(2 + j2)(-j5)}{2 + j2 - j5}$$

$$= 5 + \frac{10 - j10}{2 - j3} = 5 + \frac{14.14\angle{-45°}}{3.6\angle{-56.3°}} = 5 + 3.93\angle 11.3°$$

$$= 5 + 3.85 + j0.77 = 8.85 + j0.77 = 8.88\angle 4.97°$$

Power delivered to the circuit, $P_T = I^2 R_T = 100$ W

$$\therefore \qquad I^2 \times 8.85 = 100$$

Current in the circuit, $I = \sqrt{\dfrac{100}{8.85}} = 3.36\text{ A}$

Power factor $pf = \cos\theta = \dfrac{R}{Z}$

$$= \frac{8.85}{8.88} = 0.99$$

Since $VI \cos\theta = 100$ W

$$V \times 3.36 \times 0.99 = 100$$

$$\therefore \qquad V = \frac{100}{3.36 \times 0.99} = 30.06\text{ V}$$

The value of the voltage source, $V = 30.06$ V

Reactive power $P_r = VI \sin\theta$

$$= 30.06 \times 3.36 \times \sin(4.97°)$$

$$= 30.06 \times 3.36 \times 0.087 = 8.8 \text{ VAR}$$

Example 3.37 *For the circuit shown in Fig. 3.52, determine the circuit constants when a voltage of 100 V is applied to the circuit, and the total power absorbed is 600 W. The circuit constants are adjusted such that the currents in the parallel branches are equal and the voltage across the inductance is equal and in quadrature with the voltage across the parallel branch.*

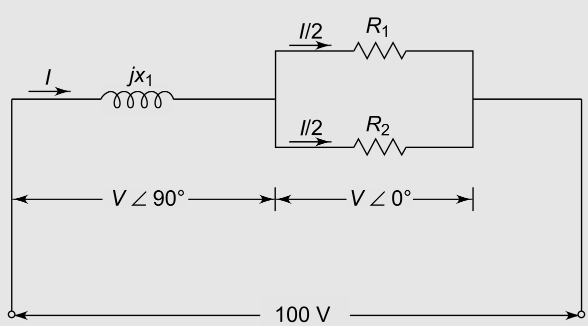


Fig. 3.52

Solution Since the voltages across the parallel branch and the inductance are in quadrature, the total voltage becomes $100\angle 45°$ as shown in Fig. 3.53.

Total voltage is $100\angle 45° = V + j0 + 0 + jV$

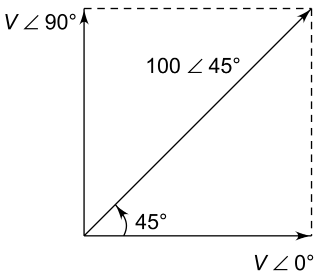


Fig. 3.53

From the above result, $70.7 + j70.7 = V + jV$

$$\therefore \quad V = 70.7$$

If we take current as the reference, then current passing through the circuit is $I\angle 0°$. Total power absorbed by the circuit $= VI\cos\theta = 600$ W

or $\quad 100 \times I \times \cos 45° = 600$ W

$$\therefore \quad I = 8.48 \text{ A}$$

Hence, the inductance, $X_1 = \dfrac{V\angle 90^\circ}{I\angle 0^\circ} = \dfrac{70.7\angle 90^\circ}{8.48} = 8.33\angle 90^\circ$

$$\therefore \quad X_1 = 8.33\ \Omega$$

Current through the parallel branch, R_1 is $I/2 = 4.24$ A

Resistance, $\quad R_1 = \dfrac{V\angle 0}{I/2\angle 0} = \dfrac{70.7}{4.24} = 16.6\,\Omega$

Current through parallel branch R_2 is $I/2 = 4.24$ A

Resistance is $\quad R_2 = \dfrac{70.7}{4.24} = 16.67\,\Omega$

Example 3.38 *Determine the average power delivered by the $500\angle 0°$ voltage source in Fig. 3.54 and also dependent source.*

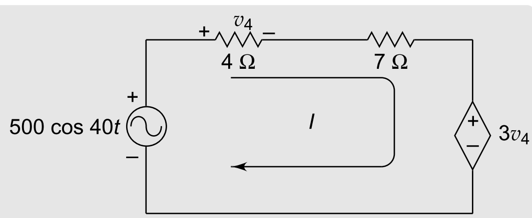


Fig. 3.54

Solution The current I can be determined by using Kirchhoff's voltage law.

$$I = \frac{500\angle 0^\circ - 3v_4}{7+4}$$

where $\quad v_4 = 4I$

$$I = \frac{500\angle 0^\circ}{11} - \frac{12I}{11}$$

$$I = 21.73\angle 0°$$

Power delivered by the $500\angle 0°$ voltage source $= \dfrac{500 \times 21.73}{2} = 5.432\,\text{kW}$

Power delivered by the dependent voltage source

$$= \frac{3v_4 \times I}{2} = \frac{3 \times 4I \times I}{2} = 2.833\,\text{kW}$$

Example 3.39 *Find the average power delivered by the dependent voltage source in the circuit shown in Fig. 3.55.*

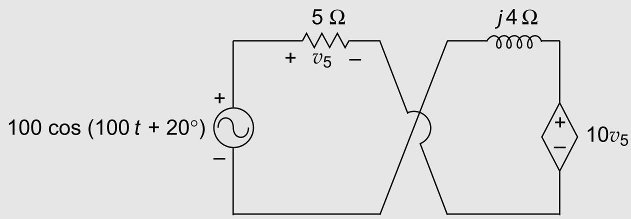


Fig. 3.55

Solution The circuit is redrawn as shown in Fig. 3.56.

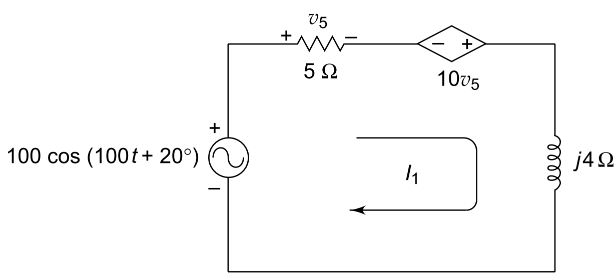


Fig. 3.56

Assume current I_1 flowing in the circuit.

The current I_1 can be determined by using Kirchhoff's voltage law.

$$I_1 = \frac{100\angle 20^{\circ} + 10 \times 5 I_1}{5 + j4}$$

$$I_1 - \frac{50\, I_1}{5 + j4} = \frac{100\angle 20^{\circ}}{5 + j4}$$

$$I_1 = 2.213 \angle -154.9^{\circ}$$

Average power delivered by the dependent source

$$= \frac{V_m\, I_m}{2} = \cos\theta$$

$$= \frac{10 V_5\, I_1}{2} \cos\theta$$

$$= \frac{50 \times (2.213)^2}{2} = 122.43\,\text{W}$$

Example 3.40 *For the circuit shown in Fig. 3.57, find the average power delivered by the voltage source.*

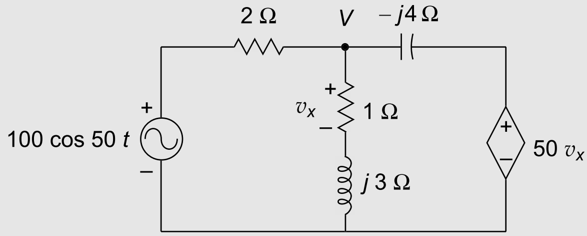


Fig. 3.57

Solution Applying Kirchhoff's current law at node

$$\frac{V-100\angle 0^\circ}{2}+\frac{V}{1+j3}+\frac{V-50V_x}{-j4}=0$$

$$V_x=\frac{V}{1+j3}\text{ volts}$$

Substituting in the above equation, we get

$$\frac{V-100\angle 0^\circ}{2}+\frac{V}{1+j3}+\frac{V}{-j4}-\frac{50\,\text{V}}{(1+j3)(-j4)}=0$$

$$V = 14.705\ \angle 157.5^\circ$$

$$I=\frac{V-100\angle 0^\circ}{2}=\frac{14.705\angle 157.5^\circ-100\angle 0^\circ}{2}=56.865\angle 177.18^\circ$$

$$\text{Power delivered by the source }=\frac{100\times 56.865\cos 177.18^\circ}{2}$$

$$= 2.834\text{ kW}$$

Example 3.41 *For the circuit shown in Fig. 3.58, find the average power delivered by the dependent current source.*

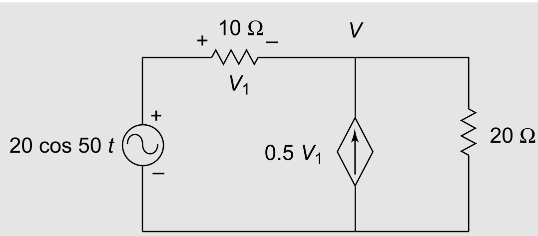


Fig. 3.58

Solution Applying Kirchhoff's current law at node

$$\frac{V-20\angle 0^\circ}{10}-0.5V_1+\frac{V}{20}=0$$

where $V_1 = 20\ \angle 0^\circ - V$

Substituting V_1 in the above equation, we get

$$V = 18.46 \angle 0°$$

$$V_1 = 1.54 \angle 0°$$

Average power delivered by the dependent source

$$\frac{V_m I_m \cos\theta}{2} = \frac{18.46 \times 0.5 \times 1.54}{2} = 7.107 \text{ W}$$

Example 3.42 *In the circuit shown in Fig. 3.59, calculate,*

(i) The total impedance

(ii) The total current

(iii) Power factor

(iv) The total S, P and Q

(v) The total admittance. Also, draw vector diagram. *[JNTU May/June 2006]*

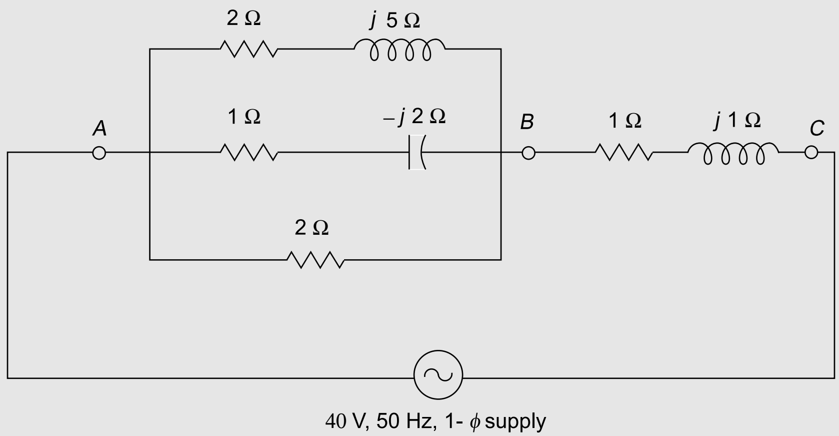


Fig. 3.59

Solution (i)

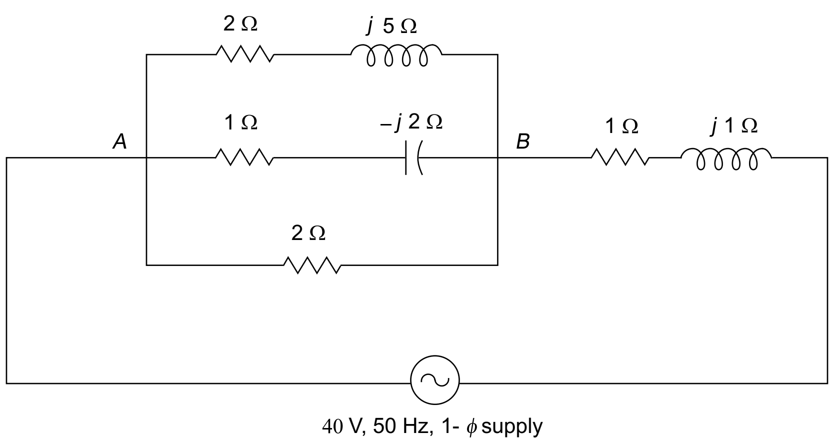


Fig. 3.60

Admittance between A and B is

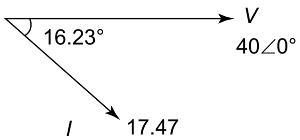


Fig. 3.61

$$\frac{1}{2+j5}+\frac{1}{1-j2}+\frac{1}{2}$$

$$=\frac{1}{5.38\angle 68.2^{\circ}}+\frac{1}{2.24\angle -63.4^{\circ}}+0.5$$

$$=0.069-j0.17+0.199+j0.399+0.5$$

$$=0.768+j0.229=0.8\angle 16.6^{\circ}$$

Impedance between A and B = $\dfrac{1}{0.8\angle 16.6^{\circ}}=1.25\angle -16.6^{\circ}$

Total impedance = $1+j1+1.198-j0.36=2.29\angle 16.23^{\circ}\,\Omega$

(ii) Total current $=\dfrac{40}{2.29\angle 16.23^{\circ}}=17.47\angle -16.23^{\circ}$ A

(iii) Power factor = cos 16.23 = 0.96 lagging

(iv) $P = VI\cos\varphi$

$= 40 \times 17.47 \cos 16.23° = 670.95$ W

$Q = VI\sin\varphi$

$= 40 \times 17.47 \sin 16.23° = 195.31$ VAR

$S = P + jQ = 640.95 + j195.31$

$= 698.798\angle 16.23°$ VA.

(v) Total admittance $=\dfrac{1}{2.29\angle 16.23°}=0.43\angle -16.23°\nu$

Example 3.43 *The voltage of a circuit is V = 200 sin (ωt + 30°) and the current is I = 50 sin (ωt + 60°). Calculate*

(i) the average power, reactive volt-amperes and apparent power

(ii) the circuit elements if ω = 100π rad/sec *[JNTU April/May 2007]*

Solution $V = 200\sin(\omega t + 30°)$

$i = 50\sin(\omega t + 60°)$

(i) Avg. power $= V_m I_m \cos\theta$

$$=\frac{200}{\sqrt{2}}\times\frac{50}{\sqrt{2}}\cos(60-30)$$

$$P_{av}=4330.127\text{ W.}$$

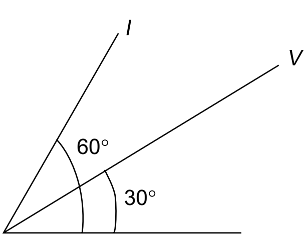


Fig. 3.62

Reactive volt ampere $= V_m I_m \sin\theta$

$$=\frac{200}{\sqrt{2}}\cdot\frac{50}{\sqrt{2}}\sin(60-30)$$

$$P_r = 2500\text{ VAR}$$

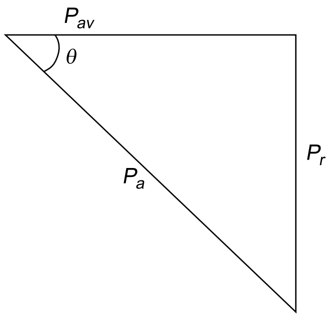


Fig. 3.63

Apparent power

$$= \frac{p_{av}}{\cos\theta} = \frac{4330.127}{\cos 30^\circ} = 5000 \text{ VA}$$

(ii) The current leads the voltage by 30°. Hence the circuit must contain R and C.

$$\tan\theta = \frac{1}{\omega RC} \Rightarrow \tan 30^\circ = \frac{1}{100\pi \times RC}$$

$$\Rightarrow RC = 0.0055 \Rightarrow C = \frac{0.0055}{R}$$

$$Z = \frac{Vm}{I_m} = \sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2}$$

$$\frac{200}{50} = \sqrt{R^2 + \left(\frac{R}{100\pi \times 0.0055}\right)^2}$$

$$4 = 1.155\text{R} \Rightarrow R = \frac{4}{1.155} = 3.46\ \Omega$$

$$\text{and } C = \frac{0.0055}{3.46} = 1.59 \text{ mF}$$

Example 3.44 *In the circuit shown in Fig. 3.64, what 50-Hz voltage is to be applied across A B terminals so that a current of 10 A will flow in the capacitor.*

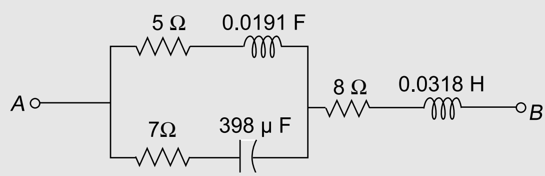


Fig. 3.64

Solution

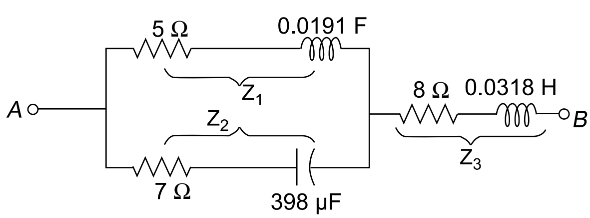


Fig.3.65

$$Z_1 = 5 + j2\pi \times 50 \times 0.0191 = 5 + j6\ \Omega$$

$$Z_2 = 7 + \frac{1}{j2\pi \times 50 \times 398\mu} = 7 - j8\ \Omega$$

$$Z_3 = 8 + j2\pi \times 50 \times 0.0318 = 8 + j10\ \Omega$$

Given that current through the capacitor is 10 A = I_2. Hence voltage across Z_2 is

$$V_1 = 10 \times Z_2 = 10(7 - j8) = 70 - j80 \text{ V}$$

The current through the other branch is

$$I_1 = \frac{V_1}{Z_1}$$

$$= \frac{70 - j80}{5 + j6} = -2.13 - j13.44\,\text{A}$$

Total current in the network is

$$I = I_1 + I_2$$
$$= -2.13 - j13.44 + 10$$
$$= 7.87 - j13.44\ A$$

Let V_2 be the voltage across Z_3.

$$V_2 = IZ_3$$
$$= (7.87 - j13.44)\,(8 + j10)$$
$$= 197.38 - j28.85\ \text{V}$$

The voltage to be applied across AB terminals so that a current of 10 A will flow in the capacitor $V = V_1 + V_2$

$$= 70 - j80 + 197.38 - j28.85$$
$$= 267.38 - j108.85$$
$$= 288.68 \angle -22.15°\ \text{V}.$$

Example 3.45 *Find the values of R and C in the circuit shown in Fig. 3.66 so that $V_b = 4V_a$ and V_a and V_b are in phase quadrature.*

[JNTU May/June 2002]

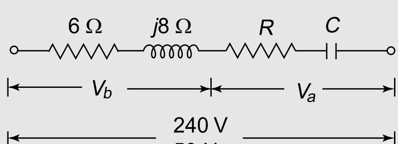


Fig. 3.66

Solution

$$V_b = I\sqrt{6^2 + 8^2} = 10\,I$$

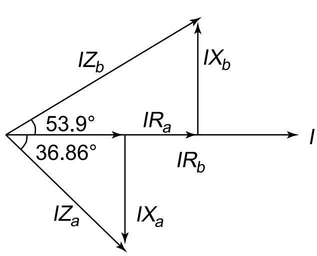


Fig. 3.67

$$Z_b = 6 + j8 - 10\angle 53.13°, \quad Z_a = \sqrt{R^2 + X_a^2}$$

$$V_a = I\sqrt{R^2 + X_a^2}, \quad V_b = 4\,V_a$$

$$10\,I = 4 \times I\sqrt{R^2 + X_a^2} \Rightarrow \sqrt{R^2 + X_a^2} - 2.5$$

$$R^2 + X_a^2 = 6.25$$

Let Z_a be at an angle θ with reference. Given that V_a and V_b are in phase quadrature.

$$\therefore \qquad \theta + 53.13^\circ = 90^\circ \Rightarrow \theta = 36.87^\circ$$

$$R = Z_a \cos\phi = 2$$

$$X_a = Z_a \sin\phi = 1.5\ \Omega$$

$$\Rightarrow \qquad C = \frac{1}{2\pi f\, X_a} = \frac{1}{2\pi \times 50 \times 1.5} = 2.12\,\text{mF}$$

Example 3.46 *Determine the branch and total active and reactive powers in the parallel circuit shown in Fig. 3.68. Use j notation.* *[JNTU May/June 2002]*

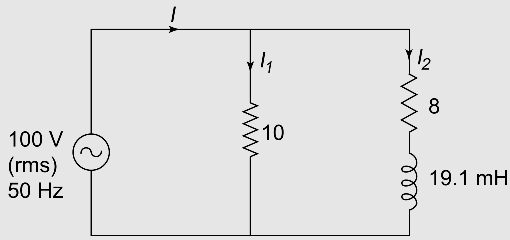


Fig. 3.68

Solution *Branch I*

$I_1 = 10\text{ A}\angle 0°$

Active power

$= I^2R = 10^2\,(10)$

$= 1\text{ kW}$

Reactive power $= 0$

Branch II

$$I_2 = \frac{100}{8 + j6} = 10\angle -36.86° = 8 - j6$$

Complex power $= VI^* = 100\,(8 + j6)$

$= 800 + j\,600$

Active power $= 0.8$ kW

Reactive power $= 0.6$ KVAR

$\therefore$ Total active power in the circuit $= 1.8$ kW

$\therefore$ Total reactive power in the circuit $= 0.6$ KVAR

Example 3.47 *Find the branch currents, total current and the total power in the circuit shown in Fig. 3.69.* *[JNTU May/June 2004]*

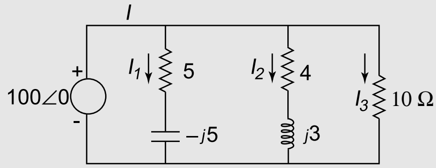


Fig. 3.69

Solution Branch currents $I_1 = \dfrac{100 + j0}{5 - j5} = 10 + j10$

$$I_2 = \frac{100 + j0}{4 + j3} = 16 - j12$$

$$I_3 = \frac{100 + j0}{10} = 10 + j0$$

$$\text{Total current } (I) = I_1 + I_2 + I_3$$
$$= 36 - j2$$
$$= 36.055\underline{|-3.179°}$$

$$\text{Total power} = VI \times \cos\phi$$
$$= 100 \times 36.055 \times \cos 3.179°$$
$$= 3599.95 \text{ watts.}$$

Example 3.48 *Find the total current and the power consumed by the circuit.*
[JNTU May/June 2004]

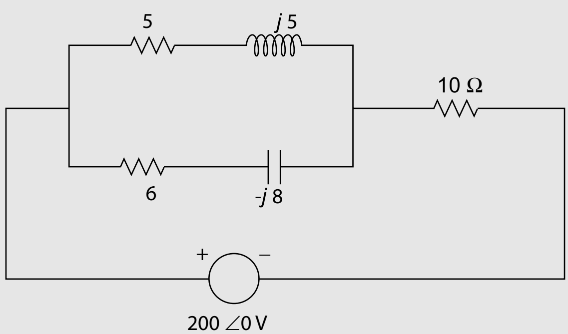


Fig. 3.70

Solution Total impedance of the circuit,

$$Z_T = (5 + j5) \,||\, (6 - j8) + 10$$

$$Z_T = 16.15 + j0.769$$

$$I = \frac{V}{Z_T} = \frac{200\angle 0}{16+5+j0.769}$$
$$= 12.35 - j0.588 \text{ A}$$
$$= 12.36\angle -2.72°$$

$$\text{Power consumed} = I^2R$$
$$= (12.36)^2 \times 16.15 = 2467 \text{ W}$$

or $$VI\cos\theta = 200 \times 12.36 \times \cos(-2.72)$$
$$= 2467 \text{ W.}$$

Example 3.49 *Find the value of R_1 and X_1 when a lagging current in the circuit gives a power of 2 kW.* *[JNTU May/June 2004]*

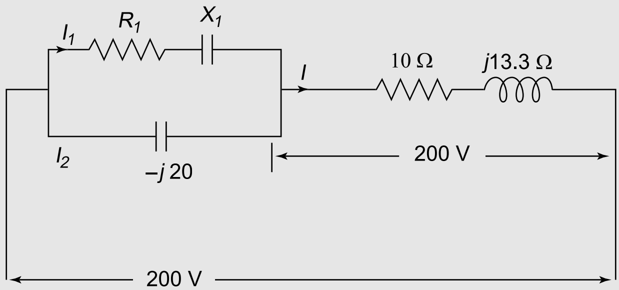


Fig. 3.71

Solution Let us take the voltage across (10 + j13.3 Ω) impedance as reference and calculate the total current I.

$$I = \frac{200\angle 0}{10 + j13.3} = 7.223 - j9.606 = 12.02\angle -53.06°\text{A}$$

Let us assume the phase angle between supply voltage and total current as ϕ which is equal to (θ + 53.06°).

Hence, real power in the circuit 2000 = 200 × 12.02 cos (θ + 53.06)

Therefore, θ = −19.5° and source voltage V = 200 ∠ −19.5°

Voltage across $R_1 + jX_1$ = 200 ∠ −19.5° − 200 ∠ 0°

$$= -11.47 - j\,66.76$$

$$I_2 = \frac{-11.47 - j66.76}{-j20} = 3.338 - j0.5735$$

$$I_1 = I - I_2$$

$$= 7.223 - j9.606 - 3.338 + j0.5735$$

$$= 9.8325\angle -66.72^{\text{v}}$$

$$Z_1 = \frac{V}{I_1} = \frac{-11.47 - j66.76}{9.8325\angle -66.72}$$

$$= 5.776 - j3.7543$$

Thus, R_1 = 5.776 Ω and X_1 = 3.7543 Ω.

Example 3.50 *A metal filament lamp, rated at 750 watts, 100 V is to be connected in series with a capacitor across a 230 V, 60 Hz supply. Calculate*

(a) the capacitance required, and

(b) the power factor. *[JNTU May/June 2008]*

Solution Given power across metal filament lamp = 750 watts

Voltage across metal filament lamp = 100 V

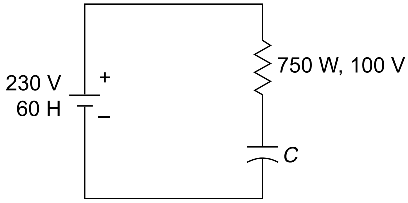


Fig. 3.72

Supply voltage = 230 V

Supply frequency = 60 Hz

Resistance across metal filament lamp =

$$R = \frac{V^2}{P} = \frac{(100)^2}{750} = 13.33\,\Omega$$

Current through metal filament lamp $= I = \dfrac{V}{R} = \dfrac{100}{13.33} = 7.5\text{ Amp}$

Impedance in the circuit

$$Z = \sqrt{R^2 + X_C^2} = \frac{V}{I}$$

$$\sqrt{R^2 + X_C^2} = \frac{230}{7.5} = 30.66$$

$$\sqrt{(13.33)^2 + X_C^2} = 30.66$$

$$X_C = 2.7.617 \Rightarrow \frac{1}{\omega c} = 27.617$$

$$\Rightarrow \qquad C = 9.605 \times 10^{-5}\,F$$

$\therefore$ Capacitance $C = 96.05\ \mu F$

$$\cos\theta = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{13.33}{30.66}$$

Power factor $= \cos\theta = 0.43468$

Example 3.51 *In the circuit (Fig. 3.73) shown, determine the voltage V_{AB} to be applied to the circuit if a current of 2.5 A is required to flow in the capacitor. Determine also total power factor and total active and reactive powers. Draw the phasor diagram.* *[JNTU May/June 2006]*

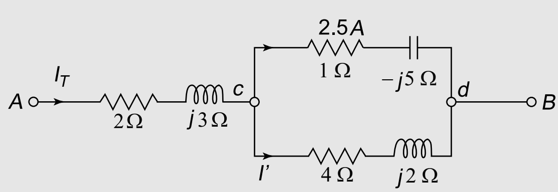


Fig. 3.73

Solution $V_{cd} = 2.5(1 - j5) = I'(4 + j2)$

(Assuming "I" is the current through '$4 + j2$' Ω)

$$I' = \frac{2.5(1-j5)}{4+j2} = 2.85\angle -105.25°$$
$$= -0.75 - j2.75$$

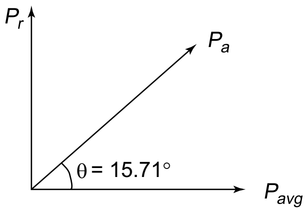


Fig. 3.74

$$I_T = 2.5 - 0.75 - j2.75 \qquad (\because I_T = 2.5 + I')$$
$$\therefore \quad = 1.75 - j2.75 = 3.26\angle -57.53°$$
$$V_{AB} = I_T(2+j3) + 2.5(1-j5)$$
$$= (1.75 - 2.75j)(2+j3) + 2.5(1-j5)$$
$$= 14.25 - 12.75j = 19.12\angle -41.82°$$
$$Z_{AB} = \frac{V_{AB}}{I_T} = \frac{19.12\angle -41.82°}{3.26\angle -57.53°} = 5.865\angle 15.71°$$
$$\theta = 15.71°$$

Total power factor $= \cos\theta = \cos 15.71° = 0.962$

Total active power $(P_{\text{avg}}) = V_{AB} I_T \cos\theta$

$$= 19.12 \times 3.26 \times 0.962 = 59.96 \text{ W}$$

Total reactive power $(P_r) = V_{AB} I_T \sin\theta$

$$= 19.12 \times 3.26 \times \sin 15.71° = 16.87 \text{ VAR}$$

Apparent power $\quad P_a = V_{AB} I_T = 19.12 \times 3.26 = 62.3112\,VA$

Example 3.52 *A current of 5 A flows through a non-inductive resistance in series with a chocking coil when supplied at 250 V, 50 Hz. If the voltage across the non inductive resistance is 125 V and that across that coil 200 V, calculate the impedance, reactance and resistance of the coil, power absorbed by the coil and the total power draw the phasor diagram.* *[JNTU May/June 2006]*

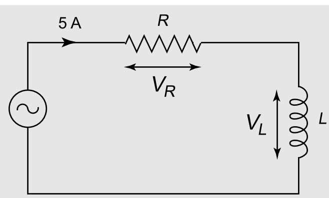


Fig. 3.75

Solution Given

$$|V_R| = 125 \text{ V}$$
$$|V_L| = 200 \text{ V}$$
$$|I| = 5 \text{ A}$$

$$|V_R| = |I|\,R = 125 \text{ V} \qquad \Rightarrow \quad R = \frac{125}{5} = 25\,\Omega \quad (\because I = 5\text{A})$$
$$|V_L| = |I|\,X_L = |I|\,(j\omega L) \quad \therefore \quad |V_L| = 200 \text{ V}$$
$$\Rightarrow |X_L| = 40 \qquad \Rightarrow \quad 5(2\pi \times 50)L = 200$$

Phasor diagram:

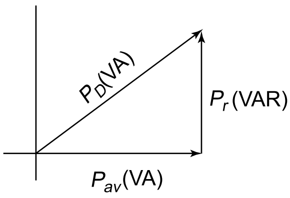


Fig. 3.76

$$\Rightarrow L = \frac{200}{500\pi} = 127.3 \text{ mH}$$

$$Z = 25 + j40 = 47.16\underline{|57.99^\circ}$$

$$\text{Power absorbed by coil } = \frac{1}{2}LI^2$$

$$= \frac{1}{2} \times 0.1273 \times 25$$

$$= 1.59 \text{ watts}$$

$$\text{True power } P_{av} = VI\cos\theta = 250 \times 5 \times \cos 57.99^\circ$$

$$= 662.58 \text{ watts}$$

$$\text{Reactive power, } P_r = I^2X_L = 25 \times 40$$

$$= 1000 \text{ VAR}$$

$$\text{Apparent power, } P_a = I^2Z = 25 \times 47.16$$

$$= 1179 \text{ VA}$$

Example 3.53 *In the following circuit (Fig. 3.77), when 220 V ac is applied across A and B, current drawn is 20 A and power input is 3000 W. Find the value of Z and its parameters.* *[JNTU May/June 2006]*

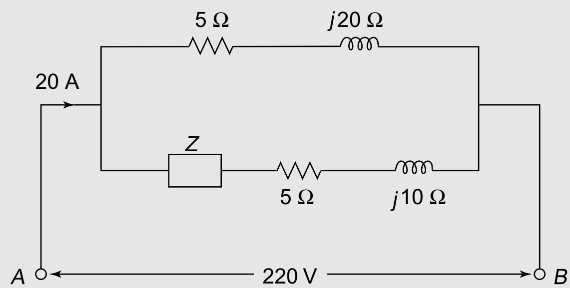


Fig. 3.77

Solution

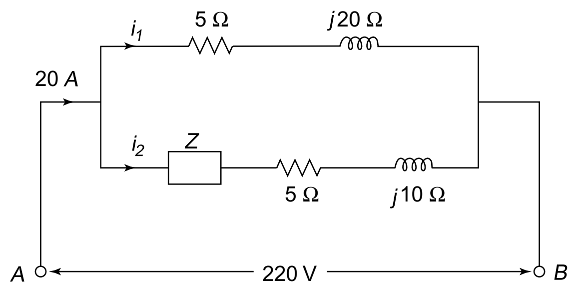


Fig. 3.78

$$i_1 = \frac{220}{5+j20}\ \text{A}$$

But $i_1 + i_2 = 20$ A

$$i_2 = 20 - \frac{220}{5+j20} \tag{1}$$

$$\text{Also, } i_2 = \frac{220}{Z+5+j10} \tag{2}$$

From (1) and (2)

$$20 - \frac{220}{5+j20} = \frac{220}{Z+5+j20}$$

$$\frac{-120+j400}{5+j20} = \frac{220}{5+Z+j20}$$

$$Z = \frac{5700+j3600}{-120+j400}$$

$$Z = -4.33 + j15.55$$

$$Z = 16.14\angle 105.56°$$

Example 3.54 *When a voltage of 220 V ac supply connected across the AB terminals, the total power input is 3.25 kW and the current is 20 A. Find the current through Z_3. (Fig. 3.79)*

[JNTU May/June 2006]

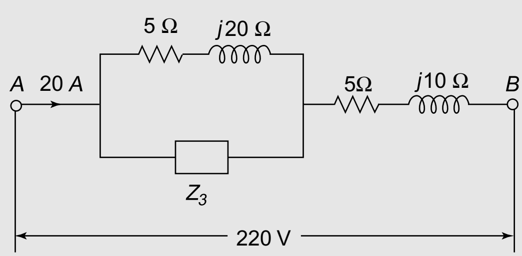


Fig. 3.79

Solution Voltage across (5 + j10) Ω branch

$$V = 20\,(5 + j10) = 223.6\ \angle 63.43° = 100 + j200$$

$$I\,(5 + j20) + 100 + j200 = 220.$$

(Let I be the current through $5 + j20\ \Omega$ branch)

$$I = \frac{120 - j200}{5+j20} = -8 - 8i$$

$$I_{Z_3} = 20 - I = 28 + 8i = 29.12\angle 15.9°$$

Example 3.55 *What is complex power? Explain in detail.*

[JNTU May/June 2006]

Solution *Complex power*

Active power (P):

The active power or real power in an a.c. circuit is given by the product of voltage, current and cosine of the phase angle. It is always positive

$$P = VI \cos\theta \text{ watts}$$

Reactive power (Q):

The reactive power in an a.c. circuit is given by the product of voltage, current and sine of the phase angle θ.
If θ is leading then reactive power is taken as +ve and it is capacitive.
If θ is lagging then reactive power is taken as −ve and it is inductive

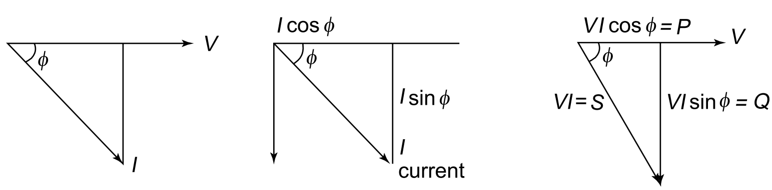
$$Q = VI \sin\theta \text{ VARs.}$$

Apparent power:

The apparent power in an a.c. circuit is the product of voltage and current. It is measured in voltamps.

$$S = VI \text{ volt amps.}$$

The component *I cos* θ = Active component or real component or in phase component of a current.
The product of voltage and the above component (active component) gives active power. The component *I sin* θ = Reactive component or quadrature component of current.



(a) Phasor diagram (b) Current (c) Power

Fig. 3.80

The produce of this component with voltage V gives the reactive power.

$$\text{Power factor } \cos\phi = \frac{\text{Real power}}{\text{Apparent power}}$$

The factor sin θ is called the reactive factor.
Complex power = (Active power) + *j* (Reactive power)

Example 3.56 *The current in a given circuit is I = (12 – j5) A when the applied voltage is V = (160 – j120)V. Determine*

(i) The complex expression for power
(ii) Power factor of the circuit
(iii) The complex expression for impedance of the circuit
(iv) Draw the phasor diagram.

[JNTU May/June 2006]

Solution (i) $P_a = V_{eff} I_{eff} VA$

$$P_{ar} = V_{eff} I_{eff} \cos\theta \text{ watts}$$

$$P_r = V_{eff} I_{eff} \sin\theta \text{ VAR}$$

$$Z = \frac{V}{I} = \frac{160 - j120}{12 - j5} = 14.91 - j3.786$$

$$|I| = 13\,\text{A}$$

$$= 15.38\angle -14.25^\circ$$

$$\therefore \quad P_{\text{avg}} = I^2 R = 2519.79\,\text{W}$$

$$P_r = I^2 X = 639.834\,\text{VAR}$$

$$P_a = I^2 Z = 2599.22\,\text{VA}$$

Complex power = 2519.79 + j 639.834

(ii) $Pf = \cos\theta = \cos(-14.25^\circ) = 0.969$

(iii) $Z = 14.91 - j3.786$

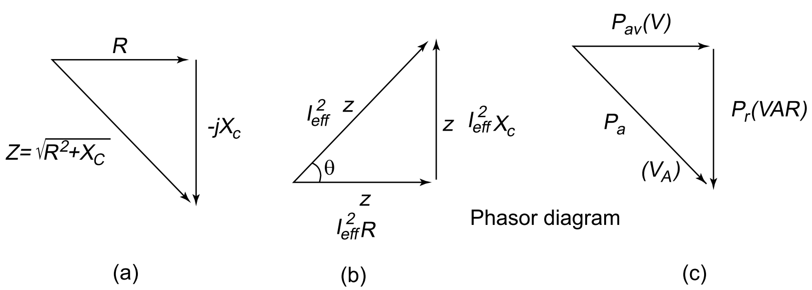


Fig. 3.81

Example 3.57 *A series RLC circuit consists of resistor of 100 Ω, an inductor of 0.318 H and a capacitor of unknown value. When this circuit is energised by a 230 V, 50 Hz ac supply, the current was found to be 23 A. Find the value of capacitor and the total power consumed.* *[JNTU June 2009]*

Solution The circuit is series RLC and is shown in Fig. 3.82

$$X_L = 2\pi f L$$

$$= 2\pi \times 50 \times 0.318 = 99.9\ \Omega$$

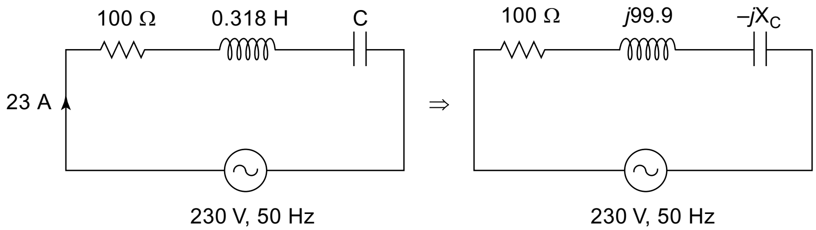


Fig. 3.82

Total impedance Z $=100 + j(99.9 - X_C)$

$$|Z| = \frac{V}{I} = \frac{230}{23} = 10\,\Omega$$

$$|Z| = \sqrt{(100)^2 + (99.9 - X_C)^2} = 10$$

$$X_C = 0.41\Omega$$

$$\frac{1}{\omega c} = 0.41\Omega$$

$$C = 7.76\,mF$$

Power consumed $= I^2R = (23)^2 \times 100$

$$= 52900 \text{ W}$$
$$= 52.9 \text{ kW}$$

Example 3.58 *Two circuits, the impedances of which $Z_1 = (10 + j15)\ \Omega$ and $Z_2 = (6 + j8)\ \Omega$ are connected in parallel. If the total current supplied is 15 A, what is the power taken by each branch?*

[JNTU Jan 2010]

Solution

Equivalent impedance $= \dfrac{Z_1\, Z_2}{Z_1 + Z_2} = \dfrac{(10 + j5)(6 + j8)}{10 + j5 + 6 + 8j} = \dfrac{(60 - 40) + j(30 + 80)}{(10 + 6) + j(5 + 8)}$ ohm

$$= \frac{20 + j110}{16 + j13} = 5.42\angle 40.60°\,\text{ohm}$$

∴ Voltage across the network $= (15 \times 5.42\angle 40.60°)$ volt

$$= 81.3\angle 40.60° \text{ volt}$$

∴ Current through $Z_1 = \dfrac{Z_2}{(Z_1 + Z_2)} \times 15\,\text{amp}$

$$= \frac{6 + j8}{16 + j13} \times 15\,\text{amp} = 7.28\angle 14.04°\text{amp}$$

∴ Current through $Z_2 = \dfrac{Z_1}{(Z_1 + Z_2)} \times 15\,\text{amp}$

$$= \frac{10 + j5}{16 + j13} \times 15\,\text{amp} = 8.13\angle -12.53°\text{amp}$$

$\therefore$ Power taken by $Z_1 = 81.3 \times 7.28 \times \cos 26.2°$ watt

$= 529.38$ watt

$\therefore$ Power taken by $Z_2 = 81.3 \times 8.13 \times \cos 53.13°$ watt

$= 396.58$ watt

3.5 STEADY STATE AC MESH ANALYSIS

We have earlier discussed mesh analysis but have applied it only to resistive circuits. Some of the ac circuits presented in this chapter can also be solved by using mesh analysis. In Chapter 2, the two basic techniques for writing network equations for mesh analysis and node analysis were presented. These concepts can also be used for sinusoidal steady-state condition. In the sinusoidal steady-state analysis, we use voltage phasors, current phasors, impedances and admittances to write branch equations, KVL and KCL equations. For ac circuits, the method of writing loop equations is modified slightly. The voltages and currents in ac circuits change polarity at regular intervals. At a given time, the instantaneous voltages are driving in either the positive or negative direction. If the impedances are complex, the sum of their voltages is found by vector addition. We shall illustrate the method of writing network mesh equations with the following example.

Consider the circuit shown in Fig. 3.83, containing a voltage source and impedances.

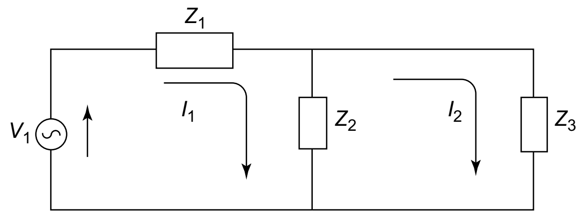


Fig. 3.83

The current in impedance Z_1 is I_1, and the current in Z_2, (assuming a positive direction downwards through the impedance) is $I_1 - I_2$. Similarly, the current in impedance Z_3 is I_2. By applying Kirchhoff's voltage law for each loop, we can get two equations. The voltage across any element is the product of the phasor current in the element and the complex impedance.

Equation for loop 1 is

$$I_1Z_1 + (I_1 - I_2)Z_2 = V_1 \tag{3.19}$$

Equation for loop 2, which contains no source is

$$Z_2(I_2 - I_1) + Z_3I_2 = 0 \tag{3.20}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(Z_1 + Z_2) - I_2Z_2 = V_1 \tag{3.21}$$

$$-I_1Z_2 + I_2(Z_2 + Z_3) = 0 \tag{3.22}$$

By solving the above equations, we can find out currents I_1 and I_2. In general, if we have M meshes, B branches and N nodes including the reference node, we assume M branch currents and write M independent equations; then the number of mesh currents is given by $M = B - (N - 1)$.

Example 3.59 *Write the mesh current equations in the circuit shown in Fig. 3.84, and determine the currents.*

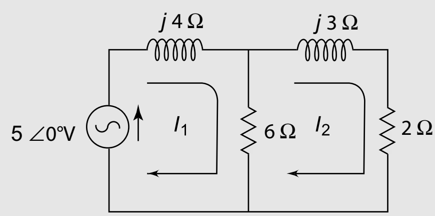


Fig. 3.84

Solution The equation for loop 1 is

$$I_1(j4) + 6(I_1 - I_2) = 5\angle 0° \tag{3.23}$$

The equation for loop 2 is

$$6(I_2 - I_1) + (j3)I_2 + (2)I_2 = 0 \tag{3.24}$$

By rearranging the above equations, the corresponding mesh current equations are

$$I_1(6 + j4) - 6I_2 = 5\angle 0° \tag{3.25}$$

$$-6I_1 + (8 + j3)I_2 = 0 \tag{3.26}$$

Solving the above equations, we have

$$I_1 = \left[\frac{(8+j3)}{6}\right] I_2$$

$$\left[\frac{(8+j3)(6+j4)}{6}\right] I_2 - 6I_2 = 5\angle 0°$$

$$I_2\left[\frac{(8+j3)(6+j4)}{6} - 6\right] = 5\angle 0°$$

$$I_2\,[10.26\angle 54.2° - 6\angle 0°] = 5\angle 0°$$

$$I_2\,[(6 + j8.32) - 6] = 5\angle 0°$$

$$I_2 = \frac{5\angle 0°}{8.32\angle 90°} = 0.6\angle -90°$$

$$I_1 = \frac{8.54\angle 20.5°}{6} \times 0.6\angle -90°$$

$I_1 = 0.855 \angle{-69.5°}$

Current in loop 1, $I_1 = 0.855\underline{|-69.5°}$

Current in loop 2, $I_2 = 0.6 \angle{-90°}$

3.5.1 Mesh Equations by Inspection

In general, mesh equations can be written by observing any network. Consider the three mesh network shown in Fig. 3.85.

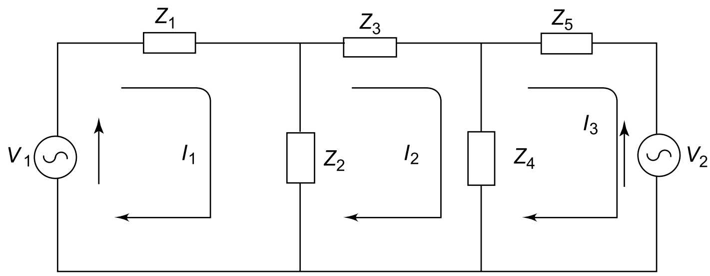


Fig. 3.85

The loop equations are

$$I_1 Z_1 + Z_2(I_1 - I_2) = V_1 \tag{3.27}$$

$$Z_2(I_2 - I_1) + Z_3 I_2 + Z_4(I_2 - I_3) = 0 \tag{3.28}$$

$$Z_4(I_3 - I_2) + Z_5 I_3 = -V_2 \tag{3.29}$$

By rearranging the above equations, we have

$$(Z_1 + Z_2)I_1 - Z_2 I_2 = V_1 \tag{3.30}$$

$$-Z_2 I_1 + (Z_2 + Z_3 + Z_4)I_2 - Z_4 I_3 = 0 \tag{3.31}$$

$$-Z_4 I_2 + (Z_4 + Z_5)I_3 = -V_2 \tag{3.32}$$

In general, the above equations can be written as

$$Z_{11} I_1 \pm Z_{12} I_2 + Z_{13} I_3 = V_a \tag{3.33}$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \tag{3.34}$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \tag{3.35}$$

If we compare the general equations with the circuit equations, we get the self impedance of loop 1

$$Z_{11} = Z_1 + Z_2$$

i.e. the sum of the impedances through which I_1 passes. Similarly, $Z_{22} = (Z_2 + Z_3 + Z_4)$, and $Z_{33} = (Z_4 + Z_5)$ are the self impedances of loops 2 and 3. This is equal to the sum of the impedances in their respective loops, through which I_2 and I_3 passes, respectively.

Z_{12} is the sum of the impedances common to loop currents I_1 and I_2. Similarly Z_{21} is the sum of the impedances common to loop currents I_2 and I_1. In the circuit shown in Fig. 3.85, $Z_{12} = -Z_2$, and $Z_{21} = -Z_2$. Here, the positive sign is used if both currents passing through the common impedance are in the same direction; and the negative sign is used if the currents are in opposite directions. Similarly, Z_{13}, Z_{23}, Z_{31}, Z_{32} are the sums of the impedances common to the mesh currents indicated in their subscripts. V_a, V_b and V_c are sums of the voltages driving their respective loops. Positive sign is used, if the direction of the loop current is the same as the direction of the source current. In Fig. 3.85, $V_b = 0$ because no source is driving loop 2. Since the source, V_2 drives against the loop current I_3, $V_c = -V_2$.

Example 3.60 *For the circuit shown in Fig. 3.86, write the mesh equations using the inspection method.*

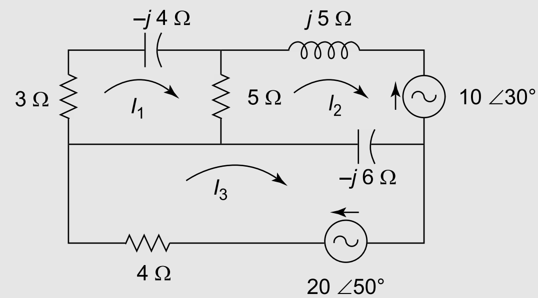


Fig. 3.86

Solution The general equations are

$$Z_{11} I_1 \pm Z_{12} I_2 \pm Z_{13} I_3 = V_a \tag{3.36}$$

$$\pm Z_{21} I_1 + Z_{22} I_2 \pm Z_{23} I_3 = V_b \tag{3.37}$$

$$\pm Z_{31} I_1 \pm Z_{32} I_2 + Z_{33} I_3 = V_c \tag{3.38}$$

Consider Eq. 3.36

Z_{11} = the self impedance of loop 1 = $(5 + 3 - j4)\ \Omega$

Z_{12} = the impedance common to both loop 1 and loop 2 = $-5\ \Omega$

The negative sign is used because the currents are in opposite directions.

$Z_{13} = 0$, because there is no common impedance between loop 1 and loop 3.

$V_a = 0$, because no source is driving loop 1.

∴ Equation 3.36 can be written as

$$(8 - j4)I_1 - 5I_2 = 0 \tag{3.39}$$

Now, consider Eq. 3.37

$Z_{21} = -5$, the impedance common to loop 1 and loop 2.

$Z_{22} = (5 + j5 - j6)$, the self impedance of loop 2.

$Z_{23} = -(-j6)$, the impedance common to loop 2 and loop 3.

$V_b = -10\ \angle 30°$, the source driving loop 2.

The negative sign indicates that the source is driving against the loop current, I_2. Hence, Eq. 3.37 can be written as

$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10\underline{|30^\circ} \tag{3.40}$$

Consider Eq. 3.38

$Z_{31} = 0$, there is no common impedance between loop 3 and loop 1
$Z_{32} = -(-j6)$, the impedance common to loop 2 and loop 3
$Z_{33} = (4 - j6)$, the self impedance of loop 3
$V_b = 20 \angle 50°$, the source driving loop 3

The positive sign is used because the source is driving in the same direction as the loop current 3. Hence, the equation can be written as

$$(j6)I_2 + (4 - j6)I_3 = 20 \angle 50° \tag{3.41}$$

The three mesh equations are

$$(8 - j4)I_1 - 5I_2 = 0$$
$$-5I_1 + (5 - j1)I_2 + (j6)I_3 = -10 \angle 30°$$
$$(j6)I_2 + (4 - j6)I_3 = 20 \angle 50°$$

Example 3.61 *For the circuits shown in Fig. 3.87, determine the line currents I_R, I_Y and I_B using mesh analysis.*

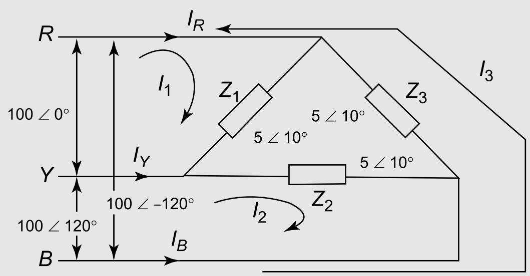


Fig. 3.87

Solution From Fig. 3.87, the three line currents are

$$I_R = I_1 - I_3$$
$$I_Y = I_2 - I_1$$
$$I_B = I_3 - I_2$$

Using the inspection method, the three loop equations are

$$5 \angle 10° \, I_1 = 100 \angle 0°$$
$$5 \angle 10° \, I_2 = 100 \angle 120°$$
$$5 \angle 10° \, I_3 = 100 \angle -120°$$

$$\therefore \qquad I_1 = \frac{100\angle 0°}{5\angle 10°} = 20\angle -10°$$

$$I_2 = \frac{100\angle 120°}{5\angle 10°} = 20\angle +110°$$

$$I_3 = \frac{100\angle -120°}{5\angle 10°} = 20\angle -130°$$

The line currents are

$$I_R = I_1 - I_3 = 20\angle -10° - 20\angle -130°$$
$$= 19.69 - j3.47 + 12.85 + j15.32$$
$$= 32.54 + j11.85 = 34.63\angle 20°$$
$$I_Y = I_2 - I_1 = 20\angle 110° - 20\angle -10°$$
$$= -6.84 + j18.79 - 19.69 + j3.47$$
$$= -26.53 + j22.26 = 34.63\angle 140°$$
$$I_B = I_3 - I_2 = 20\angle -130° - 20\angle 110°$$
$$= -12.85 - j15.32 + 6.84 - j18.79 = -6.01 - j34.11 = 34.63\angle -100°$$

Example 3.62 *For the circuit shown in Fig. 3.88, determine the value of V_2 such that the current (3 + j4) Ω impedance is zero.*

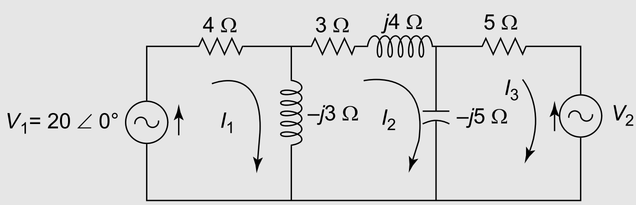


Fig. 3.88

Solution The three loop equations are

$$(4 + j3)I_1 - (j3)I_2 = 20\angle 0°$$
$$(-j3)I_1 + (3 + j2)I_2 + j5I_3 = 0$$
$$(j5)I_2 + (5 - j5)I_3 = -V_2$$

Since the current I_2 in $(3 + j4)$ Ω is zero

$$I_2 = \frac{\Delta_2}{\Delta} = 0$$

∴ $$\Delta_2 = 0$$

where $$\Delta_2 = \begin{vmatrix} (4+j3) & 20\angle 0° & 0 \\ (-j3) & 0 & j5 \\ 0 & -V_2 & (5-j5) \end{vmatrix} = 0$$

$$(4 + j3)V_2(j5) - 20\angle 0°\{(-j3)(5 - j5)\} = 0$$

$$V_2 = \frac{20\angle 0^\circ\{(-j3)(5-j5)\}}{(j5)(4+j3)}$$

$$= 20\angle 0^\circ \frac{\{-15-j15\}}{-15+j20} = 20\angle 0^\circ \times \frac{21.21\angle -135^\circ}{25\angle 126.86^\circ}$$

$$V_2 = 16.97\angle -261.86^\circ\text{V}$$

Example 3.63 *Using mesh analysis, determine the voltage V_s which gives a voltage of $30\angle 0^\circ$ V across the 30 Ω resistor shown in Fig. 3.89.*

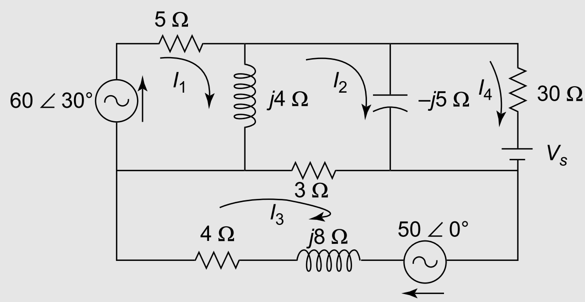


Fig. 3.89

Solution By the inspection method, we can have four equations from four loops.

$$(5+j4)I_1 - (j4)I_2 = 60\angle 30^\circ \tag{3.42}$$

$$(-j4)I_1 + (3-j1)I_2 - 3I_3 + (j5)I_4 = 0 \tag{3.43}$$

$$-3I_2 + (7+j8)I_3 = 50\angle 0^\circ \tag{3.44}$$

$$(j5)I_2 + (30-j5)I_4 = -V_s \tag{3.45}$$

Solving the above equations using Cramer's rule, we get

$$I_4 = \frac{\begin{vmatrix} (5+j4) & (-j4) & 0 & 60\angle 30^\circ \\ (-j4) & (3-j1) & -3 & 0 \\ 0 & -3 & (7+j8) & 50\angle 0^\circ \\ 0 & (j5) & 0 & -V_s \end{vmatrix}}{\begin{vmatrix} (5+j4) & (-j4) & 0 & 0 \\ (-j4) & (3-j1) & -3 & (j5) \\ 0 & -3 & (7+j8) & 0 \\ 0 & (j5) & 0 & (30-j5) \end{vmatrix}}$$

$$\Delta = (5+j4)\begin{vmatrix} (3-j1) & -3 & (j5) \\ -3 & (7+j8) & 0 \\ (j5) & 0 & (30-j5) \end{vmatrix}$$

$$+(j4)\begin{vmatrix} (-j4) & -3 & (j5) \\ 0 & (7+j8) & 0 \\ 0 & 0 & (30-j5) \end{vmatrix}$$

$$= (5 + j4)\ \{(3 - j1)\ (7 + j8)\ (30 - j5) + 3\ [(-3)\ (30 - j5)]$$
$$+ j5\ [(-j5)\ (7 + j8)]\} + (j4)\ \{(-j4)\ (7 + j8)\ (30 - j5)\}$$
$$= (5 + j4)\ \{[3.16\ \angle{-18.4°} \times 10.6\ \angle 48.8° \times 30.4\ \angle{-9.46°}]$$
$$- 9 \times 30.4\ \angle{-9.46°} + 25\ (10.6\ \angle 48.8°)\}$$
$$+ (j4)\ \{4\ \angle{-90°} \times 10.6\ \angle 48.8° \times 30.4\ \angle{-9.46°}\}$$
$$= (5 + j4)\ \{1018.27\ \angle 20.94° - 273\ .6\ \angle{-9.46°} + 265\ \angle 48.8°\}$$
$$+ j4\ \{1288.96\ \angle{-50.66°}\}$$
$$= (5 + j4)\ \{951 + j363.9 - 269.8 + j44.97 + 174.55 + j199.38\}$$
$$+ 4\ \angle 90°\ \{1288.96\ \angle{-50.66°}\}$$
$$= (5 + j4)\ \{855.75 + j608.25\} + 4\ \angle 90°\ \{1288.96\ \angle{-50.66°}\}$$
$$= 6.4\ \angle 38.6° \times 1049.9\ \angle 35.4° + 4\ \angle 90° \times 1288.96\ \angle{-50.66°}$$
$$= 6719.36\ \angle 74° + 5155.84\ \angle 39.34°$$
$$= 1852.1 + j6459 + 3987.5 + j3268.3$$
$$= 5839.6 + j9727.3$$
$$= 11345.5\ \angle 59°$$

$$\Delta_4 = (5 + j4)\begin{vmatrix} (3 - j1) & -3 & 0 \\ -3 & (7 + j8) & 50\angle 0° \\ (j5) & 0 & -V_s \end{vmatrix}$$

$$+ j4\begin{vmatrix} -j4 & -3 & 0 \\ 0 & (7 + j8) & 50\angle 0° \\ 0 & 0 & -V_s \end{vmatrix} - 60\angle 30°\begin{vmatrix} -j4 & (3 - j1) & -3 \\ 0 & -3 & 7 + j8 \\ 0 & (j5) & 0 \end{vmatrix}$$

$$= (5 + j4)\ \{[(3 - j1)\ (7 + j8)\ (-V_s)] + 3[(3V_s) - (j5)\ 50\ \angle 0°]\}$$
$$+ (j4)\ \{(-j4)\ (7 + j8)\ (-V_s)\} - 60\ \angle 30°\ \{(-j4)\ (-j5)\ (7 + j8)\}$$
$$= 6.4\ \angle 38.6°\ \{[3.16\ \angle{-\,18.4°} \times 10.6\ \angle 48.8°\ (-V_s)]$$
$$+ [9V_s - (15j)\ 50\ \angle 0°] + 4\ \angle 90°\ \{4\ \angle{-\,90°} \times 10.6\ \angle 48.8°)\ (-V_s)\}$$
$$- 60\ \angle 30°\ \{4\ \angle{-\,90°} \times 5\ \angle{-\,90°} \times 10.6\ \angle 48.8°\}$$
$$= 6.4\ \angle 38.6°\ \{-33.49\ \angle 30.4°\ V_s\} + 6.4\ \angle 38.6° \times 9V_s$$
$$+ 4\ \angle 90°\ \{-42.4\ \angle{-\,41.2°}\ V_s\} - 60\ \angle 30°\ \{212\ \angle{-\,131.2°}\}$$
$$- 6.4\ \angle 38.6°\ \{+750\ \angle 90°\}$$
$$= V_s\ \{-214.33\ \angle 69° + 57.6\ \angle 38.6° - 169.6\ \angle 48.8°\}$$
$$- \{12720\ \angle{-\,101.2°} + 4800\ \angle 128.6°\}$$
$$= V_s\ \{-76.8 - j200 + 45 + j35.93 - 111.7 - j127.6\}$$
$$- \{-2470.6 - j12477.75 - 2994.6 + j3751.2\}$$
$$= V_s\ \{-143.5 - j291.67\} - \{-5465.2 - j8726.55\}$$

$$\therefore \quad I_4 = \frac{(-143.5 - j291.67)V_s + (5465.2 + j8726.5)}{11345.5\angle 59^\circ}$$

Since voltage across the 30 Ω resistor is 30 ∠0° V. Current passing through it is $I_4 = 1\ \angle 0°$ A

$$\therefore \quad 1\angle 0^\circ = \frac{(-143.5 - j291.67)V_s + (5465.2 + j8726.5)}{11345.5\angle 59^\circ}$$

$$11345.5\ \angle 59^\circ = 325\ \angle -116.19^\circ\ V_s + 5465.2 + j8726.5$$

$$V_s = \frac{-5465.2 - j8726.5 + 5843.36 + j9724.99}{325\angle -116.19^\circ}$$

$$= \frac{378.16 + j998.49}{325\angle -116.19^\circ} = \frac{1067.7 + j69.26^\circ}{325\angle -116.19^\circ}$$

$$V_s = 3.29\ \angle 185.45^\circ.$$

Example 3.64 *Determine the voltage V which results in a zero current through the impedance in the circuit shown in Fig. 3.90*

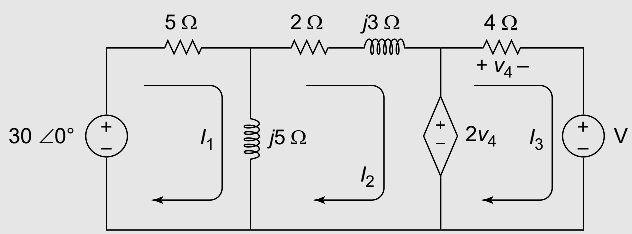


Fig. 3.90

Solution Choosing mesh currents as shown in Fig. 3.90, the three loop equations are

$$(5 + j5)I_1 - j5\ I_2 = 30\ \angle 0^\circ$$

$$-j5\ I_1 + (2 + j8)\ I_2 = -2V_4$$

$$-2V_4 + V_4 + V = 0$$

$$V_4 = V$$

Since the current (2 + *j*3) Ω is zero

$$I_2 = \frac{\Delta_2}{\Delta} = 0$$

where $$\Delta_2 = \begin{vmatrix} 5+j5 & 30\angle 0^\circ \\ -j5 & -2V \end{vmatrix} = 0$$

$$(5+j5)(-2V)+(j5)30\angle 0^\circ = 0$$

$$V = \frac{30\angle 0^\circ (j5)}{2(5+j5)} = \frac{150\angle 90^\circ}{14.14\angle 45^\circ}$$

$$V = 10.608\angle 45^\circ \text{volts}$$

3.6 STEADY STATE AC NODAL ANALYSIS

The node voltage method can also be used with networks containing complex impedances and excited by sinusoidal voltage sources. In general, in an N node network, we can choose any node as the reference or datum node. In many circuits, this reference is most conveniently chosen as the common terminal or ground terminal. Then it is possible to write (N – 1) nodal equations using KCL. We shall illustrate nodal analysis with the following example.

Consider the circuit shown in Fig. 3.91.

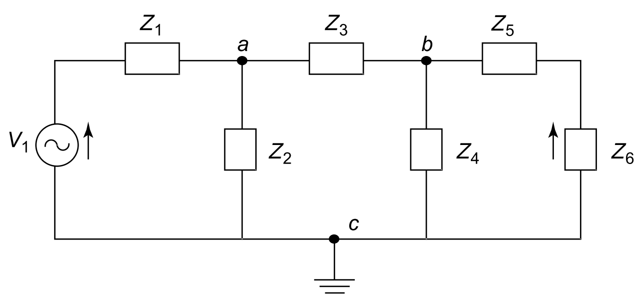


Fig. 3.91

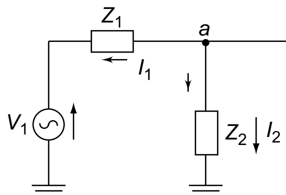


Fig. 3.92

Let us take a and b as nodes, and c as reference node. V_a is the voltage between nodes a and c. V_b is the voltage between nodes b and c. Applying Kirchhoff's current law at each node, the unknowns V_a and V_b are obtained.

In Fig. 3.92, node a is redrawn with all its branches, assuming that all currents are leaving the node a.

In Fig. 3.92, the sum of the currents leaving node a is zero.

$$\therefore \qquad I_1 + I_2 + I_3 = 0 \tag{3.46}$$

where $I_1 = \dfrac{V_a - V_1}{Z_1}, I_2 = \dfrac{V_a}{Z_2}, I_3 = \dfrac{V_a - V_b}{Z_3}$

Substituting I_1, I_2 and I_3 in Eq. 3.46, we get

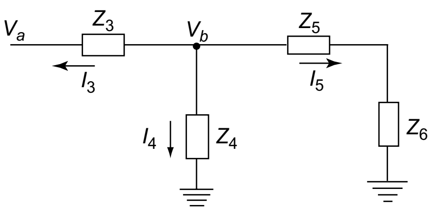


Fig. 3.93

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \qquad (3.47)$$

Similarly, in Fig. 3.93, node b is redrawn with all its branches, assuming that all currents are leaving the node b.

In Fig. 3.93, the sum of the currents leaving the node b is zero.

$$\therefore \qquad I_3 + I_4 + I_5 = 0 \qquad (3.48)$$

where $I_3 = \dfrac{V_b - V_a}{Z_3}, I_4 = \dfrac{V_b}{Z_4}, I_5 = \dfrac{V_b}{Z_5 + Z_6}$

Substituting I_3, I_4 and I_5 in Eq. 3.48

we get $$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b}{Z_5 + Z_6} = 0 \qquad (3.49)$$

Rearranging Eqs 3.47 and 3.49, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)V_a - \left(\frac{1}{Z_3}\right)V_b = \left(\frac{1}{Z_1}\right)V_1 \qquad (3.50)$$

$$-\left(\frac{1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5 + Z_6}\right)V_b = 0 \qquad (3.51)$$

From Eqs 3.50 and 3.51, we can find the unknown voltages V_a and V_b.

Example 3.65 *In the network shown in Fig. 3.94, determine V_a and V_b.*

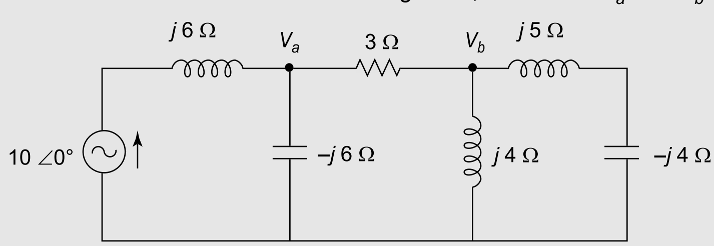


Fig. 3.94

Solution To obtain the voltage V_a at a, consider the branch currents leaving the node a as shown in Fig. 3.95(a).

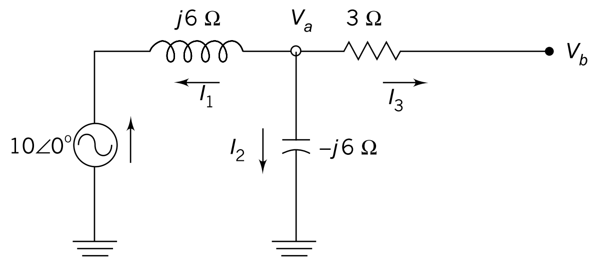


Fig. 3.95 (a)

In Fig. 3.95(a), $$I_1 = \frac{V_a - 10\angle 0°}{j6}, I_2 = \frac{V_a}{-j6}, I_3 = \frac{V_a - V_b}{3}$$

Since the sum of the currents leaving the node a is zero,

$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_a - 10\angle 0°}{j6} + \frac{V_a}{-j6} + \frac{V_a - V_b}{3} = 0 \tag{3.52}$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)V_a - \frac{1}{3}V_b = \frac{10\angle 0°}{j6}$$

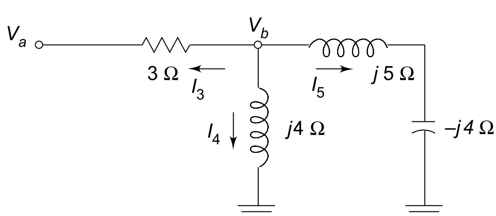


Fig. 3.95 (b)

$$\therefore \quad \frac{1}{3}V_a - \frac{1}{3}V_b = \frac{10\angle 0°}{j6} \tag{3.53}$$

To obtain the voltage V_b at b, consider the branch currents leaving node b as shown in Fig. 3.95(b).

In Fig. 3.95(b), $$I_3 = \frac{V_b - V_a}{3}, I_4 = \frac{V_b}{j4}, I_5 = \frac{V_b}{(j5 - j4)}$$

Since the sum of the currents leaving node b is zero

$$I_3 + I_4 + I_5 = 0$$

$$\frac{V_b - V_a}{3} + \frac{V_b}{j4} + \frac{V_b}{j1} = 0 \tag{3.54}$$

$$-\frac{1}{3}V_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)V_b = 0 \tag{3.55}$$

From Eqs 3.54 and 3.55, we can solve for V_a and V_b.

$$0.33V_a - 0.33V_b = 1.67\ \angle -90° \tag{3.56}$$

$$-0.33V_a + (0.33 - 0.25j - j)V_b = 0 \tag{3.57}$$

Adding Eqs 3.56 and 3.57 we get $(-1.25j)V_b = 1.67\ \angle -90°$

$$-1.25\ \angle 90°\ V_b = 1.67\ \angle -90°$$

$$V_b = \frac{1.67\angle -90°}{-1.25\angle 90°}$$

$$= -1.34\ \angle -180°$$

Substituting V_b in Eq. (3.56), we get

$$0.33V_a - (0.33)(-1.34\ \angle -180°) = 1.67\ \angle -90°$$

$$V_a = \frac{1.67\angle -90°}{0.33} = -1.31\ \text{V}$$

$$V_a = 5.22\ \angle -104.5°\ \text{V}$$

Voltages V_a and V_b are 5.22 $\angle$–104.5° V and –1.34 $\angle$–180° V respectively.

3.6.1 Nodal Equations by Inspection

In general, nodal equations can also be written by observing the network. Consider a four node network including a reference node as shown in Fig. 3.96.

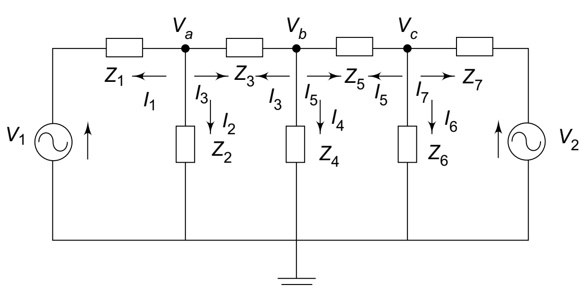


Fig. 3.96

Consider nodes a, b and c separately as shown in Figs 3.97(a), (b) and (c).

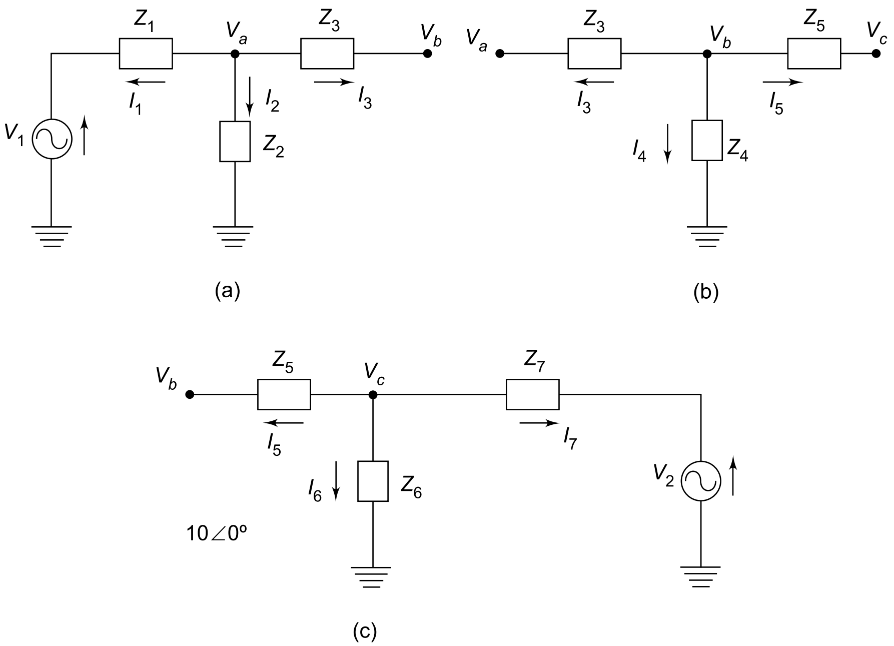


Fig. 3.97

Assuming that all the currents are leaving the nodes, the nodal equations at a, b and c are

$$I_1 + I_2 + I_3 = 0$$

$$I_3 + I_4 + I_5 = 0$$

$$I_5 + I_6 + I_7 = 0$$

$$\frac{V_a - V_1}{Z_1} + \frac{V_a}{Z_2} + \frac{V_a - V_b}{Z_3} = 0 \tag{3.58}$$

$$\frac{V_b - V_a}{Z_3} + \frac{V_b}{Z_4} + \frac{V_b - V_c}{Z_5} = 0 \tag{3.59}$$

$$\frac{V_c - V_b}{Z_5} + \frac{V_c}{Z_6} + \frac{V_c - V_2}{Z_7} = 0 \tag{3.60}$$

Rearranging the above equations, we get

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)V_a - \left(\frac{1}{Z_3}\right)V_b = \left(\frac{1}{Z_1}\right)V_1 \tag{3.61}$$

$$\left(\frac{-1}{Z_3}\right)V_a + \left(\frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}\right)V_b - \left(\frac{1}{Z_5}\right)V_c = 0 \tag{3.62}$$

$$\left(\frac{-1}{Z_5}\right)V_b + \left(\frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}\right)V_c = \left(\frac{1}{Z_7}\right)V_2 \tag{3.63}$$

In general, the above equations can be written as

$$Y_{aa}V_a + Y_{ab}V_b + Y_{ac}V_c = I_1$$
$$Y_{ba}V_a + Y_{bb}V_b + Y_{bc}V_c = I_2$$
$$Y_{ca}V_a + Y_{cb}V_b + Y_{cc}V_c = I_3$$

If we compare the general equations with the circuit equations, the self-admittance at node a is

$$Y_{aa} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

which is the sum of the admittances connected to node a.

Similarly, $Y_{bb} = \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}$, and $Y_{cc} = \frac{1}{Z_5} + \frac{1}{Z_6} + \frac{1}{Z_7}$

are the self-admittances at node b and node c, respectively. Y_{ab} is the mutual admittance between nodes a and b, i.e., it is the sum of all the admittances connecting nodes a and b. $Y_{ab} = -1/Z_3$ has a negative sign. All the mutual admittances have negative signs. Similarly, Y_{ac}, Y_{ba}, Y_{bc}, Y_{ca} and Y_{cb} are also mutual admittances. These are equal to the sums of the admittances connecting to nodes indicated in their subscripts. I_1 is the sum of all the source currents at node a. The current which drives into the node has a positive sign, while the current driving away from the node has a negative sign.

Example 3.66 *For the circuit shown in Fig. 3.98, write the node equations by the inspection method.*

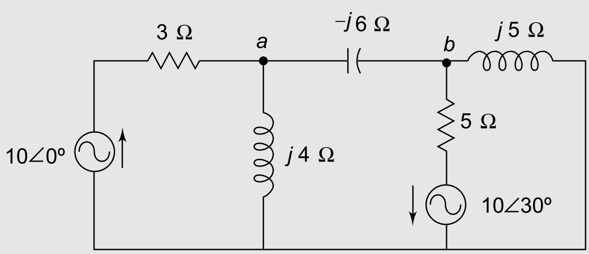


Fig. 3.98

Solution The general equations are

$$Y_{aa}V_a + Y_{ab}V_b = I_1 \tag{3.64}$$
$$Y_{ba}V_a + Y_{bb}V_b = I_2 \tag{3.65}$$

Consider Eq. 3.64.

$$Y_{aa} = \frac{1}{3} + \frac{1}{j4} + \frac{1}{-j6}$$

The self-admittance at node *a* is the sum of admittances connected to node *a*.

$$Y_{bb} = \frac{1}{-j6} + \frac{1}{5} + \frac{1}{j5}$$

The self-admittance at node *b* is the sum of admittances connected to node *b*.

$$Y_{ab} = -\left(\frac{1}{-j6}\right)$$

The mutual admittance between nodes *a* and *b* is the sum of admittances connected between nodes *a* and *b*. Similarly, $Y_{ba} = -(-1/j6)$, the mutual admittance between nodes *b* and *a* is the sum of the admittances connected between nodes *b* and *a*.

$$I_1 = \frac{10\angle 0^\circ}{3}$$

The source current at node *a*

$$I_2 = \frac{-10\angle 30^\circ}{5}$$

The source current leaving at node *b*.

Therefore, the nodal equations are

$$\left(\frac{1}{3} + \frac{1}{j4} - \frac{1}{j6}\right)V_a - \left(\frac{-1}{j6}\right)V_b = \frac{10\angle 0^\circ}{3} \tag{3.66}$$

$$-\left(\frac{-1}{j6}\right)V_a + \left(\frac{1}{5} + \frac{1}{j5} - \frac{1}{j6}\right)V_b = \frac{-10\angle 30^\circ}{5} \tag{3.67}$$

Example 3.67 *Using nodal analysis, find V_2 in Fig. 3.99.*

$I_1 = 10\underline{|0^\circ}$ A, $\quad I_2 = 10\underline{|60^\circ}$ A;

$Y_1 = (0.5 - j1.0)S$;

$Y_2 = (0.2 - j0.6)S$;

$Y_3 = (0.8 - j0.6)S$;

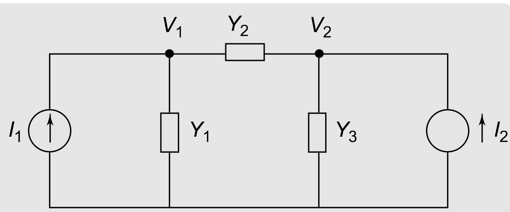


Fig. 3.99

Solution To obtain the voltage V_1, applying KCL at the node 1 considering the branch currents.

$$I_1 = V_1 Y_1 + (V_1 - V_2) Y_2$$

$$V_1(Y_1 + Y_2) - V_2 Y_2 = I_1 \tag{3.68}$$

To obtain the voltage V_2, applying KCL at the node 2 considering the branch currents.

$$I_2 = (V_2 - V_1)\, Y_2 + V_2 Y_3$$

$$-\, V_1 Y_2 + V_2(Y_2 + Y_3) = I_2 \tag{3.69}$$

For the given data, the equation (3.64) and (3.65) becomes

$$V_1(0.5 - j1.0 + 0.2 - j0.6) - V_2(0.2 - j0.6) = 10\underline{|0^\circ}$$

$$-V_1(0.2 - j0.6) + V_2(0.2 - j0.6 + 0.8 - j0.6) = 10\underline{|60^\circ}$$

Simplifying we get

$$(0.7 - j1.6)V_1 - (0.2 - j0.6)V_2 = 10\underline{|0^\circ}$$

$$-(0.2 - j0.6)V_1 + (1 - j1.2)V_2 = 10\underline{|60^\circ}$$

By using the Cramer's rule,

$$\begin{bmatrix} (0.7 - j1.6) & -(0.2 - j0.6) \\ -(0.2 - j0.6) & (1 - j1.2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10\underline{|0^\circ} \\ 10\underline{|60^\circ} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} (0.7 - j1.6) & -(0.2 - j0.6) \\ -(0.2 - j0.6) & (1 - j1.2) \end{bmatrix} = -0.9 - j2.2$$

$$\Delta_2 = \begin{bmatrix} (0.7 - j1.6) & 10\underline{|0^\circ} \\ -(0.2 - j0.6) & 10\underline{|60^\circ} \end{bmatrix} = 19.356 - j7.938$$

$$V_2 = \frac{19.356 - j7.938}{-(0.9 + j2.2)} = 8.8\underline{|89.951^\circ}$$

Example 3.68 *For the circuit shown in Fig. 3.100, write the nodal equations using the inspection method and express them in matrix form.*

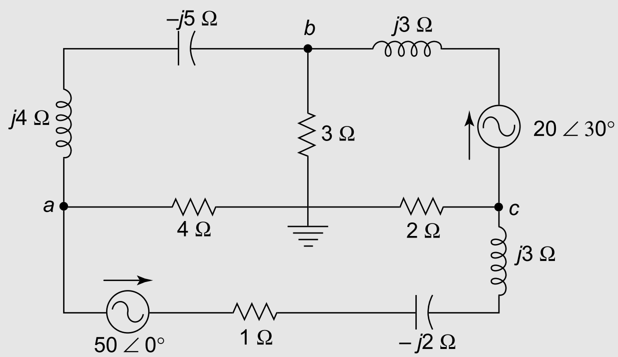


Fig. 3.100

Solution The number of nodes and reference node are selected as shown in Fig. 3.100, by assuming that all currents are leaving at each node.

At node a, $\left(\frac{1}{4} + \frac{1}{-j1} + \frac{1}{1 + j1}\right)V_a - \left(\frac{1}{-j1}\right)V_b - \left(\frac{1}{1 + j1}\right)V_c = \frac{-50\angle 0^\circ}{1 + j1}$

At node b, $-\left(\frac{1}{-j1}\right)V_a+\left(\frac{1}{3}+\frac{1}{-j1}+\frac{1}{j3}\right)V_b-\left(\frac{1}{j3}\right)V_c=\frac{20\angle 30°}{j3}$

At node c, $-\left(\frac{1}{1+j1}\right)V_a-\left(\frac{1}{j3}\right)V_b+\left(\frac{1}{2}+\frac{1}{j3}+\frac{1}{1+j1}\right)V_c=\frac{50\angle 0°}{1+j1}-\frac{20\angle 30°}{j3}$

In matrix form, the nodal equations are

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{(1+j1)}-\frac{1}{j1} & +\frac{1}{j1} & -\frac{1}{(1+j1)} \\ \frac{1}{j1} & \frac{1}{3}-\frac{1}{j1}+\frac{1}{j3} & -\frac{1}{j3} \\ -\frac{1}{(1+j1)} & -\frac{1}{(j3)} & \frac{1}{2}+\frac{1}{j3}+\frac{1}{1+j1} \end{bmatrix}\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

$$=\begin{bmatrix} \frac{-50\angle 0°}{(1+j1)} \\ \frac{20\angle 30°}{j3} \\ \left(\frac{50\angle 0°}{1+j1}-\frac{20\angle 30°}{j3}\right) \end{bmatrix}$$

Example 3.69 *For the circuit shown in Fig. 3.101, determine the voltage V_{AB}, if the load resistance R_L is infinite. Use node analysis.*

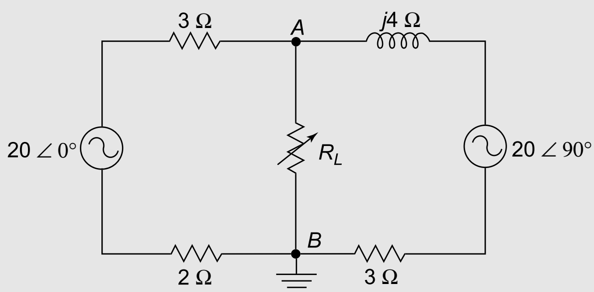


Fig. 3.101

Solution If the load resistance is infinite, no current passes through R_L. Hence R_L acts as an open circuit. If we consider A as a node and B as the reference node

$$\frac{V_A-20\angle 0°}{3+2}+\frac{V_A-20\angle 90°}{j4+3}=0$$

$$\frac{V_A}{5}+\frac{V_A}{(3+j4)}=\frac{20\angle 0°}{5}+\frac{20\angle 90°}{(3+j4)}$$

$$V_A\left[\frac{1}{5}+\frac{1}{3+j4}\right]=4\angle 0°+\frac{20\angle 90°}{5\angle 53.13°}$$

$$= 4\ \angle 0° + 4\ \angle 36.87° = 4 + 3.19 + j2.4 = 7.19 + j2.4$$

$$V_A\ [0.2 + 0.12 - j0.16] = 7.19 + j2.4$$

$$V_A=\frac{7.19+j2.4}{0.32-j0.16}=\frac{7.58\angle 18.46°}{0.35\angle -26.56°}$$

Voltage across AB is $V_{AB} = V_A = 21.66\ \angle 45.02°$ V

Example 3.70 *For the circuit shown in Fig. 3.102, determine the power output of the source and the power in each resistor of the circuit.*

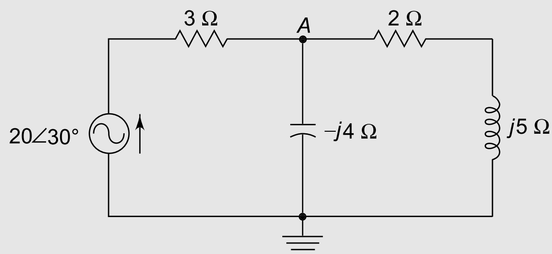


Fig. 3.102

Solution Assume that the voltage at node A is V_A. By applying nodal analysis, we have

$$\frac{V_A-20\angle 30°}{3}+\frac{V_A}{-j4}+\frac{V_A}{2+j5}=0$$

$$V_A\left[\frac{1}{3}+\frac{1}{2+j5}-\frac{1}{j4}\right]=\frac{20\angle 30°}{3}$$

$$V_A\ [0.33 + 0.068 + j0.078] = 6.67\ \angle 30°$$

$$\therefore \qquad V_A=\frac{6.67\angle 30°}{0.41\angle 11.09°}=16.27\angle 18.91°$$

Current in the 2 Ω resistor

$$I_2=\frac{V_A}{2+j5}=\frac{16.27\angle 18.91°}{5.38\angle 68.19°}$$

$$\therefore \qquad I_2 = 3.02\angle -49.28°$$

Power dissipated in the 2 Ω resistor

$$P_2 = I_2^2R = (3.02)^2 \times 2 = 18.24\ \text{W}$$

Current in the 3 Ω resistor

$$I_3=\frac{-20\angle 30°+16.27\angle 18.91°}{3}$$

$$= -6.67\angle 30° + 5.42\ \angle 18.91°$$

$$= -5.78 - j3.34 + 5.13 + j1.76 = -0.65 - j1.58$$

$$I_3 = 1.71\angle{-112°}$$

Power dissipated in the 3 Ω resistor

$$= (1.71)^2 \times 3 = 8.77 \text{ W}$$

Total power delivered by the source

$$= VI \cos \phi = 20 \times 1.71 \cos 142° = 26.95 \text{ W}$$

Example 3.71 *Determine the current in the 10 Ω resistor in the circuit shown in the Fig. 3.103 below.*

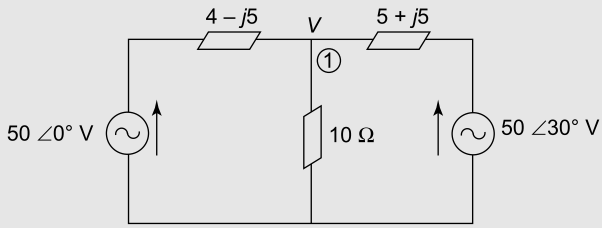


Fig. 3.103

Solution Apply nodal analysis at point (1), we get

$$\frac{V - 50\underline{|0°}}{4 - j5} + \frac{V}{10} + \frac{V - 50\underline{|30°}}{5 + j5} = 0$$

$$V\left[\frac{1}{4 - j5} + \frac{1}{10} + \frac{1}{5 + j5}\right] = \frac{50\underline{|0°}}{4 - j5} + \frac{50\underline{|30°}}{5 + j5}$$

$$V[0.297 + j0.0219] = 11.708 + j4.267$$

$$V[0.298\underline{|4.219°}] = 12.46\underline{|20.02°}$$

$$\Rightarrow V = 41.812\underline{|15.801°}$$

Current through the 10 Ω resistor $I_{10} = \dfrac{V}{R}$

$$= \frac{41.812\underline{|15.801°}}{10}$$

$$= 4.1812\underline{|15.801°} \text{ Amp}$$

Example 3.72 *Find the value of voltage V which results in $V_0 = 5\angle 0°$ V in the circuit shown in Fig. 3.104.*

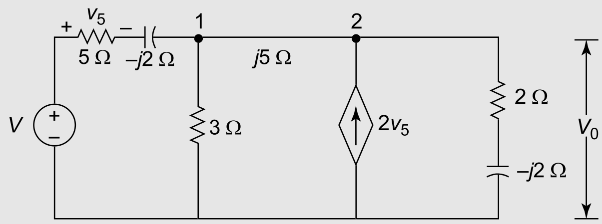


Fig. 3.104

Solution Assuming all currents are leaving the nodes, the nodal equations are

$$V_1\left[\frac{1}{5-j2}+\frac{1}{3}+\frac{1}{j5}\right]-V_2\left[\frac{1}{j5}\right]=\frac{V}{5-j2}$$

$$-V_1\left[\frac{1}{j5}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=2V_5$$

where $$V_5=5\left(\frac{V_1-V}{5-j2}\right)$$

The second equations becomes

$$V_1\left[\frac{-1}{j5}-\frac{10}{5-j2}\right]+V_2\left[\frac{1}{j5}+\frac{1}{2-j2}\right]=\frac{-10V}{5-j2}$$

$$V_0=V_2=\frac{\Delta_2}{\Delta}=5\angle 0°$$

$$\frac{\begin{vmatrix}\frac{1}{5-j2}+\frac{1}{3}+\frac{1}{j5} & \frac{V}{5-j2}\\ \frac{-1}{j5}-\frac{10}{5-j2} & \frac{-10V}{5-j2}\end{vmatrix}}{\begin{vmatrix}\frac{1}{5-j2}+\frac{1}{3}+\frac{1}{j5} & \frac{-1}{j5}\\ \frac{-1}{j5}-\frac{10}{5-j2} & \frac{1}{j5}+\frac{1}{2-j2}\end{vmatrix}}=5\angle 0°$$

The source voltage $V = 2.428\angle -88.74°$ volts.

3.7 STAR-DELTA CONVERSION

3.7.1 Star-Delta Transformation: Resistances

[JNTU Nov 2011]

In the preceding chapter, a simple technique called the *source transformation technique* has been discussed. The star delta transformation is another technique useful in solving complex networks. Basically, any three circuit elements, i.e. resistive, inductive or capacitive, may be connected in two different ways. One way of connecting these elements is called the star connection, or the *Y* connection. The other way of connecting these elements is called the delta (Δ) connection. The circuit is said to be in star connection, if three elements are connected as shown in Fig. 3.105(a), when it appears like a star (*Y*). Similarly, the circuit is said to be in delta connection, if three elements are connected as shown in Fig. 3.105(b), when it appears like a delta (Δ).

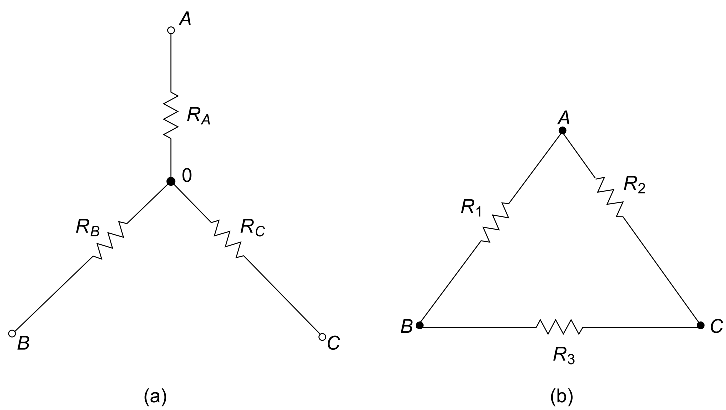


Fig. 3.105

The above two circuits are equal if their respective resistances from the terminals *AB*, *BC* and *CA* are equal. Consider the star connected circuit in Fig. 3.105(a); the resistance from the terminals *AB*, *BC* and *CA* respectively are

$$R_{AB}(Y) = R_A + R_B$$
$$R_{BC}(Y) = R_B + R_C$$
$$R_{CA}(Y) = R_C + R_A$$

Similarly, in the delta connected network in Fig. 3.105(b), the resistances seen from the terminals *AB*, *BC* and *CA*, respectively, are

$$R_{AB}(\Delta) = R_1 \| (R_2 + R_3) = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

$$R_{BC}(\Delta) = R_3 \| (R_1 + R_2) = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3}$$

$$R_{CA}(\Delta) = R_2 \| (R_1 + R_3) = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3}$$

Now, if we equate the resistances of star and delta circuits, we get

$$R_A + R_B = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3} \tag{3.70}$$

$$R_B + R_C = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} \tag{3.71}$$

$$R_C + R_A = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3} \tag{3.72}$$

Subtracting Eq. 3.71 from Eq. 3.70, and adding Eq. 3.72 to the resultant, we have

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{3.73}$$

Similarly $$R_B = \frac{R_1 R_3}{R_1 + R_2 + R_3} \tag{3.74}$$

and $$R_C = \frac{R_2 R_3}{R_1 + R_2 + R_3} \tag{3.75}$$

Thus, a delta connection of R_1, R_2 and R_3 may be replaced by a star connection of R_A, R_B and R_C as determined from Eqs 3.73, 3.74 and 3.75. Now if we multiply the Eqs 3.73 and 3.74, 3.74 and 3.75, 3.75 and 3.73, and add the three, we get the final equation as under:

$$R_A R_B + R_B R_C + R_C R_A = \frac{R_1^2 R_2 R_3 + R_3^2 R_1 R_2 + R_2^2 R_1 R_3}{(R_1 + R_2 + R_3)^2} \tag{3.76}$$

In Eq. 3.76 dividing the LHS by R_A, gives R_3; dividing it by R_B gives R_2, and doing the same with R_C, gives R_1.

Thus $$R_1 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_C}$$

$$R_2 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_B}$$

and $$R_3 = \frac{R_A R_B + R_B R_C + R_C R_A}{R_A}$$

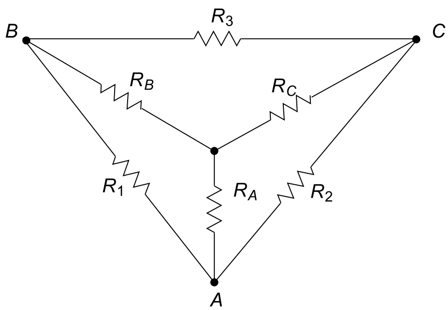


Fig. 3.106

From the above results, we can say that a star connected circuit can be transformed into a delta connected circuit and vice-versa.

From Fig. 3.106 and the above results, we can conclude that any resistance of the delta circuit is equal to the sum of the products of all possible pairs of star resistances divided by the opposite resistance of the star circuit. Similarly, any resistance of the star circuit is equal to the product of two adjacent resistances in the delta connected circuit divided by the sum of all resistances in delta connected circuit.

Example 3.73 *Obtain the star connected equivalent for the delta connected circuit shown in Fig. 3.107.*

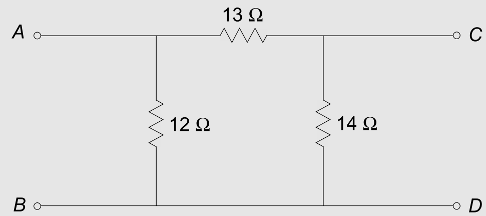


Fig. 3.107

Solution The above circuit can be replaced by a star connected circuit as shown in Fig. 3.108(a).

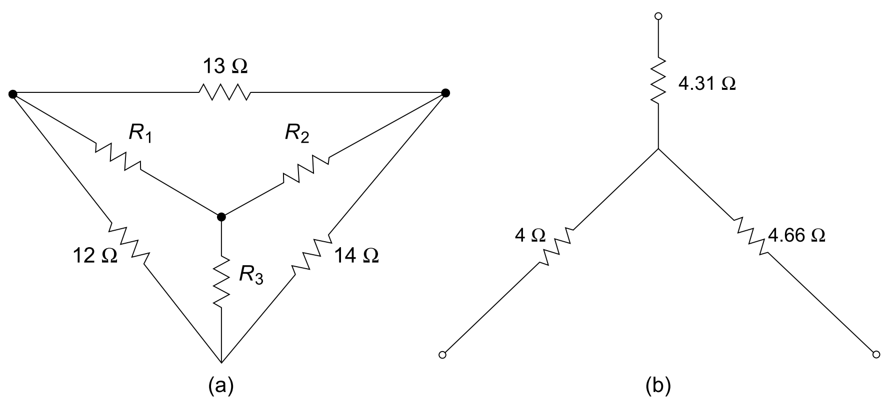


Fig. 3.108

Performing the Δ to Y transformation, we obtain

$$R_1 = \frac{13\times 12}{14+13+12}, \quad R_2 = \frac{13\times 14}{14+13+12}$$

and $$R_3 = \frac{14\times 12}{14+13+12}$$

∴ $$R_1 = 4\ \Omega,\ R_2 = 4.66\ \Omega,\ R_3 = 4.31\ \Omega$$

The star-connected equivalent is shown in Fig. 3.108(b).

Example 3.74 *Obtain the delta-connected equivalent for the star-connected circuit shown in Fig. 3.109.*

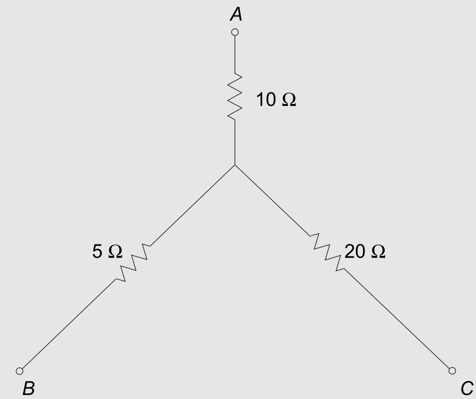


Fig. 3.109

Solution The above circuit can be replaced by a delta-connected circuit as shown in Fig. 3.110(a).

Performing the Y to Δ transformation, we get from the Fig. 3.110(a)

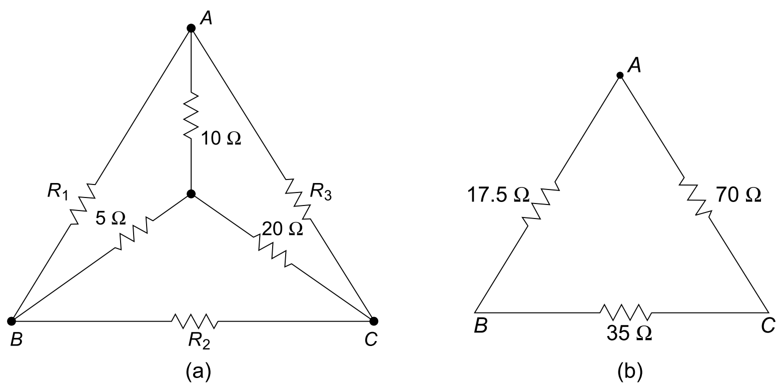


Fig. 3.110

$$R_1 = \frac{20\times10+20\times5+10\times5}{20} = 17.5\,\Omega$$

$$R_2 = \frac{20\times10+20\times5+10\times5}{10} = 35\,\Omega$$

and $$R_3 = \frac{20\times10+20\times5+10\times5}{5} = 70\,\Omega$$

The equivalent delta circuit is shown in Fig. 3.110(b).

Example 3.75 *Determine the current drawn by the circuit shown in Fig. 3.111.*

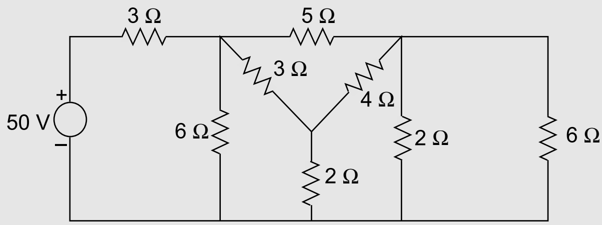


Fig. 3.111

Solution To simplify the network, the star circuit in Fig. 3.111 is converted into a delta circuit as shown under.

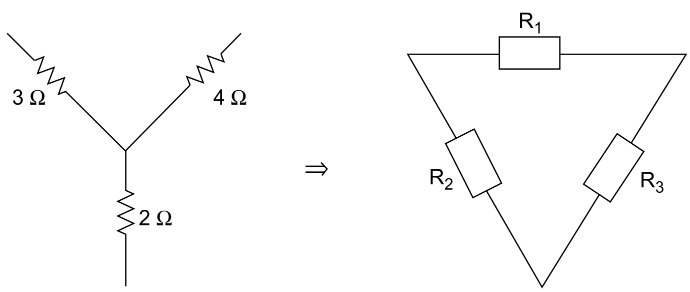


Fig. 3.112(a)

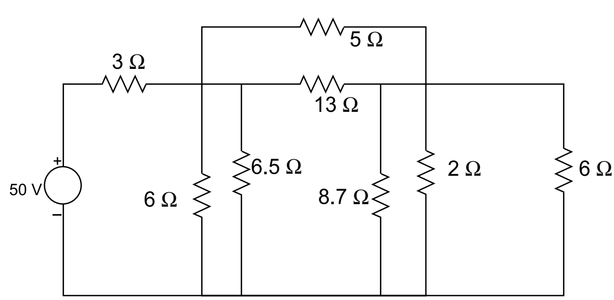


Fig. 3.112(b)

$$R_1 = \frac{4\times3+4\times2+3\times2}{2} = 13\ \Omega$$

$$R_2 = \frac{4\times3+4\times2+3\times2}{4} = 6.5\ \Omega$$

$$R_3 = \frac{4\times3+4\times2+3\times2}{3} = 8.7\ \Omega$$

The original circuit is redrawn as shown in Fig. 3.112(b).

It is further simplified as shown in Fig. 3.112(c). Here the resistors 5 Ω and 13 Ω are in parallel, 6 Ω and 6.5 Ω are in parallel, and 8.7 Ω and 2 Ω are in parallel.

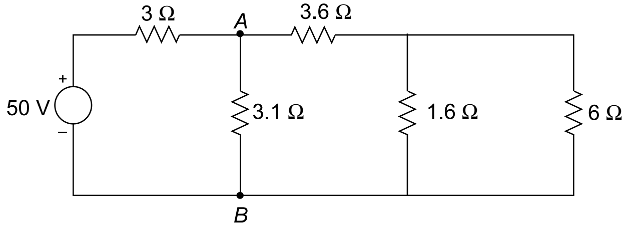


Fig. 3.112(c)

In the above circuit the resistors 6 Ω and 1.6 Ω are in parallel, the resultant of which is in series with 3.6 Ω resistor and is equal to $\left[3.6+\dfrac{6\times 1.6}{7.6}\right]=4.9\ \Omega$ as shown in Fig. 3.112(d).

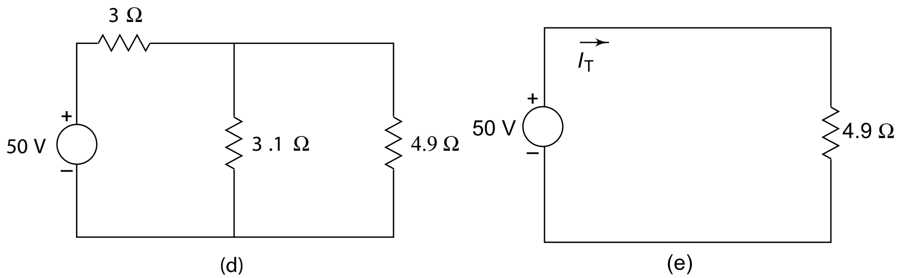


Fig. 3.112(d) and (e)

In the above circuit 4.9 Ω and 3.1 Ω resistors are in parallel, the resultant of which is in series with 3 Ω resistor.

Therefore, the total resistance $R_T = 3+\dfrac{3.1\times 4.9}{8} = 4.9\ \Omega$

The current drawn by the circuit I_T = 50/4.9 = 10.2 A (See Fig. 3.112(e)).

Example 3.76 *In Fig. 3.113 determine the equivalent resistance by using star-delta transformation.*

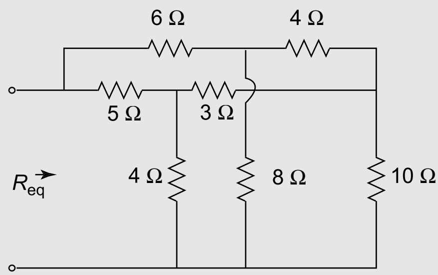


Fig. 3.113

Solution In Fig. 3.113, we have two star circuits, one consisting of 5 Ω, 3 Ω and 4 Ω resistors, and the other consisting of 6 Ω, 4 Ω and 8 Ω resistors. We convert the star circuits into delta circuits, so that the two delta circuits are in parallel.

In Fig. 3.114(a)

$$R_1 = \frac{5\times3+4\times3+5\times4}{4} = 11.75\ \Omega$$

$$R_2 = \frac{5\times3+4\times3+5\times4}{4} = 15.67\ \Omega$$

$$R_3 = \frac{5\times3+4\times3+5\times4}{5} = 9.4\ \Omega$$

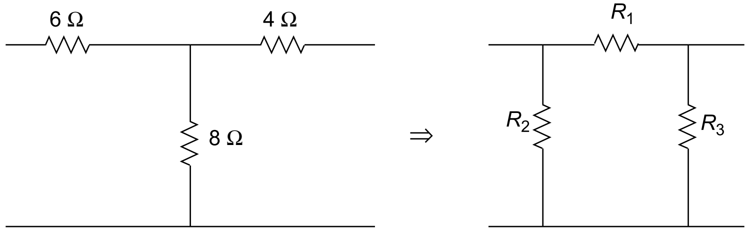


Fig. 3.114(a)

Similarly, in Fig. 3.114(b)

$$R_1 = \frac{6\times4+4\times8+8\times6}{8} = 13\ \Omega$$

$$R_2 = \frac{6\times4+4\times8+8\times6}{4} = 26\ \Omega$$

$$R_3 = \frac{6\times4+4\times8+8\times6}{6} = 17.3\ \Omega$$

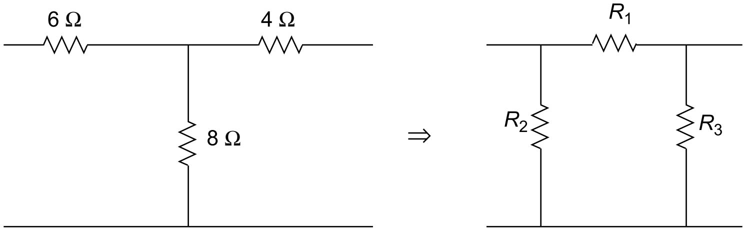


Fig. 3.114(b)

The simplified circuit is shown in Fig. 3.114(c)

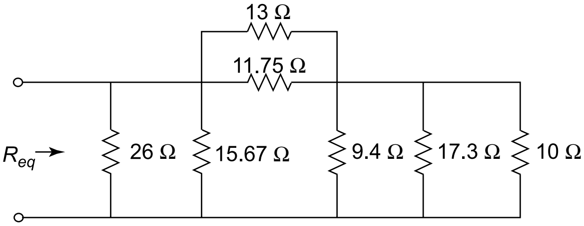


Fig. 3.114(c)

In the above circuit, the three resistors 10 Ω, 9.4 Ω and 17.3 Ω are in parallel.

Equivalent resistance = (10 || 9.4 || 17.3) = 3.78 Ω

Resistors 13 Ω and 11.75 Ω are in parallel

Equivalent resistance = (13 || 11.75) = 6.17 Ω

Resistors 26 Ω and 15.67 Ω are in parallel

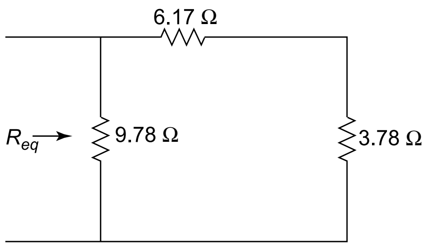


Fig. 3.114(d)

Equivalent = resistance = (26 || 15.67) = 9.78 Ω

The simplified circuit is shown in Fig. 3.114(d)

From the above circuit, the equivalent resistance is given by

$$R_{eq} = (9.78) \parallel (6.17 + 3.78)$$
$$= (9.87) \parallel (9.95) = 4.93\ \Omega$$

Example 3.77 *Find the voltage to be applied across AB in order to drive a current of 5A into the circuit by using star-delta transformation. Refer Fig. 3.115.*

[JNTU May/June 2006]

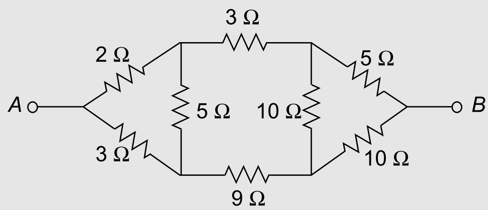


Fig. 3.115

Solution

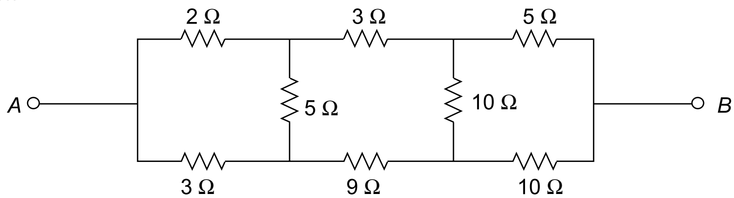


Fig. 3.116(a)

Using star-delta transformation

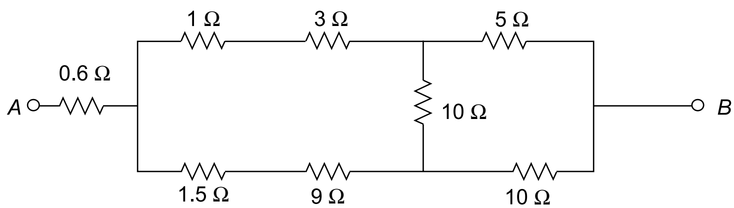


Fig. 3.116(b)

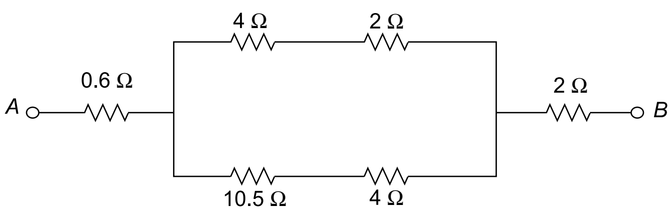
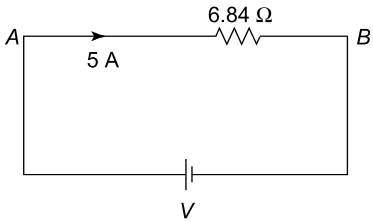


Fig. 3.116(c)

Fig. 3.116(d)



The voltage applied across V_{AB}

$$V_{AB} = 5 \times 6.84$$
$$= 34.2 \text{ V}$$

Fig. 3.116(e)

Example 3.78 *Determine the voltage appearing across terminals y_Z, when it a d.c. voltage of 100 V is appplied across x-y terminals in the figure below.*

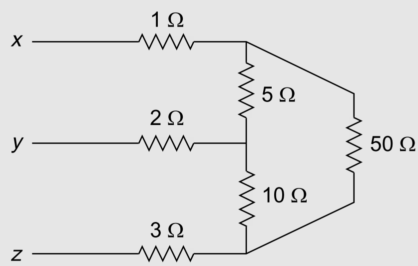


Fig. 3.117

Solution Converting delta network to star network

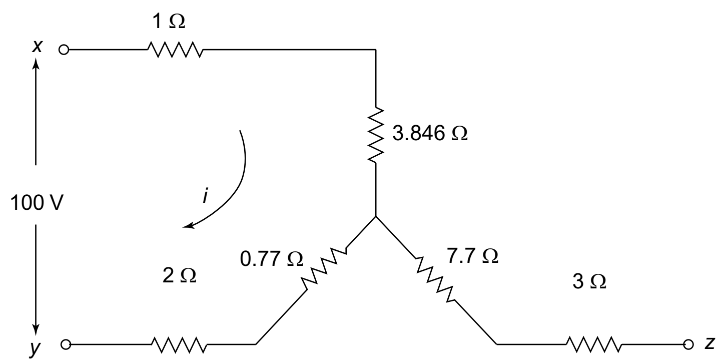


Fig. 3.118

$$\text{Current, } i = \frac{100}{1+3.846+0.77+2} = \frac{100}{7.616} = 13.13A$$

$$\text{Voltage across } y_z^{\,N}, V_z = -13.13 \times (2+0.77)$$
$$= -36.37 \text{ V}$$

Example 3.79 *Find equivalent resistance between AB in the circuit shown in the Fig. 3.119. All resistances are equal to R.*

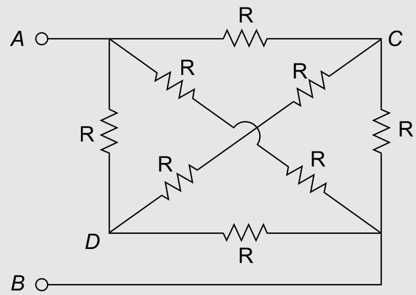


Fig. 3.119

Solution Converting the star point C into Δ.

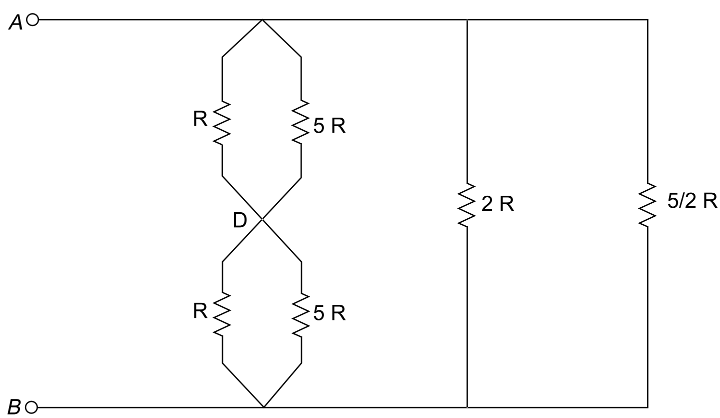


Fig. 3.120(a)

Further reducing the circuit shown in Fig. 3.120 between terminals AB

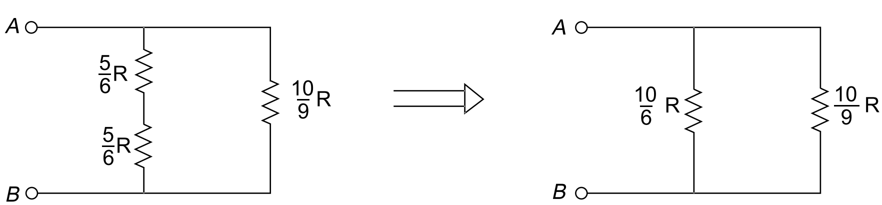


Fig. 3.120(b)

Resistance between terminals AB

$$R_{AB} = \left(\frac{10}{6}R\right) \Big/ \left(\frac{10}{9}R\right)$$

$$= \frac{10}{15}R = 0.667R$$

3.7.2 Star – Delta Transformation: Impedances

While dealing with currents and voltages in loads, it is often necessary to convert a star load to delta load, and vice-versa. Delta (Δ) connection of resistances can be replaced by an equivalent star (Y) connection and vice-versa. Similar methods can be applied in the case of networks containing general impedances in complex form. So also with ac, where the same formulae hold good, except that resistances are replaced by the impedances. These formulae can be applied even if the loads are unbalanced. Thus, considering Fig. 3.121(a), star load can be replaced by an equivalent delta-load with branch impedances as shown.

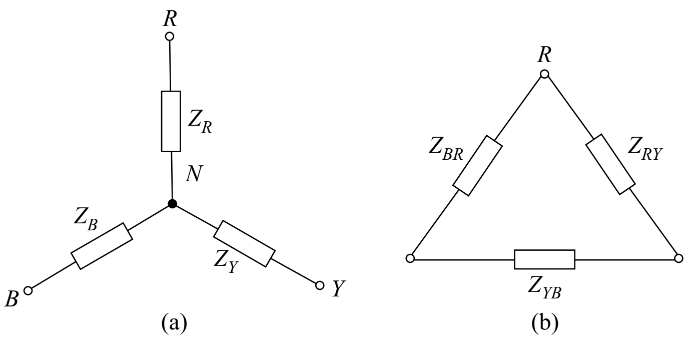


Fig. 3.121

Delta impedances, in terms of star impedances, are

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$

and
$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

The converted network is shown in Fig. 3.46(b). Similarly, we can replace the delta load of Fig. 3.121(b) by an equivalent star load with branch impedances as

$$Z_R = \frac{Z_{RY} Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$Z_Y = \frac{Z_{RY} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

and
$$Z_B = \frac{Z_{BR} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

It should be noted that all impedances are to be expressed in their complex form.

Example 3.80 *A symmetrical three-phase, three-wire 440 V supply is connected to a star-connected load as shown in Fig. 3.122(a). The impedances in each branch are Z_R = (2 +J3) Ω, Z_Y = (1 – J2) Ω and Z_B = (3 + J4) Ω. Find its equivalent delta-connected load. The phase sequence is RYB.*

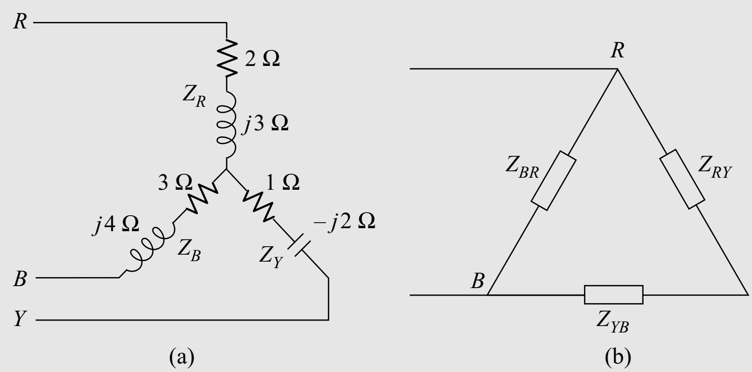


Fig. 3.122

Solution The equivalent delta network is shown in Fig. 3.122(b). From Section 9.6, we can write the equations to find Z_{RY}, Z_{YB} and Z_{BR}. We have

$$Z_{RY} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_B}$$

$$Z_R = 2 + j3 = 3.61 \angle 56.3°$$

$$Z_Y = 1 - j2 = 2.23 \angle{-63.4°}$$

$$Z_B = 3 + j4 = 5 \angle 53.13°$$

$$Z_R Z_Y + Z_Y Z_B + Z_B Z_R = (3.61 \angle 56.3°)(2.23 \angle{-63.4°}) + (2.23 \angle{-63.4°})(5 \angle 53.13°) + (5 \angle 53.13°)(3.61 \angle 56.3°)$$

$$= 8.05 \angle{-7.1°} + 11.15 \angle{-10.27°} + 18.05 \angle 109.43°$$

$$= 12.95 + j14.04 = 19.10 \angle 47.3°$$

$$Z_{RY} = \frac{19.10 \angle 47.3°}{5 \angle 53.13°} = 3.82 \angle{-5.83°} = 3.8 - j0.38$$

$$Z_{YB} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_R}$$

$$= \frac{19.10 \angle 47.3°}{3.61 \angle 56.3°} = 5.29 \angle{-9°} = 5.22 - j0.82$$

$$Z_{BR} = \frac{Z_R Z_Y + Z_Y Z_B + Z_B Z_R}{Z_Y}$$

$$= \frac{19.10\angle 47.3°}{2.23\angle -63.4°} = 8.56\angle 110.7° = -3.02 + j8$$

The equivalent delta impedances are

$$Z_{RY} = (3.8 - j0.38)\Omega$$

$$Z_{YB} = (5.22 - j0.82)\Omega$$

$$Z_{BR} = (-3.02 + j8)\Omega$$

Example 3.81 *A symmetrical three-phase, three-wire 400 V, supply is connected to a delta-connected load as shown in Fig. 3.123(a). Impedances in each branch are Z_{RY} 10 $\angle 30°\Omega$; $Z_{YB} = 10\angle -45°\Omega$ and $Z_{BR} = 2.5\angle 60°\Omega$. Find its equivalent star-connected load; he phase sequence is RYB.*

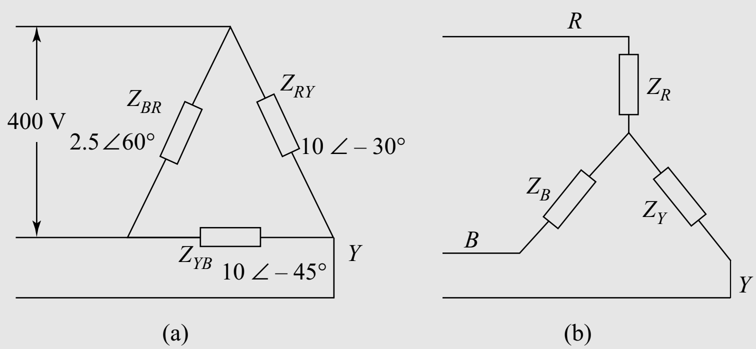


Fig. 3.123

Solution The equivalent star network is shown in Fig. 9.18(b). From Section 9.6, we can write the equations to find Z_R, Z_Y and Z_B as

$$Z_R = \frac{Z_{RY} Z_{BR}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$Z_{RY} + Z_{YB} + Z_{BR} = 10\angle 30° + 10\angle -45° + 2.5\angle 60°$$

$$= (8.66 + j5) + (7.07 - j7.07) + (1.25 + j2.17)$$

$$= 16.98 + j0.1 = 16.98\angle 0.33°\Omega$$

$$Z_R = \frac{(10\angle 30°)(2.5\angle 60°)}{16.98\angle 0.33°} = 1.47\angle 89.67°$$

$$= (0.008 + j1.47)\Omega$$

$$Z_Y = \frac{Z_{RY} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$= \frac{(10\angle 30°)(10\angle -45°)}{16.98\angle 0.33°} = 5.89\angle -15.33°\Omega$$

$$Z_B = \frac{Z_{BR} Z_{YB}}{Z_{RY} + Z_{YB} + Z_{BR}}$$

$$= \frac{(2.5\angle 60°)(10\angle -45°)}{16.98\angle 0.33°} = 1.47\angle 14.67°\Omega$$

The equivalent star impedances are

$Z_R = 1.47\ \angle 89.67°\Omega$, $Z_Y = 5.89\ \angle -15.33°\Omega$ and $Z_B = 1.47\ \angle 14.67°\Omega$

Balanced Star-Delta and Delta-Star Conversion

If the three-phase load is balanced, then the conversion formulae in Section 3.19 get simplified. Consider a balanced star-connected load having an impedance Z_1 in each phase as shown in Fig. 3.124(a).

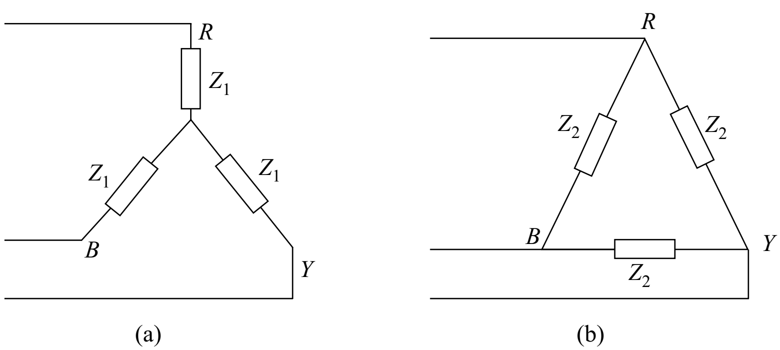


Fig. 3.124

Let the equivalent delta-connected load have an impedance of Z_2 in each phase as shown in Fig. 3.124(b). Applying the conversion formulae from Section 3.19 for delta impedances in terms of star impedances, we have

$$Z_2 = 3Z_1$$

Similarly, we can express star impedances in terms of delta $Z_1 = Z_{2/3}$.

Example 3.82 *Three identical impedances are connected in delta as shown in Fig. 3.125(a). Find an equivalent star network such that the line current is the same when connected to the same supply.*

Solution The equivalent star network is shown in Fig. 3.50(b). From Section 3.14.2.1, we can write the equation to Find $Z_1 = Z_{2/3}$

$$Z_2 = 3 + j4 = 5 \angle 53.13° \Omega$$

$$Z_1 = \frac{5}{3} \angle 53.13° = 1.66 \angle 53.15° = (1.0 + j1.33)\Omega$$

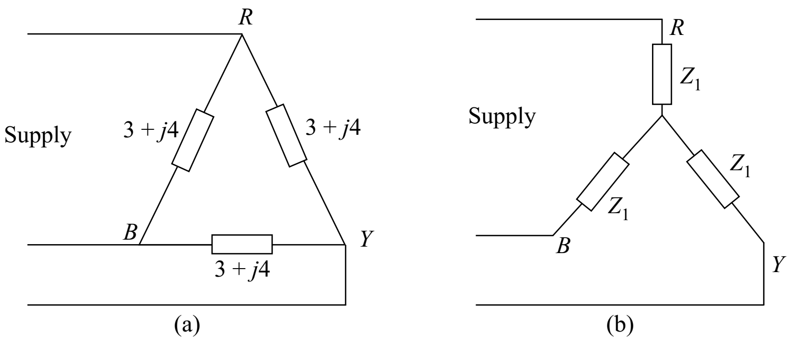


Fig. 3.125

Practice Problems

3.1 A delayed full wave rectified sine wave has an average value of half the maximum value as shown in Fig. 3.126. Find the angle θ.

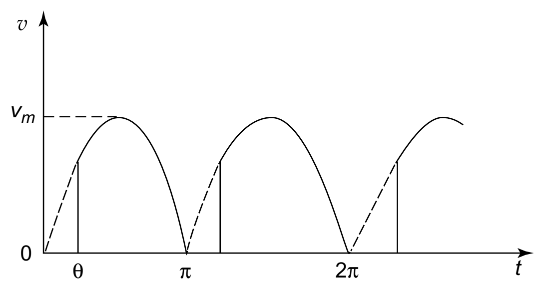


Fig. 3.126

3.2 For the circuit shown in Fig. 3.127, determine the impedance, phase angle and total current.

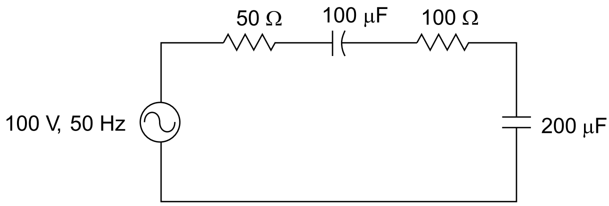


Fig. 3.127

3.3 Calculate the total current in the circuit in Fig. 3.128, and determine the voltage across resistor V_R, and across capacitor V_C.

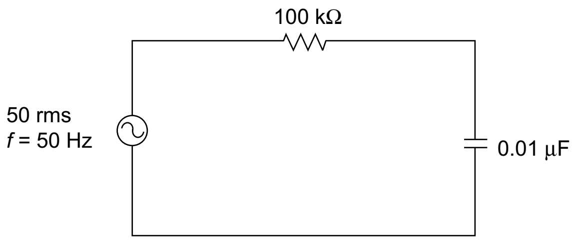


Fig. 3.128

3.4 Determine the impedance and phase angle in the circuit shown in Fig. 3.129.

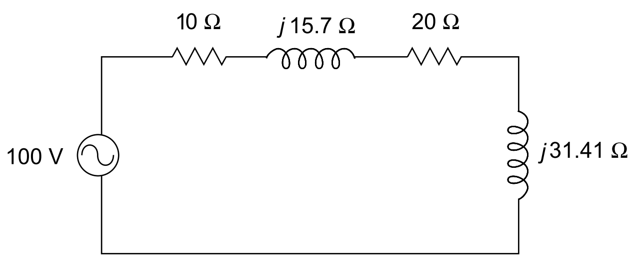


Fig. 3.129

3.5 Calculate the impedance at each of the following frequencies; also determine the current at each frequency in the circuit shown in Fig. 3.130.

(a) 100 Hz (b) 3 kHz

3.6 A signal generator supplies a sine wave of 10 V, 10 kHz, to the circuit shown in Fig. 3.131. Calculate the total current in the circuit. Determine the phase angle θ for the circuit. If the total inductance in the circuit is doubled, does θ increase or decrease, and by how many degrees?

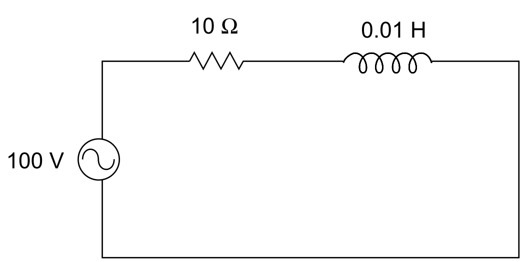


Fig. 3.130

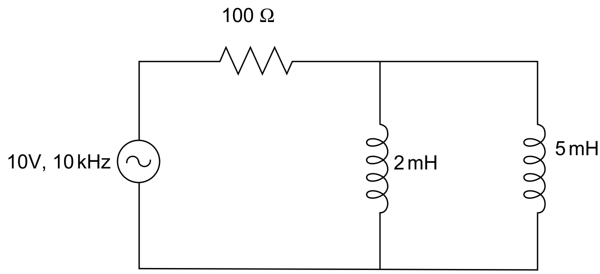


Fig. 3.131

3.7 For the circuit shown in Fig. 3.132, determine the voltage across each element. Is the circuit predominantly resistive or inductive? Find the current in each branch and the total current.

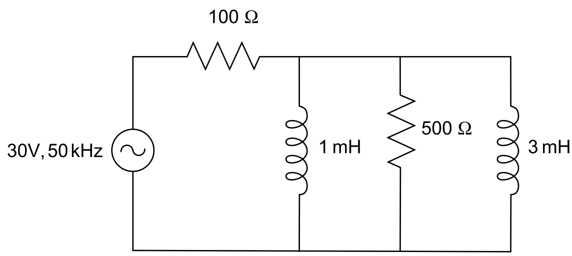


Fig. 3.132

3.8 Determine the total impedance Z_T, the total current I_T, phase angle θ, voltage across inductor L, and voltage across resistor R_3 in the circuit shown in Fig. 3.133.

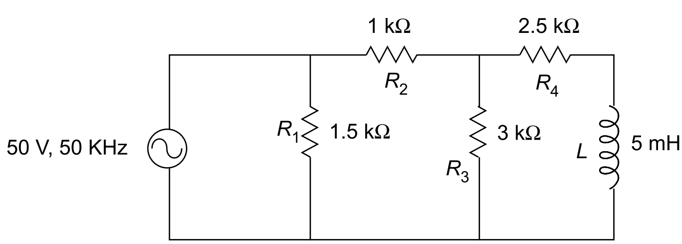


Fig. 3.133

3.9 For the circuit shown in Fig. 3.134, determine the value of frequency of supply voltage when a 100 V, 50 A current is supplied to the circuit.

3.10 A sine wave generator supplies a signal of 100 V, 50 Hz to the circuit shown in Fig. 3.135. Find the current in each branch, and total current. Determine the voltage across each element and draw the voltage phasor diagram.

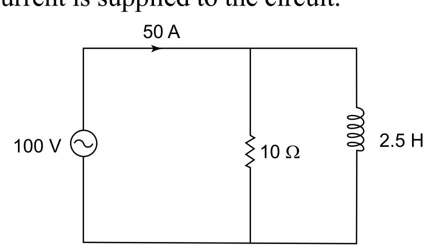


Fig. 3.134

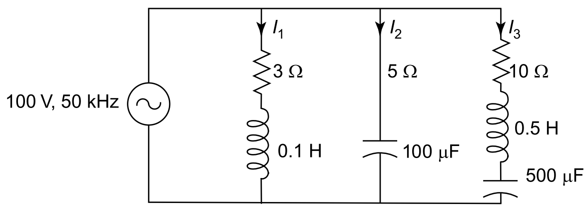


Fig. 3.135

3.11 Determine the voltage across each element in the circuit shown in Fig. 3.136. Convert the circuit into an equivalent series form. Draw the voltage phasor diagram.

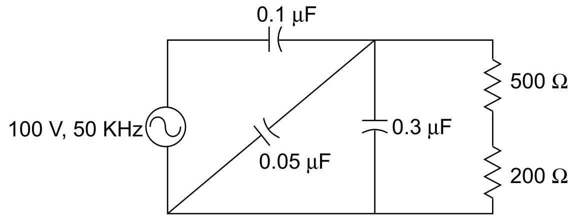


Fig. 3.136

3.12 For the circuit shown in Fig. 3.137, determine the total current I_T, phase angle θ and voltage across each element.

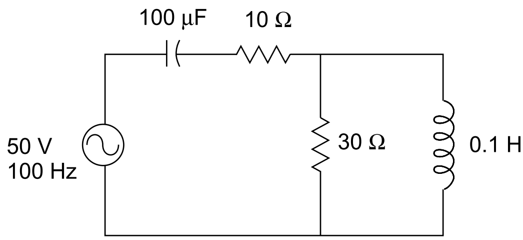


Fig. 3.137

3.13 For the circuit shown in Fig. 3.138, the applied voltage $v = V_m \cos \omega t$. Determine the current in each branch and obtain the total current in terms of the cosine function.

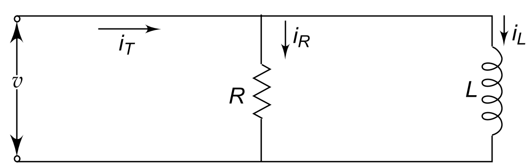


Fig. 3.138

3.14 For the circuit shown in Fig. 3.139, the voltage across the inductor is $v_L = 15 \sin 200\,t$. Find the total voltage and the angle by which the current lags the total voltage.

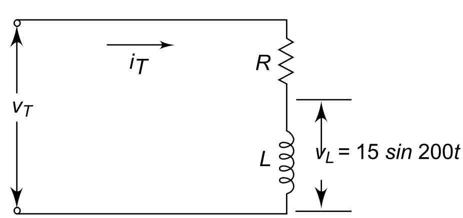


Fig. 3.139

3.15 In a parallel circuit having a resistance $R = 5\ \Omega$ and $L = 0.02$ H, the applied voltage is $v = 100 \sin (1000\ t + 50°)$ volts. Find the total current and the angle by which the current lags the applied voltage.

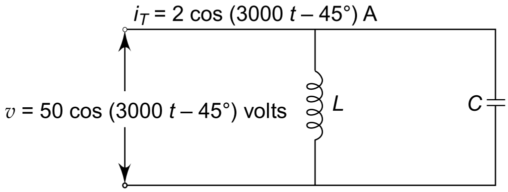


Fig. 3.140

3.16 In the parallel circuit shown in Fig. 3.140, the current in the inductor is five times greater than the current in the capacitor. Find the element values.

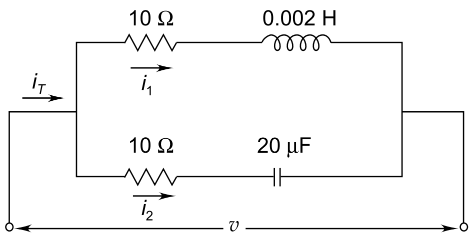


Fig. 3.141

3.17 In the parallel circuit shown in Fig. 3.141, the applied voltage is $v = 100 \sin 5000\ t$ V. Find the currents in each branch and also the total current in the circuit.

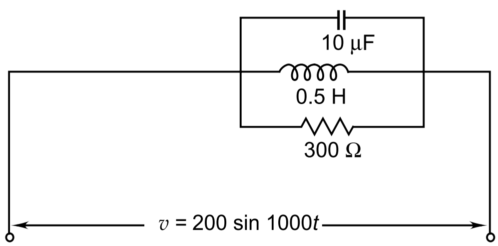


Fig. 3.142

3.18 For the circuit shown in Fig. 3.143, a voltage of $250 \sin \omega t$ is applied. Determine the power factor of the circuit, if the voltmeter readings are $V_1 = 100$ V, $V_2 = 125$ V, $V_3 = 150$ V and the ammeter reading is 5 A.

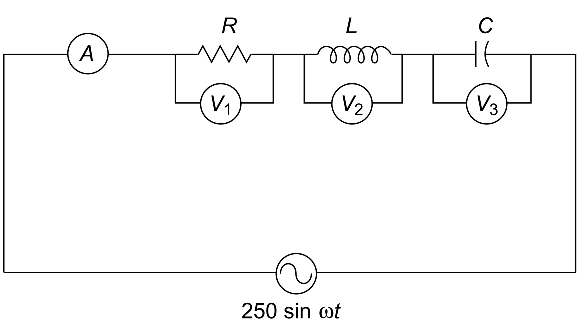


Fig. 3.143

3.19 For the circuit shown in Fig. 3.144, a voltage $v(t)$ is applied and the resulting current in the circuit $i(t) = 15 \sin (\omega t + 30°)$ amperes. Determine the active power, reactive power, power factor, and the apparent power.

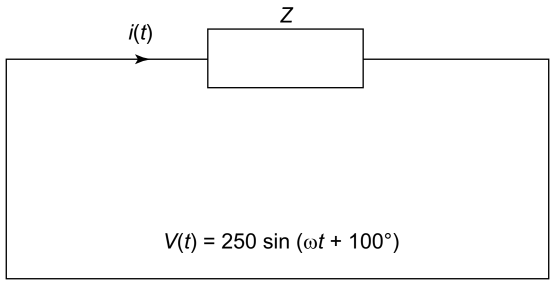


Fig. 3.144

3.20 A series RL circuit draws a current of $i(t) = 8 \sin(50t + 45°)$ from the source. Determine the circuit constants, if the power delivered by the source is 100 W and there is a lagging power factor of 0.707.

3.21 Two impedances, $Z_1 = 10 \underline{|-60°} \, \Omega$ and $Z_2 = 16 \angle 70° \, \Omega$ are in series and pass an effective current of 5 A. Determine the active power, reactive power, apparent power and power factor.

3.22 For the circuit shown in Fig. 3.145, determine the value of the impedance if the source delivers a power of 200 W and there is a lagging power factor of 0.707. Also find the apparent power.

3.23 A voltage of $v(t) = 100 \sin 500\,t$ is applied across a series R-L-C circuit where $R = 10\,\Omega$, $L = 0.05$ H and $C = 20$ μF. Determine the power supplied by the source, the reactive power supplied by the source, the reactive power of the capacitor, the reactive power of the inductor, and the power factor of the circuit.

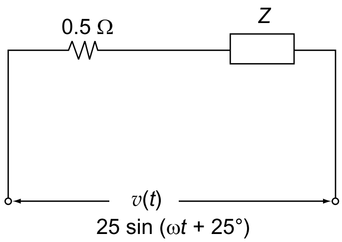


Fig. 3.145

3.24 For the circuit shown in Fig. 3.146, determine the power dissipated and the power factor of the circuit.

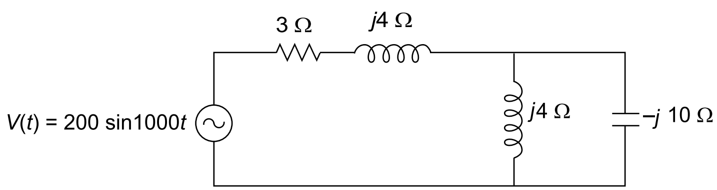


Fig. 3.146

3.25 For the circuit shown in Fig. 3.147, determine the power factor and the power dissipated in the circuit.

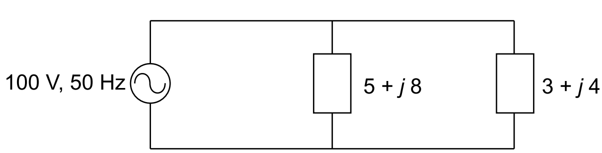


Fig. 3.147

3.26 For the circuit shown in Fig. 3.148, determine the power factor, active power, reactive power and apparent power.

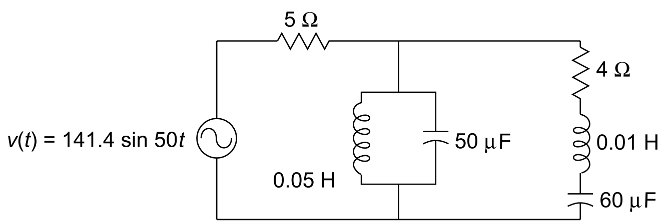


Fig. 3.148

3.27 In the parallel circuit shown in Fig. 3.149, the power in the 5 Ω resistor is 600 W and the total circuit takes 3000 VA at a leading power factor of 0.707. Find the value of impedance Z.

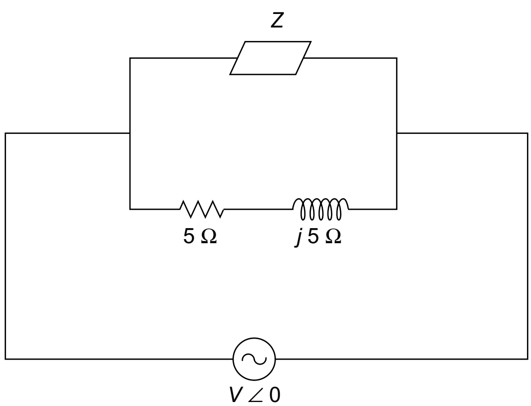


Fig. 3.149

3.28 For the parallel circuit shown in Fig. 3.150, the total power dissipated is 1000 W. Determine the apparent power, the reactive power, and the power factor.

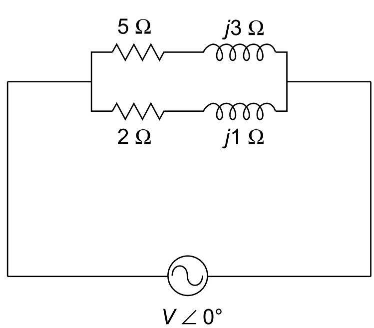


Fig. 3.150

3.29 A voltage source $v(t) = 150 \sin \omega t$ in series with 5 Ω resistance is supplying two loads in parallel, $Z_A = 60\angle 30°$, and $Z_B = 50\angle -25°$. Find the average power delivered to Z_A, the average power delivered to Z_B, the average power dissipated in the circuit, and the power factor of the circuit.

3.30 For the circuit shown in Fig. 3.151, determine the true power, reactive power and apparent power in each branch. What is the power factor of the total circuit?

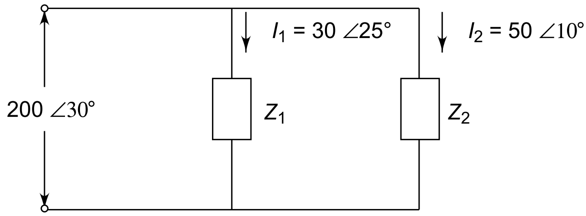


Fig. 3.151

3.31 Determine the value of the voltage source, and the power factor in the network shown in Fig. 3.152 if it delivers a power of 500 W to the circuit shown in Fig. 3.151. Also find the reactive power drawn from the source.

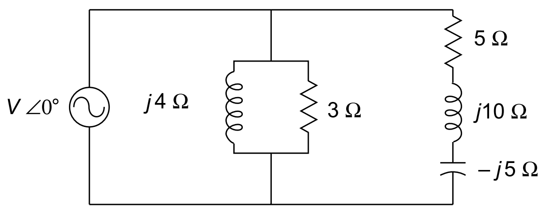


Fig. 3.152

3.32 Find the average power dissipated by the 500 Ω resistor shown in Fig. 3.153.

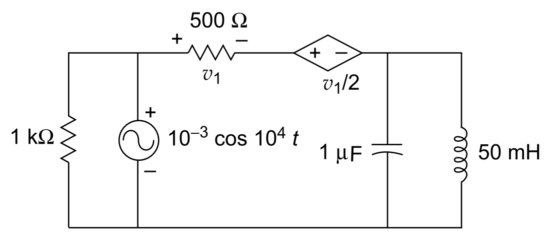


Fig. 3.153

3.33 Find the power dissipated by the voltage source shown in Fig. 3.154.

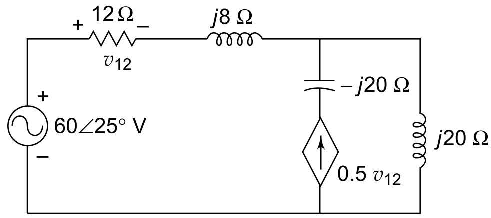


Fig. 3.154

3.34 Find the power delivered by current source shown in Fig. 3.155.

3.35 For the circuit shown in Fig. 3.156, determine the power factor, active power, reactive power and apparent power.

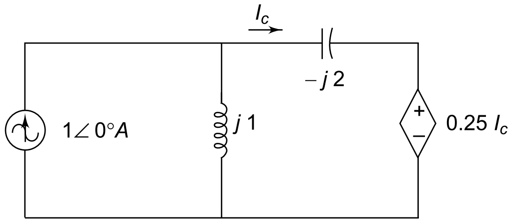


Fig. 3.155

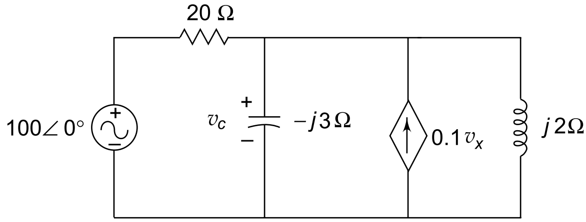


Fig. 3.156

3.36 Determine the voltage V_{ab} and V_{bc} in the network shown in Fig. 3.157 by loop analysis, where source voltage $e(t) = \sqrt{2} \times 100\cos(314t + 45°)$.

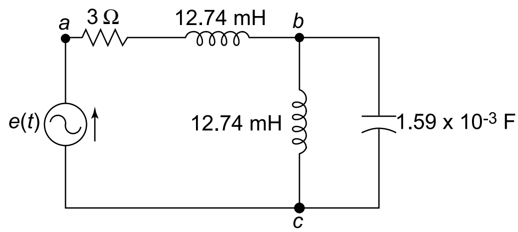


Fig. 3.157

3.37 Determine the power output of the voltage source by loop analysis for the network shown in Fig. 3.158. Also determine the power extended in the resistors.

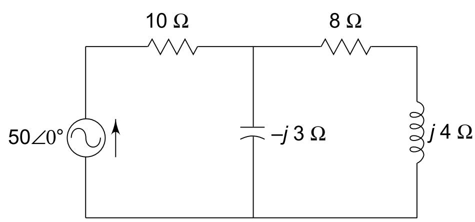


Fig. 3.158

3.38 Determine the value of source currents by loop analysis for the circuit shown in Fig. 3.159 and verify the results by using node analysis.

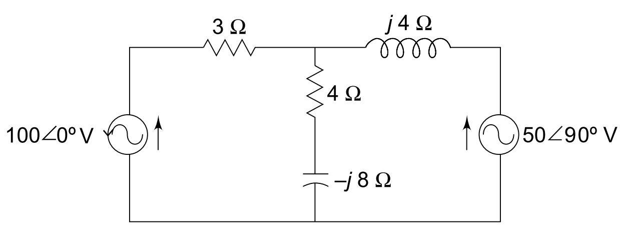


Fig. 3.159

3.39 Determine the power out of the source in the circuit shown in Fig. 3.160 by nodal analysis and verify the results by using loop analysis.

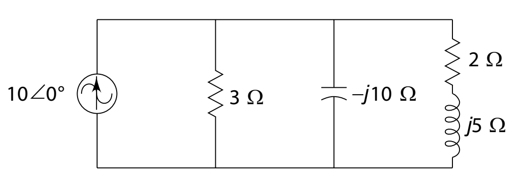


Fig. 3.160

3.40 For the circuit shown in Fig. 3.161 find the voltage across the dependent source branch by using mesh analysis.

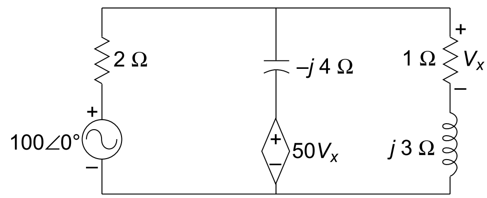


Fig. 3.161

3.41 For the circuit shown in Fig. 3.162, obtain the voltage across 500 kΩ resistor.

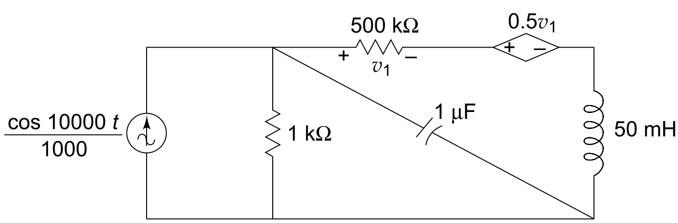


Fig. 3.162

3.42 For the circuit shown in Fig. 3.163, the load resistance R_L is adjusted until it absorbs the maximum average power. Calculate the value of R_L and the maximum average power.

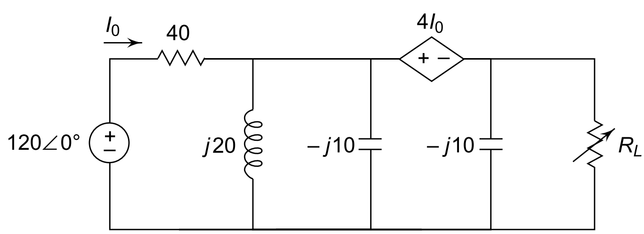


Fig. 3.163

Objective Type Questions

3.1 A 1 kHz sinusoidal voltage is applied to an RL circuit, what is the frequency of the resulting current?

(a) 1 kHz (b) 0.1 kHz (c) 100 kHz (d) 2 kHz

3.2 A series RL circuit has a resistance of 33 kΩ, and an inductive reactance of 50 kΩ. What is its impedance and phase angle?

(a) 56.58 Ω, 59.9° (b) 59.9 kΩ, 56.58°
(c) 59.9 Ω, 56.58° (d) 5.99 Ω, 56.58°

3.3 In a certain *RL* circuit, $V_R = 2$ V and $V_L = 3$ V. What is the magnitude of the total voltage?

(a) 2 V (b) 3 V (c) 5 V (d) 3.61 V

3.4 When the frequency of applied voltage in a series RL circuit is increased what happens to the inductive reactance?

(a) decreases (b) remains the same
(c) increases (d) becomes zero

3.5 In a certain parallel RL circuit, $R = 0\ \Omega$, and $X_L = 75\ \Omega$. What is the admittance?

(a) 0.024 S (b) 75 S (c) 50 S (d) 1.5 S

3.6 What is the phase angle between the inductor current and the applied voltage in a parallel RL circuit?

(a) 0° (b) 45° (c) 90° (d) 30°

3.7 When the resistance in an RC circuit is greater than the capacitive reactance, the phase angle between the applied voltage and the total current is closer to

(a) 90° (b) 0° (c) 45° (d) 120°

3.8 A series RC circuit has a resistance of 33 kΩ, and a capacitive reactance of 50 kΩ. What is the value of the impedance?

(a) 50 kΩ (b) 33 kΩ (c) 20 kΩ (d) 59.91 Ω

3.9 In a certain series RC circuit, $V_R = 4$ V and $V_C = 6$ V. What is the magnitude of the total voltage?

(a) 7.2 V (b) 4 V (c) 6 V (d) 52 V

3.10 When the frequency of the applied voltage in a series RC circuit is increased what happens to the capacitive reactance?

(a) it increases (b) it decreases (c) it is zero (d) remains the same

3.11 In a certain parallel RC circuit, $R = 50\ \Omega$ and $X_C = 75\ \Omega$. What is Y?

(a) 0.01 S (b) 0.02 S (c) 50 S (d) 75 S

3.12 The admittance of an RC circuit is 0.0035 S, and the applied voltage is 6 V. What is the total current?

(a) 6 mA (b) 20 mA (c) 21 mA (d) 5 mA

3.13 What is the phase angle between the capacitor current and the applied voltage in a parallel RC circuit?

(a) 90° (b) 0° (c) 45° (d) 180°

3.14 In a given series RLC circuit, X_C is 150 Ω, and X_L is 80 Ω, what is the total reactance? What is the type of reactance?

(a) 70 Ω, inductive (b) 70 Ω, capacitive
(c) 70 Ω, resistive (d) 150 Ω, capacitive

3.15 In a certain series RLC circuit $V_R = 24$ V, $V_L = 15$ V, and $V_C = 45$ V. What is the source voltage?

(a) 38.42 V (b) 45 V (c) 15 V (d) 24 V

3.16 When $R = 10\ \Omega$, $X_C = 18\ \Omega$ and $X_L = 12\ \Omega$, the current

(a) leads the applied voltage (b) lags behind the applied voltage
(c) is in phase with the voltage (d) is none of the above

3.17 A current $i = A \sin 500\,t$ A passes through the circuit shown in Fig. 3.164. The total voltage applied will be

(a) $B \sin 500\,t$ (b) $B \sin (500\,t - \theta°)$
(c) $B \sin (500\,t + \theta°)$ (d) $B \cos (200\,t + \theta°)$

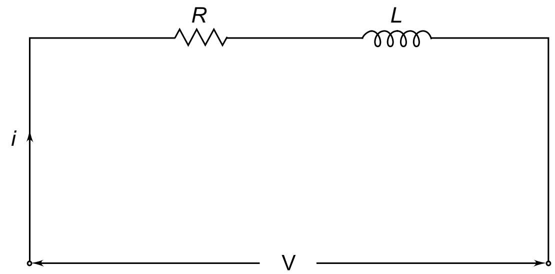


Fig. 3.164

3.18 A current of 100 mA through an inductive reactance of 100 Ω produces a voltage drop of

(a) 1 V (b) 6.28 V (c) 10 V (d) 100 V

3.19 When a voltage $v = 100 \sin 5000\,t$ volts is applied to a series circuit of $L = 0.05$ H and unknown capacitance, the resulting current is $i = 2 \sin (5000\,t + 90°)$ amperes. The value of capacitance is

(a) 66.7 pF (b) 6.67 pF (c) 0.667 μF (d) 6.67 μF

3.20 A series circuit consists of two elements has the following current and applied voltage.

$$i = 4 \cos (2000\,t + 11.32°)\ \text{A}$$
$$v = 200 \sin (2000\,t + 50°)\ \text{V}$$

The circuit elements are

(a) resistance and capacitance (b) capacitance and inductance
(c) inductance and resistance (d) both resistances

3.21 A pure capacitor of $C = 35$ μF is in parallel with another single circuit element. The applied voltage and resulting current are

$$v = 150 \sin 300\,t\ \text{V}$$
$$i = 16.5 \sin (3000\,t + 72.4°)\ \text{A}$$

The other element is

(a) capacitor of 30 μF (b) inductor of 30 mH
(c) resistor of 30 Ω (d) none of the above

3.22 The phasor combination of resistive power and reactive power is called

(a) true power (b) apparent power
(c) reactive power (d) average power

3.23 Apparent power is expressed in

(a) volt-amperes (b) watts
(c) volt-amperes or watts (d) VAR

3.24 A power factor of '1' indicates

(a) purely resistive circuit, (b) purely reactive circuit
(c) combination of both, (a) and (b) (d) none of these

3.25 A power factor of 0 indicates

(a) purely resistive element (b) purely reactive element
(c) combination of both (a) and (b) (d) none of the above

3.26 For a certain load, the true power is 100 W and the reactive power is 100 VAR. What is the apparent power?

(a) 200 VA (b) 100 VA (c) 141.4 VA (d) 120 VA

3.27 If a load is purely resistive and the true power is 5 W, what is the apparent power?

(a) 10 VA (b) 5 VA (c) 25 VA (d) $\sqrt{50}$ VA

3.28 True power is defined as

(a) $VI \cos \theta$ (b) VI (c) $VI \sin \theta$ (d) none of these

3.29 In a certain series RC circuit, the true power is 2 W, and the reactive power is 3.5 VAR. What is the apparent power?

(a) 3.5 VA (b) 2 VA (c) 4.03 VA (d) 3 VA

3.30 If the phase angle θ is 45°, what is the power factor?

(a) cos 45° (b) sin 45° (c) tan 45° (d) none of these

3.31 To which component in an RC circuit is the power dissipation due?

(a) capacitance (b) resistance (c) both (d) none

3.32 A two element series circuit with an instantaneous current $I = 4.24 \sin(5000\, t + 45°)$ A has a power of 180 watts and a power factor of 0.8 lagging. The inductance of the circuit must have the value.

(a) 3 H (b) 0.3 H (c) 3 mH (d) 0.3 mH

3.33 In the circuit shown in Fig. 3.165, if branch A takes 8 KVAR, the power of the circuit will be

(a) 2 kW (b) 4 kW
(c) 6 kW (d) 8 kW

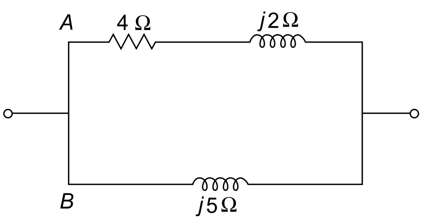


Fig. 3.165

3.34 In the circuit shown in Fig. 3.166, the voltage across 30 Ω resistor is 45 volts. The reading of the ammeter A will be

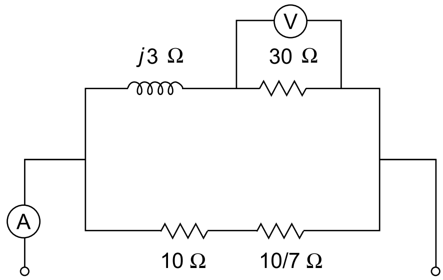


Fig. 3.166

(a) 10 A (b) 19.4 A
(c) 22.4 A (d) 28 A

3.35 In the circuit shown in Fig. 3.167, v_1 and v_2 are two identical sources of $10\angle 90°$. The power supplied by V_1 is

(a) 6 W (b) 8.8 W (c) 11 W (d) 16 W

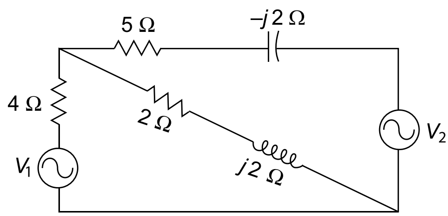


Fig. 3.167

3.36 Mesh analysis is based on

(a) Kirchhoff's current law (b) Kirchhoff's voltage law
(c) Both (d) None

4 Coupled Circuits and Resonance

4.1 INTRODUCTION TO COUPLED CIRCUITS

Two circuits are said to be ‘coupled’ when energy transfer takes place from one circuit to the other when one of the circuits is energised. There are many types of couplings like conductive coupling as shown by the potential divider in Fig. 4.1(a) inductive or magnetic coupling as shown by a two winding transformer in Fig. 4.1(b) or conductive and inductive coupling as shown by an auto transformer in Fig. 4.1(c). A majority of the electrical circuits in practice are conductively or electromagnetically coupled. Certain coupled elements are frequently used in network analysis and synthesis. Transformer, transistor and electronic pots, etc. are some among these circuits. Each of these elements may be represented as a two port network as shown in Fig. 4.1(d).

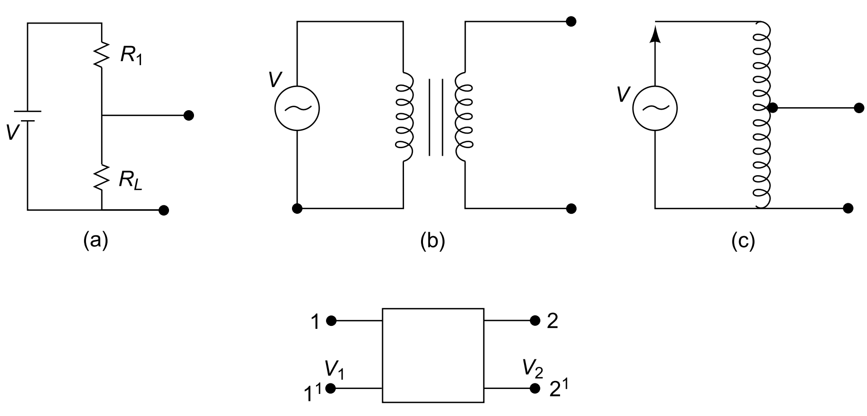


Fig. 4.1

4.1.1 Conductively Coupled Circuit and Mutual Impedance

A conductively coupled circuit which does not involve magnetic coupling is shown in Fig. 4.2(a).

In the circuit shown the impedance Z_{12} or Z_{21} common to loop 1 and loop 2 is called *mutual impedance*. It may consist of a pure resistance, a pure inductance, a pure capacitance or a combination of any of these elements. Mesh analysis, nodal analysis or Kirchhoff's laws can be used to solve these type of circuits as described in Chapter 1.

The general definition of mutual impedance is explained with the help of Fig. 4.2(b).

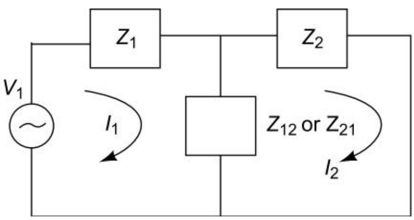


Fig. 4.2(a)

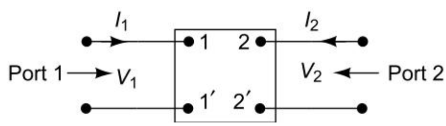


Fig. 4.2(b)

The network in the box may be of any configuration of circuit elements with two ports having two pairs of terminals 1-1′ and 2-2′ available for measurement. The mutual impedance between port 1 and 2 can be measured at 1-1′ or 2-2′. If it is measured at 2-2′. It can be defined as the voltage developed (V_2) at 2-2′ per unit current (I_1) at port 1-1′. If the box contains linear bilateral elements, then the mutual impedance measured at 2-2′ is same as the impedance measured at 1-1′ and is defined as the voltage developed (V_1) at 1-1′ per unit current (I_2) at port 2-2′.

Example 4.1 *Find the mutual impedance for the circuit shown in Fig. 4.3.*

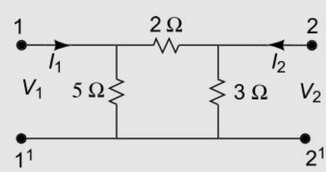


Fig. 4.3

Solution Mutual impedance is given by

$$\frac{V_2}{I_1} \text{ or } \frac{V_1}{I_2}$$

$$V_2 = \frac{3}{2} I_1 \text{ or } \frac{V_1}{I_1} = 1.5\,\Omega$$

$$V_1 = 5 \times I_2 \times \frac{3}{10} \text{ or } \frac{V_2}{I_2} = 1.5\,\Omega$$

4.1.2 Self Inductance and Mutual Inductance *[JNTU Nov 2011]*

The property of inductance of a coil was introduced in Section 1.6. A voltage is induced in a coil when there is a time rate of change of current through it. The inductance parameter L, is defined in terms of the voltage across it and the time rate of change of current through it $v(t) = L\frac{di(t)}{dt}$ where, $v(t)$ is the voltage across the coil, $I(t)$ is the current through the coil and L is the inductance of the coil. Strictly speaking, this definition is of self-inductance and this is considered as a circuit element with a pair of terminals. Whereas a circuit element "mutual inductor" does not exist. Mutual inductance is a property associated with two or more coils or inductors which are in close proximity and the presence of common magnetic flux which links the coils. A transformer is such a device whose operation is based on mutual inductance.

Let us consider two coils, L_1, and L_2 as shown in Fig. 4.4(a), which are sufficiently close together, so that the flux produced by i_1 in coil L_1, also link coil L_2. We assume that the coils do not move with respect to one another, and the medium in which the flux is established has a constant permeability. The two coils may be also arranged on a common magnetic core, as shown in Fig. 4.4(b). The two coils are said to be magnetically coupled, but act as a separate circuits. It is possible to relate the voltage induced in one coil to the time rate of change of current in the other coil. When a voltage v_1 is applied across L_1, a current i_1 will start flowing in this coil, and produce a flux ϕ. This flux also links coil L_2. If i_1 were to change with respect to time, the flux 'ϕ' would also change with respect to time. The time-varying flux surrounding the second coil, L_2 induces an emf, or voltage, across the terminals of L_2; this voltage is proportional to the time rate of change of current flowing through the first coil L_1. The two coils, or circuits, are said to be inductively coupled, because of this property they are called coupled elements or coupled circuits and the induced voltage, or emf is called the voltage/emf of mutual induction and is given by $v_2(t) = M_1\frac{di_1(t)}{dt}$ volts, where v_2 is the voltage induced in coil L_2 and M_1 is the coefficient of proportionality, and is called the coefficient of mutual inductance, or simple mutual inductance.

If current i_2 is made to pass through coil L_2 as shown in Fig. 4.4(c) with coil L_1 open, a change of i_2 would cause a voltage v_1 in coil L_1, given by $v_1(t) = M_2\frac{di_2(t)}{dt}$.

In the above equation, another coefficient of proportionality M_2 is involved. Though it appears that two mutual inductances are involved in determining the mutually induced voltages in the two coils, it can be shown from energy considerations that the two coefficients are equal and, therefore, need not be represented by two different letters. Thus $M_1 = M_2 = M$.

$$\therefore \quad v_2(t) = M\frac{di_1(t)}{dt}\text{ Volts} \tag{4.1}$$

$$v_1(t) = M\frac{di_2(t)}{dt}\text{ Volts} \tag{4.2}$$

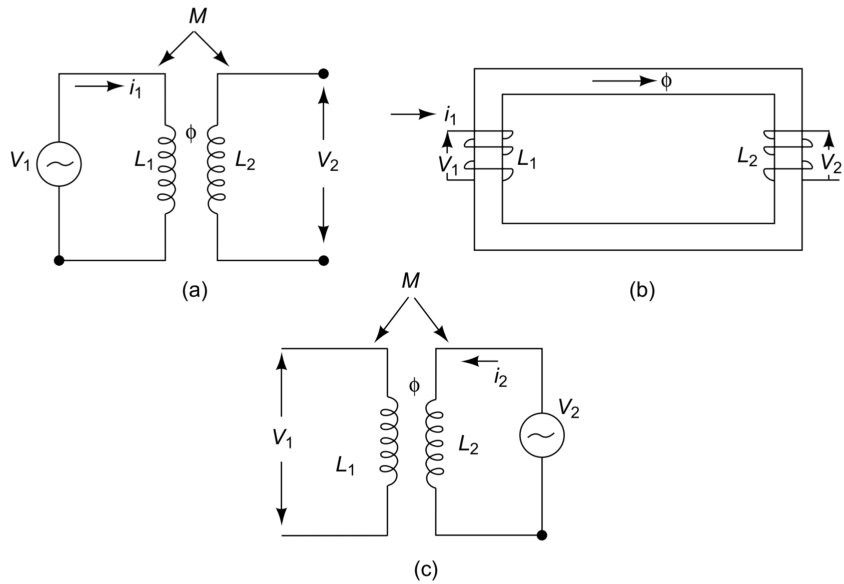


Fig. 4.4

In general, in a pair of linear time invariant coupled coils or inductors, a non-zero current in each of the two coils produces a mutual voltage in each coil due to the flow of current in the other coil. This mutual voltage is present independently of, and in addition to, the voltage due to self induction. Mutual inductance is also measured in Henrys and is positive, but the mutually induced voltage, $M\frac{di}{dt}$ may be either positive or negative, depending on the physical construction of the coil and reference directions. To determine the polarity of the mutually induced voltage (i.e. the sign to be used for the mutual inductance), the dot convention is used.

4.1.3 Dot Rule of Coupled Circuits

Dot convention is used to establish the choice of correct sign for the mutually induced voltages in coupled circuits.

Circular dot marks and/or special symbols are placed at one end of each of two coils which are mutually coupled to simplify the diagrammatic representation of the windings around its core.

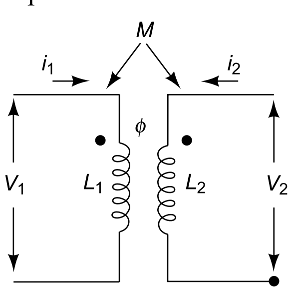


Fig. 4.5

Let us consider Fig. 4.5 which shows a pair of linear, time invariant, coupled inductors with self inductances L_1 and L_2 and a mutual inductance M. If these inductors form a portion of a network, currents i_1 and i_2 are shown, each arbitrarily assumed entering at the dotted terminals, and voltages v_1 and v_2 are developed across the inductors. The voltage across L_1 is, thus composed of two parts and is given by

$$v_1(t) = L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt} \tag{4.3}$$

The first term on the RHS of the above equation is the self induced voltage due to i_1, and the second term represents the mutually induced voltage due to i_2.

$$\text{Similarly, } v_2(t) = L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt} \tag{4.4}$$

Although the self-induced voltages are designated with positive sign, mutually induced voltages can be either positive or negative depending on the direction of the winding of the coil and can be decided by the presence of the *dots* placed at one end of each of the two coils. The convention is as follows.

If two terminals belonging to different coils in a coupled circuit are marked identically with dots then for the same direction of current relative to like terminals, the magnetic flux of self and mutual induction in each coil add together. The physical basis of the dot convention can be verified by examining Fig. 4.6. Two coils *ab* and *cd* are shown wound on a common iron core.

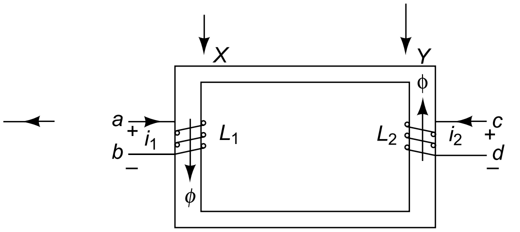


Fig. 4.6

It is evident from Fig. 4.6 that the direction of the winding of the coil *ab* is clock-wise around the core as viewed at *X*, and that of *cd* is anti-clockwise as viewed at *Y*. Let the direction of current i_1 in the first coil be from *a* to *b*, and increasing with time. The flux produced by i_1 in the core has a direction which may be found by right hand rule, and which is downwards in the left limb of the core. The flux also increases with time in the direction shown at *X*. Now suppose that the current i_2 in the second coil is from *c* to *d*, and increasing with time. The application of the right hand rule indicates that the flux produced by i_2 in the core has an upward direction in the right limb of the core. The flux also increases with time in the direction shown at *Y*. The assumed currents i_1 and i_2 produce flux in the core that are additive. The terminals *a* and *c* of the two coils attain similar polarities simultaneously. The two simultaneously positive terminals are identified by two dots by the side of the terminals as shown in Fig. 4.7.

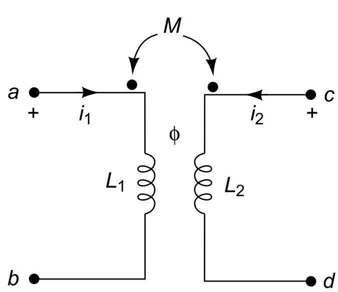


Fig. 4.7

The other possible location of the dots is the other ends of the coil to get additive fluxes in the core, i.e. at *b* and *d*. It can be concluded that the mutually induced voltage is positive when currents i_1 and i_2 both enter (or leave) the windings by the dotted terminals. If the current in one winding enters at the dot-marked terminals and the current in the other winding leaves at the dot-marked terminal, the voltages due to self and mutual induction in any coil have opposite signs.

Example 4.2 *Using dot convention, write voltage equations for the coils shown in Fig. 4.8.*

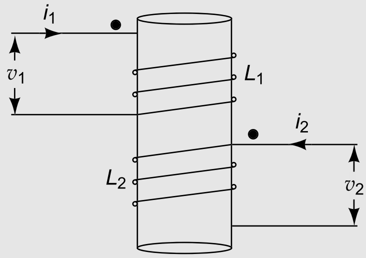


Fig. 4.8

Solution Since the currents are entering at the dot marked terminals the mutually induced voltages or the sign of the mutual inductance is positive; using the sign convention for the self inductance, the equations for the voltages are

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Example 4.3 *Write the equation for voltage v_0 for the circuits shown in Fig. 4.9.*

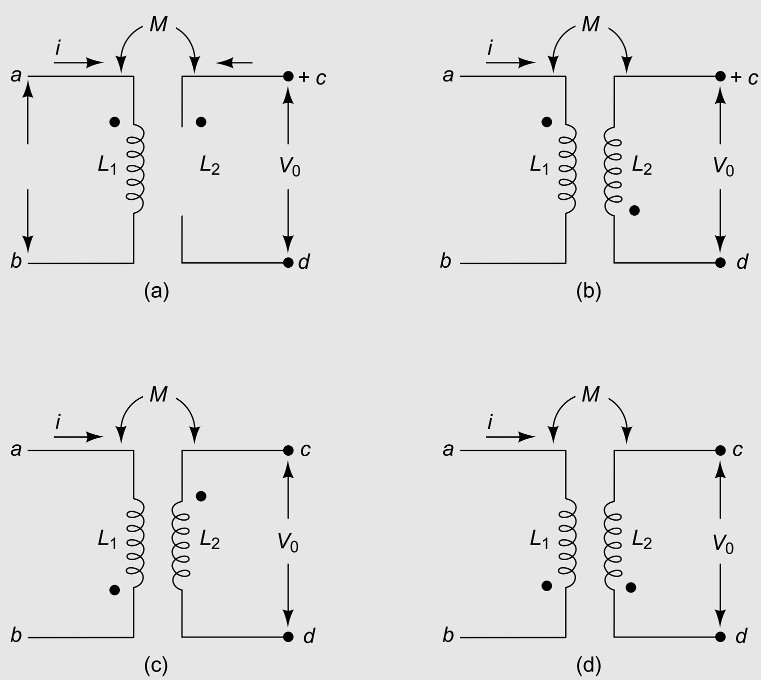


Fig. 4.9

Solution v_0 is assumed positive with respect to terminal C and the equation is given by

(a) $v_0 = M\dfrac{di}{dt}$ (b) $v_0 = -M\dfrac{di}{dt}$

(c) $v_0 = -M\dfrac{di}{dt}$ (d) $v_0 = M\dfrac{di}{dt}$

Example 4.4 *Formulate the loop equation for the network shown in Fig. 4.10.* *[JNTU 2004]*

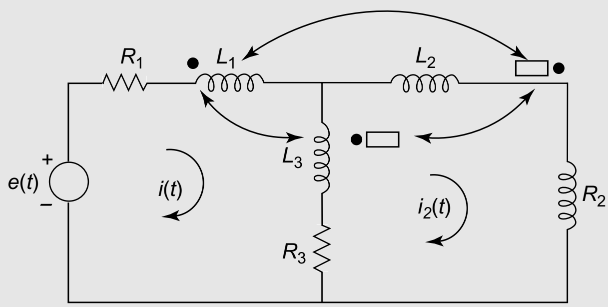


Fig. 4.10

Solution For the loop (1)

$$e(t) = i_l R_1 + L_1\frac{dl_i}{dt} + M_{31}\frac{d}{dt}(i_1 - i_2) + M_{21}\frac{d}{dt}(-i_2) + L_3\frac{d}{dt}(i_1 - i_2)$$

$$+ M_{13}\frac{d}{dt}(i_1) + M_{23}\frac{d}{dt}(-i_2) + (i_1 - i_2)R_3$$

$$e(t) = i_1 R_1 + L_1 \mathrm{S} i_1 + M_{31}\mathrm{S}(i_1 - i_2) + M_{21}(-\mathrm{S} i_2) + L_3\mathrm{S}(i_1 - i_2)$$

$$+ M_{13}\mathrm{S} i_1 + M_{23}(-\mathrm{S} i_2) + (i_1 - i_2)R_3$$

$$e(t) = i_1\left[R_1 + R_3 + \mathrm{S}(L_1 + L_3 + M_{31} + M_{13})\right] - i_1$$

$$\left[\mathrm{S}(M_{31} + M_{21}L_3 + M_{23}) + R_3\right]$$

For the loop (2)

$$L_2\frac{di_2}{dt} + M_{12}\frac{d}{dt}(-i_1) + M_{32}\frac{d}{dt} - (i_1 - i_2) + i_2 R_2 + (i_2 - i_1)R_3$$

$$+ L_3\frac{di}{dt}(i_2 - i_1) + M_{13}\frac{d}{dt}(-i_1) + M_{2/3}\frac{d}{dt}i_2 = 0$$

$$i_2\left[S(L_2+M_{32}+M_{23}+R_3+L_3)+R_2+R_3\right]$$

$$-i_1\left[S(M_{12}+M_{32}+M_{13}+L_3)+R_3\right]=0$$

Example 4.5 *In the circuit shown in Fig. 4.11, write the equation for the voltages across the coils ab and cd; also mention the polarities of the terminals.*

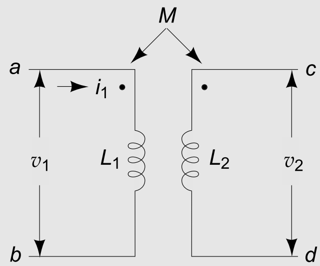


Fig. 4.11

Solution Current i_1 is only flowing in coil *ab*, whereas coil *cd* is open. Therefore, there is no current in coil *cd*. The emf due to self induction is zero on coil *cd*.

$$\therefore \quad v_2(t)=M\frac{di_1(t)}{dt} \text{ with } C \text{ being positive}$$

Similarly the emf due to mutual induction in coil *ab* is zero.

$$\therefore \quad v_1(t)=L\frac{di_1(t)}{dt}$$

Example 4.6 *In the circuit shown in Fig. 4.12, write the equation for the voltages v_1 and v_2. L_1 and L_2 are the coefficients of self inductances of coils 1 and 2, respectively, and M is the mutual inductance.*

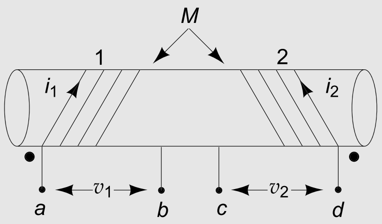


Fig. 4.12

Solution In the figure, *a* and *d* are like terminals.

Currents i_1 and i_2 are entering at dot marked terminals.

$$v_1=L_1\frac{di_1(t)}{dt}+\frac{M\,di_2(t)}{dt}$$

$$v_2=L_2\frac{di_2(t)}{dt}+\frac{M\,di_1(t)}{dt}$$

Example 4.7 *For the circuit shown in Fig. 4.13, write the mesh equations.*

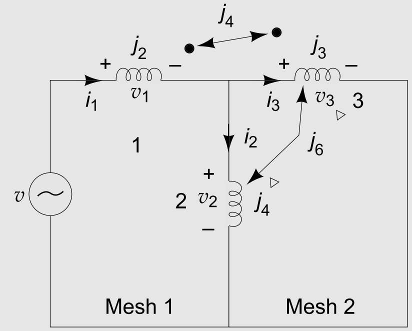


Fig. 4.13

Solution There exists mutual coupling between coil 1 and 3, and 2 and 3. Assuming branch currents i_1, i_2 and i_3 in coils 1, 2 and 3, respectively, the equation for mesh 1 is

$$v = v_1 + v_2$$

$$v = i_1 j_2 - i_3 j_4 + i_2 j_4 - i_3 j_6 \tag{4.5}$$

$j_4\ i_3$ is the mutual inductance drop between coils (1) and (3), and is considered negative according to dot convention and $i_3\ j_6$ is the mutual inductance drop between coils 2 and 3.

For the 2nd mesh $0 = -v_2 + v_3 = -(j_4 i_2 - j_6 i_3) + j_3 i_3 - j_6 i_2 - j_4 i_1$ (4.6)

$$= -j_4 i_1 - j_{10} i_2 + j_9 i_3 \tag{4.7}$$

$$i_1 = i_3 + i_2$$

Example 4.8 *Explain the Dot Convention for mutually coupled coils.*

[JNTU June 2006]

Solution *Dot Convention*

Mutual inductance is the ability of one inductor to induce voltage across the neighbouring inductor measured in Henrys (H).

The mutually induced emf $M\dfrac{di}{dt}$ may be positive (or) negative but M is always positive.

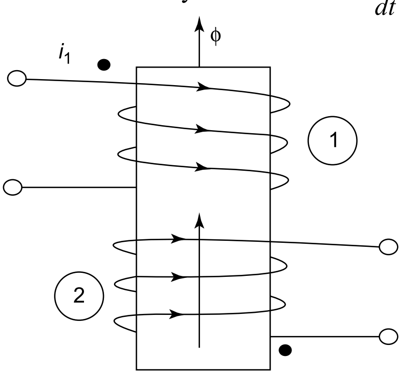


Fig. 4.14

We apply dot convention to determine the polarity of the induced emf. Consider two coils (1) and (2) as shown.

1. Place a dot at one end of coil (1) and assume that the current enters at that dotted end in coil (1).
2. Place another dot at one of the ends of coil (2) such that the current entering at that end in coil (2) establishes magnetic flux in the same direction.

In order that the flux produced by I_2 flowing in coil (2) produce flux in the same upward direction it should enter at lower end of coil (2). Hence place a dot at that end of coil (2).

4.1.4 Coefficient of Coupling *[JNTU June 2009, Nov 2011]*

The amount of coupling between the inductively coupled coils is expressed in terms of the coefficient of coupling, which is defined as

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

where M = mutual inductance between the coils
L_1 = self inductance of the first coil, and
L_2 = self inductance of the second coil

Coefficient of coupling is always less than unity, and has a maximum value of 1 (or 100%). This case, for which $K = 1$, is called perfect coupling, when the entire flux of one coil links the other. The greater the coefficient of coupling between the two coils, the greater the mutual inductance between them, and vice-versa. It can be expressed as the fraction of the magnetic flux produced by the current in one coil that links the other coil.

For a pair of mutually coupled circuits shown in Fig. 4.15, let us assume initially that i_1, i_2 are zero at $t = 0$

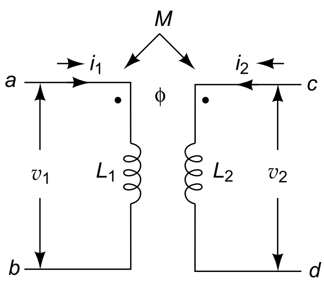


Fig. 4.15

then $v_1(t) = L_1 \dfrac{di_1(t)}{dt} + M \dfrac{di_2(t)}{dt}$

and $v_2(t) = L_2 \dfrac{di_2(t)}{dt} + M \dfrac{di_1(t)}{dt}$

Initial energy in the coupled circuit at $t = 0$ is also zero. The net energy input to the system shown in Fig. 4.15 at time t is given by

$$W(t) = \int_0^t \left[v_1(t)\ i_1(t) + v_2(t)\ i_2(t) \right] dt$$

Substituting the values of $v_1(t)$ and $v_2(t)$ in the above equation yields

$$W(t) = \int_0^t \left[L_1 i_1(t) \frac{di_1(t)}{dt} + L_2 i_2(t) \frac{di_2(t)}{dt} + M(i_1(t)) \frac{di_2(t)}{dt} + i_2(t) \frac{di_1(t)}{dt} \right] dt$$

From which we get

$$W(t) = \frac{1}{2} L_1 \left[i_1(t) \right]^2 + \frac{1}{2} L_2 \left[i_2(t) \right]^2 + M \left[i_1(t) i_2(t) \right]$$

If one current enters a dot-marked terminal while the other leaves a dot marked terminal, the above equation becomes

$$W(t) = \frac{1}{2} L_1 \left[i_1(t)\right]^2 + \frac{1}{2} L_2 \left[i_2(t)\right]^2 - M\left[i_1(t) i_2(t)\right]$$

According to the definition of passivity, the net electrical energy input to the system is non-negative. $W(t)$ represents the energy stored within a passive network, it cannot be negative.

$\therefore$ $W(t) \geq 0$ for all values of i_1, i_2; L_1, L_2 or M

The statement can be proved in the following way. If i_1 and i_2 are both positive or negative, $W(t)$ is positive. The other condition where the energy equation could be negative is

$$W(t) = \frac{1}{2} L_1 [i_1(t)]^2 + \frac{1}{2} L_2 \left[i_2(t)\right]^2 - M\left[i_1(t) i_2(t)\right] \tag{4.8}$$

The above equation can be rearranged as

$$W(t) = \frac{1}{2}\left(\sqrt{L_1} i_1 - \frac{M}{\sqrt{L_1}} i_2\right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right) i_2^2$$

The first term in the parenthesis of the right side of the above equation is positive for all values of i_1 and i_2, and, thus, the last term cannot be negative; hence

$$L_2 - \frac{M^2}{L_1} \geq 0 \tag{4.9}$$

$$\frac{L_1 L_2 - M^2}{L_1} \geq 0 \tag{4.10}$$

$$L_1 L_2 - M^2 \geq 0 \tag{4.11}$$

$$\sqrt{L_1 L_2} \geq M \tag{4.12}$$

$$M \leq \sqrt{L_1 L_2} \tag{4.13}$$

Obviously the maximum value of the mutual inductance is $\sqrt{L_1 L_2}$. Thus, we define the coefficient of coupling for the coupled circuit as

$$K = \frac{M}{\sqrt{L_1 L_2}} \tag{4.14}$$

The coefficient, K, is a non negative number and is independent of the reference directions of the currents in the coils. If the two coils are a great distance apart in space, the mutual inductance is very small, and K is also very small. For iron-core coupled circuits, the value of K may be as high as 0.99, for air-core coupled circuits, K varies between 0.4 to 0.8.

Example 4.9 *Two inductively coupled coils have self inductances $L_1 = 50$ mH and $L_2 = 200$ mH. If the coefficient of coupling is 0.5 (i), find the value of mutual inductance between the coils, and (ii) what is the maximum possible mutual inductance?*

Solution (i) $M = K\sqrt{L_1 L_2}$

$$= 0.5\sqrt{50\times 10^{-3}\times 200\times 10^{-3}} = 50\times 10^{-3}\,\text{H}$$

(ii) Maximum value of the inductance when $K = 1$,

$$M = \sqrt{L_1 L_2} = 100\ \text{mH}$$

Example 4.10 *Derive the expression for coefficient coupling between pair of magnetically coupled coils.* *[JNTU June 2006]*

Solution *Coefficient of Coupling*

It is a measure of the flux linkages between the two coils.

The coefficient of coupling is defined as the fraction of the total flux produced by one coil linking with another and it is denoted by 'k'.

Let $\phi_1 \Rightarrow$ flux produced by coil − 1

$\phi_2 \rightarrow$ flux produced by coil − 2

$\phi_{12} \rightarrow$ flux produced by coil − 1 linking with coil − 2

$\phi_{21} \rightarrow$ flux produced by coil − 2 linking with coil − 1

$\therefore$ Coefficient of coupling $k = \dfrac{\phi_{12}}{\phi_1} = \dfrac{\phi_{21}}{\phi_2}$

k value lies between 0 and 1.

we know that $\quad M_{12} = \dfrac{M_2\phi_{12}}{i_1},\ M_{21} = \dfrac{M_1\phi_{21}}{i_2}$

$$M_{12}\times M_{21} = \frac{M_2\phi_{12}\times M_1\phi_{21}}{i_1 i_2}$$

$$M^2 = \frac{M_2\times k\phi_1}{i_1}\times\frac{M_1\times k\phi_2}{i_2}$$

$$M^2 = k^2\,\frac{M_1\phi_1}{i_1}\times\frac{M_2\phi_2}{i_2} = k^2 L_1 L_2$$

$$\Rightarrow k = \frac{M}{\sqrt{L_1 L_2}}$$

Example 4.11 *Obtain the equivalent 'T' for magnetically coupled circuit shown in Fig. 4.16.* *[JNTU June 2006]*

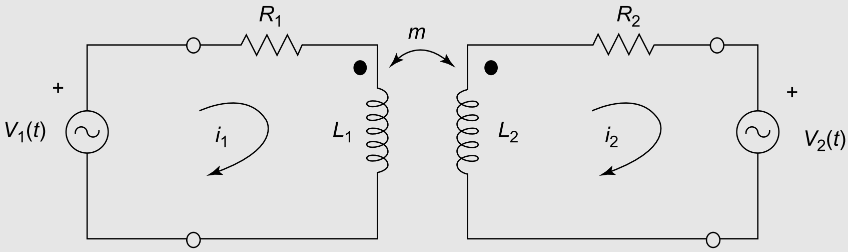


Fig. 4.16

Solution

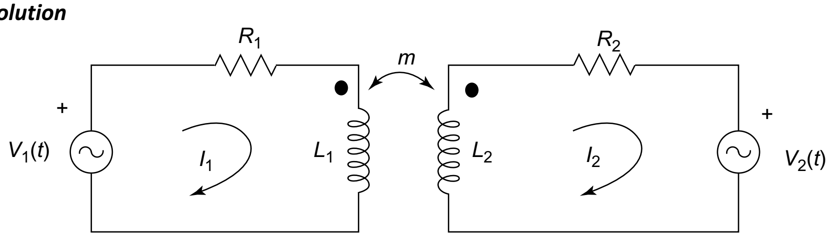


Fig. 4.17

$$V_1(t) = I_1R_1 + L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}$$

$$V_2(t) = I_2R_2 + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}$$

The equivalent 'T' for magnetically coupled circuit is

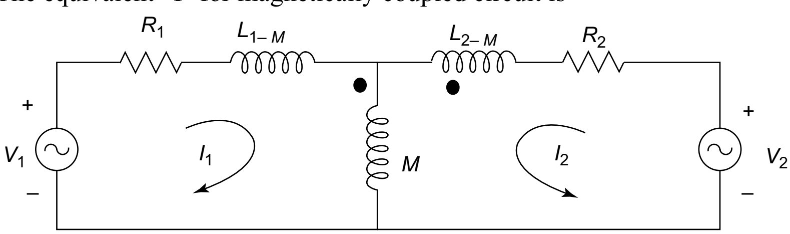


Fig. 4.18

Example 4.12 *Write down the loop equations for the network shown in Fig. 4.19.* *[JNTU June 2006]*

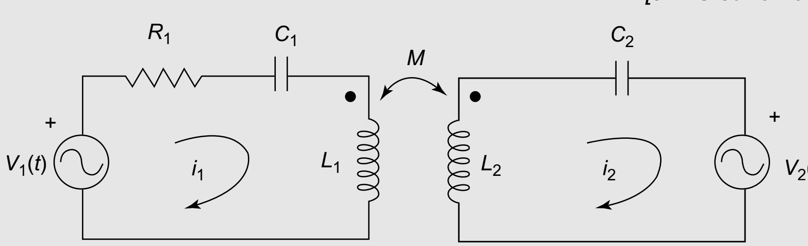


Fig. 4.19

Solution As i_1 is entering at the dot terminal, and i_2 is leaving the dot terminal, sign of M (mutual inductance) is −ve

$$i_1(R_1 - j/wC_1 + jwL_1) - i_2 jwM = V_1(t)$$

is loop equation for 1st mesh.

$$I_2(jwL_2 - j/wC_2) - i_1(jwM) = -V_2(t)$$

is loop equation for 2nd mesh

Example 4.13 *Obtain the equivalent 'T' for a magnetically coupled circuit shown in Fig. 4.20.* *[JNTU May 2007]*

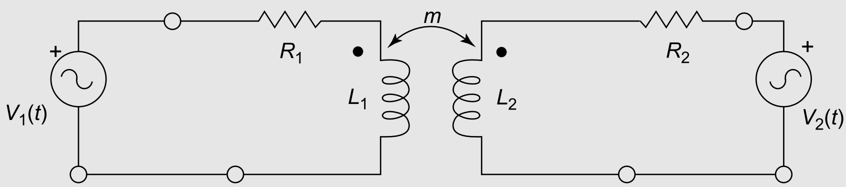


Fig. 4.20

Solution The equivalent for 'T' the given magnetically coupled circuit is

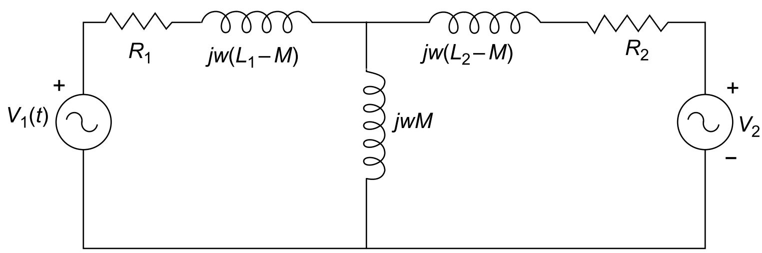


Fig. 4.21

Example 4.14 *Write down the loop equations for the network shown in Fig. 4.22.* *[JNTU May 2007]*

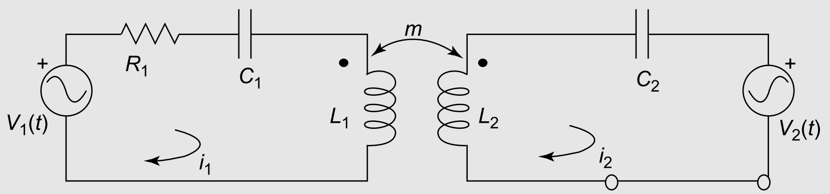


Fig. 4.22

Solution The loop equations for the given network is

$$V_1 = I_1(R_1 + jwL_1) + \frac{1}{jwc_1}(I_1) - jwMI_2$$

$$jwL_2 I_2 + \frac{1}{jwc_2}(I_2) - I_1(jwM) + V_2 = 0.$$

4.1.5 Ideal Transformer

Transfer of energy from one circuit to another circuit through mutual induction is widely utilised in power systems. This purpose is served by transformers. Most often, they transform energy at one voltage (or current) into energy at some other voltage (or current).

A transformer is a static piece of apparatus, having two or more windings or coils arranged on a common magnetic core. The transformer winding to which the supply source is connected is called the *primary*, while the winding connected to load is called the *secondary*. Accordingly, the voltage across the primary is called the primary voltage, and that across the secondary, the secondary voltage. Correspondingly i_1 and i_2 are the currents in the primary and secondary windings. One such transformer is shown in Fig. 4.23(a). In circuit diagrams, ideal transformers are represented by Fig. 4.23(b). The vertical lines between the coils represent the iron core; the currents are assumed such that the mutual inductance is positive. An ideal transformer is characterised by assuming (i) zero power dissipation in the primary and secondary windings, i.e. resistances in the coils are assumed to be zero, (ii) the self inductances of the primary and secondary are extremely large in comparison with the load impedance, and (iii) the coefficient of coupling is equal to unity, i.e. the coils are tightly coupled without having any leakage flux. If the flux produced by the current flowing in a coil links all the turns, the self inductance of either the primary or secondary coil is proportional to the square of the number of turns of the coil. This can be verified from the following results.

The magnitude of the self induced emf is given by

$$v = L\frac{di}{dt} \tag{4.15}$$

If the flux linkages of the coil with N turns and current are known, then the self induced emf can be expressed as

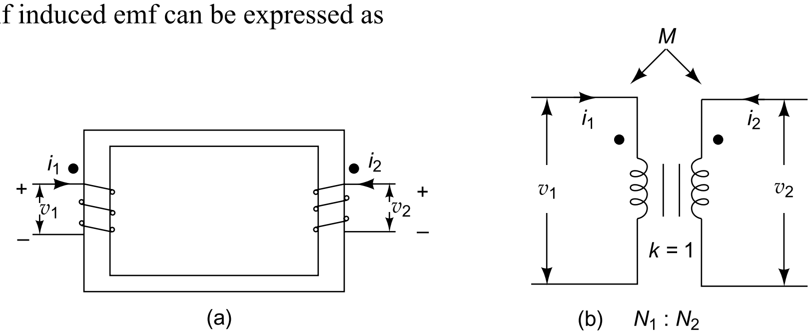


Fig. 4.23

$$L = N\frac{d\phi}{dt} \tag{4.16}$$

$$L\frac{di}{dt} = N\frac{d\phi}{dt} \tag{4.17}$$

$$L = N\frac{d\phi}{dt}$$

But $$\phi = \frac{Ni}{\text{reluctance}}$$

$\therefore$ $$L = N\frac{d}{di}\left(\frac{Ni}{\text{reluctance}}\right)$$

$$L = \frac{N^2}{\text{reluctance}}$$

$$L \alpha N^2 \tag{4.18}$$

From the above relation it follows that

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2 \tag{4.19}$$

where $a = N_2/N_1$ is called the *turns ratio* of the transformer. The turns ratio, a, can also be expressed in terms of primary and secondary voltages. If the magnetic permeability of the core is infinitely large then the flux would be confined to the core. If ϕ is the flux through a single turn coil on the core and N_1, N_2 are the number of turns of the primary and secondary, respectively, then the total flux through windings 1 and 2, respectively, are

$$\phi_1 = N_1\,\phi;\ \phi_2 = N_2\,\phi$$

Also we have $$v_1 = \frac{d\phi_1}{dt}, \text{ and } v_2 = \frac{d\phi_2}{dt} \tag{4.20}$$

so that $$\frac{V_2}{V_1} = \frac{N_2\dfrac{d\phi}{dt}}{N_1\dfrac{d\phi}{dt}} = \frac{N_2}{N_1} \tag{4.21}$$

Figure 4.17 shows an ideal transformer to which the secondary is connected to a load impedance Z_L. The turns ratio $\dfrac{N_2}{N_1} = a$.

The ideal transformer is a very useful model for circuit calculations, because with few additional elements like R, L and C, the actual behaviour of the physical transformer can be accurately represented. Let us analyse this transformer with sinusoidal excitations. When the excitations are sinusoidal voltages or currents, the steady state response will also be sinusoidal. We can use phasors for representing these voltages and currents. The input impedance of the transformer can be determined by writing mesh equations for the circuit shown in Fig. 4.24

$$V_1 = j\omega L_1 L_1 - j\omega M I_2 \tag{4.22}$$

$$0 = -\,j\omega M I_1 + (Z_L + j\omega L_2)I_2 \tag{4.23}$$

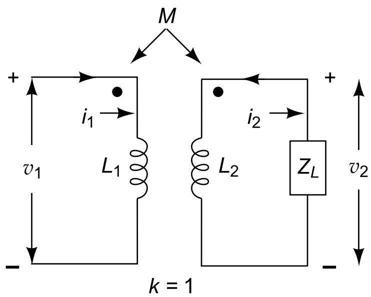


Fig. 4.24

where V_1, V_2 are the voltage phasors, and I_1, I_2 are the current phasors in the two windings. $j\omega L_1$ and $j\omega L_2$ are the impedances of the self inductances and $j\omega M$ is the impedance of the mutual inductance, ω is the angular frequency.

From Eq. 4.25, $I_2 = \dfrac{j\omega M I_1}{(Z_L + j\omega L_2)}$

Substituting in Eq. 4.24, we have

$$V_1 = I_1 j\omega L_1 + \frac{I_1\omega^2 M^2}{Z_L + j\omega L_2}$$

The input impedance $Z_{\text{in}} = \dfrac{V_1}{I_1}$

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 M^2}{(Z_L + j\omega L_2)}$$

When the coefficient of coupling is assumed to be equal to unity,

$$M = \sqrt{L_1 L_2}$$

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{(Z_L + j\omega L_2)}$$

We have already established that $\dfrac{L_2}{L_1} = a^2$

where a is the turns ratio N_2/N_1

$$\therefore \qquad Z_{\text{in}} = j\omega L_1 + \frac{\omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

Further simplification leads to

$$Z_{\text{in}} = \frac{(Z_L + j\omega L_2)\, j\omega L_1 + \omega^2 L_1^2 a^2}{(Z_L + j\omega L_2)}$$

$$Z_{\text{in}} = \frac{j\omega L_1 Z_L}{(Z_L + j\omega L_2)}$$

As L_2 is assumed to be infinitely large compared to Z_L

$$Z_{\text{in}} = \frac{j\omega L_1 Z_L}{j\omega a^2 L_1} = \frac{Z_L}{a^2} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

The above result has an interesting interpretation, that is the ideal transformers change the impedance of a load, and can be used to match circuits with different impedances in order to achieve maximum power transfer. For example, the input impedance of a loudspeaker is usually very small, say 3 to 12 Ω for connecting directly to an amplifier. The transformer with proper turns ratio can be placed between the output of the amplifier and the input of the loudspeaker to match the impedances as shown in Fig. 4.25.

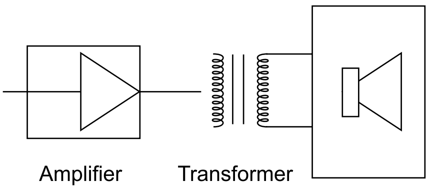


Fig. 4.25

Example 4.15 *An ideal transformer has $N_1 = 10$ turns, and $N_2 = 100$ turns. What is the value of the impedance referred to as the primary, if a 1000 Ω resistor is placed across the secondary?*

Solution The turns ratio $a = \dfrac{100}{10} = 10$

$$Z_{\text{in}} = \frac{Z_L}{a^2} = \frac{1000}{100} = 10\,\Omega$$

The primary and secondary currents can also be expressed in terms of turns ratio.

From Eq. 4.25, we have

$$I_1 jwM = I_2 (Z_L + jwL_2)$$

$$\frac{I_1}{I_0} = \frac{Z_L + j\omega L_2}{j\omega M}$$

When L_2, is very large compared to Z_L,

$$\frac{I_1}{I_2} = \frac{j\omega L_2}{j\,\omega\,M} = \frac{L_2}{M}$$

Substituting the value of $M = \sqrt{L_1 L_2}$ in the above equation $\dfrac{I_1}{I_2} = \dfrac{L_2}{M}$

$$\frac{I_1}{I_2} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}}$$

$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}} = a = \frac{N_2}{N_1}$$

Example 4.16 *An amplifier with an output impedance of 1936 Ω is to feed a loudspeaker with an impedance of 4 Ω.*

(a) Calculate the desired turns ratio for an ideal transformer to connect the two systems.

(b) An rms current of 20 mA at 500 Hz is flowing in the primary. Calculate the rms value of current in the secondary at 500 Hz.

(c) What is the power delivered to the load?

Solution (a) To have maximum power transfer the output impedance of the

$$\text{amplifier} = \frac{\text{Load impedence}}{a^2}$$

$$\therefore \qquad 1936 = \frac{4}{a^2}$$

$$\therefore \qquad a = \sqrt{\frac{4}{1936}} = \frac{1}{22}$$

$$\text{or} \qquad \frac{N_2}{N_1} = \frac{1}{22}$$

(b) $I_1 = 20$ mA

We have $\dfrac{I_1}{I_2} = a$

RMS value of the current in the secondary winding

$$= \frac{I_1}{a} = \frac{20 \times 10^{-3}}{1/22} = 0.44 \text{ A}$$

(c) The power delivered to the load (speaker)

$$= (0.44)^2 \times 4 = 0.774 \text{ W}$$

The impedance changing properties of an ideal transformer may be utilised to simplify circuits. Using this property, we can transfer all the parameters of the primary side of the transformer to the secondary side, and *vice-versa*. Consider the coupled circuit shown in Fig. 4.26(a).

To transfer the secondary side load and voltage to the primary side, the secondary voltage is to be divided by the ratio, a, and the load impedance is to be divided by a^2. The simplified equivalent circuits shown in Fig. 4.26(b).

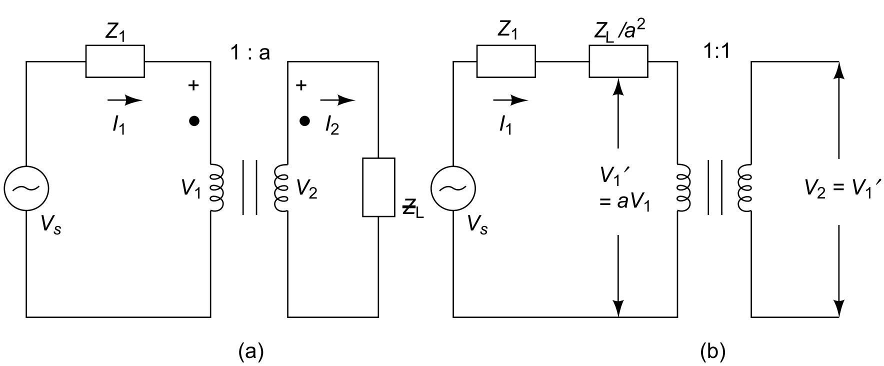


Fig. 4.26

Example 4.17 *For the circuit shown in Fig. 4.27 with turns ratio, a = 5, draw the equivalent circuit referring (a) to primary and (b) secondary. Take source resistance as 10 Ω.*

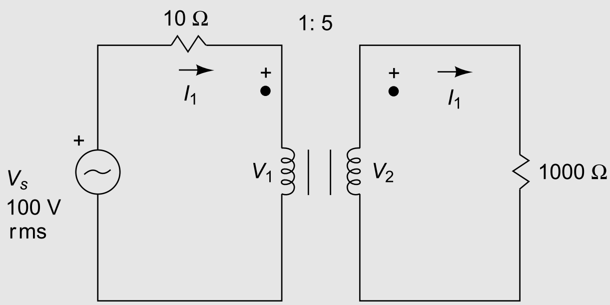


Fig. 4.27

Solution (a) Equivalent circuit referred to primary is as shown in Fig. 4.28(a).

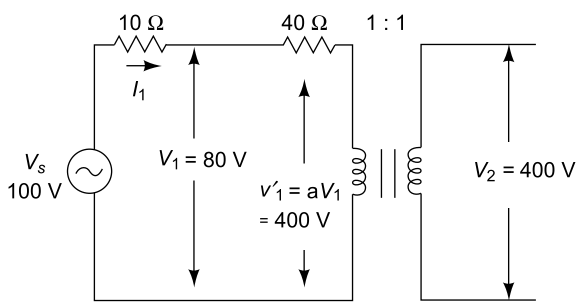


Fig. 4.28(a)

(b) Equivalent circuit referred to secondary is as shown in Fig. 4.28(b).

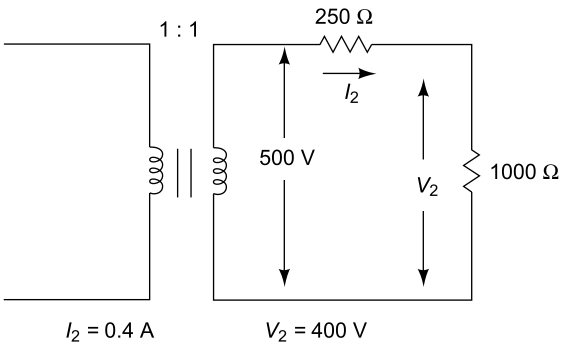


Fig. 4.28(b)

Example 4.18 *In Fig. 4.29, $L_1 = 4H$; $L_2 = 9_1$, H, K = 0.5, $i_1 = 5\cos(50t - 30°)$ A, $i_2 = 2\cos(50t - 30°)$ A. Find the values of (a) v_1, (b) v_2, and (c) the total energy stored in the system at t = 0.*

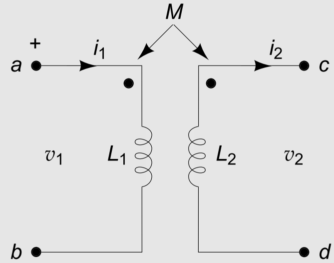


Fig. 4.29

Solution Since the current in coil *ab* is entering at the dot marked terminal, whereas in coil *cd* the current is leaving, we can write the equations as

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

$$M = K\sqrt{L_1L_2} = 0.5\sqrt{36} = 3$$

(a) $$v_1 = 4\frac{d}{dt}\left[5\cos(50t - 30°) - 3\frac{d}{dt}[2\cos(50t - 30°)]\right]$$

$$v_1 = 20[-\sin(50t - 30°)\times 50] - 6[-\sin(50t - 30°)50]$$

$$v_1 = 500 - 150 = 350 \text{ V}$$

(b) $$v_2 = -3\frac{d}{dt}[5\cos(50t - 30°)] + 9\frac{d}{dt}[2\cos(50t - 30°)]$$

$$= -15[-\sin(50t - 30°)\times 50] + 18[-\sin(50t - 30°)50]$$

at $t = 0$

$$v_2 = -375 + 450 = 75 \text{ V}$$

(c) The total energy stored in the system

$$W(t) = \frac{1}{2}L_1[i_1(t)]^2 + \frac{1}{2}L_2[i_2(t)]^2 - M[i_1(t)i_2(t)]$$

$$= \frac{1}{2}\times 4[5\cos(50t - 30°)]^2 + \frac{1}{2}\times 9[2\cos(50t - 30°)]^2$$
$$-3[5\cos(50t - 30°)\times 2\cos(50t - 30°)]$$

at $t = 0$ W(t) = 28.5 j

Example 4.19 *Solve for the currents I_1, and I_2 in the circuit shown in Fig. 4.30. Also, find the ratio of V_2/V_1.* *[JNTU June 2006]*

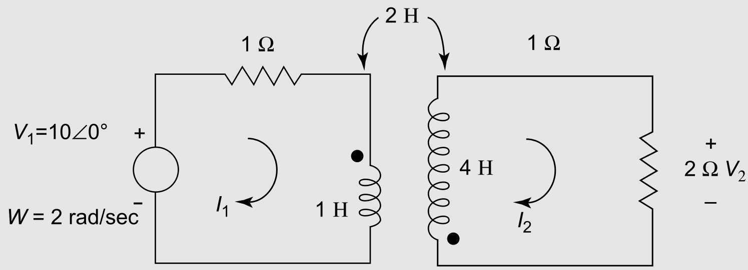


Fig. 4.30

Solution $w = 2$ rad/sec

$J \times L_1 = J_2\ \Omega$

$J \times L_2 = J(4 \times 2) = J8$

KVL to Loop 1

$$M = J_4$$

$$I_1(1 + J_2) + (J4)I_2 = V_1 \tag{4.24}$$

KVL to Loop 2

$$(J_4)I_1 + (2 + J_8)I_2 = 0$$

So the mesh equation are

$$(1 + J2)I_1 + (J4)I_2 = V_1 = 10$$

$$(J4)I_1 + (2 + J8)I_2 = 0$$

$$\begin{bmatrix} 1+J2 & J4 \\ J4 & 2+J8 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$I_1 = \frac{\begin{vmatrix} 10 & J4 \\ 0 & 2+J8 \end{vmatrix}}{\Delta} \quad I_2 = \frac{\begin{vmatrix} 1+J2 & 10 \\ J4 & 0 \end{vmatrix}}{\Delta}$$

$$\Delta = \begin{vmatrix} 1+J2 & J4 \\ J4 & 2+J8 \end{vmatrix} = 2+12i$$

$$I_1 = \frac{20+80i}{2+12i} \qquad I_2 = \frac{-40i}{2+12i}$$

$I_1 = 6.75 - 0.540i \quad I_2 = -3.243 - 0.540i$

$V_2 = 2I_2 \qquad I_2 = 3.287\angle -170.53°\text{ A}$

Ratio
$$\frac{V_2}{V_1} = \frac{2\times(3.287\angle-170.53°\ )}{10\angle 0°}$$

$$\frac{V_2}{V_1} = 0.657\angle-170.537°$$

4.2 ANALYSIS OF COUPLED CIRCUITS

Inductively coupled multi-mesh circuits can be analysed using Kirchhoff's laws and by loop current methods. Consider Fig. 4.31, where three coils are inductively coupled. For such a system of inductors we can define a inductance matrix L as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

where L_{11}, L_{22} and L_{33} are self inductances of the coupled circuits, and $L_{12} = L_{21}$; $L_{23} = L_{32}$ and $L_{13} = L_{31}$ are mutual inductances. More precisely, L_{12} is the mutual inductance between coils 1 and 2, L_{13} is the mutual inductance between coils 1 and 3, and L_{23} is the mutual inductance between coils 2 and 3. The inductance matrix has its order equal to the number of inductors and is symmetric. In terms of voltages across the coils, we have a voltage vector related to i by

$$[v] = [L]\left[\frac{di}{dt}\right]$$

where v and i are the vectors of the branch voltages and currents, respectively. Thus the branch volt-ampere relationships of the three inductors are given by

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

Using KVL and KCL, the effective inductances can be calculated. The polarity for the inductances can be determined by using passivity criteria, whereas the signs of the mutual inductances can be determined by using the dot convention.

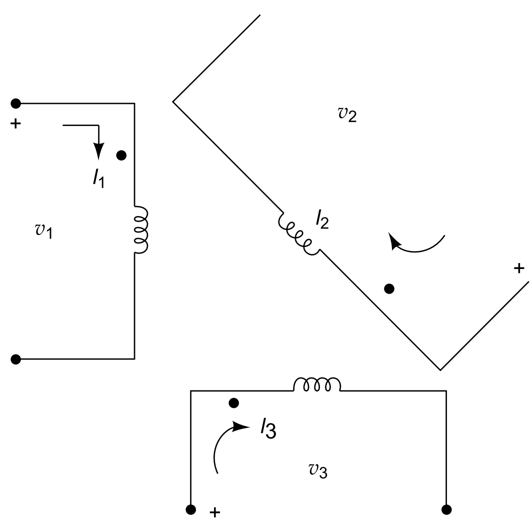


Fig. 4.31

Example 4.20 *For the circuit shown in Fig. 4.32, write the inductance matrix.*

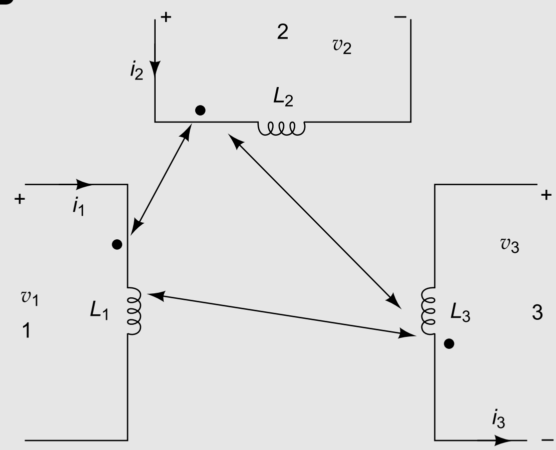


Fig. 4.32

Solution Let L_1, L_2 and L_3 be the self inductances, and $L_{12} = L_{21}$, $L_{23} = L_{32}$ and $L_{13} = L_{31}$ be the mutual inductances between coils, 1, 2, 2, 3 and 1, 3, respectively. $L_{12} = L_{21}$ is positive, as both the currents are entering at dot marked terminals, whereas $L_{13} = L_{31}$, and $L_{23} = L_{32}$ are negative.

$\therefore$ The inductance matrix is $L = \begin{bmatrix} L_1 & L_{12} & -L_{13} \\ L_{21} & L_2 & -L_{23} \\ -L_{31} & -L_{32} & L_3 \end{bmatrix}$

4.2.1 Series Connection of Coupled Circuits

Let there be two inductors connected in series, with self inductances L_1 and L_2 and mutual inductance of M. Two kinds of series connections are possible; series aiding as in Fig. 4.33(a), and series opposition as in Fig. 4.33(b).

In the case of series aiding connection, the currents in both inductors at any instant of time are in the same direction relative to like terminals as shown in Fig. 4.33(a). For this reason, the magnetic fluxes of self induction and of mutual induction linking with each element add together.

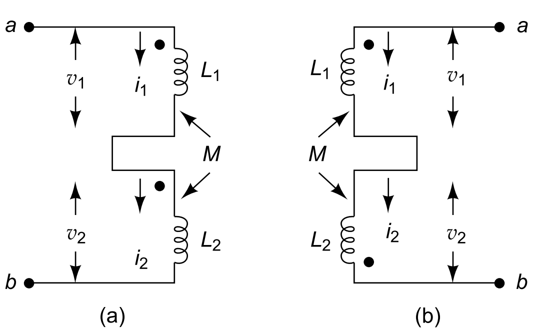


Fig. 4.33

In the case of series opposition connection, the currents in the two inductors at any instant of time are in opposite direction relative to like terminals as shown in Fig. 4.33(b). The inductance of an element is given by $L = \phi/i$ where ϕ is the flux produced by the inductor.

$\therefore \qquad \phi = Li$

For the series aiding circuit, if ϕ_1, and ϕ_2 are the flux produced by the coils 1 and 2, respectively, then the total flux

$$\phi = \phi_1 + \phi_2$$

where $\phi_1 = L_1 i_1 + M i_2$

$$\phi_2 = L_2 i_2 + M i_1$$

$\therefore$ $\phi = Li = L_1 i_1 + M i_2 + L_2 i_2 + M i_1$

Since $i_1 = i_2 = i$

$$L = L_1 + L_2 + 2M$$

Similarly, for the series opposition

$$\phi = \phi_1 + \phi_2$$

where $\phi_1 = L_1 i_1 - M i_2$

$$\phi_2 = L_2 i_2 - M i_2$$

$$\phi = Li = L_1 i_1 - M i_2 + L_2 i_2 - M i_1$$

Since $i_1 = i_2 = i$

$$L = L_1 + L_2 + 2M$$

In general, the inductance of two inductively coupled elements in series is given by $L = L_1, + L_2 \pm 2M$.

Positive sign is applied to the series aiding connection, and negative sign to the series opposition connection.

Example 4.21 *Two coils connected in series have an equivalent inductance of 0.4 H when connected in aiding, and an equivalent inductance 0.2 H when the connection is opposing. Calculate the mutual inductance of the coils.*

Solution When the coils are arranged in aiding connection, the inductance of the combination is $L_1 + L_2 + 2M = 0.4$; and for opposing connection, it is $L_1 + L_2 - 2M = 0.2$. Solving the two equations, we get

$$4M = 0.2 \text{ H}$$

$$M = 0.05 \text{ H}$$

Example 4.22 *Calculate the effective inductance of the current shown in Fig. 4.34.*

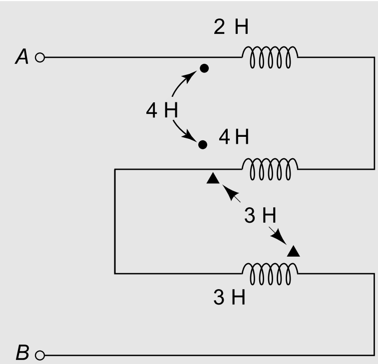


Fig. 4.34

Solution Let 'i' be the current from A to B and v be the voltage across AB.

$$v = \frac{di}{dt}[2+4+3-4-4+3+3]$$

The first three terms are self-induced terms and the later four terms are mutual terms.

$$\therefore \qquad v = 7\frac{di}{dt}$$

$$L = 7\text{H}$$

Example 4.23 *Calculate the effective inductance of the circuit shown in Fig. 4.35 across terminals a and b.*

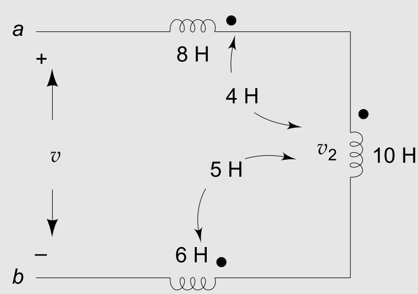


Fig. 4.35

Solution Let the current in the circuit be i

$$v = 8\frac{di}{dt} - 4\frac{di}{dt} + 10\frac{di}{dt} - 4\frac{di}{dt} + 5\frac{di}{dt} + 6\frac{di}{dt} + 5\frac{di}{dt}$$

or $$\frac{di}{dt}[34-8] = 26\frac{di}{dt} = v$$

Let L be the effective inductance of the circuit across ab. Then the voltage across $ab = v = L\frac{di}{dt} = 26\frac{di}{dt}$

Hence, the equivalent inductance of the circuit is given by 26 H.

Example 4.24 *Write down the voltage equation for the following, and determine the effective inductance.* *[JNTU June 2006]*

Solution Apply KVL in the given loop

$$V(t) = L_1\frac{di(t)}{dt} + L_2\frac{di(t)}{dt} + M_A\frac{di(t)}{dt} + M_A\frac{di(t)}{dt} + L_3\frac{di(t)}{dt} - M_B\frac{di(t)}{dt} - M_B\frac{di(t)}{dt} - M_C\frac{di(t)}{dt} - M_C\frac{di(t)}{dt}$$

$$\therefore \qquad V(t) = [L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C]\frac{di(t)}{dt}$$

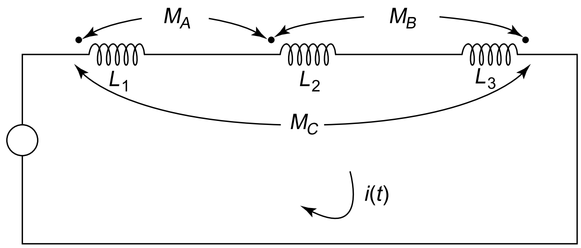


Fig. 4.36

is the required voltage equation.

We have $\quad V(t) = L\dfrac{di(t)}{dt}$

$$L\frac{di(t)}{dt} = [L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C]\frac{di(t)}{dt}$$

$\therefore \quad L = L_1 + L_2 + L_3 + 2M_A - 2M_B - 2M_C$ is the equivalent inductance.

Example 4.25 *Two identical coils connected in series gave an inductance of 800 mH and when one of the coils is reversed gave an inductance of 400 mH. Determine self-inductance, mutual inductance between the coils and the co-efficient of coupling.* *[JNTU June 2006]*

Solution Let 'L' be the self inductance of the coils and M be the mutual inductance between the coils.

Given data

Two identical coils connected in series gave an inductance of 800 mH

i.e. $\quad L + L + 2M = 800 \quad$ [$\because$ identical coils $L_1 = L_2 = L$]

$\quad 2L + 2M = 800$

When one of the coils is reversed gave an inductance of 400 mH

i.e. $\quad L + L - 2M = 400$

$\quad 2L - 2M = 400$

Add (1) and (2) we get $4L = 1200$

$L = 300$ mH

Subtracting (2) from (1) we get $4M = 400$ mH

$M = 100$ mH

$\therefore$ Self inductance of each coil $= L = 300$ mH

Mutual inductance between the coils $= M = 100$ mH

$$\text{Co-efficient of coupling} = K = \frac{M}{\sqrt{L_1 L_2}}$$

$$\therefore \quad K = \frac{M}{\sqrt{LL}} \qquad [\because L_1 = L_2 = L]$$

$$\therefore \qquad K = \frac{M}{\sqrt{L^2}} = \frac{M}{L}$$

$$\therefore \qquad K = \frac{100\,\text{mH}}{300\,\text{mH}}$$

$$\therefore \qquad K = 1/3$$

$\therefore$ Co-efficient of coupling = 1/3.

Example 4.26 *In the circuit shown in Fig. 4.37 find the voltage across the terminals A and B if the current changes at the rate of 100 A/sec. The value of L_1, L_2 and M are 1 H, 2 H, and 0.5 H respectively. [JNTU May 2007]*

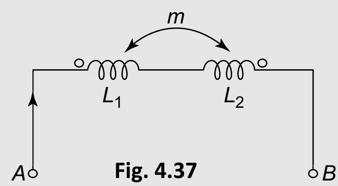


Fig. 4.37

Solution

$$V_{AB} = L_1 \frac{di}{dt} - M\frac{di}{dt} + L_2\frac{di}{dt} - M\frac{di}{dt}$$

$$= (L_1 + L_2 - 2M)\frac{di}{dt}$$

$$V_{AB} = (1 + 2 - 2(0.5))\,100$$

$$V_{AB} = 200 \text{ volts}$$

Example 4.27 *A 15 mH coil is connected in series with another coil. The total inductance is 70 mH. When one of the coils is reversed, the total inductance is 30 mH. Find the inductance of second coil, mutual inductance and coefficient of coupling. Derive the expression used. [JNTU June 2009]*

Solution Total inductance $= L_1 + L_2 + 2\text{M}$

$$= 15\,\text{mH} + x + 2\,\text{M} = 70\,\text{mH} \tag{4.25}$$

Total inductance $= L_1 + L_2 - 2\text{M}$

$$= 15\,\text{mH} + x - 2\,\text{M} = 30\,\text{mH} \tag{4.26}$$

So inductance of 2nd coil:

$$(4.25) + (4.26) \quad 15\,\text{mH} + x + 2\,\text{M} = 70\,\text{mH}$$

$$15\,\text{mH} + x - 2\,\text{M} = 30\,\text{mH}$$

or

$$30\,\text{mH} + 2x = 100\,\text{mH}$$

$$\therefore \qquad x = 35\,\text{mH}$$

Now putting this in (4.25)

$$15\,\text{mH} + 35\,\text{mH} + 2\,\text{M} = 70\,\text{mH}$$

$$2\text{M} = 20\,\text{mH}$$

$$\therefore \qquad \text{M} = 10\,\text{mH}$$

$$\text{M} = k\sqrt{L_1 L_2}$$

$$\therefore \quad 10 = k\sqrt{35 \times 15}$$

$$\therefore \quad k = 0.436$$

4.2.2 Parallel Connection of Coupled Circuits

Consider two inductors with self inductances L_1 and L_2 connected parallel which are mutually coupled with mutual inductance M as shown in Figs 4.38(a) and (b).

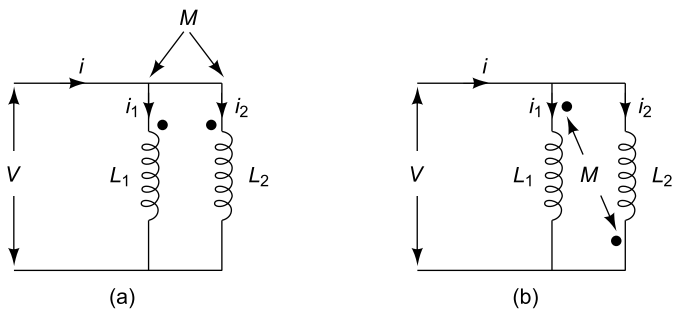


Fig. 4.38

Let us consider Fig. 4.38(a) where the self-induced emf in each coil assists the mutually induced emf as shown by the dot convention.

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \tag{4.27}$$

The voltage across the parallel branch is given by

$$v = L_1 \frac{di_2}{dt} + M\frac{di_2}{dt} \quad \text{or} \quad L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}$$

also
$$L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} = L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}$$

$$\frac{di_1}{dt}(L_1 - M) = \frac{di_2}{dt}(L_2 - M)$$

$$\therefore \quad \frac{di}{dt} = \frac{di_2}{dt}\frac{(L_2 - M)}{(L_1 - M)} \tag{4.28}$$

Substituting Eq. 4.28 in Eq. 4.27, we get

$$\frac{di}{dt} = \frac{di_2}{dt}\frac{(L_2 - M)}{(L_1 - M)} + \frac{di_2}{dt} = \frac{di_1}{dt}\left[\frac{(L_2 - M)}{L_1 - M} + 1\right] \tag{4.29}$$

If L_{eq} is the equivalent inductance of the parallel circuit in Fig. 4.38(a) then v is given by

$$v = L_{eq}\frac{di}{dt}$$

$$L_{eq}\frac{di}{dt} = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L_{eq}}\left[L_1\frac{di_1}{dt} + M\frac{di_2}{dt}\right]$$

Substituting Eq. 4.30 in the above equation we get

$$\frac{di}{dt} = \frac{1}{L_{eq}}\left[L_1\frac{di_2(L_2 - M)}{dt(L_2 - M)} + M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L_{eq}}\left[L_1\frac{(L_2 - M)}{L_1 - M} + M\right]\frac{di_2}{dt} \tag{4.30}$$

Equating Eq. 4.30 and Eq. 4.29, we get

$$\frac{L_2 - M}{L_2 - M} + 1 = \frac{1}{L_{eq}}\left[L_1\left(\frac{L_2 - M}{L_1 - M}\right) + M\right]$$

Rearranging and simplifying the above equation results in

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

If the voltage induced due to mutual inductance oppose the self induced emf in each coil as shown by the dot convention in Fig. 4.38(b), the equivalent inductance its given by

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$

Example 4.28 *For the circuit shown in Fig. 4.39, find the ratio of output voltage to the source voltage.*

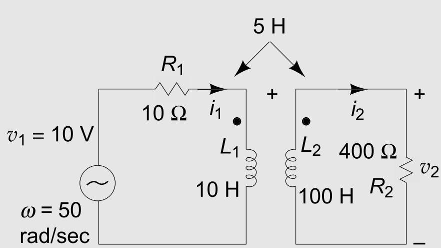


Fig. 4.39

Solution Let us consider i_1 and i_2 as mesh currents in the primary and secondary windings.

As the current i_1 is entering at the dot marked terminal, and current i_2 is leaving the dot marked terminal, the sign of the mutual inductance is to be negative. Using Kirchhoff's voltage law, the voltage equation for the first mesh is

$$i_1(R_1 + j\omega L_1) - i_2 j\omega M = v_1$$

$$i_1(10 + j500) - i_2 j250 = 10 \tag{4.31}$$

Similarly, for the 2nd mesh

$$i_1(R_2 + j\omega L_2) - i_1 j\omega M = 0$$

$$i_2(400 + j5000) - i_1 j250 = 0 \tag{4.32}$$

$$i_2 = \frac{\begin{vmatrix} (10+j500) & 10 \\ -j250 & 0 \end{vmatrix}}{\begin{vmatrix} (10+j500) & -j250 \\ -j250 & (400+j5000) \end{vmatrix}}$$

$$i_2 = 0.00102 \angle -84.13°$$

$$v_2 = i_2 \times R_2$$

$$= 0.00102 \angle -84.13° \times 400$$

$$= 0.408 \angle -84.13°$$

$$\frac{v_2}{v_1} = \frac{0.408}{10} \angle -84.13°$$

$$\frac{v_2}{v_1} = 40.8 \times 10^{-3} \angle -84.13°$$

Example 4.29 *Calculate the effective inductance of the circuit shown in Fig. 4.40 across AB.*

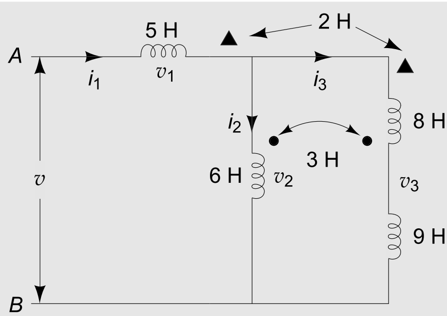


Fig. 4.40

Solution The inductance matrix is

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix}$$

From KVL $v = v_1 + v_2$ (4.33)

and $v_2 = v_3$ (4.34)

From KCL $i_1 = i_2 + i_3$ (4.35)

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & -3 \\ -2 & -3 & 17 \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ di_3/dt \end{bmatrix}$$

$$v_1 = 5\frac{di_1}{dt} - 2\frac{di_3}{dt} \tag{4.36}$$

and

$$v_2 = 6\frac{di_2}{dt} - 3\frac{di_3}{dt} \tag{4.37}$$

$$v_3 = -2\frac{di_1}{dt} - 3\frac{di_2}{dt} + 17\frac{di_3}{dt} \tag{4.38}$$

From Eq. 4.33, we have

$$v = v_1 + v_2$$

$$= 5\frac{di_1}{dt} - 2\frac{di_3}{dt} + 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

$$v = 5\frac{di_1}{dt} + 6\frac{di_2}{dt} - 5\frac{di_3}{dt} \tag{4.39}$$

From Eq. 4.35,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.40}$$

Substituting Eq. 4.40 in Eq. 4.39, we have

$$v_3 = -2\left[\frac{di_2}{dt} + \frac{di_3}{dt}\right] - 3\left[\frac{di_2}{dt}\right] + 17\left[\frac{di_3}{dt}\right]$$

or

$$-5\frac{di_2}{dt} + 15\frac{di_3}{dt} = v_3 \tag{4.41}$$

Multiplying Eq. 4.37 by 5, we get

$$30\frac{di_2}{dt} - 15\frac{di_3}{dt} = 5v_2 \tag{4.42}$$

Adding Eqs (4.41) and (4.42), we get

$$25\frac{di_2}{dt} = v_3 + 5v_2$$

$$25\frac{di_2}{dt} = 6v_2$$

$$= 6v_3, \text{ since } v_2 = v_3$$

or $$v_2 = \frac{25}{6}\frac{di_2}{dt}$$

From Eq. 4.39

$$\frac{25}{6}\frac{di_2}{dt} = 6\frac{di_2}{dt} - 3\frac{di_3}{dt}$$

from which $\frac{di_2}{dt} = \frac{18}{11}\frac{di_3}{dt}$

From Eq. 4.42

$$\frac{di_2}{dt} = \frac{di_2}{dt} + \frac{11}{18}\frac{di_2}{dt} = \frac{29}{18}\frac{di_2}{dt}$$

Substituting the values of $\frac{di_2}{dt}$ and $\frac{di_3}{dt}$ in Eq. 4.41 yields

$$v = 5\frac{di_1}{dt} + 6\frac{18}{29}\frac{di_1}{dt} - 5\frac{11}{18}\frac{di_2}{dt}$$

$$= 5\frac{di_2}{dt} + \frac{108}{29}\frac{di_1}{dt} - \frac{55}{18}\frac{18}{29}\frac{di_1}{dt}$$

$$v = \frac{198}{29}\frac{di_1}{dt} = 6.827\frac{di_1}{dt}$$

$\therefore$ equivalent inductance across AB = 6.827 H

Example 4.30 *Write the mesh equations for the network shown in Fig. 4.41.*

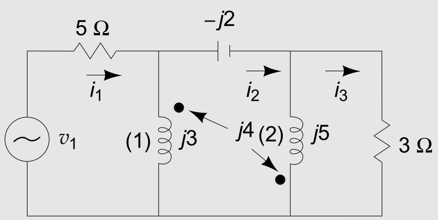


Fig. 4.41

Solution The circuit contains three meshes. Let us assume three loop currents i_1 i_2 and i_3.

For the first mesh

$$5i_1 + j3(i_1 - i_2) + j4(i_3 - i_2) = v_1 \quad (4.43)$$

The drop due to self inductance is $j3(i_1 - i_2)$ is written by considering the: Current $(i_1 - i_2)$ entering at dot marked terminal in the first coil, $j4(i_3 - i_2)$ is the mutually induced voltage in coil 1 due to current $(i_3 - i_2)$ entering at dot marked terminal of coil 2.

Similarly, for the 2nd mesh,

$$j3(i_2 - i_1) + j5(i_2 - i_3) - j2i_2 + j4(i_2 - i_3) + j4(i_2 - i_1) = 0 \quad (4.44)$$

$j4(i_2 - i_1)$ is the mutually induced voltage in coil 2 due to the current in coil 1, and $j4(i_2 - i_3)$ is the mutually induced voltage in coil 1 due to the current in coil 2.

For the third mesh,

$$3i_3 + j5(i_3 - i_2) + j4(i_1 - i_2) = 0 \tag{4.45}$$

Further simplification of Eqs 4.43, 4.44 and 4.45 leads to

$$(5 + j3)i_1 - j7i_2 + j4i_3 = v_1 \tag{4.46}$$

$$-j7i_1 + j14i_2 - j9i_3 = 0 \tag{4.47}$$

$$j4i_1 - j9i_2 + (3 + j5)i_3 = 0 \tag{4.48}$$

Example 4.31 *The inductance matrix for the circuit of three series connect coupled coils is given in Fig. 4.42. Find the inductances, and indicate the dots for the coils.*

$$L = \begin{bmatrix} 4 & -4 & 1 \\ -4 & 2 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

All elements are in Henrys.

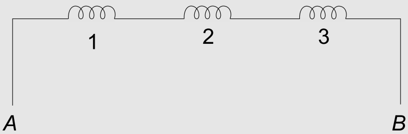


Fig. 4.42

Solution The diagonal elements (4, 2, 6) in the matrix represent the self inductances of the three coils 1, 2 and 3, respectively. The second element in the 1st row (-4) is the mutual inductance between coil 1 and 2, the negative sign indicates that the current in the first coil enters the dotted terminal, and the current in the second coil enters at the undotted terminal. Similarly, the remaining elements are fixed. The values of inductances and the dot convention is shown in Fig. 4.43.

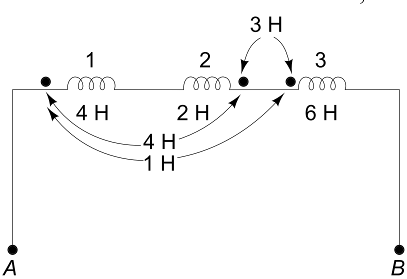


Fig. 4.43

Example 4.32 *Find the voltage across the 10 Ω resistor for the network shown in Fig. 4.44.*

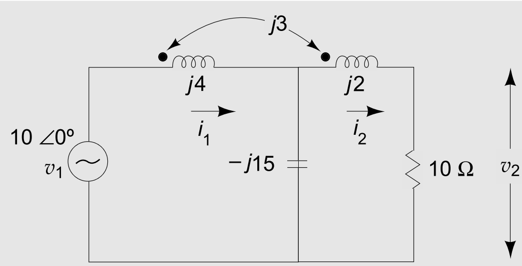


Fig. 4.44

Solution From Fig. 4.43 it is clear that

$$v_2 = i_2 10 \tag{4.49}$$

Mesh equation for the first mesh is

$$j4i_1 - j15(i_1 - i_2) + j3i_2 = 10 \angle 0°$$

$$-j11i_1 + j18i_2 = 10 \angle 0° \tag{4.50}$$

Mesh equation for the 2nd mesh is

$$j2i_2 + 10i_2 - j15(i_2 - i_1) + j3i_1 = 0$$

$$j18i_1 - j13i_2 + 10i_2 = 0$$

$$j18i_1 + i_2(10 - j13) = 0 \tag{4.51}$$

Solving for i_2 from Eqs 4.50 and 4.51, we get

$$i_2 = \begin{bmatrix} -j11 & 10\angle 0° \\ j18 & 0 \end{bmatrix} \Big/ \begin{bmatrix} -j11 & j18 \\ j18 & 10 - j3 \end{bmatrix}$$

$$= \frac{-180\angle 90°}{291 - j110}$$

$$= \frac{-180\angle 90°}{311\angle 20.70°} = -0.578\angle 110.7°$$

$$v_2 = i_2\, 10 = -\,5.78 \angle 110.7°$$

$$|v_2| = 5.78$$

Example 4.33 *Write the loop equations for the coupled circuit shown in Fig. 4.45.* *[JNTU June 2006]*

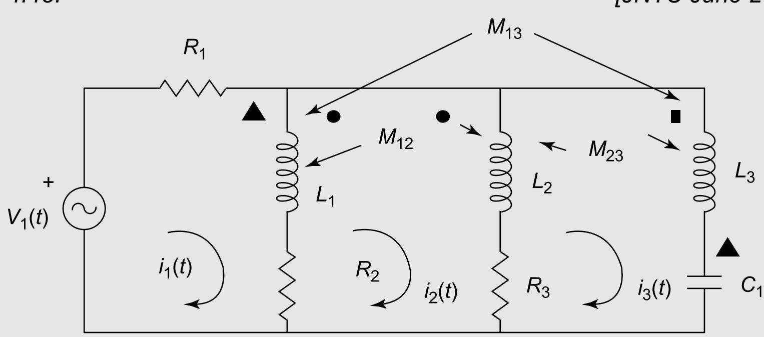


Fig. 4.45

Solution

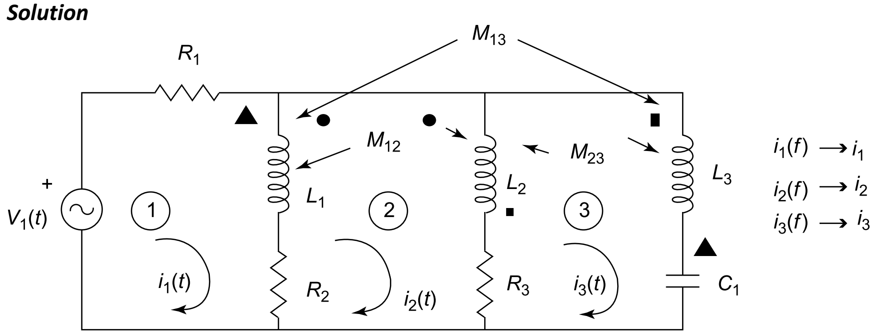


Fig. 4.46

Loop Equations: (*By Dot Rule Convention*)

$$(1) \Rightarrow V_1(t) = i_1(t)(R_1 + R_2) + L_1 \frac{di_1(t)}{dt} - i_2(t)R_2 + M_{12}\frac{di_2(t)}{dt}$$

$$- M_{13}\frac{di_3(t)}{dt} - L_1\frac{di_2(t)}{dt} - M_{12}\frac{di_3(t)}{dt}$$

$$(2) \Rightarrow R_2(i_2(t) - i_1(t)) + L_1\left(\frac{di_2(t)}{dt} - \frac{di_1(t)}{dt}\right) - M_{12}\left(\frac{di_2(t)}{dt} - \frac{di_3}{dt}\right)$$

$$+ M_{13}\frac{di_3}{dt} + L_2\left(\frac{di_2}{dt} - \frac{di_3}{dt}\right) - M_{12}\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) - M_{23}\frac{di_3}{dt}$$

$$+ R_3(i_2 - i_3) = 0$$

$$(3) \Rightarrow R_3(i_3 - i_2) + L_2\left(\frac{di_3}{dt} - \frac{di_2}{dt}\right) - M_{12}\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) + M_{23}\frac{di_3}{dt}$$

$$+ L_3\frac{di_3}{dt} - M_{13}\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) + M_{23}\left(\frac{di_3}{dt} - \frac{di_2}{dt}\right) + \frac{1}{C_1}\int i_3 dt = 0.$$

Example 4.34 *Write the loop equations for the coupled circuits shown in Fig. 4.47.* *[JNTU May 2007]*

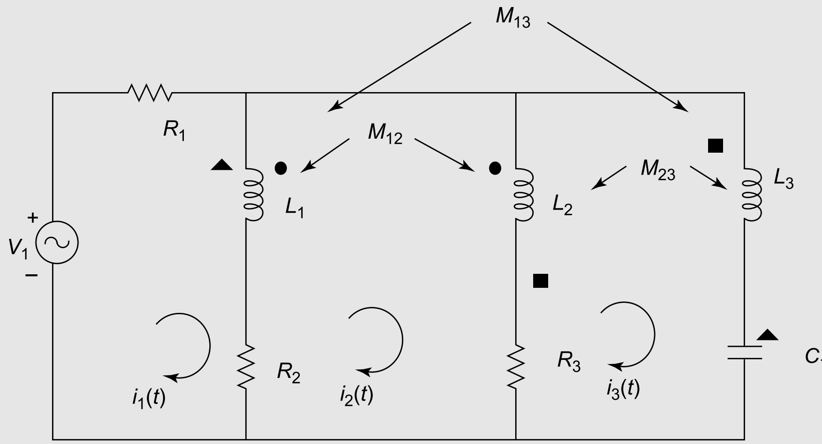


Fig. 4.47

Solution Given circuit is

The loop equations are

$$V_1(t) = R_1 i_1(t) + L_1\frac{d}{dt}\left[i_1(t) - i_2(t)\right] - M_{12}\frac{d}{dt}\left[i_2(t) - i_3(t)\right] \tag{4.52}$$

$$- M_{13}\frac{d}{dt}\left[i_3(t)\right] + R_2\left[i_1(t) - i_2(t)\right]$$

Loop 2

$$R_2\left[i_2(t) - i_1(t)\right] + L_1\left[\frac{di_2(t)}{dt} - \frac{di_i(t)}{dt}\right] - M_{12}\frac{d}{dt}\left[i_2(t) - i_3(t)\right]$$

$$+M_{13}\frac{di_3(t)}{dt}+L_2\frac{d\left[i_2(t)-i_3(t)\right]}{dt}-M_{12}\left[\frac{di_2(t)}{dt}-\frac{di_i(t)}{dt}\right]$$

$$-M_{23}\frac{di_3(t)}{dt}+R_3(i_2-i_3)=0$$

Loop 3

$$R_3(i_3-i_2)+L_2\frac{d(i_3-i_2)}{dt}-M_{12}\frac{d(i_l-i_2)}{dt}+M_{23}\frac{di_3}{dt}$$

$$+L_3\frac{di_3}{dt}-M_{13}\frac{d}{dt}-M_{23}\frac{d(i_3-i_2)}{dt}+\frac{1}{C_1}\int i_3dt=0$$

4.3 CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS—TUNED CIRCUITS

Tuned circuits are, in general, single tuned and double tuned. Double tuned circuits are used in radio receivers to produce uniform response to modulated signals over a specified bandwidth; double tuned circuits are very useful in communication system.

4.3.1 Single Tuned Circuit

Consider the circuit in Fig. 4.48. A tank circuit (i.e. a parallel resonant circuit) on the secondary side is inductively coupled to coil (1) which is excited by a source, v_i. Let R_s be the source resistance and R_1, R_2 be the resistances of coils, 1 and 2, respectively. Also let L_1, L_2 be the self inductances of the coils, 1 and 2, respectively.

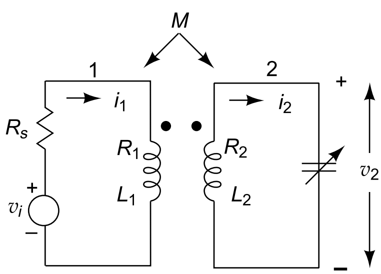


Fig. 4.48

Let $R_s + R_1 + j\omega L_1 = R_s$ with the assumption that $R_s \gg R_1 \gg j\omega L_1$

The mesh equations for the circuit shown in Fig. 4.48 are

$$i_1R_s - j\omega Mi_2 = v_i$$

$$-j\omega Mi_1+\left(R_2+j\omega L_2-\frac{j}{\omega C}\right)i_2=0$$

$$i_2=\begin{vmatrix} R_s & v_i \\ -j\omega M & 0\end{vmatrix} \Bigg/ \begin{vmatrix} R_s & (-j\omega M) \\ (-j\omega M) & \left(R_2+j\omega L_2-\frac{i}{\omega C}\right)\end{vmatrix}$$

or $$i_2=\frac{jv_i\omega M}{R_s\left(R_2+j\omega L_2-\frac{j}{\omega C}\right)+\omega^2M^2}$$

The output voltage $v_o = i_2 \cdot \frac{1}{j\omega C}$

$$v_o = \frac{jv_i\omega M}{j\omega C\left\{R_s\left[R_2 + \left(j\omega L_2 - \frac{1}{\omega C}\right)\right] + \omega^2\ M^2\right\}}$$

The voltage transfer function, or voltage amplification, is given by

$$\frac{v_o}{v_i} = A = \frac{M}{C\left\{R_s\left[R_2 + \left(j\omega L_2 - \frac{1}{\omega C}\right)\right] + \omega^2\ M^2\right\}}$$

When the secondary side is tuned, i.e. when the value of the frequency ω is such that $\omega L_2 = 1/\omega C$, or at resonance frequency ω_r, the amplification is given by

$$A = \frac{v_o}{v_i} = \frac{M}{C\left[R_s\, R_2 + \omega_r^2\ M^2\right]}$$

the current i_2 at resonance $i_2 = \frac{jv_i\omega_r M}{R_s R_2 + \omega_r^2\ M^2}$

Thus, it can be observed that the output voltage, current and amplification depends on the mutual inductance M at resonance frequency, when $M = K\sqrt{L_1 L_2}$.

The maximum output voltage or the maximum amplification depends on M. To get the condition for maximum output voltage, make $dv_o/dM = 0$.

$$\frac{dv_o}{dM} = \frac{d}{dM}\left[\frac{v_i M}{C\left[R_s R_2 + \omega_r^2 M^2\right]}\right]$$

$$= 1 - 2M^2\omega_r^2\left[R_s R_2 + \omega_r^2 M^2\right]^{-1} = 0$$

From which, $R_s R_2 = \omega_r^2 M^2$

or $$M = \sqrt{\frac{R_s R_2}{\omega_r}}$$

From the above value of M, we can calculate the maximum output voltage.

Thus $$v_{oM} = \frac{v_i}{2\omega_r C\sqrt{R_s R_2}},$$

or the maximum amplification is given by

$$A_m = \frac{1}{2\omega_r C\sqrt{R_s R_2}} \quad \text{and} \quad i_2 = \frac{jv_i}{2\sqrt{R_s R_2}}$$

The variation of the amplification factor or output voltage with the coefficient of coupling is shown in Fig. 4.49.

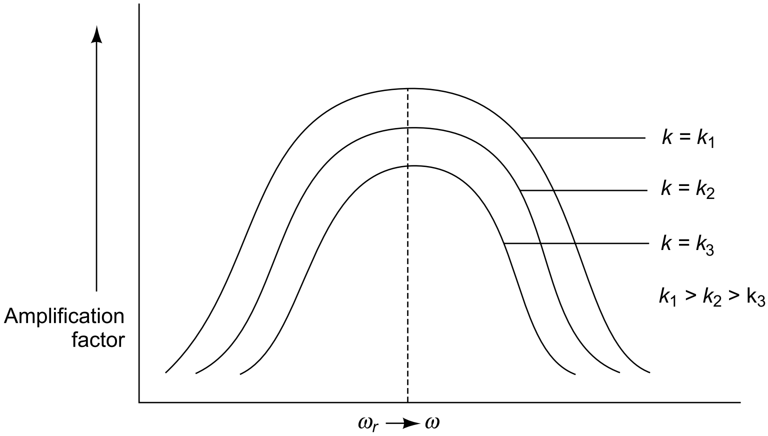


Fig. 4.49

Example 4.35 *Consider the single tuned circuit shown in Fig. 4.50 and determine (i) the resonant frequency (ii) the output voltage at resonance and (iii) the maximum output voltage. Assume $R_s >> w_rL_1$, and K = 0.9.*

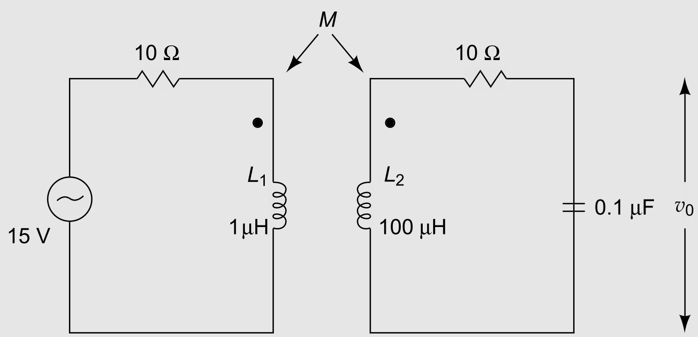


Fig. 4.50

Solution

$$M = K\sqrt{L_1L_2}$$

$$= 0.9\sqrt{1\times10^{-6}\times100\times10^{-6}}$$

$$= 9\,\mu\text{H}$$

(i) Resonance frequency

$$\omega_r = \frac{1}{\sqrt{L_2C}} = \frac{1}{\sqrt{100\times10^{-6}\times0.1\times10^{-6}}}$$

$$= \frac{10^6}{\sqrt{10}}\ \text{rad / sec.}$$

or $\quad f_r = 50.292$ kHz

The value of $\omega_r L_1 = \dfrac{10^6}{\sqrt{10}} 1\times10^{-6} = 0.316$

Thus the assumption that $\omega_r L_1 R_s \ll$ is justified,

(ii) Output voltage

$$v_o = \frac{Mv_i}{C\left[R_s R_2 + \omega_r^2 M\right]}$$

$$= \frac{9\times10^{-6}\times15}{0.1\times10^{-6}\left[10\times10+\left(\frac{10^6}{\sqrt{10}}\right)^2\times9\times10^{-6}\right]} = 1.5\,\text{mV}$$

(iii) Maximum value of output voltage

$$v_{oM} = \frac{\omega_i}{2\omega_r C\sqrt{R_s R_2}}$$

$$= \frac{15}{2\times\frac{10^6}{\sqrt{10}}\times0.1\times10^{-6}\sqrt{100}}$$

$$v_{oM} = 23.7\,\text{V}$$

Example 4.36 *The resonant frequency of the tuned circuit shown in Fig. 4.51 is 1000 rad/sec. Calculate the self inductances of the two coils and the optimum value of the mutual inductance.*

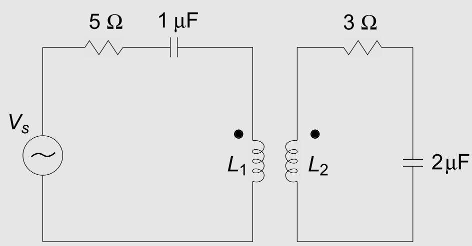


Fig. 4.51

Solution We know that

$$\omega_r^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2}$$

$$L_1 = \frac{1}{\omega_r^2C_1} = \frac{1}{(1000)^2 1\times10^{-6}} = 1\text{ H}$$

$$L_2 = \frac{1}{\omega_r^2C_2} = \frac{1}{(1000)^2\times2\times10^{-6}} = 0.5\text{ H}$$

Optimum value of the mutual inductance is given by

$$M_{\text{optimum}} = \frac{\sqrt{R_1R_2}}{\omega_r}$$

where R_1 and R_2 are the resistances of the primary and secondary coils

$$M = \frac{\sqrt{15}}{1000} = 3.87\,\text{mH}$$

4.3.2 Double Tuned Coupled Circuits

Figure 4.52 shows a double tuned transformer circuit involving two series resonant circuits.

For the circuit shown in the figure, a special case where the primary and secondary resonate at the same frequency ω_r, is considered here,

i.e, $$\omega_r^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2}$$

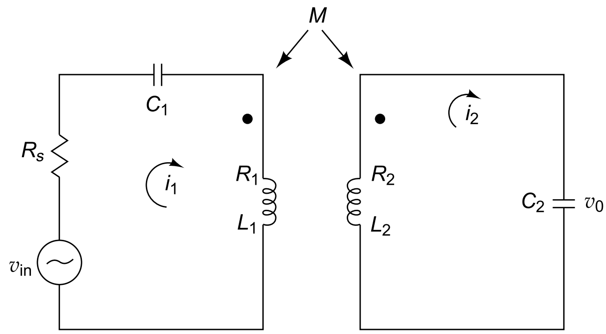


Fig. 4.52

The two mesh equations for the circuit are

$$v_{\text{in}} = i_1\left(R_s + R_1 + j\omega L_1 - \frac{j}{\omega C_1}\right) - i_2 j\omega M$$

$$0 = -j\omega M i_1 + i_2\left(R_2 + j\omega L_2 - \frac{j}{\omega C_2}\right)$$

From which

$$i_2 = \frac{V_{\text{in}} j\omega M}{\left[(R_s + R_1) + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}$$

also $$\omega_r = \frac{1}{\sqrt{L_1C_1}} = \frac{1}{\sqrt{L_2C_2}} \text{ at resonance}$$

$$v_o = \frac{V_{\text{in}} M}{C_2\left[(R_s + R_1)R_2 + \omega_r^2 M^2\right]}$$

or $$v_o = A v_{in}$$

where A is the amplification factor given by

$$A = \frac{M}{C_2\left[(R_1 + R_s)R_2 + \omega_r^2 M^2\right]}$$

The maximum amplification or the maximum output voltage can be obtained by taking the first derivative of v_o with respect to M, and equating it to zero.

$$\therefore \qquad \frac{dV_o}{dM} = 0, \text{ or } \frac{dA}{dM} = 0$$

$$\frac{dA}{dM} = (R_1 + R_s)R_2 + \omega_r^2 M^2 - 2M^2\omega_r^2 = 0$$

$$\omega_r^2 M^2 = R_2(R_1 + R_s)$$

$$M_c = \frac{\sqrt{R_2(R_1 + R_s)}}{\omega_r}$$

where M_c is the critical value of mutual inductance. Substituting the value of M_c in the equation of v_o, we obtain the maximum output voltage as

$$|v_o| = \frac{V_{\text{in}}}{2\omega_r^2 C_2 M_c}$$

$$= \frac{V_{\text{in}}}{2\omega_r C_2 \sqrt{R_2(R_1 + R_s)}}$$

and $$|i_2| = \frac{V_{\text{in}}}{2\omega_r M_c} = \frac{V_{\text{in}}}{2\sqrt{R_2(R_1 + R_s)}}$$

By definition, $M = K\sqrt{L_1 L_2}$, the coefficient of coupling, K at $M = M_c$ is called the critical coefficient of coupling, and is given by $K_c = M_c/\sqrt{L_2 L_1}$.

The critical coupling causes the secondary current to have the maximum possible value. At resonance, the maximum value of amplification is obtained by changing M, or by changing the coupling coefficient for a given value of L_1 and L_2. The variation of output voltage with frequency for different coupling coefficients is shown in Fig. 4.53.

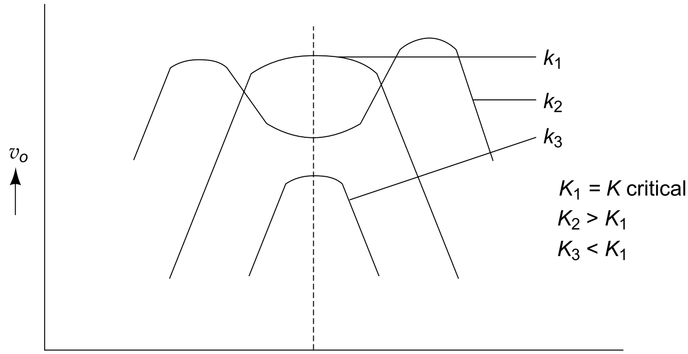


Fig. 4.53

Example 4.37 *The tuned frequency of a double tuned circuit shown in Fig. 4.54 is 10^4 rad/sec. If the source voltage is 2 V and has a resistance of 0.1Ω; calculate the maximum output voltage at resonance if $R_1 = 0.01\ \Omega$, $L_1 = 2\ \mu H$; $R_2 = 0.1\Omega$, and $L_2 = 25\ \mu H$.*

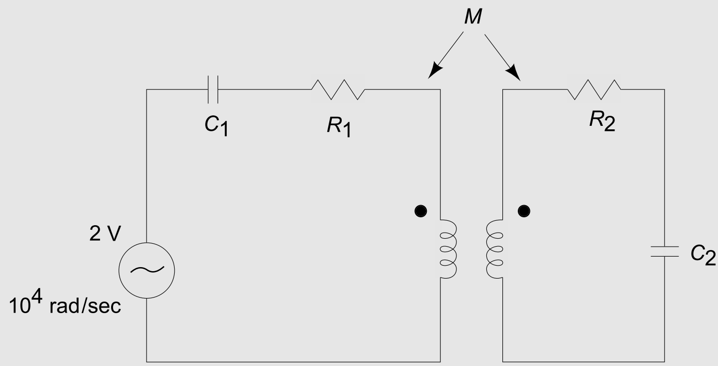


Fig. 4.54

Solution The maximum output voltage $v_o = \dfrac{v_i}{2\omega_r^2 C_2 M_c}$

where M_c is the critical value of the mutual inductance given by

$$M_c = \frac{\sqrt{R_2(R_1 + R_s)}}{\omega_r}$$

$$M_c = \frac{\sqrt{0.1(0.01+0.1)}}{10^4} = 10.48\ \mu\text{H}$$

At resonance

$$\omega_r^2 = \frac{1}{L_2 C_2}$$

$$C_2 = \frac{1}{\omega_r^2 L_2} = \frac{1}{(10^4)^2 \times 25 \times 10^{-6}} = 0.4 \times 10^{-3}\ \text{F}$$

$$v_0 = \frac{2}{2(10^4)^2 \times 0.4 \times 10^{-3} \times 10.48 \times 10^{-6}}$$

$$= 2.385\ \text{V}$$

4.4 SERIES RESONANCE

Frequency response analysis is important to us for two primary reasons. First, if we know the frequency response then we can predict the response of the circuit to any input. Sinusoidal waveforms have the elegant property that they can be combined to form other (non-sinusoidal) waveforms. Therefore the frequency response allows us to understand a circuits response to more complex inputs. Second, we are often interested in designing circuits with particular frequency characteristics. For example, in the design of an audio 3-way loud speaker system, we would like to direct low frequency signals to the woofers, high frequency signals to the tweets, and mid frequency signals to the mid range speakers. Therefore we would need a circuit that is capable of passing certain frequencies of a signal and rejecting others. The concept of resonance is highly useful in the design of basic filtering circuits for use in everyday applications such as an audio amplifiers.

Consider an AC circuit with a single voltage source and any number of resistors, capacitors and inductors. If the frequency of the source is fixed, then a complete analysis in either the time domain or the frequency domain is possible. In the time domain, a differential is extracted from the circuit and solved. In general, the order of the differential equation is equal to the number of energy storage elements in the circuit. A much easier method is to solve the circuit using phasor analysis in the frequency domain. The analysis is easier in the frequency domain because differentiation in time transforms to multiplication by $j\omega$. As a result, an algebraic equation arises rather than a differential equation. Algebraic equations are easier to solve the differential equations. If the frequency of the voltage source is varied, the impedance of each storage element changes, as the response of the circuit varies as a function of input frequency. The frequency response of a circuit is a quantitative description of its behaviour in the frequency domain.

In many electrical circuits, resonance is a very important phenomenon. The study of resonance is very useful, particularly in the area of communications. For example, the ability of a radio receiver to select a certain frequency, transmitted by a station and to eliminate frequencies from other stations is based on the principle of resonance. In a series RLC circuit, the current lags behind, or leads the applied voltage depending upon the values of X_L and X_C. X_L causes the total current to lag behind the applied voltage, while X_C causes the total current to lead the applied voltage. When $X_L > X_C$, the circuit is predominantly inductive, and when $X_C > X_L$ the circuit is predominantly capacitive. However, if one of the parameters of the series RLC circuit is varied in such a way that the current in the circuit is in phase with the applied voltage, then the circuit is said to be in resonance.

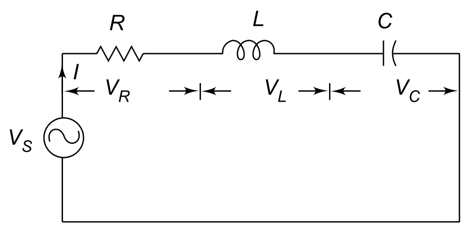


Fig. 4.55

Consider the series RLC circuit shown in Fig. 4.55.

The total impedance for the series RLC circuit is

$$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

It is clear from the circuit that the current $I = V_S/Z$

The circuit is said to be in resonance if the current is in phase with the applied voltage. In a series RLC circuit, series resonance occurs when $X_L = X_C$. The frequency at which the resonance occurs is called the *resonant frequency*.

Since $X_L = X_C$, the impedance in a series RLC circuit is purely resistive. At the resonant frequency, f_r, the voltages across capacitance and inductance are equal in magnitude. Since they are 180° out of phase with each other, they cancel each other and, hence zero voltage appears across the LC combination.

At resonance

$$X_L = X_C \text{ i.e. } \omega L = \frac{1}{\omega C}$$

Solving for resonant frequency, we get

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$f_r^2 = \frac{1}{4\pi^2 LC}$$

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

In a series RLC circuit, resonance may be produced by varying the frequency, keeping L and C constant; otherwise, resonance may be produced by varying either L or C for a fixed frequency.

Example 4.38 *For the circuit shown in Fig. 4.56, determine the value of capacitive reactance and impedance at resonance.*

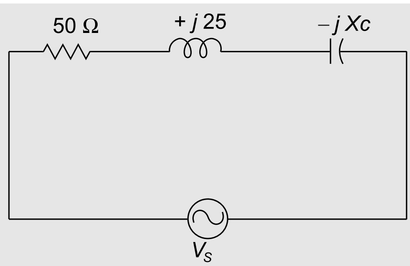


Fig. 4.56

Solution At resonance

$$X_L = X_C$$

Since $X_L = 25\ \Omega$

$$X_L = 25\ \Omega \quad \therefore \frac{1}{\omega C} = 25$$

The value of impedance at resonance is

$$Z = R$$

$$\therefore \quad Z = 50\ \Omega$$

Example 4.39 *Determine the resonant frequency for the circuit shown in Fig. 4.57.*

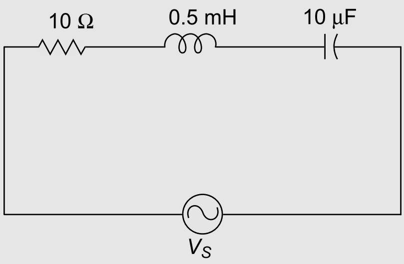


Fig. 4.57

Solution The resonant frequency is $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{10\times 10^{-6}\times 0.5\times 10^{-3}}}$$

$$f_r = 2.25\,\text{kHz}$$

Example 4.40 *A 50 Ω resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 4.58. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.*

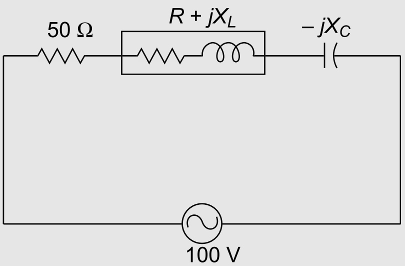


Fig. 4.58

Solution At resonance, current in the circuit is maximum

$$I = 0.7\text{ A}$$

Voltage across capacitor is $V_C = IX_C$

Since $V_C = 200, I = 0.7$

$$X_C = \frac{1}{\omega C}$$

$$\omega C = \frac{0.7}{200}$$

$$\therefore \quad C = \frac{0.7}{200\times 2\pi\times 200}$$

$$= 2.785\,\mu\text{F}$$

At resonance

$$X_L - X_C = 0$$

$$\therefore \quad X_L = X_C$$

Since $X_C = \dfrac{1}{\omega C} = \dfrac{200}{0.7} = 285.7\ \Omega$

$$X_L = \omega L = 285.7\ \Omega$$

$$\therefore \quad L = \frac{285.7}{2\pi \times 200} = 0.23\,\text{H}$$

At resonance, the total impedance

$$Z = R + 50$$

$$\therefore \quad R + 50 = \frac{V}{I} = \frac{100}{0.7}$$

$$R + 50 = 142.86\ \Omega$$

$$\therefore \quad R = 92.86\ \Omega$$

4.4.1 Impedance and Phase Angle of a Series Resonant Circuit

The impedance of a series RLC circuit is

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The variation of X_C and X_L with frequency is shown in Fig. 4.59(a).

At zero frequency, both X_C and Z are infinitely large, and X_L is zero because at zero frequency the capacitor acts as an open circuit and the inductor acts as a short circuit. As the frequency increases, X_C decreases and X_L increases. Since X_C is larger than X_L, at frequencies below the resonant frequency f_r, Z decreases along with X_C. At resonant frequency $X_C = X_L$, and $Z = R$. At frequencies above the resonant frequency f_r, X_L is larger than X_C, causing Z to increase. The phase angle as a function of frequency is shown in Fig. 4.59(b).

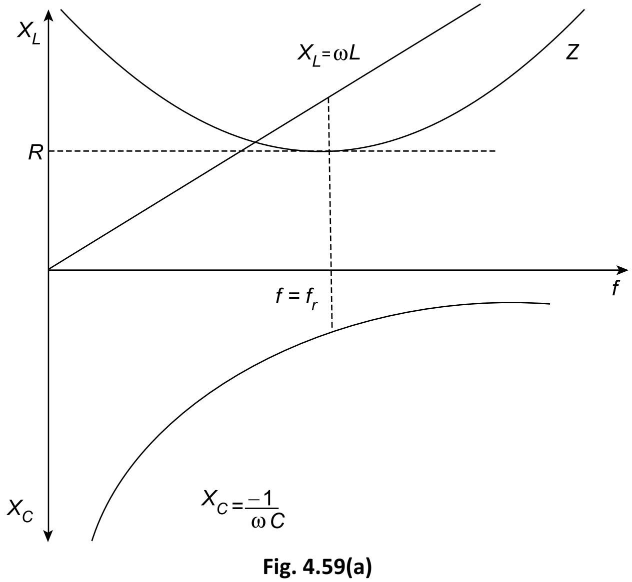


Fig. 4.59(a)

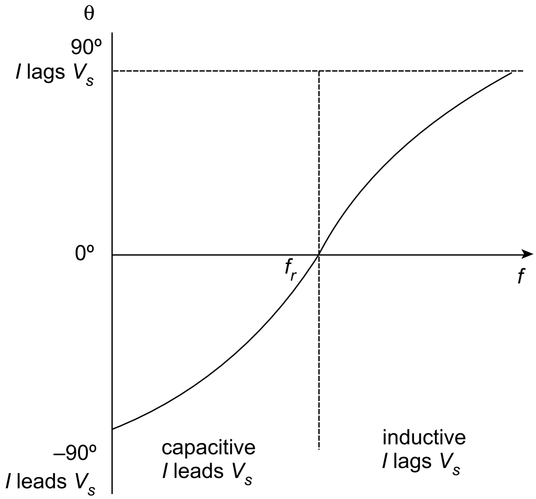


Fig. 4.59(b)

At a frequency below the resonant frequency, current leads the source voltage because the capacitive reactance is greater than the inductive reactance. The phase angle decreases as the frequency approaches the resonant value, and is 0° at resonance. At frequencies above resonance, the current lags behind the source voltage, because the inductive reactance is greater than capacitive reactance. As the frequency goes higher, the phase angle approaches 90°.

Example 4.41 *For the circuit shown in Fig. 4.60, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.*

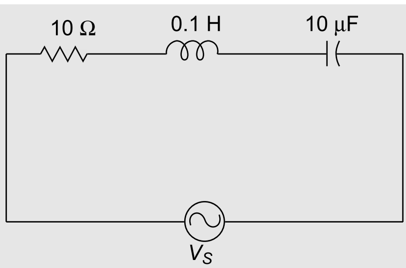


Fig. 4.60

Solution Resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.2\,\text{Hz}$$

At 10 Hz below f_r = 159.2 – 10 = 149.2 Hz

At 10 Hz below f_r = 159.2 + 10 = 169.2 Hz

Impedance at resonance is equal to R

$\therefore \quad Z = 10\ \Omega$

Capacitive reactance at 149.2 Hz is

$$X_{C_1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi \times 149.2 \times 10^{-6} \times 10}$$

$$\therefore \quad X_{C_1} = 106.6\,\Omega$$

Capacitive reactance at 169.2 Hz is

$$X_{C_2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi \times 169.2 \times 10 \times 10^{-6}}$$

$$\therefore \quad X_{C_2} = 94.06\,\Omega$$

Inductive reactance at 149.2 Hz is

$$X_{L_1} = \omega_2 L = 2\pi \times 149.2 \times 0.1 = 93.75\,\Omega$$

Inductive reactance at 169.2 Hz is

$$X_{L_2} = \omega_2 L = 2\pi \times 169.2 \times 0.1 = 106.31\,\Omega$$

Impedance at 149.2 Hz is

$$|Z| = \sqrt{R^2 + \left(X_{L_1} - X_{C_1}\right)^2}$$
$$= \sqrt{(10)^2 + (93.75 - 106.6)^2} = 16.28\,\Omega$$

Here X_{C_1} is greater than X_{L_1}, so Z is capacitive.
Impedance at 169.2 Hz is

$$|Z| = \sqrt{R^2 + (X_{L_2} - X_{C_2})^2}$$
$$= \sqrt{(10)^2 + (106.31 - 94.06)^2} = 15.81\,\Omega$$

Here X_{L_1} is greater than X_{C_1}, so Z is inductive.

Example 4.42 *A series RLC circuit consists of resistance R = 20 Ω, inductance, L = 0.01 H and capacitance, C = 0.04 μF. Calculate the frequency at resonance. If a 10 volts of frequency equal to the frequency of resonance is applied to this circuit, calculate the values of V_C and V_L across C and L respectively. Find the frequencies at which these voltages V_C and V_L are maximum? [JNTU June 2006]*

Solution $R = 20\ \Omega$; $L = 0.01$ H; $C = 0.04\ \mu$F

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01 \times 0.04 \times 10^{-6}}} = 7.957\,\text{kHz}$$

At resonance $I = \dfrac{V}{R} = \dfrac{10}{20} = 0.5\text{A}$

The voltage drop across the inductor is

$$V_L = I\,X_L$$

$$= \frac{\omega L V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{2\pi \times 7.957 \times 10^3 \times 0.01 \times 10}{\sqrt{(20)^2 + \left(2\pi \times 7.957 \times 10^3 \times 0.01 - \frac{1}{2\pi \times 7.957 \times 10^3 \times 0.04 \times 10^{-6}}\right)^2}}$$

$$= 250\,V$$

$$V_C = I\,X_C = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \times \frac{1}{\omega C}$$

$$= \frac{10 \times \frac{1}{2\pi \times 7.957 \times 10^3 \times 0.04 \times 10^{-6}}}{\sqrt{(20)^2 + \left(2\pi \times 7.957 \times 10^3 \times 0.01 - \frac{1}{2\pi \times 7.957 \times 10^3 \times 0.04 \times 10^{-6}}\right)^2}}$$

$V_C = 250$ V

The frequency at which the voltage across inductor maximum

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{1 - \frac{R^2 C}{2L}}}$$

$$= \frac{1}{2\pi\sqrt{0.01 \times 0.04 \times 10^{-6}}}\sqrt{\frac{1}{1 - \frac{(20)^2 \times 0.04 \times 10^{-6}}{2 \times 0.01}}}$$

$f_L = 7960$ Hz

The frequency at which the voltage across capacitor maximum

$$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.01 \times 0.04 \times 10^{-6}} - \frac{(20)^2}{2 \times 0.01}}$$

$$\therefore \quad = 7949\,\text{Hz}$$

The maximum voltage across the capacitor occurs below resonant frequency, and the maximum voltage across the inductor occurs above the resonant frequency.

Example 4.43 *A series circuit comprising R, L and C is supplied at 220 V, 50 Hz. At resonance, the voltage across the capacitor is 550 V. The current at resonance is 1 A. Determine the circuit parameters R, L and C.* *[JNTU May 2006]*

Solution At resonance

$$X_L = X_C$$

Current at resonance $= I = \dfrac{V}{R + j(X_L - X_C)} = \dfrac{V}{R}$

$$I = \frac{220}{R}$$

$$\therefore \quad R = 220\ \Omega$$

$$V_C = I_O X_C$$

$$550 = 1 \times \frac{1}{\omega_o c}$$

$$C = \frac{1}{550 \times 2\pi f} = \frac{1}{550 \times 2 \times \pi \times 50}$$

$$C = 5.78\ \mu\text{F}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

$$LC = \left(\frac{1}{2\pi f_o}\right)^2$$

$$L = \frac{1}{C}\left(\frac{1}{2\pi f_o}\right)^2$$

$$= \frac{1}{5.78 \times 10^{-6}}\left(\frac{1}{100\pi}\right) = 1.750\,\text{H}$$

$\therefore$ Circuit elements at resonance are

$$R = 220\ \Omega,\ L = 1.75\ \text{H},\ C = 5.78\ \mu\text{F}$$

4.4.2 Voltage and Current in a Series Resonant Circuit

The variation of impedance and current with frequency is shown in Fig. 4.61.

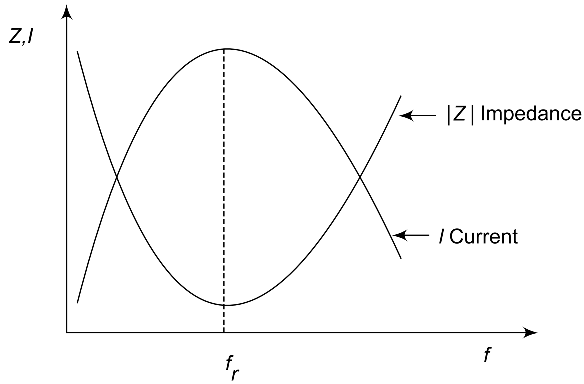


Fig. 4.61

At resonant frequency, the capacitive reactance is equal to inductive reactance, and hence the impedance is minimum. Because of minimum impedance, maximum current flows through the circuit. The current variation with frequency is plotted.

The voltage drop across resistance, inductance and capacitance also varies with frequency. At $f = 0$, the capacitor acts as an open circuit and blocks current. The complete source voltage appears across the capacitor. As the frequency increases, X_C decreases and X_L increases, causing total reactance $X_C - X_L$ to decrease. As a result, the impedance decreases and the current increases. As the current increases, V_R also increases, and both V_C and V_L increase.

When the frequency reaches its resonant value f_r, the impedance is equal to R, and hence, the current reaches its maximum value, and V_R is at its maximum value.

As the frequency is increased above resonance, X_L continues to increase and X_C continues to decrease, causing the total reactance, $X_L - X_C$ to increase. As a result there is an increase in impedance and a decrease in current. As the current decreases, V_R also decreases, and both V_C and V_L decrease. As the frequency becomes very high, the current approaches zero, both V_R and V_C approach zero, and V_L approaches V_S.

The response of different voltages with frequency is shown in Fig. 4.62.

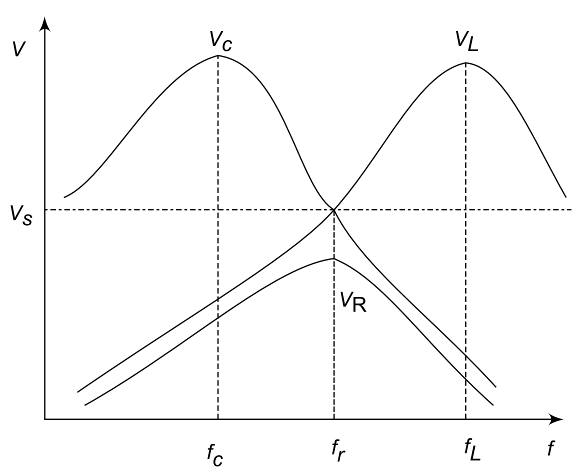


Fig. 4.62

The drop across the resistance reaches its maximum when $f = f_r$. The maximum voltage across the capacitor occurs at $f = f_c$. Similarly, the maximum voltage across the inductor occurs at $f = f_L$.

The voltage drop across the inductor is

$$V_L = IX_L$$

where $$I = \frac{V}{Z}$$

$$\therefore \quad V_L = \frac{\omega L V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

To obtain the condition for maximum voltage across the inductor, we have to take the derivative of the above equation with respect to frequency, and make it equal to zero.

$$\therefore \quad \frac{dV_L}{d\omega} = 0$$

If we solve for ω, we obtain the value of ω when V_L is maximum.

$$\frac{dV_L}{d\omega} = \frac{d}{d\omega}\left\{\omega L V\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\right\}$$

$$LV\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)^{-1/2}$$

$$\frac{-\frac{\omega L V}{2}\left(R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)\left(2\omega L^2 - \frac{2}{\omega^3 C^2}\right)}{R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}} = 0$$

From this,

$$R^2 - \frac{2L}{C} + 2/\omega^2 C^2 = 0$$

$$\therefore \quad \omega L = \sqrt{\frac{2}{2LC - R^2C^2}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{2}{2 - \frac{R^2 C}{L}}}$$

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{1 - \frac{R^2 C}{2L}}}$$

Similarly, the voltage across the capacitor is

$$V_C = IX_C = \frac{I}{\omega C}$$

$$\therefore V_C = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \times \frac{1}{\omega C}$$

To get maximum value $\frac{dV_C}{d\omega} = 0$

If we solve for ω, we obtain the value of ω when V_C is maximum.

$$\frac{dV_C}{d\omega} = \omega C \frac{1}{2}\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\left[2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)\right]$$
$$+ \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, C = 0$$

From this,

$$\omega_C^2 = \frac{1}{LC} - \frac{R^2}{2L}$$

$$\omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$\therefore \qquad f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

The maximum voltage across the capacitor occurs below the resonant frequency; and the maximum voltage across the inductor occurs above the resonant frequency.

4.4.3 Bandwidth of Series Resonance *[JNTU Nov 2011]*

The bandwidth of any system is the range of frequencies for which the current or output voltage is equal to 70.7% of its value at the resonant frequency, and it is denoted by *BW*. Figure 4.63 shows the response of a series RLC circuit.

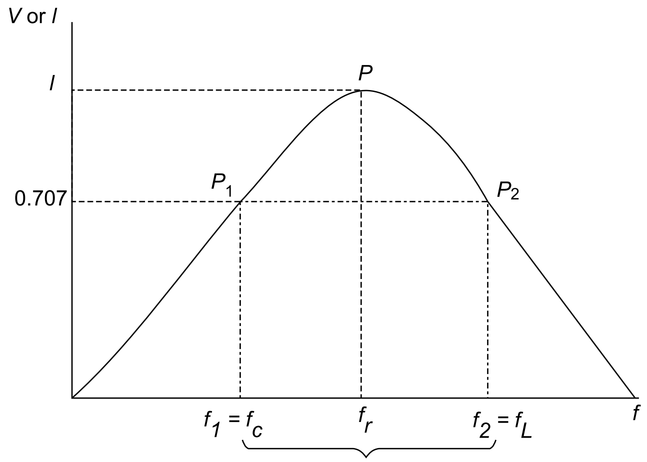


Fig. 4.63

Here the frequency f_1 is the frequency at which the current is 0.707 times the current at resonant value, and it is called the lower cut-off frequency. The frequency f_2 is the frequency at which the current is 0.707 times the current at resonant value (i.e. maximum value), and is called the *upper cut-off frequency*. The bandwidth, or BW, is defined as the frequency difference between f_2 and f_1.

$$\therefore \quad BW = f_2 - f_1$$

The unit of BW is hertz (Hz).

If the current at P_1 is 0.707 I_{max}, the impedance of the circuit at this point is $\sqrt{2}\,R$, and hence

$$\frac{1}{\omega_1 C} - \omega_1 L = R \tag{4.53}$$

Similarly,
$$\omega_2 L - \frac{1}{\omega_2 C} = R \tag{4.54}$$

If we equate both the above equations, we get

$$\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}$$

$$L(\omega_1 + \omega_2) = \frac{1}{C}\left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right) \tag{4.55}$$

From Eq. 4.55, we get

$$\omega_1 \omega_2 = \frac{1}{LC}$$

we have $\omega_r^2 = \dfrac{1}{LC}$

$$\therefore \qquad \omega_r^2 = \omega_1\omega_2 \tag{4.56}$$

If we add Eqs 4.53 and 4.54, we get

$$\frac{1}{\omega_1 C} - \omega_1 L + \omega_2 L - \frac{1}{\omega_2 C} = 2R$$

$$\left(\omega_2 - \omega_1\right)L + \frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1\omega_2}\right) = 2R \tag{4.57}$$

Since

$$C = \frac{1}{\omega_r^2 L}$$

and

$$\omega_1\omega_2 = \omega_r^2$$

$$(\omega_2 - \omega_1)L + \frac{\omega_r^2 L(\omega_2 - \omega_1)}{\omega_r^2} = 2R \tag{4.58}$$

From Eq. 4.58, we have

$$\omega_2 - \omega_1 = \frac{R}{L} \tag{4.59}$$

$$f_2 - f_1 = \frac{R}{2\pi L} \tag{4.60}$$

or $$BW = \frac{R}{2\pi L}$$

From Eq. 4.60, we have

$$f_2 - f_1 = \frac{R}{2\pi L}$$

$$\therefore \quad f_r - f_1 = \frac{R}{4\pi L}$$

$$f_2 - f_r = \frac{R}{4\pi L}$$

The lower frequency limit $f_1 = f_r - \dfrac{R}{4\pi L}$ (4.61)

The upper frequency limit $f_2 = f_r + \dfrac{R}{4\pi L}$ (4.62)

If we divide the equation on both sides by f_r, we get

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} \tag{4.63}$$

Here an important property of a coil is defined. It is the ratio of the reactance of the coil to its resistance. This ratio is defined as the Q of the coil. Q is known as a figure of merit, it is also called quality factor and is an indication of the quality of a coil.

$$Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R} \tag{4.64}$$

If we substitute Eq. 4.63 in Eq. 4.64, we get

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} \tag{4.65}$$

The upper and lower cut-off frequencies are sometimes called the *half-power frequencies*. At these frequencies the power from the source is half of the power delivered at the resonant frequency.

At resonant frequency, the power is

$P_{\max} = I^2_{\max} R$

At frequency f_1, the power is $P_1 = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 R = \frac{I^2_{\max} R}{2}$

Similarly, at frequency f_2, the power is

$$P_2 = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 R$$

$$= \frac{I^2_{\max} R}{2}$$

The response curve in Fig. 4.63 is also called the *selectivity curve* of the circuit. Selectivity indicates how well a resonant circuit responds to a certain frequency and eliminates all other frequencies. The narrower the bandwidth, the greater the selectivity.

4.4.4 The Quality Factor (*Q*) and its Effect on Bandwidth

The quality factor, Q, is the ratio of the reactive power in the inductor or capacitor to the true power in the resistance in series with the coil or capacitor.

The quality factor

$$Q = 2\pi \times \frac{\text{Maximum energy stored}}{\text{Energy dissipated per cycle}}$$

In an inductor, the max energy stored is given by $\frac{LI^2}{2}$

Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 R \times T = \frac{I^2 RT}{2}$

$\therefore$ Quality factor of the coil $Q = 2\pi \times \dfrac{\frac{1}{2}LI^2}{\frac{I^2R}{2} \times \frac{1}{f}} = \dfrac{2\pi fL}{R} = \dfrac{\omega L}{R}$

Similarly, in a capacitor, the max energy stored is given by $\dfrac{CV^2}{2}$

The energy dissipated per cycle $= (I/\sqrt{2})^2 R \times T$

The quality factor of the capacitance circuit

$$Q = \frac{2\pi \frac{1}{2} C \left(\frac{I}{\omega C}\right)^2}{\frac{I^2}{2} R \times \frac{1}{f}} = \frac{1}{\omega CR}$$

In series circuits, the quality factor $Q = \dfrac{\omega L}{R} = \dfrac{1}{\omega CR}$

We have already discussed the relation between bandwidth and quality factor, which is $Q = \dfrac{f_r}{BW}$.

A higher value of circuit Q results in a smaller bandwidth. A lower value of Q causes a larger bandwidth.

4.4.5 Magnification in Series Resonant Circuit

If we assume that the voltage applied to the series RLC circuit is V, and the current at resonance is I, then the voltage across L is $V_L = IX_L = (V/R)\,\omega_r L$

Similarly, the voltage across C

$$V_C = I X_C = \frac{V}{R\omega_r C}$$

Since $Q = 1/\omega_r CR = \omega_r L/R$

where ω_r is the frequency at resonance.

Therefore $V_L = VQ$

$V_C = VQ$

The ratio of voltage across either L or C to the voltage applied at resonance can be defined as magnification.

$\therefore$ Magnification $= Q = V_L/V$ or V_C/V

Example 4.44 *A series circuit with $R = 10\ \Omega$, $L = 0.1$ H and $C = 50\ \mu$F has an applied voltage $V = 50\angle 0°$ with a variable frequency. Find the resonant frequency, the value of frequency at which maximum voltage occurs across the inductor and the value of frequency at which maximum voltage occurs across the capacitor.*

Solution The frequency at which maximum voltage occurs across the inductor is

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{\left(1-\frac{R^2C}{2L}\right)}}$$

$$= \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}}\sqrt{\frac{1}{1-\left(\frac{(10)^2\times 50\times 10^{-6}}{2\times 0.1}\right)}}$$

$$= 72.08 \text{ Hz}$$

Similarly, $$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{2L}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times 50\times 10^{-6}}-\frac{(10)^2}{2\times 0.1}}$$

$$= \frac{1}{2\pi}\sqrt{200000-500}$$

$$= 71.08 \text{ Hz}$$

Resonant frequency $$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}} = 71.18 \text{ Hz}$$

It is clear that the maximum voltage across the capacitor occurs below the resonant frequency and the maximum inductor voltage occurs above the resonant frequency.

Example 4.45 *A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor when the capacitor is set to 500 PF, the current has the max. value, while it is reduced to one half when capacitance 600 PF, find (i) resistance (ii) inductance (iii) Q factor of inductor.*

Solution Given $f = 1$ MHz

Let the max. current be I_{max}

Given at 1 MHz, for $C = 500$ Pf

$$I = I_{\text{max}}$$

$\therefore$ Imaginary part of impedance is zero, i.e. $X_L = X_C$

$$2\pi f L = \frac{1}{2\pi f c}$$

$$6.283 \times 10^6 \times L = 318.31$$

$$L = 50.66\,\mu\text{H}$$

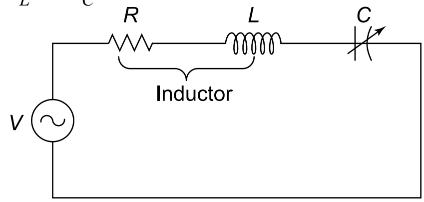


Fig. 4.64

Now also given $I = \dfrac{I_{max}}{2}$ at $C = 600$ PF

$$I = \frac{I_{max}}{2} = \frac{V}{R + j(6.283\times10^6 L - 265.25)} \quad (4.66)$$

$$\left(\because \quad X_C = \frac{1}{2\pi f c} = \frac{1}{2\pi\times10^6\times600\times10^{-12}} = 265.25\right)$$

$$\text{and } I_{max} = \frac{V}{R} \quad (4.67)$$

Dividing Eq. 4.67 by Eq. 4.66

$$Z = \frac{R + j(6.283\times10^6 L - 265.25)}{R}$$

$$\Rightarrow \quad 2R = R + j(6.283 \times 10^6 L - 265.25)$$

$$R = j(318.31 - 265.25)$$

$$R = 53.06\,\Omega$$

$\therefore$ (i) $R = 53.06\,\Omega$

(ii) $L = 50.66\,\mu\text{H}$

(iii) $Q = \dfrac{\omega L}{R} = 5.999 \approx 6$

Example 4.46 *Obtain the expression for the frequency at which maximum voltage occurs across the capacitance in series resonance circuit in terms of the Q-factor and resonance frequency.*

Solution The frequency at which V_c is maximum is given by

$$f_c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

$$f_c = \frac{1}{2\pi}\left[\sqrt{\frac{1}{LC}\left[1 - \frac{R^2 C}{2L}\right]}\right]$$

$$= \frac{1}{2\pi}\left[\sqrt{\frac{R^2}{LC}\left(\frac{1}{R^2} - \frac{C}{2L}\right)}\right]$$

$$= \frac{1}{2\pi}\frac{R}{\sqrt{LC}}\left[\sqrt{\frac{1}{R^2} - \frac{C}{2L}}\right]$$

$$= \frac{1}{2\pi}\frac{R}{\sqrt{LC}}\left[\sqrt{\frac{C}{L}\left[\frac{L}{CR^2} - \frac{1}{2}\right]}\right]$$

$$= \frac{1}{2\pi\sqrt{LC}}.R\sqrt{\frac{C}{L}}\left[\frac{L}{CR^2} - \frac{1}{2}\right]^{1/2}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}};\ Q = \frac{1}{R}\sqrt{\frac{L}{C}} \Rightarrow \frac{1}{Q} = R\sqrt{\frac{C}{L}}$$

$$\therefore \qquad f_c = \frac{f_o}{Q}\left[\frac{L}{CR^2} - \frac{1}{2}\right]^{1/2}$$

Example 4.47 *In a series RLC circuit if the applied voltage is 10 V, and resonance frequency is 1 kHz, and Q factor is 10, what is the maximum voltage across the inductance.*

Solution Resonance frequency $(f_r) = \dfrac{1}{2\pi\sqrt{LC}} = 1000$ (4.68)

Quality factor $(Q) = \dfrac{1}{R}\sqrt{\dfrac{L}{C}} = 10$ (4.69)

$$\sqrt{LC} = \frac{1}{2\pi \times 1000} = 6283.18$$

$$LC = 39.47 \times 10^6$$

From 4.68, $\dfrac{1}{2\pi} = \sqrt{LC}\ 1000$ (4.70)

From 4.69, $\dfrac{1}{R} = \sqrt{\dfrac{C}{L}}\ 10$ (4.71)

From 4.70 and 4.71,

$$\frac{1}{2\pi R} = 10^4\sqrt{LC}\sqrt{\frac{C}{L}}$$

$$\frac{1}{2\pi RC} = 10000$$

$$RC = 1.59154 \times 10^{-5} \simeq 1.6 \times 10^{-5}$$

The maximum voltage across the inductance occurs at frequency greater than the resonance frequency which is given by

$$f_L = \frac{1}{2\pi\sqrt{LC - \frac{(RC)^2}{2}}}$$

$$f_L = \frac{1}{2\pi\sqrt{39.47\times10^6 - \frac{(1.6\times10^{-5})^2}{2}}} = 1002.5$$

It can be observed that, the above frequency is approximately equal to resonance frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{39.47\times10^6}}$$

Hence we can take the voltage across the inductor $= Q \times V$

$= 10 \times 10$

$= 100$ volts

Example 4.48 *A series RLC circuit is connected across a variable frequency supply and has R = 12 ohms, L = l mH and C = l000 pF. Calculate*

(a) Resonant frequency.

(b) Q factor and

(c) Half power frequencies. Derive the formulae used. *[JNTU Jan 2010]*

Solution (a) Resonant frequency $= \frac{1}{2\pi\sqrt{\text{LC}}} = \frac{1}{2\pi\sqrt{1\times10^{-3}\times1000\times10^{-12}}}$ Hz

$= 159.155$ kHz

(b) Q-factor $= \frac{1}{R}\sqrt{\frac{L}{C}}$

$$= \frac{1}{12}\sqrt{\frac{1\times10^{-3}}{1000\times10^{-12}}} = 83.33$$

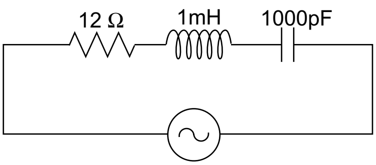


Fig. 4.65

(c) Half power frequencies are given as,

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] = 158.203 \text{ kHz}$$

and $$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] = 160.113 \text{ kHz}$$

Example 4.49 *Determine the quality factor of a coil for the series circuit consisting of R = 10 Ω, L = 0.1 H and C = 10 μF.*

Solution Quality factor $Q = \dfrac{f_r}{BW}$

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 10\times 10^{-6}}} = 159.2\,\text{Hz}$$

At lower half power frequency, $X_C > X_L$

$$\frac{1}{2\pi f_1 C} - 2\pi f_1 L = R$$

From which $f_1 = \dfrac{-R+\sqrt{R^2+4L/C}}{4\pi L}$

At upper half power frequency $X_L > X_C$

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = R$$

From which $f_2 = \dfrac{+R+\sqrt{R^2+4L/C}}{4\pi L}$

Bandwidth $BW = f_2 - f_1 = \dfrac{R}{2\pi L}$

Hence $Q_0 = \dfrac{f_r}{BW} = \dfrac{2\pi f_r L}{R} = \dfrac{2\times\pi\times 159.2\times 0.1}{10}$

$$Q_0 = \frac{f_r}{BW} = 10$$

Example 4.50 *A voltage v(t) = 10 sin ωt is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 500 V. Moreover, the bandwidth is known to be 400 rad/sec and the impedance at resonance is 100 Ω. Find the resonant frequency. Also find the values of L and C of the circuit.*

Solution The applied voltage to the circuit is

$$V_{\text{max}} = 10\text{ V}$$

$$V_{\text{rms}} = \frac{10}{\sqrt{2}} = 7.07\,\text{V}$$

The voltage across capacitor $V_C = 500$ V

The magnification factor $Q = \dfrac{V_C}{V} = \dfrac{500}{7.07} = 70.7$

The bandwidth $BW = 400$ rad/sec

$$\omega_2 - \omega_1 = 400 \text{ rad/sec}$$

The impedance at resonance $Z = R = 100\ \Omega$

Since
$$Q = \frac{\omega_r}{\omega_2 - \omega_1}$$

$$\omega_r = Q(\omega_2 - \omega_1) = 28280 \text{ rad/sec}$$

$$f_r = \frac{28280}{2\pi} = 4499\,\text{Hz}$$

The bandwidth
$$\omega_2 - \omega_1 = \frac{R}{L}$$

$$L = \frac{R}{\omega_2 - \omega_1} = \frac{100}{400} = 0.25\,\text{H}$$

Since
$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$C = \frac{1}{(2\pi f_r)^2 \times L} = \frac{1}{2\pi \times (4499)^2 \times 0.25} = 5\,n\text{F}$$

Example 4.51 *A series RLC circuit consists of a 50 Ω. resistance, 0.2 H inductance and 10 μF capacitor with an applied voltage of 20 V. Determine the resonant frequency. Find the Q factor of the circuit. Compute the lower and upper frequency limits and also find the bandwidth of the circuit.*

Solution Resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}} = 112.5\,\text{Hz}$$

Quality factor $Q = \dfrac{\omega L}{R} = \dfrac{2\pi \times 112.5 \times 0.2}{50} = 2.83$

Lower frequency limit

$$f_1 = f_r - \frac{R}{4\pi L} = 112.5 - \frac{50}{4 \times \pi \times 0.2} = 92.6\,\text{Hz}$$

Upper frequency limit

$$f_2 = f_r + \frac{R}{4\pi L} = 112.5 + \frac{50}{4\pi \times 0.2} = 112.5 + 19.89 = 132.39\,\text{Hz}$$

Bandwidth of the circuit

$$BW = f_2 - f_1 = 132.39 - 92.6 = 39.79 \text{ Hz}$$

Example 4.52 *Determine the quality factor, bandwidth and the half power frequencies of a series resonant circuit with R = 5 Ω, L = 0.05 H and C = 5 μf.*

Solution Resonance frequency

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{0.05\times 5\times 10^{-6}}}$$

$$= 318.3\,\text{Hz}$$

Quality factor $$Q = \frac{W_r L}{R} = \frac{2\pi(318.3)(0.05)}{5}$$

$$= 20$$

Bandwidth $$= \frac{f_r}{Q} = \frac{318.3}{20} = 15.915\,\text{Hz}$$

$$f_r = \sqrt{f_1 f_2}$$

$$f_r^2 = f_1 f_2 \Rightarrow f_1 = \frac{f_r^2}{f_2}$$

Also $f_2 - f_1 = 15.915$ Hz

$$f_2 - \frac{f_r^2}{f_2} = 15.915$$

$$f_2^2 - f_r^2 - 15.915 f_2 = 0$$

$$\Rightarrow \quad f_2^2 - 15.915 f_2 - 10.13\times 10^4 = 0$$

$$f_2 = 326\text{ Hz}$$

$$f_1 = 310\text{ Hz}$$

Half power points can also be calculated using

$$f_1 = f_r - \frac{R}{4\pi L} = 318.3 - \frac{5}{4\pi\times 0.05} = 310\,\text{Hz}$$

$$f_2 = f_r + \frac{R}{4\pi L} = 318.3 + \frac{5}{4\pi\times 0.05} = 326\,\text{Hz}$$

Example 4.53 *A series RLC circuit with Q = 250 is resonant at 1.5 MHz. Find the frequencies at half power points and also bandwidth.*

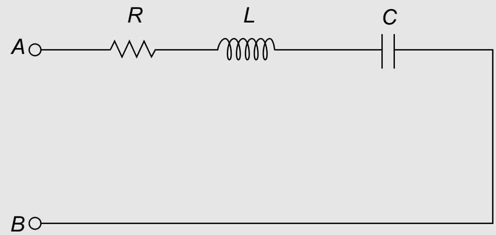


Fig. 4.66

Solution Given $Q = 250$

$$Q = \frac{\omega_o L}{R}$$

$$250 = \frac{2\pi \times f_o \times L}{R} \Rightarrow \frac{R}{L} = \frac{2\pi \times 1.5 \times 10^6}{250} = 37.7 \times 10^3$$

Lower half power frequency $f_1 = f_r - \dfrac{R}{4\pi L}$

$$= 1.5 \times 10^6 - \frac{37.7 \times 10^3}{4\pi}$$

$$= 1.5 \times 10^6 - 3 \times 10^3$$

$$= 1.496 \text{ MHz}$$

Upper half power frequency $f_2 = f_r + \dfrac{R}{4\pi L}$

$$= 1.5 \times 10^6 + \frac{37.7 \times 10^3}{4\pi}$$

$$= 1.5\text{M} + 3k = 1.503 \text{ MHz}$$

Bandwidth $= f_2 - f_1 = 1.503\text{M} - 1.496\text{ M} = 7 \text{ kHz}$

Example 4.54 *For the circuit shown in Fig. 4.67, determine the value of Q at resonance and bandwidth of the circuit.*

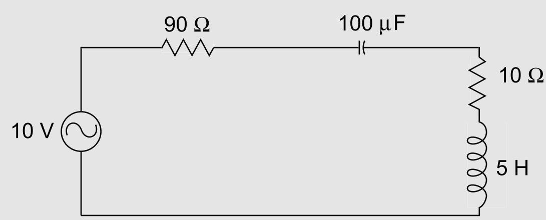


Fig. 4.67

Solution The resonant frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{5\times 100\times 10^{-6}}}$$

$$= 7.12\,\text{Hz}$$

Quality factor $Q = X_L/R = 2\pi f_r L/R$

$$= \frac{2\pi\times 7.12\times 5}{100} = 2.24$$

Bandwidth of the circuit is $BW = \dfrac{f_r}{Q} = \dfrac{7.12}{2.24} = 3.178\,\text{Hz}$

Example 4.55 *For the circuit shown in Fig. 4.68, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and the Q factor of the circuit.*

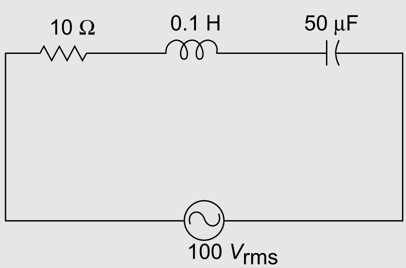


Fig. 4.68

Solution The frequency of resonance occurs when $X_L = X_C$

$$\omega L = \frac{1}{\omega C}$$

$$\therefore \qquad \omega = \frac{1}{\sqrt{LC}}\,\text{radians/sec} = \frac{1}{\sqrt{0.1\times 50\times 10^{-6}}}$$

$$= 447.2\,\text{radians/sec}$$

$$f_r = \frac{1}{2\pi}(447.2) = 71.17\text{ Hz}$$

The current passing through the circuit at resonance,

$$I = \frac{V}{R} = \frac{100}{10} = 10\text{ A}$$

The voltage drop across the inductor

$$V_L = IX_L = I\omega L$$

$$= 10\times 447.2\times 0.1 = 447.2\text{ V}$$

The quality factor $Q = \dfrac{\omega L}{R}$

$$= \frac{447.2 \times 0.1}{10} = 4.47$$

Example 4.56 *A series RLC circuit has a quality factor of 5 at 50 rad/sec. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. The total impedance of the circuit is 20 Ω. Find the circuit constants.*

Solution The quality factor $Q = 5$

At resonance the impedance becomes resistance.

The current at resonance is $I = \dfrac{V}{R}$

$$\therefore \qquad 10 = \frac{100}{R}$$

$$R = 10\ \Omega$$

$$Q = \frac{\omega L}{R}$$

Since $Q = 5, R = 10$

$$\omega L = 50$$

$$\therefore \qquad L = \frac{50}{\omega} = 1\,\text{H}$$

$$Q = \frac{1}{\omega CR}$$

$$C = \frac{1}{Q\omega R}$$

$$= \frac{1}{5 \times 50 \times 10}$$

$$C = 400\ \mu F$$

Example 4.57 *In the circuit shown in Fig. 4.69 a maximum current of 0.1 A flows through the circuit when the capacitor is at 5 μF with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, the quality factor Q and the value of resistance at resonant frequency.*

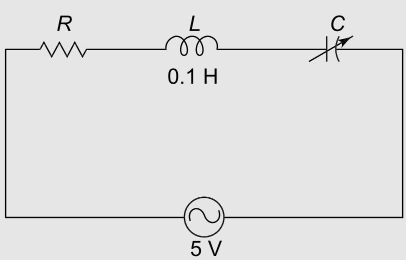


Fig. 4.69

Solution At resonance, the current is maximum in the circuits

$$I = \frac{V}{R}$$

$$\therefore \qquad R = \frac{V}{I} = \frac{5}{0.1} = 50\,\Omega$$

The resonant frequency is

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \times 5 \times 10^{-6}}} = 1414.2 \text{ rad/sec}$$

$$f_r = \frac{1414.2}{2\pi} = 225\,\text{Hz}$$

The quality factor is

$$Q = \frac{\omega L}{R} = \frac{1414.2 \times 0.1}{50} = 2.8$$

Since $\qquad \dfrac{f_r}{BW} = Q$

The bandwidth $BW = \dfrac{f_r}{Q} = \dfrac{225}{2.8} = 80.36\,\text{Hz}$

Example 4.58 *In the circuit shown in Fig. 4.70, determine the circuit constants when the circuit draws a maximum current at 10 μF with a 10 V, 100 Hz supply. When the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/sec. Also find the maximum current that flows through the circuit.*

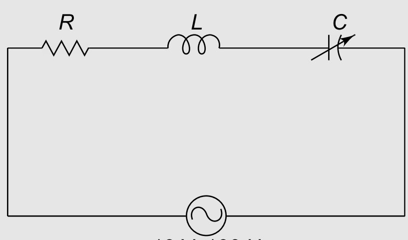


Fig. 4.70

Solution At resonant frequency, the circuit draws maximum current. So, the resonant frequency f_r = 100 Hz

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{C \times (2\pi f_r)^2}$$

$$= \frac{1}{10 \times 10^{-6}(2\pi \times 100)^2} = 0.25 \text{ H}$$

We have $$\omega L - \frac{1}{\omega C} = R$$

$$900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}} = R$$

$$\therefore \quad R = 132.4\ \Omega$$

The quality factor $Q = \dfrac{\omega L}{R} = \dfrac{900 \times 0.25}{132.4} = 1.69$

The maximum current in the circuit is $I = \dfrac{10}{132.4} = 0.075$ A

Example 4.59 *In the circuit shown in Fig. 4.71 the current is at its maximum value with capacitor value C = 20 μF and 0.707 times its maximum value with C = 30 μF. Find the value of Q at ω = 500 rad/sec, and circuit constants.*

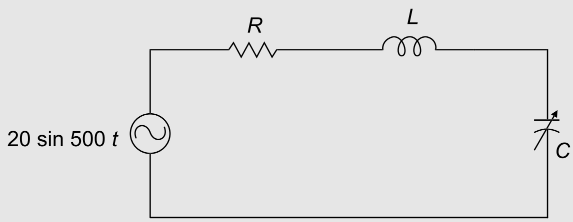


Fig. 4.71

Solution The voltage applied to the circuit is $V = 20$ V. At resonance, the current in the circuit is maximum. The resonant frequency $\omega_r = 500$ rad/sec.

Since $$\omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad L = \frac{1}{\omega_r^2 C} = \frac{1}{(500)^2 \times 20 \times 10^{-6}} = 0.2\,\text{H}$$

Since we have $$\omega L - \frac{1}{\omega C} = R$$

$$500 \times 0.2 - \frac{1}{500 \times 30 \times 10^{-6}} = R$$

$$\therefore R = 100 - 66.6 = 33.4$$

The quality factor is $Q = \dfrac{\omega L}{R} = \dfrac{500 \times 0.2}{33.4} = 2.99$

Example 4.60 *A coil having R = 15 Ω and L = 40 mH is connected in series with a capacitor across a 240 V source resonates at 350 Hz. Find the value of*

(a) capacitance *(b) power dissipated in the coil*

(c) Q factor *(d) voltage across the capacitor and coil*

Solution (a) At $f = f_r, X_L = X_C$

$$\Rightarrow \quad C = \frac{1}{4\pi^2 L f_r^2} = \frac{1}{4\times\pi^2\times 40\times 10^{-3}\times(350)^2}$$

$$= 5.17\,\mu\text{F}$$

(b) At resonance $I = \dfrac{V}{R} = \dfrac{240}{15} = 16$ A

Power dissipated $= I^2 R = (16)^2 \times 15 = 4.84$ kW

(c) $$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{15}\sqrt{\frac{40\times 10^{-3}}{5.17\times 10^{-6}}} = 5.863$$

(d) $V_c = -jQV = 5.863\times 240 = 1407.12\,\angle{-90^\circ}\ V$

Let the voltage across the inductance of the coil be $V_L = V_C$ in magnitude

$\therefore \qquad V_L = 1407.12\,\angle 90^\circ$

Let V_R is the voltage across the resistance of the coil then

$$V_R = V = 240\,\angle 0^\circ$$

The voltage across the coil $V_{\text{coil}} = V_L + V_R$

$$= 1407.12\angle 90^\circ + 240\,\angle 0^\circ$$

$$= 240 + j1407.12$$

$$= 1427.44\,\angle 80.32^\circ$$

Example 4.61 *With respect to a (resonant circuit), i.e., series resonant circuit, prove that the bandwidth is inversely proportional to the Q-factor at resonance*

Solution The bandwidth of any system is the range of frequencies for which the current (or) the output voltage equals to 70.7% of it's value at resonance.

Bandwidth $= f_2 - f_1$

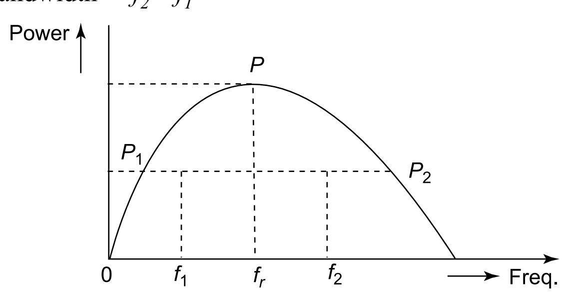


Fig. 4.72

If the current at P_1 is 0.707 Imax, the impedance of the circuit at this point is $\sqrt{2}\,R$.

$$\frac{1}{\omega_1 c} - \omega_1 L = R \tag{4.72}$$

$$\omega_2 L - \frac{1}{\omega_2 c} = R \tag{4.73}$$

By equating 4.72 and 4.73 we get,

$$\frac{1}{c}\left[\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right] = L(\omega_1 + \omega_2) \tag{4.74}$$

$$\omega_1 \omega_2 = \frac{1}{LC}$$

we have $$\omega_r^2 = \frac{1}{LC} \Rightarrow \omega_r^2 = \omega_1\,\omega_2 \tag{4.75}$$

Adding the equations 4.74 and 4.75,

$$\Rightarrow \qquad \frac{1}{\omega_1 c} - \frac{1}{\omega_2 c} + L(\omega_2 - \omega_1) = 2R$$

$$\frac{1}{c}\left[\frac{\omega_1 - \omega_2}{\omega_1 \omega_2}\right] + L(\omega_2 - \omega_1) = 2R$$

Since $c = \dfrac{1}{\omega_r{}^2 L}, \omega_1\,\omega_2 = \omega_r{}^2$

$$(\omega_2 - \omega_1)\,L + L(\omega_2 - \omega_1) = 2R$$

$$L\,(\omega_2 - \omega_1) = R$$

$$\omega_2 - \omega_1 = \frac{R}{L}$$

$$\Rightarrow \qquad f_2 - f_1 = \frac{R}{2\pi L}$$

$$\therefore \quad \frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{1}{Q}$$

$$\therefore \qquad Q = \frac{f_r}{BW} \tag{4.76}$$

Example 4.62 *A series R-L-C circuit with R = 100 V, L = 0.5 H and C = 40 μF has an applied voltage of 50 V with variable frequency. Calculate*

(a) Resonance frequency
(b) Current at resonance
(c) Voltage across R, L and C
(d) Upper and Lower half frequencies
(e) Bandwidth
(f) Q-factor of the circuit *[JNTU May 2007]*

Solution $R = 100\ \Omega, L = 0.5\ H, C = 40\ \mu\text{F}, V = 50\ V$

(a) Resonance frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{0.5\times 40\times 10^{-6}}}$

$$f_r = 35.58\ \text{Hz}$$

(b) Current at resonance, $I = \dfrac{V}{Z} = \dfrac{V}{R}$

$$I = \frac{V}{R} = \frac{50}{100} = 0.5\ \text{A}$$

(c) Voltage across resistance, $V_R = I_R = 0.5 \times 100 = 50$ volts

Voltage across inductance, $V_L = \omega L = 2\pi \times 0.5 \times 35.58 = 111.8$ volts

Voltage across capacitance,

$$V_C = \frac{1}{\omega C} = \frac{1}{2\pi\times 40\times 10^{-6}\times 35.58} = 111.8\,\text{volts}$$

(d) $f_r - f_1 = \dfrac{R}{4\pi L} \Rightarrow 35.59 - f_1 = \dfrac{100}{4\pi\times 0.5}$

$\therefore$ Lower-half frequency, $f_1 = 19.6745$ Hz

$$f_2 - f_r = \frac{R}{4\pi L} \Rightarrow f_2 - 35.59 = \frac{100}{4\pi\times 0.5}$$

$\therefore$ Upper-half frequency, $f_2 = 51.5055$ Hz

(e) Bandwidth, $BW = \dfrac{R}{2\pi L} = \dfrac{100}{2\pi\times 0.5} = 31.831\,\text{Hz}$

(f) Q-factor, $Q = \dfrac{f_r}{BW} = \dfrac{35.59}{31.831} = 1.1181$

4.5 PARALLEL RESONANCE, ANTI-RESONANCE AT ALL FREQUENCIES

[JNTU June 2009]

4.5.1 Resistance Present in Both Branches

[JNTU Nov 2011]

Basically, parallel resonance occurs when $X_C = X_L$. The frequency at which resonance occurs is called the *resonant* frequency. When $X_C = X_L$, the two branch currents are equal in magnitude and 180° out of phase with each other. Therefore, the two currents cancel each other out, and the total current is zero. Consider the circuit shown in Fig. 4.73. The condition for resonance occurs when $X_L - X_C$.

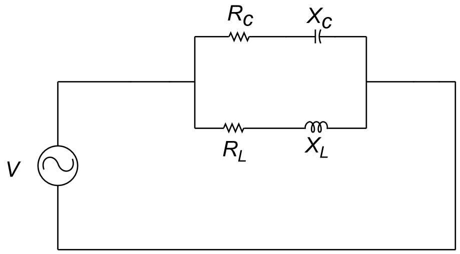


Fig. 4.73

In Fig. 4.73, the total admittance

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$

$$= \frac{R_L + j\omega L}{R_L^2 + \omega^2 L^2} + \frac{R_C + (j/\omega C)}{R_C^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \dfrac{1}{\omega^2 C^2}} + j\left\{\left[\frac{1/\omega C}{R_C^2 + \dfrac{1}{\omega^2 C^2}}\right] - \left[\frac{\omega L}{R_L^2 + \omega^2 L^2}\right]\right\} \quad (4.77)$$

At resonance the susceptance part becomes zero

$$\frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{\dfrac{1}{\omega_r C}}{R_C^2 + \dfrac{1}{\omega_r^2 C^2}} \quad (4.78)$$

$$\omega_r L\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{\omega_r C}\left[R_L^2 + \omega_r^2 L^2\right]$$

$$\omega_r^2\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{LC}\left[R_L^2 + \omega_r^2 L^2\right]$$

$$\omega_r^2 R_C^2 - \frac{\omega_r^2 L}{C} = \frac{1}{LC} R_L^2 - \frac{1}{C^2}$$

$$\omega_r^2\left[R_C^2 - \frac{L}{C}\right] = \frac{1}{LC}\left[R_L^2 - \frac{L}{C}\right]$$

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - (L/C)}{R_C^2 - (L/C)}} \quad (4.79)$$

The condition for resonant frequency is given by Eq. 4.79. As a special case, if $R_L = R_C$, then Eq. 4.79 become

$$\omega_r = \frac{1}{\sqrt{LC}}$$

Therefore

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

Example 4.63 *Find the resonant frequency in the ideal parallel LC circuit shown in Fig. 4.74.*

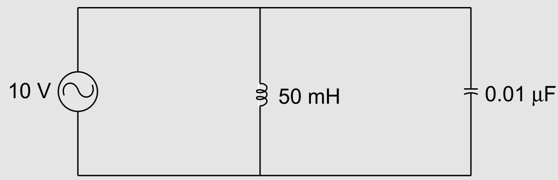


Fig. 4.74

Solution $$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{50\times10^{-3}\times0.01\times10^{-6}}} = 7117.6\,\text{Hz}$$

Example 4.64 *For the parallel circuit shown in the Fig. 4.75. Find the resonance frequency at $R_L = R_C$*

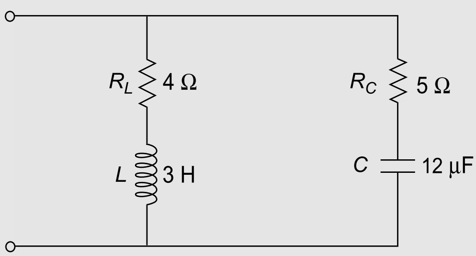


Fig. 4.75

Solution

$$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{R_L^2-(L/C)}{R_C^2-(L/C)}}$$

$$= \frac{1}{2\pi\sqrt{3\times12\times10^{-6}}}\sqrt{\frac{(4)^2-(3/12\times10^{-6})}{(5)^2-(3/12\times10^{-6})}}$$

$$= 26\sqrt{\frac{249984}{249975}} = 26\text{ Hz}$$

Example 4.65 *Compare series and parallel resonant circuits.*

[JNTU June 2009]

Solution

Series Resonant Circuit	*Parallel Resonant Circuit*
1. The applied voltage and the resulting current are in phase which also mean that the p.f. of RLC series resonant circuit is unity.	1. Power factor is unity.
2. The net reactance is zero at resonance and the impedance does have the resistive part only.	2. Net impedance at resonance of the parallel circuit is maximum and equal to (L/CR) Ω.
3. The current in the circuit is maximum and is (V/R) A. Since at resonance, the line current in the series LCR circuit is maximum hence it is called acceptor circuit.	3. Current at resonance is [V/(L/CR)] and is in phase with the applied voltage. The value of current at resonance is minimum.
4. At resonance the circuit has got minimum impedance and maximum admittance.	4. The admittance is minimum and the net susceptance is zero at resonance.
5. Frequency of resonance is given by is given by $f_o = \dfrac{1}{2\pi\sqrt{LC}}$ Hz.	5. The resonance frequency of this circuit is $f_o = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$.

4.5.2 Resonant Frequency for a Tank Circuit *[JNTU June 2009]*

The parallel resonant circuit is generally called a tank circuit because of the fact that the circuit stores energy in the magnetic field of the coil and in the electric field of the capacitor. The stored energy is transferred back and forth between the capacitor and coil and vice–versa. The tank circuit is shown in Fig. 4.76. The circuit is said to be in resonant condition when the susceptance part of admittance is zero.

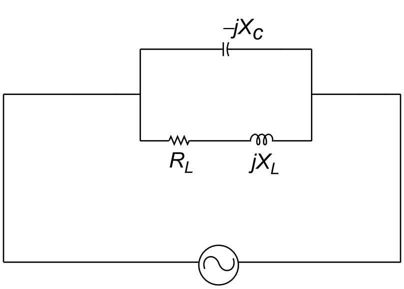


Fig. 4.76

The total admittance is

$$Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C} \tag{4.80}$$

Simplifying Eq. 4.80, we have

$$Y = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C} = \frac{R_L}{R_L^2 + X_L^2} + j\left[\frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2}\right]$$

To satisfy the condition for resonance, the susceptance part is zero.

$$\therefore \quad \frac{1}{X_C} = \frac{X_L}{R_L^2 + X_L^2} \tag{4.81}$$

$$\left(\omega C = \frac{\omega L}{R_L^2 + \omega^2 L^2}\right) \tag{4.82}$$

From Eq. 4.82, we get

$$R_L^2 + \omega^2 L^2 = \frac{L}{C}$$

$$\omega^2 L^2 = \frac{L}{C} - R_L^2$$

$$\omega^2 = \frac{1}{LC} - \frac{R_L^2}{L^2}$$

$$\therefore \quad \omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \tag{4.83}$$

The resonant frequency for the tank circuit is

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \tag{4.84}$$

Example 4.66 *In a parallel resonance circuit shown in Fig. 4.77 find the resonance frequency, dynamic resistance and bandwidth.*

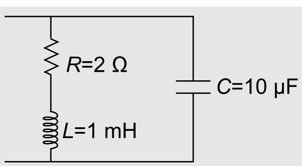


Fig. 4.77

Solution The circuit shown in the above figure is the most common form of parallel resonant circuit in practical use and is also called the tank circuit.
The admittance of the circuit is

$$Y = \frac{1}{Z} = \frac{1}{Z_C} + \frac{1}{Z_L}$$

$$
\begin{aligned}
Y &= \frac{1}{-jX_C} + \frac{1}{R + jX_L} \\
&= j\omega C + \frac{1}{R + j\omega L} \\
&= j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2} \\
&= \frac{R}{R^2 + \omega^2 L^2} + j\omega\left(C - \frac{L}{R^2 + \omega^2 L^2}\right)
\end{aligned}
$$

At resonance the susceptance part is zero.

Hence at $\omega = \omega_r, C = \dfrac{L}{R^2 + \omega_r^2 L^2} = 0$

$$R^2 + \omega_r^2 L^2 = \frac{L}{C}$$

$$\omega_r^2 L^2 = \frac{L}{C} - R^2 \Rightarrow \omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \tag{4.85}$$

Resonance frequency, $f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$ (4.86)

$$
\begin{aligned}
\therefore \qquad f_r &= \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi L}\sqrt{\frac{L}{C} - R^2} \\
&= \frac{1}{2\pi \times 1 \times 10^{-3}}\sqrt{\frac{1 \times 10^{-3}}{10 \times 10^{-6}} - 4} \\
&= 1559.4 \text{ Hz}
\end{aligned}
$$

Dynamic impedance:

The input admittance at resonance is given by

$$Y_r = \frac{R}{R^2 + \omega_r^2 L^2}$$

The impedance at resonance is

$$Z_r = \frac{1}{y_r} = \frac{R^2 + \omega_r^2 L^2}{R} = R + \frac{\omega_r^2 L^2}{R}$$

Substituting $\omega_r^2 L^2$ from Eq. 4.85 gives,

$$Z_r = R + \frac{\frac{L}{C} - R^2}{R} = R + \frac{L}{CR} - R$$

$Z_r = \dfrac{L}{CR}$ which is called dynamic impedance.

This is a pure resistance because it is independent of the frequency.

Here, dynamic resistance $= \dfrac{1\times10^{-3}}{10\times10^{-6}\times2} = 50\ \Omega$

Bandwidth of the parallel resonance circuit $= \dfrac{\omega_r}{Q}$

$$\omega_r = \frac{1}{L}\sqrt{\frac{L}{C} - R^2} = 9797.95$$

$$Q_o = \frac{\omega o L}{R} = \frac{9797.95\times1\times10^{-3}}{2} = 4.898$$

Bandwidth $= \dfrac{9797.5}{4.898} = 2000.4$

Example 4.67 *For the tank circuit shown in Fig. 4.78, find the resonant frequency.*

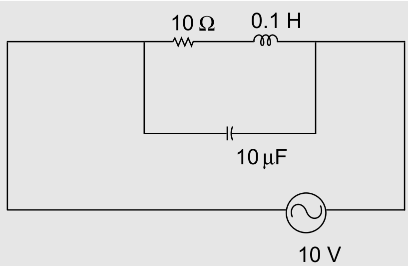


Fig. 4.78

Solution The resonant frequency

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times10\times10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$

$$= \frac{1}{2\pi}\sqrt{(10)^6 - (10)^2} = \frac{1}{2\pi}(994.98) = 158.35\,\text{Hz}$$

Example 4.68 *Find the value of L at which the circuit resonates at a frequency of 1000 rad/sec in the circuit shown in Fig. 4.79.*

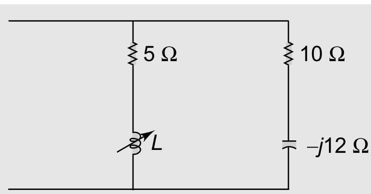


Fig. 4.79

Solution $Y = \dfrac{1}{10 - j12} + \dfrac{1}{5 + jX_L}$

$$Y = \frac{10+j12}{10^2+12^2} + \frac{5-jX_L}{25+X_L^2}$$

$$= \frac{10}{10^2+12^2} + \frac{5}{25+X_L^2} + j\left[\frac{12}{10^2+12^2} - \frac{X_L}{25+X_L^2}\right]$$

At resonance the susceptance becomes zero.

Then $$\frac{X_L}{25+X_L^2} = \frac{12}{10^2+12^2}$$

$$12X_L^2 - 244\,X_L + 300 = 0$$

From the above equation

$$X_L^2 - 20.3\,X_L + 25 = 0$$

$$X_L = \frac{+20.3 \pm \sqrt{(20.3)^2 - 4\times 25}}{2}$$

$$= \frac{20.3+\sqrt{412-100}}{2} \text{ or } \frac{20.3-\sqrt{412-100}}{2}$$

$$= 18.98\ \Omega \text{ or } 1.32\ \Omega$$

$\therefore$ $$X_L = \omega L = 18.98 \text{ or } 1.32\ \Omega$$

$$L = \frac{18.98}{1000} \text{ or } \frac{1.32}{1000}$$

$$L = 18.98 \text{ mH or } 1.32 \text{ mH}$$

Example 4.69 *Two impedances $Z_1 = 20 + j10$ and $Z_2 = 10 - j30$ are connected in parallel and this combination is connected in series with $Z_3 = 30 + jX$. Find the value of X which will produce resonance.*

Solution Total impedance is

$$Z = Z_3 + (Z_1 \parallel Z_2)$$

$$= (30+jX) + \left\{\frac{(20+j10)(10-j30)}{20+j10+10-j30}\right\}$$

$$= (30+jX) + \frac{200-j600+j100+300}{30-j20}$$

$$= 30+jX+\left(\frac{500-j500}{30-j20}\right)$$

$$= 30+jX+\left[\frac{500(1-j)(30+j20)}{(30)^2+(20)^2}\right]$$

$$= (30 + jX) + \left[\frac{500(30 + j20 - j30 + 20)}{900 + 400}\right]$$

$$= 30 + jX + \frac{5}{13}(50 - j10)$$

$$= \left(30 + \frac{5}{13} \times 50\right) + j\left(X - \frac{5}{13} \times 10\right)$$

At resonance, the imaginary part is zero

$$\therefore \quad X - \frac{50}{13} = 0$$

$$X = \frac{50}{13} = 3.85\ \Omega$$

Example 4.70 *For the circuit shown in Fig. 4.80, find the value of capacitance which results in resonance, when $f_r = 2000/\pi$.*

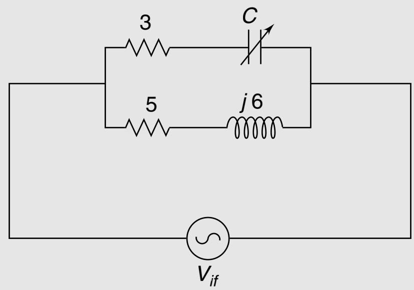


Fig. 4.80

Solution At resonance, the imaginary part of the admittance is zero. Hence, the complex admittance is a real number

$$Y = \frac{1}{5 + j6} + \frac{1}{3 - jx_c}$$

$$= \frac{5 - j6}{61} + \frac{3 + jx_c}{(3 - jx_c)(3 + jx_c)}$$

$$= \frac{5 - j6}{61} + \frac{3 + jx_c}{9 + x_c^2}$$

Separating the real and imaginary parts

$$Y = \left(\frac{5}{61} + \frac{3}{9 + x_c^2}\right) + j\left(\frac{x_c}{9 + x_c^2} - \frac{6}{61}\right)$$

Equating the *j* term to zero.

$$\frac{x_c}{9 + x_c^2} = \frac{6}{61}$$

$$6x_c^2 - 61x_c + 54 = 0$$

From which X_c = 9.18 (or) 0.979 Ω

$$\frac{1}{\omega C} = 9.18 \text{ (or) } \frac{1}{\omega C} = 0.979$$

Example 4.71 *An impedance $Z_1 = 10 + j10\ \Omega$ is connected in parallel with another impedance of 8.5 Ω resistance and a variable capacitance connected in series. Find C such that the circuit is in resonance at 5 kHz.*

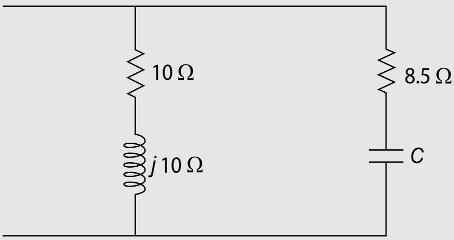


Fig. 4.81

Solution Considering the admittance

$$Y = \frac{1}{10 + j10} + \frac{1}{8.5 - jX_c}$$

$$= \frac{10 - j10}{10^2 + 10^2} + \frac{8.5 + jX_c}{(8.5)^2 + X_c^2}$$

$$= \frac{10}{10^2 + 10^2} + \frac{8.5}{(8.5)^2 + X_c^2} + j\left(\frac{X_c}{(8.5)^2 + X_c^2} - \frac{10}{200}\right)$$

At resonance the susceptance becomes zero.

$$\frac{X_C}{(8.5)^2 + X_c^2} = \frac{1}{20}$$

$$20\, X_c = X_c^2 + 72.25$$

$$X_c^2 - 20X_c + 72.25 = 0$$

$$X_c = 15.267 \text{ or } 4.732$$

$$\frac{1}{\omega c} = X_c = 15.267 \text{ or } 4.732$$

$$c = \frac{1}{2 \times \pi \times 5000 \times 15.267} \text{ or } \frac{1}{2 \times \pi \times 5000 \times 4.732}$$

$$c = 2.084\ \mu\text{F or } 6.726\ \mu\text{F}$$

Example 4.72 *For the parallel network shown in Fig. 4.82, determine the value of R at 10 Ω resonance.*

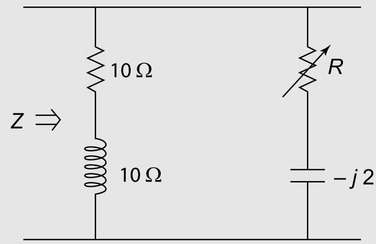


Fig. 4.82

Solution $Z = (10 + j10) \parallel (R - j2)$

$$= \frac{(10 + j10)(R - j2)}{10 + j10 + R - j2}$$

$$= \frac{10R - j20 + j10R + 20)}{10 + R + 8j}$$

$$= \frac{10R + 20 + j(10R - 20)}{10 + R + 8j}$$

$$= \frac{[(10R + 20) + j(10R - 20)][10 + R - j8]}{(10 + R)^2 + 64}$$

$$= [(10R + 20)(10 + R) + 8(10R - 20) - j8(10R + 20) + j(10 + R)(10R - 20)]\frac{1}{(10 + R)^2 + 64}$$

At resonance imaginary part = 0

$$\Rightarrow \quad 8(10R + 20) - (10 + R)(10R - 20) = 0$$

$$10R^2 = 360$$

$$R = 6\ \Omega$$

Example 4.73 *An impedance $Z_1 = 10 + j10\ \Omega$ is connected in parallel with another impedance of resistance 8.5 Ω and a variable capacitance connected in series. Find C such that the circuit is in resonance at 5 kHz.*

[JNTU Jan 2010]

Solution

$$Z_1 = 10 + j10\ \Omega$$

$$Z_2 = 8.5 - jX_C$$

$$X_C^2(R_2^2 + X_C^2) = X_L(R_1^2 + X_C^2)$$

or $\quad X_C^2(10^2 + 10^2) = 10(8.5^2 + X_C^2)$

or $\quad 200X_C^2 = 722.5 + 10X_C^2$

or $$X_C = 3.8$$

or $$2 \times \pi \times 5 \times 1000 \times C = \frac{1}{3.8}$$

or $$C = 8.5\ \mu\text{F}$$

4.5.3 Condition for Maximum Impedance

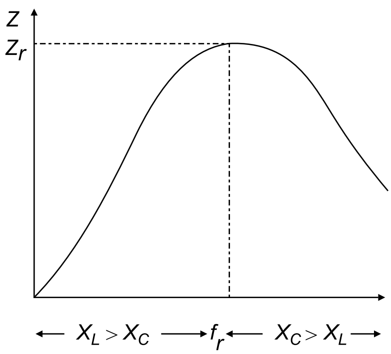


Fig. 4.83

The impedance of a parallel resonant circuit is maximum at the resonant frequency and decreases at lower and higher frequencies as shown in Fig. 4.83.

At very low frequencies, X_L is very small and X_C is very large, so the total impedance is essentially inductive. As the frequency increases, the impedance also increases, and the inductive reactance dominates until the resonant frequency is reached. At this point $X_L = X_C$ and the impedance is at its maximum. As the frequency goes above resonance, capacitive reactance dominates and the impedance decreases.

Example 4.74 *Determine the value of the capacitance C in order that the circuit in Fig. 4.84 is resonant at 6366 Hz.*

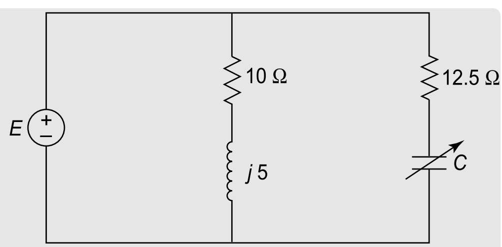


Fig. 4.84

Solution The admittance considered is

$$Y = \frac{1}{10 + j5} + \frac{1}{12.5 - jX_C}$$

$$= \frac{10 - j5}{10^2 + 5^2} + \frac{12.5 + jX_C}{(12.5)^2 + X_C^2}$$

$$= \frac{10}{10^2 + 5^2} + \frac{12.5}{12.5^2 + X_C^2} + j\left(\frac{X_C}{(12.5)^2 + X_C^2} - \frac{5}{102 + 52}\right)$$

At resonance the susceptance becomes zero.

$$\frac{X_C}{(12.5)^2 + X_C^2} = \frac{5}{10^2 + 5^2}$$

$$5X_c^2 + 5(12.5)^2 = (10^2 + 5^2)X_C$$

$$5X_c^2 - 125X_C + 781.25 = 0$$

$$X_C = 125 \pm \frac{\sqrt{(125)^2 - 4(781.25)5}}{2 \times 5} = 125$$

$$\frac{1}{\omega C} = 12.5$$

$$C = \frac{1}{2 \times \pi \times 6366 \times 12.5}$$

$$= 2 \times 10^{-6}\, F = 2\, \mu F$$

4.5.4 Bandwidth and *Q* Factor of Parallel Resonance *[JNTU Jan 2010]*

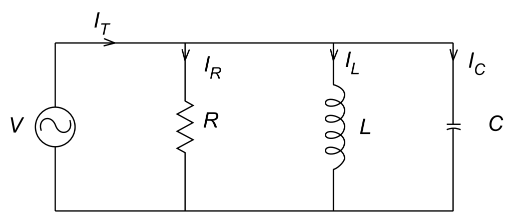


Fig. 4.85

Consider the parallel RLC circuit shown in Fig. 4.85.

In the circuit shown, the condition for resonance occurs when the susceptance part is zero.

Admittance $Y = G + jB$ (4.87)

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \tag{4.88}$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

The frequency at which resonance occurs is

$$\omega_r C - \frac{1}{\omega_r L} = 0 \tag{4.89}$$

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{4.90}$$

The voltage and current variation with frequency is shown in Fig. 4.86. At resonant frequency, the current is minimum.

The bandwidth, $BW = f_2 - f_1$

For parallel circuit, to obtain the lower half power frequency,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R} \tag{4.91}$$

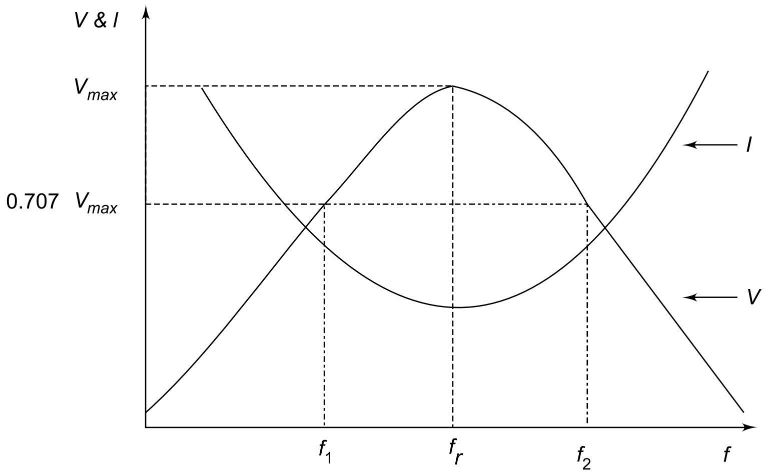


Fig. 4.86

From Eq. 4.91, we have

$$\omega_1^2 + \frac{\omega_1}{RC} - \frac{1}{LC} = 0 \tag{4.92}$$

If we simplify Eq. 4.92, we get

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{4.93}$$

Similarly, to obtain the upper half power frequency

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R} \tag{4.94}$$

From Eq. 4.94, we have

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{4.95}$$

Bandwidth

$$BW = \omega_2 - \omega_1 = \frac{1}{RC}$$

The quality factor is defined as $Q_r = \dfrac{\omega_r}{\omega_2 - \omega_1}$

$$Q_r = \frac{\omega_r}{1/RC} = \omega_r RC$$

In other words,

$$Q = 2\pi \times \frac{\text{Maximum energy stored}}{\text{Energy dissipated/cycle}}$$

In the case of an inductor,

The maximum energy stored $= \frac{1}{2}LI^2$

Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 \times R \times T$

The quality factor $Q = 2\pi \times \dfrac{1/2(LI^2)}{\dfrac{I^2}{2}R \times \dfrac{1}{f}}$

$$\therefore \qquad Q = 2\pi \times \frac{\frac{1}{2}L\left(\frac{V}{\omega L}\right)^2 R}{\frac{V^2}{2} \times \frac{1}{f}} = \frac{2\pi f L R}{\omega^2 L^2} = \frac{R}{\omega L}$$

For a capacitor, maximum energy stored $= 1/2(CV^2)$

Energy dissipated per cycle $= P \times T = \dfrac{V^2}{2 \times R} \times \dfrac{1}{f}$

The quality factor $Q = 2\pi \times \dfrac{1/2(CV^2)}{\dfrac{V^2}{2R} \times \dfrac{1}{f}} = 2\pi f CR = \omega CR$

4.5.5 Magnification in Parallel Resonant Circuit

Current magnification occurs in a parallel resonant circuit. The voltage applied to the parallel circuit, $V = IR$

Since $I_L = \dfrac{V}{\omega_r L} = \dfrac{IR}{\omega_r L} = IQ_r$

For the capacitor, $I_C = \dfrac{V}{1/\omega_r C} = IR\omega_r C = IQ_r$

Therefore, the quality factor $Q_r = I_L/I$ or I_C/I

4.5.6 Reactance Curves in Parallel Resonance

The effect of variation of frequency on the reactance of the parallel circuit is shown in Fig. 4.87.

The effect of inductive susceptance,

$$B_L = \frac{-1}{2\pi f L}$$

Inductive susceptance is inversely proportional to the frequency or ω. Hence it is represented by a rectangular hyperbola, *MN*. It is drawn in fourth quadrant, since B_L is negative. Capacitive susceptance, $B_C = 2\pi fC$. It is directly proportional to the frequency f or ω. Hence it is represented by OP, passing through the origin. Net susceptance $B = B_C - B_L$. It is represented by the curve JK, which is a hyperbola. At point ω_r, the total susceptance is zero, and resonance takes place. The variation of the admittance Y and the current I is represented by curve VW. The current will be minimum at resonant frequency.

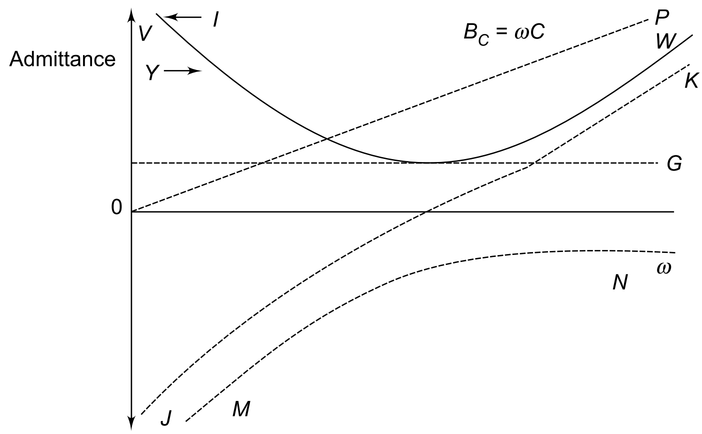


Fig. 4.87

Example 4.75 *From the parallel circuit shown in Fig. 4.88*

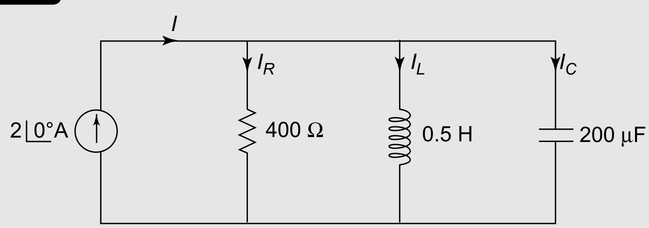


Fig. 4.88

(a) Find the resonance frequency
(b) Find the currents in all the branches at resonance
(c) Quality factor

Solution (a) $f_r = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{0.5\times 200\times 10^{-6}}} = 15.91\text{ Hz}$

(b) At resonance, the current through the resistance is same as the current from the source

$\therefore\ I_R = I = 2\text{ A}$

The voltage across the parallel branch $= I_R R$

$\Rightarrow V(t) = 2\times 400 = 800\ \underline{|0^\circ}$

$$\therefore \qquad I_L(t) = \frac{800\angle 0}{JWL} = \frac{800\angle 0}{100 \times 0.5\angle 90} = 16\angle -90°$$

$$I_C(t) = \frac{800\angle 0}{-J/WC} = \frac{800\angle 0}{50\angle -90°} = 16\angle 90°$$

(c) The quality factor $= \dfrac{I_L}{I}$ (or) $\dfrac{I_C}{I} = \dfrac{16}{2} = 8$

Example 4.76 *In the circuit shown in Fig. 4.89, an inductance of 0.1 H having a Q of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at resonant frequency of 500 rad/sec.*

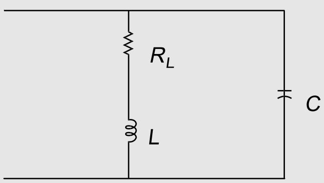


Fig. 4.89

Solution The quality factor $Q = \dfrac{\omega_r L}{R}$

Since $L = 0.1$ H, $Q = 5$ and

$$\omega_r = 500 \text{ rad/sec}$$

$$Q = \frac{500 \times 0.1}{R}$$

$$\therefore \qquad R = \frac{500 \times 0.1}{5} = 10\,\Omega$$

Since $$\omega_r^2 = \frac{1}{LC}$$

$$(500)^2 = \frac{1}{0.1 \times C}$$

$\therefore$ The capacitance value $C = \dfrac{1}{0.1 \times (500)^2} = 40\ \mu$F

Example 4.77 *In the parallel resonant circuit, determine the resonance frequency, dynamic resistance and bandwidth for the circuit shown in the Fig. 4.90.*

[JNTU May/June 2006]

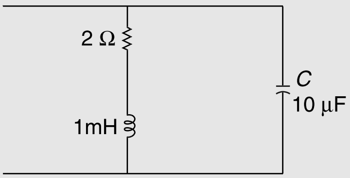


Fig. 4.90

Solution Total admittance (tank circuit)

$$Y = \frac{1}{R + j\omega L} + \frac{1}{-j/\omega C}$$

$$= \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$= \frac{R}{R^2 + \omega^2 C^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$

At resonance, the susceptance part (B) becomes zero.
Reactance

$$\begin{array}{ccc} Y = G + jB & & Z = R + jX \\ \swarrow \qquad \searrow & & \downarrow \\ \text{Conductance} \qquad \text{Susceptance} & & \text{Resistance} \end{array}$$

$$\omega_r C = \frac{\omega_r L}{R^2 + \omega_r^2 L^2}$$

$$R^2 + \omega^2{}_r L^2 = \frac{L}{C} \Rightarrow \omega^2{}_r = \frac{1}{L^2}\left(\frac{L}{C} - R^2\right)$$

$$\Rightarrow \qquad \omega^2{}_r = \frac{1}{LC} - \frac{R^2}{L^2} \Rightarrow \omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Here $R = 2\ \Omega$, $L = 1\,\text{mH}$, $C = 10\ \mu\text{F}$

$$\omega_r = \sqrt{\frac{1}{10^{-8}} - \frac{4}{10^{-6}}} = \sqrt{10^6 \times 96} = 9.79 \times 10^3\,\text{Hz}$$

$$f_r = \frac{\omega_r}{2\pi} = 1.559\ \text{kHz}$$

Dynamic resistance $(R) = \dfrac{R^2 + \omega^2{}_r L^2}{R}$

$$= \left.\frac{R^2 + \omega^2 L^2}{R}\right|_{\omega=\omega_r} = 2 + \frac{96 \times 10^6 \times 10^{-6}}{2} = 50\ \Omega$$

Bandwidth $= \dfrac{1}{RC}$ (for lid resonant ckt)

$$= \frac{1}{50 \times 10\,\mu f} = 2\,\text{kHz}$$

$$BW = \frac{R}{L} = \frac{2}{1\,mH} = 2\,\text{kHz.}$$

Practice Problems

4.1 Using the dot convention, write the voltage equations for the coils shown in Fig. 4.91.

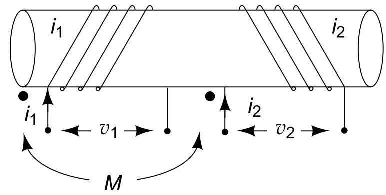


Fig. 4.91

4.2 Two inductively coupled coils have self inductances $L_1 = 40$ mH and $L_2 = 150$ mH. If the coefficient of coupling is 0.7, (i) find the value of mutual inductance between the coils, and (ii) the maximum possible mutual inductance.

4.3 For the circuit shown in Fig. 4.92 write the inductance matrix.

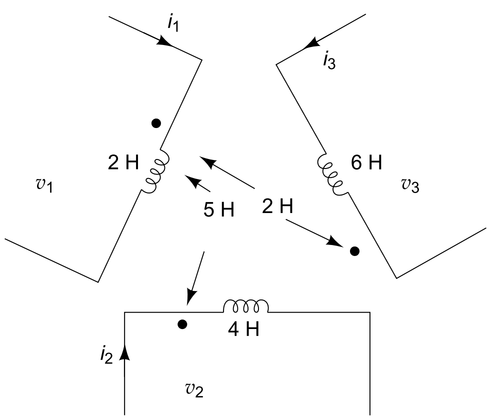


Fig. 4.92

4.4 Two coils connected in series have an equivalent inductance of 0.8 H when connected in aiding, and an equivalent inductance of 0.5 H when the connection is opposing. Calculate the mutual inductance of the coils.

4.5 In Fig. 4.93, L_1, $= 2$ H; $L_2 = 6$ H; $K = 0.5$; $i_1 = 4 \sin (40t - 30°)$ A; $i_2 = 2 \sin (40t - 30°)$ A. Find the values of (i) v_1, and (ii) v_2.

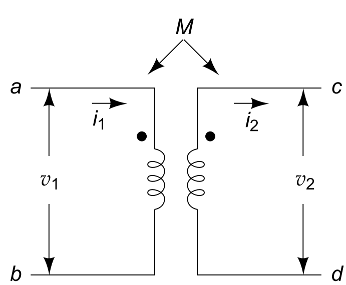


Fig. 4.93

4.6 For the circuit shown in Fig. 4.94, write the mesh equations.

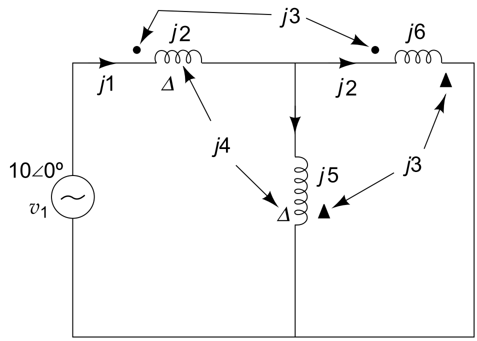


Fig. 4.94

4.7 Calculate the effective inductance of the circuit shown in Fig. 4.95 across *XY*.

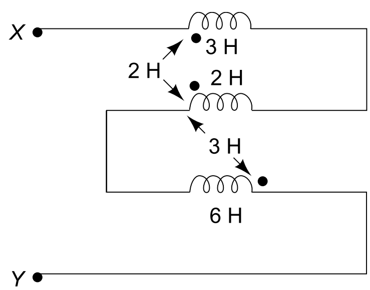


Fig. 4.95

4.8 For the circuit shown in Fig. 4.96, find the ratio of output voltage to the input voltage.

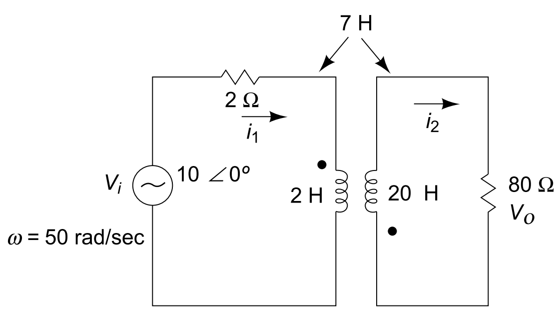


Fig. 4.96

4.9 Calculate the effective inductance of the circuit shown in Fig. 4.97.

4.10 Write the mesh equations for the network shown in Fig. 4.98.

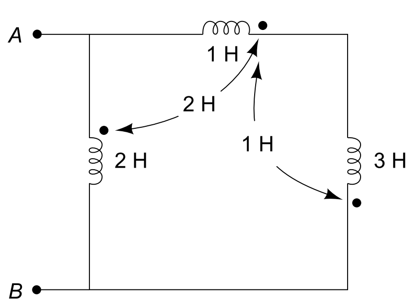


Fig. 4.97

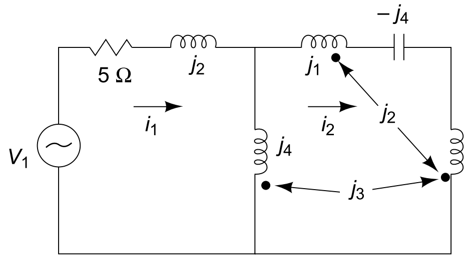


Fig. 4.98

4.11 Find the source voltage if the voltage across the 100 ohms is 50 V for the network in the Fig. 4.99.

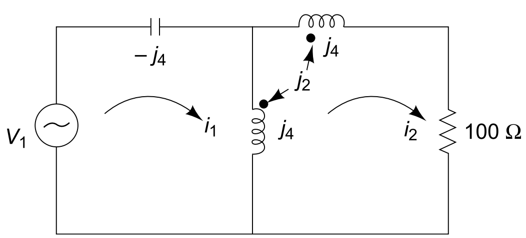


Fig. 4.99

4.12 The inductance matrix for the circuit of a three series connected coupled coils is given below. Find the inductances and indicate the dots for the coils.

$$L = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 4 & -6 \\ 1 & -6 & 6 \end{bmatrix}$$

4.13 For the circuit shown in Fig. 4.100 determine the frequency at which the circuit resonates. Also find the voltage across the capacitor at resonance, and the Q factor of the circuit.

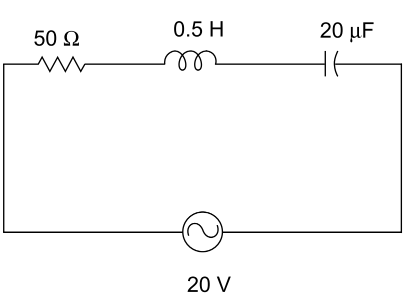


Fig. 4.100

4.14 A series RLC circuit has a quality factor of 10 at 200 rad/sec. The current flowing through the circuit at resonance is 0.5 A and the supply voltage is 10 V. The total impedance of the circuit is 40 Ω. Find the circuit constants.

4.15 The impedance $Z_1 = (5 + j3)\ \Omega$, and $Z_2 = (10 - j30)\ \Omega$. are connected in parallel as shown in Fig. 4.101. Find the value of X_3 which will produce resonance at the terminals *a* and *b*.

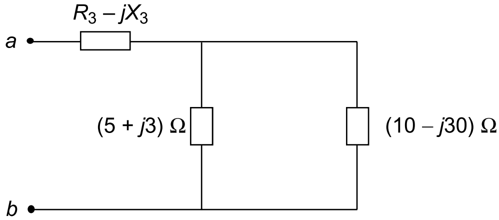


Fig. 4.101

4.16 A RLC series circuit is to be chosen to produce a magnification of 10 at 100 rad/sec. The source can supply a maximum current of 10 A and the supply voltage is 100 V. The power frequency impedance of the circuit should not be more than 14.14 Ω. Find the values of *R*, *L* and *C*.

4.17 A voltage $v(t) = 50 \sin \omega t$ is applied to a series RLC circuit. At the resonant frequency of the circuit, the maximum voltage across the capacitor is found to be 400 V. The bandwidth is known to be 500 rad/sec and the impedance at resonance is 100 Ω. Find the resonant frequency, and compute the upper and lower limits of the bandwidth. Determine the values of *L* and *C* of the circuit.

4.18 A current source is applied to the parallel arrangement of *R*, *L* and *C* where $R = 12\ \Omega$, $L = 2$ H and $C = 3\ \mu$F. Compute the resonant frequency in rad/sec. Find the quality factor. Calculate the value of bandwidth. Compute the lower and upper frequency of the bandwidth. Compute the voltage appearing across the parallel elements when the input signal is $i(t) = 10 \sin 1800\, t$.

4.19 For the circuit shown in Fig. 4.102, determine the value of R_C for which the given circuit resonates.

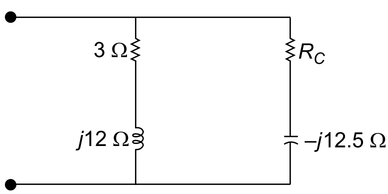


Fig. 4.102

4.20 For the circuit shown in Fig. 4.103 the applied voltage $v(t) = 15 \sin 1800t$. Determine the resonant frequency. Calculate the quality factor and bandwidth. Compute the lower and upper limits of the bandwidth.

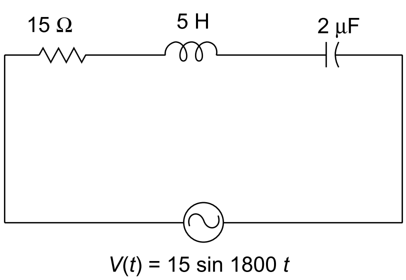


Fig. 4.103

4.21 In the circuit shown in Fig. 4.104, the current is at its maximum value with inductor value $L = 0.5$ H, and 0.707 times of its maximum value with $L = 0.2$ H. Find the value of *Q* at $\omega = 200$ rad/sec and circuit constants.

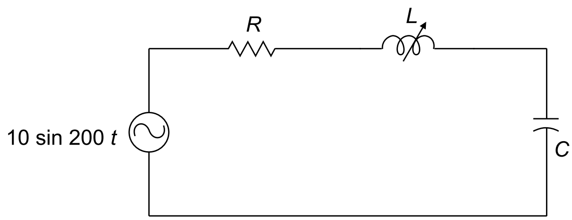


Fig. 4.104

4.22 The voltage applied to the series RLC circuit is 5 V. The Q of the coil is 25 and the value of the capacitor is 200 PF. The resonant frequency of the circuit is 200 kHz. Find the value of inductance, the circuit current and the voltage across the capacitor.

Objective Type Questions

4.1 Mutual inductance is a property associated with
(a) only one coil
(b) two or more coils
(c) two or more coils with magnetic coupling

4.2 Dot convention in coupled circuits is used
(a) to measure the mutual inductance
(b) to determine the polarity of the mutually induced voltage in coils
(c) to determine the polarity of the self induced voltage in coils

4.3 Mutually induced voltage is present independently of, and in addition to, the voltage due to self induction.
(a) true (b) false

4.4 Two terminals belonging to different coils are marked identically with dots, if for the different direction of current relative to like terminals the magnetic flux of self and mutual induction in each circuit add together.
(a) true (b) false

4.5 The maximum value of the coefficient of coupling is
(a) 100% (b) more than 100% (c) 90%

4.6 The case for which the coefficient of coupling $K = 1$ is called perfect coupling
(a) true (b) false

4.7 The maximum possible mutual inductance of two inductively coupled coils with self inductances $L_1 = 25$ mH and $L_2 = 100$ mH is given by
(a) 125 mH (b) 75 mH (c) 50 mH

4.8 The value of the coefficient of coupling is more for aircored coupled circuits compared to the iron core coupled circuits.
(a) true (b) false

4.9 Two inductors are connected as shown in Fig. 4.105. What is the value of the effective inductance of the combination.

(a) 8 H (b) 10 H
(c) 4 H

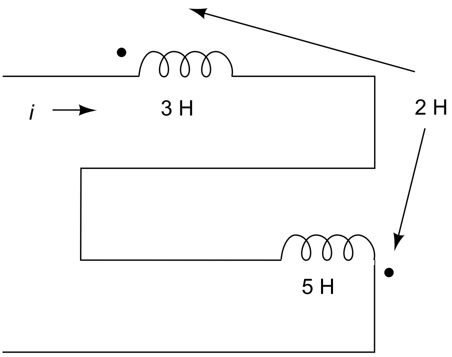


Fig. 4.105

4.10 Two coils connected in series have an equivalent inductance of 3 H when connected in aiding. If the self inductance of the first coil is 1 H, what is the self inductance of the second coil (Assume $M = 0.5$ H)

(a) 1 H (b) 2 H (c) 3 H

4.11 For Fig. 4.106 shown below, the inductance matrix is given by

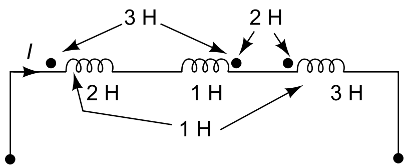


Fig. 4.106

(a) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

4.12 What is the total reactance of a series RLC circuit at resonance?

(a) equal to X_L (b) equal to X_c (c) equal to R (d) zero

4.13 What is the phase angle of a series RLC circuit at resonance?

(a) zero (b) 90° (c) 45° (d) 30°

4.14 In a series circuit of $L = 15$ mH and $C = 0.015$ μF and $R = 80\ \Omega$, what is the impedance at the resonant frequency?

(a) (15 mH) ω (b) (0.015 F) ω (c) 80 Ω (d) 1/(ω × (0.015))

4.15 In a series RLC circuit operating below the resonant frequency, the current

(a) I leads V_S (b) I lags behind V_S (c) I is in phase with V_S

4.16 In a series RLC circuit, if C is increased, what happens to the resonant frequency?

(a) It increases (b) It decreases
(c) It remains the same (d) It is zero

4.17 In a certain series resonant circuit, $V_c = 150$ V, $V_L = 150$ V and $V_R = 50$ V. What is the value of the source voltage?

(a) zero (b) 50 V (c) 150 V (d) 200 V

4.18 A certain series resonant circuit has a bandwidth of 1000 Hz. If the existing coil is replaced by a coil with a lower Q, what happens to the bandwidth?

(a) It increases (b) It decreases
(c) It is zero (d) It remains the same

4.19 In a parallel resonance circuit, why does the current lag behind the source voltage at frequencies below resonance?

(a) because the circuit is predominantly resistive
(b) because the circuit is predominantly inductive
(c) because the circuit is predominantly capacitive
(d) none of the above

4.20 In order to tune a parallel resonant circuit to a lower frequency, the capacitance must

(a) be increased (b) be decreased
(c) be zero (d) remain the same

4.21 What is the impedance of an ideal parallel resonant circuit without resistance in either branch?

(a) zero (b) inductive (c) capacitive (d) infinite

4.22 If the lower cut-off frequency is 2400 Hz and the upper cut-off frequency is 2800 Hz, what is the bandwidth?

(a) 400 Hz (b) 2800 Hz (c) 2400 Hz (d) 5200 Hz

4.23 What values of L and C should be used in a tank circuit to obtain a resonant frequency of 8 kHz? The bandwidth must be 800 Hz. The winding resistance of the coil is 10 V.

(a) 2mH, 1 μF (b) 10 H, 0.2 μF
(c) 1.99 mH, 0.2 μF (d) 1.99 mH, 10 μF

5 Network Theorems

5.1 NETWORK THEOREMS WITH DC EXCITATION

5.1.1 Thevenin's Theorem

[JNTU May/June 2008]

In many practical applications, it is always not necessary to analyse the complete circuit; it requires that the voltage, current, or power in only one resistance of a circuit be found. The use of this theorem provides a simple, equivalent circuit which can be substituted for the original network. Thevenin's theorem states that any two terminal linear network having a number of voltage current sources and resistances can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance, where the value of the voltage source is equal to the open circuit voltage across the two terminals of the network, and resistance is equal to the equivalent resistance measured between the terminals with all the energy sources are replaced by their internal resistances. According to Thevenin's theorem, an equivalent circuit can be found to replace the circuit in Fig. 5.1.

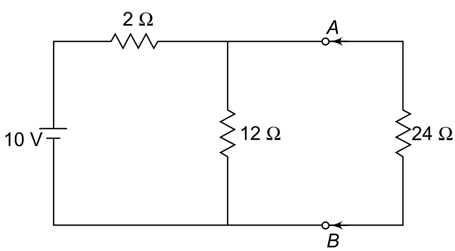


Fig. 5.1

In the circuit, if the load resistance 24 Ω is connected to Thevenin's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experienced in the original circuit. To verify this, let us find the current passing through the 24 Ω resistance due to the original circuit.

$$I_{24} = I_T \times \frac{12}{12+24}$$

where $$I_T = \frac{10}{2+(12 \| 24)} = \frac{10}{10} = 1\,\text{A}$$

$\therefore$ $$I_{24} = 1 \times \frac{12}{12+24} = 0.33\,\text{A}$$

The voltage across the 24 Ω resistor = 0.33 × 24 = 7.92 V. Now let us find Thevenin's equivalent circuit.

The Thevenin voltage is equal to the open circuit voltage across the terminals 'AB', i.e. the voltage across the 12 Ω resistor. When the load resistance is disconnected from the circuit, the Thevenin voltage

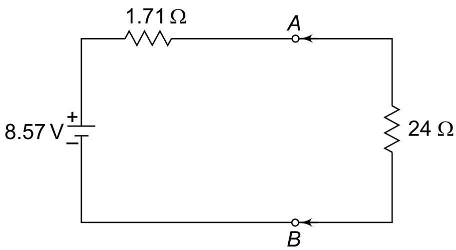


Fig. 5.2

$$V_{\text{Th}} = 10 \times \frac{12}{14} = 8.57\text{ V}$$

The resistance into the open circuit terminals is equal to the Thevenin resistance

$$R_{\text{Th}} = \frac{12 \times 2}{14} = 1.71\,\Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.2.

Now let us find the current passing through the 24 Ω resistance and voltage across it due to Thevenin's equivalent circuit.

$$I_{24} = \frac{8.57}{24+1.71} = 0.33\text{ A}$$

The voltage across the 24 Ω resistance is equal to 7.92 V. Thus, it is proved that R_L (= 24 Ω) has the same values of current and voltage in both the original circuit and Thevenin's equivalent circuit.

Example 5.1 *Determine the Thevenin's equivalent circuit across 'AB' for the given circuit shown in Fig. 5.3.*

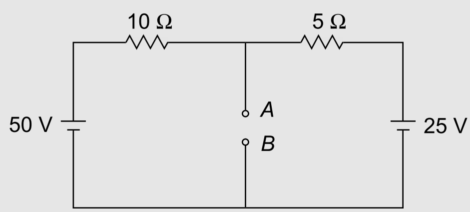


Fig. 5.3

Solution The complete circuit can be replaced by a voltage source in series with a resistance as shown in Fig. 5.4(a)

where V_{Th} is the voltage across terminals AB and

R_{Th} is the resistance seen into the terminals AB.

To solve for V_{Th}, we have to find the voltage drops around the closed path as shown in Fig. 5.4(b).

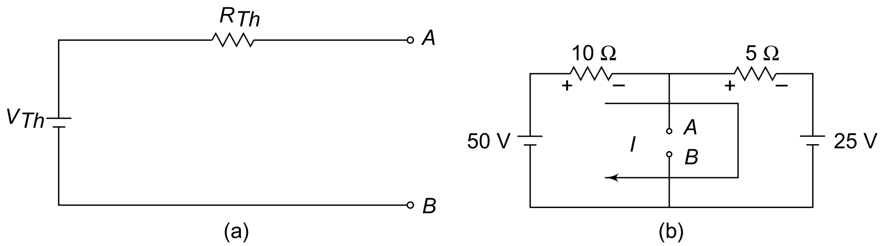


Fig. 5.4

We have $50 - 25 = 10I + 5I$

or $15I = 25$

$\therefore$ $$I = \frac{25}{15} = 1.67\text{ A}$$

Voltage across $10\ \Omega = 16.7$ V
Voltage drop across $5\ \Omega = 8.35$ V

or $$V_{Th} = V_{AB} = 50 - V_{10}$$
$$= 50 - 16.7 = 33.3\text{ V}$$

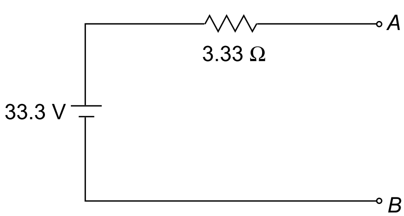


Fig. 5.4(c)

To find R_{Th}, the two voltage sources are removed and replaced with short circuit. The resistance at terminals AB then is the parallel combination of the $10\ \Omega$ resistor and $5\ \Omega$ resistor; or

$$R_{Th} = \frac{10 \times 5}{15} = 3.33\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.4(c).

Example 5.2 *Use Thevenin's theorem to find the current in 3 Ω resistor in Fig. 5.5.*

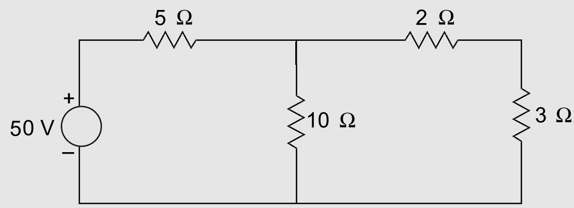


Fig. 5.5

Solution Current in the $3\ \Omega$ resistor can be found by using Thevenin's theorem.

In circuit shown in Fig. 5.6(a) can be replaced by a single voltage source in series with a resistor as shown in Fig. 5.6(b).

$$V_{Th} = V_{AB} = \frac{50}{15} \times 10 = 33.3\text{ V}$$

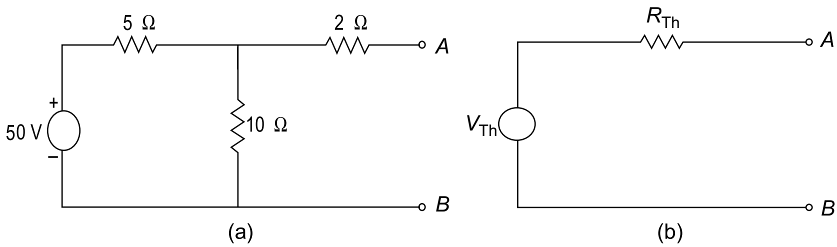


Fig. 5.6

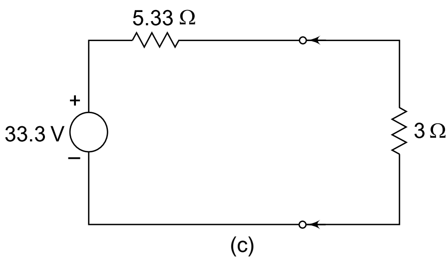


Fig. 5.6

$R_{Th} = R_{AB}$, the resistance seen into the terminals AB

$$R_{AB} = 2 + \frac{5 \times 10}{15} = 5.33\ \Omega$$

The 3 Ω resistor is connected to the Thevenin equivalent circuit as shown in Fig. 5.6.

Current passing through the 3 Ω resistor

$$I_3 = \frac{33.3}{5.33 + 3} = 4.00\ \text{A}$$

Example 5.3 *Use Thevenin's theorem to find the current through the 5 Ω resistor in Fig. 5.7.*

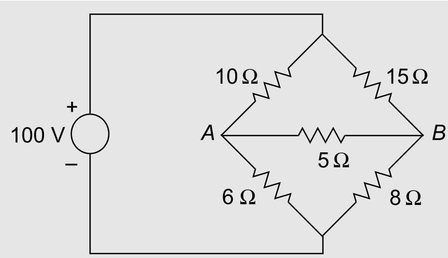


Fig. 5.7

Solution Thevenin's equivalent circuit can be formed by obtaining the voltage across terminals AB as shown in Fig. 5.8(a).

Current in the 6 Ω resistor $\quad I_6 = \frac{100}{16} = 6.25$ A

Voltage across the 6 Ω resistor $V_6 = 6 \times 6.25 = 37.5$ V

Current in the 8 Ω resistor $I_8 = \frac{100}{23} = 4.35$ A

Voltage across the 8 Ω resistor is $V_8 = 4.35 \times 8 = 34.8$ V

Voltage across the terminals AB is $V_{AB} = 37.5 - 34.8 = 2.7$ V

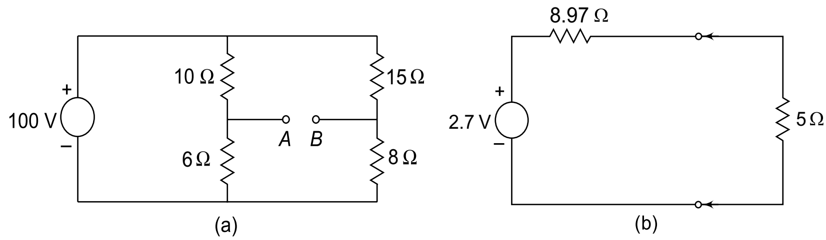


Fig. 5.8

The resistance as seen into the terminals R_{AB}

$$= \frac{6\times10}{6+10} + \frac{8\times15}{8+15}$$

$$= 3.75 + 5.22 = 8.97\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.8(b).

Current in the 5 Ω resistor $I_5 = \dfrac{2.7}{5+8.97} = 0.193$ A

Example 5.4 *Find Thevenin's equivalent circuit for the circuit shown in Fig. 5.9.*

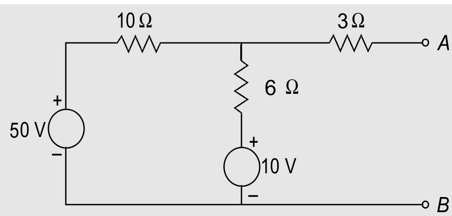


Fig. 5.9

Solution Thevenin's voltage is equal to the voltage across the terminals *AB*.

$$\therefore \qquad V_{AB} = V_3 + V_6 + 10$$

Here the current passing through the 3 Ω resistor is zero.
Hence $\qquad V_3 = 0$
By applying Kirchhoff's law we have

$$50 - 10 = 10I + 6I$$

$$I = \frac{40}{16} = 2.5\text{ A}$$

The voltage across 6 Ω is V_6 with polarity as shown in Fig. 5.10(a), and is given by

$$V_6 = 6 \times 2.5 = 15\text{ V}$$

The voltage across terminals *AB* is $V_{AB} = 0 + 15 + 10 = 25$ V.
The resistance as seen into the terminals *AB*

$$R_{AB} = 3 + \frac{10\times6}{10+6} = 6.75\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.10(b).

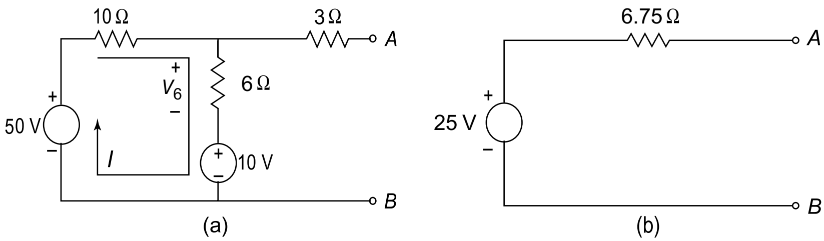


Fig. 5.10

Example 5.5 *Determine the Thevenin's equivalent circuit across terminals AB for the circuit in Fig. 5.11.*

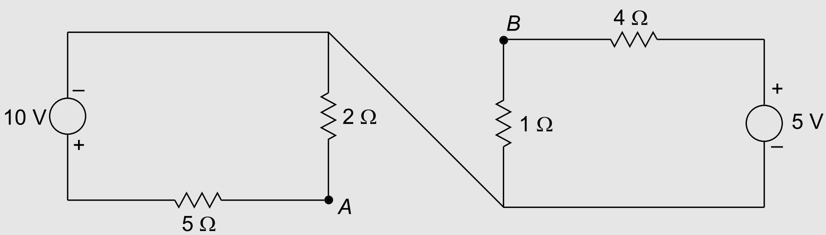


Fig. 5.11

Solution The given circuit is redrawn as shown in Fig. 5.12(a).

Voltage $V_{AB} = V_2 + V_1$

Applying Kirchhoff's voltage law to loop 1 and loop 2, we have the following

Voltage across the 2 Ω resistor $\quad V_2 = 2 \times \frac{10}{7} = 2.85\,\text{V}$

Voltage across the 1 Ω resistor $\quad V_1 = 1 \times \frac{5}{5} = 1\,\text{V}$

$$\therefore \qquad V_{AB} = V_2 + V_1$$

$$= 2.85 - 151.85 \text{ V}$$

The resistance seen into the terminals *AB*

$$R_{AB} = (5 \,||\, 2) + (4 \,||\, 1)$$

$$= \frac{5 \times 2}{5+2} + \frac{4 \times 1}{4+1}$$

$$= 1.43 + 0.8 = 2.23\ \Omega$$

Thevenins's equivalent circuit is shown in Fig. 5.12(b).

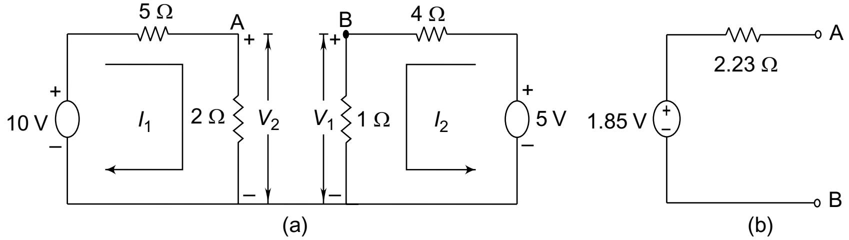


Fig. 5.12

Example 5.6 *For the circuit shown in Fig. 5.13, obtain Thevenin's equivalent circuit.*

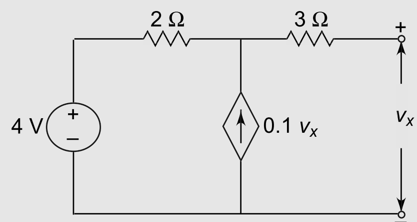


Fig. 5.13

Solution The circuit consists of a dependent source. In the presence of dependent source R_{Th} can be determined by finding v_{OC} and i_{SC}

$$\therefore \qquad R_{\text{Th}} = \frac{v_{OC}}{i_{SC}}$$

Open circuit voltage can be found from the circuit shown in Fig. 5.14(a).
Since the output terminals are open, current passes through the 2 Ω branch only.

$$v_x = 2 \times 0.1\, v_x + 4$$

$$v_x = \frac{4}{0.8} = 5\,\text{V}$$

Short circuit current can be calculated from the circuit shown in Fig. 5.14(b).
Since $v_x = 0$, dependent current source is opened.

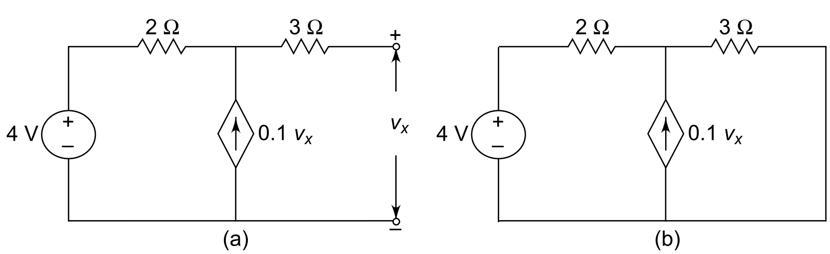
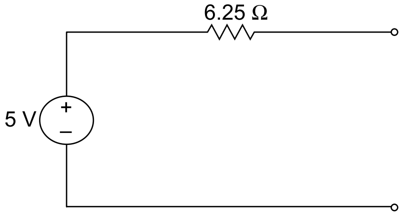


Fig. 5.14



(c)

Fig. 5.14

The current $i_{SC} = \dfrac{4}{2+3} = 0.8$ A

$$\therefore \qquad R_{\text{Th}} = \frac{v_{OC}}{i_{SC}} = \frac{5}{0.8} = 6.25\ \Omega$$

The Thevenin's equivalent circuit is shown in Fig. 5.14(c).

Example 5.7 *For the circuit shown in Fig. 5.15, find the current i_2 in the 2 Ω resistor by using Thevenin's theorem.*

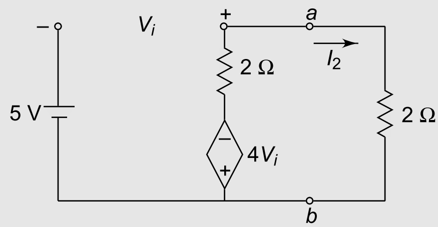


Fig. 5.15

Solution From the circuit, there is open voltage at terminals *AB* which is

$$V_{OC} = -4V_i$$

where $\qquad V_i = -4V_i - 5$

$\therefore \qquad V_i = -1$

Thevenin's voltage $V_{OC} = 4$ V

From the circuit, short circuit current is determined by shorting terminals *a* and *b*. Applying Kirchhoff's voltage law, we have

$$4V_i + 2i_{SC} = 0$$

We know $\qquad V_i = -5$

Substituting V_i in the above equation, we get

$$i_{SC} = 10 \text{ A}$$

$$\therefore \qquad R_{\text{Th}} = \frac{V_{OC}}{i_{SC}} = \frac{4}{10} = 0.4\ \Omega$$

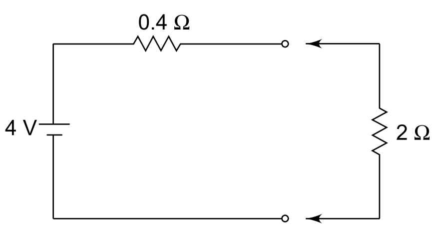


Fig. 5.16

The Thevenin's equivalent circuit is as shown in Fig. 5.16.

The current in the 2 V resistor

$$i_2 = \frac{4}{2.4} = 1.67 \text{ A}$$

Example 5.8 *Find Thevenin's equivalent circuit at terminal AB for the network shown in Fig. 5.17.* *[JNTU April/May 2002]*

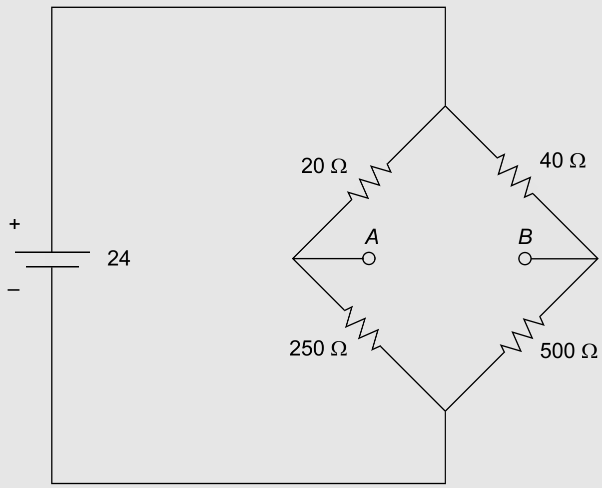


Fig. 5.17

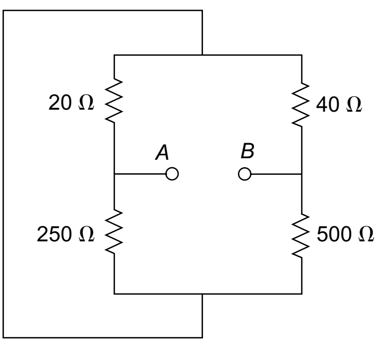


Fig. 5.18

Solution For finding R_{Th}

Remove voltage source and short the terminals

$$R_{AB} = 20 \,||\, 250 + 40 \,||\, 500$$

$$= \frac{20 \times 250}{20 + 250} + \frac{40 \times 500}{40 + 500} = 55.54\ \Omega$$

For finding $V_{OC}\ (V_{AB})$

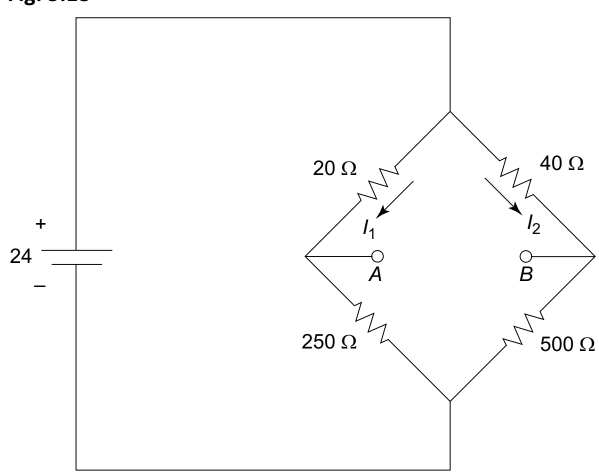


Fig. 5.19

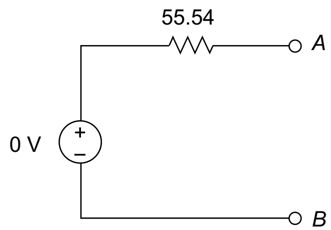


Fig. 5.20

$$I_1 = \frac{24}{20+250} = 0.0888 \text{ A}$$

$$I_2 = \frac{24}{40+500} = 0.0444 \text{ A}$$

For loop OAB apply KVL

$$V_{AB} = 20 \times 0.0888 - 40 \times 0.0444 = 0$$

$$V_{AB} = 0$$

Thevenin's equivalent circuit is given in Fig. 5.20.

Example 5.9 *Find the Thevenin's equivalent for the circuit in Fig. 5.21.*

[JNTU April/May 2003]

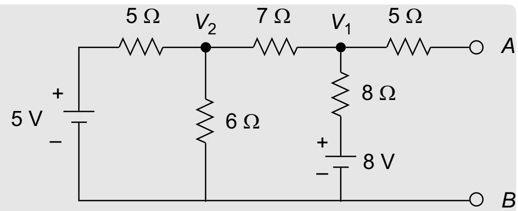


Fig. 5.21

Solution The Thevenin's equivalent resistance is calculated assuming all voltage sources shorted and as seen from *AB*, the circuit will be as shown below:

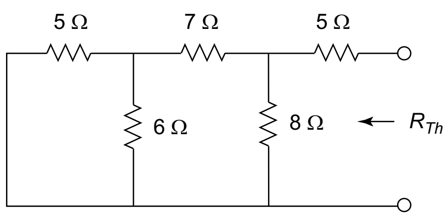


Fig. 5.22

$$R_{Th} = \left[\left\{(5\,//\,6 + 7)\right\}//8\right] + 5$$

$$\left[\left\{\frac{30}{11}+7\right\}//8\right]+5 = \left[\frac{\frac{107}{11}\times 8}{\frac{107}{11}+8}\right]+5$$

$$= 4.389 + 5$$

$$= 9.389 \ \Omega$$

Let us assume voltages at nodes (1) and (2) be V_1 and V_2.
Now writing node equations.

$$\frac{V_1 - 8}{8} + \frac{V_1 - V_2}{7} = 0$$

$$7V_1 - 56 + 8V_1 - 8V_2 = 0 \Rightarrow 15\,V_1 - 8V_2 = 56 \tag{1}$$

$$\frac{V_2}{6} + \frac{V_2 - V_1}{7} + \frac{V_2 - 5}{5} = 0 \Rightarrow -30V_1 + 107V_2 = 210 \tag{2}$$

on solving equations (1) and (2) we get

$V_1 = 5.6 \qquad \Rightarrow \qquad V_{OC} = 5.6$

∴ Thevenin's equivalent circuit is

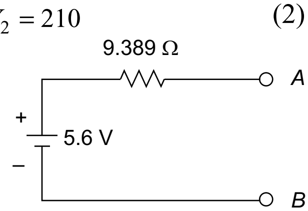


Fig. 5.23

Example 5.10 *Find the current through 10 Ω resistor using Thevenin's theorem.* *[JNTU May/June 2004]*

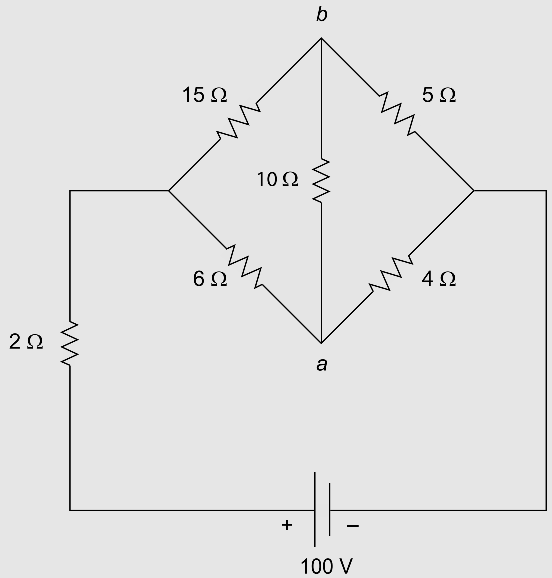


Fig. 5.24

Solution Let us redraw the circuit by removing 10 Ω

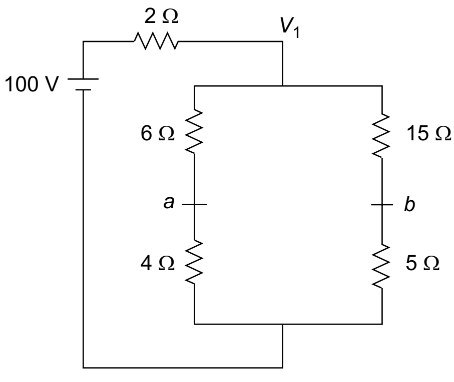


Fig. 5.25

Applying KCL at V_1

$$\frac{V_1-100}{2}+\frac{V_1}{10}+\frac{V_1}{20}=0$$

from which

$$V_1 = 76.92 \text{ V}$$

$$V_{\text{Th}} = V_a - V_b$$

$$=\frac{V_1}{6+4}\times 4-\frac{V_1}{15+5}\times 5$$

$$=11.538 \text{ V}$$

$$R_1=\frac{6\times 15+15\times 2+2\times 6}{2}=66$$

$$R_2=\frac{132}{15}=8.8$$

$$R_3=\frac{132}{6}=22$$

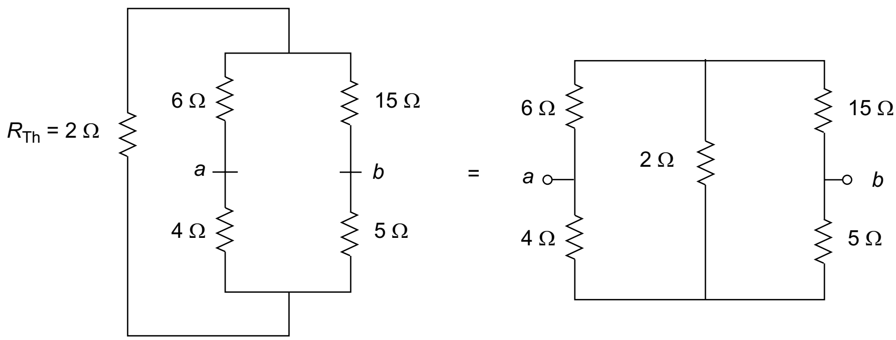


Fig. 5.26

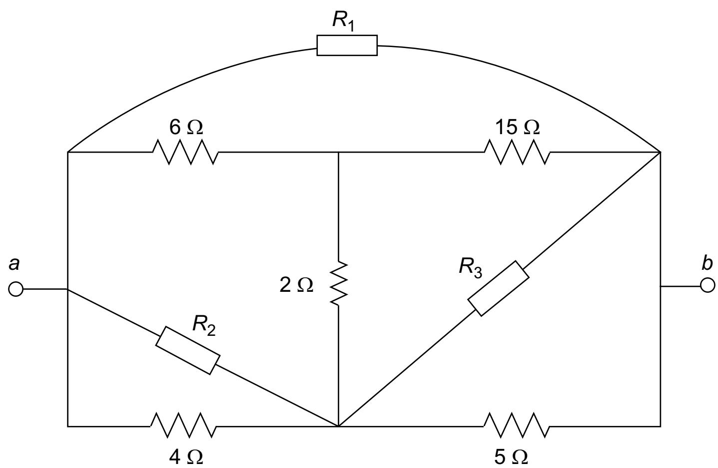


Fig. 5.27

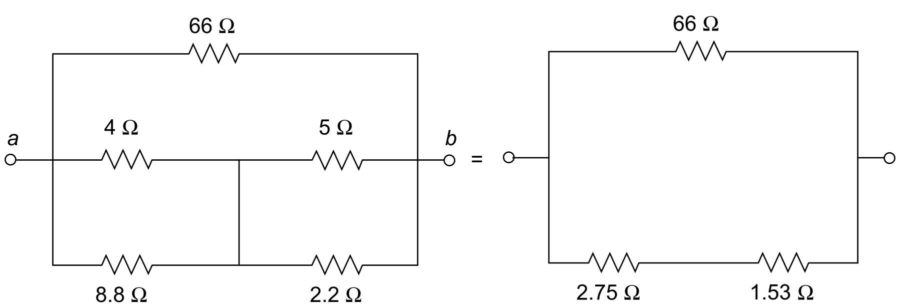


Fig. 5.28

$$R_{ab} = R_{th} = \frac{66 \times 4.28}{70.28} = 4.02\ \Omega$$

Thevenin's equivalent circuit is given by

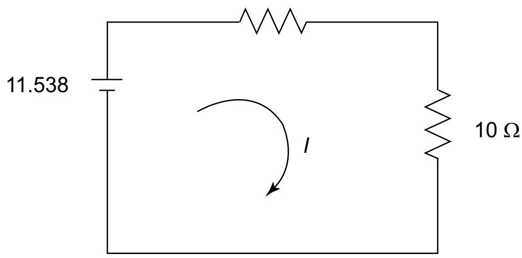


Fig. 5.29

where

$$I = \frac{11.538}{14.02} = 0.823 \text{ A}$$

Example 5.11 *What are the limitations of Thevenin's Theorem?* *[JNTU May/June 2008]*

Solution *Limitations of Thevenin's theorem*:

If there are two sub-networks which are connected between the terminals *AB*, at which we have to replace the Thevenin's network then the independent sources on one network do not depend on the voltages and currents in the other network.

Example 5.12 *Explain the steps to apply Thevenin's theorem and draw the Thevenin's equivalent circuit.* *[JNTU May/June 2008]*

Solution *Steps to apply Thevenin's theorem*:

Let us consider the given circuit.

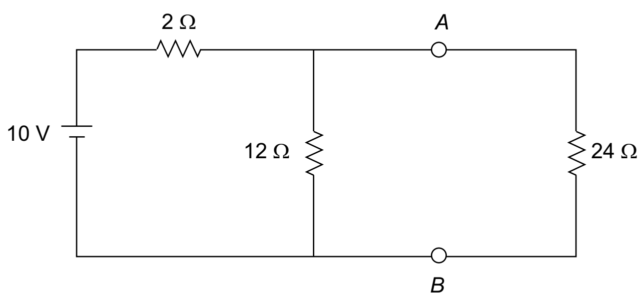


Fig. 5.30

An equivalent circuits should be replaced across *AB*.

In the circuit, if the load resistance of 24 Ω is connected to Thevenin's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experienced in the original circuit. To verify this, let us find the current passing through the 24 Ω resistance due to the original circuit.

$$I_{24} = I_T \times \frac{12}{12+24}$$

$$I_T = \frac{10}{2+(12\|24)} = \frac{10}{10} = 1\text{ A}$$

$$I_{24} = 1 \times \frac{12}{12+24} = 0.33\text{ A}$$

The voltage across the 24 Ω resistor = 0.33 × 24 = 7.92 V.

The Thevenin's voltage is equal to the open circuit voltage across the terminals '*A*' i.e., the voltage across the 12 Ω resistor. When the load resistance is disconnected from the circuit, the Thevenin's voltage

$$V_{th} = 10 \times \frac{12}{14} = 8.57\text{ V}$$

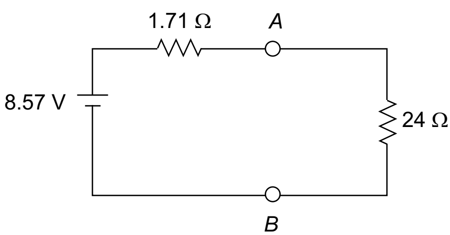


Fig. 5.31

The resistance into the open circuit terminals is equal to the Thevenin resistance

$$R_{\text{Th}} = \frac{12 \times 2}{14} = 1.71\ \Omega$$

Thevenin's equivalent circuits is shown in Fig. 5.31.

The current passing through the 24 Ω resistance and voltage across it due to Thevenin's equivalent circuit

$$I_{24} = \frac{8.57}{24+1.71} = 0.33\text{ A}$$

The voltage across the 24 Ω resistance is equal to 7.92 V. Thus, it is proved that R_L (= 24 Ω) has the same values of current and voltage in both the original circuit and Thevenin equivalent circuit.

Example 5.13 *Using Thevenin's theorem, find the current through 1 Ω resistor in the circuit shown in Fig. 5.32.* *[JNTU May/June 2006]*

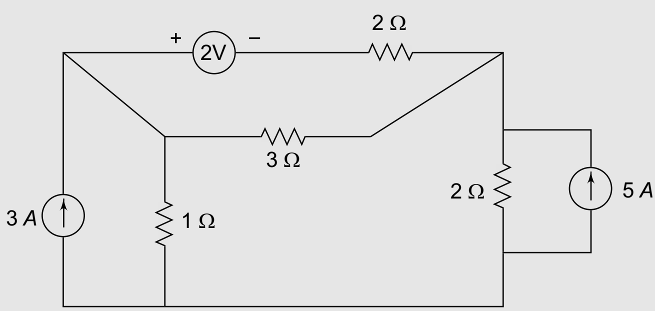


Fig. 5.32

Solution The given circuit is

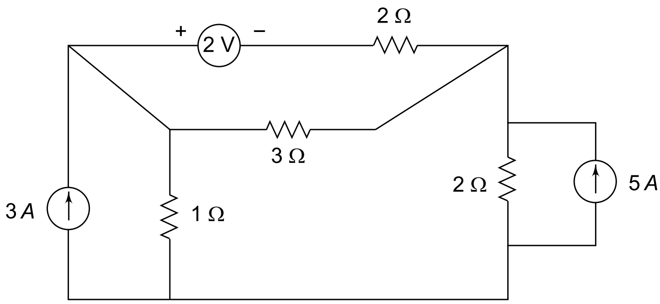


Fig. 5.33

To find R_{Th}

By keeping all the sources to zero, the circuit reduces to

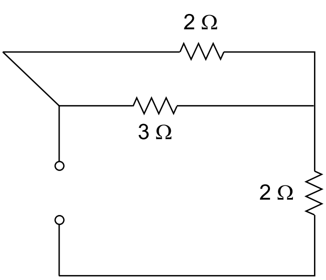


Fig. 5.34

$$R_{Th} = 2||3 + 2$$

$$R_{Th} = \frac{6}{5} + 2$$

$$R_{Th} = \frac{16}{5}$$

To find V_{Th}

Transforming current source of 5 A to voltage source the circuit reduces to

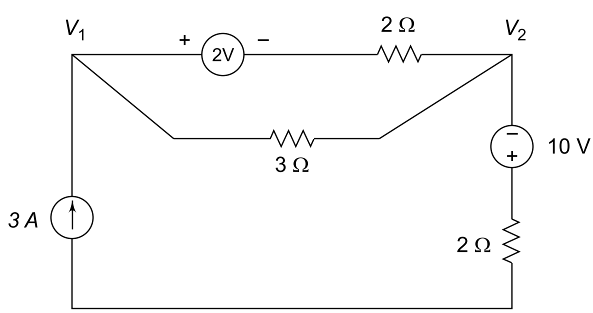


Fig. 5.35

Applying nodal analysis,

$$\frac{V_1 - V_2 - 2}{2} + \frac{V_1 - V_2}{3} = 3$$

$$V_1 - V_2 = \frac{24}{5} \qquad (1)$$

$$\frac{V_2 - V_1 + 2}{2} + \frac{V_2 - V_1}{3} + \frac{V_2 + 10}{2} = 0$$

$$-10\,V_1 + 16\,V_2 = -72 \qquad (2)$$

From (1) and (2)

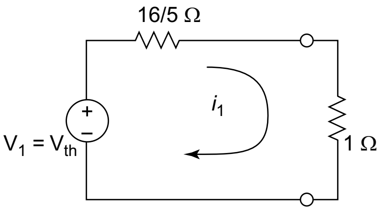


Fig. 5.36

$$V_1 = 0.8 \text{ V}$$

$$V_2 = -4 \text{ V}$$

The Thevenin's circuit with 1 Ω resistance is shown in figure

∴ The current through 1 Ω resistor

$$i_1 = \frac{0.8}{\frac{16}{5} + 1} = 0.19 \text{ A}$$

Example 5.14 *Determine the Thevenin's equivalent across the terminals A and B as shown in Fig. 5.37.* *[JNTU June 2009]*

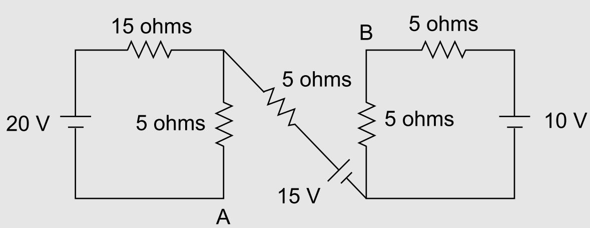


Fig. 5.37

Solution

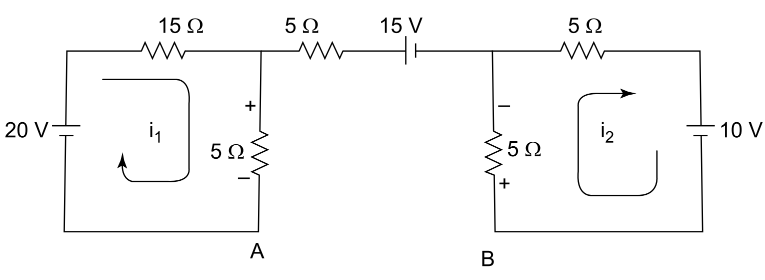


Fig. 5.38

$$i_1 = \frac{20 \text{ V}}{20\ \Omega} = 1 \text{ A}, \qquad i_2 = 1 \text{ A}$$

$$\therefore \quad V_{AB} = -5 \text{ V} + 15 \text{ V} - 5 \text{ V} = +5 \text{ V} = V_{\text{Th}}$$

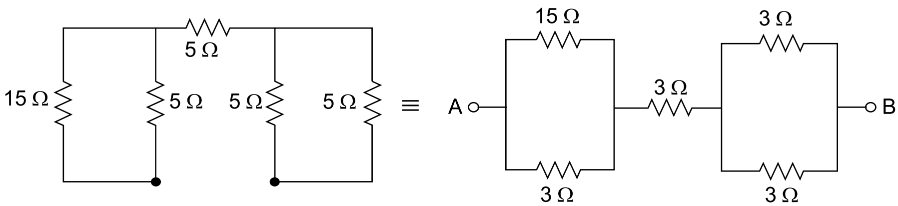


Fig. 5.39

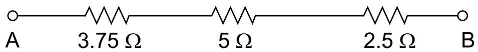


Fig. 5.40

$\therefore \quad R_{AB} = 11.25\,\Omega = R_{\text{Th}}$

The Thevenin's equivalent circuit is

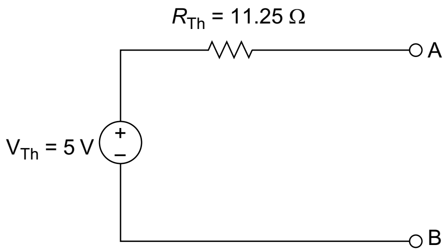


Fig. 5.41

5.1.2 Norton's Theorem

[JNTU Jan 2010]

Another method of analysing the circuit is given by *Norton's theorem*, which states that any two terminal linear network with current sources, voltage sources and resistances can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance. The value of the current source is the short circuit current between the two terminals of the network and the resistance is the equivalent resistance measured between the terminals of the network with all the energy sources are replaced by their internal resistance.

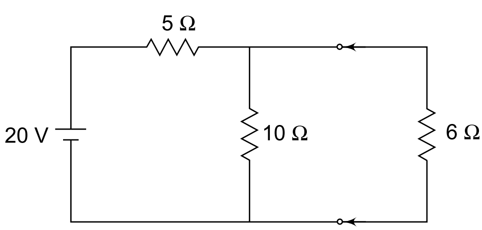


Fig. 5.42

According to Norton's theorem, an equivalent circuit can be found to replace the circuit in Fig. 5.42.

In the circuit if the load resistance $6\,\Omega$ is connected to Norton's equivalent circuit, it will have the same current through it and the same voltage across its terminals as it experiences in the original circuit. To verify this, let us find the current passing through the $6\,\Omega$ resistor due to the original circuit.

$$I_6 = I_T \times \frac{10}{10+6}$$

where $\quad I_T = \dfrac{20}{5+(10\,\|\,6)} = 2.285 \text{ A}$

$$\therefore \quad I_6 = 2.285 \times \frac{10}{16} = 1.43 \text{ A}$$

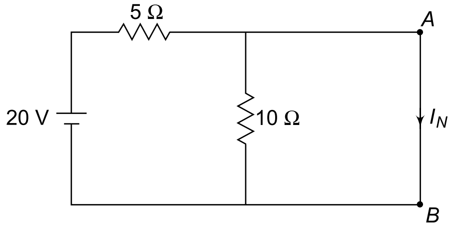


Fig. 5.43

i.e. the voltage across the 6 Ω resistor is 8.58 V. Now let us find Norton's equivalent circuit. The magnitude of the current in the Norton's equivalent circuit is equal to the current passing through short circuited terminals as shown in Fig. 5.43.

Here $$I_N = \frac{20}{5} = 4\text{ A}$$

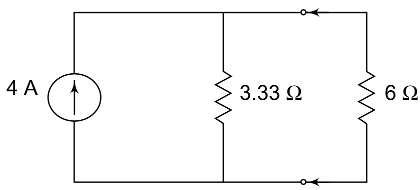


Fig. 5.44

Norton's resistance is equal to the parallel combination of both the 5 Ω and 10 Ω resistors

$$R_N = \frac{5\times 10}{15} = 3.33\ \Omega$$

The Norton's equivalent source is shown in Fig. 5.44.

Now let us find the current passing through the 6 Ω resistor and the voltage across it due to Norton's equivalent circuit.

$$I_6 = 4\times\frac{3.33}{6+3.33} = 1.43\text{ A}$$

The voltage across the 6 Ω resistor = 1.43 × 6 = 8.58 V

Thus, it is proved that R_L (=6 Ω) has the same values of current and voltage in both the original circuit and Norton's equivalent circuit.

Example 5.15 *Determine Norton's equivalent circuit at terminals AB for the circuit shown in Fig. 5.45.*

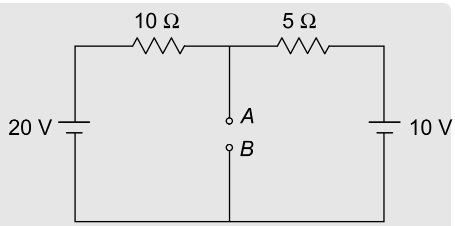
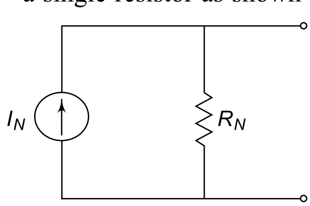


Fig. 5.45

Solution The complete circuit can be replaced by a current source in parallel with a single resistor as shown in Fig. 5.46(a), where I_N is the current passing through the short circuited output terminals AB and R_N is the resistance as seen into the output terminals.



(a)

Fig. 5.46

To solve for I_N, we have to find the current passing through the terminals AB as shown in Fig. 5.46(b).

From Fig. 5.46(b), the current passing through the terminals AB is 4 A. The resistance at terminals AB is the parallel combination of the 10 Ω resistor and the 5 Ω resistor,

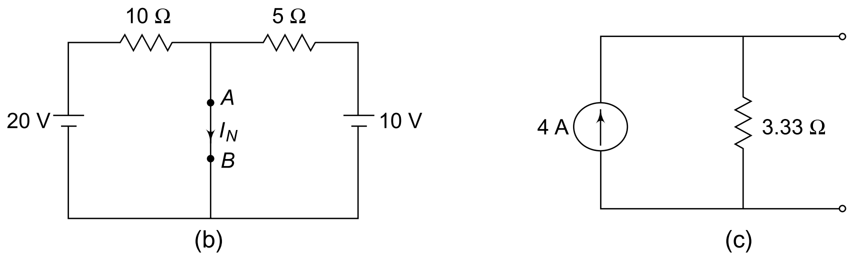


Fig. 5.46

$$\text{or} \qquad R_N = \frac{10 \times 5}{10 + 5} = 3.33\ \Omega$$

Norton's equivalent circuit is shown in Fig. 5.46(c).

Example 5.16 *Determine Norton's equivalent circuit for the circuit shown in Fig. 5.47.*

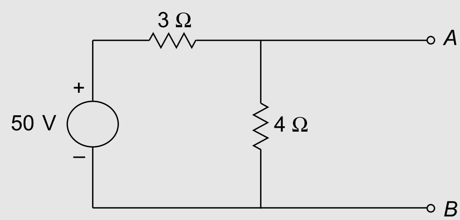


Fig. 5.47

Solution Norton's equivalent circuit is given by Fig. 5.48(a).

where I_N = Short circuit current at terminals AB

R_N = Open circuit resistance at terminals AB

The current I_N can be found as shown in Fig. 5.48(b).

$$I_N = \frac{50}{3} = 16.7\ \text{A}$$

Norton's resistance can be found from Fig. 5.48(c).

$$R_N = R_{AB} = \frac{3 \times 4}{3 + 4} = 1.71\Omega$$

Norton's equivalent circuit for the given circuit is shown in Fig. 5.48(d).

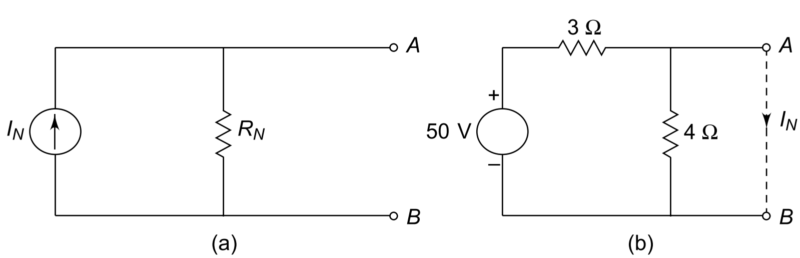


Fig. 5.48

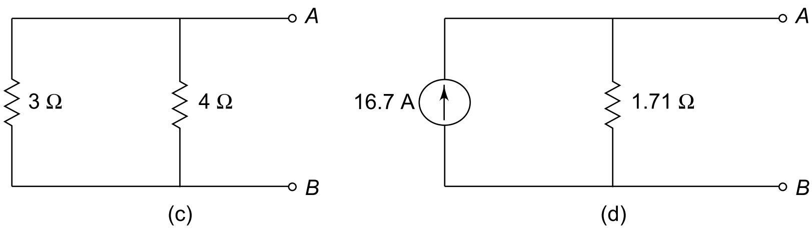


Fig. 5.48

Example 5.17 *Determine Norton's equivalent circuit for the given circuit shown in Fig. 5.49.*

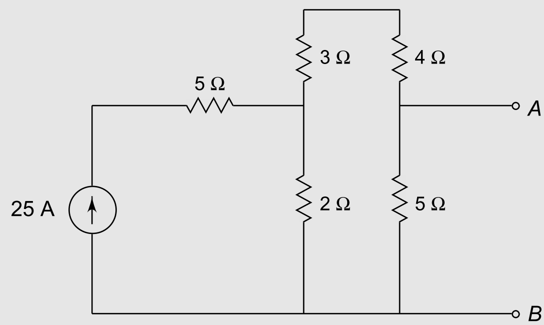


Fig. 5.49

Solution The short circuit current at terminals *AB* can be found from Fig. 5.50(a) and Norton's resistance can be found from Fig. 5.50(b).

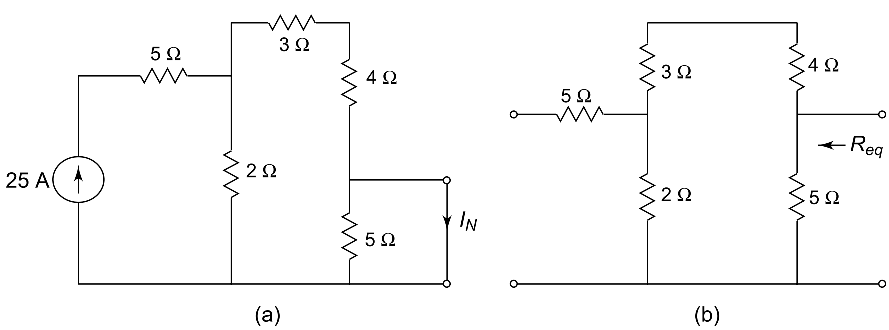


Fig. 5.50

The current I_N is same as the current in the 3 Ω resistor or 4 Ω resistor.

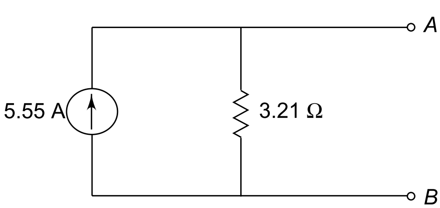


Fig. 5.50(c)

$$I_N = I_3 = 25 \times \frac{2}{7+2} = 5.55\,\text{A}$$

The resistance as seen into the terminals *AB* is

$$R_{AB} = 5 \,||\, (4 + 3 + 2)$$

$$= \frac{5 \times 9}{5 + 9} = 3.21\,\Omega$$

Norton's equivalent circuit is shown in Fig. 5.50(c).

Example 5.18 *Determine the current flowing through the 5 Ω resistor in the circuit shown in Fig. 5.51 by using Norton's theorem.*

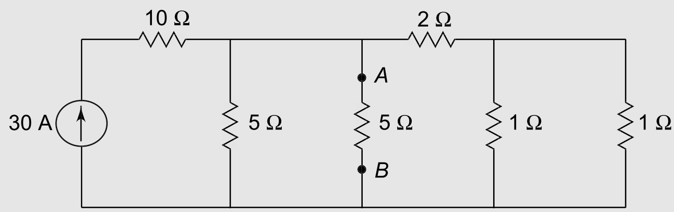


Fig. 5.51

Solution The short circuit current at terminals AB can be found from the circuit as shown in Fig. 5.52(a). Norton's resistance can be found from Fig. 5.52(b).

In Fig. 5.52(a), the current I_N = 30 A.

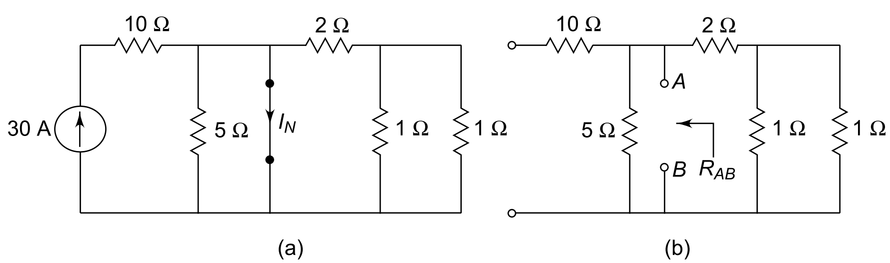


Fig. 5.52

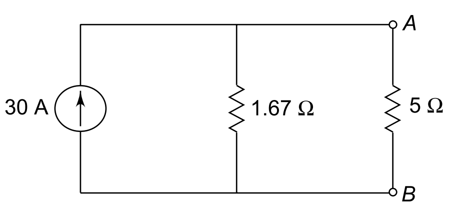


Fig. 5.52(c)

The resistance in Fig. 5.52(b)

$$R_{AB} = 5 \,||\left(2 + \frac{1 \times 1}{2}\right)$$

$$= 5 \,||\, (2.5) = \frac{5 \times 2.5}{7.5} = 1.67\,\Omega$$

Norton's equivalent circuit is shown in Fig. 5.52(c).

$\therefore$ The current in the $5\,\Omega$ resistor

$$I_5 = 30 \times \frac{1.67}{6.67} = 7.51 \text{ A}$$

Example 5.19 *Replace the given network shown in Fig. 5.53 by a single current source in parallel with a resistance.*

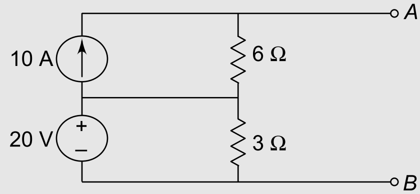


Fig. 5.53

Solution Here, using superposition technique and Norton's theorem, we can convert the given network.

We have to find a short circuit current at terminals AB in Fig. 5.54(a) as shown

The current I'_N is due to the 10 A source $I'_N = 10$ A

The current I''_N is due to the 20 V source (See Figs 5.54(b) and (c))

$$I''_N = \frac{20}{6} = 3.33 \text{ A}$$

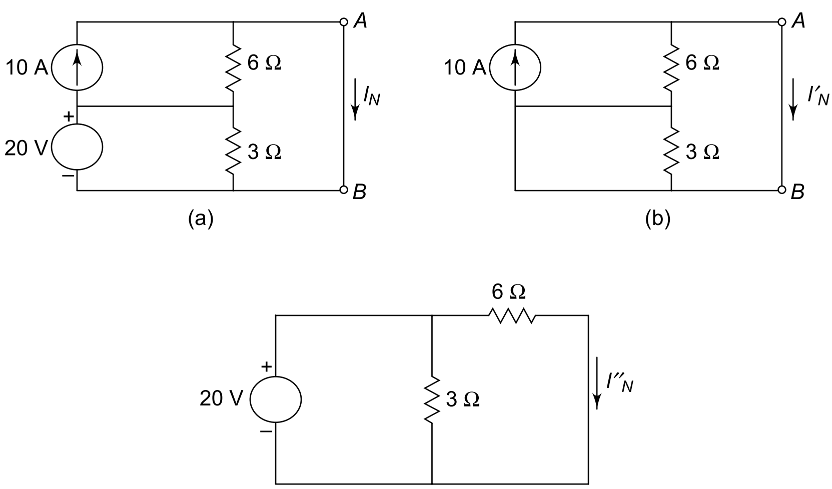


Fig. 5.54

The current I_N is due to both the sources

$$I_N = I'_N + I'_N$$

$$= 10 + 3.33 = 13.33 \text{ A}$$

The resistance as seen from terminals AB

$$R_{AB} = 6\ \Omega \text{ (from the Fig. 5.54(d))}$$

Hence, the required circuit is as shown in Fig. 5.54(e).

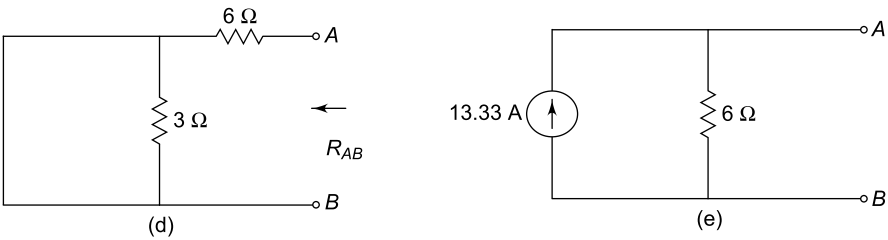


Fig. 5.54

Example 5.20 *For the circuit shown in Fig. 5.55, find Norton's equivalent circuit.*

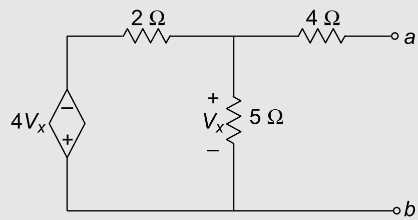


Fig. 5.55

Solution In the case of circuit having only dependent sources (without independent sources), both V_{OC} and i_{SC} are zero. We apply a 1 A source externally and determine the resultant voltage across it, and then find $R_{\text{Th}} = \dfrac{V}{1}$ or we can also apply the 1 V source externally and determine the current through it and then we find $R_{\text{Th}} = 1/i$.

By applying the 1 A source externally as shown in Fig. 5.56(a).

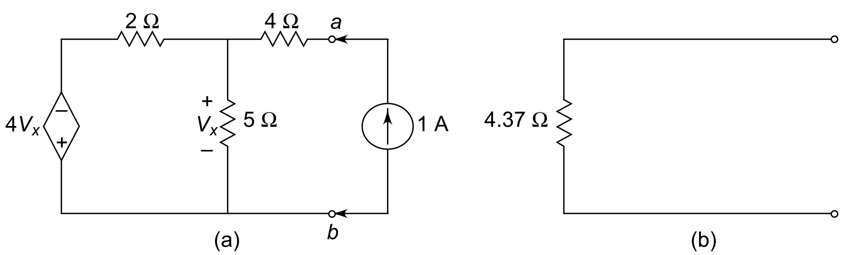


Fig. 5.56

and application of Kirchhoff's current law, we have

$$\frac{V_x}{5} + \frac{V_x + 4V_x}{2} = 1$$

$$V_x = 0.37 \text{ V}$$

The current in the 4 Ω branch is

$$\frac{V_x - V}{4} = -1$$

Substituting V_x in the above equation, we get

$$V = 4.37 \text{ V}$$

$$\therefore \qquad R_{\text{Th}} = \frac{V}{1} = 4.37 \ \Omega$$

If we short circuit the terminals *a* and *b* we have

$$\frac{V_x - 4V_x}{2} = 0$$

$$V_x = 0$$

$$I_{SC} = \frac{V_x}{4} = 0$$

Therefore, Norton's equivalent circuit is as shown in Fig. 5.56(b).

Example 5.21 *Obtain the Norton's equivalent circuit at the terminals A, B for the following Fig. 5.57.*

[JNTU April/May 2003]

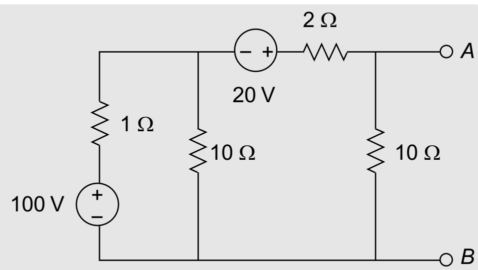


Fig. 5.57

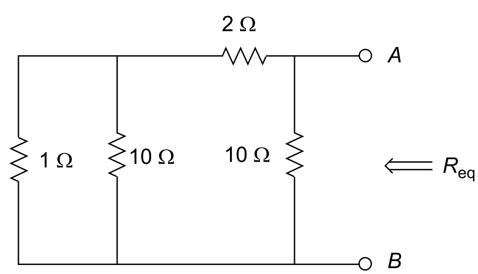


Fig. 5.58

Solution For finding the Norton's resistance, replace the voltage sources by the short circuit.

$$R_{eq} = \{[(1||10) + 2]||10\}$$

$$= 2.253 \ \Omega$$

For finding the I_N short the terminals *A* and *B* find current I_N. Apply superposition

(i) With 100 V source

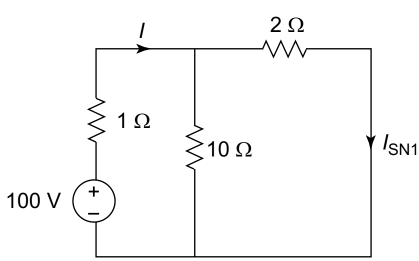


Fig. 5.59

$$Z = [(2||10) + 1] = 2.67$$

$$I_{SN1} = \frac{100}{Z}$$

$$= \frac{100}{2.67} = 37.45 \text{ A}$$

(ii) With 20 V source

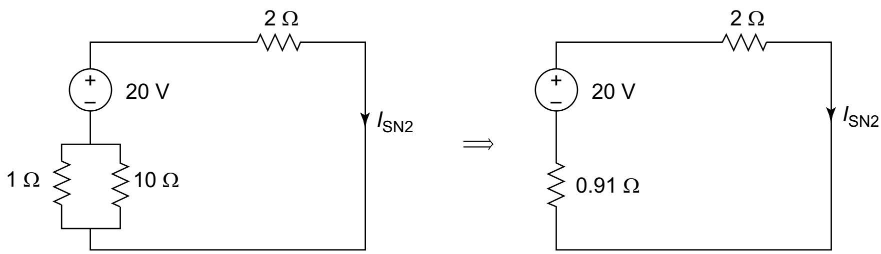


Fig. 5.60

$$I_{SN2} = \frac{20}{2.91} = 6.872 \text{ A}$$

$$\therefore \quad I_{SN} = I_{SN1} + I_{SN2}$$

$$= 37.45 + 6.872$$

$$= 44.322 \text{ A}$$

∴ Nortons equivalent circuit is shown in Fig. 5.61.

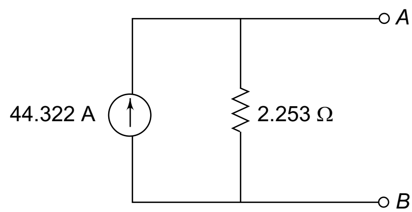


Fig. 5.61

Example 5.22 *Find the Norton's equivalent across the terminals ab as shown in Fig. 5.62. Hence find current through 10 ohms.* *[JNTU June 2009]*

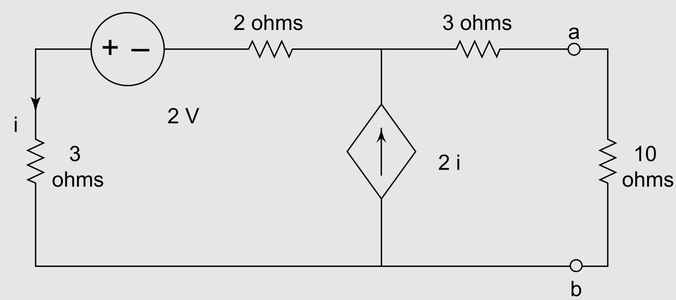


Fig. 5.62

Solution Short circuiting a-b terminal–

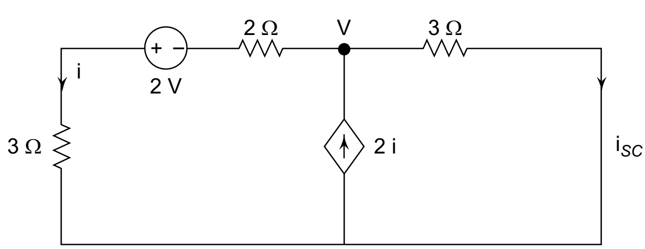


Fig. 5.63

$$2i = i + i_{SC} \Rightarrow i_{SC} = i$$

$$i = \frac{2+V}{5} = \frac{V}{3}$$

or, $6 + 3\,V = 5\,V$

or, $6 = 2\,V$

or, $V = 3$

$$\therefore \quad i = \frac{2+3}{5} = 1 \text{ amp}$$

$$\therefore \quad i_{SC} = 1 \text{ amp}$$

Open circuiting a-b terminal and deactivating independent voltage source–

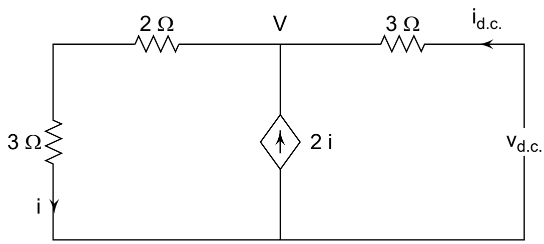


Fig. 5.64

$$2i + i_{d.c.} = i$$

$$\therefore \quad i_{d.c.} = -i$$

Now, $\frac{V}{5} = i$ or, $V = 5i$

Now, $\frac{V_{d.c.} - V}{3} = i_{d.c.}$

or, $\frac{V_{d.c.} - 5i}{3} = -i$

or, $V_{d.c.} - 5i = -3i$

or, $V_{d.c.} = 2i = -2i_{d.c.}$

$$\therefore \quad R_{\text{int}} = 2 \text{ ohm}$$

5.1.3 Millman's Theorem *[JNTU June 2009]*

Millman's theorem states that in any network, if the voltage sources $V_1, V_2, \ldots V_n$ in series with internal resistances $R_1, R_2, \ldots R_n$, respectively, are in parallel, then these sources may be replaced by a single voltage source V' in series with R' as shown in Fig. 5.65.

where $$V' = \frac{V_1G_1 + V_2G_2 + \cdots V_nG_n}{G_1 + G_2 + \ldots + G_n}$$

Here G_n is the conductance of the nth branch,

and $$R' = \frac{1}{G_1 + G_2 + \cdots + G_n}$$

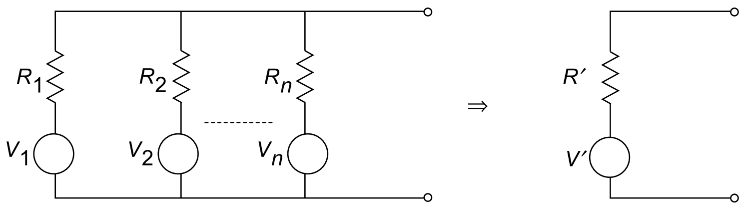


Fig. 5.65

A similar theorem can be stated for n current sources having internal conductances which can be replaced by a single current source I' in parallel with an equivalent conductance.

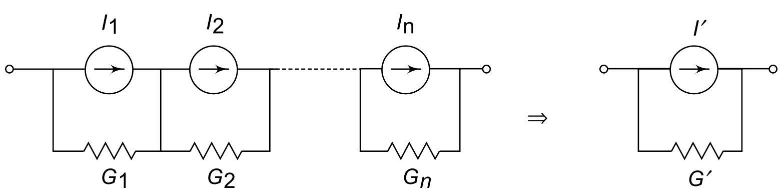


Fig. 5.66

where $$I' = \frac{I_1R_1 + I_2R_2 + \cdots I_nR_n}{R_1 + R_2 + \cdots + R_n}$$

and $$G' = \frac{1}{R_1 + R_2 + \cdots + R_n}$$

Example 5.23 *Calculate the current I shown in Fig. 5.67 using Millman's Theorem.*

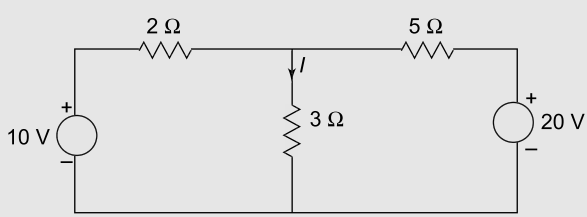


Fig. 5.67

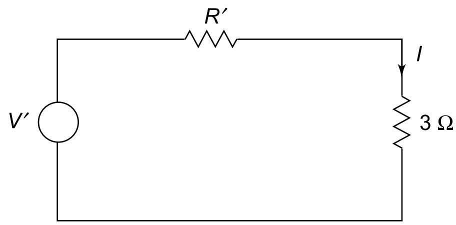


Fig. 5.68

Solution According to Millman's theorem, the two voltage sources can be replaced by a single voltage source in series with resistance as shown in Fig. 5.68.

we have $V' = \dfrac{V_1G_1 + V_2G_2}{G_1 + G_2}$

$$= \frac{[10(1/2) + 20(1/5)]}{1/2 + 1/5} = 12.86 \text{ V}$$

and $R' = \dfrac{1}{G_1 + G_2} = \dfrac{1}{1/2 + 1/5} = 1.43\ \Omega$

Therefore, the current passing through the 3 Ω resistor is

$$I = \frac{12.86}{3 + 1.43} = 2.9 \text{ A}$$

Example 5.24 *Find the current I_L. Use Millman's theorem.*
[JNTU May/June 2004]

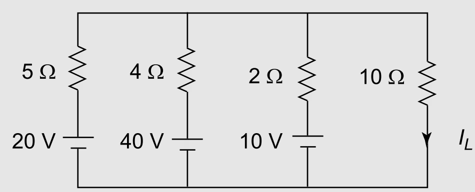


Fig. 5.69

Solution From Millman's theorem,

$$V' = \frac{V_1G_1 + V_2G_2 + \ldots + V_nG_n}{G_1 + G_2 + \ldots + G_n}$$

$$R' = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

$$\therefore\ V' = \frac{20 \times \frac{1}{5} + 40 \times \frac{1}{4} + \left(-10 \times \frac{1}{2}\right)}{\frac{1}{5} + \frac{1}{4} + \frac{1}{2}} = 9.47736$$

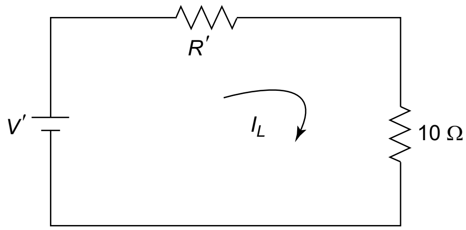


Fig. 5.70

$$R' = \frac{1}{\frac{1}{5}+\frac{1}{4}+\frac{1}{2}} = 1.0526$$

$$I_L = \frac{9.4736}{1.0526+10} = 0.857\text{A}$$

5.1.4 Reciprocity Theorem

[JNTU June 2009]

In any linear bilateral network, if a single voltage source V_a in branch 'a' produces a current I_b in branch 'b', then if the voltage source V_a is removed and inserted in branch 'b' will produce a current I_b in branch 'a'. The ratio of response to excitation is same for the two conditions mentioned above. This is called the *reciprocity theorem*.

Consider the network shown in Fig. 5.71. AA' denotes input terminals and BB' denotes output terminals.

The application of voltage V across AA' produces current I at BB'. Now if the positions of the source and responses are interchanged, by connecting the voltage source across BB', the resultant current I will be at terminals AA'. According to the reciprocity theorem, the ratio of response to excitation is the same in both cases.

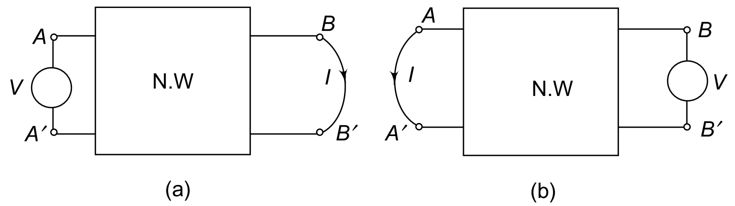


Fig. 5.71

Example 5.25 *Verify the reciprocity theorem for the network shown in Fig. 5.72.*

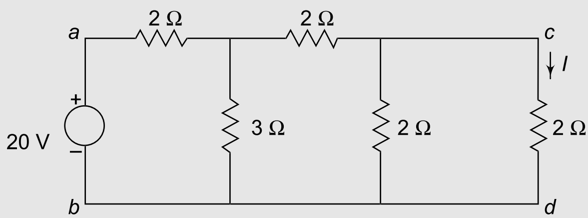


Fig. 5.72

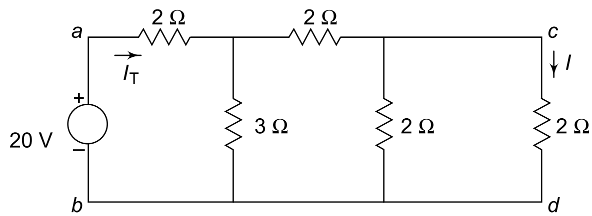
Solution Total resistance in the circuit = 2 + [3 || (2 + 2 || 2)] = 3.5 Ω.

The current drawn by the circuit (See Fig. 5.73(a)).

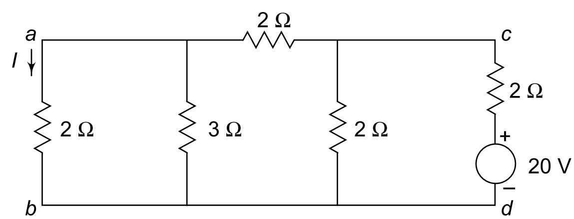
$$I_T = \frac{20}{3.5} = 5.71\ \Omega$$

The current in the 2 Ω branch *cd* is I = 1.43 A.

Applying the reciprocity theorem, by interchanging the source and response we get (See Fig. 5.73(b)).



(a)



(b)

Fig. 5.73

Total resistance in the circuit = 3.23 Ω.

$$\text{Total current drawn by the circuit } = \frac{20}{3.23} = 6.19\,\text{A}$$

The current in the branch *AB* is I = 1.43 A

If we compare the results in both cases, the ratio of input to response is the same, i.e. (20/1.43) = 13.99.

Example 5.26 *Verify the reciprocity theorem for the given circuit shown in Fig. 5.74.*

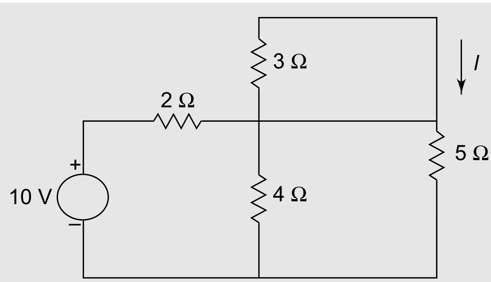


Fig. 5.74

Solution In Fig. 5.74, the current in the 5 Ω resistor is

$$I_5 = I_2 \times \frac{4}{8+4} = 2.14 \times \frac{4}{12} = 0.71\ \text{A}$$

where $I_2 = \dfrac{10}{R_T}$

and $R_T = 4.67$

$\therefore \quad I_2 = \dfrac{10}{4.67} = 2.14 \text{ A}$

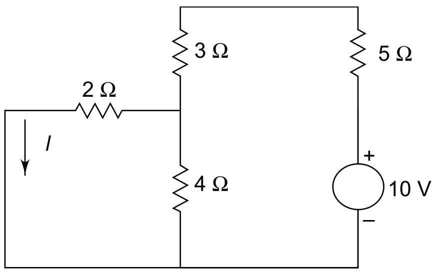


Fig. 5.75

We interchange the source and response as shown in Fig. 5.75.

In Fig. 5.75, the current in 2 Ω resistor is

$$I_2 = I_3 \times \frac{4}{4+2}$$

where $I_3 = \dfrac{10}{R_T}$ and $R_T = 9.33\ \Omega$

$$\therefore \quad I_3 = \frac{10}{9.33} = 1.07 \text{ A}$$

$$I_2 = 1.07 \times \frac{4}{6} = 0.71 \text{ A}$$

In both cases, the ratio of voltage to current is $\dfrac{10}{0.71} = 14.08$.

Hence the reciprocity theorem is verified.

Example 5.27 *Verify the reciprocity theorem in the circuit shown in Fig. 5.76.*

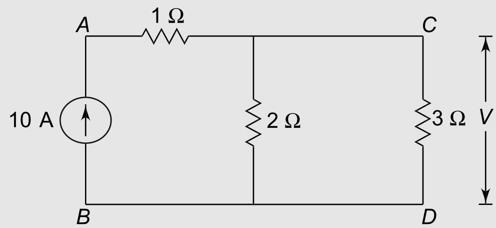


Fig. 5.76

Solution The voltage V across the 3 Ω resistor is

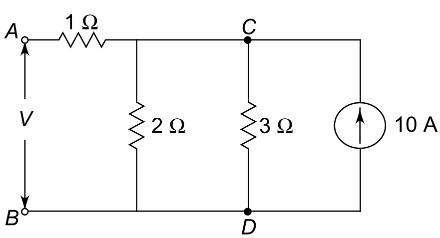


Fig. 5.77

$$V = I_3 \times R$$

where $I_3 = 10 \times \dfrac{2}{2+3} = 4 \text{ A}$

$$\therefore \quad V = 4 \times 3 = 12 \text{ V}$$

We interchange the current source and response as shown in Fig. 5.77.

To find the response, we have to find the voltage across the 2 Ω resistor

$$V = I_2 \times R$$

where $$I_2 = 10 \times \frac{3}{5} = 6 \text{ A}$$

$\therefore$ $$V = 6 \times 2 = 12 \text{ V}$$

In both cases, the ratio of current to voltage is the same, i.e. it is equal to 0.833. Hence the reciprocity theorem is verified.

Example 5.28 *Verify reciprocity theorem in circuit shown in Fig. 5.78.* *[JNTU April/May 2003]*

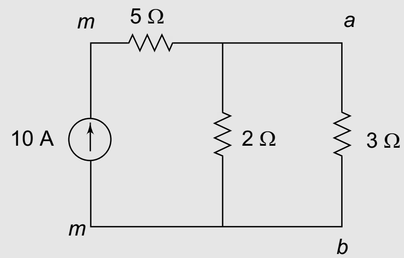


Fig. 5.78

Solution Let us find current in 3 V resistor.

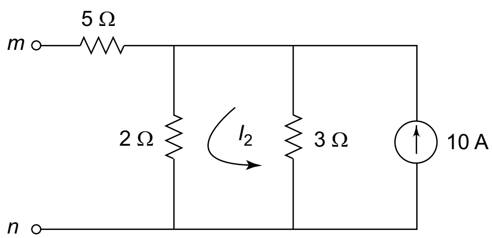


Fig. 5.79

$$I_3 = 10 \times \frac{2}{2+3} = 4 \text{ A}$$

$$V_{ab} = 3 \times 4 = 12$$

According to reciprocity theorem the voltage across AB $V_{ab} = 12$

Now connect the current source across AB and find the voltage across m and n.

$$I_2 = 10 \times \frac{3}{5} = 6 \text{ A}$$

The voltage across $mn = 2 \times 6 = 12$ volts, same as V_{ah}. Hence, the reciprocity theorem is proved.

Example 5.29 *Verify reciprocity theorem for the network shown in Fig. 5.80.* *[JNTU May/June 2006]*

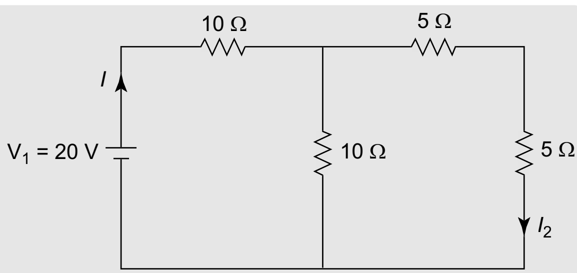


Fig. 5.80

Solution Reciprocity theorem states that in any passive linear bilateral single source network interchanging the positions of ideal voltage source and an ammeter does not change the ammeter reading (current) and interchanging the positions of current source and voltmeter does not change voltmeter reading (voltmeter).

Verifying theorem for the circuit shown in Fig. 5.81

$$I = \frac{20}{10+5} = \frac{4}{3} \qquad \therefore\ I_2 = \frac{2}{3}$$

(Current divider rule)

Interchanging the voltage source,

$$I = \frac{20}{15} = \frac{4}{3} \qquad \Rightarrow \quad I_1 = \frac{2}{3}$$

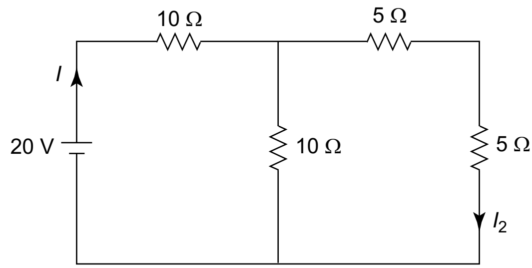


Fig. 5.81

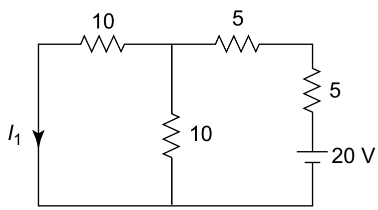


Fig. 5.82

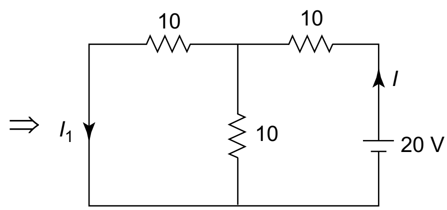


Fig. 5.83

∴ The ratio of excitation to response when only one excitation is applied is constant when positions of excitation and response are interchanged. Hence reciprocity theorem is verified.

Example 5.30 *Verify reciprocity theorem for the voltage V and current I in the network shown in Fig. 5.84.*

[JNTU Jan 2010]

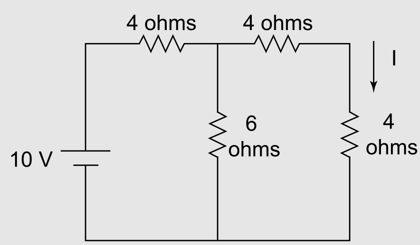


Fig. 5.84

Solution

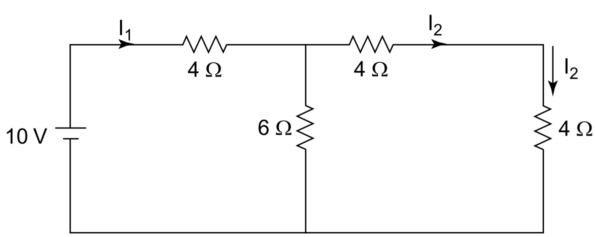


Fig. 5.85

$$\therefore \quad R_{eq} = [(4+4) \| 6] + 4$$

$$\therefore \quad R_{eq} = \frac{8 \times 6}{14} + 4$$

$$R_{eq} = \frac{48}{14} + 4$$

$$R_{eq} = \frac{24}{7} + 4$$

$$R_{eq} = \frac{52}{7}\,\Omega$$

$$I_1 = \frac{10 \times 7}{52} = \frac{70}{52} = \frac{35}{76}$$

$$= 1.346\,\text{A}$$

$$\therefore \quad I_2 = 1.346 \times \frac{6}{6+4+4}$$

$$I_2 = 1.346 \times \frac{6}{14}$$

$$= 0.576\,\text{A}$$

$$I_2 = I = 0.576\text{ A}$$

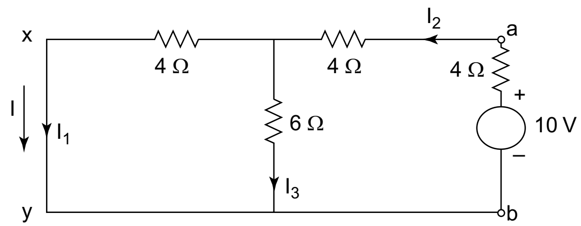


Fig. 5.86

$$R_{eq} = (4||6) + 4 + 4$$

$$\therefore \quad R_{eq} = \frac{24}{10} + 8$$

$$\therefore \quad R_{eq} = 2.4 + 8 = 10.4\ \Omega$$

$$I_2 = \frac{10}{10.4} = 0.961\,\text{A}$$

$$\therefore \quad I_1 = 0.961 \times \frac{6}{6+4}$$

$$= 0.961 \times \frac{3}{5}$$

$$I_1 = 0.576\text{ A}$$

$$\therefore \qquad I_1 = I = 0.576\text{ A}$$

5.1.5 Compensation and Substitution Theorem

The *compensation theorem* states that any element in the linear, bilateral network, may be replaced by a voltage source of magnitude equal to the current passing through the element multiplied by the value of the element, provided the currents and voltages in other parts of the circuit remain unaltered. Consider the circuit shown in Fig. 5.87(a). The element R can be replaced by voltage source V, which is equal to the current I passing through R multiplied by R as shown in Fig. 5.87(b).

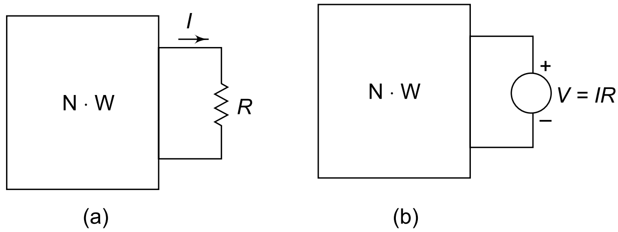


Fig. 5.87

This theorem is useful in finding the changes in current or voltage when the value of resistance is changed in the circuit. Consider the network containing a resistance R shown in Fig. 5.88(a). A small change in resistance R, that is $(R + \Delta R)$, as shown in Fig. 5.88(b) causes a change in current in all branches. This current increment in other branches is equal to the current produced by the voltage source of voltage I. ΔR which is placed in series with altered resistance as shown in Fig. 5.88(c).

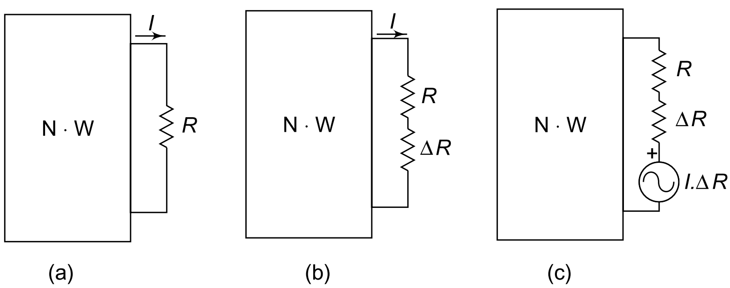


Fig. 5.88

Example 5.31 *Determine the current flowing in the ammeter having 1 Ω internal resistance connected in series with a 3 Ω resistor as shown in Fig. 5.89.*

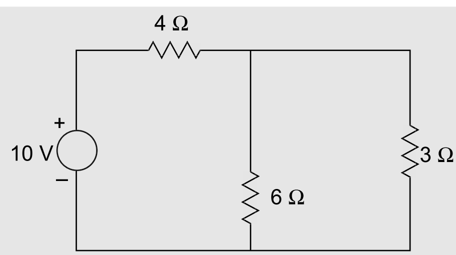


Fig. 5.89

Solution The current flowing through the 3 Ω branch is $I_3 = 1.11$ A. If we connect the ammeter having 1 Ω resistance to the 3 Ω branch, there is a change in resistance. The changes in currents in other branches then result as if a voltage source of voltage $I_3\ \Delta R = 1.11 \times 1 = 1.11$ V is inserted in the 3 Ω branch as shown in Fig. 5.90.

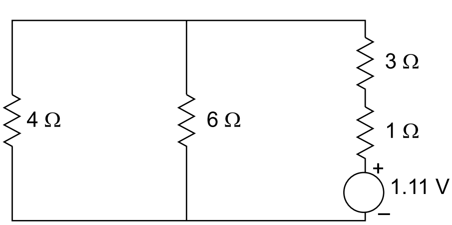


Fig. 5.90

Current due to this 1.11 V source is calculated as follows.

$$\text{Current } I'_3 = 0.17 \text{ A}$$

This current is opposite to the current I_3 in the 3 Ω branch.

Hence the ammeter reading = (1.11 − 0.17) = 0.94 A.

Example 5.32 *Using the compensation theorem, determine the ammeter reading where it is connected to the 6 Ω resistor as shown in Fig. 5.91. The internal resistance of the ammeter is 2 Ω.*

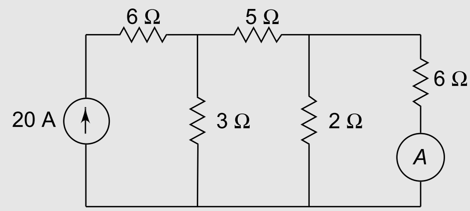


Fig. 5.91

Solution The current flowing through the 5 Ω branch

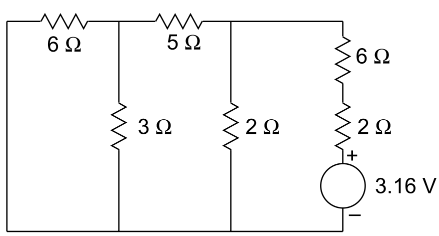


Fig. 5.92

$$I_5 = 20 \times \frac{3}{3+6.5} = 6.315 \text{ A}$$

So the current in the 6 Ω branch

$$I_6 = 6.315 \times \frac{2}{6+2} = 1.58 \text{ A}$$

If we connect the ammeter having 2 Ω internal resistance to the 6 Ω branch, there is a change in resistance. The changes in currents in other branches results if a voltage source of voltage $I_6 \Delta R = 1.58 \times 2 = 3.16$ V is inserted in the 6 Ω branch as shown in Fig. 5.92.

The current due to this 3.16 V source is calculated.

The total impedance in the circuit

$$R_T = \{[(6 \,||\, 3) + 5] \,||\, [2]\} + \{6 + 2\}$$
$$= 9.56\ \Omega$$

The current due to 3.16 V source

$$I'_6 = \frac{3.16}{9.56} = 0.33 \text{ A}$$

This current is opposite to the current I_6 in the 6 Ω branch.

Hence, the ammeter reading = (1.58 − 0.33)
= 1.25 A

5.1.6 Superposition Theorem

The superposition theorem states that in any linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone, while the other sources are non-operative; that is, while considering the effect of individual sources, other ideal voltage sources and ideal current sources in the network are replaced by short circuit and open circuit across their terminals. This theorem is valid only for linear systems. This theorem can be better understood with a numerical example.

Consider the circuit which contains two sources as shown in Fig. 5.93.

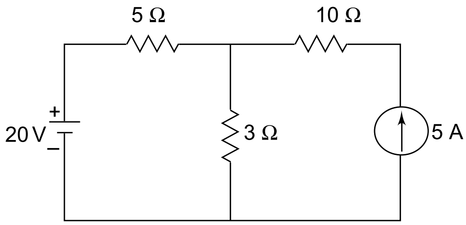


Fig. 5.93

Now let us find the current passing through the 3 Ω resistor in the circuit. According to superposition theorem, the current I_2 due to the 20 V voltage source with 5 A source open circuited $= 20/(5 + 3) = 2.5$ A.

(See Fig. 5.94)

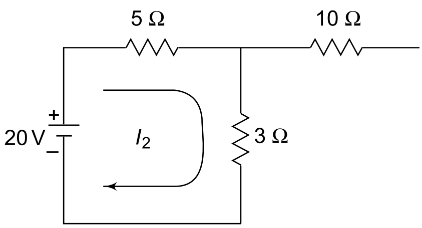


Fig. 5.94

The current I_5 due to 5 A source with 20 V source short circuited is

$$I_5 = 5 \times \frac{5}{(3+5)} = 3.125 \text{ A}$$

The total current passing through the 3 Ω resistor is

$$(2.5 + 3.125) = 5.625 \text{ A}$$

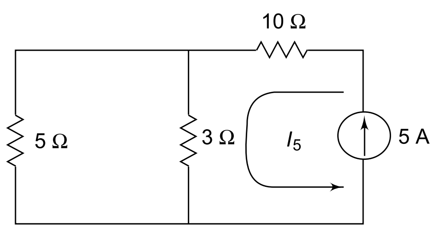


Fig. 5.95

Let us verify the above result by applying nodal analysis.

The current passing in the 3 Ω resistor due to both sources should be 5.625 A.

Applying nodal analysis to Fig. 5.96, we have

$$\frac{V-20}{5} + \frac{V}{3} = 5$$

$$V\left[\frac{1}{5} + \frac{1}{3}\right] = 5 + 4$$

$$V = 9 \times \frac{15}{8} = 16.875 \text{ V}$$

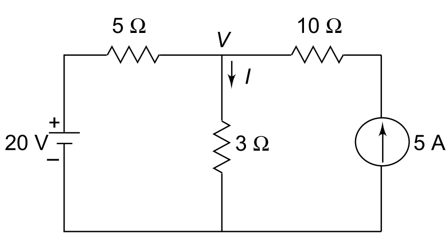


Fig. 5.96

The current passing through the 3 Ω resistor is equal to $V/3$

i.e. $$I = \frac{16.875}{3} = 5.625 \text{ A}$$

So the superposition theorem is verified.

Let us now examine the power responses.

Power dissipated in the 3 Ω resistor due to voltage source acting alone

$$P_{20} = (I_2)^2 R = (2.5)^2\, 3 = 18.75 \text{ W}$$

Power dissipated in the 3 Ω resistor due to current source acting alone

$$P_5 = (I_5)^2 R = (3.125)^2\, 3 = 29.29 \text{ W}$$

Power dissipated in the 3 Ω resistor when both the sources are acting simultaneously is given by

$$P = (5.625)^2 \times 3 = 94.92 \text{ W}$$

From the above results, the superposition of P_{20} and P_5 gives

$$P_{20} + P_5 = 48.04 \text{ W}$$

which is not equal to $P = 94.92$ W

We can, therefore, state that the superposition theorem is not valid for power responses. It is applicable only for computing voltage and current responses.

Example 5.33 *Find the voltage across the 2 Ω resistor in Fig. 5.97 by using the super-position theorem.*

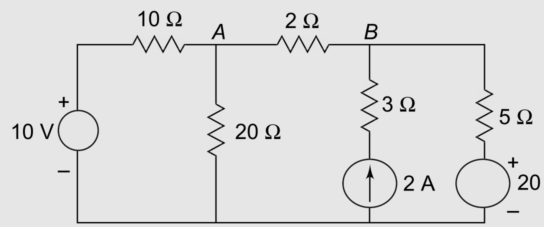


Fig. 5.97

Solution Let us find the voltage across the 2 Ω resistor due to individual sources. The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor.

Our first step is to find the voltage across the 2 Ω resistor due to the 10 V source, while other sources are set equal to zero.

The circuit is redrawn as shown in Fig. 5.98(a).

Assuming a voltage V at node 'A' as shown in Fig. 5.98(a), the current equation is

$$\frac{V-10}{10} + \frac{V}{20} + \frac{V}{7} = 0$$

$$V[0.1 + 0.05 + 0.143] = 1$$

or

$$V = 3.41 \text{ V}$$

The voltage across the 2 Ω resistor due to the 10 V source is

$$V_2 = \frac{V}{7} \times 2 = 0.97 \text{ V}$$

Our second step is to find out the voltage across the 2 Ω resistor due to the 20 V source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 5.98(b).

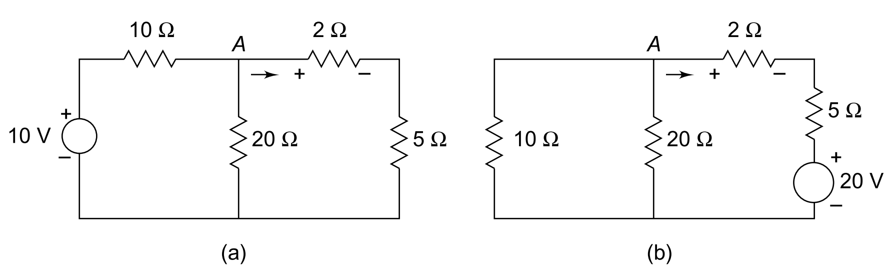


Fig. 5.98

Assuming voltage V at node A as shown in Fig. 5.98(b), the current equation is

$$\frac{V-20}{7}+\frac{V}{20}+\frac{V}{10}=0$$

$$V[0.143+0.05+0.1]=2.86$$

or
$$V=\frac{2.86}{0.293}=9.76\,\text{V}$$

The voltage across the 2 Ω resistor due to the 20 V source is

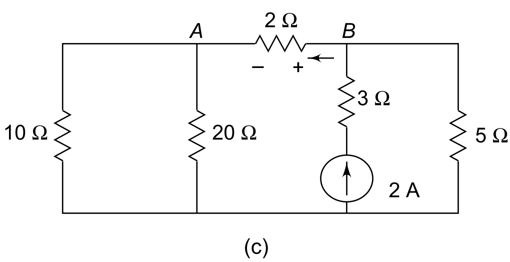
$$V_2=\left(\frac{V-20}{7}\right)\times 2=-2.92\text{ V}$$

The last step is to find the voltage across the 2 Ω resistor due to the 2 A current source, while the other sources are set equal to zero. The circuit is redrawn as shown in Fig. 5.98(c).

$$\text{The current in the 2 }\Omega\text{ resistor}=2\times\frac{5}{5+8.67}$$

$$=\frac{10}{13.67}=0.73\,\text{A}$$

The voltage across the 2 Ω resistor = 0.73 × 2 = 1.46 V



(c)

Fig. 5.98

The algebraic sum of these voltages gives the total voltage across the 2 Ω resistor in the network

$$V=0.97-2.92-1.46$$
$$=-3.41\text{ V}$$

The negative sign of the voltage indicates that the voltage at 'A' is negative.

Example 5.34 *For the resistive network shown in Fig. 5.99, find the current in each resistor, using the superposition principle.*

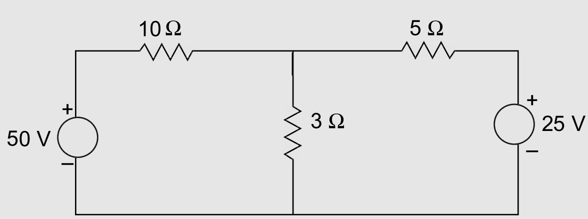


Fig. 5.99

Solution The current due to the 50 V source can be found in the circuit shown in Fig. 5.100(a).

Total resistance $R_T = 10 + \dfrac{5\times 3}{8} = 11.9\,\Omega$

Current in the 10 Ω resistor $I_{10} = \dfrac{50}{11.9} = 4.2\,\text{A}$

Current in the 3 Ω resistor $I_3 = 4.2 \times \dfrac{5}{8} = 2.63\,\text{A}$

Current in the 5 Ω resistor $I_5 = 4.2 \times \dfrac{3}{8} = 1.58\,\text{A}$

The current due to the 25 V source can be found from the circuit shown in Fig. 5.100(b).

Total resistance

$$R_T = 5 + \frac{10\times 3}{13} = 7.31\,\Omega$$

Current in the 5 Ω resistor

$$I_5' = \frac{25}{7.31} = 3.42\,\text{A}$$

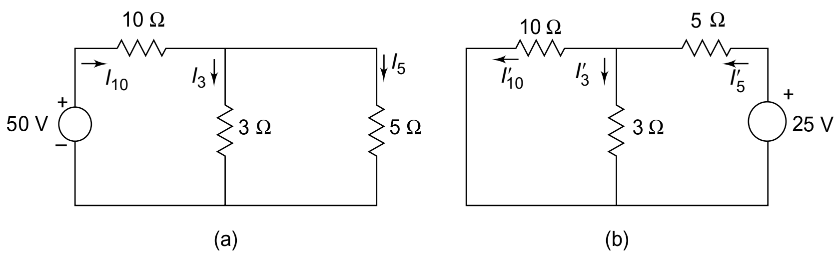


Fig. 5.100

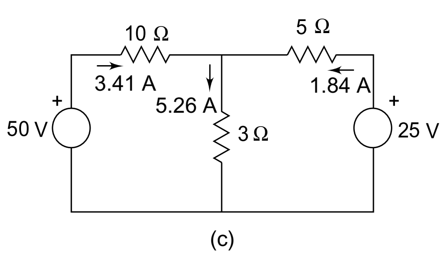
Current in the 3 Ω resistor $I_3' = 3.42 \times \dfrac{10}{13} = 2.63\,\text{A}$

Current in the 10 Ω resistor $I_{10}' = 3.42 \times \dfrac{3}{13} = 0.79\,\text{A}$

According to superposition principle

Current in the 10 Ω resistor

$= I_{10} - I'_{10} = 4.2 - 0.79 = 3.41$ A



(c)

Fig. 5.100

Current in the 3 Ω resistor

$= I_3 + I'_3 = 2.63 + 2.63 = 5.26$ A

Current in the 5 Ω resistor

$= I'_5 - I_5 = 3.42 - 1.58 = 1.84$ A

When both sources are operative, the directions of the currents are shown in Fig. 5.100(c).

Example 5.35 *Determine the voltage across the terminals AB in the circuit shown in Fig. 5.101.*

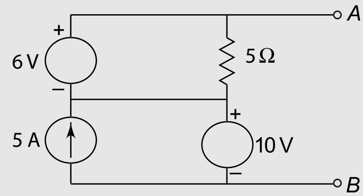


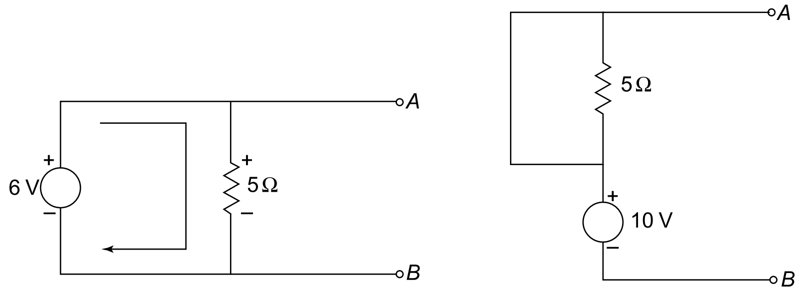
Fig. 5.101

Solution Voltage across AB is $V_{AB} = V_{10} + V_5$.

To find the voltage across the $5\,\Omega$ resistor, we have to use the superposition theorem.

Voltage across the $5\ \Omega$ resistor V_5 due to the 6 V source, when other sources are set equal to zero, is calculated using Fig. 5.102(a).

$$V_5 = 6\text{ V}$$

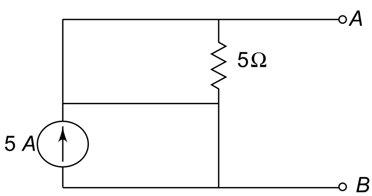


(a) (b)

Fig. 5.102

Voltage across the $5\,\Omega$ resistor V_5' due to the 10 V sources, when other sources are set equal to zero, is calculated using Fig. 5.102(b).

$$V_5' = 0$$



(c)

Fig. 5.102

Voltage across the $5\,\Omega$ resistor V'_5 due to the 5 A source only, is calculated using Fig. 5.102(c)

$$V''_5 = 0$$

According to the superposition theorem, Total voltage across the $5\ \Omega$ resistor $= 6 + 0 + 0 = 6\text{V}$.

So the voltage across terminals AB is $V_{AB} = 10 + 6 = 16$ V.

Example 5.36 *For the circuit shown in Fig. 5.103, find the current i_4 using the superposition principle.*

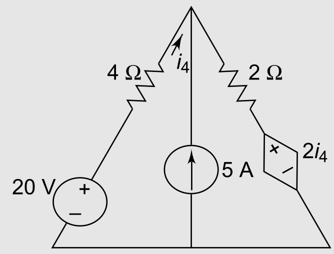


Fig. 5.103

Solution The circuit can be redrawn as shown in Fig. 5.104(a).

The current i_4' due to the 20 V source can be found using the circuit shown in Fig. 5.104(b).

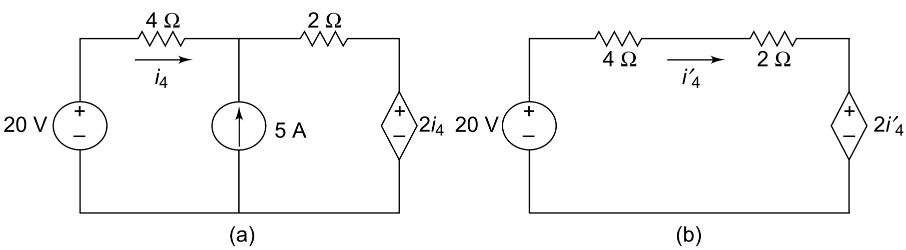


Fig. 5.104

Applying Kirchhoff's voltage law,

$$-20 + 4i_4' + 2i_4' + 2i_4' = 0$$

$$i_4' = 2.5 \text{ A}$$

The current i_4' due to the 5 A source can be found using the circuit shown in Fig. 5.104(c).

By assuming V' at node shown in Fig. 5.104(c) and applying Kirchhoff's current law

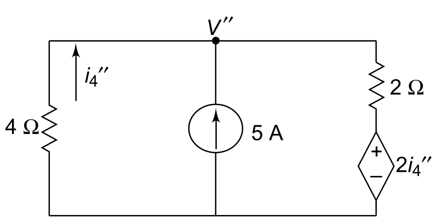


Fig. 5.104(c)

$$\frac{V''}{4} - 5 + \frac{V'' - 2i_4''}{2} = 0$$

$$i_4'' = \frac{-V''}{4}$$

From the above equations

$$i_4' = -1.25 \text{ A}$$

$$\therefore \text{Total current } \; i_4 = i_4' + i_4' = 1.25 \text{ A}$$

Example 5.37 *Determine the current through the 2 Ω resistor as shown in the Fig. 5.105 by using the superposition theorem.*

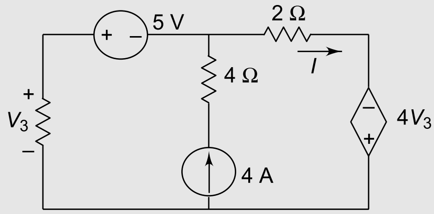


Fig. 5.105

Solution The current I' due to the 5 V source can be found using the circuit shown in Fig. 5.106(a).

By applying Kirchhoff's voltage law, we have

$$3I' + 5 + 2I' - 4V_3' = 0$$

we know $\qquad V_3' = -3I'$

From the above equations

$$I' = -0.294 \text{ A}$$

The current I' due to the 4 A source can be found using the circuit shown in Fig. 5.106(b).

By assuming node voltage V_3', we find

$$I'' = \frac{V_3'' + 4V_3''}{2}$$

By applying Kirchhoff's current law at node, we have

$$\frac{V_3''}{3} - 4 + \frac{V_3'' + 4V_3''}{2} = 0$$

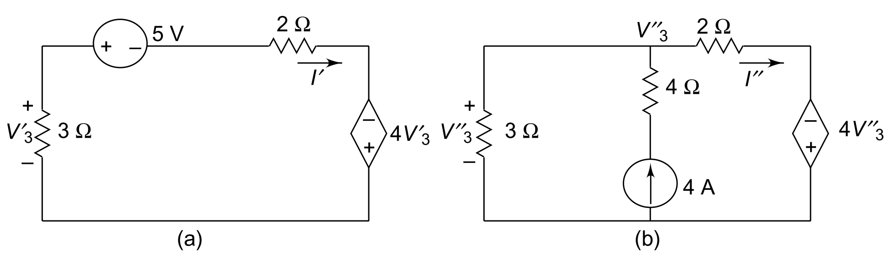


Fig. 5.106

$$V_3' = 1.55 \text{ V}$$

$$\therefore \qquad I'' = \frac{V_3'' + 4V_3''}{2} = 3.875 \text{ A}$$

Total current in the 2 Ω resistor $I = I' + I'' = -\,0.294 + 3.875$

$$\therefore \qquad I = 3.581 \text{ A}$$

Example 5.38 *Find the current I the circuit shown in Fig. 5.107.*

[JNTU May/June 2006]

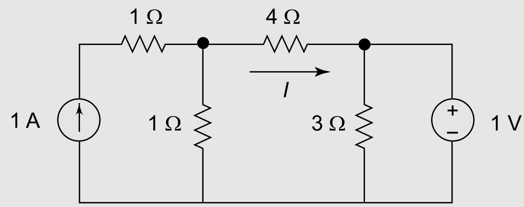


Fig. 5.107

Solution Applying superposition

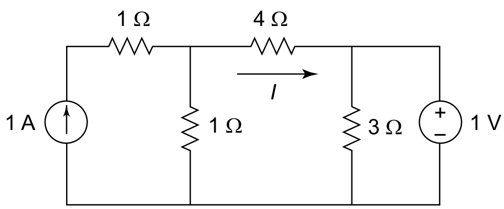


Fig. 5.108

Open circuit the current source

$$I_1 = \frac{1}{5}$$

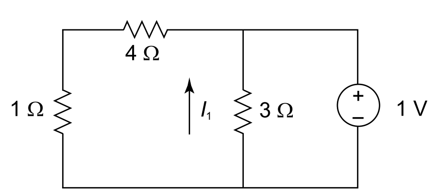


Fig. 5.109

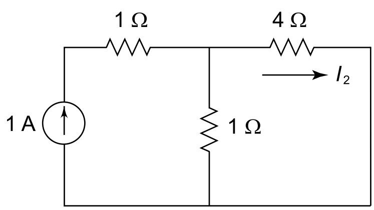


Fig. 5.110

Short the voltage source

$$I_2 = 1 \times \frac{1}{5} = \frac{1}{5}$$

Total current through $4\ \Omega = \frac{1}{5} - \frac{1}{5} = 0$

Example 5.39 *Find the current i in the circuit shown in Fig. 5.111 using superposition theorem.* *[May/June 2006 Network Analysis]*

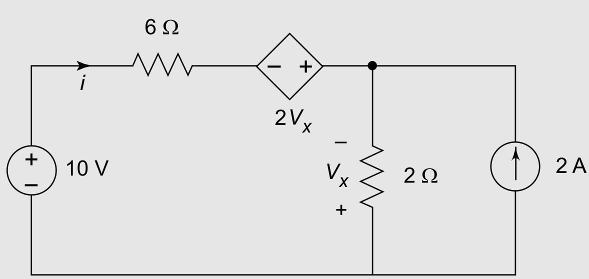


Fig. 5.111

Solution Consider 2 A current source acting alone by short circuiting voltage source 10 V as shown in Fig. 5.112(a) $6i_1 - 2V_x - V_x = 0$

$$i_2 = -2\text{ A}$$
$$V_x = -2(i_1 - i_2)$$
$$= -2i_1 - 4$$

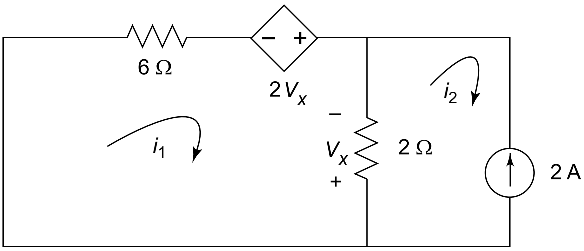


Fig. 5.112(a)

$$6i_1 - 3V_x = 0 \Rightarrow 6i_1 - 3(-2i_1 - 4) = 0$$
$$6i_1 + 6i_1 + 12 = 0 \Rightarrow i_1 = -1\text{ A}$$

Consider 10 V voltage source acting alone by opening 2 A current source in Fig. 5.112(b)

$$-10 + 6i_1' - 2V_x - V_x = 0$$
$$V_x = -2i_1'$$

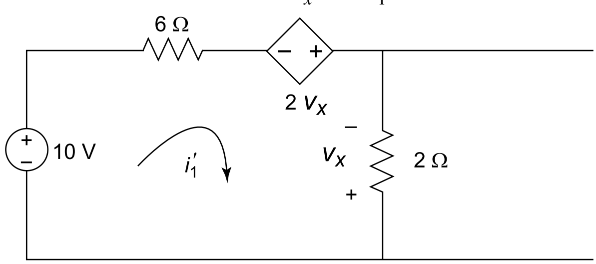


Fig. 5.112(b)

$$-10 + 6i_1' - 3V_x = 0 \Rightarrow -10 + 6i_1' + 6i_1' = 0 \Rightarrow i_1' = 5/6$$
$$i = i_1 + i_1' = -1 + 5/6 = -1/6\ A$$

Example 5.40 *Is superposition valid for power? Explain.*
[JNTU May/June 2004]

Solution Superposition theorem is valid only for linear systems.

Superposition cannot be applied for power because the equation for power is non linear.

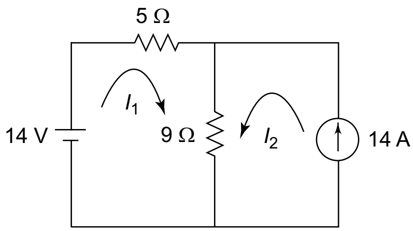


Fig. 5.113

Let us consider a network with a voltage source and current source as shown below and find the power consumed in 9 Ω resistor by super position.

When 14 V source is acting, the current in 9 Ω is 1 A

The power $= i^2 \times 9 = 9$ watts

When 14 A source is acting, the current in 9 Ω is 5 A

The power $= i^2 \times 9 = 225$ watts

Total power $= 225 + 9 = 234$ watts

When both are acting the KVL for loop 1 and 2

are $\quad 14 = 5i_1 + 9(i_1 + i_2)$

$$14i_1 = -112$$

$$i_1 = -8\text{ A};\ i_2 = 14\text{ A}$$

Current in 9 Ω resistor is $i_1 + i_2 = 6$ A

Power $= (6)^2 \times 9 = 324$ watts

Since power is not the same in both the cases, the superposition theorem does not hold true.

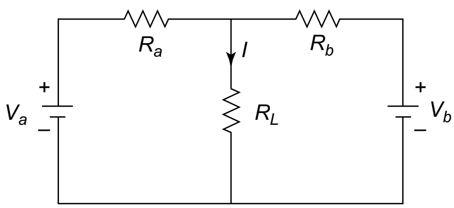


Fig. 5.114

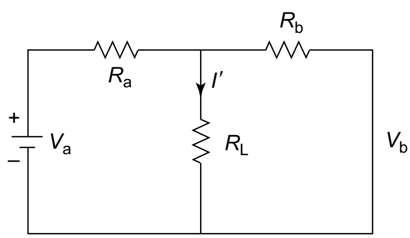


Fig. 5.115

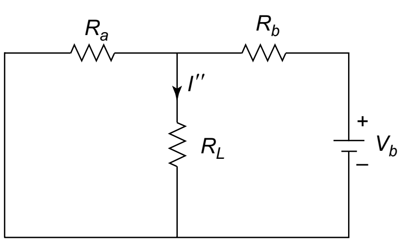


Fig. 5.116

Consider the circuit shown below.

When V_a is acting.

I' be the current through R_L: and Power $= (I')^2 R_L$

When V_b is acting I'' be the current through R_L and Power $= (I'')^2 R_L$

Total current: Through R_L by superposition

$I = I' + I''$ and power $= I^2 R_L$

$(I')^2 R_L + (I'')^2, R_L \neq I^2 R_L$

because $I^2 = (I' + I'')^2 = (I')^2 + (I'')^2 + 2I'I''$

Hence $(I')^2 + (I'')^2 \neq I^2$ and therefore superposition theorem is not valid for power.

Example 5.41 *Using superposition theorem, find V_{AB}.*
[JNTU May/June 2004]

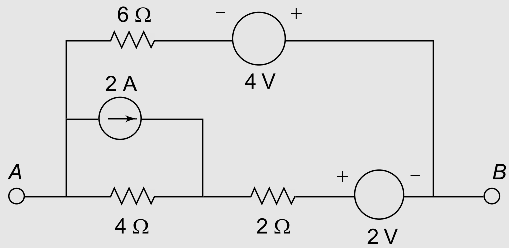


Fig. 5.117

Solution When 4 V source is acting alone, the circuit becomes

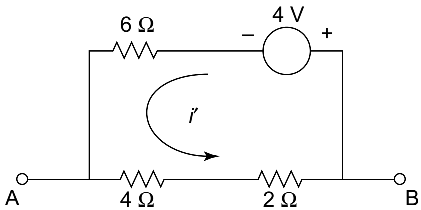


Fig. 5.118

Current through the circuit

$$i' = \frac{-1}{3}\text{ A}$$

$$\therefore \qquad V_{AB1} = i' \times 6 = -2\text{ V}$$

When 2 V source is acting alone, the circuit becomes

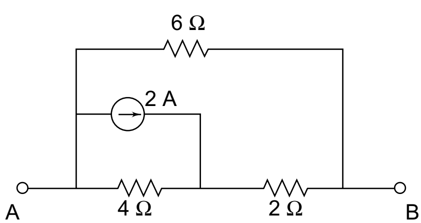


Fig. 5.119

Current through the circuit

$$i' = \frac{2}{12} = \frac{1}{6}\text{ A}$$

$$\therefore \qquad V_{AB2} = -i'' \times 6 + 2$$

$$= -1 + 2 = 1\text{ V}$$

When 2 A source is acting alone, the circuit becomes

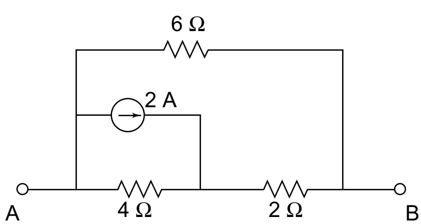


Fig. 5.120

Current in 4 Ω resistor $= 2 \times \frac{8}{12} = \frac{4}{3}$ A

Voltage across 4 Ω resistor $= \frac{16}{3}$ V

Current in 2 Ω resistor $= 2 \times \frac{4}{12} = \frac{2}{3}$ A

Voltage across 2 Ω resistor $= \frac{4}{3}$ V

$$\therefore \qquad V_{AB3} = -V_4 + V_2 = \frac{-16}{3} + \frac{4}{3} = -4\text{ V}$$

Voltage across AB

$$V_{AB} = V_{AB1} + V_{AB2} + V_{AB3}$$

$$= -2 + 1 - 4 = -5 \text{ volts.}$$

Example 5.42 *Solve for current in 5 ohms resistor by principle of super position theorem shown in Fig. 5.121.* *[JNTU June 2009]*

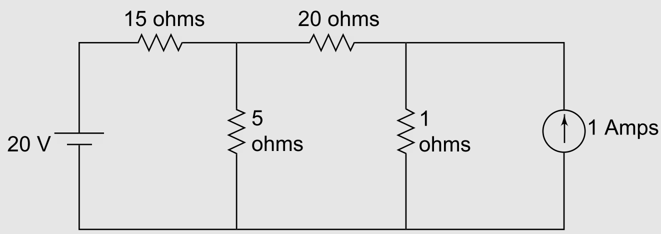


Fig. 5.121

Solution Open circuiting current source

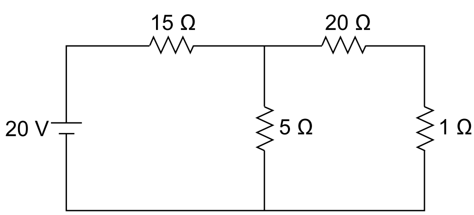


Fig. 5.122

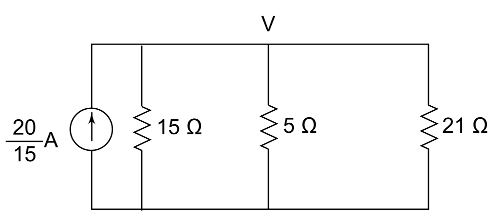


Fig. 5.123

Replacing series combination of 20 Ω and 1 Ω by (20 + 1) Ω = 21 Ω and 20 V voltage source with series resistance of 15 Ω by current source of $\left(\frac{20}{15}\right)$ amp with parallel resistance of 15 Ω.

$$\therefore \quad \frac{20}{15} = \frac{V}{15} + \frac{V}{5} + \frac{V}{21}$$

or, $\quad V = 4.232$ volt

$\therefore$ Current in 5 Ω

$$= \frac{4.232}{5} \text{ amp}$$

$$= 0.846 \text{ amp}$$

Short circuiting voltage source

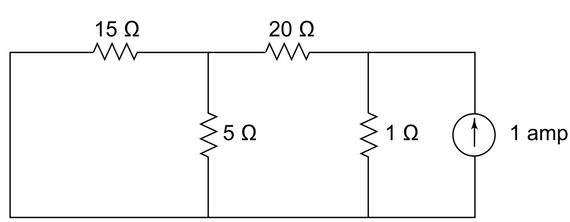


Fig. 5.124

Replacing 1 amp current source with parallel resistance of 1 Ω by a voltage source of 1 V with series resistance of 1 Ω

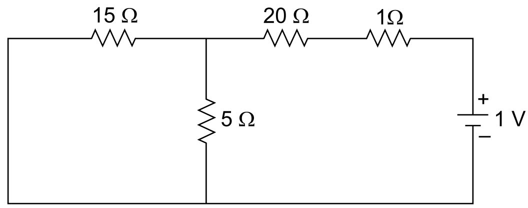


Fig. 5.125

Replacing series combination of 20 Ω and 1 Ω by (20 + 1) Ω = 21 Ω

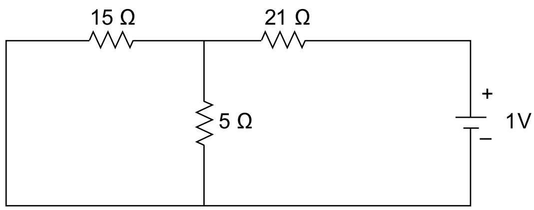


Fig. 5.126

Replacing voltage source of 1 V with series resistance of 21 Ω by a current source of (1/21) amp with a parallel resistance of 21 Ω

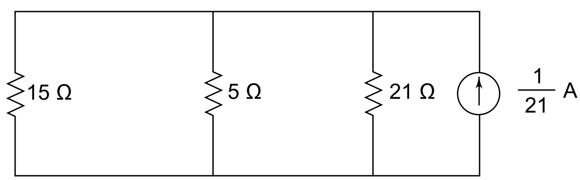


Fig. 5.127

$$\therefore \qquad \frac{1}{21} = \frac{V}{15} + \frac{V}{5} + \frac{V}{21}$$

$$\therefore \qquad \text{V} = 0.151 \text{ volt}$$

$$\therefore \quad \text{Current through } 5\,\Omega = \frac{0.151}{5} \text{ amp}$$

$$= 0.03 \text{ amp}$$

$$\therefore \quad \text{Total current in } 5\ \Omega = (0.846 + 0.03) \text{ amp}$$

$$= 0.876 \text{ amp.}$$

5.1.7 Maximum Power Transfer Theorem *[JNTU Jan 2010]*

Many circuits basically consist of sources, supplying voltage, current, or power to the load; for example, a radio speaker system, or a microphone supplying the input signals to voltage pre-amplifiers. Sometimes it is necessary to transfer maximum voltage, current or power from the source to the load. In the simple resistive circuit shown in Fig. 5.128, R_S is the source resistance. Our aim is to find the necessary conditions so that the power delivered by the source to the load is maximum.

It is a fact that more voltage is delivered to the load when the load resistance is high as compared to the resistance of the source. On the other hand, maximum current is transferred to the load when the load resistance is small compared to the source resistance.

For many applications, an important consideration is the maximum power transfer to the load; for example, maximum power transfer is desirable from the output amplifier to the speaker of an audio sound system. The maximum Power Transfer Theorem states that maximum power is delivered from a source to a load when the load resistance is equal to the source resistance. In Fig. 5.128, assume that the load resistance is variable.

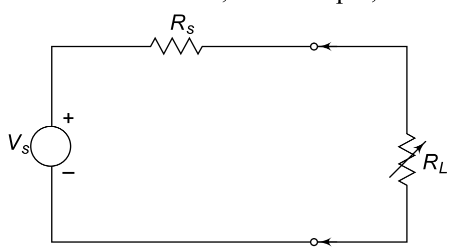


Fig. 5.128

Current in the circuit is $I = V_S/(R_S + R_L)$

Power delivered to the load R_L is $P = I^2 R_L = V^2{}_S R_L/(R_S + R_L)^2$

To determine the value of R_L for maximum power to be transferred to the load, we have to set the first derivative of the above equation with respect to R_L, i.e. when $\dfrac{dP}{dR_L}$ equals zero.

$$\frac{dP}{dR_L} = \frac{d}{dR_L}\left[\frac{V_S^2}{(R_S + R_L)^2} R_L\right]$$

$$= \frac{V_S^2\left\{(R_S + R_L)^2 - (2R_L)(R_S + R_L)\right\}}{(R_S + R_L)^4}$$

$$\therefore \qquad (R_S + R_L)^2 - 2R_L(R_S + R_L) = 0$$

$$R^2{}_S + R^2{}_L + 2R_S R_L - 2R^2{}_L - 2R_S R_L = 0$$

$$\therefore \qquad R_S = R_L$$

So, maximum power will be transferred to the load when load resistance is equal to the source resistance.

Example 5.43 *In the circuit shown in Fig. 5.129 determine the value of load resistance when the load resistance draws maximum power. Also find the value of the maximum power.*

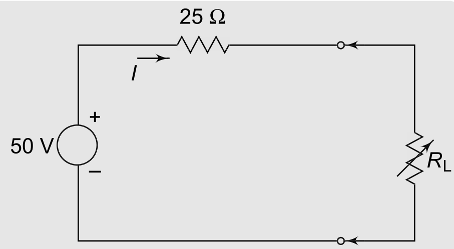


Fig. 5.129

Solution In Fig. 5.129, the source delivers the maximum power when load resistance is equal to the source resistance.

$$R_L = 25\ \Omega$$

The current $I = 50/(25 + R_L) = 50/50 = 1$ A

The maximum power delivered to the load $P = I^2R_L$

$$= 1 \times 25 = 25 \text{ W}$$

Example 5.44 *Determine the maximum power delivered to the load in the circuit shown in Fig. 5.130.*

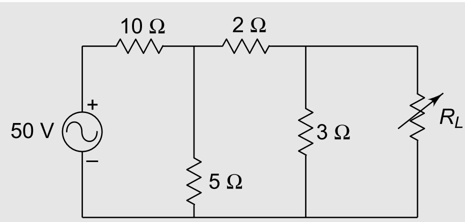


Fig. 5.130

Solution For the given circuit, let us find out the Thevenin's equivalent circuit across *AB* as shown in Fig. 5.131(a).

The total resistance is

$$R_T = [\{(3 + 2) \parallel 5\} + 10]$$

$$= [2.5 + 10] = 12.5\ \Omega$$

Total current drawn by the circuit is

$$I_T = \frac{50}{12.5} = 4 \text{ A}$$

The current in the 3 V resistor is

$$I_3 = I_T \times \frac{5}{5+5} = \frac{4 \times 5}{10} = 2 \text{ A}$$

Thevenin's voltage $V_{AB} = V_3 = 3 \times 2 = 6$ V

Thevenin's resistance $R_{\text{Th}} = R_{AB} = [((10 \parallel 5) + 2) \parallel 3]\ \Omega = 1.92\ \Omega$

Thevenin's equivalent circuit is shown in Fig. 5.131(b).

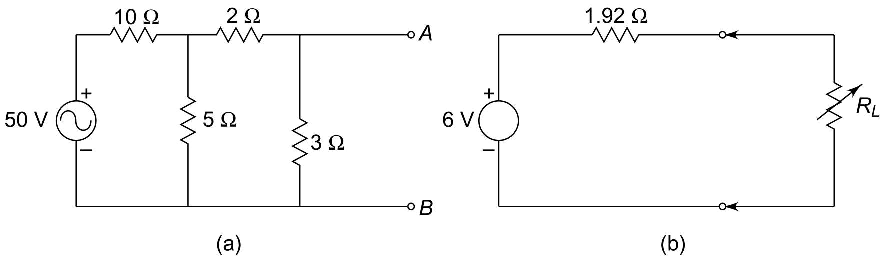


Fig. 5.131

From Fig. 5.131(b), and maximum power transfer theorem

$$R_L = 1.92\ \Omega$$

∴ Current drawn by load resistance R_L

$$I_L = \frac{6}{1.92+1.92} = 1.56 \text{ A}$$

Power delivered to the load $= I^2_L R_L$

$$= (1.56)^2 \times 1.92 = 4.67 \text{ W}$$

Example 5.45 *Determine the load resistance to receive maximum power from the source; also find the maximum power delivered to the load in the circuit shown in Fig. 5.132.*

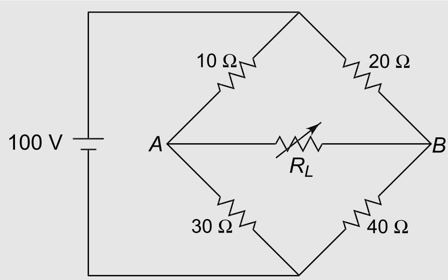


Fig. 5.132

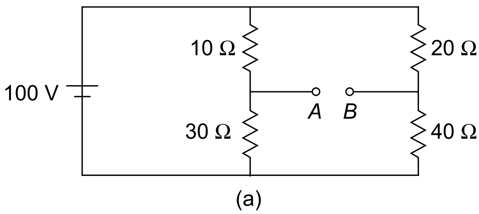


Fig. 5.133

Solution For the given circuit, we find out the Thevenin's equivalent circuit.

Thevenin's voltage across terminals A and B

$$V_{AB} = V_A - V_B$$

Voltage at point A is

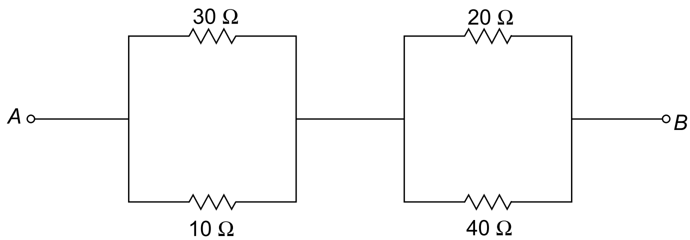
$$V_A = 100 \times \frac{30}{30+10} = 75 \text{ V}$$

Voltage at point B is

$$V_B = 100 \times \frac{40}{40+20} = 66.67 \text{ V}$$

$$\therefore V_{AB} = 75 - 66.67 = 8.33 \text{ V}$$

To find Thevenin's resistance, the circuit in Fig. 5.133(a) can be redrawn as shown in Fig. 5.133(b).



(b)

Fig. 5.133

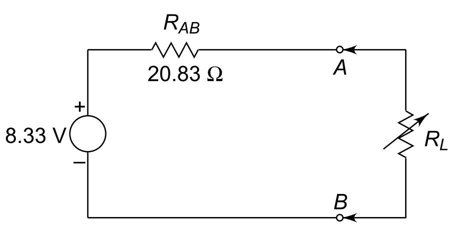
From Fig. 5.133(b), Thevenin's resistance

$$R_{AB} = [(30 \parallel 10) + (20 \parallel 40)]$$
$$= [7.5 + 13.33] = 20.83\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.133(c).

According to maximum power transfer theorem

$$R_L = 20.83\ \Omega$$



(c)

Fig. 5.133

Current drawn by the load resistance

$$I_L = \frac{8.33}{20.83 + 20.83} = 0.2\text{ A}$$

∴ Maximum power delivered to load $= I^2_L R_L$

$$= (0.2)^2\,(20.83) = 0.833\text{ W}$$

Example 5.46 *The circuit shown in the Fig. 5.134 below has resistance R which absorbs maximum power. Compute the value of R and maximum power.*

[JNTU April/May 2003]

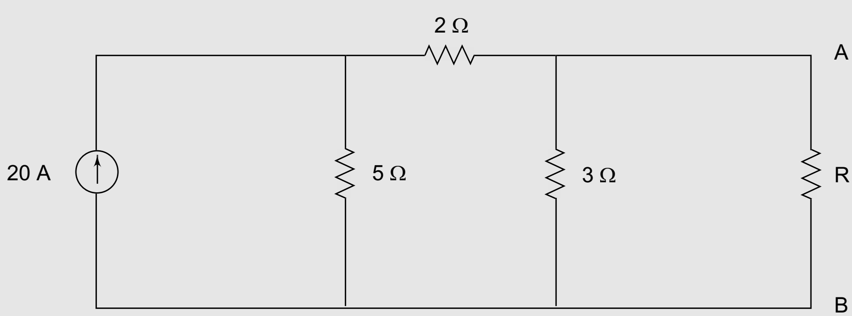


Fig. 5.134

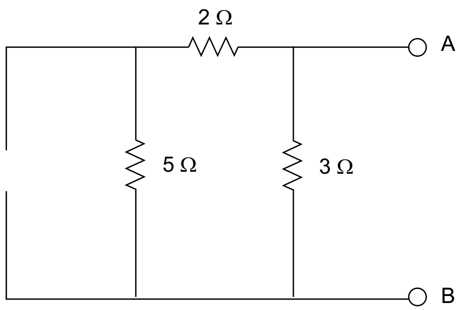


Fig. 5.135(a)

Solution According to maximum power transfer theorem, maximum power can be transferred when load resistance is equal to the interval resistance of the source which can be calculated as the resistance seen from AB with source open.

$$\therefore \quad R_{\text{Th}} = (5 + 2)//3$$

$$\frac{21}{10} = 2.1\,\Omega$$

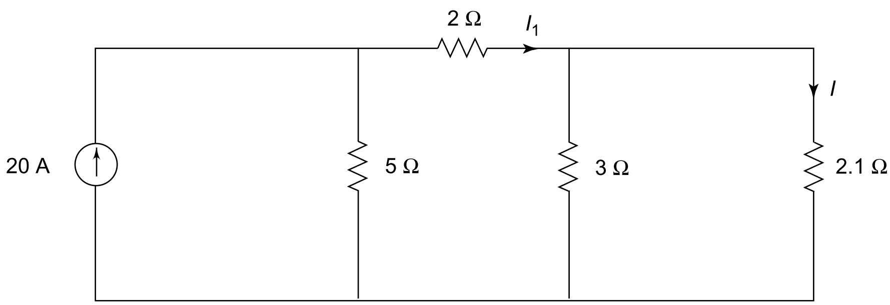


Fig. 5.135(b)

Now the circuit can be drawn as
According to current dividing rule

$$I_1 = \frac{20 \times 5}{(5 + 3.235)} = 12.14 \text{ A}$$

$$I_2 = \frac{I_1 \times 3}{5.1} = \frac{12.14 \times 3}{5} = 7.14 \text{ A}$$

So the maximum power that can be delivered to resistor R is

$$I^2R = (7.14)^2 \times 2.1 = 107 \text{ watts.}$$

5.1.8 Tellegen's Theorem *[JNTU Jan 2010]*

Tellegen's theorem is valid for any lumped network which may be linear or non linear, passive or active, time-varying or time-invarient. This theorem states that in an arbitrary lumped network, the algebraic sum of the powers in all branches at any instant is zero. All branch currents and voltages in that network must satisfy Kirchhoff's laws. Otherwise, in a given network, the algebraic sum of the powers delivered by all sources is equal to the algebraic sum of the powers absorbed by all elements. This theorem is based on Kirchhoff's two laws, but not on the type of circuit elements.

Consider two networks N_1 and N_2, having the same graph with different types of elements between the corresponding nodes.

Then $$\sum_{K=1}^{b} v_{1K} i_{2K} = 0$$

and $$\sum_{K=1}^{b} v_{2K} i_{1K} = 0$$

To verify Tellegen's theorem, consider two circuits having same graphs as shown in Fig. 5.136.

In Fig. 5.136(a)

$$i_1 = i_2 = 2 \text{ A}; i_3 = 2 \text{ A}$$

and $$v_1 = -2 \text{ V}, v_2 = -8 \text{ V}, v_3 = 10 \text{ V}$$

In Fig. 5.136(b)

$$i_1^1 = i_2^1 = 4 \text{ A}; i_3^1 = 4 \text{ A}$$

and $$v_1^1 = -20 \text{ V}; v_2^1 = 0 \text{ V}; v_3^1 = 20 \text{ V}$$

Now $$\sum_{K=1}^{3} v_K i_K^1 = v_1 i_1^1 + v_2 i_2^1 + v_3 i_3^1$$

$$= (-2)(4) + (-8)(4) + (10)(4) = 0$$

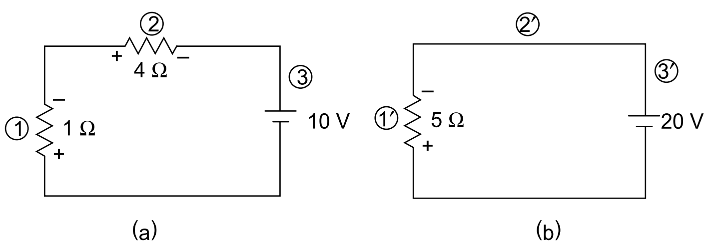


Fig. 5.136

and $$\sum_{K=1}^{3} v_K^1 i_K = v_1^1 i_1 + v_2^1 i_2 + v_3^1 i_3$$

$$= (-20)(2) + (0)(2) + (20)(2) = 0$$

Similarly,

$$\sum_{K=1}^{3} v_K i_K = v_1 i_1 + v_2 i_2 + v_3 i_3$$

$$= (-2)(2) + (-8)(2) + (10)(2) = 0$$

and $$\sum_{K=1}^{3} v_K^1 i_K^1 = (-20)(4) + (0)(4) + (20)(4) = 0$$

This verifies Tellegen's theorem.

Example 5.47 *Verify Tellegen's theorem in the network shown in the Fig. 5.137.*
[JNTU May/June 2006]

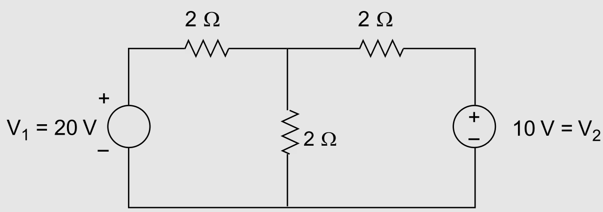


Fig. 5.137

Solution Tellegens theorem states that in any arbitrary lumped network, the algebraic sum of the powers in all the branches at any instant is zero and all the branch currents and voltages must satisfy Kirchoff's law.

Verifying Tellegens theorem for the above circuit.

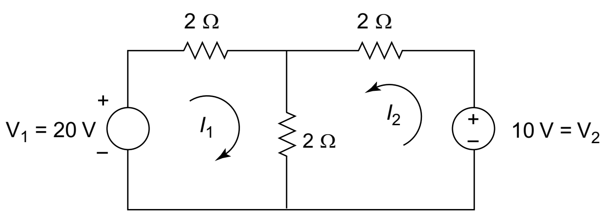


Fig. 5.138

There are 5 elements in the above circuit.

Applying mesh equations.

$$4i_1 + 2i_2 = 20$$

$$\Rightarrow 2i_1 + i_2 = 10$$

$$2i_1 + 4i_2 = 10 \tag{1}$$

$$i_1 + 2i_2 = 5 \tag{2}$$

Solving (1) and (2)

$$i_1 = 5, i_2 = 0$$

$\sum_{k=1}^{5} V_k I_k$ for this circuit is

$$-100 + 50 + 50 + (0)^2 (2) - (0)(10) = 0$$

Hence, verified.

Example 5.48 *Verify Tellegen's Theorem for the network shown in Fig. 5.139.* *[JNTU Jan 2010]*

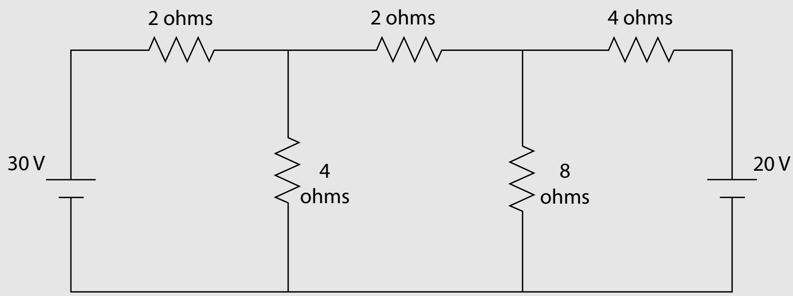


Fig. 5.139

Solution

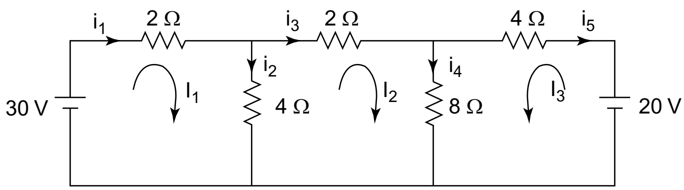


Fig. 5.140

$$30 = 6I_1 - 4I_2$$

$$0 = -4I_1 + 14I_2 + 8I_3$$

$$20 = 8I_2 + 12I_3$$

$$\Delta = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 14 & 8 \\ 0 & 8 & 12 \end{bmatrix} = 624 - 192 = 432$$

$$\Delta_1 = \begin{bmatrix} 30 & -4 & 0 \\ 0 & 14 & 8 \\ 20 & 8 & 12 \end{bmatrix} = 3120 - 640 = 2480$$

$$\Delta_2 = \begin{bmatrix} 6 & 30 & 0 \\ -4 & 0 & 8 \\ 0 & 20 & 12 \end{bmatrix} = -960 + 1440 = 480$$

$$\Delta_3 = \begin{bmatrix} 6 & -4 & 30 \\ -4 & 14 & 0 \\ 0 & 8 & 20 \end{bmatrix} = 1680 - 320 - 960 = 400$$

$\therefore$ $I_1 = 5.74$ amp, $I_2 = 1.11$ amp, $I_3 = 0.93$ amp

$\therefore$
$$i_1 = I_1 = 5.74 \text{ amp}$$

$$i_2 = I_1 - I_2 = 4.63 \text{ amp}$$

$$i_3 = I_2 = 1.11 \text{ amp}$$
$$i_4 = I_2 + I_3 = 2.04 \text{ amp}$$
$$i_5 = -I_3 = -0.93 \text{ amp}$$

$\therefore$ Total power supplied $= (30 \times 5.74) + (20 \times 0.93)$ watt

$\therefore$ Total power dissipated $= (5.74^2 \times 2) + (4.63^2 \times 4) + (1.11^2 \times 2) + (2.04^2 \times 8) + (0.93^2 \times 4)$ watt

$\therefore$ Total power in all branches = Power supplied – Power dissipated $= 0$

$\therefore$ Tellegeni's theorem is verified.

5.2 NETWORK THEOREMS WITH AC EXCITATIONS

5.2.1 Thevenin's Theorem

[JNTU Jan 2010]

Thevenin's theorem gives us a method for simplifying a given circuit. The Thevenin equivalent form of any complex impedance circuit consists of an equivalent voltage source V_{Th}, and an equivalent impedance Z_{Th}, arranged as shown in Fig. 5.141. The values of equivalent voltage and impedance depend on the values in the original circuit.

Though the Thevenin equivalent circuit is not the same as its original circuit, the output voltage and output current are the same in both cases. Here, the Thevenin voltage is equal to the open circuit voltage across the output terminals, and impedance is equal to the impedance seen into the network across the output terminals.

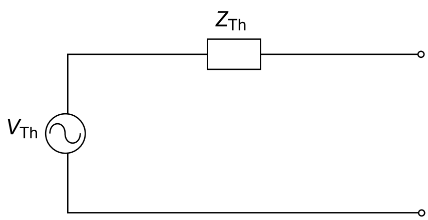


Fig. 5.141

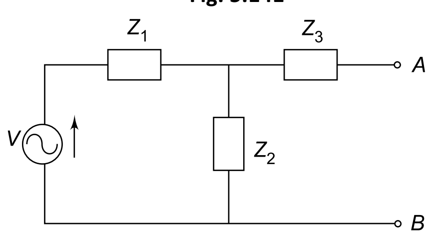


Fig. 5.142

Consider the circuit shown in Fig. 5.142.

Thevenin equivalent for the circuit shown in Fig. 5.142 between points A and B is found as follows.

The voltage across points A and B is the Thevenin equivalent voltage. In the circuit shown in Fig. 5.142, the voltage across A and B is the same as the voltage across Z_2 because there is no current through Z_3.

$$\therefore \quad V_{\text{Th}} = V\left(\frac{Z_2}{Z_1 + Z_2}\right)$$

The impedance between points A and B with the source replaced by short circuit is the Thevenin equivalent impedance. In Fig. 5.142, the impedance from A to B is Z_3 in series with the parallel combination of Z_1 and Z_2.

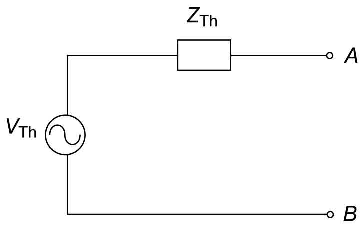


Fig. 5.143

$$\therefore \quad Z_{Th} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}$$

The Thevenin equivalent circuit is shown in Fig. 5.143.

Thevenin's theorem is especially useful in analyzing power systems and other circuits where load resistance/impedance is subject to change, and re-calculation of the circuit is necessary with each trial value of load resistance, to determine voltage across it and current through it. Many circuits are only linear over a certain range of values, thus Thevenin's equivalent is valid only within this linear range and may not be valid outside the range. The Thevenin's equivalent has an equivalent I-V characteristic only from the point of view of the load. Since power is not linearly dependent on voltage or current, the power dissipation of the Thevenin's equivalent is not identical to the power dissipation of the real system.

Example 5.49 *For the circuit shown in Fig. 5.144, determine Thevenin's equivalent between the output terminals.*

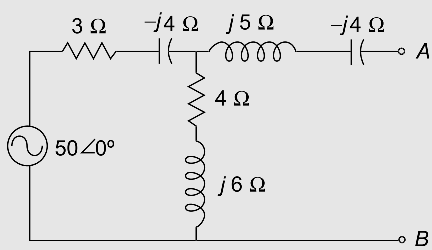


Fig. 5.144

Solution The Thevenin voltage, V_{Th}, is equal to the voltage across the $(4 + j6)\ \Omega$ impedance. The voltage across $(4 + j6)\ \Omega$ is

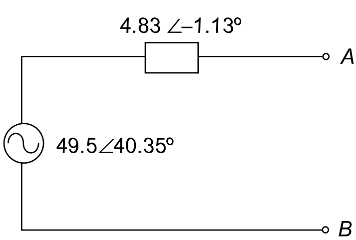


Fig. 5.145

$$V = 50\angle 0° \times \frac{(4+j6)}{(4+j6)+(3-j4)}$$

$$= 50\angle 0° \times \frac{4+j6}{7+j2}$$

$$= 50\angle 0° \times \frac{7.21\angle 56.3°}{7.28\angle 15.95°}$$

$$= 50\angle 0° \times 0.99\angle 40.35°$$

$$= 49.5\angle 40.35° \text{ V}$$

The impedance seen from terminals A and B is

$$Z_{Th} = (j5 - j4) + \frac{(3-j4)(4+j6)}{3-j4+4+j6}$$

$$= j1 + \frac{5\angle 53.13° \times 7.21\angle 56.3°}{7.28\angle 15.95°}$$

$= j1 + 4.95\angle -12.78° = j1 + 4.83 - j1.095$

$= 4.83 - j0.095$

$$\therefore \quad Z_{Th} = 4.83\angle -1.13° \ \Omega$$

The Thevenin equivalent circuit is shown in Fig. 5.145.

Example 5.50 *For the circuit shown in Fig. 5.146, determine the load current by applying Thevenin's theorem.*

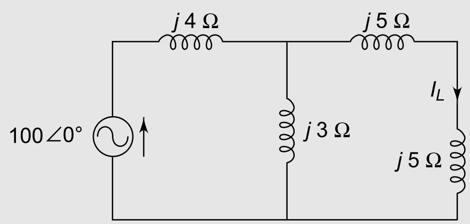


Fig. 5.146

Solution Let us find the Thevenin equivalent circuit for the circuit shown in Fig. 5.147(a).

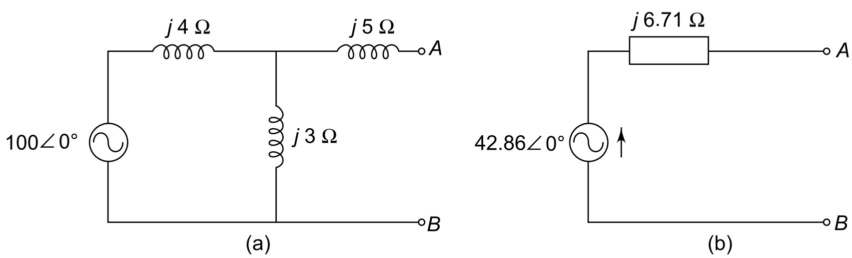


Fig. 5.147

Voltage across AB is the voltage across $(j3)\ \Omega$

$$\therefore \qquad V_{AB} = 100\ \angle 0° \times \frac{(j3)}{(j3)+(j4)}$$

$$= 100\ \angle 0° \frac{(j3)}{j7} = 42.86\angle 0°$$

Impedance seen from terminals AB

$$Z_{AB} = (j5) + \frac{(j4)(j3)}{j7}$$

$$= j5 + j1.71 = j6.71\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.147(b).

If we connect a load to Fig. 5.147(b), the current passing through $(j5)\ \Omega$ impedance is

$$I_L = \frac{42.86\angle 0°}{(j6.71 + j5)} = \frac{42.86\angle 0°}{11.71\angle 90°} = 3.66\angle -90°$$

Example 5.51 *For the circuit shown in Fig. 5.148, determine Thevenin's equivalent circuit.*

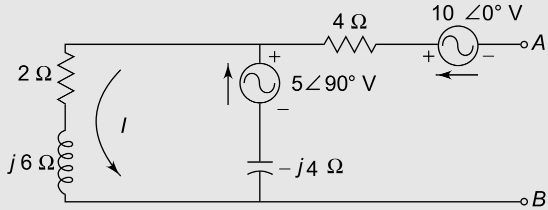


Fig. 5.148

Solution Voltage across $(-j4)$ Ω is

$$V_{-j4} = \frac{5\angle 90^\circ}{(2+j2)}(-j4)$$

$$= \frac{20\angle 0^\circ}{2.83\angle 45^\circ} = 7.07\angle -45^\circ$$

Voltage across AB is $V_{AB} = -V_{10} + V_5 - V_{-j4}$

$$= -10\angle 0^\circ + 5\angle 90^\circ - 7.07\angle -45^\circ$$

$$= j5 - 10 - 4.99 + j4.99$$

$$= -14.99 + j9.99$$

$$V_{AB} = 18\angle 146.31^\circ$$

The impedance seen from terminals AB, when all voltage sources are short circuited is

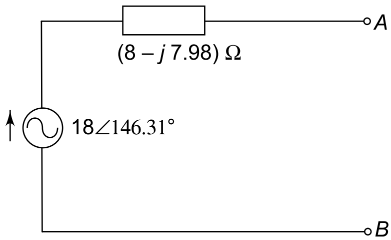


Fig. 5.149

$$Z_{AB} = 4 + \frac{(2+j6)\,(-j4)}{2+j2}$$

$$= 4 + \frac{6.32\angle 71.56^\circ \times 4\angle -90^\circ}{2.83\angle 45^\circ}$$

$$= 4 + 8.93\angle -63.44^\circ$$

$$= 4 + 4 - j7.98 = (8 - j7.98)\ \Omega$$

Thevenin's equivalent circuit is shown in Fig. 5.149.

Example 5.52 *Convert the active network shown in Fig. 5.150 by a single voltage source in series with impedance.*

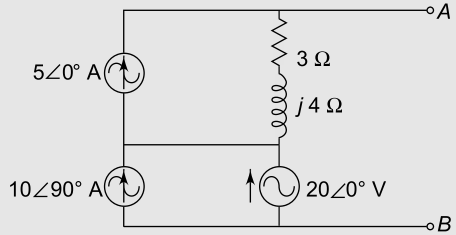


Fig. 5.150

Solution Using the superposition theorem, we can find Thevenin's equivalent circuit. The voltage across AB, with $20\angle 0^\circ$ V source acting alone, is V'_{AB}, and can be calculated from Fig. 5.151(a).

Since no current is passing through the $(3 + j4)\ \Omega$ impedance, the voltage

$$V'_{AB} = 20\angle 0°$$

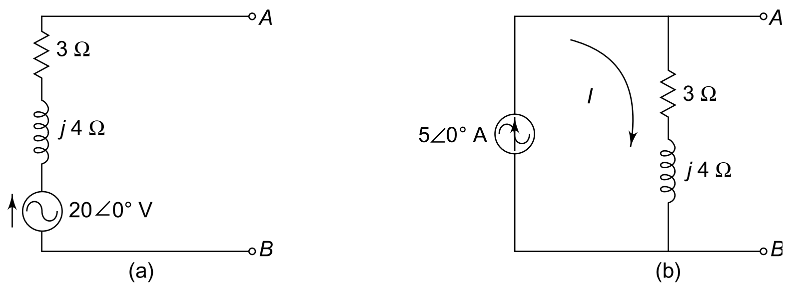


Fig. 5.151

The voltage across AB, with $5\angle 0°$ A source acting alone, is V'_{AB}, and can be calculated from Fig. 5.151(b).

$$V''_{AB} = 5\angle 0°\ (3 + j4) = 5\angle 0° \times 5\angle 53.13° = 25\angle 53.13°\ \text{V}$$

The voltage across AB, with $10\angle 90°$ A source acting alone, is V'''_{AB}, and can be calculated from Fig. 5.151(c).

$$V'''_{AB} = 0$$

According to the superposition theorem, the voltage across AB due to all sources is

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB}$$

$\therefore$ $$V_{AB} = 20\angle 0° + 25\angle 53.13° = 20 + 15 + j19.99$$

$$= (35 + j19.99)\ \text{V} = 40.3\angle 29.73°\ \text{V}$$

The impedance seen from terminals AB

$$Z_{\text{Th}} = Z_{AB} = (3 + j4)\ \Omega$$

$\therefore$ The required Thevenin circuit is shown in Fig. 5.151(d).

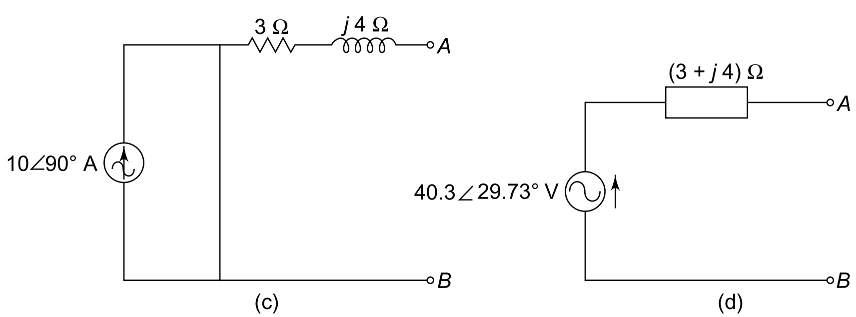


Fig. 5.151

Example 5.53 *For the circuit shown in Fig. 5.152, find the current in the $j5\,\Omega$ inductance by using Thevenin's theorem.*

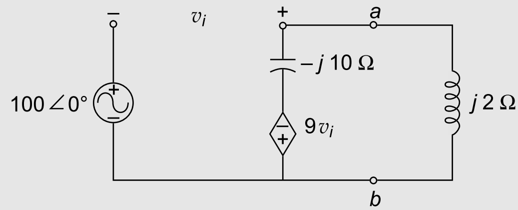


Fig. 5.152

Solution From the circuit shown in Fig. 5.152 the open circuit voltage at terminals a and b is

$$V_{oc} = -9\ V_i$$

where
$$V_i = -9V_i - 100\angle 0°$$
$$10V_i = -100\angle 0°$$
$$V_i = -10\angle 0°$$

Thevenin's voltage $V_{oc} = 90\angle 0°$

From the circuit, short circuit current is determined by shorting terminals a and b. Applying Kirchhoff's voltage law, we have

$$9V_i - j10\ i_{sc} = 0$$
$$i_{sc} = 9\angle 90°$$
$$\therefore \quad Z_{\text{Th}} = \frac{V_{oc}}{I_{sc}} = \frac{90\angle 0°}{9\angle 90°} = 10\angle -90°$$
$$Z_{\text{Th}} = -j10\ \Omega$$

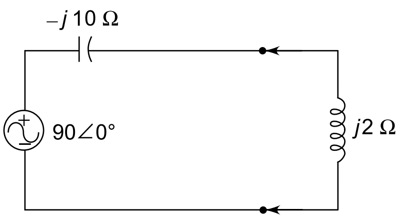


Fig. 5.153

The Thevenin's equivalent circuit is shown in Fig. 5.153.

The current in the $j2\ \Omega$ inductor is $= \dfrac{90\angle 0°}{j8}$

$= 11.25\angle 90°$

Example 5.54 *Use Thevenin's Theorem and find the current through (5 + j4) ohms impedance, for the network as shown in Fig. 5.154.*

[JNTU May/June 2008]

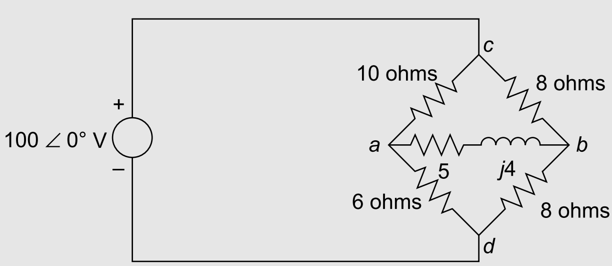


Fig. 5.154

Solution The given circuit is Thevenin's equivalent circuit can be obtained across the terminals *ab*.

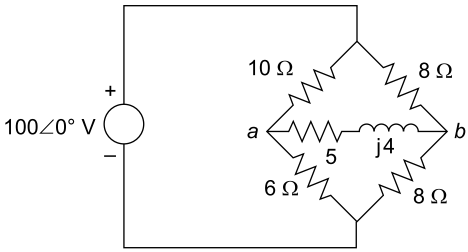


Fig. 5.155(a)

Current in the 6 Ω resistor

$$I_6 = \frac{100}{16} = 6.25\,\text{A}$$

Voltage across the 6 Ω resistor

$$V_6 = 6 \times 6.25 = 37.5\text{ V}$$

Current in the 8 Ω resistor $\quad I_8 = \dfrac{100}{16} = 6.25\,\text{A}$

Voltage across the 8 Ω resistor $\quad V_8 = 8 \times 6.25 = 50\text{ V}$

Voltage across the terminals *AB* $\quad V_{AB} = 37.5 - 50$

$$= -12.5\text{ V}$$

The resistance as seen through the terminals

$$R_{AB} = \frac{6\times10}{6+10} + \frac{8\times8}{8+8} = \frac{60}{16} + 4$$

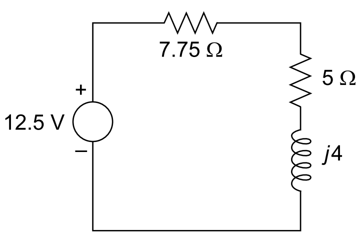


Fig. 5.155(b)

$3.75 + 4 = 7.75\ \Omega$

Equivalent circuit is

The current flowing in $(5 + j4)\ \Omega$ is

$$= \frac{12.5}{7.75 + 5 + j4} = \frac{12.5}{12.75 + j4}$$

$$= \frac{12.5}{13.362\angle 17.41} = 0.935\angle{-17.41}\,\text{A}$$

Example 5.55 *Find the current in load impedance Z_L of the network shown in Fig. 5.156, by applying Thevenin's theorem.*

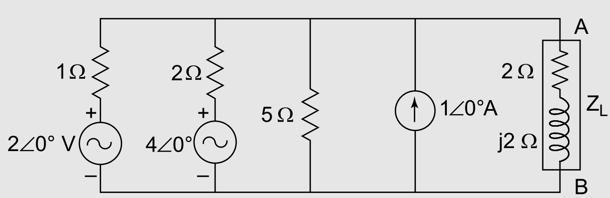


Fig. 5.156

Solution The current source has been replaced by the voltage source and the load impedance is removed from the network.

Then the network becomes as shown as Fig. 5.157(a)

The mesh equations are

$$\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -2\angle 0^\circ \\ -1\angle 0^\circ \end{bmatrix}$$

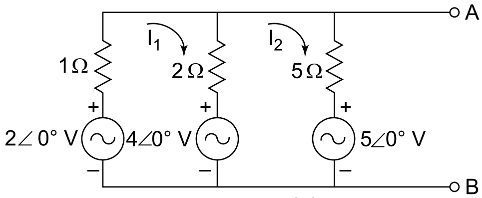


Fig. 5.157(a)

$$\therefore \quad I_2 = \frac{\begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & 7 \end{vmatrix}} = \frac{-7}{17}$$

Then $\quad V_{AB} = 5I_2 + 5\angle 0^\circ$

$$= 5 \times \frac{-7}{17} + 5$$

$$= \frac{50}{17} \text{ volts}$$

$$\mathrm{R}_{Th} = (1)\,\|\,(2)\,\|\,(5) = \frac{10}{17}\ \Omega$$

The Thevenin's circuit is

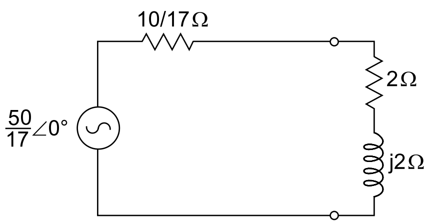


Fig. 5.157(b)

Hence the current i through the load impedance is

$$i = \frac{50/17}{\frac{10}{17} + 2 + j2}$$

$$= 0.899\,\underline{|-37.69^\circ}\ \text{A}$$

Example 5.56 *Find the current through the branch A-B of the network shown in Fig. 5.158 using Thevenin's theorem.* *[JNTU Jan 2010]*

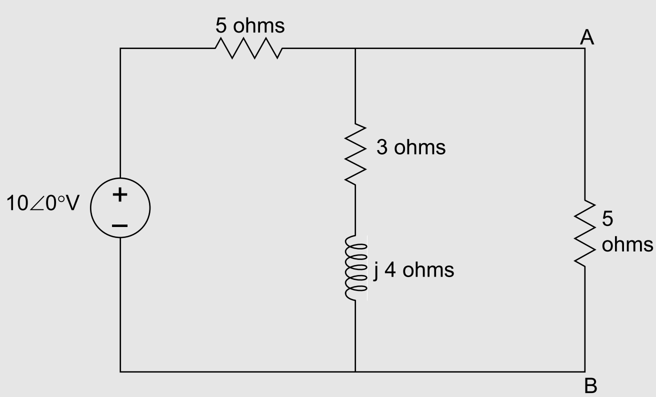


Fig. 5.158

Solution

V_{Th} is calculated by open circuiting AB terminal

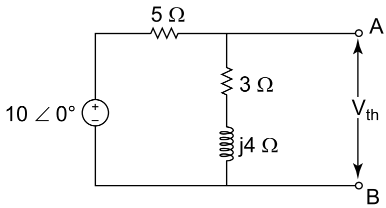


Fig. 5.159(a)

$$\therefore V_{Th} = \frac{10}{5+3+j4} \times (3+j4) \text{ volt}$$

$$= \frac{10 \times 5}{8.94} \angle 26.56° \text{ volt}$$

$$= 5.6\angle 26.56° \text{ volt.}$$

Z_{int} is determined by open circuiting A-B terminal and short circuiting voltage source

$$Z_{int} = \frac{(3+4j)5}{3+4j+5} \text{ ohm}$$

$$= 2.8\angle 26.56° \text{ ohm}$$

$$\therefore \text{Current through } AB = \frac{V_{th}}{Z_{int}+Z_L}$$

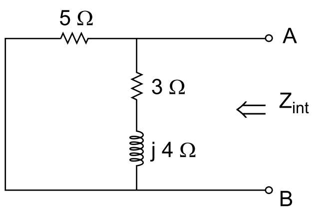


Fig. 5.159(b)

$$= \frac{5.6\angle 26.56°}{2.8\angle 26.56° + 5} \text{ amp}$$

$$= \frac{5.6\angle 26.56°}{7.5+1.252j} \text{ amp}$$

$$= 0.74\angle 17.083° \text{ amp}$$

5.2.2 Norton's Theorem

[JNTU Jan 2010]

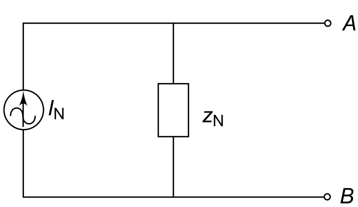


Fig. 5.160

Another method of analysing a complex impedance circuit is given by Norton's theorem. The Norton equivalent form of any complex impedance circuit consists of an equivalent current source I_N and an equivalent impedance Z_N, arranged as shown in Fig. 5.160. The values of equivalent current and impedance depend on the values in the original circuit.

Though Norton's equivalent circuit is not the same as its original circuit, the output voltage and current are the same in both cases; Norton's current is equal to the current passing through the short circuited output terminals and the value of impedance is equal to the impedance seen into the network across the output terminals.

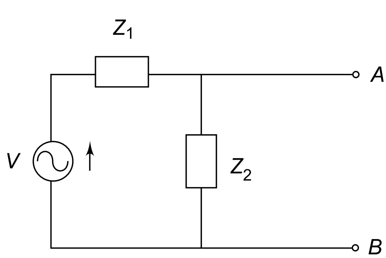


Fig. 5.161

Consider the circuit shown in Fig. 5.161.

Norton's equivalent for the circuit shown in Fig. 5.161 between points A and B is found as follows. The current passing through points A and B when it is short-circuited is the Norton's equivalent current, as shown in Fig. 5.162.

Norton's current $I_N = V/Z_1$

The impedance between points A and B, with the source replaced by a short circuit, is Norton's equivalent impedance. In Fig. 5.161, the impedance from A to B, Z_2 is in parallel with Z_1.

$$\therefore \qquad Z_N = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

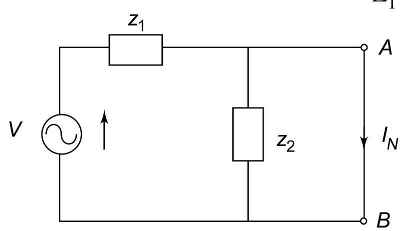


Fig. 5.162

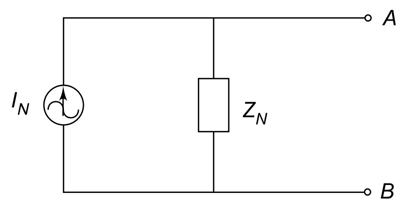


Fig. 5.163

Norton's equivalent circuit is shown in Fig. 5.163.

The advantages seen with Thevenin's theorem apply to Norton's theorem. If we wish to analyze load resistor voltage and current over several different values of load resistance, we can use the Norton's equivalent circuit again and again, applying nothing more complex than simple parallel circuit analysis to determine what's happening with each trial load. This theorem is not applicable to circuits consisting of nonlinear elements and not valid to unilateral circuits. This theorem is not valid where the magnetic coupling exists between load and the circuit.

Example 5.57 *For the circuit shown in Fig. 5.164, determine Norton's equivalent circuit between the output terminals, AB.*

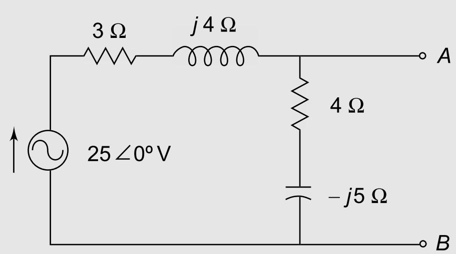


Fig. 5.164

Solution Norton's current I_N is equal to the current passing through the short circuited terminals AB as shown in Fig. 5.165.

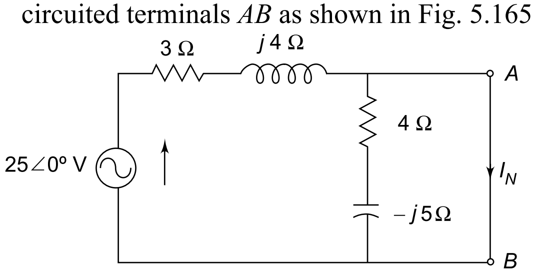


Fig. 5.165

The current through terminals AB is

$$I_N = \frac{25\angle 0^\circ}{3 + j4} = \frac{25\angle 0^\circ}{5\angle 53.13^\circ}$$

$$= 5\angle -53.13^\circ$$

The impedance seen from terminals AB is

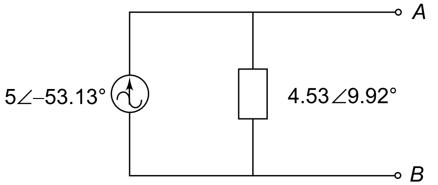


Fig. 5.166

$$Z_N = \frac{(3+j4)(4-j5)}{(3+j4)+(4-j5)}$$

$$= \frac{5\angle 53.13^\circ \times 6.4\angle -51.34^\circ}{7.07\angle -8.13^\circ}$$

$$= 4.53\angle 9.92^\circ$$

Norton's equivalent circuit is shown in Fig. 5.166.

Example 5.58 *For the circuit shown in Fig. 5.167, determine the load current I_L by using Norton's theorem.*

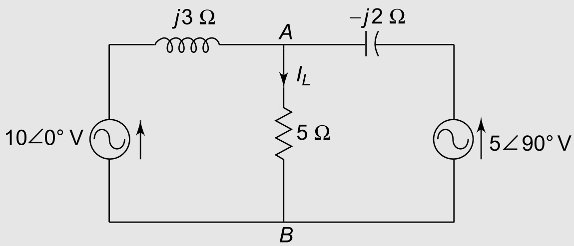


Fig. 5.167

Solution Norton's impedance seen from terminals AB is

$$Z_{AB} = \frac{(j3)(-j2)}{(j3)-(j2)} = \frac{6}{j1}$$

$$\therefore \qquad Z_{AB} = 6\angle -90^\circ$$

Current passing through AB, when it is shorted

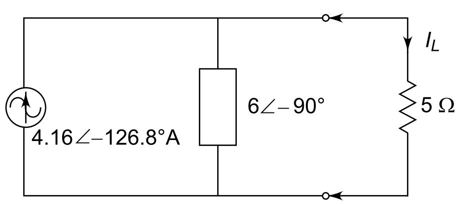


Fig. 5.168

$$I_N = \frac{10\angle 0^\circ}{3\angle 90^\circ} + \frac{5\angle 90^\circ}{2\angle -90^\circ}$$

$$\therefore \quad I_N = 3.33\angle -90^\circ + 2.5\angle 180^\circ$$

$$= -j3.33 - 2.5$$

$$I_N = 4.16\angle -126.8^\circ$$

Norton's equivalent circuit is shown in Fig. 5.168.

$$\text{Load current is } I_L = I_N \times \frac{6\angle -90^\circ}{5+6\angle -90^\circ}$$

$$= 4.16\angle -126.8^\circ \times \frac{6\angle -90^\circ}{5-j6}$$

$$= \frac{4.16\times 6\angle -216.8^\circ}{7.81\angle -50.19^\circ}$$

$$= 3.19\angle -166.61^\circ$$

Example 5.59 *For the circuit shown in Fig. 5.169, determine Norton's equivalent circuit.*

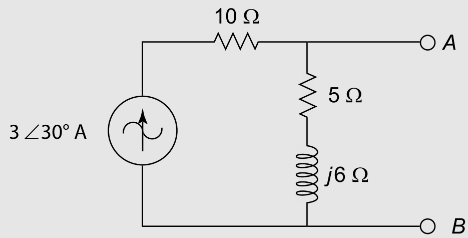


Fig. 5.169

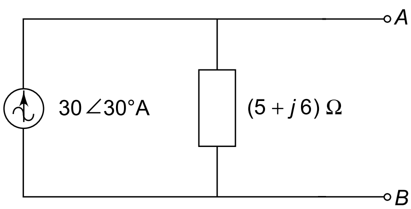


Fig. 5.170

Solution The impedance seen from the terminals when the source is reduced to zero

$$Z_{AB} = (5 + j6)\ \Omega$$

Current passing through the short circuited terminals, *A* and *B*, is

$$I_N = 30\ \angle 30°\ \text{A}$$

Norton's equivalent circuit is shown in Fig. 5.170.

Example 5.60 *Determine the current through the load impedance $Z_L = (8 + j6)\ \Omega$ connected across AB in the network shown in Fig. 5.171 by applying Norton's theorem.* *[JNTU April/May 2002]*

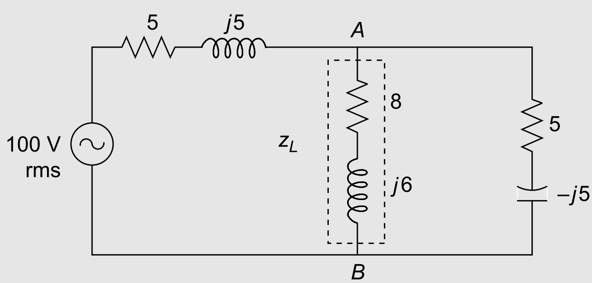


Fig. 5.171

Solution

(i) *To find the Norton's current*

Short the load terminals as shown in Fig. 5.172.

$$I_N = \frac{100}{5 + j5} = 14.142\angle -45^{\circ}\ \text{A}$$

(ii) *To find R_N*

Open the load terminals and replace the source with short circuit as shown in Fig. 5.173.

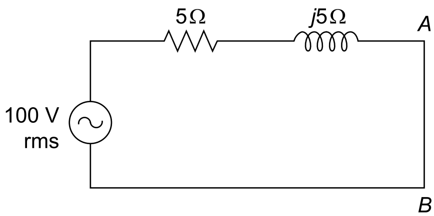


Fig. 5.172

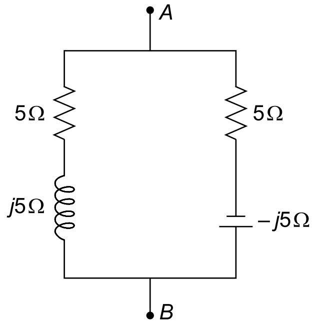


Fig. 5.173

$$R_N = \frac{(5+j5)(5-j5)}{10} = \frac{25+25}{10} = 5\ \Omega$$

$$I_L = \frac{(14.14\angle-45^\circ)5}{13+j6} = \frac{70.7\angle-45^\circ}{14.317\angle24.77^\circ}$$

$$I_L = 4.93\angle-69.77^\circ\ \text{A}$$

$$I_L = 1.704 - j\,4.625\ \text{A}$$

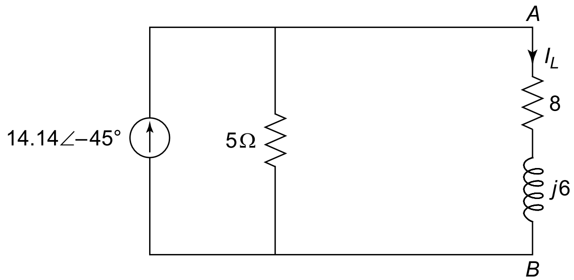


Fig. 5.174

Example 5.61 *Using Norton's theorem, find the current through the load impedance Z_L, for the network as shown in Fig. 5.175.* *[JNTU May/June 2008]*

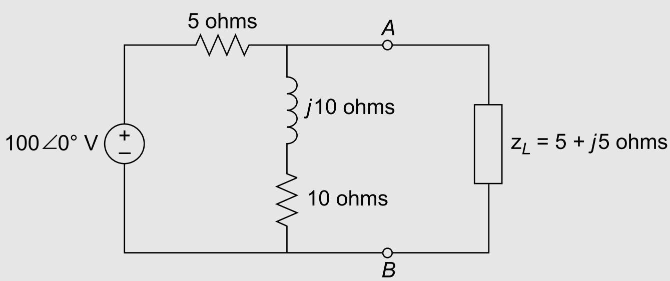


Fig. 5.175

Solution The given network is

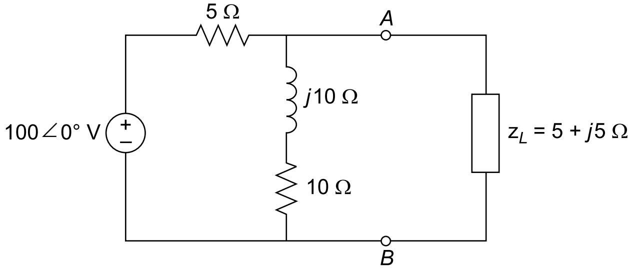


Fig. 5.176(a)

First replace with Norton's equivalent across the terminals *AB*. Norton's current I_N is equal to the current passing through the short-circuited terminals *AB*.

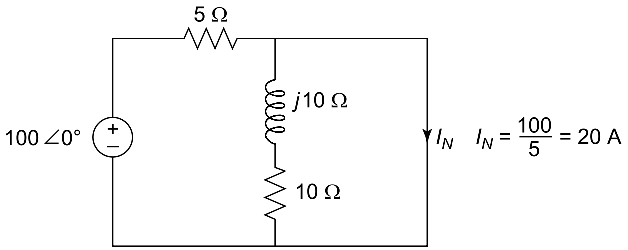


Fig. 5.176(b)

The impedance across the terminals *AB*

$$Z_n = 5 \,||\, (10 + j10)$$

$$= \frac{5 \times (10 + j10)}{15 + j10} = \frac{5 \times 10 \times (1 + j)}{3 + j2} = 3.92\underline{|11.31^\circ}$$

The circuit when replaced is shown below.

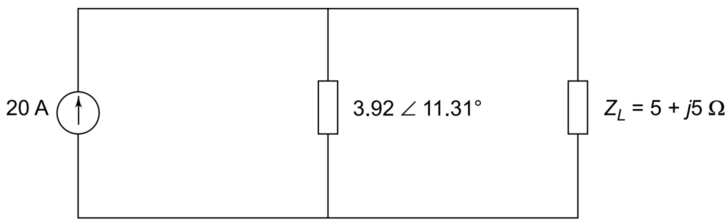


Fig. 5.176(c)

The current flowing through Z_L

$$I_1 = \frac{20 \times 3.92\underline{|11.31^\circ}}{3.92\underline{|11.31^\circ} + 7.071\underline{|45^\circ}}$$

$$= \frac{20 \times 3.92\underline{|11.31^\circ}}{3.843 + j0.768 + 5 + j5}$$

$$= \frac{20 \times 3.92 \angle 11.31^\circ}{8.843 + j5.768}$$

$$= \frac{20 \times 3.92 \angle 11.31^\circ}{10.557 \angle 33.11^\circ} = 7.426 \angle -21.8^\circ$$

Example 5.62 *Using Norton's theorem, find the current through the load impedance Z_L as shown in Fig. 5.177.* *[JNTU June 2009]*

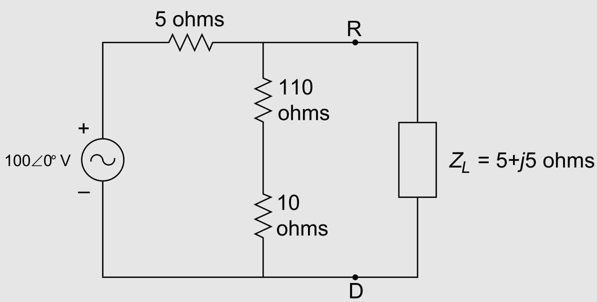


Fig. 5.177

Solution

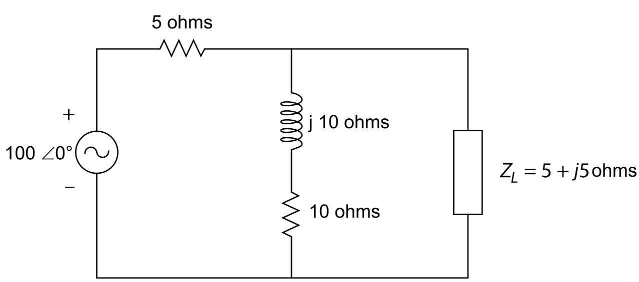


Fig. 5.178(a)

Short circuiting load terminal

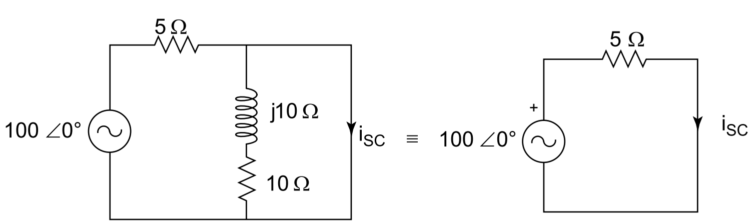


Fig. 5.178(b)

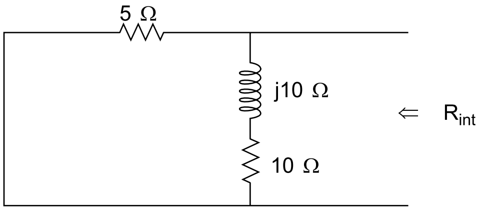


Fig. 5.178(c)

$$\therefore \quad i_{sc} = \frac{100}{5} \text{ amp} = 20 \text{ amp.}$$

To determine the equivalent resistance of the circuit looking through load terminal, the constant source is deactivated as shown

$$\therefore \quad R_{int} = \frac{(10+10j)5}{10+10j+5} \text{ ohm} = \frac{50(1+j)}{5(3+2j)} \text{ ohm} = \frac{10(1+j)}{(3+2j)} \text{ ohms.}$$

So, Norton's equivalent circuit is given as

$$\therefore \quad \text{Current through load } = I_L = i_{sc} \times \frac{R_{int}}{R_{int} + z_L}$$

$$= 20 \times \frac{10(1+j)/(3+2j)}{10(1+j)/(3+2j) + 5(1+j)}$$

$$= 20 \times \frac{10(1+j)}{10(1+j) + 5(1+j)(3+2j)}$$

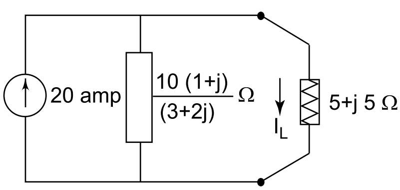


Fig. 5.178(d)

$$= \frac{20 \times 10}{5(1+j)} \times \frac{(1+j)}{2+3+2j}$$

$$= \frac{40}{5+2j} \text{ amp} = 7.428\angle -21.8° \text{amp}$$

$$= 6.897 - 2.758\,j$$

Example 5.63 *Obtain Norton's equivalent across terminals A and B for network shown in Fig. 5.179.* *[JNTU June 2009]*

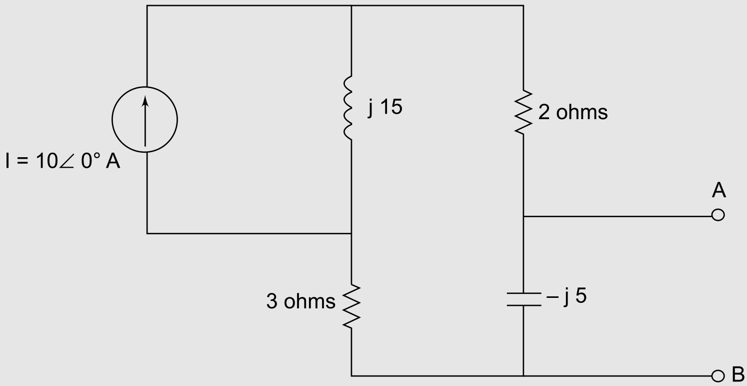
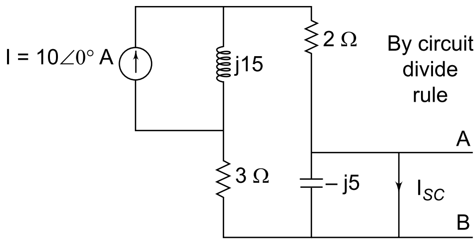


Fig. 5.179

Solution



By circuit divide rule

$$I_{SC} = \frac{(j15)10\angle 0^{\circ}}{(5+j15)}\,\text{A}$$

$$= 9 + 3j$$

$$= 9.486\angle 18.435^{\circ}$$

Fig. 5.180(a)

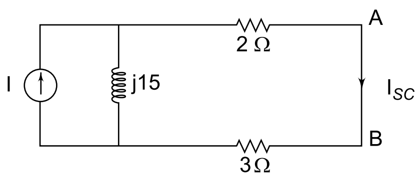


Fig. 5.180(b)

Now deactivating the source

$$= (-j5)||(5+15j)$$

$$= 1 - j7.$$

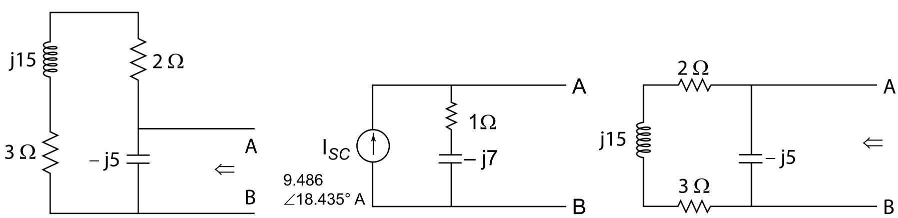


Fig. 5.180(c)

Example 5.64 *Using Norton's theorem, find the current through the load impedance Z_L, for the network as shown in Fig. 5.181.* *[JNTU Jan 2010]*

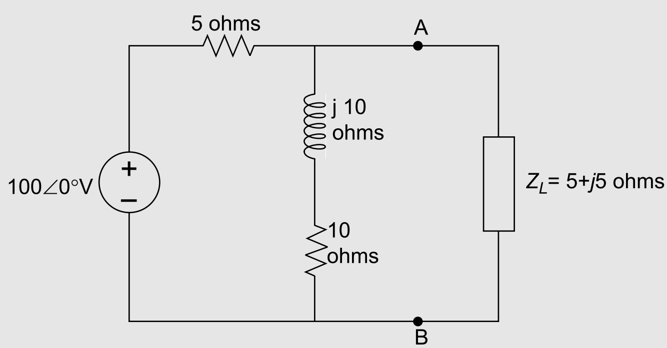


Fig. 5.181

Solution

To measure internal resistance Z_L is removed and voltage source is short circuited giving

$$\therefore \quad R_{int} = \frac{(10+10j)5}{10+10j+5}\Omega$$

$$= \frac{\cancel{5}\times 10(1+j)}{\cancel{5}(3+2j)} = \frac{10\sqrt{1^2+1^2}}{\sqrt{3^2+2^2}}\angle \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{2}{3}\right)$$

$$= 3.92\angle 11.31°\Omega$$

$$\therefore \quad Y_{11} = \frac{\Delta 11}{\Delta} = \frac{2+\frac{4}{s}+\frac{1}{s^2}}{\frac{4}{s}+\frac{2}{s^2}} = \frac{2s^2+4s+1}{2s(1+2s)}$$

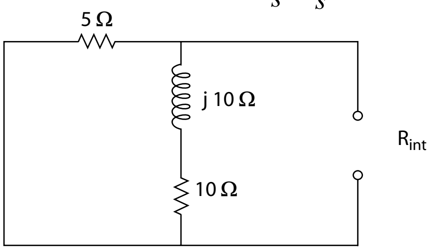


Fig. 5.182

$$Y_{12} = -\frac{\Delta 21}{\Delta} = \frac{1+2s+2s^2}{2s(1+2s)}$$

$$Y_{21} = -\frac{\Delta 12}{\Delta} = \frac{1+2s+2s^2}{2s(1+2s)}$$

$$Y_{22} = \frac{\Delta 22}{\Delta} = \frac{2s^2+4s+1}{2s(1+2s)}$$

In this problem, $\Delta_{11} = \Delta_{22}, \Delta_{12} = \Delta_{21}$

$Y_{11} = Y_{22}, Y_{12} = Y_{21}$

$\therefore$ The network is symmetrical and reciprocal.

5.2.3 Millman's Theorem

[JNTU June 2009, Jan 2010]

Millman's Theorem states that in any network, if the voltage sources $V_1, V_2, \ldots V_n$ in series with internal impedances $Z_1, Z_2, \ldots Z_n$, respectively, are in parallel, then these sources may be replaced by single voltage source V' in series with an impedance Z' as shown in Fig. 5.183.

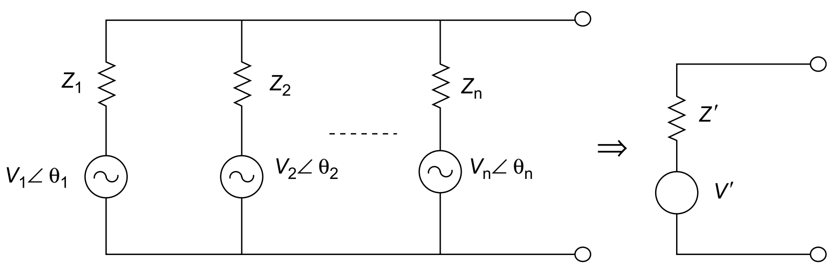


Fig. 5.183

where $V' = \dfrac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i}$ and $Z' = \dfrac{1}{\sum_{i=1}^{n} Y_i}$

A similar theorem can be stated for n current sources having internal admittances which can be replaced by a current source I' in parallel with an equivalent admittance.

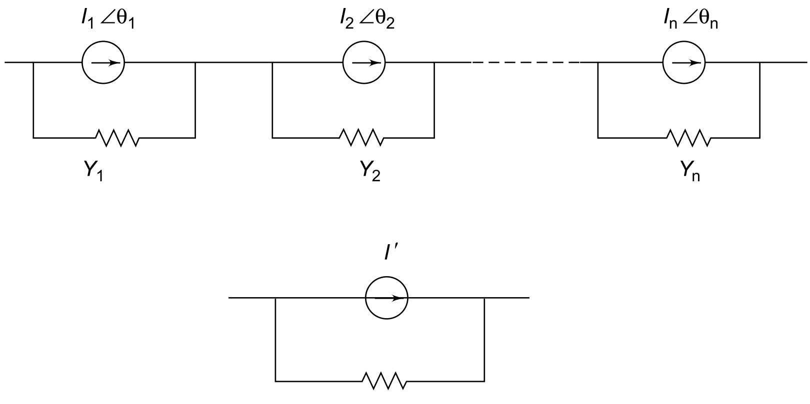


Fig. 5.184

where $$I' = \frac{\sum_{i=1}^{n} I_i Z_i}{\sum_{i=1}^{n} Z_i} \quad \text{and} \quad Y' = \frac{1}{\sum_{i=1}^{n} Z_i}$$

$$Z_i = \frac{1}{Y_i}$$

Millman's theorem is very convenient for determining the voltage across a set of parallel branches, where there are enough voltage sources present to preclude solution via regular series – parallel reduction method. It doesn't require the use of simultaneous equations. However, it is limited in that it only applied to circuits which can be redrawn to fit this form. It can not be used to solve an unbalanced bridge circuit.

Example 5.65 *Calculate the current flowing in 2 Ω resistance shown in Fig. 5.185.*

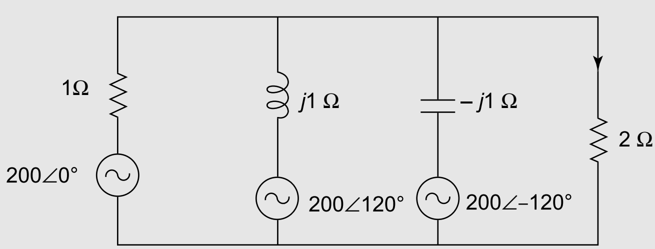


Fig. 5.185

Solution The above circuit can be redrawn as shown in Fig. 5.186.

$\therefore$ From Millman's theorem, the equivalent impedance is given by

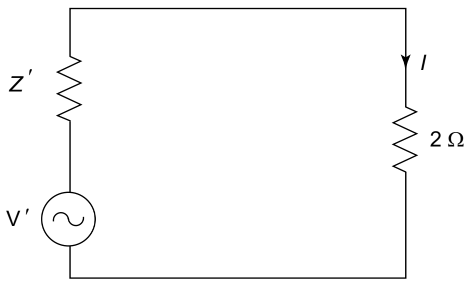


Fig. 5.186

$$Z' = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{1 - \frac{1}{j1} + \frac{1}{j1}} = 1\,\Omega$$

$$V' = \frac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y}$$

$$= 200 + 200\underline{|-120^\circ}(-j1) + 200\underline{|120^\circ}(j1)$$

$$= 200 - 200\underline{|-30^\circ} + 200\underline{|210^\circ}$$

$$= -146.5\underline{|0^\circ}\ \text{V}$$

The current in 2 Ω resistance

$$I = \frac{V^1}{Z^1 + 2} = \frac{-146.5\underline{|0^\circ}}{3}$$

$$= -48.67\underline{|0^\circ}\ \text{A}$$

Example 5.66 *Use Millman's theorem to find the current in the load Z_L in the circuit shown in Fig. 5.187.*

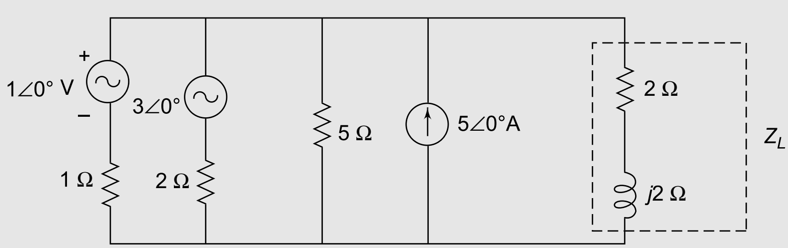


Fig. 5.187

Solution First converting the current source $5\underline{|0^\circ}$ A in parallel with 5 Ω resistance is connected into the voltage source in series with resistance.

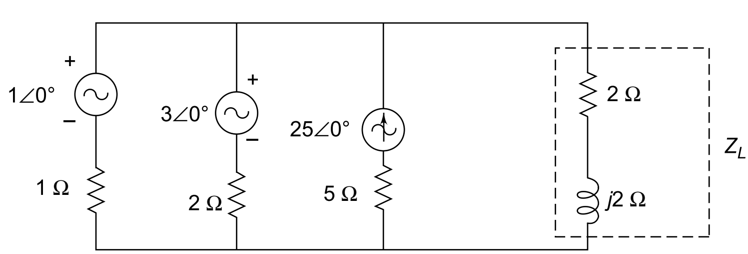


Fig. 5.188

The above circuit can be redrawn as shown in Fig. 5.188.

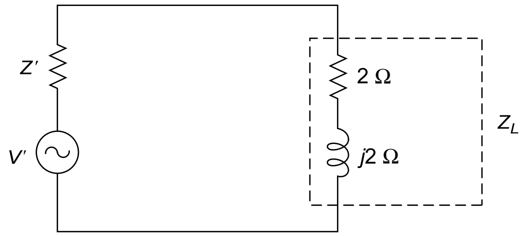


Fig. 5.189

$\therefore$ From Millman's theorem, the equivalent impedance is given by

$$Z' = \frac{1}{Y_1 + Y_2 + Y_3}$$

$$= \frac{1}{\frac{1}{1}+\frac{1}{2}+\frac{1}{5}} = 0.59\ \Omega$$

Voltage source $\quad V' = \dfrac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y^1}$

where $\quad Y' = \dfrac{1}{Z^1} = \dfrac{1}{0.59} = 1.695\ \mho$

$$\therefore \quad V' = \frac{1\angle 0^\circ \times 1 + 3\angle 0^\circ \times \frac{1}{2} + 25\angle 0^\circ \times \frac{1}{5}}{1.695}$$

$$V' = \frac{7.5\angle 0^\circ}{1.695}$$

$$= 4.42\angle 0^\circ \text{ Volts}$$

The load current $\quad I_L = \dfrac{V'}{Z' + Z_L}$

$$= \frac{4.42\angle 0^\circ}{0.59 + 2 + j2\ \Omega}$$

$$I_L = \frac{4.42\angle 0^\circ}{3.27\angle 37.67^\circ} = 1.35\angle -37.67^\circ \text{ A}$$

5.2.4 Reciprocity Theorem

[JNTU Jan 2009]

In a linear bilateral single source network, if a single voltage (current) source in one branch 'a' of the network produces a current (voltage) in branch 'b', then if the voltage (current) source is shifted to branch 'b' will produce a current (voltage) in branch 'a'. The ratio of excitation and response is same in both the cases. This theorem is valid for networks comprising of linear, bilateral, passive elements energised by a single voltage or current source.

The above theorem can be verified by a simple example.

Example 5.67 *Verify the reciprocity theorem for the network shown in Fig. 5.190.*

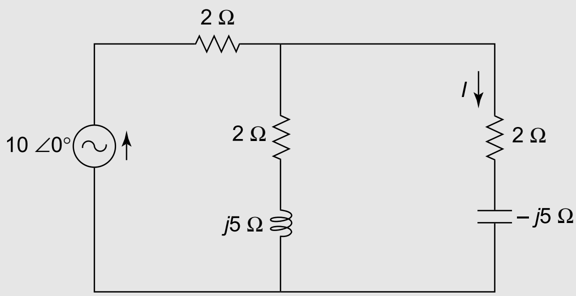


Fig. 5.190

Solution Total impedance in the circuit = $2+[(2+j5) \parallel (2-j5)] = 9.25\ \Omega$

The current drawn by the circuit

$$I_T = \frac{10\underline{|0^\circ}}{9.25} = 1.08\underline{|0^\circ}\ \text{A}$$

The current in the $(2 - j5)\ \Omega$ branch

$$I = I_T \times \frac{2+j5}{2+j5+2-j5}$$

$$= 1.08\underline{|0^\circ} \times \frac{2+j5}{4} = 1.45\underline{|68.2^\circ}$$

Applying the reciprocity theorem, by interchanging the source and response, we get

Total Impedance in the circuit = $[2 \parallel (2 + j5)] + 2 - j5$

$$= \frac{2(2+j5)}{2+2+j5} + 2 - j5$$

$$= 5.8\underline{|-50.1^\circ}\ \Omega$$

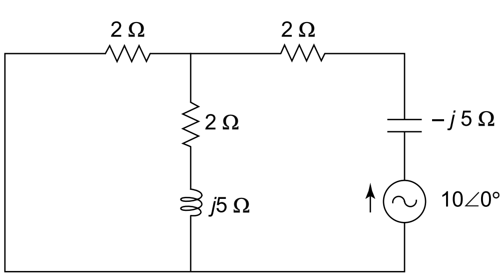


Fig. 5.191

$$\text{Total current drawn by the circuit} = \frac{10\angle 0^\circ}{5.8\angle -50.1^\circ}$$

$$= 1.72\angle 50.1^\circ \text{ A}$$

$$\text{The current in the } 2\,\Omega \text{ branch is } = 1.72\angle 50.1^\circ \times \frac{2+j5}{4+j5}$$

$$= 1.45\angle 67^\circ \text{ A}$$

If we compare the results in both cases, the ratio of input to response is same.

Example 5.68 *Verify reciprocity theorem for the voltage source and the current I in the circuit shown in Fig. 5.192.*

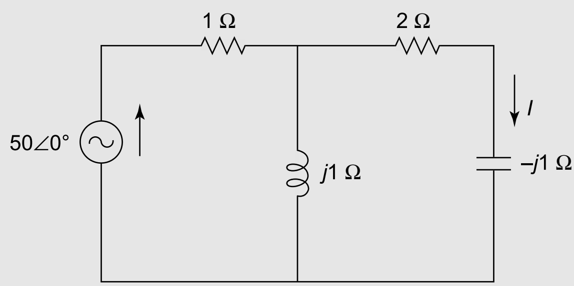


Fig. 5.192

Solution Total impedance in the circuit $Z_T = [1 + [(j1) \parallel (2 - j1)]]$

$$Z_T = 1.81\angle 33.69^\circ\ \Omega$$

Total current drawn by the circuit

$$I_T = \frac{5\angle 0^\circ}{1.81\angle 33.69^\circ}$$

$$= 2.76\angle -33.69^\circ \text{ A}$$

The current in the $(2 - j1)\ \Omega$ branch

$$I = I_T \times \frac{j1}{2 - j1 + j1}$$

$$= 2.76\angle -33.69^\circ \times \frac{1\angle 90^\circ}{2}$$

$$= 1.38\angle 56.31^\circ \text{ A}$$

Applying the reciprocity theorem, by interchanging the source and response, we get
Total impedance in the circuit $Z_T = [1 \parallel (j1) + (2 - j1)]$

$$Z_T = 2.55\angle -11.30^\circ\ \Omega$$

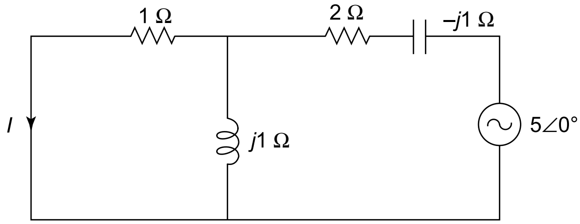


Fig. 5.193

Total current drawn by the circuit

$$I_T = \frac{5\angle 0^\circ}{2.55\angle -11.30^\circ}$$

$$= 1.96\angle 11.30^\circ \text{ A}$$

The current I in 1 Ω branch

$$I = \frac{1.96\angle 11.30^\circ \times 1\angle 90^\circ}{1.414\angle 45^\circ}$$

$$= 1.38\angle 56.36^\circ \text{ A}$$

The voltage to current ratio is same in both the circuits.

Example 5.69 *In a single current source circuit shown in Fig. 5.194, find the voltage V. Verify the reciprocity theorem for the circuit.*

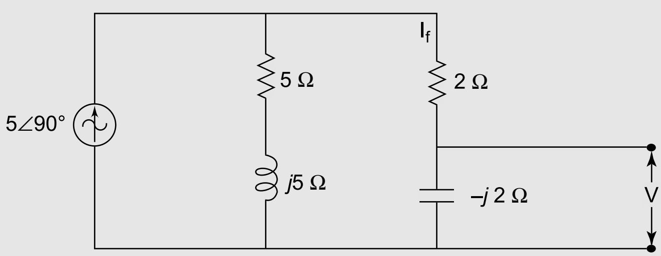


Fig. 5.194

Solution The voltage across $(-j2)$ Ω impedance

$$V = I(-j2) \text{ volts}$$

where the current passing through $(-j2)$ Ω is

$$I = 5\angle 90^\circ \times \frac{5 + j5}{5 + j5 + 2 - j2}$$

$$= 4.65\angle 111.8^\circ \text{ A}$$

$$\therefore \text{ The voltage } V = 4.65\angle 111.8^\circ \times 2\angle -90^\circ$$

$$= 9.24\angle 21.8^\circ \text{ Volts}$$

Applying reciprocity theorem by interchanging source and response as shown in the circuit of Fig. 5.195.

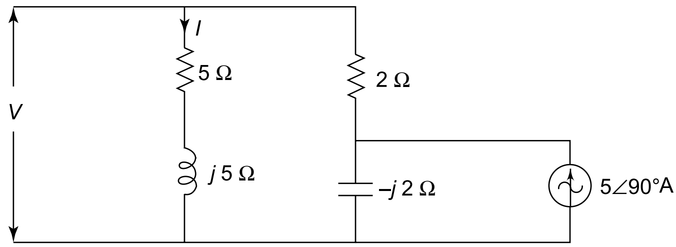


Fig. 5.195

The current passing through (5 + j5) Ω impedance

$$I = 5\underline{|90^\circ} \times \frac{-j2}{7 + j5 - j2}$$

$$I = 1.31\underline{|-23.2^\circ} \text{ A}$$

The voltage across (5 + j5) Ω impedance

$$V = (5 + j5) \times 1.31\underline{|-23.2^\circ}$$

$$= 9.25\underline{|21.8^\circ} \text{ voltage}$$

The response to excitation ratio is same in both the circuits.

5.2.5 Compensation and Substitution Theorems

The compensation theorem states that any impedance having voltage across its terminal in the linear, bilateral network, may be replaced by a voltage source of zero internal impedance equal to the current passing through the impedance multiplied by the value of the impedance, provided the currents and voltages in other part of the network remain unaltered.

Let a branch of a network contain impedance Z_1 and Z_2. If the current in this branch is I, the voltage drop across Z_1 is IZ_1 with polarity as shown in Fig. 5.196(a). Fig. 5.196(b) shows the compensation source $V_C = IZ_1$ which replace Z_1. However V_C must have polarity as shown in Fig. 5.196(b). If any chance which should effect I occurs in the network then the compensation source must be changed accordingly. The compensation is often referred as substitution theorem. This theorem is of use, when it is required to evaluate the changes in magnitudes of currents and voltages in the different branches of a network, due to a small change in the impedance of one of the branches.

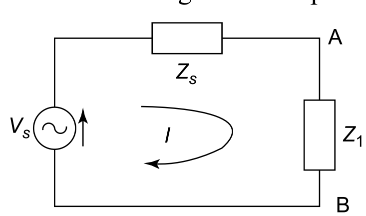


Fig. 5.196(a)

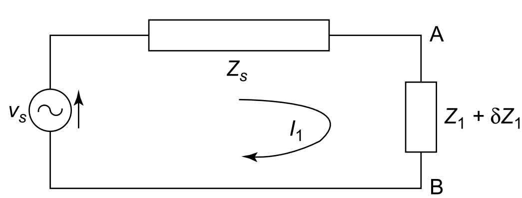


Fig. 5.196(b)

Consider a network shown in Fig. 5.196(a).

The current in the circuit is $I = \dfrac{V_s}{Z_s + Z_1}$

Let the impedance of branch AB change from Z_1 to $(Z_1 + \delta Z_1)$. Let I_1 be the new current.

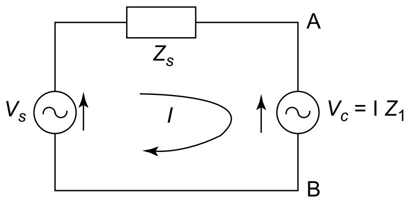


Fig. 5.196(c)

The current $I_1 = \dfrac{V_s}{Z_s + Z_1 + \delta Z_1}$

The impedance Z_1 of the network shown in Fig. 5.196(a) may be replaced by a voltage source, V_C. By substitution theorem $V_C = IZ_1$ with polarity as shown in Fig. 5.196(c).

Similarly, the network shown in Fig. 5.196(b) can be replaced by the network shown in Fig. 5.196(d).

Let δI_1 denote the small change in current, due to the small change in the impedance value by δZ_1.

$$\therefore \qquad \delta I_1 = I - I_1 = \frac{V_s}{Z_s + Z_1} - \frac{V_s}{Z_s + Z_1 + \delta Z_1}$$

$$= \frac{V_s \cdot \delta Z_1}{(Z_s + Z_1)(Z_s + Z_1 + \delta Z_1)}$$

$$\delta I_1 = I \cdot \frac{\delta Z_1}{Z_s + Z_1 + \delta Z_1}$$

$$\text{since} \qquad I = \frac{V_s}{Z_s + Z_1}$$

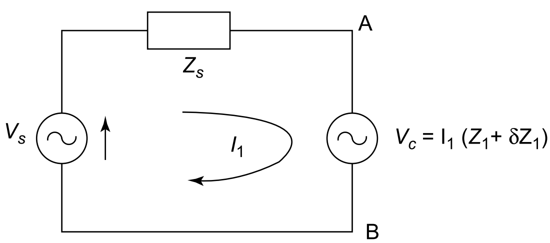


Fig. 5.196(d)

The network for which the above relationship holds good is as shown in Fig. 5.196(e).

By compensation theorem the small change in the magnitude of current due to a small change in a branch impedance is given by

$$\delta I_1 = \frac{I\,\delta Z_1}{Z_s + Z_1 + \delta Z_1}$$

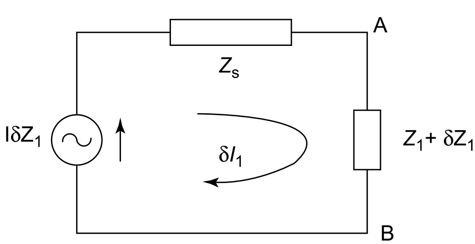


Fig. 5.196(e)

Therefore, the original voltage source should be set equal to zero and a new voltage source $I\delta Z_1$ must be introduced with correct polarity.

Example 5.70 *For the circuit shown in Fig. 5.197 find the change in the current by using compensation theorem when the reactance has changed to j5 Ω.*

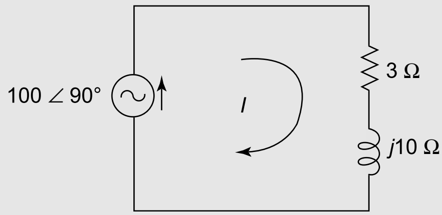


Fig. 5.197

Solution The current in the circuit shown is $I = \dfrac{100\angle 90^\circ}{3+j10} = 9.58\angle 16.7^\circ$ A

The inductive reactance is changed from $j10\ \Omega$ to $j5\ \Omega$

$\therefore$ Change in impedance $\delta Z = j5\,\Omega$.

The new circuit is shown in Fig. 5.198.

The change in current due to change in impedance

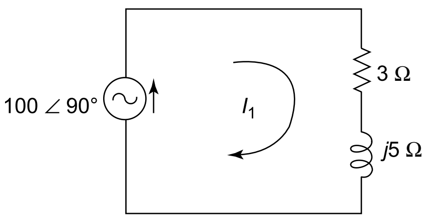


Fig. 5.198

$$\delta I = \frac{I\cdot\delta Z}{Z_{\text{total}}} = \frac{9.58\angle 16.7^\circ \times j5}{3+j5}$$

$$\delta I = 8.22\angle 47.7^\circ \text{ A.}$$

Example 5.71 *In the network shown in Fig. 5.199, the 2 Ω resistor is changed to 4 Ω. Determine the resulting change in current through the load impedance, using compensation theorem.*

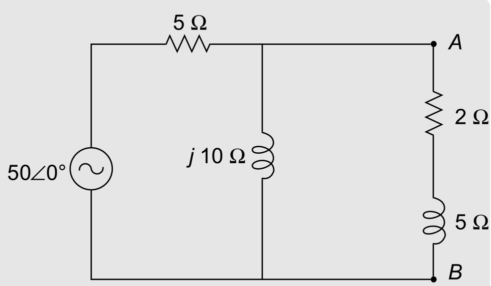


Fig. 5.199

Solution The thevenin's equivalent circuit of a given network with open circuit terminals shown in Fig. 5.200.

Open circuit voltage across terminals AB

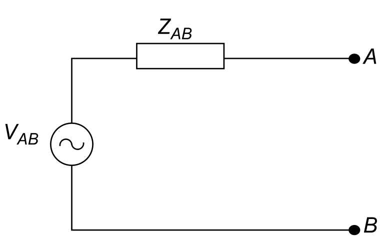


Fig. 5.200

$$V_{AB} = (j10)\,\frac{50\angle 0^\circ}{5+j10} = \frac{10\angle 90^\circ\ 50\angle 0^\circ}{11.18\angle 63.43^\circ}$$

$$V_{AB} = 44.72\angle 26.57^\circ \text{ Volts}$$

The impedance seen into the terminals AB

$$Z_{AB} = 5\,\|\,(j10) = \frac{5(10\angle 90^\circ)}{5+j10}$$

$$Z_{AB} = 4.472\angle 26.57^\circ = (4+j2)\ \Omega$$

The Thevenin's equivalent circuit is shown in Fig. 5.201.

Current $$I = \frac{44.72\angle 26.57^\circ}{6 + j7} = 4.86\angle -22.93^\circ \text{ A}$$

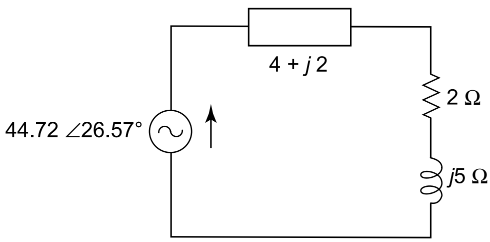


Fig. 5.201

when the impedance of $2\,\Omega$ is change to $4\,\Omega$, the Thevenin's equivalent circuit with new load impedance is shown in Fig. 5.202.

Change in impedance $\delta Z = 4 - 2 = 2\,\Omega$

Total impedance $= (8 + j7)\,\Omega$ $= 10.63\angle 41.18^\circ\,\Omega$

By compensation theorem, we have
Change in current

$$\delta I = \frac{I \cdot \delta Z}{Z_{\text{total}}} = \frac{4.86\angle -22.93^\circ \times 2}{10.63\angle 41.18^\circ}$$

$$\therefore \quad \delta I = 0.914\angle -64.11^\circ \text{ A}$$

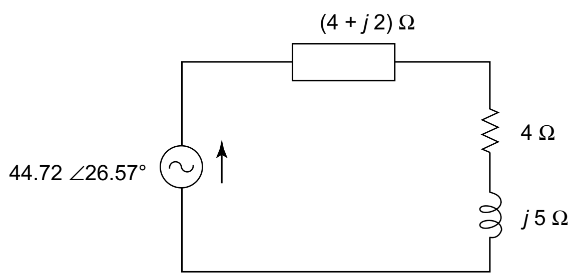


Fig. 5.202

5.2.6 Superposition Theorem

[JNTU Jan 2010]

The superposition theorem can be used to analyse ac circuits containing more than one source. The superposition theorem states that the response in any element in a circuit is the vector sum of the responses that can be expected to flow if each source acts independently of other sources. As each source is considered, all of the other sources are replaced by their internal impedances, which are mostly short circuits in the case of a voltage source, and open circuits in the case of a current source. This theorem is valid only for linear systems. In a network containing complex impedance, all quantities must be treated as complex numbers.

Consider a circuit which contains two sources as shown in Fig. 5.203.

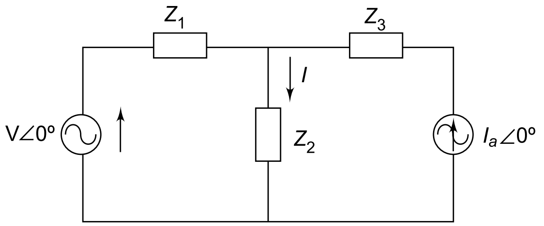


Fig. 5.203

Now let us find the current I passing through the impedance Z_2 in the circuit. According to the superposition theorem, the current due to voltage source V $\angle 0°$ V is I_1 with current source $I_a \angle 0°$ A open circuited.

$$I_1 = \frac{V \angle 0°}{Z_1 + Z_2}$$

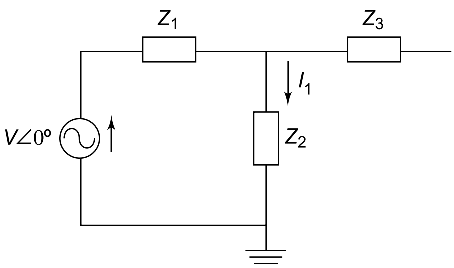


Fig. 5.204

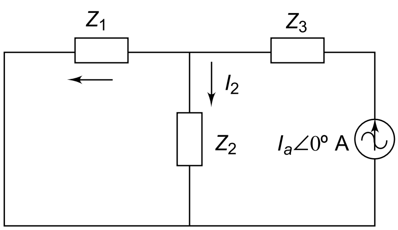


Fig. 5.205

The current due to $I_a \angle 0°$ A is I_2 with voltage source $V\angle 0°$ short circuited.

$$\therefore \qquad I_2 = I_a \angle 0° \times \frac{Z_1}{Z_1 + Z_2}$$

The total current passing through the impedance Z_2 is

$$I = I_1 + I_2$$

The superposition theorem finds use in the study of AC circuits, amplifier circuits, where sometimes AC is often superimposed with DC. This theorem defines the behaviour of a linear circuit. Within the context of linear circuit analysis, this theorem provides the basis for all other theorems. Given a linear circuit, it is easy to see how mesh analysis and nodal analysis make use of the principle of superposition.

It is not possible to apply superposition theorem directly to determine power associated with an element. In addition, application of superposition theorem does not normally lead to simplification of analysis. It is not best technique to determine all currents and voltages in a circuit, driven by multiple sources. Superposition theorem works only for circuits that are reducible to series/ parallel combinations for each of the sources at a time. This theorem is useless for analyzing an unbalanced bridge circuit. Networks containing components like lamps or varistors could not be analyzed.

Example 5.72 *Determine the voltage across (2 + j5) Ω impedance as shown in Fig. 5.206 by using the superposition theorem.*

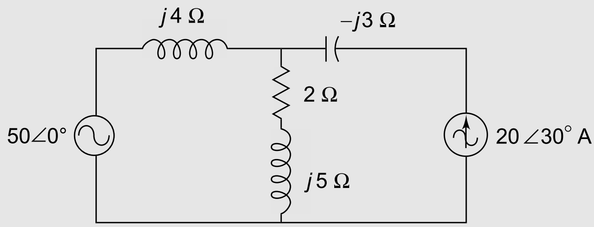


Fig. 5.206

Solution According to the superposition theorem, the current due to the 50∠0° V voltage source is I_1 as shown in Fig. 5.207 with current source 20∠30° A open circuited.

$$\text{Current} \qquad I_1 = \frac{50\angle 0^\circ}{2+j4+j5} = \frac{50\angle 0^\circ}{(2+j9)}$$

$$= \frac{50\angle 0^\circ}{9.22\angle 77.47^\circ} = 5.42\angle -77.47^\circ \text{ A}$$

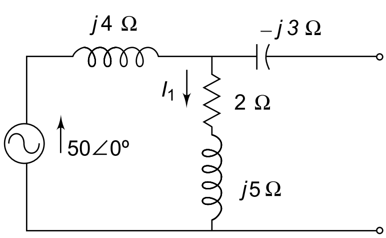


Fig. 5.207

Voltage across (2 + j5) Ω due to current I_1 is

$$V_1 = 55.42\angle -77.47^\circ\,(2+j5)$$

$$= (5.38)(5.42)\angle -77.47^\circ + 68.19^\circ$$

$$= 29.16\angle -9.28^\circ$$

The current due to 20∠30° A current source is I_2 as shown in Fig. 5.208, with voltage source 50 ∠0° V short circuited.

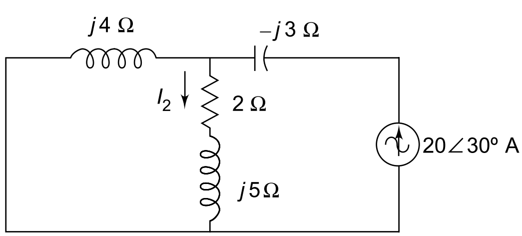


Fig. 5.208

$$\text{Current} \qquad I_2 = 20\angle 30^\circ \times \frac{(j4)\,\Omega}{(2+j9)\,\Omega}$$

$$= \frac{20\angle 30^\circ \times 4\angle 90^\circ}{9.22\angle 77.47^\circ}$$

$$\therefore \qquad I_2 = 8.68\angle 120^\circ - 77.47^\circ = 8.68\angle 42.53^\circ$$

Voltage across (2 + $j5$) Ω due to current I_2 is

$$V_2 = 8.68\angle 42.53° \,(2 + j5)$$
$$= (8.68)\,(5.38)\,\angle 42.53° + 68.19°$$
$$= 46.69\,\angle 110.72°$$

Voltage across (2 + $j5$) Ω due to both sources is

$$V = V_1 + V_2$$
$$= 29.16\,\angle -9.28° + 46.69\angle 110.72°$$
$$= 28.78 - j4.7 - 16.52 + j43.67$$
$$= (12.26 + j38.97)\text{ V}$$

Voltage across (2 + $j5$) Ω is $V = 40.85\,\angle 72.53°$.

Example 5.73 *For the circuit shown in Fig. 5.209, determine the voltage V_{AB} using the superposition theorem.*

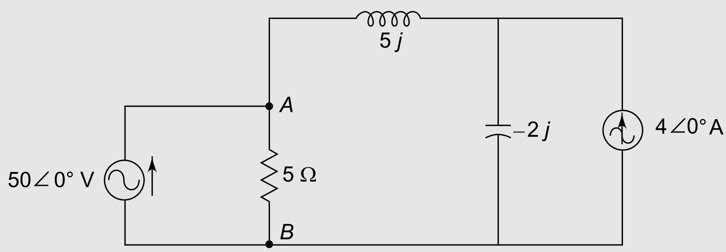


Fig. 5.209

Solution Let source 50∠0° V act on the circuit and set the source 4∠0° A equal to zero. If the current source is zero, it becomes open-circuited. Then the voltage across AB is V_{AB} = 50∠0°.

Now set the voltage source 50∠0° V is zero, and is short circuited, or the voltage drop across AB is zero.

The total voltage is the sum of the two voltages.

∴ $$V_T = 50\angle 0°$$

Example 5.74 *For the circuit shown in Fig. 5.210, determine the current in (2+j3) Ω by using the superposition theorem.*

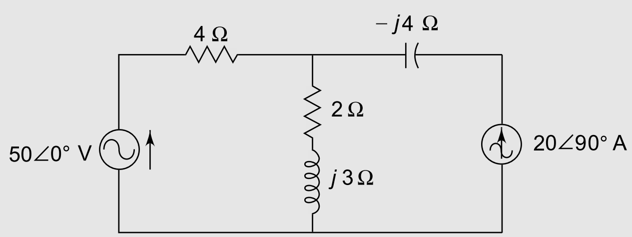


Fig. 5.210

Solution The current in (2 + j3) Ω, when the voltage source $50\angle 0°$ acting alone is

$$I_1 = \frac{50\angle 0°}{(6+j3)} = \frac{50\angle 0°}{6.7\angle 26.56°}$$

$$\therefore \qquad I_1 = 7.46\angle -26.56° \text{ A}$$

Current in (2 + j3) Ω, when the current source $20\angle 90°$ A acting alone is

$$I_2 = 20\angle 90° \times \frac{4}{(6+j3)}$$

$$= \frac{80\angle 90°}{6.7\angle 26.56°} = 11.94\angle 63.44° \text{ A}$$

Total current in (2 + j3) Ω due to both sources is

$$I = I_1 + I_2$$

$$= 7.46\angle -26.56° + 11.94\angle 63.44°$$

$$= 6.67 - j3.33 + 5.34 + j10.68$$

$$= 12.01 + j7.35 = 14.08\angle 31.46°$$

Total current in (2 + j3) Ω is $I = 14.08\angle 31.46°$

Example 5.75 *Find the current in the 6 Ω resistor using superposition theorem as shown in Fig. 5.211.* *[JNTU May/June 2006]*

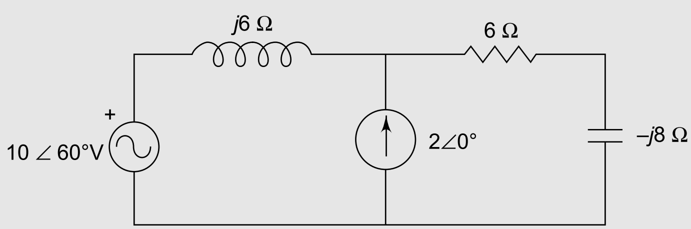


Fig. 5.211

Solution

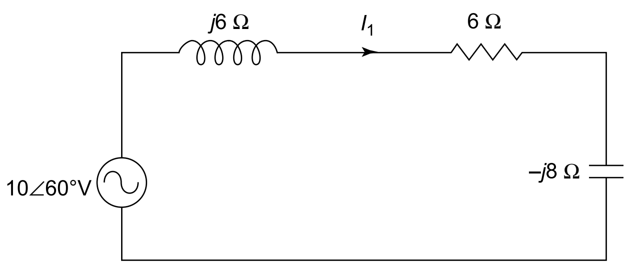


Fig. 5.212 (a)

$$I_1 = \frac{10\angle 60°}{6+j6-j8} = \frac{10\angle 60°}{6-j2} = \frac{10\angle 60°}{6.32\angle -18.43°} = 1.58\angle 78.43° \text{ A}$$

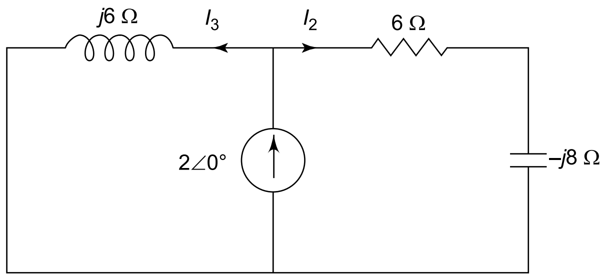


Fig. 5.212 (b)

$$I_2 = 2\angle 0^\circ \times \frac{j6}{6+j6-j8} = 2\angle 0^\circ \times \frac{j6}{6-j2}$$

$$= 1\angle 0^\circ \times \frac{j6}{3-j1} = \frac{1\angle 0^\circ \times 6\angle 90^\circ}{3.16\angle -18.43^\circ} = 1.899\angle 108.43^\circ \text{ A}$$

By superposition theorem, current through

$$6\ \Omega = I_1 + I_2$$
$$= 1.58\ \angle 78.43^\circ + 1.899\angle 108.43^\circ$$
$$= 0.317 + j1.548 + [-0.6 + j1.8]$$
$$= -0.283 + j3.348 = 3.36\angle 94.83^\circ\text{A}$$

Example 5.76 *Determine the current I in the circuit shown in Fig. 5.213 using superposition theorem:* *[JNTU May/June 2002]*

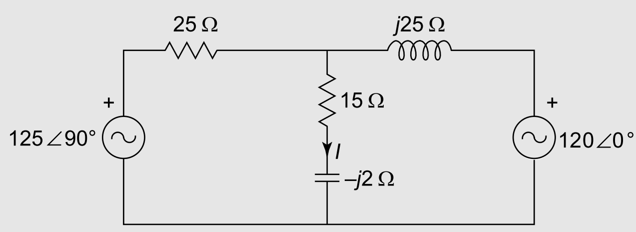


Fig. 5.213

Solution Consider $125\angle 90^\circ$ volt voltage source and short circuiting the other voltage source.

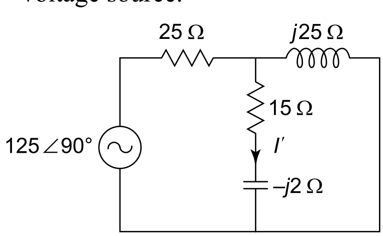


Fig. 5.214 (a)

$$R_{eq} = \frac{j25(15-j2)}{j25+15-j2} + 25 = 37.9\underline{|9.01}\ \Omega$$

$$I_s = \frac{125\underline{|90^\circ}}{37.9\underline{|9.01}}$$

$$= 3.29\underline{|80.99}\ \text{A}$$

$$I' = I_s \times \frac{j25}{15 - j2 + j25} = 3.29\underline{|80.99} \times \frac{25\underline{|90^\circ}}{27.45\underline{|56.88^\circ}}$$

$$I' = 2.99\underline{|114.11}\,\text{A}$$

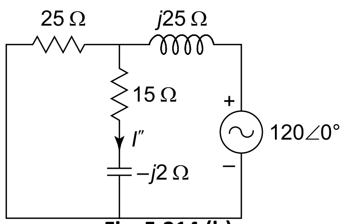


Fig. 5.214 (b)

Now consider $120\angle 0^\circ$ V voltage source and short circuit the other voltage source.

$$R_{\text{eq}} = \frac{25(15 - j2)}{25 + 15 - j2} + j25$$

$$R_{\text{eq}} = 25.99\underline{|68.75^\circ}$$

$$I_s = \frac{120\underline{|0^\circ}}{25.99\underline{|68.75^\circ}} = 4.617\underline{|-68.75^\circ}\,\text{A}$$

$$I'' = I_s \times \frac{25}{15 - j2 + 25} = I_s \times \frac{25}{40 - j2}$$

$$I'' = 4.617\underline{|-68.75} \times \frac{25}{40 - j2}$$

$$I'' = 2.88\underline{|-65.89^\circ}$$

$$I = I' + I''$$

$$= 2.99\underline{|114.11^\circ} + 2.88\underline{|-65.89^\circ}$$

$$= -1.22 + j2.729 + 1.176 - j2.62$$

$$= -0.044 + j0.1 = 0.1\underline{|113.74^\circ}\,\text{A}$$

5.2.7 Maximum Power Transfer Theorem *[JNTU Jan 2010]*

The maximum power transfer theorem has been discussed for resistive loads. The maximum power transfer theorem states that the maximum power is delivered from a source to its load when the load resistance is equal to the source resistance. It is for this reason that the ability to obtain impedance matching between circuits is so important. For example, the audio output transformer must match the high impedance of the audio power amplifier output to the low input impedance of the speaker. Maximum power transfer is not always desirable, since the transfer occurs at a 50 per cent efficiency. In many situations, a maximum voltage transfer is desired which means that unmatched impedances are necessary. If maximum power transfer is required, the load resistance should equal the given source resistance. The maximum power transfer theorem can be applied to complex impedance circuits. If the source impedance is complex, then the maximum power transfer occurs when the load impedance is the complex conjugate of the source impedance.

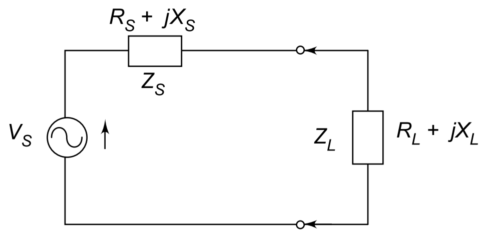


Fig. 5.215

Consider the circuit shown in Fig. 5.215, consisting of a source impedance delivering power to a complex load.

Current passing through the circuit shown

$$I = \frac{V_s}{(R_s + jX_s) + (R_L + jX_L)}$$

Magnitude of current $I = |I| = \dfrac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$

Power delivered to the circuit is

$$P = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

In the above equation, if R_L is fixed, the value of P is maximum when

$$X_s = -X_L$$

Then the power $$P = \frac{V_s^2 R_L}{(R_s + R_L)^2}$$

Let us assume that R_L is variable. In this case, the maximum power is transferred when the load resistance is equal to the source resistance (already discussed in Chapter 3). If $R_L = R_s$ and $X_L = -X_s$, then $Z_L = Z^*_s$. This means that the maximum power transfer occurs when the load impedance is equal to the complex conjugate of source impedance Z_s.

Maximum power transfer does not coincide with maximum efficiency. Application of the maximum power transfer theorem to AC power distribution will not result in max or even high efficiency. The goal of high efficiency is more important for AC power distribution, which dictates a relatively low generator impedance compared to load impedance. Maximum power transfer does not coincide with the goal of lowest noise. The low level radio frequency amplifier between the antenna and a radio receiver is often designed for lowest possible noise. This often requires a mismatch of the amplifier input impedance to the antenna as compared with that dictated by the maximum power transfer theorem.

Example 5.77 *For the circuit shown in Fig. 5.216, find the value of load impedance for which the source delivers maximum power. Calculate the value of the maximum power.*

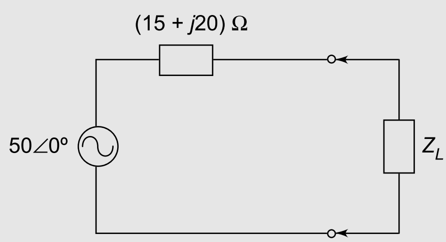


Fig. 5.216

Solution In the circuit shown in Fig. 5.216, the maximum power transfer occurs when the load impedance is complex conjugate of the source impedance

$$\therefore \qquad Z_L = Z^*_s = 15 - j20$$

When $Z_L = 15 - j20$, the current passing through circuit is

$$I = \frac{V_s}{R_s + R_L} = \frac{50\angle 0°}{15 + j20 + 15 - j20} = \frac{50\angle 0°}{30} = 1.66\angle 0°$$

The maximum power delivered to the load is

$$P = I^2 R_L = (1.66)^2 \times 15 = 41.33 \text{ W}$$

Example 5.78 *For the circuit shown in Fig. 5.217, find the value Z that will receive maximum power, also determine this power.*

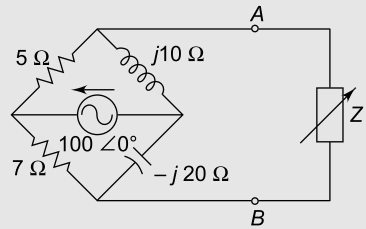


Fig. 5.217

Solution The equivalent impedance at terminals *AB* with the source set equal to zero is

$$Z_{AB} = \frac{5(j10)}{5 + j10} + \frac{7(-j20)}{(7 - j20)}$$

$$= \frac{50\angle 90°}{11.18\angle 63.43°} + \frac{140\angle -90°}{21.19\angle -70.7°}$$

$$= 4.47\angle 26.57° + 6.6\angle -19.3°$$

$$= 3.99 + j1.99 + 6.23 - j2.18$$

$$= 10.22 - j0.19$$

The Thevenin equivalent circuit is shown in Fig. 5.218(a).

The circuit in Fig. 5.218(b) is redrawn as shown in Fig. 5.218(b).

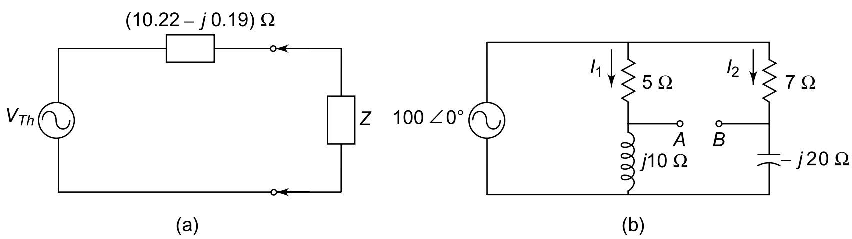


Fig. 5.218

$$\text{Current} \quad I_1 = \frac{100\angle 0°}{5 + j10}$$

$$= \frac{100\angle 0°}{11.18\angle 63.43°} = 8.94\angle -63.43° \text{A}$$

$$\text{Current} \quad I_2 = \frac{100\angle 0°}{7 - j20} = \frac{100\angle 0°}{21.19\angle -70.7°} = 4.72\angle 70.7°$$

Voltage at A, $V_A = 8.94\ \angle -63.43° \times j10 = 89.4\ \angle -26.57°$
Voltage at B, $V_B = 4.72\ \angle 70.7° \times -j20 = 94.4\ \angle -19.3°$
Voltage across terminals AB

$$\begin{aligned} V_{AB} &= V_A - V_B \\ &= 89.4\ \angle 26.57° - 94.4\ \angle -19.3° \\ &= 79.96 + j39.98 - 89.09 + j31.2 \\ &= -9.13 + j71.18 \\ V_{\text{Th}} &= V_{AB} = 71.76\ \angle 97.3°\ \text{V} \end{aligned}$$

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ Load $Z = 10.22 + j0.19$

Current passing through the load Z

$$I = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z} = \frac{71.76\angle 97.3°}{20.44} = 3.51\angle 97.3°$$

Maximum power delivered to the load is

$$= (3.51)^2 \times 10.22 = 125.91\ \text{W}$$

Example 5.79 *For the circuit shown in Fig. 5.219, the resistance R_s is variable from 2 Ω to 50 Ω. What value of R_s results in maximum power transfer across the terminals AB?*

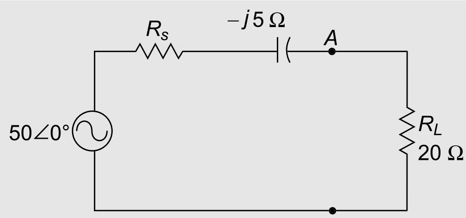


Fig. 5.219

Solution In the circuit shown the resistance R_L is fixed. Here, the maximum power transfer theorem does not apply. Maximum current flows in the circuit when R_s is minimum. For the maximum current

$$R_s = 2$$

But $$\begin{aligned} Z_T &= R_s - j5 + R_L = 2 - j5 + 20 = (22 - j5) \\ &= 22.56\ \angle -12.8° \end{aligned}$$

$$\therefore \quad I = \frac{V_s}{Z_T} = -\frac{50\ \angle 0°}{22.56\angle -12.8°} = 2.22\angle 12.8°$$

Maximum power $P = I^2R = (2.22)^2 \times 20 = 98.6$ W

Example 5.80 *For the circuit shown in Fig. 5.220, find the value of Z that will receive maximum power; also determine this power.*

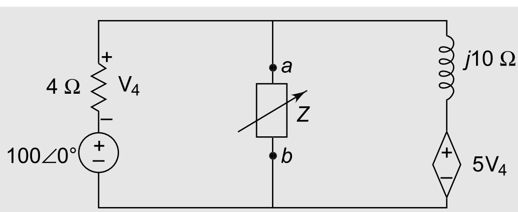


Fig. 5.220

Solution The equivalent impedance can be obtained by finding V_{oc} and i_{sc} at terminals a b. Assume that current i is passing in the circuit.

$$i = \frac{100\angle 0° - 5V_4}{4 + j10}$$

$$= \frac{100\angle 0°}{4 + j10} - \frac{5 \times 4i}{4 + j10}$$

$$i = 3.85\ \angle -22.62°$$

$$V_{oc} = 100\ \angle 0° - 4 \times 3.85\ \angle -22.62°$$

$$= 86\ \angle 3.94°$$

$$i_{sc} = 25 + j50 = 56\ \angle 63.44°$$

Thevenin's equivalent impedance

$$Z_{\text{Th}} = \frac{V_{oc}}{i_{sc}} = 1.54\angle -59.5°$$

$$= 0.78 - j1.33$$

The circuit is drawn as shown in Fig. 5.221.

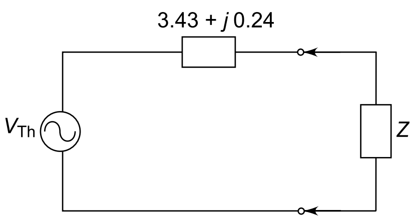


Fig. 5.221

To get maximum power, the load must be the complex conjugate of the source impedance.

$\therefore$ Load $Z = 0.78 + j1.33$

Current passing through load Z

$$I = \frac{V_{\text{Th}}}{Z_{\text{Th}} + Z} = \frac{86\angle 3.94°}{1.56} = 55.13\angle 3.94°$$

Maximum power delivered to the load is $(55.13)^2 \times (0.78) = 2370.7$ W.

Example 5.81 *In the network shown in Fig. 5.222, find the value of Z_L so that the power transfer from the source is maximum. Also find P_{max}.*

[JNTU May/June 2006]

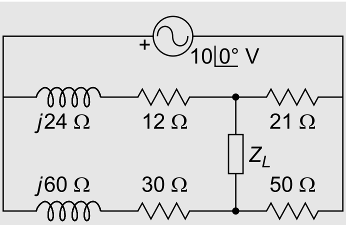


Fig. 5.222

Solution Let us remove 'z_L'. The Internal impedance of the circuit looking through $x - y$ is given by

$$z_{in} = \frac{(21)(12 + j24)}{21 + 12 + j24} + \frac{50(30 + j60)}{50 + 30 + j60}$$

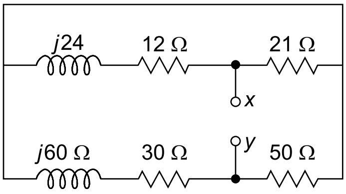


Fig. 5.223(a)

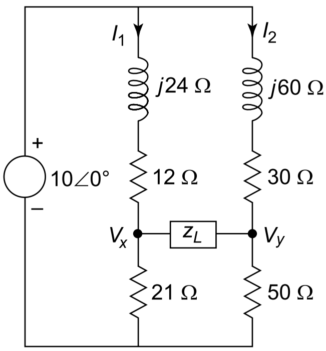


Fig. 5.223(b)

$$= \frac{563.44\angle 63.43^\circ}{40.8\angle 36^\circ} + \frac{3354.10\angle 63.43^\circ}{100\angle 36.87^\circ}$$

$$= 13.81\angle 27.43^\circ + 33.54\angle 26.56^\circ$$

$$z_{in} = 42.19 + j21.49\ \Omega$$

As per maximum power transfer theorem, Z_L should be the complex of z_{in}

$$Z_L = z_{in}^* = (42.19 - j21.49)\ \Omega$$

$$V_{OC} = V_{xy}$$

$$V_x = \frac{12 + j24}{12 + j24 + 21} \times 10\angle 0^\circ$$

$$= 6.577\angle 27.43^\circ\ \text{V}$$

$$V_y = \frac{30 + j60}{30 + j60 + 50} \times 10\angle 0^\circ$$

$$= 6.71\angle 26.56^\circ$$

$$V_{OC} = V_x - V_y = 6.577\angle 27.43^\circ - 6.71\angle 26.56^\circ$$

$$= -0.163 + j0.029$$

$$V_{OC} = 0.1657\angle 170^\circ\ \text{V}$$

$$P_{max} = \frac{V_{OC}^2}{4R_L} = \frac{(0.1657)^2}{4 \times 42.19} = 0.1627\ \text{mW}$$

$$P_{max} = 0.1627\ \text{mW}$$

Example 5.82 *In the circuit shown in the given Fig. 5.224, find the value of R_L which results in max power transfer. Calculate the value of the maximum power.*

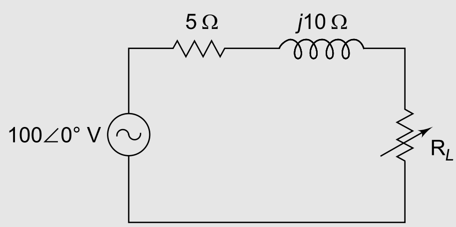


Fig. 5.224

Solution The value of R_L for which the maximum power transfer

$$R_L = |5 + j10| = \sqrt{5^2 + 10^2} = 11.18\ \Omega$$

Then the circuit current is

$$I = \frac{100\angle 0^\circ}{11.18 + 5 + j10} = \frac{100\angle 0^\circ}{19.02\angle 31.78^\circ}$$

$$= 5.26\angle -31.718^\circ \text{ A}$$

The maximum power across R_L is

$$P_{max} = I^2 R = (5.26)^2\ 11.18 = 309 \text{ watts}$$

5.2.8 Tellegen's Theorem

[JNTU Jan 2010]

The Tellegen's theorem states that the summation of instantaneous power or summation of complex power of sinusoidal sources in a network is zero. The network power may be linear or non linear, passive or active and time invariant or variant.

The Tellegen's theorem is used to design filters in signal processing applications. The assumptions for electrical circuits are generalized for dynamic systems obeying the laws of irreversible thermodynamics. Topology and structure of reaction networks can be analyzed using the Tellegen's theorem. Another application of Tellegen's theorem is to determine stability and optimality of complex process systems.

Consider a network shown in Fig. 5.225.

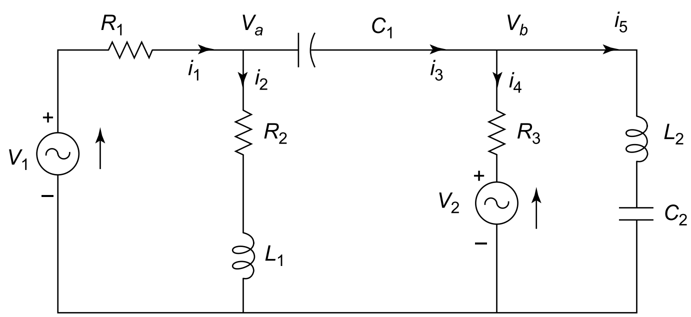


Fig. 5.225

Applying Kirchhoff's current law at nodes, we get

At node a,

$$i_1 - i_2 - i_3 = 0$$

At node b,

$$i_3 - i_4 - i_5 = 0$$

Total instantaneous powers delivered by the voltage sources

$$= V_1\, i_1 - V_2\, i_4 \tag{1}$$

Total instantaneous power absorbed by all the passive element

$$= i_1(-V_1 - V_a) + V_a\, i_2 + (V_a - V_b)\, i_3 + (V_b + V_2)\, i_4 + V_b\, i_5 \tag{2}$$

$\therefore$ Summation of all instantaneous powers $= (1) + (2)$

$$V_1 i_1 - V_2 i_4 - V_1 i_1 - V_a i_1 + V_a i_2 + V_a i_3 - V_b i_3 + V_b i_4 + V_2 i_4 + V_b i_5$$

$$V_1(i_1 - i_1) + V_2(-i_4 + i_4) + V_a(-i_1 + i_2 + i_2) + V_b(-i_3 + i_4 + i_5) = 0$$

Since the algebraic sum of the currents at each of the nodes is zero.

Example 5.83 *Verify Tellegen's theorem for the network shown in Fig. 5.226.*

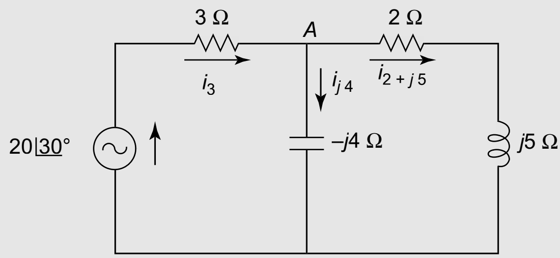


Fig. 5.226

Solution Assume that the voltage at node a is V_A. By applying modal analysis, we have

$$\frac{20\angle 30° - V_A}{3} = \frac{V_A}{-j4} + \frac{V_A}{2+j5}$$

$$V_A\left[\frac{1}{3} + \frac{1}{2+j5} - \frac{1}{j4}\right] = \frac{20\angle 30°}{3}$$

$$V_A = \frac{6.67\angle 30^\circ}{0.41\angle 11.09^\circ} = 16.27\angle 18.91^\circ \text{ V}$$

Current in 3 Ω branch

$$I_3 = \frac{20\angle 30^\circ - V_A}{3} = \frac{20\angle 30^\circ - 16.27\angle 18.91^\circ}{3}$$

$$I_3 = 1.7\angle 67.8^\circ \text{ A}$$

Current in $-j4$ Ω branch

$$I_{-j4} = \frac{16.27\angle 18.91^\circ}{4\angle -90^\circ} = 4.067\angle 108.91^\circ \text{ A}$$

Current in $(2 + j5)$ Ω branch

$$I_{2+j5} = \frac{16.27\angle 18.91^\circ}{5.385\angle 68.198^\circ} = 3.021\angle -49.3^\circ \text{ A}$$

Power in 3 Ω branch $P_3 = V_3 \times I_3$

where voltage $V_3 = I_3 \times 3 = 1.7\angle 67.8^\circ \times 3 = 5.1\angle 67.8^\circ$ V

$$\therefore \qquad P_3 = 5.1\angle 67.8^\circ \times 1.7\angle 67.8^\circ = 8.67\angle 135.6^\circ \text{ W}$$

$$= -6.2 + j6.066 \text{ W}$$

Power in $(-j4)$ Ω branch

$$P_{-j4} = 16.27\angle 18.91^\circ \times 4.067\angle 108.91^\circ$$

$$= 66.161\angle 127.82^\circ$$

$$= -40.6 + j52.26 \text{ W}$$

Power in $(2 + j5)$ Ω branch

$$P_{2+j5} = 16.27\angle 18.91^\circ \times 3.02\angle -49.3^\circ$$

$$= 49.135\angle -30.39^\circ$$

$$= 42.2 - j24.86 \text{ W}$$

Power delivered by the source

$$P_{20} = 20\angle 30^\circ \times 1.7\angle 67.8^\circ$$

$$= 34\angle 97.8^\circ = -4.61 + j33.68 \text{ W}$$

Sum of the powers in the circuit is zero, which proves Tellegen's theorem.

Practice Problems

5.1 Find the Thevenin's and Norton's equivalents for the circuit shown in Fig. 5.227 with respect to terminals *ab*.

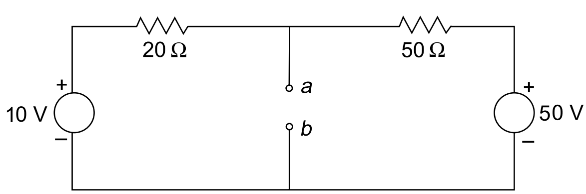


Fig. 5.227

5.2 Determine the Thevenin and Norton's equivalent circuits with respect to terminals *AB* for the circuit shown in Fig. 5.228.

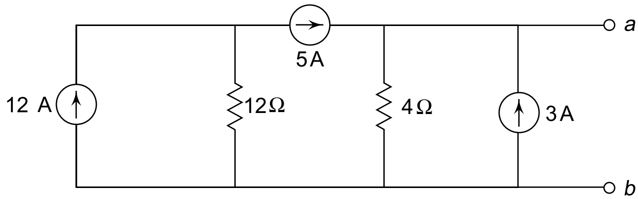


Fig. 5.228

5.3 By using source transformation or any other technique, replace the circuit shown in Fig. 5.229 between terminals *AB* with the voltage source in series with a single resistor.

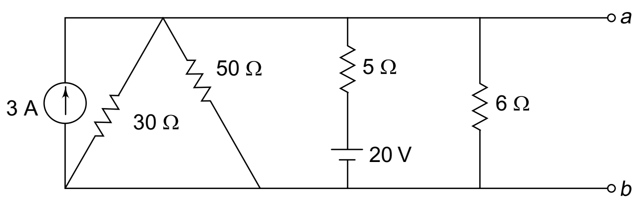


Fig. 5.229

5.4 For the circuit shown in Fig. 5.230, what will be the value of R_L to get the maximum power? What is the maximum power delivered to the load? What is the maximum voltage across the load? What is the maximum current in it?

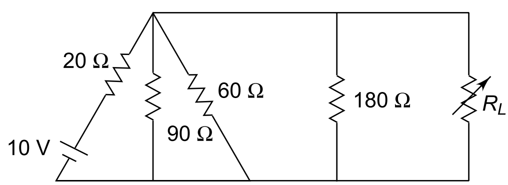


Fig. 5.230

5.5 For the circuit shown in Fig. 5.231 determine the value of R_L to get the maximum power. Also find the maximum power transferred to the load.

5.6 Determine the current passing through 2 Ω resistor by using Thevenin's theorem in the circuit shown in Fig. 5.232.

5.7 Find Thevenin's equivalent circuit for the network shown in Fig. 5.233 and hence find the current passing through the 10 Ω resistor.

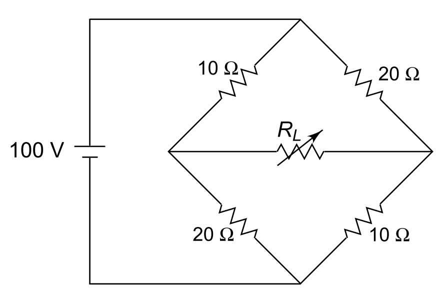


Fig. 5.231

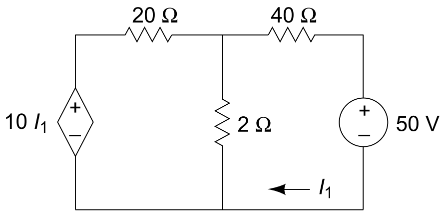


Fig. 5.232

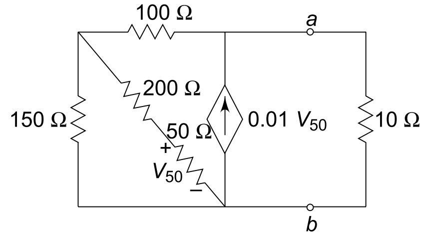


Fig. 5.233

5.8 Obtain Norton's equivalent circuit of the network shown in Fig. 5.234.

5.9 Determine (i) the equivalent voltage generator and (ii) the equivalent current generator which may be used to represent the given network in Fig. 5.235 at the terminals *AB*.

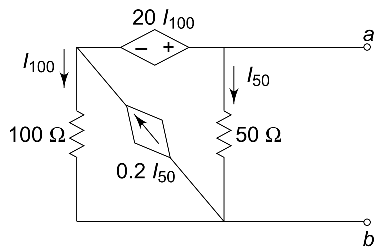


Fig. 5.234

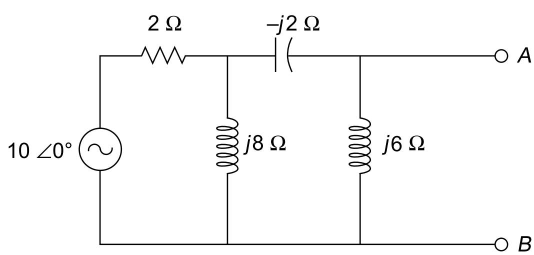


Fig. 5.235

5.10 For the circuit shown in Fig. 5.236, find the value of Z that will receive the maximum power. Also determine this power.

5.11 Determine the voltage V_{ab} and V_{bc} in the network shown in Fig. 5.237 by Thevenin's theorem, where source voltage

$$e(t) = \sqrt{2} \times 100 \cos(314t + 45°).$$

5.12 Find the current in the 15 Ω resistor in the network shown in Fig. 5.238 by Thevenin's theorem.

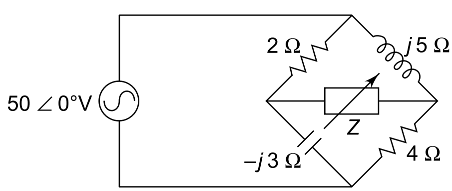


Fig. 5.236

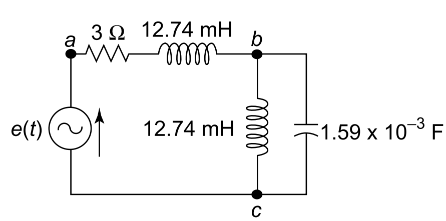


Fig. 5.237

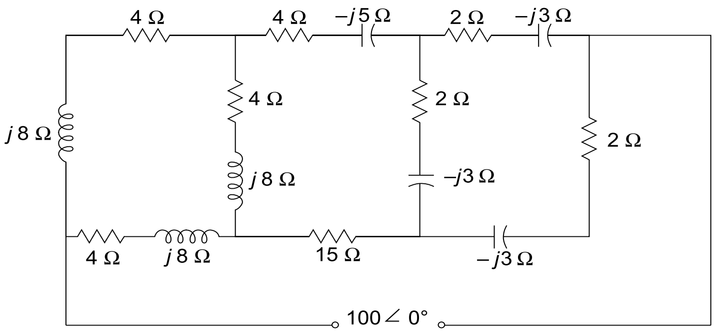


Fig. 5.238

5.13 Determine the power output of the voltage source by loop analysis for the network shown in Fig. 5.239. Also determine the power extended in the resistors.

5.14 In the circuit shown in Fig. 5.240, determine the power in the impedance $(2+j5)\,\Omega$ connected between *A* and *B* using Norton's theorem.

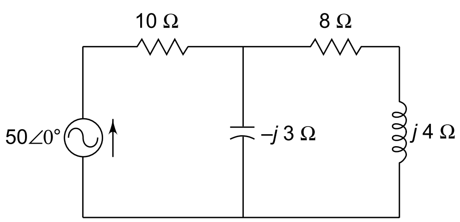


Fig. 5.239

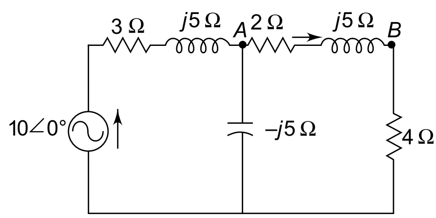


Fig. 5.240

5.15 Convert the active network shown in Fig. 5.241 by a single voltage source in series with an impedance, and also by a single current source in parallel with the impedance.

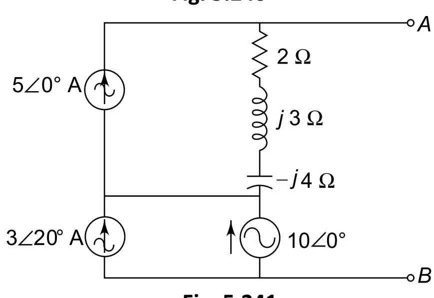


Fig. 5.241

5.16 Determine the power out of the source in the circuit shown in Fig. 5.242 by Thevenin's theorem and verify the results by using Norton's theorem.

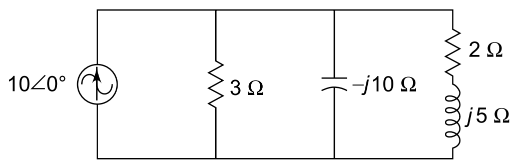


Fig. 5.242

5.17 Use Thevenin's theorem to find the current through the $(5 + j4)\ \Omega$ impedance in Fig. 5.243. Verify the results using Norton's theorem.

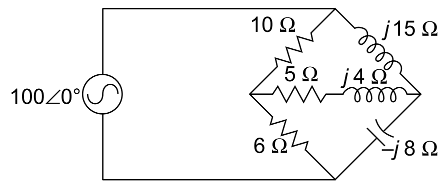


Fig. 5.243

5.18 Determine Thevenin's and Norton's equivalent circuits across terminals *AB*, in Fig. 5.244.

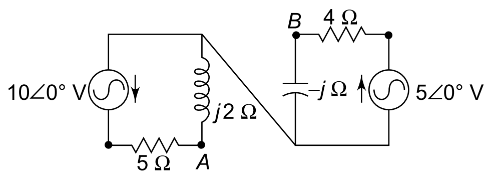


Fig. 5.244

5.19 Determine Norton's and Thevenin's equivalent circuits for the circuit shown in Fig. 5.245.

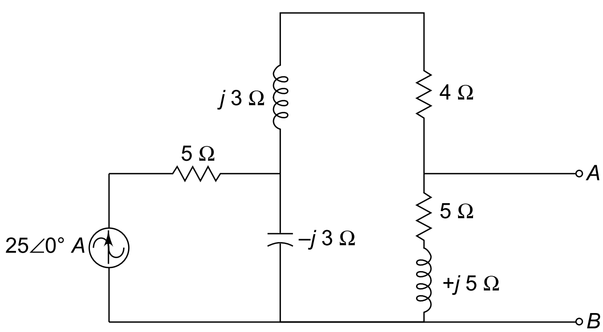


Fig. 5.245

5.20 Determine the maximum power delivered to the load in the circuit shown in Fig. 5.246.

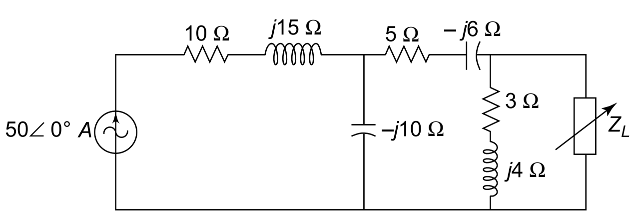


Fig. 5.246

5.21 For the circuit shown in Fig. 5.247, find the voltage across the dependent source branch by using Norton's theorem.

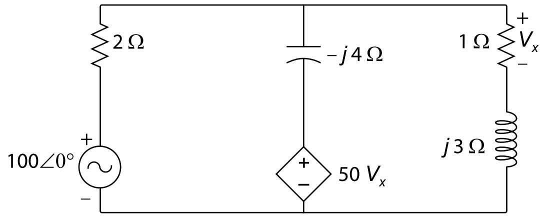


Fig. 5.247

5.22 Find Thevenin's equivalent for the network shown in Fig. 5.248.

5.23 For the circuit shown in Fig. 5.249, obtain the voltage across 500 kΩ resistor.

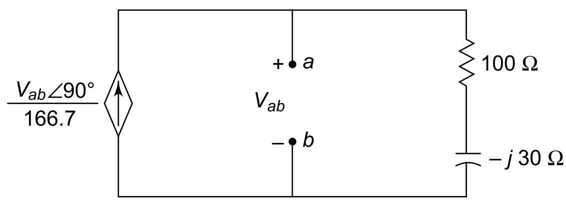


Fig. 5.248

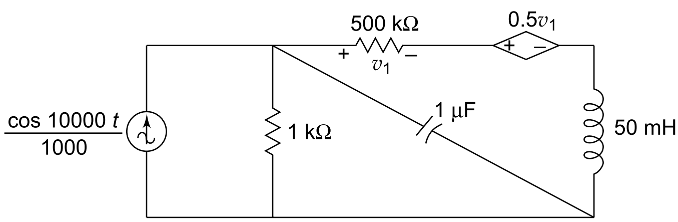


Fig. 5.249

5.24 For the circuit shown in Fig. 5.250, obtain the Thevenin's equivalent circuit at terminals *ab*.

5.25 Find the current *I* in the circuit shown in Fig. 5.251 by using the superposition theorem.

5.26 Determine the current *I* in the circuit shown in Fig. 5.252 by using the superposition theorem.

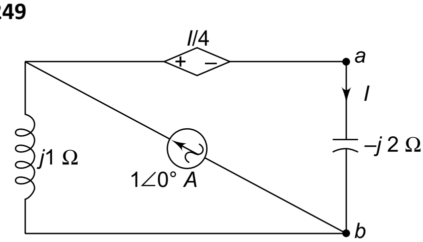


Fig. 5.250

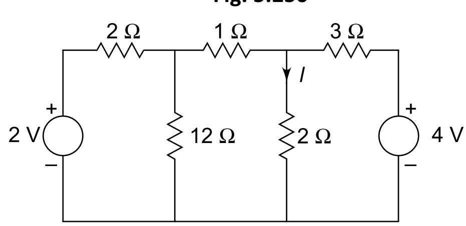


Fig. 5.251

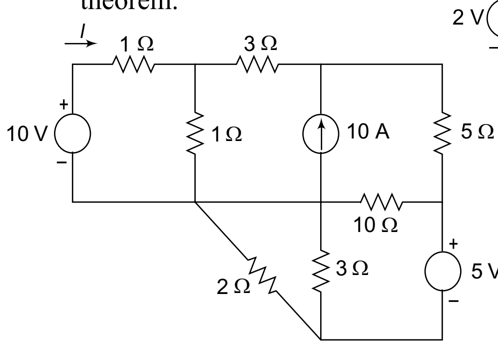


Fig. 5.252

5.27 Calculate the new current in the circuit shown in Fig. 5.253 when the resistor R_3 is increased by 30%.

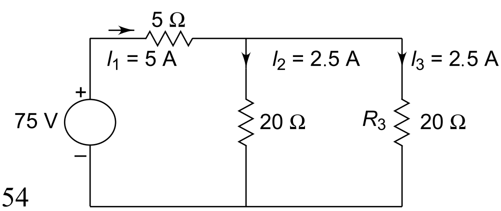


Fig. 5.253

5.28 The circuit shown in Fig. 5.254 consists of dependent source Use the superposition theorem to find the current I in the 3 Ω resistor.

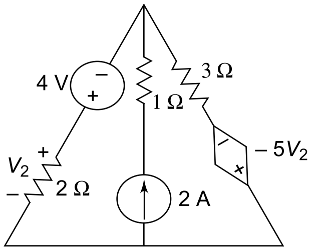


Fig. 5.254

5.29 Obtain the current passing through 2 Ω resistor in the circuit shown in Fig. 5.255 by using the superposition theorem.

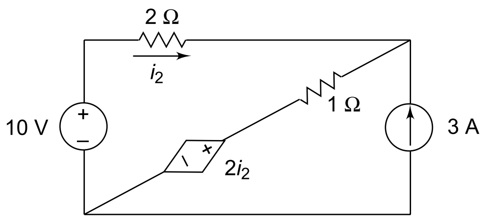


Fig. 5.255

5.30 For the circuit shown in Fig. 5.256, find the current in each resistor using the superposition theorem.

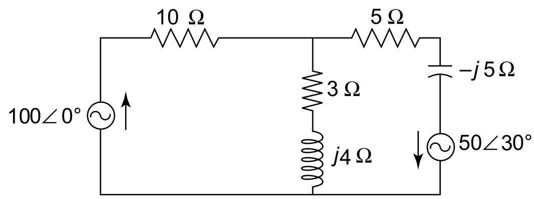


Fig. 5.256

5.31 Determine the value of source currents by superposition theorem for the circuit shown in Fig. 5.257 and verify the results by using nodal analysis.

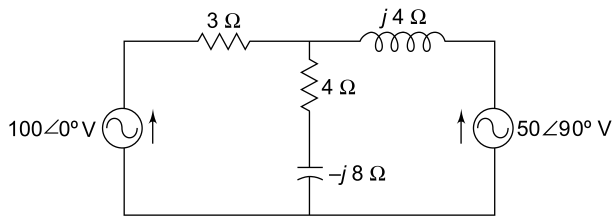


Fig. 5.257

Objective Type Questions

5.1 Reduce the circuit shown in Fig. 5.258 to its Thevenin equivalent circuit as viewed from terminal A and B.

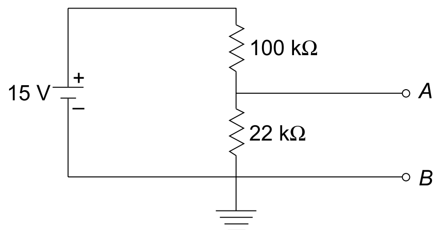


Fig. 5.258

(a) The circuit consists of 15 V battery in series with 100 kΩ
(b) The circuit consists of 15 V battery in series with 22 kΩ
(c) The circuit consists of 15 V battery in series with parallel combination of 100 kV and 22 kΩ
(d) None of the above

5.2 Norton's equivalent circuit consists of

(a) voltage source in parallel with resistance
(b) voltage source in series with resistance
(c) current source in series with resistance
(d) current source in parallel with resistance

5.3 Maximum power is transferred when load impedance is

(a) equal to source resistance
(b) equal to half of the source resistance
(c) equal to zero
(d) none of the above

5.4 In the circuit shown in Fig. 5.259, what is the maximum power transferred to the load

(a) 5 W (b) 2.5 W
(c) 10 W (d) 25 W

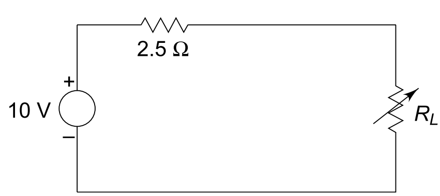


Fig. 5.259

5.5 Thevenins voltage in the circuit shown in Fig. 5.260 is

(a) 3 V (b) 2.5 V
(c) 2 V (d) 0.1 V

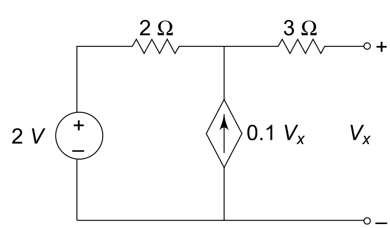


Fig. 5.260

5.6 Norton's current in the circuit shown in Fig. 5.261 is

(a) $\frac{2i}{5}$ (b) zero

(c) infinite (d) None

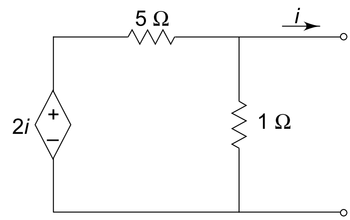


Fig. 5.261

5.7 A dc circuit shown in Fig. 5.262 has a voltage V, a current source I and several resistors. A particular resistor R dissipates a power of 4 W when V alone is active. The same resistor dissipates a power of 9 W when I alone is active. The power dissipated by R when both sources are active will be

(a) 1 W (b) 5 W

(c) 13 W (d) 25 W

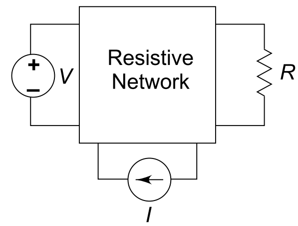


Fig. 5.262

5.8 While applying Thevenin's theorem, the Thevenin's voltage is equal to

(a) short circuit voltage at the terminals
(b) open circuit voltage at the terminals
(c) voltage of the source
(d) total voltage available in the circuit

5.9 Thevenin impedance Z_{Th} is found

(a) by short-circuiting the given two terminals
(b) between any two open terminals
(c) by removing voltage sources along with the internal resistances
(d) between same open terminals as for V_{Th}

5.10 Thevenin impedance of the circuit at its terminals A and B in Fig. 5.263 is

(a) 5 H
(b) 2 Ω
(c) 1.4 Ω
(d) 7 H

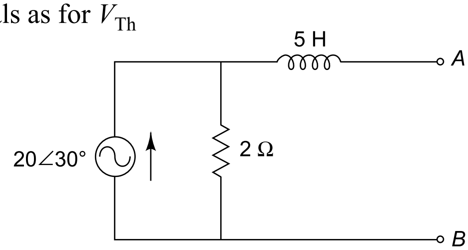


Fig. 5.263

5.11 Norton's equivalent form in any complex impedance circuit consists of

(a) an equivalent current source in parallel with an equivalent resistance.
(b) an equivalent voltage source in series with an equivalent conductance.
(c) an equivalent current source in parallel with an equivalent impedance.
(d) None of the above.

5.12 The maximum power transfer theorem can be applied

(a) only to dc circuits (b) only to ac circuits
(c) to both dc and ac circuits (d) neither of the two

5.13 Maximum power transfer occurs at a

(a) 100% efficiency (b) 50% efficiency
(c) 25% efficiency (d) 75% efficiency

5.14 In the circuit shown in Fig. 5.264, the power supplied by the 10 V source is

(a) 6.6 W
(b) 21.7 W
(c) 30 W
(d) 36.7 W

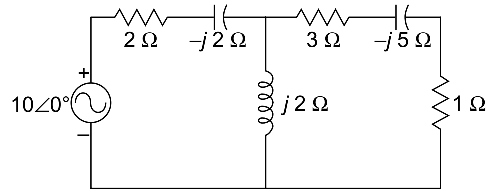


Fig. 5.264

5.15 A source has an emf of 10 V and an impedance of $500 + j100\ \Omega$. The amount of maximum power transferred to the load will be

(a) 0.5 mW
(b) 0.05 mW
(c) 0.05 W
(d) 0.5 W

5.16 For the circuit shown in Fig. 5.265, find the voltage across the dependent source.

(a) $8\ \angle 0°$
(b) $4\ \angle 0°$
(c) $4\ \angle 90°$
(d) $8\ \angle -90°$

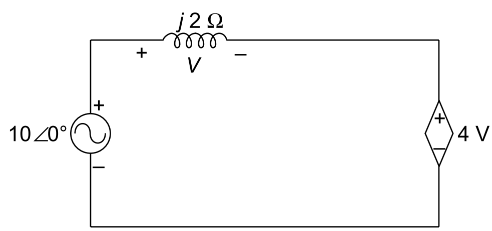


Fig. 5.265

5.17 Superposition theorem is valid only for

(a) linear circuits (b) non-linear circuits
(c) both linear and non-linear (d) neither of the two

5.18 Superposition theorem is not valid for

(a) voltage responses (b) current responses
(c) power responses (d) all the three

5.19 Determine the current *I* in the circuit shown in Fig. 5.266. It is

(a) 2.5 A (b) 1 A
(c) 3.5 A (d) 4.5 A

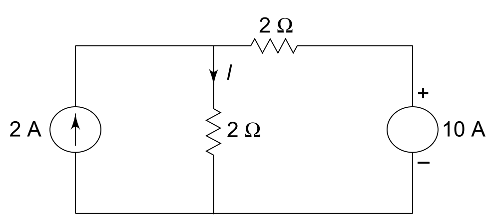


Fig. 5.266

5.20 The reciprocity theorem is applicable to
(a) linear networks only
(b) bilateral networks only
(c) linear/bilateral networks
(d) neither of the two

5.21 Compensation theorem is applicable to
(a) linear networks only
(b) non-linear networks only
(c) linear and non-linear networks
(d) neither of the two

5.22 When the superposition theorem is applied to any circuit, the dependent voltage source in that circuit is always
(a) opened (b) shorted (c) active (d) none of the above

5.23 Superposition theorem is not applicable to networks containing.
(a) non-linear elements
(b) dependent voltage sources
(c) dependent current sources
(d) transformers

5.24 The superposition theorem is valid
(a) only for ac circuits
(b) only for dc circuits
(c) For both, ac and dc circuits
(d) neither of the two

5.25 When applying the superposition theorem to any circuit
(a) the voltage source is shorted, the current source is opened
(b) the voltage source is opened, the current source is shorted
(c) both are opened
(d) both are shorted

5.26 In a complex impedance circuit, the maximum power transfer occurs when the load impedance is equal to
(a) complex conjugate of source impedance
(b) source impedance
(c) source resistance
(d) none of the above

5.27 The Thevenin equivalent impedance of the circuit in Fig. 5.267 is
(a) $(1 + j5)\ \Omega$
(b) $(2.5 + j25)\ \Omega$
(c) $(6.25 + j6.25)\ \Omega$
(d) $(2.5 + j6.25)\ \Omega$

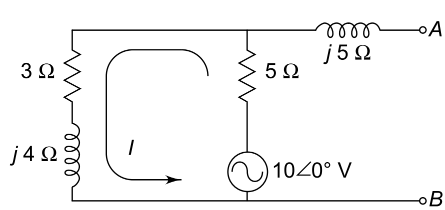
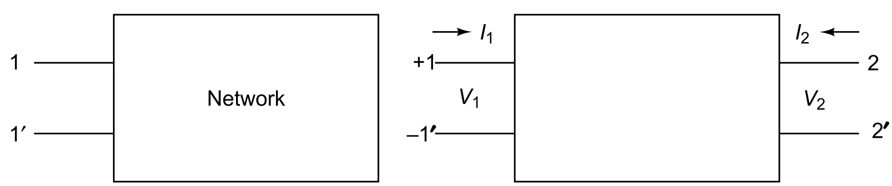


Fig. 5.267

6 Two-Port Networks

6.1 RELATIONSHIP OF TWO-PORT NETWORK

Generally any network may be represented schematically by a rectangular box. A network may be used for representing either source or load, or for a variety of purposes. A pair of terminals at which a signal may enter or leave a network is called a port. A *port* is defined as any pair of terminals into which energy is supplied, or from which energy is withdrawn, or where the network variables may be measured. One such network having only one pair of terminals $(1 - 1')$ is shown in Fig. 6.1(a).



(a) **(b)**

Fig. 6.1

A two-port network is simply a network inside a black box, and the network has only two pairs of accessible terminals; usually one pair represents the input and the other represents the output. Such a building block is very common in electronic systems, communication systems, transmission and distribution systems. Figure 6.1(b) shows a two-port network, or two terminal pair network, in which the four terminals have been paired into ports 1–1′ and 2–2′. The terminals

1–1′ together constitute a port. Similarly, the terminals 2–2′ constitute another port. Two ports containing no sources in their branches are called *passive ports*; among them are power transmission lines and transformers. Two ports containing sources in their branches are called *active ports*. A voltage and current assigned to each of the two ports. The voltage and current at the input terminals are V_1 and I_1; whereas V_2 and I_2 are specified at the output port. It is also assumed that the currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2, respectively. The variables of the two-port network are V_1, V_2, and I_1, I_2. Two of these are dependent variables, the other two are independent variables. The number of possible combinations generated by the four variables, taken two at a time, is six. Thus, there are six possible sets of equations describing a two-port network.

6.2 OPEN CIRCUIT IMPEDANCE (Z) PARAMETERS

A general linear two-port network defined in Section 6.1 which does not contain any independent sources is shown in Fig. 6.2.

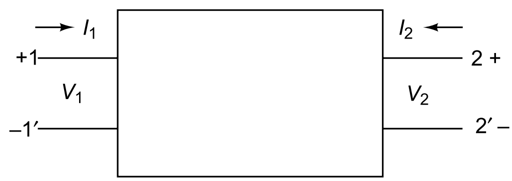


Fig. 6.2

The Z parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port voltages V_1 and V_2 in terms of the currents I_1 and I_2. Here V_1 and V_2 are dependent variables, and I_1, I_2 are independent variables. The voltage at port 1–1′ is the response produced by the two currents I_1 and I_2. Thus

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \tag{6.1}$$

Similarly,

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \tag{6.2}$$

Z_{11}, Z_{12}, Z_{21} and Z_{22} are the network functions, and are called impedance (Z) parameters, and are defined by Eqs 6.1 and 6.2. These parameters can be represented by matrices.

We may write the matrix equation $[V] = [Z]\,[I]$

where V is the column matrix $= \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Z is the square matrix $= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$

and we may write $|I|$ in the column matrix $= \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

Thus, $$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero. Suppose port 2–2′ is left open-circuited, then $I_2 = 0$

Thus $$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

where Z_{11} is the driving-point impedance at port 1–1′ with port 2–2′ open circuited. It is called the open circuit input impedance

Similarly, $$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

where Z_{21} is the transfer impedance at port 1–1′ with port 2–2′ open circuited. It is also called the open circuit forward transfer impedance. Suppose port 1–1′ is left open circuited, then $I_1 = 0$

Thus, $$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

where Z_{12} is the transfer impedance at port 2–2′, with port 1–1′ open circuited. It is also called the open circuit reverse transfer impedance.

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

where Z_{22} is the open circuit driving point impedance at port 2–2′ with port 1–1′ open circuited. It is also called the open circuit output impedance. The equivalent circuit of the two-port networks governed by the Eqs 6.1 and 6.2, i.e. open circuit impedance parameters is shown in Fig. 6.3.

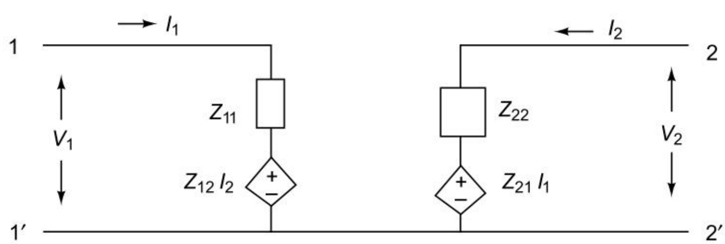


Fig. 6.3

If the network under study is reciprocal or bilateral, then in accordance with the reciprocity principle

$$\left.\frac{V_2}{I_1}\right|_{I_2=0} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

or $$Z_{21} = Z_{12}$$

It is observed that all the parameters have the dimensions of impedance. Moreover, individual parameters are specified only when the current in one of the ports is zero. This corresponds to one of the ports being open circuited from which the Z parameters also derive the name *open circuit impedance parameters*.

Example 6.1 *Find the Z parameters for the circuit shown in Fig. 6.4.*

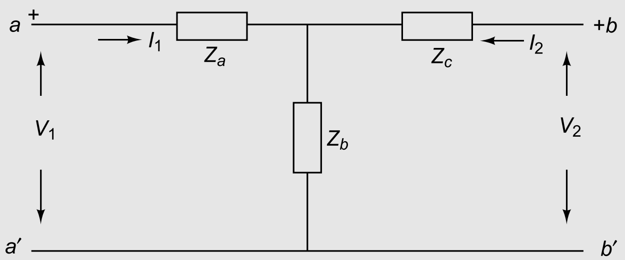


Fig. 6.4

Solution

The circuit in the problem is a T network. From Eqs 6.1 and 6.2 we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

When port b-b' is open circuited, $Z_{11} = \dfrac{V_1}{I_1}$

where $$V_1 = I_1(Z_a + Z_b)$$
∴ $$Z_{11} = (Z_a + Z_b)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

where $$V_2 = I_1 Z_b$$
∴ $$Z_{21} = Z_b$$

When port a-a' is open circuited, $I_1 = 0$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

where $$V_2 = I_2(Z_b + Z_c)$$
∴ $$Z_{22} = (Z_b + Z_c)$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where $\qquad V_1 = I_2 Z_b$

$\therefore \qquad Z_{12} = Z_b$

It can be observed that $Z_{12} = Z_{21}$, so the network is a bilateral network which satisfies the principle of reciprocity.

6.3 SHORT CIRCUIT ADMITTANCE (*Y*) PARAMETERS

A general two-port network which is considered in Section 6.2 is shown in Fig. 6.5.

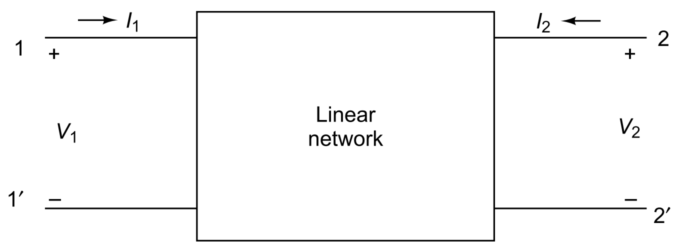


Fig. 6.5

The Y parameters of a two-port for the positive directions of voltages and currents may be defined by expressing the port currents I_1 and I_2 in terms of the voltages V_1 and V_2. Here I_1, I_2 are dependent variables and V_1 and V_2 are independent variables. I_1 may be considered to be the superposition of two components, one caused by V_1 and the other by V_2.

Thus,

$$I_1 = Y_{11} V_1 + Y_{12} V_2 \tag{6.3}$$

Similarly,

$$I_2 = Y_{21} V_1 + Y_{22} V_2 \tag{6.4}$$

Y_{11}, Y_{12}, Y_{21} and Y_{22} are the network functions and are also called the admittance (Y) parameters. They are defined by Eqs 6.3 and 6.4. These parameters can be represented by matrices as follows:

$$[I] = [Y]\,[V]$$

where $\qquad I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix};\ Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$

and $\qquad V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Thus, $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The individual Y parameters for a given network can be defined by setting each port voltage to zero. If we let V_2 be zero by short circuiting port 2–2′, then

$$Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2 = 0}$$

Y_{11} is the driving point admittance at port 1–1′, with port 2–2′ short circuited. It is also called the short circuit input admittance.

$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2 = 0}$$

Y_{21} is the transfer admittance at port 1–1′ with port 2–2′ short circuited. It is also called short circuited forward transfer admittance. If we let V_1 be zero by short circuiting port 1–1′, then

$$Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1 = 0}$$

Y_{12} is the transfer admittance at port 2–2′ with port 1–1′ short circuited. It is also called the short circuit reverse transfer admittance.

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1 = 0}$$

Y_{22} is the short circuit driving point admittance at port 2–2′ with port 1–1′ short circuited. It is also called the short circuit output admittance. The equivalent circuit of the network governed by Eqs 6.3 and 6.4 is shown in Fig. 6.6.

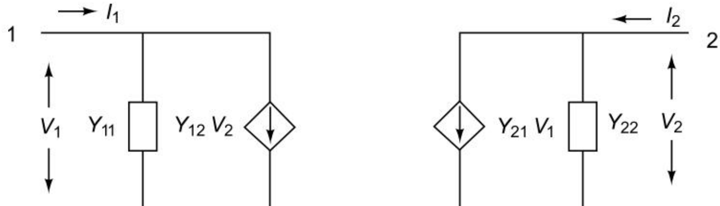


Fig. 6.6

If the network under study is reciprocal, or bilateral, then

$$\left. \frac{I_1}{V_2} \right|_{V_1 = 0} = \left. \frac{I_2}{V_1} \right|_{V_2 = 0}$$

or $$Y_{12} = Y_{21}$$

It is observed that all the parameters have the dimensions of admittance which are obtained by short circuiting either the output or the input port from which the parameters also derive their name, i.e. the *short circuit admittance parameters*.

Example 6.2 *Find the Y parameters for the network shown in Fig. 6.7.*

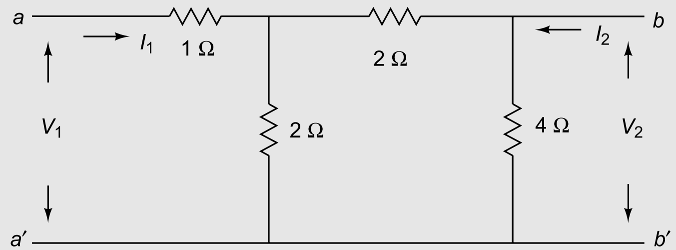


Fig. 6.7

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

When b-b' is short circuited, $V_2 = 0$ and the network looks as shown in Fig. 6.8(a).

$$V_1 = I_1 Z_{eq}$$

$$Z_{eq} = 2\ \Omega$$

$\therefore$ $$V_1 = I_1 2$$

$$Y_{11} = \frac{I_1}{V_1} = \frac{1}{2}\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_2}\right|_{V_2=0}$$

With port b-b' short circuited, $-I_2 = I_1 \times \frac{2}{4} = \frac{I_1}{2}$

$\therefore$ $$-I_2 = \frac{V_1}{4}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{1}{4}\ \mho$$

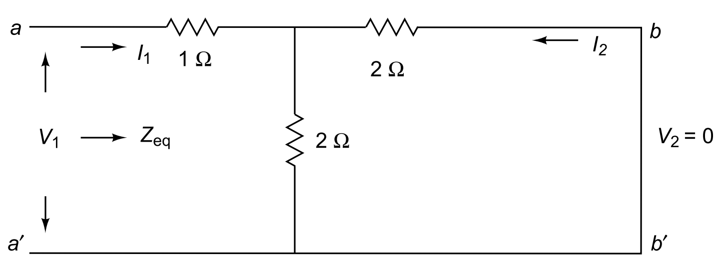


Fig. 6.8(a)

Similarly, when port a-a' is short circuited, $V_1 = 0$ and the network looks as shown in Fig. 6.8(b).

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

$$V_2 = I_2 Z_{eq}$$

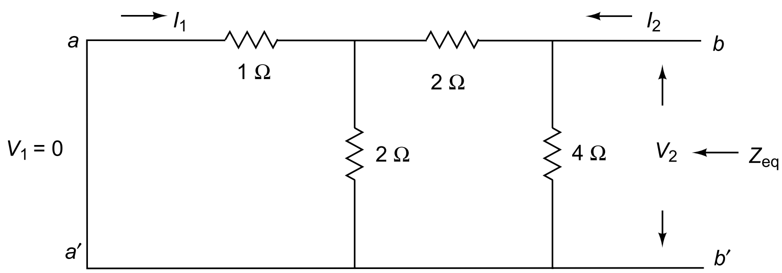


Fig. 6.8(b)

where Z_{eq} is the equivalent impedance as viewed from b-b'.

$$Z_{eq} = \frac{8}{5}\ \Omega$$

$$V_2 = I_2 \times \frac{8}{5}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{5}{8}\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

With a-a' short circuited, $-I_1 = \frac{2}{5} I_2$

Since $$I_2 = \frac{5V_2}{8}$$

$$-I_1 = \frac{2}{5} \times \frac{5}{8} V_2 = \frac{V_2}{4}$$

$\therefore$ $$Y_{12} = \frac{I_1}{V_2} = -\frac{1}{4}\ \mho$$

The describing equations in terms of the admittance parameters are

$$I_1 = 0.5\ V_1 - 0.25\ V_2$$
$$I_2 = -0.25\ V_1 + 0.625\ V_2$$

6.4 TRANSMISSION (*ABCD*) LINE PARAMETERS

Transmission parameters, or *ABCD* parameters, are widely used in transmission line theory and cascade networks. In describing the transmission parameters, the input variables V_1 and I_1 at port 1-1′, usually called the *sending end*, are expressed in terms of the output variables V_2 and I_2 at port 2-2′, called the *receiving end*. The transmission parameters provide a direct relationship between input and output. Transmission parameters are also called general circuit parameters, or chain parameters. They are defined by

$$V_1 = AV_2 - BI_2 \tag{6.5}$$

$$I_1 = CV_2 - DI_2 \tag{6.6}$$

The negative sign is used with I_2, and not for the parameter B and D. Both the port currents I_1 and $-I_2$ are directed to the right, i.e. with a negative sign in Eqs 6.5 and 6.6 the current at port 2-2′ which leaves the port is designated as positive. The parameters A, B, C and D are called the *transmission parameters*. In the matrix form, Eqs 6.5 and 6.6 are expressed as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

The matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called the *transmission matrix*.

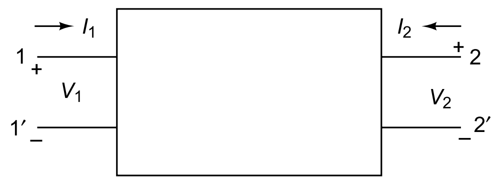


Fig. 6.9

For a given network, these parameters can be determined as follows. With port 2-2′ open, i.e. $I_2 = 0$; applying a voltage V_1 at the port 1-1′, using Eq. 6.5, we have

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} \quad \text{and} \quad C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

$$\frac{1}{A} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \left. g_{21}\right|_{I_2=0}$$

$1/A$ is called the open circuit voltage gain, a dimensionless parameter. And $\dfrac{1}{C} = \left.\dfrac{V_2}{I_1}\right|_{I_2=0} = Z_{21}$, which is the open circuit transfer impedance.

With port 2-2′ short circuited, i.e. with $V_2 = 0$, applying voltage V_1 at port 1-1′, from Eq. 6.6, we have

$$-B = \left.\frac{V_1}{I_2}\right|_{V_2=0} \quad \text{and} \quad -D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

$$-\frac{1}{B} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = Y_{21}, \text{ which is the short circuit transfer admittance}$$

$-\dfrac{1}{D} = \left.\dfrac{I_2}{I_1}\right|_{V_2=0} = \alpha_{21}\big|_{V_2=0}$, which is the short circuit current gain, a dimensionless parameter.

6.4.1 Cascade Connection

The main use of the transmission matrix is in dealing with a cascade connection of two-port networks as shown in Fig. 6.10.

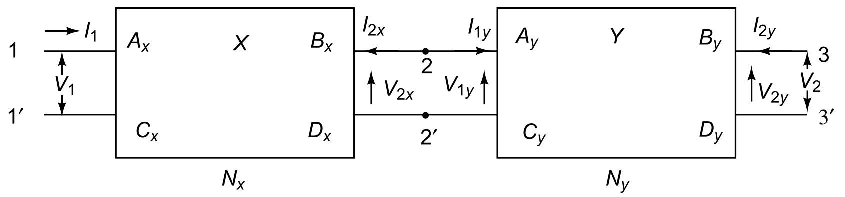


Fig. 6.10

Let us consider two two-port networks N_x and N_y connected in cascade with port voltages and currents as indicated in Fig. 6.10. The matrix representation of *ABCD* parameters for the network *X* is as under.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} V_{2x} \\ -I_{2x} \end{bmatrix}$$

And for the network *Y*, the matrix representation is

$$\begin{bmatrix} V_{1y} \\ I_{1y} \end{bmatrix} = \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_{2y} \\ -I_{2y} \end{bmatrix}$$

It can also be observed that at for 2–2′

$$V_{2x} = V_{1y} \text{ and } I_{2x} = -I_{1y}.$$

Combining the results, we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ C_x & D_x \end{bmatrix} \begin{bmatrix} A_y & B_y \\ C_y & D_y \end{bmatrix} \begin{bmatrix} V_2 \\ -I_1 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the transmission parameters matrix for the overall network.

Thus, the transmission matrix of a cascade of a two-port networks is the product of transmission matrices of the individual two-port networks. This property is used in the design of telephone systems, microwave networks, radars, etc.

Example 6.3 *Find the transmission or general circuit parameters for the circuit shown in Fig. 6.11.*

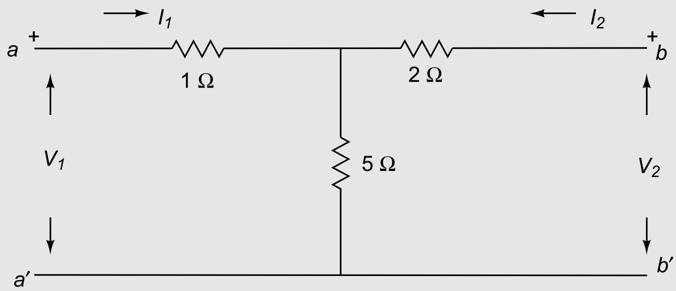


Fig. 6.11

Solution

From Eqs 6.5 and 6.6 in Section 6.4, we have

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

When b-b' is open, $\quad I_2 = 0;\ A = \left.\dfrac{V_1}{V_2}\right|_{I_2 = 0}$

where $\quad V_1 = 6I_1 \quad \text{and} \quad V_2 = 5I_1$

$\therefore \quad A = \dfrac{6}{5}$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \frac{1}{5}\ \mho$$

When b-b' is short circuited; $V_2 = 0$ (See Fig. 6.12)

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2 = 0};\ D = \left.\frac{-I_1}{I_2}\right|_{V_2 = 0}$$

In the circuit, $\quad -I_2 = \dfrac{5}{17}\ V_1$

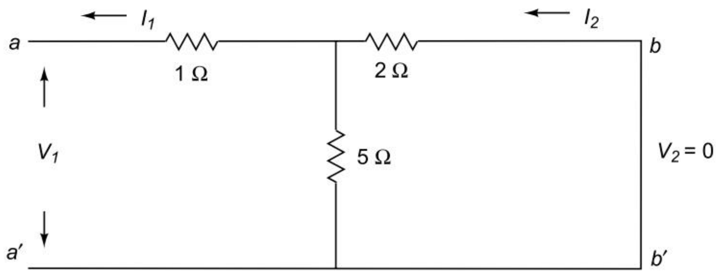


Fig. 6.12

$$\therefore \qquad B = \frac{17}{5}\ \Omega$$

Similarly, $$I_1 = \frac{7}{17}V_1 \text{ and } -I_2 = \frac{5}{17}V_1$$

$$\therefore \qquad D = \frac{7}{5}$$

6.5 INVERSE TRANSMISSION ($A'B'C'D'$) LINE PARAMETERS

In the preceding section, the input port voltage and current are expressed in terms of output port voltage and current to describe the transmission parameters. While defining the transmission parameters, it is customary to designate the input port as the sending end and output port as receiving end. The voltage and current at the receiving end can also be expressed in terms of the sending end voltage and current. If the voltage and current at port 2-2′ is expressed in terms of voltage and current at port 1-1′, we may write the following equations.

$$V_2 = A'V_1 - B'I_1 \tag{6.7}$$
$$I_2 = C'V_1 - D'I_1 \tag{6.8}$$

The coefficients A', B', C' and D' in the above equations are called inverse transmission parameters. Because of the similarities of Eqs 6.7 and 6.8 with Eqs 6.5 and 6.6 in Section 6.4, the A', B', C', D' parameters have properties similar to *ABCD* parameters. Thus when port 1-1′ is open, $I_1 = 0$.

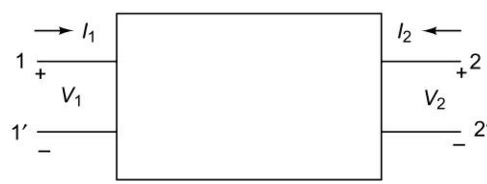


Fig. 6.13

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0} ; C' = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

If port 1-1′ is short circuited, $V_1 = 0$

$$B' = \left.\frac{-V_2}{I_1}\right|_{V_1=0} ; D = \left.\frac{-I_2}{I_1}\right|_{V_1=0}$$

6.6 HYBRID (*h*) PARAMETERS

Hybrid parameters, or h parameters find extensive use in transistor circuits. They are well suited to transistor circuits as these parameters can be most conveniently measured. The hybrid matrices describe a two-port, when the voltage of one port and the current of other port are taken as the independent variables. Consider the network in Fig. 6.14.

Fig. 6.14

If the voltage at port 1-1′ and current at port 2-2′ are taken as dependent variables, we can express them in terms of I_1 and V_2.

$$V_1 = h_{11} I_1 + h_{12} V_2 \tag{6.9}$$
$$I_2 = h_{21} I_1 + h_{22} V_2 \tag{6.10}$$

The coefficients in the above equations are called hybrid parameters. In matrix notation

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eqs 6.9 and 6.10, the individual h parameters may be defined by letting $I_1 = 0$ and $V_2 = 0$.

When $V_2 = 0$, the port 2-2′ is short circuited.

Then $h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$ Short circuit input impedance $\left(\frac{1}{Y_{11}}\right)$

$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$ Short circuit forward current gain $\left(\frac{Y_{21}}{Y_{11}}\right)$

Similarly, by letting port 1-1′ open, $I_1 = 0$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \quad \text{Open circuit reverse voltage gain} \left(\frac{Z_{12}}{Z_{22}}\right)$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \quad \text{Open circuit output admittance} \left(\frac{1}{Z_{22}}\right)$$

Since the h parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called hybrid parameters. An equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig. 6.15.

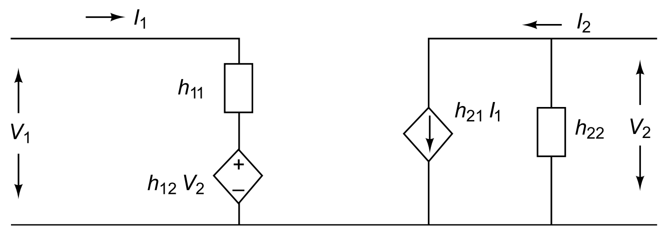


Fig. 6.15

Example 6.4 *Find the h parameters of the network shown in Fig. 6.16.*

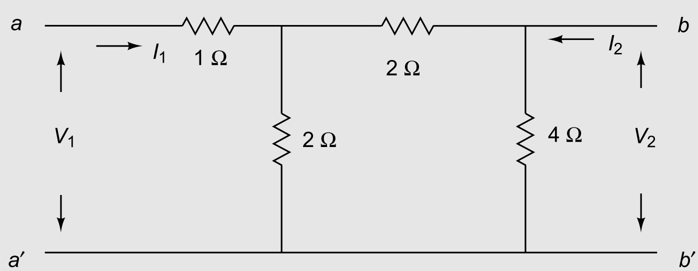


Fig. 6.16

Solution

From Eqs 6.9 and 6.10, we have

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}; \; h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}; \; h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

If port b-b' is short circuited, $V_2 = 0$. The circuit is shown in Fig. 6.17(a).

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}; \; V_1 = I_1 Z_{eq}$$

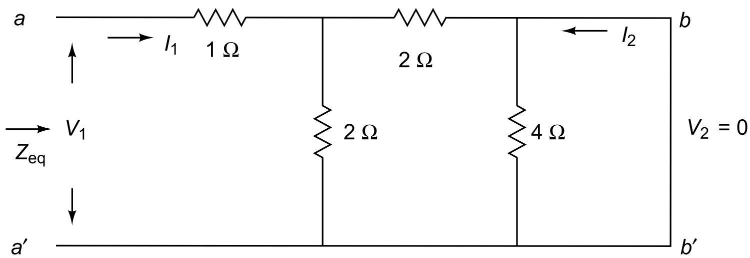


Fig. 6.17(a)

Z_{eq} the equivalent impedance as viewed from the port a-a' is 2 Ω

$$\therefore \qquad V_1 = I_1 2 \text{ V}$$

$$h_{11} = \frac{V_1}{I_1} = 2\ \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \quad \text{when } V_2 = 0; -I_2 = \frac{I_1}{2}$$

$$\therefore \qquad h_{21} = -\frac{1}{2}$$

If port a-a' is let open, $I_1 = 0$. The circuit is shown in Fig. 6.17(b).

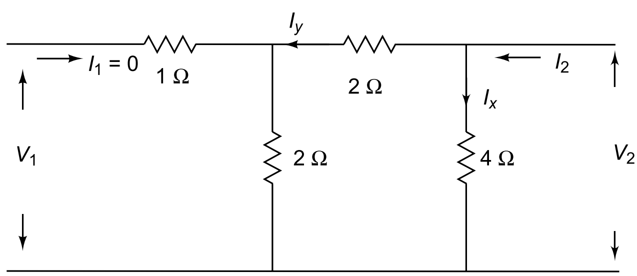


Fig. 6.17(b)

Then
$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$V_1 = I_Y 2; \ I_Y = \frac{I_2}{2}$$

$$V_2 = I_X 4; \ I_X = \frac{I_2}{2}$$

$$\therefore \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{2}\ \mho$$

6.7 INVERSE HYBRID (*g*) PARAMETERS

Another set of hybrid matrix parameters can be defined in a similar way as was done in Section 6.6. This time the current at the input port I_1 and the voltage at the output port V_2 can be expressed in terms of I_2 and V_1. The equations are as follows.

$$I_1 = g_{11} V_1 + g_{12} I_2 \tag{6.11}$$

$$V_2 = g_{21} V_1 + g_{22} I_2 \tag{6.12}$$

The coefficients in the above equations are called the inverse hybrid parameters. In matrix notation

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

It can be verified that $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$

The individual g parameters may be defined by letting $I_2 = 0$ and $V_1 = 0$ in Eqs 6.11 and 6.12.

Thus, when $I_2 = 0$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2 = 0} = \text{Open circuit input admittance}\left(\frac{1}{Z_{11}}\right)$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2 = 0} = \text{Open circuit voltage gain}$$

When $V_1 = 0$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1 = 0} = \text{Short circuit reverse current gain}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1 = 0} = \text{Short circuit output impedance}\left(\frac{1}{Y_{22}}\right)$$

6.8 RELATIONSHIP BETWEEN PARAMETER SETS

6.8.1 Expression of *Z*-parameters in Terms of *Y*-parameters and Vice-versa

From Eqs 6.1, 6.2, 6.3 and 6.4, it is easy to derive the relation between the open circuit impedance parameters and the short circuit admittance parameters by means of two matrix equations of the respective parameters. By solving Eqs 6.1 and 6.2 for I_1 and I_2, we get

$$I_1 = \begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix} \Big/ \Delta_z \text{ ; and } I_2 = \begin{vmatrix} Z_{11} & V_1 \\ V_{21} & V_2 \end{vmatrix} \Big/ \Delta_z$$

where Δ_z is the determinant of Z matrix

$$\Delta_z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$I_1 = \frac{Z_{22}}{\Delta_z} V_1 - \frac{Z_{12}}{\Delta_z} V_2 \tag{6.13}$$

$$I_2 = \frac{-Z_{21}}{\Delta_z} V_1 + \frac{Z_{11}}{\Delta_z} V_2 \tag{6.14}$$

Comparing Eqs. 6.13 and 6.14 with Eqs. 6.3 and 6.4 we have

$$Y_{11} = \frac{Z_{22}}{\Delta_z} \text{ ; } Y_{12} = \frac{-Z_{12}}{\Delta_z}$$

$$Y_{21} = \frac{Z_{21}}{\Delta_z} \text{ ; } Y_{22} = \frac{Z_{11}}{\Delta_z}$$

In a similar manner, the Z parameters may be expressed in terms of the admittance parameters by solving Eqs 6.3 and 6.4 for V_1 and V_2

$$V_1 = \begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix} \Big/ \Delta_y \text{ and } V_2 = \begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix} \Big/ \Delta_y$$

where Δ_y is the determinant of the Y matrix

$$\Delta_y = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}$$

$$V_1 = \frac{Y_{22}}{\Delta_y} I_1 - \frac{Y_{12}}{\Delta_y} I_2 \tag{6.15}$$

$$V_2 = \frac{-Y_{21}}{\Delta_y} I_1 + \frac{Y_{11}}{\Delta_y} I_2 \tag{6.16}$$

Comparing Eqs 6.15 and 6.16 with Eqs 6.1 and 6.2, we obtain

$$Z_{11} = \frac{Y_{22}}{\Delta_y} \text{ ; } Z_{12} = \frac{-Y_{12}}{\Delta_y}$$

$$Z_{21} = \frac{-Y_{21}}{\Delta_y} \text{ ; } Z_{22} = \frac{Y_{11}}{\Delta_y}$$

Example 6.5 *For a given, Z_{11} = 3 Ω, Z_{12} = 1 Ω; Z_{21} = 2 Ω and Z_{22} = 1 Ω, find the admittance matrix, and the product of Δ_y and Δ_z.*

Solution

$$\text{The admittance matrix } = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{22}}{\Delta_z} & \dfrac{-Z_{12}}{\Delta_z} \\ \dfrac{-Z_{21}}{\Delta_z} & \dfrac{Z_{11}}{\Delta_z} \end{bmatrix}$$

given $$Z = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$\therefore$ $$\Delta_z = 3 - 2 = 1$$

$\therefore$ $$\Delta_y = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = 1$$

$$(\Delta_y)\,(\Delta_z) = 1$$

6.8.2 General Circuit Parameters or *ABCD* Parameters in Terms of *Z* Parameters and *Y* Parameters

We know that

$$V_1 = AV_2 - BI_2;\ V_1 = Z_{11}\,I_1 + Z_{12}\,I_2;\ I_1 = Y_{11}\,V_1 + Y_{12}\,V_2$$
$$I_1 = CV_2 - DI_2;\ V_2 = Z_{21}\,I_1 + Z_{22}\,I_2;\ I_2 = Y_{21}\,V_1 + Y_{22}\,V_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0};\ C = \left.\frac{I_1}{V_2}\right|_{I_2=0};\ B = \left.\frac{-V_1}{I_2}\right|_{V_2=0};\ D = \left.\frac{-I_1}{I_2}\right|_{V_2=0}$$

Substituting the condition $I_2 = 0$ in Eqs 6.1 and 6.2 we get

$$\left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{Z_{11}}{Z_{21}} = A$$

Substituting the condition $I_2 = 0$ in Eq. 6.4 we get,

$$\left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{-Y_{22}}{Y_{21}} = A$$

Substituting the condition $I_2 = 0$ in Eq. 6.2

we get $$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{Z_{21}} = C$$

Substituting the condition $I_2 = 0$ in Eqs 6.3 and 6.4, and solving for V_2 gives $\dfrac{-I_1\,Y_{21}}{\Delta y}$

where Δy is the determinant of the admittance matrix

$$\left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{-\Delta y}{Y_{21}} = C$$

Substituting the condition $V_2 = 0$ in Eq. 6.4, we get

$$\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{1}{Y_{21}} = B$$

Substituting the condition $V_2 = 0$ in Eqs 6.1 and 6.2 and solving for $I_2 = \dfrac{-V_1 Z_{21}}{\Delta_z}$

$$-\left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{\Delta_z}{Z_{21}} = B$$

where Δ_z is the determinant of the impedance matrix.

Substituting $V_2 = 0$ in Eq. 6.2

we get $\left.\dfrac{-I_1}{I_2}\right|_{V_2=0} = \dfrac{Z_{22}}{Z_{21}} = D$

Substituting $V_2 = 0$ in Eqs 6.3 and 6.4, we get

$$\left.\frac{-I_1}{I_2}\right|_{V_2=0} = \frac{-Y_{11}}{Y_{21}} = D$$

The determinant of the transmission matrix is given by

$$-AD + BC$$

Substituting the impedance parameters in A, B, C and D, we have

$$BC - AD = \frac{\Delta z}{Z_{21}}\frac{1}{Z_{21}} - \frac{Z_{11}}{Z_{21}}\frac{Z_{22}}{Z_{21}}$$

$$= \frac{\Delta z}{(Z_{21})^2} - \frac{Z_{11}Z_{22}}{(Z_{21})^2}$$

$$BC - AD = \frac{-Z_{12}}{Z_{21}}$$

For a bilateral network, $Z_{12} = Z_{21}$

$\therefore \quad BC - AD = -1$

or $\quad AD - BC = 1$

Therefore, in a two-port bilateral network, if three transmission parameters are known, the fourth may be found from equation $AD - BC = 1$.

In a similar manner the h parameters may be expressed in terms of the admittance parameters, impedance parameters or transmission parameters. Transformations of this nature are possible between any of the various parameters. Each parameter has its own utility. However, we often find that it is necessary to convert from one set of parameters to another. Transformations between different parameters, and the condition under which the two-port network is reciprocal are given in Table 6.1.

Table 6.1

	Z	Y	$ABCD$	$A'B'C'D'$	h	g
Z	$\begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix}$	$\begin{matrix} \frac{Y_{22}}{\Delta_y} & \frac{-Y_{12}}{\Delta_y} \\ \frac{-Y_{21}}{\Delta_y} & \frac{Y_{11}}{\Delta_y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D'}{C'} & \frac{1}{C'} \\ \frac{\Delta_{T'}}{C'} & \frac{A'}{C'} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{22}} & \frac{h_{22}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & \frac{-g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\Delta_g}{g_{11}} \end{matrix}$
Y	$\begin{matrix} \frac{Z_{22}}{\Delta_z} & \frac{-Z_{12}}{\Delta_z} \\ \frac{-Z_{21}}{\Delta z} & \frac{Z_{11}}{\Delta z} \end{matrix}$	$\begin{matrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{A'}{B'} & \frac{-1}{B'} \\ \frac{-\Delta_{T'}}{B'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{-g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
AB CD	$\begin{matrix} \frac{Z_{11}}{Z_{21}} & \frac{\Delta z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{matrix}$	$\begin{matrix} \frac{-Y_{22}}{Y_{21}} & \frac{-1}{Y_{21}} \\ \frac{\Delta Y}{Y_{21}} & \frac{-Y_{11}}{Y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{D'}{\Delta_{T'}} & \frac{B'}{\Delta_{T'}} \\ \frac{C'}{\Delta_{T'}} & \frac{A'}{\Delta_{T'}} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{21}} & \frac{h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta_g}{g_{21}} \end{matrix}$
$A'\,B'$ $C'\,D'$	$\begin{matrix} \frac{Z_{22}}{Z_{12}} & \frac{\Delta z}{Z_{12}} \\ \frac{1}{Z_{12}} & \frac{Z_{11}}{Z_{12}} \end{matrix}$	$\begin{matrix} \frac{-Y_{11}}{Y_{12}} & \frac{-1}{Y_{12}} \\ \frac{-\Delta_Y}{Y_{12}} & \frac{-Y_{22}}{Y_{12}} \end{matrix}$	$\begin{matrix} \frac{D}{\Delta_T} & \frac{B}{\Delta_T} \\ \frac{C}{\Delta_T} & \frac{A}{\Delta_T} \end{matrix}$	$\begin{matrix} A' & B' \\ C' & D' \end{matrix}$	$\begin{matrix} \frac{1}{h_{12}} & \frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & \frac{\Delta_h}{h_{12}} \end{matrix}$	$\begin{matrix} \frac{-\Delta_g}{g_{12}} & \frac{-g_{22}}{g_{12}} \\ \frac{-g_{11}}{g_{12}} & \frac{-1}{g_{12}} \end{matrix}$
h	$\begin{matrix} \frac{\Delta_z}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ \frac{-Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{Y_{11}} & \frac{-Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & \frac{\Delta_Y}{Y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} \frac{B'}{A'} & \frac{1}{A'} \\ \frac{\Delta_{T'}}{A'} & \frac{C'}{A'} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\Delta_g} & \frac{-g_{12}}{\Delta_g} \\ \frac{-g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{matrix}$
g	$\begin{matrix} \frac{1}{Z_{11}} & \frac{-Z_{12}}{Z_{11}} \\ \frac{Z_{21}}{Z_{11}} & \frac{\Delta_Z}{Z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_Y}{Y_{22}} & \frac{Y_{12}}{Y_{22}} \\ \frac{-Y_{21}}{Y_{22}} & \frac{1}{Y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & \frac{-\Delta_T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{C'}{D'} & \frac{-1}{D'} \\ \frac{\Delta_{T'}}{D'} & \frac{B'}{D'} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta_h} & \frac{-h_{12}}{\Delta_h} \\ \frac{-h_{21}}{\Delta_h} & \frac{h_{11}}{\Delta_h} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$
The two port is reciprocal If	$Z_{12} = Z_{21}$	$Y_{12} = Y_{21}$	The determinant of the transmission matrix = 1 ($\Delta_T = 1$)	The determinant of the inverse transmission matrix = 1	$h_{12} = -h_{21}$	$g_{12} = -g_{21}$

Example 6.6 *The impedance parameters of a two port network are $Z_{11} = 6\,\Omega$; $Z_{22} = 4\,\Omega$; $Z_{12} = Z_{21} = 3\,\Omega$. Compute the Y parameters and ABCD parameters and write the describing equations.*

Solution

ABCD parameters are given by

$$A = \frac{Z_{11}}{Z_{21}} = \frac{6}{3} = 2;\ B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} = 5\ \Omega$$

$$C = \frac{1}{Z_{21}} = \frac{1}{3}\ \mho;\quad D = \frac{Z_{22}}{Z_{21}} = \frac{4}{3}$$

Y parameters are given by

$$Y_{11} = \frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{4}{15}\ \mho;\ Y_{12} = \frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{-1}{5}\ \mho$$

$$Y_{21} = Y_{12} = \frac{-Z_{12}}{\Delta z} = \frac{-1}{5}\ \mho;\ Y_{22} = \frac{Z_{11}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{2}{5}\ \mho$$

The equations, using *Z* parameters are

$$V_1 = 6I_1 + 3I_2$$
$$V_2 = 3I_1 + 4I_2$$

Using *Y* parameters

$$I_1 = \frac{4}{15}V_1 - \frac{1}{5}V_2$$

$$I_2 = \frac{-1}{5}V_1 + \frac{2}{5}V_2$$

Using *ABCD* parameters

$$V_1 = 2V_2 - 5I_2$$

$$I_1 = \frac{1}{3}V_2 - \frac{4}{3}I_2$$

6.9 INTER CONNECTION OF TWO-PORT NETWORKS

6.9.1 Series Connection of Two-port Network

It has already been shown in Section 6.4.1 that when two-port networks are connected in cascade, the parameters of the interconnected network can be conveniently expressed with the help of *ABCD* parameters. In a similar way, the *Z*-parameters can be used to describe the parameters of series connected two-port networks; and Y parameters can be used to describe parameters of parallel connected two-port networks. A series connection of two-port networks is shown in Fig. 6.18.

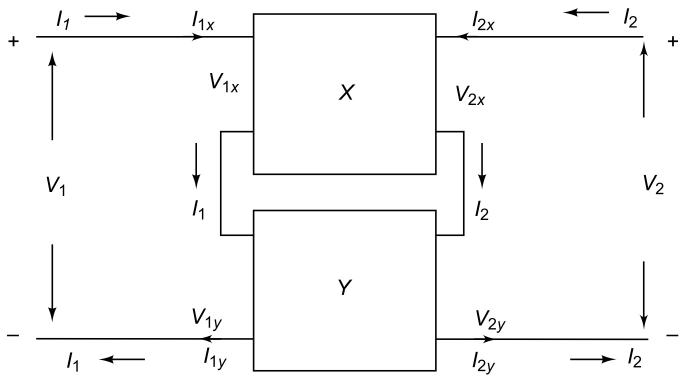


Fig. 6.18

Let us consider two, two-port networks, connected in series as shown. If each port has a common reference node for its input and output, and if these references are connected together then the equations of the networks X and Y in terms of Z parameters are

$$V_{1X} = Z_{11X} I_{1X} + Z_{12X} I_{2X}$$
$$V_{2X} = Z_{21X} I_{1X} + Z_{22X} I_{2X}$$
$$V_{1Y} = Z_{11Y} I_{1Y} + Z_{12Y} I_{2Y}$$
$$V_{2Y} = Z_{21Y} I_{1Y} + Z_{22Y} I_{2Y}$$

From the inter-connection of the networks, it is clear that

$$I_1 = I_{1X} = I_{1Y};\ I_2 = I_{2X} = I_{2Y}$$

and $$V_1 = V_{1X} + V_{1Y};\ V_2 = V_{2X} + V_{2Y}$$

$\therefore$ $$V_1 = Z_{11X} I_1 + Z_{12X} I_2 + Z_{11Y} I_1 + Z_{12Y} I_2$$
$$= (Z_{11X} + Z_{11Y}) I_1 + (Z_{12X} + Z_{12Y}) I_2$$
$$V_2 = Z_{21X} I_1 + Z_{22X} I_2 + Z_{21Y} I_1 + Z_{22Y} I_2$$
$$= (Z_{21X} + X_{21Y}) I_1 + (Z_{22X} + Z_{22Y}) I_2$$

The describing equations for the series connected two-port network are

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

where $$Z_{11} = Z_{11X} + Z_{11Y};\ Z_{12} = Z_{12X} + Z_{12Y}$$
$$Z_{21} = Z_{21X} + Z_{21Y};\ Z_{22} = Z_{22X} + Z_{22Y}$$

Thus, we see that each Z parameter of the series network is given as the sum of the corresponding parameters of the individual networks.

6.9.2 Parallel Connection of Two Two-port Networks

Let us consider two two-port networks connected in parallel as shown in Fig. 6.19. If each two-port has a reference node that is common to its input and output port, and if the two ports are connected so that they have a common reference node, then the equations of the networks X and Y in terms of Y parameters are given by

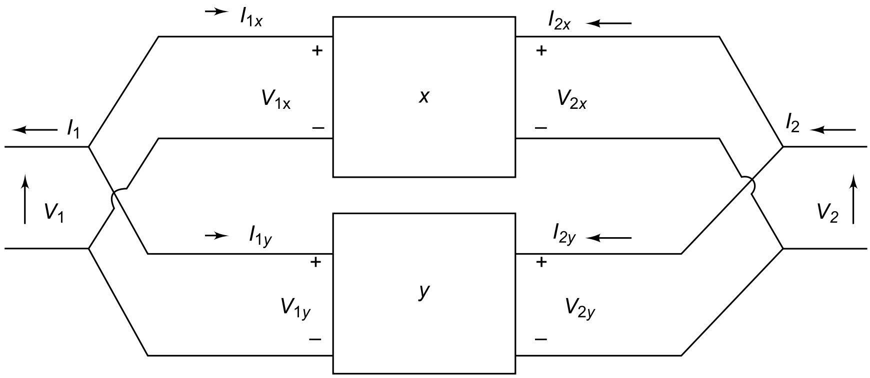


Fig. 6.19

$$
\begin{aligned}
I_{1X} &= Y_{11X} V_{1X} + Y_{12X} V_{2X} \\
I_{2X} &= Y_{21X} V_{1X} + Y_{22X} V_{2X} \\
I_{1Y} &= Y_{11Y} V_{1Y} + Y_{12Y} V_{2Y} \\
I_{2Y} &= Y_{21Y} V_{1Y} + Y_{22Y} V_{2Y}
\end{aligned}
$$

From the interconnection of the networks, it is clear that

$$V_1 = V_{1X} = V_{1Y};\ V_2 = V_{2X} = V_{2Y}$$

and $$I_1 = I_{1X} + I_{1Y};\ I_2 = I_{2X} + I_{2Y}$$

$\therefore$
$$
\begin{aligned}
I_1 &= Y_{11X} V_1 + Y_{12X} V_2 + Y_{11Y} V_1 + Y_{12Y} V_2 \\
&= (Y_{11X} + Y_{11Y}) V_1 + (Y_{12X} + Y_{12Y}) V_2 \\
I_2 &= Y_{21X} V_1 + Y_{22X} V_2 + Y_{21Y} V_1 + Y_{22Y} V_2 \\
&= (Y_{21X} + Y_{21Y}) V_1 + (Y_{22X} + Y_{22Y}) V_2
\end{aligned}
$$

The describing equations for the parallel connected two-port networks are

$$
\begin{aligned}
I_1 &= Y_{11} V_1 + Y_{12} V_2 \\
I_2 &= Y_{21} V_1 + Y_{22} V_2
\end{aligned}
$$

where
$$
\begin{aligned}
Y_{11} &= Y_{11X} + Y_{11Y};\ Y_{12} = Y_{12X} + Y_{12Y} \\
Y_{21} &= Y_{21X} + Y_{21Y};\ Y_{22} = Y_{22X} + Y_{22Y}
\end{aligned}
$$

Thus we see that each Y parameter of the parallel network is given as the sum of the corresponding parameters of the individual networks.

Example 6.7 *Two networks shown in Figs. 6.20(a) and (b) are connected in series. Obtain the Z parameters of the combination. Also verify by direct calculation.*

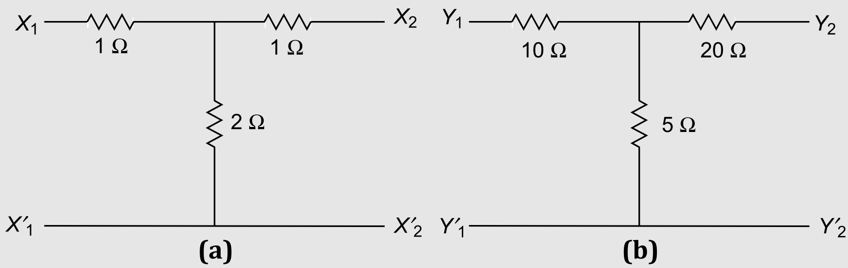


Fig. 6.20

Solution

The Z parameters of the network in Fig. 6.20(a) are

$$Z_{11X} = 3\ \Omega\ Z_{12X} = Z_{21X} = 2\ \Omega\ Z_{22X} = 3\ \Omega$$

The Z parameters of the network in Fig. 6.20(b) are

$$Z_{11Y} = 15\ \Omega\ Z_{21Y} = 5\ \Omega\ Z_{22Y} = 25\ \Omega\ Z_{12Y} = 5\ \Omega$$

The Z parameters of the combined network are

$$Z_{11} = Z_{11X} + Z_{11Y} = 18\ \Omega$$
$$Z_{12} = Z_{12X} + Z_{12Y} = 7\ \Omega$$
$$Z_{21} = Z_{21X} + Z_{21Y} = 7\ \Omega$$
$$Z_{22} = Z_{22X} + Z_{22Y} = 28\ \Omega$$

Check If the two networks are connected in series as shown in Fig. 6.20(c), the Z parameters are

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 18\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 7\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 28\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 7\ \Omega$$

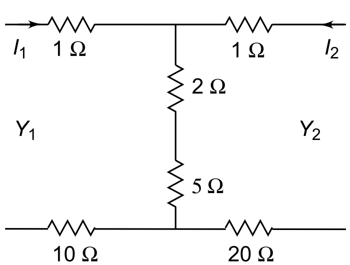


Fig. 6.20(c)

Example 6.8 *Two identical sections of the network shown in Fig. 6.21 are connected in parallel. Obtain the Y parameters of the combination.*

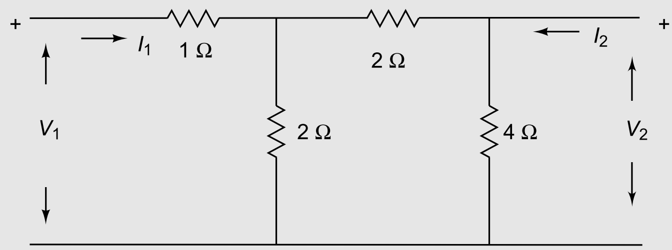


Fig. 6.21

Solution

The Y parameters of the network in Fig. 6.21 are (See Ex. 6.2).

$$Y_{11} = \frac{1}{2}\,\mho \quad Y_{21} = \frac{-1}{4}\,\mho \quad Y_{22} = \frac{5}{8}\,\mho \quad Y_{12} = \frac{-1}{4}\,\mho$$

If two such networks are connected in parallel then the Y parameters of the combined network are

$$Y_{11} = \frac{1}{2} + \frac{1}{2} = 1\,\mho \qquad Y_{21} = \frac{-1}{4} \times 2 = \frac{-1}{2}\,\mho$$

$$Y_{22} = \frac{5}{8} \times 2 = \frac{5}{4}\,\mho \qquad Y_{12} = \frac{-1}{4} \times 2 = \frac{-1}{2}\,\mho$$

6.10 T AND π REPRESENTATIONS

A two-port network with any number of elements may be converted into a two-port three-element network. Thus, a two-port network may be represented by an equivalent T network, i.e. three impedances are connected together in the form of a T as shown in Fig. 6.22.

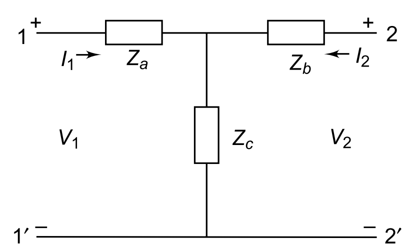


Fig. 6.22

It is possible to express the elements of the T-network in terms of Z parameters, or $ABCD$ parameters as explained below.

Z parameters of the network

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = Z_a + Z_c$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = Z_c$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = Z_b + Z_c$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = Z_c$$

From the above relations, it is clear that

$$Z_a = Z_{11} - Z_{21}$$
$$Z_b = Z_{22} - Z_{12}$$
$$Z_c = Z_{12} = Z_{21}$$

ABCD parameters of the network

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \frac{Z_a + Z_c}{Z_c}$$

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2 = 0}$$

When 2-2′ is short circuited

$$-I_2 = \frac{V_1 Z_c}{Z_b Z_c + Z_a (Z_b + Z_c)}$$

$$B = (Z_a + Z_b) + \frac{Z_a Z_b}{Z_c}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \frac{1}{Z_c}$$

$$D = \left.\frac{-I_1}{I_2}\right|_{V_2 = 0}$$

When 2-2′ is short circuited

$$-I_2 = I_1 \frac{Z_c}{Z_b + Z_c}$$

$$D = \frac{Z_b + Z_c}{Z_c}$$

From the above relations we can obtain

$$Z_a = \frac{A-1}{C}; \quad Z_b = \frac{D-1}{C}; \quad Z_c = \frac{1}{C}$$

Example 6.9 *The Z parameters of a two-port network are $Z_{11} = 10\ \Omega$; $Z_{22} = 15\ \Omega$; $Z_{12} = Z_{21} = 5\ \Omega$. Find the equivalent T network and ABCD parameters.*

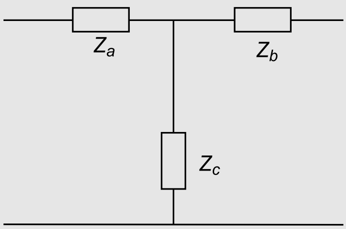


Fig. 6.23

Solution

The equivalent *T* network is shown in Fig. 6.23,

where $$Z_a = Z_{11} - Z_{21} = 5\ \Omega$$
$$Z_b = Z_{22} - Z_{12} = 10\ \Omega$$
and $$Z_c = 5\ \Omega$$

The *ABCD* parameters of the network are

$$A = \frac{Z_a}{Z_c} + 1 = 2;\ B = (Z_a + Z_b) + \frac{Z_a\,Z_b}{Z_c} = 25\ \Omega$$

$$C = \frac{1}{Z_c} = 0.2\ \mho\ D = 1 + \frac{Z_b}{Z_c} = 3$$

In a similar way, a two-port network may be represented by an equivalent π-network, i.e. three impedances or admittances are connected together in the form of π as shown in Fig. 6.24.

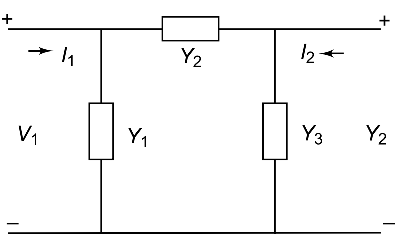


Fig. 6.24

It is possible to express the elements of the π-network in terms of *Y* parameters or *ABCD* parameters as explained below.

Y parameters of the network

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = Y_1 + Y_2$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -\,Y_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = Y_3 + Y_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} = -\,Y_2$$

From the above relations, it is clear that

$$Y_1 = Y_{11} + Y_{21}$$

$$Y_2 = -Y_{12}$$
$$Y_3 = Y_{22} + Y_{21}$$

Writing *ABCD* parameters in terms of *Y* parameters yields the following results.

$$A = \frac{-Y_{22}}{Y_{21}} = \frac{Y_3 + Y_2}{Y_2}$$

$$B = \frac{-1}{Y_{21}} = \frac{1}{Y_2}$$

$$C = \frac{-\Delta y}{Y_{21}} = Y_1 + Y_3 + \frac{Y_1 Y_3}{Y_2}$$

$$D = \frac{-Y_{11}}{Y_{21}} = \frac{Y_1 + Y_2}{Y_2}$$

From the above results, we can obtain

$$Y_1 = \frac{D-1}{B}$$

$$Y_2 = \frac{1}{B}$$

$$Y_3 = \frac{A-1}{B}$$

Example 6.10 *The port currents of a two-port network are given by*

$$I_1 = 2.5V_1 - V_2$$
$$I_2 = -V_1 + 5V_2$$

Find the equivalent π-network.

Solution

Let us first find the *Y* parameters of the network

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 2.5\ \mho;\ Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -1\ \mho$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -1\ \mho;\ Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 5\ \mho$$

The equivalent π-network is shown in Fig. 6.25.

where $Y_1 = Y_{11} + Y_{21} = 1.5\ \mho;$

$Y_2 = -Y_{12} = -1\ \mho$

and $Y_3 = Y_{22} + Y_{12} = 4\ \mho$

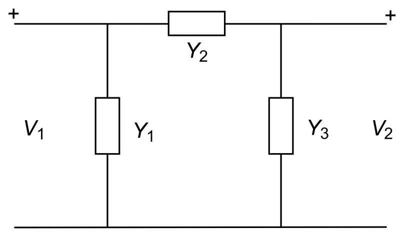


Fig. 6.25

6.11 TERMINATED TWO-PORT NETWORK

6.11.1 Driving Point Impedance at the Input Port of a Load Terminated Network

Figure 6.26 shows a two-port network connected to an ideal generator at the input port and to a load impedance at the output port. The input impedance of this network can be expressed in terms of parameters of the two port network.

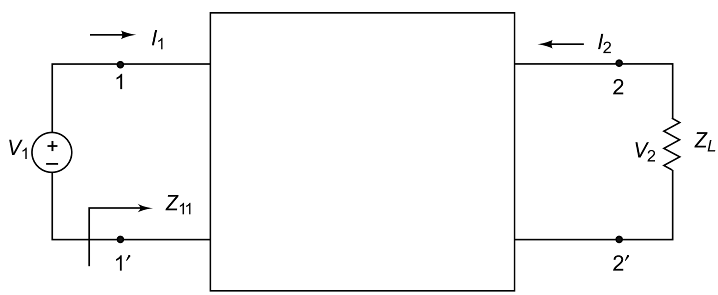


Fig. 6.26

(i) In Terms of Z Parameters The load at the output port 2-2′ impose the following constraint on the port voltage and current,

i.e., $$V_2 = -Z_L I_2$$

Recalling Eqs 6.1 and 6.2, we have

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Substituting the value of V_2 in Eq. 6.2, we have

$$-Z_L I_2 = Z_{21}I_1 + Z_{22}I_2$$

from which $$I_2 = \frac{-I_1 Z_{21}}{Z_L + Z_{22}}$$

Substituting the value of I_2 in Eq. 6.1 gives

$$V_1 = Z_{11}I_1 - \frac{Z_{12}Z_{21}I_1}{Z_L + Z_{22}}$$

$$V_1 = I_1\left(Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}\right)$$

Hence, the driving point impedance at 1-1′ is

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_L + Z_{22}}$$

If the output port is open, i.e. $Z_L \to \infty$, the input impedance is given by $V_1/I_1 = Z_{11}$

If the output port is short circuited, i.e. $Z_L \to 0$,

The short circuit driving point impedance is given by

$$\frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{22}} = \frac{1}{Y_{11}}$$

(ii) In Terms of Y Parameters If a load admittance Y_L is connected across the output port. The constraint imposed on the output port voltage and current is

$$-I_2 = V_2 Y_L, \text{ where } Y_L = \frac{1}{Z_L}$$

Recalling Eqs 6.3 and 6.4 we have

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Substituting the value of I_2 in Eq. 6.4, we have

$$-V_2 Y_L = Y_{21}V_1 + Y_{22}V_2$$

$$V_2 = -\left(\frac{Y_{21}}{Y_L + Y_{22}}\right)V_1$$

Substituting V_2 value in Eq. 6.3, we have

$$I_1 = Y_{11}V_1 - \frac{Y_{12}Y_{21}V_1}{Y_L + Y_{22}}$$

From which $$\frac{I_1}{V_1} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}$$

Hence the driving point impedance is given by

$$\frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}(Y_1 + Y_{22}) - Y_{12}Y_{21}}$$

Table 6.2

			In terms of			
Driving point impedance at input port, or input impedance $\left(\frac{V_1}{I_1}\right)$	Z parameters $\frac{\Delta_z + Z_{11} Z_L}{Z_{22} + Z_L}$	Y parameters $\frac{Y_{22} + Y_L}{\Delta_y + Y_{11} Y_L}$	$ABCD$ $\frac{AZ_L + B}{CZ_L + D}$	$A'B'C'D'$ $\frac{B' - D'Z_L}{C'Z_L - A'}$	h parameter $\frac{\Delta_h Z_L + h_{11}}{1 + h_{22} Z_L}$	g parameter $\frac{1 + g_{22} Y_L}{\Delta_{gYL} + g_{11}}$
Driving point impedance at output port, or output impedance $\left(\frac{V_2}{I_2}\right)$	$\frac{\Delta_z + Z_{22} Z_s}{Z_1 + Z_{11}}$	$\frac{Y_{11} + Y_s}{\Delta_y + Y_s Y_{22}}$	$\frac{DZ_s + B}{CZ_s + A}$	$\frac{A'Z_s + B'}{C'Z_s + D'}$	$\frac{h_{11} + Z_s}{\Delta_h + h_{22} Z_s}$	$\frac{g_{22} + \Delta_s}{1 + g_{11} Z_s}$

Note: The above relations are obtained, when $V_s = 0$ and $I_s = 0$ at the input port.

If the output port is open, i.e., $Y_L \to 0$

$$\frac{V_1}{I_1} = \frac{Y_{22}}{\Delta_y} = Z_{11}$$

If the output port is short circuited, i.e. $Y_L \to \infty$

Then $$Y_{\text{in}} = Y_{11}$$

In a similar way, the input impedance of the load terminated two port network may be expressed in terms of other parameters by simple mathematical manipulations. The results are given in Table 6.2.

6.11.2 Driving Point Impedance at the Output Port with Source Impedance at the Input Port

Let us consider a two-port network connected to a generator at input port with a source impedance Z_s as shown in Fig. 6.27. The output impedance, or the driving point impedance, at the output port can be evaluated in terms of the parameters of two-port network.

(i) In terms of Z parameters If I_1 is the current due to V_s at port 1-1′

From Eqs 6.1 and 6.2, we have

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

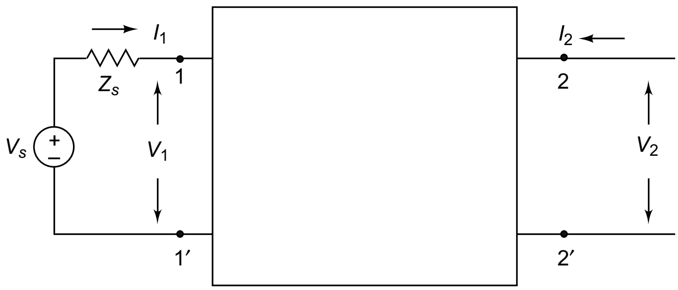


Fig. 6.27

$$V_1 = V_s - I_1 Z_s$$
$$= Z_{11} I_1 + Z_{12} I_2 - (I_1)(Z_s + Z_{11}) = Z_{12} I_2 - V_s$$

$$-I_1 = \frac{Z_{12} I_2 - V_s}{Z_s + Z_{11}}$$

Substituting I_1 in Eq. 6.2, we get

$$V_2 = -Z_{21} \frac{(Z_{12} I_2 - V_s)}{Z_s + Z_{11}} + Z_{22} I_2$$

With no source voltage at port 1-1′, i.e. if the source V_s is short circuited

$$V_2 = \frac{-Z_{21} Z_{12}}{Z_s + Z_{11}} I_2 + Z_{22} I_2$$

Hence the driving point impedance at port 2-2′ $= \dfrac{V_2}{I_2}$

$$\frac{V_2}{I_2} = \frac{Z_{22} Z_s + Z_{22} Z_{11} - Z_{21} Z_{12}}{Z_s + Z_{11}} \text{ or } \frac{\Delta_z + Z_{22} Z_s}{Z_s + Z_{11}}$$

If the input port is open, i.e. $Z_s \to \infty$

Then
$$\frac{V_2}{I_2} = \left[\frac{\dfrac{\Delta_Z}{Z_s} + Z_{22}}{1 + \dfrac{Z_{11}}{Z_s}} \right]_{Z_s = \infty} = Z_{22}$$

If the source impedance is zero with a short circuited input port, the driving point impedance at output port is given by

$$\frac{V_2}{I_2} = \frac{\Delta_Z}{Z_{11}} = \frac{1}{Y_{22}}$$

(ii) In terms of Y parameters Let us consider a two-port network connected to a current source at input port with a source admittance Y_s as shown in Fig. 6.28.

At port 1-1′ $\quad I_1 = I_s - V_1 Y_s$

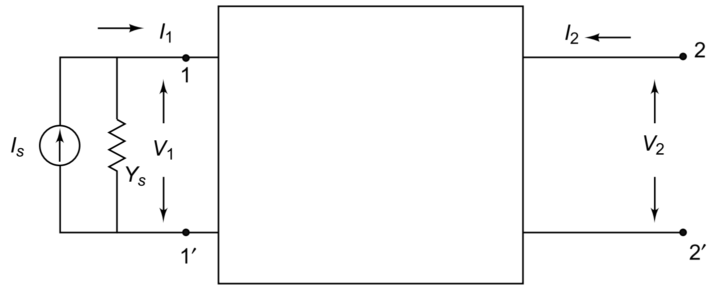


Fig. 6.28

Recalling Eqs 6.3 and 6.4, we have

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Substituting I_1 in Eq. 6.3, we get

$$I_s - V_1 Y_s = Y_{11} V_1 + Y_{12} V_2$$
$$-V_1(Y_s + Y_{11}) = Y_{12} V_2 - I_s$$
$$-V_1 = \frac{Y_{12} V_2 - I_s}{Y_s + Y_{11}}$$

Substituting V_1 in Eq. 6.4, we get

$$I_2 = -Y_{21} \left(\frac{Y_{12} V_2 - I_s}{Y_s + Y_{11}} \right) + Y_{22} V_2$$

With no source current at 1-1′, i.e. if the current source is open circuited

$$I_2 = \frac{-Y_{21} Y_{12} V_2}{Y_s + Y_{11}} + Y_{22} V_2$$

Hence the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{Y_{22} Y_s + Y_{22} Y_{11} - Y_{21} Y_{12}}{Y_s + Y_{11}} \text{ or } \frac{\Delta_y + Y_{22} Y_s}{Y_s + Y_{11}}$$

If the source admittance is zero, with an open circuited input port, the driving point admittance at the output port is given by

$$\frac{I_2}{V_2} = \frac{\Delta_y}{Y_{11}} = \frac{1}{Z_{22}} = Y_{22}$$

In a similar way, the output impedance may be expressed in terms of the other two port parameters by simple mathematical manipulations. The results are given in Table 6.2.

Example 6.11 *Calculate the input impedance of the network shown in Fig. 6.29.*

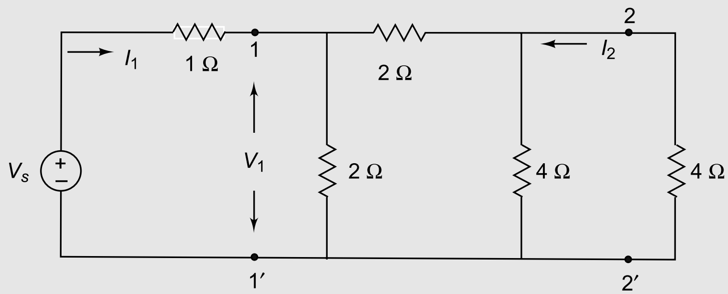


Fig. 6.29

Solution

Let us calculate the input impedance in terms of Z parameters. The Z parameters of the given network (see Solved Problem 6.1) are $Z_{11} = 2.5\ \Omega$; $Z_{21} = 1\ \Omega$; $Z_{22} = 2\ \Omega$; $Z_{12} = 1\ \Omega$

From section 6.11.1 we have the relation

$$\frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}\,Z_{21}}{Z_L + Z_{22}}$$

where Z_L is the load impedance $= 2\ \Omega$

$$\frac{V_1}{I_1} = 2.5 - \frac{1}{2+2} = 2.25\ \Omega$$

The source resistance is 1 Ω

$\therefore$ $$Z_{\text{in}} = 1 + 2.25 = 3.25\ \Omega$$

Example 6.12 *Calculate the output impedance of the network shown in Fig. 6.30 with a source admittance of 1 ℧ at the input port.*

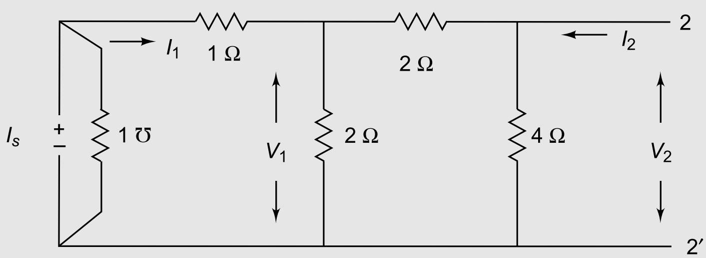


Fig. 6.30

Solution

Let us calculate the output impedance in terms of Y parameters. The Y parameters of the given network (see Ex. 6.2) are

$$Y_{11} = \frac{1}{2}\ \mho;\ Y_{22} = \frac{5}{8}\ \mho,\ Y_{21} = Y_{12} = \frac{-1}{4}\ \mho$$

From Section 6.11.2, we have the relation

$$\frac{I_2}{V_2} = \frac{Y_{22}\,Y_s + Y_{22}\,Y_{11} - Y_{21}\,Y_{12}}{Y_s + Y_{11}}$$

where Y_s is the source admittance = 1 mho

$$Y_{22} = \frac{I_2}{V_2} = \frac{\frac{5}{8}\times 1 + \frac{5}{8}\times\frac{1}{2} - \frac{1}{16}}{1+\frac{1}{2}} = \frac{7}{12}\ \mho$$

or $$Z_{22} = \frac{12}{7}\ \mho$$

6.12 LATTICE NETWORKS

One of the common four-terminal two-port network is the lattice, or bridge network shown in Fig. 6.31(a). Lattice networks are used in filter sections and are also used as attenuaters. Lattice structures are sometimes used in preference to ladder structures in some special applications. Z_a and Z_d are called series arms, Z_b and Z_c are called the diagonal arms. It can be observed that, if Z_d is zero, the lattice structure becomes a π-section. The lattice network is redrawn as a bridge network as shown in Fig. 6.31(b).

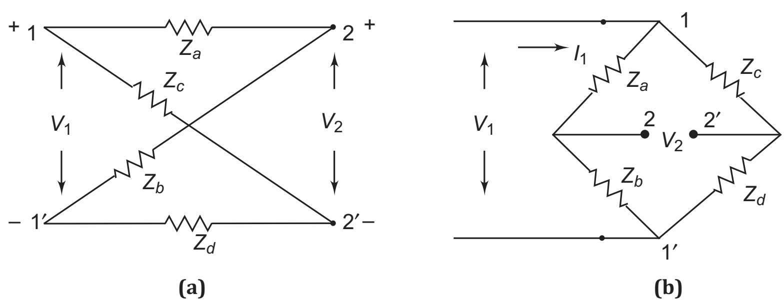


Fig. 6.31

Z Parameters

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

When $I_2 = 0$; $$V_1 = I_1 \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} \tag{6.17}$$

$$\therefore \qquad Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, then $Z_a = Z_d$ and $Z_b = Z_c$

$$\therefore \qquad Z_{11} = \frac{Z_a + Z_b}{2}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

When $I_2 = 0$, V_2 is the voltage across 2–2′

$$V_2 = V_1\left[\frac{Z_b}{Z_a + Z_b} - \frac{Z_d}{Z_c + Z_d}\right]$$

Substituting the value of V_1 from Eq. 6.17, we have

$$V_2 = \left[\frac{I_1(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d}\right]\left[\frac{Z_b(Z_c + Z_d) - Z_d(Z_a + Z_b)}{(Z_a + Z_b)(Z_c + Z_d)}\right]$$

$$\frac{V_2}{I_1} = \frac{Z_b(Z_c + Z_d) - Z_d(Z_a + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{Z_bZ_c - Z_aZ_d}{Z_a + Z_b + Z_c + Z_d}$$

$$\therefore \qquad Z_{21} = \frac{Z_bZ_c - Z_aZ_d}{Z_a + Z_b + Z_c + Z_d}$$

If the network is symmetric, $Z_a = Z_d$, $Z_b = Z_c$

$$Z_{21} = \frac{Z_b - Z_a}{2}$$

When the input port is open, $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

The network can be redrawn as shown in Fig. 6.31(c).

$$V_1 = V_2\left[\frac{Z_c}{Z_a + Z_c} - \frac{Z_d}{Z_b + Z_d}\right] \tag{6.18}$$

$$V_2 = I_2\left[\frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d}\right] \tag{6.19}$$

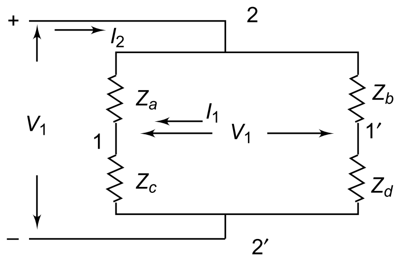


Fig. 6.31 (c)

Substituting the value of V_2 in Eq. 6.18, we get

$$V_1 = I_2\left[\frac{Z_c(Z_b+Z_d)-Z_d(Z_a+Z_c)}{Z_a+Z_b+Z_c+Z_d}\right]$$

$$\frac{V_1}{I_2} = \frac{Z_c Z_b - Z_a Z_d}{Z_a+Z_b+Z_c+Z_d}$$

If the network is symmetric, $Z_a = Z_d$; $Z_b = Z_c$

$$\frac{V_1}{I_2} = \frac{Z_b^2 - Z_a^2}{2(Z_a+Z_b)}$$

$\therefore$
$$Z_{12} = \frac{Z_b - Z_a}{2}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_2=0}$$

From Eq. 6.19, we have

$$\frac{V_2}{I_2} = \frac{(Z_a+Z_c)(Z_d+Z_b)}{Z_a+Z_b+Z_c+Z_d}$$

If the network is symmetric,

$$Z_a = Z_d;\ Z_b = Z_c$$

$$Z_{22} = \frac{Z_a+Z_b}{2} = Z_{11}$$

From the above equations, $Z_{11} = Z_{22} = \dfrac{Z_a+Z_b}{2}$

and
$$Z_{12} = Z_{21} = \frac{Z_b - Z_a}{2}$$

$\therefore$
$$Z_b = Z_{11} + Z_{12}$$
$$Z_a = Z_{11} - Z_{12}$$

Example 6.13 *Obtain the lattice equivalent of a symmetrical T network shown in Fig. 6.32.*

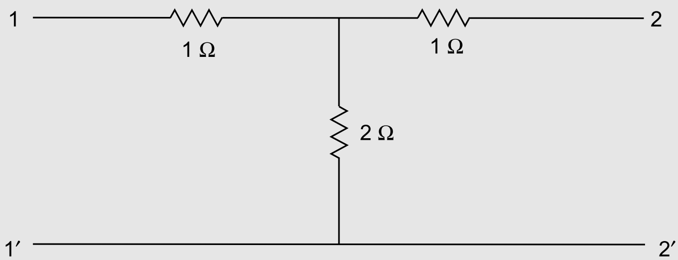


Fig. 6.32

Solution

A two-port network can be realised as a symmetric lattice if it is reciprocal and symmetric. The Z parameters of the network are (see Ex. 6.1). $Z_{11} = 3\ \Omega$; $Z_{12} = Z_{21} = 2\ \Omega$; $Z_{22} = 3\ \Omega$.

Since $Z_{11} = Z_{22}$; $Z_{12} = Z_{21}$, the given network is symmetrical and reciprocal.

$\therefore$ The parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 1\ \Omega$$

$$Z_b = Z_{11} + Z_{12} = 5\ \Omega$$

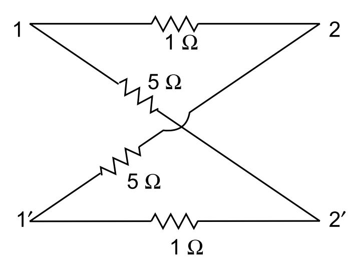


Fig. 6.33

The lattice network is shown in Fig. 6.33.

Example 6.14 *Obtain the lattice equivalent of a symmetric π-network shown in Fig. 6.34.*

Solution

The Z parameters of the given network are

$$Z_{11} = 6\ \Omega = Z_{22};\ Z_{12} = Z_{21} = 4\ \Omega$$

Hence the parameters of the lattice network are

$$Z_a = Z_{11} - Z_{12} = 2\ \Omega$$
$$Z_b = Z_{11} + Z_{12} = 10\ \Omega$$

The lattice network is shown in Fig. 6.35.

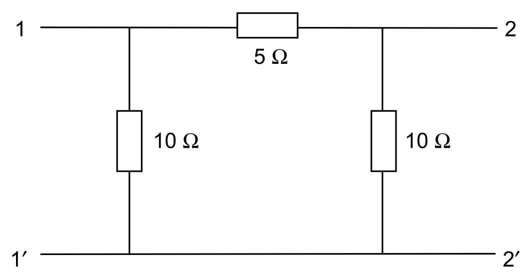


Fig. 6.34

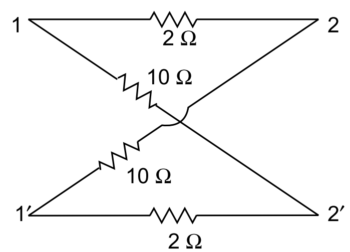


Fig. 6.35

6.13 IMAGE PARAMETERS

The image impedance Z_{I1} and Z_{I2} of a two-port network shown in Fig. 6.36 are two values of impedance such that, if port 1–1′ of the network is terminated in Z_{I1}, the input impedance of port 2-2′ is Z_{I2}; and if port 2-2′ is terminated in Z_{I2}, the input impedance at port 1-1′ is Z_{I1}.

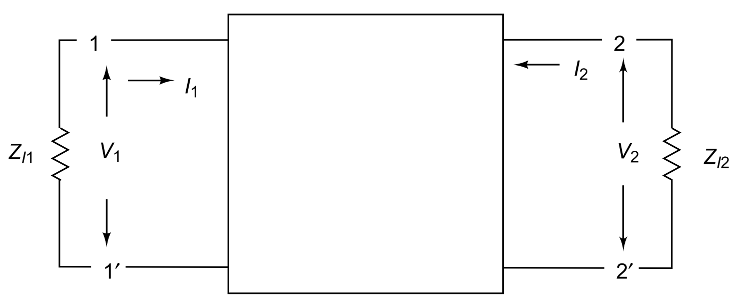


Fig. 6.36

Then, Z_{I1} and Z_{I2} are called image impedances of the two port network shown in Fig. 6.36. These parameters can be obtained in terms of two-port parameters. Recalling Eqs 6.5 and 6.6 in Section 6.4, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

If the network is terminated in Z_{I2} at 2-2′ as shown in Fig. 6.37.

$$V_2 = -I_2 Z_{I2}$$

$$\frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2} = Z_{I1}$$

$$Z_{I1} = \frac{-AI_2 Z_{I2} - BI_2}{-CI_2 Z_{I2} - DI_2}$$

$$Z_{I1} = \frac{-AZ_{I2} - B}{-CZ_{I2} - D}$$

or
$$Z_{I1} = \frac{AZ_{I2} + B}{CZ_{I2} + D}$$

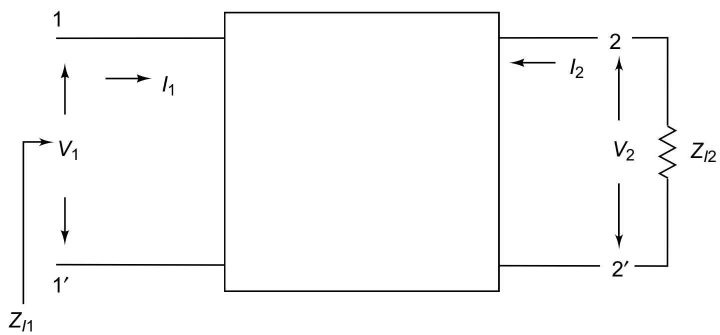


Fig. 6.37

Similarly, if the network is terminated in Z_{I1} at port 1-1′ as shown in Fig. 6.38, then

$$V_1 = -I_1 Z_{I1}$$

$$\frac{V_2}{I_2} = Z_{I2}$$

∴
$$-Z_{I1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

$$-Z_{I1} = \frac{AI_2 Z_{I2} - BI_2}{CI_2 Z_{I2} - DI_2}$$

$$-Z_{I1} = \frac{AZ_{I2} - B}{CZ_{I2} - D}$$

From which
$$Z_{I2} = \frac{DZ_{I1} + B}{CZ_{I1} + A}$$

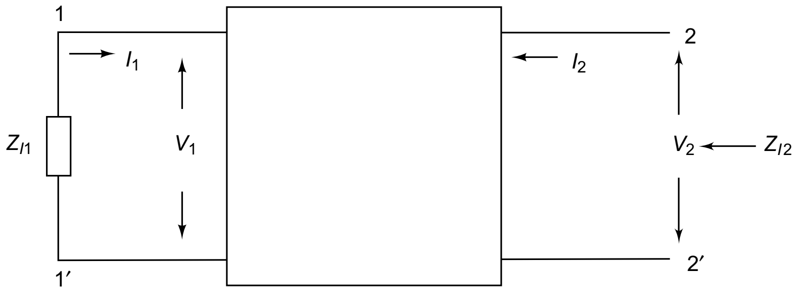


Fig. 6.38

Substituting the value of Z_{I1} in the above equation

$$Z_{I2}\left[C\frac{(-AZ_{I2}+B)}{(CZ_{I2}-D)}+A\right]=D\left[\frac{-AZ_{I2}+B}{CZ_{I2}-D}\right]+B$$

From which $Z_{I2}=\sqrt{\dfrac{BD}{AC}}$

Similarly, we can find $Z_{I1}=\sqrt{\dfrac{AB}{CD}}$

If the network is symmetrical, then $A = D$

$$\therefore \qquad Z_{I1}=Z_{I2}=\sqrt{\frac{B}{C}}$$

If the network is symmetrical, the image impedances Z_{I1} and Z_{I2} are equal to each other; the image impedance is then called the *characteristic* impedance, or the *iterative* impedance, i.e. if a symmetrical network is terminated in Z_L, its input impedance will also be Z_L, or its impedance transformation ratio is unity. Since a reciprocal symmetric network can be described by two independent parameters, the image parameters Z_{I1} and Z_{I2} are sufficient to characterise reciprocal symmetric networks. Z_{I1} and Z_{I2} the two image parameters do not completely define a network. A third parameter called *image transfer constant* ϕ is also used to describe reciprocal networks. This parameter may be obtained from the voltage and current ratios.

If the image impedance Z_{I2} is connected across port 2-2′, then

$$V_1 = AV_2 - BI_2 \tag{6.20}$$
$$V_2 = -I_2 Z_{I2} \tag{6.21}$$
$$\therefore \qquad V_1=\left[A+\frac{B}{Z_{I2}}\right]V_2 \tag{6.22}$$
$$I_1 = CV_2 - DI_2 \tag{6.23}$$
$$I_1 = -[CZ_{I2}+D]I_2 \tag{6.24}$$

From Eq. 6.22

$$\frac{V_1}{V_2} = \left[A + \frac{B}{Z_{I2}}\right] = A + B\sqrt{\frac{AC}{BD}}$$

$$\frac{V_1}{V_2} = A + \sqrt{\frac{ABCD}{D}} \tag{6.25}$$

From Eq. 6.24

$$\frac{-I_1}{I_2} = [CZ_{I2} + D] = D + C\sqrt{\frac{BD}{AC}}$$

$$\frac{-I_1}{I_2} = D + \sqrt{\frac{ABCD}{A}} \tag{6.26}$$

Multiplying Eqs 6.25 and 6.26 we have

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\frac{AD + \sqrt{ABCD}}{D}\right)\left(\frac{AD + \sqrt{ABCD}}{A}\right)$$

$$\frac{-V_1}{V_2} \times \frac{I_1}{I_2} = \left(\sqrt{AD} + \sqrt{BC}\right)^2$$

or $$\sqrt{AD} + \sqrt{BC} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}}$$

$$\sqrt{AD} + \sqrt{AD - 1} = \sqrt{\frac{-V_1}{V_2} \times \frac{I_1}{I_2}} \qquad (\because AD - BC = 1)$$

Let $$\cos h\,\phi = \sqrt{AD}\,;\ \sin h\,\phi = \sqrt{AD - 1}$$

$$\tan h\,\phi = \frac{\sqrt{AD - 1}}{\sqrt{AD}} = \sqrt{\frac{BC}{AD}}$$

$\therefore$ $$\phi = \tan h^{-1}\sqrt{\frac{BC}{AD}}$$

Also $$e^{\phi} = \cos h\,\phi + \sin h\,\phi = \sqrt{-\frac{V_1 I_1}{V_2 I_2}}$$

$$\phi = \log_e \sqrt{\left(-\frac{V_1 I_1}{V_2 I_2}\right)} = \frac{1}{2}\log_e\left(\frac{V_1}{V_2}\,\frac{I_1}{I_2}\right)$$

Since $$V_1 = Z_{I1} I_1;\ V_2 = -I_2 Z_{I2}$$

$$\phi = \frac{1}{2}\log_e\left[\frac{Z_{I1}}{Z_{I2}}\right] + \log\left[\frac{I_1}{I_2}\right]$$

For symmetrical reciprocal networks, $Z_{I1} = Z_{I2}$

$$\phi = \log_e \left[\frac{I_1}{I_2} \right] = \gamma$$

where γ is called the *propagation constant*.

Example 6.15 *Determine the image parameters of the T network shown in Fig. 6.39.*

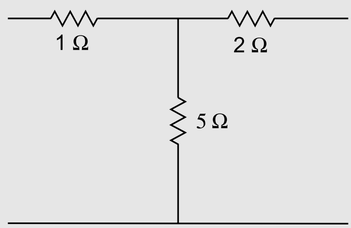


Fig. 6.39

Solution

The *ABCD* parameters of the network are

$$A = \frac{6}{5};\ B = \frac{17}{5};\ C = \frac{1}{5};\ D = \frac{7}{5} \quad \text{(See Ex. 6.3)}$$

Since the network is not symmetrical, ϕ, Z_{I1} and Z_{I2} are to be evaluated to describe the network.

$$Z_{I1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{\frac{6}{5} \times \frac{17}{5}}{\frac{1}{5} \times \frac{7}{5}}} = 3.817\ \Omega$$

$$Z_{I2} = \sqrt{\frac{BC}{AC}} = \sqrt{\frac{\frac{17}{5} \times \frac{7}{5}}{\frac{6}{5} \times \frac{1}{5}}} = 4.453\ \Omega$$

$$\phi = \tan h^{-1} \sqrt{\frac{BC}{AD}} = \tan h^{-1} \sqrt{\frac{17}{42}}$$

or

$$\phi = \text{In}\left[\sqrt{AD} + \sqrt{AD - 1}\right]$$

$$\phi = 0.75$$

Solved Problems

6.1 Find the Z parameters for the circuit shown in Fig. 6.40.

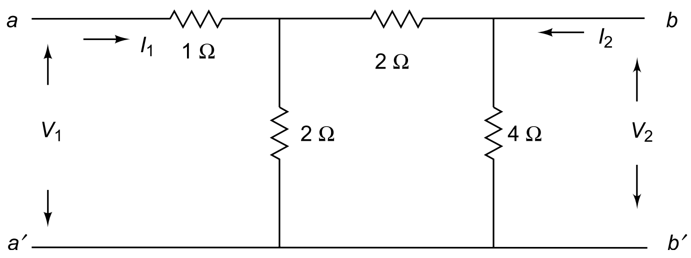


Fig. 6.40

Solution

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

When $I_2 = 0$; V_1 can be expressed in terms of I_1 and the equivalent impedance of the circuit looking from the terminal a-a' as shown in Fig. 6.41(a).

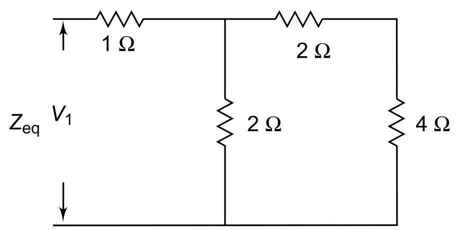


Fig. 6.41(a)

$$Z_{eq} = 1 + \frac{6 \times 2}{6 + 2} = 2.5\ \Omega$$

$$V_1 = I_1\, Z_{eq} = I_1\, 2.5$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = 2.5\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

V_2 is the voltage across the 4 Ω impedance as shown in Fig. 6.41(b).

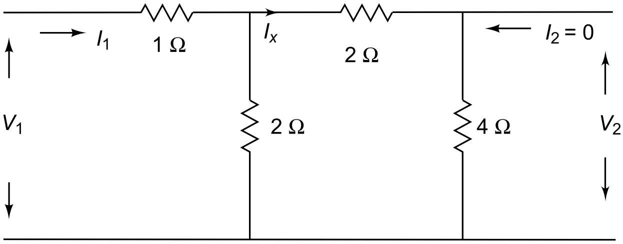


Fig. 6.41(b)

Let the current in the 4 Ω impedance be I_x

$$I_x = I_1 \times \frac{2}{8} = \frac{I_1}{4}$$

$$V_2 = I_x 4 = \frac{I_1}{4} \times 4 = I_1$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0} = 1\ \Omega$$

$$Z_{22} = \left.\frac{I_2}{I_2}\right|_{I_1 = 0}$$

When port a-a′ is open circuited the voltage at port b-b′ can be expressed in terms of I_2, and the equivalent impedance of the circuit viewed from b-b′ as shown in Fig. 6.41(c).

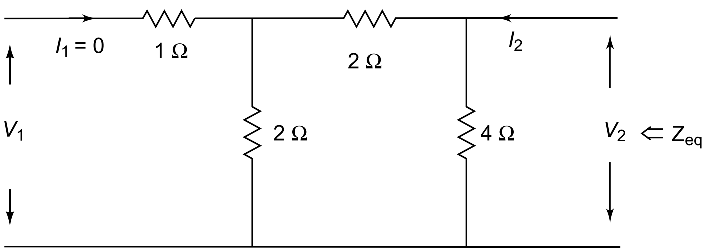


Fig. 6.41 (c)

$$V_2 = I_2 \times 2$$

$$\therefore \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = 2\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

V_1 is the voltage across the 2 Ω (parallel) impedance, let the current in the 2 Ω impedance is I_Y as shown in Fig. 6.41(d).

$$I_Y = \frac{I_2}{2}$$

$$V_1 = 2\,I_Y$$

$$V_1 = 2\,\frac{I_2}{2}$$

$$\therefore \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = 1\ \Omega$$

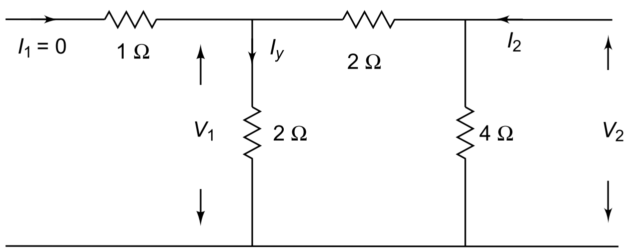


Fig. 6.41(d)

Here $Z_{12} = Z_{21}$, which indicates the bilateral property of the network. The describing equations for this two-port network in terms of impedance parameters are

$$V_1 = 2.5I_1 + I_2$$
$$V_2 = I_1 + 2I_2$$

6.2 Find the short circuit admittance parameters for the circuit shown in Fig. 6.42.

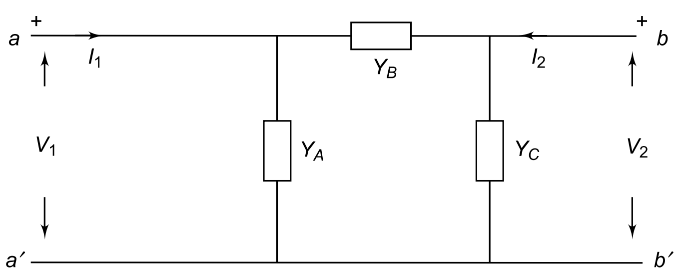


Fig. 6.42

Solution

The elements in the branches of the given two-port network are admittances. The admittance parameters can be determined by short circuiting the two-ports.

When port b-b' is short circuited, $V_2 = 0$. This circuit is shown in Fig. 6.43(a).

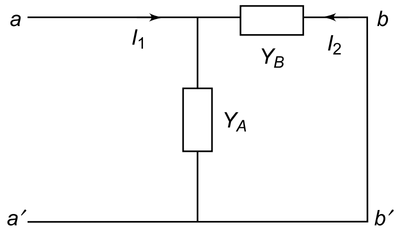


Fig. 6.43(a)

$$V_1 = I_1 Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from a-a'.

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_A + Y_B$$

$$V_1 = \frac{I_1}{Y_A + Y_B}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = (Y_A + Y_B)$$

With port b-b' short circuited, the nodal equation at node 1 gives

$$-I_2 = V_1 Y_B$$

$$\therefore \qquad Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -Y_B$$

when port a-a' is short circuited; $V_1 = 0$ this circuit is shown in Fig. 6.43(b).

$$V_2 = I_2 Z_{eq}$$

where Z_{eq} is the equivalent impedance as viewed from b-b'

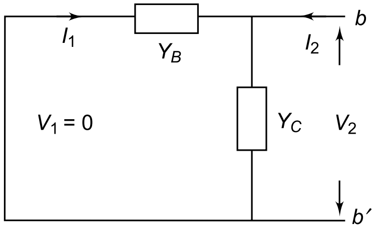


Fig. 6.43(b)

$$Z_{eq} = \frac{1}{Y_{eq}}$$

$$Y_{eq} = Y_b + Y_c$$

$$\therefore \qquad V_2 = \frac{I_2}{Y_B + Y_C}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = (Y_B + Y_C)$$

With port a-a' short circuited, the nodal equation at node 2 gives

$$-I_1 = V_2\, Y_B$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_B$$

The describing equations in terms of the admittance parameters are

$$I_1 = (Y_A + Y_B)V_1 - Y_B V_2$$
$$I_2 = -Y_B V_1 + (Y_C + Y_B)V_2$$

6.3 Find the Z parameters of the RC ladder network shown in Fig. 6.44.

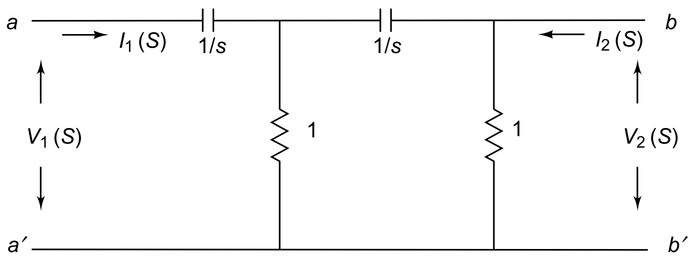


Fig. 6.44

Solution

With port b-b' open circuited and assuming mesh currents with $V_1(S)$ as the voltage at a-a', the corresponding network is shown in Fig. 6.45(a).

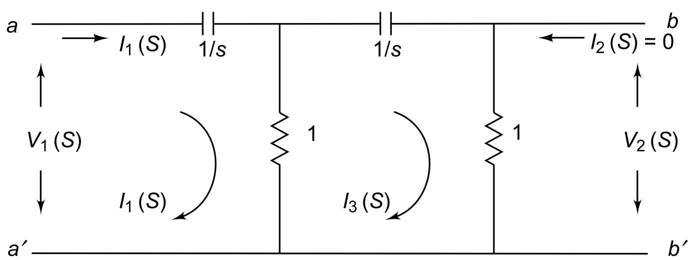


Fig. 6.45(a)

The KVL equations are as follows

$$V_2(S) = I_3(S) \tag{6.27}$$

$$I_3(S) \times \left(2 + \frac{1}{S}\right) = I_1(S) \tag{6.28}$$

$$\left(1 + \frac{1}{S}\right) I_1(S) - I_3(S) = V_1(S) \tag{6.29}$$

From Eq. 6.28, $I_3(S) = I_1(S)\left(\dfrac{S}{1+2S}\right)$

From Eq. 6.29 $$\left(\frac{S+1}{S}\right) I_1(S) - I_1(S)\,\frac{S}{1+2S} = V_1(S)$$

$$I_1(S)\left(\frac{1+S}{S} - \frac{S}{1+2S}\right) = V_1(S)$$

$$I_1(S)\left(\frac{S^2+3S+1}{S(1+2S)}\right) = V_1(S)$$

$$Z_{11} = \left.\frac{V_1(S)}{I_1(S)}\right|_{I_2=0} = \frac{(S^2+3S+1)}{S(1+2S)}$$

Also $$V_2(S) = I_3(S) = I_1(S)\,\frac{S}{1+2S}$$

$$Z_{21} = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2=0} = \frac{S}{1+2S}$$

With port a-a' open circuited and assuming mesh currents with $V_2(S)$ as the voltage as b-b', the corresponding network is shown in Fig. 6.45(b).

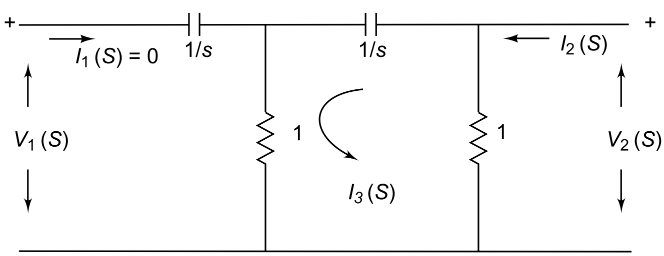


Fig. 6.45(b)

The KVL equations are as follows:

$$V_1(S) = I_3(S) \tag{6.30}$$

$$\left(2+\frac{1}{S}\right) I_3(S) = I_2(S) \tag{6.31}$$

$$V_2(S) = I_2(S) - I_3(S) \tag{6.32}$$

From Eq. 6.31 $I_3(S) = I_2(S)\left(\dfrac{S}{2S+1}\right)$

From Eq. 6.32 $V_2(S) = I_2(S) - I_2(S)\left(\dfrac{S}{2S+1}\right)$

$$V_2(S) = I_2(S)\left(1 - \frac{S}{2S+1}\right)$$

$$Z_{22} = \left.\frac{V_2(S)}{I_2(S)}\right|_{I_1(S)=0} = \frac{S+1}{2S+1}$$

Also
$$V_1(S) = I_3(S) = I_2(S)\left(\frac{S}{2S+1}\right)$$

$$Z_{12} = \left.\frac{V_1(S)}{I_2(S)}\right|_{I_1(S)=0} = \left(\frac{S}{2S+1}\right)$$

The describing equations are

$$V_1(S) = \left[\frac{S^2+3S+1}{3(2S+1)}\right] I_1 + \left[\frac{S}{2S+1}\right] I_2$$

$$V_2(S) = \left[\frac{S}{2S+1}\right] I_1 + \left[\frac{S+1}{2S+1}\right] I_2$$

6.4 Find the transmission parameters for the circuit shown in Fig. 6.46.

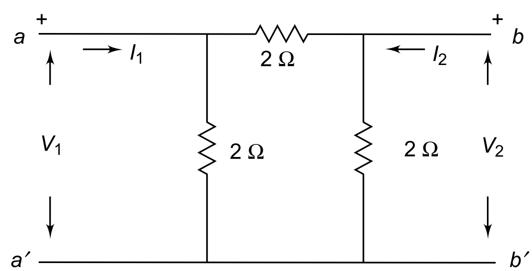


Fig. 6.46

Solution

Recalling Eqs 6.5 and 6.6, we have

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

When port b-b' is short circuited with V_1 across a-a', $V_2 = 0$, $B = \frac{-V_1}{I_2}$ and the circuit is as shown in Fig. 6.47(a).

$$-I_2 = \frac{V_1}{2} \quad I_1 = V_1$$

∴
$$B = 2\ \Omega$$

$$D = \frac{-I_1}{I_2} = 2$$

When port b-b' is open with V_1 across a-a', $I_2 = 0$

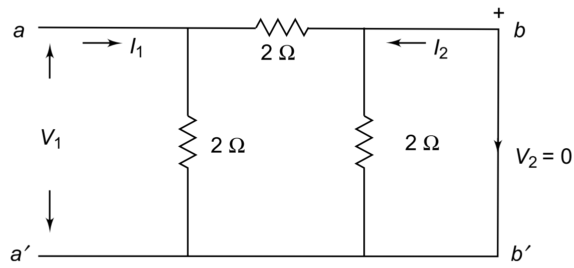


Fig. 6.47 (a)

$A = V_1/V_2$ and the circuit is as shown in Fig. 6.47(b), where V_1 is the voltage across the 2 Ω resistor across port *a-a′* and V_2 is the voltage across the 2 Ω resistor across port *b-b′* when $I_2 = 0$.

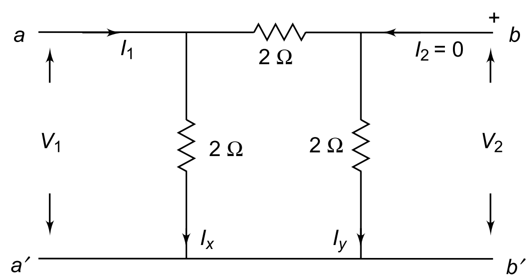


Fig. 6.47(b)

From Fig. 6.47(b), $I_Y = \dfrac{V_1}{4}$

$$V_2 = 2 \times I_Y = \frac{V_1}{2}$$

$$A = 2$$

From Fig. 6.47(b) $I_x = \dfrac{V_1}{2}$

$$C = \frac{I_1}{V_2}$$

where $I_1 = \dfrac{3V_1}{4}$

Therefore $C = \dfrac{3}{2}$ ℧

6.5 Find h parameters for the network in Fig. 6.48.

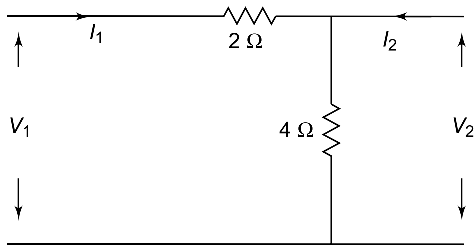


Fig. 6.48

Solution

When $V_2 = 0$ the network is as shown in Fig. 6.49.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2 = 0} = 2\ \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0} ; I_2 = -I_1$$

$\therefore$ $$h_{21} = -1$$

When $$I_1 = 0; h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0} ; h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1 = 0}$$

$$V_1 = I_2\, 4,\ V_2 = I_2\, 4$$

$\therefore$ $$h_{12} = 1\ h_{22} = \frac{1}{4}\ \mho$$

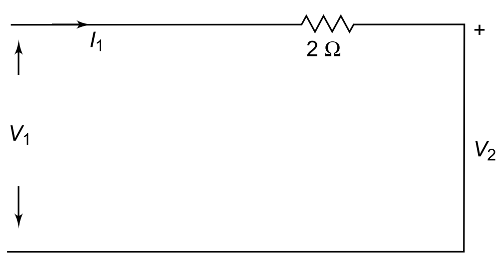


Fig. 6.49

6.6 For the hybrid equivalent circuit shown in Fig. 6.50, (a) determine the current gain, and (b) determine the voltage gain.

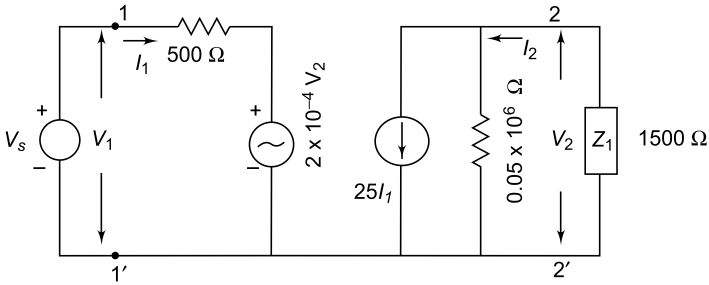


Fig. 6.50

Solution

From port 2-2′ we can find

$$I_2 = \frac{(25I_1)(0.05 \times 10^6)}{(1500 + 0.05 \times 10^6)}$$

(a) current gain $\dfrac{I_2}{I_1} = \dfrac{1.25 \times 10^6}{0.0515 \times 10^6} = 24.3$

(b) applying KVL at port 1-1′

$$V_1 = 500\, I_1 + 2 \times 10^{-4}\, V_2$$

$$I_1 = \frac{V_1 - 2 \times 10^{-4}\, V_2}{500} \tag{6.33}$$

Applying KCL at port 2-2′

$$I_2 = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

also $$I_2 = \frac{-V_2}{1500}$$

$$\frac{-V_2}{1500} = 25I_1 + \frac{V_2}{0.05} \times 10^{-6}$$

Substituting the value of I_1 from Eq. 6.33, in the above equation, we get

$$\frac{-V_2}{1500} = 25\left(\frac{V_1 - 2 \times 10^{-4}\, V_2}{500}\right) + \frac{V_2}{0.05} \times 10^{-6}$$

$$-6.6 \times 10^{-4}\, V_2 = 0.05V_1 - 0.1 \times 10^{-4}\, V_2 + 0.2 \times 10^{-4}\, V_2$$

$\therefore$ $$\frac{V_2}{V_1} = -73.89$$

The negative sign indicates that there is a 180° phase shift between input and output voltage.

6.7 The hybrid parameters of a two-port network shown in Fig. 6.51 are h_{11} = 1 K; h_{12} = 0.003; h_{21} = 100; h_{22} = 50 μ℧. Find V_2 and Z parameters of the network.

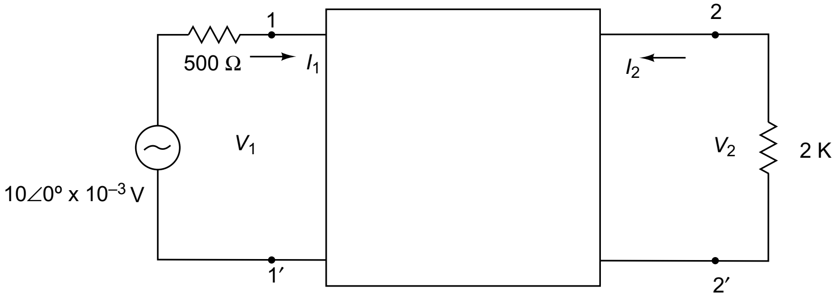


Fig. 6.51

Solution

$$V_1 = h_{11} I_1 + h_{12} V_2 \qquad (6.34)$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \qquad (6.35)$$

At port 2-2′ $\qquad V_2 = -I_2\, 2000$

Substituting in Eq. 6.35, we have

$$I_2 = h_{21}I_1 - h_{22}I_2\, 2000$$

$$I_2\,(1 + h_{22}\, 2000) = h_{21}\, I_1$$

$$I_2(1 + 50 \times 10^{-6} \times 2000) = 100\, I_1$$

$$I_2 = \frac{100\, I_1}{1.1}$$

Substituting the value of V_2 in Eq. 6.34, we have

$$V_1 = h_{11}\, I_1 - h_{12}\, I_2\, 2000$$

Also at port 1-1′, $\qquad V_1 = V_S - I_1\, 500$

$$\therefore \qquad V_S - I_1\, 500 = h_{11}\, I_1 - h_{12}\, \frac{100\, I_1}{1.1} \times 2000$$

$$(10 \times 10^{-3}) - 500\, I_1 = 1000\, I_1 - 0.003 \times \frac{100}{1.1}\, I_1 \times 2000$$

$$954.54 I_1 = 10 \times 10^{-3}$$

$$I_1 = 10.05 \times 10^{-6}\ \text{A}$$

$$V_1 = V_S - I_1 \times 500$$

$$= 10 \times 10^{-3} - 10.5 \times 10^{-6} \times 500 = 4.75 \times 10^{-3}\ \text{V}$$

$$V_2 = \frac{V_1 - h_{11}\, I_1}{h_{12}}$$

$$V_2 = \frac{4.75 \times 10^{-3} - 1000 \times 10.5 \times 10^{-6}}{0.003} = -\,1.916\ \text{V}$$

(b) Z parameters of the network can be found from Table 6.1.

$$Z_{11} = \frac{\Delta_h}{h_{22}} = \frac{h_{11}h_{22} - h_{21}h_{12}}{h_{22}} = \frac{1\times 10^3 \times 50\times 10^{-6} - 100\times 0.003}{50\times 10^{-6}}$$

$$= -5000\ \Omega$$

$$Z_{12} = \frac{h_{12}}{h_{22}} = \frac{0.003}{50\times 10^{-6}} = 60\ \Omega$$

$$Z_{21} = \frac{-h_{21}}{h_{22}} = \frac{-100}{50\times 10^{-6}} = -2\times 10^6\ \Omega$$

$$Z_{22} = \frac{1}{h_{22}} = 20\times 10^3\ \Omega$$

6.8 The Z parameters of a two port network shown in Fig. 6.52 are $Z_{11} = Z_{22} = 10\ \Omega$; $Z_{21} = Z_{12} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, V_2, I_2 and input impedance.

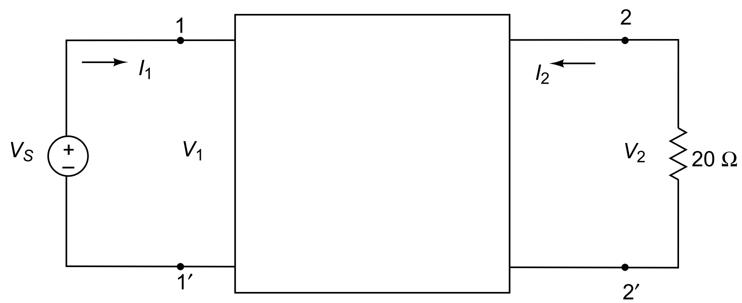


Fig. 6.52

Solution

Given $\quad V_1 = V_S = 20$ V

From Section 6.11.1, $\quad V_1 = I_1\left(Z_{11} - \dfrac{Z_{12}\,Z_{21}}{Z_L + Z_{22}}\right)$

where $\quad Z_L = 20\ \Omega$

$\therefore \quad 20 = I_1\left(10 - \dfrac{4\times 4}{20+10}\right)$

$$I_1 = 2.11\ \text{A}$$

$$I_2 = -I_1\frac{Z_{21}}{Z_L + Z_{22}} = -2.11\times\frac{4}{20+10} = -0.281\ \text{A}$$

At port 2-2′

$$V_2 = -I_2 \times 20 = 0.281 \times 20 = 5.626 \text{ V}$$

Input impedance $\quad = \dfrac{V_1}{I_1} = \dfrac{20}{2.11} = 9.478\ \Omega$

6.9 The Y parameters of the two-port network shown in Fig. 6.53 are $Y_{11} = Y_{22} = 6$ ℧; $Y_{12} = Y_{21} = 4$ ℧

(a) determine the driving point admittance at port 2-2′ if the source voltage is 100 V and has an impedance of 1 ohm.

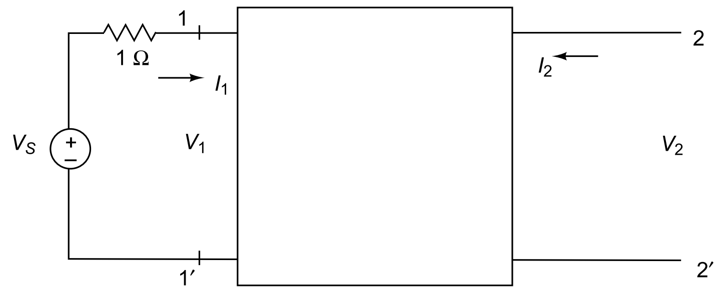


Fig. 6.53

Solution

From Section 6.11.2,

$$\frac{I_2}{V_2} = \frac{Y_{22}\,Y_S + Y_{22}Y_{11} - Y_{21}Y_{12}}{Y_S + Y_{11}}$$

where Y_S is the source admittance = 1 ℧

$\therefore$ The driving point admittance $= \dfrac{6\times 1 + 6\times 6 - 4\times 4}{1+6} = 3.714$ ℧

Or the driving point impedance at port 2-2′ $= \dfrac{1}{3.714}\ \Omega$

6.10 Obtain the Z parameters for the two-port unsymmetrical lattice network shown in Fig. 6.54.

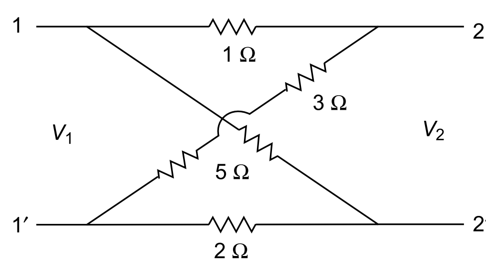


Fig. 6.54

Solution

From Section 6.12, we have

$$Z_{11} = \frac{(Z_a + Z_b)(Z_d + Z_c)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+3)(2+5)}{1+3+5+2} = 2.545\ \Omega$$

$$Z_{21} = \frac{Z_b Z_c - Z_a Z_d}{Z_a + Z_b + Z_c + Z_d} = \frac{3\times 5 - 1\times 2}{11} = 1.181\ \Omega$$

$$Z_{21} = Z_{12}$$

$$Z_{22} = \frac{(Z_a + Z_c)(Z_d + Z_b)}{Z_a + Z_b + Z_c + Z_d} = \frac{(1+5)(2+3)}{11} = 2.727\ \Omega$$

6.11 For the ladder two-port network shown in Fig. 6.55, find the open circuit driving point impedance at port 1-2.

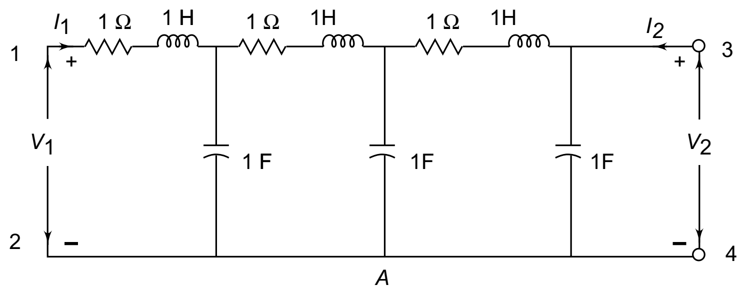


Fig. 6.55

Solution

The Laplace transform of the given network is shown in Fig. 6.56.

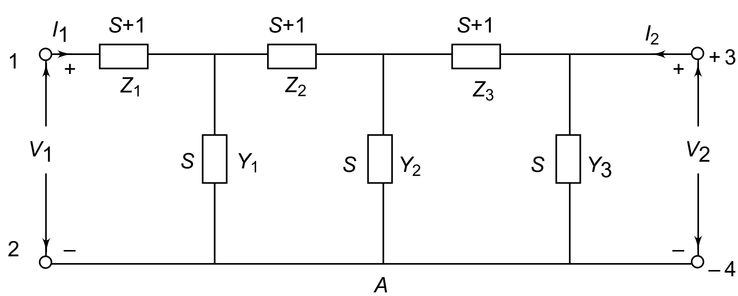


Fig. 6.56

Then the open circuit driving point impedance at port 1-2 is given by

$$Z_{11} = (s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s}}}}}$$

$$= \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

6.12 For the bridged T network shown in Fig. 6.57, find the driving point admittance y_{11} and transfer admittance y_{21} with a 2 Ω load resistor connected across port 2.

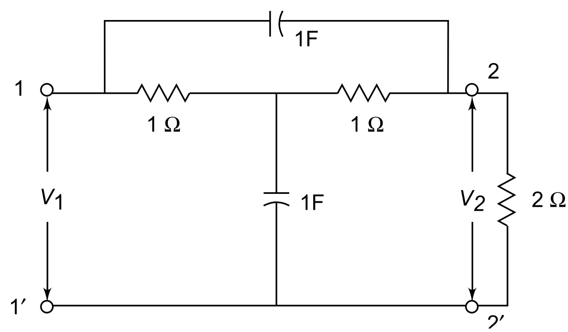


Fig. 6.57

Solution

The corresponding Laplace transform network is shown in Fig. 6.58.

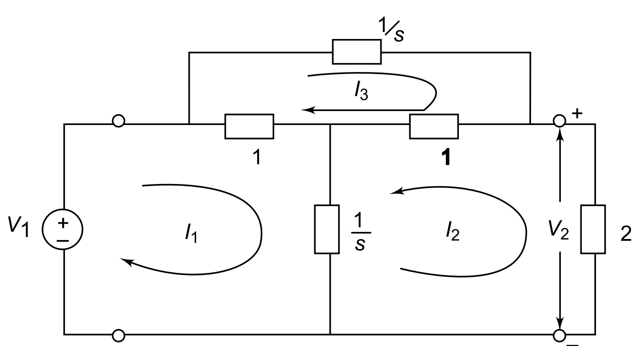


Fig. 6.58

The loop equations are

$$I_1\left(1+\frac{1}{s}\right)+I_2\left(\frac{1}{s}\right)-I_3 = V_1$$

$$I_1\left(\frac{1}{s}\right)+I_2\left(1+\frac{1}{s}\right)+I_3 = 0$$

$$I_1(-1)+I_2+I_3\left(2+\frac{1}{s}\right) = 0$$

Therefore,

$$\Delta = \begin{vmatrix} \left(1+\frac{1}{s}\right) & \frac{1}{s} & -1 \\ \frac{1}{s} & 1+\frac{1}{s} & 1 \\ -1 & 1 & 2+\frac{1}{s} \end{vmatrix} = \frac{s+2}{s^2}$$

Similarly, $$\Delta_{11} = \begin{vmatrix} \left(1+\frac{1}{s}\right) & \frac{1}{s} \\ 1 & \left(2+\frac{1}{s}\right) \end{vmatrix} = \frac{s^2+3s+1}{s^2}$$

and $$\Delta_{12} = \begin{vmatrix} \frac{1}{s} & +1 \\ +1 & \left(2+\frac{1}{s}\right) \end{vmatrix} = \frac{s^2+2s+1}{s^2}$$

Hence, $$y_{11} = \frac{\Delta_{11}}{\Delta} = \frac{s^2+3s+1}{s+2}$$

and $$y_{21} = \frac{\Delta_{12}}{\Delta} = \frac{-\left(s^2+2s+1\right)}{s+2}$$

6.13 For the two port network shown in Fig. 6.59, determine the h-parameters. Using these parameters calculate the output (Port 2) voltage, V_2, when the output port is terminated in a 3 Ω resistance and a 1 V(dc) is applied at the input port ($V_1 = 1$ V).

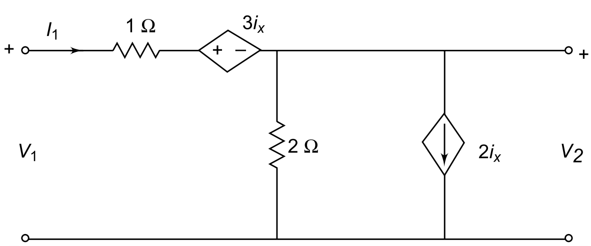


Fig. 6.59

Solution

The h parameters are defined as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For $V_2 = 0$, the circuit is redrawn as shown in Fig. 6.60(a).

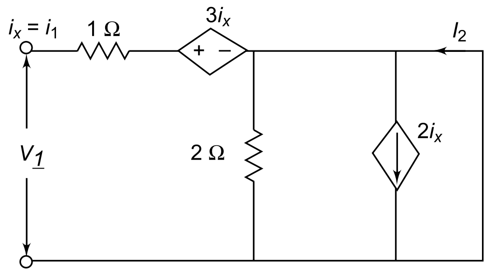


Fig. 6.60(a)

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{i_1 \times 1 + 3i_1}{i_1} = 4$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{i_2}{i_1} = \frac{2i_1 - i_1}{i_1} = 1$$

For $I_1 = 0$, the circuit is redrawn as shown in Fig. 6.60(b).

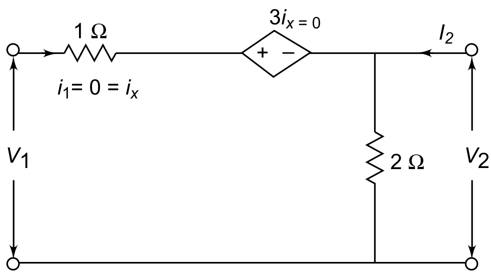


Fig. 6.60(b)

$$h_{12} = \frac{V_1}{V_2} = 1;\ h_{22} = \frac{I_2}{V_2} = \frac{1}{2} = 0.5$$

Hence,
$$h = \begin{bmatrix} 4 & 1 \\ 1 & 0.5 \end{bmatrix}$$

$$V_1 = 1\text{ V}$$
$$V_1 = 4I_1 + V_2$$
$$I_2 = I_1 + 0.5\,V_2$$

Eliminating I_1 from the above equations and putting

$$V_1 = 1 \text{ and } I_2 = \frac{-V_2}{3} \text{ we get, } V_2 = \frac{-3}{7} \text{ V}$$

6.14 Find the current transfer ratio $\frac{I_2}{I_1}$ for the network shown in Fig. 6.61.

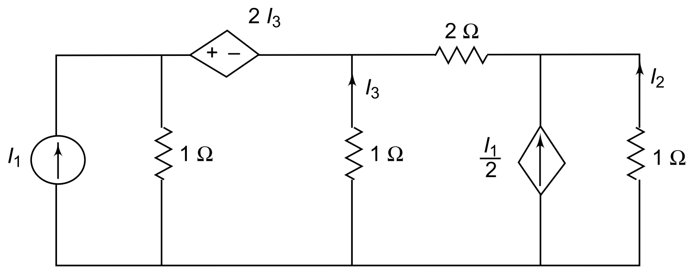


Fig. 6.61

Solution

By transforming the current source into voltage source, the given circuit can be redrawn as shown in Fig. 6.62.

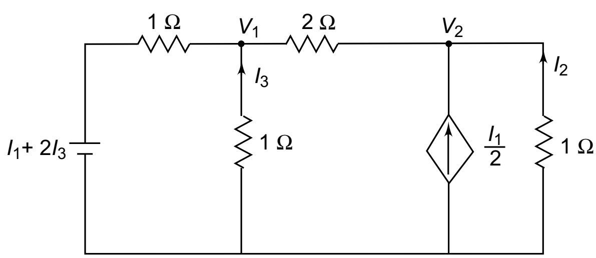


Fig. 6.62

Applying Kirchhoff's nodal analysis

$$\frac{V_1 - (I_1 + 2I_3)}{1} + \frac{V_1}{1} + \frac{V_1 - V_2}{2} = 0$$

and
$$\frac{V_2 - V_1}{2} - \frac{I_1}{2} - I_2 = 0$$

Putting $\qquad V_1 = -I_3 \text{ and } V_2 = -I_2$

The above equations become

$$-I_3 - I_1 - 2I_3 - I_3 + \frac{I_2 - I_3}{2} = 0$$

and $$\frac{I_2 - I_3}{2} - \frac{I_1}{2} - I_2 = 0$$

or $$I_1\ 0.5I_2 - 4.5I_3 = 0$$

and $$-0.5I_1 - 1.5I_2 + 0.5I_3 = 0$$

By eliminating I_3, we get

$$\frac{I_2}{I_1} = \frac{-5.5}{13} = -0.42$$

Practice Problems

6.1 Find the Z parameters of the network shown in Fig. 6.63.

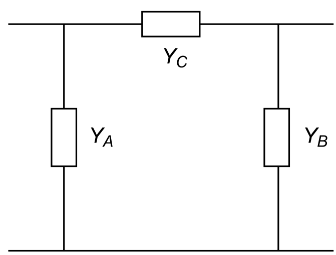


Fig. 6.63

6.2 Find the transmission parameters for the *R–C* network shown in Fig. 6.64.

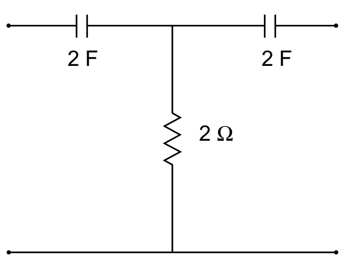


Fig. 6.64

6.3 Find the inverse transmission parameters for the network in Fig. 6.65.

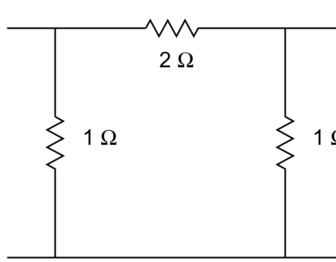


Fig. 6.65

6.4 Calculate the overall transmission parameters for the cascaded network shown in Fig. 6.66.

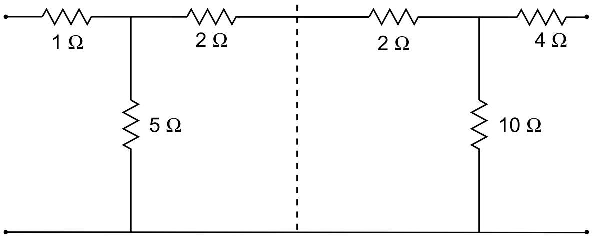


Fig. 6.66

6.5 For the two-port network shown in Fig. 6.67, find the *h* parameters and the inverse *h* parameters.

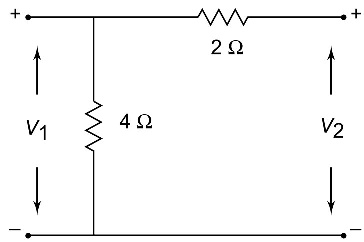


Fig. 6.67

6.6 Determine the impedance parameters for the *T* network shown in Fig. 6.68 and draw the *Z* parameter equivalent circuit.

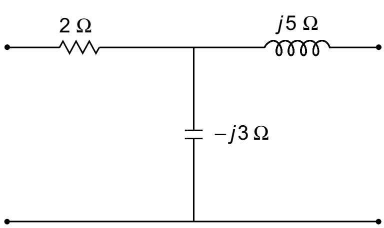


Fig. 6.68

6.7 Determine the admittance parameters for the π-network shown in Fig. 6.69 and draw the *Y* parameter equivalent circuit.

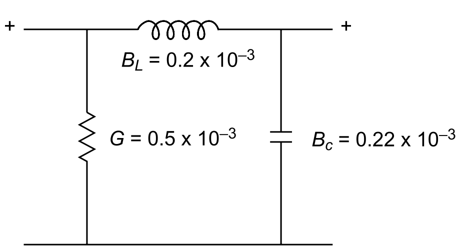


Fig. 6.69

6.8 Determine the impedance parameters and the transmission parameters for the network in Fig. 6.70.

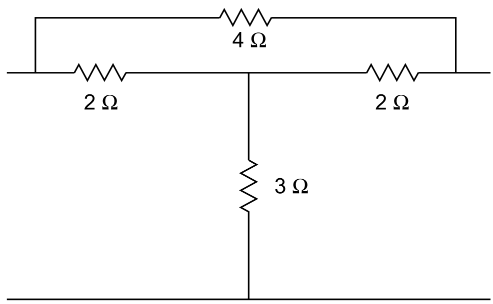


Fig. 6.70

6.9 For the hybrid equivalent circuit shown in Fig. 6.71, determine (a) the input impedance (b) the output impedance.

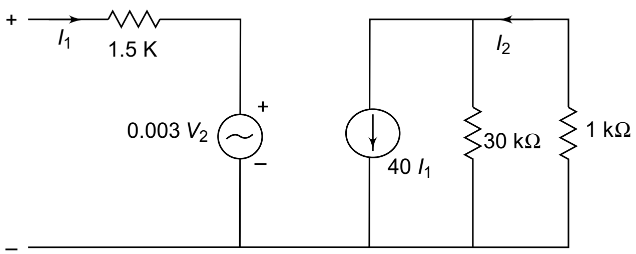


Fig. 6.71

6.10 Determine the input and output impedances for the Z parameter equivalent circuit shown in Fig. 6.72.

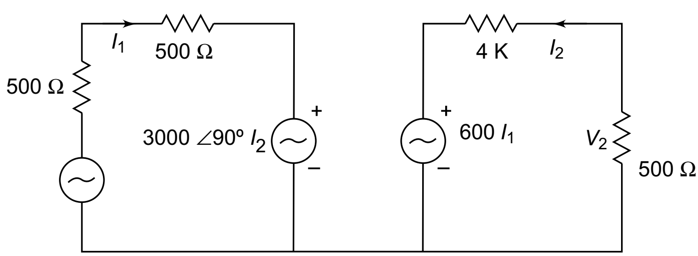


Fig. 6.72

6.11 The hybrid parameters of a two-port network shown in Fig. 6.73 are h_{11} = 1.5 K; $h_{12} = 2 \times 10^{-3}$; $h_{21} = 250$; $h_{22} = 150 \times 10^{-6}$ ℧ (a) Find V_2 (b). Draw the Z parameter equivalent circuit.

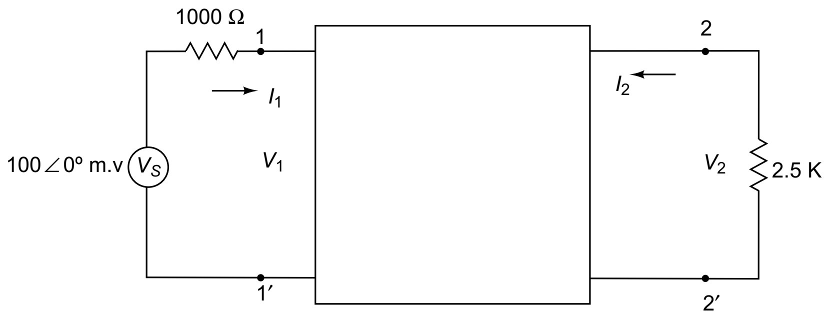


Fig. 6.73

6.12 The Z parameters of a two-port network shown in Fig. 6.74 are $Z_{11} = 5\ \Omega$; $Z_{12} = 4\ \Omega$; $Z_{22} = 10\ \Omega$; $Z_{21} = 5\ \Omega$. If the source voltage is 25 V, determine I_1, V_2 I_2, and the driving point impedance at the input port.

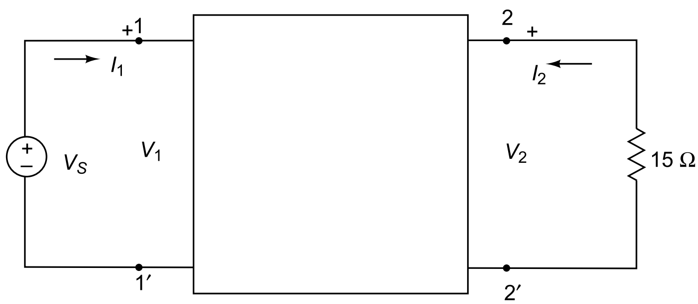


Fig. 6.74

6.13 Obtain the image parameters of the symmetric lattice network given in Fig. 6.75.

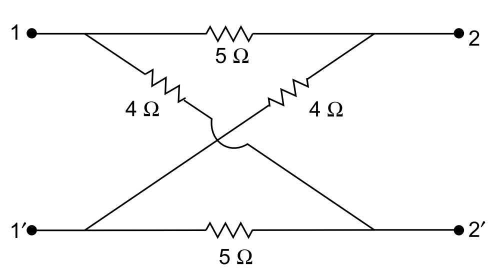


Fig. 6.75

6.14 Determine the Z parameters and image parameters of a symmetric lattice network whose series arm impedance is 10 Ω and diagonal arm impedance is 20 Ω.

6.15 For the network shown in Fig. 6.76, determine all four open circuit impedance parameters.

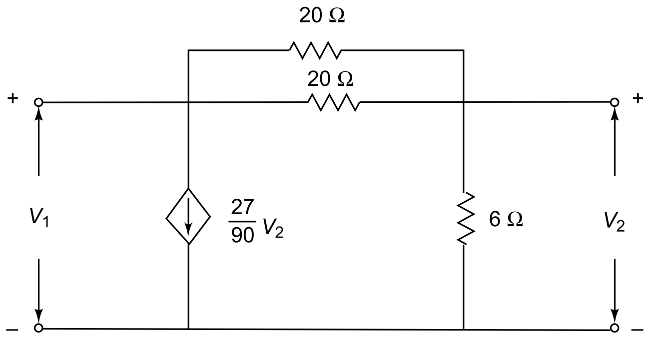


Fig. 6.76

6.16 For the network shown in Fig. 6.77, determine y_{12} and y_{21}.

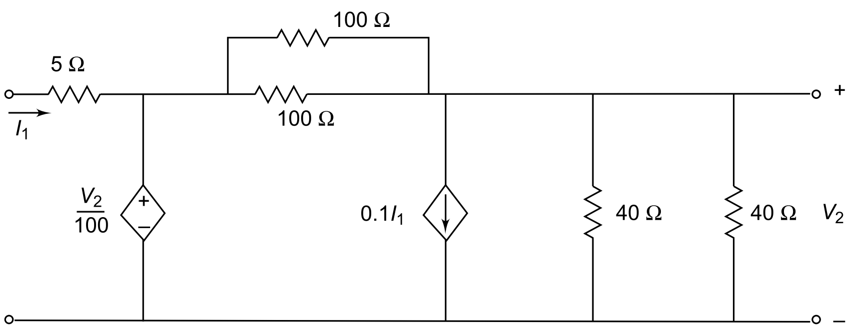


Fig. 6.77

6.17 For the network shown in Fig. 6.78, determine h parameters at $\omega = 10^8$ rad/sec.

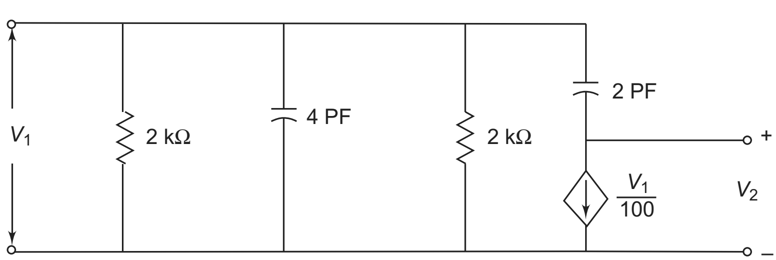


Fig. 6.78

6.18 For the network shown in Fig. 6.79, determine y parameters.

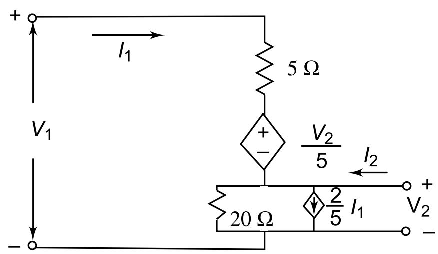


Fig. 6.79

Objective Type Questions

6.1 A two-port network is simply a network inside a black box, and the network has only
(a) two terminals
(b) two pairs of accessible terminals
(c) two pairs of ports

6.2 The number of possible combinations generated by four variables taken two at a time in a two-port network is
(a) four (b) two (c) six

6.3 What is the driving-point impedance at port one with port two open circuited for the network in Fig. 6.80?

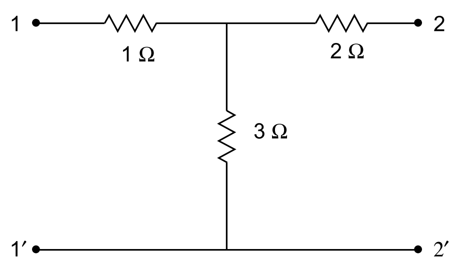


Fig. 6.80

(a) 4 Ω (b) 5 Ω (c) 3 Ω

6.4 What is the transfer impedance of the two-port network shown in Fig. 6.87?
(a) 1 Ω (b) 2 Ω (c) 3 Ω

6.5 If the two-port network in Fig. 6.87 is reciprocal or bilateral then
(a) $Z_{11} = Z_{22}$ (b) $Z_{12} = Z_{21}$ (c) $Z_{11} = Z_{12}$

6.6 What is the transfer admittance of the network shown in Fig. 6.81.

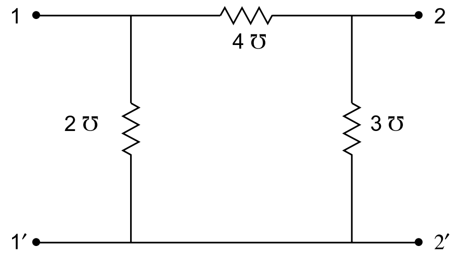


Fig. 6.81

(a) –2 ℧ (b) –3 ℧ (c) –4 ℧

6.7 If the two-port network in Fig. 6.88 is reciprocal then

(a) $Y_{11} = Y_{22}$ (b) $Y_{12} = Y_{22}$ (c) $Y_{12} = Y_{11}$

6.8 In describing the transmission parameters

(a) the input voltage and current are expressed in terms of output voltage and current.

(b) the input voltage and output voltage are expressed in terms of output current and input current.

(c) the input voltage and output current are expressed in terms of input current and output voltage.

6.9 If $Z_{11} = 2\ \Omega$; $Z_{12} = 1\ \Omega$; $Z_{21} = 1\ \Omega$ and $Z_{22} = 3\ \Omega$, what is the determinant of admittance matrix.

(a) 5 (b) 1/5 (c) 1

6.10 For a two-port bilateral network, the three transmission parameters are given

by $A = \frac{6}{5}$; $B = \frac{17}{5}$ and $C = \frac{1}{5}$, what is the value of D?

(a) 1 (b) $\frac{1}{5}$ (c) $\frac{7}{5}$

6.11 The impedance matrices of two, two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 5 & 25 \end{bmatrix}$. If the two networks are connected in series. What is the impedance matrix of the combination?

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$

6.12 The admittance matrices of two two-port networks are given by $\begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 5/8 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$. If the two networks are connected in parallel, what is the admittance matrix of the combination?

(a) $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -1 & 5/2 \end{bmatrix}$

(c) $\begin{bmatrix} 3/2 & -3/4 \\ -3/4 & 15/8 \end{bmatrix}$

6.13 If the Z parameters of a two-port network are $Z_{11} = 5\ \Omega$ $Z_{22} = 7\ \Omega$; $Z_{12} = Z_{21} = 3\ \Omega$ then the A, B, C, D parameters are respectively given by

(a) $\frac{5}{3}; \frac{26}{3}; \frac{1}{3}; \frac{7}{3}$ (b) $\frac{10}{3}; \frac{52}{3}; \frac{2}{3}; \frac{14}{3}$

(c) $\frac{15}{3}; \frac{78}{3}; \frac{3}{3}; \frac{21}{3}$

6.14 For a symmetric lattice network the value of the series impedance is 3 Ω and that of the diagonal impedance is 5 Ω, then the Z parameters of the network are given by

(a) $Z_{11} = Z_{22} = 2\ \Omega$
$Z_{12} = Z_{21} = 1/2\ \Omega$

(b) $Z_{11} = Z_{22} = 4\ \Omega$
$Z_{12} = Z_{21} = 1\ \Omega$

(c) $Z_{11} = Z_{22} = 8\ \Omega$
$Z_{12} = Z_{21} = 2\ \Omega$

6.15 For a two-port network to be reciprocal.

(a) $Z_{11} = Z_{22}$ (b) $y_{21} = y_{22}$ (c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

6.16 Two-port networks are connected in cascade. The combination is to be represented as a single two port network. The parameters of the network are obtained by adding the individual

(a) Z parameter matrix (b) h parameter matrix

(c) $A^1\ B^1\ C^1\ D^1$ matrix (d) $ABCD$ parameter matrix

6.17 The h parameters h_{11} and h_{12} are obtained

(a) By shorting output terminals (b) By opening input terminals

(c) By shorting input terminals (d) By opening output terminals

6.18 Which parameters are widely used in transmission line theory

(a) Z parameters (b) Y parameters

(c) $ABCD$ parameters (d) h parameters

7 Transients

7.1 TRANSIENT RESPONSE—DIFFERENTIAL EQUATION APPROACH

A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to other state. The behaviour of the voltage or current when it is changed from one state to another is called the *transient state*. The time taken for the circuit to change from one steady state to another steady state is called the *transient time*. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic, equations. When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the *natural response*. Storage elements deliver their energy to the resistances. Hence the response changes with time, gets saturated after some time, and is referred to as the *transient response*. When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called *forced response*. In other words, the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods can be used to find out the complete solution.

7.2 FIRST ORDER DIFFERENTIAL EQUATIONS—RL, RC CIRCUITS WITH DC EXCITATION—TIME CONSTANTS

7.2.1 DC Response of R-L Circuit

Consider a circuit consisting of a resistance and inductance as shown in Fig. 7.1. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch S is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

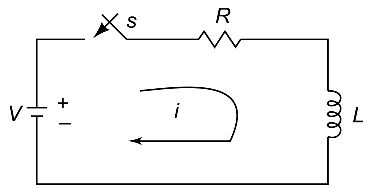


Fig. 7.1

$$V = Ri + L\frac{di}{dt} \tag{7.1}$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \tag{7.2}$$

In the above equation, the current i is the solution to be found and V is the applied constant voltage. The voltage V is applied to the circuit only when the switch S is closed. The above equation is a linear differential equation of first order. Comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K \tag{7.3}$$

whose solution is

$$x = e^{-pt}\int Ke^{+Pt}\,dt + ce^{-Pt} \tag{7.4}$$

where c is an arbitrary constant. In a similar way, we can write the current equation as

$$i = ce^{-(R/L)t} + e^{-(R/L)t}\int \frac{V}{L}e^{(R/L)t}\,dt$$

$$\therefore \qquad i = ce^{-(R/L)t} + \frac{V}{R} \tag{7.5}$$

To determine the value of c in Eq. 7.5, we use the initial conditions. In the circuit shown in Fig. 7.1, the switch S is closed at $t = 0$. At $t = 0^-$, i.e. just before closing the switch S, the current in the inductor is zero. Since the inductor does not allow sudden changes in currents, at $t = 0^+$ just after the switch is closed, the current remains zero.

Thus at $\qquad t = 0, i = 0$

Substituting the above condition in Eq. 7.5, we have

$$0 = c + \frac{V}{R}$$

Hence
$$c = -\frac{V}{R}$$

Substituting the value of c in Eq. 5, we get

$$i = \frac{V}{R} - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)$$

$$i = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right) \tag{7.6}$$

Equation 7.6 consists of two parts, the steady state part V/R, and the transient part $(V/R)e^{-(R/L)t}$. When switch S is closed, the response reaches a steady state value after a time interval as shown in Fig. 7.2.

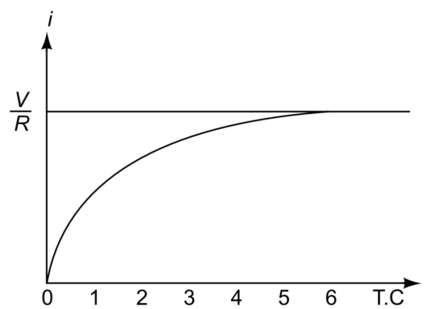


Fig. 7.2

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value. In the transient part of the solution, the quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R. The time constant of a function $\frac{V}{R}e^{-\left(\frac{R}{L}\right)t}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the *time constant* and is denoted by τ

$$\therefore \qquad \tau = \frac{L}{R} \text{ sec}$$

$\therefore$ the transient part of the solution is

$$i = -\frac{V}{R}\exp\left(-\frac{R}{L}t\right) = -\frac{V}{R}e^{-t/\tau}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498\frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067\frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its final value.

In Fig. 7.1, we can find out the voltages and powers across each element by using the current.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

$$\therefore \qquad v_R = V\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

Similarly, the voltage across the inductance is

$$v_L = L\frac{di}{dt}$$

$$= L\frac{V}{R} \times \frac{R}{L}\exp\left(-\frac{R}{L}t\right) = V\exp\left(-\frac{R}{L}t\right)$$

The response are shown in Fig. 7.3.

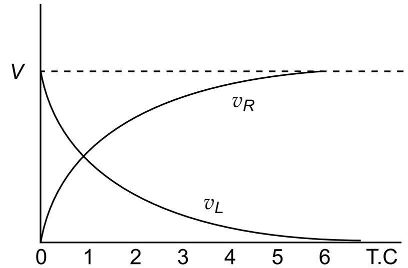


Fig. 7.3

Power in the resistor is

$$p_R = v_R i = V\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\left(1 - \exp\left(-\frac{R}{L}t\right)\right)\frac{V}{R}$$

$$= \frac{V^2}{R}\left(1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right)\right)$$

Power in the inductor is

$$p_L = v_L i = V \exp\left(-\frac{R}{L}t\right) \times \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)$$

$$= \frac{V^2}{R}\left(\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right)$$

The responses are shown in Fig. 7.4.

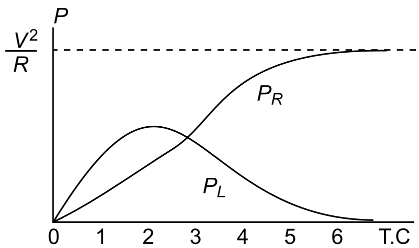


Fig. 7.4

Example 7.1 *A series RL circuit with R = 30 Ω and L = 15 H has a constant voltage V = 60 V applied at t = 0 as shown in Fig. 7.5. Determine the current i, the voltage across resistor and the voltage across the inductor.*

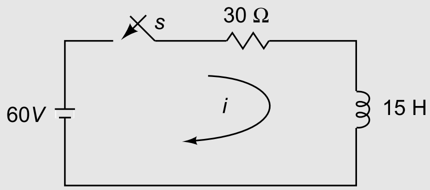


Fig. 7.5

Solution

By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\therefore \quad \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt}\int Ke^{Pt}\,dt$$

where $P = 2, K = 4$

$\therefore$ $$i = ce^{-2t} + e^{-2t} \int 4e^{2t}\, dt$$

$\therefore$ $$i = ce^{-2t} + 2$$

At $t = 0$, the switch S is closed.

Since the inductor never allows sudden changes in currents. At $t = 0^+$ the current in the circuit is zero.

Therefore at $t = 0^+, i = 0$

$\therefore$ $$0 = c + 2$$

$\therefore$ $$c = -2$$

Substituting the value of c in the current equation, we have

$$i = 2(1 - e^{-2t})\text{ A}$$

Voltage across resistor $v_R = iR$

$$= 2(1 - e^{-2t}) \times 30 = 60(1 - e^{-2t})\text{ V}$$

Voltage across inductor $v_L = L\dfrac{di}{dt}$

$$= 15 \times \frac{d}{dt}\, 2(1 - e^{-2t}) = 30 \times 2e^{-2t} = 60e^{-2t}\text{ V}$$

7.2.2 DC Response of R-C Circuit

Consider a circuit consisting of resistance and capacitance as shown in Fig. 7.6. The capacitor in the circuit is initially uncharged, and is in series with a resistor. When the switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.

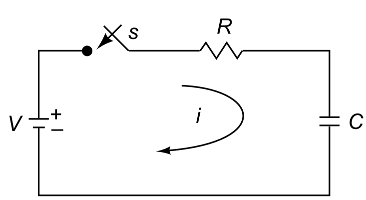


Fig. 7.6

$$V = Ri + \frac{1}{C}\int i\, dt \tag{7.7}$$

By differentiating the above equation, we get

$$0 = R\frac{di}{dt} + \frac{i}{C} \tag{7.8}$$

or $$\frac{di}{dt} + \frac{1}{RC}i = 0 \tag{7.9}$$

Equation 7.9 is a linear differential equation with only the complementary function. The particular solution for the above equation is zero. The solution for this type of differential equation is

$$i = ce^{-t/RC} \tag{7.10}$$

Here, to find the value of c, we use the initial conditions.

In the circuit shown in Fig. 7.6, switch S is closed at $t = 0$. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t = 0^+$. So, the current in the circuit at $t = 0^+$ is V/R

$\therefore$ At $t = 0$, the current $i = \dfrac{V}{R}$

Substituting this current in Eq. 7.10, we get

$$\frac{V}{R} = c$$

$\therefore$ The current equation becomes

$$i = \frac{V}{R} e^{-t/RC} \tag{7.11}$$

When switch S is closed, the response decays with time as shown in Fig. 7.7.

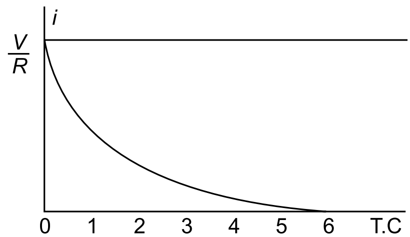


Fig. 7.7

In the solution, the quantity RC is the time constant, and is denoted by τ,

where $\tau = RC$ sec

After 5 TC, the curve reaches 99 per cent of its final value. In Fig. 7.6, we can find out the voltage across each element by using the current equation.

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R} e^{-(1/RC)t}; \; v_R = Ve^{-\frac{t}{RC}}$$

Similarly, voltage across the capacitor is

$$v_C = \frac{1}{C} \int i \, dt$$

$$= \frac{1}{C}\int \frac{V}{R}\, e^{-t/RC}\, dt$$

$$= -\left(\frac{V}{RC} \times RC\, e^{-t/RC}\right) + c = -Ve^{-t/RC} + c$$

At $t = 0$, voltage across capacitor is zero

$$\therefore \qquad c = V$$

$$\therefore \qquad v_C = V(1 - e^{-t/RC})$$

The responses are shown in Fig. 7.8.

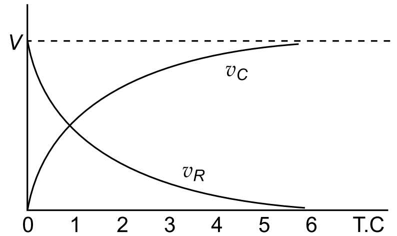


Fig. 7.8

Power in the resistor

$$p_R = v_R i = Ve^{-t/RC} \times \frac{V}{R}\, e^{-t/RC} = \frac{V^2}{R}\, e^{-2t/RC}$$

Power in the capacitor

$$p_C = v_C i = V(1 - e^{-t/RC})\, \frac{V}{R}\, e^{-t/RC}$$

$$= \frac{V^2}{R}\,(e^{-t/RC} - e^{-2t/RC})$$

The responses are shown in Fig. 7.9.

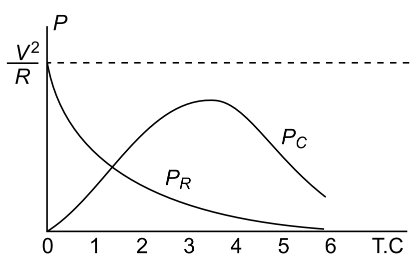


Fig. 7.9

Example 7.2 *A series RC circuit consists of resistor of 10 Ω and capacitor of 0.1 F as shown in Fig. 7.10. A constant voltage of 20 V is applied to the circuit at t = 0. Obtain the current equation. Determine the voltages across the resistor and the capacitor.*

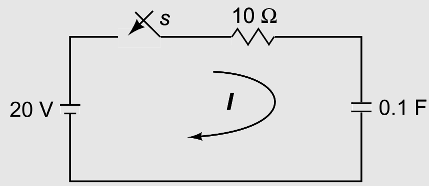


Fig. 7.10

Solution

By applying Kirchhoff's law, we get

$$10i + \frac{1}{0.1}\int i\,dt = 20$$

Differentiating with respect to t we get

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \quad \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t = 0$, switch S is closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is $i = V/R = 20/10 = 2$ A.

At $t = 0$, $i = 2$ A.

$\therefore$ The current equation $i = 2e^{-t}$

Voltage across the resistor is $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}$ V

Voltage across the capacitor is $v_C = V\left(1 - e^{-\frac{t}{RC}}\right)$

$$= 20\,(1 - e^{-t})\text{ V}$$

7.3 FIRST ORDER DIFFERENTIAL EQUATIONS—RL, RC CIRCUITS WITH AC EXCITATION—TIME CONSTANTS

7.3.1 Sinusoidal Response of R-L Circuit

Consider a circuit consisting of resistance and inductance as shown in Fig. 7.11. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V\cos(\omega t + \theta)$ is applied to the series R-L circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

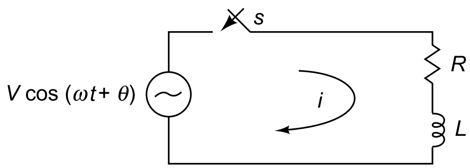


Fig. 7.11

$$V\cos(\omega t+\theta) = Ri + L\frac{di}{dt} \tag{7.12}$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t+\theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right)i = \frac{V}{L}\cos(\omega t+\theta) \tag{7.13}$$

For the above equation, the solution consists of two parts, viz. complementary function and particular integral.

The complementary function of the solution i is

$$i_c = ce^{-t(R/L)} \tag{7.14}$$

The particular solution can be obtained by using undetermined co-efficients.

By assuming

$$i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \tag{7.15}$$

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \tag{7.16}$$

Substituting Eqs 7.15 and 7.16 in Eq. 7.13, we have

$$\{-A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) + \frac{R}{L}\{A\cos(\omega t+\theta)$$

$$+ B\sin(\omega t+\theta)\} = \frac{V}{L}\cos(\omega t+\theta)$$

or $$\left(-A\omega + \frac{BR}{L}\right)\sin(\omega t+\theta) + \left(B\omega + \frac{AR}{L}\right)\cos(\omega t+\theta) = \frac{V}{L}\cos(\omega t+\theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$

$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

$$A = V\frac{R}{R^2 + (\omega L)^2}$$

$$B = V \frac{\omega L}{R^2 + (\omega L)^2}$$

Substituting the values of A and B in Eq. 7.15, we get

$$i_p = V \frac{R}{R^2 + (\omega L)^2} \cos(\omega t + \theta) + V \frac{\omega L}{R^2 + (\omega L)^2} \sin(\omega t + \theta) \qquad (7.17)$$

Putting $\qquad M \cos \phi = \dfrac{VR}{R^2 + (\omega L)^2}$

and $\qquad M \sin \phi = V \dfrac{\omega L}{R^2 + (\omega L)^2}$,

to find M and ϕ, we divide one equation by the other

$$\frac{M \sin \phi}{M \cos \phi} = \tan \phi = \frac{\omega L}{R}$$

Squaring both equations and adding, we get

$$M^2 \cos^2 \phi + M^2 \sin^2 \phi = \frac{V^2}{R^2 + (\omega L)^2}$$

or $\qquad M = \dfrac{V}{\sqrt{R^2 + (\omega L)^2}}$

$\therefore$ The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right) \qquad (7.18)$$

The complete solution for the current $i = i_c + i_p$

$$i = ce^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents, at $t = 0$, $i = 0$

$\therefore \qquad c = -\dfrac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1} \dfrac{\omega L}{R}\right)$

The complete solution for the current is

$$i = e^{-(R/L)t}\left[\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1} \frac{\omega L}{R}\right)\right] + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right)$$

Example 7.3 *In the circuit shown in Fig. 7.12, determine the complete solution for the current, when switch S is closed at t = 0. Applied voltage is $v(t) = 100 \cos (10^3 t + \pi/2)$. Resistance R = 20 Ω and inductance L = 0.1 H.*

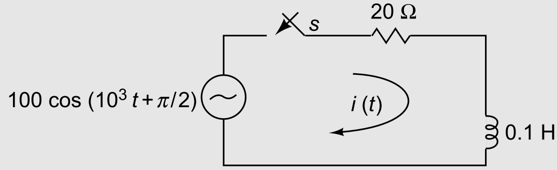


Fig. 7.12

Solution

By applying Kirchhoff's voltage law to the circuit, we have

$$20i + 0.1\frac{di}{dt} = 100 \cos (10^3\, t + \pi/2)$$

$$\frac{di}{dt} + 200i = 1000 \cos (1000t + \pi/2)$$

$$(D = 200)i = 1000 \cos (1000t + \pi/2)$$

The complementary function $i_c = ce^{-200t}$

By assuming particular integral as

$$i_p = A \cos (\omega t + \theta) + B \sin (\omega t + \theta)$$

we get

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

where $\omega = 1000$ rad/sec $V = 100$ V

$$\theta = \pi/2$$

$$L = 0.1 \text{ H}, R = 20\ \Omega$$

Substituting the values in the above equation, we get

$$i_p = \frac{100}{\sqrt{(20)^2 + (1000 \times 0.1)^2}} \cos\left(1000t + \frac{\pi}{2} - \tan^{-1}\frac{100}{20}\right)$$

$$= \frac{100}{101.9} \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

$$= 0.98 \cos\left(1000t + \frac{\pi}{2} - 78.6°\right)$$

The complete solution is

$$i = ce^{-200t} + 0.98\cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

At $t = 0$, the current flowing through the circuit is zero, i.e. $i = 0$

$$\therefore \qquad c = -0.98\cos\left(\frac{\pi}{2} - 78.6^\circ\right)$$

$\therefore$ The complete solution is

$$i = \left[-0.98\cos\left(\frac{\pi}{2} - 78.6^\circ\right)\right]e^{-200t} + 0.98\cos\left(1000t + \frac{\pi}{2} - 78.6^\circ\right)$$

7.3.2 Sinusoidal Response of R-C Circuit

Consider a circuit consisting of resistance and capacitance in series as shown in Fig. 7.13. The switch, S, is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V\cos(\omega t + \theta)$ is applied to the R-C circuit, where V is the amplitude of the wave and θ is the phase angle. Applying Kirchhoff's voltage law to the circuit results in the following differential equation.

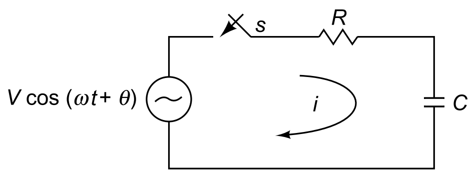


Fig. 7.13

$$V\cos(\omega t + \theta) = Ri + \frac{1}{C}\int i\,dt \tag{7.19}$$

$$R\frac{di}{dt} + \frac{i}{C} = -V\omega\sin(\omega t + \theta)$$

$$\left(D + \frac{1}{RC}\right)i = -\frac{V\omega}{R}\sin(\omega t + \theta) \tag{7.20}$$

The complementary function $i_C = ce^{-t/RC}$ (7.21)

The particular solution can be obtained by using undetermined coefficients.

$$i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta) \tag{7.22}$$

$$i'_P = -A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta) \tag{7.23}$$

Substituting Eqs 7.22 and 7.23 in Eq. 7.20, we get

$$\{-A\omega\sin(\omega t + \theta) + B\omega\cos(\omega t + \theta)\}$$

$$+\frac{1}{RC}\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\} = -\frac{V\omega}{R}\sin(\omega t + \theta)$$

Comparing both sides, $-A\omega + \dfrac{B}{RC} = -\dfrac{V\omega}{R}$

$$B\omega + \frac{A}{RC} = 0$$

From which,

$$A = \frac{VR}{R^2 + \left(\dfrac{1}{\omega c}\right)^2}$$

and
$$B = \frac{-V}{\omega C\left[R^2 + \left(\dfrac{1}{\omega c}\right)^2\right]}$$

Substituting the values of A and B in Eq. 7.22, we have

$$i_p = \frac{VR}{R^2 + \left(\dfrac{1}{\omega c}\right)^2}\cos(\omega t + \theta) + \frac{-V}{\omega C\left[R^2 + \left(\dfrac{1}{\omega C}\right)^2\right]}\sin(\omega t + \theta)$$

Putting
$$M\cos\phi = \frac{VR}{\left[R^2 + \left(\dfrac{1}{\omega C}\right)^2\right]}$$

and
$$M\sin\phi = \frac{V}{\omega C\left[R^2 + \left(\dfrac{1}{\omega C}\right)^2\right]}$$

To find M and ϕ, we divide one equation by the other,

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{1}{\omega CR}$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{\left[R^2 + \left(\dfrac{1}{\omega C}\right)^2\right]}$$

$\therefore$
$$M = \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \qquad (7.24)$$

The complete solution for the current $i = i_c + i_p$

$$\therefore \qquad i = ce^{-(t/RC)} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \qquad (7.25)$$

Since the capacitor does not allow sudden changes in voltages at $t = 0$, $i = \frac{V}{R}$ $\cos\theta$

$$\therefore \qquad \frac{V}{R}\cos\theta = c + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$c = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

The complete solution for the current is

$$i = e^{-(t/RC)}\left[\frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{\omega CR}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right) \qquad (7.26)$$

Example 7.4 *In the circuit shown in Fig. 7.14, determine the complete solution for the current when switch S is closed at t = 0. Applied voltage is $v(t)$ = 50 cos $\left(10^2 t + \frac{\pi}{4}\right)$. Resistance R = 10 Ω and capacitance C = 1 μF.*

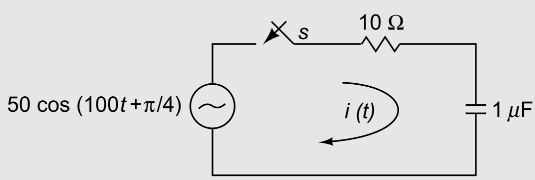


Fig. 7.14

Solution

By applying Kirchhoff's voltage law to the circuit, we have

$$10i + \frac{1}{1\times 10^{-6}}\int i\,dt = 50\cos\left(100t + \frac{\pi}{4}\right)$$

$$10\frac{di}{dt} + \frac{i}{1\times 10^{-6}} = -5(10)^3\sin\left(100t + \frac{\pi}{4}\right)$$

$$\frac{di}{dt} + \frac{i}{10^{-5}} = -500\sin\left(100t + \frac{\pi}{4}\right)$$

$$\left(D + \frac{1}{10^{-5}}\right)i = -500\sin\left(100t + \frac{\pi}{4}\right)$$

The complementary function is $i_C = ce^{-t/10^{-5}}$. By assuming particular integral as $i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$,

we get $$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

where $\omega = 100$ rad/sec $\qquad \theta = \pi/4$

$C = 1\mu$F $\qquad R = 10\ \Omega$

Substituting the values in the above equation, we have

$$i_p = \frac{50}{\sqrt{(10)^2 + \left(\frac{1}{100\times 10^{-6}}\right)^2}}\cos\left(\omega t + \frac{\pi}{4} + \tan^{-1}\frac{1}{100\times 10^{-6}\times 10}\right)$$

$$i_p = 4.99\times 10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

At $t = 0$, the current flowing through the circuit is

$$\frac{V}{R}\cos\theta = \frac{50}{10}\cos\pi/4 = 3.53\text{ A}$$

$$i = \frac{V}{R}\cos\theta = 3.53\text{ A}$$

$\therefore$ $$i = ce^{-t/10^{-5}} + 4.99\times 10^{-3}\cos\left(100t + \frac{\pi}{4} + 89.94^\circ\right)$$

At $\qquad t = 0$

$$c = 3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94°\right)$$

Hence the complete solution is

$$i = \left[3.53 - 4.99 \times 10^{-3} \cos\left(\frac{\pi}{4} + 89.94°\right)\right] e^{-(t/10^{-5})}$$

$$+ 4.99 \times 10^{-3} \cos\left(100t + \frac{\pi}{4} + 89.94°\right)$$

7.4 SECOND ORDER DIFFERENTIAL EQUATIONS—HOMOGENEOUS, NON-HOMOGENEOUS—RLC CIRCUITS WITH DC AND AC EXCITATION

7.4.1 DC Response of RLC Circuit

Consider a circuit consisting of resistance, inductance and capacitance as shown in Fig. 7.15. The capacitor and inductor are initially uncharged, and are in series with a resistor. When switch S is closed at $t = 0$, we can determine the complete solution for the current. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

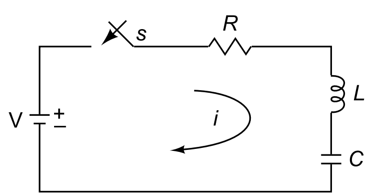


Fig. 7.15

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{7.27}$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i \tag{7.28}$$

or

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0 \tag{7.29}$$

The above equation is a second order linear differential equation, with only complementary function. The particular solution for the above equation is zero. Characteristic equation for the above differential equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0 \tag{7.30}$$

The roots of Eq. 7.30 are

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $$K_1 = -\frac{R}{2L} \text{ and } K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

Here K_2 may be positive, negative or zero.

K_2 is positive, when $\left(\frac{R}{2L}\right)^2 > 1/LC$

The roots are real and unequal, and give the over damped response as shown in Fig. 7.16. Then Eq. 7.29 becomes

$$[D - (K_1 + K_2)]\ [D - (K_1 - K_2)]\ i = 0$$

The solution for the above equation is

$$i = c_1 e^{(K_1 + K_2)t} + c_2\, e^{(K_1 - K_2)t}$$

The current curve for the overdamped case is shown in Fig. 7.16.

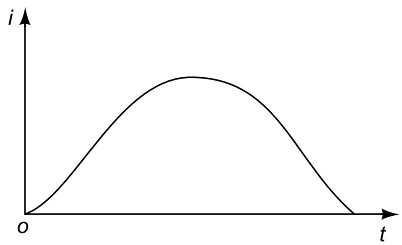


Fig. 7.16

K_2 is negative, when $(R/2L)^2 < 1/LC$

The roots are complex conjugate, and give the underdamped response as shown in Fig. 7.17. Then Eq. 7.29 becomes

$$[D - (K_1 + jK_2)]\ [D - (K_1 - jK_2)]i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

The current curve for the underdamped case is shown in Fig. 7.17.

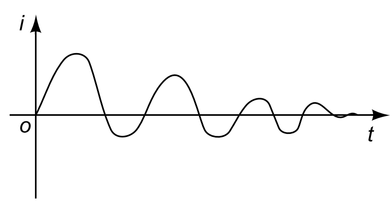


Fig. 7.17

K_2 is zero, when $(R/2L)^2 = 1/LC$

The roots are equal, and give the critically damped response as shown in Fig. 7.18. Then Eq. 7.29 becomes

$$(D - K_1)\ (D - K_1)i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}(c_1 + c_2 t)$$

The current curve for the critically damped case is shown in Fig. 7.18.

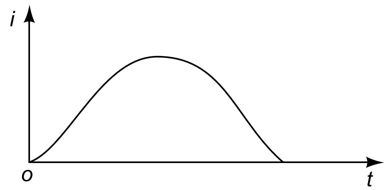


Fig. 7.18

Example 7.5 *The circuit shown in Fig. 7.19 consists of resistance, inductance and capacitance in series with a 100 V constant source when the switch is closed at t = 0. Find the current transient.*

Solution

At $t = 0$, switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20\times 10^{-6}}\int i dt \tag{7.31}$$

Differentiating the Eq. 7.31, we get

$$0.05\frac{d^2 i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20\times 10^{-6}} i = 0$$

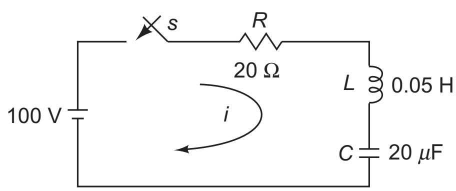


Fig. 7.19

$$\frac{d^2 i}{dt^2} + 400\frac{di}{dt} + 10^6 i = 0$$

$$\therefore \quad (D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{+k_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t)]$$

$$i = e^{-200t}\,[c_1 \cos 979.8t + c_2 \sin 979.8t)]\text{ A}$$

At $t = 0$, the current flowing through the circuit is zero

$$i = 0 = (1)\,[c_1 \cos 0 + c_2 \sin 0]$$

$\therefore$ $$c_1 = 0$$

$\therefore$ $$i = e^{-200t}\,c_2 \sin 979.8t\text{ A}$$

Differentiating, we have

$$\frac{di}{dt} = c_2\,[e^{-200t}\,979.8 \cos 979.8t + e^{-200t}\,(-200) \sin 979.8t)]$$

At $t = 0$, the voltage across inductor is 100 V

$\therefore$ $$L\frac{di}{dt} = 100$$

or $$\frac{di}{dt} = 2000$$

At $t = 0$ $$\frac{di}{dt} = 2000 = c_2\,979.8 \cos 0$$

$\therefore$ $$c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is

$$i = e^{-200t}\,(2.04 \sin 979.8t)\text{ A}$$

7.4.2 Sinusoidal Response of RLC Circuit

Consider a circuit consisting of resistance, inductance and capacitance in series as shown in Fig. 7.20. Switch S is closed at $t = 0$. At $t = 0$, a sinusoidal voltage $V \cos(\omega t + \theta)$ is applied to the RLC series circuit, where V is the amplitude of the wave and θ is the phase angle. Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

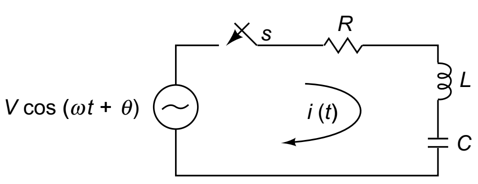


Fig. 7.20

$$V\cos(\omega t+\theta) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{7.32}$$

Differentiating the above equation, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + i/C = -V\omega\sin(\omega t+\theta)$$

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = -\frac{V\omega}{L}\sin(\omega t+\theta) \tag{7.33}$$

The particular solution can be obtained by using undetermined coefficients. By assuming

$$i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \tag{7.34}$$

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \tag{7.35}$$

$$i_p'' = -A\omega^2\cos(\omega t+\theta) - B\omega^2\sin(\omega t+\theta) \tag{7.36}$$

Substituting ip, i_p' and i_p'' in Eq. 7.33, we have

$$\{-A\omega^2\cos(\omega t+\theta) - B\omega^2\sin(\omega t+\theta)\} + \frac{R}{L}\{-A\omega\sin(\omega t+\theta)$$

$$+ B\omega\cos(\omega t+\theta)\} + \frac{1}{LC}\{A\cos(\omega t+\theta) + B\sin(\omega t$$

$$+ \theta)\} = -\frac{V\omega}{L}\sin(\omega t+\theta) \tag{7.37}$$

Comparing both sides, we have

Sine coefficients

$$-B\omega^2 - A\frac{\omega R}{L} + \frac{B}{LC} = -\frac{V\omega}{L}$$

$$A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = \frac{V\omega}{L} \tag{7.38}$$

Cosine coefficients

$$-A\omega^2 + B\frac{\omega R}{L} + \frac{A}{LC} = 0$$

$$A\left(\omega^2 - \frac{1}{LC}\right) - B\left(\frac{\omega R}{L}\right) = 0 \tag{7.39}$$

Solving Eqs 7.38 and 7.39, we get

$$A = \frac{V \times \dfrac{\omega^2 R}{L^2}}{\left[\left(\dfrac{\omega R}{L}\right)^2 - \left(\omega^2 - \dfrac{1}{LC}\right)^2\right]}$$

$$B = \frac{\left(\omega^2 - \frac{1}{LC}\right)V\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2\right]}$$

Substituting the values of A and B in Eq. 7.34, we get

$$i_p = \frac{V\frac{\omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \cos(\omega t + \theta)$$

$$+ \frac{\left(\omega^2 - \frac{1}{LC}\right)V\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \sin(\omega t + \theta) \qquad (7.40)$$

Putting $$M\cos\phi = \frac{V\frac{\omega^2 R}{L^2}}{\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2}$$

and $$M\sin\phi = \frac{V\left(\omega^2 - \frac{1}{LC}\right)\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 - \left(\omega^2 - \frac{1}{LC}\right)^2\right]}$$

To find M and ϕ we divide one equation by the other

or $$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

$$\phi = \tan^{-1}\left[\left(\omega L - \frac{1}{\omega C}\right)\Big/R\right]$$

Squaring both equations and adding, we get

$$M^2\cos^2\phi + M^2\sin^2\phi = \frac{V^2}{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}$$

$$\therefore \qquad M = \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}}$$

The particular current becomes

$$i_p = \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\frac{\left(\dfrac{1}{\omega C} - \omega L\right)}{R}\right] \qquad (7.41)$$

The complementary function is similar to that of DC series RLC circuit.

To find out the complementary function, we have the characteristic equation

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0 \qquad (7.42)$$

The roots of Eq. 7.42, are

$$D_1, D_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\dfrac{R}{2L}$ and $K_2 = \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$

$$\therefore \qquad D_1 = K_1 + K_2 \text{ and } D_2 = K_1 - K_2$$

K_2 becomes positive, when $(R/2L)^2 > 1/LC$

The roots are real and unequal, which gives an overdamped response. Then Eq. 7.42 becomes

$$[D - (K_1 + K_2)]\,[D - (K_1 - K_2)]i = 0$$

The complementary function for the above equation is

$$i_c = c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t}$$

Therefore, the complete solution is

$$\begin{aligned} i &= i_c + i_p \\ &= c_1 e^{(K_1 + K_2)t} + c_2 e^{(K_1 - K_2)t} \\ &\quad + \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right] \end{aligned}$$

K_2 becomes negative, when $\left(\dfrac{R}{2L}\right)^2 < \dfrac{1}{LC}$

Then the roots are complex conjugate, which gives an underdamped response. Equation 7.42 becomes

$$[D-(K_1+jK_2)]\,[D-(K_1-jK_2)]i=0$$

The solution for the above equation is

$$i_c = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t]$$

Therefore, the complete solution is

$$i = i_c + i_p$$

$$\therefore \quad i = e^{K_1 t}\,[c_1 \cos K_2 t + c_2 \sin K_2 t] + \frac{V}{\sqrt{R^2+\left(\dfrac{1}{\omega C}-\omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR}-\frac{\omega L}{R}\right)\right]$$

K_2 becomes zero, when $\left(\dfrac{R}{2L}\right)^2 = 1/LC$

Then the roots are equal which gives critically damped response. Then, Eq. 7.42 becomes $(D-K_1)\,(D-K_1)i=0$.

The complementary function for the above equation is

$$i_c = e^{K_1 t}\,(c_1 + c_2 t)$$

Therefore, the complete solution is $i = i_c + i_p$

$$\therefore \quad i = e^{K_1 t}\,[c_1 + c_2 t] + \frac{V}{\sqrt{R^2+\left(\dfrac{1}{\omega C}-\omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR}-\frac{\omega L}{R}\right)\right]$$

Example 7.6 *In the circuit shown in Fig. 7.21, determine the complete solution for the current, when the switch is closed at t = 0. Applied voltage is* $v(t) = 400 \cos\left(500t + \dfrac{\pi}{4}\right)$. *Resistance R = 15 Ω, inductance L = 0.2 H and capacitance C = 3μF.*

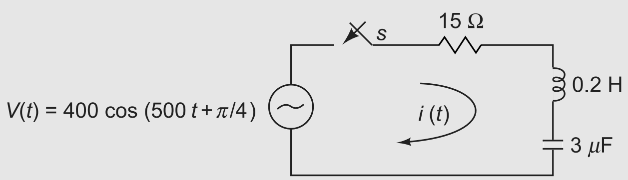


Fig. 7.21

Solution

By applying Kirchhoff's voltage law to the circuit,

$$15i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3\times 10^{-6}}\int i(t)dt = 400 \cos\left(500t + \frac{\pi}{4}\right)$$

Differentiating the above equation once, we get

$$15\frac{di}{dt} + 0.2\frac{d^2 i}{dt} + \frac{i}{3\times 10^{-6}} = -2\times 10^5 \sin\left(500t + \frac{\pi}{4}\right)$$

$$(D^2 + 75D + 16.7\times 10^5)i = \frac{-2\times 10^5}{0.2} \sin\left(500t + \frac{\pi}{4}\right)$$

The roots of the characteristic equation are

$$D_1 = -37.5 + j1290 \text{ and } D_2 = -37.5 - j1290$$

The complementary current

$$i_c = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t)$$

Particular solution is

$$i_p = \frac{V}{\sqrt{R^2 + \left(\dfrac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right]$$

$$\therefore \quad i_p = 0.71 \cos\left(500t + \frac{\pi}{4} + 88.5°\right)$$

The complete solution is

$$i = e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t) + 0.71 \cos(500t + 45° + 88.5°)$$

At $t = 0$, $i_0 = 0$

$$\therefore \quad c_1 = -0.71 \cos(133.5°) = +0.49$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-37.5t}(-1290c_1 \sin 1290t + c_2\, 1290 \cos 1290t) - 37.5e^{-37.5t}(c_1 \cos 1290t + c_2 \sin 1290t) - 0.71\times 500 \sin(500t + 45° + 88.5°)$$

At $t = 0$, $\dfrac{di}{dt} = 1414$

$$\therefore \quad 1414 = 1290c_2 - 37.5\times 0.49 - 0.71\times 500 \sin(133.5°)$$

$$1414 = 1290c_2 - 18.38 - 257.5$$

$$\therefore \quad c_2 = 1.31$$

The complete solution is

$$i = e^{-37.5t}(0.49 \cos 1290t + 1.31 \sin 1290t) + 0.71 \cos(500t + 133.5°)$$

7.5 LAPLACE TRANSFORM METHODS

7.5.1 Definition of Laplace Transforms

The Laplace transform is used to solve differential equations and corresponding initial and final value problems. Laplace transforms are widely used in engineering, particularly when the driving function has discontinuities and appears for a short period only.

In circuit analysis, the input and output functions do not exist forever in time. For casual functions, the function can be defined as $f(t)\,u(t)$. The integral for the Laplace transform is taken with the lower limit at $t = 0$ in order to include the effect of any discontinuity at $t = 0$.

Consider a function $f(t)$ which is to be continuous and defined for values of $t \geq 0$. The Laplace transform is then

$$\mathscr{L}[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, u(t)\, dt = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

$f(t)$ is a continuous function for $t \geq 0$ multiplied by e^{-st} which is integrated with respect to t between the limits 0 and ∞. The resultant function of the variable s is called Laplace transform of $f(t)$. Laplace transform is a function of independent variable s corresponding to the complex variable in the exponent of e^{-st}. The complex variable s is, in general, of the form $s = \sigma + j\omega$ and σ and ω being the real and imaginary parts, respectively. For a function to have a Laplace transform, it must satisfy the condition $\int_{0}^{\infty} f(t)\, e^{-st}\, dt < \infty$. Laplace transform changes the time domain function $f(t)$ to the frequency domain function $F(s)$. Similarly, inverse Laplace transformation converts frequency domain function $F(s)$ to the time domain function $f(t)$ as shown below.

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{-j}^{+j} F(s)\, e^{st}\, ds$$

Here, the inverse transform involves a complex integration. $f(t)$ can be represented as a weighted integral of complex exponentials. We will denote the transform relationship between $f(t)$ and $F(s)$ as

$$f(t) \xleftrightarrow{\mathscr{L}} F(s)$$

7.5.2 Properties of Laplace Transforms

Laplace transforms have the following properties.

(a) *Superposition Property*—The Laplace transform of the sum of the two or more functions is equal to the sum of transforms of the individual function,

i.e. if $$f_1(t) \xleftrightarrow{\mathscr{L}} F_1(s) \text{ and}$$

$$f_2(t) \xleftrightarrow{\mathscr{L}} F_2(s), \text{ then}$$

$$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Consider two functions $f_1(t)$ and $f_2(t)$. The Laplace transform of the sum or difference of these two functions is given by

$$\begin{aligned}\mathscr{L}\{f_1(t) \pm f_2(t)\} &= \int_0^\infty \{f_1(t) \pm f_2(t)\} e^{-st}\, dt \\ &= \int_0^\infty f_1(t) e^{-st}\, dt \pm \int_0^\infty f_2(t) e^{-st}\, dt \\ &= F_1(s) \pm F_2(s)\end{aligned}$$

$$\therefore \qquad \mathscr{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$$

(b) *Linearity property*—If K is a constant, then

$$\mathscr{L}[Kf(t)] = K\,\mathscr{L}[f(t)] = K\,F(s)$$

Consider a function $f(t)$ multiplied by a constant K. The Laplace transform of this function is given by

$$\begin{aligned}\mathscr{L}[Kf(t)] &= \int_0^\infty Kf(t) e^{-st}\, dt \\ &= K\int_0^\infty f(t) e^{-st}\, dt = KF(s)\end{aligned}$$

If we can use these two properties jointly, we have

$$\begin{aligned}\mathscr{L}[K_1 f_1(t) + K_2 f_2(t)] &= K_1\,\mathscr{L}[f_1(t)] + K_2\,\mathscr{L}[f_2(t)] \\ &= K_1 F_1(s) + K_2 F_2(s)\end{aligned}$$

7.5.3 Laplace Transform of Some Useful Functions

(i) The unit step function $f(t) = u(t)$

where
$$\begin{aligned}u(t) &= 1 \text{ for } t > 0 \\ &= 0 \text{ for } t < 0\end{aligned}$$

$$\begin{aligned}\mathscr{L}[f(t)] &= \int_0^\infty u(t) e^{-st}\, dt \\ &= \int_0^\infty 1 e^{-st}\, dt = \frac{-1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}\end{aligned}$$

$$\mathscr{L}[u(t)] = \frac{1}{s}$$

(ii) Exponential function $f(t) = e^{-at}$

$$\mathscr{L}(e^{-at}) = \int_0^\infty e^{-at}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+a)t} = \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= \frac{1}{s+a}$$

$$\therefore \qquad \mathscr{L}[e^{-at}] = \frac{1}{s+a}$$

(iii) The cosine function: $\cos \omega t$

$$\mathscr{L}(\cos \omega t) = \int_0^\infty \cos \omega t\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t}\, dt + \int_0^\infty e^{-(s+j\omega)t}\, dt\right]$$

$$= \frac{1}{2}\left[-\frac{e^{-(s-j\omega)t}}{s-j\omega}\right]_0^\infty + \frac{1}{2}\left[-\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

$$\therefore \qquad \mathscr{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

(iv) The sine function: $\sin \omega t$

$$\mathscr{L}(\sin \omega t) = \int_0^\infty \sin \omega t\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, \frac{1}{2j}\, [e^{j\omega t} - e^{-j\omega t}] dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}\, dt - \int_0^\infty e^{-(s+j\omega)t}\, dt\right]$$

$$= \frac{1}{2j}\left\{\left[-\frac{e^{-(s-j\omega)t}}{(s-j\omega)}\right]_0^{\infty} + \left[\frac{e^{-(s+j\omega)t}}{(s+j\omega)}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

$$\therefore \qquad \mathscr{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

(v) The function t^n, where n is a positive integer

$$\mathscr{L}(t^n) = \int_0^{\infty} t^n\, e^{-st}\, dt$$

$$= \left[\frac{t^n\, e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s}\, nt^{n-1}\, dt$$

$$= \frac{n}{s}\int_0^{\infty} e^{-st}\, t^{n-1}\, dt$$

$$= \frac{n}{s}\, \mathscr{L}(t^{n-1})$$

Similarly, $\qquad \mathscr{L}(t^{n-1}) = \dfrac{n-1}{s}\, \mathscr{L}(t^{n-2})$

By taking Laplace transformations of t^{n-2}, t^{n-3},.... and substituting in the above equation, we get

$$\mathscr{L}(t^n) = \frac{n}{s}\,\frac{n-1}{s}\,\frac{n-2}{s}\cdots\frac{2}{s}\,\frac{1}{s}\,\mathscr{L}(t^{n-n})$$

$$= \frac{\angle n}{s^n}\,\mathscr{L}(t^0) = \frac{\angle n}{s^n}\times\frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

$$\therefore \qquad \mathscr{L}(t) = 1/s^2$$

(vi) The hyperbolic sine and cosine function

$$\mathscr{L}(\cos h\, at) = \int_0^{\infty} \cos h\, at\, e^{-st}\, dt$$

$$= \int_0^{\infty}\left[\frac{e^{at}+e^{-at}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s-a)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\,\frac{1}{(s-a)} + \frac{1}{2}\,\frac{1}{(s+a)} = \frac{s}{s^2-a^2}$$

Similarly,

$$\mathscr{L}(\sin h\ at) = \int_0^\infty \sin h\ (at) e^{-st}\, dt$$

$$= \int_0^\infty \left[\frac{e^{at} - e^{-at}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2}$$

Example 7.7 *Find the Laplace transform of the function*

$$f(t) = 4t^3 + t^2 - 6t + 7$$

Solution

$$\mathscr{L}(4t^3 + t^2 - 6t + 7) = 4\,\mathscr{L}(t^3) + \mathscr{L}(t^2) - 6\mathscr{L}(t) + 7\mathscr{L}(1)$$

$$= 4 \times \frac{\angle 3}{s^4} + \frac{\angle 2}{s^3} - 6\frac{\angle 1}{s^2} + 7\frac{1}{s}$$

$$= \frac{24}{s^4} + \frac{2}{s^3} - \frac{6}{s^2} + \frac{7}{s}$$

Example 7.8 *Find the Laplace transform of the function $f(t) = \cos^2 t$*

Solution

$$\mathscr{L}(\cos^2 t) = \mathscr{L}\left(\frac{1 + \cos 2t}{2}\right)$$

$$= \mathscr{L}\left(\frac{1}{2}\right) + \mathscr{L}\left(\frac{\cos 2t}{2}\right) = \frac{1}{2}\,[\mathscr{L}(1) + \mathscr{L}(\cos 2t)]$$

$$= \frac{1}{2s} + \frac{s}{2\left(s^2 + 4\right)} = \frac{2s^2 + 4}{2s\left(s^2 + 4\right)}$$

Example 7.9 *Find the Laplace transform of the function*

$$f(t) = 3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$$

Solution

$$\mathscr{L}(3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$$

$$= 3\mathscr{L}(t^4) - 2\mathscr{L}(t^3) + 4\mathscr{L}(e^{-3t}) - 2\mathscr{L}(\sin 5t) + 3\mathscr{L}(\cos 2t)$$

$$= 3\,\frac{\angle 4}{s^5} - 2\frac{\angle 3}{s^4} + 4\frac{1}{s+3} - 2 \times \frac{5}{s^2+25} + 3 \times \frac{s}{s^2+4}$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$$

7.5.4 Laplace Transform Theorems

(a) *Differentiation Theorem* If a function $f(t)$ is piecewise continuous, then the Laplace transform of its derivative $\frac{d}{dt}[f(t)]$ is given by

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

Proof By definition,

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,d\{f(t)\}$$

Integrating by parts, we get

$$= \left[e^{-st}\,f(t)\right]_0^\infty + \int_0^\infty se^{-st} f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sF(s)$$

Hence we have

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

This is applicable to higher order derivatives also. The Laplace transform of second derivative of $f(t)$ is

$$\mathcal{L}[f''(t)] = \mathcal{L}\left[\frac{d}{dt}\left(f'(t)\right)\right]$$

$$= s\,\mathcal{L}[f'(t)] - f'(0) = s\{sF(s) - f(0)\} - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

where $f'(0)$ is initial value of first derivative of $f(t)$

Similarly,

$$\mathcal{L}[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

In general, the nth order derivative is given by

$$\mathscr{L}(f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{n-1}(0)$$

Example 7.10 *Using the formula for Laplace transform of derivatives, obtain the Laplace transform of (a) sin 3t, (b) t^3*

Solution

(a) Let $f(t) = \sin 3t$

Then
$$f'(t) = 3\cos 3t, f''(t) = -9\sin 3t$$
$$\mathscr{L}[f''(t)] = s^2[\mathscr{L}f(t)] - sf(0) - f'(0) \qquad (7.43)$$
$$f(0) = 0, f'(0) = 3$$
$$\mathscr{L}[f''(t)] = \mathscr{L}[-9\sin 3t]$$

Substituting in Eq. 7.43, we get
$$\mathscr{L}[-9\sin 3t] = s^2\,\mathscr{L}[f(t)] - 3$$
$$\mathscr{L}[-9\sin 3t] - s^2\,[\mathscr{L}(\sin 3t)] = -3$$
$$\mathscr{L}[(s^2+9)\sin 3t] = 3$$
$$\therefore \quad \mathscr{L}(\sin 3t) = \frac{3}{s^2+9}$$

(b) Let $f(t) = t^3$

Differentiating successively, we get
$$f'(t) = 3t^2, f''(t) = 6t, f'''(t) = 6$$

By using differentiation theorem, we get
$$\mathscr{L}[f'''(t)] = s^3\,\mathscr{L}[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

Substituting all initial conditions, we get
$$\mathscr{L}[f'''(t)] = s^3\,\mathscr{L}[f(t)]$$
$$\mathscr{L}[6] = s^3\,\mathscr{L}[f(t)]$$
$$\therefore \quad \frac{6}{s} = s^3\,\mathscr{L}[f(t)]$$
$$F(s) = \mathscr{L}[f(t)] = \frac{6}{s^4}$$

(b) *Integration Theorem* If a function $f(t)$ is continuous, then the Laplace transform of its integral $\int f(t)dt$ is given by

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$$

Proof By definition

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-st}\,dt$$

Integrating by parts, we get

$$= \left[\frac{e^{-st}}{-s} \int_0^t f(t)\, dt \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} f(t)\, dt$$

Since, the first term is zero, we have

$$\mathscr{L}\left[\int_0^t f(t)\, dt \right] = \frac{1}{s} \mathscr{L}[f(t)] = \frac{F(s)}{s}$$

Example 7.11 *Find the Laplace transform of ramp function r(t) = t.*

Solution

$$\text{We know that } \int_0^t u(t) = r(t) = t$$

Integration of unit step function gives the ramp function.

$$\mathscr{L}[r(t)] = \mathscr{L}\left[\int_0^t u(t)\, dt \right]$$

Using the integration theorem, we get

$$\mathscr{L}\left[\int_0^t u(t)\, dt \right] = \frac{1}{s} \mathscr{L}[u(t)] = \frac{1}{s^2}$$

since $$\mathscr{L}[u(t)] = \frac{1}{s}$$

(c) *Differentiation of Transforms* If the Laplace transform of the function $f(t)$ exists, then the derivative of the corresponding transform with respect to s in the frequency domain is equal to its multiplication by t in the time domain.

i.e. $$\mathscr{L}[tf(t)] = \frac{-d}{ds} F(s)$$

Proof By definition,

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty f(t)\, e^{-st}\, dt$$

Since s and t are independent of variables, and the limits 0, ∞ are constants not depending on s, we can differentiate partially with respect to s within the integration and then integrate the function obtained with respect to t.

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty [f(t)\, e^{-st}]\, dt$$

$$= \int_0^\infty f(t)\,[-te^{-st}]dt = -\int_0^\infty \{tf(t)\}e^{-st}\,dt = -\mathscr{L}[tf(t)]$$

Hence $\quad \mathscr{L}[tf(t)] = -\dfrac{d}{ds}F(s)$

Example 7.12 *Find the Laplace transform of function*

f (t) = t sin 2t

Solution

Let $\quad f_1(t) = \sin 2t$

$$\mathscr{L}[f_1(t)] = \mathscr{L}[\sin 2t] = F_1(s)$$

where $\quad F_1(s) = \dfrac{2}{s^2+4}$

$$\mathscr{L}(tf_1(t)) = \mathscr{L}(t\sin 2t) = \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] = +\frac{4s}{(s^2+4)^2}$$

(d) *Integration of transforms* If the Laplace transform of the function $f(t)$ exists, then the integral of corresponding transform with respect to s in the complex frequency domain is equal to its division by t in the time domain.

i.e. $\quad \mathscr{L}\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty F(s)ds$

Proof If $f(t) \leftrightarrow F(s)$

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Integrating both sides from s to ∞

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty f(t)e^{-st}\,dt\right]ds$$

By changing the order of integration, we get

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{t}\right)dt$$

$$= \int_0^\infty \left[\frac{f(t)}{t}\right]e^{-st}\,dt = \mathscr{L}\left[\frac{f(t)}{t}\right]$$

$\therefore \quad \displaystyle\int_0^\infty F(s)ds = \mathscr{L}\left[\frac{f(t)}{t}\right]$

Example 7.13 *Find the Laplace transform of the function*

$$f(t) = \frac{2 - 2e^{-t}}{t}$$

Solution

$$\text{Let} \qquad f_1(t) = 2 - 2e^{-2t} \text{ then}$$

$$\mathscr{L}[f_1(t)] = \mathscr{L}(2 - 2e^{-2t}) = \mathscr{L}(2) - \mathscr{L}(2e^{-2t}) = \frac{2}{s} - \frac{2}{s+2}$$

$$= \frac{2s + 4 - 2s}{s(s+2)} = \frac{4}{s(s+2)}$$

$$\text{Hence} \qquad \mathscr{L}\left[\frac{2 - 2e^{-2t}}{t}\right] = \int_s^\infty F_1(s)\, ds$$

$$= \int_s^\infty \frac{4}{s(s+2)}\, ds$$

By taking partial fraction expansion,

$$\text{we get} \qquad \frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{2}{s} - \frac{2}{s+2}$$

$$\therefore \qquad \mathscr{L}\left[\frac{2 - e^{-2t}}{t}\right] = \int_s^\infty \mathscr{L}\left[2 - 2e^{-2t}\right] ds = \int_s^\infty \frac{2}{s}\, ds - \int_s^\infty \frac{2}{s+2}\, ds$$

$$= \left[2 \log s - 2 \log(s+2)\right]_s^\infty$$

$$= \left[2 \log \frac{1}{1 + 2/s}\right]_s^\infty = -2 \log\left(\frac{s}{s+2}\right)$$

$$\mathscr{L}\left(\frac{2 - 2e^{-2t}}{t}\right) = 2 \log\left(\frac{s+2}{s}\right)$$

(e) *First Shifting* Theorem If the function $f(t)$ has the transform $F(s)$, then the Laplace transform of $e^{-at} f(t)$ is $F(s + a)$

Proof By definition, $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

and, therefore,

$$F(s + a) = \int_0^\infty f(t) e^{-(s+a)t}\, dt$$

$$= \int_0^\infty e^{-at} f(t) e^{-st}\, dt = \mathscr{L}[e^{-at} f(t)]$$

$$\therefore \qquad F(s+a) = \mathcal{L}[e^{-at} f(t)]$$

Similarly, we have

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Example 7.14 *Find the Laplace transform of e^{at} sin bt*

Solution

Let
$$f(t) = \sin bt$$
$$\mathcal{L}[f(t)] = \mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}$$
since
$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$
$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

Example 7.15 *Find the Laplace transform of $(t + 2)^2 e^t$*

Solution

Let
$$f(t) = (t+2)^2 = t^2+2t+4$$
$$\mathcal{L}[f(t)] = \mathcal{L}[t^2+2t+4] = \frac{2}{s^3}+\frac{2}{s^2}+\frac{4}{s}$$
since
$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$
$$\mathcal{L}[e^{t} f(t)] = \frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}+\frac{4}{s-1}$$

(f) *Second Shifting Theorem* If the function $f(t)$ has the transform $F(s)$, then the Laplace transform of $f(t-a)u(t-a)$ is $e^{-as}F(s)$.

Proof By definition, $\mathcal{L}[f(t-a)\,u(t-a)]$

$$= \int_0^\infty [f(t-a)\,u(t-a)]e^{-st}\,dt$$

Since
$$f(t-a)\,u(t-a) = 0 \text{ for } t<a$$
$$= f(t-a) \text{ for } t>a$$

$$\therefore \quad \mathcal{L}[f(t-a)\,u(t-a)] = \int_0^\infty f(t-a)e^{-st}\,dt$$

Put
$$t-a = \tau \text{ then } \tau+a=t$$
$$dt = d\tau$$

Therefore, the above becomes

$$\mathscr{L}[f(t-a)\,u(t-a)] = \int_0^{\infty} f(\tau)e^{-s(\tau+a)}d\tau$$

$$= e^{-as}\int_0^{\infty} f(\tau)e^{-s\tau}\,d\tau = e^{-as}\,F(s)$$

$$\therefore \quad \mathscr{L}[f(t-a)\,u(t-a)] = e^{-as}\,F(s)$$

Example 7.16 *If u(t) = 1, for t ≥ 0 and u(t) = 0 for t < 0, determine the Laplace transform of [u(t) – u(t – a)].*

Solution

The function $f(t) = u(t) - u(t-a)$ is shown in Fig. 7.22.

$$\mathscr{L}[f(t)] = \mathscr{L}[u(t) - u(t-a)]$$

$$= \mathscr{L}[u(t)] - \mathscr{L}[u(t-a)]$$

$$= \frac{1}{s} - e^{-as}\frac{1}{s} = \frac{1}{s}\,(1-e^{-as})$$

$$\mathscr{L}[f(t)] = \frac{1}{s}(1-e^{-as})$$

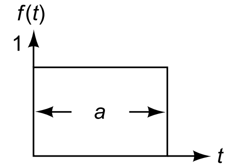


Fig. 7.22

(g) *Initial Value Theorem* If the function $f(t)$ and its derivative $f'(t)$ are Laplace transformable then $\operatorname*{Lt}_{t\to 0} f(t) = \operatorname*{Lt}_{s\to\infty} sF(s)$

Proof We know that

$$\mathscr{L}[f'(t)] = s[\mathscr{L}(f(t))] - f(0)$$

By taking the limit $s \to \infty$ on both sides

$$\operatorname*{Lt}_{s\to\infty} \mathscr{L}[f'(t)] = \operatorname*{Lt}_{s\to\infty} [sF(s) - f(0)]$$

$$\operatorname*{Lt}_{s\to\infty} \int_0^{\infty} f'(t)e^{-st}\,dt = \operatorname*{Lt}_{s\to\infty} [sF(s) - f(0)]$$

At $s \to \infty$ the integration of LHS becomes zero

i.e. $$\int_0^{\infty} \operatorname*{Lt}_{s\to\infty} [f'(t)\,e^{-st}]\,dt = 0$$

$$0 = \operatorname*{Lt}_{s\to\infty} sF(s) - f(0)$$

$$\therefore \quad \operatorname*{Lt}_{s\to\infty} sF(s) = f(0) = \operatorname*{Lt}_{t\to 0} f(t)$$

Example 7.17 *Verify the initial value theorem for the following functions*

(i) $5e^{-4t}$ *(ii) $2 - e^{5t}$*

Solution

(i) Let
$$f(t) = 5e^{-4t}$$
$$F(s) = \frac{5}{s+4}$$
$$sF(s) = \frac{5s}{s+4}$$
$$\underset{s \to \infty}{\text{Lt}}\ sF(s) = \underset{s \to \infty}{\text{Lt}}\ \frac{5}{1+4/s} = 5$$
$$\underset{t \to 0}{\text{Lt}}\ f(t) = \underset{t \to 0}{\text{Lt}}\ 5e^{-4t} = 5$$

Hence the theorem is proved.

(ii) Let
$$f(t) = 2 - e^{5t}$$
$$F(s) = \mathscr{L}(2 - e^{5t}) = \mathscr{L}(2) - \mathscr{L}[e^{5t}]$$
$$= \frac{2}{s} - \frac{1}{s-5} = \frac{s-10}{s(s-5)}$$
$$sF(s) = \frac{s-10}{s-5}$$
$$\underset{s \to \infty}{\text{Lt}}\ sF(s) = \underset{s \to \infty}{\text{Lt}}\ \left(\frac{1-10/s}{1-5/s}\right) = 1$$
$$\underset{t \to 0}{\text{Lt}}\ (2 - e^{5t}) = 1$$

Hence initial value theorem is proved.

(h) *Final Value Thorem* If $f(t)$ and $f'(t)$ are Laplace transformable, then $\underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{s \to 0}{\text{Lt}}\ sF(s)$

Proof We know that
$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$

By taking the limit $s \to 0$ on both sides, we have
$$\underset{s \to 0}{\text{Lt}}\ \mathscr{L}[f'(t)] = \underset{s \to 0}{\text{Lt}}\ [sF(s) - f(0)]$$
$$\underset{s \to 0}{\text{Lt}} \int_0^\infty f'(t) e^{-st}\, dt = \underset{s \to 0}{\text{Lt}}\ [sF(s) - f(0)]$$
$$\therefore \quad \int_0^\infty f'(t) dt = \underset{s \to 0}{\text{Lt}}\ [sF(s) - f(0)]$$
$$\left[f(t)\right]_0^\infty = \underset{t \to \infty}{\text{Lt}}\ f(t) - \underset{t \to 0}{\text{Lt}}\ f(t) = \underset{s \to 0}{\text{Lt}}\ sF(s) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides.

$$\therefore \qquad \underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{s \to 0}{\text{Lt}}\ sF(s)$$

Example 7.18 *Verify the final value theorem for the following functions.*

(i) $2 + e^{-3t} \cos 2t$ *(ii)* $6(1 - e^{-t})$

Solution

(i) Let

$$f(t) = 2 + e^{-3t} \cos 2t$$

$$F(s) = \frac{2}{s} + \frac{(s+3)}{(s+3)^2 + 4}$$

$$sF(s) = 2 + \frac{s^2}{(s+3)^2 + 4^2} + \frac{3s}{(s+3)^2 + 4^2}$$

$$\underset{s \to 0}{\text{Lt}}\ sF(s) = \underset{s \to 0}{\text{Lt}} \left[2 + \frac{(s+3)s}{(s+3)^2 + 4^2} \right] = 2$$

$$\underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{t \to \infty}{\text{Lt}}\ (2 + e^{-3t} \cos 2t) = 2$$

Hence the final value theorem is proved.

(ii) Let

$$f(t) = 6(1 - e^{-t})$$

$$F(s) = \frac{6}{s} - \frac{6}{s+1} = \frac{6}{s(s+1)}$$

$$sF(s) = \frac{6}{s+1}$$

$$\underset{s \to 0}{\text{Lt}}\ sF(s) = 6$$

$$\underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{t \to \infty}{\text{Lt}}\ 6(1 - e^{-t}) = 6$$

Hence the final value theorem is proved.

7.5.5 The Inverse Transformation

So far, we have discussed Laplace transforms of a functions $f(t)$. If the function in frequency domain $F(s)$ is given, the inverse Laplace transform can be determined by taking the partial fraction expansion which will be recognisable as the transform of known functions.

Example 7.19 *If* $F(s) = \dfrac{2}{(s+1)(s+5)}$, *find the function* $f(t)$.

Solution

First we divide the given function into partial fractions

$$F(s) = \frac{2}{(s+1)(s+5)}$$

$$\frac{2}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$

$$2 = A(s+5) + B(s+1)$$

Comparing both sides

$$A + B = 0$$

$$5A + B = 2$$

From which $$A = \frac{1}{2}, B = -\frac{1}{2}$$

Hence $$\frac{2}{(s+1)(s+5)} = \frac{1}{2(s+1)} + \frac{-1}{2(s+5)}$$

$$\mathscr{L}^{-1}\left[\frac{2}{(s+1)(s+5)}\right] = \mathscr{L}^{-1}\left[\frac{1}{2(s+1)}\right] - \mathscr{L}^{-1}\left[\frac{1}{2(s+5)}\right]$$

We know that $$\mathscr{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

and $$\mathscr{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}$$

$\therefore$ $$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2}\,e^{-t} - \frac{1}{2}\,e^{-5t}$$

7.5.6 Laplace Transform of Periodic Functions

Periodic functions appear in many practical problems. Let function $f(t)$ be a periodic function which satisfies the condition $f(t) = f(t+T)$ for all $t > 0$ where T is period of the function.

$$\mathscr{L}[f(t)] = \int_0^T f(t)\,e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \cdots + \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt + \cdots$$

$$= \int_0^T f(t)\,e^{-st}\,dt + \int_0^T f(t)\,e^{-st}\,e^{-sT}\,dt + \cdots + \int_0^T f(t)\,e^{-st}\,e^{-nsT}\,dt + \cdots$$

$$= (1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \cdots)\int_0^T f(t)\,e^{-st}\,dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$$

Example 7.20 *Find the transform of the waveform shown in Fig. 7.23*

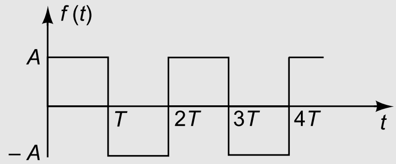


Fig. 7.23

Solution

Here the period is $2T$

$$\therefore \qquad \mathscr{L}[f(t)] = \frac{1}{1-e^{-2sT}}\left[\int_0^{2T} f(t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[\int_0^{T} Ae^{-st}\,dt + \int_T^{2T} (-A)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[\left(\frac{-A}{s}e^{-st}\right)_0^T + \left(\frac{A}{s}e^{-st}\right)_T^{2T}\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}\left(e^{-sT}-1\right) + \frac{A}{s}\left(e^{-2sT}-e^{-sT}\right)\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[\frac{A}{s}\left(1-e^{-sT}\right)^2\right] = \frac{A}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}}\right)$$

$$\therefore \qquad \mathscr{L}[f(t)] = \frac{A}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}}\right)$$

7.5.7 The Convolution Integral

If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, then the product of $F(s)\,G(s) = H(s)$, where $H(s)$ is the Laplace transform of $h(t)$ given by $f(t) * g(t)$ and defined by

$$h(t) = f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\,d\tau$$

Proof Let $\int_0^t f(\tau)\, g(t-\tau)\, d\tau = h(t)$

By definition

$$\mathscr{L}[h(t)] = \int_0^\infty e^{-st}\, h(t)\, dt$$

$$= \int_0^\infty e^{-st} \int_0^t f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau\, dt$$

By changing the order of integration of the above equation, we have

$$\mathscr{L}[h(t)] = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty f(\tau) \left[\int_\tau^\infty e^{-st}\, g(t-\tau)\, dt \right] d\tau$$

Put $t - \tau = y$, and we get

$$\mathscr{L}[h(t)] = \int_0^\infty f(\tau) \left[\int_0^\infty e^{-s(y+\tau)}\, g(y)\, dy \right] d\tau$$

$$= \int_0^\infty f(\tau)\, e^{-st}[G(s)]\, d\tau$$

$$= G(s) \cdot F(s)$$

Therefore, $\qquad \mathscr{L}[h(t)] = H(s) = G(s) \cdot F(s)$

$h(t) = \int_0^t f(\tau)\, g(t-\tau) d\tau$ defines the convolution of functions $f(t)$ and $g(t)$ and is expressed symbolically as

$$h(t) = f(t) * g(t)$$

This theorem is very useful in frequency domain analysis.

Example 7.21 *By using the convolution theorem, determine the inverse Laplace transform of the following functions.*

(i) $\dfrac{1}{s^2\left(s^2 - a^2\right)}$ $\qquad$ *(ii)* $\dfrac{1}{s^2\,(s+1)}$

Solution

(i) Let $$H(s) = \frac{1}{s^2\left(s^2 - a^2\right)} \text{ and}$$

let $$F(s) = \frac{1}{s^2} \text{ and } G(s) = \frac{1}{s^2 - a^2}$$

We know $$f(t) = \mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$g(t) = \mathscr{L}^{-1}[G(s)] = \mathscr{L}^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a} \sin h\,(at)$$

Hence

$$L^{-1}\left[\frac{1}{s^2\left(s^2 - a^2\right)}\right] = \int_0^t g(\tau) f(t-\tau) d\tau$$

$$= \frac{1}{a}\int_0^t (t-\tau) \sin h\,(a\,\tau) d\tau$$

$$\frac{1}{a}\left[\left(t-\tau\right)\int_0^t \sin h\left(a\tau\right) d\tau - \int_0^t \left(-1\right)\int \sin h\left(a\tau\right) d\tau\right]$$

$$= \frac{1}{a}\left[\left(t-\tau\right)\frac{\cos ha\tau}{a}\Bigg)_0^t + \int_0^t \frac{\cos ha\tau}{a} d\tau\right]$$

$$= \frac{1}{a}\left[\frac{-t}{a} + \frac{\sin ha\tau}{a}\Bigg)_0^t\right]$$

$$= \frac{1}{a^2}\,[\sin h\, at - t]$$

(ii) Let $$H(s) = \frac{1}{s^2\,(s+1)} \text{ and } F(s) = \frac{1}{s^2} \quad G(s) = \frac{1}{s+1}$$

We know that $$f(t) = \mathscr{L}^{-1}\,[F(s)] = t$$

$$g(t) = \mathscr{L}^{-1}\,[G(s)] = e^{-t}$$

$$h(t) = \mathscr{L}^{-1}[H(s)] = \int_0^t g(\tau) f(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\tau}\,(t-\tau)\, d\tau$$

$$= (t-\tau)(-e^{-\tau})_0^t - \int_0^t (-1)\,(-e^{-\tau})d\tau$$

$$= t - \int_0^t e^{-\tau}\,d\tau$$

$$= t - (-e^{-\tau})_0^t = t + e^{-t} - 1$$

7.5.8 Partial Fractions

Most transform methods depend on the partial fraction of a given transform function. Given any solution of the form $N(s) = P(s)/Q(s)$, the inverse Laplace transform can be determined by expanding it into partial fractions. The partial fractions depend on the type of factor. It is to be assumed that $P(s)$ and $Q(s)$ have real coefficients and contain no common factors. The degree of $P(s)$ is lower than that of $Q(s)$.

Case 1 When roots are real and simple

In this case $\quad N(s) = P(s)/Q(s)$

where $\quad Q(s) = (s-a)(s-b)(s-c)$

Expanding $N(s)$ into partial fractions, we get

$$N(s) = \frac{A}{(s-a)} + \frac{B}{(s-b)} + \frac{C}{(s-c)} \tag{7.44}$$

To obtain the constant A, multiplying Eq. 7.44 with $(s-a)$ and putting $s = a$, we get

$$N(s)(s-a)\,|_{s=a} = A$$

Similarly, we can get the other constants

$$B = (s-b)N(s)\,|_{s=b}$$

$$C = (s-c)N(s)\,|_{s=c}$$

Example 7.22

Determine the partial fraction expansion for $N(s) = \dfrac{s^2+s+1}{s(s+5)(s+3)}$.

Solution

$$N(s) = \frac{s^2+s+1}{s(s+5)(s+3)}$$

$$\frac{s^2+s+1}{s(s+5)(s+3)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s+3}$$

$$A = sN(s)|_{s=0} = \left.\frac{s^2+s+1}{(s+5)(s+3)}\right|_{s=0} = \frac{1}{15}$$

$$B = (s+5)\,N(s)\,|_{s=-5} = \left.\frac{s^2+s+1}{s(s+3)}\right|_{s=-5}$$

$$= \frac{(25)+(-5)+1}{(-5)(-5+3)} = \frac{21}{10} = 2.1$$

$$C = (s+3)\,N(s)|_{s=-3} = \left.\frac{s^2+s+1}{s(s+5)}\right|_{s=-3}$$

$$= \frac{9-3+1}{(-3)(-3+5)} = \frac{7}{-6} = -1.17$$

Case 2 When roots are real and multiple

In this case $N(s) = P(s)/Q(s)$

where $Q(s) = (s-a)^n\,Q_1(s)$

The partial fraction expansion of $N(s)$ is

$$N(s) = \frac{A_0}{(s-a)^n} + \frac{A_1}{(s-a)^{n-1}} + \cdots + \frac{A_{n-1}}{(s-a)} + \frac{P_1(s)}{Q_1(s)} \qquad (7.45)$$

where $\dfrac{P_1(s)}{Q_1(s)} = R(s)$ represents the remainder terms of expansion. To obtain the constant $A_0, A_1, ..., A_{n-1}$, let us multiply both sides of Eq. 7.45 by $(s-a)^n$

Thus

$$(s-a)^n\,N(s) = N_1(s) = A_0 + A_1(s-a)$$
$$+ A_2(s-a)^2 + ... + A_{n-1}\,(s-a)^{n-1} + R(s)\,(s-a)^n \qquad (7.46)$$

where $R(s)$ indicates the remainder terms.

Putting $s = a$, we get

$$A_0 = (s-a)^n\,N(s)|_{s=a}$$

Differentiating Eq. 7.46 with respect to s, and putting $s = a$, we get

$$A_1 = \left.\frac{d}{ds}\,N_1(s)\right|_{s=a}$$

Similarly, $$A_2 = \left.\frac{1}{2!}\,\frac{d^2}{ds^2}\,N_1(s)\right|_{s=a}$$

In general, $$A_n = \left.\frac{1}{n!}\,\frac{d^n\,N_1(s)}{ds^n}\right|_{s=a}$$

Example 7.23 *Determine the partial fraction expansion for*

$$N(s) = \frac{s-5}{s(s+2)^2}$$

Solution

$$N(s) = \frac{s-5}{s(s+2)^2}$$

$$N(s) = \frac{s-5}{s(s+2)^2} = \frac{A}{s} + \frac{B_0}{(s+2)^2} + \frac{B_1}{s+2}$$

$$A = N(s)s\,|_{s=0} = \left.\frac{s-5}{(s+2)^2}\right|_{s=0} = \frac{-5}{4} = -1.25$$

$$N_1(s) = (s+2)^2\,N(s) = \frac{s-5}{2}$$

$$B_0 = N(s)\,(s+2)^2\,|_{s=-2} = \left.\frac{s-5}{2}\right|_{s=-2}$$

$$= \frac{-7}{-2} = 3.5$$

$$B_1 = \left.\frac{d}{ds}N_1(s)\right|_{s=-2}$$

$$= \left.\frac{d}{ds}\left(1-\frac{5}{s}\right)\right|_{s=-2}$$

$$= +\left.\frac{5}{s^2}\right|_{s=-2} = \frac{5}{4} = 1.25$$

Case 3 When roots are complex

Consider a function $N(s) = \dfrac{P(s)}{Q_1(s)(s-\alpha+j\beta)(s-\alpha-j\beta)}$

The partial fraction expansion of $N(s)$ is

$$N(s) = \frac{A}{s-\alpha-j\beta} + \frac{b}{s-\alpha+j\beta} + \frac{P_1(s)}{Q_1(s)} \tag{7.47}$$

where $P_1(s)/Q_1(s)$ is the remainder term.

Multiplying Eq. 7.47 by $(s-\alpha-j\beta)$ and putting $s=\alpha+j\beta$, we get

$$A = \frac{P(\alpha+j\beta)}{Q_1(\alpha+j\beta)(+2j\beta)}$$

Similarly, $$B = \frac{P(\alpha - j\beta)}{(-2j\beta)\,Q_1(\alpha - j\beta)}$$

In general, $B = A^*$ where A^* is complex conjugate of A.

If we denote the inverse transform of the complex conjugate terms as $f(t)$

$$f(t) = \mathscr{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{B}{s-\alpha+j\beta}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{A}{s-\alpha-j\beta} + \frac{A^*}{s-\alpha+j\beta}\right]$$

where A and A^* are conjugate terms.

If we denote $A = C + jD$, then

$$B = C - jD = A^*$$

$$\therefore \qquad f(t) = e^{\alpha t}\,(Ae^{j\beta t} + A^*\,e^{-j\beta t})$$

Example 7.24 *Find the inverse transform of the function*

$$F(s) = \frac{s+5}{s\left(s^2+2s+5\right)}$$

Solution

$$F(s) = \frac{s+5}{s\left(s^2+2s+5\right)}$$

By taking partial fractions, we have

$$F(s) = \frac{s+5}{s\left(s^2+2s+5\right)} = \frac{A}{s} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2}$$

$$A = F(s)s|_{s=0} = \frac{s+5}{\left(s^2+2s+5\right)} = 1$$

$$B = F(s)(s+1-j2)\,|_{s=-1+j2} = \left.\frac{s+5}{s(s+1+j2)}\right|_{s=-1+j2}$$

$$= \frac{4+j2}{(-1+j2)\,j4}$$

$$= \frac{2+j}{2j(-1+j2)} = \frac{2+j}{-2j-4} = \frac{-1}{2}$$

$$B^* = F(s)(s + 1 + j2)\,|_{s=-1-j2}$$

$$= \left.\frac{s+5}{s(s+1-j2)}\right|_{s=-1-j2}$$

$$= \frac{-1-j2+5}{(-1-j2)(-1-j2+1-j2)}$$

$$= \frac{4-j2}{+(1+j2)(j4)} = \frac{4-j2}{4j-8} = \frac{2(2-j)}{-4(2-j)} = \frac{-1}{2}$$

$$\therefore \qquad F(s) = \frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)}$$

The inverse transform of $F(s)$ is $f(t)$

$$f(t) = \mathscr{L}^{-1}\,[F(s)] = \mathscr{L}^{-1}\left[\frac{1}{s} - \frac{1}{2(s+1-j2)} - \frac{1}{2(s+1+j2)}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{(s+1-j2)}\right] - \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s+1+j2}\right]$$

$$= 1 - \frac{1}{2}\,e^{(-1+j2)t} - \frac{1}{2}\,e^{(-1-j2)t}$$

7.5.9 Transient Response Related to S-Plane—Solution Using Laplace Transform Method

Laplace transform methods are used to find out transient currents in circuits containing energy storage elements. To find these currents, first the differential equations are formed by applying Kirchhoff's laws to the circuit, then these differential equations can be easily solved by using Laplace transformation methods.

Consider a series RL circuit shown in Fig. 7.24.

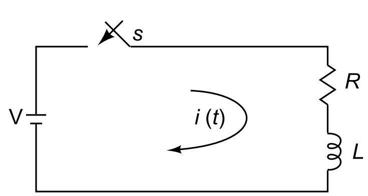


Fig. 7.24

When the switch is closed at $t = 0$, the voltage V is applied to the circuit.

By applying Kirchhoff's laws, we get

$$Ri(t) + L\frac{di}{dt} = V \tag{7.48}$$

Now, application of Laplace transform to each term gives,

$$RI(s) + L[sI(s) - i(0)] = \frac{V}{s}$$

$$RI(s) + sL\,I(s) - Li(0) = \frac{V}{s} \tag{7.49}$$

$i(0)$ is the current passing through the circuit just before the switch is closed. When $i(0) = 0$, Eq. 7.49, becomes

$$RI(s) + sLI(s) = \frac{V}{s}$$

$$I(s) = \frac{V/L}{s\left(s + \frac{R}{L}\right)}$$

The current $i(t)$ can be determined by taking inverse Laplace transform.

$$i(t) = \mathscr{L}^{-1}[I(s)] = \frac{V}{L}\mathscr{L}^{-1}\left[\frac{1}{s(s + R/L)}\right]$$

To find the constants, let

$$\frac{1}{s(s + R/L)} = \frac{A}{s} + \frac{B}{s + R/L}$$

$$A = \frac{1}{s(s + R/L)} \times s\Big|_{s=0} = \frac{L}{R}$$

$$B = \frac{1}{s(s + R/L)} \times \left(s + \frac{R}{L}\right)\Big|_{s=-R/L} = \frac{-L}{R}$$

$$\therefore \qquad i(t) = \mathscr{L}^{-1}[I(s)] = \frac{V}{L}\mathscr{L}^{-1}\left[\frac{L}{Rs} - \frac{L}{R(s + R/L)}\right]$$

$$= \frac{V}{L} \times \left[\frac{L}{R}(1) - \frac{L}{R}e^{-(R/L)t}\right]$$

$$= \frac{V}{L} \times \frac{L}{R}\,[1 - e^{-(R/L)t}]$$

Current

$$i(t) = \frac{V}{L}\,[1 - e^{-(R/L)t}]$$

Example 7.25 *In the circuit shown in Fig. 7.25, determine the current i(t) when the switch is changed from position 1 to 2. The switch is moved from position 1 to 2 at time t = 0.*

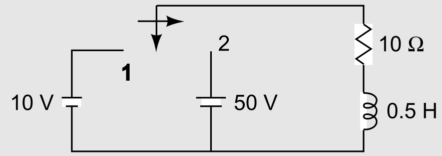


Fig. 7.25

Solution

When the switch is at position 2, application of Kirchhoff's law gives

$$10i(t) + 0.5\frac{di}{dt} = 50 \tag{7.50}$$

Taking Laplace transforms on both sides

$$10I(s) + 0.5[sI(s) - i(0)] = \frac{50}{s} \tag{7.51}$$

Where $i(0)$ is the current passing through *RL* circuit when switch is at position 1.

Therefore, the initial current is 10/10 = 1 A

$$i(0) = 1 \text{ A}$$

Then Eq. 7.51, becomes

$$10I(s) + 0.5[sI(s) - 1] = \frac{50}{s}$$

$$I(s)[10 + 0.5s] - 0.5 = \frac{50}{s}$$

$$I(s) = \frac{50/s + 0.5}{10 + 0.5s} = \frac{0.5}{s}\frac{(s+100)}{0.5(s+20)}$$

$$= \frac{s+100}{s(s+20)}$$

$$i(t) = \mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}\left[\frac{s+100}{s(s+20)}\right]$$

$$\frac{s+100}{s(s+20)} = \frac{A}{s} + \frac{B}{s+20}$$

$$A + B = 1$$

$$20A = 100$$
$$A = 5, B = -4$$
$$i(t) = \mathscr{L}^{-1}\left[\frac{5}{8}\right] + \mathscr{L}^{-1}\left[\frac{-4}{s+20}\right]$$
$$i(t) = 5 - 4e^{-20t}$$

Example 7.26 *In the circuit shown in Fig. 7.26, obtain the equations for $i_1(t)$ and $i_2(t)$ when the switch is closed at t = 0.*

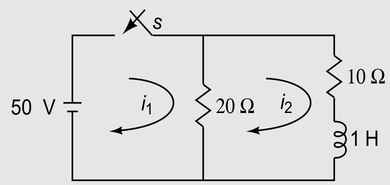


Fig. 7.26

Solution

When the switch is closed, 50 V source is applied to the circuit. By applying Kirchhoff's law, we have

$$20i_1(t) - 20i_2(t) = 50 \tag{7.52}$$
$$30i_2(t) + 1\frac{di_2}{dt} - 20i_1(t) = 0 \tag{7.53}$$

Taking Laplace transform on both sides, we get

$$20I_1(s) - 20I_2(s) = \frac{50}{s}$$
$$-20I_1(s) + (30+s)\,I_2(s) = i_2(0)$$

Since the current passing through the inductance just after the switch closed is zero, $i_2(0) = 0$

$$\begin{bmatrix} 20 & -20 \\ -20 & (30+s) \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{50}{s} \\ 0 \end{bmatrix}$$

$$I_1(s) = \frac{\begin{vmatrix} \dfrac{50}{s} & -20 \\ 0 & (30+s) \end{vmatrix}}{\begin{vmatrix} 20 & -20 \\ -20 & 30+s \end{vmatrix}} = \frac{50/s\,(30+s)}{20(s+10)}$$

$$= \frac{2.5(s+30)}{s(s+10)}$$

$$I_2(s) = \frac{\begin{vmatrix} 20 & \dfrac{50}{s} \\ -20 & 0 \end{vmatrix}}{\begin{vmatrix} 20 & -20 \\ -20 & 30+s \end{vmatrix}} = \frac{\dfrac{50}{s} \times 20}{20(s+10)}$$

$$= \frac{50}{s(s+10)}$$

Taking partial fractions, we get

$$I_1(s) = \frac{2.5(s+30)}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$= \frac{+7.5}{s} - \frac{5}{s+10}$$

Taking inverse transform, we get

$$i_1(t) = \mathscr{L}[I_1(s)] = \mathscr{L}^{-1}\left[\frac{+7.5}{s}\right] - \mathscr{L}^{-1}\left[\frac{5}{s+10}\right]$$

$$i_1(t) = +7.5 - 5e^{-10t}$$

Similarly, $$I_2(s) = \frac{50}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$I_2(s) = \frac{+5}{s} - \frac{5}{s+10}$$

Taking inverse transform, we have

$$i_2(t) = \mathscr{L}^{-1}[I_2(s)] = \mathscr{L}^{-1}\left(\frac{+5}{s}\right) - \mathscr{L}^{-1}\left(\frac{5}{s+10}\right)$$

$$i_2(t) = +5 - 5e^{-10t}$$

Solved Problems

7.1 For the circuit shown in Fig. 7.27, find the current equation when the switch is changed from position 1 to position 2 at $t = 0$.

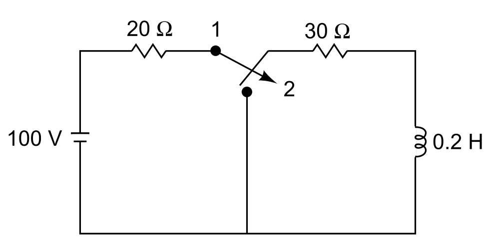


Fig. 7.27

Solution

When the switch is at position 2, the current equation can be written by using Kirchhoff's voltage law as

$$30i(t) + 0.2\frac{di(t)}{dt} = 0$$

$$\left(D + \frac{30}{0.2}\right)i = 0$$

$$(D + 150)i = 0$$

$$\therefore \qquad i = c_1 e^{-150t}$$

At $t = 0$, the switch is changed to position 2, i.e. $i(0) = c_1$.

At $t = 0$, the initial current passing through the circuit is the same as the current passing through the circuit when the switch is at position 1. At $t = 0^-$, the switch is at position 1, and the current passing through the circuit $i = 100/50 = 2$ A.

At $t = 0^+$, the switch is at position 2. Since the inductor does not allow sudden changes in current, the same current passes through the circuit. Hence the initial current passing through the circuit, when the switch is at position 2 is $i(0^+) = 2$A.

$$\therefore \qquad c_1 = 2 \text{ A}$$

Therefore, the current $\;i = 2e^{-150t}$

7.2 For the circuit shown in Fig. 7.28, find the current equation when the switch is opened at $t = 0$.

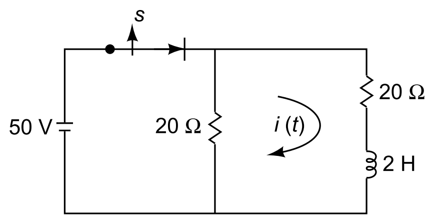


Fig. 7.28

Solution

At $t = 0$, switch S is opened. By using Kirchhoff's voltage law, the current equation can be written as

$$20i + 20i + 2\frac{di}{dt} = 0$$

$$40i + 2\frac{di}{dt} = 0$$

$$\therefore \qquad D + 20i = 0$$

The solution for the above equation is

$$i = c_1 e^{-20t}$$

When the switch has been closed for a time, since the inductor acts as short circuit for dc voltages, the current passing through the inductor is 2.5 A.

That means, just before the switch is opened, the current passing through the inductor is 2.5 A. Since the current in the inductor cannot change instantaneously, $i(0^+)$ is also equal to 2.5 A.

At $t = 0$ $$c_1 = i(0^+) = 2.5$$

Therefore, the final solution is $i(t) = 2.5e^{-20t}$

7.3 For the circuit shown in Fig. 7.29, find the current equation when the switch is opened at $t = 0$.

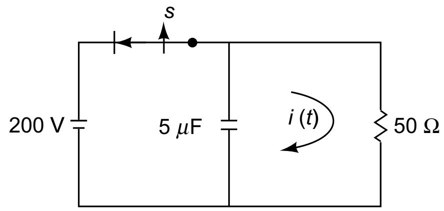


Fig. 7.29

Solution

By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{5 \times 10^{-6}} \int i dt + 50i = 0$$

Differentiating the above equation once, we get

$$50 \frac{di}{dt} + \frac{1}{5 \times 10^{-6}} i = 0$$

$$\therefore \quad \left(D + \frac{1}{250 \times 10^{-6}} \right) i = 0$$

$$\therefore \quad i = c_1 \exp \left(\frac{-1}{250 \times 10^{-6}} t \right) \tag{7.54}$$

At $t = 0^-$, just before the switch S is opened, the voltage across the capacitor is 200 V. Since the voltage across the capacitor cannot change instantly, it remains equal to 200 V at $t = 0^+$. At that instant, the current through the resistor is

$$i(0^+) = \frac{200}{50} = 4 \text{ A}$$

In Eq. 7.54, the current is $i(0^+)$ at $t = 0$

$\therefore \qquad c_1 = 4$ A

Therefore, the current equation is

$$i = 4 \exp\left(\frac{-1}{250 \times 10^{-6}} t\right) \text{A}$$

7.4 For the circuit shown in Fig. 7.30, find the current equation when the switch S is opened at $t = 0$.

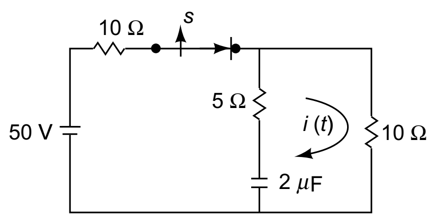


Fig. 7.30

Solution

By using Kirchhoff's voltage law, the current equation is given by

$$\frac{1}{2 \times 10^{-6}} \int i dt + 5i + 10i = 0$$

Differentiating the above equation, we have

$$15 \frac{di}{dt} + \frac{i}{2 \times 10^{-6}} = 0$$

$$\left(D + \frac{1}{30 \times 10^{-6}}\right) i = 0$$

$$\therefore \qquad i = c_1 \exp\left(\frac{-1}{30 \times 10^{-6}}\right) t$$

At $t = 0^-$, just before switch S is opened, the current through 10 ohms resistor is 2.5 A. The same current passes through 10 Ω at $t = 0^+$

$\therefore \qquad i(0^+) = 2.5$ A

At $t = 0 \qquad i(0^+) = 2.5$ A

$\therefore \qquad c_1 = 2.5$

The complete solution is $i = 2.5 \exp\left(\dfrac{-1}{30 \times 10^{-6}} t\right)$

7.5 For the circuit shown in Fig. 7.31, find the complete expression for the current when the switch is closed at $t = 0$.

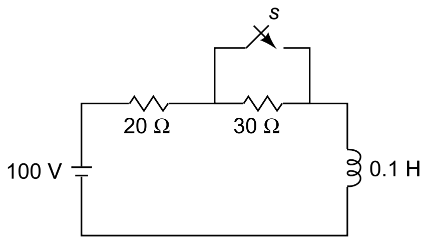


Fig. 7.31

Solution

By using Kirchhoff's law, the differential equation when the switch is closed at $t = 0$ is given by

$$20i + 0.1\frac{di}{dt} = 100$$

$$(D + 200)i = 1000$$

$$i = c_1 e^{-200t} + e^{-200t}\int 1000e^{200t}\,dt$$

$$\therefore \qquad i = c_1\, e^{-200t} + 5$$

At $t = 0^-$, the current passing through the circuit is $i(0^-) = \dfrac{100}{50} = 2$ A. Since, the inductor does not allow sudden changes in currents, at $t = 0^+$, the same current passes through circuit.

$$\therefore \qquad i(0^+) = 2 \text{ A}$$

At $t = 0$ $\qquad i(0^+) = 2$

$$\therefore \qquad c_1 = -3$$

The complete solution is $i = -3e^{-200t} + 5$ A

7.6 The circuit shown in Fig. 7.32, consists of series RL elements with $R = 150\ \Omega$ and $L = 0.5$ H. The switch is closed when $\phi = 30°$. Determine the resultant current when voltage $V = 50\cos(100t + \phi)$ is applied to the circuit at $\phi = 30°$.

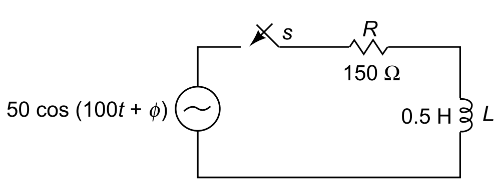


Fig. 7.32

Solution

By using Kirchhoff's laws, the differential equation, when the switch is closed at $\phi = 30°$ is

$$150i + 0.5\frac{di}{dt} = 50\cos(100t + \phi)$$
$$0.5Di + 150i = 50\cos(100t + 30°)$$
$$(D + 300)i = 100\cos(100t + 30°)$$

The complementary current $i_c = ce^{-300t}$

To determine the particular current, first we assume a particular current

$$i_p = A\cos(100t + 30°) + B\sin(100t + 30°)$$

Then $$i'_p = -100A\sin(100t + 30°) + 100B\cos(100t + 30°)$$

Substituting i_p and i'_p in the differential equation and equating the coefficients, we get

$$-100A\sin(100t + 30°) + 100B\cos(100t + 30°) + 300A\cos(100t + 30°) + 300B\sin(100t + 30°) = 100\cos(100t + 30°)$$
$$-100A + 300B = 0$$
$$300A + 100B = 100$$

From the above equation, we get

$$A = 0.3 \text{ and } B = 0.1$$

The particular current is

$$i_p = 0.3\cos(100t + 30°) + 0.1\sin(100t + 30°)$$

$\therefore$ $$i_p = 0.316\cos(100t + 11.57°)\text{ A}$$

The complete equation for the current is $i = i_p + i_c$

$\therefore$ $$i = ce^{-300t} + 0.316\cos(100t + 11.57°)$$

At $t = 0$, the current $i_0 = 0$

$\therefore$ $$c = -0.316\cos(11.57°) = -0.309$$

Therefore, the complete solution for the current is

$$i = -0.309e^{-300t} + 0.316\cos(100t + 11.57°)\text{ A}$$

7.7 The circuit shown in Fig. 7.33, consists of series RC elements with $R = 15\ \Omega$ and $C = 100\ \mu$F. A sinusoidal voltage $v = 100\sin(500t + \phi)$ volts is applied to the circuit at time corresponding to $\phi = 45°$. Obtain the current transient.

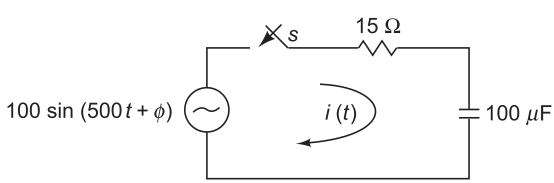


Fig. 7.33

Solution

By using Kirchhoff's laws, the differential equation is

$$15i + \frac{1}{100 \times 10^{-6}} \int i dt = 100 \sin (500t + \phi)$$

Differentiating once, we have

$$15\frac{di}{dt} + \frac{1}{100 \times 10^{-6}} i = (100)(500) \cos (500t + \phi)$$

$$\left(D + \frac{1}{1500 \times 10^{-6}}\right) i = 3333.3 \cos (500t + \phi)$$

$$(D + 666.67)i = 3333.3 \cos (500t + \phi)$$

The complementary function $i_c = ce^{-666.67t}$

To determine the particular current, first we assume a particular current

$$i_p = A \cos (500t + 45°) + B \sin (500t + 45°)$$
$$i'_p = -500 A \sin (500t + 45°) + 500 B \cos (500t + 45°)$$

Substituting i_p and i'_p in the differential equation, we get

$$-500A \sin (500t + 45°) + 500B \cos (500t + 45°) + 666.67A \cos (500t + 45°) + 666.67B \sin (500t + 45°) = 3333.3 \cos (500t + \phi)$$

By equating coefficients, we get

$$500B + 666.67A = 3333.3$$
$$666.67B - 500A = 0$$

From which, the coefficients

$$A = 3.2; B = 2.4$$

Therefore, the particular current is

$$i_p = 3.2 \cos (500t + 45°) + 2.4 \sin (500t + 45°)$$
$$i_p = 4 \sin (500t + 98.13°)$$

The complete equation for the current is

$$i = i_c + i_p$$
$$i = ce^{-666.67t} + 4 \sin (500t + 98.13°)$$

At $t = 0$, the differential equation becomes

$$15i = 100 \sin 45°$$

$$i = \frac{100}{15} \sin 45° = 4.71 \text{ A}$$

$\therefore$ At $t = 0$

$$4.71 = c + 4 \sin (98.13°)$$

$\therefore$ $$c = 0.75$$

The complete current is

$$i = 0.75\, e^{-666.67t} + 4 \sin (500t + 98.13°)$$

7.8 The circuit shown in Fig. 7.34 consisting of series RLC elements with $R = 10\ \Omega$, $L = 0.5$ H and $C = 200\ \mu$F has a sinusoidal voltage $v = 150 \sin (200t + \phi)$. If the switch is closed when $\phi = 30°$, determine the current equation.

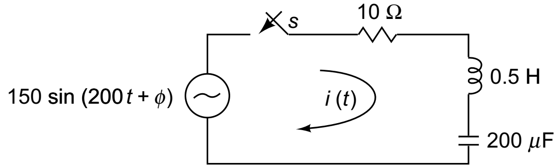


Fig. 7.34

Solution

By using Kirchhoff's laws, the differential equation is

$$10i + 0.5\frac{di}{dt} + \frac{1}{200 \times 10^{-6}} \int i\,dt = 150 \sin (200t + \phi)$$

Differentiating once, we have

$$(D^2 + 20D + 10^4)i = 60000 \cos (200t + \phi)$$

The roots of the characteristics equation are

$$D_1 = -10 + j99.49 \text{ and } D_2 = -10 - j99.49$$

The complementary function is

$$i_c = e^{-10t} (c_1 \cos 99.49t + c_2 \sin 99.49)$$

We can find the particular current by using the undetermined coefficient method.

Let us assume

$$i_p = A \cos (200t + 30°) + B \sin (200t + 30°)$$
$$i'_p = -200\, A \sin (200t + 30°) + 200\, B \cos (200t + 30°)$$
$$i''_p = -(200)^2 A \cos (200t + 30°) - (200)^2 B \sin (220t + 30°)$$

Substituting these values in the equation, and equating the coefficients, we get

$$A = 0.1 \quad B = 0.067$$

Therefore, the particular current is

$$i_p = 1.98 \cos (200t - 52.4°) \text{ A}$$

The complete current is

$$i = e^{-10t} (c_1 \cos 99.49t + c_2 \sin 99.49t) + 1.98 \cos (200t - 52.4°) \text{ A}$$

From the differential equation at $t = 0$, $i_0 = 0$ and $\dfrac{di}{dt} = 300$

$\therefore$ At $t = 0$

$$c_1 = -1.98 \cos(-52.4°) = -1.21$$

Differentiating the current equation, we have

$$\frac{di}{dt} = e^{-10t}(-99.49c_1 \sin 99.49t + 99.49c_2 \cos 99.49t)$$
$$-200\,(1.98) \sin(200t - 52.4°) - 10e^{-10t}(c_1 \cos 99.49t + c_2 \sin 99.49t)$$

At $t = 0$, $\dfrac{di}{dt} = 300$ and $c_1 = -1.21$

$$300 = 99.49\, c_2 - 396 \sin(-52.4°) - 10\,(-1.21)$$
$$300 = 99.49\, c_2 + 313.7 + 12.1$$
$$c_2 = -25.8$$

Therefore, the complete current equation is

$$i = e^{-10t}(0.07 \cos 99.49t - 25.8 \sin 99.49t) + 1.98 \cos(200t - 52.4°) \text{ A}$$

7.9 For the circuit shown in Fig. 7.35, determine the transient current when the switch is moved from position 1 to position 2 at $t = 0$. The circuit is in steady state with the switch in position 1. The voltage applied to the circuit is $v = 150 \cos(200t + 30°)$ V.

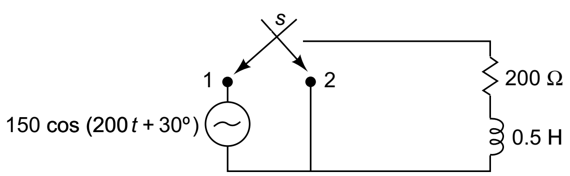


Fig. 7.35

Solution

When the switch is at position 2, by applying Kirchhoff's law, the differential equation is

$$200i + 0.5\frac{di}{dt} = 0$$
$$(D + 400)i = 0$$

$\therefore$ The transient current is

$$i = ce^{-400t}$$

At $t = 0$, the switch is moved from position 1 to position 2. Hence the current passing through the circuit is the same as the steady state current passing through the circuit when the switch is in position 1.

When the switch is in position 1, the current passing through the circuit is

$$i = \frac{v}{z} = \frac{150\angle 30°}{R + j\omega L}$$

$$= \frac{150 \angle 30°}{200 + j(200)(0.5)} = \frac{150 \angle 30°}{223.6 \angle 26.56°} = 0.67 \angle 3.44°$$

Therefore, the steady state current passing through the circuit when the switch is in position 1 is

$$i = 0.67 \cos(200t + 3.44°)$$

Now substituting this equation in transient current equation, we get

$$0.67 \cos(200t + 3.44°) = ce^{-400t}$$

At $t = 0$; $c = 0.67 \cos(3.44°) = 0.66$

Therefore, the current equation is $i = 0.66e^{-400t}$

7.10 In the circuit shown in Fig. 7.36, determine the current equations for i_1 and i_2 when the switch is closed at $t = 0$.

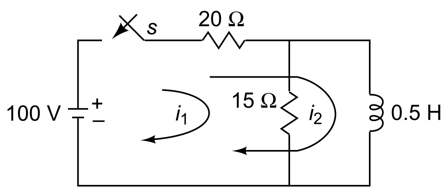


Fig. 7.36

Solution

By applying Kirchhoff's laws, we get two equations

$$35i_1 + 20i_2 = 100 \tag{7.55}$$

$$20i_1 + 20i_2 + 0.5\frac{di_2}{dt} = 100 \tag{7.56}$$

From Eq. 7.55, we have

$$35i_1 = 100 - 20i_2$$

$$i_1 = \frac{100}{35} - \frac{20}{35}i_2$$

Substituting i_1 in Eq. 7.56, we get

$$20\left(\frac{100}{35} - \frac{20}{35}i_2\right) + 20i_2 + 0.5\frac{di_2}{dt} = 100 \tag{7.57}$$

$$57.14 - 11.43i_2 + 20i_2 + 0.5\frac{di_2}{dt} = 100$$

$$(D + 17.14)i_2 = 85.72$$

From the above equation,

$$i_2 = ce^{-17.14t} + 5$$

Loop current i_2 passes through inductor and must be zero at $t = 0$

At $t = 0$, $i_2 = 0$

$\therefore \quad c = -5$

$\therefore \quad i_2 = 5(1 - e^{-17.14t})$ A

and the current $\quad i_1 = 2.86 - \{0.57 \times 5(1 - e^{-17.14t})\}$

$= (0.01 + 2.85\, e^{-17.14t})$ A

7.11 For the circuit shown in Fig. 7.37, find the current equation when the switch is changed from position 1 to position 2 at $t = 0$.

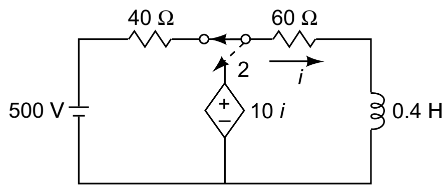


Fig. 7.37

Solution

By using Kirchhoff's voltage law, the current equation is given by

$$60i + 0.4\frac{di}{dt} = 10i$$

At $t = 0^-$, the switch is at position 1, the current passing through the circuit is

$$i(0^-) = \frac{500}{100} = 5 \text{ A}$$

$$0.4\frac{di}{dt} + 50i = 0$$

$$\left(D + \frac{50}{0.4}\right)i = 0$$

$$i = ce^{-125t}$$

At $t = 0$, the initial current passing through the circuit is same as the current passing through the circuit when the switch is at position 1.

At $\quad t = 0,\ i(0) = i(0^-) = 5$ A

At $\quad t = 0,\ c = 5$ A

$\therefore$ The current $\quad I = 5e^{-125t}$

7.12 For the circuit shown in Fig. 7.38, find the current equation when the switch S is opened at $t = 0$.

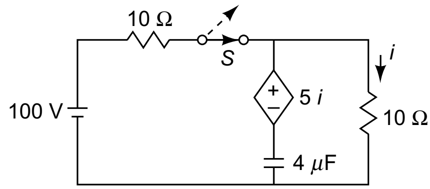


Fig. 7.38

Solution

When the switch is closed for a long time,

At $t = 0^-$, the current $i(0^-) = \dfrac{100}{20} = 5$ A

When the switch is opened at $t = 0$, the current equation by using Kirchhoff's voltage law is given by

$$\frac{1}{4\times10^{-6}}\int i\,dt + 10i = 5i$$

$$\frac{1}{4\times10^{-6}}\int i\,dt + 5i = 0$$

Differentiating the above equation

$$5\frac{di}{dt} + \frac{1}{4\times10^{-6}}i = 0$$

$$\left(D + \frac{1}{20\times10^{-6}}\right)i = 0$$

$\therefore$ $$i = ce^{\frac{-1}{20\times10^{-6}}t}$$

At $t = 0^-$, just before switch S is opened, the current passing through the 10 Ω resistor is 5 A. The same current passes through 10 Ω at $t = 0$.

$\therefore$ At $t = 0$, $i(0) = 5\ A$

At $t = 0$, $c_1 = 5\ A$

The current equation is $$i = 5e^{\frac{-t}{20\times10^{-6}}}$$

7.13 For the circuit shown in Fig. 7.39, find the current in the 20 Ω when the switch is opened at $t = 0$.

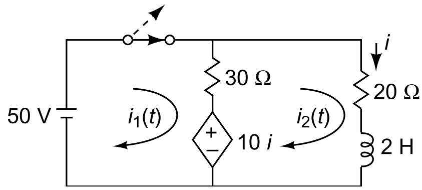


Fig. 7.39

Solution

When the switch is closed, the loop current i_1 and i_2 are flowing in the circuit.

The loop equations are $30(i_1 - i_2) + 10i_2 = 50$

$$30(i_2 - i_1) + 20i_2 = 10i_2$$

From the above equations, the current in the 20 Ω resistor $i_2 = 2.5$ A.

The same initial current is flowing when the switch is opened at $t = 0$.

When the switch is opened the current equations

$$30i + 20i + 2\frac{di}{dt} = 10i$$

$$40i + \frac{2di}{dt} = 0$$

$$(D + 20)i = 0$$

$$i = ce^{-20t}$$

At $t = 0$, the current $i(0) = 2.5$ A

$\therefore$ At $t = 0$, $c = 2.5$

The current in the 20 Ω resistor is $i = 2.5\, e^{-20t}$.

7.14 For the circuit shown in Fig. 7.40, find the current equation when the switch is opened at $t = 0$.

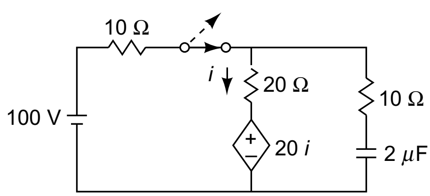


Fig. 7.40

Solution

When the switch is closed, the current in the 20 Ω resistor i can be obtained using Kirchhoff's voltage law.

$$10i + 20i + 20i = 100$$
$$50i = 100, \therefore\ i = 2\text{ A}$$

The same initial current passes through the 20 Ω resistor when the switch is opened at $t = 0$.

The current equation is

$$20i + 10i + \frac{1}{2\times 10^{-6}}\int i dt = 20i$$
$$10i + \frac{1}{2\times 10^{-6}}\int i dt = 0$$

Differentiating the above equation, we get

$$10\frac{di}{dt} + \frac{1}{2\times 10^{-6}} i = 0$$
$$\left(D + \frac{1}{20\times 10^{-6}}\right) i = 0$$

The solution for the above equation is

$$i = ce^{\frac{-1}{20\times 10^{-6}}t}$$

At $t = 0$, $i(0) = i(0^-) = 2$ A

$\therefore$ At $t = 0$, $c = 2$ A

The current equation is

$$i = 2e^{\frac{-1}{20\times 10^{-6}}t}$$

7.15 For the waveform shown in Fig. 7.41, find the Laplace transform.

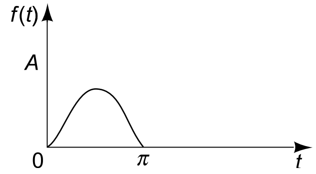


Fig. 7.41

Solution

The function for the waveform shown in Fig. 7.41 is

$$f(t) = A \sin t \qquad \text{for } 0 < t < \pi$$
$$= 0 \qquad t > \pi$$

By definition, we have

$$\mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt$$

$$= \int_0^{\pi} f(t)\, e^{-st}\,dt + \int_{\pi}^{\infty} f(t)\, e^{-st}\,dt$$

Since $f(t) = 0$ for $t > \pi$, the second term becomes zero

$$\therefore \qquad \mathscr{L}[f(t)] = \int_0^{\pi} f(t)e^{-st}\,dt$$

$$= \int_0^{\pi} A \sin t\, e^{-st}\,dt$$

$$= A\frac{e^{-st}}{(s^2+1)}\,[-s \sin t - \cos t]_0^{\pi}$$

$$= 2A\,\frac{e^{-s\pi}-1}{(s^2+1)}$$

7.16 Find the Laplace transform of

$$\begin{aligned} f(t) &= t \text{ for } 0 < t < 1 \\ &= 0 \text{ for } t > 1 \end{aligned}$$

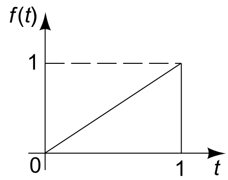


Fig. 7.42

Solution

By definition,

$$\mathscr{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)dt$$

$$= \int_0^{1} f(t)e^{-st}\,dt + \int_1^{\infty} f(t)\, e^{-st}\,dt$$

Since $f(t) = 0$ for $t > 1$, the second term becomes zero

$$\mathscr{L}[f(t)] = \int_0^{1} f(t)e^{-st}\,dt$$

$$= \int_0^1 te^{-st}\,dt$$

$$= t\int_0^1 e^{-st}\,dt - \int_0^1 \frac{e^{-st}}{-s}\,dt$$

$$= t\left(\frac{e^{-st}}{-s}\right)_0^1 - \left(\frac{e^{-st}}{s^2}\right)_0^1$$

$$= \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} - e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right]$$

7.17 Verify the initial and final value theorems for the function

$$f(t) = e^{-t}(\sin 3t + \cos 5t).$$

Solution

$$f(t) = e^{-t}(\sin 3t + \cos 5t)$$
$$F(s) = \mathscr{L}[f(t)] = \mathscr{L}[e^{-t}(\sin 3t + \cos 5t)]$$

Since $\qquad \mathscr{L}(e^{-t}\sin 3t) = \dfrac{3}{(s+1)^2 + 3^2}$

and $\qquad \mathscr{L}(e^{-t}\cos 5t) = \dfrac{s+1}{(s+1)^2 + 5^2}$

$\therefore \qquad F(s) = \mathscr{L}[f(t)] = \dfrac{3}{(s+1)^2 + 3^2} + \dfrac{s+1}{(s+1)^2 + 5^2}$

According to the initial value theorem,

$$\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ sF(s)$$

$$F(s) = \frac{3}{s^2 + 2s + 10} + \frac{s+1}{s^2 + 2s + 26}$$

$$sF(s) = \frac{3s}{s^2\left(1 + \frac{2}{s} + \frac{10}{s^2}\right)} + \frac{s^2 + s}{s^2\left(1 + \frac{2}{s} + \frac{26}{s^2}\right)}$$

$$= \frac{3}{s\left(1 + \frac{2}{s} + \frac{10}{s^2}\right)} + \frac{1}{1 + \frac{2}{s} + \frac{26}{s^2}} + \frac{1}{s\left(1 + \frac{2}{s} + \frac{26}{s^2}\right)}$$

$$\underset{s\to\infty}{\text{Lt}}\ sF(s) = 1$$

$$f(t) = e^{-t}(\sin 3t + \cos 5t)$$

$$\underset{t \to 0}{\text{Lt}}\ f(t) = 1$$

Hence the initial value theorem is proved. According to the final value theorem,

$$\underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{s \to 0}{\text{Lt}}\ sF(s)$$

$$\underset{s \to 0}{\text{Lt}}\ sF(s) = 0$$

$$\underset{t \to \infty}{\text{Lt}}\ f(t) = 0$$

Hence the final value theorem is proved.

7.18 Determine the inverse Laplace transform of the function

$$F(s) = \frac{s-3}{s^2+4s+13}$$

Solution

$$F(s) = \frac{s-3}{s^2+4s+13} = \frac{s-3}{(s+2)^2+9} = \frac{(s+2)-5}{(s+2)^2+9}$$

We can write the above equation as

$$\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

By taking the inverse Laplace transforms, we get

$$\mathscr{L}^{-1}F(s) = \mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - \mathscr{L}^{-1}\left[\frac{5}{(s+2)^2+9}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}\,e^{-2t}\sin 3t = \frac{e^{-2t}}{3}\,[3\cos 3t - 5\sin 3t]$$

7.19 Find the inverse transform of the following

(a) $\log\left(\dfrac{s+5}{s+6}\right)$ (b) $\dfrac{1}{(s^2+5^2)^2}$

Solution

(a) Let $F(s) = \log\left(\dfrac{s+5}{s+6}\right)$

Then $$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\log\left(\frac{s+5}{s+6}\right)\right] = \frac{1}{s+5} - \frac{1}{s+6}$$

We know that $\mathscr{L}^{-1}\left[\dfrac{d}{ds}F(s)\right] = -t f(t)$

$\therefore$ $\mathscr{L}^{-1}\left[\dfrac{d}{ds}F(s)\right] = \mathscr{L}^{-1}\left[\dfrac{1}{s+5} - \dfrac{1}{s+6}\right] = e^{-5t} - e^{-6t}$

Hence $-tf(t) = e^{-5t} - e^{-6t}$

$$f(t) = \frac{e^{-6t} - e^{-5t}}{t}$$

(b) Let $F(s) = \dfrac{1}{\left(s^2+5^2\right)^2}$

$$\frac{1}{\left(s^2+5^2\right)^2} = \frac{1}{s}\,\frac{s}{\left(s^2+5^2\right)^2}$$

Therefore $\mathscr{L}^{-1}\left[\dfrac{1}{\left(s^2+5^2\right)^2}\right] = \mathscr{L}^{-1}\left[\dfrac{1}{s}\,\dfrac{s}{\left(s^2+5^2\right)^2}\right]$

According to the integration theorem,

$$\mathscr{L}^{-1}\left[\frac{1}{s}\,\frac{s}{\left(s^2+5^2\right)^2}\right] = \int_0^t \left[\mathscr{L}^{-1}\frac{s}{\left(s^2+5^2\right)^2}\right] dt$$

If $\mathscr{L}[f(t)] = F(s)$, then $\mathscr{L}\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty F(s)\,ds$

Here $\displaystyle\int_s^\infty \frac{s}{\left(s^2+5^2\right)^2}\,ds = \frac{-1}{2}\left[\frac{1}{s^2+5^2}\right]_s^\infty = \frac{1}{2}\,\frac{1}{s^2+5^2}$

Therefore $\dfrac{f(t)}{t} = \mathscr{L}^{-1}\left(\dfrac{1}{2}\cdot\dfrac{1}{s^2+5^2}\right) = \dfrac{1}{10}\sin 5t$

$\therefore$ $f(t) = \dfrac{t\sin 5t}{10}$

or $\mathscr{L}^{-1}\left[\dfrac{1}{s}\,\dfrac{s}{\left(s^2+5^2\right)^2}\right] = \displaystyle\int_0^t \frac{t\sin 5t}{10}\,dt$

$$= \frac{1}{10}\left[t\left(\frac{-\cos 5t}{5}\right) + \frac{\sin 5t}{25}\right]_0^t$$

$$= \frac{1}{250}\,[\sin 5t - 5t\cos 5t]$$

7.20 Find the Laplace transform of the full wave rectified output as shown in Fig. 7.43.

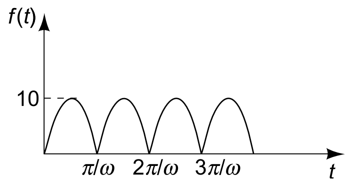


Fig. 7.43

Solution

We have

$$f(t) = 10 \sin \omega t \text{ for } 0 < t < \frac{\pi}{\omega}$$

Hence
$$\mathscr{L}[f(t)] = \frac{\int_0^{\pi/\omega} \left(e^{-st} f(t)\right) dt}{1 - e^{-s\pi/\omega}}$$

$$= \frac{\int_0^{\pi/\omega} \left(e^{-st}\, 10 \sin \omega t\right) dt}{1 - e^{-s\pi/\omega}}$$

$$= \frac{10}{1 - e^{-s\pi/\omega}} \left[\frac{e^{-st}}{s^2 + \omega^2}\left(-s \sin \omega t - \omega \cos \omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{10}{\left(1 - e^{-s\pi/\omega}\right)\left(s^2 + \omega^2\right)} \; [\omega e^{-s\pi/\omega} + \omega]$$

$$= \frac{10\omega}{s^2 + \omega^2} \frac{\left(1 + e^{-s\pi/\omega}\right)}{\left(1 - e^{-s\pi/\omega}\right)}$$

$$= \frac{10\omega}{s^2 + \omega^2} \frac{e^{s\pi/2\omega} + e^{-s\pi/2\omega}}{e^{s\pi/2\omega} - e^{-s\pi/2\omega}}$$

$$= \frac{10\omega}{s^2 + \omega^2} \cos h\left(\frac{s\pi}{2\omega}\right)$$

7.21 Find the Laplace transform of the square wave shown in Fig. 7.44.

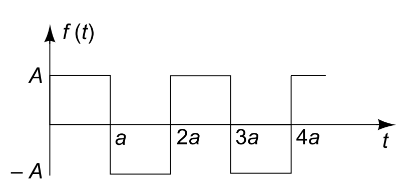


Fig. 7.44

Solution

We have

$$f(t) = A \quad 0 < t < a$$
$$= -A \quad a < t < 2a$$

$$\mathscr{L}[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a Ae^{-st}\,dt + \int_a^{2a}(-A)e^{-st}\,dt\right]$$

$$= \frac{A}{s}\frac{\left(1-2e^{-as}+e^{-2as}\right)}{1-e^{-2as}}$$

$$= \frac{A}{s}\frac{\left(1-e^{-as}\right)^2}{\left(1+e^{-as}\right)\left(1-e^{-as}\right)} = \frac{A}{s}\tanh\left(\frac{as}{2}\right)$$

7.22 Obtain the inverse transform of $F(s) = \dfrac{1}{s(s+2)}$ by using the convolution integral.

Solution

Let $\quad F_1(s) = \dfrac{1}{s}$ and $F_2(s) = \dfrac{1}{s+2}$

We have $\quad f_1(t) = \mathscr{L}^{-1}[F_1(s)] = \mathscr{L}^{-1}\left(\dfrac{1}{s}\right) = 1$

Similarly, $\quad f_2(t) = \mathscr{L}^{-1}[F_2(s)] = \mathscr{L}^{-1}\left(\dfrac{1}{s+2}\right) = e^{-2t}$

According to the convolution integral,

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)\, f_2(\tau)\,d\tau$$

Since $\quad f_1(t-\tau) = 1$ and $f_2(t) = e^{-2\tau}$

$\therefore$
$$f_1(t) * f_2(t) = \int_0^t 1\cdot e^{-2\tau}\,d\tau$$

$$= \left(\frac{e^{-2\tau}}{-2}\right)_0^t = \frac{e^{-2t}}{-2} + \frac{1}{2} = \frac{1}{2}\,[1-e^{-2t}]$$

$\therefore$
$$\mathscr{L}^{-1}\left[\frac{1}{s(s+2)}\right] = \frac{1}{2}\,[1-e^{-2t}]$$

7.23 Determine the convolution integral when $f_1(t) = e^{-2t}$ and $f_2(t) = 2t$.

Solution

We have

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

Then

$$f_1(t) * f_2(t) = \int_0^t 2\tau e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_0^t 2\tau e^{2\tau}\, d\tau$$

$$= 2e^{-2t}\left[\tau \frac{e^{2\tau}}{2} - \int 1 \cdot \frac{e^{2\tau}}{2}\, d\tau\right]_0^t$$

$$= 2e^{-2t}\left[\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right]$$

$$= \left[t - \frac{1}{2} + \frac{e^{-2t}}{2}\right]$$

7.24 The circuit shown in Fig. 7.45 consists of series R-L elements. The sine wave is applied to the circuit when the switch s is closed at $t = 0$. Determine the current $i(t)$.

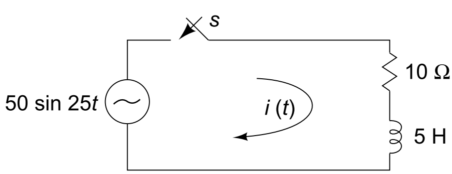


Fig. 7.45

Solution

In the circuit, the current $i(t)$ can be determined by using Kirchhoff's law.

$$5\frac{di}{dt} + 10i = 50 \sin 25t$$

Applying Laplace transform on both sides

$$5[sI(s) - i(0)] + 10I(s) = 50 \times \frac{25}{s^2 + (25)^2}$$

where $i(0)$ is the initial current passing through the circuit. Since the inductor does not allow sudden changes in currents, the current $i(0) = 0$.

$$\therefore \qquad 5sI(s) + 10I(s) = \frac{50 \times 25}{s^2 + (25)^2}$$

$$I(s) = \frac{1250}{(s^2 + 625)(5s + 10)} = \frac{250}{(s^2 + 625)(s + 2)}$$

By taking partial fractions, we have

$$I(s) = \frac{250}{(s+2)(s+j25)(s-j25)}$$

$$I(s) = \left[\frac{A}{s+2} + \frac{B}{s+j25} + \frac{C}{(s-j25)}\right]$$

where $A = (s+2)\, I(s)\,|_{s=-2}$

$$= (s+2)\,\frac{250}{(s+2)\left[s^2 + (25)^2\right]}\Bigg|_{s=-2}$$

$$= \frac{250}{629} = 0.397$$

$$B = (s+j25)\, I(s)\,|_{s=-j25}$$

$$= (s+j25)\,\frac{250}{(s+2)(s+j25)(s-j25)}\Bigg|_{s=-j25}$$

$$= \frac{250}{(2-j25)(-j50)} = \frac{-5}{(25+j2)}$$

$$C = (s-j25)\, I(s)|_{s=j25}$$

$$= (s-j25)\,\frac{250}{(s+2)(s+j25)(s-j25)}\Bigg|_{s=j25}$$

$$= \frac{250}{(2+j25)(j50)} = \frac{5}{(25-j2)}$$

Substituting the values of A, B, C in $I(s)$, we get

$$I(s) = \frac{0.397}{s+2} - \frac{5}{(25+j2)(s+j25)} + \frac{5}{(25-j2)(s-j25)}$$

By taking the inverse transform on both sides, we get

$$i(t) = 0.397\, e^{-2t} - \frac{5}{(25+j2)}\, e^{-j25t} + \frac{5}{(25-j2)}\, e^{j25t}$$

7.25 For the circuit shown in Fig. 7.46, determine the current $i(t)$ when the switch is at position 2. The switch s is moved from position 1 to position 2 at time $t = 0$. The switch has been in position 1 for a long time.

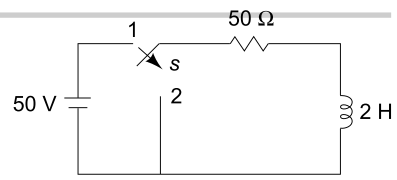


Fig. 7.46

Solution

When the switch s is at position 2, by applying Kirchhoff's voltage law, we get

$$2\frac{di}{dt} + 50i = 0$$

$$\frac{di}{dt} + 25i = 0$$

Taking Laplace transform on both sides

$$s\,I(s) - i(0) + 25\,I(s) = 0$$

where $i(0)$ is the initial current passing through circuit just after the switch is at position 2. Since the inductor does not allow sudden changes in currents, $i(0)$ is the same as the steady state current when the switch is at position 1.

$\therefore$ $$i(0) = \frac{50}{50} = 1\text{ A}$$

Hence $$s\,I(s) - 1 + 25\,I(s) = 0$$

$\therefore$ $$I(s) = \frac{1}{s+25}$$

By taking inverse transform of the above equation, we have the current

$$i(t) = e^{-25t}$$

7.26 For the circuit shown in Fig. 7.47, find the voltage across the 0.5 Ω resistor when the switch, s, is opened at $t = 0$. Assume there is no charge on the capacitor and no current in the inductor before switching.

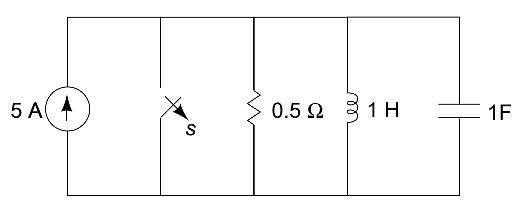


Fig. 7.47

Solution

By applying Kirchhoff's current law to the circuit, we have

$$2v + 1\int_{-\infty}^{t} v\,dt + \frac{dv}{dt} = 5$$

$$2v + 1\int_{-\infty}^{0} v\,dt + 1\int_{0}^{t} v\,dt + \frac{dv}{dt} = 5$$

Taking Laplace transforms on both sides, we get

$$2V(s) + \mathscr{L}\left[\int_{-\infty}^{0} v\,dt\right] + \frac{V(s)}{s} + [sV(s) - v(0)] = \frac{5}{s}$$

Since the initial voltage across the capacitor and the initial current in the inductor is zero, the above equation becomes

$$2V(s) + \frac{V(s)}{s} + sV(s) = \frac{5}{s}$$

$$V(s)\,[2s + s^2 + 1] = 5$$

$$V(s) = \frac{5}{s^2 + 2s + 1}$$

$$\therefore \qquad V(s) = \frac{5}{(s+1)^2}$$

Taking inverse transforms on both sides, we have

$$v(t) = +5te^{-t}$$

7.27 For the circuit shown in Fig. 7.48, determine the current in the 10 Ω resistor when the switch is closed at $t = 0$. Assume initial current through the inductor is zero.

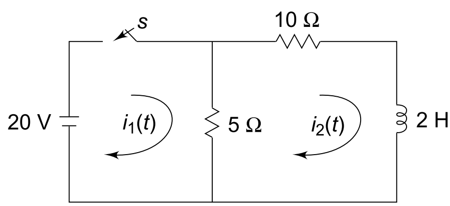


Fig. 7.48

Solution

By taking mesh currents when the switch is closed at $t = 0$, we have

$$20 = 5i_1(t) - 5i_2(t)$$

and $$-5i_1(t) + 15i_2(t) + 2\frac{di_2}{dt} = 0$$

Taking Laplace transforms on both sides, we have

$$5I_1(s) - 5I_2(s) = \frac{20}{s}$$

$$-5I_1(s) + 15I_2(s) + 2[sI_2(s) - i(0)] = 0$$

Since the initial current through the inductor is zero $i(0) = 0$

$$\therefore \quad \begin{bmatrix} 5 & -5 \\ -5 & (2s+15) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 20/s \\ 0 \end{bmatrix}$$

$$\therefore \quad I_2(s) = \frac{\begin{vmatrix} 5 & 20/s \\ -5 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & -5 \\ -5 & (2s+15) \end{vmatrix}} = \frac{20/s \times 5}{5(2s+15)-25}$$

$$I_2(s) = \frac{100}{5s[2s+10]}$$

Taking partial fractions, we get

$$\frac{10}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

Solving for the constants

$$A = \left.\frac{10}{s(s+5)}s\right|_{s=0} = 2$$

$$B = \left.\frac{10}{s(s+5)}(s+5)\right|_{s=-5} = -2$$

$$\therefore \quad I_2(s) = \frac{2}{s} - \frac{2}{s+5}$$

Taking inverse transform on both sides, we have

$$i_2(t) = 2 - 2e^{-5t}$$

Therefore, the current passing through the 10 Ω resistor is $(2 - 2e^{-5t})$ A

7.28 For the circuit shown in Fig. 7.49, determine the current when the switch is moved from position 1 to position 2 at $t = 0$. The switch has been in position 1 for a long time to get steady state values.

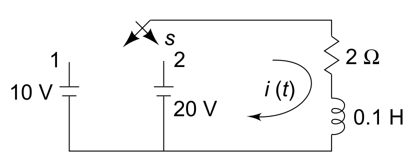


Fig. 7.49

Solution

When the switch is at position 2, by applying Kirchhoff's law, the current equation is

$$0.1\frac{di}{dt} + 2i = 20$$

Taking Laplace transform on both sides, we get

$$0.1[sI(s) - i(0)] + 2I(s) = \frac{20}{s}$$

$i(0)$ is the current passing through the circuit just after the switch is at position 2. Since the inductor does not allow sudden changes in currents, this current is equal to the steady state current when the switch was at position 1.

Therefore, $$i(0) = \frac{10}{2} = 5\text{ A}$$

Substituting $i(0)$, in the equation, we get

$$0.1[sI(s) - 5] + 2I(s) = \frac{20}{s}$$

$$I(s)[0.1s + 2] = \frac{20}{s} + 0.5$$

$$I(s) = \frac{5(s+40)}{s(s+20)}$$

By taking partial fractions, we have

$$\frac{5(s+40)}{s(s+20)} = \frac{A}{s} + \frac{B}{s+20}$$

$$A = \left.\frac{5(s+40)}{s(s+20)} \times s\right|_{s=0} = 10$$

$$B = \left.\frac{5(s+40)}{s(s+20)} \times (s+20)\right|_{s=-20} = -5$$

$\therefore$ $$I(s) = \frac{10}{s} - \frac{5}{s+20}$$

Taking inverse transforms on both sides, we have

$$i(t) = 10 - 5e^{-20t}\text{ A}$$

7.29 For the circuit shown in Fig. 7.50, determine the current when the switch is closed at a time corresponding to $\phi = 0$. Assume initial charge on the capacitor is $q_0 = 2$ coulombs with polarity shown.

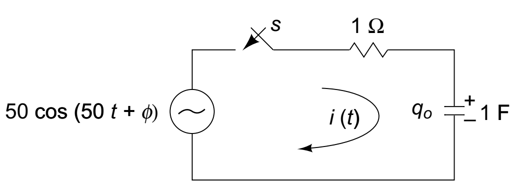


Fig. 7.50

Solution

By applying Kirchhoff's voltage law, we have

$$i(t) + \frac{1}{1}\int_{-\infty}^{t} i\,dt = 50\cos(50t)$$

$$i(t) + \int_{-\infty}^{0} \frac{dq}{dt}\,dt + \int_{0}^{t} i\,dt = 50\cos(50t)$$

Taking Laplace transforms on both sides, we have

$$I(s) + \frac{I(s)}{s} + \frac{q_0}{s} = \frac{50s}{s^2 + 50^2}$$

$$I(s)\left[1 + \frac{1}{s}\right] + \frac{2}{s} = \frac{50s}{s^2 + 50^2}$$

$$I(s) = \left[\frac{50s}{s^2 + 50^2} - \frac{2}{s}\right]\frac{s}{s+1}$$

$$= \frac{\left[50s^2 - 2s^2 - 2(50)^2\right]}{\left[s^2 + (50)^2\right]\left[s+1\right]}$$

$$= \frac{48s^2 - 2(50)^2}{\left[s^2 + (50)^2\right]\left[s+1\right]}$$

By taking partial fractions, we have

$$I(s) = \frac{A}{(s + j50)} + \frac{B}{(s - j50)} + \frac{C}{s+1}$$

$$A = I(s)(s + j50)|_{s=-j50}$$

$$= \left.\frac{48s^2 - 2(50)^2}{(s - j50)(s+1)}\right|_{s=-j50}$$

$$= \frac{1250}{(j + 50)}$$

Similarly, $B = I(s)(s - j50)\,|_{s=j50}$

$$= \left.\frac{48s^2 - 2(50)^2}{(s + j50)(s+1)}\right|_{s=j50} = \frac{1250}{50 - j}$$

and $C = I(s)(s + 1)|_{s=-1}$

$$= \left.\frac{48s^2 - 2(50)^2}{s^2 + (50)^2}\right|_{s=-1} = -1.98$$

Substituting the values of A, B, C, we get

$$I(s) = \frac{1250}{(50+j)(s+j50)} + \frac{1250}{(50-j)(s-j50)} - \frac{1.98}{s+1}$$

Taking inverse transforms

$$i(t) = \left[\frac{1250}{(50+j)} e^{-j50t} + \frac{1250}{50-j} e^{+j50t} - 1.98e^{-t}\right] \text{A}$$

7.30 For the circuit shown in Fig. 7.51, determine the current in the circuit when the switch is closed at $t = 0$. Assume that there is no initial charge on the capacitor or current in the inductor.

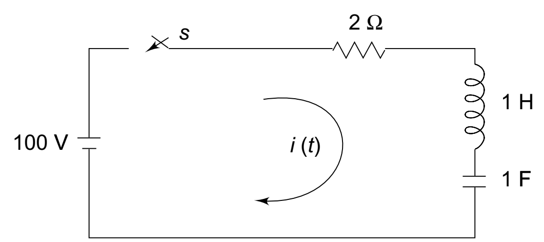


Fig. 7.51

Solution

When the switch is closed, by applying Kirchhoff's voltage law, we have

$$2i(t) + \frac{di}{dt} + 1\int i dt = 100$$

Taking Laplace transforms on both sides

$$2I(s) + [sI(s) - i(0)] + \frac{I(s)}{s} + \frac{q_0}{s} = \frac{100}{s}$$

Since the initial current in the inductor and initial charge on the capacitor is zero, the above equation reduces to

$$2I(s) + sI(s) + \frac{I(s)}{s} = \frac{100}{s}$$

$$I(s)\left[2 + s + \frac{1}{s}\right] = \frac{100}{s}$$

$$I(s) = \frac{100}{s^2 + 2s + 1}$$

$$\therefore \quad I(s) = \frac{100}{(s+1)^2}$$

Taking inverse transforms on both sides, we get

$$i(t) = 100\, te^{-t} \text{ A}$$

7.31 For the circuit shown in Fig. 7.52, determine the total current delivered by the source when the switch is closed at $t = 0$. Assume no initial charge on the capacitor.

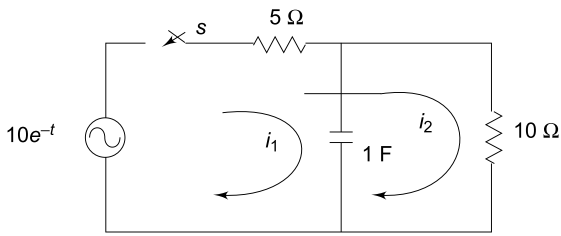


Fig. 7.52

Solution

By applying Kirchhoff's law, the two mesh equations are

$$5i_1 + \frac{1}{1}\int_{-\infty}^{t} i_1\,dt + 5i_2 = 10e^{-t}$$

$$5i_1 + 5i_2 + 10i_2 = 10e^{-t}$$

Taking Laplace transforms on both sides, we get

$$5I_1(s) + \frac{I_1(s)}{s} + \frac{q_0}{s} + 5I_2(s) = \frac{10}{s+1}$$

Since the initial charge on the capacitor is zero, the equation becomes

$$5I_1(s) + \frac{I_1(s)}{s} + 5I_2(s) = \frac{10}{s+1}$$

Similarly, $$5I_1(s) + 15I_2(s) = \frac{10}{s+1}$$

By forming a matrix, we have

$$\begin{bmatrix} (5+1/s) & 5 \\ 5 & 15 \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{10}{s+1} \\ \dfrac{10}{s+1} \end{bmatrix}$$

$$I_1(s) = \frac{\begin{vmatrix} \dfrac{10}{s+1} & 5 \\ \dfrac{10}{s+1} & 15 \end{vmatrix}}{\begin{vmatrix} 5+\dfrac{1}{s} & 5 \\ 5 & 15 \end{vmatrix}} = \frac{\left(\dfrac{150}{s+1} - \dfrac{50}{s+1}\right)}{15\left(5+\dfrac{1}{s}\right) - 25}$$

$$I_1(s) = \frac{\left[100/(s+1)\right]s}{50(s+0.3)} = \frac{2s}{(s+0.3)(s+1)}$$

By taking partial fractions, we have

$$I_1(s) = \frac{A}{s+0.3} + \frac{B}{s+1}$$

To get

$$A = I_1(s)(s+0.3)|_{s=-0.3}$$

$$= \left.\frac{2s}{s+1}\right|_{s=-0.3} = \frac{-(0.6)}{0.7} = -0.86$$

Similarly,

$$B = I_1(s)(s+1)|_{s=-1}$$

$$= \left.\frac{2s}{s+0.3}\right|_{s=-1} = \frac{-2}{-(0.7)} = 2.86$$

$\therefore$

$$I_1(s) = \frac{-0.86}{s+0.3} + \frac{2.86}{s+1}$$

Taking inverse transforms on both sides, we have

$$i_1(t) = (2.86\, e^{-t} - 0.86\, e^{-0.3t})\text{A}$$

Similarly

$$I_2(s) = \frac{\begin{vmatrix} \left(5+\frac{1}{s}\right) & \frac{10}{s+1} \\ 5 & \frac{10}{s+1} \end{vmatrix}}{\begin{vmatrix} \left(5+\frac{1}{s}\right) & 5 \\ 5 & 15 \end{vmatrix}} = \frac{0.2}{(s+1)(s+0.3)}$$

By taking partial fractions, we have

$$I_2(s) = \frac{A}{s+0.3} + \frac{B}{s+1}$$

To get

$$A = I_2(s)(s+0.3)|_{s=-0.3} = \left.\frac{0.2}{s+1}\right|_{s=-0.3}$$

$$A = 0.286$$

$$B = I_2(s)(s+1)|_{s=-1}$$

$$= \left.\frac{0.2}{s+0.3}\right|_{s=-1} = \frac{0.2}{-0.7} = -0.286$$

$\therefore$

$$I_2(s) = \frac{0.286}{s+0.3} - \frac{0.286}{s+1}$$

By taking inverse transforms, we have

$$i_2(t) = (0.286e^{-0.3t} - 0.286e^{-t})\text{A}$$

Hence, the total current delivered by the source

$$i(t) = i_1(t) + i_2(t)$$

$$\therefore \quad i(t) = (2.574e^{-t} - 0.574e^{-0.3t})\text{A}$$

7.32 For the circuit shown in Fig. 7.53, determine the current delivered by the source when the switch is closed at $t = 0$. Assume that there is no initial charge on the capacitor and no initial current through the inductor.

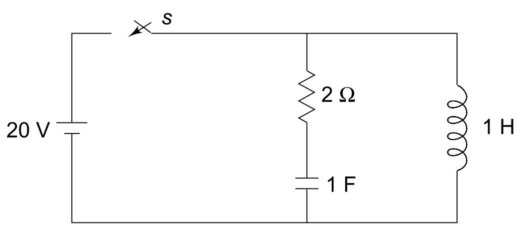


Fig. 7.53

Solution

The circuit is redrawn in the s domain in impedance form as shown in Fig. 7.54.

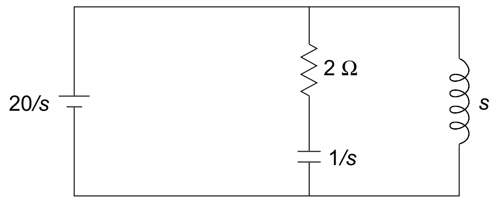


Fig. 7.54

The equivalent impedance in the s domain

$$Z(s) = \frac{\left(2 + \frac{1}{s}\right)s}{\left(2 + \frac{1}{s} + s\right)} = \frac{2s(s + 0.5)}{s^2 + 2s + 1}$$

The current $\quad I(s) = \dfrac{V(s)}{Z(s)}$

$$= \frac{\frac{20}{s}(s^2 + 2s + 1)}{2s(s + 0.5)} = \frac{10(s^2 + 2s + 1)}{s^2(s + 0.5)}$$

By taking partial fractions, we have

$$I(s) = \frac{A}{s^2} + \frac{A'}{s} + \frac{B}{s+0.5}$$

The constant B for the simple root at $s = -0.5$ is

$$B = (s + 0.5)\, I(s)|_{s=-0.5} = 10$$

To obtain the constants of multiple roots, we first find $I_1(s)$.

$$I_1(s) = s^2 I(s) = \frac{10\left(s^2 + 2s + 1\right)}{(s + 0.5)}$$

Using the general formula for multiple root expansion, we get

$$A = \frac{1}{0!}\frac{d^0}{ds^0}\left[\frac{10\left(s^2 + 2s + 1\right)}{s + 0.5}\right]\Bigg|_{s=0} = 20$$

$$A' = \frac{1}{1!}\frac{d'}{ds'}\left[\frac{10\left(s^2 + 2s + 1\right)}{s + 0.5}\right]\Bigg|_{s=0} = 0$$

Therefore, $$I(s) = \frac{20}{s^2} + \frac{10}{s + 0.5}$$

By taking inverse transform on both sides, we have

$$i(t) = (20t + 10\, e^{-0.5t})\text{ A}$$

7.33 Find the value of $i(0^+)$ using the initial value theorem for the Laplace transform given below.

$$I(s) = \frac{2s + 3}{(s + 1)\,(s + 3)}$$

Verify the result by solving it for $i(t)$.

Solution

The initial value theorem is given by

$$\underset{t \to 0}{\text{Lt}}\; i(t) = \underset{s \to \infty}{\text{Lt}}\; SI(s)$$

$$= \underset{s \to \infty}{\text{Lt}}\; \frac{s\left(2s + 3\right)}{\left(s + 1\right)\left(s + 3\right)}$$

Bringing s in the denominator and putting $s = \infty$, we get

$$\underset{s \to \infty}{\text{Lt}}\; \frac{s^2\left(2 + \frac{3}{s}\right)}{s^2\left(1 + \frac{1}{s}\right)\left(1 + \frac{3}{s}\right)} = 2$$

To verify the result, we solve for $i(t)$ and put $t \to \infty$.

Taking partial fractions

$$I(s) = \frac{A}{s + 1} + \frac{B}{s + 3}$$

where,
$$A = (s+1)\,\frac{2s+3}{(s+1)(s+3)}\bigg|_{s=-1} = \frac{1}{2}$$
$$B = (s+3)\,\frac{2s+3}{(s+1)(s+3)}\bigg|_{s=-3} = \frac{3}{2}$$

Taking inverse transform, we get
$$i(t) = \frac{1}{2}\,e^{-t} + \frac{3}{2}\,e^{-3t}$$

By putting $t = 0$, we have $i(0) = 2$. The result is verified.

7.34 Find $\mathscr{L}^{-1}\{F_1(s)F_2(s)\}$ by using the convolution of the following functions.
$$F_1(s) = \frac{1}{s+1} \quad \text{and} \quad F_2(s) = \frac{1}{s+2}$$

Solution

Taking inverse transforms
$$f_1(t) = 5\,e^{-t}$$
$$f_2(t) = e^{-2t}$$

Convolution theorem is given by
$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$$
$$= \int_0^t 5e^{-(t-\tau)}\, e^{-2\tau}\, d\tau$$
$$= 5e^{-t} \int_0^t e^{\tau} \cdot e^{-2\tau}\, d\tau$$
$$= 5e^{-t} \int_0^t e^{-\tau}\, d\tau$$
$$= 5e^{-t}\,[1 - e^{-t}]$$

7.35 In the circuit shown in Fig. 7.55, determine the voltage $v(t)$. The capacitor and inductor are initially de-energised.

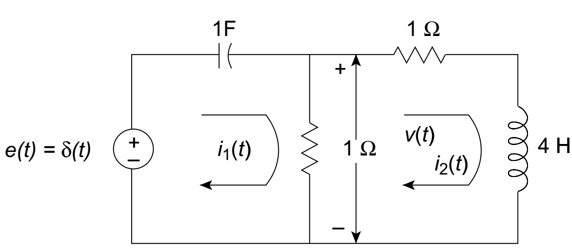


Fig. 7.55

Solution

The transform of the given circuit will be as shown in Fig. 7.56.

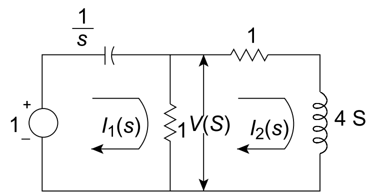


Fig. 7.56

Applying Kirchhoff's voltage law, we get

$$1 = I_1(s)\ \frac{1}{s} + \{I_1(s) - I_2(s)]1$$

and $$O = I_2(s)(1 + 4s) + [I_2(s) - I_1(s)] \times 1$$

or $$1 = I_1(s)\left(\frac{1}{s}+1\right) - I_2(s)$$

and $$O = 2\ I_2(s)(1 + 2s) - I_1(s)$$

Solving the above equations for $I_1(s)$ and $I_2(s)$, we get

$$I_2(s) = \frac{s}{(s+1)\left(s-\frac{1}{2}\right)}$$

$$I_1(s) = 2 - \frac{2}{s-\frac{1}{2}}$$

Taking inverse transform, we get

$$i_2(t) = \frac{2}{3}\ e^{-t} + \frac{1}{3}\ e^{\frac{1}{2}t}$$

$$i_1(t) = 2\delta(t) - 2e^{1/2t}$$

$\therefore$ $$v(t) = [i_1(t) - i_2(t)] \times 1$$

$$= 2\delta(t) - \frac{2}{3}\ e^{-t} + \frac{7}{3}\ e^{\frac{7}{2}t}$$

7.36 Find the current in the circuit shown in Fig. 7.57 at an instant t, after opening the switch if a current of 1 A had been passing through the circuit at the instant of opening.

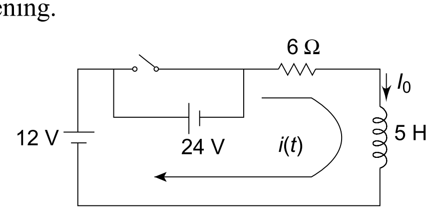


Fig. 7.57

Solution

Applying Kirchhoff's voltage law in the circuit, we get

$$6i(t) + 5\,\frac{di(t)}{dt} = 12 + 24$$

Taking Laplace transform both sides

$$6I(s) + 5[sI(s) - i(0)] = \frac{36}{s}$$

where, $i(0) = 1$ A

$$I(s)\,[6 + 5s] = \frac{36}{s} + 5$$

$$I(s) = \frac{36 + 5s}{s(6 + 5s)}$$

Taking partial fractions

$$I(s) = \frac{6}{8} - \frac{5}{s + \frac{6}{5}}$$

Taking inverse transform, we have

$$i(t) = 6 - 5e^{-\frac{6}{5}t}$$

Practice Problems

7.1 (a) What do you understand by transient and steady state parts of response? How can they be identified in a general solution?

(b) Obtain an expression for the current $i(t)$ from the differential equation

$$\frac{d^2i(t)}{dt^2} + 10\,\frac{di(t)}{dt} + 25i(t) = 0$$

with initial conditions

$$i(0^+) = 2\;\frac{di(0^+)}{dt} = 0$$

7.2 A series circuit shown in Fig. 7.58, comprising of resistance 10 Ω and inductance 0.5 H, is connected to a 100 V source at $t = 0$. Determine the complete expression for the current $i(t)$.

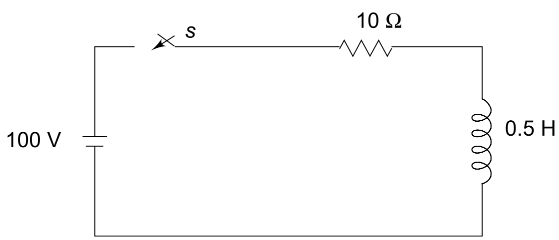


Fig. 7.58

7.3 In the network shown in Fig. 7.59, the capacitor c_1 is charged to a voltage of 100 V and the switch S is closed at $t = 0$. Determine the current expression i_1 and i_2.

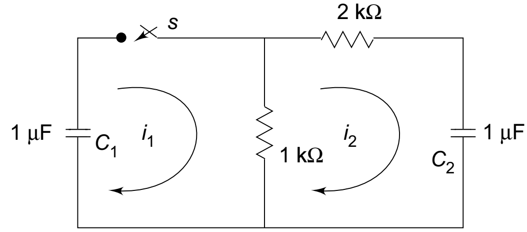


Fig. 7.59

7.4 A series RLC circuit shown in Fig. 7.60, comprising $R = 10\ \Omega$, $L = 0.5$ H and $C = 1\ \mu$F, is excited by a constant voltage source of 100 V. Obtain the expression for the current. Assume that the circuit is relaxed initially.

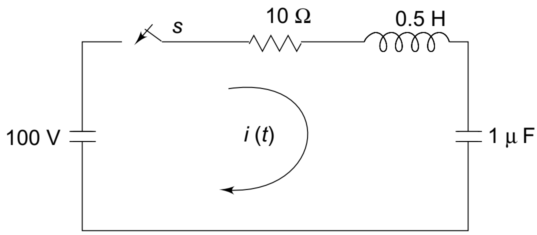


Fig. 7.60

7.5 In the circuit shown in Fig. 7.61, the initial current in the inductance is 2 A and its direction is as shown in the figure. The initial charge on the capacitor is 200 C with polarity as shown when the switch is closed. Determine the current expression in the inductance.

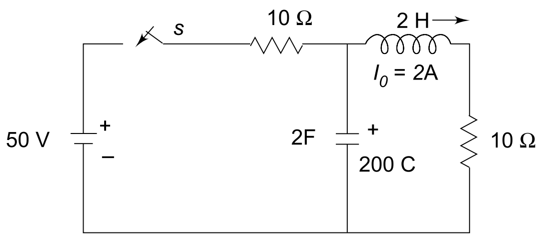


Fig. 7.61

7.6 In the circuit shown in Fig. 7.62, the switch is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Determine V_1 and V_2 at $t = 0^+$.

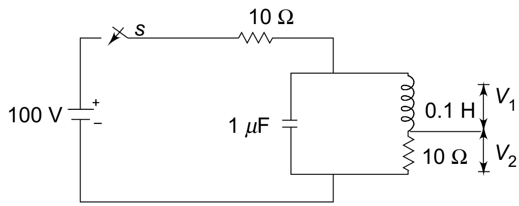


Fig. 7.62

7.7 In the network shown in Fig. 7.63, the switch is moved from position 1 to position 2 at $t = 0$. The switch is in position 1 for a long time. Determine the current expression $i(t)$.

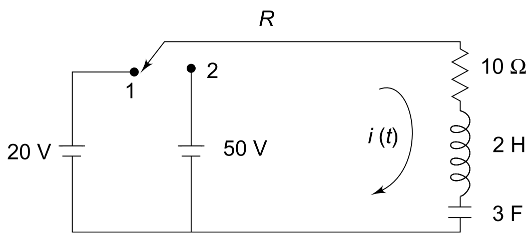


Fig. 7.63

7.8 In the network shown in Fig. 7.64, determine the current expression for $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$. The network has no initial energy.

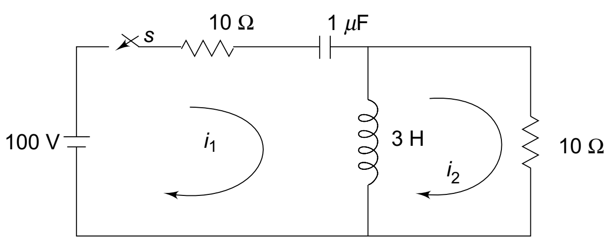


Fig. 7.64

7.9 In the network shown in Fig. 7.65, the switch is closed at $t = 0$ and there is no initial charge on either of the capacitances. Find the resulting current $i(t)$.

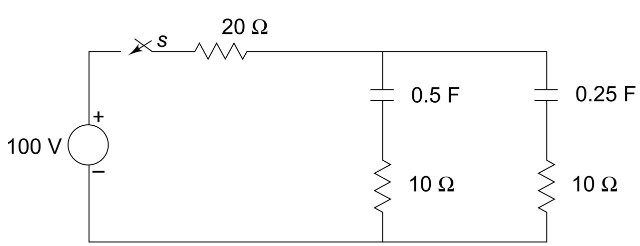


Fig. 7.65

7.10 In the RC circuit shown in Fig. 7.66, the capacitor has an initial charge $q_0 = 25 \times 10^{-6}$ C with polarity as shown. A sinusoidal voltage $v = 100 \sin (200t + \phi)$ is applied to the circuit at a time corresponding to $\phi = 30°$. Determine the expression for the current $i(t)$.

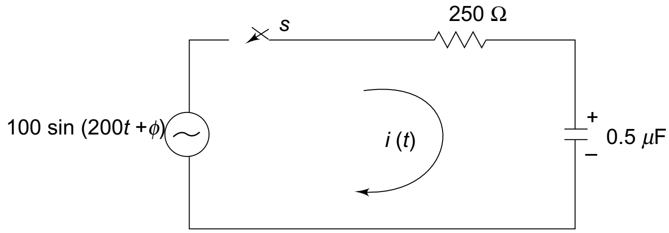


Fig. 7.66

7.11 In the network shown in Fig. 7.67, the switch is moved from position 1 to position 2 at $t = 0$. The switch is in position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression $i(t)$.

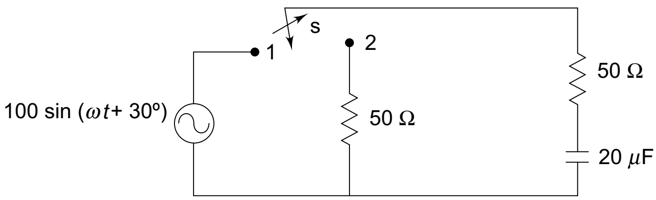


Fig. 7.67

7.12 In the network shown in Fig. 7.68, the switch is moved from position 1 to position 2 at $t = 0$. Determine the current expression.

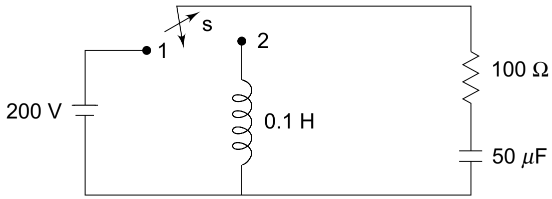


Fig. 7.68

7.13 In the network shown in Fig. 7.69, find $i_2(t)$ for $t > 0$, if $i_1(0) = 5$ A.

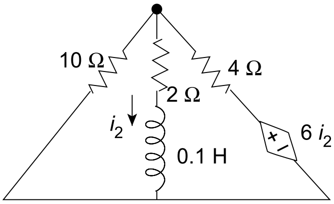


Fig. 7.69

7.14 For the circuit shown in Fig. 7.70, find v_5, if the switch is opened for $t > 0$.

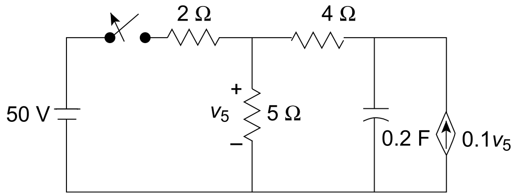


Fig. 7.70

7.15 Calculate the voltage $v_1(t)$ across the inductance for $t > 0$ in the circuit shown in Fig. 7.71.

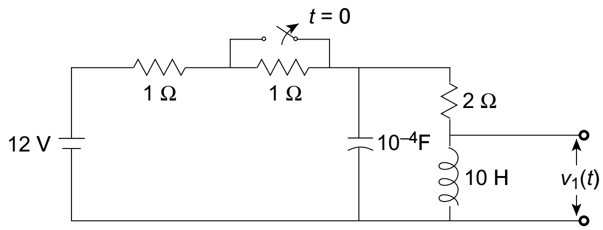


Fig. 7.71

7.16 The network shown in Fig. 7.72 is initially under steady state condition with the switch in position 1. The switch is moved from position 1 to position 2 at $t \neq 0$. Calculate the current $i(t)$ through R_1 after switching.

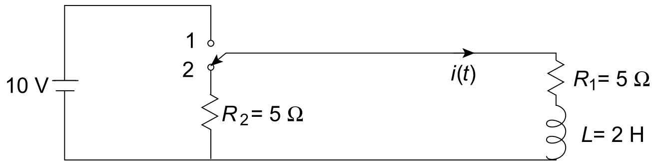


Fig. 7.72

7.17 Find the Laplace transforms of the following functions.

(a) $t^3 + at^2 + bt + 3$ (b) $\sin^2 5t$

(c) e^{5t+6} (d) $\cos h^2\, 3t$

7.18 Find the inverse transforms of the following functions.

(a) $\dfrac{1}{s^2 + 9}$ (b) $\dfrac{2\pi}{s + \pi}$

(c) $\dfrac{8}{(s+3)(s+5)}$ (d) $\dfrac{5}{s^2+9}$

(e) $\dfrac{k_1}{s}+\dfrac{k_2}{s^2}+\dfrac{k_3}{s^3}$

7.19 Find the inverse transforms of the following functions.

(a) $\dfrac{5s+4}{(s-1)(s^2+2s+5)}$ (b) $\dfrac{4s+2}{s^2+2s+5}$

(c) $\dfrac{s}{s^2-2s+5}$ (d) $\dfrac{s(s+1)}{s^2+4s+5}$

7.20 Find the transforms of the following functions.

(a) $te^{-2t}\sin 2t+\dfrac{\cos 2t}{t}$ (b) $\log\left[\dfrac{s^2-1}{s(s+1)}\right]$

(c) $(1+2t\,e^{-5t})^3$ (d) $\dfrac{s+4}{(s^2+5s+12)^2}$

7.21 Using the convolution theorem, determine the inverse transform of the following functions.

(a) $\dfrac{5}{s^2(s+2)^2}$ (b) $\dfrac{s}{(s^2+25)^2}$

(c) $\dfrac{s}{(s^2+9)(s^2+25)}$

7.22 Find the Laplace transform of the periodic square wave shown in Fig. 7.73.

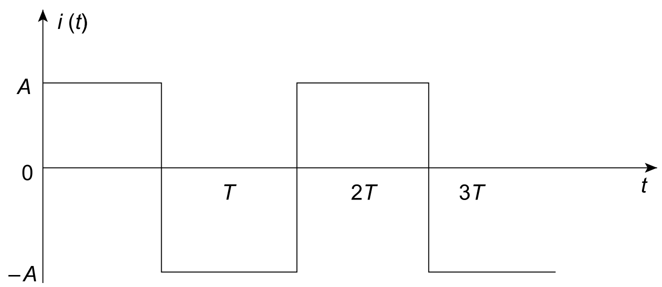


Fig. 7.73

7.23 Find the Laplace transform of a sawtooth waveform $f(t)$ which is periodic, with period equal to unity, and is given by $f(t)=\alpha t$ for $0<t<1$.

7.24 Find the Laplace transform of the periodic wave form shown in Fig. 7.74.

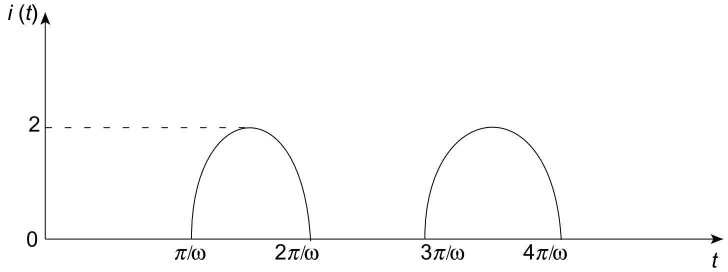


Fig. 7.74

7.25 For the circuit shown in Fig. 7.75, determine the current when the switch is closed at $t = 0$. Assume zero charge on the capacitor initially.

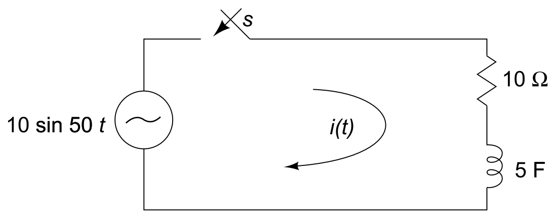


Fig. 7.75

7.26 For the circuit shown in Fig. 7.76, determine the current when the switch is closed at $t = 0$.

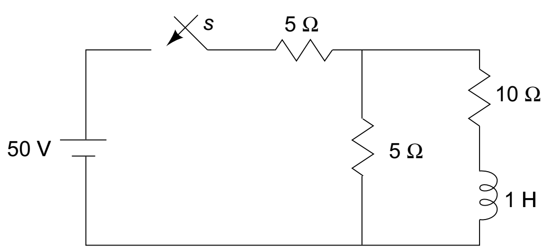


Fig. 7.76

7.27 For the circuit shown in Fig. 7.77 determine the total current when the switch S is closed at $t = 0$.

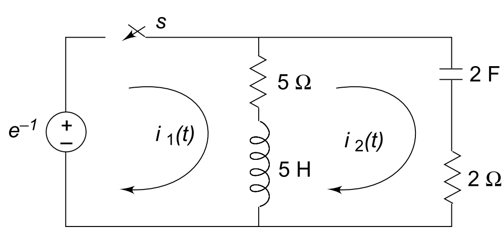


Fig. 7.77

7.28 For the circuit shown in Fig. 7.78, determine the voltage across the output terminals when the input is unit step function. Assume no initial charge on the capacitor.

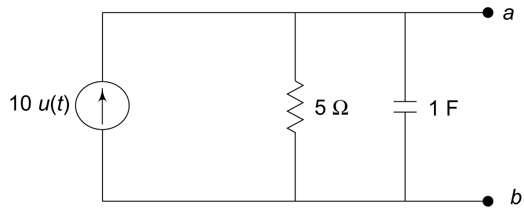


Fig. 7.78

7.29 For the circuit shown in Fig. 7.79, determine the current through the circuit, when the switch is moved from position 1 to position 2.

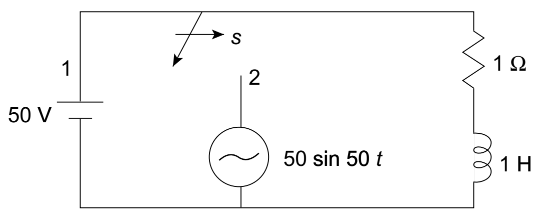


Fig. 7.79

7.30 For the circuit shown in Fig. 7.80, determine the current through the resistor when the switch is moved from position 1 to position 2. Assume that initial charge on the capacitor is 5 C.

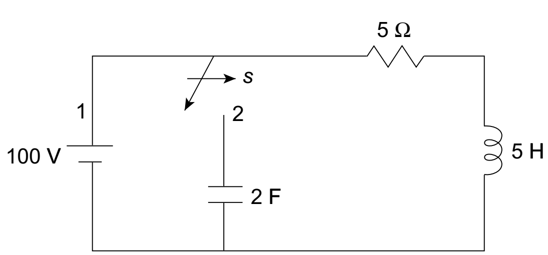


Fig. 7.80

7.31 For the circuit shown in Fig. 7.81, determine the current when the switch is closed at $t = 0$.

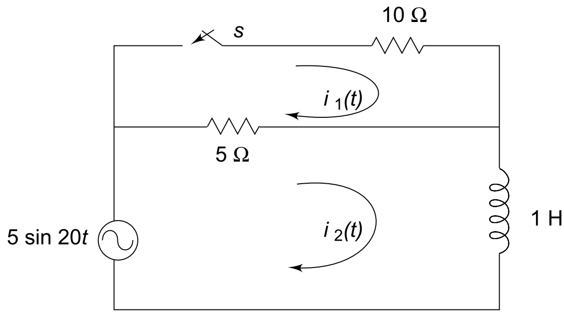


Fig. 7.81

7.32 For the given function $f(t) = 3u(t) + 2e^{-t}$, find its final value $f(\infty)$ using final value theorem.

7.33 An exponential voltage $v(t) = 10e^{-t}$ is suddenly applied at $t = 0$ to the circuit shown in Fig. 7.82. Obtain the particular solution for current $i(t)$ through the circuit.

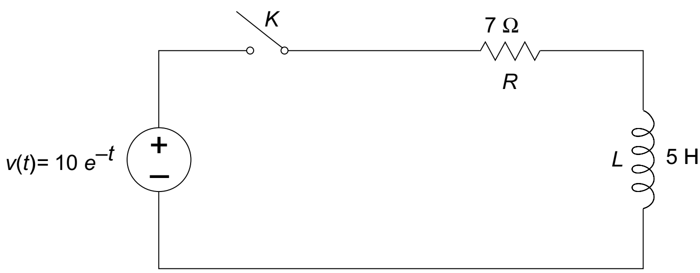


Fig. 7.82

7.34 For the circuit shown in Fig. 7.83, the switch is closed at $t = 0$. Determine $i_1(t)$ and $i_2(t)$. The initial currents $i_1(0) = 1$ A and $i_2(0) = 2$ A.

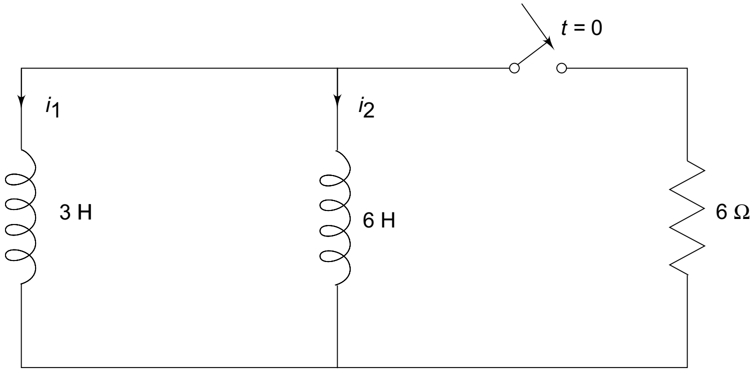


Fig. 7.83

7.35 In the circuit shown in Fig. 7.84, the switch is changed from position 1 to 2 at $t = 0$. A steady state position is existing in position 1 before $t = 0$. Determine the current $i(t)$ using Laplace transform method.

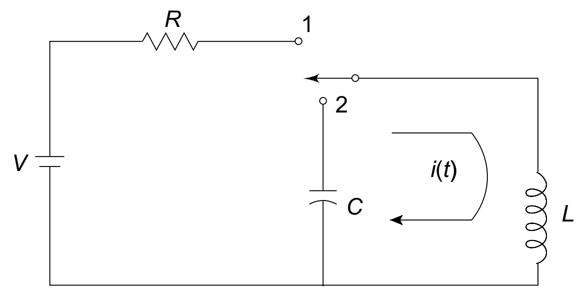


Fig. 7.84

Objective Type Questions

7.1 Transient behaviour occurs in any circuit when
- (a) there are sudden changes of applied voltage.
- (b) the voltage source is shorted.
- (c) the circuit is connected or disconnected from the supply.
- (d) all of the above happen.

7.2 The transient response occurs
- (a) only in resistive circuits
- (b) only in inductive circuits
- (c) only in capacitive circuits
- (d) both in (b) and (c).

7.3 Inductor does not allow sudden changes
- (a) in currents
- (b) in voltages
- (c) in both (a) and (b)
- (d) in none of the above

7.4 When a series RL circuit is connected to a voltage V at $t = 0$, the current passing through the inductor L at $t = 0^+$ is

(a) $\frac{V}{R}$ (b) infinite (c) zero (d) $\frac{V}{L}$

7.5 The time constant of a series RL circuit is

(a) LR (b) $\frac{L}{R}$ (c) $\frac{R}{L}$ (d) $e^{-R/L}$

7.6 A capacitor does not allow sudden changes
- (a) in currents
- (b) in voltages
- (c) in both currents and voltages
- (d) in neither of the two

7.7 When a series RC circuit is connected to a constant voltage at $t = 0$, the current passing through the circuit at $t = 0^+$ is

(a) infinite (b) zero (c) $\frac{V}{R}$ (d) $\frac{V}{\omega C}$

7.8 The time constant of a series RC circuit is

(a) $\dfrac{1}{RC}$ (b) $\dfrac{R}{C}$ (c) RC (d) e^{-RC}

7.9 The transient current in a loss-free LC circuit when excited from an ac source is an ____________ sine wave.

(a) undamped (b) overdamped
(c) underdamped (d) critically damped

7.10 Transient current in an RLC circuit is oscillatory when

(a) $R = 2\sqrt{L/C}$ (b) $R = 0$
(c) $R > 2\sqrt{L/C}$ (d) $R < 2\sqrt{L/C}$

7.11 The initial current in the circuit shown in Fig. 7.85 when the switch is opened for $t > 0$ is

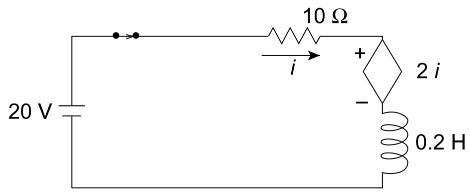


Fig. 7.85

(a) 1.67 A (b) 3 A (c) 0 A (d) 2 A

7.12 The initial current in the circuit shown in Fig. 7.86 below when the switch is opened for $t > 0$ is

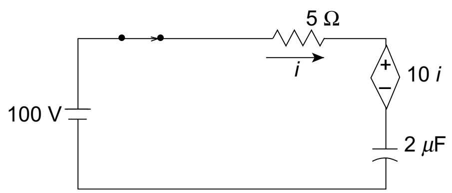


Fig. 7.86

(a) 1.5 A (b) 0 A (c) 2 A (d) 10 A

7.13 For the circuit shown in Fig. 7.87 the current in the 10 Ω resistor when the switch is changed from 1 to 2 is

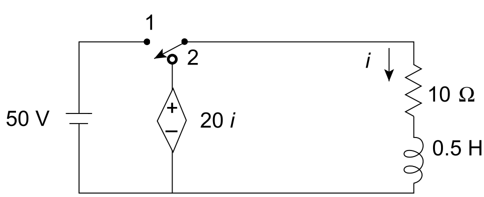


Fig. 7.87

(a) $5\,e^{+20t}$ (b) $5\,e^{-20t}$ (c) $20\,e^{+5t}$ (d) $20\,e^{-5t}$

Subject Code: R13211/313 **Set 1**

I B. Tech II Semester Regular Examinations August - 2014

NETWORK ANALYSIS

(Common to ECE, EIE, E Com. E.E Branches)

Time: 3 hours **Max. Marks: 70**

Question Paper Consists of **Part-A** and **Part-B**
Answering the question in **Part-A** is Compulsory,
Three Questions should be answered from **Part-B**

PART-A

1. **(i) Define electric potential, electric current and electric energy.**

Ans. Refer Sections 1.2, 1.3 and 1.4.

(ii) A certain inductive coil takes 15 A when the supply voltage is 230 V, 50 Hz. If the frequency is changed to 40 Hz, the current increases to 17.2 A. Calculate resistance and inductance of the coil.

Ans. Given: If $V = 230$ V, 50 Hz then $I = 15$ A

If $V = 230$ V, 40 Hz then $I = 17.2$ A

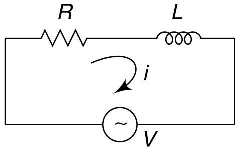


Fig. 1

We know that $V = I \times Z$

From first condition, $230 = i(R + j\omega L)$

$$230 = 15(R + j(50)L) \quad (1)$$

From second condition, $230 = 17.2(R + j(40)L)$ (2)

Solving eqns (1) and (2), we get

$$\boxed{R = 8.87\ \Omega; \quad L = 0.25\ \text{H}}$$

(iii) Write the differences between series and parallel resonance.

Ans. Refer Sections 4.4 and 4.5.

(iv) State compensation theorem.

Ans. Refer Sections 5.1.5 and 5.2.5.

(v) Write the *Z*-parameters of the following network (Fig. 2):

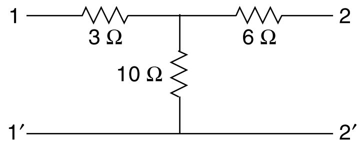


Fig. 2

Ans. To find the z-parameters

We know that

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

In the first loop, current is I_1, but $I_2 = 0$

$$\Rightarrow \quad \boxed{z_{11} = \frac{V_1}{I_1} = 13\,\Omega}$$

$$\boxed{z_{21} = \frac{V_2}{I_1} = 10\,\Omega}$$

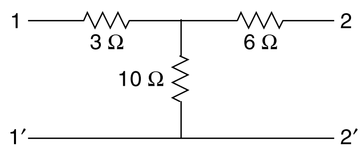


Fig. 3

Similarly, in the second loop, current $I_1 = 0$

$$\Rightarrow \quad \boxed{z_{22} = 16\,\Omega}$$

$$\boxed{z_{12} = 10\,\Omega}$$

(vi) What is the time constant? What are the time constant of series *R*-*L* and *R*-*C* circuits?

Ans. Refer Sections 7.2.1 and 7.2.2.

(vii) A series *R*-*L* circuit has *R* = 20 ohms and *L* = 8 H. The circuit is connected across a dc voltage source of 120 V at *t* = 0. Calculate the time at which the voltage drops across *R* and *L* are the same.

[2 + 4 + 3 + 2 + 4 + 3 + 4]

Ans. Given: $R = 20\ \Omega$; $L = 8$ H

We know that for an RL circuit,

$$i(t) = \frac{V}{R}(1 - e^{-t/\tau}) \quad \text{where } i = \frac{L}{R}$$
$$= \frac{120}{20}(1 - e^{\frac{-t(R)}{L}})$$

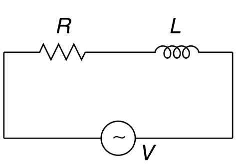


Fig. 4

Given that $V_R = V_L \quad \Rightarrow \quad i \cdot R = L\dfrac{di}{dt}$

$$\Rightarrow \quad \cancel{20} \times \cancel{6}(1 - e^{-R(t/L)}) = \cancel{8} \times \cancel{6} \times \left(\frac{\cancel{20}}{\cancel{8}}\right) e^{-R(t/L)}$$

$$\Rightarrow \qquad 1 = 2e^{-\left(\frac{Rt}{L}\right)}$$

$$\Rightarrow \qquad e^{\frac{Rt}{L}} = 2$$

$$t = \frac{L}{R}\ln(2)$$

$$\boxed{t = 0.2777 \text{ second}}$$

PART-B

2. (a) State and explain Kirchhoff's voltage and current law with an example.

Ans. Refer Sections 1.10.1 and 1.10.4

(b) Find the voltage $V(t)$ in the network shown in Fig. 5 using nodal technique. All impedances are in ohms. **[6 + 10]**

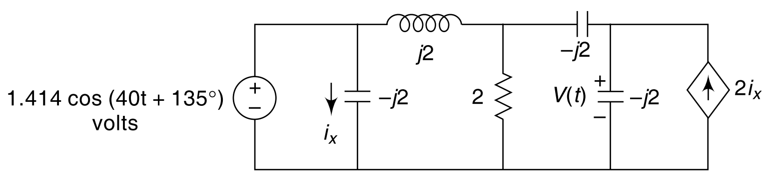


Fig. 5

Ans. Let the node voltages at nodes 1, 2, 3 be V_1, V_2 and V_3 respectively.

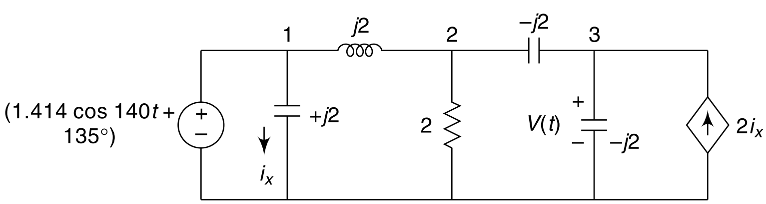


Fig. 6

Node 1 $\quad i_x = \dfrac{V_1}{-j2}$ (1)

But $\quad V_1 = 1.414\cos(40t + 135°)$

Node 2 $\quad$ Applying KCL,

$$\frac{V_2 - V_1}{j2} + \frac{V_2}{2} + \frac{V_2 - V_3}{-j2} = 0 \tag{2}$$

Node 3 $\quad$ Applying KCL again,

$$\frac{V_3 - V_2}{-j2} + \frac{V_3}{-j2} = 2i_x \tag{3}$$

Substituting Eq. (1) in (3),

$$\Rightarrow \qquad \frac{V_3 - V_2}{-j2} + \frac{V_3}{-j2} = \frac{2V_1}{-j2}$$

$$\Rightarrow \qquad 2V_3 - V_2 = 2V_1$$

$$\Rightarrow \qquad 2V_1 + V_2 - 2V_3 = 0 \qquad (4)$$

From Eq. (2),

$$-jV_2 + jV_1 + V_2 + jV_2 - jV_3 = 0$$

$$jV_1 + V_2 - jV_3 = 0$$

$$\Rightarrow \qquad j(V_1 - V_3) + V_2 = 0 \qquad (5)$$

From Eq. (4), $2(V_1 - V_3) + V_2 = 0$

In the above two equations, substitute $(V_1 - V_3) = x$

We get $2x + V_2 = 0$ (6); $jx + V_2 = 0$ (7)

(6)-(7) $\Rightarrow \quad x(2 - j) = 0 \quad \Rightarrow \quad x = 0$

Substituting this in the assumption gives $V_1 = V_3$

But $\quad V_1 = 1.414 \cos(40t + 135°)$

$\therefore \quad V_1 = V_3 = 1.414\angle 135°$

$$\boxed{V(t) = V_3 = 1.414\angle 135°}$$

3. (a) A sinusoidal 50 Hz voltage of 200 V supplies three parallel circuits as shown in Fig. 7. Find the current in each circuit and the total current. Draw the vector diagram. Assume supply voltage V = 200 V, 50 Hz.

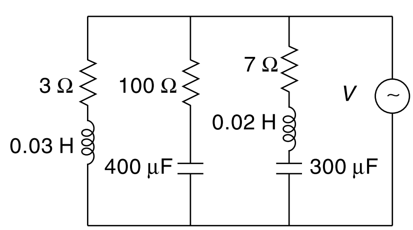


Fig. 7

Ans. Given: $R_1 = 3\ \Omega$; $R_2 = 100\ \Omega$; $R_3 = 7\ \Omega$

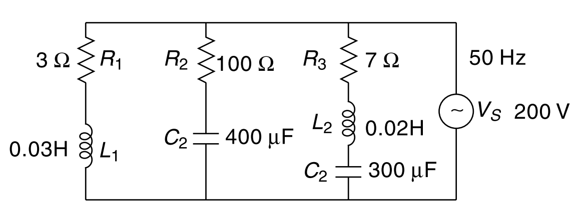


Fig. 8

$$L_1 = 0.03 \text{ H}; L_2 = 0.02 \text{ H}$$
$$C_1 = 400 \ \mu\text{F}; C_2 = 300 \ \mu\text{F}$$
$$V_S = 200 \text{ V, 50 Hz}$$

The inductive reactance across L_1 is given as

$$X_{L1} = 2\pi f L_1 = 2\pi \times 50 \times 0.03$$
$$= 9.424 \ \Omega$$

Capacitive reactance $X_{C1} = \dfrac{1}{2\pi f C_1} = \dfrac{1}{2\pi \times 50 \times 400 \times 10^{-6}}$

$$= 7.957 \ \Omega$$

$$X_{L2} = 2\pi f L_2 = 2\pi \times 50 \times 0.02$$
$$= 6.2831 \ \Omega$$

Capacitive reactance $X_{C2} = \dfrac{1}{2\pi f C_2} = \dfrac{1}{2\pi \times 50 \times 300 \times 10^{-6}}$

$$= 10.6103 \ \Omega$$

Impedance of each branch is denoted as z_i

$$\therefore \quad z_1 = \sqrt{R_1^2 + X_{L1}^2} = \sqrt{3^2 + (9.424)^2} = 9.889 \ \Omega$$
$$z_2 = \sqrt{R_2^3 + X_{C1}^2} = \sqrt{(100)^2 + (7.957)^2} = 100.316 \ \Omega$$
$$z_3 = \sqrt{R_3^2 + X_{L2}^2 + X_{C2}^2}$$
$$z_3 = \sqrt{7^2 + (10.6103 - 6.2831)^2} = 8.229 \ \Omega$$

Current in each branch is given by I_i:

$$I_1 = \frac{V_S}{z_1} = \frac{200}{9.889} = 20.225 \text{ A}$$
$$I_2 = \frac{V_S}{z_2} = \frac{200}{100.316} = 1.99 \approx 2 \text{ A}$$
$$I_3 = \frac{V_S}{z_3} = \frac{200}{8.229} = 24.304 \text{ A}$$

Phase angle θ is given as

$$\theta_1 = \tan^{-1}\left(\frac{X_{L1}}{R_1}\right) = \tan^{-1}\left(\frac{9.424}{3}\right) = 72.34°$$
$$\theta_2 = \tan^{-1}\left(-\frac{X_{C1}}{R_2}\right) = \tan^{-1}\left(-\frac{7.957}{100}\right) = -4.54°$$
$$\theta_3 = \tan^{-1}\left(\frac{X_{L2} - X_{C2}}{R_3}\right) = \tan^{-1}\left(\frac{6.2831 - 10.6103}{7}\right)$$
$$= -31.723°$$

Phasor Diagram:

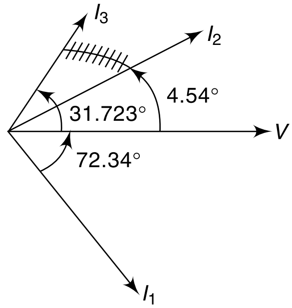


Fig. 9

(b) The impedance of a parallel circuit are Z_1 = (6 + j8) ohms and Z_2 = (8 – j6) ohms. If the applied voltage is 120 V, find (i) current and power factor of each branch, (ii) overall current and power factor of the combination, and (iii) power consumed by each impedance. Draw a phasor diagram. [8 + 8]

Ans. Given $z_1 = (6 + j8)\ \Omega$;

$z_2 = (8 - j6)\ \Omega$

$V_S = 120$ V

From the given, the circuit can be drawn as shown in Figure 7.

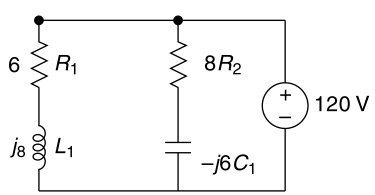


Fig. 10

Branch 1 consists of R_1 and L_1 whereas branch 2 consists of R_2 and C_1.

Impedance in Branch 1

$$z_1 = \sqrt{R_1^2 + L_1^2}$$

$$= \sqrt{6^2 + 8^2} = 10\ \Omega$$

Current $I_1 = \dfrac{V_S}{z_1} = \dfrac{120}{10} = 12$ A

Phase angle $\theta_1 = \tan^{-1}\left(\dfrac{V_{L1}}{V_{R1}}\right)$

But $\quad V_{L1} = \dfrac{120 \times j8}{6 + j8} \quad$ and $\quad V_{R1} = \dfrac{120 \times 6}{6 + j8}$

$\therefore \qquad \theta_1 = \tan^{-1}\left(\dfrac{8}{6}\right) = 53.13$

$\therefore \qquad$ Power factor = cos $\theta_1 = 0.6$

Power consumed by first branch $= \dfrac{V^2}{z_1} = \dfrac{120 \times 120}{10}$

$= 1440$ W

In Branch 2,

Impedance $z_2 = \sqrt{R_2^2 + C_2^2} = \sqrt{8^2 + 6^2} = 10\ \Omega$

Current $I_2 = \dfrac{V_S}{z_2} = \dfrac{120}{10} = 12$ A

Phase $\theta_2 = \tan^{-1}\left(\dfrac{V_{C1}}{V_{R2}}\right) = \tan^{-1}\left(\dfrac{6}{8}\right)$

$= -36.869°$

Power factor = cos $\theta_2 = 0.8$

Power consumed $= \dfrac{V^2}{z_2} = \dfrac{120 \times 120}{10} = 1440$ W

Phasor diagram:

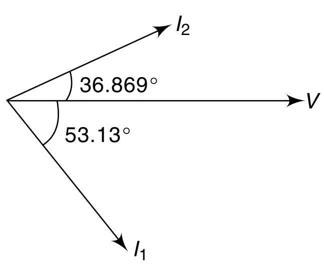


Fig. 11

Let overall impedance be denoted by Z_{eq}.

$$\frac{1}{Z_{\text{eq}}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$\Rightarrow \qquad Z_{\text{eq}} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(6 + j8)(8 - 6j)}{14 + 2j}$$

$$= \frac{96 + 28j}{14 + 2j} = \frac{48 + j14}{7 + j}$$

$$Z_{\text{eq}} = \frac{7}{5}(3 + j)$$

Overall phase angle $\phi = \tan^{-1}\left(\frac{1}{3}\right) = 18.43°$

Overall power factor $= \cos \phi = \cos(18.43)$

$= 0.948$

4. (a) Obtain an expression for efficient of coupling.

Ans. Refer Sections 4.1.4.

(b) Two similar coils connected in series gave a total inductance of 600 mH and when one of the coil is reserved, the total inductance is 300 mH. Determine the mutual inductance between the coils and coefficient of coupling.

Ans. Given:

(i) When two coils are connected in series with inductances L_1 and L_2, the total inductance is 600 mH,

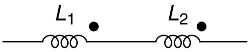


Fig. 12(a)

i.e., $$L_{total} = L_1 + L_2 + 2M = 600 \text{ mH} \quad (1)$$

where M = mutual inductance

(ii) When one coil is reversed inductance is 300 mH

i.e., $$L_{total} = L_1 + L_2 - 2M = 300 \text{ mH} \quad (2)$$

Taking ratio of Eq. (1) to Eq. (2),

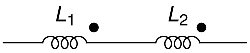


Fig. 12(b)

$$\Rightarrow \quad \frac{600}{300} = \frac{L_1 + L_2 + 2M}{L_1 + L_2 - 2M}$$

$$\Rightarrow \quad 2(L_1 + L_2) - 4M = L_1 + L_2 + 2M$$

$$\Rightarrow \quad L_1 + L_2 = 6M$$

But $$M = K\sqrt{L_1 L_2}$$

$$\therefore \quad L_1 + L_2 = 6K\sqrt{L_1 L_2}$$

It is given that $L_1 = L_2$

$$\Rightarrow \quad 2L = 6KL$$

$$\boxed{K = \frac{1}{3}}$$

$$\Rightarrow \quad M = \frac{1}{3}L$$

Substitute this in Eq. (2),

$$\Rightarrow \quad 2L - 2M = 300$$

i.e., $$2L - \frac{2L}{3} = 300$$

$$\Rightarrow \quad \boxed{L = 225 \text{ mH}}$$

$$\boxed{M = 75 \text{ mH}}$$

(c) State and explain maximum power transfer theorem. **[5 + 5 + 6]**

Ans. Refer Sections 5.1.7 and 5.2.7.

5. (a) For a series resonance circuit with constant voltage and variable frequency, obtain the frequency at which voltage across the inductor is maximum. Calculate this maximum voltage when *R* = 50 ohms, *L* = 0.05 H, *C* = 20 microfarads and *V* = 100 volts.

Ans. Given: $R = 50\ \Omega$; $L = 0.05$ H

$C = 20\ \mu$F; $V = 100$ V

Maximum voltage across inductor occurs at frequency f_L

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{\left(1-\frac{R^2C}{2L}\right)}}$$

$$= \frac{1}{2\pi\sqrt{0.05\times 20\times 10^{-6}}}\sqrt{\frac{1}{1-\frac{(50)^2\times 20\times 10^{-6}}{2\times 0.05}}}$$

$$= 225.079 \text{ Hz}$$

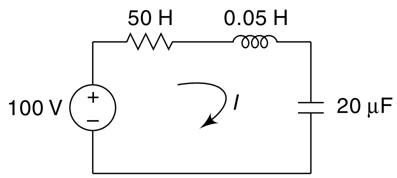


Fig. 13

The current I through the circuit is given by

$$I = \frac{V}{z_{\text{eq}}}$$

$$= \frac{100}{50 + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{100}{50 + j(70.7 - 35.353)}$$

$$= \frac{100}{50 + 35.34j}$$

The voltage across the inductor V_L is

$$V_L = IX_L = \frac{100}{50 + 35.34j}\times 70.7j$$

$$= 115.47\angle 54.74^\circ$$

(b) Determine the current through R_L = 10 Ω resistor as shown in Fig. 14 using Thevenin's theorem. Verify the same with Norton's theorem. **[6 + 10]**

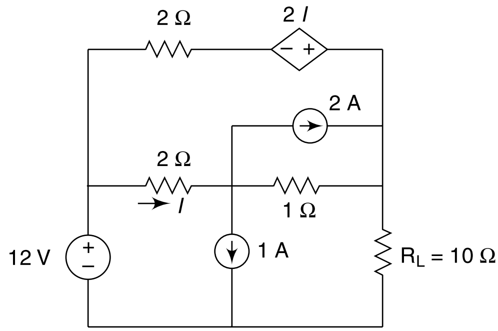


Fig. 14

Ans. *Using Thevenin's theorem:*

To find V_{th}:

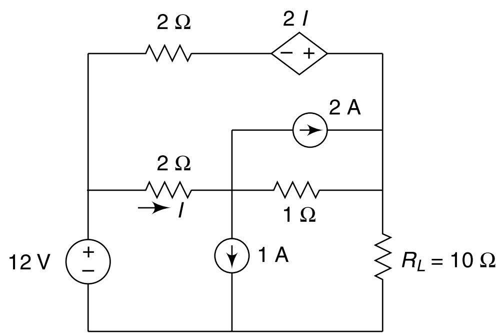


Fig. 15

The circuit can be redrawn in the following way:

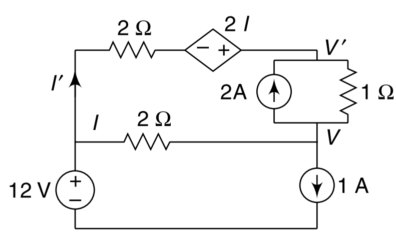


Fig. 16

And there are assumptions being made let the current flowing be I' and node voltage be V'.

Current through resistor 2 Ω I is

$$I = \frac{12 - V}{2} \quad \Rightarrow \quad V = 12 - 2I \tag{1}$$

$$I' = \frac{12 + 2I - V'}{2} \tag{2}$$

Applying KCL at V,

$$I + I' = 1 \quad (3)$$

Substituting (2) in (3),

$$6 + I - \frac{V'}{2} + I = 1$$

$$2I - \frac{V'}{2} + 5 = 0$$

$$I - \frac{V'}{4} = \frac{-5}{2} \quad (4)$$

Applying KCL at V'

$$I' + 2 = \frac{V' - V}{1} \quad (5)$$

Substituting (3) and (1) in Eq. (5),

$$\Rightarrow \quad (1 - I) + 2 = V' - (12 - 2I)$$

$$V' + 3I = 15 \quad (6)$$

Solving eqns (4) and (5) gives

$$\boxed{I = 0.7142\text{ A};\ V' = 12.857\ V = V_{th}}$$

To find I_{SC}:

Let I_{SC} be the current flowing through the 10 Ω resistor and the equivalent circuit to determine, it is drawn below:

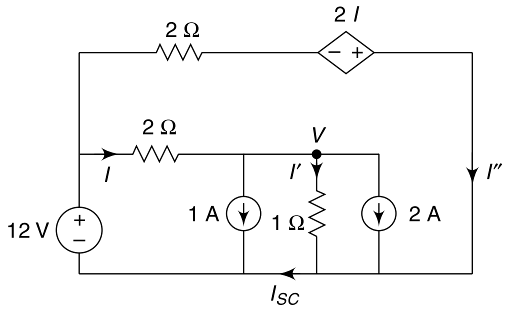


Fig. 17

From the above figure/circuit,

$$I_{SC} = I' + I'' + 2$$

Applying KCL at the node V,

$$\Rightarrow \quad I = 1 + 2 + \frac{V}{1}$$

$$I = V + 3 \quad (1)$$

I can be also determined as

$$I = \frac{12 - V}{2} \tag{2}$$

Substituting (1) in (2),

$$6 - \frac{V}{2} = V + 3 \quad \Rightarrow \quad 3 = \frac{3V}{2}$$

$$\boxed{V = 2\text{ V}}$$

and $\quad I = 3 + V = 3 + 2 = 5\text{ A}$

Applying Ohm's law across the resistor of 2Ω gives

$$I'' = \frac{12 + 2I}{2} = \frac{12 + 10}{2}$$
$$= 11\text{ A}$$
$$I' = \frac{V}{1} = 2\text{ A}$$

But $\quad I_{SC} = I' + I'' + 2$

$$= 2 + 11 + 2$$
$$I_{SC} = 15\text{ A}$$
$$R_{th} = \frac{V_{\text{th}}}{I_{SC}} = \frac{12.857}{15} = 0.857\ \Omega$$

6. (a) Derive the symmetry and reciprocity conditions for *ABCD*-parameters and *h*-parameters.

Ans. Refer Section 6.13.

(b) Determine *Y*-parameters of the network shown in Fig. 18. [8 + 8]

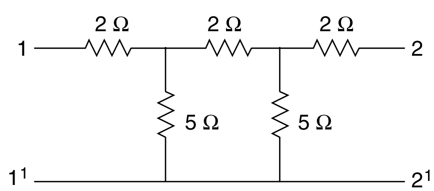


Fig. 18

Ans. The below π-configuration can be changed to T-configuration and the circuit changes to

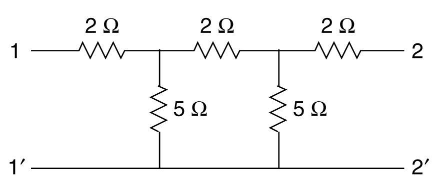


Fig. 19

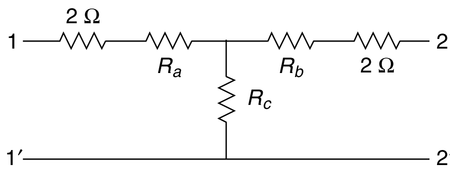


Fig. 20

$$R_a = \frac{5(2)}{5+5+2} = \frac{10}{12}; \quad R_b = \frac{5(2)}{5+5+2} = \frac{10}{12}$$

$$R_c = \frac{5(5)}{5+5+2} = \frac{25}{12}$$

After substituting these values in the circuit, the above one reduces to

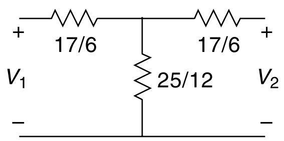


Fig. 21

The above T-configuration can be changed to π-configuration as

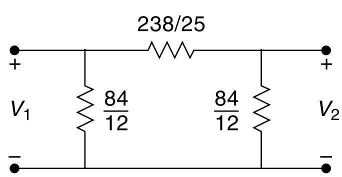


Fig. 22

$$Y = \begin{bmatrix} Y_a + Y_c & -Y_c \\ -Y_c & Y_b + Y_c \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{12}{84} + \frac{25}{238} & \frac{-25}{238} \\ \frac{-25}{238} & \frac{12}{84} + \frac{25}{238} \end{bmatrix}$$

$$= \begin{bmatrix} 0.24 & -0.105 \\ -0.105 & 0.24 \end{bmatrix}$$

7. A series R-C circuit with R = 10 ohms and C = 2 F has a sinusoidal voltage source 200 sin (500t + ϕ) applied at time when ϕ = 0. (i) Find the expression for the current. (ii) At what value of ϕ must the switch be closed so that the current directly enters steady state? [16]

Ans. $R = 10\ \Omega$, $C = 2\text{F}$, $V_s(t) = 200 \sin(500t + \phi)$

From the given data, the circuit that is to be inferred is

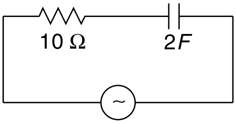


Fig. 23

i.e. $\omega = 500$

Capacitive reactance

$$X_C = \frac{1}{\omega C} = \frac{1}{50 \times 2} = 0.001\,\text{F}$$

Applying KVL around the loop gives

$$10i + \frac{1}{2}\int i dt = 200 \sin(500t + \phi)$$

Given $\phi = 0$

$$\Rightarrow \qquad \frac{10di}{dt} + \frac{i}{2} = 200 \times 500 \cos(500\,t)$$

$$\Rightarrow \qquad \left(D + \frac{1}{20}\right) i = 10^4 \cos(500\,t)$$

CF; $i_c = Ce^{-t/20}$

Finding particular integral:

$$\text{PI} = \frac{1}{D + \frac{1}{20}} \cdot 10^4 \cos(500\,t)$$

$$= 10^4 \left(\frac{D - \frac{1}{20}}{D^2 - \frac{1}{400}} \right) \cos(500\,t)$$

$$= 10^4 \left[\frac{D - \frac{1}{20}}{-(500)^2 - \left(\frac{1}{20}\right)^2} \right] \cos 500\,t$$

Approximating the denominator,

$$= -\frac{10^4}{25 \times 10^4}\left(-\sin 500\,t \cdot 500 - \frac{1}{20}\cos 500\,t\right)$$

Particular current i_p,

$$i_p = 20 \sin 500\,t + \frac{1}{500} \cos 500\,t$$

$$i = Ce^{-t/20} + 20 \sin 500\,t + \frac{1}{500} \cos 500\,t$$

To find C

Given $\qquad t = 0 \quad \Rightarrow \quad i = 0$

$$0 = C + 0 + \frac{1}{500} \quad \Rightarrow \quad C = \frac{-1}{500}$$

$$i(t) = \frac{-1}{500} e^{-t/20} + 20 \sin 500t + \frac{1}{500} \cos 500\, t$$

(ii) To find value of ϕ at which switch must be closed so that the current directly enters steady state:

$$\omega t_0 + \phi = \tan^{-1}(\omega RC)$$

At $t = 0$,

$$\phi = \tan^{-1}(500 \times 2 \times 10)$$
$$= \tan^{-1}(10000)$$
$$= 89.99°$$

Subject Code: R13211/313 **Set 2**

I B. Tech II Semester Regular Examinations August - 2014

NETWORK ANALYSIS

(Common to ECE, EIE, E Com. E.E Branches)

Time: 3 hours **Max. Marks: 70**

Question Paper Consists of **Part-A** and **Part-B**
Answering the question in **Part-A** is Compulsory,
Three Questions should be answered from **Part-B**

PART-A

1. **(i) Define average value, RMS value and form factor for an alternating quantity.**

Ans. Refer Sections 2.2.4, 2.2.5 and 2.2.7.

(ii) Determine the source voltage and phase angle, if the voltage across the resistance is 70 V and across an inductive reactance is 20 V, in an *R-L* series circuit.

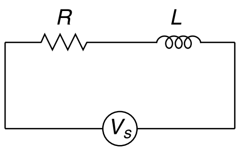


Fig. 1

Ans. Given: $V_R = 70$ V and $V_L = 20$ V

To find: Source voltage V_S, phase angle ϕ

$$V_S = \sqrt{V_R^2 + V_L^2} = \sqrt{70^2 + 20^2} = 72.8 \text{ V}$$

$$\text{phase angle } \phi = \tan^{-1}\left(\frac{V_L}{V_R}\right) = \tan^{-1}\left(\frac{20}{70}\right)$$

$$= 15.945°$$

(iii) For the circuit shown in Fig. 2, determine the value of capacitive reactance, impedance and current at resonance.

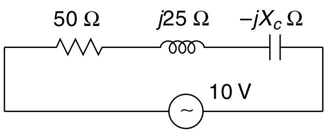


Fig. 2

Ans. At resonance, imaginary part of impedance is zero,

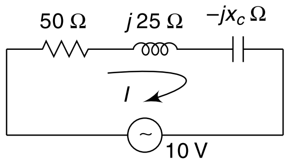


Fig. 3

i.e., $X_C = X_L$

$\Rightarrow$ $X_C = 25\ \Omega$

$\therefore$ $z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2} = R$

$= 50\ \Omega$

Current $I = \dfrac{V}{z} = \dfrac{10}{50} = \dfrac{1}{5}\text{A} = 0.2\text{ A}$

(iv) State the maximum power transfer theorem.

Ans. Refer Sections 5.1.7 and 5.2.7.

(v) Write the condition of symmetry and reciprocity for transmission, inverse transmission and inverse *h*-parameters.

Ans. Refer Sections 6.4 and 6.5.

(vi) What is meant by natural and forced response?

Ans. Refer Section 7.1.

(vii) In a series R-L circuit, the application of dc voltage results in a current of 0.741 times the final steady-state value of current after one second. However, after the current has reached its final value, the source is short-circuited. What would be the value of the current after one second? [3 + 3 + 3 + 2 + 4 + 3 + 4]

Ans. Let the final steady-state current be I_o.

Given: $I = 0.741\ I_o$

We know that $I = I_o(1 - e^{-t/\tau})$

$\Rightarrow$ $0.741 I_o = I_o(1 - e^{-t/\tau})$

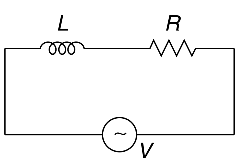


Fig. 4

$t = 1 \Rightarrow \tau = \dfrac{L}{R} = 0.740$

$\boxed{R/L = 1.3509}$

PART-B

2. (a) For the circuit shown in Fig. 5, find all the branch currents using nodal analysis. Also show that total power delivered is equal to total power dissipated.

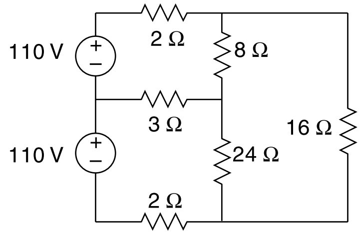


Fig. 5

Ans. Let the node voltages be V_1, V_2 and V_3.

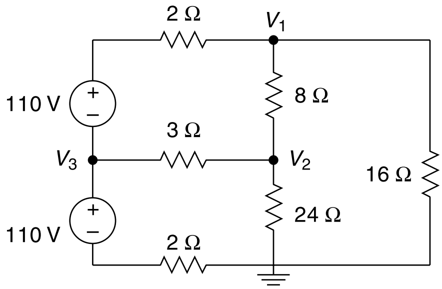


Fig. 6

Applying KCL at the node 1 gives.

At V_1, $$\frac{V_1 - V_3 - 110}{2} + \frac{V_1 - V_2}{8} + \frac{V_1}{16} = 0$$

$$\Rightarrow \quad 11V_1 - 8V_3 - 2V_2 = 880 \qquad (1)$$

At V_2, $$\frac{V_2}{24} + \frac{V_2 - V_3}{3} + \frac{V_2 - V_1}{8} = 0$$

$$12V_2 - 8V_3 - 3V_1 = 0 \qquad (2)$$

At V_3, $$\frac{V_3 + 110 - V_1}{2} + \frac{V_3 - V_2}{3} + \frac{V_3 - 110}{2} = 0$$

$$-\frac{V_1}{2} + \frac{V_3}{2} + \frac{V_3}{3} + \frac{V_3}{2} - \frac{V_2}{3} = 0$$

$$-\frac{V_1}{2} - \frac{V_2}{3} + \frac{4V_3}{3} = 0 \qquad (3)$$

Solving Eqns (1), (2) and (3) gives

$$\boxed{V_1 = 157.14 \text{ V}; V_2 = 94.28 \text{ V}; V_3 = 82.50 \text{ V}}$$

The following circuit shows current through all branches.

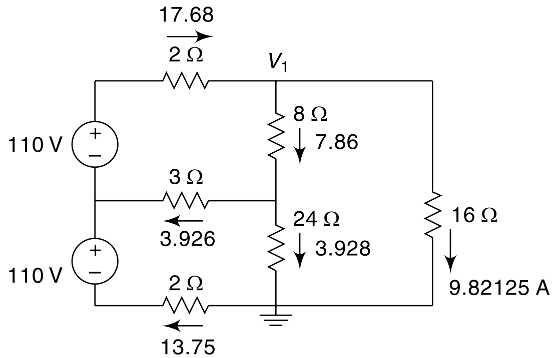


Fig. 7

Power dissipated $= 110 \times 17.68 + 110 \times 13.75$

$= 3457.3$ W

Power consumed $= 2(17.68)^2 + 8(7.86)^2 + 3(3.926)^2 + 24(3.93)^2 + 16(9.82)^2 + 2(13.75)^2 = 3457.36$ W

Since power dissipated is equal to power consumed, Tellegen's is theorem is verified.

(b) A current of 5 A flows through a non-inductive resistance in series with a chocking coil when supplied at 250 V, 50 Hz. If the voltage across the non-inductive resistance is 12 5 V and that across the coil is 200 V, calculate impedance, reactance and resistance of the coil, and power absorbed by the soil. Also draw the phasor diagram.

[8 + 8]

Ans. The current flowing through the circuit is 5 A.

$$R = \frac{125}{5} = 25\ \Omega$$

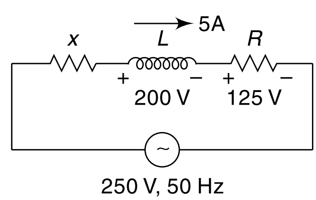


Fig. 8

The equivalent impedance is

$$z_{eq} = \sqrt{(25 + x)^2 + (\omega L)^2}$$

Applying ohm's law, $V = IZ_{eq}$

$$\Rightarrow \qquad 250 = 5 \cdot \sqrt{(25 + x)^2 + (\omega L)^2}$$

$$\sqrt{(25 + x)^2 + (\omega L)^2} = 50 \qquad (1)$$

Similarly for coil,

$$V = IZ\text{eq}$$

$$200 = 5\sqrt{x^2 + (\omega L)^2}$$

$$x^2 + (\omega L)^2 = 1600 \qquad (2)$$

From (1), $625 + x^2 + 50x + (\omega L)^2 = 2500$

$$625 + 1600 + 50x = 2500 \qquad \text{[from (2)]}$$

$$x = 5.5\ \Omega$$

Substituting the value of x in Eqn (2),

$$(\omega L)^2 = 40^2 - x^2$$

We know $f = 50$ Hz

$\Rightarrow$ $L = 0.126$ H

$$\omega L = 39.62$$

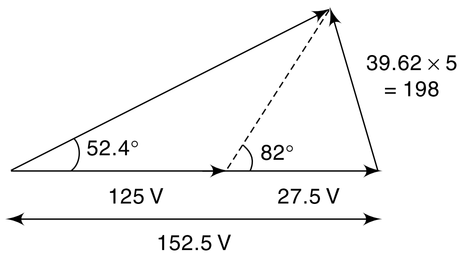


Fig. 9

Impedance of coil = 40 Ω

Reactance of coil = 39.62 Ω (= ωL)

Resistance of coil = x = 5.5 Ω

Power absorbed by coil = 5 × 200 = 1000 W = 1 kW

3. (a) Define incidence matrix. For the graph shown in Fig. 10, find the complete incidence matrix.

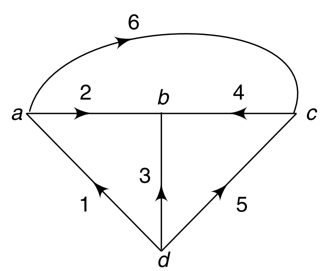


Fig. 10

Ans. Given graph is

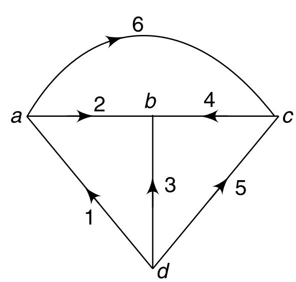


Fig. 11

The incidence matrix for the graph can be written as

$$A = \begin{array}{c} \\ \text{nodes} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \text{branches} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) Two impedance $Z_1 = 10 + j31.4$ ohms and $Z_1 = (10 + R) + j(31.4 - X_c)$ ohms are connected in parallel across a single-phase ac supply. The current taken by the two impedance branches are equal in magnitude and the phase angle between them is 90°. Calculate the value of R and X_C and phase difference of the branch currents with respect to the applied voltage. **[8 + 8]**

Ans. Given: $z_1 = 10 + j31.4$

$z_2 = (10 + R) + j(31.4 - X_C)$

$I_1 = I_2$

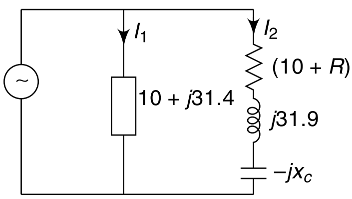


Fig. 12

Since the same voltage is applied to both impedances,

$\therefore$ $|Z_1| = |Z_2|$ and

it is given that $\tan^{-1}\phi_1 - \tan^{-1}\phi_2 = 90°$

From, $|Z_1| = |Z_2| \Rightarrow \sqrt{R^2 + X^2} = \sqrt{R_1^2 + X_1^2}$

$$R^2\left(1 + \frac{X^2}{R^2}\right) = X_1^2\left(1 + \frac{R_1^2}{X_1^2}\right) \quad (1)$$

and $\tan^{-1}\phi_1 - \tan^{-1}\phi_2 = 90°$ (given)

$$\tan^{-1}\left(\frac{\frac{X}{R} - \frac{X'}{R'}}{1 + \frac{X}{R}\cdot\frac{X'}{R'}}\right) = 90°$$

$$\Rightarrow \quad 1 + \frac{X}{R}\cdot\frac{X'}{R'} = 0$$

$$\Rightarrow \quad \frac{X}{R} = -\frac{R'}{X'} \quad (2)$$

Substituting Eq. (2) in (1),

$\Rightarrow$ $R = -X_1$

and $R_1 = X$

$\Rightarrow$ $R = 10 = -X_1$

and also $X = 31.4\ \Omega$

$\therefore$ $R_1 = 31.4\ \Omega$

$\Rightarrow$ $X_C = 41.4\ \Omega$

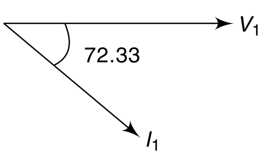


Fig. 13(a)

Phase difference $\phi_1 = \tan^{-1}\left(\frac{31.4}{10}\right) = 72.33°$

$$\phi_2 = \tan^{-1}\left(\frac{31.4 - 41.4}{21.4 + 10}\right)$$

$= -17.66°$

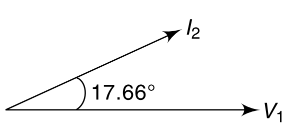


Fig. 13(b)

4. (a) State and explain Tellegen's theorem.

Ans. Refer Sections 5.1.8 and 5.2.8.

(b) For the network shown in Fig. 14, determine (i) resonance frequency, (ii) input admittance at resonance, (iii) quality factor, and (iv) bandwidth.

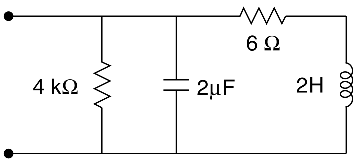


Fig. 14

Ans. Let $L = 2\text{H}$, $R = 6\ \Omega$, $C = 2\mu\text{F}$

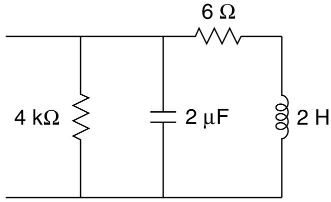


Fig. 15

$$\omega_o = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

$$LC = 2 \times 2 \times 10^{-6}$$

$$= 4 \times 10^{-6}$$

$$\frac{1}{LC} = 2{,}50{,}000$$

$$\left(\frac{R}{L}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$$

Substituting these in ω_o equation,

$$\omega_o \approx 500$$

$$f_o = 79.576 \text{ Hz}$$

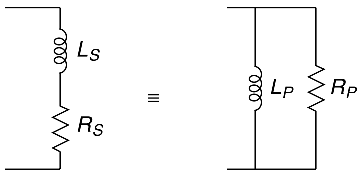


Fig. 16

(ii) $$Q_C = \frac{\omega_o L}{R}$$

$$= \frac{500 \times 2}{6}$$

$$= 166.67$$

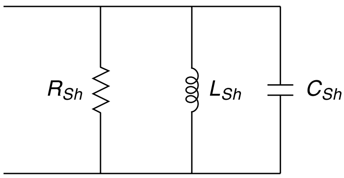


Fig. 17

$$L_P = L_S\left(1 + \frac{1}{Q_C^2}\right) \approx 2 \text{ H}$$

$$R_P = R_S(1 + Q_C^2) = 166.67 \text{ k}\Omega$$

$$R_{Sh} = R_P \,||\, 4k = \frac{166.67k \times 4k}{170.67k} = 3.906 \text{ k}\Omega$$

$$L_P = L_{Sh} = 2 \text{ H}$$

$$C_{Sh} = C = 2\ \mu\text{F}$$

Quality factor of the entire circuit,

$$Q_e = \frac{R_{Sh}}{\omega_o L_{Sh}} = \frac{3.906k}{500 \times 2} = 3.906$$

$$\text{Bandwidth} = \frac{f_o}{Q_e} = \frac{79.576}{3.906} = 20.37 \text{ Hz}$$

$$\begin{aligned}\text{Admittance} &= \frac{1}{R_{Sh}} - \frac{j}{X_L} + \frac{j}{X_C} \\ &= \frac{1}{R_{Sh}} - \frac{j}{\omega L_{Sh}} + j\omega C_{Sh} \\ &= 0.256 \times 10^{-3} - j \times 10^{-3} + j \times 10^{-3} \\ &= 0.256 \times 10^{-3} \text{ (at resonance)}\end{aligned}$$

5. (a) Two coils *A* and *B* having turns 100 and 1000 respectively are wound side by side on closed circuit coil of mean length 80 cm and 80 cm² X-section area. The relative permeability of iron is 900. Calculate the mutual inductance between the coils.

Ans. Given $N_1 = 100$; $N_2 = 1000$

X-section area $A = 8 \text{ cm}^2 = 8 \times 10^{-4} \text{ m}^2$

and mean length $l = 80 \text{ cm} = 0.8 \text{ m}$

Relative permeability $\mu_r = 900$

and $\mu_o = 4\pi \times 10^{-7}$

Mutual inductance between the coils

$$\begin{aligned}\mu &= \frac{N_1 N_2 \mu_o \mu_r a}{l} \\ &= \frac{100 \times 1000 \times 900 \times 4\pi \times 10^{-7} \times 8 \times 10^{-4}}{0.8} \\ &= 113.09 \times 10^{-3} \text{ H} \\ &= 0.113 \text{ H}\end{aligned}$$

5. (b) Determine the current through load resistance $R_L = 5\ \Omega$ for the circuit shown in Fig. 18 using Thevenin theorem. Also find the maximum power transfer to the resistance R_L. **[7 + 9]**

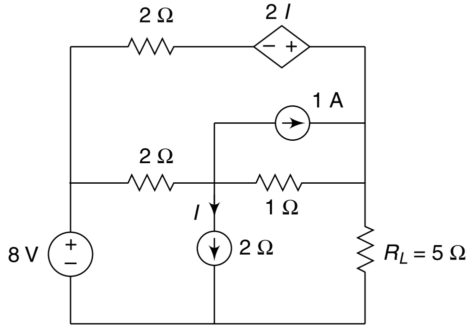


Fig. 18

Ans. *To find* V_{th}:

The equivalent circuit is

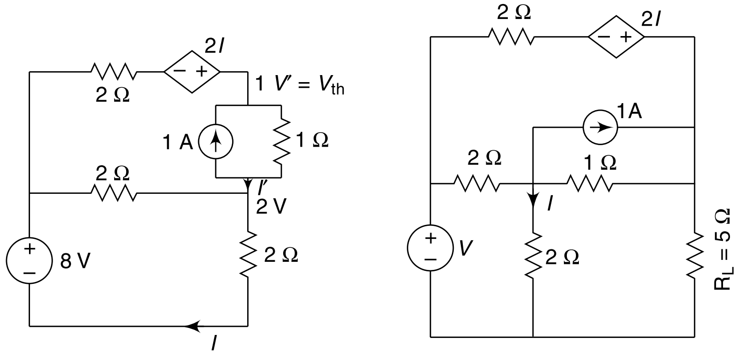


Fig. 19

Let the node voltages at nodes 1 and 2 be V_{th} and V'.

$$V = 2I \tag{1}$$

Applying KCL at Node (1),

$$I' = \frac{V' - V}{1} - 1$$

$$\Rightarrow \qquad I' + 1 = V' - V \tag{2}$$

Applying KCL at Node (2),

$$\frac{V-8}{2} + \frac{V}{2} = I'$$

$$I' = V - 4 \tag{3}$$

Substituting (3) and (1) in (2),

$$\frac{8 - V' + 2I}{2} + 1 = V' - V$$

$$3V' - 3V = 10 \tag{4}$$

$$V - 4 + 1 = V' - V$$

$$V' - 2V = -3 \tag{5}$$

Solving (4) and (5),

$$3V' = 29$$

$$\Rightarrow \qquad V' = \frac{29}{3}$$

$$\Rightarrow \qquad V_{th} = 9.667 \text{ V}$$

$$R_{th} = \frac{V_{th}}{I_{SC}}$$

To find I_{SC}.

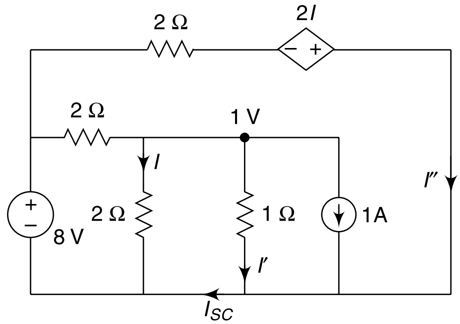


Fig. 20

Applying KCL at Node 1,

$$\frac{V-8}{2}+\frac{V}{2}+\frac{V}{1}+1=0$$

$$2V=3 \Rightarrow V=3/2=I'$$

$$I''=\frac{8+2I}{2}=\frac{8+V}{2}=\frac{8+1.5}{2}=4.75\text{ A}$$

$$I_{SC}=I'+1+I'' \text{ (KCL at ground)}$$

$$=1.5+1+4.75=7.25\text{ A}$$

$$R_{\text{th}}=\frac{V_{\text{th}}}{I_{SC}}=\frac{9.667}{7.25}=1.33\ \Omega$$

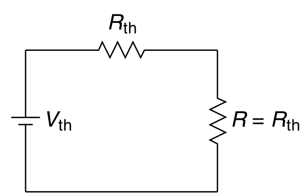


Fig. 21

For maximum power transfer,

$$\text{Max power transfer } =\frac{V_{\text{th}}^2}{4R_{\text{th}}}$$

$$=17.56\text{ W}$$

6. (a) Express *Y*-parameters in terms of *ABCD* and *Z*-parameters.

Ans. Refer Sections 6.8.1 and 6.8.2.

(b) Determine the *h*-parameters of the following network as shown in Fig. 22. **[7 + 9]**

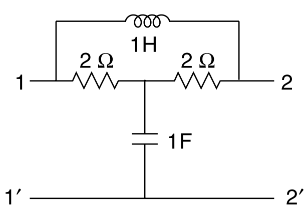


Fig. 22

Ans.

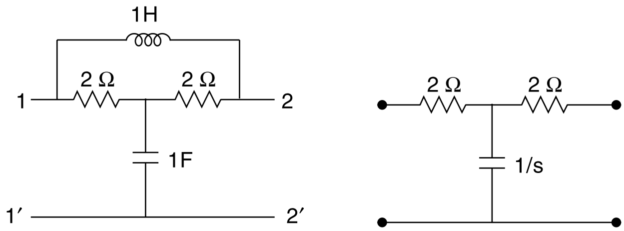


Fig. 23

First, we'll consider the T-network:

To simplify the computation, we'll transform it to a π-network:

$$R_1 = \frac{2\left(\frac{1}{s}\right) + 2\left(\frac{1}{s}\right) + 4}{\frac{1}{s}}$$

$$= 4 + 4s$$

$$R_2 = \frac{4 + 4/s}{2} = 2 + \frac{2}{8}$$

$$R_3 = 2 + \frac{2}{s}$$

$$R_P = R_1 \parallel s$$

$$= \frac{(4+4s)s}{4+4s+s}$$

$$= \frac{4s+4s^2}{4+5s}$$

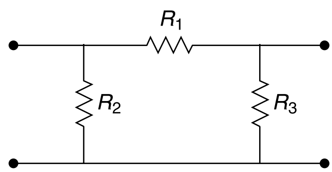


Fig. 24

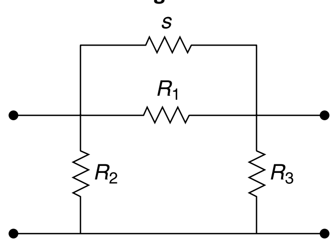


Fig. 25

The equivalent circuit is

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_a + Y_c & -Y_c \\ -Y_c & Y_b + Y_c \end{bmatrix}$$

$$Y_a + Y_c = Y_b + Y_c = \frac{2s+2}{s} + \frac{4s+4s^2}{4+5s}$$

$$= (2s+2)\left(\frac{1}{s} + \frac{2s}{4+5s}\right)$$

$$= (2s+2)\left[\frac{4+5s+2s^2}{s(4+5s)}\right]$$

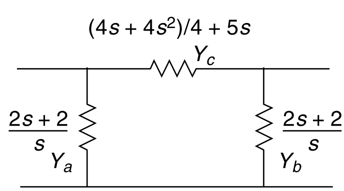


Fig. 26

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{(2s+2)(4+5s+2s^2)}{s(4+5s)} & \dfrac{(4+4s)s}{4+5s} \\ \dfrac{-(4+4s)s}{4+5s} & \dfrac{(2s+2)(4+5s+2s^2)}{s(4+5s)} \end{bmatrix}$$

$$h_{11} = \frac{1}{Y_{11}} = \frac{s(4+5s)}{(2s+2)(4+5s+2s^2)}$$

$$h_{12} = \frac{-Y_{12}}{Y_{11}} = \frac{-(4+4s)s}{(4+5s)}\left[\frac{s(4+5s)}{(2s+2)(4+5s+2s^2)}\right]$$

$$= \frac{-2s^2}{2s^2+5s+4}$$

$$h_{21} = \frac{Y_{12}}{Y_{11}} = \frac{2s^2}{2s^2+5s+4}$$

$$h_{22} = \frac{\Delta Y}{Y_{11}} = \frac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_{11}} = Y_{22} - \left(\frac{Y_{12}}{Y_{11}}\right)Y_{21}$$

$$= \frac{(2s+2)(4+5s+2s^2)}{s(4+5s)} - \frac{2s \times 2s}{2s^2+5s+4}\left(\frac{2+2s}{4+5s}\right)$$

$$= \frac{2+2s}{4+5s}\left[\frac{4+5s+2s^2}{s} - \frac{4s^2}{2s^2+5s+4}\right]$$

$$= \frac{2+2s}{4+5s}\left[\frac{(4+5s+2s^2)^2 - 4s^3}{s(2s^2+5s+4)}\right]$$

7. In a series *RLC* circuit, *R* = 6, ohms, *L* = 1 H, *C* = 1 F. A dc voltage of 40 V is applied at *t* = 0. Obtain the expression for *i*(*t*) using differential equation approach. Explain the procedure to evaluate conditions. [16]

Ans. Let the current through the circuit be $i(t)$.

Applying KVL around the loop,

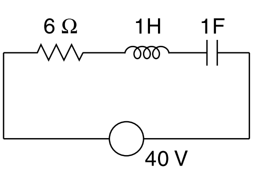


Fig. 27

$$40 = i(t)R + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

Differentiating on both sides,

$$i_n = R\frac{d}{dt}i(t) + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}i(t) = 0$$

$$\frac{d^2 i(t)}{dt^2} + 6\frac{di(t)}{dt} + i(t) = 0$$

$$(D^2 + 6D + 1)\, i(t) = 0$$

Solving the equation gives

$$D = -0.171, -5.828$$

$$i_f = \frac{40}{6} = 6.66$$

$$i(t) = Ae^{-0.171t} + Be^{-5.828t} + 6.66$$

$$i(0) = 0 \quad \Rightarrow \quad A + B + 6.66 = 0$$

$$A + B = -6.66$$

$$L\frac{di(t)}{dt}\bigg|_{t=0} = 0 \quad \Rightarrow \quad A(-0.171) + B(-5.828) = 0$$

$$A = 226.98,\ B = -6.66$$