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Gray and Nongray Planetary Atmospheres

Structure, Convective Instability, and Greenhouse Effect

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Analytic expressions are derived for the atmospheric structure, greenhouse effect, and level of onset of convective instability in window gray and nongray planetary atmospheres. Where pressure-induced transitions or dust are principal sources of infrared opacity, gray approximations may have a strict relevance; but, even when permitted transitions are the opacity sources, the appropriate expressions for the nongray atmosphere are quite similar to those for the gray case. Constant lapse rates should occur in deep radiative equilibrium as well as deep convective equilibrium atmospheres. Atmospheres with polyatomic, condensable gases, pressure-sensitive opacities, and long-wavelength windows tend to have higher tropopauses and more extensive convective tropospheres than atmospheres with converse properties. The outer Jovian planets may have very extensive outer envelopes in radiative rather than convective equilibrium. The mean infrared optical depth, τ, for a nongray convective atmospheric greenhouse on the Earth is about 2; for Venus, several hundred. In the window gray case, T increases with τ about twice as fast for convective as for radiative atmospheres; in the nongray case, about four times as fast.

INTRODUCTION

The line character of molecular rotation–vibration spectra implies that a real planetary atmosphere is distinctly nongray in the infrared region of the spectrum central to studies of heat balance and atmospheric structure. Attempts to apply the gray approximation uncritically can lead, at least in principle, to significant errors. Nevertheless, there are a number of applications in which appropriate gray approximations can give a useful first estimate of the physical circumstances; there are also certain conceivable types of planetary atmospheres which can more rigorously be considered gray. In the study of stellar atmospheres, also, the gray assumption has proved a useful first approximation.

In some applications of the present paper, we consider mean opacities of the Rosseland type

$$(\bar{k}_R)^{-1} = \int_0^\infty (k_\nu)^{-1} \frac{dB_\nu}{dT} d\nu \bigg/ \int_0^\infty \frac{dB_\nu}{dT} d\nu, \qquad (1)$$

where k_ν is the mass extinction coefficient ($cm^2\ g^{-1}$), ν the frequency, T the absolute temperature, and B_ν the Planck distribution function for blackbody radiation. From (1) it is evident that $\bar{k}_R$ heavily weights window regions of the spectrum. Where, as is often the case, there is one dominant window in the absorption spectrum of a planetary atmosphere, $\bar{k} \simeq k_w$, the extinction coefficient of the window region. In the Earth's atmosphere, such a dominant window region exists between 8 and 13 μ, due to an absence of absorption by CO_2 and H_2O. A similar, somewhat wider, window exists in the Martian atmosphere. In the lower atmosphere of Venus, CO_2 hot bands such as those at 8.4 and at 10.4 μ absorb in the 8–13 μ interval, and the principal window region moves to slightly shorter wavelengths (Sagan, 1960; Pollack, 1969*a*). In the Jovian atmosphere, NH_3 strongly absorbs at 8–13 μ, and the principal expected windows are at wavelengths of 14–30 μ, approximately (Trafton, 1967).

However, the Rosseland mean opacity is useful only when the corresponding optical depth exceeds unity. The clearest demonstration of the inapplicability of $\bar{k}_R$ for small τ occurs when $k_\nu = 0$ at one frequency; then $\bar{k}_R = 0$. Thus, for small optical depths, $\bar{k}_R$ overemphasizes window regions, leads to smaller opacities, and, therefore, to smaller radiative equilibrium temperature gradients than will actually obtain. For small τ, another mean opacity, e.g., the Chandrasekhar mean, or the Planck mean, will be more appropriate.

In this paper, we distinguish among (1) the pure gray case, in which k_ν is closely independent of ν everywhere; (2) the window gray case, in which radiative transport occurs primarily through one window region, w, which can be considered gray; and (3) the nongray case, in which the atmosphere is not as cooperative as in cases (1) and (2). Where permitted transitions in the vibration–rotation spectra of gases dominate the absorption spectrum, the nongray case applies. Thus, for such planets as the Earth and Mars, the gray and window gray approximations are least applicable. Nevertheless, as we show later, even for these planets the gray approximations provide a fair description of the observed atmospheres. Where pressure-induced dipole or quadrupole transitions are the dominant opacity source, the variation of k_ν with ν will be quite smooth, and the window gray approximation, $\bar{k} = k_w$, will be appropriate. This circumstance has some applicability to the Jovian planets, where pressure-induced transitions of H_2 are dominant (although permitted transitions of NH_3 and CH_4 also occur), and, to a lesser extent, to the lower atmosphere of Venus.

Gray and window gray approximations are also valid in several circumstances in which dust is a principal source of extinction. Consider the circumstance that $x \equiv 2\pi a/\lambda > 1$ for all wavelengths of interest; here a is the mean particle radius, and λ the wavelength of light. Then the extinction cross section will be independent of wavelength (see van de Hulst, 1957). If, further, the particle is large enough that $ak_\nu \gg 1$—so that the frequency variation of opacity has little influence on the particle transmissivity—then the absorption cross section will also be wavelength-independent. We note that the scattering component, a diffraction term, can be ignored. Thus, this large particle circumstance lends to a pure gray case.

For very small particles, $x \ll 1$, a window gray case arises independent of the absorption coefficient, provided only that there is some absorption. In this circumstance (see van de Hulst, 1957, p. 66), the absorption cross section is directly proportional to $\lambda^{-1}a^3$, while the scattering cross section is directly proportional to $\lambda^{-4}a^6$. For sufficiently small particles, the former dominates, and the extinction cross section has a smooth λ^{-1} dependence, superimposed on which is the absorption spectrum intrinsic to the wavelength dependence of the imaginary part of the particle's refractive index. This dependence is generally smooth. The convolution of these two effects often yields a dominant window region, and a valid window gray approximation.

Thus there are a sufficient number of cases of interest to warrant a fairly thorough investigation of the gray and window gray cases in a form which can be applied conveniently to real planetary atmospheres. In the following discussion we investigate the structure of gray and window gray atmospheres in radiative-convective equilibrium, the question of the onset of convection, and the characterization of the atmospheric region at which convection begins, and then apply the results to several illustrative examples. We finally compare gray with nongray solutions in the transmissivity-average approach and consider limiting cases.

Radiative Equilibrium in Pure Gray and Window Gray Atmospheres

Radiative equilibrium calculations for pure gray planetary atmospheres date back to the work of K. Schwarzschild, E. Gold, and R. Emden. Here we wish to generalize the discussion to the window gray case, allowing for the dependence of optical depth on pressure and, to a small extent, on path.

In the following discussion, we assume that the optical depth in a frequency bandwidth w is

$$\tau_w = C_w W^r P^s. \tag{2}$$

Here W is the reduced path of absorbers or scatterers, P is the ambient pressure, and r and s are dimensionless constants. Howard, Burch, and Williams (1956), and Burch, Gryvnak, Singleton, France, and Williams (1962) give numerous examples of gases which—at least over a limited range of pathlengths and pressures—behave approximately as in Eq. (2). A detailed discussion of the validity of Eq. (2) for CO_2 and H_2O is presented in the accompanying paper by Pollack (1969*a*); see also Bartko and Hanel (1968).

The usual equation of radiative transfer implicitly assumes $r = 1$, as in the standard subsidiary condition

$$d\tau = -\rho k\, dz, \tag{3}$$

where ρ is the atmospheric density, and z, the altitude above the planetary surface. We might be tempted to generalize (3) to

$$d\tau = -k' \rho^r z^{r-1}\, dz \propto d(W^r). \tag{4}$$

However, if we integrate (4) between two atmospheric levels, A and B, we find $\tau \propto W_B{}^r - W_A{}^r$. But Eq. (2) implies that $\tau \propto (W_B - W_A)^r$. Thus, the observations imply a failure of superposition (see Pollack, 1969*b*), which the formulation of Eq. (4) in general fails to reproduce. The exceptions are when $W_B \gg W_A$, or when $r \simeq 1$. In the case of these exceptions, it is possible to write down the appropriate equation of transfer

$$\mu^r(dI_\nu/d\nu) = I_\nu - \mathscr{S}_\nu{}', \qquad |r-1| \ll 1, \tag{5}$$

where I_ν is the specific intensity, and $\mathscr{S}'_\nu$, the source function, and then make the usual Eddington approximation to the equation of radiative transfer for the pure gray case. Taking care to preserve real values only we find

$$T^4 = T_e{}^4[\tfrac{1}{2} + \tfrac{1}{4}(r+2)\,\bar{\tau}], \qquad |r-1| \ll 1, \tag{6}$$

where T_e is the effective radiating or equilibrium temperature of the planet, and $\bar{\tau}$ is the mean optical depth and is proportional to $\bar{k}'$. Thus when r is very close to unity, the departure from the usual Eddington solution is small; when r is not very close to unity, the Eddington formulation is invalid. The latter situation is typical of the nongray case, and will be considered further in the last sections of the present paper. In the following pages, we carry $r \neq 1$ for completeness, but the results apply only when $|r-1| \ll 1$. We also employ the Eddington approximation, considering fluxes only; the approximation is known to be quite accurate, even to rather small optical depths (Sobolev, 1963; Goody, 1964; Irvine, 1968).

The usual pure gray Eddington solution assumes that the source function can be represented as σT^4, where σ is the Stefan-Boltzmann constant. But for the window gray case this assumption fails. It is nevertheless possible to write the source function in the window region in the same form:

$$B_w \propto T^n, \tag{7}$$

where n is a positive constant, not necessarily an integer. This has proved to be a useful approximation in several other radiative-transfer problems for planetary atmospheres, provided very large changes in T do not occur (see, e.g., Pollack and Sagan, 1965). An estimate of its range of validity can be obtained from the following successive approximations argument: We define n (appropriately normalized) in terms of logarithmic derivatives

$$n \equiv d \ln B / d \ln T. \tag{8}$$

Explicitly evaluating the Planck function, we find

$$n = b\, e^b/(e^b - 1), \qquad b = h\nu_w/kT, \tag{9}$$

where h and k are Planck's and Boltzmann's constants, respectively. In all applications of the present paper the Wien approximation to the Planck distribution roughly holds, $e^b \gg 1$, and

$$n = h\nu_w/kT. \tag{10}$$

Since, then, for any small variation in T, n varies by a proportionate amount, in what sense can Eq. (7) with n constant be considered valid? We take $B = B_{ref}$ $(T/T_{ref})^n$, and substitute in (8), taking care not to set terms in dn equal to zero. In the zeroth approximation, $dn/dT = -h\nu/kT^2$ from (10), and a new value of n is obtained. Proceeding by successive approximations we find

$$n_j = (h\nu_w/kT)\{1 + [\ln(T/T_{ref})]^j\}, \tag{11}$$

where j is the order of the approximation. Thus, the iteration converges to Eq. (10) provided $T/T_{ref} < e$; the approximation is valid over a substantial range in absolute temperatures. For larger ranges, the atmosphere can be divided into layers characterized by different values of n, each with its own window frequency, ν_w, and mean temperature, $\bar{T}$. Accordingly, we rewrite (6) as

$$T^n = T_e{}^n[\tfrac{1}{2} + \tfrac{1}{4}(r+2)\,\bar{\tau}]. \tag{12}$$

The temperature at the top of the atmosphere, where $\bar{\tau} = 0$, is

$$T_0 = 2^{-1/n}T_e. \tag{13}$$

For an absorber or scatterer distributed in a planetary atmosphere with a constant mixing ratio, $W = qP$, where q is a proportionality constant; thus

$$\tau_w = C_w q^r P^{r+s}. \tag{14}$$

Throughout the discussion, q, r, and s refer to the dominant source of extinction in the window region. If there are several sources, mean values of these parameters must be estimated. The exponent s is a measure of pressure-broadening for a gas, or, for a cloud, of the vertical distribution of cloud particles; this latter use of s has been made, e.g., in Pollack and Sagan (1965). Are there lower limits on s? Combining (14) with the equation of hydrostatic equilibrium,

$$dP = +\rho g\,dz, \tag{15}$$

where g is the local acceleration due to gravity (P increasing with z, measured from a reference point near the top of the atmosphere) for the dust case ($r = 1$), we find

$$\frac{d\bar{\tau}}{dz} = (s+1)\frac{g\bar{\tau}}{RT}, \tag{16}$$

where R is the universal gas constant. Since $d\bar{\tau}/dz$ is positive definite, $s > -1$. From the equation of continuity for mass transport one can demonstrate (R. Wattson, 1968, private communication) the stronger theorem that $s > 0$ everywhere. Therefore, unless additional physics—e.g., condensation—is introduced, no cloud stratification is possible; if the visible clouds of a planet are dust, there should be continuous dust clouds down to the surface.

We are now in a position to calculate the atmospheric structure in radiative equilibrium for the case that the principal source of extinction is uniformly mixed. We obtain $dP/d\bar{\tau}$ from Eq. (14) and $d\bar{\tau}/dT$ from Eq. (12). The resulting expression for $dP/dT = (dP/d\bar{\tau})\,(d\bar{\tau}/dT)$ is integrated subject to the boundary condition that $T \to T_0$ as $P \to 0$. Thus,

$$P^{r+s} = \frac{4}{C_w q^r(r+2)}\left[\left(\frac{T}{T_e}\right)^n - \frac{1}{2}\right]. \tag{17}$$

We then evaluate $dT/dz = (dT/dP)(dP/dz)$, using the equation of hydrostatic equilibrium. The result is

$$\left(\frac{dT}{dz}\right)_{rad} = +\frac{g}{c_p}\frac{\gamma(r+s)}{n(\gamma-1)}\left[1 - \left(\frac{T_0}{T}\right)^n\right]. \tag{18}$$

Here c_p is the specific heat capacity at constant pressure, and $\gamma \equiv c_p/c_v$ is the ratio of specific heats. In deep atmospheres, g, c_p, γ, r, and s will all depend on z, but in most applications they can be taken as constants. We have derived Eq. (18) by an integral argument in order to display, e.g., P–τ, T–τ, and P–T relations. Alternatively we could have derived (18) more rigorously by using only a derivative formulation involving dB_w/dT, and avoiding T–τ relations.

Equation (18) indicates that the radiative temperature gradient is independent of the absolute value of the extinction coefficient or mixing ratio, and depends instead on pressure- and abundance-broadening and the vertical distribution of aerosols. If we define ψ by $T \propto P^{\psi}$, the same remark is true for ψ. Since $n > 3$ for almost all cases of interest, $T^n \gg T_0{}^n$ at relatively modest depths. Thus, from (17), with the approximations already adopted, $\psi \simeq (r+s)/n$; except at the very top of a gray radiative equilibrium atmosphere, the structure is independent of C_w and q, and depends instead on r, s, and ν_w.

From (18) we find that as $T \to T_0$, $(dT/dz)_{rad} \to 0$; towards the top of the atmosphere an approximately isothermal region should exist. Such a region, the stratosphere, is of course known from micro-wave occultation experiments for Mars and Venus, and has long been known for the Earth. For the reasons described earlier, Rosseland mean opacities do not apply for this case of small optical depth. Towards high altitudes, the integrated absorption declines, and new atmospheric windows open. Accordingly, n will change —probably markedly [see Eq. (10)]—and, by (18), this will alter the value of the temperature gradient for small $\bar{\tau}$, and the level at which isothermality is reached. But the trend towards an isothermal stratosphere is clear. Equation (18) also implies that as T becomes larger then T_0, $(dT/dz)_{rad} \to$ const; below the isothermal region, a region of approximately constant temperature gradient should exist if radiative equilibrium prevails. This argument suggests that constant subadiabatic lower atmospheric lapse rates may in certain cases be a consequence of radiative rather than convective equilibrium. However, at some point in the lower atmosphere, if it is sufficiently deep, convective equilibrium will prevail—as discussed in the following section.

Convective Equilibrium in a Window Gray Atmosphere

The Schwarzschild instability criterion indicates that convection breaks out when the local or structural temperature gradient exceeds the adiabatic temperature gradient; in our case, when

$$|(dT/dz)_{rad}| > |(dT/dz)_{ad}|. \qquad (19)$$

We adopt

$$(dT/dz)_{ad} = -g/\eta c_p, \qquad (20)$$

where η, the subadiabatic index, is a measure of departures from the dry adiabatic lapse rate due, e.g., to the release of latent heat of condensation (see Sagan, 1962). Strictly speaking, η is also a function of z, but Eq. (20) is a convenient first approximation; in the terrestrial troposphere, $\eta = 1.6$ fairly closely. Combining (18), (19), and (20), we find the instability condition to be

$$\frac{\eta}{n(\gamma-1)}\frac{(r+s)}{}\left[1-\left(\frac{T_0}{T}\right)^n\right] > 1. \qquad (21)$$

In the case of classic astronomical interest ($n = 4$, $\eta = 1$, $r = 1$, $s = 0$, $T \gg T_{\bullet}$), Eq. (21) reduces to $\gamma < \frac{4}{3}$, as it should. Equation (21) can be put in neater form by combining with Eqs. (12) and (13); instability occurs when the weighted mean optical depth, $\bar{\tau}$, exceeds some critical value, $\bar{\tau}_t$ (the subscript indicating tropopause):

$$\bar{\tau}_t = \frac{2}{r+2}[(1-\Xi)^{-1}-1], \qquad (22)$$

where

$$\Xi \equiv \frac{n(\gamma-1)}{\eta\gamma(r+s)} = \frac{h\nu_w(\gamma-1)}{\eta\gamma(r+s)\,k\bar{T}}; \qquad (23)$$

$\bar{T}$ is a mean temperature over the atmospheric depths of interest.

In most applications, n is reasonably fixed; therefore $\bar{\tau}_t$ is approximately independent of the atmospheric depth chosen for $\bar{T}$. In Fig. 1 is a plot of $\bar{\tau}_t$ vs. Ξ. The parameter Ξ is constrained to the range $0 < \Xi < 1$ for convective equilibrium. In fact from Eq. (21), deep in an atmosphere the instability condition is $\Xi < 1$. By comparing Eqs. (18) and (20) we see that, for the deep atmosphere,

$$\Xi = \frac{(dT/dz)_{ad}}{(dT/dz)_{rad}}.$$

According to the classical theory of specific heats, $(\gamma-1)/\gamma = 2/(f+2)$, where f is the

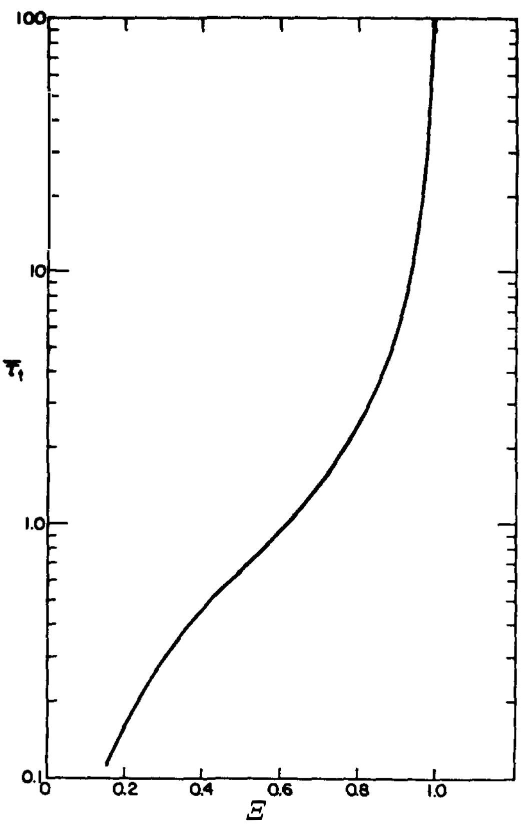

FIG. 1. Mean optical depth of the tropopause as a function of the parameter Ξ, for $r = 1$.

number of degrees of freedom of the molecule in question. It is evident that increasing f, η, or s, or decreasing ν_w decreases Ξ and therefore decreases $\bar{\tau}_t$: an atmosphere with polyatomic condensable gases, sources of extinction which are pressure-sensitive, and long-wavelength windows will tend to have a high tropopause and an extensive convective troposphere; conversely, an atmosphere with monatomic or diatomic noncondensable gases, sources of extinction which are not pressure-sensitive, and short-wavelength windows will tend to have a deep tropopause and a relatively modest troposphere. Because of modifications introduced below for the nongray case, it is not permissible to conclude that sources of extinction with large r will necessarily lead to high tropopauses.

For window gray calculations for Venus, Earth, Mars, and Jupiter, we take the appropriate values for the various parameters, adopt $r \simeq 1$ as our approximations demand, and $s \simeq \frac{1}{2}$ (see Pollack, 1969*b*); we find in all cases $\bar{\tau}_t \simeq 1$. Convective equilibrium occurs only below the level at which significant opacity develops. An interesting case exists for the outer Jovian planets, Saturn, Uranus, and Neptune. We invoke the fact that the chemical compositions of the atmospheres of the Jovian planets are very similar. With ν_w fixed, the equilibrium temperatures are so low for the outer Jovian planets that Ξ approaches unity, and $\bar{\tau}_t$ becomes very large; such planets have extremely extensive stratospheres, and tropopauses—if they exist at all—only at such great optical depths that they cannot be observed in the infrared. The received solar flux is too low to drive convection. It may be for this reason that the prominent atmospheric turbulence which characterizes Jupiter is less often observed in the more distant members of the Jovian group, even after allowance is made for the greater difficulty in observing these planets. Studies of the microwave spectra of the outer Jovian planets will be useful in elucidating their lower atmospheric structures.

In the case that the principal source of extinction is dust, which need not be uniformly mixed even in the absence of condensation, the dust must clearly be carried regularly into the atmosphere; otherwise it will fall out of the atmosphere, as described by the Stokes-Cunningham equation, in time scales short compared with the age of the solar system. This would seem to require a convective lower atmosphere. Hence, the dust will be vertically distributed in such a way that an adiabatic lapse rate is maintained; the dust will act to minimize the radiative *vis-a-vis* the convective energy flux. Thus, from (22) and (23) with $\eta = 1$, and $r = 1$,

$$s > \frac{n(\gamma - 1)}{\gamma} \frac{\bar{\tau} + \frac{2}{3}}{\bar{\tau}} - 1, \qquad (24)$$

specifying the relation between the vertical distribution and the optical depth. Typical lower limits on s obtained in this way for $\bar{\tau} > 0.5$ range from a few tenths (typical for the pure gray case, $n = 4$) to more than unity, values characteristic of gaseous opacity sources. In some cases of practical interest (e.g., deep CO_2 atmospheres) the right-hand side of inequality (24) becomes negative. In this case, s will take on larger values than the minimum necessary to maintain convective equilibrium. The absurdly high values of s required by Eq. (24) to maintain convection for small optical depths imply that (a) some other opacity sources besides dust (e.g., the free atmosphere) are important for small $\bar{\tau}$, and/or (b) convection is not maintained for small $\bar{\tau}$. We know on other grounds that both (a) and (b) are true.

At this point we wish to write the Schwarzschild instability criterion in terms of a critical tropopause temperature, T_t, instead of a critical tropopause mean optical depth, $\bar{\tau}_t$. From (13) and (21) convection breaks out when $T > T_t$, where

$$T_t = T_0[1 - \Xi]^{-1/n} = T_e[2(1 - \Xi)]^{-1/n}. \tag{25}$$

Some limiting cases are of interest: When $\Xi = 0$, $T = T_0$, and the entire atmosphere is in convective equilibrium. However, it is clear from the definition of Ξ [Eq. (23)] that this never occurs; accordingly, a nonconvective upper atmospheric region must always exist. When $\Xi = 1$, $T_t = \infty$, and the entire calculated atmosphere is in radiative equilibrium, with no convection to great depths. As we have seen $\Xi \geqslant 1$ is not excluded, particularly for low-temperature planets. Since, for the terrestrial planets and Jupiter a typical value of $\Xi \simeq 0.5$, we find that convective instability characteristically breaks out on such planets where

$$T = T_t = T_e.$$

This condition is of course connected with our previous result that $\bar{\tau}_t \simeq 1$ for these planets; we know from Eq. (12) that $T = T_e$ at $\bar{\tau} \simeq \frac{2}{3}$. Since convective clouds will, in general, exist only below the tropopause, the cloudtop temperature of an extensive planetary cloud layer will often have a temperature almost as low as $T_t \simeq T_e$. For Jupiter, however,

$$T_t \simeq T_e \simeq 120°\mathrm{K}$$

implies that the atmosphere is in convective equilibrium far above the "visible cloudtops," where the temperature is determined by spectroscopic techniques as 150–200°K. Thus 8–13 μ limb-darkening on Jupiter should be explicable by assuming pure absorption by ammonia in a convective atmosphere, the principal absorber in this wavelength interval. This in fact proves to be the case (Veverka and Sagan, 1969). For Venus, the deep subadiabatic region below the apparent tropopause (see Sagan and Pollack, 1969) might be thought due to the persistence of a constant lapse rate radiative equilibrium atmosphere down to the 400°K level [see Eq. (18)]. However, the foregoing considerations, and similar nongray calculations make this suggestion implausible. Deposition of sunlight in the atmosphere below the clouds is a possible alternative explanation of the subadiabacity [see Pollack, 1969*b*].

It is remarkable, considering the caveats appropriate to gray and window gray atmospheres, how well they account for the gross features of planetary atmospheres —particularly of the terrestrial planets where they are known to be least applicable. The reason for this success appears to lie in a convergence between the analytic expressions for window gray and nongray atmospheres for realistic special cases, as detailed below.

Structure and Convective Instability in Nongray Planetary Atmospheres

We now wish to compare the results derived up to this point for a window gray atmosphere with those implied for a nongray atmosphere in which there is one dominant opacity source at each frequency interval, and one window region, w, which makes the major contribution to the net flux. The relative extinction in the window region will be weak. In a companion paper, Pollack (1969*b*) has developed an approach, utilizing transmission-averaged opacities, to this nongray radiative-transfer problem which takes explicit account of the fact that superposition of optical depths no longer holds. One consequence of superposition failure is a greater tendency towards convective instability than applies in an otherwise comparable gray problem. From Pollack's Eq. (23), we can write

$$\psi_{rad}(z) \simeq \frac{3F(z)}{4\pi} \frac{C_w^{1/r_w} P^{(s_w/r_w+1)} \theta G^{1/r_w}}{n_w B_w \Gamma(1 + r_w^{-1})}. \quad (26)$$

Most of these quantities have counterparts in the gray formulation of the present paper: $T \propto P^{\psi(z)}$; $F(z)$ is the net thermal flux; $\tau_w \equiv C_w W^{r_w} P^{s_w}$; $\theta \equiv dW/dP$; $G(T)$, related to the Boltzmann factor in molecular spectroscopy, allows for the temperature dependence of optical depth: $\tau_w \propto G(T)$; $B_w \propto T^{n_w}$; and Γ is the usual gamma function. Seeking an asymptotic power law solution, we now scale the right-hand side of Eq. (26) in pressure: By assumption, $F \propto P^{-u}$ and $G \propto P^v$. In general, $u > 0$ and will be small, and $v > 0$. Accordingly,

$$\psi_{rad} \propto P^{(s_w/r_w)+1+(v/r_w)-u-\psi_{\mathrm{rad}} n_w}. \quad (27)$$

The presence of ψ_{rad} with a negative coefficient in the exponent implies that at sufficient depth ψ_{rad} will approach a constant value. This may be seen by evaluating Eq. (27) at some starting point and proceeding to levels of increasing atmospheric pressure. For u small, beginning with any value of ψ_{rad} brings us to larger ψ_{rad} as we proceed deeper into the atmosphere. At some point the entire exponent in Eq. (27) will reach zero. Thereafter ψ_{rad} will remain constant. Setting the exponent of Eq. (27) equal to zero, we obtain an asymptotic value for a deep planetary atmosphere:

$$\psi_{rad} \simeq \frac{s_w + r_w + v - ur_w}{n_w r_w}. \quad (28)$$

In actuality, ψ_{rad} will vary slowly because n_w is a slowly varying function of temperature, and u and v are not quite constants.

In the discussion following Eq. (18) for the window gray case we found

$$\psi_{rad} = (r + s)/n,$$

and $r \simeq 1$, the equivalent of Eq. (28) with $u = v = 0$. Thus, in both the window gray and the nongray cases, we have found the deep atmospheric structure of a planetary atmosphere to depend on pressure—and abundance—broadening, and not on the absolute values of extinction coefficients.

The structure of an adiabatic atmosphere can be represented as

$$\psi_{ad} = \frac{\gamma - 1}{\eta\gamma}. \quad (29)$$

Comparing with ψ_{rad} for the deep window gray atmosphere, we rederive the Schwarzschild convective instability criterion, Eq. (21), for $T^n \gg T_0{}^n$. In just the same way, a nongray instability condition for deep atmospheres can be derived by comparing (28) and (29):

$$r_w \Xi_w \equiv \frac{n_w r_w(\gamma - 1)}{\eta\gamma(s_w + r_w + v - ur_w)} < 1. \quad (30)$$

We see that convection is promoted by large v and small u.

We now consider a few explicit cases, to determine the likelihood that the lower portions of nongray planetary atmospheres will be in convective equilibrium.

In the case of the Earth, $\eta = 1.6$, $\gamma \simeq 1.4$. Detailed calculations indicate u is small and $s_w/r_w \simeq 1$. Therefore convective instability requires $n_w \lesssim 11[1 + \frac{1}{2}(v/r_w)]$, implying convective instability in the deep terrestrial atmosphere. For Venus, with $\eta \simeq 1.2$, at least in the upper troposphere (see Sagan and Pollack, 1969), and $\gamma/(\gamma - 1) \simeq 5$,

$$n_w \lesssim 13[1 + \tfrac{1}{2}(v/r_w)],$$

and $\lambda_w \gtrsim 4\mu$, again a condition we expect to be fulfilled. For Jupiter,

$$n_w \lesssim 7[1 + \tfrac{1}{2}(v/r_w)],$$

and $\lambda_w \gtrsim 20\ \mu$. Here convective instability is marginal, a situation enhanced as we proceed to the more distant Jovian planets. As in the gray case, there is a real question whether the deep atmospheres of Saturn, Uranus, and Neptune are in adiabatic equilibrium.

Particularly when there are few hot bands in the principal window region, but even in cases where hot bands are present, $v - r_w u \simeq 0$. Then, the structure of a nongray radiative requilibrium atmosphere and the nongray Schwarzschild instability criterion are identical to their window gray counterparts, with the exceptions: (1) n is replaced everywhere by $n_w r_w$, and (2) r is no longer restricted to values close to unity ($r_w \simeq \frac{1}{2}$ is more realistic for many gases). This then suggests that the principal analytic and numerical results of previous sections—including Eqs. (18), (22), (24), and (25)—remain valid if these two alterations are noted; and provides some explanation for the applicability of window gray calculations to distinctly nongray atmospheres.

The Greenhouse Effects in Radiative and in Convective Atmospheres

We are now in a position to calculate and compare the so-called greenhouse effects in radiative and convective atmospheres, and in gray and nongray atmospheres, using our previous approximations. We will then make several specific applications. The exact solution for the pure gray greenhouse problem has recently been discussed by Wildt (1966).

From $\tau_w = C_w P^{s_w} W^{r_w}$ for the nongray case, and the definition of ψ we have

$$\tau \propto T^{(s_w + r_w)/\psi}. \tag{31}$$

From (28) and (29), Eq. (31) specifies the $T - \tau$ relations for radiative and convective equilibrium:

$$\tau_{rad}/\tau_{ref} = (T/T_{ref})^{n_w r_w (s_w + r_w)/(s_w + r_w + r - u r_w)} \tag{32}$$

$$\tau_{ad}/\tau_{ref} = (T/T_{ref})^{\eta\gamma(s_w + r_w)/(\gamma - 1)} \tag{33}$$

These are the nongray counterparts of Eq. (12) for the deep atmosphere and of a similar adiabatic equation in the window gray case. In the following, we make the approximation $v - u r_w \ll s_w + r_w$. Defining E by $\bar{\tau} \propto T^E$ we find the results of Table I.

Table I is in accord with the relation between window gray and nongray cases discussed in the previous section; we note that $\Xi_w \propto n_w$. Typical values for Ξ for the planets out to Jupiter are 0.5 to 0.7; characteristically, $r_w \simeq \frac{1}{2}$. Thus, in the window gray case, T increases with τ about twice as fast for convective as for radiative atmospheres; in the nongray case, about four times as fast. This implies that the use of, e.g., the Eddington solution to the transfer equation—derived under assumption of radiative equilibrium—to calculate the optical depth needed in a given greenhouse context (as, e.g., used by Jastrow and Rasool, 1963) seriously underestimates the required optical depths. A similar conclusion has been reached by Wattson (1968), using the discrete ordinate method for the solution of the transfer equation. Table I also implies that the radiative slope of the T–τ relation is about twice as steep in the gray as in the nongray case. While Table I superficially suggests identical convective slopes of the T–τ relation for gray and nongray cases, this is not correct, because r takes on different values in the two cases. For example, for $s = s_w = \frac{1}{2}$, $r = 1$, and $r_w = \frac{1}{2}$, $(n/\Xi) \simeq 1.5(n_w/\Xi_w)$. Thus the convective slope is less steep in the nongray than in the gray case. Accordingly, smaller optical depths are required to reach a given surface temperature in the nongray than in the gray case, a result of relevance to the Venus greenhouse problem. As a final comment on Table I, we note that the slopes are generally steeper in window gray than in pure gray circumstances, because, in most applications, $n > 4$.

TABLE I

Exponents, E, of the T–$\bar{\tau}$ Relation

	Radiative	Convective
Window gray	n	n/Ξ
Nongray	$r_w n_w$	n_w/Ξ_w

If we differentiate $T \propto P^\psi$ with respect to z, and utilize the equations of state and of hydrostatic equilibrium, we find that $dT/dz \propto (s + r)/n$ for the window gray radiative case, and $dT/dz \propto (s_w + r_w)/n_w r_w$ for the nongray radiative case. But $(s + r) \simeq 1.5$ for the window gray case, while $(s_w + r_w)/r_w \simeq 2$ for the nongray case. Thus, there is a slight tendency for steeper temperature gradients for nongray than for gray radiative equilibrium. In convective equilibrium—the case holding for the deep atmosphere of the terrestrial planets—the temperature gradient is independent of whether the atmosphere is gray. By contrast T increases with $\bar{\tau}$ faster in the gray than in the nongray convective case, as we have seen.

As a specific application, we consider the terrestrial greenhouse. The mean surface temperature of the Earth is about 10°C; from current values of the bolometric albedo, the equilibrium temperature of the Earth is about 30°C cooler. The optical depth at T_e is approximately $\frac{2}{3}$, or $2/(2+r)$, as suggested by Eq. (12). Taking $\gamma \simeq 1.4$, $\eta \simeq 1.6$, $\lambda_w \simeq 10\ \mu$ and $r_w + s_w$ around 1.5, we find $n \simeq 5.5$, $\Xi_w \simeq 0.7$, and $\bar{\tau} \simeq 2$. Considering ambiguities in the definition of $\bar{\tau}$ and in integrating over slant paths, this estimate is in rather good agreement with other calculations, e.g., the result that about 10% of the Earth's thermal emission escapes directly through the atmosphere to space (Byers, 1954, p. 307).

In the case of Venus, $T_s/T_e \simeq 3$, $\eta \simeq 1$ for the lower atmosphere, $\gamma \simeq 1.3$, $\lambda_w \simeq 8\ \mu$, and $\bar{T} \simeq 400$°K; hence, $n \simeq 4.5$, and Ξ ranges from 1.0 to 0.67 for $r_w + s_w$ ranging from 1 to 1.5. The nongray adiabatic result appropriate to the problem is

$$100 \leqslant \tau_{ad} \leqslant 500.$$

Had we assumed pressure-induced dipole transitions ($r_w + s_w \simeq 2$) as the principal source of opacity, we would have found $\tau_{ad} \simeq 1300$; pressure-induced opacity sources make relatively inefficient greenhouses. These convective greenhouses can be compared with corresponding values expected for radiative equilibrium: $\tau_{rad} \simeq 100$ for the window gray case, and $\tau_{rad} \simeq 9$ for the nongray case. This last value is close to that originally estimated for Venus, assuming radiative equilibrium, some years ago (Sagan, 1960). In that paper, it was pointed out that permitted transitions of CO_2 could not supply the required opacity, and that if we are restricted to permitted transitions (see above), absorption by an abundant asymmetric-top molecule was indicated, particularly to provide absorption at long wavelengths. It was then argued that water was the most likely such molecule. In a companion paper Pollack (1969*a*) has recalculated the nongray CO_2–H_2O greenhouse effect for Venus, and confirms that plausible quantities of carbon dioxide and water vapor can account for the high surface temperatures.

Similar calculations could, in principle, be performed for the atmospheres of the Jovian planets. For a given $\bar{\tau}$, higher temperatures are reached in radiative than in convective atmospheres. With the deep radiative regions suggested in the present paper for the outer Jovian planets, rather high temperatures should be reached not very far below the visible clouds. We have no estimates at the present time of $\bar{\tau}$ for these planets, or of a surface temperature, if, indeed, such planets have surfaces in the usual sense. Nevertheless, whether the atmospheres are in convective or in radiative equilibrium, it seems likely that temperatures which are clement by terrestrial standards are to be found below the visible cloudtops of all the Jovian planets (see Sagan, 1961). At sufficiently great depths—say, hundreds of atmospheres pressure—Rayleigh scattering alone will reduce the incident solar flux to negligible quantities. If the clouds absorb in the visible and near-infrared, as the colors on Jupiter imply, the reduction to negligible flux will occur at lower pressures. When the greenhouse becomes sufficiently "dirty" the atmospheric structure will approach isothermality, for either convective or radiative atmospheres. The isothermal regime will tend to continue towards great depths, until the effects of the intrinsic heat of the planet—ultimately due to gravitational potential energy and radioactive decay—begin to make themselves felt.

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References

Bartko, F., and Hanel, R. (1968). Nongrey equilibrium temperature distributions above the clouds of Venus. *Astrophys. J.* **151**, 365.

Burch, D. E., Gryvnak, D., Singleton, E. B., France, W. L., and Williams, D. (1962). "Infrared Absorption by Carbon Dioxide, Water Vapor and Minor Atmospheric Constituents." Report 62-698, Air Force Cambridge Research Laboratory, Bedford, Massachusetts.

Byers, H. R. (1954). The atmosphere up to thirty kilometers. *In* "The Earth as a Planet" (G. P. Kuiper, ed.), Chap. 7. Univ. of Chicago Press, Chicago, Illinois.

Goody, R. M. (1964). "Atmospheric Radiation", Vol. I, "Theoretical Basis." Clarendon Press, Oxford, England.

Howard, T. N., Burch, T. E., and Williams, D. (1956). Infrared transmission of synthetic atmospheres. *J. Opt. Soc. Am.* **46**, 189–190, 237–245, 334–338, 452–455.

Irvine, W. M. (1968). Multiple scattering by large particles. II. Optically thick layers. *Astrophys. J.* **152**, 823.

Jastrow, R., and Rasool, S. I. (1963). Radiative transfer in the atmospheres of Venus and Mars. *In* "Space Research. III" (W. Priester, ed.), Proceedings of the COSPAR symposium, p. 1036. North-Holland Publ. Co., Amsterdam, Holland.

Pollack, J. B. (1969*a*). A nongray CO_2–H_2O greenhouse model of Venus. *Icarus* **10**, 314.

Pollack, J. B. (1969*b*). Temperature structure of nongray planetary atmospheres. *Icarus* **10**, 301.

Pollack, J. B., and Sagan, C. (1965). The infrared limb-darkening of Venus. *J. Geophys. Res.* **70**, 4403.

Sagan, C. (1960). The radiation balance of Venus. *Jet Propulsion Lab., Calif. Inst. Technol., Tech. Rep.* **32–34**.

Sagan, C. (1961). On the origin and planetary distribution of life. *Radiation Res.* **15**, 174.

Sagan, C. (1962). Structure of the lower atmosphere of Venus. *Icarus* **1**, 151.

Sagan, C., and Pollack, J. B. (1969). On the structure of the Venus atmosphere. *Icarus*, **10** 274.

Sobolev, V. V. (1963). "A Treatise on Radiative Transfer." D. Van Nostrand, Princeton, New Jersey.

Trafton, L. M. (1967). Model atmospheres of the major planets. *Astrophys. J.* **147**, 765.

Van de Hulst, H. C. (1957). "Light Scattering by Small Particles." Wiley, New York.

Veverka, J., and Sagan, C. (1969). The infrared limb-darkening of Jupiter. To be published.

Wattson, R. B. (1968). Investigation of a grey optically-thick planetary atmosphere in convective-radiation equilibrium. *Astrophys. J.* **154**, 987.

Wildt, R. (1966). The greenhouse effect in a gray planetary atmosphere. *Icarus* **5**, 24.